Essays on Real Options with Managerial Controls and Optimal Capital Structure

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PhD DISSERTATION

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Abstract

In this study we extend the real options framework to include managerial interacting learning and control options i.e. actions that are expected to enhance value but have uncertain outcome. We allow the history of decisions to affect the impact (expected profitability, variance and cost) of future decisions and we show the optimal timing and optimal decision regions by also allowing early exercise and abandonment options. This framework allows the study of the effect on the value of firm's investment opportunities of options to change the distribution of future payoffs through for example marketing research and advertisement (or product redesign or repositioning), basic research or exploration actions and product attribute or quality enhancing actions. The framework also allows the analysis of optimal timing of such actions, optimal timing of introduction of pilot projects, early development of the complete project and abandonment options. We provide analytic compound formulas for sequential options with embedded control and learning actions under the assumption that project value follows either diffusion or a jump diffusion process. We also extend the model to complex multistage problems with path dependent actions, by developing a numerical lattice based model. We illustrate the importance of this theoretical framework through applications in R&D projects and the valuation of new products. Building on recent theories based on the contingent claim approach we also model the determination of optimal investment policy (with respect to timing of investment) and the simultaneous determination of optimal capital structure and we study the impact of debt financing constraints on firm value, the optimal timing of investment and other important variables like the credit spreads. We also explore the social welfare implications of financing constraints. Finally we incorporate managerial learning and control actions in this more general framework that can be interpreted as pre-investment risky growth options (e.g. R&D, or pilot projects) using the methodology of this study and we investigate their effect on firm value and other important variables like leverage, equity and the credit spreads.

1. Introduction

The real options theory has been mostly associated with investment appraisal and firm valuation by taking into consideration managerial flexibility to react under uncertainty. The option to defer investment for the potential of favourable conditions in the value of the investment, the option to abandon when conditions become unfavourable (thus limiting losses), the option to expand or contract capacity depending on demand conditions, are only few examples of the valuable flexibility that exists in managerial hands and is analyzed in the real options literature. Conceptually the real options approach can be seen as an Expanded version of the NPV approach (see Trigeorgis, 1996) that takes into account managerial flexibility (and also possible flexibility under competitive interactions)- the traditional Net Present Value (NPV) approach can be seen as a special case of this more general approach where this flexibility value zero (see also Dixit and Pindyck, 1994, for an overview of the theoretical framework and applications and Copeland and Antikarov, 2001, for a more applied approach to real options). The purpose of this study is to add another dimension in these problems, namely introduce managerial actions to improve the value of a project (or the firm) that have action specific uncertainty beyond that introduced by exogenous competition. For example the price of the a particular product might be out of the control of management and move stochastically over time; on the other hand the firm may engage in actions to improve its market share (quantity of sales) or efforts to reduce it's costs. Some examples include the introduction of new products, the improvement of attributes or quality enhancement of a current product, the adoption of technological innovations in operations (e.g. new software or operational processes). The exercise of these plans/options would be made at a cost, will be targeting an increase in demand but may have a random outcome (this uncertainty exists beyond exogenous uncertainty). A firm that does not take these actions will simply abandon a potentially valuable option to improve revenues. The value of these actions are due to the expected improvements in cash flows but also from the additional (resolved) volatility that combined with the option to invest and other managerial flexibility (e.g. abandonment option) further enhances flexibility value. Our approach has similarities with Pindyck (1993) who presents a framework for the analysis of options where costs are driven by two components regarding uncertainty: exogenous uncertainty (e.g. prices of materials or inputs into a process) modeled as a continuous diffusion process and additional uncertainty that is purposefully (and at a cost) being resolved by the firm's management. The latter component of uncertainty can be used to model technical uncertainty (see also Schwartz and Moon, 2000 for further applications of this framework in new drug development). In this study we explore an alternative approach where actions are taken in discrete points in time and have an impulse outcome. Our analysis in the first part of the thesis focuses on the analysis of these "managerial control" actions with random outcome on the value side and allows for positive expected impact on value (in contrast to Pindyck, 1993 where the expected impact is zero) using the framework of Martzoukos (2000). Childs and Triantis (1999) also consider a situation where completion of a research project resolves uncertainty but does not affect the expected impact. We specifically investigate path-dependency in the characteristics (average impact, volatility and costs) of these managerial control actions which adds economically and methodologically interesting issues. This feature is important, since in many situations encountered in practice the sequence of actions affects the outcome; an interesting example is learning-by-doing where the impact of a follow-up investment may be higher than the previous or the costs may be less. Path-dependency introduces methodological challenges that once met allow for other interesting issues that are also analyzed in this study like time-to-learn (lags in realization of impact and volatility of controls) and convex adjustment costs (abandonment values with path-dependent recovery amounts).

Another dimension that is investigated in this study is that of learning actions. Investment in pure learning actions like research, experimentation, and marketing research is difficult to be motivated and explained in a real option setting. This is because learning is often thought to decrease volatility while option values are increasing in volatility-why would then a manager ever invest in learning actions? Learning is essential when real assets may exhibit specific uncertainty (noise) due to incomplete markets or unique physical, contractual locational characteristics (this motivation is provided in Childs et al., 2001 and Childs et.al., 2002); see also Martzoukos 2003, and Martzoukos and Trigeorgis, (2001). Effectively, under these cases the firm may find it valuable to resolve more uncertainty regarding the true value of the project. Our model is consistent with Childs et. al. (2001) and (2002) filtering approach and is extended in a multi-stage setting with path-dependency. We make explicit and we distinguish between options to enhance value with random outcome- for example quality enhancement, R&D, or advertisement-and learning actions like pure research, marketing or experimentation. The methodological treatment of learning and managerial control actions is similar thus keeping the problem tractable and practical for implementation. Our results elaborate and provide further insights on the importance of information acquisition an issue that is absent in most real option models. We show how learning may improve option values and how it affects the probability of development. The investigation of learning and control actions in a unified framework is demonstrated in the second part of the thesis where analytic compound option formulas with embedded learning or control actions, early exercise, abandonment and path-dependency are provided. The compound option formula of Geske (1979) and the extendible option of Longstaff (1990) are special cases of this more general formulation. Competitive interactions that may limit or sometimes destroy some forms of flexibility are often for simplicity modelled using a competitive erosion parameter- in analogy to the dividend yield modelling of financial options. This modelling approach cannot handle cases of random arriving changes that can be accommodated in the form of jumps. For this reason we furthermore demonstrate the jump-diffusion case implementation with managerial control and learning actions. In general, our setting (with managerial control and learning) captures the notions described in Weitzman and Roberts (1981) while also maintaining the correct adjustment for risk in the real options framework. The learning and the control actions are induced endogenously by the firm by optimally weighing the expected benefits (in terms of additional option value) compared with the additional costs. Our assumption throughout the study regarding the risk of controls or learning volatility is that is uncorrelated with market risk and is not priced (alternatively a risk-neutral agent approach could be implemented like in Childs et al, 2001).

An additional important feature that is only recently being explored in the literature is the integration of investment options and financing decisions in a unified framework. Financing choices may affect the value of investment opportunities and investment timing. For example, the benefits of debt financing due to tax advantages arising from tax deductibility of interest payments may be exposed to the risk of being lost if the firm waits due to unfavourable movements in demand; this in general pushes investment earlier. Including debt into the picture also brings issues of optimal capital structure, default risk, and the determination of credit spreads and the selection of optimal default trigger by equity holders. We build on Mauer and Sarkar (2005) (that in turn has extended Leland, 1994) to include optimal capital structure and optimal investment timing in a unified framework and we analyze the important issue of debt financing constraints. Debt holders may reduce the provision of credit due to moral hazard or asymmetric information (see Jensen and Meckling, 1976 and Myers and Majluf, 1984, for discussion of these issues). Asymmetric information has also been provided as a reason justifying why the suppliers of credit engage in credit rationing (see Fazzari et al., 1988, Stiglitz and Weiss, 1981). Boyle and Guthrie (2003) analyze the effect of financing constraints on investment policy but their model focuses on liquidity/cash constraints while ours on constraints on the level of debt financing. Furthermore, our model shares the good characteristics of the Mauer and Sarkar (2005) model in that it explicitly models optimal capital structure decisions (with the added constraint in our setting), the tax benefits of debt, credit spread determination and the optimal default policy of the firm.

The thesis consists of three parts. In the first part, we maintain the traditional real option setting (with the options to react under uncertainty incorporated in prior literature), and we introduce active managerial actions to control (enhance) cash flows, albeit with random outcome. The first part of the thesis extends prior literature results as follows:

- A multi-stage investment setting with path dependency between value-enhancing managerial control actions with random outcome
- Delays in the realization of managerial control's impact (time-to-learn)
- Accelerated versus sequential investment policies, learning-by-doing and decreasing marginal reversibility of capital invested (Convexity in adjustment costs)
- Applications in new product development and innovation adoption and discussion of implications

In the second part we extend the analysis of the first chapter and do the following:

- Incorporate managerial actions to learn besides option to enhance
- Extend stochastic process dynamics for project value that accommodates jumps (discontinuities) coming from intense competition, political, or regulatory reasons

- Provide analytic solutions for sequential (compound) options with embedded learning and value-enhancing controls and exogenous jumps and describe the factors affecting the determination of decision regions
- Discuss numerical applications in new product development (including pilot projects, learning actions before development).

In the third part of the thesis, we build on recent theories of the capital structure that use the contingent claim approach and provide a natural extension of the real option theory developed in the first two parts to provide for the potential of debt financing. This new setting provides the environment for the analysis of new interesting issues highlighted below:

- Simultaneous determination of optimal investment and capital structure decisions under exogenous uncertainty
- Discussion of the factors affecting the trade-off between investment and financing flexibility
- The valuation of corporate securities, equity and debt, the determination of credit spreads and investment and default triggers
- The effect of financing constraints on debt on firm value, the investment and default triggers, the credit spreads and the values of equity
- The effect of managerial control actions with random outcome on firm value, the value of corporate securities, the credit spreads and investment and default triggers (also simultaneously considering the effect of financing constraints on debt)

Each chapter in the thesis separately provides all the relevant literature review, the models, the main findings and contributions and applications.

References

Boyle, Glenn W., and Graeme A. Guthrie. (2003). "Investment, Uncertainty, and Liquidity," Journal of Finance 58, 2143-2166.

Childs, Paul D., and Alexander J.Triantis. (1999). "Dynamic R&D Investment Policies," Management Science 45, 1359-1377.

Childs, Paul D., Ott, Steven H., and Timothy J. Riddiough, (2001). "Valuation and Information Acquisition Policy for Claims Written on Noisy Assets," Financial Management 30, 45-75.

Childs, Paul D., Ott, Steven H., and Timothy J. Riddiough. (2002). "Optimal Valuation of Claims on Noisy Assets: Theory and an Application," Real Estate Economics 30, 415-444.

Copeland, Thomas E., and Vladimir Antikarov. (2001). "Real Options: A Practitioner's Guide. W.W. Norton & Company; 1st edition

Dixit, Avinash , and Robert Pindyck. (1994). Investment under Uncertainty. Princeton, New Jersey: Princeton University Press.

Fazzari, Steven M., Glenn R. Hubbard, and Bruce C. Petersen. (1988). "Financing Constraints and Corporate Investment," Brookings Papers on Economic Activity 1, 141-195.

Geske, Robert (1979). "The Valuation of Compound Options" Journal of Financial Economics 7, 63-81.

Jensen, Michael C. and William H. Meckling, 'Theory of the Firm: Managerial Behavior, Agency Costs and Ownership Structure,' Journal of Financial Economics, Vol. 3, No. 4 (1976).

Pindyck, Robert S. (1993). “Investments of Uncertain Cost,” Journal of Financial Economics 34, 53-76.

Leland, Hayne (1994). “Corporate Debt Value, Bond Covenants, and Optimal Capital Structure,” Journal of Finance 49, 1213-1252.

Longstaff, Francis A. (1990). “Pricing Options with Extendible Maturities: Analysis and Applications,” Journal of Finance, 45, 935-957.

Martzoukos, Spiros H. (2000). “Real Options with Random Controls and the Value of Learning,” Annals of Operations Research 99, 305-323.

Martzoukos, Spiros H. (2003). “Real R&D Options with Endogenous and Exogenous Learning,” In D. A. Paxson (ed.) Real R&D Options, 111-129. Butterworth-Heinemann Quantitative Finance Series.

Martzoukos, Spiros, and Lenos Trigeorgis. (2001). “Resolving a Real Options Paradox with Incomplete Information: After all, Why Learn?,” Working Paper, University of Cyprus.

Mauer, David C., and Sudipto Sarkar. (2005). “Real Options, Agency Conflicts, and Optimal Capital Structure,” Journal of Banking and Finance 26, 1405-1428.

Myers, Steward C. and Nicholas Majluf. (1984). “Corporate Financing and Investment Decisions when Firms have Information that Investors do not have” Journal of Financial Economics 13, 187-221.

Schwartz, Eduardo S., and Mark Moon. (2000). “Evaluating Research and Development Investments.” In M. J. Brennan and L. Trigeorgis (eds.), Project Flexibility, Agency and Competition. New Developments in the Theory and Applications of Real Options. Oxford University Press.

Stiglitz, Joseph and Andrew M. Weiss. (1981). "Credit Rationing in Markets with Imperfect Information," American Economic Review 71, 393-410.

Trigeorgis, Lenos (1996). Real Options: Managerial Flexibility and Strategy in Resource Allocation. Cambridge: The MIT Press.

Weitzman, Martin L., and Kevin Roberts. (1981). "Funding Criteria for Research, Development, and Exploration Projects," Econometrica 49,1261-1288.

2. Real R&D Options with Time-to-learn and Learning-by-doing

Abstract

R&D actions are implemented as optional, costly and interacting control actions expected to enhance value but with uncertain outcome. We examine the interesting issues of the optimal timing of R&D, the impact of lags in the realization of the R&D outcome, and the choice between accelerated versus staged (sequential) R&D. These issues are also especially interesting since the history of decisions affects future decisions and the distributions of asset prices and induces path-dependency. We show that the existence of optional R&D efforts enhances the investment option value significantly. The impact of a dividend-like payout rate or of project volatility on optimal R&D decisions may be different with R&D timing flexibility than without. The attractiveness of sequential strategies is enhanced in the presence of learning-by-doing and decreasing marginal reversibility of capital effects.

2.1. Introduction

We adopt a real options framework (see Dixit and Pindyck, 1994, Trigeorgis, 1996) to model optional and costly R&D actions. The real options literature has analyzed the impact of uncertainty in making optimal investment timing decisions. Our model helps expand the insights derived in evaluating such R&D efforts. We analyze a firm which can select among a number of optional actions: defer investment in costly value-enhancing control actions (R&D), invest in a control action (or select from mutually exclusive ones), invest in several control actions sequentially, develop the project early (exercise an investment option), or abandon it for a resale value (see Myers and Majd, 1990). Our approach can be applied in cases where firms, before making the final capital-intensive investment decision to bring a product to market, can invest in efforts to enhance its market appeal or lower the cost of production. Consider, for example, a leading car manufacturer contemplating bringing a new model to the market. Before doing so, the firm has the option to delay the commercialization phase and invest (via R&D or by adopting existing technological innovations) in improving the attributes of the initial design. Such improvements may affect the aerodynamic performance, the looks, the engine design, the brake or the suspension system, etc. Some of these improvement efforts may occur at a reduced total cost if they take place not sequentially but together in a focused (accelerated) effort. Other efforts may occur more effectively if they take place sequentially, due to the firm´s learning during the early stages. Such actions can be seen as R&D investments (or adoption of existing innovations) intended to enhance the value of existing investment opportunities. The outcome of such improvement efforts is uncertain, and in some cases the new product may even prove to be less valuable than the original one.[1]

An early treatment of controls with random outcome is Korn (1997). Impulse-type random controls were introduced in real options by Martzoukos (2000) (see also Martzoukos, 2003). These authors treated independent controls available at predetermined times only, while we focus on the optimal timing of interacting controls. Control interactions introduce path-dependency since the history of decisions affects future decisions and the distribution of asset

[1] Such may be the case, for example, in the redesign of a product with the intention to strengthen the price and/or market share, potentially with a negative customer response or reduced productivity (see Brynjolfsson and Hitt, 2000).

prices. Path-dependencies are important in new product development or in innovation adoption. Efforts to improve the features of software, for example, may have lower cost and higher expected impact if they follow earlier R&D to introduce basic features and functionality to the software. Similarly, the effect of a new innovation may be different if it follows a prior innovation, reflecting learning-by-doing or other synergies. These additional features via path-dependencies can substantially affect investment option values and optimal investment thresholds.

R&D actions were previously studied in a statistical framework by Roberts and Weitzman (1981), but without proper adjustment for risk (as in a real options framework). Pennings and Lint (1997) present a real options model for the valuation of an R&D project where the arrival time of new information regarding profitability is random and exogenous. Schwartz and Moon (2000) discuss information revelation regarding the level of costs in the case of development of a new drug in a compound-option framework without path-dependency. They concentrate on the effect of technical and input-cost uncertainty and use assumptions similar to those in Pindyck (1993) that costs follow a controlled diffusion process. Schwartz and Gorostiza (2000) value information technology in development and acquisition projects. Childs and Triantis (1999) consider a situation where completion of a research project resolves uncertainty (learning-by-doing). They focus on the choice between accelerated versus staged (sequential) R&D, but assume completion of R&D is mandatory before any realization of profits, whereas in our case this is optional. Effectively, they focus on the sequential development process of a new project whereas we focus on optional R&D efforts to enhance the value of an existing investment opportunity. Grenadier and Weiss (1997) analyze alternative innovation adoption strategies for firms confronted with a sequence of randomly arriving innovations. Innovations in their setting arrive at random times and the firm can follow different strategies regarding their adoption. They focus on the uncertainty regarding the arrival of new technologies while in our framework the firm has an option to adopt an existing innovation with unknown impact and it must select the best alternative at an optimal timing. We also allow for a multistage setting with other potential strategies (like abandonment or early exercise of the investment option), and incorporate path-dependencies between actions[2]. R&D investments may also take time to complete, an effect we call "time-to-learn" due to lag in the realization of the control's impact (e.g., due to delayed response of consumers to the new features or time needed to build them). Lags in the development process of construction projects were previously analyzed by Majd and Pindyck (1987), who used the term "time-to-build" (see also Bar-Ilan, Sulem, and Zanello, 2002).

We consider as a benchmark the case of a single R&D action without timing flexibility in its activation. Then we study numerically the general case with optimal timing of R&D, and the impact that lags in the realization of the research outcome have on investment option value. The presence of optional R&D actions can significantly enhance investment option value and can affect the critical decision thresholds. We investigate the dividend-like payout rate and asset volatility effects on project value and optimal thresholds. Results differ significantly with timing flexibility for the control actions than without. We also show that lags (time-to-learn) reduce option value, and that the sensitivity of thresholds to parameter values of the stochastic process depends on the degree of timing flexibility in controls. Finally, in a more general setting we analyze a complex situation with two potential R&D strategies: an accelerated strategy with higher cost and higher average impact, versus a (flexible) staged strategy involving two sequential control actions. Contrary to what one might expect, the sequential strategy does not dominate the accelerated strategy. The appeal of the sequential strategy is enhanced if there are learning-by-doing effects or decreasing marginal reversibility of capital invested in research.

The chapter is organized as follows. In the next section we discuss the (benchmark) special case of R&D without timing flexibility, and then describe the general framework for optimal activation and timing of control actions involving path-dependency. Section 3 presents our numerical results. The last section concludes. An Appendix provides numerical evidence concerning the accuracy and convergence of our numerical scheme.

[2] Our setting is one of active control, unlike the passive learning case studied in Majd and Pindyck (1989) where efficiencies in production accumulate simply while the firm is in operation.

2.2 A general framework with control actions and path-dependency

We assume that project returns follow a risk-neutral process of the form:

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dz_t + k_j dq_j \,. \tag{1}$$

In the above equation r is the riskless rate, δ is a dividend-like payout rate representing an opportunity cost of waiting to invest, and σ is the standard deviation of the rate of change in project value. The term dz_t is an increment to a standard Wiener process describing the uncertainty of project value in the absence of control actions. Equation (1) is similar to that of jump-diffusion, but this equation does not refer to jump-diffusion. The similarity exists because the control is of an impulse-type, multiplicative nature (with random outcome). We denote by k_j the impact on project returns of control action j, specified by its cost, I_C, and the distribution of $1+k_j$. Counter dq_j takes the value one if control j is activated by the decision-maker and zero if not. dq_j is a control variable, not a random variable (unlike the case of jump-diffusion). The decision maker has the option to activate controls at a cost, solving the optimization problem discussed in section 2.2. Since the system is stochastic with project value S being the primary driver, at each point in time when a control is available the level of S helps determine whether control activation is optimal or not. Other determinants of control activation are the parameter values of the stochastic process of S, the remaining time to option maturity, and the control characteristics (distribution of the impact, parameter values of the distribution, and the cost of the control). Control activation is also affected by whether there is timing flexibility in control activation, and whether other mutually-exclusive controls also exist. The cost of the control and the parameter values of its distribution may be dependent on time and previous activation of other controls.

We assume that an equilibrium model like the continuous-time CAPM (see Merton, 1973) holds and that controls have firm-specific risk which is uncorrelated with the market portfolio and is thus not priced. We use risk-neutral valuation as established in Constantinides (1978), Harrison and Pliska (1981), and Cox, Ingersoll, and Ross (1985). The dividend-like payout rate (or opportunity cost of waiting to invest) δ can be deducted from the equilibrium-required rate of return as in McDonald and Siegel (1984)[3]. Denoting the accumulated (Brownian) noise from t = 0 to T by Z_T, we can rewrite equation (1) in the following form

$$\frac{S_T}{S_0} = \exp\left[(r - \delta - \frac{\sigma^2}{2})T + \sigma Z_T \right] \prod_j (1 + k_j dq_j) . \tag{1a}$$

$\prod_j (1+k_j)$ denotes the impact of j multiplicative controls that have been activated (dq_j = 1) before time T. Equations (1-1a) imply that the underlying asset in the absence of controls is log-normally distributed. The multiplicative effect of control action $1+k_j$ is assumed to be log-normally (log$N$(.)) distributed:

$$1+k_j \sim \log N\left(\exp(\gamma_j), \exp(\gamma_j)\left(\exp(\sigma_j^2)-1\right)^{0.5} \right), \tag{2}$$

$$\ln[S_T] - \ln[S_t] = \int_0^T \left(r - \delta - 0.5\sigma^2 \right) dt + \int_0^T \sigma dZ_t + \sum_j \left(\ln(1+k_j) dq_j \right) . \tag{2a}$$

The assumption of log-normally distributed controls is convenient. It not only ensures non-negative asset values, but also conditional on control activation asset values retain their general distributional properties since the product of two log-normal distributions is log-normal. Control j is characterized by its average impact and volatility parameters, γ_j and σ_j^2 respectively, and by its cost I_C. Parameter $\gamma_j > 0$ represents efforts to enhance the value of project returns. When a control is activated, immediate return to R&D equals $(k_j S - I_C)/I_C$ and it can be either positive or negative. This return will be realized only when (and if) the final investment is made. In general we consider an investment opportunity to pay a capital cost X and realize net value $S - X$. The control problem involves efforts to enhance project value S before the final investment decision takes place. The objective is to activate available controls optimally by choosing the best among several alternative controls at an optimal time taking into account potential path-dependencies in the parameter values of controls. The parameter values (including the cost) of these control actions may depend on the sequence in which they are activated. Even though control actions are

[3] It may also account for competitive erosion to the project's cash flows (e.g., Childs and Triantis, 1999, Trigeorgis, 1996, ch.9, and Trigeorgis, 1991).

expected to enhance project value S (at a cost), the final outcome is uncertain. Of course, we expect that the availability of such "optional" actions will enhance real option value.

The parameter values of controls may depend on activation of previous controls, but not on the exact value outcome of the activation, neither on the level of S. The outcome of previous activation, however, will influence future optimal actions since optimal decisions depend on the level of asset value S at each time. Due to the numerical nature of the solution it is feasible to allow controls have characteristics (cost and parameter values of the distribution of its impact) that are functions of the level of state-variable S. The assumption that the outcome of a control is independent of the outcome of other previously-activated controls is reasonable when each control in the sequence treats different aspects of the problem under consideration. In some situations (not treated in this paper) this may not be the case.

2.2.1. Simple case (without timing flexibility): A benchmark

We first consider a special (single-period) case without flexibility in the timing of the control that has an analytic solution. This will allow us to investigate the possible action regions at $t = 0$ for different levels of project value S (where for notational convenience we drop the dependence of S on time). It will also serve as a benchmark for testing the accuracy of our numerical results. The benchmark relates to the simple case of a European investment (call) option *conditional* on activation of a single control (at a cost I_C), without timing flexibility. The solution to the option value in this special case becomes:

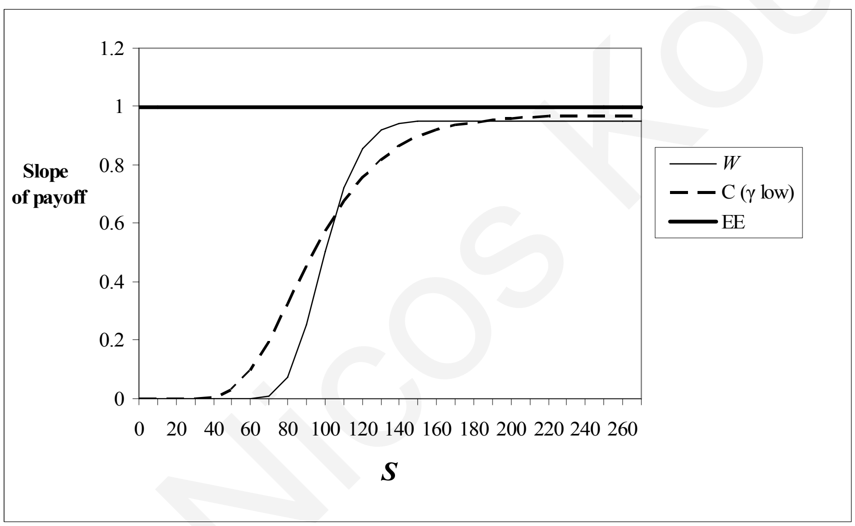
$$V^C = Se^{-\delta T+\gamma_1} N(d_1) - Xe^{-rT} N(d_2) - I_C \qquad (3)$$

with $$d_1 = \frac{\ln(S/X) + (r-\delta)T + \gamma_1 + 0.5\sigma^2 T + 0.5\sigma_1^2}{[\sigma^2 T + \sigma_1^2]^{1/2}},\ d_2 = d_1 - [\sigma^2 T + \sigma_1^2]^{1/2},$$

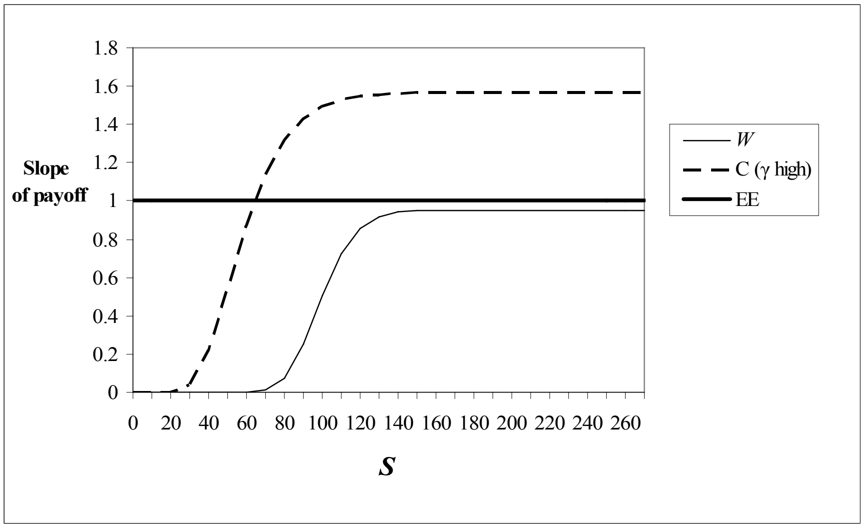
where I_C, γ_1, and σ_1^2 are, respectively, the cost, average impact, and variance of the impact of the control action. If the control were costless ($I_C = 0$), it would always increase option value as long as the average impact is positive ($\gamma_1 > 0$) for a call option. In this simple case the firm owns (has monopoly power over) an investment option, and can (at t = 0) take any of the following actions. First, it can simply wait (*W*) and keep the investment option alive (but sacrifice the embedded optional control). In this case the value of waiting V^W equals the simple Black and Scholes (1973) solution (V^C with $\gamma_1 = \sigma_1 = 0$ and $I_C = 0$). Second, it can take the single control action (*C*) and get V^C. Finally, it can exercise early (EE) the investment option for an immediate value $V^{EE} = S - X$. Thus, the optimal value V^* is the best among the three possible alternatives V^C, V^W, and V^{EE}.

Figure 1. Payoff function slopes and optimal decisions for a European option involving control and possible early exercise at *t* = 0

Panel A



Panel B



Notes: We illustrate typical examples for the determination of optimal decision regions for a single-stage investment problem where the firm can either invest early (*EE*) at $t = 0$, Wait (*W*), or activate a managerial control (*C*) and then decide whether to invest at $t = T$. We examine the slope of the payoff functions for each strategy (partial derivatives of option value with respect to *S*) using the analytic formulas in equations (3) and (4). The average impact of control is denoted by γ.

To examine the optimal decisions among these three action choices as a function of the level of the underlying project value S, it is useful to compare (see Figure 1) the slopes of the respective claim values from the following equations:

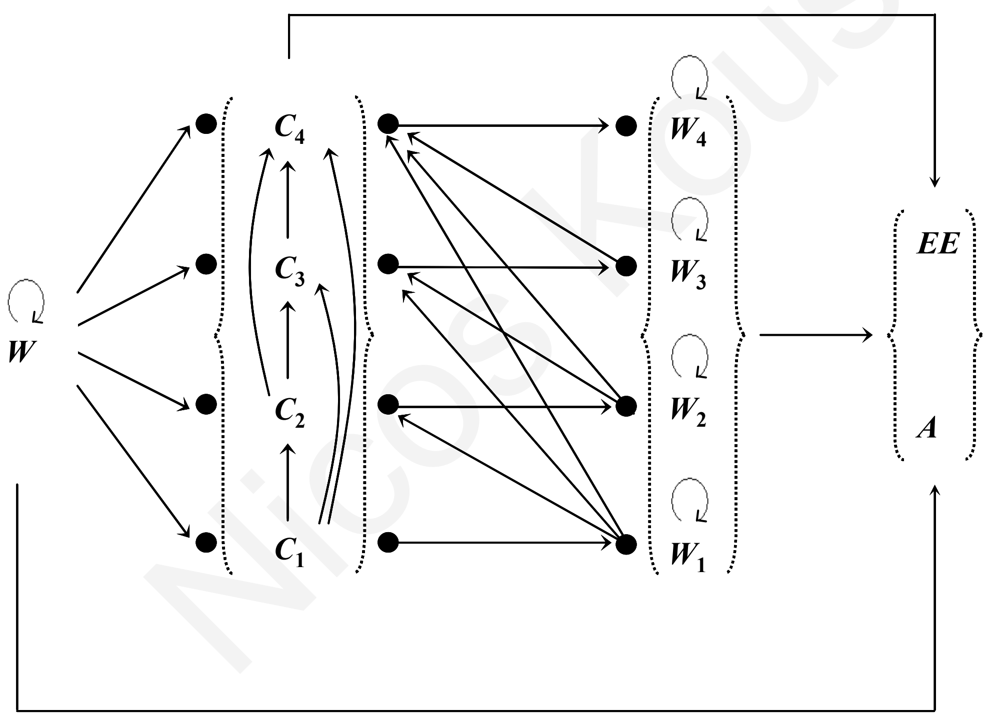
$$\frac{\partial V^{W}}{\partial S} = e^{-\delta T} N(d_1) \underset{S>>X}{\rightarrow} e^{-\delta T}, \frac{\partial V^{C}}{\partial S} = e^{-\delta T+\gamma} N(d_1) \underset{S>>X}{\rightarrow} e^{-\delta T+\gamma}, \frac{\partial V^{EE}}{\partial S} = 1, \qquad (4)$$

where *S*>>*X* means *S* sufficiently higher than *X* (so that $N(d_1) \rightarrow 1$). Panel A in Figure 1 shows the case where for very large project values (*S*>>*X*) the slope in case of early exercise remains always above the other slopes so the optimal decision is to exercise early (*EE*). Panel B in Figure 1 shows the case where the slope with control activation remains the highest and for high values of *S*, activating the control (*C*) is the optimal decision. For low values of *S*, neither the costly control nor the early exercise decision would have much value, and thus the wait (*W*) decision will prevail. At higher values of *S* (assuming the average control impact γ_1 is positive) the increase in value is higher for control activation, and after some critical threshold the wait decision (W) gets dominated by the control decision (C). But at certain values of $S > X$, the decision can also be early exercise (EE). Thus, in Panel B (at t = 0) for increasingly higher project values S the regions of optimal decisions can be $\{W,C,EE,C\}$, $\{W,EE,C\}$ or $\{W,C\}$. If the slopes of the (W) and (C) cases were to cross, we could have $\{W,C,W,EE,C\}$, $\{W,C,W,C\}$ or $\{W,EE,C\}$, etc. In the first panel, the order could be $\{W,C,W,C,EE\}$, $\{W,C,EE\}$, or $\{W,EE\}$, etc. For each Panel some regions may vanish and the regions that actually prevail would also depend on other option parameter values like project volatility and the opportunity cost of waiting to invest. The regions of optimal decisions at t = 0 derive from the simple case where an analytic solution exists (which in the presence of control timing flexibility is equivalent to the decision stage just before option maturity). They can provide insights for the more general case where the control(s) can be activated at an optimal time. The exact level of thresholds that separate the various regions as well as the actual regions that result may also depend on the additional flexibility to time the control activation, the number and order of path-dependent controls, etc. These can be analyzed with precision only as part of a numerical investigation. We do so right after we discuss in the next subsection the more general problem, structure and solution methodology.

2.2.2. Multi-stage decisions with optimal timing of path-dependent R&D actions

This section discusses our more general framework. Consider the managerial investment decision problem with n_c control actions shown in Figure 2 (with $n_c = 4$ for illustration purposes). The firm has an option to invest an amount X to obtain project value S. It can also invest in n_c control actions to improve the level of cash flows (or alternatively to reduce costs) before the project development/commercialization decision. This (optional and costly) investment in controls may occur in the context of mutually exclusive actions or sequences of actions involving path-dependency. The information regarding the firm's actions at each decision point t is captured by the operating mode m_t at that time. In general, there is a starting mode reflecting the decision to wait (*W*) before project development or any control decisions are made. There are n_c possible controls $\{C_1, C_2, \ldots, C_{n_c}\}$ and n_c intermediate wait modes between control actions $\{W_1, W_2, \ldots, W_{n_c}\}$. Finally, there are two terminal boundary conditions: an early exercise of the investment option involving a development/commercialization (*EE*) decision, and an abandonment mode for capital recovery (*A*). Modes {*EE*, *A*} are absorbing states. In early exercise mode (*EE*) the firm obtains $S - X$, while in mode (*A*) the firm recovers a percentage α of total investment *TC* in prior control actions[4].

Figure 2. A general decision framework with path-dependent R&D control actions



Notes: The firm starts in a wait (*W*) mode and has the option to defer investment in controls, or invest in controls (C_1)-(C_4) either sequentially or by skipping some control actions. After investing in a control action the firm can also move to wait (W_1)-(W_4) modes, invest in other remaining control actions, exercise the early investment option (*EE*) or abandon (*A*) the project for a recovery amount. The modes (*EE*) and (*A*) are accessible at any time.

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[4] This recovery factor is usually below 100% depending on recovery (the extent of partial reversibility) of capital, but sometimes it may be above 100% due to accumulated know-how, sale of patents, etc.

We divide the time to option maturity T into n_s equally-spaced decision points, with $n_s = \{1,2,3...\}$. Thus $t = 0, \frac{T}{n_s}, \frac{2T}{n_s}, ..., \frac{(n_s - 1)T}{n_s}$ represent the corresponding action times with $n_s + 1$ being the terminal decision point at time *T*. At maturity *T* the firm has a last chance to decide whether to develop or abandon the project. The effect of the *j*th control action, $1 + k_j$, is assumed to be log-normally distributed with average impact γ_j and volatility σ_j. In the more general specification $\gamma(h, j)$ and $\sigma(h, j)$ are conditional on the previous state *h* since there is path-dependency between actions; we use this notation for simplicity, although the parameter values may depend not only on the previous action but also on a whole sequence of actions. Following activation of control action *j* at time *t*, $j \in \{C_1, C_2, ..., C_{n_c}\}$, and conditional on the previous action *h*, project log-returns are normally distributed:

$$\ln\left(\frac{S_{t+\Delta t}}{S_t} \mid h, j\right) \sim N\left((r - \delta - \frac{1}{2}\sigma^2)\Delta t + \gamma(h, j), \sigma^2 \Delta t + \sigma^2(h, j)\right). \tag{5}$$

If no control is activated, $j \in \{W, W_1, W_2, ..., W_{n_c}\}$, the terms involving the average impact $\gamma(h, j)$ and the volatility of the control´s impact $\sigma^2(h, j)$ in equation (5) vanish.

The average impact and volatility of controls, and their cost, is determined by the sequence (path) in which the controls are being activated. For example, a particular control C_2 may represent an expensive new design. This new design alone may have a different average impact (if it is activated directly from *W*), $\gamma(W,C_2)$, than if it follows another recently introduced similar design C_1, $\gamma(C_1,C_2)$. The cost of each control action j may also be path-dependent. For example, suppose C_A is an accelerated control strategy of high impact with cost $I(W,C_A)$. If $\{C_1,C_2\}$ is a sequential control strategy, each individual control with half the impact of the accelerated strategy, total costs may differ from $I(W,C_A)$. Pre-specified switching matrices provide the parameter values (average impact, volatility, and cost) of controls for every feasible decision sequence (transition). These switching matrices must be economically (or logically) consistent. If, for example, C_A and the sequence $\{C_1, C_2\}$ are mutually exclusive alternatives, we should compare $I(W,C_A)$ with $I(W,C_1) + I(C_1,C_2)$. If $I(W,C_A) > I(W,C_1) + I(C_1,C_2)$ cost efficiencies that favor the sequential strategy may be achieved due to learning-by-doing. If $I(W,C_A) < I(W,C_1) + I(C_1,C_2)$, there may be scale efficiencies.

When path-dependency is involved we can only define payoffs conditional on prior decisions and search for the optimal sequence of actions. We assume decisions can be revised at periodic intervals Δt. $V^{m_t}(.)$ is the payoff under decision m_t. This payoff is a function of the level of project value S at that point determined by the path of actions followed, including the switching costs $I(h, j)$, the average impact of controls, the development cost X, the recovery rate α in case of abandonment, etc. Superset M includes all information about admissible actions, action sequences, and the parameter values of controls in each case. At each time t, stochastic subset M_t^- describes the history of actions up to time t, and stochastic subset M_t^+ defines the remaining admissible actions and relevant parameter values. More specifically, the problem of finding the optimal value function $V^*(.)$ involves maximizing $V^{m_t}(.)$ by choosing the optimal action at t given the past decisions:

$$V^*(S_t, t \mid M, M_t^+, M_t^-) = \max_{M_t^+}\{V^{m_t}\}. \tag{6}$$

We differentiate the following cases.

For $m_t \in \{C_1, C_2, ..., C_{n_c}\}$: $V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) =$

$$e^{(-r\Delta t)} E_t\left[V^*(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_{t+\Delta t}^+, M_{t+\Delta t}^-)\right] - I(m_{t-\Delta t}, m_t) \ . \tag{6a}$$

For $m_t \in \{EE\}$: $$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) = S_t - X \ . \tag{6b}$$

For $m_t \in \{A\}$, with $TC\left(M_t^-\right)$ being the total control costs paid until t:

$$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) = \alpha TC\left(M_t^-\right). \tag{6c}$$

For $m_t \in \{W_1, W_2, ..., W_{n_c}\}$:

$$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) = e^{(-r\Delta t)} E_t\left[V^*(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_{t+\Delta t}^+, M_{t+\Delta t}^-)\right]. \tag{6d}$$

Equations (6a-6d) incorporate various path-dependent factors like costs $I(m_{t-\Delta t}, m_t)$, average impact and volatility, early development options, and abandonment to recover a percentage of total past investment in controls *TC*. The expectation operator $E[.]$ is taken with respect to the distribution of log-returns given earlier.

At the last decision point, $n_s + 1$, we have the terminal condition:

$$V^{m_T}(S_T, T \mid M, M_T^+, M_T^-) = \max\left(S_T - X, \alpha TC\left(M_T^-\right)\right). \tag{7}$$

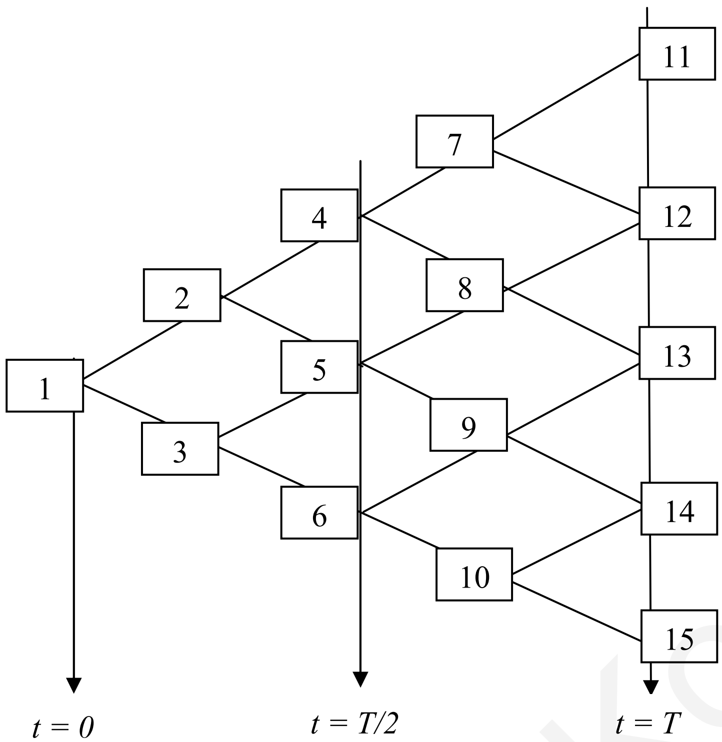
To find the optimal project value at $t = 0$, equations (6) are evaluated for each decision mode at each decision point and (after discretization) for each state of the realization of the underlying asset S. Due to path-dependencies, V^* cannot be evaluated in the usual backward dynamic programming manner. Rather, we must take into account all feasible combinations of actions and paths of project value.

We use the above framework in order to study the optimal timing of controls, and optimal activation of controls with delay in the realization of their outcome (time-to-learn). We also study the optimal activation of mutually exclusive sequences of controls, specifically the choice between an accelerated strategy with high impact and cost and a sequential strategy in two controls each having lower impact and cost.

