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CONCEPTUAL AND REPRESENTATIONAL UNDERSTANDING RELATED TO THE CONCEPT OF FUNCTION: A COMPARATIVE STUDY BETWEEN CYPRUS AND ITALY

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To my husband, Demetris
my son, Iakovos
and my parents Kyriacos and Christa
Thank you for being by my side in this journey

ABSTRACT

The concept of function is of fundamental importance in the learning of mathematics (Eisenberg, 1992). The understanding of the concept of function have been a main concern of mathematics educators and of mathematics education research community (Dubinsky & Harel, 1992; Sierpinska, 1992). Functions have a key place in the mathematics curriculum, at all levels of schooling. Being fundamental for the study of mathematics, the function concept has been identified as the single most important notion from kindergarten to graduate school (Dubinsky & Harel, 1992). Nevertheless, students of secondary or even tertiary education, in any country, seem to encounter difficulties in conceptualizing the notion of function (Elia & Spyrou, 2006; Hitt, 1998; Markovits, Eylon, & Bruckheimer, 1986). A vast number of studies have used different approaches to explore the understanding of the concept of function in mathematics teaching and learning.

This study attempted to synthesize the aforementioned ideas of the literature in a model, so as to investigate pre-service teachers' understanding of function in a more comprehensive manner. Specifically, the general aim of this research study was to explore Cypriot and Italian pre-service teachers' display of behavior, cognitive structures and performance in different aspects of the understanding of function. In particular, pre-service teachers' algebraic or coordinated approaches used when solving simple function tasks, definition of function, examples of function, recognizing functions given in different forms, transferring functions from one mode of representation to another and effectiveness in solving complex problems were explored. Furthermore, the results emerged were validated by quantitative and qualitative data taken from two different countries (Cyprus and Italy).

The research study was conducted in three phases. In the first phase four groups of pre-service teachers participated. Particularly, the data of the first group (Group A) were collected in 2005 and the participants were 135 Cypriot pre-service teachers. The data of the second group (Group B) were collected two years later, in 2007 and the participants were 153 Cypriot pre-service teachers. The data of the third group (Group C) were collected in 2008 and the participants were 260 Cypriot pre-service teachers. Finally, the data of the fourth group (Group D) were collected in 2009 and the participants were 200 Italian pre-service teachers. A test consisted of seven tasks was given to the participants. The participants of the second phase were 279 Cypriot and 206 Italian pre-service teachers. There were given two tests consisted of nine and fourteen tasks, respectively. In the third

phase nine of the second phase's participants were chosen for task-based interviews, which were consisted of nine tasks.

From the data analysis important finding were emerged. Particularly, a structural model was constructed and verified in order to determine and document the importance of multiple representational flexibility and problem solving ability in the conceptual understanding of function. The various dimensions of the multiple representational flexibility and problem solving ability were furthermore examined. Particularly, it was emerged that multiple representational flexibility is a multidimensional concept that involves the concept image which consists of the definition and examples of the concept, the recognition of the concept given in various representations (graphical and diagrammatic representation, symbolic and verbal expression) and the conversions from an algebraic to a graphical representation and vice versa. Furthermore, the model highlighted the important role of the coordinated approach and complex problem solving tasks in the conceptual understanding of function and their relation with the problem solving ability. Despite the differences exist in the performance of the Cypriot and Italian pre-service teachers this structure remained invariant for both groups.

As far as the approaches pre-service teachers use in order to solve simple function tasks their stability was verified. Furthermore, Cypriot and Italian pre-service teachers were divided into three groups according to the approach they follow and use. The relation of the coordinated approach with the other dimensions of the understanding of function (concept image, recognition, conversions, problem solving) was examined. The coordinated approach group outperformed the other groups in all the dimensions of the understanding of the concept.

Cypriot and Italian pre-service teachers' behavior and performance were investigated in the various dimensions of the conceptual understanding of function as it was emerged from the structural model. The lowest level of success was observed in problem solving on functions. Furthermore, the compartmentalization that exists in the thinking processes of Cypriot and Italian pre-service teachers was evident. In addition, several misunderstandings and ideas held by the pre-service teachers for the concept of function were emerged.

ΠΕΡΙΛΗΨΗ

Η έννοια της συνάρτησης είναι πολύ σημαντική στα μαθηματικά και τις εφαρμογές τους (Eisenberg, 1992). Η κατανόηση της έννοιας αυτής είναι ένα θέμα που συγκεντρώνει την προσοχή των εκπαιδευτικών αλλά και της μαθηματικής εκπαιδευτικής κοινότητας γενικότερα (Dubinsky & Harel, 1992; Sierpinska, 1992). Οι συναρτήσεις κατέχουν μια ιδιαίτερα σημαντική θέση στο αναλυτικό πρόγραμμα των μαθηματικών, σε όλα τα επίπεδα της εκπαίδευσης. Αποτελούν μια σημαντική και θεμελιώδη έννοια για τη μελέτη των μαθηματικών, από το νηπιαγωγείο μέχρι και τη μέση εκπαίδευση (Dubinsky & Harel, 1992). Παρόλο αυτά η κατανόηση των συναρτήσεων είναι αρκετά δύσκολη. Μαθητές δευτεροβάθμιας εκπαίδευσης αλλά και φοιτητές, σε κάθε χώρα, έχουν δυσκολίες στην εννοιολογική κατανόηση της έννοιας της συνάρτησης (Elia & Spyrou, 2006; Hitt, 1998; Markovits, Eylon, & Bruckheimer, 1986). Ένας μεγάλος αριθμός ερευνών χρησιμοποίησε διαφορετικές προσεγγίσεις για να διερευνήσει την κατανόηση της έννοιας της συνάρτησης στη διδασκαλία και μάθηση των μαθηματικών.

Στη παρούσα μελέτη επιχειρείται η σύνθεση των ιδεών που υπάρχουν στη βιβλιογραφία, με στόχο τη διερεύνηση της κατανόησης της έννοιας της συνάρτησης από φοιτητές. Συγκεκριμένα, ο γενικός στόχος της παρούσας έρευνας είναι να διερευνήσει τη συμπεριφορά, τις γνωστικές δομές και την επίδοση Κύπριων και Ιταλών φοιτητών σε διαφορετικές πτυχές της έννοιας της συνάρτησης. Συγκεκριμένα, διερευνήθηκε η προσέγγιση (αλγεβρική ή ολιστική) που ακολουθούν οι φοιτητές όταν λύνουν έργα συναρτήσεων, ο ορισμός και τα παραδείγματα που δίνουν για την έννοια, η αναγνώριση της έννοιας όταν δίνεται σε διαφορετικές αναπαραστάσεις, η μετάφραση από τη μια αναπαράσταση στην άλλη και η επιτυχία στην επίλυση προβλήματος. Επιπλέον, τα αποτελέσματα που προέκυψαν επιβεβαιώθηκαν από ποσοτικά και ποιοτικά δεδομένα που πάρθηκαν από δύο χώρες (Κύπρο και Ιταλία).

Η παρούσα έρευνα διεξήχθη σε τρεις ερευνητικές φάσεις. Στην πρώτη ερευνητική φάση έλαβαν μέρος τέσσερεις ομάδες φοιτητών. Συγκεκριμένα, η πρώτη ομάδα αποτελείτο από 135 Κύπριους φοιτητές και τα δεδομένα συλλέχτηκαν το 2005, η δεύτερη ομάδα αποτελείτο από 153 Κύπριους φοιτητές και τα δεδομένα συλλέχτηκαν το 2007, η τρίτη ομάδα αποτελείτο από 260 Κύπριους φοιτητές και τα δεδομένα συλλέχτηκαν το 2008 και η τέταρτη ομάδα αποτελείτο από 200 Ιταλούς φοιτητές και τα δεδομένα συλλέχτηκαν το 2009. Ένα δοκίμιο που αποτελείτο από εφτά έργα δόθηκε στους φοιτητές. Οι συμμετέχοντες της δεύτερης ερευνητικής φάσης ήταν 279 Κύπριοι και 206 Ιταλοί

φοιτητές. Στη φάση αυτή δόθηκαν δύο δοκίμια τα οποία αποτελούνταν από εννιά και δεκατέσσερα έργα αντίστοιχα. Στην τρίτη ερευνητική φάση πήραν μέρος εννιά άτομα που επιλέγηκαν από τη δεύτερη φάση για να συμμετάσχουν σε συνεντεύξεις οι οποίες αποτελούνταν από εννιά έργα.

Από την ανάλυση των δεδομένων προέκυψαν σημαντικά αποτελέσματα. Συγκεκριμένα προέκυψε ένα δομικό μοντέλο που επιβεβαιώνει και υπογραμμίζει το σημαντικό ρόλο που διαδραματίζει η ευελιξία χρήσης πολλαπλών αναπαραστάσεων και η ικανότητα επίλυσης προβλήματος στην εννοιολογική κατανόηση της έννοιας της συνάρτησης. Επιπλέον διερευνήθηκαν οι διαστάσεις της ευελιξίας χρήσης πολλαπλών αναπαραστάσεων και της επίλυσης προβλήματος. Συγκεκριμένα, η ευελιξία χρήσης πολλαπλών αναπαραστάσεων είναι μια πολυδιάστατη έννοια που περιλαμβάνει την εικόνα της έννοιας η οποία περαιτέρω προκύπτει από τον ορισμό και τα παραδείγματα που δίνονται για τη συγκεκριμένη έννοια, την αναγνώριση της έννοιας όταν δίνεται σε ποικιλία αναπαραστάσεων (γραφικές παραστάσεις, διαγράμματα, συμβολικές και λεκτικές εκφράσεις) και τις μεταφράσεις από αλγεβρική σε γραφική αναπαράσταση και το αντίθετο. Επιπλέον, από το μοντέλο προέκυψε ο σημαντικός ρόλος που διαδραματίζει η ολιστική προσέγγιση και η επίλυση πολύπλοκων προβλημάτων στη γενική ικανότητα επίλυσης προβλήματος. Παρά τις διαφορές που υπήρχαν στην επίδοση των Κύπριων και Ιταλών φοιτητών η δομή αυτή ήταν η ίδια και για τις δύο ομάδες.

Σχετικά με τις προσεγγίσεις που χρησιμοποιούν οι φοιτητές για να επιλύσουν απλά έργα συναρτήσεων επιβεβαιώθηκε η σταθερότητα τους. Επιπλέον οι Κύπριοι και Ιταλοί φοιτητές χωρίστηκαν σε τρεις ομάδες ανάλογα με την προσέγγιση που χρησιμοποίησαν. Η σχέση της ολιστικής προσέγγισης με τις άλλες διαστάσεις της κατανόησης της συνάρτησης διερευνήθηκε. Συγκεκριμένα τα αποτελέσματα έδειξαν ότι η ομάδα των φοιτητών που χρησιμοποίησαν ολιστική προσέγγιση είχαν καλύτερα αποτελέσματα σε όλες τις διαστάσεις της κατανόησης της έννοιας.

Διερευνήθηκε περαιτέρω, η συμπεριφορά και η επίδοση των Κύπριων και Ιταλών φοιτητών σε όλες τις διαστάσεις της εννοιολογικής κατανόησης της συνάρτησης έτσι όπως αυτές προέκυψαν από το δομικό μοντέλο. Χαμηλότερη επίδοση είχαν στα έργα επίλυσης προβλήματος. Επιπλέον παρατηρήθηκε το φαινόμενο της στεγανοποίησης τόσο για τους Κύπριους όσο και για τους Ιταλούς φοιτητές. Επιπρόσθετα διαφάνηκαν οι διάφορες παρανοήσεις και οι ιδέες που έχουν οι φοιτητές για την έννοια της συνάρτησης .

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CHAPTER I

THE PROBLEM

Introduction

The concept of function is of fundamental importance in the learning of mathematics (Eisenberg, 1992). It emerges from the general inclination of humans to connect two quantities, which is as ancient as mathematics. The understanding of the concept of function as well as the complexity of the teaching of this concept have been a main concern of mathematics educators and of mathematics education research community (Dubinsky & Harel, 1992; Sierpinska, 1992). Functions have a key place in the mathematics curriculum, at all levels of schooling. Being fundamental for the study of mathematics, the function concept has been identified as the single most important notion from kindergarten to graduate school (Dubinsky & Harel, 1992). Nevertheless, students of secondary or even tertiary education, in any country, seem to encounter difficulties in conceptualizing the notion of function (Elia & Spyrou, 2006; Hitt, 1998; Markovits, Eylon, & Bruckheimer, 1986).

An important factor influencing the learning of functions is the diversity of representations related to this concept. A representation is considered to be a configuration of some kind that, as a whole or part by part, corresponds to, is referentially associated with, stands for, symbolizes, interacts in a special manner with, or otherwise represents something else (Palmer, 1977). Representations can be considered as useful tools for constructing meaning and for communicating information and understanding (Greeno & Hall, 1997). It is very important for students to be involved in choosing representations and constructing representations in forms that help them see patterns and perform calculations, taking advantage of the fact that different forms provide different supports for inference and calculation. According to Duval (2006) no kind of mathematical processing can be performed without using a semiotic system of representation, because mathematical processing always involves substituting some semiotic representation for another.

The need for a variety of semiotic representations in the teaching and learning of mathematics is usually explained through reference to the mental cost of processing, the limited representation affordances for each domain of symbolism and the ability to transfer knowledge from one representation to another (Duval, 1987; Duval, 1993; Gagatsis, 1997).

A representation cannot describe fully a mathematical construct and each representation has different advantages, so using various representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2002; Lesh, Behr, & Post, 1987a). Ainsworth, Bibby and Wood (1997) suggested that the use of multiple representations can help students develop different ideas and processes, constrain meanings and promote deeper understanding.

A substantial number of research studies have examined the role of different representations on the understanding and interpretation of functions (Gagatsis & Shiakalli, 2004; Hitt, 1998; Markovits et al., 1986). The concept of function admits a variety of representations and consequently has the capability of being taught using diverse representations, each of which offers information about particular aspects of the concept without being able to describe it completely. The literature illustrates functions in several ways, such as mapping diagrams, tables, graphs and equations. Using multiple representations to teach functions (numeric, graphic, and symbolic) intends to promote and enhance a broad and deep understanding of the concept. Furthermore, the standard representational forms of the concept of function are not enough for students to construct the whole meaning and grasp the whole range of its applications. Several researchers (Evangelidou, Spyrou, Elia, & Gagatsis, 2004; Gagatsis & Shiakalli, 2004; Gagatsis, Elia, & Mougi, 2002; Mousoulides & Gagatsis, 2004; Sfard 1992; Sierpinska, 1992) indicated the significant role of different representations of function and the conversion from one representation to another on the understanding of the concept.

In addition, a useful way to characterize a person's thinking about functions is in terms of the notions of concept definition and concept image (Lloyd & Wilson, 1998). These constructs point to a distinction between the formal definition an individual holds for a given concept and the way that he or she thinks about the concept. Concept image and concept definitions are two terms that have been discussed extensively in the literature concerning students’ conceptions of function (Vinner & Dreyfus, 1989; Tall & Vinner, 1981; Vinner & Hershkowitz, 1980). In addition to the concept definition - concept image scheme a third aspect of concept understanding is concept usage, "which refers to the ways one operates with a concept in generating or using examples or in doing proofs" (Moore, 1994). The centrality of examples in teaching and learning mathematics has been long acknowledged. Examples are an integral part of mathematical thinking, learning and teaching, particularly with respect to conceptualization, generalization, abstraction, argumentation, and analogical thinking (Zodik & Zaslavsky, 2008). Furthermore, it can be used to access students' image for a particular concept.

According to Moschkovich, Schoenfeld and Arcavi (1993), there are two fundamentally different perspectives from which a function is viewed: the process perspective and the object perspective. From the process perspective, a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value. Students who view functions under this perspective could substitute a value for x into an equation and calculate the resulting value for y or could find pairs of values for x and y to draw a graph. In contrast, from the object perspective, a function or relation and any of its representations are thought of as entities for example, algebraically as members of parameterized classes, or in the plane as graphs that are thought of as being 'picked up whole' and rotated or translated. Students who view functions under this perspective could recognize that equations of lines with the form $y = 3x + b$ are parallel or could draw these lines without calculations if they have already drawn one line or they can fill a table of values for two functions (e.g., $f(x) = 2x$, $g(x) = 2x + 2$) using the relationship between them (e.g., $g(x) = f(x) + 2$) (Knuth, 2000).

The development of students' ability to solve problems is considered to be the basic aim of mathematics teaching for all the educators, regardless the learning theory they follow. Problem solving is an ability that reflects the level of mathematical thinking and includes creative and critical thinking also. In addition problem solving or "dealing with a problematic situation" has overcome the limits of mathematical science and it is included in many areas of the social sciences (Philippou & Christou, 1995, p. 128). In relation with the concept of function, many researchers (Gagatsis & Shiakalli, 2004; Mousoulides & Gagatsis, 2004) have dealt with the relationship between success in, solving direct translation tasks and success in solving problems by articulating different representations of the concept of function. The results showed that the solution of a particular problem requires the combination of different abilities and among them the ability to coordinate various representations of a function.

The above mentioned areas that have a significant impact on the conceptual understanding of function have been examined until now rather separately and in isolation from one another. Limited attention has been given to interrelations among students' concept image of function that involves concept's definition and examples, use of different representations of the mathematical concept, students' approaches when dealing with simple function tasks and problem solving. In this research study we attempt to synthesize the above mentioned research domains and construct a model concerning the conceptual understanding of function that it will be validated by using qualitative and quantitative data from two countries (Cyprus and Italy). The above investigation is conducted in two countries in order to explore if there are differences between the Cypriot and Italian pre-service teachers concerning their behaviour, their performance and the cognitive structure of the various dimensions of the conceptual understanding of function. The participation of pre-service teachers from two countries will give further validation to the model. The fact that it is a comparative study is quite important, since these two countries have cultural similarities. It is noteworthy the fact that the impact of cultural tradition is highly relevant to mathematics learning (Leung, Graf, & Lopez-Real, 2006). However, despite cultural similarities, differences are observed in the educational systems of the two countries.

The Problem and the Purpose of the Study

The general aim of this research study was to explore Cypriot and Italian pre-service teachers' display of behavior, cognitive structures and performance in different aspects of the understanding of function. Basic dimensions explored were the multiple representational flexibility and problem solving ability.

The research study was conducted in three phases. The goal of the first phase was to contribute to the understanding of the algebraic and coordinated approaches pre-service teachers develop and use in solving function tasks and to examine which approach is more correlated with their ability in solving complex problems. Furthermore an important goal of this phase was to investigate the stability of these approaches and the stability of their relation.

The goal of the second phase was to explore pre-service teachers' display of behavior, cognitive structures and performance in six aspects of the understanding of function: effectiveness in solving complex problems with functions, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic expression, verbal expression), transferring function from one mode of representation to another and the approach when dealing with simple function tasks. A main concern was also to examine problem solving in relation to the other types of displayed behavior.

The goal of the third phase was to triangulate the quantitative data regarding teachers' understanding of the concept of function and to further investigate their behavior as well as their ideas and misunderstandings in the above mentioned aspects of the understanding of function: concept definition, examples of function, the recognition of functions given in various representations, "coordinated" or algebraic approaches when dealing with simple function tasks and effectiveness in problem solving.

Particularly, the modes of representations (symbolic, diagrammatic, graphical, verbal, tabular) and the functions of the different representations, concerning the concept of function, included in the Cypriot mathematics textbooks of the 9th, 10th, 11th and 12th grade were explored in order to gain insights to the way the concept is taught in middle and high school.

A structural model was constructed and verified in order to determine and document the importance of multiple representational flexibility and problem solving ability in the conceptual understanding of function. The various dimensions of the multiple representational flexibility and problem solving ability were furthermore examined. Particularly, it was investigated if the multiple representational flexibility is a multidimensional concept that involves the concept image which consists of the definition and examples of a concept, the recognition of the concept given in various representations (graphical and diagrammatic representation, symbolic and verbal expression) and the conversions from an algebraic to a graphical representation and vice versa. Furthermore, the role of the coordinated approach and problem solving tasks and their relation with the problem solving ability was examined. In addition, it was also examined if this structure is the same for the Cypriot and Italian pre-service teachers despite the differences that exist in the curricula of the two countries.

As far as the algebraic and coordinated approaches pre-service teachers use in order to solve simple function tasks, their stability and their relation with problem solving was examined. Cypriot and Italian pre-service teachers were divided into three groups according to the approach they follow and use in order to examine the relation of the coordinated approach with the other dimensions of the understanding of function (concept image, recognition, conversions, problem solving).

Furthermore, Cypriot and Italian pre-service teachers' behavior and performance were investigated in the various dimensions of the conceptual understanding of function as it emerged from the structural model.

Additionally, several misunderstandings and ideas held by the pre-service teachers for the concept of function were analyzed and discussed.