2.2.3. Numerical lattice implementation and accuracy investigation

To evaluate the expectation operator in equations (6) we discretize the state-space using a numerical lattice scheme. This is in contrast to Martzoukos (2000) who used a finite-difference (rectangular) scheme with Markov-chain methods to solve for sequential (independent) controls; that method could easily handle sequential controls, but not path-dependency. We approximate the log-normal distribution between decision points (stages) with a binomial lattice with N_s steps (the total number of steps *N* being $n_s\, N_s$). The terminal nodes of each sub-lattice serve as starting nodes for new ones. Figure 3 illustrates a simple case where decisions are allowed at t = 0, t = *T*/2, and *t* = *T*. Controls are allowed only at *t* = 0 and t = *T*/2 (a two-stage decision problem, i.e. $n_s = 2$). The very first sub-lattice starts at node 1 with terminal nodes points 4, 5, and 6. From each of these terminal nodes starts a different sub-lattice. For example, the sub-lattice that starts from node 4 ends at nodes 11, 12, and 13. In this example, each sub-lattice is constructed with two steps ($N_s = 2$). At node 1 all decisions are examined successively. For each decision, the value is provided by the option value given by the sub-lattice. At the terminal nodes 4, 5, 6 we evaluate each possible decision and keep the one with the highest value; allowing for more in-between steps enhances accuracy. For decisions to be evaluated at nodes 4, 5, and 6 each time a new sub-lattice is constructed, and so on. Effectively this defines the state and decision space over which a forward-backward exhaustive search method is applied. Despite the fact that the figure for simplicity shows an overall recombining lattice, this will not generally be the case: each sub-lattice is a recombining one, but the overall lattice is not. Only in the absence of path-dependencies, would the approach reduce to using a simple backwards induction and the overall lattice would be a recombining one. In the absence of path dependency and with the optional controls available at predetermined times, the numerical solution would become similar to that for sequential (compound) options (see Geske, 1979).

Figure 3. Example of sub-lattice construction



Notes: In this example of sub-lattice construction, there are decision stages at t = 0, t = T/2, and at t = T; and two steps per sub-lattice. Since controls can be activated at t = 0 and at t = T/2, this is a 2-stage problem. Sub-lattices are defined by nodes (1-4-6), (4-11-13), (5-12-14), and (6-13-15).

The overall lattice may not be recombining because, the distribution of outcomes at $i + 1$ will differ, depending on the decision at decision point i. We thus employ different volatility, up and down probabilities, and up and down jumps for the sub-lattice implementation depending on the decision. The conditional volatilities $v^2(m_t, m_{t+\Delta t})$ between decision points are as follows:

For $m_t \in \{C_1, C_2, ..., C_{n_c}\}$:

$$v^2(m_t, m_{t+\Delta t}) = \sigma^2 \frac{T_s}{N_s} + \frac{\sigma^2(m_t, m_{t+\Delta t})}{N_s}. \tag{8}$$

When controls are not activated and $m_t \in \{W, W_1, ..., W_{n_c}\}$, the conditional volatilites simply reduce to the diffusion volatility $v^2(m_t, m_{t+\Delta t}) = \sigma^2 \frac{T_s}{N_s}$. Between stages, we use the following up and down project value steps:

$$u(m_t, m_{t+\Delta t}) = \exp\left(v(m_t, m_{t+\Delta t})\right), \qquad d = \frac{1}{u(m_t, m_{t+\Delta t})}.$$

The probabilities of an up and down move for the case of controls, i.e., for $m_t \in \{C_1, C_2, ..., C_{n_c}\}$, are :

$$\begin{aligned} p_u(m_t, m_{t+\Delta t}) &= \frac{\exp\left((r-\delta)\frac{T_s}{N_s} + \frac{\gamma(m_t, m_{t+\Delta t})}{N_s} \right) - d(m_t, m_{t+\Delta t})}{u(m_t, m_{t+\Delta t}) - d(m_t, m_{t+\Delta t})}, \\ p_d(m_t, m_{t+\Delta t}) &= 1 - p_u(m_t, m_{t+\Delta t}). \end{aligned} \tag{9}$$

For $m_t \in \{W, W_1, ..., W_{n_c}\}$, the term involving γ in equation (9) vanishes. The above specification allows us to evaluate the expectation in equations (6). Note that in the special case discussed earlier involving a single ($n_s = 1$) control decision at t = 0 without any timing flexibility, the whole lattice is just the single sub-lattice. In the following applications we use a discretization scheme with one step per month.

Evaluation of real (investment) claims with path-dependency is generally a complex and computationally intensive problem. Such problems rarely allow for analytic-type solutions[5]. The numerical complexity in the case of real option problems was discussed in Kulatilaka (1988) and Ritchken and Kamrad (1991). Hull and White (1993) demonstrated that it is feasible to solve American (or semi-American) option problems with path-dependency using a lattice framework[6]. Thompson (1995) analyzes a specialized contract called "take or pay" where the path-dependency was also due to past decisions. Being able to describe the path with one value and due to the specific structure of his problem he was able to solve it with backwards induction, with the additional use of an auxiliary variable that describes the possible path realizations. In our model path-dependency involves both the history of decisions and (due to the random outcome of decisions) the history of the asset price. Furthermore, we allow a rich set of alternative actions the firm can take at each point in time. Each action not only affects the future distribution of the state variable but also affects the set of remaining actions. Problems with such features of path-dependency can be solved with a backward-forward approach of exhaustive search. At each decision point a (finite) set of alternatives is investigated by taking the induced path-dependencies into account.

We investigate the accuracy of our lattice scheme using one step per month. This is quite feasible in the simple case with a single control available at t = 0 (that has an analytic solution). We have derived numerical results with several parameterizations and compare them against the known analytic solution (see Table A0 of the Appendix). In the next section we analyze more complex problems with no analytic solutions, for which the high accuracy observed earlier may not be achieved. For those problems we investigate how the solution behaves as the number of steps in the lattice implementation increases. Our calculations show (see tables A1-A3 in the appendix) that the numerical model appears to converge relatively quickly and the chosen lattice scheme with one step per month is adequate.

[5] For exceptions with analytic solutions, see Bar-Ilan and Strange (1998) and Hartman and Hendrickson (2002).

[6] They actually solved exotic derivative problems where dependency was due to the history of the asset price and evaluated American lookback and Asian options using interpolation techniques.

2.3. Applications, numerical results and discussion

In this section we apply our framework to real option problems with optional R&D actions at the pre-investment stage. We study option valuation and optimal decisions in problems with increasing level of complexity. We start with the optimal timing of a single control action, compared to the case where there is no timing flexibility. We also consider the ability to abandon operations for recovery of capital invested in controls. An additional complexity involves a possible delay in the effect of the control (time-to-learn). We then consider mutually exclusive control strategies, namely the choice between an accelerated and a sequential strategy, and examine cases involving learning-by-doing or diminishing marginal recovery rates. We provide sensitivity results to parameter values of the controls as well as to the standard option parameters, namely the project (Brownian) volatility and the opportunity cost of waiting to invest ("dividend-like payout rate") δ[7].

For our numerical results we use as base case a single control action with cost I_1 =10, average impact (on project value) γ_1 = 0.20, and volatility of impact σ_1 = 0.30. The other base case parameters are: $r = \delta = 0.05$, volatility of the diffusion process $\sigma = 0.15$, investment horizon T = 5 years, and project development cost X = 100. In the case involving timing flexibility, we assume $n_s = 5$ decision points, whereas if there is no timing flexibility $n_s = 1$. The choice of parameter values is consistent with the literature. The level of X defines the investment scale, and approximates the value of the single-project firm (before adjusting for the net present value of the investment). A cost for the control I_1 = 10 is consistent with the empirical observation that R&D expenditures are about 9% of the market value of equity (Amir, Guan, and Livne, 2004). Chan, Martin, and Kensinger (1990) find that event announcements to increase R&D expenditures have a positive and statistically significant impact on share value. Such announcements increase return variance by a factor of about 100% on the announcement day. Grabowski and Vernon (1990) document returns to Pharmaceutical R&D of around 15%-30%. As they demonstrate, the return distribution is highly skewed, with the top decile providing a

[7] Sensitivity to the dividend-like rate is important, since it may also indirectly (in a non-game theoretic framework) capture effects like value erosion due to actions of competitors, etc. (see Trigeorgis, 1991).

return around 400%-500%. Kothari, Laguerre, and Leone (2002) present empirical evidence for a positive relation between R&D expenditures and uncertainty of future benefits. They also point out that if R&D expenditures were to replace all capital expenditures, earnings variance would increase by 30-70%. Childs and Triantis (1999) use a standard deviation of 0.40 assuming it is completely attributed to R&D and that the volatility attributed to the Brownian motion is zero. Our choice of control volatility σ_1 = 0.30 thus seems plausible (if one control is activated within a year, the total volatility would equal 33.5%). For brevity we present selected numerical results, but our insights are supported by sensitivity on a much wider range of parameter values[8].

2.3.1. Optimal timing of a single R&D control action

The simplest case with induced path-dependency involves the optimal timing of a single control action. This case exhibits path-dependency because control activation affects the forward lattice construction, so simple backwards induction cannot take into account the possibility of earlier control activation. We present results for various cases at different levels of the initial project value S. In contrast to Grenadier and Weiss (1997), we focus on the flexibility in the timing of innovation adoption; in general we show how the results differ according to the level of the stochastic variable, the parameters of the control actions and the parameter values of the exogenous stochastic process (which is absent in their model).

For comparison purposes we present numerical results for cases involving both timing and no timing flexibility. Option values and optimal decision thresholds are presented in Tables 1 – 2. Table 1 describes the case with no timing flexibility. With timing flexibility (Table 2), the single control may be activated at any of five decision stages before option maturity. In both tables the first column refers to the base case. Note that due to the specific option and control parameter values in Table 1, for much higher than reported values of S,

[8] We have investigated (partly only reported) parameter values for a cost I_1 = 5, 10, 20, 30, 40; for a mean impact γ_1 = 0.10, 0.20, 0.30, 0.40, 0.80; and for a volatility of impact σ_1 = 0.10, 0.30, 0.50. For the parameter values of the stochastic process, we have made numerical investigations for a dividend-like payout rate δ = 0, 0.03, 0.05, 0.08, 0.10, 0.20; and for a Brownian volatility σ = 0.05, 0.15, 0.30, 0.50.

$$\frac{\partial V^C}{\partial S} = e^{-\delta T + \gamma} N(d_1) = e^{-0.05} N(d_1) \underset{S>>X}{<} \frac{\partial V^{EE}}{\partial S} = 1,$$

and the dominant strategy is early exercise (EE). This is not the case in Table 2, where the presence of flexibility to delay R&D decisions may make decisions other than early exercise (EE) dominant at high project values.

Table 1. Option value and optimal decisions:
Single R&D action *without* timing flexibility

	Base Case		**$I_1 = 40$**		**$\delta = 0.1$**		**$\sigma = 0.50$**		**$\gamma_1 = 0.30$**		**$\sigma_1 = 0.50$**	
S	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**
260	160.00	*EE*	160.00	*EE*	160.00	*EE*	171.52	C_1	185.50	C_1	160.27	C_1
251	**151.00**	***EE***	151.00	*EE*	151.00	*EE*	163.37	C_1	176.05	C_1	151.83	C_1
250	150.05	C_1	150.00	*EE*	150.00	*EE*	162.47	C_1	175.00	C_1	150.90	C_1
240	140.58	C_1	140.00	*EE*	140.00	*EE*	153.57	C_1	164.51	C_1	141.57	C_1
230	131.11	C_1	130.00	*EE*	130.00	*EE*	144.83	C_1	154.02	C_1	132.25	C_1
220	121.67	C_1	120.00	*EE*	120.00	*EE*	136.09	C_1	143.55	C_1	122.92	C_1
210	112.24	C_1	110.00	*EE*	110.00	*EE*	127.34	C_1	133.09	C_1	113.72	C_1
200	102.83	C_1	100.00	*EE*	100.00	*EE*	118.60	C_1	122.63	C_1	104.55	C_1
190	93.49	C_1	90.00	*EE*	90.00	*EE*	109.86	C_1	112.23	C_1	95.38	C_1
180	84.15	C_1	80.00	*EE*	80.00	*EE*	101.19	C_1	101.82	C_1	86.36	C_1
170	74.93	C_1	70.00	*EE*	70.00	*EE*	92.89	C_1	91.51	C_1	77.44	C_1
160	65.74	C_1	60.00	*EE*	60.00	*EE*	84.58	C_1	81.20	C_1	68.52	C_1
150	56.75	C_1	50.00	*EE*	50.00	*EE*	76.28	C_1	71.05	C_1	59.94	C_1
140	47.85	C_1	40.00	*EE*	40.00	*EE*	67.97	C_1	60.96	C_1	51.38	C_1
130	39.23	C_1	30.00	*EE*	30.00	*EE*	59.95	C_1	51.10	C_1	43.14	C_1
123	33.35	C_1	**23.00**	***EE***	23.00	*EE*	54.54	C_1	44.32	C_1	37.50	C_1
120	30.91	C_1	20.81	*W*	20.00	*EE*	52.23	C_1	41.48	C_1	35.09	C_1
110	22.92	C_1	15.20	*W*	10.00	*EE*	44.51	C_1	32.15	C_1	27.48	C_1
104	18.42	C_1	12.25	*W*	**4.00**	***EE***	39.88	C_1	26.80	C_1	23.04	C_1
100	15.42	C_1	10.33	*W*	2.98	*W*	36.79	C_1	23.24	C_1	20.08	C_1
90	8.70	C_1	6.45	*W*	1.61	*W*	29.81	C_1	15.05	C_1	13.48	C_1
83	**4.53**	$\mathbf{C_1}$	4.28	*W*	0.94	*W*	24.92	C_1	9.86	C_1	9.10	C_1
80	3.54	*W*	3.54	*W*	0.74	*W*	22.82	C_1	7.67	C_1	7.39	C_1
79	3.29	*W*	3.29	*W*	0.67	*W*	**22.12**	$\mathbf{C_1}$	6.96	C_1	6.82	C_1
71	1.72	*W*	1.72	*W*	0.29	*W*	18.10	*W*	**1.93**	$\mathbf{C_1}$	2.49	C_1
70	1.58	*W*	1.58	*W*	0.26	*W*	17.66	*W*	1.58	*W*	2.02	C_1
69	1.46	*W*	1.46	*W*	0.24	*W*	17.22	*W*	1.46	*W*	**1.55**	$\mathbf{C_1}$
60	0.55	*W*	0.55	*W*	0.07	*W*	13.23	*W*	0.55	*W*	0.55	W

Notes: We show option values and optimal decisions (Dec). This is the case of no timing flexibility of the control, and we use time to maturity (T = 5) with decision at t = 0 only (no flexibility to delay). Admissible actions (Dec.): Wait (W), R&D Control (C_1), and early exercise of investment option (*EE*). Base case parameter values are $r = \delta = 0.05$, $\sigma = 0.15$, development cost X =100. For R&D control: average impact γ_1= 0.20, volatility σ_1 = 0.30, and cost I_1 = 10. Results are numerically derived using one step per month. Sensitivity is with respect to project value *S*, control cost I_1, opportunity cost of waiting δ, exogenous project volatility σ, average impact γ_1 and volatility of control impact σ_1.

Table 2. Option value and optimal decisions: Single R&D action *with* timing flexibility

	Base Case		**$I_1 = 40$**		**$\delta = 0.1$**		**$\sigma = 0.50$**		**$\gamma_1 = 0.30$**		**$\sigma_1 = 0.50$**		**Abandonment**	
S	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**
220	150.59	C_1	120.59	C_1	138.07	C_1	160.07	C_1	177.41	C_1	151.92	C_1	150.63	C_1
217	147.12	C_1	**117.12**	$\mathbf{C_1}$	134.76	C_1	156.87	C_1	173.56	C_1	148.51	C_1	147.16	C_1
210	139.02	C_1	110.00	*EE*	127.05	C_1	149.40	C_1	164.59	C_1	140.61	C_1	139.06	C_1
200	127.45	C_1	100.00	*EE*	116.03	C_1	138.75	C_1	151.77	C_1	129.33	C_1	127.51	C_1
190	115.94	C_1	90.00	*EE*	105.05	C_1	128.14	C_1	138.99	C_1	118.11	C_1	116.06	C_1
180	104.49	C_1	80.00	*EE*	94.13	C_1	117.62	C_1	126.24	C_1	106.92	C_1	104.63	C_1
170	93.08	C_1	70.00	*EE*	83.23	C_1	107.48	C_1	113.51	C_1	96.01	C_1	93.25	C_1
160	81.72	C_1	60.00	*EE*	72.37	C_1	**97.55**	$\mathbf{C_1}$	100.81	C_1	85.23	C_1	82.03	C_1
150	70.62	C_1	50.00	*EE*	61.80	C_1	88.20	*W*	88.28	C_1	74.59	C_1	70.99	C_1
140	59.63	C_1	40.00	*EE*	51.30	C_1	79.08	*W*	75.83	C_1	64.11	C_1	60.10	C_1
136	55.28	C_1	36.00	*EE*	47.11	C_1	75.48	*W*	70.87	C_1	59.95	C_1	55.80	C_1
135	54.19	C_1	**35.00**	***EE***	46.06	C_1	74.58	*W*	69.63	C_1	58.91	C_1	54.76	C_1
130	48.87	C_1	30.86	*W*	41.02	C_1	70.22	*W*	63.53	C_1	53.86	C_1	49.60	C_1
123	41.67	C_1	25.49	*W*	34.24	C_1	64.25	*W*	55.19	C_1	47.04	C_1	42.51	C_1
120	38.62	C_1	23.27	*W*	31.35	C_1	61.71	*W*	51.64	C_1	44.16	C_1	39.54	C_1
110	28.71	C_1	16.63	*W*	21.90	C_1	53.37	*W*	40.00	C_1	34.81	C_1	29.95	C_1
104	**23.23**	$\mathbf{C_1}$	13.23	*W*	16.90	C_1	48.48	*W*	33.44	C_1	29.39	C_1	24.66	C_1
100	19.98	*W*	11.07	*W*	13.67	C_1	45.27	*W*	29.16	C_1	25.84	C_1	21.28	C_1
99	19.24	*W*	10.62	*W*	12.88	C_1	44.51	*W*	28.11	C_1	24.97	C_1	**20.45**	$\mathbf{C_1}$
94	15.66	*W*	8.39	*W*	9.02	C_1	40.77	*W*	**22.98**	$\mathbf{C_1}$	20.86	C_1	16.78	*W*
93	14.96	*W*	7.96	*W*	8.25	C_1	40.03	*W*	22.07	*W*	20.06	C_1	16.10	*W*
92	14.30	*W*	7.52	*W*	7.48	C_1	39.30	*W*	21.17	*W*	**19.27**	$\mathbf{C_1}$	15.44	*W*
90	13.03	*W*	6.79	*W*	**5.99**	$\mathbf{C_1}$	37.82	*W*	19.49	*W*	17.72	*W*	14.15	*W*
80	7.58	*W*	3.67	*W*	2.68	*W*	30.60	*W*	12.01	*W*	11.10	*W*	8.50	*W*
70	3.71	*W*	1.62	*W*	0.94	*W*	23.94	*W*	6.31	*W*	6.12	*W*	4.41	*W*
60	1.40	*W*	0.55	*W*	0.25	*W*	17.83	*W*	2.71	*W*	2.71	*W*	1.79	*W*

Notes: We show option values and optimal decisions (Dec). This is the case with timing flexibility of the control, and we use time to maturity ($T = 5$) with five yearly decision stages ($n_s = 5$). Admissible actions (Dec.): Wait (*W*), R&D Control (C_1), and early exercise of investment option (EE); in the last two columns we also allow Abandonment (A) to recover 50% of R&D expenditures. Base case parameter values are $r = \delta = 0.05$, $\sigma = 0.15$, development cost X = 100. For R&D control: average impact $\gamma_1 = 0.20$, volatility $\sigma_1 = 0.30$, and cost $I_1 = 10$. Sensitivity is with respect to project value *S*, control cost I_1, opportunity cost of waiting δ, exogenous volatility σ, average impact γ_1 and volatility of control impact σ_1.

In both tables we observe that option value increases and control activation thresholds decrease when the control is more valuable (e.g., when it has a higher average impact and/or higher volatility of impact or when it has a lower cost). At the limit, for reasonably high values of the control cost, the model effectively becomes equivalent to the one in the absence of controls where only the actions of early exercise (EE) or wait (W) are possible. For example, when I = 40, control activation occurs very rarely, and option value is substantially lower.

The other option parameters (dividend-like payout rate, δ, and Brownian volatility, σ) provide interesting differences worth of discussing. In both cases (with and without timing flexibility of control action) an increase in the payout rate or a decrease in project volatility results in lower option values, consistent with standard real options theory. Their impact on threshold levels, though, is different: With no timing flexibility an increase in the payout rate δ or a decrease in the volatility of project value may increase the control threshold. In contrast, with timing flexibility an increase in δ or a decrease in project volatility decreases the control activation threshold (in which case the traditional real options intuition still holds). For $\delta = 0$ (results not reported for brevity) and timing flexibility all actions are deferred and waiting (*W*) prevails for all levels of *S*. This means that early exercise (*EE*) would occur at the end of the time horizon, but investing in a control action (*C*) could occur at the decision stage just before option maturity, since at that point the control action cannot be further deferred.

We also examine in the last column of Table 2 the case where capital invested in control action(s) can be recovered via later abandonment. In the table we show results assuming partial reversibility of invested capital (including "rights resale" value) equal to the total capital invested in control action(s), i.e., a known recovery factor α = 50%[9]. With the ability to abandon (with capital reversibility), option values are higher and investment in the control action occurs earlier. For higher recovery values, option values increase further and critical activation thresholds further decrease. These results are consistent with the intuition deriving from the literature on partial reversibility of investment. For example, Abel et. al. (1996) in a two-period option model of investment with partial reversibility of invested capital show that the option to disinvest raises the incentive to invest. Similarly, Abel and Eberly (1997) study a firm that faces revenue uncertainty and compare the case with and without capital reversibility. They find that the case of reversible investment increases the fundamental value of the firm.

[9] We have also tried (not shown) α = 150%, 125%, 100%, 75%, 25% without affecting the derived insights.

2.3.2. The effect of time-to-learn

Time-to-complete before realizing the impact of an R&D action can also be an important feature in many investment projects[10]. In Table 3 we investigate the impact of time-to-learn constraints for different periods needed for the impact (and volatility) of control to materialize. We assume the general case where during the delay early development is still possible at any time. Sensitivity to several parameters is provided in groups of three columns, first with no delay (no delay), then with a delay of one decision stage (delay = 1), and finally with a delay of three decision stages (delay = 3) (recall the total number of decision stages is five). The first upper three columns report results using the base case parameters. They show that such restrictions (delays) in the realization of the control reduce option value. Comparing the base case to the lower part of the table we see that, as expected, a higher control cost reduces option value and defers control exercise, whereas more valuable controls (having higher average impact or higher impact volatility) increase option value and lead to earlier control activation.

[10] For example, Schwartz and Moon (2000) mention that the stages needed for a drug development based on US Federal Drug Administration (FDA) standards may take more than 11 years to complete.

Table 3. Option value and optimal decisions: Single R&D action with time-to-learn (delay) effects

	Base case						**δ= 0.10**						**σ = 0.50**					
	No delay		***Delay* = 1**		***Delay* = 3**		***No delay***		***Delay* = 1**		***Delay* = 3**		***No delay***		***Delay* = 1**		***Delay* = 3**	
S	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**
220	150.59	C_1	142.82	C_1	128.38	C_1	138.07	C_1	120.00	*EE*	120.00	*EE*	160.07	C_1	154.91	C_1	142.75	C_1
210	139.02	C_1	131.82	C_1	118.47	C_1	127.05	C_1	110.00	*EE*	110.00	*EE*	149.40	C_1	144.77	C_1	133.57	C_1
200	127.45	C_1	120.87	C_1	108.58	C_1	116.03	C_1	100.00	*EE*	100.00	*EE*	138.75	C_1	134.69	C_1	124.42	C_1
190	115.94	C_1	109.93	C_1	98.72	C_1	105.05	C_1	**90.00**	***EE***	90.00	*EE*	128.14	C_1	124.67	C_1	115.31	C_1
180	104.49	C_1	99.06	C_1	88.93	C_1	94.13	C_1	80.20	C_1	80.00	*EE*	117.62	C_1	114.73	C_1	106.28	C_1
170	93.08	C_1	88.25	C_1	79.19	C_1	83.23	C_1	70.48	C_1	70.00	*EE*	107.48	C_1	104.93	C_1	97.39	C_1
160	81.72	C_1	77.54	C_1	69.56	C_1	72.37	C_1	60.88	C_1	60.00	*EE*	**97.55**	$\mathbf{C_1}$	95.31	C_1	88.62	C_1
150	70.62	C_1	66.97	C_1	60.05	C_1	61.80	C_1	51.47	C_1	50.00	*EE*	88.20	*W*	85.82	C_1	79.95	C_1
140	59.63	C_1	56.58	C_1	50.71	C_1	51.30	C_1	42.26	C_1	40.00	*EE*	79.08	*W*	76.44	C_1	71.33	C_1
135	54.19	C_1	51.48	C_1	46.12	C_1	46.06	C_1	37.77	C_1	35.00	*EE*	74.58	*W*	**71.82**	$\mathbf{C_1}$	67.09	C_1
130	48.87	C_1	46.44	C_1	41.60	C_1	41.02	C_1	33.35	C_1	30.00	*EE*	70.22	*W*	67.45	*W*	62.90	C_1
120	38.62	C_1	36.67	C_1	32.80	C_1	31.35	C_1	24.89	C_1	20.00	*EE*	61.71	*W*	59.20	*W*	54.69	C_1
110	28.71	C_1	27.38	C_1	24.43	C_1	21.90	C_1	17.00	C_1	**10.00**	***EE***	53.37	*W*	51.16	*W*	46.65	C_1
104	**23.23**	$\mathbf{C_1}$	22.10	C_1	19.66	C_1	16.90	C_1	12.62	C_1	7.06	*W*	48.48	*W*	46.42	*W*	41.89	C_1
103	22.35	*W*	21.25	C_1	18.89	C_1	16.09	C_1	11.92	C_1	6.62	*W*	47.67	*W*	45.65	*W*	**41.10**	$\mathbf{C_1}$
101	20.75	*W*	**19.55**	$\mathbf{C_1}$	17.36	C_1	14.47	C_1	10.52	C_1	5.75	*W*	46.07	*W*	44.09	*W*	39.65	*W*
100	19.98	*W*	18.71	*W*	16.59	C_1	13.67	C_1	9.83	C_1	5.33	*W*	45.27	*W*	43.31	*W*	38.94	*W*
97	17.78	*W*	16.56	*W*	11.54	C_1	11.33	C_1	7.85	C_1	4.41	*W*	43.01	*W*	41.10	*W*	36.91	*W*
93	14.96	*W*	13.85	*W*	**11.54**	$\mathbf{C_1}$	8.25	C_1	**5.38**	$\mathbf{C_1}$	3.26	*W*	40.03	*W*	38.19	*W*	34.25	*W*
90	13.03	*W*	11.98	*W*	9.72	*W*	5.99	C_1	4.20	*W*	2.55	*W*	37.82	*W*	36.06	*W*	32.26	*W*
84	9.55	*W*	8.60	*W*	6.61	*W*	**3.81**	$\mathbf{C_1}$	2.54	W	1.50	*W*	33.45	*W*	31.85	*W*	28.34	*W*
80	7.58	*W*	6.71	*W*	4.89	*W*	2.68	*W*	1.76	*W*	1.05	*W*	30.60	*W*	29.08	*W*	25.81	*W*
70	3.71	*W*	3.09	*W*	1.92	*W*	0.94	*W*	0.56	*W*	0.34	*W*	23.94	*W*	22.62	*W*	19.97	*W*
60	1.40	*W*	1.04	*W*	0.58	*W*	0.25	*W*	0.13	*W*	0.08	*W*	7.59	*W*	16.75	*W*	14.62	*W*

Table 3 (continued)

	$I_1 = 30$						**$\gamma_1 = 0.30$**						**$\sigma_1 = 0.50$**					
	No delay		***Delay* = 1**		***Delay* = 3**		***No delay***		***Delay* = 1**		***Delay* = 3**		***No delay***		***Delay* = 1**		***Delay* = 3**	
S	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**
220	130.59	C_1	122.82	C_1	120.00	*EE*	177.41	C_1	168.30	C_1	151.40	C_1	151.92	C_1	144.16	C_1	129.72	C_1
210	119.02	C_1	111.82	C_1	110.00	*EE*	164.59	C_1	156.12	C_1	140.39	C_1	140.61	C_1	133.40	C_1	120.00	C_1
200	107.45	C_1	100.87	C_1	100.00	*EE*	151.77	C_1	143.95	C_1	129.41	C_1	129.33	C_1	122.70	C_1	110.34	C_1
191	97.08	C_1	**91.02**	**C_1**	91.00	*EE*	140.26	C_1	133.01	C_1	119.54	C_1	119.23	C_1	113.13	C_1	101.70	C_1
190	95.94	C_1	90.00	*EE*	90.00	*EE*	138.99	C_1	131.80	C_1	118.44	C_1	118.11	C_1	112.07	C_1	100.74	C_1
180	84.49	C_1	80.00	*EE*	80.00	*EE*	126.24	C_1	119.69	C_1	107.51	C_1	106.92	C_1	101.54	C_1	91.24	C_1
170	73.08	C_1	70.00	*EE*	70.00	*EE*	113.51	C_1	107.61	C_1	96.63	C_1	96.01	C_1	91.11	C_1	81.83	C_1
160	61.72	C_1	60.00	*EE*	60.00	*EE*	100.81	C_1	95.60	C_1	85.81	C_1	85.23	C_1	80.82	C_1	72.54	C_1
150	50.62	C_1	50.00	*EE*	50.00	*EE*	88.28	C_1	83.69	C_1	75.08	C_1	74.59	C_1	70.71	C_1	63.41	C_1
145	**45.10**	**C_1**	45.00	*EE*	45.00	*EE*	82.03	C_1	77.77	C_1	69.76	C_1	69.33	C_1	65.73	C_1	58.92	C_1
141	**41.00**	***EE***	41.00	*EE*	41.00	*EE*	77.07	C_1	73.07	C_1	65.53	C_1	65.15	C_1	61.78	C_1	55.35	C_1
140	40.03	*W*	40.00	*EE*	40.00	*EE*	75.83	C_1	71.90	C_1	64.48	C_1	64.11	C_1	60.80	C_1	54.47	C_1
136	36.42	*W*	**36.00**	***EE***	36.00	*EE*	70.87	C_1	67.24	C_1	60.29	C_1	59.95	C_1	56.91	C_1	50.95	C_1
135	35.54	*W*	35.00	*W*	**35.00**	***EE***	69.63	C_1	66.08	C_1	59.25	C_1	58.91	C_1	55.94	C_1	50.07	C_1
130	31.28	*W*	30.86	*W*	30.86	*W*	63.53	C_1	60.30	C_1	54.06	C_1	53.86	C_1	51.15	C_1	45.75	C_1
120	23.47	*W*	23.27	*W*	23.27	*W*	51.64	C_1	49.00	C_1	43.89	C_1	44.16	C_1	41.83	C_1	37.33	C_1
110	16.73	*W*	16.63	*W*	16.63	*W*	40.00	C_1	38.08	C_1	34.07	C_1	34.81	C_1	32.91	C_1	29.26	C_1
100	11.13	*W*	11.07	*W*	11.07	*W*	29.16	C_1	27.71	C_1	24.73	C_1	25.84	C_1	24.48	C_1	21.62	C_1
94	8.42	*W*	8.39	*W*	8.39	*W*	22.98	C_1	21.86	C_1	19.45	C_1	20.86	C_1	19.72	C_1	17.31	C_1
92	7.55	*W*	7.52	*W*	7.52	*W*	**21.17**	**C_1**	**19.97**	**C_1**	17.74	C_1	**19.27**	**C_1**	18.18	C_1	15.91	C_1
90	6.80	*W*	6.79	*W*	6.79	*W*	19.49	*W*	18.18	*W*	16.06	C_1	17.72	*W*	16.68	C_1	14.55	C_1
86	5.41	*W*	5.40	*W*	5.40	*W*	16.30	*W*	15.09	*W*	12.84	C_1	14.80	*W*	**13.75**	**C_1**	11.89	C_1
83	4.46	*W*	4.45	*W*	4.45	*W*	14.04	*W*	12.88	*W*	**10.51**	**C_1**	12.90	*W*	11.88	*W*	9.97	C_1
80	3.68	*W*	3.67	*W*	3.67	*W*	12.01	*W*	10.90	*W*	8.55	*W*	11.10	*W*	10.15	*W*	**8.11**	**C_1**
70	1.62	*W*	1.62	*W*	1.62	*W*	6.31	*W*	5.41	*W*	3.43	*W*	6.12	*W*	5.33	*W*	3.55	*W*
60	0.56	*W*	0.55	*W*	0.55	*W*	2.71	*W*	2.06	*W*	0.88	*W*	2.71	*W*	2.15	*W*	0.98	*W*

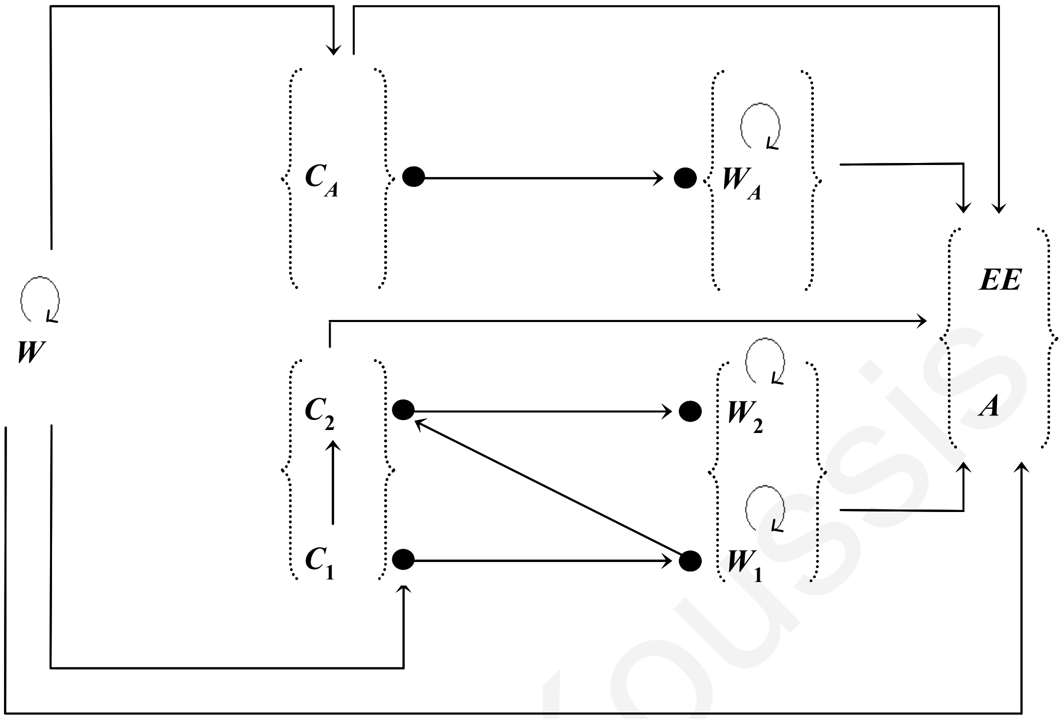
Notes: We show option values and optimal decisions (Dec). This is the case with timing flexibility of the control, and we use time to maturity ($T = 5$) with five yearly decision stages ($n_s = 5$). Time-to-learn (delay) refers to the delay periods for the realization of the control's impact. Admissible actions (Dec.): Wait (W), R&D Control (C_1), and early exercise of investment option (*EE*). Base case parameter values are $r = \delta = 0.05$, $\sigma = 0.15$, development cost $X = 100$. Parameter values for R&D control: average impact $\gamma_1 = 0.20$, volatility $\sigma_1 = 0.30$, and cost $I_1 = 10$. Sensitivity is with respect to project value S, control cost I_1, opportunity cost of waiting δ, exogenous volatility σ, average impact γ_1 and volatility of control impact σ_1.

As expected, a higher payout rate reduces option value whereas a higher volatility enhances option value. The effect of these two parameters on control activation is more subtle though. The earlier discussion in Tables 1 – 2 about the impact of timing flexibility is relevant here. When there is no delay, timing flexibility has bigger impact and increasing the opportunity cost of waiting to invest or lowering project volatility shifts the activation threshold earlier. When there is time delay (of several decision stages out of a total of five), this effect is reduced (with delay = 1) or even reversed (with delay = 3); increasing the opportunity cost of waiting to invest or lowering volatility shifts the activation threshold to higher project values S. This impact of δ is similar to the time-to-build effect in Majd and Pindyck (1987). A higher level of δ when it takes time to complete development of a new construction means higher erosion of value and therefore a higher level of project value S is needed to induce investment. Furthermore, since we allow the investment to be made without activation of the (optional) control, with higher δ the early exercise region in our case becomes more attractive and may even dominate.

2.3.3. Optimal choice between accelerated and sequential control strategies

In this section we investigate the more involved case where a firm can invest in a more costly (accelerated) R&D action with higher expected impact, or in a sequential strategy where a first-stage low-cost R&D investment can be followed by another small-scale R&D investment. We investigate the critical control thresholds for the single high-cost high-impact accelerated action versus the low-cost low-impact sequential strategy. Figure 4 describes the set of possible decisions.

Figure 4. Choice between accelerated (C_A) and sequential (C_1, C_2) R&D strategies



Notes: The firm starts in a wait mode (*W*) and has the option to defer investment in controls, or invest in either of two mutually exclusive strategies: an Accelerated control strategy (C_A) that gives a high expected impact at a high cost, and a Sequential (staged) control strategy (C_1, C_2) where each control has less impact, less volatility, and less cost. The firm can exercise early the investment option (*EE*), or abandon the project (*A*) at any time.

At the start, the firm can either wait (W), implement an accelerated control action C_A by paying I_A, invest in the first stage C_1 of the sequential strategy by paying I_1, or move directly to exercise early (EE) the development/commercialization option. If it chooses the accelerated action C_A, it can later develop the project or abandon it by recovering αI_A, or delay further actions (W_A). Similarly, development and abandonment options also exist after first-stage R&D action C_1, while the firm can proceed with second-stage R&D action C_2 by paying I_2. After C_2, it can continue with project development (EE), delay (W_2), or abandonment to recover $\alpha(I_1 + I_2)$. Abandonment depends on the recovery factor, α, as well as the preceding control actions. Childs and Triantis (1999) study R&D investments by examining the choice between accelerated versus staged (sequential)

R&D. In their treatment, completion of the research project is mandatory before any realization of profits, whereas in our case it is not. In contrast to Childs and Triantis (1999), we focus our investigation more on factors that affect our control strategy choice. For the base case, we use the same option parameter values as before, and for the accelerated control strategy the same parameter values as in the previous case of a single control. For the sequential strategy, we assume initially that each control action involves half the cost, half the average impact, and half the variance of the accelerated strategy, i.e., $I_1 = I_2 = 5$, $\gamma_1 = \gamma_2 = 0.10$, and $\sigma_1 = \sigma_2 = 0.30/\sqrt{2}$. With this specification, if both controls of the sequential strategy were simultaneously activated, the total costs and expected benefits of the sequential strategy would match those of the accelerated one.

The results are illustrated in Table 4. Those for the base case (first column) are rather typical. Often it is not beneficial to follow a sequential strategy. The value of the flexibility to stage the R&D investment in this case is not so important, since the total impact of the accelerated strategy (and its high volatility) can be realized sooner. The accelerated strategy may have advantages (especially at large project values) since the investment option can be exercised soon after (whereas the full impact of a sequential strategy might not be realized). The sequential strategy may still be advantageous at lower project values, as in the 2nd column when the total expected impact is high. We investigate this case further, focusing on the additional option to abandon and recover R&D costs, as well as on the sensitivity to the opportunity cost of waiting to invest, and project volatility. The presence of abandonment enhances option value, and can justify earlier control activation. In addition, it may also enhance the range where the sequential strategy prevails. Consistent with the discussions in previous subsections, lower project volatility and higher opportunity cost of waiting to invest will also justify earlier control activation.

Table 4. Accelerated (C_A) versus sequential (C_1/C_2) control strategy

	$\gamma_A = 0.2$ $\gamma_1 = \gamma_2 = 0.1$ **(Base Case)**		$\gamma_A = 0.8$ $\gamma_1 = \gamma_2 = 0.4$		$\gamma_A = 0.8$ $\gamma_1 = \gamma_2 = 0.4$ **& Abandon**		$\gamma_A = 0.8$ $\gamma_1 = \gamma_2 = 0.4$ $\sigma = 0.05$		$\gamma_A = 0.8$ $\gamma_1 = \gamma_2 = 0.4$ $\sigma = 0.3$		$\gamma_A = 0.8$ $\gamma_1 = \gamma_2 = 0.4$ $\delta = 0.03$		$\gamma_A = 0.8$ $\gamma_1 = \gamma_2 = 0.4$ $\delta = 0.08$	
S	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**
200	127.45	C_A	318.28	C_A	318.28	C_A	318.28	C_A	318.35	C_A	326.83	C_A	305.76	C_A
190	115.94	C_A	297.11	C_A	297.11	C_A	297.11	C_A	297.20	C_A	305.23	C_A	285.22	C_A
180	104.49	C_A	275.94	C_A	275.94	C_A	275.94	C_A	276.05	C_A	283.64	C_A	264.68	C_A
170	93.08	C_A	254.77	C_A	254.77	C_A	254.77	C_A	254.94	C_A	262.05	C_A	244.13	C_A
160	81.72	C_A	233.60	C_A	233.60	C_A	233.60	C_A	233.87	C_A	240.45	C_A	223.59	C_A
150	70.62	C_A	212.43	C_A	212.43	C_A	212.43	C_A	212.81	C_A	218.86	C_A	203.04	C_A
140	59.63	C_A	191.26	C_A	191.27	C_A	191.26	C_A	191.76	C_A	197.29	C_A	182.50	C_A
130	48.87	C_A	170.10	C_A	170.12	C_A	170.09	C_A	170.87	C_A	175.73	C_A	161.96	C_A
120	38.62	C_A	148.95	C_A	148.98	C_A	148.92	C_A	150.11	C_A	154.19	C_A	141.43	C_A
110	28.71	C_A	127.81	C_A	127.88	C_A	127.75	C_A	129.39	C_A	132.74	C_A	120.91	C_A
104	**23.23**	$\boldsymbol{C_A}$	115.16	C_A	115.26	C_A	115.05	C_A	117.17	C_A	119.89	C_A	108.62	C_A
100	19.99	*W*	106.74	C_A	106.86	C_A	106.59	C_A	109.14	C_A	111.36	C_A	100.43	C_A
90	13.03	*W*	85.73	C_A	86.04	C_A	85.44	C_A	89.23	C_A	90.26	C_A	80.05	C_A
84	9.55	*W*	73.30	C_A	73.71	C_A	72.78	C_A	77.42	C_A	**77.71**	$\boldsymbol{C_A}$	67.91	C_A
80	7.58	*W*	65.06	C_A	65.54	C_A	64.35	C_A	69.89	C_A	69.78	C_1	59.84	C_A
76	5.83	*W*	56.86	C_A	57.58	C_A	55.95	C_A	62.52	C_A	**61.98**	$\boldsymbol{C_1}$	51.88	C_A
74	5.08	*W*	52.78	C_A	53.70	C_A	51.77	C_A	**58.86**	$\boldsymbol{C_A}$	58.16	***W***	47.99	C_A
70	3.72	*W*	44.92	C_A	46.03	C_A	43.47	C_A	52.12	*W*	50.70	*W*	40.25	C_A
66	2.61	*W*	37.17	C_A	**38.51**	$\boldsymbol{C_A}$	35.23	C_A	45.83	*W*	43.46	*W*	32.59	C_A
64	2.16	*W*	**33.37**	$\boldsymbol{C_A}$	35.06	C_1	31.16	C_A	42.80	*W*	39.94	*W*	28.78	C_A
60	1.40	*W*	26.58	C_1	28.79	C_1	23.28	C_A	36.84	*W*	33.13	*W*	21.83	C_A
59	1.26	*W*	24.98	C_1	27.27	C_1	**21.35**	$\boldsymbol{C_A}$	35.38	*W*	31.49	*W*	20.13	C_A
56	0.87	*W*	**20.39**	$\boldsymbol{C_1}$	22.85	C_1	16.29	C_1	31.25	*W*	26.70	*W*	15.09	C_A
53	0.57	*W*	16.33	*W*	18.84	C_1	11.60	C_1	27.33	*W*	22.17	*W*	**10.66**	$\boldsymbol{C_A}$
50	0.35	*W*	12.69	*W*	**15.10**	$\boldsymbol{C_1}$	7.35	C_1	23.52	*W*	17.95	*W*	7.68	*W*
48	0.25	*W*	10.56	*W*	12.85	*W*	**4.80**	$\boldsymbol{C_1}$	21.15	*W*	15.33	*W*	6.03	*W*
40	0.04	*W*	4.08	*W*	5.87	*W*	0.25	*W*	12.81	*W*	6.76	*W*	1.79	*W*

Notes: We show option values and optimal decisions (Dec). This is the case with timing flexibility of the control, and we use time to maturity ($T = 5$) with five yearly decision stages ($n_s = 5$). Admissible actions (Dec.): Wait (W), accelerated R&D control (C_A), 1st stage of the sequential R&D Control (C_1), early exercise of investment option (*EE*). Base case parameter values are $r = \delta = 0.05$, $\sigma = 0.15$, development cost X = 100. Parameter values for the accelerated strategy: $\gamma_A = 0.20$, $\sigma_A = 0.30$, and $I_A = 10$; and for the sequential strategy: $\gamma_1 = \gamma_2 = 0.10$, $\sigma_1 = \sigma_2 = \sigma_A/\sqrt{2}$, and $I_1 = I_2 = 5$. Sensitivity is with respect to project value S, control cost I_1, opportunity cost of waiting δ, exogenous volatility σ, average impact γ_A, and abandonment.

Table 5. Accelerated (C_A) versus sequential (C_1/C_2) control strategy:

Learning-by-doing effects and convexity of adjustment costs

	Base case and abandonment effect						Learning by doing				Volatility effect	
	Base Case		Abandonment (linear)		Abandonment (convex)		Lower cost $I_1 = I_2 = 4$		Higher impact $\gamma_2 = 0.12$		$\sigma_1 = \sigma_2 = 0.30$	
S	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**	**Value**	**Dec.**
200	127.45	C_A	127.51	C_A	127.51	C_A	127.45	C_A	127.45	C_A	127.45	C_A
190	115.94	C_A	116.06	C_A	116.06	C_A	115.94	C_A	115.94	C_A	115.94	C_A
180	104.49	C_A	104.63	C_A	104.63	C_A	104.49	C_A	104.49	C_A	104.49	C_A
170	93.08	C_A	93.25	C_A	93.25	C_A	93.08	C_A	93.08	C_A	93.08	C_A
160	81.72	C_A	82.03	C_A	82.03	C_A	81.72	C_A	81.72	C_A	81.72	C_A
150	70.62	C_A	70.99	C_A	70.99	C_A	70.62	C_A	70.62	C_A	70.62	C_A
145	65.10	C_A	65.53	C_A	65.53	C_A	65.10	C_A	**65.10**	$\boldsymbol{C_A}$	65.10	C_A
140	59.63	C_A	60.10	C_A	60.10	C_A	59.63	C_A	59.81	C_1	59.63	C_A
137	56.36	C_A	56.86	C_A	56.86	C_A	56.36	C_A	56.67	C_1	**56.36**	$\boldsymbol{C_A}$
130	48.87	C_A	49.60	C_A	49.60	C_A	48.87	C_A	49.42	C_1	49.53	C_1
128	46.80	C_A	47.56	C_A	47.56	C_A	**46.80**	$\boldsymbol{C_A}$	47.40	C_1	47.64	C_1
120	38.62	C_A	39.54	C_A	39.54	C_A	39.06	C_1	39.47	C_1	40.21	C_1
110	28.71	C_A	29.95	C_A	29.95	C_A	29.86	C_1	30.01	C_1	31.31	C_1
104	**23.23**	$\boldsymbol{C_A}$	24.66	C_A	24.66	C_A	24.71	C_1	24.68	C_1	26.28	C_1
101	20.75	*W*	22.11	C_A	**22.11**	$\boldsymbol{C_A}$	22.22	C_1	22.12	C_1	23.86	C_1
100	19.99	*W*	21.28	C_A	21.31	C_1	21.40	C_1	21.28	C_1	23.06	C_1
99	19.24	*W*	**20.45**	$\boldsymbol{C_A}$	20.54	C_1	20.61	C_1	20.46	C_1	22.27	C_1
95	16.36	*W*	17.48	*W*	**17.67**	$\boldsymbol{C_1}$	17.52	C_1	17.26	C_1	19.22	C_1
94	15.66	*W*	16.79	*W*	16.97	*W*	16.77	C_1	**16.48**	$\boldsymbol{C_1}$	18.47	C_1
91	13.66	*W*	14.80	*W*	14.98	*W*	**14.60**	$\boldsymbol{C_1}$	14.41	*W*	16.28	C_1
90	13.03	*W*	14.16	*W*	14.34	*W*	13.92	*W*	13.77	*W*	15.57	C_1
86	11.29	*W*	11.70	*W*	11.95	*W*	12.47	*W*	11.29	*W*	**12.85**	$\boldsymbol{C_1}$
80	7.58	*W*	8.50	*W*	8.77	*W*	8.23	*W*	8.06	*W*	9.40	*W*
70	3.72	*W*	4.41	*W*	4.65	*W*	4.11	*W*	3.97	*W*	4.91	*W*
60	1.40	*W*	1.79	*W*	1.97	*W*	1.57	*W*	1.50	*W*	2.00	*W*
50	0.35	*W*	0.50	*W*	0.59	*W*	0.40	*W*	0.37	*W*	0.54	*W*
40	0.04	*W*	0.07	*W*	0.09	*W*	0.05	*W*	0.05	*W*	0.07	*W*

Notes: We show option values and optimal decisions (Dec). This is the case with timing flexibility of the control, and we use time to maturity ($T = 5$) with five yearly decision stages ($n_s = 5$). Admissible actions (Dec.): Wait (*W*), accelerated R&D control (C_A), 1st stage of the sequential R&D Control (C_1), early exercise of investment option (*EE*). Base case parameter values are $r = \delta = 0.05$, $\sigma = 0.15$, development cost $X = 100$. Parameter values for the accelerated strategy: $\gamma_A = 0.20$, $\sigma_A = 0.30$, and $I_A = 10$; and for the sequential strategy: $\gamma_1 = \gamma_2 = 0.10$, $\sigma_1 = \sigma_2 = \sigma_A/\sqrt{2}$, and $I_1 = I_2 = 5$. For learning-by-doing: in the lower-cost case $I_1 = I_2 = 4$; in the higher impact case $\gamma_2 = 0.12$; in the higher volatility case $\sigma_1 = \sigma_2 = 0.30$; abandonment (linear) is for $\alpha = 0.50$ of R&D expenditures; and abandonment (convex) is for $\alpha_1 = 0.90 > \alpha_A = 0.50 > \alpha_2 = 0.10$.

2.3.4. The effects of learning-by-doing, and convexity of adjustment costs

The sequential strategy may often offer certain advantages. We will consider two general cases, the case of learning-by-doing, and the case of diminishing marginal reversibility of R&D capital (convexity of adjustment costs). For example, the sequential strategy may offer more than half the average impact and/or volatility per stage, or the combined effects can be achieved at a lower total cost. This is the case with learning-by-doing effects, similar to that analyzed extensively in the case of manufacturing to model efficiencies in production (e.g., Majd and Pindyck, 1987). When learning-by-doing efficiencies appear in the form of reduced total costs, the firm invests an initial low amount I_1 = 4 in the first stage. It can then implement the second stage by incurring another cost I_2 = 4, for a total of 8 (instead of 10 for the accelerated strategy). For similar reasons we can also justify a higher average impact in the second only stage of a sequential strategy (γ_2 = 0.12). This may be the case when the particular technology is disruptive, having low impact at the beginning but a greater one subsequently (e.g., Schwartz and Gorostiza, 2000b).

We also provide the example where $\sigma_1 = \sigma_2 = 0.30$ (whereas the accelerated strategy has a combined volatility of σ_A = 0.30). We have finally considered recovery of capital invested in sequential R&D in the case of convexity of adjustment costs (decreasing marginal recovery of investment, as in Abel and Eberly, 1997). We have investigated the case where after C_1 we can recover $\alpha_1 I_1$, after C_2 we can recover $\alpha_1 I_1 + \alpha_2 I_2$, and after C_A we can recover $\alpha_A I_A$. To capture convex adjustment costs, we use $\alpha_1 = 0.90 > \alpha_A = 0.50 > \alpha_2 = 0.10$ (and like before we assume $I_A = I_1 + I_2$) and we compare it to the linear adjustment cost function with $\alpha_1 = \alpha_A = \alpha_2 = 0.50$.

Table 5 confirms that learning-by-doing and convexity in adjustment costs can play an important role in R&D project decisions. With learning-by-doing option values are enhanced and control is activated earlier. The presence of such effects also alters the optimal decisions: for low values of S, the first stage of the sequential strategy may be chosen instead of the costly accelerated strategy; at higher values of S, the sequential strategy does not offer any advantages. Similarly, when the combined uncertainty of the sequential control actions is higher than that of the accelerated strategy (last column), option values are higher and the sequential strategy may again become the preferred choice. The second column in Table 5 shows the case with abandonment. Abandonment increases the value of the investment opportunity, and leads to earlier investment in R&D. The third column presents the results for the case of convexity in the adjustment costs when R&D capital is reversible. Sensitivity results (not reported) confirm that convexity increases option value and can enhance the relative attractiveness of the sequential R&D.

2.4. Conclusions

This paper has investigated control actions (R&D and innovation adoption) with uncertain outcome in a real options framework with features of path-dependency. In the special case involving optimal timing of a single action, our results confirm the significant impact of time-to-learn effects and capital recovery of realized costs. We investigate the critical threshold to activate the control action (rather than wait, or proceed directly to project development) and the impact of key parameters on this threshold. When there is flexibility to activate a control action at an optimal time, an increase in the dividend-like payout rate or a decrease in the volatility of project value induces earlier control activation (i.e., lowers the threshold level). The opposite is often observed in the case with no timing flexibility in R&D. With time-to-learn effects, the impact of such parameters is more subtle since there are varying degrees of timing flexibility. We also investigate the case of two mutually-exclusive alternative strategies, one with a single (accelerated) control action versus a sequential strategy. Interestingly, the assumed flexibility in a sequential strategy is not always that valuable in this setting. The sequential strategy often has less appeal unless there is abandonment (with recovery of R&D invested capital), presence of learning-by-doing effects, or decreasing marginal recovery of capital invested in research. Our approach can be beneficial in other areas beyond R&D, such as in marketing research and advertisement actions prior to the introduction of a new product.

Appendix

This Appendix investigates the convergence of our numerical lattice scheme. We have investigated the accuracy of our lattice scheme with one step per month, in the simple case with a single control available at t = 0 that has an analytic solution. To make exposition simpler we consider the case where the cost of control is zero and control is always activated ($\gamma_1 > 0$). The accuracy of the numerical scheme can be established by how closely it approximates the known benchmark with analytic solution (equation 3). We provide numerical results with several parameterizations and compare against the known analytic solution. Table A0 summarizes the results.

Table A0. Accuracy of numerical lattice against analytic benchmark (European option with single control without timing flexibility)

		Numerical			**Analytic**			**% Difference (Num/Anal)**		
		***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
δ=0.05, σ=0.15	**γ_1=0, σ_1=0**	3.536	10.329	20.807	3.511	10.372	20.822	0.007	-0.004	-0.001
	γ_1=0.1, σ_1=0.3	8.994	18.979	31.874	9.026	19.041	31.825	-0.004	-0.003	0.002
	γ_1=0.5, σ_1=0.3	31.258	53.405	77.569	31.374	53.493	77.570	-0.004	-0.002	0.000
	γ_1=0.1, σ_1=0.5	13.255	23.669	36.367	13.199	23.751	36.362	0.004	-0.003	0.000
	γ_1=0.5, σ_1=0.5	35.925	57.081	80.286	35.924	57.197	80.314	0.000	-0.002	0.000
δ=0.10, σ=0.15	**γ_1=0, σ_1=0**	0.739	2.979	7.639	0.731	3.023	7.686	0.010	-0.015	-0.006
	γ_1=0.1, σ_1=0.3	3.254	8.115	15.447	3.270	8.171	15.407	-0.005	-0.007	0.003
	γ_1=0.5, σ_1=0.3	15.079	29.148	45.996	15.138	29.221	45.962	-0.004	-0.002	0.001
	γ_1=0.1, σ_1=0.5	6.245	12.160	20.033	6.200	12.233	20.036	0.007	-0.006	0.000
	γ_1=0.5, σ_1=0.5	19.803	33.736	49.976	19.756	33.829	49.980	0.002	-0.003	0.000
δ=0.05, σ=0.5	**γ_1=0, σ_1=0**	22.481	32.873	44.632	22.449	33.009	44.531	0.001	-0.004	0.002
	γ_1=0.1, σ_1=0.3	27.777	39.912	53.463	27.732	40.060	53.350	0.002	-0.004	0.002
	γ_1=0.5, σ_1=0.3	52.922	73.777	96.213	52.903	73.969	96.113	0.000	-0.003	0.001
	γ_1=0.1, σ_1=0.5	29.430	41.735	55.429	29.366	41.891	55.307	0.002	-0.004	0.002
	γ_1=0.5, σ_1=0.5	54.900	75.823	98.311	54.856	76.023	98.199	0.001	-0.003	0.001

Notes: Comparative results are provided for the case of a European call option with a single embedded managerial control using the numerical lattice and the analytic solution of equation (3). Parameter values are X = 100, r = 0.05, T = 5, and cost of control I_C = 0. We provide sensitivity with respect to the level of S (80, 100, 120), the average impact γ_1 (0.10, 0.50), the volatility of impact σ_1 (0.30, 0.50), the opportunity cost of waiting δ (0.05, 0.10), and the project volatility σ (0.15, 0.50). The case where γ_1 = σ_1 = 0 is equivalent to the absence of control.

Sensitivity results are given with respect to the moneyness of the option (S being above, equal to, or lower than X), the average impact and volatility of the impact, the opportunity cost of delaying (δ), and project volatility (σ). We value a five-year investment option (using a discretization scheme with one step per month). The results confirm that very reasonable accuracy levels can be achieved (with numerical error in all cases investigated being less than 1.5%). This comparison is not a proof of accuracy since only a special case (without optimal R&D timing) allows for an analytic solution, but it provides evidence that high accuracy levels can be achieved.

We examine all cases considered in the applications with a different number of steps and check for numerical differences when more steps are used in the lattice implementation. Table A1 presents results for the optimal control timing case, Table A2 for the time-to-learn case, and Table A3 for the choice between accelerated versus sequential actions (without and with learning-by-doing). We have checked from 3 to 24 steps per year (3, 6, 9, 12, 15, 18, 21, and 24-step schemes), but report only a subset due to space constraints. Convergence is rather fast and the chosen 12-step scheme is very adequate; from 12 steps and beyond oscillations are of minimal magnitude, and the difference between our scheme and ones that are somewhat more accurate but highly more intensive computationally is negligible. Further, optimal decisions are generally not affected (except in just a few cases). To save space we report option values for 12, 21, 24, and the average of 21 and 24 steps, and show the % difference between the 12-step scheme used and either the 24-step scheme or the average of the 21 and 24 steps (an odd and a nearby even number). The chosen 12-step scheme differs marginally from the more accurate results provided by the 21/24-step schemes. Our calculations show that the numerical model appears to converge relatively quickly and the chosen lattice scheme with one step per month is adequate.

Table A1. Investigation of numerical accuracy/convergence: Optimal timing of a single R&D action

	γ = 0.2 (Base case)			**γ = 0**			**γ = 0.4**		
Number of yearly steps	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
12	7.584	19.980	38.618	3.668	11.071	23.265	18.033	40.517	66.689
21	7.576	20.010	38.585	3.634	11.128	23.270	18.139	40.624	66.719
24	7.593	20.003	38.596	3.654	11.098	23.271	18.163	40.599	66.733
Avg(21,24)	7.584	20.007	38.591	3.644	11.113	23.271	18.151	40.611	66.726
% Diff.(24-12)	0.001	0.001	-0.001	-0.004	0.002	0.000	0.007	0.002	0.001
% Diff.(Avg(21,24)-12)	0.000	0.001	-0.001	-0.007	0.004	0.000	0.007	0.002	0.001

	σ_1 = 0.3 (Base case)			**σ_1 = 0.1**			**σ_1 = 0.5**		
Number of yearly steps	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
12	7.584	19.980	38.618	6.081	17.586	35.329	11.104	25.835	44.160
21	7.576	20.010	38.585	6.085	17.604	35.390	11.105	25.908	44.069
24	7.593	20.003	38.596	6.090	17.579	35.401	11.107	25.896	44.133
Avg(21,24)	7.584	20.007	38.591	6.087	17.591	35.395	11.106	25.902	44.101
% Diff.(24-12)	0.001	0.001	-0.001	0.002	0.000	0.002	0.000	0.002	-0.001
% Diff.(Avg(21,24)-12)	0.000	0.001	-0.001	0.001	0.000	0.002	0.000	0.003	-0.001

Notes: Time to maturity (T = 5) with five yearly decision stages (n_s = 5). For the base case we use $r = \delta = 0.05$, $\sigma = 0.15$, development cost X =100. Parameter values for R&D control: average impact γ_1= 0.20 with volatility σ_1 = 0.30, and cost I_1 = 10. Admissible actions: Wait (W), R&D Control (C_1), and early exercise of investment option (EE). Sensitivity is with respect to the average impact γ_1 and the volatility of control impact σ_1.

Table A2. Investigation of numerical accuracy/convergence:

Time-to-learn

	***γ* = 0.2 (Base case)**			***γ* = 0**			***γ* = 0.4**		
Number of yearly steps	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
12	5.813	17.686	34.746	3.668	11.071	23.265	15.550	36.452	59.916
21	5.819	17.711	34.755	3.634	11.128	23.270	15.727	36.567	59.978
24	5.826	17.715	34.755	3.654	11.098	23.271	15.759	36.585	59.988
Avg(21,24)	5.822	17.713	34.755	3.644	11.113	23.271	15.743	36.576	59.983
% Diff.(24-12)	0.002	0.002	0.000	-0.004	0.002	0.000	0.013	0.004	0.001
% Diff.(Avg(21,24)-12)	0.002	0.002	0.000	-0.007	0.004	0.000	0.012	0.003	0.001

	σ_1 = 0.3 (Base case)			**σ_1 = 0.1**			**σ_1 = 0.5**		
Number of yearly steps	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
12	5.813	17.686	34.746	4.612	15.011	32.052	9.128	23.054	39.565
21	5.819	17.711	34.755	4.585	15.054	32.080	9.143	23.052	39.542
24	5.826	17.715	34.755	4.588	15.052	32.085	9.136	23.046	39.541
Avg(21,24)	5.822	17.713	34.755	4.586	15.053	32.083	9.140	23.049	39.541
% Diff.(24-12)	0.002	0.002	0.000	-0.005	0.003	0.001	0.001	0.000	-0.001
% Diff.(Avg(21,24)-12)	0.002	0.002	0.000	-0.006	0.003	0.001	0.001	0.000	-0.001

Notes: Time to maturity (*T* = 5) with five yearly decision stages (n_s = 5). For the base case we use r = δ = 0.05, σ = 0.15, development cost X = 100. Parameter values for R&D control: average impact γ_1 = 0.20 with volatility σ_1 = 0.30, and cost I_1 = 10. Admissible actions: Wait (*W*), R&D Control (C_1), and early exercise of investment option (*EE*). Time-to-learn (delay) refers to the delay periods for the realization of the control's impact. Here we assume delay = 2. Sensitivity is with respect to the average impact γ_1 and the volatility of control impact σ_1.

Table A3. Investigation of numerical accuracy/convergence:

Accelerated versus sequential strategy with learning-by-doing

	No learning by doing								
	γ = 0.2 (Base case)			**γ = 0**			**γ = 0.4**		
Number of yearly steps	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
12	7.584	19.986	38.618	3.668	11.071	23.265	18.035	40.517	66.689
21	7.576	20.010	38.585	3.634	11.128	23.270	18.139	40.624	66.719
24	7.593	20.003	38.596	3.654	11.098	23.271	18.166	40.599	66.733
Avg(21,24)	7.584	20.007	38.591	3.644	11.113	23.271	18.152	40.611	66.726
% Diff.(24-12)	0.001	0.001	-0.001	-0.004	0.002	0.000	0.007	0.002	0.001
% Diff.(Avg(21,24)-12)	0.000	0.001	-0.001	-0.007	0.004	0.000	0.007	0.002	0.001
	σ_1 = 0.3 (Base case)			**σ_1 = 0.1**			**σ_1 = 0.5**		
Number of yearly steps	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
12	7.584	19.986	38.618	6.081	17.586	35.329	11.170	25.835	44.160
21	7.576	20.010	38.585	6.085	17.604	35.390	11.176	25.908	44.069
24	7.593	20.003	38.596	6.090	17.579	35.401	11.176	25.896	44.133
Avg(21,24)	7.584	20.007	38.591	6.087	17.591	35.395	11.176	25.902	44.101
% Diff.(24-12)	0.001	0.001	-0.001	0.002	0.000	0.002	0.001	0.002	-0.001
% Diff.(Avg(21,24)-12)	0.000	0.001	-0.001	0.001	0.000	0.002	0.001	0.003	-0.001
	Learning by doing (Lower costs of sequential strategy)								
	γ = 0.2 (Base case)			**γ = 0**			**γ = 0.4**		
Number of yearly steps	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
12	8.226	21.403	39.057	3.676	11.094	23.275	19.408	40.971	66.689
21	8.225	21.416	39.034	3.649	11.140	23.279	19.498	41.010	66.719
24	8.231	21.409	39.037	3.661	11.117	23.286	19.514	41.020	66.733
Avg(21,24)	8.228	21.412	39.035	3.655	11.128	23.282	19.506	41.015	66.726
% Diff.(24-12)	0.001	0.000	-0.001	-0.004	0.002	0.000	0.005	0.001	0.001
% Diff.(Avg(21,24)-12)	0.000	0.000	-0.001	-0.006	0.003	0.000	0.005	0.001	0.001
	σ_1 = 0.3 (Base case)			**σ_1 = 0.1**			**σ_1 = 0.5**		
Number of yearly steps	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**	***S*= 80**	***S*=100**	***S*=120**
12	8.226	21.403	39.057	6.300	18.051	36.105	12.732	27.100	44.229
21	8.225	21.416	39.034	6.284	18.077	36.110	12.712	27.102	44.206
24	8.231	21.409	39.037	6.294	18.059	36.110	12.715	27.090	44.191
Avg(21,24)	8.228	21.412	39.035	6.289	18.068	36.110	12.713	27.096	44.199
% Diff.(24-12)	0.001	0.000	-0.001	-0.001	0.000	0.000	-0.001	0.000	-0.001
% Diff.(Avg(21,24)-12)	0.000	0.000	-0.001	-0.002	0.001	0.000	-0.002	0.000	-0.001

Notes: Time to maturity (T = 5) with five yearly decision stages (n_s = 5). For the base case we use r = δ = 0.05, σ = 0.15, development cost X = 100. Admissible actions: Wait (W), accelerated R&D control (C_A), 1st stage of the sequential R&D Control (C_1), early exercise of investment option (EE). Parameter values for the benchmark case: Accelerated strategy: $\gamma_A = 0.20$, $\sigma_A = 0.30$, and I_A = 10; Sequential strategy: $\gamma_1 = \gamma_2 = 0.10$, $\sigma_1 = \sigma_2 = \sigma_A/\sqrt{2}$, and $I_1 = I_2 = 5$. For learning-by-doing the sequential strategy has lower costs per stage ($I_1 = I_2 = 4$).

References

Abel, Andrew B., Avinash K. Dixit, Janice C. Eberly, and Robert S. Pindyck. (1996). "Options, the Value of Capital, and Investment," The Quarterly Journal of Economics 111, 753-777.

Abel, Andrew B., and Janice C. Eberly. (1997). "An Exact Solution for the Investment and Value of a Firm Facing Uncertainty, Adjustment Costs, and Irreversibility," Journal of Economic Dynamics and Control 21, 831-852.

Amir, Eli., Yanling Guan, and Gilad Livne. (2004). "The Association between the Uncertainty of Future Economic Benefits and Current R&D and Capital Expenditures: An Industry Analysis," Working Paper, London Business School.

Bar-Ilan, Avner, and William C. Strange. (1998). "A Model of Sequential Investment," Journal of Economic Dynamics & Control 22, 437-463.

Bar-Illan, Avner, Agnes Sulem, and Alessandro Zanello. (2002). "Time-to-Build and Capacity Choice," Journal of Economic Dynamics & Control 26, 69-98.

Black, Fischer, and Myron Scholes. (1973). "The Pricing of Options and Corporate Liabilities," Journal of Political Economy 81, 637–659.

Brynjolfsson, Erik, and Lorin M. Hitt. (2000). "Beyond Computation: Information Technology, Organizational Transformation and Business Performance," Journal of Economic Perspectives 14, 23-48.

Chan, Su Han, John D. Martin, and John W. Kensinger. (1990). "Corporate Research and Development Expenditures and Share Value," Journal of Financial Economics 26, 255-276.

Childs, Paul D., and Alexander J. Triantis. (1999). "Dynamic R&D Investment Policies," Management Science 45, 1359-1377.

Constantinides, George (1978). "Market Risk Adjustment in Project Valuation," Journal of Finance 33, 603–616.

Cox, John, Jonathan E. Ingersoll Jr, and Stephen Ross. (1985). "An Intertemporal General Equilibrium Model of Asset Prices," Econometrica 53, 363–384.

Dixit, Avinash, and Robert Pindyck. (1994). Investment under Uncertainty. Princeton, New Jersey: Princeton University Press.

Geske, Robert (1979). "The Valuation of Compound Options," Journal of Financial Economics 7, 63-81.

Grabowski, Henry, and John Vernon. (1990). “A New Look at the Returns and Risks to Pharmaceutical R&D,” Management Science 36, 804-821.

Grenadier, Steven R., and Allen M. Weiss. (1997). “Investment in Technological Innovations: An Option Pricing Approach,” Journal of Financial Economics 44, 397-416.

Harrison, J. Michael, and Stanley Pliska. (1981). “Martingales and Stochastic Integrals in the Theory of Continuous Trading,” Stochastic Processes and Their Applications 11, 313–316.

Hartman, Richard, and Michael Hendrickson. (2002). “Optimal Partially Reversible Investment,” Journal of Economic Dynamics & Control 26, 483-508.

Hull, John, and Alan White. (1993). “Efficient Procedures for Valuing European and American Path-Dependent Options,” Journal of Derivatives 1, 21-31.

Korn, Ralf (1997). “Optimal Impulse Control when Control Actions Have Random Consequences,” Mathematics of Operations Research 22, 639-667.

Kothari, S. P., Ted E. Laguerre, and Andrew J. Leone. (2002). “Capitalization versus Expensing: Evidence on the Uncertainty of Future Earnings from Capital Expenditures versus R&D Outlays,” Review of Accounting Studies 7, 355-382.

Kulatilaka, Nalin (1988). “Valuing the Flexibility of Flexible Manufacturing Systems,” IEEE Transactions in Engineering Management 35, 250-257.

Majd, Saman, and Robert Pindyck. (1987). “Time to Build, Option Value, and Investment Decisions,” Journal of Financial Economics 18, 7-27.

Majd, Saman, and Robert Pindyck. (1989). “The Learning Curve and Optimal Production under Uncertainty,” RAND Journal of Economics 20, 331-343.

Martzoukos, Spiros H. (2000). “Real Options with Random Controls and the Value of Learning,” Annals of Operations Research 99, 305-323.

Martzoukos, Spiros H. (2003). “Real R&D Options with Endogenous and Exogenous Learning.” In D. A. Paxson (ed.), Real R&D Options. Butterworth-Heinemann Quantitative Finance Series.

McDonald, Robert and Daniel Siegel. (1984). “Option Pricing When the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note,” Journal of Finance 39, 261–265.

Merton, Robert C. (1973). “An Intertemporal Capital Asset Pricing Model,” Econometrica 41, 867–887.

Myers, Stewart C., and Saman Majd. (1990). "Abandonment Value and Project Life," Advances in Futures and Options Research 4,1-21.

Pennings, Enrico, and Onno Lint. (1997). "The Option Value of Advanced R&D," European Journal of Operational Research 103, 83-94.

Pindyck, Robert S. (1993). "Investments of Uncertain Cost," Journal of Financial Economics 34, 53-76.

Ritchken, Peter, and Bardia Kamrad. (1991). "A Binomial Contingent Claims Model for Valuing Risky Ventures," European Journal of Operational Research 53, 106-118.

Roberts, Kevin, and Martin Weitzman. (1981). "Funding Criteria for Research, Development, and Exploration Projects," Econometrica 49,1261-1288.

Schwartz, Eduardo S., and Carlos Zozaya-Gorostiza. (2003). "Investment Under Uncertainty in Information Technology: Acquisition and Development Projects," Management Science, 49, 57-70.

Schwartz, Eduardo S., and Carlos Zozaya-Gorostiza. (2000b). "Evaluating Investments in Disruptive Technologies," UCLA Working Paper.

Schwartz, Eduardo S., and Mark Moon. (2000). "Evaluating Research and Development Investments." In M. J. Brennan and L. Trigeorgis (eds.), Project Flexibility, Agency and

Competition. New Developments in the Theory and Applications of Real Options. Oxford University Press.

Thompson, Andrew C. (1995). "Valuation of Path-dependent Contingent Claims with Multiple Exercise Decisions over Time: The Case of Take-or-Pay," Journal of Financial and Quantitative Analysis 30, 271-293.

Trigeorgis, Lenos (1991). "Anticipated Competitive Entry and Early Preemptive Investment in Deferrable Projects," Journal of Economics and Business 43, 143-156.

Trigeorgis, Lenos (1996). Real Options: Managerial Flexibility and Strategy in Resource Allocation. Cambridge: The MIT Press.

3. Sequential options with exogenous jumps and active interacting managerial control actions

Abstract

We study the interaction between learning and value-enhancing actions with random outcome before irreversible investment decisions are made. We employ diffusion process for the value of the project and we add endogenous, optimally determined, costly managerial controls to learn or enhance value. This framework allows the study of the effect on the value of firm's investment opportunities of options to change the distribution of future payoffs through for example marketing research and advertisement (or product redesign or repositioning), basic research or exploration actions and product attribute or quality enhancing actions. The framework also allows the analysis of optimal timing of such actions, optimal timing of introduction of pilot projects, early development of the complete project and abandonment options. We provide analytic formulas for sequential options with embedded control and learning actions under the assumption that project value follows either diffusion or a jump diffusion process and we investigate the decision regions that will appear under different parametization of the model. We also extend the model to complex multistage problems with path dependent actions, by developing a numerical lattice based model. The implementation for the jump-diffusion process case is provided in the appendix. We illustrate the importance of this theoretical framework through an application for the valuation of new product development.

3.1. Introduction

The real options approach to firm and project valuation extends traditional NPV analysis and accommodates managerial flexibility to react under uncertainty. For example, McDonald and Siegel (1986) value an irreversible investment opportunity when the value of the project and its costs are uncertain and the firm has the flexibility to delay investment. They show that the value of the investment opportunity can be substantially higher than the NPV with the extra value reflecting the value of waiting. There are several other cases where the real option flexibility is important like the case of flexible manufacturing systems (e.g. Kulatilaka (1988)), construction (e.g. Majd and Pindyck (1987)) R&D investments (e.g. Pennings and Lint (1997) or Childs and Triantis (1999)) and the adoption of technological innovations (Grenadier and Weiss (1997)). Recently, there is a tendency to also incorporate game theoretic interactions, optimal capital structure and other corporate policy features in these models (see for example Lambrecht, 2001, and Mauer and Sarkar, 2005)

An aspect not incorporated in many real options models is that of the value of learning and managerial intervention to enhance value or reduce costs that may have action-specific uncertainty. In the present paper we make this step and introduce managerial "control" options to enhance project value through learning or direct value enhancing actions (or efforts to reduce costs). Learning options prior to investment include investments in marketing research, R&D or exploration activities and pilot projects or experimentation of new production processes. These actions resolve uncertainty about true project value or cost enabling management to have valuable information before irreversible investment is undertaken. Childs et al. (2001) (see also Childs et al., 2002) also model information acquisition for options on noisy claims and Epstein et al. (1999) discuss the value of market research in a real options model. Impulse-type random controls were introduced in real options by Martzoukos (2000) who also analyzes learning. Direct value-enhancing (control) actions include advertising, efforts to improve the attributes or the quality of a product or efforts to reduce cost through adoption of new technologies in production. These actions are targeting to an increase in project value albeit have a random outcome. Traditionally, the way to incorporate sequential actions to improve value in the real options framework has been through compound-growth options. Differentiation among alternative strategies is done through different growth factors. Our approach captures the action-specific uncertainty of these actions and at the same time captures interactions in the form of path-dependency i.e. one action affecting the expected average impact and volatility of another action. Abraham and Taylor (1997) create an option pricing model that incorporates the uncertain impact of exogenous events; in our case action specific uncertainty is endogenously determined in the model at optimal time. Another possible method of capturing synergies between different actions is to analyze their values separately and incorporate correlations linking the effect of different actions (e.g. in Childs et al. (1998) where they compare sequential versus parallel development).

Our setting captures the notions described in Weitzman and Roberts (1981) while also maintaining the correct adjustment for risk in the real options framework. The learning and the control actions are induced endogenously by the firm by optimally weighing the expected benefits (in terms of additional option value) compared with the additional costs; the additional induced risk is assumed to be firm specific and thus not priced. Other related papers is Childs and Triantis (1999) and Grenadier and Weiss (1997).

We first develop analytic formulas for compound-growth options with embedded learning and attribute improvement control actions in a two stage model. Our analytic model includes Geske (1979) and Longstaff (1990) as special cases and is not limited to the standard call on call case but extend to other cases like call on put, put on call and put on put. We similarly provide formulas for the case where the underlying asset follows jump diffusion with multiple sources of jumps. Furthermore, we show how to incorporate path-dependency using the analytic formulas. We then focus on the compound-growth option (call on call) case that has interesting applications in real investment problems. The analytic formulas show how learning and control actions affect the value of an investment opportunity, the probability to proceed to next stage, the probability to proceed and develop in the final stage and the probability to develop early.

Real life investment problems include multiple stages decisions with the potential for early development, optimal timing of actions, and interactions between learning and control actions. For these reasons we extend the analysis by implementing a numerical model that can be used for the evaluation of such complex cases with path dependencies. Our theoretical model is then applied in the context of new product development showing the importance of marketing research, attribute or quality improvement actions, advertisement, pilot projects, etc.

3.2. Model assumptions

In this section we set up the framework and assumptions that we will also use to develop the analytic formulas of the next section and the more general multistage model in section IV. Our first assumption relates to the stochastic process of the underlying asset (present value of project cash flows). Our results in the main text are based on diffusion case and the jump diffusion with multiple classes of jumps case is discussed in the appendix. In the possible presence of $i = MC_1, MC_2, ...MC_{N_{MC}}$ optional managerial learning or control actions the process is defined as:

$$\frac{dS_t}{S_t} = adt + \sigma dz + k_i dq_i \qquad (1)$$

Parameter a denotes the expected rate of return (capital gain) of the project (including the impact of jumps). For managerial controls we assume that they induce an additional effect on expected returns[11]. Parameter σ is the standard deviation of the rate of return, and dz is an increment to a standard Wiener process describing the exogenous uncertainty of the state variable. Parameter k_i is a random variable that represents the effect on

[11] This effect can be thought to be an effect not captured by historical information but based on managerial discretion. Effectively, the realization of this return is on managerial discretion by weighing the expected benefits with the expected costs of control actions.

project returns of control or learning action i and dq_i is a control variable that takes the value one if the action is activated and zero if not.

The PDE that the option should satisfy is

$$rV = \frac{1}{2}\sigma^2 S^2 V_{SS} + (r-\delta)SV_S + V_t + + \sum_{i=1}^{N_{MC}} E\left[[V(SY_i,t) - V(S,t)]dq_i\right] \qquad (2)$$

To derive the PDE one can follow Merton's (1976) replication argument, which imposes two further assumptions, that the intertemporal CAPM of Merton (1973b) holds and that managerial controls have firm specific risks, which are uncorrelated with the market portfolio and thus not priced. Alternatively, we can use the framework developed in Garman (1976), Cox, Ingersoll, and Ross (1985) and Hull and White (1988) that use a complete markets framework and no arbitrage arguments.

For real options valuation the latter approach is probably more suitable and avoids assumptions about the existence of a "twin" security that mimics the risk of the project cash flows and is used to replicate the option. The no-arbitrage approach maintains the expected returns should be adjusted to their certainty equivalent measure and their payoffs discounted at the risk free rate (see for example Constandinides (1978) for an application of this idea in project appraisal)[12].An opportunity cost of waiting (δ) that should be deducted from the equilibrium required rate of return (see McDonald and Siegel, 1984) is also incorporated, which may also be used to model exogenous competitive erosion to the project's cash flows (e.g., Childs and Triantis, (1999), and Trigeorgis (1996) ch.9)

Denoting the accumulated (Brownian) noise from $t = 0$ to T by Z_t we then have that asset values at a future period T will be determined by:

[12] As stated in Constantinides (1978) the project value S may or may not be traded in capital markets but the results will still hold by requiring only that assets earn the equilibrium rate of return as defined by the intertemporal CAPM.

$$\frac{S_T}{S_t} = \exp\left[(r - \delta - \frac{\sigma^2}{2})T + \sigma Z_T\right]\left[\prod_i (1 + k_i dq_i)\right] \qquad (3)$$

We assume that the effect of control actions are log-normally distributed. Each control, learning action has impact $Y = 1 + k_i$ that follows a lognormal distribution:

$$Y = (1 + k_i) \sim \log N\left(\exp(\gamma_i), \exp(\gamma_i)(\exp(\sigma_i^2) - 1)^{0.5}\right) \qquad (4)$$

The assumption of log-normally distributed controls is adopted since it allows non-negative asset values, and also, conditional on control, asset values retain log-normality. We will use (γ_i, σ_i) to denote characteristics of control or learning actions. We use $\gamma_i > 0$ to describe efforts to enhance value with random outcome. Alternatively, if *S* was interpreted as a cost , $\gamma_i < 0$ would mean efforts to reduce costs. In this study we focus on efforts to enhance project value. The special case of $\gamma_i = 0$ with $\sigma_i^2 > 0$ while methodologically similar to the control case, it is nevertheless used to capture costly learning actions i.e. resolution of uncertainty about the true project value. These formulation is consistent with a Bayesian approach and the above parameters of the lognormal distribution can be estimated as the parameters of the preposterior distribution (see Kaufman,1963, and a recent application by Davis and Samis, 2005).Alternative approach called empirical based on Hui and Berger (1983) for estimating the volatility of option prices is presented in Karolyi (1993) and is applied for financial options.

The risk neutral distribution of *S* at *T* conditional on the activation of control *i* and on the is given by:

$$\ln\left(\frac{S_T}{S_t} \mid i, n=\{n_1, n_2, \ldots, n_{N_j}\}\right) \sim N\left((r-\delta-\frac{1}{2}\sigma^2)(T-t)+\gamma_i, \sigma^2(T-t)+\sigma_i^2\right) \quad (5)$$

The distribution of returns conditional on no activation of control is found by setting $\gamma_i = \sigma_i^2 = 0$.

The boundary condition at the maturity of the option T is the maximum of the values of the decision to exercise (EE) and getting $S_T - X$ (X is the final/development cost) or abandon (A) the project for some recovery amount α of the total costs TC that have been paid until that point. The recovery amount may be a function of the costs that have been paid for enhancing the product features or resale of expertise obtained. In the most general setting we allow the firm to take decisions at N_{dec} discrete points in time before expiration of the option. The decision points can be at any point in time but for convenience we use equally spaced decision points for the numerical model with denote $t_0 = 0, t_1 = \frac{T}{N_{dec}}, t_2 = \frac{2T}{N_{dec}}, \ldots, t_{T-1} = \frac{(N_{dec}-1)T}{N_{dec}}$ to be the corresponding time where actions can be taken before the maturity of the option. For the analytic solutions with two stage problems we have decisions at $t_0 = 0$ and $t_1 < T$.Note that at time T the decision would either be exercising or abandoning the project. Growth options like pilot projects are incorporated by allowing the firm to acquire a fraction m of the project value. Growth factors can in general exist under different decisions where the firm can continue to the next stage i.e. in modes $\{W, MC_1, MC_2, \ldots, MC_{N_{MC}}\}$ and can be path-dependent. The set of all possible actions is denoted by $M = \{W, A, EE, MC_1, MC_2, \ldots MC_{NC}, W_1, W_2, \ldots W_{N_{MC}}\}$; it includes wait (W), abandon (A), exercise investment option (EE) , a set of managerial controls (MC_i) and a set of possible states of inaction after a control has been performed (W_i). Note that W_i denotes the "wait" mode after managerial enhancement action i has taken place and is used a separate action to keep track of the realized path. At any decision point in time t, and depending on the problem, the set of available choices M_t will not necessarily include all decisions and will be a subset of the superset M i.e.

$M_t \subseteq M$. The set of actions at t as well as their characteristics (e.g. impact and volatility of control and costs) is affected by the action history M_t^- ; this is practically monitored by keeping track of the previous action d_{t-1}. This allows the incorporation of path-depedency in the problem specification so that the value of the project under decision d_t, V^{d_t} is a function of the history of actions (M_t^- or d_{t-1}). The modes {EE, A} are absorbing states since under both cases the decision process stops (no further actions are performed).

3.3. Analytic formulas

We now derive valuation formulas for investment options with embedded multiple managerial controls with different characteristics in a two stage framework. We also analyze the optimal timing issue by considering early exercise and extendible options. Multiperiod extensions of the valuation formulas are feasible but involve keeping track of the decision paths and the evaluation of multivariate normal integral. For these reasons we present a numerical multistage lattice solution in the next section that can also accommodate path dependency between actions.

Here we present the compound call on call case with embedded learning, control, early exercise and abandonment at $t_0 = 0$ and $t_1 > 0$ (the appendix, section A provides other interesting cases of call on put, put on call and put on put and appendix B solutions for the jump diffusion case). At $t_0 = 0$ if the firm early exercises the investment option will get $S_0 - X$ and if it decides to abandon zero; with respect to the latter the firm is at least in the same position by deciding to wait. If the firm decides wait or managerial control action i.e. $d_0 \in M_0 \setminus \{EE, A\}$ then the payoff at an intermediary point $t_1 < T$ under decision d_1 and conditional on decision d_0 is generally defined by:

$$V^{d_1}(S_{t_1}, t_1 \mid M, M_t^-) = m(d_0, d_1)S_{t_1} + e^{-r(T-t_1)}E_{t_1}^{d_1}(S_{t_1} \mid M, M_t^-) - X(d_0, d_1) + aX_{d_0} \quad (6)$$

where $d_1 \in M_1 = \{W, EE, MC_1, MC_2, ...MC_{N_{MC}}\}$. With $d_1 = W$ the second term is basically a standard call option. With $d_1 = MC_i$ the second term is given by a modified version of the standard call option formula that we describe below (see also Martzoukos, 2003). Note that the value function at the intermediary point is a function of previous decisions; the growth factors, expected impact and volatility but also the costs are functions of previous decision. Furthermore, the model allows the incorporation of recovery of *a* percent of past paid costs. The model of Geske (1979) and Longstaff (1990) are special cases of this specification. Geske (1979) model applies zero growth factors and abandonment option and no managerial controls and Longstaff (1990) does not include growth options, managerial controls and abandonment options.