Research Questions of the Study

The main research questions of the first phase of the study as retrieved from the aim and the goals of this research study were the following:

- What approach (algebraic or coordinated) do pre-service teachers prefer to use when they solve simple function tasks?
- How able are pre-service teachers to solve complex function problems?
- Which approach (algebraic or coordinated) is more correlated with pre-service teachers' ability in solving complex problems?
- How stable and long lasting are these approaches and their relation with problem solving?

The following research questions were examined in the second phase:

- How able are Cypriot and Italian pre-service teachers to give a right definition, examples of the concept of function, to recognize functions given in different representations, to make conversions from an algebraic to a graphical representation of function and vice versa and to solve complex function problems?
- What conceptions, ideas and misunderstandings do Cypriot and Italian pre-service teachers have of function on the basis of their concept definitions, of the examples of the notion and of the justifications given in the tasks involving the recognition of function given in various representations?
- What approach (algebraic or coordinated) do Cypriot and Italian pre-service teachers prefer to use when they solve simple function tasks?
- How do Cypriot and Italian pre-service teachers behave during the solution of tasks involving the definition, the examples, the recognition, the conversions, the approaches and problem solving of function?

- What are the structure and the relationships between the definition, the examples, the approaches, the recognition, the conversions and problem solving?
- What are the similarities between Cypriot and Italian pre-service teachers in regard to the structure of their function understanding?
- What differences exist in the behavior of the coordinated, algebraic and various approaches groups of pre-service teachers use during the solution of tasks involving the definition, the examples, the recognition, the conversions and the problem solving of function?

The following research questions were examined in the third phase:

- How able are the participants to give a right definition, to give examples of the concept, to recognize functions given in different representations and to solve problems involving linear, quadratic and exponential functions?
- What approach do the participants use (algebraic or coordinated) in order to solve simple function tasks and what is the relation of their approach with their performance to the other tasks?
- What differences exist between the three groups of pre-service teachers (high, medium and low performance group) concerning their performance in the nine tasks of the interview?
- What conceptions, ideas and misunderstandings do pre-service teachers have of function?

Significance of the Study

The concept of function is of fundamental importance in the learning of mathematics (Eisenberg, 1992). Functions have a key place in the mathematics curriculum, at all levels of schooling. Furthermore, the function concept has been identified as the single most important notion from kindergarten to graduate school (Dubinsky & Harel, 1992).

Although functions have been recognised as an important mathematical concept, limited attention has been given to the interrelations among pre-service teachers' approaches used when solving simple function tasks, definition of function, examples of function, recognizing functions given in different forms, transferring functions from one mode of representation to another and effectiveness in solving complex problems. This research study aims to synthesize most of the ideas discussed in the studies of the aforementioned research domains, i.e. the different ways of constructing mental images of function, using a diversity of representations of the concept of function and problem solving, in a composite model, so as to investigate pre-service teachers' understanding of function in a more comprehensive manner. Furthermore, this model will be validated by quantitative and qualitative data taken from two different countries (Cyprus and Italy).

We attempt to contribute to mathematics education research understanding with respect to the concept of function by investigating the relationship among the aforementioned components that are constitutive of the meaning of function. We anticipate that this model will provide a coherent picture of pre-service teachers' construction of the meaning of function that is desirable for current approaches of instruction that aim at the development of the understanding of this concept. Furthermore, the findings would be enlightening for mathematics educators about the importance of using a composite model constituted by these types of behaviour as a means not only to examine and explain how the function concept is understood by students, but to teach functions at secondary school. At the school didactic activities should be implemented, that are not restricted in limited and separately taught aspects, but interconnected with each other on the basis of the above forms of understanding of the notion. It is anticipated that these activities will contribute to the development of a global and coherent understanding of function and successful problem solving.

The role of representations in the teaching and learning of mathematics was examined by a number of researchers (Duval, 2006; Elia, Gagatsis, & Demetriou, 2007). However, many questions remain unanswered in this field. Furthermore, the previous research studies have examined only one dimension of the multiple representational flexibility i.e. the recognition (Niemi, 1996) or the conversions (Gagatsis & Shiakalli, 2004), in order to examine the role and impact of multiple representations in problem solving. Taking into consideration the fact that the ability to recognize a concept given in various representations and the ability to translate a concept from one representation to another, contribute to the conceptual understanding and improve students' performance in problem solving (Even, 1998), the need for a careful investigation emerges. In this research study the multiple representational flexibility is viewed in a completed way as a multidimensional concept involving the concept image, the ability to recognize the concept in various representations and the ability to make conversions from one representation to another.

As far as the concept image and concept definition are concerned, these are two terms that have been discussed extensively in the literature concerning students' conceptions of function (Vinner & Dreyfus, 1989; Tall & Vinner, 1981; Vinner & Hershkowitz, 1980) and it is now an old construct in mathematics education. However, these terms continue to be cited in the literature (e.g. Przenioslo 2004; Giraldo 2006; Nardi 2006) since concept image is an important construct. In this research study the term of concept image is enriched by involving not only the definition of a concept but examples of the concept of function since example usage that involves example generation and verification is crucial for the understanding of a concept (Dahlberg & Housman, 1997) and plays a significant role in the formation of the concept's image. In addition, as was mentioned before we consider that concept image plays a significant role and it is one of the dimensions of the multiple representational flexibility.

Previous studies indicated students' ineffective use of the geometric approach, which originates within the object perspective (Knuth, 2000) and their systematic use of the algebraic approach (process perspective). In this research study the terms algebraic and coordinated approach are employed. Furthermore, the stability of these approaches is investigated and their relation not only with problem solving but with all the dimensions of function conceptual understanding is explored.

The above investigation is conducted in two countries (Cyprus and Italy) in order to explore if there are differences between the Cypriot and Italian pre-service teachers concerning their behaviour, their performance and the cognitive structure of the various dimensions of the conceptual understanding of function. The fact that it is a comparative study is quite important, since despite cultural similarities, differences are observed in their educational systems. In addition it is important the fact that in this research study there is a combination of quantitative and qualitative data.

Limitations of the Study

There is a number of limitations resulted from the methodology employed in the present study. The study was conducted between Cypriot and Italian pre-service teachers attending at the Universities of Cyprus, Bologna and Palermo. Therefore the results are only representative of that group of participants. There is a need to verify the structural model emerged in this study in middle and high school students. The decision for selecting pre-service teachers was based on the fact that there was a need to explore prospective teachers' understanding in a fundamental mathematical concept such as the concept of function.

A second set of limitations is related to the tasks used in the two tests. The tasks involved in the two tests were appropriate to the subjects and were based on the exercises and examples involved in the mathematics textbooks of the 9^{th}, 10^{th}, 11^{th} and 12^{th} grade of middle and high school. In addition, the representations included in the tasks were the ones included in the textbooks (graphical and diagrammatic representation, verbal and symbolic expression, tabular representation). However, the fact that most of the participants were not mathematically oriented made it difficult for them to work successfully with the tasks.

A third limitation is related to the time needed for working with the tasks and the number of tasks included in the tests. Particularly, two tests were constructed the one involving nine tasks and the other fourteen tasks. The tests were administered to the teachers by researchers in two 90 minutes sessions. The solution of the tasks was time demanding for the pre-service teachers and their professor. Although in some cases there was a need to insert more tasks in order to enrich and describe better a particular dimension of the understanding of function, this was not possible due to time limitations.

Another limitation is related to the participants of the tasks-based interviews. The purpose of the interviews was to triangulate the quantitative data regarding teachers' understanding of the concept of function and to further investigate their behavior as well as their ideas and misunderstandings in various aspects of the understanding of the concept. In addition task-based interviews were used because they are powerful means to focus on "subjects' purposes of addressing mathematical tasks, rather than just on patterns of correct and incorrect answers in the results they produce" (Goldin, 2000).

The participants were nine Cypriot pre-service teachers. The participants were chosen according to their performance to the two tests of the second phase and their willingness to participate in an interview. Particularly, three of them had a high performance, three a medium performance and three low performance to the two tests. It would have been desirable to be able to interview Italian pre-service teachers also but this was not possible due to practical reasons.

Thesis Structure and Summary

The general aim of this research study was to explore Cypriot and Italian pre-service teachers' display of behavior, cognitive structures and performance in different aspects of the understanding of function. In particular, six aspects of the understanding of function were explored: effectiveness in solving complex problems with functions, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic expression, verbal expression), transferring function from one mode of representation to another and the algebraic or coordinated approaches when dealing with simple function tasks.

The next chapters describe more thoroughly the results of other research studies and theories related to the above subject, the research process, analyses and findings.

Particularly, chapter two provides a review of the literature relevant to this study. Different definitions of the concept of representations are analyzed and discussed. The importance of representations in mathematics teaching and learning is highlighted. The role of multiple representations in mathematics learning is emphasized. Several research studies indicating the significant role of different representations of function are presented. The terms concept image and concept definition are discussed, as well as the importance of the centrality of examples in teaching and learning mathematics. Furthermore, the role of the concept definition and examples in the formation of concept image and in the conceptual understanding of the concept of function is examined. The terms algebraic and coordinated approach are explained and defined. Finally, the importance of problem solving in the conceptual understanding of functions is highlighted.

Chapter three explains the methodology and the experimental design that involved three phases. The goal of the first phase was to contribute to the understanding of the algebraic and coordinated approaches pre-service teachers develop and use in solving function tasks and to examine which approach is more correlated with their ability in solving complex problems. Furthermore an important goal of this phase was to investigate the stability of these approaches and the stability of their relation. The goal of the second phase was to explore teachers' display of behavior, cognitive structure and performance in the various aspects of the understanding of function. The goal of the third phase was to triangulate the quantitative data regarding teachers understanding of the concept of function and to further investigate pre-service teachers' behavior in several aspects of the understanding of function.

In Chapter three the participants of each phase, the instruments used, the variables and the scoring of the tasks and the statistical techniques used in order to analyze the data emerged from the study are presented. Furthermore, a brief analysis of the Cypriot curriculum and the mathematics textbooks of the 9th, 10th, 11th and 12th grades in relation to the modes of representations and the function of representations concerning the particular concept are presented.

Chapter four presents the results of the three phases based on the research questions posed. The stability of the coordinated approach is verified and its relation with the other dimensions of the understanding of function (concept image, recognition, conversions, problem solving) is documented. A structural model is presented that documents the importance of multiple representational flexibility and problem solving ability in the conceptual understanding of function. The fact that multiple representational flexibility is a multidimensional concept that involves the concept image which consists of the definition and examples of a concept, the recognition of the concept given in various representations (graphical and diagrammatic representation, symbolic and verbal expression) and the conversions from an algebraic to a graphical representation and vice versa is discussed. Furthermore, the role of the coordinated approach and problem solving tasks and their relation with the problem solving ability is examined. It is also examined if the above mentioned structure is the same for the Cypriot and Italian pre-service teachers despite the differences exist in the curriculum of the two countries. Cypriot and Italian pre-service teachers' behavior and performance is investigated in the various dimensions of the conceptual understanding of function as they emerged from the structural model. Furthermore, several misunderstandings and ideas held by the pre-service teachers for the concept of function are analyzed and discussed.

Chapter five discusses the findings of the study and provides didactical implications suggestions for future research.

Operational Definitions

Concept definition is the form of words/symbols used by the tutor/course notes/textbook to define a mathematical concept (Bingolbali & Monaghan, 2007). It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole.

Concept usage refers to the ways one operates with a concept in generating or using examples or in doing proofs (Moore, 1994). In addition, learner-generated examples is a teaching strategy of asking learners to construct their own examples of mathematical objects under given constraints (Watson & Mason, 2005).

Concept image is the total cognitive structure associated with a concept in an individual's mind. It includes mental pictures, associated properties and processes as well as strings of words and symbols (Tall & Vinner, 1981). In this study we consider that access is gained in teachers' *concept image* through the *definition* and *examples* given for the concept of function.

Conversions are transformations of representation that consist of changing a register without changing the objects being denoted: for example, passing from the algebraic notation of an equation to its graphic representation, passing from the natural language statement of a relationship to its notation using letters, etc (Duval, 2006).

Multiple representational flexibility is the ability to switch mental sets in response to within- and between- representation alterations (recognition, treatment, conversion) of the same mathematical object (Gagatsis, Deliyianni, Elia, & Panaoura, 2010). In this study this definition includes recognition and conversions and it is also extended by including the concept image that is based on concept definition and concept usage (examples).

Algebraic perspective/approach is similar with the pointwise approach described by Even (1998) and the one described by Mousoulides and Gagatsis (2004). In this perspective, a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value (Moschkovich et al., 1993).

"Coordinated" perspective/approach combines the algebraic and the graphical approach. In this perspective, the function is thought from a local and a global point of view at the same time. The students' can "coordinate" (flexibly manipulate) two systems of representation, the algebraic and the graphical one.

Problem solving refers to the process of associating prior experiences, knowledge, information and intuition in order to determine the outcome or a solution of a situation for which the procedure for determining the outcome is not directly known (Charles, Lester, & O' Daffer, 1987).

Conceptual understanding of mathematical concepts is defined with reference to internal networks of representations. It appears when the representations are connected in constantly developed and structured nets. The relations between the different representations can be based on similarities, differences or inclusion (Hiebert & Carpenter, 1992). The conceptual understanding can be considered as a network of relations between: (a) already existing knowledge and information and (b) existing knowledge and new knowledge (Hiebert & Lefevre, 1986).

CHAPTER II

THEORETICAL FRAMEWORK

Introduction

In this chapter in an attempt to review the related literature and provide a coherent framework for the research investigation, the literature review is organized into four major strands.

The first strand discusses the role of representations in mathematics teaching and learning and particularly their connection with the concept of function. Firstly, the concept of representation is presented as well as the definitions given by several researchers. The importance of representations in mathematics teaching and learning is highlighted and the role of multiple representations in mathematics learning and problem solving is emphasized. The relation of multiple representations with the concept of function is presented.

The second strand discusses the terms of concept image and concept definition, as well as the importance of the centrality of examples in teaching and learning mathematics. Furthermore, it examines the role of the concept definition and examples in the formation of students' concept image and their relation with the conceptual understanding of the concept of function.

In the third stand the terms algebraic and coordinated approaches are explained and defined. Furthermore, the important role of the coordinated approach in problem solving and in the conceptual understanding of function is highlighted.

Finally in the fourth strand, definitions for problem solving are given and the importance of problem solving in the conceptual understanding of functions is documented.

In this research study in an effort to investigate the conceptual understanding of the concept of function by pre-service teachers the above mentioned areas were explored. It is considered that the recognition of the concept given in various representations, the conversions from one mode to the other and the concept image which consists of the concept definition and examples of the concept are the basic dimensions of the multiple representational flexibility. The coordinated approach and the solution of complex problems are the basic dimensions of the problem solving ability. Multiple representational flexibility and problem solving ability are the basic dimensions of the conceptual understanding of function.

Representations, mathematics learning and the understanding of function

The concept of representation

Representation is a slippery and ambiguous concept, and it has been for a long time (Goldin & Kaput, 1996; Roth & McGinn, 1998; Seeger, 1998). There are different connotations of the concept. The difficulty in articulating an accurate definition for the term representation is worth stressing. Different psychological theories dealt with the concept of representation and as a result different definitions were formed.

Kaput (1987) maintained that there is a need for a systematical theoretical framework for the representational systems which will contribute to the co-frontal of several practical problems concerning representations. Particularly, Kaput (1987) claimed that emphasis should be given to the way symbols are used and to their combinations in the mathematical representational systems and in the way these systems are correlated. According to Kaput (1987) the concept of representation involved the following components:

- a representational entity
- the entity that it represents
- particular aspects of the representational entity
- the particular aspects of the entity that it represents that form the representation
- and finally the correspondence between the two entities

For the aims of this research study the representational entity was considered to be the concept of function and the entities that they represent were the diagrams, the graphs, the verbal and symbolic expressions. The particular aspects of the representational entity were the characteristics of a function. The particular aspects of the entity that it represents were for example in the case of the graph the x and y axes, the numbers on the axes and the points. Finally, in order to use correctly the various representations of function it is necessary to have a correspondence between the two entities.

Brousseau (2004) gave a definition for the concept of representation that is similar to the one given by Kaput. He assumed that a representation has at least three things: an object represented, representing an object which is often called "representation" and a certain relationship that connects the representative to the object. This relationship is also called "Representation". In fact, a representation is more in the presence of both worlds: the one world contains the thing represented and the other contains the representative thing. The representative and the represented have relationships with their respective worlds. According to Brousseau (2004) the representation is the triple <S, f, S '> and not only S'. Furthermore, the representation must also satisfy a commutative property: the image f (R (a, b)) of a relationship R between two objects a and b represented in the universe world must coincide with the relationship fR applied to the images f(a) and f(b) of the a and b. The means that we can study the relations in the world represented, and we can translate them into the representative world or start by translating the objects and the relationships and look at the relations in the representative images. This condition plays a fundamental role in the research of representations since it allows to re-import a meaning or an outcome of the representative world in the represented world, and as a result of the use of the representation.

Several researches gave simpler definitions for the concept of representation. Palmer (1977) stated that a representation is a configuration of some kind that, as a whole or part by part, corresponds to, is referentially associated with, stands for, symbolizes, interacts in a special manner with, or otherwise represents something else. Goldin and Kaput (1996) added that representations do not occur in isolation. They usually belong to highly structured systems, either personal and idiosyncratic or cultural and conventional. These have been termed "symbol schemes" (Kaput, 1987) or "representational systems" (Goldin, 1987; Lesh, Landau & Hamilton, 1983). Furthermore, the representing relationship is in general not fixed, nor is its specific nature a necessary feature of the representation. This is because, inevitably and intrinsically, an interaction or act of interpretation is involved in the relation between that which is representing and that which is represented (von Glasersfeld, 1987).

McKendree, Small, Stenning and Conlon (2002) stated that a representation is a structure that stands for something else: a word for an object, a sentence for a state-of-affairs, a diagram for an arrangement of things, a picture for a scene. According to McKendree et al. (2002) the second critical concept is transformation of representations. In a system of representations, particular operations can be used to manipulate the structures into new ones which are systematically related to the one they were derived from. The meaningful transformations of representations are at the core of understanding human information processing. Reasoning with an abstract representation of a situation can be much more effective than reasoning with a concrete situation alone. Thus, a good representation system captures exactly the features of a problem that are important rather than representing everything. The operations that create meaningful transformations can manipulate those critical features in useful ways, and the pay-off is a more effective and efficient reasoning about a particular problem or situation.

Von Glasersfeld (1987) approached the matter of representations from a constructivist perspective according to which learning is an active process. Furthermore, perceiving, from a constructivist point of view, is always an active making, rather than a passive receiving. He stated that the term of representation is referring to: (a) iconic representations, (b) symbols and (c) mental representations. The first two are external representations while the third is internal. The iconic representation is a rebuild of an empirical object. Its interpretation is not a part of the external world, but it is the combination of preexisting elements in the repertoire of a human’s experiences. Every human perceives and interprets an external representation according to the mental representations already constructed as a result of his/her knowledge and his/her experiences. The symbolic representations refer to the signs, the symbols and other semiotic objects such as the letters of the alphabet, the words and the graphs. The mental representations are produced internally and are replayed, shelved, or discarded according to their usefulness and applicability in experiential contexts organized. The more often they turn out to be viable, the more solid and reliable they seem. But no amount of usefulness or reliability can alter their internal, conceptual origin. They are always made up of elements that first arose on the sensorimotor level of experience. Thus, they are made up of elements that the experiencing subject already has, though they may, of course, be novel combinations or they may exemplify some abstraction — but if they do, they do so by applying the abstraction to quite specific sensorimotor material. Furthermore, according to the constructivist view they are dynamic since they are not conceived as postcards that can be retrieved from some file, but rather as relatively self-contained programs or production routines that can be called up and run.

According to Roth and McGinn (1998) the term representation has been used to refer to graphical displays. However, this term is ambiguous because it has also been used to refer to mental content. This second meaning arose because a high degree of similarity was thought to exist between mental content and materially embodied representations (e.g., Larkin & Simon, 1987). In the recent literature on the sociology of scientific knowledge, the term inscription was introduced to distinguish representations, which exist in material form (e.g., paper, computer screen) and can therefore be shared by several agents, from mental representations, which are not accessible. When representations are investigated as inscriptions, including inscription production, maintenance, transformation, and roles in shared activity, a different view of cognition arises than when the emphasis is on mental representations. Inscription is a term that recently gained considerable currency in science and technology studies. Inscriptions are signs that are materially embodied in some medium, such as paper or computer monitors. Graphs, tables, lists, photographs, diagrams, spreadsheets, and equations are characteristically classified as inscriptions (Latour, 1987). Because of their material embodiment, inscriptions (in contrast to mental representations) are publicly and directly available, so that they are primarily social objects. Knowledgeability with respect to inscriptions is indicated by the degree to which individuals participate in purposive, authentic, inscription-related activities.