We next define a general two stage compound option valuation formula that accommodates early exercise $\{EE\}$, wait or extend $\{W\}$ and N_{MC} managerial controls at the intermediary point t_1 is as follows. X_W defines cost of delaying the option to next stage; $X_W = 0$ will be used for standard wait and $X_W > 0$. X_{EE} defines the cost of early exercise which can in general be different than X and X_i , $i = MC_1, MC_2, ...MC_{N_{MC}}$ denotes the costs for the each of N_{MC_i} controls. Additionally, define R_{d_1}, $d_1 \in \{W, EE, MC_1, MC_2, ...MC_{N_{MC}}\}$ to be the number of regions that decision d_1 optimally appears at t_1 and use L to denote the lower boundary of that region and H to denote the high boundary of that region. Then the value of the general sequential two stage option is given by:

$$
\begin{aligned}
&Call\_on\_\,Call\,(.\,|\,d_o \in M \setminus \{EE, A\} = \{W, MC_1, MC_2, ...MC_{N_{MC}}\}) = \\
&\qquad Se^{-\delta t_1+\gamma_0}\left[\sum_{l=1}^{R_{EE}}[N_l(a_{EE,1}^{L}) - N_l(a_{EE,1}^{H})]\right] \\
&\qquad - X_{EE}e^{-rt_1}\left[\sum_{l=1}^{R_{EE}}[N_l(a_{EE,2}^{L}) - N_l(a_{EE,2}^{H})]\right] \\
&\qquad + m(d_0, W)Se^{-\delta t_1+\gamma_0}\left[\sum_{l=1}^{R_W}[N_l(a_{W,1}^{L}) - N_l(a_{W,1}^{H})]\right] \\
&\qquad - X(d_0, W)e^{-rt_1}\left[\sum_{l=1}^{R_W}[N_l(a_{W,2}^{L}) - N_l(a_{W,2}^{H})]\right] \\
&\qquad + \sum_{i=1}^{N_{MC}}\left[\begin{array}{l} m(d_0, i)Se^{-\delta t_1+\gamma_0}\displaystyle\sum_{l=1}^{R_{MC_i}}[N_l(a_{\iota,1}^{L}) - N_l(a_{i,1}^{H})] \\ - X(d_0, i)e^{-\delta t_1}\displaystyle\sum_{l=1}^{R_{MC_i}}[N_l(a_{\iota,2}^{L}) - N_l(a_{i,2}^{H})] \end{array}\right] \\
&\qquad + e^{-rt_1}aX_{d_0}N(-a_{A,2}) \\
&\qquad + Se^{-\delta T+\gamma_0}\left[\sum_{l=1}^{R_W}[N_l(a_{W,1}^{L}, b_{W,1}, \rho_W) - N_l(a_{W,1}^{H}, b_{W,1}, \rho_W)]\right] \\
&\qquad - Xe^{-rT}\left[\sum_{l=1}^{R_W}[N_l(a_{W,2}^{L}, b_{W,2}, \rho_W) - N_l(a_{W,2}^{H}, b_{W,2}, \rho_W)]\right] \\
&\qquad + \sum_{i=1}^{N_{MC}}\left[Se^{-\delta T+\gamma_0+\gamma(d_0,i)}\sum_{l=1}^{R_{MC_i}}[N_l(a_{i,1}^{L}, b_{i,1}, \rho_i) - N_l(a_{i,1}^{H}, b_{i,1}, \rho_i)]\right] \\
&\qquad - \sum_{i=1}^{N_{MC}}\left[Xe^{-rT}\sum_{l=1}^{R_{MC_i}}[N_l(a_{i,2}^{L}, b_{i,2}, \rho_i) - N_l(a_{i,2}^{H}, b_{i,2}, \rho_i)]\right] \\
&\qquad + e^{-rT}\sum_{l=1}^{R_W}a(X_{d_0} + X(d_0, W))[N_l(a_{W,2}^{L}, -b_{W,2}, -\rho_W)] - N_l(a_{W,2}^{H}, -b_{W,2}, -\rho_W)] \\
&\qquad + e^{-rT}\sum_{i=1}^{N_{MC}}\sum_{l=1}^{R_{MC_i}}a(X_{d_0} + X(d_0, i))[N_l(a_{i,2}^{L}, -b_{i,2}, -\rho_i)] - N_l(a_{i,2}^{H}, -b_{i,2}, -\rho_i)]
\end{aligned}
$$

where $$a_{d_1,1}^{(L,H)} = \frac{\ln(S / S_{t_1}^{*(L,H)}(d_0, d_1)) + (r - \delta + 0.5\sigma^2)t_1 + \gamma_{d_0} + 0.5\sigma_{d_0}^2}{(\sigma^2 t_1 + \sigma_{d_0}^2)^{1/2}},$$

(7)

$$a_{d_1,2}^{(L,H)} = a_{d_1,1}^{(L,H)} - (\sigma^2 t_1 + \sigma_{d_0}^2)^{1/2}$$

$$b_{d_1,1} = \frac{\ln(S / S_T^*(d_0, d_1)) + (r-\delta)T + (\gamma_0 + \gamma(d_0, d_1)) + 0.5\sigma^2 T + 0.5(\sigma_{d_0}^2 + \sigma^2(d_0, d_1))}{(\sigma^2 T + \sigma_{d_0}^2 + \sigma^2(d_0, d_1))^{1/2}}$$

$$b_{d_1,2} = b_{d_1,1} - (\sigma^2 T + \sigma_{d_0}^2 + \sigma^2(d_0, d_1))^{1/2}$$

$\rho_{d_1} = \sqrt{\dfrac{(\sigma^2 t_1 + \sigma_{d_0}^2)}{(\sigma^2 T + \sigma_{d_0}^2 + \sigma^2(d_0, d_1))}}$ where for decision $d_1 = W$, $\sigma^2(d_0, W) = 0$ and

$$\rho_W = \sqrt{\frac{(\sigma^2 t_1 + \sigma_0^2)}{(\sigma^2 T + \sigma_0^2)}}$$

Note that $\left[\sum_{l=1}^{R_i}[N_l(a_{i,2}^L) - N_l(a_{i,2}^H)]\right], i \in M_1$ can be interpreted as the probability of reaching a particular region i at t_1, while $\sum_{l=1}^{R_i}[N_l(a_{i,2}^L, b_{i,2}, \rho_i) - N_l(a_{i,2}^H, b_{i,2}, \rho_i)], i \in M \setminus \{EE\} = \{W, MC_1, MC_2, ...MC_{N_{MC}}\}$ gives the probability of reaching region i at t_1 and also exercising investment option at T. Furthermore,

$$\sum_{l=1}^{R_i}[N_l(a_{i,2}^L, -b_{i,2}, -\rho_i) - N_l(a_{i,2}^H, -b_{i,2}, -\rho_i)], i \in M \setminus \{EE\} = \{W, MC_1, MC_2, ...MC_{N_{MC}}\}$$

denotes the probability to reach region $i \in M_1$ at t_1 and abandoning the project at T.

At the same time $m_i S e^{-\delta t_1 + \gamma_0}\left[\sum_{l=1}^{R_i}[N_l(a_{i,1}^L) - N_l(a_{i,1}^H)]\right]$ gives the expected value that the optionholder gets if it enters region $i \in M_1$ at t_1 (note that for EE $m_i = 1$) and

$\left[Se^{-\delta T+\gamma_0+\gamma(d_0,d_1)} \sum_{l=1}^{R_i} [N_l(a_{i,1}^L, b_{i,1}, \rho_i) - N_l(a_{i,1}^H, b_{i,1}, \rho_i)] \right]$ gives the expected value the optionholders gets at T given that he has passed through decision $i \in M_1 \setminus \{EE\}$ (note that for W $\gamma(d_0, d_1) = 0$) at t_1. In section A of the appendix we provide similar formulas for compound-growth options of call on put, put on call and put on put. Appendix B provides formulas for all the cases for the special case of two sequential controls using the jump diffusion assumption where all the information needed for numerical evaluation is provided.

Equation (7) as well as the generic formulas in appendix A have some "abstract" features. There are two pieces of information that need to be determined (1) the number of optimal regions for each action (2) the critical point for switching from one region to the another. The discussion that follows focuses exactly on the determination of this information for the call on call case. Similar discussion applies for other cases but we avoid it for brevity.

The critical threshold at maturity is determined by applying the value matching condition:

$$S_T^*(d_0.d_1) - X = a(X(d_0, d_1) + X_{d_0})$$

Note that depending on the path the critical trigger point at maturity will differ.

At the intermediary point, we need a graphical inspection for finding all possible regions. Although there might be special cases where we know the regions a priori-some mentioned in next section-in general there is no easy way to determine the optimal payoff (the optimal decisions might interchange at different S_{t_1}). The exact critical point of S where we switch from optimal decision i to j would be determined by solving for S that equates the payoff of the current optimal decision with the new one i.e. by applying a value matching condition of the form:

$$V^i(S_{t_1}^*(d_0, d_i), t_1 \mid M, M_t^-) = V^j(S_{t_1}^*(d_0, d_j), t_1 \mid M, M_t^-) \quad i, j \in M_1$$

Note that in general the payoff at t_1 will have the following form:

$$V^i(S,t_1 \mid M, M_t^-) = I_{M_t \in \{W, MC_i\}}\left[m_i S + S e^{-\delta(T-t_1)+\gamma(d_0,d_1)} N(\nu_1) - X_2 e^{-r(T-t_1)} N(\nu_2)\right]$$
$$- I_{M_t \in \{W, MC_i\}} X(d_0, d_i) + a(X_{d_0} + I_{M_t \in \{W, MC_i\}} X_i) N(-\nu_2)$$

with

$$v_1 = \frac{\ln(S / S_T^*(d_0,d_1)) + (r-\delta)(T-t_1) + \gamma(d_0,d_1) + 0.5\sigma^2(T-t_1) + 0.5\sigma^2(d_0,d_1)}{[\sigma^2 T + \sigma^2(d_0,d_1)]^{1/2}}$$

$$v_2 = v_1 - [\sigma^2 T + \sigma^2(d_0,d_1)]^{1/2}$$

$$I_{M_t \in \{W, MC_i\}} = 1, \text{ zero otherwise}$$

Theoretically S_{t_1} can take any values from $[0,\infty)$ and so this searching process would be practically infeasible if the payoffs intertwine as S increases with no payoff dominating the other. Fortunately we are able to determine which payoffs dominate at the two limits and so this searching process can be accomplished by only searching within a finite interval. For $S_{t_1} \to 0$ obviously the optimal decision would be to abandon the project[13]. For the upper decision region suppose for example that we find that decision i dominates other decisions for a high value of S, S^h and we want to ensure that i dominates all other decisions for any $S > S^h$. To ensure this we only need to show that the rate at which the payoff of i increases is higher than any other slope for $S > S^h$. Since the slope of the payoff function shows the increase in the payoff though an incremental increase in *S*, this

[13] Another choice would be to wait until maturity but as a practical matter this would also give a zero payoff. Furthermore if the abandonment value is some positive amount then we will always choose to abandon.

means that decision i will be preferred for a high value of S_h if its slope for any $S > S_h$ is higher than the other payoffs. From equation 6 we find that:

$$\text{Slope}_{d_1} = \frac{\partial V^{d_1}(S_{t_1} \mid .)}{\partial S} = m(d_0, d_1) + e^{-\delta(T-t_1)+\gamma(d_0,d_1)} N(a_{d_1,1})$$

Also note that for a sufficiently large *S* we have $N(a_{d_1,1}) \to 1$ for *W* or *MC* payoffs. If by moving from slope at S^h to slope evaluated at $S > S^h$ the increment in $N(a_{d_1,1})$ is negligible this means that decision will be determined from growth factor and expected impact *γ*. The optimal decision for high values of *S* thus in fact depends on $m(d_0, d_1) + e^{-\delta(T-t_{d_1})+\gamma_i}$. For decision $\{A\}$ the slope is zero for any *S* so abandonment will not be preferred for high values of *S* over all other decisions (wait for example will be preferred). Exercise of the investment option $\{EE\}$ gives a slope equal to one for all *S* i.e. there is a one to one translation of value of *S* to payoff to the option holder while for $\{W\}$ the slope would be $e^{-\delta(T-t_{d_1})} < 1$ which means that $\{EE\}$ will dominate {W} for high values of S.Our arguments make intuitive sense since as S increases the optimal decision will depend on the way each decision translates *S* into value rather than the costs of following a particular decision. In turn this means that growth factors and value enhancing impact factors (γ's) will play the most important role for high values of *S*.

Unfortunately, there is no easy answer to the question of which decision dominates for the intermediary values of *S* in the general framework we have just described since for low values of *S* all factors including costs, growth components, volatility of learning and enhancing options and the impact of controls will play their role in determining the optimal decision. The determined regions are in this cases determined though a graphical inspection of the regions. This difficulty is alleviated in the numerical solution procedure that we describe in the next section since the regions are automatically determined. The slope argument we have just discussed can give us some intuition on when some actions and in particular learning and enhancement options are likely to be more important- we discuss this, some applications of the formula as well as special cases in the next section.

3.3.1. Some special cases illustrated

The formula we have described in previous section (see equation 7) encompasses managerial control actions with random outcome, learning, early development and path-dependency in both the impact and volatility of control/learning actions and the recovery amount of abandonment. Naturally it can encompass many other cases appearing in the literature as special cases. First, the case of Geske (1979) can be calculated by applying the following parameters:

$$\begin{aligned}
&\gamma_{d_0} = \sigma_{d_0}^2 = 0 \\
&N_{MC} = R_{EE} = 0 \\
&m_W = 0 \\
&d_0 = W \\
&X(d_0, W) > 0 \\
&a = 0
\end{aligned}$$

Note that in the intermediary decision point there are two regions appearing *A*, *W* and $S_{t_1}^{*H}(d_0, W) = \infty \Rightarrow N(a_{W,1}^H, b_{W,1}; \rho) = N(a_{W,2}^H, b_{W,2}; \rho) = 0$. There are also alternative ways to get the same result, using for example a costly control action with zero impact and volatility. Also note that with $X(d_0, W) = 0$ we reduce to the case of simple European call option (if we additionally allow for $\gamma_{d_0} = \sigma_{d_0}^2 \neq 0$ we have a formula for European call option with embedded managerial control). The case of the extendible option of Longstaff (1990) which is more complex than the Geske (1979) case is obtained as follows:

$$\gamma_{d_0} = \sigma^2_{d_0} = 0$$
$$R_{EE} = R_W = 1$$
$$N_{MC} = 0$$
$$m_W = 0$$
$$d_0 = W$$
$$X(d_0, W) > 0$$
$$a = 0$$

In this case there are three regions appearing in the intermediary decision point, A¸ W, EE (the Wait mode here is equivalent with extension option in Longstaff, 1990). There are two trigger points from A to W and from W to EE that can be calculated by applying appropriate value matching conditions described in previous section: the value of Wait equated with Abandon and the value of Early Exercise with Wait; note that additionally $S^{*H}_{t_1}(d_0, EE) = \infty \Rightarrow N(a^H_{EE,1}) = N(a^H_{EE,2}) = 0$.

The interesting special case of the compound-growth option with two sequential controls, (optionally) activated at $t = 0$ and/or at the intermediate date $t = t_1$ is discussed below. The first control MC_0 has mean impact and variance of impact characteristics $(\gamma_{0,} \sigma_0)$ and can be activated at $t = 0$ at a cost X_0, and the second control (MC_1) has distributional characteristics $(\gamma_{1,} \sigma_1)$ and can be activated at $t = t_1$ at a cost X_1. Using previous notation the set of available decisions are $M_0 = \{W, MC_0\}$, $M_1 = \{A, MC_1\}$ with abandonment value for simplicity set to zero. For MC_1 we also allow a growth option i.e. if MC_1 is activated the firm gets a fraction of S equal to m_1 (e.g. a pilot project). The value of the compound-growth option conditional on the activation of control d_0 at $t = 0$ is given by:

$$Call\_on\_Call(.\,|\, d_0 \in M \setminus \{E, A\} = \{W, MC_1, MC_2, \ldots MC_{N_{MC}}\}) =$$
$$S e^{-\delta T + \gamma_0 + \gamma_1} N(a_1, b_1, \rho) - X e^{-rT} N(a_2, b_2, \rho) + m_1 S e^{-\delta t_1 + \gamma_0} N(a_1) - X_1 e^{-r t_1} N(a_2) \tag{8}$$

where

$$a_1 = \frac{\ln(S/S^*) + (r - \delta + 0.5\sigma^2)t_1 + \gamma_0 + 0.5\sigma_0^2}{(\sigma^2 t_1 + \sigma_0^2)^{1/2}},\ a_2 = a_1 - (\sigma^2 t_1 + \sigma_0^2)^{1/2}$$

$$b_1 = \frac{\ln(S/X) + (r - \delta)T + (\gamma_0 + \gamma_1) + 0.5\sigma^2 T + 0.5(\sigma_0^2 + \sigma_1^2)}{(\sigma^2 T + \sigma_0^2 + \sigma_1^2)^{1/2}},$$

$$b_2 = b_1 - (\sigma^2 T + \sigma_0^2 + \sigma_1^2)^{1/2}$$

$$\rho = \sqrt{\frac{(\sigma^2 t_1 + \sigma_0^2)}{(\sigma^2 T + \sigma_0^2 + \sigma_1^2)}}$$

The value of the option assuming MC_0 is not activated at $t = 0$ but waiting is decided is given by setting $\gamma_0 = \sigma_0 = 0$. The value of the project at $t = 0$ equals $\max\left(Call\_on\_Call(.|MC_0) - X_0, Call\_on\_Call(.|W)\right)$. The compound call option of Geske (1979) is a special case by setting $\gamma_0 = \sigma_0 = \gamma_1 = \sigma_1 = 0$, $X_0 = 0$ and $m_1 = 0$.

This case is a simple case of the general model discussed in the previous section where there is only one region at t_1 above which the managerial control will be activated. Note that the upper critical boundary is ∞ and this is what makes some terms from equation 7 for the upper boundary to disappear. Obviously in this case the managerial control payoff will lie above the abandonment payoff for all values of S where $m_1 S + S e^{-\delta(T-t_1)+\gamma_1} N(d_1) - X e^{-r(T-t_1)} N(d_2) - X_1 > 0$.

So there is only one critical value, $S^*_{MC_1}$, which is found by solving numerically the value matching condition:

$$m_1 S + S e^{-\delta(T-t_1)+\gamma_1} N(v_1) - X_2 e^{-r(T-t_1)} N(v_2) - X_1 = 0 \tag{9}$$

with

$$v_1 = \frac{\ln(S/X) + (r-\delta)(T-t_1) + \gamma_1 + 0.5\sigma^2 (T-t_1) + 0.5\sigma_1^2}{[\sigma^2 T + \sigma_1^2]^{1/2}}$$

$$v_2 = d_1 - [\sigma^2 T + \sigma_1^2]^{1/2}$$

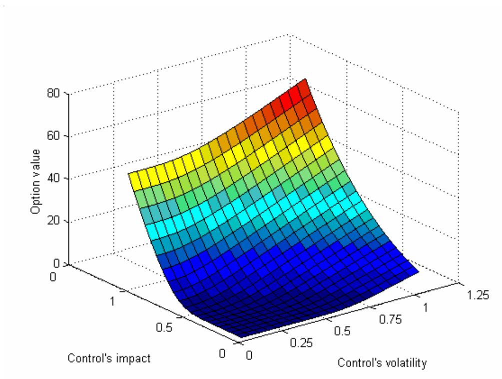
The analytic solutions for the European compound-growth call on put, put on call and put on put are similarly derived and provided in section B of the appendix.

3.3.2. Insights on region determination and the value of learning and control

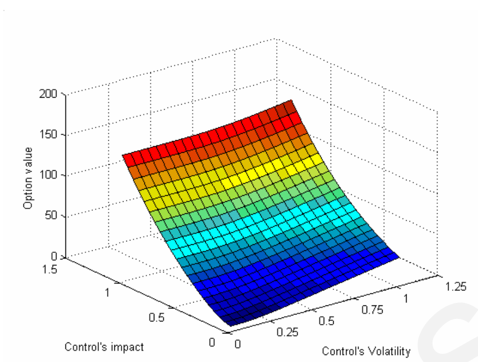
Focusing on the special case of the control-growth option described above we see in figure 1 a numerical example of the *joint* effect of the changes in impact of a managerial action γ_0 and volatility σ_0 will affect the value of the compound-growth option. As we can see the marginal impact of these managerial actions is more profound for out of the money options where a small increase in the values of either impact or volatility can give a high increase in the value of the option. For the at the money and in the money cases, the increase in value due to an increase in either impact or volatility increases more smoothly, almost linearly. Later on we show some more numerical results using the above formula for selected parameter values and discuss some implications.

Figure 1: Sensitivity of compound call option with embedded learning and control with respect to changes in control's impact and volatility

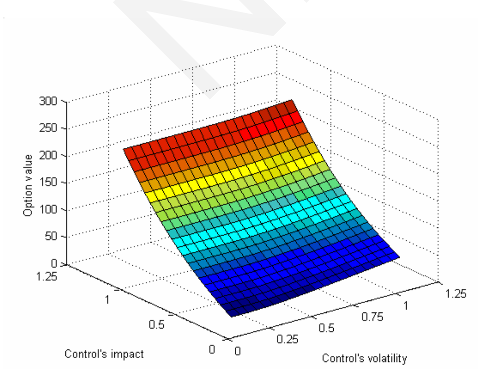
Panel a: Out of the money (*S* = 60)



Panel b: At the money (*S* = 100)



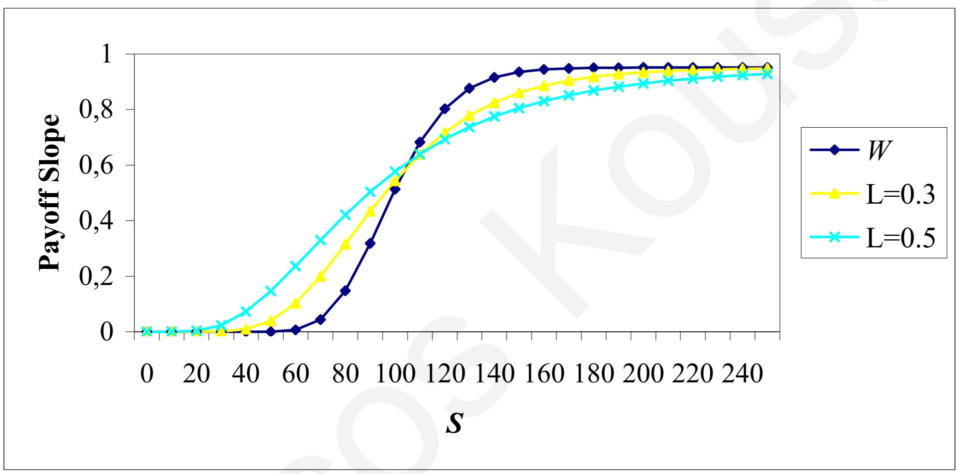
Panel c: In the money (*S* = 140)

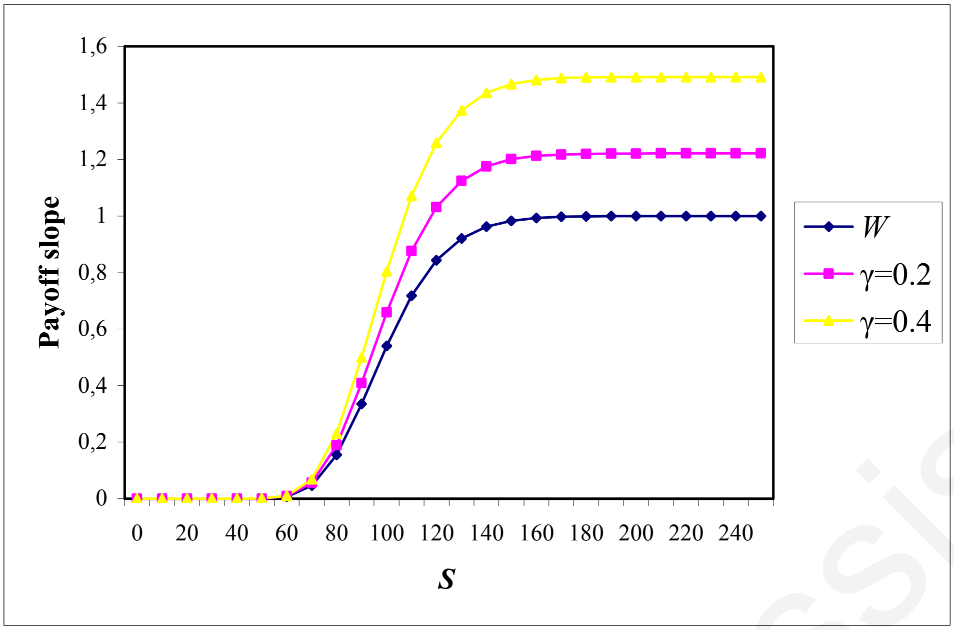


Notes: Numerical results using analytic formula for the compound option with controls (equation 8 of the main text) by varying γ_0 and σ_0. Parameters are development cost $X = 100$, cost of control $X_1 = 5$, $r = \delta = 0.05$, $\sigma = 0.1$, $t_1 = 1$ and $T = 2$.

Interesting insights can also be gained though an investigation of the slope of the paoff functions. In order to concentrate on the learning and enhancement effect let's assume $m(d_0, d_1) = 0 \ \forall d_1 \in M_1$ and δ=0. Other parameters are as follows $r = 0.05, X = 100, \sigma = 0.2, T = 2, t_1 = 1$. Figure 2, panel *a* shows the behaviour of the slope of learning payoff as a function of *S* for different values of volatility versus the slope of the wait payoff.

Figure 2: Slope of payoffs for learning versus wait as function of the underlying value





Notes: We analyze the slope of payoff function (see equation 6 and the discussion that follows in the main text). The slope shows the rate at which the payoff increases under wait strategy versus learning or value enhancing strategy. For d_1 = *W, L* use $m(d_0,d_1) = 0$. Other parameters are r = 0.05, δ = 0, X = 100, σ = 0.2, t_1=1 and T =2 .For learning use σ_L = 0.3 or σ_L = 0.5 and for the value enhancement control action use σ_{MC} = 0.3 and $\gamma = 0.2$ or $\gamma = 0.4$.

The figure clearly shows that learning is important for low values of S and reduces in importance for higher values of S where the wait decision starts to payoff more. Depending on the costs of the learning action it is then likely that a wait region at low values of S could be followed by a learning region and wait will again be optimal in the upper region. The results indicate there is a higher incremental value of learning when the uncertainty resolution is substantial and when the project is not deep in the money. In figure 1 panel *b* we focus on the behaviour of the slope for value enhancement action relative to the passive strategy of waiting. We see that the value enhancement action increases more rapidly than the wait payoff for all values of S. This means that a value enhancement action will at some point be preferred (but how valuable this strategy will be depends on the cost of the particular action). The exact regions will be determined by a graphical inspection of the payoff functions to determine the optimal regions for each decision (R_i for each $i \in M_1$) and though the use of appropriate value matching conditions to get accurate values for all "trigger" points $S^*_{t_1}$.

3.3.3. A complex example with multiple learning and managerial control actions

Now we will see how we would practically use equation 7 to evaluate a two stage problem with multiple control actions, wait and early exercise features and path-dependency. Assume we have the set of actions at $t = 0$ are $M_0 = \{W, EE, L_0, MC_0\}$. This means that the firm can either wait, early exercise, perform a learning action (for example R&D or marketing research) or engage in a managerial action to enhance value (for example advertisement or improve attributes of a product). Then we assume that at $t = t_1$ the firm can choose from the set of actions $M_1 = \{W, EE, L_1, MC_1\}$. This means that it can wait again until maturity, early exercise the investment option, perform a second learning action (L_1) or perform a second enhancement option (MC_1). Notice that in this case we allow that the firm makes all possible combinations of actions between t=0 and $t = t_1$ i.e.

Combination of actions:

$(W,W),(W,L_1),(W,MC_1),(W,EE),$
$(L_0,W),(L_o,EE),(L_0,L_1),(L_0,MC_1),$
$(MC_0,W),(MC_0,EE),(MC_0,L_1),(MC_0,MC_1)$

To be able to solve for the value of the project we need to determine the optimal regions at the intermediate point t_1 and then evaluate equation 7 three times, conditional on *W*, conditional on L_0 and conditional on MC_0. The value of the project at $t=0$ would then be calculated as the maximum of the value of *S-X*, $Call\_on\_Call(.|\,d_0=W)$, $Call\_on\_Call(.|\,d_0=MC_0)-X_{MC_0}$ and $Call\_on\_Call(.|\,d_0=L_0)-X_{L_0}$. Notice that the optimal regions and the critical points at t_1 are not affected by the decision at $t=0$; however several other variables are affected: the probabilities of reaching a region ($N(a_{i,2}^L)-N(a_{i,2}^H)\ \ i\in M_1$), the probabilities to reach a region *and* develop in the final stage ($N(a_{i,2}^L,b_{i,2},\rho_i)-N(a_{i,2}^H,b_{i,2},\rho_i)\ \ i\in M_1\,|\,\{EE\}$), the risk neutral expected value of the project if it ends in the money at t_1 (given by $m_i S e^{-\delta t_1+\gamma_0}N(a_{i,1})$) and the risk neutral expected value at *T* given that *S* pass through region *i* at t_1 (given by $\left[Se^{-\delta T+\gamma_0+\gamma_i}\sum_{l=1}^{R_i}[N_l(a_{i,1}^L,b_{i,1},\rho_i)-N_l(a_{i,1}^H,b_{i,1},\rho_i)]\right]$).

To perform a numerical investigation, assume that the initial value of the project is $S_0=100$, the development costs are $X=100$, $r=\delta=0.05$, $\sigma=0.2$ with time to maturity of the option of $T=2$ years and intermediate managerial decision point at t_1=1. For the first learning action assume $\gamma_{L_0}=0,\sigma_{L_0}=0.5$ and cost $X_{L_0}=2.5$, and for managerial enhancement option at $t=0$ assume $\gamma_{MC_0}=0.1,\sigma_{MC_0}=0.3$ with cost $X_{MC_0}=5$. For the second stage managerial control actions assume that they have the same characteristics but double costs i.e. $\gamma_{L_1}=0,\sigma_{L_1}=0.5$ and cost $X_{L_1}=5$ and

$\gamma_{MC_1} = 0.1, \sigma_{MC_1} = 0.3$ with cost $X_{MC_1} = 10$. Given these parameter values the optimal regions at t_1 are obtained by comparing the payoffs for different possible values of the realization of the value of the stochastic value of the project cash flows. This is illustrated in figure 3.

Figure 3: The payoff functions of the compound-growth option with two managerial controls, wait and early exercise at the intermediate point t_1

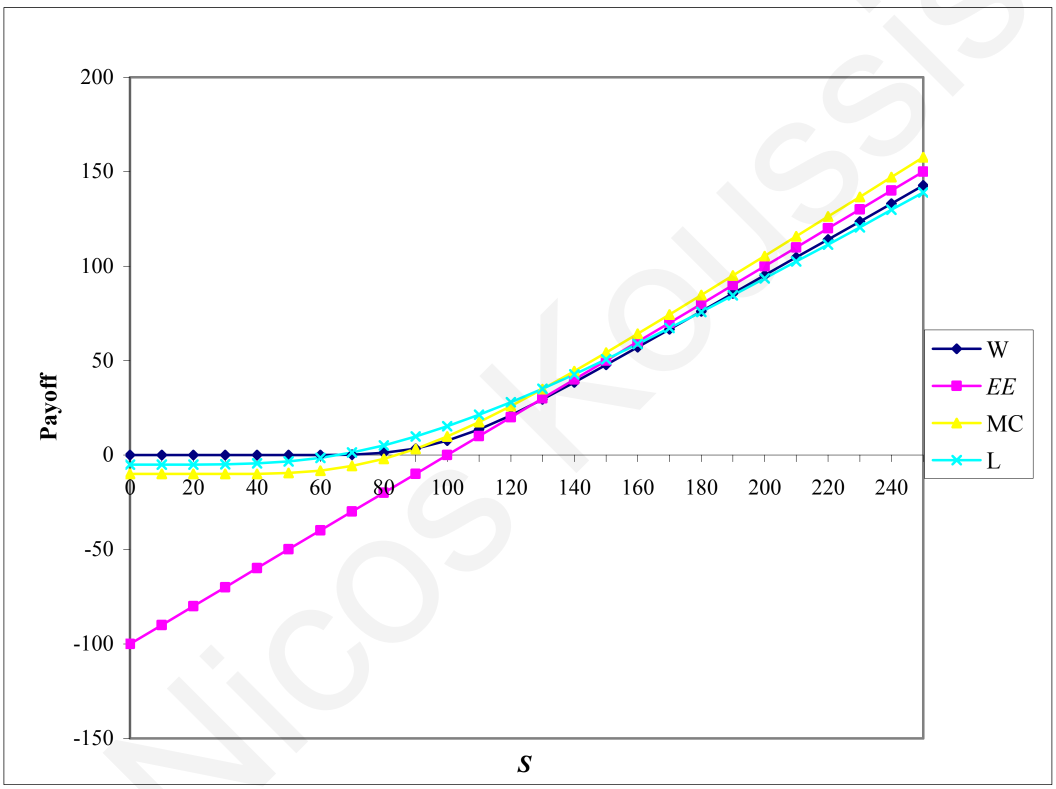
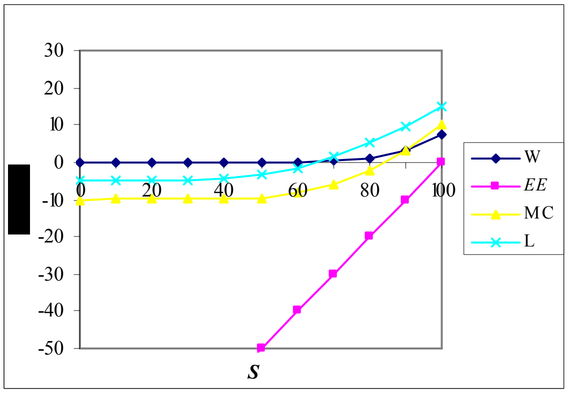
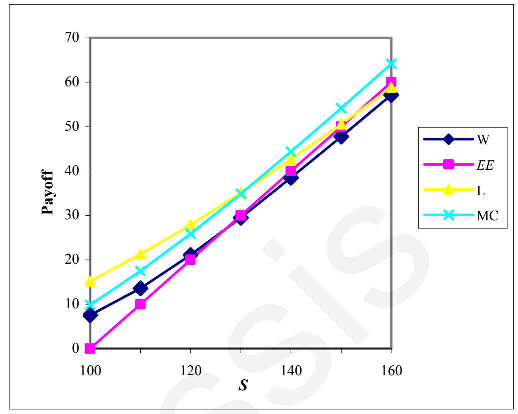


Figure 3 (cont.)

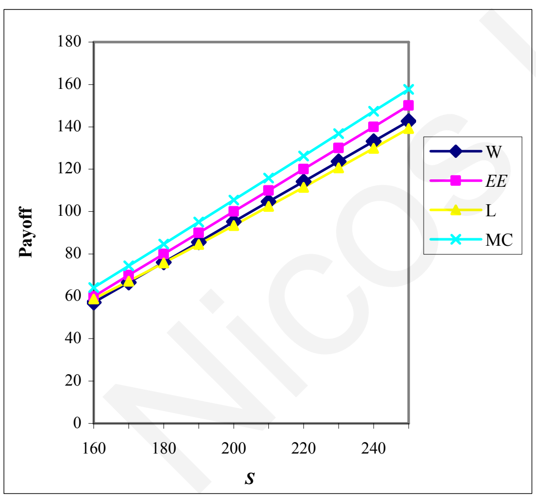
Panel a:



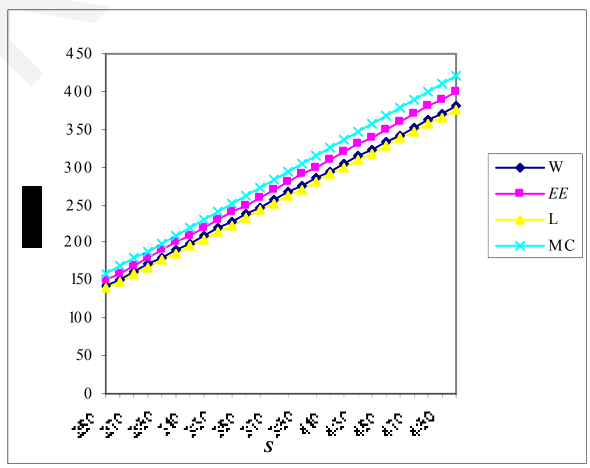
Panel b:



Panel c:



Panel d:



Notes: We investigate the payoffs for alternative decisions at $t_1 = 1$ for an investment option with maturity $T = 2$. The set of possible actions at t_1 is Wait (W), Early Exercise (EE), Managerial Control 1 (MC) or Learning 1(L). The general parameters for the problem is $S = 100$, $r = \delta = 0.05$, $\sigma = 0.2$. For learning 1 use $\sigma_L = 0.5$, $X_L = 5$ while for managerial enhancement option use $\sigma_{MC} = 0.3$, $\gamma_{MC} = 0.1$, $X_{MC} = 10$.

Figure 3 shows that there will be three possible regions of actions at t_1, W, L_1, MC_1 (the three different panels take a closer look over the regions). This means that the decision to EE is a dominated strategy and will not appear at t_1 regardless of the decision at $t = 0$. Notice that at low values of realization of the value of the project S, W will be the optimal strategy while for very high values of S MC_1 is the dominating strategy. The MC_1 payoff grows at a higher rate than any other payoff for any high values of S and so no other payoff will surpass it. This is shown by the slopes of the payoffs. For example at S = 250 we have that $\text{Slope}_{MC_1}\left|S = 250 = e^{-\delta(T-t_{d_1})+\gamma_i} N(a_{MC_1,1}) = 1.05\right.$ while for wait we have $\text{Slope}_{W}\left|S = 250 = e^{-\delta(T-t_{d_1})} N(a_{W_1,1}) = 0.951\right.$ and for learning we have $\text{Slope}_{L_1}\left|S = 250 = e^{-\delta(T-t_{d_1})} N(a_{L_1,1}) = 0.928\right.$. Given that the cumulative normal terms have almost reached their limits of one ($N(a_{W,1}) = 1$, $N(a_{MC_1,1}) = 0.998$, $N(a_{L_1,1}) = 0.976$) and that $e^{-\delta(T-t_1)+\gamma_i} > e^{-\delta(T-t_1)}$ for any $\gamma > 0$ then we should not expect the payoff of W or L_1 to surpass the payoff of MC_1 [14]. Note that the payoff of EE would dominate both the payoffs of W, and L_1 because it crosses them at some point and then it is impossible for the other payoffs to surpass it again (since they do not grow more than the EE payoff which equals one). This happens because of the positive opportunity cost δ but even if $\delta = 0$ the payoffs of W and L_1 could only grow at a rate equal to one at the limit as S tends to infinity. Finally note that the payoff of W dominates L_1 for high values of S since the learning payoff does not grow faster in the upper range. All these observations are confirmed if we plot the payoffs for even higher values of S as figure 3 panel d illustrates.

[14] Even if the slope of learning goes to one for an incremental increase in S (i.e. added 0.024) while the slope of managerial enhancement action does not change, it is still not be possible for the learning payoff to surpass the managerial enhancement payoff.

In order to determine the critical points with accuracy we apply the value matching conditions. First note that the lower boundary where W is activated is $S_{t_1}^{*L}(d_0, W) = 0$. To find the critical point where we switch from wait to learning we solve:

$$
\begin{aligned}
&V^{L_1}(S_{t_1}^{*L}(d_0, L_1), t_1 \mid M, M_t^-) = V^W(S_{t_1}^{*H}(d_0, W), t_1 \mid M, M_t^-) \\
&\Rightarrow S_{t_1}^{*H}(d_0, W) = S_{t_1}^{*L}(d_0, L_1) = 65.844
\end{aligned}
$$

Then we need to determine the highest boundary for the decision L_1. This is found by solving:

$$
\begin{aligned}
&V^{MC_1}(S_{t_1}^{*L}(d_0, MC_1), t_1 \mid M, M_t^-) = V^L(S_{t_1}^{*H}(d_0, L_1), t_1 \mid M, M_t^-) \\
&\Rightarrow S_{t_1}^{*H}(d_0, L_1) = S_{t_1}^{*L}(d_0, MC_1) = 131.096
\end{aligned}
$$

With this information we can use equation 7 to get:

$$
\begin{aligned}
&Call\_on\_Call(. \mid d_0 = W) \quad = 15.888 \\
&Call\_on\_Call(. \mid d_0 = L_0) \quad = 24.827 \\
&Call\_on\_Call(. \mid d_0 = MC_0) = 26.159
\end{aligned}
$$

After considering the costs of each strategy at $t = 0$ the optimal decision would be to perform learning at $t = 0$ and the value of the complete project at $t = 0$ is $Call\_on\_Call(. \mid d_0 = L_0) - X_{L_0} = 24.827 - 2.5 = 22.327$.

Table 1 provides additional information about the above case. Specifically it shows the probability of reaching a particular region at t_1 , the present value of costs expected to be paid at t_1, the joint probability of reaching a region at t_1 and developing or abandoning at T and the present value of the project and it's costs under each scenario expected at T.

These calculations are provided for each possible decision at $t = 0$ and may proof useful in practical applications since they provide further insights for managerial planning and budgeting purposes[15].

Table 1: Probabilities to reach regions, expected costs and expected values using the analytic formula

Panel a: Conditional on d_0 = *W*

Region at t_1	Marginal Prob.	Expected Cost at t_1	Region at t_1 & Region at *T* Joint prob. *A*	*EE*	Sum	Expected Value at *T*	Expected Cost at *T*
W	0.023	0.000	1.000	0.000	1.000	0.015	0.014
L_1	0.904	4.298	0.664	0.336	1.000	47.284	30.446
MC_1	0.073	0.694	0.937	0.063	1.000	9.709	5.667
Sum	1						

Panel b: Conditional on $d_0 = L_0$

Region at t_1	Marginal Prob.	Expected Cost at t_1	Region at t_1 & Region at *T* Joint prob. *A*	*EE*	Sum	Expected Value at *T*	Expected Cost at *T*
W	0.306	0.000	0.999	0.001	1.000	0.059	0.055
L_1	0.474	2.253	0.836	0.164	1.000	22.894	14.810
MC_1	0.220	2.093	0.796	0.204	1.000	39.503	18.417
Sum	1						

Panel c: Conditional on $d_0 = MC_0$

Region at t_1	Marginal Prob.	Expected Cost at t_1	Region at t_1 & Region at *T* Joint prob. *A*	*EE*	Sum	Expected Value at *T*	Expected Cost at *T*
W	0.105	0.000	1.000	0.000	1.000	0.036	0.036
L_1	0.639	3.038	0.763	0.237	1.000	33.408	21.401
MC_1	0.257	2.441	0.768	0.232	1.000	40.662	21.032
Sum	1						

Notes: Parameters are $S = 100$, $r = \delta = 0.05$, $T = 2$, $t_1 = 1$, $\gamma_{MC_0} = \gamma_{MC_1} = 0.1$, $\sigma_{MC_0} = \sigma_{MC_1} = 0.3$, $\sigma_{L_0} = \sigma_{L_1} = 0.5$, $X_{L_1} = 2 \cdot X_{L_0} = 5$, $X_{MC_1} = 2 \cdot X_{MC_0} = 10$. This case is evaluated using equation 7 of the main text.