A considerable amount of research indicates that interpreting inscriptions such as graphs—that is, reconstructing possible worlds from inscriptions—is very difficult for many students (Leinhardt, Zaslavsky, & Stein, 1990). From an inscription perspective, students' difficulties come as no surprise, for at least two reasons. First the relationship between any inscription and its referent is a matter of social practice. It is difficult to reconstruct the canonical meaning of an inscription without having participated in the relevant community of practice. Second, interpretation means that participants have to reconstitute a real-world situation for which the graphic could be a representation. That requires participants to construct contexts that have previously been abstracted in the process of creating the inscription. The learning process has to enable future users of inscriptions by setting the stage for them to use inscriptions in flexible ways. For this to occur, students must be afforded means to use inscriptions in open-ended and flexible ways rather than to apply algorithmic prescriptions for how inscriptions are to be used. Furthermore, in everyday out-of-school situations, inscriptions are crafted for particular purposes and always serve interests. Learning environments therefore need to make provisions so that students can pursue their own goals and, in the process, learn to use inscriptions for their own intentions.

Goldin and Kaput (1996) and Goldin (1998) made an important distinction for the psychology of learning and doing mathematics and that is between internal and external systems of representation (see Fig. 2.1).

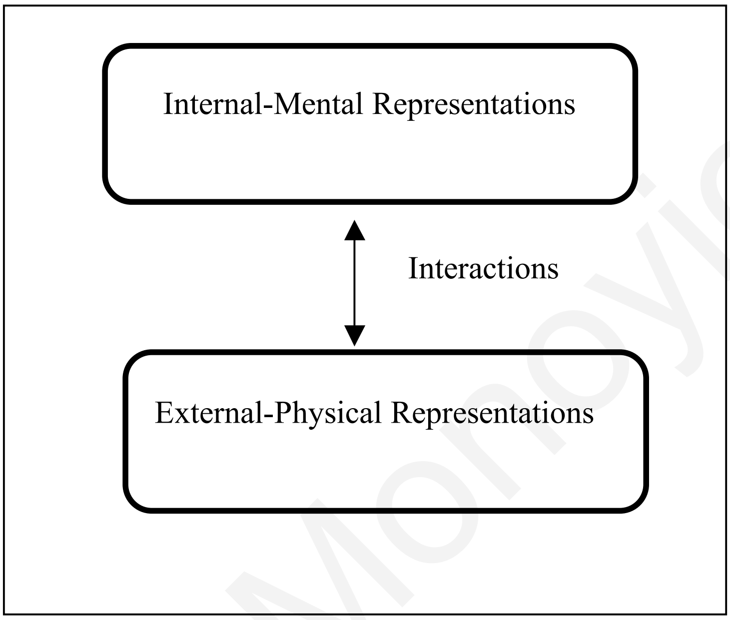


Figure 2.1. Internal versus external representations (Goldin & Kaput, 1996, p.399)

According to Goldin (1998) a representational system consists first of primitive characters or signs. These can be discrete entities drawn from a fairly well-defined set, such as characters in symbolic logic, spoken words, letters in an alphabet, punctuation marks, numerals, arithmetic operation symbols or components of a logic circuit. They can also be abstract mathematical or physical entities such as numbers, vectors, matrices, velocities, or forces. An important feature of these characters, when they are regarded as the elementary entities within a representational system, is that we do not yet ascribe to them further meaning or interpretation. In addition to the elementary signs, a representational system includes (sometimes ambiguous) rules for combining the signs into permitted configurations.

Furthermore, Goldin (1998) proposed a unifying theoretical model of external and internal representations in which the internal representations have symbolic relations with the external representations and with other internal representations. Actions on external task environments (e.g., steps within external representational systems, translations from one representational system to another, or construction of entirely new representations) are viewed here as mediated by internal representational systems. These may or may not bear some structural resemblance to external systems.

The term external representations was used to refer to physically embodied, observable configurations such as words, graphs, pictures, equations, or computer microworlds. These are in principle accessible to observation by anyone with suitable knowledge. The interpretation of external representations as belonging to structured systems, and the interpretation of their representing relationships, is not “objective” or “absolute” but depends on the internal representations of the individual(s) doing the interpreting. The external representations in mathematics are necessary for the presentation and the communication of mathematical ideas and they may have one or more forms. They cannot be understood and or operate in isolation. For example, an equation or a graphical representation have meaning only if they are part of a larger system. In the semiotic system is assigned worth to the representation in comparison to the other representations (Goldin & Shteingold, 2001; Duval, 2006). The representational systems have a particular structure and as a result the different representations in the system are closely related (Goldin & Steingold, 2001).

The term internal representation was used to refer to possible mental configurations of individuals, such as learners or problem solvers. Such configurations are not directly observable but are inferred from what students or subjects say or do, that is, from their external behavior (Goldin & Kaput, 1996). The internal representation of mathematical ideas is necessary for their understanding (Hiebert & Carpenter, 1988).

Goldin (1998) proposed five types of mature, internal cognitive representational systems. These are (a) the verbal/syntactic systems, (b) imagistic systems, (c) formal notional systems, (d) a system of planning, monitoring, and executive control, and finally (e) a system of affective representation. A verbal/syntactic system of representation describes the individual’s capabilities for processing natural language, on the level of words, phrases, and sentences (only). Input channels for such a system include hearing and reading; output channels include speaking and writing. Imagistic systems include several different non-verbal, non-notational cognitive systems. The most important of these for the psychology of mathematics education seem to be visual/spatial, auditory/rhythmic, and tactile/kinesthetic systems of representation. The conventional, formal notations of mathematics are highly structured, symbolic systems-numeration systems, arithmetic algorithms, rational number and algebraic notations, rules for symbol manipulation, etc. Their structure as external representations for mathematical and logical problem solving is readily explored. A cognitive representational system that includes strategic planning, monitoring, and decision-making (executive control) can be regarded as directing or guiding the problem solving process. This system includes competencies for: (1) keeping track of the state of affairs in the other systems, and in itself; (2) deciding the steps to be taken, or moves to be made, within all of the internal representational systems, including itself; and (3) modifying the other systems-deciding to improve the formal notational conventions, invent new words, adopt changed affect, and so forth. A fifth type of internal, representational system, a system of affective representation, is needed not only to model learning and problem solving effectively, but to discuss educational goals that include enjoyment and positive self-concept as well as cognitive competencies.

The model proposed by Goldin (1998) has consequences not just for theory, but for practice. Most fundamentally, the present model suggests that our goals in mathematics education should not be limited to conveying specific mathematical content to students, or even to teaching them specific problem-solving processes. Rather the overarching goal should be to foster in students the construction of powerful, internal systems of representation of the kinds discussed.

Much of school mathematics is still devoted explicitly to the manipulation of formal notational systems, despite considerable effort to change classroom practice to emphasize problem solving strategies, visualization, pattern recognition, and other more conceptually-oriented techniques. The main goal must be to develop internal representational systems of many kinds and to provide an equivalent level of attention to imagistic representation, planning and executive control, and affective representation, without sacrificing the learning of mathematical techniques. Developing power in these representational systems must become explicit objective, and methods of assessment must be developed that can evaluate our success.

Specifically, the external systems of representation include conventional representations that are usually symbolic in nature. Our numeration system, mathematical equations, algebraic expressions, graphs, geometric figures, and number lines are examples of external representations. These representations have been developed over time and are widely used. External representations also include written and spoken language (Goldin & Shteingold, 2001).

Lesh et al. (1987b) have identified five distinct types of external representation systems that occur in mathematics learning and problem solving. They are: (a) experience-based "scripts" in which knowledge is organized around "real world" events that serve as general contexts for interpreting and solving other kinds of problem situations, (b) manipulatable models like Cuisenaire rods, arithmetic blocks, fraction bars, number lines in which the "elements" in the system have little meaning per se but the "built in" relationships and operations fit many everyday situations, (c) pictures or diagrams-static figural models that, like manipulatable models, can be internalized as "images", (d) spoken languages including specialized sublanguages related to domains like logic and (e) written symbols which like spoken languages, can involve specialized sentences and phrases.

Figure 2.2 presents the five types of external representation systems and the relations among them. From the diagram we can infer that all the representation systems are important for the understanding and learning of information. Furthermore, the modes of external representation are interrelated. Irrespective of whether initial instruction focuses on the combination of manipulatives and real world situations or written symbols and pictures, or other combinations of external representations, students need to build their understanding across the various modes (Cathcart, Pothier, Vance, & Bezuk, 2006).

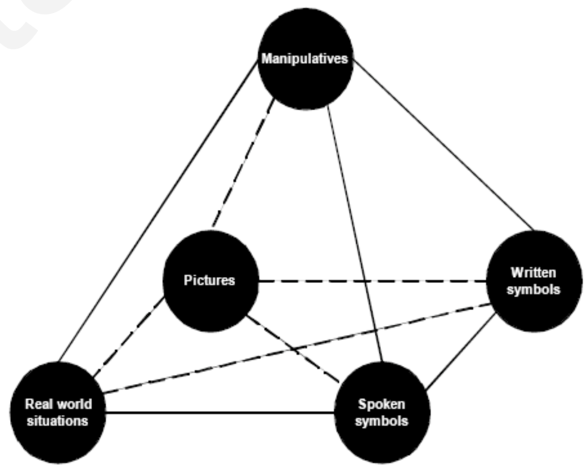


Figure 2.2. The five distinct types of external representation systems (Lesh et al., 1987b, p. 33)

Fundamental to effective teaching and learning is the interaction between external and internal representation, with students interpreting teachers' external representations in a way which makes sense to them (Goldin & Shteingold, 2001; Gould, 2005). As suggested by Halford (1993, p. 7), "To understand a concept entails having an internal, cognitive representation or mental model that reflects the structure of that concept. The representation defines the workspace for problem solving and decision making with respect to the concept". Goldin and Shteingold (2001) consider that a basic aim of teaching mathematics should be the development of effective internal representational systems that will interact efficiently with the external mathematical systems. As a result the person will externalize the actions that came from internal structures. At the same time the person will internalize actions after the interaction with the external natural structures of a symbolic system. The internal representations are not immediate observable. Their existence and their structure are declared by the behavior of the subjects, mainly with their interaction with the external representations (Hiebert & Carpenter, 1992). In addition, the way a person perceives and interprets a representation is based on his/her mental representations that have been constructed on the basis of previous experience and knowledge (Gagatsis, Michaelidou, & Shiakalli, 2001).

The way that students develop their representational thinking was studied by Pape and Tchoshanov (2001). The authors stated that this is a two-sided process, one is the internalization of external representations and the other one is the externalization of mental images of mathematical objects. Pape and Tchoshanov (2001) argue that there is a mutual influence between the internal and the external forms of representation. For instance, simple graphs produce simple understandings, while complex graphs, such as a layered graph, allow the understanding of more complex phenomena. The authors also emphasize the social component of representation. Representations are used to communicate with others and engender feedback and discussions. Thus representations should not be static, but evolve as their use and understanding evolve likewise.

Duval (2006) maintained that there is also another essential difference between the representational systems that is very often missed. Some semiotic systems can be used for only one cognitive function: mathematical processing. On the other hand, other semiotic systems can fulfill a large range of cognitive functions: communication, information processing, awareness, imagination. This functional difference between the various semiotic representation systems used in mathematics is essential because it is intrinsically connected with the way mathematical processes run: within a monofunctional semiotic system most processes take the form of algorithms, while within a multifunctional semiotic system the processes can never be converted into algorithms (see fig. 2.3.). For example, in elementary geometry, there is no algorithm for using figures in an heuristic way and the way a mathematical proof runs in natural language cannot be formalized but by using symbolic systems. Proofs using natural language cannot be understood by most.

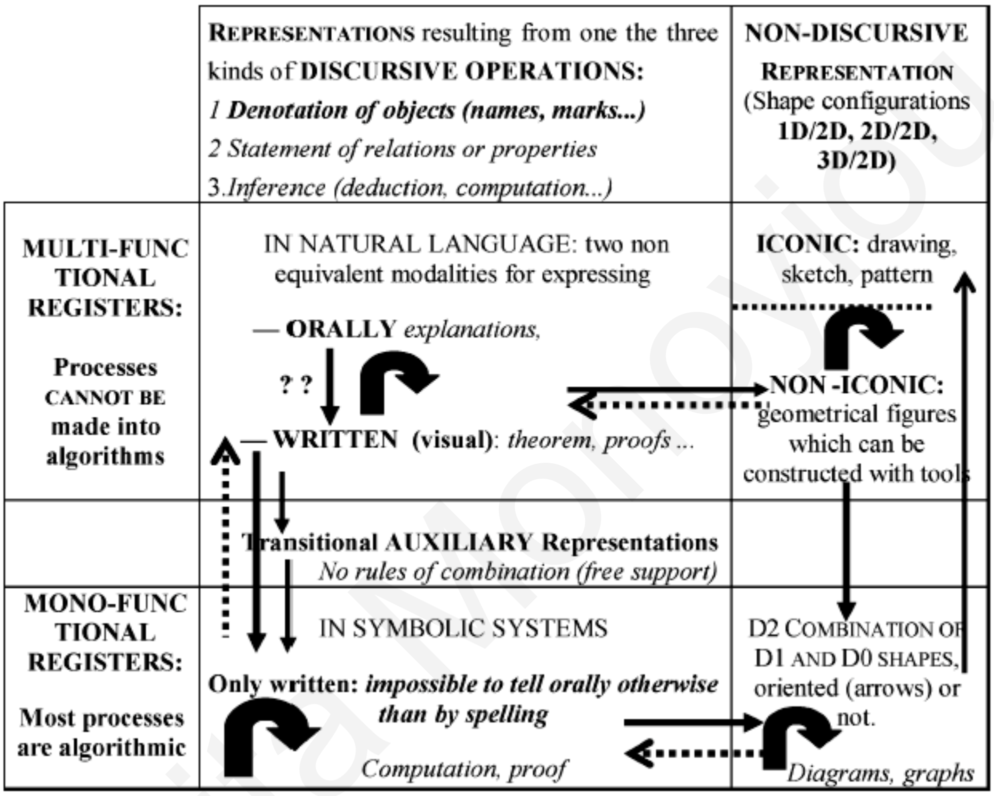


Figure 2.3. Classification of the registers that can be mobilized in mathematical processes (Duval, 2006, p. 110)

What matters for understanding the thinking processes involved in any mathematical activity is to focus on the level of semiotic representation systems and not on the particular representation produced. And the following two points are essential. Firstly, it is only at this level that the basic property of semiotic representation and its significance for mathematics can be grasped: the fact that they can be exchanged one for another, while keeping the same denotation. Secondly, a mark cannot function as a sign outside of the semiotic system in which its meaning takes on value in opposition to other signs within that system. That means, too, that there are rules for producing relevant semiotic representations. That can be easily checked for any numeric notation system or for Cartesian graphs (Duval, 2006).

Of course, some representations that do not depend on a semiotic system are used in mathematical activity. The best example is the matchstick use for representing small integers. They have neither rules of formation nor specific possibilities of transformation. These are used like a material for free manipulations (Duval, 2006).

In this research study various external representations for the concept of function were used. Furthermore, a combination of multifunctional (verbal expressions) and monofunctional (symbolic, diagrammatic and graphical representations) semiotic representations is been used taking into account that the one semiotic representation can be used in the place of the other.

Representations and mathematics learning and teaching

Through the study of the history of the development of mathematics we realize that the development of semiotic representations was an essential condition for the development of mathematical thought (Duval, 2006). Representations are important tools for the mathematical thought (Cheng, 1999). Greeno and Hall (1997) maintain that representations can be considered as useful tools for constructing meaning and for communicating information and understanding. They underline the importance of students' engaging in choosing representations and constructing representations in forms that help them see patterns and perform calculations, taking advantage of the fact that different forms provide different supports for inference and calculation. According to Duval (2006) no kind of mathematical processing can be performed without using a semiotic system of representation, because mathematical processing always involves substituting some semiotic representation for another. Furthermore what matters in the learning of mathematics is not representations but their transformation.

From an epistemological point of view there is a basic difference between mathematics and the other domains of scientific knowledge. Mathematical objects, in contrast to phenomena of astronomy, physics, chemistry, biology, etc., are never accessible by perception or by instruments (microscopes, telescopes, measurement apparatus) (Duval, 2006). The only way to have access to them and deal with them is using signs and semiotic representations (Dreyfus & Eisenberg, 1996; Duval, 2006). It is worth mentioning that in mathematics there is the possibility of using a variety of semiotic representations. Some of the semiotic representations are common to any kind of thinking such as natural language and some are specific to mathematics such as algebraic and formal notations (Duval, 2006).

A crucial problem of mathematics comprehension for learners is that in some cases the mathematical object is confused with the semiotic representation (Duval, 2006). According to the results of several researches (Dufour-Janvier, Bednarz, & Belanger, 1987) although the students can produce and use the mathematical representation on demand, they do not have the attitude of turning to these as tools to help them solve problems. These representations are rather perceived as mathematical objects in themselves. Students' difficulties concerning the understanding of the relation that exists between the representation and the entity that is represented is probably attributed to the need for dual representation (DeLoache, 2000; DeLoache, Uttal, & Pierroutsakos, 1998). The students must confront the representation as an entity with its own existence, characteristics and properties and at the same time as the representation of another entity (DeLoache, 2000), which in the particular case is the mathematical concept. However, DeLoache et al. (1998) maintained that a representation can be used successfully if three conditions are fulfilled. First, one must realize that a symbol-referent relation exists. Second, one must understand something about how the symbol is related to its referent. Finally, using an informational symbol requires computing specific relations between the symbol and its referent.

Similarly, Ainsworth (2006) claimed that the benefits of appropriate representations do not come for free. Learners are faced with complex learning tasks when they are first presented with a novel representation. They must come to understand how it encodes information and how it relates to the domain it represents. In the case of a graph, the format would be attributes such as lines, labels, and axes. They must also learn what the "operators" are for a given representation. For a graph, operators to be learnt include how to find the gradients of lines, maxima and minima, and intercepts. Very important is also the relation between the representation and the represented field (DeLoache et al., 1998). The students must realize the relation between the representation and the represented field (Ainsworth, 2006) since in the process of the interpretation of the representation the content plays an important role (Roth & Bowen, 2001). A related categorization is that proposed by Lesh et al. (1987a) between transparent or opaque representational systems. A transparent representation would have no more nor less meaning than the idea(s) or structure(s) they represent. An opaque representation would emphasize some aspects of the idea(s) or structure(s) and de-emphasize others (Lesh et al., 1987a).

Very important is the fact that the use of external representations must lead to the construction of internal actions. The internal actions, which must develop in a students' mind, require a kind of abstraction from the content of the representation. The more interesting and attractive is the content the more difficult is the abstraction is achieved (DeLoache et al., 1998; Seeger, 1998). According to DeLoache et al. (1998), in Japan where students excel in mathematics, a single, small set of manipulatives is used throughout the elementary school years. Because the objects are used repeatedly in various contexts, they presumably become less interesting as things in themselves. Moreover, children become accustomed to use the same manipulatives to represent different kinds of mathematical problems. For these reasons, they are not faced with the necessity of treating an object simultaneously as something interesting in its own right and a representation of something else. In contrast, American teachers use a variety of objects in a variety of contexts. This practice may have the unexpected consequence of focusing children's attention on the objects rather than on what the objects represent (DeLoache et al., 1998).

In many situations learners may be required to construct a representation rather than interpret a presented representation (Ainsworth, 2006). There is evidence that creating representations can lead to a better understanding of the situation. Grossen and Carnine (1990) found that children learned to solve logic problems more effectively if they drew their responses to problems rather than selected a pre-drawn diagram. Van Meter (2001) found that drawing was most helpful in learning from science texts when students were prompted with guidance questions whilst creating diagrams. The NCTM's Principles and Standards for School Mathematics (2000) document includes a new process standard that addresses representations and stresses the importance of the use and construction of multiple representations in mathematics learning. The students must be encouraged to represent the mathematical concepts in a way that the representations will have meaning for them even though these representations may not be the conventional one. At the same time, they must learn when to use the conventional forms of representation in a way that will facilitate the learning of mathematics and the communication concerning the mathematical concepts (NCTM, 2000).

Additionally the learners must encouraged to select a representation that they find most appropriate, and so they may have to consider such aspects of the situation as the representation and task characteristics as well as individual preferences (Ainsworth, 2006), something that entails meta-representational efficacy. There is evidence in certain situations that learners can select effective representations. Zacks and Tversky (1999) found that people were successful at choosing bar graphs to represent discrete comparisons between data points and line graphs to depict trends. Novick, Hurley and Francis (1999) found that students were able to choose which of hierarchical, matrix or network representations was most appropriate to represent the structure of a story problem, presumably based upon abstract schematic knowledge about when best to apply particular representations. Similarly, McKendree et al. (2002) stated that what is much less often taught is how to search for a representation for a given problem or to understand the abstract properties of the representation that makes it a useful one in a particular instance. Being able to think about why a representation may or may not be good in a particular context is a big part of being a critical thinker. If a student can realize that the problem they are working on is best represented in a particular way, it can help them identify the most important aspects of a situation and analyze what should be done next. If they know what the ‘accepted’ transformations of a problem type are, then they can begin to think at a higher level about why those are the ones that are used and why, perhaps, it might be interesting to try a different representation (McKendree et al., 2002).