[15] The results are based on calculations based on risk-neutral world. In practice, calculation of the real growth rate will be necessary.

Interestingly, if the firm decides to wait at $t = 0$ then most likely will move to learning at t_1. Exercise of learning at $t = 0$ reduces the probability to exercise the second learning (and its associate expected costs) but increases the likelihood of exercising a control action, the overall probability of development and expected value received at *T*. Compared with the decision to exercise learning at , exercise of managerial control at $t = 0$ increases the probability to exercise the second learning and control and the overall probability of development and expected value received at *T*; due to the higher cost however will not be preferred at $t = 0$.

We have also implemented the same case assuming a recovery amount in case of abandonment of 50% of past paid costs. For example, if learning is exercised at $t = 0$ the firm can recover 1,25 at t_1 while if additionally a control is exercise at t_1 the firm may recover 6.25 (=1,25+5) at *T*. Analogously path-dependent recovery costs hold if the firm exercises control at $t = 0$. If the firm decides to wait and then exercise learning or control at t_1 then it can recover 50% of the paid costs at *T*. Under this specification we have the following results (net of associated costs of each action):

$$Call\_on\_Call(.|d_0 = W) = 17{,}264$$
$$Call\_on\_Call(.|d_0 = L_0) = 23.951$$
$$Call\_on\_Call(.|d_0 = MC_0) = 23.515$$

Again, the optimal decision at $t = 0$ will be to exercise the learning action. The decision regions will be *A, W*, L_1, MC_1 at t_1 (abandon now appears in the lower region). Similar calculations for the probabilities of reaching each region and expected values and costs are possible but are avoided for brevity.

Now consider a slightly modified version of the above problem where we keep all the parameters of the problem the same and we double the costs for L_1 to $X_{L_1} = 10$ and of MC_1 to $X_{MC_1} = 20$. In this case we will have the following regions appearing at t_1: *W*,

L_1, EE, MC_1 i.e. we will also have a region where EE appears to be optimal region as figure 4 illustrates.

Figure 4: The payoff functions of the compound-growth option with two managerial controls, wait and early exercise at the intermediate point t_1: Higher costs for learning and control at t_1

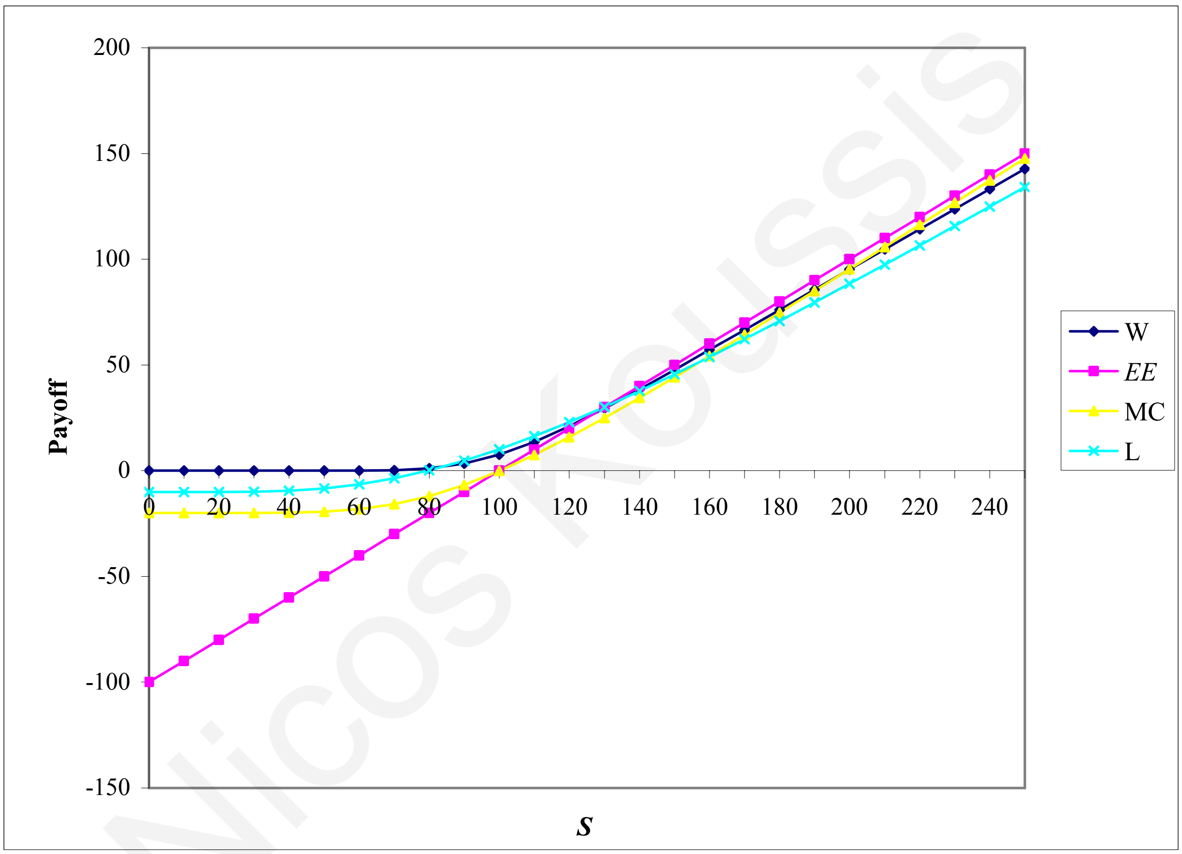
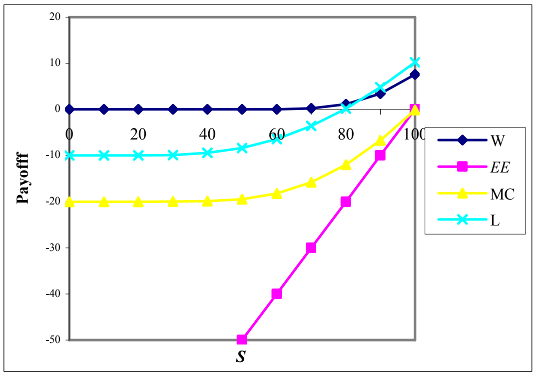
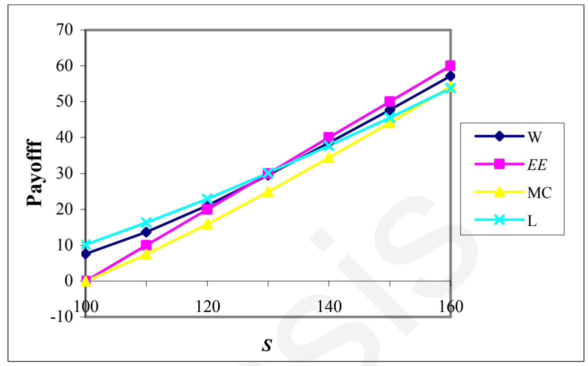


Figure 4 (cont.)

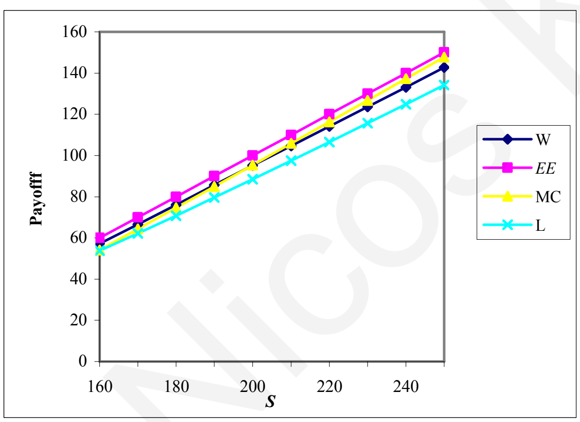
Panel a:



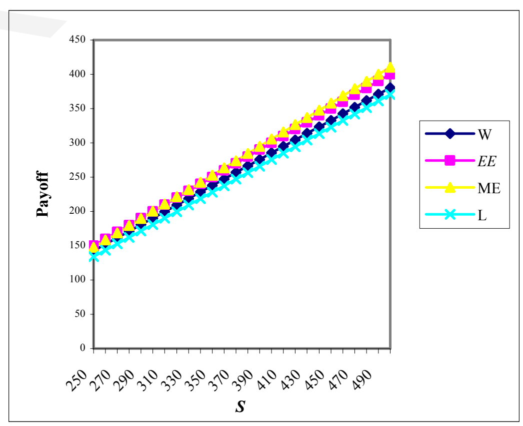
Panel b:



Panel c:



Panel d:



Notes: We investigate the payoffs for alternative decisions at $t_1 = 1$ for an investment option with maturity $T = 2$. The set of possible actions at t_1 is Wait, Early Exercise, Managerial Enhancement 1 or Learning 1 i.e. $M_1 = \{W, EE, ME_1, L_1\}$. The general parameters for the problem is $S_0 = 100, r = \delta = 0.05, \sigma = 0.2$. For learning 1 use $\sigma_{L_1} = 0.5, X_{L_1} = 10$ while for managerial enhancement option use $\sigma_{ME_1} = 0.3, \gamma_{ME_1} = 0.1, X_{ME_1} = 20$.

We know from the slope criterion that for high values of S $\text{Slope}_{MC_1} = e^{-\delta(T-t_1)+\gamma_{MC_1}} = 1.051$ while for *EE* the slope is everywhere equal to one, so that the payoff of MC_1 will surpass the *EE* payoff at some point. This is indeed confirmed in the panel d of figure 4.

Our analytic formulas where confined to a two stage investment problem. The results can be extended to multi-period problems but one would then have to evaluate the cumulative multivariate normal functions. Furthermore, one would have to check for the various combinations of actions that are increasing as the number of stages increase. Another issue is that of path dependency of the characteristics of actions. For example a managerial enhancement action may have higher impact if activated after a learning action than if activated after a passive wait decision. The analytic model we have just developed allows for path dependency between actions of period 0 and that of period at t_1. For instance, in the examples above we may allow that γ_{MC_1} is higher if the previous action was MC_0 than anything else. In the context of R&D this could reflect an increasing effectiveness of the second R&D action if some fundamental research development has first taken place.

In the next section we extend the framework to multi-period sequential managerial actions with possible path dependencies using a numerical lattice model. The numerical model allows a more effective way of searching for the best alternative and allowing complex path dependence structures and at the same time as we will see is relatively accurate.

3.4. A sequential numerical model with interacting learning and control actions

3.4.1. A generalization to multiple stages with path dependencies between actions

Now we consider an extended version of the investment problem discussed earlier to allow for multiple stages, multiple interacting learning and control actions with path depedencies, growth options, abandonment options and early development in a unified framework. We discuss a numerical method that can be used to evaluate these complex cases.

We now use the more general specification $\gamma(h,i)$ and $\sigma(h,i)$ for the description of the impact and volatility of control, conditional on the previous state *h*. With activation of action $d_t = MC_i$ at *t*, log-returns for the diffusion process will follow:

$$\ln\left(\frac{S_{t+\Delta t}}{S_t} \mid h\right) \sim N\left((r-\delta-\frac{1}{2}\sigma^2)\Delta t + \gamma(h,i), \sigma^2\Delta t + \sigma^2(h,i)\right) \qquad (10)$$

The information regarding the expected impact and volatility of controls will be determined by the sequence (path) in which the controls are being activated. Note also that we may allow the cost of each control action to be path dependent. We define the information regarding the path dependency of controls' costs in a matrix where we define the costs $x(h,i)$ that need to be paid for switching from decision h to i.

An example of path-dependency is discussed in Koussis, Martzoukos and Trigeorgis 2005. First, interpret MC_1 to be an accelerated control strategy of high impact then

$x(W,MC_1)$ it's associated cost. Additionally assume that $\{MC_2,MC_3\}$ are the first and second of two sequential investments in controls with total outcome of impact and volatility comparable to that of the first control. Total costs might however differ from $x(\mathrm{W},\mathrm{MC}_1)$ due for example to learning by doing. In general $X(\mathrm{W},\mathrm{ME}_1) > X(\mathrm{W},\mathrm{ME}_2) + X(\mathrm{ME}_2,\mathrm{ME}_3)$ can be used to model implies cost efficiencies achieved due to learning by doing. The opposite, $X(\mathrm{W},\mathrm{ME}_1) < X(\mathrm{W},\mathrm{ME}_2) + X(\mathrm{ME}_2,\mathrm{ME}_3)$, would imply scale efficiencies. Similarly, the impact or volatility might be different for the sequence compared with the accelerated strategy.

3.4.2. A lattice based numerical solution framework

We allow decisions to be made sequentially at Δt (assumed for simplicity equal) intervals. We define $V^{d_t}(.)$ the payoff the firm gets under decision $d_t = i$. This payoff is a function of the level of cash flows S at that decision point, the characteristics of available controls, the development cost X, the switching (path-dependent) control costs $X(h,i)$, the recovery rate α for the case of abandonment options, growth factors m_i of S, etc. At each decision point $t < T$ we wish to maximize the value of the investment by making the optimal pre-investment learning/exploration and/or control actions:

$$V^*(S_t, t \mid M, M_t^+, M_t^-) = \max_{M_t}\{V^{d_t}\} \tag{12}$$

We have the following cases for $V^{d_t}(.)$:

$$V^{d_t}(S_t, t \mid M, M_t^+, M_t^-) =$$
$$e^{(-r\Delta t)} E_t^{d_t}\left[V^*(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-)\right] + m_{d_t} S_t - X(d_{t-\Delta t}, d_t) \tag{13a}$$

for $d_t \in \{MC_1, MC_2, ..., MC_{N_{ME}}\}$,

$$V^{d_t}(S_t, t \mid M, M_t^+, M_t^-) = S_t - X \tag{13b}$$

for $d_t \in \{EE\}$,

$$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) = \alpha_t TC_t(M_t^-) \tag{13c}$$

for $d_t \in \{A\}$,

where $TC(M_t^-)$ are the total costs paid for learning or value enhancing actions until t and

$$V^{d_t}(S_t, t \mid M, M_t^+, M_t^-) = e^{(-r\Delta t)} E_t\left[V^*(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-)\right] \tag{13d}$$

for $d_t \in \{W_1, W_2, ..., W_{N_c}\}$.

Finally, at the last decision point at t = T, the optimal values are given by the terminal condition:

$$V^{d_T}(S_T, T \mid M, M_T^+, M_T^-) = \max(S_T - X, a\, TC(M_T^-)) \tag{13e}$$

We can see that equations (13) incorporate path dependent costs, impact and volatility of controls and learning, early development options and abandonment options to recover a fraction α of the total investments in controls and pilot projects. Expectation when $d_t \in \{MC_1, MC_2, ..., MC_{N_{ME}}\}$ is taken with respect to the distribution of log-returns that depend on the specification chosen for the exogenous process and the impact of controls; for the case of no control with $d_t \in \{W, W_1, W_2, ..., W_{N_c}\}$, expectation is taken excluding the impact of controls. Note further that for $d_t \in \{EE, A\}$ the expectation operator returns zero (these are terminal/absorbing states with no feasible continuation of decisions)[16].

In order to find project value at t = 0, we should the value functions in equation (13) is evaluated for each decision mode, at each decision point in time and for each state of the underlying asset S. Due to the presence of path dependency, $V^{m_i^*}$ cannot be evaluated in the usual backward solution method of dynamic programming. Instead, we must take into account all alternative combinations of actions and paths of the state-variable. We thus implement a forward-backward looking algorithm of exhaustive search (alternatively, see Hull and White, 1993, or Thompson, 1995), and the optimal decision will determine today's option value.

In order to evaluate the expectation operator defined in equations (13) we need a discretized state-space and we use a numerical lattice scheme. From equation (10) the underlying asset S has a lognormal distribution between decision points. We approximate this distribution between steps with a binomial lattice with N_{sub} number of steps, with total number of steps N equal to $N_s\, N_{sub}$. The conditional volatilities $v^2(d_t, d_{t+\Delta t})$ between decision points for the diffusion case are:

$$v^2(d_t, d_{t+\Delta t}) = \sigma^2 \frac{T_{sub}}{N_{sub}} + \frac{\sigma^2(d_t, d_{t+\Delta t})}{N_s}, \tag{14}$$

for $d_t \in \{MC_1, MC_2, ..., MC_{N_{ME}}\}$

[16] We demonstrate the implementation for the jump diffusion case in the appendix.

The specification in (14) allocates the volatility of control actions and jumps to N_{sub} points for a total uncertainty of $\sigma^2(m_t, m_{t+\Delta t})$. When controls are not activated we just set the volatility of controls to zero.

Furthermore we use the following up and down moves for the lattice between stages:

$$up(d_t, d_{t+\Delta t}) = \exp\left(v(d_t, d_{t+\Delta t})\right), \qquad down(d_t, d_{t+\Delta t}) = \frac{1}{u(d_t, d_{t+\Delta t})}$$

Finally the probabilities for an up and down move (diffusion case) for $d_t \in \{MC_1, MC_2, ..., MC_{N_{ME}}\}$ are:

$$p_u(d_t, d_{t+\Delta t}) = \frac{\exp\left((r-\delta)\frac{T_{sub}}{N_{sub}} + \frac{\gamma(d_t, d_{t+\Delta t})}{N_{sub}}\right) - down(d_t, d_{t+\Delta t})}{up(d_t, d_{t+\Delta t}) - down(d_t, d_{t+\Delta t})},$$

$$p_d(d_t, d_{t+\Delta t}) = 1 - p_u(d_t, d_{t+\Delta t})$$

while for $d_t \in \{W, W_1, ..., W_{N_c}\}$ we set the γ and σ parameters of controls to zero.

With this specification between decision points for the sub-lattice construction we are able to incorporate the asset price and embedded control actions and evaluate the expectation in equations (13).

In the next section we test the accuracy of the numerical model with the analytic solutions provided in the section II. Then, we discuss the importance of options to learn and enhance value by analyzing the new product development case.

3.5. Numerical results and applications

Our first set of numerical results shows the accuracy of the numerical model by comparing with the results of the analytic formula for the compound option with learning or control. In the next section we analyze a realistic multi stage application for new product development.

3.5.1. A comparison of the analytic and numerical model

Table 2 shows the comparison between the analytic and lattice based numerical model for the case of a compound-growth option with learning. At the intermediate date T_1, besides the value of the option to invest at the terminal date, the firm may also acquire a fraction m of the project value (a pilot project). In our problem specification, the firm cannot take the investment option unless it pays X_1, which we interpret as the cost of getting the pilot project cash flows plus resolving uncertainty for the final project. The first panel provides results for the case where no growth option is available (only learning) and the second panel considers the case with an option to acquire a fraction $m = 0.1$ of the project value, plus learning. We can see that the numerical model provides a very good approximation to the analytic formulas in both cases. Focusing on the first panel we note that the case of zero volatility of control and zero impact reflects the case of the compound option of Geske (1979). The results show that when the volatility are positive the value of learning options embedded in investment options can be extremely important (this result will be even more considerable with positive expected impact). Taking for example the case where S = 100 we see that compared to the case of a simple compound option with no learning, the value of the compound option with a learning potential (volatility) of 0.1 increases by more than 50%, while a learning potential (volatility) of 0.2 increases value by 242%. In the second panel we see that the availability of growth options besides learning can further enhance project values. It captures the realistic case where a pilot project provides learning. Overall, the results indicate that project value can be substantially underestimated if learning, control, and other project attributes like growth options are neglected. If we interpret the learning action as marketing research, the higher the uncertainty that marketing research will resolve for a given cost the more likely that it will be performed. In the next section we investigate more complex investment decision scenarios in the context of new product development.

Table 2: Compound option with learning: Comparison of numerical and analytic values

Time	Vol. of Control	S = 80 Analytic	S = 80 Numerical	S = 100 Analytic	S = 100 Numerical	S = 120 Analytic	S = 120 Numerical
		Growth option factor *m* = 0					
***T* = 1**	0.000	0.000	0.000	1.103	1.094	14.320	14.315
	0.100	0.001	0.001	1.656	1.662	14.839	14.838
	0.200	0.016	0.015	3.773	3.774	16.864	16.863
	0.300	0.282	0.282	7.079	7.093	19.883	19.892
	0.400	1.743	1.753	10.660	10.678	23.341	23.357
	0.500	4.424	4.438	14.266	14.288	26.991	27.012
***T* = 2**	0.000	0.013	0.013	2.123	2.118	14.100	14.094
	0.100	0.027	0.026	2.648	2.654	14.675	14.675
	0.200	0.126	0.127	4.406	4.410	16.547	16.550
	0.300	0.616	0.618	7.203	7.214	19.310	19.319
	0.400	2.038	2.050	10.447	10.461	22.506	22.519
	0.500	4.400	4.416	13.792	13.811	25.906	25.924
***T* = 5**	0.000	0.302	0.300	3.860	3.859	13.635	13.643
	0.100	0.378	0.380	4.244	4.244	14.091	14.094
	0.200	0.668	0.664	5.427	5.439	15.462	15.464
	0.300	1.338	1.338	7.339	7.348	17.560	17.566
	0.400	2.552	2.563	9.747	9.758	20.091	20.102
	0.500	4.319	4.328	12.402	12.418	22.850	22.866
		Growth option factor *m* = 0.1					
Time	**Vol. of Control**	**S = 80 Analytic**	**S = 80 Numerical**	**S = 100 Analytic**	**S = 100 Numerical**	**S = 120 Analytic**	**S = 120 Numerical**
***T* = 1**	0.000	2.964	2.963	8.670	8.662	25.992	25.989
	0.100	3.220	3.218	10.239	10.243	26.537	26.536
	0.200	4.511	4.511	13.344	13.349	28.567	28.567
	0.300	6.707	6.712	16.827	16.841	31.587	31.595
	0.400	9.357	9.368	20.413	20.431	35.045	35.061
	0.500	12.223	12.237	24.019	24.041	38.695	38.716
***T* = 2**	0.000	3.133	3.134	9.857	9.846	25.407	25.401
	0.100	3.506	3.505	11.001	11.005	26.045	26.044
	0.200	4.796	4.797	13.576	13.584	27.957	27.961
	0.300	6.838	6.843	16.674	16.685	30.725	30.734
	0.400	9.299	9.310	19.957	19.972	33.921	33.934
	0.500	11.973	11.989	23.304	23.323	37.321	37.339
***T* = 5**	0.000	3.946	3.951	11.345	11.331	23.977	23.986
	0.100	4.320	4.320	12.004	12.009	24.510	24.512
	0.200	5.400	5.403	13.699	13.705	25.991	25.995
	0.300	7.029	7.033	15.970	15.978	28.137	28.144
	0.400	9.006	9.014	18.527	18.540	30.679	30.691
	0.500	11.187	11.199	21.220	21.236	33.440	33.456

Notes: Parameters are $r = \delta = 0.05$, $\sigma = 0.10$, $t_1 = T/2$, $\gamma = 0$ and cost of control $X_1 = 5$. For the numerical lattice we use $N_{sub} = 60$ steps.

3.5.2. The new product development case

In this section we employ the numerical model and we discuss the case of new product development by incorporating more complex and realistic features than in the previous cases. First, we take the scenario where $T = 5$ and $\sigma_{MC} = 0.3$ as base case and we extend it in several dimensions while maintaining only two decision points (at *t* = 0 and $t = T/2$). Then we will extend the framework adding more decision points and more path-dependency.

The first two columns of Table 2 provide the project´s option value at *t* = 0 for a simple base case, where the firm can only choose to activate a learning action (*L*) at T_1, and it can only wait (*W*) at *t* = 0. The first extension we consider is the optimal timing of learning when early development is also possible. The set of all possible sequence of actions are given in panel a of figure 5.

Figure 5: A two stage investment problem with learning (*L*), managerial control (*MC*), and early development (*EE*) in: The set of possible actions

Panel (a): Optimal timing of learning (*L*) and early development (*EE*) with <u>no</u> control action (*MC*)

t = 0	*t* = T1
W	*W*
W	*EE*
W	*L*
L	*W*
L	*EE*

Panel (b): Optimal timing of learning (*L*), early development (*EE*), and control action (*MC*)

t = 0	t = T1
W	*W*
W	*EE*
W	*L*
W	*C*
L	*W*
L	*EE*
L	*MC*
MC	*W*
MC	*EE*

In column 3 and 4 of Table 3 we provide numerical results for this scenario of the optimal timing of learning and development case. In comparison with the results of the base case we see that optimal values are enhanced and optimal decisions may differ; *L* and *EE* may now be optimal at $t = 0$. Another extension concerns the availability of other actions to learn or enhance value. For example the firm may have the option to activate two learning actions sequentially at $t = 0$ and $t = T/2$. Alternatively, the firm may have the option to learn initially and then enhance project value by a control action. We concentrate on the case where the firm can activate both a learning action and a control. The set of all possible combinations of marketing research (learning), improvement actions (controls) and early development that can be made are given in panel b of figure 5.Note that the case described in panel b is substantially more complex. For example it allows the firm to perform *L* and then choose at the intermediary decision point between *MC*, *W*, or *EE*. Columns 5 and 6 of Table 3 provide results for this case where the characteristics of control are $\gamma_{MC} = 0.1$ with σ_{MC} =0.3.

Table 3: Project value for four different scenarios with increasing flexibility and impact

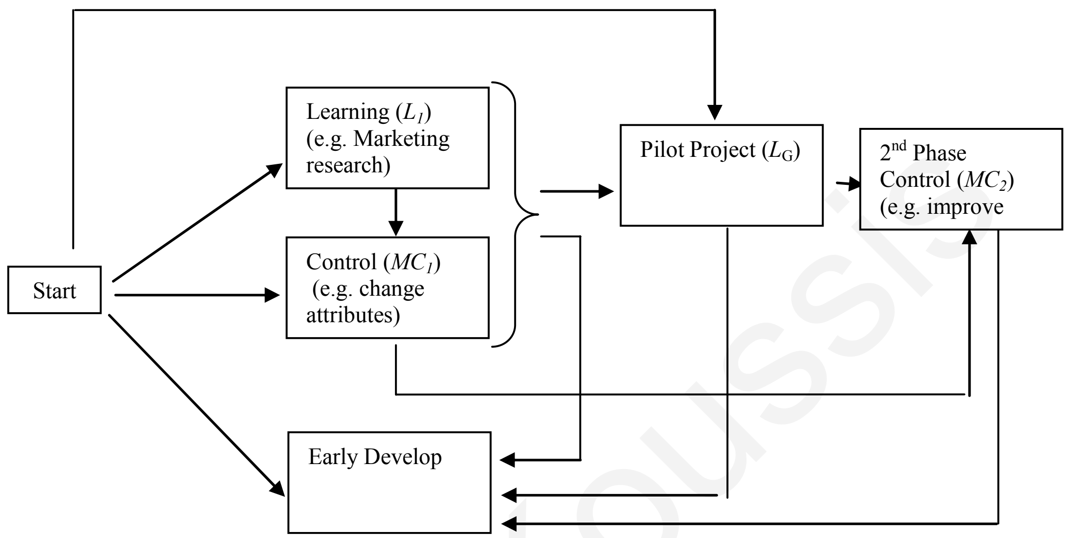
	Learning only				Learning and control			
	(I)		(II)		(III)		(IV)	
	L only at t_1		Timing of L and EE		Timing of L, MC and EE		Diff. impact of MC after L	
S	Value	Dec. at t=0	Value	Dec. at t=0	Value	Dec. at t=0	Value	Dec. at t=0
240	109.032	*W*	140.000	*EE*	140.000	*EE*	140.000	*EE*
230	101.245	*W*	130.000	*EE*	130.000	*EE*	130.000	*EE*
220	93.457	*W*	120.000	*EE*	120.000	*EE*	120.000	*EE*
210	85.671	*W*	110.000	*EE*	110.000	*EE*	110.000	*EE*
200	77.886	*W*	100.000	*EE*	100.000	*EE*	100.000	*EE*
190	70.106	*W*	90.000	*EE*	90.000	*EE*	90.000	*EE*
180	62.336	*W*	80.000	*EE*	80.000	*EE*	80.690	*L*
170	54.589	*W*	70.000	*EE*	70.000	*EE*	71.520	*L*
160	46.891	*W*	60.000	*EE*	60.000	*EE*	62.472	*L*
150	39.292	*W*	50.000	*EE*	50.000	*EE*	53.567	*L*
140	31.889	*W*	40.000	*EE*	40.480	*MC*	44.896	*L*
130	24.806	*W*	30.000	*EE*	31.670	*MC*	36.417	*L*
120	18.223	*W*	20.000	*EE*	23.362	*MC*	28.458	*L*
110	12.401	*W*	13.272	*L*	15.592	*MC*	20.911	*L*
100	7.503	*W*	7.824	*W*	8.656	*W*	13.954	*L*
90	3.833	*W*	3.912	*W*	4.322	*W*	7.934	*L*
80	1.489	*W*	1.499	*W*	1.618	*W*	2.999	*L*
70	0.377	*W*	0.378	*W*	0.395	*W*	0.395	*W*
60	0.052	*W*	0.052	*W*	0.053	*W*	0.053	*W*
50	0.003	*W*	0.003	*W*	0.003	*W*	0.003	*W*

Notes: Parameters are $r = \delta = 0.05$, $\sigma = 0.10$, $T = 5$ and $t_1 = T / 2$, $\sigma_L = 0.30$ with $X_L = 5$ for all cases. For case I there is no early development or timing of learning; learning is available only at t_1. For case II there is optimal timing of learning and development option. For case III there is optimal timing of learning, control and development option with $\gamma(W,MC) = \gamma(L,MC) = 0.1$ and $\sigma(W,MC) = \sigma(L,MC) = 0.30$. Case IV is the same as case III but the control characteristics are different if prior action is L i.e $\gamma(W,MC) = 0.1$, $\sigma(W,MC) = 0.30$, $\gamma(L,MC) = 0.2$, $\sigma(L,MC) = 0.30$. For the numerical lattice we use $N_{sub} =$ 30 steps.

The results show that option values change and more importantly that there is a large region where it pays to proceed with further improvement actions (*MC*) immediately. The last two columns of the table provide numerical results for the case where the characteristics of control after learning action are different from the case where the firm proceeds directly to control). Specifically, if the firm performs control directly it gains $\gamma(W,MC) = 0.1$, and if the firm performs control after marketing research it gains $\gamma(L,MC) = 0.2$ while the volatility of control remains the same in both cases to $\sigma(W,MC) = \sigma(L,MC) = 0.3$. The results indicate a large change in optimal values and optimal decisions. Under this scenario there is a large region where it is optimal to go for marketing research first so that the firm can later capture a higher effectiveness of control.

Next, we consider a complex scenario with 5 decision points, 2 learning actions and 2 controls with optimal timing and path dependency. Figure 6 gives a general description of the problem and a base case specification of the parameters of the problem. There are two learning (L_1, L_G) and two control actions (MC_1, MC_2). In the first phase we can activate either L_1 or MC_1, then we can proceed with a pilot project that will give a fraction m of the project cash flows S and at the same time will create a learning effect (L_G). The first phase actions can be skipped altogether and the firm can move directly to the pilot project or even to early development. Furthermore, we allow the firm to also activate a second phase of actions The firm can move from a learning action to a control, specifically from L_1 to MC_1, and from L_G to MC_2. The volatility of the pilot project is set to be double the volatility of the first learning action (reflecting the fact that the pilot project is expected to be more effective in revealing the true demand level). If instead a first phase of learning action has already been activated then the first action resolves half of the total uncertainty and the other half can be optionally revealed through the pilot project. The volatilities of control actions are all set to 0.30. For the impact of controls we assume that the impact of controls doubles if learning has been performed.

Figure 6: A multi stage investment option with multiple interacting learning and control actions for the new product development problem



Notes: The firm has the option to invest in a first phase of learning or controls (L_1 , C_1), develop the project early (*EE*), invest in a pilot project (L_G) and invest in a second phase control action C_2 . Base case parameters are $r = \delta = 0.05$, $\sigma = 0.1$ and $T = 5$ and the cost for each action is $X_{L_1} = X_{MC_1} = X_{MC_2} = 10$ and $X_G = 20$. Growth factor of pilot project is $m = 0.1$. The average impact and volatility of learning and control actions are given in the following matrices.

Volatility matrix of learning and control actions

From \ To	L_1	L_G	MC_1	MC_2
W	$(0.3)^2$	$2(0.3)^2$	$(0.3)^2$	-
L_1	-	$(0.3)^2$	$(0.3)^2$	-
L_G	-	-	-	$(0.3)^2$
MC_1	-	-	-	$(0.3)^2$

Mean impact matrix of control actions

From \ To	L_1	L_G	MC_1	MC_2
W	0	0	0.1	-
L_1	-	0	0.2	-
L_G	-	-	-	0.2
MC_1	-	-	-	0.1

The costs of each the actions are $X_{L_1} = X_{MC_1} = X_{MC_2} = 10, \quad X_G = 20$ and the maturity of the option is $T = 5$. Our numerical results provide sensitivities with respect to the growth option parameter m, and the importance of learning actions to enhance the impacts of controls that are reflected in parameters $\gamma(L_1, MC_1)$ and $\gamma(L_G, MC_2)$. Table 4 provides sensitivity with respect to the effectiveness of learning action while keeping the growth option potential to $m = 0.1$.

Table 4: Multistage investment program with a pilot project option and two phases of learning and controls: Sensitivity with respect to the effectiveness of learning actions

	Growth m = 0.1					
	$\gamma(L_G, MC_2) = 0.1$				$\gamma(L_G, MC_2) = 0.2$	
	$\gamma(L_1, MC_1) = 0.1$		$\gamma(L_1, MC_1) = 0.2$		$\gamma(L_1, MC_1) = 0.2$	
S	Value	Dec. at t=0	Value	Dec. at t=0	Value	Dec. at t=0
240	155.495	MC_1	164.770	L_1	170.096	L_G
230	144.523	MC_1	153.281	L_1	158.230	L_G
220	133.564	MC_1	141.814	L_1	146.400	L_G
210	122.629	MC_1	130.381	L_1	134.615	L_G
200	111.747	MC_1	118.995	L_1	122.890	L_G
190	100.925	MC_1	107.689	L_1	111.274	L_G
180	90.153	MC_1	96.446	L_1	99.761	L_G
170	79.471	MC_1	85.296	L_1	88.365	L_G
160	68.965	MC_1	74.308	L_1	77.109	L_G
150	58.589	MC_1	63.483	L_1	66.080	L_G
140	48.468	MC_1	52.891	L_1	55.347	L_G
130	38.700	MC_1	42.644	L_1	44.903	L_G
120	29.294	MC_1	32.784	L_1	34.831	L_G
110	20.575	MC_1	23.535	L_1	25.390	L_G
100	12.514	MC_1	14.982	L_1	16.521	L_G
90	6.599	W	7.840	W	9.006	W
80	2.720	W	3.198	W	3.951	W
70	0.755	W	0.849	W	1.180	W
60	0.113	W	0.119	W	0.179	W
50	0.006	W	0.006	W	0.009	W
40	0.000	W	0.000	W	0.000	W

Notes: see problem description and base case parameters in figure 6

The first two columns show the results when the learning actions cannot improve the impact of the controls that are activated in next stages; columns 3 and 4 provide option values and optimal decisions for the case where only L_1 can improve the impact for MC_1 and the last two columns when both L_1 and L_G can improve the impact, for MC_1 and MC_2 respectively. The results show that if learning does not provide any additional value-enhancement for the control actions, then it is likely that it will be skipped and the firm will proceed to the controls immediately. If instead, L_1 provides a better impact for the control actions then it is likely that the firm will proceed with learning at $t = 0$. The pilot project L_G will not be preferred over L_1 at $t = 0$ unless it also provides an improved impact for the second phase control MC_2 as well. In Table 5 we provide sensitivity with respect to the level of the growth factor.

Table 5: New product development (investment program with a pilot project option and two phases of learning and control): Sensitivity with respect to the level *m* of pilot project cash flows

	$\gamma(L_G,MC_2) = 0.2$, $\gamma(L_1,MC_1) = 0.2$					
	Growth $m = 0$		**Growth $m = 0.1$**		**Growth $m = 0.2$**	
S	**Value**	**Dec. at t=0**	**Value**	**Dec. at t=0**	**Value**	**Dec. at t=0**
240	164.770	L_1	170.096	L_G	194.096	L_G
230	153.281	L_1	158.230	L_G	181.230	L_G
220	141.814	L_1	146.400	L_G	168.400	L_G
210	130.381	L_1	134.615	L_G	155.615	L_G
200	118.995	L_1	122.890	L_G	142.890	L_G
190	107.689	L_1	111.274	L_G	130.274	L_G
180	96.446	L_1	99.761	L_G	117.761	L_G
170	85.296	L_1	88.365	L_G	105.365	L_G
160	74.308	L_1	77.109	L_G	93.109	L_G
150	63.483	L_1	66.080	L_G	81.080	L_G
140	52.891	L_1	55.347	L_G	69.347	L_G
130	42.644	L_1	44.903	L_G	57.903	L_G
120	32.784	L_1	34.831	L_G	46.831	L_G
110	23.535	L_1	25.390	L_G	36.390	L_G
100	14.982	L_1	16.521	L_G	26.521	L_G
90	7.840	*W*	9.006	*W*	17.430	L_G
80	3.198	*W*	3.951	*W*	9.360	*W*
70	0.849	*W*	1.180	*W*	3.920	*W*
60	0.119	*W*	0.179	*W*	1.002	*W*
50	0.006	*W*	0.009	*W*	0.105	*W*
40	0.000	*W*	0.000	*W*	0.002	*W*

Notes: see problem description and base case parameters in figure 6

As expected, the higher the growth factor the more likely that we will proceed with the pilot project immediately.

3.6. Conclusions

We analyze investment options with embedded learning (explorative research, marketing research, etc,) and control (attribute or quality improvement, advertisement etc.) actions. The paper extends the analysis of investment options to provide analytic solutions for compound options with embedded optional pilot project, learning, and control actions, early development and abandonment when the project value follows diffusion or jump diffusion process. Geske (1979) and Longstaff (1990) are special cases. We show that the availability of options to learn and control can substantially affect project option values and optimal decisions. We demonstrate the incorporation of path-dependency in the impact, volatility and cost of actions and we extend the results for multiperiod sequential options using a numerical lattice. Within this extended framework we demonstrate the importance of learning actions like exploration activities, investigative R&D, marketing research that can be launched prior to value-enhancing investments (attribute enhancing R&D, advertising activities, etc.).

Appendices

Appendix A: General valuation formulas for the call on put, put on put and put on put cases

In this section we provide analytic valuation formulas for the general case of the European compound-growth options (under Geometric Brownian motion assumptions) conditional on controls activated at $t = 0$ and with optional actions ($W, MC_1, MC_2, ...MC_{N_{MC}}$) at t_1. The formulas are for the cases of call on put, put on call and put on put. For the case of the call on put the optionholder has the option to pay a fixed amount X_i and a fraction m_i of S at t_1 and acquire a put option that expires at T or early exercise and get $X - S$. S should in this case be interpreted as a cost and controls are directed towards a decrease in S (*γ's* are expected to be negative in this case). For the case of a put on call, the optionholder has the option at t_1 to give up a call (short a call), pay a fraction m_i of the underlying asset value and get X_i or early exercise and get $X - S$. S should again be interpreted as a cost and controls increase optionholder's value when are directed towards a decrease in S (*γ's* are negative). For the case of a put on put the optionholder has the option at t_1 to get X_i and a fraction m_i of the underlying and give up a put or early exercise and get $S - X$. S in this case should be interpreted as project value and the controls are directed towards an increase in project value (*γ*' s are positive). Note that readjustments on the signs of certain variables like X_i's and m_i's can give rise to other special cases with possible different economic interpretations. The formulas are generic assuming that multiple regions exist at t_1 and the actual thresholds are not specified. In general they could be found by applying value matching conditions depending on the regions (see discussion in main text). Also note that we do not explicitly consider path-dependency and abandonment options in the notation although these features can be easily incorporated in direct analogy to equation 7 of the main text.

The formulas are:

$$
\begin{aligned}
&Call\_on\_Put(.|\, d_o \in M \setminus \{E,A\} = \{W, MC_1, MC_2, ...MC_{N_{MC}}\}) = \\
&\quad X_{EE} e^{-rt_1}\left[\sum_{l=1}^{R_{EE}}[N_l(-a_{EE,2}^L) - N_l(-a_{EE,2}^H)]\right] - Se^{-\delta t_1 + \gamma_0}\left[\sum_{l=1}^{R_{EE}}[N_l(-a_{EE,1}^L) - N_l(-a_{EE,1}^H)]\right] \\
&\quad + Xe^{-rT}\left[\sum_{l=1}^{R_W}[N_l(-a_{W,2}^L, -b_{W,2}, \rho_W) - N_l(-a_{W,2}^H, -b_{W,2}, \rho_W)]\right] \\
&\quad - Se^{-\delta T + \gamma_0}\left[\sum_{l=1}^{R_W}[N_l(-a_{W,1}^L, -b_{W,1}, \rho_W) - N_l(-a_{W,1}^H, -b_{W,1}, \rho_W)]\right] \\
&\quad + \sum_{i=1}^{N_{MC}}\left[Xe^{-rT}\sum_{l=1}^{R_{MC_i}}[N_l(-a_{i,2}^L, -b_{i,2}, \rho_i) - N_l(-a_{i,2}^H, -b_{i,2}, \rho_i)]\right] \\
&\quad - \sum_{i=1}^{N_{MC}}\left[Se^{-\delta T + \gamma_0 + \gamma_i}\sum_{l=1}^{R_{MC_i}}[N_l(-a_{i,1}^L, -b_{i,1}, \rho_i) - N_l(-a_{i,1}^H, -b_{i,1}, \rho_i)]\right] \\
&\quad - X_W e^{-rt_1}\left[\sum_{l=1}^{R_W}[N_l(-a_{W,2}^L) - N_l(-a_{W,2}^H)]\right] - m_W Se^{-\delta t_1 + \gamma_0}\left[\sum_{l=1}^{R_W}[N_l(-a_{W,1}^L) - N_l(-a_{W,1}^H)]\right] \\
&\quad - \sum_{i=1}^{N_{MC}}\left[X_i e^{-\delta t_1}\sum_{l=1}^{R_{MC_i}}[N_l(-a_{\iota,2}^L) - N_l(-a_{i,2}^H)] + m_i Se^{-\delta t_1 + \gamma_0}\sum_{l=1}^{R_{MC_i}}[N_l(-a_{\iota,1}^L) - N_l(-a_{i,1}^H)]\right]
\end{aligned}
$$

$$Put\_on\_Put\,(.\,|\,d_o \in M \setminus \{E,A\} = \{W, MC_1, MC_2, ...MC_{N_{MC}}\}) =$$

$$Se^{-\delta t_1+\gamma_0}\left[\sum_{l=1}^{R_{EE}}[N_l(a^L_{EE,1}) - N_l(a^H_{EE,1})]\right] - X_{EE}e^{-rt_1}\left[\sum_{l=1}^{R_{EE}}[N_l(a^L_{EE,2}) - N_l(a^H_{EE,2})]\right]$$

$$+ Se^{-\delta T+\gamma_0}\left[\sum_{l=1}^{R_W}[N_l(a^L_{W,1},-b_{W,1},-\rho_W) - N_l(a^H_{W,1},-b_{W,1},-\rho_W)]\right]$$

$$- Xe^{-rT}\left[\sum_{l=1}^{R_W}[N_l(a^L_{W,2},-b_{W,2},-\rho_W) - N_l(a^H_{W,2},-b_{W,2},-\rho_W)]\right]$$

$$+ \sum_{i=1}^{N_{MC}}\left[Se^{-\delta T+\gamma_0+\gamma_i}\sum_{l=1}^{R_{MC_i}}[N_l(a^L_{i,1},-b_{i,1},-\rho_i) - N_l(a^H_{i,1},b_{i,1},\rho_i)]\right]$$

$$- \sum_{i=1}^{N_{MC}}\left[Xe^{-rT}\sum_{l=1}^{R_{MC_i}}[N_l(a^L_{i,2},-b_{i,2},-\rho_i) - N_l(a^H_{i,2},-b_{i,2},-\rho_i)]\right]$$

$$+ m_W Se^{-\delta t_1+\gamma_0}\left[\sum_{l=1}^{R_W}[N_l(a^L_{W,1}) - N_l(a^H_{W,1})]\right] - X_W e^{-rt_1}\left[\sum_{l=1}^{R_W}[N_l(a^L_{W,2}) - N_l(a^H_{W,2})]\right]$$

$$+ \sum_{i=1}^{N_{MC}}\left[m_i Se^{-\delta t_1+\gamma_0}\sum_{l=1}^{R_{MC_i}}[N_l(a^L_{i,1}) - N_l(a^H_{i,1})] - X_i e^{-\delta t_1}\sum_{l=1}^{R_{MC_i}}[N_l(a^L_{i,2}) - N_l(a^H_{i,2})]\right]$$

$$Put\_on\_Call(.\,|\,d_o \in M \setminus \{E,A\} = \{W, MC_1, MC_2, ...MC_{N_{MC}}\}) =$$

$$X_{EE}e^{-rt_1}\left[\sum_{l=1}^{R_{EE}}[N_l(-a^L_{EE,2}) - N_l(-a^H_{EE,2})]\right] - Se^{-\delta t_1+\gamma_0}\left[\sum_{l=1}^{R_{EE}}[N_l(-a^L_{EE,1}) - N_l(-a^H_{EE,1})]\right]$$

$$+ Xe^{-rT}\left[\sum_{l=1}^{R_W}[N_l(-a^L_{W,2},b_{W,2},-\rho_W) - N_l(-a^H_{W,2},b_{W,2},-\rho_W)]\right]$$

$$- Se^{-\delta T+\gamma_0}\left[\sum_{l=1}^{R_W}[N_l(-a^L_{W,1},b_{W,1},-\rho_W) - N_l(-a^H_{W,1},b_{W,1},-\rho_W)]\right]$$

$$+ \sum_{i=1}^{N_{MC}}\left[Xe^{-rT}\sum_{l=1}^{R_{MC_i}}[N_l(-a^L_{i,2},b_{i,2},-\rho_i) - N_l(-a^H_{i,2},b_{i,2},-\rho_i)]\right]$$

$$- \sum_{i=1}^{N_{MC}}\left[Se^{-\delta T+\gamma_0+\gamma_i}\sum_{l=1}^{R_{MC_i}}[N_l(-a^L_{i,1},b_{i,1},\rho_i) - N_l(-a^H_{i,1},b_{i,1},\rho_i)]\right]$$

$$+ X_W e^{-rt_1}\left[\sum_{l=1}^{R_W}[N_l(-a^L_{W,2}) - N_l(-a^H_{W,2})]\right] - m_W Se^{-\delta t_1+\gamma_0}\left[\sum_{l=1}^{R_W}[N_l(-a^L_{W,1}) - N_l(-a^H_{W,1})]\right]$$

$$+ \sum_{i=1}^{N_{MC}}\left[X_i e^{-\delta t_1}\sum_{l=1}^{R_{MC_i}}[N_l(-a^L_{i,2}) - N_l(-a^H_{i,2})] - m_i Se^{-\delta t_1+\gamma_0}\sum_{l=1}^{R_{MC_i}}[N_l(-a^L_{i,1}) - N_l(-a^H_{i,1})]\right]$$

The parameters of the univariate and multivariate cumulative Normal are defined in equation (7) of the main text.