DiSessa, Hammer, Sherin, and Kolpakowski (1991) studied students’ competencies in participating in various representational practices. They showed that students, as young as elementary school students, have many competencies for creating, critiquing and inventing new representations. Developing these competencies, they conjecture, is important in enhancing students’ representational innovation, as well as deepening their understanding of any kind of representation (diSessa, 2004). Furthermore, diSessa (2004) argued that students often have a deep meta-representational competence which includes their abilities to judge the value of representations along such dimensions as epistemic fidelity, compactness, parsimony, systematicity and conventionality. McKendree et al. (2002) maintained that what is often lacking is the emphasis on the skills of selection of, construction with, and reasoning about commonly occurring forms of representation. A repertoire of such representational skills can give learners new ways to manipulate and transform information for themselves. Also, it becomes possible to infer properties of the information from the structures of the form of representation in which the information is being expressed, making transfer of learning more likely. Representational skill thus provides an essential dimension to critical thinking, a cognitive flexibility that can become a powerful tool for transforming raw information into fruitful and personally meaningful knowledge (McKendree et al., 2002).

Representations enhance the understanding of mathematical concepts and problem solving if the students can understand their function and they can use them with fluency. According to Dufour-Janvier et al. (1987) the premature use of a representation may cause difficulties for the students. The use of representations that are as abstract as the concept that is being taught may have as a result the students to handle rules and symbols without a meaning. As a consequence, their attention is focused on the understanding of the particular representations and not to the mathematical knowledge and skills that are the goal of the teaching. In other words, the teaching of the representation became an end in itself instead of being the mediator that will help students to understand and organize their mathematical concepts.

In addition Dufour-Janvier et al. (1987) claimed that the application of a representation in an appropriate context can lead children to develop misconceptions that will hinder them in later learning. An example is the use of number line during the learning of positive integers. The students develop the notion of the number line as a series of “stepping stones”. Each step is conceived as a rock, and between two successive rocks there is a hole. This is very far from the concept of the density of the real numbers as illustrated by the number line.

Finally, it is very important that the external representations being used in the teaching should be related with the previous knowledge end experiences of the students. The imposition of external representations that are too distant from the child results in having the child react negatively or causes him difficulties. If one wants to use an external representation in teaching, he needs to take into consideration that it should be as close as possible to children’s internal representations (Dufour-Janvier et al, 1987; DeLoache et al., 1998; von Glaserfeld, 1987). However, one frequently imposes external representations without realizing the gap that can exist between those used and the one envisaged by the child of the problem situation based on his knowledge and experiences. As a result obstacles are being put in the process of the understanding of the concept that are the result of well structured nets of representations (Hiebert & Carpenter, 1992).

The education pre-service teachers receive plays an important role in the choices they will make concerning the use of the appropriate representations in their teaching. Ward, Anhalt and Vinson (2003) analyzed the lessons plans made by candidate teachers and studied the representations the teachers chose to use for teaching a mathematic topic in the most effective way as possible that is so that students' understanding is maximized. In this study the concept of pedagogical content knowledge was evolved that is defined as the transformation of a teacher's own content knowledge into pedagogical representations that connect with the prior knowledge and disposition of the learner. Serving as a lens during the analyses and coding of these lessons plans were the five representations defined by Lesh et al. (1987b). It is worth mentioning that the language representations were divided into two subcategories: (a) the language that is used to talk about mathematical procedures and defining mathematical terms and (b) the language referring to mathematical discourse. Thought the semester, the candidate teachers were taught in a constructivist manner using manipulatives, technology, problem solving, hand-on exploration, writing, discourse, making real world connections and they had the opportunity to observe teaching in an elementary classroom. The results of the research showed improvement of the teachers' pedagogical content knowledge that was evident through their choices of representations in the lesson plans. Specifically, the teachers gradually used with increasing facility and flexibility a variety of representations. The results also showed that there is not an ideal number of representations for a successful lesson plan and it is not important which representation will be used but how it will be used and if it will be used by the teacher or the student (Ward et al., 2003).

Ball (1993) stated that teachers already have orientations to their role, to the nature and substance of mathematics, to what helps students learn. They already have patterns of reasoning and concerns that drive the kinds of decisions and compromises they make as they teach mathematics. These patterns are often quite different from what might be entailed in trying to interweave consideration of students' thinking with close analysis of the content to create productive representational contexts that can help students to develop mathematical understandings. For instance, a focus on making mathematics fun will justify some representations that are not grounded in meaning, that offer little opportunity for exploration or connections. Similarly, an orientation to and understanding of mathematics as rules and algorithms does not support a search for or use of conceptually grounded representational contexts (Ball, 1993).

In an effort to investigate the pedagogical content knowledge for the understanding, the construction and the use of appropriate representations, Ball (1993) presents the construction of representational context for the learning of multiplication and division by teachers and third graders. Students enter the representational context that the teacher has set up and, in dealing with a specific problem, they generate alternative ways to represent or check their understandings. Together, students and teacher develop language and conventions that enable them to connect and use particular representations in situations. They also develop ways of reaching beyond and across specific situations to abstract and generalize emergent understandings. The representations are tools to be wielded in mathematical investigations, in framing and solving problems, in making and proving general claims. The tools themselves are sharpened and developed through these processes. Students also sometimes invent or introduce representations independently. In helping students learn to understand a mathematical concept, teachers must take justifiable decisions about representations, their construction, use, and adaptation. The choice of representations must be the product of a thinking process that demands deep understanding of the particular concept as well as the mathematical reasoning of the students in the particular age (Ball, 1993).

In this research study an effort is made to investigate the way pre-service teachers behave towards different modes of representations that are being used in the teaching or are presented at the school textbooks of the middle and high school and facilitate the conceptual understanding of function.

Multiple representations and mathematics learning

As was mentioned in a previous part mathematical activity intrinsically consists in the transformation of representations. There are two types of transformations of semiotic representations that are radically different: treatments and conversions. Treatments are transformations of representations that happen within the same register: for example, carrying out a calculation while remaining strictly in the same notation system for representing the numbers, solving an equation or system of equations, completing a figure using perceptual criteria of connectivity or symmetry, etc. That gives importance to the intrinsic role of semiotic systems in mathematical processes. The treatments, which can be carried out, depend mainly on the possibilities of semiotic transformation, which are specific to the register used (Duval, 2006).

Conversions are transformations of representation that consist of changing a register without changing the objects being denoted: for example, passing from the algebraic notation for an equation to its graphic representation, passing from the natural language statement of a relationship to its notation using letters, etc. Conversion is a representation transformation, which is more complex than treatment because any change of register first requires recognition of the same represented object between two representations whose contents have very often nothing in common. It is like a gap that depends on the starting register and the target register. Too often, conversion is classified as translation or encoding (Duval, 2006).

Duval (2006) gives an interesting example of a register for which a rule of conversion can be explicitly given. To construct a graph it suffices to have only the following rule: to every ordered pair of numbers one can associate a point on a coordinate plane with given increments on the two axes. And the construction of graphs corresponding to linear functions appears to give students no difficulties whatever. But one has only to reverse the direction of the change of register to see this rule cease to be operational and sufficient (Figure 2.4.).

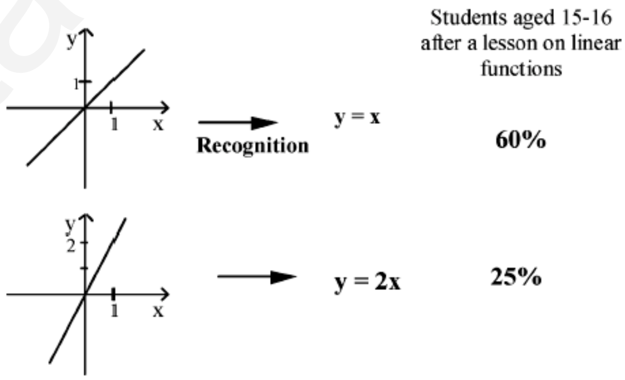


Figure 2.4. A recognition task (Duval, 2006, p. 113)

The task proposed by Duval (2006) was a task of simple recognition, not one of construction or of reading coordinates of points: choose among many possible expressions (for example, among $y = x$, $y = -x$, $y = x + 1$) the one which corresponds to the graph (Duval, 1988). In standard teaching, the tasks offered are never recognition, but simply reading tasks that require only a process of placing points guided by local understanding and not a process of global interpretation guided by understanding of qualitative visual variables. Converting a semiotic representation into another one cannot be considered either as an encoding or a treatment. In these two examples, conversion is explicitly required and it appears that it can be confined to transitory situations for solving some particular problem. But most often it is implicitly required whenever two, or even three, registers must be used together in an interactive way. Through the various kinds of conversions more than through treatments we touch on the cognitive complexity of comprehension in learning mathematics and on the specific thinking processes required by mathematical activity (Duval, 2006). In addition, Lesh et al. (1987b) indicate that treatments and conversions are interrelated.

A conversion (translation) involves two modes of representation. From the modes equation and graph (used also in this study) we have the translations “graph to equation” and “equation to graph”. The translation processes are developed effectively if the students are asked to make translations from the source to the target and vice versa in a symmetrical way (Janvier, 1987). In addition Duval (2006) maintains that it is only by investigating representation variations in the source register and representation variations in a target register that students can at the same time realize what is mathematically relevant in a representation, achieve its conversion in another register and dissociate the represented object from the content of these representations.

Students’ success in conversion tasks or their systematic mistakes depend on the cognitive distance between the source representation content and the target representation content. In some cases, it is like a one-to-one mapping and the source representation is transparent to the target representation. In these cases, conversion seems nothing more than a simple coding. But in other cases between a source representation and its converted representation in a target register, there is either congruence or non-congruence. The non-congruent conversions are for many students an impassable barrier in their mathematics comprehension and therefore for their learning. Facing non-congruent representation conversion, learners are trapped in a conflict between mathematical knowledge requirement and cognitive impossibility. The conversion of representation requires the cognitive dissociation of the represented object and the content of the particular semiotic representation through which it has been first introduced and used in teaching. But there is a cognitive impossibility of dissociating any semiotic representation content and its first represented object when there is no other possible access to mathematical object than semiotic. That conflict leads to the consideration of two representations of the same object as being two mathematical objects. The consequence is then the inability to change register and to use knowledge outside of narrow learning contexts. The registers of the representations remain compartmentalized, and only fragmentary and monoregistral comprehension is possible (Duval, 2006). Vinner and Dreyfus (1989) also use the term compartmentalization and define it as the phenomenon that occurs when a person has two different, potentially conflicting schemes in his or her cognitive structure. Certain situations stimulate one scheme, and other situations stimulate the other. Inconsistent behavior is not the only indication of compartmentalization. Sometimes, a given situation does not stimulate the scheme that is the most relevant to the situation.

The need for a variety of semiotic representations in the teaching and learning of mathematics is usually explained through reference to the cost of processing, the limited representation affordances for each domain of symbolism and the ability to transfer knowledge from one representation to another (Duval, 1987; Duval, 1993; Gagatsis, 1997; D' Amore, 1998). A representation cannot describe fully a mathematical construct and each representation has different advantages, so using various representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2002). Ainsworth, Bibby and Wood (1997) suggested that the use of multiple representations can help students develop different ideas and processes, constrain meanings and promote deeper understanding. By combining representations students are no longer limited by the strengths and weaknesses of one particular representation. Kaput (1992) claimed that the use of more than one representation or notation system helps students to obtain a better picture of a mathematical concept. Seufert (2003) maintained that multiple representations can serve many functions for learning. First, multiple representations may complement each other with regard to their content. A second function is that multiple representations can complement each other with regard to their representational and computational efficiency, as different forms of representation may be differently useful for different purposes. Third, one representation may constrain the interpretation of another representation. The combination of representations that both complement and constrain each other enables learners to deal with the material from different perspectives and with different strategies, and therefore can have synergetic effects on the construction of coherent knowledge structures (Seufert, 2003).

Concerning the use of multiple representations referring to the same concept the student must realize the common properties of different representations, their common mathematical structure and manage to achieve the goal of teaching which is to construct the concept successfully (Gagatsis et al., 2001). In order to accomplish understanding through multiple representations learners must create referential connections between corresponding elements and corresponding structures in different representations (Seufert, 2003). The coordination of different semiotic systems is not a consequence but a requirement for conceptual understanding in mathematics (Duval, 2006).

Pape and Tchoshanov (2001) in order to establish the importance of multiple representations in the understanding of mathematical concepts, took into consideration results from the cognitive theory and the investigation of human mind. They suggested that the cognitive capacity of the human brain is aligned with multiple representational patterns. Therefore, students' thinking skills are developed if they study in a multiple representational environment.

For the design of multi-representational systems in educational software a set of dimensions is taken into consideration that are: (a) the number of representations employed; (b) the way that information is distributed over the representations; (c) the form of the representational system; (d) the sequence of representations; and (e) support for translation between representations. Specifically by definition, multi-representational systems employ at least two representations. Multi-representational systems can allow flexibility in the way that information is distributed between the representations. A typical multi-media system can display pictures, text, animations, sound, equations, and graphs, often simultaneously. If not all representations are drawn upon simultaneously, a number of further issues arise. The first issue is the sequence in which the representations should be presented or constructed. Even if a sequence has been predetermined, the learner or the system still needs to decide at what point to add a new representation or switch between the representations. Computerized environments have made possible a wide variety of ways to indicate to learners the relation between representations (Ainswoth, 2006). These dimensions can be taken into consideration in the process of designing tasks in mathematics that involve the use of different representations.

According to Janvier (1987) most mathematics textbooks today make use of a variety of representations more extensively than ever before, in order to promote understanding. The use of different modes of representations and connections between them represents an initial point in mathematics education at which pupils use one symbolic system to expand and understand another (Leinhardt et al., 1990). Thus, the ability to identify and represent the same concept in different representations is considered as a presupposition for the understanding of the particular concept (Duval, 2002; Even, 1998). Besides recognizing the same concept in multiple systems of representations, the ability to manipulate flexibly the concept within these representations as well as the ability to "translate" the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh et al., 1987b) and allow students to see rich relationships (Even, 1998).

Moving a step forward Hitt (1998) identified the following levels in the construction of the concept of function:

Level 1: Imprecise ideas about the concept (incoherent mixture of different representations of the concept).

Level 2: Identification of different representations of the concept. Identification of systems of representation.

Level 3: Translation with preservation of meaning from one system of representation to another.

Level 4: Coherent articulation between two systems of representation.

Level 5: Coherent articulation of different systems of representation in the solution of a problem.

According to Hitts' levels the ability to translate a concept from one system of representation to another constitutes a presupposition for the combination of different representational systems that leads to successful problem solving.

Recently, Gagatsis et al. (2010) and Gagatsis, Deliyianni, Elia, Monoyiou, and Panaoura (2009) suggested the term multiple-representational flexibility. This term is also used in this research study. According to Demetriou (2004) flexibility refers to the quantity of variations that can be introduced by a person in the concepts and mental operations he or she already possesses. Excelling and developing in understanding, learning, reasoning and problem solving are to a considerable extend a function of increase in flexibility (Demetriou, 2004). Krems (1995) defines cognitive flexibility as a person's ability to adjust his or her problem solving as task demands are modified. Similarly, Chevalier and Blaye (2008) consider cognitive flexibility as the ability to switch mental sets in response to changing relevant cues in the environment. Shafrir (1999) proposed the term "representational competence". Representational competence refers to the individual's awareness and understanding that an instance can be represented in various forms and still retain its essential meaning. This definition consists of two hierarchical levels: In Level 1 the representational competence is conceptualized as the ability to convey and receive equivalent meaning through multiple representations within and/or across different sign systems. In Level 2 representational competence is defined as the ability to re-present equivalent meaning by incorporating higher order relation within and/or across different sign systems.

Adjusting the previous definitions Gagatsis et al. (2010) consider multiple-representation flexibility as the ability to switch mental sets in response to within- and between-representation alterations (recognition, treatment, conversion) of the same mathematical object. In other words, Gagatsis et al. (2010) assume that multiple-representation flexibility refers to switching between different systems of representations of a concept (inter-representation flexibility), as well as to recognizing and manipulating the concept within multiple representations (intra-representation flexibility).

From a statistical perspective and based on the idea that flexibility refers to de-compartmentalization, flexibility is the phenomenon of the association of distinct variables or clusters that correspond to different mathematical conceptualizations or different problem-solving strategies or different representations of, or different cognitive processes related to, the same concept which have strong statistical relation (correlation, implication, similarity) between them. In other words, experimental-operational flexibility is a phenomenon contrary to the experimental-operational compartmentalization as it is described in Gagatsis et al. (2010).

Furthermore, Deliyianni, Elia, Panaoura and Gagatsis (2009) tested whether flexibility in multiple representations and problem-solving ability has an effect on decimal number addition understanding and they investigate its factorial structure within the framework of Confirmatory Factor Analysis, across students of primary and secondary school. The findings provided a strong case for the important role of multiple-representational flexibility and problem-solving ability in primary and secondary school students' decimal number addition understanding. According to the results the ability to recognize decimal number addition, to symbolically manipulate decimal number addition and to convert flexibly from one decimal number addition representation to another differentially affected multiple-representational flexibility. In fact, the findings revealed that multiple-representational flexibility constitutes a multi-faceted construct in which the representation transformations interact with the modes of representation.

In this research study the multiple-representational flexibility will be investigated concerning the concept of function. Based on the definition given by Gagatsis et al. (2010) we also consider multiple-representational flexibility as the ability to switch mental sets in response to within- and between-representation alterations (recognition, treatment, conversion) of the same mathematical object but also we consider that representational flexibility also involves the ability to give a right definition and examples of a concept using various representations since these aspects are highly connected with students' ability to recognize the concept.

Representations and functions

The concept of function is central in mathematics and its applications. The function concept is a complex, multifaceted idea whose power and richness permeate almost all areas of mathematics (Christou, Pitta-Pantazi, Souyoul, & Zachariades, 2005). It emerges from the general inclination of humans to connect two quantities, which is as ancient as mathematics. The didactical metaphor of this concept seems difficult, since it involves three different aspects: the epistemological dimension as expressed in the historical texts; the mathematics teachers' views and beliefs about function; and the didactical dimension which concerns students' knowledge and the restrictions implied by the educational system (Evangelidou et al., 2004).

On this basis, it seems natural for students of secondary or even tertiary education, in any country, to have difficulties in conceptualizing the notion of function. The complexity of the didactical metaphor and the understanding of the concept of function have been a main concern of mathematics educators and a major focus of attention for the mathematics education research community (Dubinsky & Harel, 1992; Sierpinska, 1992). An additional factor that influences the learning of functions is the diversity of representations related to this concept (Hitt, 1998). Yerushalmy (1997) concentrated on students' abilities to deal with these different kinds of representations and on translations among them. An important educational objective in mathematics, as was mentioned, is for pupils to identify and use efficiently various forms of representation of the same mathematical concept and move flexibly from one system of representation of the concept to another. Similarly, Dreyfus (1991) posited that the learning process proceeds through four stages, namely, using one representation, using more than one representation in parallel, making links between parallel representations and integrating representations and flexibly moving between them. The influence of different representations on the understanding and interpretation of functions has been examined by a substantial number of research studies (Hitt, 1998; Markovits et al., 1986).

Several researchers (Evangelidou et al., 2004; Gagatsis & Shiakalli 2004; Gagatsis et al., 2002; Gagatsis, Elia, &Andreou, 2003; Mousoulides & Gagatsis, 2004; Sfard 1992; Sierpinska, 1992) indicated the significant role of different representations of function and the conversion from one representation to another on the understanding of the concept. Thus, the standard representational forms of the concept of function are not enough for students to construct the whole meaning and grasp the whole range of its applications. Mathematics instructors, at the secondary level, traditionally have focused their instruction on the use of algebraic representations of functions. Eisenberg and Dreyfus (1991) pointed out that the way knowledge is constructed in schools favours mostly the analytic elaboration of the notion which deteriorates the approach of function from the graphical point of view. Kaldrimidou and Ikonomou (1998) showed that teachers and students pay much more attention to algebraic symbols and problems than to pictures and graphs. A reason for this is that in many cases the iconic (visual) representations can cause cognitive difficulties, because perceptual analysis and synthesis of mathematical information presented implicitly in a diagram often make greater demands on a student that any other aspect of a problem (Aspinwall, Shaw, & Presmeg, 1997).

In addition, most of the aforementioned studies have shown that students tend to have difficulties in transferring information gained in one context to another (e.g., Gagatsis & Shiakalli, 2004). Sfard (1992) showed that students are unable to bridge the algebraic and graphical representations of functions, while Markovits et al. (1986) observed that translation from graphical to algebraic form was more difficult than vice-versa. Norman (1992) found that even mathematicians studying for a master's degree tend to use just one kind of representation, the graphical one. Christou, Zachariades and Papageorgiou (2002) identified hierarchical levels among the graphical and symbolic representations of mathematical functions and verified an association with students' ability to identify various representations of the mathematical functions. Sierpinska (1992) maintained that students have difficulties in making the connections between different representations of functions, in interpreting graphs and manipulating symbols related to functions. A possible reason for these kinds of behavior is that most instructional practices limit the representations of functions to the translation of the algebraic form of a function to its graphic form.

Lack of competence in coordinating multiple representations of the same concept can be seen as an indication for the existence of compartmentalization, which may result in inconsistencies and delay in mathematics learning at school. The particular phenomenon reveals a cognitive difficulty that arises from the need to accomplish flexible and competent translation back and forth between different kinds of mathematical representations (Duval, 2002).

Elia, Gagatsis and Gras (2005) investigated students' understanding of function based on their performance in mathematical activities that integrated both types of transformation of representations proposed by Duval (2002), i.e. treatment and conversion. It was revealed that success in one type of conversion of an algebraic relation did not necessarily imply success in another mode of conversion of the same relation. Lack of implications or connections among different types of conversion (i.e., with different starting representations or even with different target representations) of the same mathematical content indicates the difficulty in handling two or more representations in mathematical tasks. This incompetence provided a strong case for the existence of the phenomenon of compartmentalization among different registers of representation in students' thinking, even in tasks involving the same relations or functions. The differences among students' scores in the various conversions from one representation to another, referring to the same algebraic relation or function, provides support to the different cognitive demands and distinctive characteristics of different modes of representation. This inconsistent behavior can be also seen as an indication of students' conception that different representations of the same concept are completely distinct and autonomous mathematical objects and not just different ways of expressing the meaning of a particular notion. Inconsistencies were also observed in students' responses with reference to the different conceptual features of the mathematical relations involved in the conversions, i.e. functions or not.

Gagatsis, Elia and Andreou (2003) found that 14-year-old students were not in a position to change systems of representation of the same mathematical content of functions in a coherent way, indicating that systems of representations remained compartmentalized and mathematical thinking was fragmentary. Similarly, Even (1998) indicated that students had difficulties when they needed to link different representations of functions flexibly.

Elia and Spyrou (2006) examined students' performance in recognition and conversion tasks involving different modes of representation of function. Higher success rates were observed in the tasks which involved algebraic representations, relative to the tasks involving verbal and graphic representations (either Cartesian graphs or arrow diagrams). This finding can be attributed to the fact that mathematics instruction in schools focuses on the use of algebraic representations of functions, thus hindering the approach of function in other representational modes (e.g., Kaldrimidou and Ikonomou, 1998). In addition, students responded in tasks involving the same type of representation in a consistent and coherent manner. Nevertheless, they approached in a distinct way the different forms of representation of functions, providing support for the existence of the compartmentalization phenomenon (Gagatsis et al., 2003). Students probably considered the different systems of representation as different and autonomous mathematical objects and not as distinct means of representing the same concept (Duval, 1993). This conception was apparent also from students' failure in a conversion task of representations that was not transparent. Since a concept is not acquired when some components of mathematical thought are compartmentalized, teaching needs to accomplish the breach of compartmentalization, i.e., de-compartmentalization and coordination among different types of representations. One way to achieve this is by giving students the opportunity to engage in conversions of representation that can be congruent or not in different directions (Duval, 2002).

Additionally, Elia and Gagatsis (2008) investigated the structure of students' abilities to carry out conversions from one mode of representation to another in the context of functions. Data were obtained from 587 students in grades 9 and 11. Using Confirmatory Factor Analysis, a model that provides information about the significant role of the initial representation of a conversion in students' processes was developed and validated. Using the hierarchical clustering of variables and the implicative statistical analysis, evidence was provided for the phenomenon of compartmentalization among representations in students' thinking. All analyses indicated that the structure of the abilities and the relative inherent nature of difficulties of the conversions of functions and of other types of algebraic relations remain stable in the two age groups. In general, the outcomes of the three methods of data analysis were found to coincide and to be open to complementary use in capturing the ways in which students use different representations of functions.

In this research study emphasis will be given to the recognition of functions given in various representations and the conversions from an algebraic to a graphical representation and vice versa. The relation of these aspects with the conceptual understanding of function and with problem solving will be examined.

Concept image

Concept image and concept definition

Compared with other fields of human endeavour, mathematics is usually regarded as a subject of great precision in which concepts can be defined accurately to provide a firm foundation for the mathematical theory. The psychological realities are somewhat different. Many concepts we meet in mathematics have been encountered in some form or other before they are formally defined and a complex cognitive structure exists in the mind of every individual, yielding a variety of personal mental images when a concept is evoked (Tall & Vinner, 1981).

A useful way to characterize a person's thinking about functions is in terms of the notions of concept definition and concept image (Lloyd & Wilson, 1998). These constructs point to a distinction between the formal definition an individual holds for a given concept and the way that he or she thinks about the concept.

Concept image and concept definitions are two terms that have been discussed extensively in the literature concerning students' conceptions of function (Vinner & Dreyfus, 1989; Tall & Vinner, 1981; Vinner & Hershkowitz, 1980). It is now an old construct in mathematics education but it has weathered the years well and continues to be cited in the literature (e.g. Przenioslo, 2004; Giraldo, 2006; Nardi, 2006) since it is an important construct. It appears that when it was first introduced there was a widespread belief that if mathematics teachers/lecturers got their definitions right, then the concepts behind the definitions would, by careful tutor explanation and student diligence, become transparent to the student. If this interpretation is correct, then the authors of the construct contributed to our current understanding that while a tutor's definition of a concept may evoke correct associations for some students, many students will generate, amongst some intended associations, unintended concept images. Whatever the interpretation, students' concept images became an object of study (Bingolbali & Monaghan, 2007). Although formal definitions of mathematical concepts are introduced to high school or college students, students do not essentially use them when asked to identify or construct a mathematical object either concerning this concept or not. They are frequently based on a concept image. Consequently, students' responses to tasks or questions related to the concept depend on these conceptions and deviate from teachers' expectations.

Concept definition is straightforward to describe, it is the form of words/symbols used by the tutor/course notes/textbook to define a mathematical concept (Bingolbali & Monaghan, 2007). It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole. It may also be a personal reconstruction by the student of a definition. It is then the form of words that the student uses for his own explanation of his (evoked) concept image. Whether the concept definition is given to him or constructed by himself, he may vary it from time to time. In this way a personal concept definition can differ from a formal concept definition, the latter being a concept definition which is accepted by the mathematical community at large (Tall & Vinner, 1981).

But what is an appropriate formal definition of function for the classroom? Because of developments within mathematics, over the years the generally accepted notion of function has changed from a means to describe dependence relationships between quantities to the highly abstract mathematical construct that now comprises the modern conception of function (Cooney & Wilson, 1993; Dreyfus, 1990). Since the 1930s the official definition of function (known as the Dirichlet-Bourbaki definition) has emphasized univalent correspondence over covariation, largely because of the many problems this definition can be employed to solve and the numerous useful relationships it can be used to describe, relationships not possible to describe under the more restrictive covariation definition (Cooney & Wilson, 1993; Thompson, 1994; Vinner & Dreyfus, 1989). Analogous development has occurred in the treatment of the function concept in school textbooks in this century (Cooney & Wilson, 1993), with the Dirichlet-Bourbaki definition currently dominating most high school curricula (Dreyfus, 1990). The Dirichlet-Bourbaki concept is that of a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second test (the codomain). The Dirichlet-Bourbaki approach defined as functions many correspondences that were not recognized as functions by previous generations of mathematicians (Malik, 1981). Among these correspondences are discontinuous functions, functions defined on split domains (i.e., by different rules on different subdomains), functions with a finite number of exceptional points, and functions defined by means of a graph (Vinner & Dreyfus, 1989).

The notion of concept image in contrast with the concept definition is less straightforward to describe. Tall and Vinner (1981) describe it as the total cognitive structure associated with a concept in an individual's mind. It includes mental pictures, associated properties and processes as well as strings of words and symbols. It is a dynamic entity that develops, differentially over students, through a multitude of experiences. Some of these will, from a mathematical viewpoint, be incorrect, e.g. squaring a number could be defined as "multiplying a number by itself" and an associated property of squaring, grounded in students' experiences with natural numbers, might be "squaring makes the number bigger". It is unlikely that all knowledge that forms a concept image (mental pictures, associated properties and processes) will be simultaneously brought to bear by students in their actions in mathematical tasks. Tall and Vinner (1981) call the knowledge that is brought to bear by a particular student, at a particular time, and on a particular task, the evoked concept image. Of course, as for most important constructs in mathematics education, it is extremely difficult to gain insight into students' evoked concept images. All we can do is to find what we regard as valid ways to observe students and interpret the data. For instance the concept of subtraction is usually first met as a process involving positive whole numbers. At this stage children may observe that subtraction of a number always reduces the answer. For such a child this observation is part of his concept image and may cause problems later on when the subtraction of negative numbers will be encountered. For this reason all mental attributes associated with a concept, whether they be conscious or unconscious, should be included in the concept image; they may contain the seeds of future conflict (Tall & Vinner, 1981).

For each individual a concept definition generates its own concept image which might be called the "concept definition image". This is, of course, part of the concept image. In some individuals it may be empty, or virtually non-existent. In others it may, or may not, be coherently related to other parts of the concept image. For instance the concept definition of a mathematical function might be taken to be "a relation between two sets A and B in which each element of A is related to precisely one element in B." But individuals who have studied functions may or may not remember the concept definition and the concept image may include many other aspects, such as the idea that a function is given by a rule or a formula, or perhaps that several different formulae may be used on different parts of the domain A. There may be other notions, for instance the function may be thought of as an action which maps a in A to f(a) in B, or as a graph, or a table of values. All or none of these aspects may be in an individual's concept image. But a teacher may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure. Initially the student in this position can operate quite happily with his restricted notion adequate in its restricted context. He may even have been taught to respond with the correct formal definition whilst having an inappropriate concept image. Later, when he meets functions defined in a broader context he may be unable to cope (Tall & Vinner, 1981).

Furthermore, Tall and Vinner (1981) claimed that in many cases we may learn to recognize concepts from experience, and then appropriately use them in different contexts without knowing a precise definition for them at all. In other words, the absence of a concept definition does not impede the development of a rich and flexible concept. On the other hand, a concept definition may be meaningfully learnt or learnt in a rote manner, and it may or may not be coherent with the formal definition (that is, a concept definition which is accepted by the mathematical community). Moreover, the concept definition may or may not be consistently related with the whole concept image. An individual can successfully memorize a concept definition and know it by heart without having a rich concept image associated. In this case however, it is likely to be meaningless. Therefore, a broad concept image does not necessarily include an accurate concept definition (or any definition at all) and, conversely, the ability to state a concept definition is not necessarily associated with a rich concept image. This theory suggests that effective teaching approaches must not only focus on the formal mathematical structure of the concepts, but also aim to enrich students' whole set of ideas related to them. Many authors have pointed out that formal definitions may embody obstacles to students in early stages of learning (e.g. Sierpinska, 1992). One first obstacle arises from the language itself: definitions are built upon words that have meanings in current language, but, in order to understand definitions, one must abstract from those meanings. Another – and perhaps stronger – obstacle is that most definitions are thoroughly unfamiliar to students. Hence, teaching approaches based on the logical order of concepts – axioms, definitions, propositions and theorems – are actually anti-pedagogical inversions (Giraldo, 1996).

Thus, Vinner (1992) notes that the teacher plays an important role in the formation of students' concept images, though he does not explore this interaction/development empirically. To facilitate the construction of rich concept images that are consistent with and highlight for students key features of definitions, teachers must use, and encourage students to use, images and definitions dialectically (Vinner, 1992). Furthermore, a teacher's ability to efficiently make sense of students' concept images, which may seem strange and unfamiliar, depends on his or her own concept image (Even, 1990). For teachers to develop informed intuition to guide their instruction and students' learning, their images should explicitly illuminate the essential features of the formal concept definition.

Researchers have pointed to the position that many students and some prospective teachers do not hold a modern conception of functions (Dreyfus & Eisenberg, 1983, 1987; Even, 1993; Ferrini-Mundy & Graham, 1991; Markovits et al., 1983, 1986; Marnyanskii, 1975; Vinner, 1983; Vinner & Dreyfus, 1989). In a study conducted by Vinner and Dreyfus (1989), students were asked to define the term "function" and the responses generated were categorized into six classifications:

1. The Formal Dirichlet-Bourbaki definition;
2. A dependence relation between two variables (y depends on x);
3. A rule which requires a certain amount of regularity;
4. An operation or process;
5. A formula, algebraic expression, or equation; or
6. A representation typically in a meaningless graphical or symbolic form.

Students generally appear to hold to the requirement that the function is reasonable and describable by a formula (Graham & Ferrini-Mundy, 1990). One possible reason proffered for this has been that the general experience students and prospective teachers have with functions is almost exclusively built around functions whose rule of correspondence is given by a formula (Even, 1993; Vinner & Dreyfus, 1989). The result, however, has been that students tend to exclude many of the functions that are acceptable under the modern definition but were not acceptable under historical definitions such as: Two variables may be so related that a change in the value of one produces a change in the value of the other. In this case the second variable is said to be a function of the first; or Any mathematical expression containing a variable x, that has a definite value when a number is substituted for x, is a function of x (Hight, 1968). Leinhardt et al. (1990) and Vinner and Dreyfus (1989) pointed out that many of the errors produced during student classification of relations as functions or not functions do not occur as a result of the lack of acceptance of the Dirichlet-Bourbaki definition. They found students accept the Dirichlet-Bourbaki definition of the function but when involved in classification tasks, students rely upon personal experience that is closely connected to historical definitions of function. Therefore, these errors may not be a consequence of a misconception but rather a "missed" concept. Furthermore, these authors also argued that students' common images of the function concept have direct implications for instruction since they can be used as the starting point of any future teaching of the concept of function to similar populations. Furthermore, they raised doubts whether a formal definition of function should be taught in courses where it is not intensively needed. Rather, they claimed that formal definitions should be only a conclusion of the various examples introduced to the students.

The evolution of the function concept is often portrayed as a move from an operational notion as a process to a structural notion as an object (Sfard, 1991). A body of recent research, much of it applying the process-object distinction, supports the idea that the historical evolution of the function concept may provide a pedagogical sequence that better supports the students' development of deep conceptual understandings of functions (Eisenberg, 1991; Markovits et al., 1986; Sfard, 1991; Thompson, 1994; Vinner & Dreyfus, 1989). Recent calls for reform have begun to reflect a growing consensus that a covariation approach furnishes an indispensable first step in the direction of a deep comprehension of the function concept in general (NCTM, 1989). In light of the historical tension between covariation and correspondence in the school curriculum and in students' understandings of functions, it is advantageous for teachers to recognize and appreciate features of both definitions.

A vast number of studies have used different approaches to explore the concept of function in mathematics teaching and learning. Sierpinska (1992) explored pupils' conceptions of functions, how these are formulated in the context of teaching, and their relationship with the historical development of function. Sfard (1992) introduced the structural and operational understanding of the notion of function, while Dubinsky and Harel (1992) suggested four different stages in the understanding of functions (prefunction, action, process and object). Markovits et al. (1986) identified that students encountered difficulties in various tasks involving four particular types of function: the constant function, a piecewise function, a function represented by a discrete set of points and functions defined by constraints.

Elia and Spyrou (2006), in their study, revealed some of the ideas that university students have about function. Such an idea is the identification of “function” by a large percentage of students with the narrow concept of one-to-one function. This finding is in accord with the results of previous studies indicating that one-valued ness is a dominating criterion that students use for deciding whether a given correspondence is a function or not (Vinner & Dreyfus, 1989). This idea is also associated with the process of enumeration, which involves one-to-one correspondence as a matter of routine for the students. Another idea was that function is an analytic relation between two variables (as it worked historically, initially with Bernoulli’s definition, and more clearly with Euler’s). A number of students have even stated this explicitly in their justifications when attempting to identify functions among other algebraic relations. Moreover, students’ dominating idea that a graph of a function must be connected or “continuous” caused difficulties in recognition and conversion tasks involving disconnectedness of a function’s graph.

A study by Blanton and Kaput (2004) on early algebraic thinking examined how elementary school students developed and expressed functions. Findings suggested that with age development different representational and linguistic tools (tables, graphs, pictures, words and symbols) became a growing part of students’ repertoire of doing mathematics, that is of making sense of and expressing functional relationships.

In this study pre-service teachers’ concept image of function not only based on their constructed concept definition but also based on the examples given for the concept is investigated (Even, 1990; Vinner & Dreyfus, 1989).

Examples of a concept

In addition to the concept definition - concept image scheme discussed and explained in the previous section, Moore (1994) discovered a third aspect of concept understanding: concept usage, "which refers to the ways one operates with a concept in generating or using examples or in doing proofs". These three aspects - concept definition, concept image, and concept usage - constitute what Moore calls the concept-understanding scheme. Concept usage is important in learning and working with a new concept. Specifically the example usage which involves example generation and verification is crucial for understanding a new concept (Dahlberg & Housman, 1997).

The centrality of examples in teaching and learning mathematics has been long acknowledged. Examples are an integral part of mathematical thinking, learning and teaching, particularly with respect to conceptualization, generalization, abstraction, argumentation, and analogical thinking (Zodik & Zaslavsky, 2008). It has long been acknowledged that people learn mathematics principally through engagement with examples, rather than through formal definitions and techniques. Indeed, it is only through examples that definitions have any meaning, since the technical words of mathematics describe classes of objects or relations with which the learner has to become familiar (Watson & Mason, 2005).

Sowder (1980) used “examples” to describe illustrations of concepts and also manifestations of principles. Zodik and Zaslavsky (2008) maintained that an example is a particular case of a larger class, from which one can reason and generalize. They also noticed that examples may differ in their nature and purpose. An example of a concept (e.g., a rational number) is quite different in nature from an example of how to carry out a procedure (e.g., finding the least common denominator) (Zodik & Zaslavsky, 2008). In their treatment of examples, they also included non-examples, that are associated with conceptualization and definitions, and serve to highlight critical features of a concept; as well as counter-examples that are associated with claims and their refutations. There has been much debate about the usefulness of including counter-examples in students’ experience. In some studies, counter-examples appear to be helpful in focusing students on what is relevant and what is irrelevant; in other studies, the role of counter-examples appears to confuse students who do not understand how or what it refutes (Zaslavsky & Ron, 1998). Many researches claimed that both non-examples and counter-examples can serve to sharpen distinctions and deepen understanding of mathematical entities. Furthermore, Sowder (1980) concluded that the inclusion of non-examples has an unpredictable cognitive role. Charles (1980) suggests that “one conjecture worthy of investigation is that non-examples are more instructive for learning difficult concepts, whereas examples are more instructive for learning “easy'” concepts.” (p.19). Zaslavsky and Ron (1998) suggested that students often feel that a counter-example is an exception that does not really refute the statement in question.

Watson and Mason (1998) investigating such examples moved a step forward and developed the more general notion of boundary examples. For example: a general straight line may be given as $y = mx + c$ but this formulation may exclude lines such as $x = a$ from the student’s experience. Because they are not encompassed by this general form, teachers find themselves having to introduce them deliberately in some way, as something extra to remember. They used the word "boundary" because they see students' experiences of examples in terms of spaces: families of related objects which collectively satisfy a particular situation, or answer a particular mathematics question, or deserve the same label. Such spaces appear to cluster around dominant central images.

Watson and Mason (2002a) maintained that "example" includes anything used as raw material for intuiting relationships and inductive reasoning: illustrations of concepts and principles; contexts which illustrate or motivate a particular topic in mathematics; and particular solutions where several are possible. Furthermore, they used the term exemplification to describe any situation in which something specific is being offered to represent a general class with which the student is expected to become familiar. Thus according to them learning mathematics can be seen as a process of generalizing from specific examples: learning to add involves generalizing a process which works for given examples so that it can be applied to examples one has not met before; learning about quadrilaterals involves understanding what types are possible, and what their properties are. The broader the range of examples, the richer the possibilities for generalizations and connections to be made, so the extent of the set of familiar examples is influential in construction of conceptual understanding.