Section B: Analytic formulas and the numerical lattice implementation for the jump-diffusion case

The assumptions and PDE for the jump-diffusion case

In the possible presence of $i = MC_1, MC_2, ...MC_{N_{MC}}$ optional managerial learning or control actions the process and N_j independent classes of jumps the value of the project is defined as:

$$\frac{dS_t}{S_t} = (a - \sum_{j=1}^{N_j} \lambda \bar{k}_j)dt + \sigma dz + k_i dq_i + \sum_{j=1}^{N_j} k_j d\pi_j \tag{A1}$$

Jumps have an impact k_j of j=1,2,…,N_j jumps with $d\pi_j$ denoting Poisson processes with frequency of arrival λ_j per year. The diffusion case with controls is simply obtained by setting the last term in equation (A1) equal to zero. This applies also for the discussion that follows.

The PDE that the option should satisfy is

$$\begin{aligned} rV = {} & \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \delta - \sum_{j=1}^{N_j} \lambda_j \bar{k}_j) SV_S + V_t + \sum_{j=1}^{N_j} \lambda_j E\left[V(SY_j, t) - V(S,t)\right] \\ & + \sum_{i=1}^{N_{MC}} E\left[[V(SY_i, t) - V(S,t)]dq_i\right] \end{aligned} \tag{A2}$$

To derive the PDE one can follow Merton's (1976) replication argument, which imposes two further assumptions, that the intertemporal CAPM of Merton (1973b) holds and that managerial controls *and jumps* have firm specific risks, which are uncorrelated with the market portfolio and thus not priced. Alternatively, we can use the framework developed in Garman (1976), Cox, Ingersoll, and Ross (1985) and Hull and White (1988) that use a complete markets framework and no arbitrage arguments.

Denoting the accumulated (Brownian) noise from t = 0 to T by Z_t we then have that asset values at a future period T will be determined by:

$$\frac{S_T}{S_t} = \exp\left[(r-\delta-\frac{\sigma^2}{2})T+\sigma Z_T\right]\left[\prod_i (1+k_i dq_i)+\prod_j (1+k_j dq_j)\right] \qquad \text{(A3)}$$

We assume that the effect of control actions and jumps are log-normally distributed. Each control, learning action or exogenous jump has impact Y = $1+k_p$ that follows a lognormal distribution:

$$Y=(1+k_p)\sim \log N\left(\exp(\gamma_p),\exp(\gamma_p)(\exp(\sigma_p^2)-1)^{0.5}\right) \quad p=i,j \qquad \text{(A4)}$$

The assumption of log-normally distributed controls and jumps is adopted since it allows non-negative asset values, and also, conditional on control or jump activation, asset values retain log-normality. We will use (γ_i,σ_i) to denote characteristics of control or learning actions and (γ_j,σ_j^2) to denote the characteristics of randomly arriving jumps, with j = 1,2,…,N_j jump classes. We use $\gamma_i>0$ to describe efforts to enhance value with random outcome. Alternatively, if S was interpreted as a cost, $\gamma_i<0$ would mean efforts to reduce costs.

The risk neutral distribution of *S* at *T* conditional on the activation of control *i* and on the realization $n=\{n_1,n_2,...,n_{N_j}\}$ of jumps is given by:

$$\ln\left(\frac{S_T}{S_t} \mid i, n=\{n_1,n_2,...n_{N_j}\}\right) \sim N\left((r-\delta-\frac{1}{2}\sigma^2-\sum_{j=1}^{N_j}\lambda_j \bar{k}_j)(T-t)+\gamma_i+\sum_{j=1}^{N_i} n_j\gamma_j, \sigma^2(T-t)+\sigma_i^2+\sum_{j=1}^{N_j} n_j\sigma_j^2\right) \quad (A5)$$

The distribution of returns conditional on no activation of control is found by setting $\gamma_i=\sigma_i^2=0$ and the diffusion by setting the $\sum_{j=1}^{N_j}\lambda_j \bar{k}_j=\sum_{j=1}^{N_i} n_j\gamma_j=\sum_{j=1}^{N_i} n_j\sigma_j=0$.

Analytic formulas

Due to the complexity of the notation we present the formulas for the case of compound-growth options. The formulation can be easily generalized to multiple actions and regions but is not reported for brevity. Our results are consistent with Gukhal (2004) who prices compound options for the jump-diffusion case but here we also allow for endogenous controls and learning. In the case where project value follows jump diffusion with $j=1,2,...N_j$ sources of jumps with impact γ_j and volatility σ_j, and like before there exist two controls at $t=0$ and $t=t_1$, the compound-growth option conditional on activation of control action at $t=0$ is given by:

$$Call\_on\_Call(.\mid d_0 \in M\setminus\{E,A\}=\{W,MC_1,MC_2,...MC_{N_{MC}}\})=$$
$$\sum_{n(t_1)_1=0}^{\infty}...\sum_{n(t_1)_{N_j}=0}^{\infty}\sum_{n(t_2)_1=0}^{\infty}...\sum_{n(t_2)_{N_j}=0}^{\infty}\{p\left(n(t_1)_1,...,n(t_1)_{N_j}\right)p\left(n(t_2)_1,...,n(t_2)_{N_j}\right)$$
$$[Se^{-\delta T-\sum_{j=1}^{N_j}(\lambda_j \bar{k}_j T)+\sum_{j=1}^{N_j}\left(n(t_1)_j+n(t_2)_j\right)\gamma_j+(\gamma_0+\gamma_1)} N(a_{1,n(t_1)},b_{1,n(t_2)},\rho_{n(t_1),n(t_2)})-Xe^{-rT}N(a_{2,n(t_1)},b_{2,n(t_2)},\rho_{n(t_1),n(t_2)})]\}$$
$$+\sum_{n(t_1)_1=0}^{\infty}...\sum_{n(h_1)_{N_j}=0}^{\infty}\{p\left(n(t_1)_1,...,n(t_1)_{N_j}\right)[m_1 Se^{-\delta T-\sum_{j=1}^{N_j}(\lambda_j \bar{k}_j t_1)+\sum_{j=1}^{N_j} n(t_1)_j\gamma_j+\gamma_0} N(a_{1,n(t_1)})-X_1e^{-rt_1}N(a_{2,n(t_1)})]\}$$

(9)

where

$$a_{1,n(t_1)} = \frac{\ln(S/S^*) + (r - \delta - \sum_{i=1}^{N_j}(\lambda_i \bar{k}_i) + 0.5\sigma^2)t_1 + \sum_{j=1}^{N_j} n(t_1)_j \gamma_j + \gamma_0 + 0.5\sum_{j=1}^{N_j} n(t_1)_j \sigma_j^2 + 0.5\sigma_0^2}{\left(\sigma^2 t_1 + \sum_{j=1}^{N_j} n(t_1)_j \sigma_j^2 + \sigma_0^2\right)^{1/2}}$$

$$a_{2,n(t_1)} = a_{1,n(t_1)} - \left(\sigma^2 t_1 + \sum_{j=1}^{N_j} n(t_1)_j \sigma_j^2 + \sigma_0^2\right)^{1/2}$$

$$b_{1,n(t_2)} = \frac{\ln(S/X) + (r - \delta - \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j) + 0.5\sigma^2)T + (\gamma_0 + \gamma_1) + \sum_{j=1}^{N_j}\left(n(t_1)_j + n(t_2)_j\right)\gamma_j + 0.5\sum_{j=1}^{N_j}\left(n(t_1)_j + n(t_2)_j\right)\sigma_j^2 + 0.5(\sigma_0^2 + \sigma_1^2)}{\left(\sigma^2 T + \sigma_0^2 + \sigma_1^2 + \sum_{j=1}^{N_j} n(t_2)_j \sigma_j^2\right)^{1/2}}$$

$$b_{2,n(t_2)} = b_{1,n(t_2)} - \left(\sigma^2 T + \sigma_0^2 + \sigma_1^2 + \sum_{j=1}^{N_j} n(t_2)_j \sigma_j^2\right)^{0.5}$$

$$\rho_{n(t_1),n(t_2)} = \sqrt{\frac{\left(\sigma^2 t_1 + \sigma_0^2 + \sum_{j=1}^{N_j} n(t_1)_j \sigma_j^2\right)}{\left(\sigma^2 T + \sigma_0^2 + \sigma_1^2 + \sum_{j=1}^{N_j}\left(n(t_1)_j + n(t_2)_j\right)\sigma_j^2\right)}}$$

$$p(n(t)_1, ..., n(t)_{N_j}) = \prod_{j=1}^{N_j}[e^{-(\lambda_j t)}(\lambda_j t)^{n(t)_j} / n(t)_j!],$$

for $t = t_1$ and $t = t_2 = T - t_1$, $j = (1,2,..., N_j)$

In this case we weight the value of the compound option with the probabilities of occurrence of all combinations of jumps that can be realized until t_1, $n(t_1) = \left(n(t_1)_1,..,n(t_1)_{N_j}\right)$, and those realized from t_1 to T, i.e., $n(t_2) = \left(n(t_2)_1,..,n(t_2)_{N_j}\right)$, $t_2 = T\text{-}t_1$.

The critical value $S^*_{MC_1}$ is found by solving numerically the equation:

$$m_1 S + \sum_{n(t_2)_1}^{\infty} \ldots \sum_{n(t_2)_{N_j}}^{\infty} \{p\left(n(t_2)_1, n(t_2)_2, \ldots, n(t_2)_{N_j}\right)$$

$$[Se^{-\delta(T-t_1)-\sum_{j=1}^{N_j}(\lambda_j \bar{k}_j (T-t_1))+\sum_{j=1}^{N_j}(n(t_2)_j \gamma_j)+\gamma_1} N(d_{1,n(t_2)}) - Xe^{-r(T-t_1)}N(d_{2,n(t_2)})]\} - X_1 = 0$$

$$d_{1,n(t_2)} = \frac{\ln(S/X) + (r - \delta - \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j) - 0.5\sigma^2)(T - t_1) + \sum_{j=1}^{N_j} n(t_2)_j \gamma_j + \gamma_1 + 0.5\sum_{j=1}^{N_j} n(t_2)_j \sigma_j^2 + 0.5\sigma_1^2}{[\sigma^2(T - t_1) + \sum_{j=1}^{N_j} n(t_2)_j \sigma_j^2 + \sigma_1^2]^{1/2}}$$

$$d_{2,n(t_2)} = d_{1,n(t_2)} - [\sigma^2(T - t_1) + \sum_{j=1}^{N_j} n(t_2)_j \sigma_j^2 + \sigma_1^2]^{1/2}$$

We also provide the analytic valuation formulas the cases of call on put, put on call and put on put. The parameters of the univariate and bivariate Normal and the probability of occurrence of jumps are given in equation above for the call on call case.

Compound-growth call on put:

$$Call\_on\_put(.|k_0)=\sum_{n(t_1)_1=0}^{\infty}\ldots\sum_{n(t_1)_{N_j}=0}^{\infty}\sum_{n(t_2)_1=0}^{\infty}\ldots\sum_{n(t_2)_{N_j}=0}^{\infty}\{p\left(n(t_1)_1,\ldots,n(t_1)_{N_j}\right)p\left(n(t_2)_1,\ldots,n(t_2)_{N_j}\right)$$

$$[Xe^{-rT}N(-a_{2,n(t_1)},-b_{2,n(t_2)},\rho_{n(t_1),n(t_2)}]-Se^{-\delta T\sum_{j=1}^{N_j}(\lambda_j\bar{k}_jT)+\sum_{j=1}^{N_j}\left(n(t_1)_j+n(t_2)_j\right)\gamma_j+(\gamma_0+\gamma_1)}N(-a_{1,n(t_1)},-b_{1,n(t_2)},\rho_{n(t_1),n(t_2)})\}$$

$$-\sum_{n(t_1)_1=0}^{\infty}\ldots\sum_{n(t_1)_{N_j}=0}^{\infty}\{p\left(n(t_1)_1,\ldots,n(t_1)_{N_j}\right)[X_1e^{-rt_1}N(-a_{2n(t_1)})+mSe^{-\delta t_1-\sum_{j=1}^{N_j}(\lambda_j\bar{k}_jt_1)+\sum_{j=1}^{N_j}n(t_1)_j\gamma_j+\gamma_0}N(-a_{1,n(t_1)})]\}$$

Compound-growth put on call:

$$Put\_on\_Call(.|k_0)=\sum_{n(t_1)_1=0}^{\infty}\ldots\sum_{n(t_1)_{N_j}=0}^{\infty}\sum_{n(t_2)_1=0}^{\infty}\ldots\sum_{n(t_2)_{N_j}=0}^{\infty}\{p\left(n(t_1)_1,\ldots,n(t_1)_{N_j}\right)p\left(n(t_2)_1,\ldots,n(t_2)_{N_j}\right)$$

$$[Xe^{-rT}N(-a_{2,n(t_1)},b_{2,n(t_2)},-\rho_{n(t_1),n(t_2)}-Se^{-\delta T\sum_{j=1}^{N_j}(\lambda_j\bar{k}_jT)+\sum_{j=1}^{N_j}\left(n(t_1)_j+n(t_2)_j\right)\gamma_j+(\gamma_0+\gamma_1)}N(-a_{1,n(t_1)},b_{1,n(t_2)},-\rho_{n(t_1),n(t_2)})]\}$$

$$+\sum_{n(t_1)_1=0}^{\infty}\ldots\sum_{n(t_1)_{N_j}=0}^{\infty}\{p\left(n(t_1)_1,\ldots,n(t_1)_{N_j}\right)[X_1e^{-rt_1}N(-a_{2,n(t_1)})]-mSe^{-\delta T-\sum_{j=1}^{N_j}(\lambda_j\bar{k}_jt_1)+\sum_{j=1}^{N_j}n(t_1)_j\gamma_j+\gamma_0}N(-a_{1,n(t_1)})]\}$$

Compound-growth put on put:

$$Put\_on\_Put(.|k_0)=\sum_{n(t_1)_1=0}^{\infty}\ldots\sum_{n(t_1)_{N_j}=0}^{\infty}\sum_{n(t_2)_1=0}^{\infty}\ldots\sum_{n(t_2)_{N_j}=0}^{\infty}\{p\left(n(t_1)_1,\ldots,n(t_1)_{N_j}\right)p\left(n(t_2)_1,\ldots,n(t_2)_{N_j}\right)$$

$$[Xe^{-rT}N(-a_{2,n(t_1)},-b_{2,n(t_2)},\rho_{n(t_1),n(t_2)})-Se^{-\delta T\sum_{j=1}^{N_j}(\lambda_j\bar{k}_jT)+\sum_{j=1}^{N_j}\left(n(t_1)_j+n(t_2)_j\right)\gamma_j+(\gamma_0+\gamma_1)}N(-a_{1,n(t_1)},-b_{1,n(t_2)},\rho_{n(t_1),n(t_2)})]\}$$

$$+\sum_{n(t_1)_1=0}^{\infty}\ldots\sum_{n(t_1)_{N_j}=0}^{\infty}\{p\left(n(t_1)_1,\ldots,n(t_1)_{N_j}\right)[X_1e^{-rt_1}N(a_{2,n(t_1)})+mSe^{-\delta T-\sum_{j=1}^{N_j}(\lambda_j\bar{k}_jt_1)+\sum_{j=1}^{N_j}n(t_1)_j\gamma_j+\gamma_0}N(a_{1,n(h_1)})]\}$$

The numerical solution for the jump-diffusion case

For the jump diffusion case the conditional on the realization of $n=\{n_1,n_2,...,n_{N_j}\}$ jumps log-returns follow:

$$\ln\left(\frac{S_T}{S_t}\,|\,h,n=\{n_1,n_2,..n_{N_j}\}\right)\sim N\left((r-\delta-\frac{1}{2}\sigma^2)T+\gamma(h,i)+\sum_{j=1}^{N_j}n_j\gamma_j,\ \sigma^2T+\sigma^2(h,i)+\sum_{j=1}^{N_j}n_j\sigma_j^2\right) \quad (11)$$

In the cases where no control is activated, $d_t=\{W,W_i\}$, we have $\gamma(h,i)=\sigma^2(h,i)=0$ regardless of the previous action *h*.

Equations (13b), (13c), and (13e) stay the same, and we have the following adjustments to equations (13a) and (13d):

$$V^{d_t}(S_t,t\,|\,M,M_t^+,M_t^-)=$$

$$e^{(-r\Delta t)}\left[\sum_{n_1=0}^{\infty}...\sum_{n_{N_j}}^{\infty}\left[p(n_1,...,n_{N_j})E_t\left[V^*\left(S_{t+\Delta t},t+\Delta t\,|\,S_t,M,M_t^+,M_t^-,n=(n_1,n_2,...,n_{N_j})\right)\right]\right]\right]$$

$$+m_{d_t}S_t-X(d_{t-\Delta t},d_t)$$

for $d_t\in\{ME_1,ME_2,....,ME_{N_{ME}}\}$

(13'a)

$$V^{d_t}(S_t, t \mid M, M_t^+, M_t^-) =$$
$$e^{(-r\Delta t)}\left[\sum_{n_1=0}^{\infty}...\sum_{n_{N_j}}^{\infty} p(n_1,...,n_{N_j})\Big[E_t\Big[V*\Big(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-, n=(n_1,n_2,...,n_{N_j})\Big)\Big]\Big]\right]$$
(13'd)

We note that for the jump diffusion case the expectations should be taken over all possible realizations of jumps, weighted by the probability of occurrence as the term $\left[\sum_{n_1=0}^{\infty}...\sum_{n_{N_j}}^{\infty} p(n_1,...,n_{N_j})\right]$ demonstrates.

The volatility for the jump diffusion conditional on the realization of $n=(n_1,n_2,...,n_{N_j})$ jumps is:

$$v^2(d_t, d_{t+\Delta t} \mid n=(n_1,n_2,...,n_{N_j})) = \sigma^2\frac{T_{sub}}{N_{sub}} + \frac{\sigma^2(d_t,d_{t+\Delta t})}{N_{sub}} + \frac{1}{N_{sub}}\sum_{j=1}^{N_j} n_j\sigma_j^2 \qquad (14')$$

The up and down steps stay like in the diffusion case (with the above specification of volatility used) and the up and down probabilities for the jump-diffusion case are given by:

$$p_u(d_t,d_{t+\Delta t}) = \frac{\exp\left((r-\delta)\frac{T_{sub}}{N_{sub}} + \frac{\gamma(d_t,d_{t+\Delta t})}{N_{sub}} + \frac{1}{N_{sub}}\sum_{j=1}^{N_j} n_j\gamma_j\right) - down(d_t,d_{t+\Delta t})}{up(d_t,d_{t+\Delta t}) - down(d_t,d_{t+\Delta t})},$$

$$p_d(d_t,d_{t+\Delta t}) = 1 - p_u(d_t,d_{t+\Delta t})$$

References

Bernado, Antonio E, and Bhagwan Crowdhry. (2002). "Resources, real options, and corporate strategy," Journal of Financial Economics 63, 211-234.

Black, Fischer, and Myron Scholes. (1973). "The Pricing of Options and Corporate Liabilities," Journal of Political Economy 81, 637–659.

Carr, Peter (1988). "The Valuation of Sequential Exchange Opportunities," Journal of Finance 43, 1235-1256.

Childs, Paul D., Ott, Steven H., and Alexander J.Triantis. (1998). "Capital Budgeting for Interrelated Projects: A Real Options Approach," Journal of Financial and Quantitative Analysis 33, 305-334.

Childs, Paul D., Ott, Steven H., and Timothy J. Riddiough, (2001). "Valuation and Information Acquisition Policy for Claims Written on Noisy Assets," Financial Management 30, 45-75.

Childs, Paul D., Ott, Steven H., and Timothy J. Riddiough. (2002). "Optimal Valuation of Claims on Noisy Assets: Theory and an Application," Real Estate Economics 30, 415-444.

Childs, Paul D., and Alexander J. Triantis. (1999). "Dynamic R&D Investment Policies." Management Science 45, 1359-1377.

Constantinides, George (1978). "Market Risk Adjustment in Project Valuation," Journal of Finance 33, 603–616.

Cox, John, Jonathan Ingersoll, and Stephen Ross. (1985). "An Intertemporal General Equilibrium Model of Asset Prices," Econometrica 53, 363–384.

Davis G. A., and M. Samis. (2005). "Using Real Options to Value and Manage Exploration," Working Paper, Colorado School of Mines.

Dixit, Avinash, and Robert Pindyck, (1994) Investments under Uncertainty. Princeton, New Jersey: Princeton University Press.

Epstein, David, Mayor, Nick, Schonbucher, Philipp, Whaley, Elizabeth, and Paul Wilmot. (1999). "The Value of Market Research When a Firm is Learning: Real Options Pricing and Optimal Filtering," in L. Trigeorgis ed., *Real Options and Business Strategy. Applications to Decision Making* (Risk Books)

Garman, Mark (1976). "A General Theory of Asset Valuation under Diffusion State Processes," Working Paper, University of California, Berkeley.

Geske, Robert (1979). "The Valuation of Compound Options," Journal of Financial Economics 7, 63-81.

Grenadier, Steven R., and Allen M. Weiss. (1997). "Investment in Technological Innovations: An Option Pricing Approach," Journal of Financial Economics 44, 397-416.

Gukhal, Chandrasekhar R. (2004). "The compound option approach to American options on jump-diffusions," Journal of Economic Dynamics and Control, 28, 2055-2074.

Harrison, J. Michael, and Stanley Pliska. (1981). "Martingales and Stochastic Integrals in the Theory of Continuous Trading," Stochastic Processes and Their Applications 11, 313–316.

Hui, Siu and James Berger. (1983). "Empirical Bayes Estimation of Rates in Longitudinal Studies," Journal of the American Statistical Association, 78, 753-759.

Hull, John, and Alan White. (1993). "Efficient Procedures for Valuing European and American Path-Dependent Options," Journal of Derivatives 1, 21-31.

Hull, John, and Alan White. (1998). "An Overview of Contingent Claims Pricing." Canadian Journal of Administrative Sciences, 5, 55-61.

Karolyi, G. Andrew (1993). "A Bayesian Approach to Modeling Stock Return Volatility for Option Valuation," The Journal of Financial and Quantitative Analysis, 28, 579-594.

Kauffman, Gordon M. (1963). "Statistical Decision and Related techniques in Oil and Gas Exploration. Englewood Cliff, N.J.: Prentice Hall.

Korn, Ralf (1997). "Optimal Impulse Control when Control Actions Have Random Consequences," Mathematics of Operations Research 22, 639-667.

Koussis, Nicos, Martzoukos, Spiros and Lenos Trigeorgis. (2005). "Real R&D options with Time-to- Learn and Learning-by-Doing", accepted in Annals of Operations Research.

Kulatilaka, Nalin (1988). "Valuing the Flexibility of Flexible Manufacturing Systems," IEEE Transactions in Engineering Management 35, 250-257.

Lambrecht, Bart (2001). "The Impact of Debt Financing on Entry and Exit in a Duopoly," Review of Financial Studies, 14, 765–804.

Longstaff, Francis A. (1990). "Pricing Options with Extendible Maturities: Analysis and Applications," Journal of Finance, 45, 935-957.

Majd, Saman, and Robert Pindyck. (1987). "Time to Build, Option Value, and Investment Decisions," Journal of Financial Economics 18, 7-27.

Majd, Saman., and Robert Pindyck. (1989). “The Learning Curve and Optimal Production under Uncertainty.” RAND Journal of Economics 20, 331-343.

Martzoukos, Spiros H. (2000). “Real Options with Random Controls and the Value of Learning,” Annals of Operations Research 99, 305-323.

Martzoukos, Spiros H. (2003). “Real R&D Options with Endogenous and Exogenous Learning,” In D. A. Paxson (ed.) Real R&D Options, 111-129. Butterworth-Heinemann Quantitative Finance Series.

Martzoukos, Spiros, and Lenos Trigeorgis. (2001). “Resolving a Real Options Paradox with Incomplete Information: After all, Why Learn?,” Working Paper, University of Cyprus.

McDonald, Robert and Daniel Siegel. (1984). “Option Pricing When the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note,” Journal of Finance 39, 261–265.

McDonald, Robert and Daniel Siegel. (1986). “The Value of Waiting to Invest,” Quarterly Journal of Economics 101, 707–727.

Mauer, David C., and Sudipto Sarkar. (2005). “Real Options, Agency Conflicts, and Optimal Capital Structure,” Journal of Banking and Finance 26, 1405-1428.

Merton, Robert (1973a). “The Theory of Rational Option Pricing,” Bell Journal of Economics and Management Science 4, 141–183.

Merton, Robert (1973b). “An Intertemporal Capital Asset Pricing Model,” Econometrica, 867–887.

Merton, Robert (1974). “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates,” Journal of Finance, 449–470.

Merton, Robert (1976). “Option Pricing When the Underlying Stock Returns are Discontinuous,” Journal of Financial Economics, 3, 125-144.

Pennings Enrico, and Onno Lint. (1997). “The Option Value of Advanced R&D,” European Journal of Operational Research 103, 83-94.

Pindyck, Robert S. (1993). “Investments of Uncertain Cost,” Journal of Financial Economics 34, 53-76.

Thompson, Andrew C. (1995). “Valuation of Path-dependent Contingent Claims with Multiple Exercise Decisions over Time: The Case of Take-or-Pay,” Journal of Financial and Quantitative Analysis 30, 271-293.

Trigeorgis, Lenos (1996). Real Options: Managerial Flexibility and Strategy in Resource Allocation. Cambridge: The MIT Press.

Weitzman, Martin L., and Kevin Roberts. (1981). “Funding Criteria for Research, Development, and Exploration Projects,” Econometrica 49,1261-1288.

4. Investment Options with Debt Financing Constraints

Abstract

Building on the Mauer and Sarkar (2005) model that captures both investment flexibility and optimal capital structure and risky debt, we study the impact of debt financing constraints on firm value, the optimal timing of investment and other important variables like the credit spreads. We also explore the social welfare implications of financing constraints. Interestingly, we show that under some circumstances financing constraints will be beneficial for social welfare, i.e., a socially optimum level of financing constraints may exist but in other cases it might be harmful for social welfare (e.g. when imposed on firms with high growth rates). The importance of debt financing constraints regarding firm value and investment policy depends largely on the relative importance of investment timing flexibility and debt financing gains. In cases where investment flexibility has high relative importance the firm can mitigate the effects of debt financing constraints by adjusting its investment policy. We show that these adjustments are non-monotonic and may create a U shape of the investment trigger as a function of the degree that debt is constrained. We show that for shorter investment horizon, constraints have a more significant impact on firm value. We also consider managerial pre-investment risky growth options (e.g. R&D, or pilot projects). We see that they reduce the maturity effect, and (in contrast to the Brownian volatility) they tend to reduce expected credit spreads.

4.1. Introduction

The main purpose of this study is to investigate the effect and importance of debt financing constraints on firm's timing of investment decision, firm value and some other important variables like the credit spreads. The study of these issues are also important for policy makers since some parameters like the tax rate, the risk-free rate, but also the level of debt constraints themselves, can be potentially (directly or indirectly) be controlled by policy makers. For this reason we also explore the social welfare implications of debt financing constraints.

We build on the contingent claim approach to investigate these issues. Since the initial contingent claims approach of valuing equity and debt was set by Merton (1974), several papers generalized and extended this idea into new dimensions including coupon payments, the tax benefits of debt and bankruptcy costs (for example, Kane et al., 1984, and 1985). Leland (1994) uses a perpetual horizon assumption and derives closed form expressions for the value of levered equity, debt and the firm in the presence of taxes and bankruptcy costs. Security values are contingent on the uncertain unlevered value of the firm. He abstracts from the investment decision and he analyzes equity holders optimal trigger point of default (unprotected debt case). Leland and Toft (1996) extend Leland (1994) to allow the firm to choose the optimal maturity of the debt, and debt level.

The above papers do not incorporate equity holders investment option decisions. Brennan and Schwartz (1984) present a finite horizon model for the valuation of the levered firm when equity holders optimally choose *both* the investment and financial policy continuously over time. Bankruptcy is triggered by bond covenant provisions when the value of the firm is less than the face value of debt that matures at the end of the time horizon. Mauer and Triantis (1994) analyze interactions of investment and financing decisions. The model allows for dynamic change in capital structure and default is triggered through a positive net worth bond covenant restriction. Gamba et al. (2005) analyze investment options with exogenous debt policy and both corporate and personal taxes. Mauer and Sarkar (2005) include optimal capital structure, optimal default and the investment option of the firm and discuss agency issues[17].

We adopt the contingent claims framework of Mauer and Sarkar (2005) and we study debt financing constraints which may exist due to exogenous regulatory restrictions set to financial institutions[18]. Debt holders may also wish to reduce their stakes in a firm due to moral hazard or asymmetric information (see Jensen and Meckling, 1976 and Myers and Majluf, 1984, for discussion of these issues). Asymmetric information also can justify why the suppliers of credit may engage in credit rationing (see Fazzari et al., 1988, Stiglitz and Weiss, 1981 and Pawlina and Renneboog, 2005, for analysis of financing constraints and credit rationing issues). In contrast to Boyle and Guthrie (2003) our model does not focus on liquidity/cash constraints but on constraints on the level of debt financing. Furthermore, we explore the effect of debt constraints in a model that allows endogenous capital structure decisions, endogenous default and valuation of risky debt (using the Mauer and Sarkar, 2005 setting), issues not considered explicitly in that paper[19].In our model investment can be launched with sufficient equity and debt funds, the latter being constrained, even in the absence of available internal financing. This situation might be particularly relevant in closely own private firms or where the information asymmetries on the equity side are of lesser importance. Other related work is that of Uhrig-Homburg (2004) that explores costly equity issue that can lead to a cash-flow shortage restriction. In relation to Mauer and Triantis (1994) the model we use here (prior to imposing the constraints) captures optimal default decisions rather than default based on bond covenant restrictions. Since our focus is on the effect of financing constraints we however avoid issues of recapitalization (financing flexibility) like they

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[17] Fries et al (1997) explore the valuation of corporate securities (debt and equity) incorporating the tax benefits, bankruptcy costs and the agency costs of debt in a competitive industry with entry and exit decisions. Valuation of corporate securities in a duopoly with entry and exit decisions has been studied by Lambrecht (2001). In this paper we do not explicitly model competition but we allow for exogenous competitive erosion.

[18] Such restrictions may implicitly arise due to compliance to minimum capital requirements.

[19] Boyle and Guthrie (2003) modelling approach of external financing constraints does not distinguish between debt or equity financing. Effectively in this way they ignore the issues involved with respect to optimal capital structure, the tax benefits of debt, and endogenous default decisions that lead to risky debt. Furthermore their model implies immediate repayment as opposed to coupon paying debt that is explicitly modelled here.

do. Gamba and Triantis (2005) consider personal and corporate taxes, capital issuance costs and liquidity constraints in a dynamic model, without the endogenous (optimal) default determination in the analytic framework of Leland and Mauer and Sarkar that we use.

We study the effect of debt financing constraints with respect to the risk-free rate, dividend yield (competitive erosion), volatility of the value of unlevered assets, bankruptcy costs and taxes. The importance of financing constraints under different parametarizations of the model depends on the relative importance of investment flexibility versus the net benefits of debt. Further insights are provided through a comparison of the Mauer and Sarkar model with Leland (1994) and the McDonald and Siegel (1986). Leland's (1994) model includes only the financing decision (with no investment timing) while McDonald and Siegel (1986) is an all-equity model that focuses on the investment option decision. Using this comparison we clearly demonstrate the trade-off between investment timing and the net benefits of debt and explain the importance of debt financing constraints under different parameter values. This analysis also provides insights on the observed U shape of the investment trigger with respect to the level of financing constraint. In the numerical sensitivity we also show the effect of financing constraints on equity value, the bankruptcy triggers, the optimal leverage, and the credit spreads. Additionally, we implement the models with finite maturity horizon for the investment option using a numerical lattice scheme and investigate the effect of financing constraints depending on the maturity of the investment option. In this section we also analyze the welfare effects of debt financing constraints. Interestingly, we show that under some circumstances financing constraints will be beneficial for social welfare, i.e., a socially optimum level of financing constraints may exist; but in other cases it might be harmful for social welfare (e.g. when imposed on firms with high growth rates). We also explore the effect of financing constraints in the components of social welfare (firm value and government taxes).

Finally, we introduce at the pre-investment stage the (growth) option to enhance the value of the unlevered asset, but in our setting the exercise of this option has random outcome.

This assumption is similar to Martzoukos (2000) (see also Martzoukos, 2003 for the special case with analytic solution) where an all-equity framework was used. Koussis, Martzoukos and Trigeorgis (2005) have extended it to include path-dependency between actions, and optimal timing of the exercise of growth options. Our assumption of growth options that when exercised have a random outcome differs from the growth option component of Childs, Mauer and Ott (2005) and Mauer and Ott (2000) in that the potential exercise of the (equity financed) pre-investment growth option affects the distribution of project value before investment is made and uncertainty reverts to "normal" once the full investment is in place. This situation is particularly relevant for risky start-up ventures. Leland (1998) investigates alternative modes of riskiness of the project but he uses this to investigate equity holders ability to engage in "asset substitution" i.e. engage in riskier strategies ex-post to debt agreement thus transferring wealth from bond holders to equity holders. Equity holders in that model can switch between low risk and high risk strategies. Our emphasis is on the study of the interaction between these pre-investment managerial actions and investment options and financing decisions with borrowing constraints. We find that a managerial decision to exercise these growth option increases firm value, mostly by increasing the value of the option on the unlevered assets; their effect on the expected net benefits of debt is of lesser importance. We also find that exercise of these growth options decrease leverage ratios and expected credit spreads in the presence of constraints, in contrast to the case of no constraints where managerial actions have no effect on leverage ratios and expected credit spreads.

In the next section, we present the theoretical framework of Leland (1994) and its extension based on Mauer and Sarkar (2005) and we then introduce the borrowing constraints. We also implement the model with finite investment horizon using a numerical binomial tree approach to study the effect of investment horizon. In section 3 we study numerically and discuss the model with investment option and optimal capital structure, and the impact of the financing constraints on firm value, the optimal threshold to invest, and other interesting variables like credit spreads. We also include a section on the welfare effects of financing constraints. In section 4 we consider pre-investment managerial growth actions with random outcome and their interaction with borrowing constraints.

4.2. The Leland and Mauer and Sarkar model with financing constraints

In this section, we review the theoretical framework of Leland (1994) that allows for optimal default policy and optimal capital structure and its extension by Mauer and Sarkar (2005) that also incorporates the optimal investment timing decision. Then we incorporate and discuss the debt financing constraints (studied numerically in section 3). The control-growth option will be added in the model and its numerical investigation will be discussed separately in section 4.

We assume that the firm's unlevered assets follow a Geometric Brownian Motion

$$\frac{dV}{V} = \mu dt + \sigma dZ \tag{1}$$

where μ denotes the capital gains of this asset, σ denotes its volatility, dZ is an increment of a standard Weiner process.

We also consider a dividend-like payout rate in the form of opportunity cost of waiting to invest δ that can be used to model coupon payments on debt and may also have the interpretation of competitive erosion on the value of assets (e.g., Childs and Triantis (1999), Trigeorgis (1996) ch.9, and Trigeorgis (1991)). Similarly to Leland (1994) we avoid using the first interpretation and we assume that V is unaffected by the firm's capital structure: any coupon payments on debt are financed by new equity leaving the value of unlevered assets unaffected. Leland (1994) has shown that liquidation of assets to meet debt coupon obligation is inefficient (reduces firm value) compared to equity financed payments. Using either a replication argument of Black and Scholes-Merton or the risk-neutral valuation as established in Constantinides (1978) we know that any contingent claim f on V should satisfy the following PDE:

$$\frac{1}{2}\sigma^2 V^2 f_{VV} + (r-\delta)V f_V - f_t - rf = 0 .$$

Figure 1 shows the sequence of decisions in our model. Working backwards and in the absence of a control, or after the control has been activated, we refer to $F(V)$ as the value of the firm. $F(V)$ is the value of an option (see figure 1) to invest capital I (potentially with borrowing) at the optimal time t_I and acquire a levered position $E(V)$. The money the firm actually needs to pay (the equity financing, not to be confused with equity value) equals $I - D(V)$. Thus the firm has the option on $\max(E(V)-(I-D(V)),0)$ which is equivalent to $\max(E(V)+D(V)-I,0)$. This demonstrates that optimal exercise of the investment option is by using the first best approach to maximize the total value of the levered firm. The maturity T of the investment option can be either finite (in which case a binomial lattice will be implemented) or infinite (in which case the analytic solution of the following equation 2 holds). The investment option is exercised when V hits the optimal investment trigger V_I which is determined by simultaneously finding optimal capital structure (through coupon payment R) and the optimal default trigger V_B. To retain an analytic component for the values of $E(V)$ and $D(V)$, default can be triggered after t_I and at any time up to infinity (following Leland, 1994).

Figure 1: Extended Leland model with growth option, investment option, and debt financing constraints

Time 0: **Control decision ($F^*(V)$)**

- Exercise of growth options, or
- exercise investment option, or
- wait

Time $t \in [0, T]$ (T is infinite in the analytic solution case): **Investment and capital structure decision ($F(V)$)**

- Wait, or
- exercise investment option at t_I when V hits optimal investment trigger V_I; determine optimal coupon subject to financing constraints, and optimal default trigger V_B

Time $t > t_I$ until ∞: **Default decision ($E(V)$)**

- Default if $V \leq V_B$

When both the investment and the default horizons are infinite we use Mauer and Sarkar (2005) to get the following equation which is a variant of their model[20] more consistent with Leland and a focus on the value of unlevered assets (see Appendix for a review of the steps followed):

[20] In their model the underlying asset equals the present value of a stochastic yearly revenue flow minus the present value of constant costs. We make an assumption consistent with Leland (and McD&S) that the underlying unlevered asset does not have a fixed component and follows a geometric Brownian motion. Because of the absence of the fixed yearly costs, the abandonment option treated in Mauer and Sarkar (2005) is meaningless in our version of the model.

$$F(V)=\left[E(V_I)+D(V_I)-I\right]\left(\frac{V}{V_I}\right)^a \text{ where}$$

$$E(V_I)=V_I-(1-\tau)\frac{R}{r}+\left[(1-\tau)\frac{R}{r}-V_B\right]\left[\frac{V_I}{V_B}\right]^\beta,$$

$$D(V_I)=\frac{R}{r}+\left[(1-b)V_B-\frac{R}{r}\right]\left[\frac{V_I}{V_B}\right]^\beta, \tag{2}$$

$$V_B=\frac{-\beta}{(1-\beta)}(1-\tau)\frac{R}{r},$$

$$\beta=\frac{1}{2}-\frac{(r-\delta)}{\sigma^2}-\sqrt{\left(\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}<0$$

$$a=\frac{1}{2}-\frac{(r-\delta)}{\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}>1$$

By I,τ,R,r we denote the investment cost, tax rate, coupon, and the risk free rate respectively. The term b denotes proportional (to V) bankruptcy costs and V_B the bankruptcy trigger point that will be optimally selected by equity holders in order to maximize equity value $E(V_I)$. $E(V_I)$ is equity holders position once investment is initiated which can be re-written in the form

$$E(V_I)=V_I-V_B\left[\frac{V_I}{V_B}\right]^\beta-\frac{R}{r}+\left[\frac{R}{r}\right]\left[\frac{V_I}{V_B}\right]^\beta+\tau\frac{R}{r}-\left[\tau\frac{R}{r}\right]\left[\frac{V_I}{V_B}\right]^\beta$$

and has the following interpretation: conditional on investment, equity holders will obtain the value of unlevered assets V_I minus the expected value of unlevered assets at bankruptcy (second term) minus a perpetual stream of coupon payments (third term) that is netted with the payments that will not be made after bankruptcy (fourth term) plus the tax benefits (fifth term) also netted in the event of bankruptcy (sixth term). At the investment trigger, debt can also be re-written as

$$D(V_I) = \frac{R}{r} - \left[\frac{R}{r}\right]\left[\frac{V_I}{V_B}\right]^{\beta} + (1-b)V_B\left[\frac{V_I}{V_B}\right]^{\beta}$$

which equals a perpetual stream of coupons received (first term) netted with the expected coupon payments not received after bankruptcy trigger (second term) plus the expected value of the firm received at the bankruptcy trigger netted for the potential bankruptcy costs (third term). The derivations of the above formulas are discussed in Leland (1994).