Teachers frequently use examples in order to demonstrate and communicate the essence of mathematical concepts and techniques (Tall & Vinner 1981), since mathematical knowledge is abstract and often difficult to acquire and even more difficult to apply to novel situations (Gick & Holyoak, 1983; Novick, 1988; Reed, Dempster & Ettinger, 1985). In order to overcome these obstacles teachers present the learners with multiple concrete and highly familiar examples of the to-be-learned concept. For instance, a mathematics instructor teaching simple probability theory may present probabilities by randomly choosing a red marble from a bag containing red and blue marbles and by rolling a six-sided die. These concrete, familiar examples instantiate the concept of probability and may facilitate learning by connecting the learner's existing knowledge with new, to-be-learned knowledge. The belief in the effectiveness of multiple concrete instantiations is reasonable: A student who sees a variety of instantiations of a concept may be more likely to recognize a novel analogous situation and apply what was learned. Learning multiple instantiations of a concept may result in an abstract, schematic knowledge representation (Gick & Holyoak, 1983; Novick & Holyoak, 1991), which, in turn, promotes knowledge transfer, or application of the learned concept to novel situations (Gick & Holyoak, 1983;

Catrambone & Holyoak, 1989). Atkinson, Derry, Renkl and Wortham (2000) point to the contribution of multiple examples, with varying formats that support the appreciation of deep structures instead of excessive attention to surface features.

In addition, studies dealing with concept formation highlight the role of carefully selected and sequenced examples and non-examples in supporting the distinction between critical and non-critical features and the construction of rich concept images and example spaces (e.g., Vinner,1983; Zaslavsky & Peled,1996; Petty & Jansson, 1987; Watson & Mason, 2005).

Michener (1978), in looking at the roles played by examples of mathematical objects for mathematicians, elaborated distinctions between: startup examples, used to arouse interest and to suggest how the theory would develop; reference examples which are learnt and used to test future conjectures or to revisit concepts; model examples, which are paradigmatic and generic and counter-examples, which demonstrate the boundaries of concepts or techniques, or which contribute to counteracting standard misconceptions.

As Watson and Mason (2002a) claimed it is almost always the responsibility of authorities to produce examples and of students' to make sense of them. Generating examples is typical of a mathematician's work, in different activities and with many different objectives. The literature has underlined the importance of this activity also in mathematics education, as learning and teaching strategy (Zaslavsky, 1995; Dahlberg & Housman, 1997; Watson & Mason, 2002b), with the relation to the construction of concepts (Hazzan & Zazkis, 1997; Zaslavsky & Shir, 2005), to the production of conjectures (Boero, Garuti & Lemut, 1999; Antonini, 2003; Allcock, 2004) and of proofs (Balacheff, 1987; Harel & Sowder, 1998).

Watson and Mason (2005) introduced the notion of learner-generated examples which is a teaching strategy of asking learners to construct their own examples of mathematical objects under given constraints. Watson and Mason claimed, and provided ample evidence, that examples generated by learners serve as a powerful pedagogical tool for enhancing the learning of mathematics at a variety of levels (Zazkis & Leikin, 2008). Zazkis and Leikin (2007) suggested that this pedagogical tool can be used also as a research tool. By examining examples generated by participants, researchers may draw inferences about their knowledge, both subject matter knowledge and pedagogical content knowledge.

In problem solving, the role of examples is considered crucial, because they allow to perform exploration and to reach generalization and abstraction (Polya, 1945). Attention has been also paid to the activity of generating examples per se, as a special case of problem solving. Zaslavsky and Peled (1996) claimed that "the state of generating examples can be seen as a problem solving situation, for which different people employ different strategies". Thus, the generation of examples is a sort of an open-ended problem, in which one must decide whether the required example exists or not. When the example does not exist, it is required to justify, why the example does not exist (Antonini, 2006).

A number of researchers believe that a useful pedagogical strategy to help students overcome the difficulties that arise when they make a transition from the elementary to the advanced mathematical thinking may be to encourage them to generate examples of mathematical concepts (Alcock & Simpson, 2004; Dahlberg & Housman, 1997; Hazzan & Zazkis, 1997; Mason, 1998; Watson & Mason, 2001, 2002a, 2002b, 2005).

In UK classrooms, learners are often given a context which generates a sequence of numbers, and asked to "find a formula". Hewitt (1992) points out that such tasks can easily become mechanistic, but if the source of a sequence is questioned at a structural level, then learner-generated examples can provide starting points for substantial mathematical work. For instance, students can be asked to identify special examples from their results, and discuss what it is that makes them special.

Sowder (1980) reports using the prompt "Give me an example, if possible, of ...", with the teacher taking responsibility for guiding students towards peculiar examples. Learner-generated examples can therefore be used as starting examples for work on new concepts, rather than merely for revisiting familiar ones.

Sadovsky (1999) challenged her students to exemplify division operations which give a dividend of 32 and a remainder of 27. She asked "How many are there? If you think there are less than three write them all down, and explain why there are no other ones. If you think there are more than three write down at least four of them and explain how other solutions can be found". One outcome of her study was her conclusion that "… these problems are simultaneously a chance to find the limits of the arithmetic practices and enrich the conception of Euclidean division". But concept development through exemplification need not be incidental or unexpected; it can be an explicit pedagogic aim.

Zaslavsky (1995) describes an effective use of generating student examples. She asked the subjects to "find an equation of a straight line that has two intersection points with the parabola $y = x^2 + 4x + 5$". Attention was thus directed to features of a parabola and a straight line, rather than to algebraic or trial-and-error techniques for finding intersections. Each strategy led to further questions such as "find an equation of a straight line which does not intersect twice with the parabola….". Students encountered most of the analytical geometric syllabus, engaging with the structures and equations of straight lines and parabola through their own examples.

Dahlberg and Housman (1997) were interested in how a student's "concept image" (Tall & Vinner, 1981) initially develops when presented with a new definition. To this end, 11 undergraduates were presented with the following definition: A function is called fine if it has a root (zero) at each integer. After studying the definition, each student was asked to generate an example and a non-example of a fine function, reformulate the definition, verify if some given functions were fine, and determine the truth of some given conjectures. When initially presented with the concept definition, students either generated examples, reformulated the statement, decomposed and synthesized the statement, or simply tried to memorized it. The authors recommended encouraging students to generate their own examples when presented with a new concept since they found that: Students who consistently employed example generation had more learning events, were able to encapsulate more examples into their concept image of fine function, and were more able to use these examples than those who primarily used other learning strategies. They suggested that it may be beneficial to introduce students to new concepts by requiring them to generate their own examples or have them verify and work with instances of a concept before providing them with examples and commentary. Perhaps engaging students in example usage activities while introducing students to new concepts can promote this strategy and encourage students to consider more carefully the meaning of definitions.

In a study of first year university students in a proof-based Analysis course, Alcock and Simpson (2002) identified two criteria necessary for students' success in mathematical reasoning – a good understanding of what objects belong to a given set defined by a formal definition; and, the ability and inclination to use the formal definition in presenting an argument. They suggest that example generation exercises will not only help students get a better idea of what objects belong to a given set, but will help them create a link between the objects and the formal definition.

In their investigation of how learners provide examples, Hazzan and Zazkis (1997) found that being asked to generate examples is a relatively difficult task for students. They recommend giving students this type of task since to construct an example, the learner has to engage with the concept's properties, and not just perform some routine operations that may not require an understanding of the underlying concept.

In this study pre-service teachers are asked to generate examples of functions. We believe that by examining examples generated by participants, we will draw inferences about their knowledge concerning the particular concept and we shall have a better "access" to their concept image.

An algebraic and a "coordinated" approach related to the concept of function

According to Moschkovich et al. (1993), there are two fundamentally different perspectives from which a function is viewed: the process perspective and the object perspective. From the process perspective, a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value. Students who view functions under this perspective could substitute a value for x into an equation and calculate the resulting value for y or could find pairs of values for x and y to draw a graph.

In contrast, from the object perspective, a function or relation and any of its representations are thought of as entities for example, algebraically as members of parameterized classes, or in the plane as graphs that are thought of as being "picked up whole" and rotated or translated. Students who view functions under this perspective could recognize that equations of lines with the form $y = 3x + b$ are parallel or could draw these lines without calculations if they have already drawn one line or they can fill a table of values for two functions (e.g., $f(x) = 2x$, $g(x) = 2x + 2$) using the relationship between them (e.g., $g(x) = f(x) + 2$) (Knuth, 2000). In addition, Presmeg (2008) maintained that the nature of the former perspective was characterized as algorithmic, involving a process of algebraic substitution, whereas the latter could be seen as a more holistic apprehension of a function in its entirety, including its iconic relationships. The algebraic approach is relatively more effective in making salient the nature of the function as a process while the geometric approach is relatively more effective in making salient the nature of function as an object (Yerushalmy & Schwartz, 1993).

Mousoulides and Gagatsis (2004) investigated students' performance in mathematical activities that involved conversion between systems of representation of the same function, and concentrated on students' approaches as regards the use of representations of functions and their connection with students' problem solving processes. The most important finding of this study was that two distinct groups were formatted with consistency that is the algebraic and the geometric approaches groups. The majority of students' work with functions was restricted to the domain of the algebraic approach. This method, which is a point to point approach giving a local image of the concept of function, was followed with consistency in all of the tasks by the students.

Many students had not mastered even the fundamentals of the geometric approach in the domain of functions. Most of students' understanding was limited to the use of algebraic representations and approach, while the use of graphical representations was fundamental in solving geometric problems. Only a few students used an object perspective and approached a function holistically, as an entity, by observing and using the association of it with the closely related function that was given. It is indicated that only these students developed the ability to employ and select graphical representations, thus the geometric approach (Mousoulides & Gagatsis, 2004).

This study's findings are in line with the results of previous studies indicating that students cannot use effectively the geometric approach, which originates within the object perspective (Knuth, 2000). The fact that most of the students chose an algebraic approach (process perspective) and also demonstrated consistency in their selection of this approach, even in tasks and problems in which the geometric approach (object perspective) seemed more efficient or that they failed to suggest a graphical approach at all, is a strong indication of the phenomenon of compartmentalization in the processes that students followed in tasks and problems on functions involving graphical and algebraic representations.

Moreover, an important finding of the study was the relation between the graphical approach and geometric problem solving. This finding is consistent with the results of previous studies (Knuth, 2000; Moschkovich et al., 1993), indicating that the geometric approach enables students to manipulate functions as an entity, and thus students are capable of finding the connections and relations between the different representations involved in problems. Students who had a coherent understanding of the concept of functions (geometric approach) could easily understand the relationships between symbolic and graphic representations in problems and were able to provide successful solutions.

Even (1998) focused on the intertwining between the flexibility in moving from one representation to another and other aspects of knowledge and understanding. The participants were 152 college mathematics students who were also prospective secondary mathematics teachers. In the first phase of the study they completed an open-ended questionnaire. In the second phase ten of them were interviewed. This study indicated that subjects had difficulties when they needed to link different representations of functions flexibly. An important finding of this study was that many students deal with functions pointwise (they can plot and read points) but cannot think of a function in a global way. The data also suggested that subjects who can easily and freely use a global analysis of changes in the graphic representation have a better and more powerful understanding of the relationships between graphic and symbolic representations than people who prefer to check some local and specific characteristics. This finding cannot be generalized since in some cases the pointwise approach proved to be more powerful. In the case of problem solving a combination of the two methods is most powerful.

In this research study the concept of function is viewed from two different perspectives, the algebraic and the "coordinated" perspective. The algebraic perspective is similar to the pointwise approach described by Even (1998) and the one described by Mousoulides and Gagatsis (2004). In this perspective, a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value (Moschkovich et al., 1993). The "coordinated" perspective combines the algebraic and the graphical approach. In this perspective, the function is thought from a local and a global point of view at the same time. The students' can "coordinate" (flexibly manipulate) two systems of representation, the algebraic and the graphical one.

Particularly, in this study we gave pre-service teachers four simple function tasks. In each task, there were two linear or quadratic functions. Both functions were in algebraic form and one of them was also in graphical representation. There was always a relation between the two functions (e.g., $f(x) = 2x$, $g(x) = 2x + 1$). Teachers were asked to interpret graphically the second function. Teachers' responses in these tasks were categorized as algebraic or coordinated based on the definitions given above for the two approaches.

It is worth mentioning that the above tasks involve transformations of functions. While teaching, learning, or understanding of functions has been an important focus in mathematics education research in the past decades and a variety of research reports focusing on functions exist, little attention has been given to the transformations of functions (Zazkis, Liljedahl, & Gadowsky, 2003). Some researchers have worked on the concept of transformations of functions (Baker, Hemenway, & Trigueros, 2001a; Baker, Hemenway, & Trigueros, 2001b; Cuoco, 1994; Eisenberg & Dreyfus, 1994; Goldenberg, 1988; Zazkis, 2003) and have found specific difficulties students have when working with a few particular problems where transformations are involved.

A common treatment of transformations of functions in pre-calculus courses involves a consideration of a graph of a function $f(x)$ on a Cartesian plane. Functions $f(ax)$, $f(-x)$, $f(x)+k$, and $f(x+k)$ correspond to a dilation, reflection, vertical translation and horizontal translation of $f(x)$, respectively. Discussion of transformations of functions traditionally begins with a consideration of a parabola and then proceeds to graphs of other quadratic relations and other functions.

Furthermore traditionally, transformations of functions have been taught with a strong emphasis on algebraic symbolism and in relative isolation from the visual transformational topics in geometry. The typical approach to transformation in most conventional textbooks varies the coefficients of a function and examines the resulting changes in the graph (Borba & Confrey, 1996). For instance, the question is asked: if $y = x^2 + 5$ is changed to $y = 2x^2 + 5$, how does the graph of the transformed function change? Borba and Confrey (1996) call this a template approach and documents student difficulties including a tendency to memorize rules without understanding their genesis and failure to make the subtle distinctions among different symbolic forms. Other studies in which such an approach is used document students' difficulties in generalizing the patterns, including problems with moving among different families of functions, and problems of visual screen constraints and scaling issues (Borba & Confrey, 1996).

Eisenberg and Dreyfus (1994) conducted an extensive exploration on students' understanding of function transformations, focusing on visualization of transformations. They acknowledged the difficulty in visualizing a horizontal translation in comparison to a vertical one, suggesting that "there is much more involved in visually processing the transformation of $f(x)$ to $f(x+k)$ than in visually processing the transformation of $f(x)$ to $f(x)+k$" (p. 58).

Baker, Hemenway, and Trigueros (2001b) and Lage and Gaisman-Trigueros (2006) have investigated the understanding of transformations of various functions from a perspective of Action–Process–Object–Schema (APOS) theory (Asiala et al., 1996; Dubinsky & MacDonald, 2001). Students who act at an action conception of transformation of functions can perform operations on functions and variables step by step, and these operations can be applied either in the analytical or graphical representation context; rely on memorized facts or external signs, as for example the exponents in the expressions or the apparent form of the graph; recognize differences between a function and its transformations only in terms of the syntax of the rule that defines the function, and recognize similarities between a function and its transformations, or between transformations, only in terms of some global property of the graph. When these actions are repeated on the analytical or graphical representation of a function, and students reflect upon them, they interiorize the actions into a process.

Students who act at a process level are able to describe changes in the basic functions as a consequence of the application of the transformation without the need to perform each step of the transformation or move the graph of a function step by step. They are able to look at the graph of the transformed function and describe the changes that result from the transformation. These students are also able to reverse the process to identify the function on which a set of transformations was applied. Students at this level show, however, difficulties in coordinating the information obtained from different representational contexts, and in flexibly translating information from one representational context to another. When students reflect on all of these processes, and are able to think of them as a whole, in any representational context, working flexibly in different representational contexts as well, it is considered that they have encapsulated the process of applying a transformation to any function into an object.

It is considered that students have an object conception of transformation, if they are able to apply actions on transformed functions and coordinate their properties in terms of possible changes in the original function, At this level, students are able to de-encapsulate any transformed function object into the process involved in its construction, and they are able to identify the basic function on which it is based and compare different transformed functions in terms of their properties in any representational context.

Lage and Gaisman-Trigueros (2006) showed that students' difficulties with the concept of transformation of functions are strongly related to their understanding of the concept of function. It was found that students who were classified at an action level show a weak understanding of the concept of function. In particular, these students showed conceptual problems when discussing the graphical representation of functions and when looking for their domain and range. Only a few of the students were capable of flexibly using those functions that were presented in a graphical context. The results of this study demonstrated that when teaching transformations it is important to consider a wider variety of functions and to explicitly demonstrate the result of applying a transformation both at the level of what happens to the function in general, and what happens to different points in its domain. Furthermore the results of this study showed that flexibility with the use of different representations is necessary when teaching transformations of functions.

Baker et al. (2001b) pointed out that vertical transformations appear easier for the students than horizontal transformations. They explained this based on their theoretical perspective, claiming that "vertical transformations are actions performed directly on the basic functions, while horizontal transformations consist of actions that are performed on the independent variable of the function and further action is needed on the object resulting from the first action to get the result of the transformation" (Baker et al., 2001b, p. 47).

Furthermore, students' difficulty with function transformation was attributed, at least in part, to their incomplete understanding of the concept of function. Baker et al. (2001b) agree with Eisenberg and Dreyfus in their observation that an object conception of function may be a prerequisite to the effective understanding of transformations of functions.

Borba and Confrey (1996) presented a detailed case study of a 16-year-old student, Ron, working on transformations of functions in a computer based multi-representational environment. Their study intended to investigate vertical and horizontal translations, reflections around vertical and horizontal lines, and vertical and horizontal stretches of functions. Ron's attempts to interpret the horizontal translation of a parabola were presented as "problematic." This was followed by his investigation to coordinate visual actions with changes in other representations.

Ponte (1984) (as cited in Schwarz, Dreyfus, & Bruckheimer, 1990) has shown that most Grade 11 students take a discrete, static approach to functions. They conceive of them as consisting of points, and thus as fundamentally discrete in nature; they are weak on the aspect of function that describes variation and they are, on the whole, unable to cope with the variation of variation. When different representations are involved in the same task, even Grade 12 students have been found to have considerable difficulty relating graphs to formulae when presented with tasks of function transformations such as shifts, $f(x)$ to $f(x)+k$ or $f(x)$ to $f(x+k)$, and stretches, $f(x)$ to $kf(x)$ or $f(x)$ to $f(kx)$. Such tasks presuppose that the function is conceived of as an object.

In this study we have included both vertical and horizontal translations of function. Furthermore, it is considered that teachers who used a coordinated approach, that is an object perspective, show also a conceptual understanding of function. Furthermore, students' coordinated approach is closely linked to the understanding of the concept and problem solving.

Problem solving and functions

The development of students' ability to solve problems is considered to be the basic aim of mathematics teaching for all the educators, regardless the learning theory they follow. Problem solving is an ability that reflects the level of mathematical thinking and includes creative and critical thinking also. In addition problem solving or "dealing with a problematic situation" has overcome the limits of mathematical science and it is included in many areas of the social sciences (Philippou & Christou, 1995, p. 128).

Problem solving refers to the process of associating prior experiences, knowledge, information and intuition in order to determine the outcome or a solution of a situation for which the procedure for determining the outcome is not directly known (Charles et al., 1987). Similar to the above definition, problem solving as proposed by Stanic and Kilpatrick (1988) and Schoenfeld (1992), is equated with Dewey's reflective thinking (Dewey, 1933). Demetriou (1998) suggests that understanding is closely linked to problem solving, which refers to processes that are employed to generate responses aiming at the transformation of a situation such that a goal is achieved or a gap is bridged.

In mathematics there is often confusion between what is called problem and an exercise. Polya (1945) moved to a clear and very useful distinction between the routine problems and the novel (non-routine) problems. Particularly, he stated that students lose their interest and their mental development is constrained when the teachers give them routine problems. In contrast, when the students are confronted with non-routine problems their curiosity is stimulated and their critical thinking is developed (Polya, 1945). Many researchers (Reys et al.,, 1989; Schoenfeld, 1985) mentioned that a problem refers to a situation that a person seeks for something or puts a goal and does not know what way to follow in order to discover or in order to achieve it. When the procedure is already known from previous experience, then it is not a problem but an exercise (Philippou & Christou, 1995). In order to reach a solution to a problem the student must connect his previous knowledge with the problematic situation. Students' mental schemas are developed and become more complicated when they are dealing with problems that demand reasoning, connections and elaboration. Therefore, learning through problem solving leads to the conceptual understanding of mathematical concepts (Lambdin, 2003).

Many researchers proposed strategies that will facilitate students to solve a problem. Polya (1945) proposed a procedure of four stages for the solution of a problem:

1st stage-Understanding the problem: This stage includes extracting and assimilating the relevant and valuable information from the given, determining the goal of the problem, reconstructing the problem if necessary, and introducing suitable notations whenever possible for easy reference and manipulation.

2nd stage-Devising a plan: This stage is to make a general plan and select relevant methods, or more appropriately, heuristics, that might be useful for solving the problem based on the understanding of the problem at the first stage.

3rd stage-Carrying out the plan: This stage is to carry out the plan, which has been decided at the preceding stage, and to keep the track to obtain the answer.