For the optimal investment threshold we use a "first best" rule throughout the paper numerical results where V_I is selected to maximize the levered value of the firm (equity plus debt) as opposed to the "second best" of equity maximization. The first order condition is (see the appendix):

$$1 + \beta\left((1-\tau)\frac{R}{r} - V_B\right)\left(\frac{V_I}{V_B}\right)^{\beta}\left(\frac{1}{V_I}\right) + \beta\left((1-b)V_B - \frac{R}{r}\right)\left(\frac{V_I}{V_B}\right)^{\beta}\left(\frac{1}{V_I}\right) - \alpha\left(\frac{1}{V_I}\right)\left(E(V_I) + D(V_I) - I\right) = 0 \tag{3}$$

Equation (6) is solved numerically by simultaneously searching for optimal *R*.

Effectively, the model presented here so far is a special case of Mauer and Sarkar (2005) and we will call it the extended-Leland/MS model. It includes Leland (1994) and McDonald and Siegel (1986) (McD&S thereon) as special cases. Leland's model can be obtained by setting $V = V_I$ in equation (2) (immediate development with no investment timing). McD&S model can also be obtained by setting coupons R equal to zero (all-equity firm with an investment option), effectively imposing a zero debt restriction and that the firm never defaults ($V_B = 0$). Furthermore, applying $R = 0$ in equation 3 we get the McD&S investment trigger that equals $V_I = \frac{a}{(a-1)} I$.

Replacing for $E(V_I)$ and $D(V_I)$ into $F(V)$ (see equation 2) the firm value can also be written as:

$$F(V) = (V_I - I)\left(\frac{V}{V_I}\right)^a + \frac{\tau R}{r}\left(1 - \left(\frac{V_I}{V_B}\right)^{\beta}\right)\left(\frac{V}{V_I}\right)^a - bV_B\left(\frac{V_I}{V_B}\right)^{\beta}\left(\frac{V}{V_I}\right)^a = \\ = E(V - I) + E(TB) - E(BC) \qquad (4)$$

where E in the last line now reads "expected value". The last line effectively shows that the value of the firm can be written as the expected value of the unlevered assets (option on unlevered assets) plus the expected value of tax benefits minus the expected value of bankruptcy costs (as in Mauer and Sarkar, 2005, but with emphasis on the value of the unlevered assets). The net benefits of debt are defined as the difference between the expected tax benefits and the expected bankruptcy costs i.e. $NB = E(TB) - E(BC)$. As we will show in the next section, this decomposition proves useful since it is shown that optimal coupon and investment trigger selection involves a trade-off between obtaining higher option on unlevered assets (the investment flexibility that the McD&S model studies) versus higher *NB* of debt (debt financing gains that the Leland model studies).

Before moving to the discussion of financing constraints that is our main issue of analysis we show how $F(V)$ in the extended-Leland/MS model in finite investment option horizon can be obtained by implementing a numerical lattice scheme. With *N* lattice steps we have that up and down lattice moves and the probabilities of up and down equal:

$$u = \exp(\sigma\sqrt{\frac{T}{N}}), \qquad d = 1/u \\ p_u = \frac{\exp\left((r-\delta)T\right) - d}{u - d}, \quad p_d = 1 - p_u \qquad (5)$$

For optimal coupon selection for a given V value we apply the condition $\frac{\partial V^L(V)}{\partial R}=0$ which gives:

$$\frac{\tau}{r}\left(1-\left(\frac{V}{V_B}\right)^{\beta}+\beta\frac{1}{r}\left(\frac{V}{V_B}\right)^{\beta}\right)+b\frac{\beta}{(1-\beta)}\frac{(1-\tau)}{r}\left(\frac{V}{V_B}\right)^{\beta}+\beta bV_B\left(\frac{V}{V_B}\right)^{\beta}\frac{1}{R}=0 \qquad (6)$$

with V_B given in equation (2).

We apply equation 6 at *each* node of the lattice and we additionally allow for the early exercise of the investment option. At exercise, option value equals $E(V)+D(V)-I$ with $E(V)$ and $D(V)$ given in equation (2).

We now make the above framework more realistic by adding financing constraints that may exist for example due to asymmetric information, moral hazard or even by internal or regulatory constraints set to the banks. Debt financing constraints set a cap $D^{\max}$ to the level of debt financing so that $D(V_I)\le D^{\max}$. Without the constraint, $D(V_I)$ could even be higher than the required level of investment, which is rather unrealistic in practical applications. Furthermore, we could have percentage constraints i.e. $D(V)\le cV^L(V)$, $V^L(V)=E(V)+D(V)$ which can be interpreted as a cap on the maximum allowable leverage ratio (e.g. imposed by debt holders). In this paper we discuss the effects of the constant value $D^{\max}$. We now effectively face a constrained maximization problem. When we use the analytic solution of equation 5 we impose the constraint by running a numerical search for the coupon that satisfies the first order condition of the investment trigger *and* at the same time satisfies that debt does not exceed $D^{\max}$. Our approach is consistent with the "first-best" strategy for the firm value maximization. In the cases where the lattice framework is used the constraint is applied and must be satisfied at each lattice node. In the following section we discuss how the firm will adjust its investment and optimal default strategies in the face of financial constraints and control-growth options.

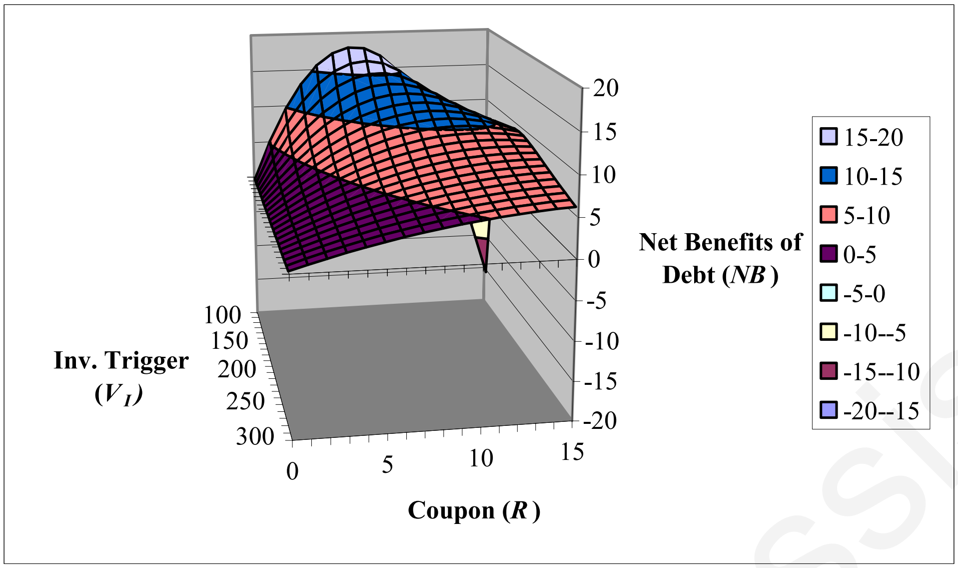
4.3. Numerical results and discussion

In this section we provide numerical results for the extended-Leland/MS model described earlier. In subsection 3.1 we provide insights on the trade-off between investment timing flexibility and the net benefits of debt that will be useful in the subsequent discussion of financing constraints. The effects of financing constraints from firm's perspective will be discussed in subsection 3.2. The social welfare effects of financing constraints are discussed in section 3.3.

4.3.1. Insights on the trade-off between investment timing flexibility and the net benefits of debt

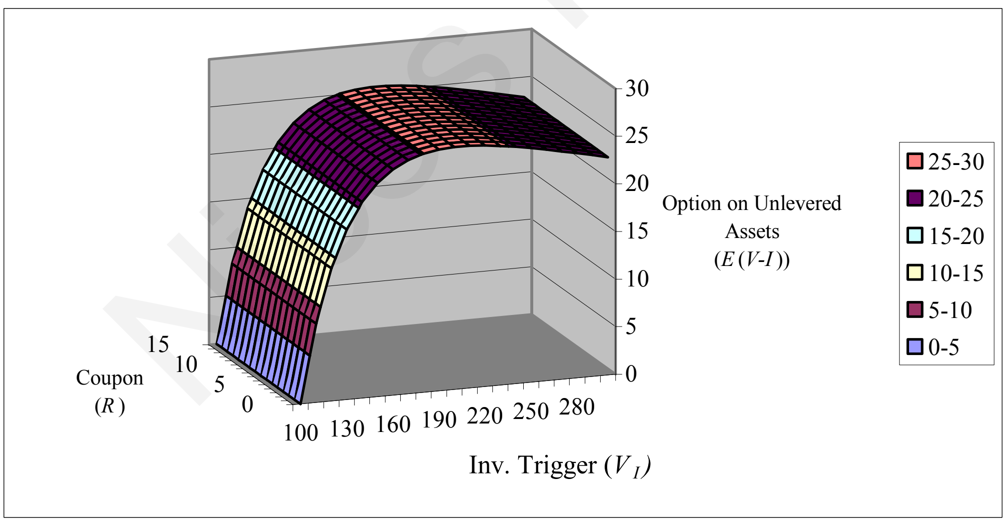
In order to illustrate the trade-off between investment flexibility and debt financing gains, we first use the decomposition of firm value from equation (7). Figures 2 and 3 use arbitrary (not optimal) values for the investment trigger. Figure 2 shows that the net benefits of debt, are decreasing in the investment threshold, while there is an optimal coupon at immediate exercise that maximizes firm value. It can be seen in figure 3, that the option on unlevered assets is invariant to the coupon and there is an investment trigger higher than the current value of unlevered assets that maximizes option value. It is thus expected that optimal investment trigger and coupon decisions involve a trade off between investment option benefits and the net benefits of debt financing.

Figure 2: Net benefits of debt as a function of the coupon and investment trigger



Notes: Net benefits of debt (*NB*) are defined as the tax benefits minus bankruptcy costs (see equation 7 of the main text). We use a value of unlevered assets V =100, a risk-free rate r = 0.06, an opportunity cost δ = 0.06, an investment cost I = 100, a volatility of unlevered assets σ = 0.25, a tax rate τ = 0.35 and a bankruptcy costs level of b =0.5.

Figure 3: Option on Unlevered Assets as a function of the coupon and investment trigger



Notes: Option on unlevered assets is defined as the option to pay I and get V (see equation 7 of the main text). We use a value of unlevered assets V =100, a risk-free rate r = 0.06, an opportunity cost δ = 0.06, an investment cost I = 100, a volatility of unlevered assets σ = 0.25, a tax rate τ = 0.35 and a bankruptcy costs level of b=0.5.

This tradeoff can be further seen in Table 1, where we compare the extended-Leland/MS model (that has both investment and financing options), with the McDonald and Siegel (1986) model (with the investment only option) and the Leland (1994) model (with the financing only option). It provides the firm values, and then the (%) net gain that has the following decomposition in (%) gain of investment flexibility and (%) gain in net benefits of debt:

$$\%\text{ Net Gain} = \frac{F(V) - F^i(V)}{F^i(V)} = \frac{\left[E(V-I) - E^i(V-I)\right]}{F^i(V)} + \frac{\left[NB - NB^i\right]}{F^i(V)} \tag{7}$$

where i = {McD&S, Leland}. We keep the base case of parameters of Leland (1994) plus a positive opportunity cost δ of 6%. Other parameters are as follows: value of unlevered assets V =100, risk-free rate r = 0.06, investment cost I =100. For the extended-Leland/MS and the Leland models bankruptcy costs b = 0.5 and tax rate τ = 0.35. The table provides sensitivity analysis for the risk-free rate r, the opportunity cost δ, the volatility of unlevered assets σ, the bankruptcy costs b, and the tax rate τ and the investment cost I. Note that the different components for Leland's model are found by applying $V_I = V$ in equation (2). When we compare the extended-Leland/MS model with the McD&S, we see that the net gain is due to the net benefits of debt only (at a loss in investment flexibility). When comparing it to the Leland model, the net gain is due to investment flexibility only (at a loss in the net benefits of debt).

Table 1: Comparison of three models with various levels of flexibility: firm value and investment and debt financing gains analysis

	Firm Value			Ext.-Leland/MS vs McD&S			Ext.-Leland/MS vs Leland		
	Ext.-Leland/MS	McD&S	Leland	% Gain $E(V\text{-}I)$	% Gain *NB*	% Net Gain	% Gain $E(V\text{-}I)$	% Gain *NB*	% Net Gain
Base	35.42	25.48	18.18	-0.03	0.42	0.39	1.36	-0.41	0.95
$r = 0.02$	23.92	18.28	11.19	-0.03	0.33	0.31	1.59	-0.46	1.14
$r = 0.04$	29.48	21.74	14.73	-0.03	0.39	0.36	1.43	-0.43	1.00
$r = 0.08$	41.38	29.27	21.34	-0.03	0.45	0.41	1.33	-0.39	0.94
$\delta = 0.02$	68.30	53.27	21.95	-0.01	0.29	0.28	2.41	-0.30	2.11
$\delta = 0.04$	47.29	35.49	19.96	-0.02	0.35	0.33	1.75	-0.38	1.37
$\delta = 0.08$	28.05	19.28	16.68	-0.05	0.51	0.45	1.10	-0.42	0.68
$\sigma = 0.05$	35.99	5.30	35.99	-1.00	6.79	5.79	0.00	0.00	0.00
$\sigma = 0.15$	28.88	15.69	23.76	-0.17	1.01	0.84	0.55	-0.33	0.22
$\sigma = 0.35$	43.09	34.40	15.04	-0.01	0.26	0.25	2.26	-0.40	1.87
$b = 0.05$	39.93	25.48	25.58	-0.06	0.63	0.57	0.93	-0.37	0.56
$b = 0.25$	37.51	25.48	21.67	-0.04	0.52	0.47	1.12	-0.39	0.73
$b = 0.75$	33.94	25.48	15.65	-0.02	0.36	0.33	1.59	-0.42	1.17
$\tau = 0.15$	27.30	25.48	3.57	0.00	0.07	0.07	7.12	-0.48	6.64
$\tau = 0.25$	30.41	25.48	9.38	-0.01	0.20	0.19	2.69	-0.45	2.24
$\tau = 0.45$	43.43	25.48	31.04	-0.09	0.80	0.70	0.75	-0.35	0.40
$I = 60$	58.23	41.88	58.18	-0.03	0.42	0.39	0.01	-0.01	0.00
$I = 80$	44.01	31.65	38.18	-0.03	0.42	0.39	0.28	-0.13	0.15
$I = 120$	29.66	21.33	0.00	-0.03	0.42	0.39	-	-	-

Notes: "Ext.-Leland/MS" refers to the main model used with investment and debt financing gains. "McD&S" refers to McDonald and Siegel (1986) model of the perpetual investment option and "Leland" to the Leland (1994) model with optimal debt financing and no investment flexibility. Base case used for all models: value of unlevered assets V =100, risk-free rate r = 0.06, opportunity cost δ = 0.06, volatility σ = 0.25, investment cost I = 100. For the Ext.-Leland/MS and the Leland model we use bankruptcy costs b = 0.5, tax rate τ = 0.35. The notation "% gain $E(V\text{-}I)$" refers to the % change in value of the option on unlevered assets and "% gain NB" refers to the % change in the net benefits of debt relative to the other two models. Sensitivity analysis is with respect to the risk-free rate r, opportunity cost δ, volatility of unlevered assets σ, bankruptcy costs b, and the tax rate τ, investment cost I.

The comparison will provide insights on the effect of financing constraints that is studied in detail in the next subsection. It is expected that when debt financing gains are relatively more important, the effect of financing constraints will be more severe. First note that, as expected, the firm value in the extended model is higher than in both other models. The (%) differences between the extended and the McD&S (Leland) models are at a maximum (minimum) at higher opportunity cost δ, higher risk-free rate r, lower volatility σ, lower bankruptcy costs b, and higher tax rate τ. In absolute values, this relation is reversed for the comparison with the Leland model in the case of the risk-free rate r, and for the comparison with the McD&S model in the case of the opportunity cost δ.

Another interesting observation is the effect on firm value of changes in the above parameters. Higher risk-free rate r and lower opportunity cost δ increase firm value in all models (both investment flexibility and net benefits of debt are affected positively). Taxes and bankruptcy costs affect the extended model only through the effect on net benefits similarly with the Leland model. A significant observation is on the effect of volatility. An increase in volatility increases the firm value in the McD&S model (investment flexibility increases) but it decreases firm value in the Leland model (net benefits of debt decrease). In the extended-Leland/MS model, these opposite forces result in a U-shape in firm value (a result not reported in Mauer and Sarkar, 2005). Finally, at higher investment cost I, firm value decreases in all models. Investment flexibility to delay is relatively more important than the net benefits of debt and thus the differences between the extended-Leland/MS and the Leland model are increasing. At high investment costs it is possible (i.e., I =120) that immediate investment is not feasible since firm value will be negative (so in the Leland model firm value equals zero).

Table 2 shows additional information with respect to the three models. The investment triggers and the bankruptcy triggers are reported first[21]. The other columns show for all models, equity and debt values, optimal coupon and credit spreads, reported at the optimal investment trigger (note that for the standard Leland model, investment takes place immediately at time zero). We first see that the investment triggers in the extended model are in all cases lower than in the McD&S model. This result is driven by the existence in the extended model of the benefits of debt which are decreasing in the investment trigger (see discussion in the previous subsection and figure 2). Note that the comparison is for two extreme cases, the extended model at optimal debt, and the McD&S which is effectively a model constrained to zero debt. As we will see in the next subsection, for in-between cases (with arbitrary levels of debt constraint) this relationship is not monotonic. We also note that the bankruptcy triggers in the extended-Leland/MS

[21] Note that in the case of low volatility, $\sigma = 0.05$, we report the theoretical trigger although the current value of V is higher than that; the investment option is exercised immediately so that firm value reported is equal to that of the Leland model.

model are higher than in the Leland model. It can be seen that the optimal coupon that is actually paid is higher in the extended model than in the Leland model. The optimal leverage and the credit spreads are the same in the extended-Leland/MS and in the Leland model, despite the differences in the equity and debt values and in the optimal coupon.

The sensitivity results for the Leland model are consistent with the analysis in Leland (1994). For the extended model, we can see that the bankruptcy trigger at the investment trigger may exhibit a U-shape with respect to the volatility. Also, as we know from Leland, the optimal capital structure is invariant to the level of unlevered assets V, and the same holds for the extended model.

Table 2: Comparison of three alternative with various levels of flexibility: Investment and bankruptcy triggers, optimal leverage, optimal coupons and credit spreads

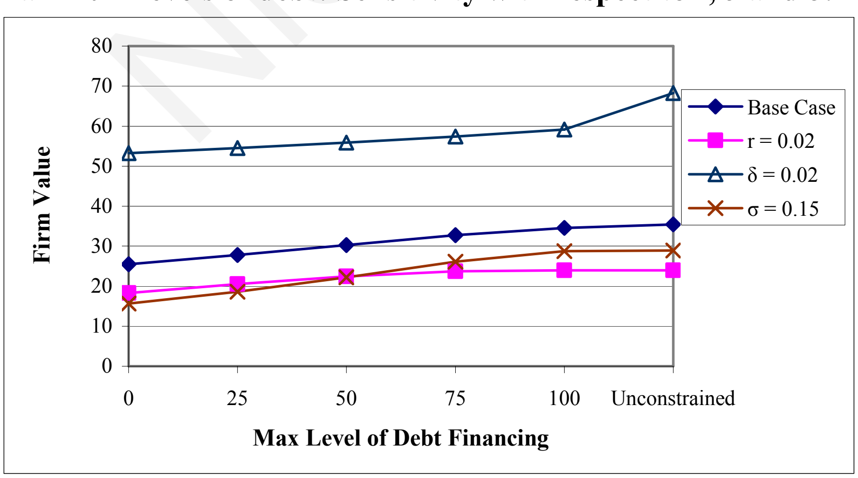
	Inv. Trigger (V_I)		Bankr. Trigger (V_B)		Optimal Capital Structure at Investment Trigger V_I									
					Equity		Debt		Optimal Leverage		Optimal Coupon		Credit Spread	
	Ext. – Leland/MS	McD&S	Ext.- Leland/MS	Leland	Ext.- Leland/MS	Leland	Ext.- Leland/MS	Leland	Ext.- Leland/MS	Leland	Ext.- Leland/MS	Leland	Ext.- Leland/MS	Leland
Base	171.57	202.77	57.92	33.76	74.82	43.60	127.94	74.57	0.63	0.63	10.84	6.32	0.0247	0.0247
$r = 0.02$	148.61	165.24	30.88	20.78	77.69	52.27	87.55	58.92	0.53	0.53	4.71	3.17	0.0338	0.0338
$r = 0.04$	158.75	182.15	43.42	27.36	75.71	47.68	106.43	67.05	0.58	0.58	7.30	4.60	0.0286	0.0286
$r = 0.08$	186.71	226.57	73.97	39.62	74.78	40.04	151.77	81.29	0.67	0.67	15.47	8.29	0.0219	0.0219
$\delta = 0.02$	406.51	495.73	165.73	40.77	159.98	39.36	335.77	82.60	0.68	0.68	25.28	6.22	0.0153	0.0153
$\delta = 0.04$	227.75	273.23	84.39	37.06	94.73	41.59	178.47	78.37	0.65	0.65	14.19	6.23	0.0195	0.0195
$\delta = 0.08$	145.64	169.93	45.14	30.98	66.01	45.34	103.92	71.34	0.61	0.61	9.44	6.48	0.0308	0.0308
$\sigma = 0.05$	84.93	115.51	56.74	66.83	20.05	23.57	95.45	112.42	0.83	0.83	6.05	7.13	0.0034	0.0034
$\sigma = 0.15$	124.17	153.68	54.77	44.12	46.40	37.36	107.27	86.40	0.70	0.70	7.77	6.26	0.0124	0.0124
$\sigma = 0.35$	229.71	264.24	64.16	27.93	108.65	47.30	155.61	67.73	0.59	0.59	15.65	6.81	0.0406	0.0406
$b = 0.05$	161.48	202.77	76.72	47.50	44.13	27.34	158.65	98.24	0.78	0.78	14.36	8.89	0.0305	0.0305
$b = 0.25$	166.65	202.77	67.05	40.24	59.10	35.46	143.66	86.21	0.71	0.71	12.55	7.53	0.0274	0.0274
$b = 0.75$	175.34	202.77	50.97	29.06	87.73	50.05	115.05	65.60	0.57	0.57	9.54	5.44	0.0229	0.0229
$\tau = 0.15$	195.76	202.77	39.61	20.25	124.03	63.34	78.72	40.24	0.39	0.39	5.67	2.90	0.0120	0.0120
$\tau = 0.25$	185.38	202.77	52.22	28.16	95.15	51.34	107.63	58.04	0.53	0.53	8.47	4.57	0.0187	0.0187
$\tau = 0.45$	154.75	202.77	58.73	37.94	59.18	38.25	143.61	92.79	0.71	0.71	12.99	8.39	0.0305	0.0304
$I = 60$	102.96	121.66	34.78	33.76	44.87	43.60	76.81	74.57	0.63	0.63	6.51	6.32	0.0248	0.0247
$I = 80$	137.25	162.21	46.32	33.76	59.86	43.60	102.33	74.57	0.63	0.63	8.67	6.32	0.0247	0.0247
$I = 120$	205.89	243.32	69.51	33.76	89.78	43.60	153.54	74.57	0.63	0.63	13.01	6.32	0.0247	0.0247

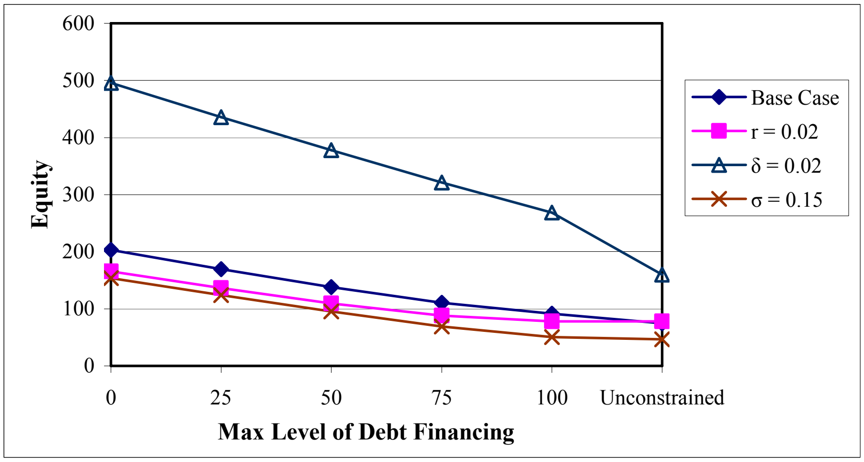
Notes: "Ext.-Leland/MS" refers to the model developed with both investment timing flexibility and debt financing gains. "McD&S" refers to McDonald and Siegel (1986) model of the perpetual investment option and "Leland" to the Leland (1994) model with optimal debt financing and no investment flexibility. Base case used for all models: value of unlevered assets V =100, risk-free rate r = 0.06, opportunity cost δ = 0.06, volatility σ = 0.25, investment cost I = 100. For the Ext. Leland and Leland model use bankruptcy costs b = 0.5, tax rate τ = 0.35.Equity, debt, optimal leverage, optimal coupons and the credit spread are calculated at the investment trigger. Sensitivity analysis with respect to the risk-free rate r, opportunity cost δ, volatility of unlevered assets σ, bankruptcy costs b, and the tax rate τ, investment cost I.

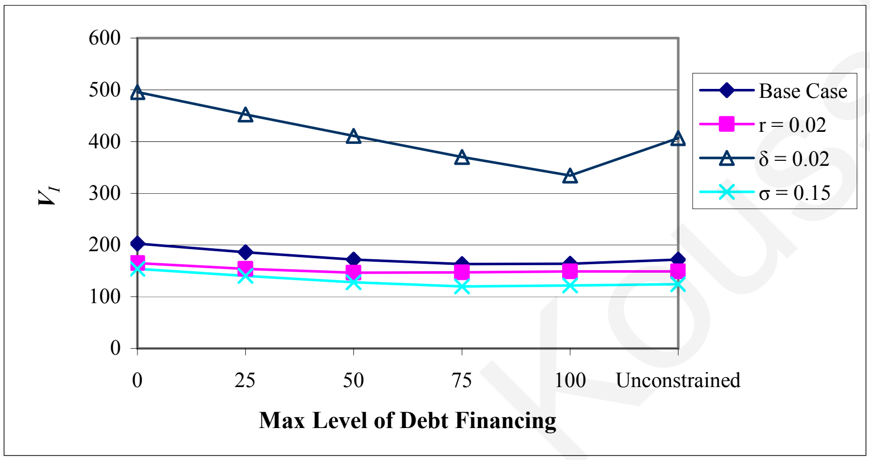
4.3.2. The effect of financing constraints on firm value and optimal firm decisions

In this section we explore the effect of financing constraints on firm and equity value, bankruptcy and investment thresholds, and on leverage and the credit spreads. In the following figures, firm values are reported at time zero. All other information about equity values, etc. is for a value of V equal to the optimal investment trigger V_I. Figures 4 and 4A show the implications of financing constraints on firm and equity value, the investment and bankruptcy triggers, leverage and the credit spread at different levels of risk-free rate, opportunity cost δ and volatility. Our discussion will concentrate on realistic levels of debt equal to the total required investment (= 100) and below. We compare the base case with ones reflecting *lower* parameter values. As can be seen from the figures, the truly unconstrained case often leads to unrealistic debt levels above 100% of the required investment capital, with an unrealistically high firm value. This is an important observation that shows the significance of our constrained borrowing approach, since to even remain at 100% debt, we need to apply the constraints. Similarly unrealistic is the high investment and bankruptcy trigger values for the truly unconstrained case.

Figure 4: Firm value, equity values, and investment trigger as a function of maximum levels of debt: Sensitivity with respect to r, δ and σ.

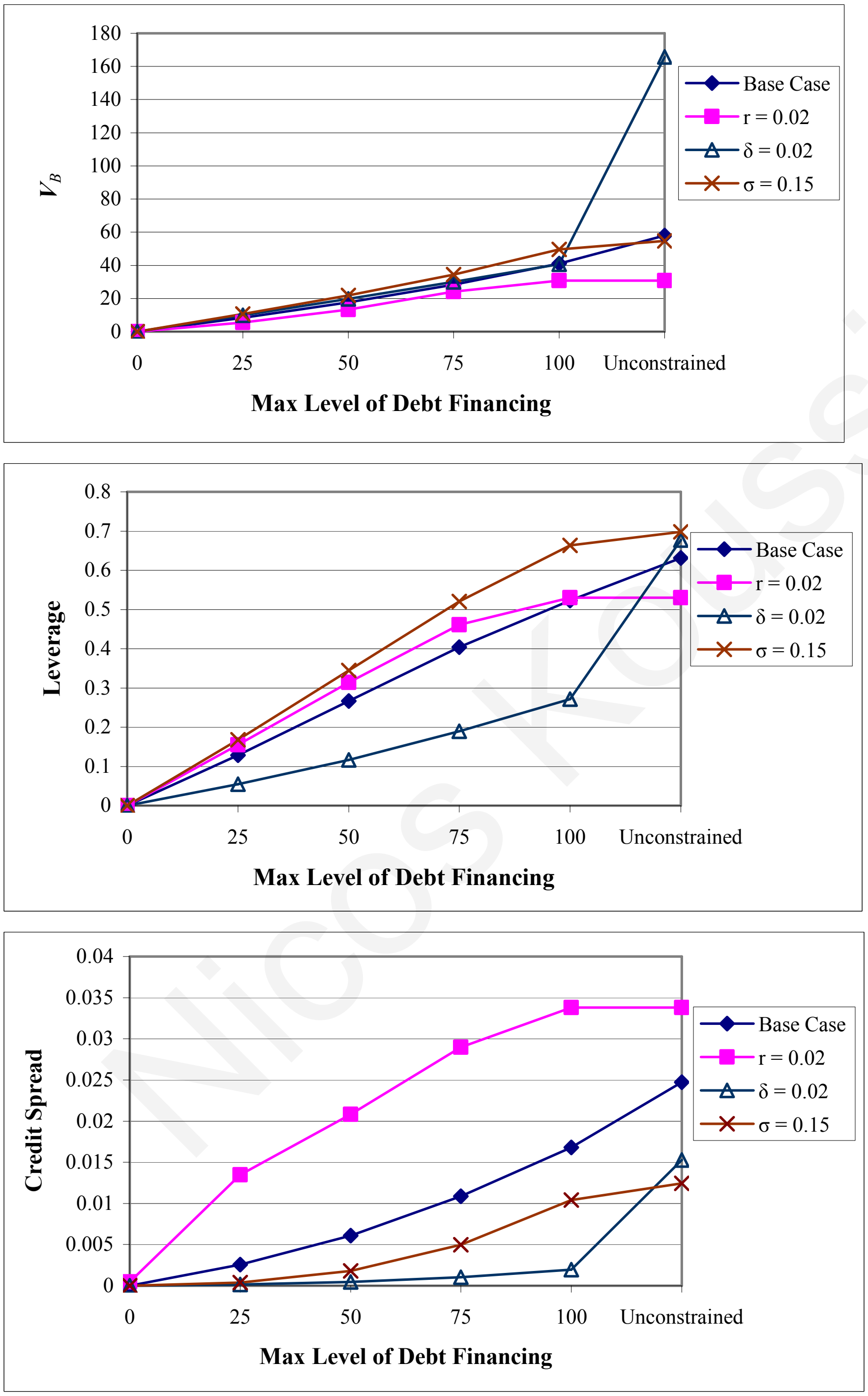






Notes: Base case used: Value of unlevered assets V =100, risk-free rate r =0.06, oopportunity cost δ = 0.06, investment cost I =100, volatility of unlevered assets σ = 0.25, tax rate τ = 0.35 and bankruptcy costs b= 0.5. Sensitivity with respect to the risk free rate r, opportunity cost δ, and volatility σ.

Figure 4A: Bankruptcy trigger, leverage and credit spreads as a function of maximum levels of debt: Sensitivity with respect to *r*, *δ* and *σ*.



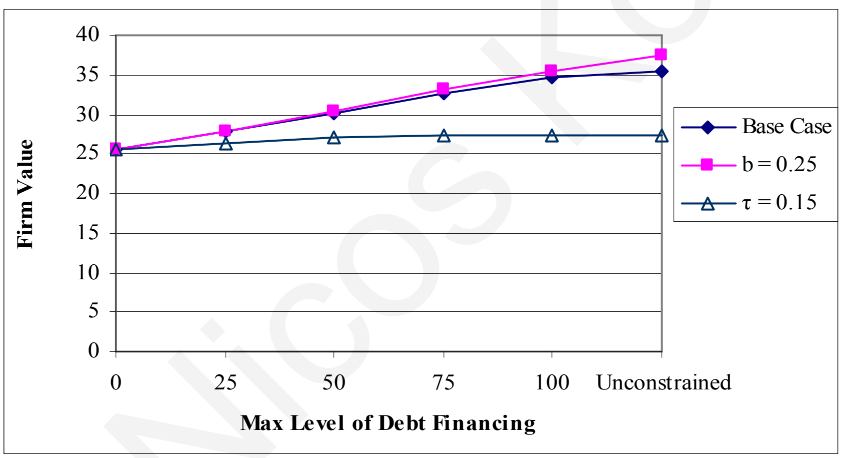
Notes: Base case used: Value of unlevered assets V =100, risk-free rate r =0.06, oopportunity cost δ = 0.06, investment cost I =100, volatility of unlevered assets σ = 0.25, tax rate τ = 0.35 and bankruptcy costs b= 0.5. Sensitivity with respect to the risk free rate r, opportunity cost δ, and volatility σ.

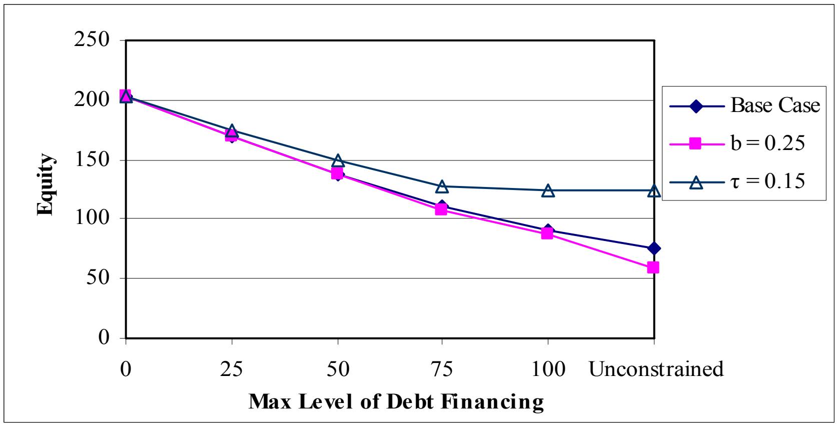
In figure 4 as expected we see that financing constraints decrease firm values and increase equity values. An interesting observation is that they often produce a U-shape in the investment trigger. This result differs from Boyle and Guthrie (2003) since they effectively focus on constraints on cash balances and we focus on constraints on debt. We interpret this U-shape as follows: when the firm is unconstrained, it will use debt at a maximum. As constraints start to become binding, the firm will adjust its investment policy by lowering the investment trigger so as to capture net benefits of debt (as we have discussed in the previous subsection, the net benefits of debt are decreasing in the investment trigger). When constraints become much more binding, the effect of net benefits of debt becomes less important, and the firm gives priority to the exploitation of its investment timing flexibility by increasing the investment trigger. After careful inspection, we also see that a small dividend yield results in a less pronounced (%) decrease in firm value (due to the higher importance of investment flexibility at lower δ discussed in subsection 3.1). A small volatility results in a more pronounced (%) decrease in firm value (reducing thus the larger financial flexibility benefits of low volatility discussed in subsection 3.1).

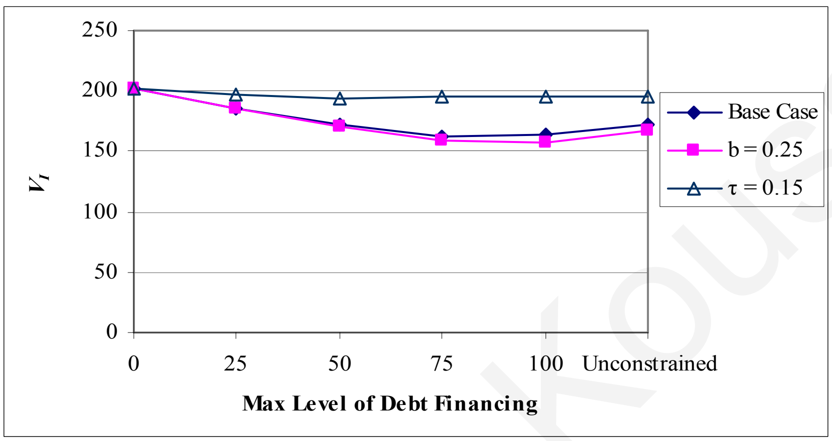
In figure 4A we see that bankruptcy trigger and leverage ratios are decreasing. The fact that lines on the figures may cross, shows that firms with different characteristics (i.e., different parameter values) will adjust optimal leverage differently in respect to imposed constraints. The last figure shows the impact of constraints on credit spreads, which is far from linear. Compared to the base case, for lower δ credit spreads are lower (see table 2 of subsection 3.1). This in general reflects lower bankruptcy risk, since investment trigger is higher, the bankruptcy trigger is lower, and the (risk-neutral) drift is higher. With stricter constraints, the difference between the levels of the bankruptcy and the investment triggers is larger, thus the credit spreads are further reduced. Again compared to the base case, for lower interest rates credit spreads are higher (see table 2 of subsection 3.1). This now reflects higher bankruptcy risk, since although both the investment and the bankruptcy trigger are somewhat lower, the (risk-neutral) drift is lower. With stricter constraints, the investment trigger goes up and the bankruptcy trigger goes down, thus further decreasing bankruptcy risk and credit spreads. The case of volatility is more complex. Lower volatility reduces the gap between the two triggers, which would increase bankruptcy risk, but with lower volatility the probability of hitting the bankruptcy trigger may be reduced and apparently this latter effect is more important.

In figures 5 and 5A we similarly see the implications of financing constraints on firm and equity value, the investment and bankruptcy triggers, leverage and the credit spread at different levels of bankruptcy costs and tax rates. In figure 5 and to the left, all values for zero debt converge to the same point which corresponds to the McD&S case, since the bankruptcy costs and tax rates affect the net benefits of debt only.

Figure 5: Firm value, equity values, and investment trigger as a function of maximum levels of debt: Sensitivity with respect to *τ* and *b*.

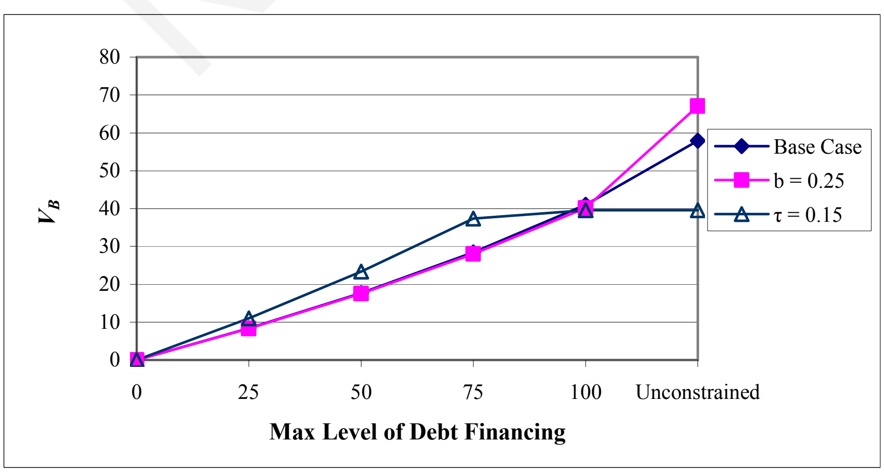


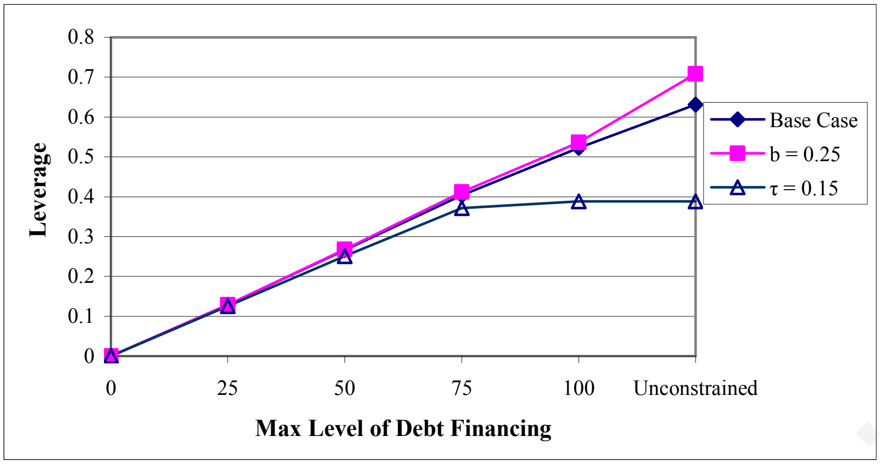


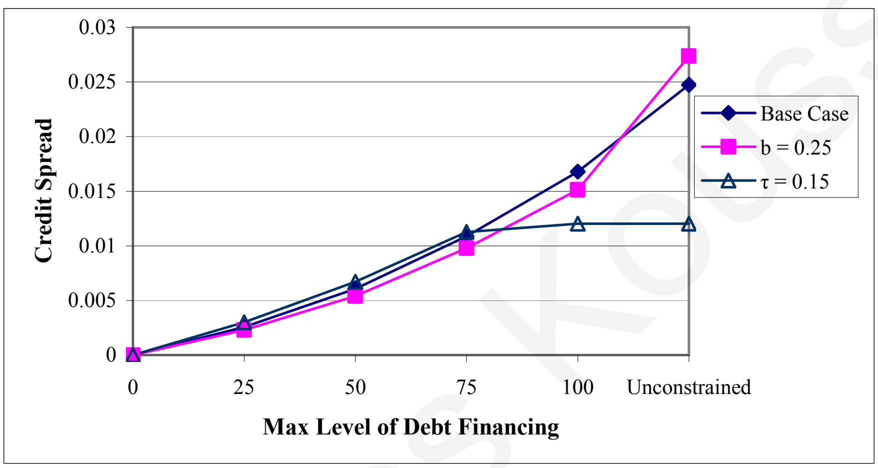


Notes: Base case parameters used: Value of unlevered assets V =100, risk-free rate r = 0.06, opportunity cost δ = 0.06, investment cost I=100, volatility of unlevered assets σ = 0.25, tax rate τ = 0.35 and bankruptcy costs b=0.5. Sensitivity with respect to bankruptcy cost b and tax rate

Figure 5A: Bankruptcy trigger, leverage and the credit spread as a function of maximum levels of debt: Sensitivity with respect to *τ* and *b*.







Notes: Base case parameters used: Value of unlevered assets V =100, risk-free rate r = 0.06, opportunity cost δ = 0.06, investment cost I =100, volatility of unlevered assets σ = 0.25, tax rate τ = 0.35 and bankruptcy costs b=0.5. Sensitivity with respect to bankruptcy cost b and tax rate τ.