4th stage-Looking back: This stage includes checking the correctness of the solutions, reflecting on key ideas and processes of problem solutions, and generalizing or extending the methods or the results.

Even if problem solving is a complicated task, three independet categories of variables that are interrelated and influence problem solving can be mentioned. These variables are related to the person who solves the problem, to the problem/task itself and to the context that the problem solving take place (Kilpatrick, as cited in Kulm, 1984). Concerning the person who solves the problem, the variables which affect the procedure are:

1. Invariable- external variables referring to the personal characteristics of the solver such as the age and the sex.

2. Variables that can be changed by teaching or by other procedures such as the learning style, the interest, the motives and the beliefs.

3. Variables that refer to the educational background of the solver such as the kind of school or the lessons he/she is attending to.

Concerning the task (problem), the variables which affect its solution are:

1. The content of the problem, that refers to the mathematical concepts that are included in the problem.

2. The context of the problem, that refers to the non-mathematical elements of the problem.

3. Syntax of the problem, that refers to the place of the words and symbols included in the problem.

4. The structure of the problem, that refers to the mathematical relations between the elements of the problem and to the mathematical operations that are necessary for its solution.

Particularly important is considered to be the influence of the variables emerged from the environment in the process of problem solving, that are related to the natural, psychological and social context (D' Amore & Zan, 1996).

The representations used in order to present the data in a problem are related to the non-mathematical concepts involved in it. Many researchers, in the last years, have examined the important role of the different modes of representation in problem solving.

Particularly, Lesh et al. (1987a) and Lesh et al. (1987b) taking into consideration the results of previous researchers (Behr, Lesh, Post, & Silver, 1983; Lesh et al., 1983), examined the role of different representations and translation abilities in problem solving. They have found that these "translation (dis)abilities" are significant factors influencing both mathematical learning and problem-solving performance, and that strengthening or remediating these abilities facilitates the acquisition and use of elementary mathematical ideas. Particularly, they highlighted the facilitating role played by the spoken language and the manipulatives models in problem solving.

According to the research of Lesh et al. (1983) involving concrete/realistic versions of typical text book word problems, students seldom work through solution in a single representational mode. Instead students frequently use several representational systems, in series and/or in parallel, with each depicting only a portion of the given situation. In fact, man realistic problem-solving situations are inherently multimodal from the outset.

Good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process.

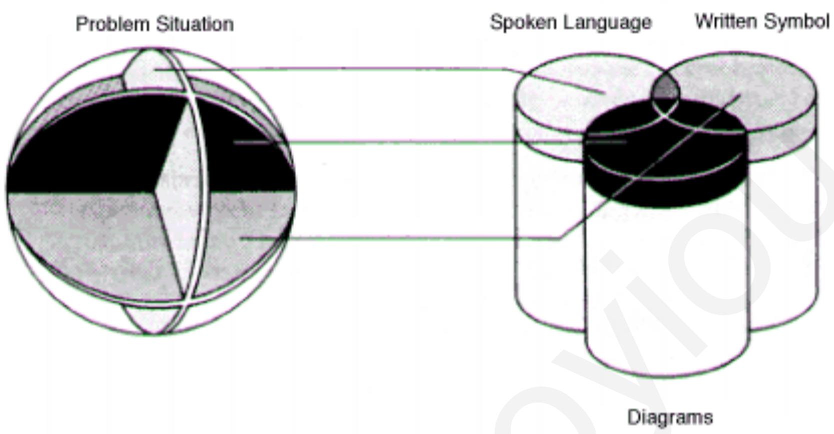


Figure 2.5. The act of representation – First interpretation (Lesh et al., 1987b, p. 38)

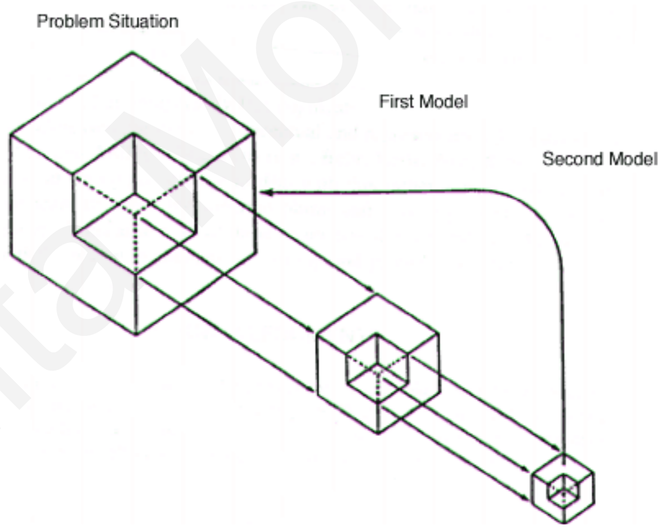


Figure 2.6. The act of representation – Second interpretation (Lesh et al., 1987b, p. 39)

Figure 2.5 suggests one way that the act of representation tends to be plural; that is, solutions often are characterized by several partial mappings from parts of the given situation to parts of several (often partly incompatible) representational systems. Each partial mapping represents a "slice" of the problem situation, using only part of the available representational system. It is not a mapping from the whole "given" situation to only a single representational system. The act of representation also may be plural in a second sense; that is, a student may begin a solution by translating to one representational system and may then map from this system to yet another system, as illustrated in Figure 2.6 (Lesh et al., 1987b).

In fact, for concrete or realistic versions of textbook word problems the actual solutions students have tended to use often combine features depicted in both Fig. 2.5 and 2.6 preceding, as well as a third aspect of representational plurality; that is, a given representational system often appears to be related (in a given student's mind) to several distinct clusters of mathematical ideas (Lesh et al., 1987b).

Niemi (1996) constructed an evaluation test for students' conceptual understanding of fractions based on their ability to use various representations, their problem solving ability, explanation and justification ability. Their flexibility in using multiple representations that was evident from their performance in fraction recognition tasks given in a variety of diagrammatical representations (number line, objects, are), was a predictive factor of their success in symbolic problems and explanation tasks. Students with representational flexibility gave more correct solution in contrast with the other students and in addition gave more diagrammatic and verbal explanations (Niemi, 1996).

In addition, Stenning, Cox and Oberlander (1995) in a research study with university students found that the participants who were able to handle different representation and make conversions between them had better results in a test regarding their thinking ability than the students who did not use representations. Similarly, Monaghan, Stenning, Oberlander and Sontrod (1999), maintained that the success in problem solving is closely related with the flexibility in using various representations.

The fact is noteworthy that a way to help students to understand a problem solving situation, is by giving them similar problems including multiple representations of the mathematical concepts (Lesh et al., 1987b). Using multiple representations the students can develop intuitive, procedural and conceptual knowledge in mathematics (Moyer, 2001). Cifarelli (1998) maintained that the successful problem solvers can construct the appropriate representations and use them as tools for the understanding of relations and information. The representation of a problem or a situation in a way that has meaning for the students helps them to organize their thinking and evoke multiple ways of approaching the problems— something that leads to better understanding (Fennell & Rowan, 2001).

In relation with the concept of function, Gagatsis and Shiakalli (2004) in a research study with university students focused on the relationship between success in solving direct translation tasks and success in solving problems by articulating different representations of the concept of function. In the direct translation tasks the source representation was either in a verbal or graphical form and the students were asked to pass to the graphical and algebraic representation or to the verbal and algebraic representation respectively. In this study were also used word problems that involved indirect translation to algebraic and graphical representations. The results of the study showed that university students' ability to translate from one representation of the concept of function to another is related to problem solving success. Students who have a better and more powerful understanding of the relationships between different kinds of representations are more successful in problem solving (Gagatsis & Shiakalli, 2004).

Elia, Panaoura, Eracleous and Gagatsis (2007) explored pupils' constructed definitions of the concept of function in relation to their abilities in dealing with tasks of functions involving different forms of representations and problem solving tasks that required the ability to flexibly use and translate between various representations of the concept. A major concern was also to examine the interrelations between these three ways of thinking about or dealing with the concept of function. Findings revealed pupils' difficulties in giving a proper definition for the concept of function and resolving problems on functions involving conversions between diverse modes of representation. Several inconsistencies among pupils' constructed definitions, their competence to use different representations of functions and their problem solving ability, were also uncovered, indicating lack of flexibility between different ways of approaching functions. Pupils' conception that different representations of a function are distinct and autonomous mathematical objects and not just different ways of expressing the meaning of the particular notion were uncovered. In other words, most representations of function remained compartmentalized and mathematical thinking was fragmentary and monoregistral (Duval, 2002) for the pupils of this study.

Similarly, Elia, Panaoura, Gagatsis, Gravvani and Spyrou (2008) in their study found that students' problem solving effectiveness had a predictive role in whether they would successfully employ the concept of function in various forms of representation, in giving a definition and examples of the concept. An interpretation for this finding is that problem solving is a complex process that involves various abilities and in this case probably skills referring to the other three aspects of the understanding of function. For instance, the solution of the particular problem of the test required among other abilities the coordination of various representations of function, i.e., verbal, algebraic and graphic, as well as acquisition of what a function is (definition) and of different types of functions (example).

Although many researches recognize the importance of translation, Presmeg and Nenduradu (2005) maintained that the flexibility in translation from one representation to another does not necessarily imply conceptual understanding of a concept. Their result emerged from the case study of a pre-service teacher who while he could flexibly use various representations (tabular, algebraic and graphical representation) in order to solve an algebraic problem he could not recognize the exponential relation involved in it and gave a linear interpretation showing lack of conceptual understanding concerning the concept of function.

Hitt (1998) stated that while the pre-service teachers solved successfully tasks that involved algebraic representations of the concept of function, they did not use the definition of the concept. Furthermore, they were unable to combine different representations of the concept and confront more complicated activities.

From the above research studies it is evident that the role of representations in problem solving is fundamental. In this study in order to reach a solution to the verbal problems given the pre-service teachers should employ a variety of representations (tabular, algebraic, graphical, verbal) and make successful translations between these representations. Their ability in problem solving will be an indication of their conceptual understanding of the concept of function. Furthermore, it is noteworthy the fact that it will be examined the interrelation between the definition of the concept, examples of the concept, recognition of function given in various representations, conversions from one mode of representation to the other and the coordinated approach with problem solving.

Summary

The concept of function is of fundamental importance in the learning of mathematics (Eisenberg, 1992). Functions have a key place in the mathematics curriculum, at all levels of schooling. Being fundamental for the study of mathematics, the function concept has been identified as the single most important notion from kindergarten to graduate school (Dubinsky & Harel, 1992). Nevertheless, students of secondary or even tertiary education, in any country, seem to encounter difficulties in conceptualizing the notion of function (Elia & Spyrou, 2006; Hitt, 1998; Markovits et al., 1986).

An important factor influencing the learning of functions is the diversity of representations related to this concept. A representation is considered to be a configuration of some kind that, as a whole or part by part, corresponds to, is referentially associated with, stands for, symbolizes, interacts in a special manner with, or otherwise represents something else (Palmer, 1977). Representations can be considered as useful tools for constructing meaning and for communicating information and understanding (Greeno & Hall, 1997). It is very important for students to be involved in choosing representations and constructing representations in forms that help them see patterns and perform calculations, taking advantage of the fact that different forms provide different supports for inference and calculation. According to Duval (2006) no kind of mathematical processing can be performed without using a semiotic system of representation, because mathematical processing always involves substituting some semiotic representation for another.

The need for a variety of semiotic representations in the teaching and learning of mathematics is usually explained through reference to the cost of processing, the limited representation affordances for each domain of symbolism and the ability to transfer knowledge from one representation to another (Duval, 1987; Duval, 1993; Gagatsis, 1997). A representation cannot describe fully a mathematical construct and each representation has different advantages, so using various representations for the same mathematical situation is at the core of mathematical understanding (Duval, 2002). Ainsworth et al. (1997) suggested that the use of multiple representations can help students develop different ideas and processes, constrain meanings and promote deeper understanding.

A substantial number of research studies have examined the role of different representations on the understanding and interpretation of functions (Gagatsis & Shiakalli, 2004; Hitt, 1998; Markovits et al., 1986). The concept of function admits a variety of representations and consequently has the capability of being taught using diverse representations, each of which offers information about particular aspects of the concept without being able to describe it completely. The literature illustrates functions in several ways, such as mapping diagrams, tables, graphs and equations. Using multiple representations to teach functions (numeric, graphic, and symbolic) intends to promote and enhance a broad and deep understanding of the concept. Furthermore several researchers (Evangelidou et al., 2004; Gagatsis & Shiakalli 2004; Gagatsis et al., 2002; Mousoulides &

Gagatsis, 2004; Sfard, 1992; Sierpinska, 1992) indicated the significant role of different representations of function and the conversion from one representation to another on the understanding of the concept.

In addition, a useful way to characterize a person's thinking about functions is in terms of the notions of concept definition and concept image (Lloyd & Wilson, 1998). These constructs point to a distinction between the formal definition an individual holds for a given concept and the way that he or she thinks about the concept. Concept image and concept definitions are two terms that have been discussed extensively in the literature concerning students' conceptions of function (Vinner & Dreyfus, 1989; Tall & Vinner, 1981; Vinner & Hershkowitz, 1980). In addition to the concept definition - concept image scheme a third aspect of concept understanding is concept usage, "which refers to the ways one operates with a concept in generating or using examples or in doing proofs" (Moore, 1994). The centrality of examples in teaching and learning mathematics has been long acknowledged. Examples are an integral part of mathematical thinking, learning and teaching, particularly with respect to conceptualization, generalization, abstraction, argumentation, and analogical thinking (Zodik & Zaslavsky, 2008). Furthermore, they can be used to access students' image for a particular concept.

According to Moschkovich et al. (1993), there are two fundamentally different perspectives from which a function is viewed: the process perspective and the object perspective. From the process perspective, a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value. Students who view functions under this perspective could substitute a value for x into an equation and calculate the resulting value for y or could find pairs of values for x and y to draw a graph. In contrast, from the object perspective, a function or relation and any of its representations are thought of as entities for example, algebraically as members of parameterized classes, or in the plane as graphs that are thought of as being 'picked up whole' and rotated or translated. Students who view functions under this perspective could recognize that equations of lines with the form $y = 3x + b$ are parallel or could draw these lines without calculations if they have already drawn one line or they can fill a table of values for two functions (e.g., $f(x) = 2x$, $g(x) = 2x + 2$) using the relationship between them (e.g., $g(x) = f(x) + 2$) (Knuth, 2000). In this research study the concept of function is viewed from two different perspectives, the algebraic and the "coordinated" perspectives. In the algebraic perspective, a function is perceived of as linking x and y

values: For each value of x , the function has a corresponding y value (Moschkovich et al., 1993). The "coordinated" perspective combines the algebraic and the graphical approach. In this perspective, the function is thought from a local and a global point of view at the same time. The students' can "coordinate" (flexibly manipulate) two systems of representation, the algebraic and the graphical one.

The development of students' ability to solve problems is considered to be the basic aim of mathematics teaching for all the educators, regardless the learning theory they follow. Problem solving is an ability that reflects the level of mathematical thinking and includes creative and critical thinking also. In addition problem solving or "dealing with a problematic situation" has overcome the limits of mathematical science and it is included in many areas of the social sciences (Philippou & Christou, 1995, p. 128). In relation with the concept of function, many researchers (Gagatsis & Shiakalli, 2004) have dealt with the relationship between success in conversion tasks and success in solving problems by articulating different representations of the concept of function. The results showed that the solution of a particular problem requires the combination of different abilities and among them the ability to coordinate various representations of a function.

The above mentioned areas that have a significant impact on the conceptual understanding of function have examined rather separately and in isolation the one to the other. Limited attention has been given to the interrelations among students' concept image of function which consists of the definition and examples of the concept, use of different representations of the mathematical concept in recognition and conversion tasks, students' approaches and problem solving. In this research study we attempt to synthesize the above mentioned research domains and construct a model concerning the conceptual understanding of function.

CHAPTER III

METHODOLOGY

Introduction

The concept of function is central in mathematics and its applications. In the school curriculum, function is an advanced topic, which is typically not explored in detail until the secondary level. Due to the unifying role of the function concept in mathematics and its ability to provide meaningful representations of complex, real world situations, current reform recommendations call for an emphasis on functions to be integrated throughout the school curriculum, beginning in the elementary grades (Christou et al., 2005). Given the importance of functions in mathematics and in the curriculum, it is crucial for researchers to explore the nature of students' knowledge of functions and especially of pre-service teachers' knowledge.

Thus, the general aim of this research study was to explore Cypriot and Italian pre-service teachers' display of behavior, cognitive structures and performance in different aspects of the understanding of function. Basic dimensions explored were the multiple representational flexibility and problem solving, since these two dimensions are crucial for mathematics learning.

The research study was conducted in three phases. The goal of the first phase was to contribute to the understanding of the algebraic and coordinated approaches pre-service teachers develop and use in solving function tasks and to examine which approach is more correlated with their ability in solving complex problems. Furthermore an important goal of this phase was to investigate the stability of these approaches and the stability of their relation. It is considered that the coordinated approach and complex problems are strongly related and are the basic dimensions of the problem solving ability that it is considered to be crucial in the conceptual understanding of every concept.

The goal of the second phase was to explore teachers' display of behavior, cognitive structures and performance in six aspects of the understanding of function: effectiveness in solving complex problems with functions, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic and verbal expression), transferring function from one mode of representation to another and the approach when dealing with simple function tasks. A main concern was also to examine problem solving in relation to the other types of displayed behavior. It is considered that the above mentioned aspects can contribute to the construction of a comprehensive model that will address pre-service teachers' understanding of functions.

The goal of the third phase was to triangulate the quantitative data regarding teachers' understanding of the concept of function and to further investigate pre-service teachers' behavior in the above mentioned aspects of the understanding of function: concept definition, examples of function, the recognition of functions given in various representations, "coordinated" or algebraic approaches when dealing with simple function tasks and effectiveness in problem solving.

This chapter contains a description of the research design applied in this study. In particular, it describes the participants of each phase, the instruments used, the variables and the scoring of the tasks and the statistical techniques used in order to analyze the data that emerged from the study.

Furthermore, a brief analysis of the Cypriot curriculum and the mathematics textbooks of the 9^{th}, 10^{th}, 11^{th} and 12^{th} grades in relation to the modes of representation and the function of representations concerning the particular concept are presented. The analysis was performed in order to gain further insight to the way the concept of function is approached to Cypriot middle and high school. Furthermore, this analysis contributed to the construction of the research's instruments and to the interpretation of the results since many of the difficulties pre-service teachers' face with the concept can be attributed to the way the concept is taught in middle and high school.

Participants

The research study was conducted in three phases. In the first phase four groups of pre-service teachers participated. Particularly, the data of the first group (Group A) were collected in 2005 and the participants were 135 Cypriot pre-service teachers. The data of the second group (Group B) were collected two years later, in 2007 and the participants were 153 Cypriot pre-service teachers. The data of the third group (Group C) were collected in 2008 and the participants were 260 Cypriot pre-service teachers. Finally, the data of the fourth group (Group D) were collected in 2009 and the participants were 200 Italian pre-service teachers. The Cypriot pre-service teachers of the second and third group graduated from a slightly different type of high school with different textbooks and different procedures for the selection of lessons as a result of the major changes that happened in the educational system, at high school in Cyprus. The participants, of all phases, were pre-service teachers. The subjects were for the most part students of high academic performance admitted to the University of Cyprus and to the Universities of Bologna and Palermo on the basis of competitive examination scores.

The participants of the second phase were 279 Cypriot and 206 Italian pre-service teachers. The subjects were admitted to the University of Cyprus and to the Universities of Bologna and Palermo on the basis of competitive examination scores. The sample of the first and second phase was randomly selected.

In the third phase nine of the second phase's Cypriot participants were chosen for task-based interviews. The sampling for this phase was purposive (typical case and convenience sampling at the same time). Particularly, the participants were chosen according to their performance in the two tests of the study and their willingness to participate in an interview. Specifically, three high performance, three medium performance and three low performance Cypriot pre-service teachers were chosen.

Instruments

In the first phase a test (Test A_1) was given in order to explore the algebraic and coordinated approaches pre-service teachers develop and use in solving function tasks and to examine which approach is more correlated with their ability in solving complex problems (Monoyiou & Gagatsis, 2008a; Monoyiou & Gagatsis, 2008b). The test consisted of seven tasks. The first four tasks were simple tasks with functions. In each task, there were two linear or quadratic functions. Both functions were in algebraic form and one of them was also in graphical representation. There was always a relation between the two functions (e.g. $f(x) = 2x$, $g(x) = 2x + 1$). Teachers were asked to interpret graphically the second function.