We observe that for low taxes, stricter constraints have a small effect on firm value and the investment trigger since for low taxes the net benefits of debt are low. In figure 5A we see that leverage and more importantly credit spreads tend to converge in the constrained region, whereas in the unconstrained region there can be significant differences for different levels of bankruptcy costs and tax rates. In the constrained region the optimal bankruptcy trigger for low tax rates is higher than in the base case. In both figures we see that reducing bankruptcy costs in the constrained region has a negligible effect.

We have also implemented a numerical lattice with 2 steps per year (figures not shown for brevity). The lattice captures a finite investment horizon. We have observed that for stricter constraints, the decrease in firm value is more pronounced when option maturity is lower. For looser constraints, the decrease in option value is rather insensitive to option maturity.

4.3.3. Welfare effects of debt financing constraints

In this section we investigate the welfare implications of financing constraints on debt. We use the definition of welfare described in Mauer and Sarkar (2005) as the sum of firm value and the expected taxes of the government. The analysis could prove useful for policy makers when they wish to examine the total effect of a policy on financing constraints for a particular industry but the analysis of this section also provides useful information with respect to the tax raising potentials of such policies.

Taxes are contingent on the continuous flow of revenues that are generated by the firm. Keeping as underlying source of uncertainty the value of unlevered assets in this case would not allow for the analysis of welfare effects since we would not be able to calculate the continuous flow of tax revenues. For this reason in this section we use the price of the product as the driving source of uncertainty like in Mauer and Sarkar (2005) and we use the relationship:

$$V = \frac{P}{\delta}(1-\tau)$$

Effectively, the value of unlevered assets is the present value of after tax income stream (we set operational costs to zero and we exclude the option to abandon that where used in the Mauer and Sarkar, 2005 model). To maintain the same initial values for V like in our previous base case (remember that $V = 100$) we then invert to get $P = \frac{V}{(1-\tau)}\delta$ to be the starting price of the product. Note that to maintain consistency $P_B = -\frac{\beta}{(1-\beta)}\delta\frac{R}{r}$ so that $V_B = \frac{P_B}{\delta}(1-\tau)$ is like before. P_I is selected using first best approach where V_I is replaced with $V_I = \frac{P_I}{\delta}(1-\tau)$ in the first order condition. Our results are now fully consistent with previous section's results defined with respect to the value of unlevered assets i.e. the price investment and price default triggers (and optimal coupon) will generate the same investment and default triggers with respect to the value of unlevered assets and the same firm, equity and debt values like before. This formulation allows the calculation of taxes as:

$$T(P_I) = \frac{\tau P_I}{r}\left(1 - \left(\frac{P_I}{P_D}\right)^{\beta}\right)$$

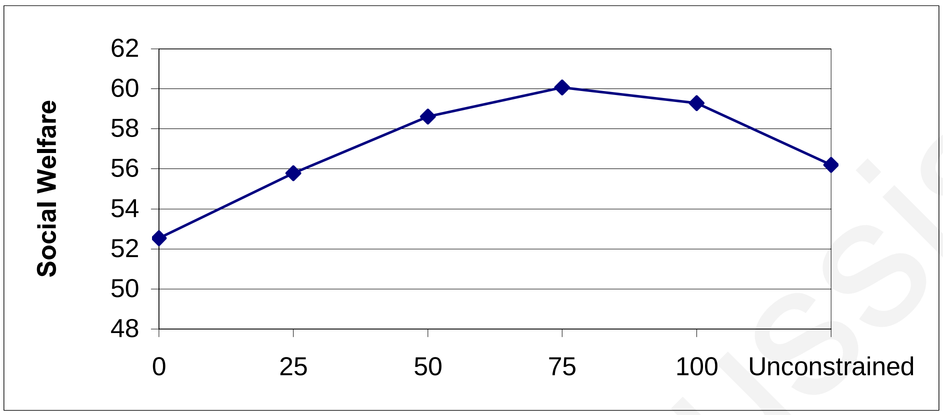
Social welfare at time zero is calculated as:

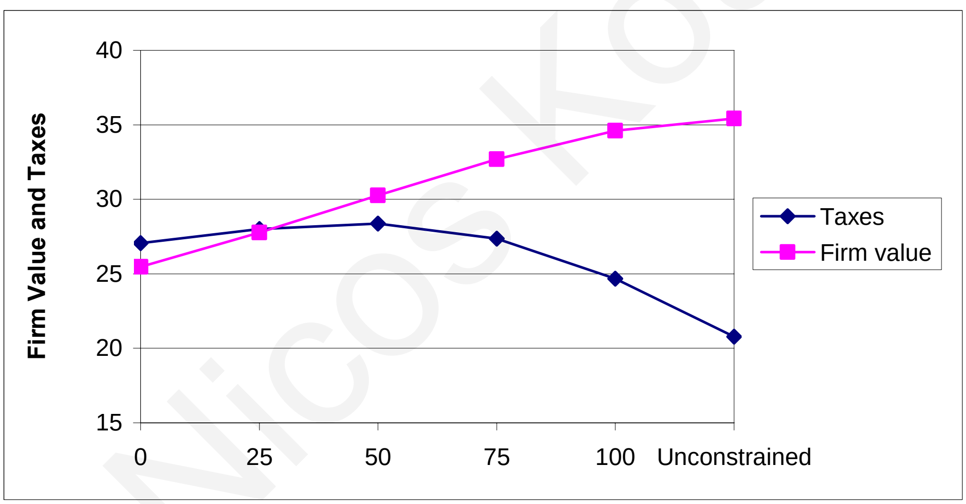
$$SW = \left(F(P_I) + T(P_I)\right)\left(\frac{P}{P_I}\right)^{\alpha}$$

Note that the function of firm value *F*(.) is the same like before evaluated with respect to the price of the commodity.

In contrast to Mauer and Sarkar (2005) we do not allow the social planner (government) to control the investment trigger but only to possibly control the level of debt financing. While obviously firms will benefit by being unconstrained on the level of debt financing, it is less clear in advance that the government will also find that unconstrained debt financing is optimal from the social point of view. In particular, the government is also interested in optimizing the level of taxes received that can be used to finance projects of public interest. Taxes received will be affected by the investment trigger and the level of default trigger which are beyond the control of government (can only be indirectly controlled by selecting the level of debt constraint).

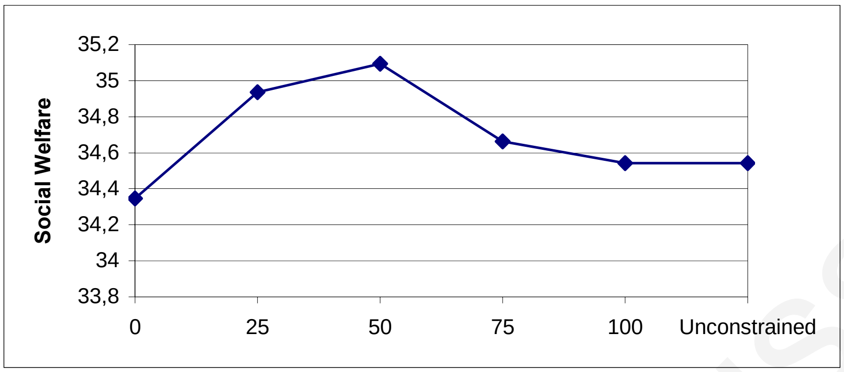
Figure 6: Social Welfare and it's components, firm value and taxes as a function of debt financing constraints

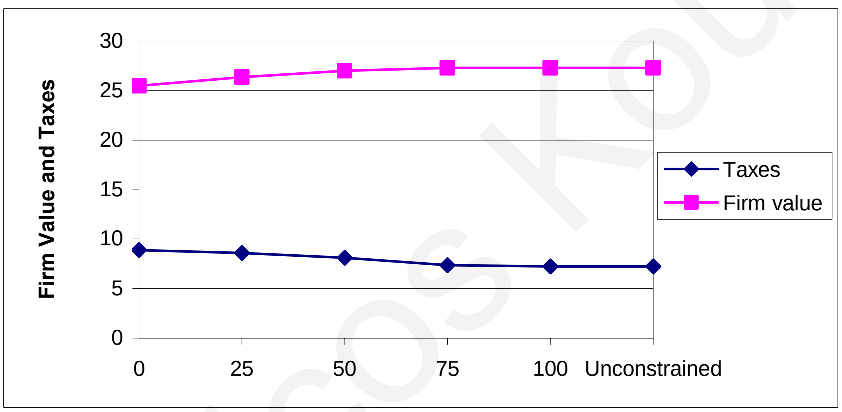




Notes: Base case used: Price of the product P = 9.231 which is equivalent to a value of unlevered assets V =100. Risk-free rate r =0.06, opportunity cost δ = 0.06, investment cost I =100, volatility of unlevered assets σ = 0.25, tax rate τ = 0.35 and bankruptcy costs b= 0.5.

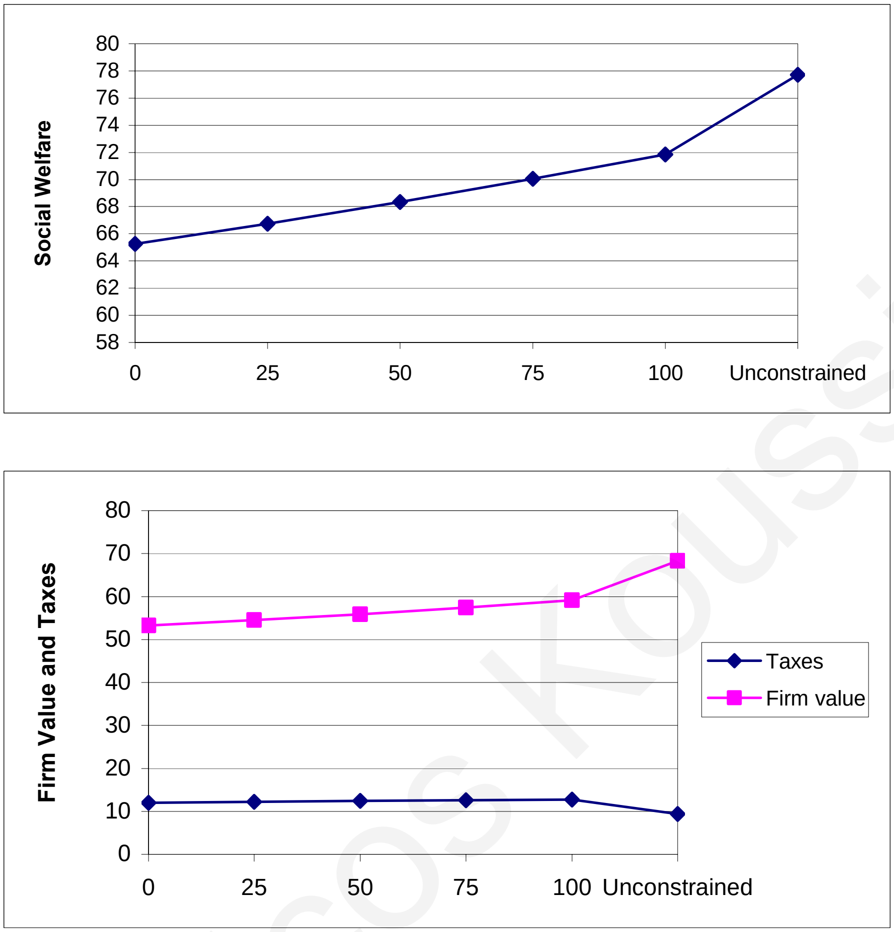
Figure 6A: Social Welfare and it's components, firm value and taxes as a function of debt financing constraints: Lower tax rate ($\tau = 0.15$)





Notes: Base case used: Price of the product P = 7.059 which is equivalent to a value of unlevered assets V =100. Risk-free rate r =0.06, opportunity cost δ = 0.06, investment cost I =100, volatility of unlevered assets σ = 0.25, tax rate τ = 0.15 and bankruptcy costs b= 0.5.

Figure 6B: Social Welfare and it's components, firm value and taxes as a function of debt financing constraints: Lower opportunity cost ($\delta = 0.02$)



Notes: Base case used: Price of the product P = 3.077 which is equivalent to a value of unlevered assets V =100. Risk-free rate r =0.06, opportunity cost δ = 0.02, investment cost I =100, volatility of unlevered assets σ = 0.25, tax rate τ = 0.35 and bankruptcy costs b= 0.5.

In figures 6 we see the effects of financing constraints on welfare and its components, firm value and taxes for the base case parameters used in the previous section. Figure 6A shows the results for a lower tax rate and lower opportunity cost δ. Interestingly, using the base case parameters we find that social welfare is maximized when the government sets a constraint of 75% on total investment cost. This result is driven by the fact that government taxes exhibit a concave shape i.e. increase initially but as more debt financing is allowed they tend to decrease (firm value is always increasing in the level of available debt financing). The behaviour of taxes is driven by the behaviour of the investment and default trigger and will vary depending on the parameters used. For the base case parameters, remember that the investment trigger exhibits a U shape and the default trigger is higher as more debt can be used. Starting from zero debt and allowing for some debt to be used will decrease the investment trigger and will start generating taxes earlier. The earlier receipt of taxes seems to dominate the decreased taxes that will be received once investment is initiated at the investment trigger (the decreased taxes at the investment trigger is due to the lower revenues generated at a lower trigger and because of the earlier default after investment is launched). The opposite forces seem to dominate as the financing constraint is relaxed even more. Interestingly for a tax rate of 15% (see figure 6A) social welfare is maximized at around 50% level of constraint. With lower taxes it is shown that taxes decrease as more debt financing is allowed. This is because the investment trigger is rather stable (see previous section) while the default trigger increases as the level of debt financing increases.

It should be emphasised that corner solutions are possible. In particular, for lower interest rates (r = 0.02) (figures for lower r and σ not shown to preserve space) social welfare increases as the financing constraint on debt increases reaching a maximum at zero debt. Taxes are thus increasing as the constraints become stricter. On the hand, for low rate of competitive erosion ($\delta = 0.02$) social welfare increases as debt financing constraints are relaxed, reaching a maximum in the unconstrained case. This is an interesting observation showing that higher debt financing potentials allowed for growth firms can also be beneficial from social welfare perspective (see figure 6C). For a lower volatility (σ = 0.15) and lower bankruptcy costs (b = 0.25) the results regarding the maximum point of the base case are not affected i.e. social welfare reaches a maximum at a 75% level of debt financing constraint.

4.4. The effect of managerial control/growth options with random outcome

In this section we use the previous models and we introduce managerial control/growth options that exist prior to the exercise of the investment option (see Martzoukos, 2000). These controls may be costly, they have an instantaneous (impulse) and random outcome and they are assumed to be equity financed (a reasonable assumption for start-up growth firms). Control characteristics are their volatility, expected impact and cost. Such actions may represent managerial growth options to engage in R&D, product redesign, advertisement, marketing, or any other actions that are targeted towards an increase in value, albeit have a random outcome. We wish to study the effect of such actions on firm value and its components (option on unlevered assets and the net benefits of debt), and on the expected at development optimal leverage, equity and debt value, and credit spreads. Of particular interest is the effect of the volatility of such actions on the aforementioned variables in contrast to the effect of Brownian volatility. Changes in the Brownian volatility that we discussed in the previous section hold both before and after the investment decision; they thus affect both the investment timing option, and the risk of debt and debt capacity of the firm. The effect (increase) of uncertainty due to the control/growth actions is action-specific and thus affects volatility before the investment decision and not after[22].

We assume that the control can be activated at time zero at a cost I_C. Its effect on the unlevered asset will have a random outcome $(1+k)$ where:

[22] Merton (1974) uses the Black and Scholes model to value equity and risky debt. In that model, increases in volatility create the so-called asset-substitution effect by transferring value from debt holders to equity holders. The assumption is that the investment and the level of debt have been already decided upon, and then there is a change in volatility. In the Leland (1994) model asset substitution can be studied by first deciding on the coupon level, and then changing volatility given the coupon level decision (see also Leland, 1998). In section 3 we discussed the sensitivity to volatility for the Leland and the extended Leland model. In our implementation of both models the new volatility level holds both before and after the investment decision. In our implementation thus of the Leland model, optimal coupon is decided given the new level of volatility. In section 4, the action-specific volatility has a direct effect on uncertainty before the investment decision and not after.

$$\ln(1+k) \sim N\left(\gamma - \frac{1}{2}\sigma_C^2, \sigma_C^2\right). \tag{8}$$

The assumption of a lognormal distribution is convenient since we retain the lognormality of the asset values when controls are activated. The expected multiplicative impact of control on V is $1+\bar{k} = \exp(\gamma)$ with a variance $\exp(\gamma)\left(\exp(\sigma_C^2)-1\right)^{0.5}$ (from lognormal distribution). We assume that an equilibrium continuous-time CAPM (see Merton, 1973) holds and that controls have firm-specific risks which are uncorrelated with the market portfolio and are thus not priced.

In general we may have optimal timing of controls and issues of path dependency (see Koussis, Martzoukos, and Trigeorgis, 2005, for an all-equity model with control/growth actions). For simplicity here we assume that controls are available only at $t = 0$, although optimal timing of those actions could be added in the present capital structure framework but at a significant expense of computational intensity and without offering any important additional insights.

Optimal firm value, $F^*(V)$ is calculated as the option to invest capital I_C in control-growth action at time zero that will potentially enhance V but has a random outcome. This gives the investment option $F(V)$ to pay capital cost I and acquire a potentially levered position $V^L(V) = E(V) + D(V)$. Note that $E(V)$ and $D(V)$ denote the stochastic values of equity and debt respectively (see section 2). Optimal firm value at $t = 0$ can be defined as follows:

$$F^*(V) = \max_{\varphi_{I_C}}\{E^C[F(V)] - I_C, F(V)]\tag{9}$$

where φ_{I_C} = {exercise of growth option, no exercise of growth option} and $E^C[.]$ is expectation under the managerial control distribution. For the evaluation of the expectation under control activation we use a Markov chain implementation. Effectively, we create a grid of V values with attached probabilities consistent with the distribution of control-growth option activation described in equation (8). In the Tables 3 and 4 that follow, all the values reported are expected due to the presence of control uncertainty, since we report them conditional on control activation. They are calculated with the use of the Markov-Chain that approximates the lognormal distribution of the multiplicative effect of the control as discussed earlier. In the extended model where delay is possible, the values are the expected ones given control activation of the expected values at the investment trigger given the uncertainty coming from the Wiener process.

Martzoukos (2000) and Koussis, Martzoukos and Trigeorgis (2005) have shown that these managerial control actions increase investment option value for an all-equity firm. Here we investigate their effect with both investment timing flexibility and net benefits of debt. Table 3 shows numerical results for the effect of controls on firm value and its two components, the expected value of unlevered assets and the expected net benefits of debt. In the same table we explore the effect of control actions with random outcome in the presence of financing constraints on debt. We assume that the cost of the control is zero to concentrate on the effect of the control distributional characteristics. Effectively, the control can be activated if its cost is less than the increase in added value it provides. For example, the firm value in the extended Leland model equals 35.42 without any control activated, and 55.18 when a control with volatility 0.60 and mean impact 0.10 is activated. Thus, an equity-financed cost up to 55.18 – 35.42 = 19.76 could be paid for this control action. Concentrating on the first panel (the case with no constraints) we see that in all models firm values are increasing in both the volatility of control and the expected impact. This is in contrast to the effect of an increase in the Brownian volatility (see discussion on Table 1) that decreases firm value in the Leland model (and creates a U-shape in the extended model). In both the extended Leland and the Leland models, an increase in the mean impact has a positive effect on both the option on unlevered asset and the net benefits of debt. An increase in volatility though, increases the option on unlevered assets, but may decrease the net benefits of debt. The net effect though of an increased control volatility is still positive, since the effect of higher volatility on the option on unlevered assets is strong enough to counterbalance a negative effect on the net benefits of debt.

Table 3: The effect of managerial control actions and financing constraints on firm value and its components (option on unlevered assets and expected net benefits of debt)

	Firm value			Option on Unlevered Assets E(*V*-*I*)		Net Benefits of Debt (*NB*)	
	Ext.-Leland/MS	McD&S	Leland	Ext.-Leland/MS	Leland	Ext.-Leland/MS	Leland
No constraints							
No control	35.42	25.48	18.18	24.67	0.00	10.75	18.18
$\gamma = 0.10$							
$\sigma_C = 0.2$	44.81	32.24	31.56	31.23	13.37	13.58	18.18
$\sigma_C = 0.4$	49.34	35.86	37.50	35.02	21.23	14.32	16.26
$\sigma_C = 0.6$	55.18	41.01	44.94	40.35	30.29	14.83	14.65
$\sigma_C = 0.2$							
$\gamma = 0.1$	44.81	32.24	31.56	31.23	13.37	13.58	18.18
$\gamma = 0.3$	66.25	47.74	59.60	46.41	35.30	19.84	24.30
$\gamma = 0.5$	96.90	70.46	94.85	69.17	64.88	27.73	29.96
Max Debt = 75							
No control	32.70	25.48	18.18	24.02	0.00	8.68	18.18
$\gamma = 0.10$							
$\sigma_C = 0.2$	41.36	32.24	30.41	30.45	13.37	10.92	17.04
$\sigma_C = 0.4$	45.24	35.86	35.07	34.41	21.23	10.84	13.84
$\sigma_C = 0.6$	50.06	41.01	41.10	39.88	30.29	10.18	10.81
$\sigma_C = 0.2$							
$\gamma = 0.1$	41.36	32.24	30.41	30.45	13.37	10.92	17.04
$\gamma = 0.3$	61.02	47.74	56.16	45.46	35.30	15.57	20.86
$\gamma = 0.5$	88.66	70.46	87.40	68.43	64.88	20.22	22.52
Max Debt = 50							
No control	30.25	25.48	14.87	24.67	0.00	5.59	14.87
$\gamma = 0.10$							
$\sigma_C = 0.2$	38.28	32.24	26.58	31.23	13.37	7.05	13.21
$\sigma_C = 0.4$	42.13	35.86	31.76	35.01	21.23	7.11	10.53
$\sigma_C = 0.6$	47.08	41.01	38.23	40.35	30.29	6.74	7.94
$\sigma_C = 0.2$							
$\gamma = 0.1$	38.28	32.24	26.58	31.23	13.37	7.05	13.21
$\gamma = 0.3$	56.58	47.74	50.71	46.40	35.30	10.18	15.40
$\gamma = 0.5$	82.74	70.46	80.93	69.16	64.88	13.58	16.05

Notes: "Ext.-Leland/MS" refers to the model with both investment timing flexibility and debt financing gains. "McD&S" refers to McDonald and Siegel (1986) model of the perpetual investment option and "Leland" to the Leland (1994) model with optimal debt financing and no investment flexibility. Base case used for all models: value of unlevered assets V =100, risk-free rate r = 0.06, opportunity cost δ = 0.06, volatility σ = 0.25, investment cost I = 100. For the Ext. Leland and Leland model use bankruptcy costs b = 0.5, tax rate τ = 0.35. Managerial control parameters have expected impact γ and volatility σ_C and are implemented using a Markov-chain with N =50 states. Max. Debt refers to constraints on the total amount of debt that can be issued.

Table 4: The effect of managerial control actions and financing constraints on optimal capital structure, expected costs, expected leverage ratio and on expected credit spreads.

	Optimal capital structure									
	Expected Equity		Expected Debt		Expected Cost		Expected Leverage		Expected Credit Spread	
	Ext. - Leland/MS	Leland	Ext.- Leland/MS	Leland	Ext.- Leland/MS	Leland	Ext.- Leland/MS	Leland	Ext.- Leland/MS	Leland
No constraints										
No control	25.79	43.60	44.10	74.57	34.47	100.00	0.63	0.63	0.0247	0.0247
$\gamma = 0.10$										
$\sigma_C = 0.2$	32.58	43.61	55.71	74.58	43.47	86.63	0.63	0.63	0.0247	0.0247
$\sigma_C = 0.4$	34.35	39.01	58.74	66.71	43.75	68.23	0.63	0.63	0.0247	0.0247
$\sigma_C = 0.6$	35.58	35.14	60.84	60.09	41.24	50.29	0.63	0.63	0.0247	0.0247
$\sigma_C = 0.2$										
$\gamma = 0.1$	32.58	43.61	55.71	74.58	43.47	86.63	0.63	0.63	0.0247	0.0247
$\gamma = 0.3$	47.59	58.28	81.38	99.68	62.73	98.35	0.63	0.63	0.0247	0.0247
$\gamma = 0.5$	66.53	71.87	113.76	122.91	83.39	99.93	0.63	0.63	0.0247	0.0247
Max Debt = 75										
No control	42.24	43.60	28.61	74.57	38.15	100.00	0.40	0.63	0.0109	0.0247
$\gamma = 0.10$										
$\sigma_C = 0.2$	53.35	53.11	35.96	63.93	47.95	86.63	0.40	0.55	0.0108	0.0194
$\sigma_C = 0.4$	57.05	53.07	35.43	50.23	47.24	68.23	0.38	0.49	0.0103	0.0173
$\sigma_C = 0.6$	61.04	53.72	32.96	37.67	43.95	50.29	0.35	0.41	0.0096	0.0147
$\sigma_C = 0.2$										
$\gamma = 0.1$	53.35	53.11	35.96	63.93	47.95	86.63	0.40	0.55	0.0108	0.0194
$\gamma = 0.3$	78.07	81.02	51.12	73.50	68.17	98.35	0.40	0.48	0.0106	0.0154
$\gamma = 0.5$	110.56	112.41	65.71	74.93	87.61	99.93	0.37	0.40	0.0098	0.0115
Max Debt = 50										
No control	47.50	64.87	17.25	50.00	34.50	100.00	0.27	0.44	0.0061	0.0122
$\gamma = 0.10$										
$\sigma_C = 0.2$	60.03	69.90	21.75	43.32	43.50	86.63	0.27	0.38	0.0061	0.0105
$\sigma_C = 0.4$	64.01	65.88	21.89	34.11	43.77	68.23	0.25	0.34	0.0058	0.0096
$\sigma_C = 0.6$	67.71	63.37	20.63	25.15	41.26	50.29	0.23	0.28	0.0055	0.0080
$\sigma_C = 0.2$										
$\gamma = 0.1$	60.03	69.90	21.75	43.32	43.50	86.63	0.27	0.38	0.0061	0.0105
$\gamma = 0.3$	87.96	99.88	31.38	49.18	62.77	98.35	0.26	0.33	0.0060	0.0086
$\gamma = 0.5$	124.45	130.90	41.71	49.97	83.42	99.93	0.25	0.28	0.0057	0.0067

Notes: "Ext.-Leland/MS" refers to the model with both investment timing flexibility and debt financing gains. "McD&S" refers to McDonald and Siegel (1986) model of the perpetual investment option and "Leland" to the Leland (1994) model with optimal debt financing and no investment flexibility. Base case used for all models: value of unlevered assets V =100, risk-free rate r = 0.06, opportunity cost δ = 0.06, volatility σ = 0.25, investment cost I = 100. For the Ext. Leland and Leland model use bankruptcy costs b = 0.5, tax rate τ = 0.35. Managerial control parameters have expected impact γ and volatility σ_C and are implemented using a Markov-chain with N =50 states. All values reported are time zero expected values. Max. Debt refers to constraints on the total amount of debt that can be issued.

The second and third panel of table 3 show the effect of different levels of financing constraints on firm value and its components. For a given debt constraint, the effect of controls is like before. Comparing the panels with increasingly strict debt constraints, we still see (as expected) a decrease in firm values. The driver of the decrease in firm value is mostly due to the decrease in the net benefits of debt. But now, we do not necessarily observe a decrease in expected option on unlevered assets. This is because of the often observed U-shape of the investment trigger (see discussion on figure 4) where the firm adjusts its investment policy to stricter constraints.

Table 4 presents more information for the expected optimal capital structure (expected leverage) and the expected credit spread. Note that firm values (of Table 3) are equal to expected equity plus expected debt minus the expected investment cost. We see that (in both the unconstrained and the constrained cases) expected equity is increasing in both control volatility and control mean impact in the extended model, while in Leland´s model it is only increasing in the mean impact (but may be decreasing in control volatility). In the unconstrained case, expected leverage and expected credit spreads stay unchanged in the presence of controls (and expected debt is affected positively in the impact and volatility of the control). With the simultaneous presence of controls and stricter debt constraints we see a decrease in expected optimal leverage and an accompanying decrease in expected credit spreads. This is to be contrasted with the case of an increase in Brownian volatility that would increase credit spreads.

Expected costs reflect the probability of development. We see that, in both the unconstrained and the constrained cases, an increase in control volatility decreases expected cost while an increase in its mean impact increases expected cost.

Table 5: The effect of controls and financing constraints with finite investment option maturity

	Firm value				
	$T = 2$	$T = 5$	$T = 10$	$T = 20$	$T = 50$
No constraints					
No control	24.83	29.06	32.17	34.34	35.22
$\gamma = 0.10$					
$\sigma_C = 0.2$	36.02	39.38	41.99	43.79	44.52
$\sigma_C = 0.4$	41.38	44.33	46.71	48.38	49.08
$\sigma_C = 0.6$	48.05	50.54	52.70	54.32	55.03
$\sigma_C = 0.2$					
$\gamma = 0.1$	36.02	39.38	41.99	43.79	44.52
$\gamma = 0.3$	61.08	62.83	64.38	65.51	65.97
$\gamma = 0.5$	95.07	95.55	96.09	96.53	96.71
Max Debt = 50					
No control	21.03	24.74	27.44	29.33	30.08
$\gamma = 0.10$					
$\sigma_C = 0.2$	30.62	33.57	35.84	37.39	38.03
$\sigma_C = 0.4$	35.22	37.79	39.86	41.30	41.91
$\sigma_C = 0.6$	40.90	43.07	44.95	46.34	46.96
$\sigma_C = 0.2$					
$\gamma = 0.1$	30.62	33.57	35.84	37.39	38.03
$\gamma = 0.3$	52.07	53.62	54.97	55.95	56.34
$\gamma = 0.5$	81.14	81.57	82.04	82.42	82.57

Notes: Base case used models: value of unlevered assets V =100, risk-free rate r = 0.06, opportunity cost δ = 0.06, volatility σ = 0.25, investment cost I = 100, bankruptcy cost b = 0.5 and tax rate τ = 0.35. Firm values are calculated using a Markov-chain implementation with N =50 states for the controls (with average impact γ and volatility σ_C) and a numerical lattice scheme for the investment option with dt = 0.5 years. Max. Debt refers to constraints on the total amount of debt that can be issued.

In the results for the numerical implementation shown in table 5, we see the impact on the firm value in the extended Leland model of investment maturity, constraints and controls. Note that with very high maturities (T = 50) the numerical solution approximates the analytic model (see first column of table 3). Reduced maturity obviously results in a decreased firm value. This result appears in both constrained and unconstrained case, and both in the presence and in the absence of controls. An interesting observation is that in the presence of controls, the effect of maturity on firm values is lessened.

4.5. Summary

We use the Mauer and Sarkar (2005) contingent claims model of firm value with the option for optimal investment timing and net benefits of risky debt (that allows for optimal capital structure and endogenously determined optimal bankruptcy), with an adaptation so that it is consistent with Leland (1994). We make the interesting observation that in this extended model firm value exhibits a U-shape in volatility (not reported in previous research).

To this (extended Leland/MS) model we add financing constraints, and with the use of a Markov-Chain method we also accommodate the presence of pre-investment control/growth options with random outcome. Beyond the analytic solution for a perpetual horizon, we also implement the investment option in a finite horizon on a binomial lattice, while maintaining the analytic structure for the capital structure decisions. The scope is to study the effect of capital constraints on firm, equity and debt value, optimal investment and bankruptcy trigger, leverage and credit spreads.

A comparison of the extended model with the McD&S model that does not include a debt financing option and the Leland (1994) model that does not include an investment option provides insights on the trade-off between investment timing flexibility and the net benefits of debt. We show that financing constraints have a more significant relative impact on firm values at higher opportunity cost (dividend yield), riskless rate of interest and taxes, and lower volatility and bankruptcy costs. The effect of financing constraints is more severe when investment option maturity is lower. Financing constraints also reduce leverage and credit spreads in a nonlinear fashion. An important observation is a U-shape of the investment trigger as a function of the constraint. This result is driven by the trade-off between investment timing flexibility and the net benefits of debt.

We also explore the social welfare implications of financing constraints on debt. Our analysis shows the effects of constraints on the components of welfare (firm value and taxes). We show that there are cases where the government can maximize social welfare by setting a constraint. In some other case social welfare is maximized when debt financing is not constrained-the most interesting case being when firms have large growth rates (lower parameter δ).

Exercise of pre-investment managerial growth options increase firm value, although they may decrease expected net benefits of debt. In contrast to the Brownian volatility, the volatility of the managerial growth options does not create a U-shape on the firm value. This action-specific volatility affects uncertainty prior to the investment decision and has no effect in the absence of constraints (and a very small reduction effect in the presence of constraints) on expected credit spreads after development. The probability of investment increases in the mean impact and decreases in the volatility of the growth option; however, firm value always increases in the mean and the volatility of the growth options. Reduced maturity results in a decreased firm value, with and without constraints. In the presence of controls, this maturity effect on firm value tends to disappear.

Appendix:

In this appendix, we show the derivation of the analytic solution for the extended Leland/MS model (see equations 2 and 3) with the embedded investment option. Although the model is a special case of Mauer and Sarkar (2005), we retain the derivation in order to demonstrate the exact form of the first order condition we use in the paper. Similarly with Leland (1994), and conditional on investment, the optimal default point V_B is found by solving for the following smooth-pasting condition:

$$\left.\frac{\partial E}{\partial V}\right|_{V=V_B} = 0 \tag{A1}$$

which is equivalent to maximizing $E(V_I)$ at $V = V_I$. The optimal bankruptcy trigger is:

$$\begin{aligned} V_B &= \frac{-\beta}{(1-\beta)}(1-\tau)\frac{R}{r} \\ \beta &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0 \end{aligned} \tag{A2}$$

Equation (A2), compared to the one in Leland, includes dividend-like competitive erosion (included in term β). Since $\beta < 0$, this means that $V_B > 0$ for any positive level of coupons R.

The general solution of the option to invest $F(V)$ can be written as:

$$F(V) = A_1 V^a + A_2 V^\beta \tag{A3}$$

The option also satisfies the usual ordinary deferential equation (since the investment horizon is perpetual):

$$rF = \frac{1}{2}\sigma^2 V^2 F_{VV} + (r-\delta)VF_V \quad \text{(A4)}$$

By applying the general solution (A3) to the differential equation we find the solution for parameters a to be:

$$a = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \quad \text{(A5)}$$

Consistently with Mauer and Sarkar (2005) we apply three boundary conditions to obtain the values for A_1, A_2 and the investment threshold V_I. In particular we have the following boundary conditions:

$$F(0) = 0 \quad \text{(A6)}$$

$$F(V_I) = E(V_I) + D(V_I) - I \quad \text{(A7)}$$

$$\left.\frac{\partial F}{\partial V}\right|_{V=V_I} = \left.\frac{\partial E}{\partial V}\right|_{V=V_I} \text{ (Second best) or } \left.\frac{\partial F}{\partial V}\right|_{V=V_I} = \left.\frac{\partial V^L}{\partial V}\right|_{V=V_I} \text{ (First best)} \quad \text{(A8)}$$

where $E(.)$ and $D(.)$ functional forms are given in equation 5 (derived in Leland, 1994) and are evaluated at V_I and $V^L(V_I) = E(V_I) + D(V_I)$ is the value of the levered firm at the investment trigger. Using (A6) we find that $A_2 = 0$ (since $\beta < 0$). With (A7) we find $A_1 = \left[E(V_1) + D(V_I) - I\right]\left(\frac{1}{V_I}\right)^a$ so replacing into (A3) we find equation (5) for the firm value:

$$F(V) = \left[E(V_I) + D(V_I) - I\right]\left(\frac{V}{V_I}\right)^{\alpha} \tag{A9}$$

Finally, we use (A8) to find the investment threshold. If the second best (equity maximization) approach is used we arrive at the following non-linear first order condition that can be solved (numerically) for V_I:

$$\begin{aligned} &\left[1 + \beta\left((1-\tau)\frac{R}{r} - V_B\right)\right]\left(\frac{V_I}{V_B}\right)^{\beta}\left(\frac{1}{V_I}\right) \\ &-\alpha\left(\frac{1}{V_I}\right)\left[V_I - (1-\tau)\frac{R}{r} + \left((1-\tau)\frac{R}{r} - V_B\right)\left(\frac{V_I}{V_B}\right)^{\beta} + D(V_I) - I\right] = 0 \end{aligned} \tag{A10}$$

Alternatively, if the first best (firm value maximization) approach is used we have the first order condition:

$$\begin{aligned} &1 + \beta\left((1-\tau)\frac{R}{r} - V_B\right)\left(\frac{V_I}{V_B}\right)^{\beta}\left(\frac{1}{V_I}\right) \\ &+ \beta\left((1-b)V_B - \frac{R}{r}\right)\left(\frac{V_I}{V_B}\right)^{\beta}\left(\frac{1}{V_I}\right) - \alpha\left(\frac{1}{V_I}\right)\left(E(V_I) + D(V_I) - I\right) = 0 \end{aligned} \tag{A11}$$

For optimal capital structure, when coupon is also a choice variable, we solve the first-order condition for the investment trigger by simultaneously searching for the optimal coupon *R*. In this paper, we use the first best approach and we implement equation A11.

References

Boyle, Glenn W., and Graeme A. Guthrie. (2003). “Investment, Uncertainty, and Liquidity,” Journal of Finance 58, 2143-2166.

Brennan, Michael J., and Eduardo S. Schwartz. (1984). “Optimal Financial Policy and Firm Valuation,”Journal of Finance 39, 593-607.

Brennan, Michael J., and Eduardo S. Schwartz. (1978). “Corporate Income Taxes, Valuation, and the Problem of Optimal Capital Structure,” Journal of Business 51, 103-114.

Brennan, Michael J., and Eduardo S. Schwartz. (1985). “Evaluating Natural Resource Investments,” Journal of Business 58, 103-114.

Childs, Paul D., David C. Mauer, and Steven H. Ott. (2000). “Interactions of Corporate Financing and Investment Decisions: The Effect of Growth Options to Exchange or Expand,”Working Paper.

Childs, Paul D., David C. Mauer, and Steven H. Ott. (2005). “Interactions of Corporate Financing and Investment Decisions: The Effect of Agency Conflicts,” Journal of Financial Economics 76, 667-670.

Childs, Paul D., and Alexander J.Triantis. (1999). “Dynamic R&D Investment Policies,” Management Science 45, 1359-1377.

Constantinides, George (1978). “Market Risk Adjustment in Project Valuation,” Journal of Finance 33, 603–616.

Dixit, Avinash, and Robert Pindyck. (1994). Investment under Uncertainty. Princeton, New Jersey: Princeton University Press.

Fazzari, Steven M., Glenn R. Hubbard, and Bruce C. Petersen. (1988). “Financing Constraints and Corporate Investment,” Brookings Papers on Economic Activity 1, 141-195.

Fries, Steven, Marcus Miller, and William Perraudin. (1997). “Debt in Industry Equilibrium,” Review of Financial Studies 10, 39-67.

Gamba, Andrea, and Alexander J. Triantis. (June 2005). “The Value of Financial Flexibility,” Working Paper, presented at the Financial Management Association European Conference, Siena.

Gamba, Andrea, Gordon A. Sick, and Carmen A. Leon. (June 2005). “Investment under Uncertainty, Debt and Taxes,” Working Paper, presented at the Annual International Conference on Real Options, Paris.

Geske, Robert (1977). “The Valuation of Corporate Liabilities as Compound Options,” Journal of Financial and Quantitative Analysis 12, 541-552.

Geske, Robert (1979). “The Valuation of Compound Options” Journal of Financial Economics 7, 63-81.

Jensen, Michael C. and William H. Meckling, 'Theory of the Firm: Managerial Behavior, Agency Costs and Ownership Structure,' *Journal of Financial Economics,* Vol. 3, No. 4 (1976).

Goldstein, Robert, Nengjiu Ju, and Hayne Leland. (2001). "An EBIT-Based Model of Dynamic Capital Structure," Journal of Business 74, 483-512.

Kane, Alex, Alan J. Marcus, and Robert L. McDonald. (1984). "How Big is the Tax Advantage to Debt?," Journal of Finance 39, 841-853.

Kane, Alex, Alan J. Marcus, and Robert L. McDonald. (1985). "Debt Policy and the Rate of Return Premium to Debt," Journal of Financial and Quantitative Analysis 20, 479-499.

Koussis, Nicos, Spiros Martzoukos, and Lenos Trigeorgis. (2005). "Real R&D Options with Time-to-learn and Learning-by-doing," accepted, Annals of Operations Research.

Lambrecht, Bart M. (2001). "The Impact of Debt Financing on Entry and Exit in a Duopoly," Review of Financial Studies 14, 765-804.

Leland, Hayne (1994). "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure," Journal of Finance 49, 1213-1252.

Leland, Hayne, and Klaus B. Toft. (1996). "Optimal Capital Structure, Endogenous Bankruptcy, and the Term Structure of Credit Spreads," Journal of Finance 51, 987-1019.

Leland, Hayne (1998). "Agency Costs, Risk Management, and Capital Structure," Journal of Finance 53, 1213-1243.

Martzoukos, Spiros H. (2000). "Real Options with Random Controls and the Value of Learning," Annals of Operations Research 99, 305-323.

Martzoukos, Spiros H. (2003). “Real R&D Options with Endogenous and Exogenous Learning.” In D. A. Paxson (ed.) Real R&D Options, 111-129. Butterworth-Heinemann Quantitative Finance Series.

Mauer, David C., and Steven H. Ott. (2000). “Agency Costs, Underinvestment, and Optimal Capital Structure: the Effect of Growth Options to Expand,” In: Brennan, M.J and Trigeorgis, L. (eds), Project Flexibility, Agency, and Competition: New Developments in the Theory and Application of Real Options. Oxford University Press, Oxford, 151-179.

Mauer, David C., and Sudipto Sarkar. (2005). “Real Options, Agency Conflicts, and Optimal Capital Structure,” Journal of Banking and Finance 26, 1405-1428.

Mauer, David C., and Alexander J. Triantis. (1994). “Interactions of Corporate Financing and Investment Decisions: A Dynamic Framework,” Journal of Finance 49, 1253-1257.

McDonald, Robert and Daniel Siegel. (1984). “Option Pricing When the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note”, Journal of Finance 39, 261–265.

McDonald, Robert and Daniel Siegel. (1986). "The Value of Waiting to Invest," Quarterly Journal of Economics 101, 707-727.

Merton Robert C. (1974). “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates,” Journal of Finance 29, 449-470.

Merton, Robert C. (1973). “An Intertemporal Capital Asset Pricing Model,” Econometrica 867–887.

Myers, Steward C. (1984). “The Capital Structure Puzzle,” Journal of Finance 39, 575-592.

Myers, Steward C. and Nicholas Majluf. (1984). "Corporate Financing and Investment Decisions when Firms have Information that Investors do not have" Journal of Financial Economics 13, 187-221.

Pawlina, Grzegorz, and Luc Renneboog. (2005). "Is Investment-Cash Flow Sensitivity Caused by Agency Costs or Asymmetric Information? Evidence from the UK", European Financial Management, 11, 483-513.

Stiglitz, Joseph and Andrew M. Weiss. (1981). "Credit Rationing in Markets with Imperfect Information," American Economic Review 71, 393-410.

Trigeorgis, Lenos (1991). "Anticipated Competitive Entry and Early Preemptive Investment in Deferrable Projects," Journal of Economics and Business 43, 143-156.

Trigeorgis, Lenos (1996). Real Options: Managerial Flexibility and Strategy in Resource Allocation. Cambridge: The MIT Press.

Uhrig-Homburg, Marliese (2005). "Cash-Flow Shortage as an Endogenous Bankruptcy Reason," Journal of Banking and Finance 29, 1509-1534.