The other three tasks were complex problems. The first problem consisted of textual information about a tank containing an initial amount of petrol (600 L) and a tank car filling the tank with petrol. The tank car contains 2000 L of petrol and the rate of filling is 100 L per minute. Teachers were asked to use the information in order to give the two equations, to draw the graphs of the two linear functions and to find when the amounts of petrol in the tank and in the car would be equal. The second problem consisted of textual and algebraic information about an ant colony. The number of ants (A) increases according to the function: $A = t^2 + 1000$ (t = the number of days). The amount of seeds that, the ants save in the colony, increases according to the function $S = 3t + 3000$ (t = the number of days). Teachers were asked to use the information in order to draw the graphs of the quadratic and linear functions and to find when the number of ants in the colony and the number of seeds would be equal. The third problem consisted of a function in a general form of $f(x) = ax^2 + bx + c$. Numbers a , b and c were real numbers and the $f(x)$ was equal to 4 when $x = 2$ and $f(x)$ was equal to -6 when $x = 7$. Teachers were asked to find how many real solutions the equation $ax^2 + bx + c$ had and explain their answer. The test was administered to teachers by researches in a 60 minutes session.

Concerning the second phase two tests (Test A_2 and Test B) were given (Monoyiou & Gagatsis, 2009; Monoyiou & Gagatsis, 2010; Monoyiou & Gagatsis, in press). The tasks of the two tests developed in order to explore teachers' display of behavior in five aspects of the understanding of function: effectiveness in solving a word problem, concept definition, examples of function, recognizing functions in different representations, transferring function from one mode of representation to another and the approaches when dealing with simple function tasks.

The test A_2 was similar with the test given in the first phase but it was enriched with two more tasks. In the first task the teachers were asked to give a simple definition of function, while in the second task they were asked to give two simple examples from the applications of functions in everyday life. The other four tasks were simple tasks with functions and the other three tasks were complex problems, the same as the ones included in the Test A_1 of the first phase.

Test B consisted of fourteen tasks. The first three tasks involved the definition of the concept. In order to give a correct response to these tasks the teachers must take into consideration the correct definition of function. In the first task the teachers were asked if there is a function all of whose values are equal to each other. The other two tasks were Right or Wrong tasks: Can f be a function, if $f(-2) = 3$ and $f(-2) = 0$? and Can f be a function, if $f(-2) = f(3) = 4$?. Four tasks involved six conversions, three of them from an algebraic expression to a graphical representation and the other three from a graphical representation to an algebraic expression.

In the other four tasks the pre-service teachers were asked to recognize whether mathematical relations in different modes of representation (Venn diagrams, graphs, algebraic expressions and verbal expressions) were functions or not, by applying the definition of function. Teachers chose between a yes or no answer and furtherore were asked to provide an explanation for their option.

There were also three complex problems. The first problem consisted of information about a parachutist who jumps from an airplane which is at 3000 m height (above the earth) and falls with stable speed 30 m/s. The teachers were asked to express the parachutist's height as function of time, to draw the graph of the function, to find the parachutist's height (from earth) 1 minute after his/her fall and to find in what height the parachutist will be 20 minutes after his/her fall. In the second problem a square with side 1 cm was given and inside the square there was another square A. The teachers were asked to express the area of the square A as a function of x, to draw the graph of the function and to find the value of x for which the square A has maximum area. The third problem was similar to the one given in Test A and consisted of the function $f(x) = ax^2 + bx + 7$ (a and b are real numbers). When $x = 0$, $f(x)$ had a maximum value. The $f(x)$ was equal to -5 when $x = 2$. The teachers were asked to find how many real solutions the equation $ax^2 + bx + 7$ has and explain their answer. The tests were administered to the teachers by the researches in two 90 minutes sessions.

Nine teachers were chosen according to their performance in the two tests and the approach they followed in simple function tasks for a task-based interview. The interview consisted of nine tasks. In the first task the teachers were asked to give a definition of the concept of function and a simple example from the applications of functions in everyday life. In the second task they were asked if $f(-2) = f(3) = 4$ is a function or not. In the third task they were asked to give an example of a function f such that for any real numbers x, y in the domain of f the following equation holds: $f(x + y) = f(x) + f(y)$ (Sajka, 2003). In the fourth task they were asked to recognize the concept given in different representations (arrow diagrams, graphs, symbolic, verbal). The fifth and sixth tasks were transformation tasks. In the fifth task the function $y = x^2 + x$ was given. The teachers were asked to draw the function $y = x^2 + x + 1$. In the seventh task the function $\sin(x)$ was given. The teachers were asked to draw the function $\sin(x + 1)$. The other three tasks were complex problems with function. The first problem involved a linear function, the second a quadratic function and the third an exponential relationship. The problems examined teachers' ability to use various representations, their ability to translate one representation to the other and their conceptual understanding of the concept.

Variables of the research and scoring of the tasks

In the following section the tasks of the three tests (A_1, A_2, B) and of the task-based interview will be presented as well as the variables used in order to represent teachers' correct solutions and strategies.

The results concerning students' answers to the Test A_1 of the first phase were codified and marked as follows. The four simple function tasks were codified by an uppercase T (task), followed by the number indicating the exercise number. Following is the letter that signifies the way teachers solved the task: (a) "a" was used to represent "algebraic approach – function as a process" to the tasks, (b) "c" stands for students who adopted a "coordinated approach – function as an entity". A solution was coded as "algebraic" if students did not use the information provided by the graph of the first function and they proceeded by constructing the graph of the second function by finding pairs of values for x and y. On the contrary, a solution was coded as coordinated if students observed and used the relation between the two functions in constructing the graph of the second function. In this case students used and coordinated two systems of representation, namely the algebraic and the graphical one. They noticed the relationship between the two equations given and they interpreted this relationship graphically by manipulating the function as an entity. The following symbols were used to represent students' solutions to the problems: Pr1a, Pr1b, Pr1c, Pr2a, Pr2b and Pr3. Right and wrong or no answers to the problems were scored as 1 and 0, respectively.

Concerning the tasks of the Test A of the second phase, the first task required teachers to give a simple definition of function. Teachers' answers to this task were coded with the variable "D1". A score 1 was assigned if the teachers gave a correct response, 0 if they gave a wrong answer and 9 for an incomplete task. Furthermore, for this task we categorized teachers' definitions in five categories: "D1c1" a correct "typical" definition, "D1c2" a definition of a special kind of function (e.g. real function, function one-to-one, continuous function), "D1c3" correct reference to the relation between variables but without the definition of the domain and range, "D1c4" reference to an ambiguous relation (answers that made reference to a relation between variables or elements of sets, or a verbal or symbolic example were included in this group), “D1c5” other answers (this type of answers made reference to sets but without relation, or reference to relation without sets or elements of sets).

The second task required from teachers to give two simple examples from the applications of functions in everyday life. Teachers’ answers to this task coded with the variable “Ex1” and “Ex2”, for example 1 and example 2 respectively. A score 1 was assigned if the teachers gave a correct example, 0 if they gave an incorrect example and 9 for an incomplete task. Furthermore, for this task we categorized teachers’ examples in six categories: “Ex1c1-Ex2c1” example of a function with the use of discrete elements of sets, “Ex1c2-Ex2c2” example of a continuous function from physics, “Ex1c3-Ex2c3” example of one-to-one function, “Ex1c4-Ex2c4” example presenting an ambiguous relation between elements of sets, “Ex1c5-Ex2c5” example of an equation and “Ex1c6-Ex2c6” example presenting an uncertain transformation of the real world.

The other four tasks were simple tasks with functions the same as the ones involved in Test A_1 e.g. Task 1: In the following diagram $y = 2x$ is given. Draw the function $y = 2x + 1$. The variables “Ap1”, “Ap2”, “Ap3” and “Ap4” were used in order to code teachers’ right or wrong answers to each of the four tasks. A score 1 was assigned if the teachers gave a correct response, 0 if they gave an incorrect answer and 9 for an incomplete task. Furthermore, we categorized teachers’ responses as algebraic or coordinated. A solution was coded as “algebraic” if students did not use the information provided by the graph of the first function and they proceeded constructing the graph of the second function by finding pairs of values for x and y. On the contrary, a solution was coded as coordinated if students observed and used the relation between the two functions in constructing the graph of the second function. In this case students used and coordinated two systems of representation namely, the algebraic and the graphical one. They noticed the relationship between the two equations given and they interpreted this relationship graphically by manipulating the function as an entity. The variables “Ap1ral” and “Ap1wal” were used to code the right and wrong algebraic solutions of the first task. The variables “Ap1rco” and “Ap1wco” were used to code the right and wrong coordinated solutions to task 1. Furthermore, for these tasks we categorized students’ responses in six categories: “Ap1c1” the teachers found the points of intersection with the axes, “Ap1c2” the teachers found two random points, “Ap1c3” they made a table of values for x and y, “Ap1c4” verbal explanation only, “Ap1c4” they gave a verbal explanation but they also found one or more points and “Ap1c6” other answers.

The other three tasks were complex problems. Concerning, problem 1: "A tank contains 600 L of petrol (initial amount). A tank car is filling the tank with petrol. The tank car contains 2000 L of petrol and the rate of filling is 100 L per minute. Use the information in order to give the two equations. Draw the two graphs (the volume of the petrol in the tank as a function of time t and the volume of the petrol in the tank car as a function of time t). Find when the amounts of petrol in the tank and in the car would be equal". The variable "Pr1a" was used to code teachers' answers to the first question, "Pr1b" was used to code their answer to the second question and "Pr1c" to the third question. A score 1 was assigned if the teachers gave a correct response, 0 if they gave an incorrect answer and 9 for an incomplete task. The strategies employed by the teachers in order to solve the problem were also categorized: "Pr1s1" was used for an algebraic solution while "Pr1s2" was used if the teachers construct a table of values for x and y.

In problem 2: "In an ant colony the number of ants (A) increases according to the function: $A = t^2 + 1000$ (t = the number of days). The amount of seeds, the ants save in the colony, increases according to the function $S = 3t + 3000$ (t = the number of days). Use the information in order to draw the graphs. Find when the number of ants in the colony and the number of seeds would be equal". The variable "Pr2a" was used to code teachers' answers to the first question and "Pr2b" was used to code their answer to the second question. A score 1 was assigned if the teachers gave a correct response, 0 if they gave an incorrect answer and 9 for an incomplete task. The strategies employed by the teachers in order to solve the problem were also categorized: "Pr2s1" was used for an algebraic solution while "Pr2s2" was used if the teachers construct a table of values for x and y.

Concerning the third problem: "The function $f(x) = ax^2 + bx + c$ is given. Numbers a, b and c are real numbers and the $f(x)$ is equal to 4 when $x = 2$ and $f(x)$ is equal to -6 when $x = 7$. Find how many real solutions the equation $ax^2 + bx + c$ has and explain your answer". The variable "Pr3" was used to code teachers' correct responses. A score 1 was assigned if the teachers gave a correct response, 0 if they gave an incorrect answer and 9 for an incomplete task. The strategies employed by the teachers in order to solve the problem were also categorized into seven categories: "Pr3s1" was used for a graphical solution (They put the points in the axes and tried to draw the function), "Pr3s2" they made a reference to the positive discriminant (or the Bolzano theorem), "Pr3s3" they gave a correct solution but wrong justification (The function has two solution because is a second degree equation), "Pr3s4" they referred that there is not a solution, "Pr3s5" they referred that there is only one solution, “Pr3s6” they mentioned that there is an infinite number of solutions, “Pr3s7” they tried to solve it algebraically but this was not possible.

Concerning Test B of the second phase, Task 1 asked teachers to mention if there exists a function all of whose values are equal to each other and explain their answer. Teachers’ answers to this task coded with the variable “D2”. A score 1 was assigned if the teachers gave a correct response, 0 if they gave a wrong answer and 9 for an incomplete task. Furthermore, for this task we categorized teachers’ justifications in three categories: “D2c1” teachers gave a correct justification $y = a$, “D2c2” they gave a wrong justification $y = x$ and “D2c3” they gave other answers. Task 2 and task 8 were Yes or No tasks concerning the definition of function: Can f be a function, if $f(-2) = 3$ and $f(-2) = 0$? and Can f be a function, if $f(-2) = f(3) = 4$?. The variables “D3” and “D4” were used to code teachers’ responses to the above tasks. A score 1 was assigned if the teachers gave a correct response, 0 if they gave a wrong answer and 9 for an incomplete task.

Task 3, 9, 10 and 11 request teachers to make six conversions from an algebraic to a graphical representation of function and vice versa. The variables “Coag1”, “Coag2”, “Coag3”, “Coga1”, “Coga2”, “Coga3” were used to code teachers’ responses to the above tasks. A score 1 was assigned if the teachers gave a correct response, 0 if they gave a wrong answer and 9 for an incomplete task.

Tasks 4, 5, 6 and 7 were recognition tasks. In task 4 five Venn diagrams were given and teachers were asked to examine which of the correspondences are functions. They had to circle the right answer and give an explanation. The variables “Red1”, “Red2”, “Red3”, “Red4” and “Red5” were used to code teachers’ responses to this task . A score 1 was assigned if the teachers gave a correct response, 0 if they gave a wrong answer and 9 for an incomplete task. Teachers’ justifications were furthermore grouped according to the categories presented in Table 3.1.

Table 3.1

Teachers' categorization of justifications to the fourth task of the second test involving recognition of functions given in a diagrammatic representation (Venn diagrams)

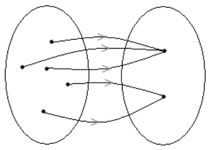
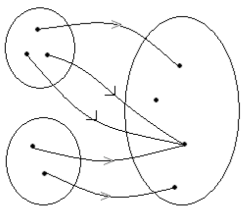
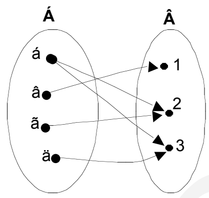
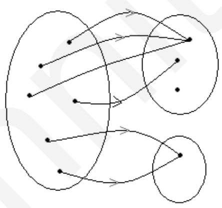
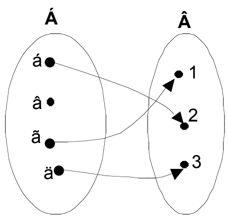
Tasks	Categories
 Red1	c1: Right justification: For every x there is a y c2: There are many x for some y c3: Other justifications c4: No justification
 Red2	c1: Right justification: For every x there is a y c2: There are many x for some y c3: There is an y without an x c4: The domain splits into two subdomains c5: Other justifications c6: No justification
 Red3	c1: Right justification: There is an x with two y c2: For every x there is a y c3: Other justifications c4: No justification
 Red4	c1: Right justification: For every x there is a y c2: There are many x for some y c3: There is an y without an x c4: The range splits into two sets c5: Other justifications c6: No justification
 Red5	c1: Right justification: There is an x without y c2: For every x there is a y c3: Other justifications c4: No justification

Table 3.2

Teachers' categorization of justifications to the fifth task of the second test involving recognition of functions given in a graphical representation

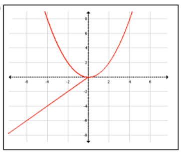
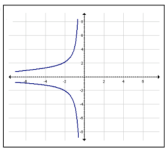
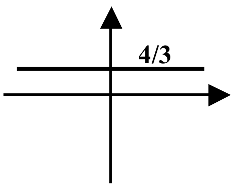
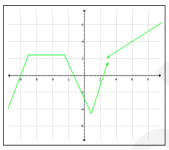
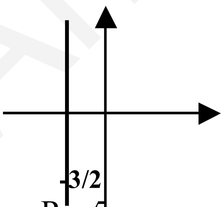
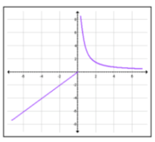
Tasks	Categories
 Reg1	c1: Right justification: For some x there are two y c2: It is a function because it is a parabola c3: Other justifications c4: No justification
 Reg2	c1: Right justification: For every x in the domain there two y c2: It is a function because it is a hyperbola c3: Other justifications c4: No justification
 Reg3	c1: Right justification: For every x there is a y c2: It is a function because it is a straight line c3: It is not a function because there is not an x in the equation c4: Other justifications c5: No justification
 Reg4	c1: Right justification: For every x there is a y c2: It is not continuous c3: Other justifications c4: No justification
 Reg5	c1: Right justification: There are many y for an x c2: It is not a function because there is not a y in the equation c3: It is a function because is a straight line c4: Other justifications c5: No justification
 Reg6	c1: Right justification: For every x there is a y c2: It is not continuous c3: Other justifications c4: No justification

Table 3.3

Teachers' categorization of justifications to the sixth task of the second test involving recognition of functions given in a symbolic expression

Tasks	Categories
$5x+3=0$ Res1	c1: Right justification: There are many y for an x c2: It is not a function because there is not a y in the expression c3: It is a function because is a straight line c4: Other justifications c5: No justification
$2x+y=0$ Res2	c1: Right justification: For every x there is a y c2: It is a function because there is an x and a y c3: Other justifications c4: No justification
$4y+1=0$ Res3	c1: Right justification: For every x there is a y c2: It is not a function because there is not an x in the expression c3: It is a function because is a straight line c4: Other justifications c5: No justification
$x^2+y^2=25$ Res4	c1: Right justification: For every x there are two y c2: It is a function because there is an x and a y c3: Other justifications c4: No justification
$x^3-y=0$ Res5	c1: Right justification: For every x there is a y c2: It is a function because there is an x and a y c3: Other justifications c4: No justification
$f(x)=\begin{cases} x & \gamma\iota\alpha \quad \chi \geq 0 \\ -x & \gamma\iota\alpha \quad \chi < 0 \end{cases}$ Res6	c1: Right justification: For every x there is a y c2: It is not continuous c3: Other justifications c4: No justification

In task 5 six graphs were given and the teachers were asked to examine which of the correspondences are functions. They had to circle the right answer and give an explanation. The variables “Reg1”, “Reg2”, “Reg3”, “Reg4”, “Reg5” and “Reg6” were used to code teachers’ responses to this task . A score 1 was assigned if the teachers gave a correct response, 0 if they gave a wrong answer and 9 for an incomplete task. Teachers’ justifications were grouped according to the categories presented in Table 3.2.

In task 6 six symbolic expressions were given and teachers were asked to examine which of the correspondences are functions. They had to circle the right answer and give an explanation. The variables “Res1”, “Res2”, “Res3”, “Res4”, “Res5” and “Res6” were used to code teachers’ responses to this task. A score 1 was assigned if the teachers gave a correct response, 0 if they gave a wrong answer and 9 for an incomplete task. Teachers’ justifications were grouped according to the categories presented in Table 3.3.

In task 7 four verbal expressions were given and teachers were asked to examine which of the correspondences are functions. They had to circle the right answer and give an explanation. The variables “Rev1”, “Rev2”, “Rev3” and “Rev4” were used to code teachers’ responses to this task. A score 1 was assigned if the teachers gave a correct response, 0 if they gave a wrong answer and 9 for an incomplete task. Teachers’ justifications were grouped according to the categories presented in Table 3.4.

Table 3.4

Teachers' categorization of justifications to the seventh task of the second test involving recognition of functions given in a verbal expression

Tasks	Categories
In the set of the girls of a class, we correspond a girl with different classmates of hers (George, Homer, Jason, Thanasis, etc.) with whom she will probably dance at a party. Rev1	c1: Right justification: For every x there are many y c2: It is a function because there is a relationship between two variables c3: Other justifications c4: No justification
In a football championship, we correspond every football game to the score achieved. Rev2	c1: Right justification: For every x there is a y c2: For one y there are many x c3: Other justifications c4: No justification
In the set of the scripts at the university entrance examinations, we correspond every script to the couple of marks given by the first and the second examiner. Rev3	c1: Right justification: For every x there is a y c2: It is not a function because one x has two y c3: Other justifications c4: No justification
In the set of the candidates for employment in different work positions in a big organization, we correspond every candidate with the posts for which she/he applies for work in the organization (candidates may apply for more than one post). Rev4	c1: Right justification: For every x there are many y c2: It is a function because there is a relationship between two variables c3: Other justifications c4: No justification

The other three tasks (12, 13, 14) were complex problems with functions. Concerning, problem 1: "A parachutist jumps from an airplane which is in 3000 m height (above the earth). The parachutist falls with stable speed 30 m/s. Express the parachutist's height as function of time. Draw the graph of the above function. Find the parachutist's height (from earth) 1 minute after his/her fall. In what height the parachutist will be 20 minutes after his/her fall? (Give an explanation)". The variable "Pr4a' was used to code teachers' answers to the first question, "Pr4b" was used to code their answer to the second question, "Pr4c" to the third question and "Pr4d" to the fourth question. A score 1 was assigned if the teachers gave a correct response, 0 if they gave an incorrect answer and 9 for an incomplete task. The strategies employed by the teachers in order to solve the problem were also categorized: "Pr4s1" was used for an algebraic solution while "Pr4s2" was used if the teachers construct a table of values for x and y.

In problem 2: "In the next figure a square with side 1 cm is given. Inside the square we draw another square A. The marked segments have the same length x. Express the area of the square A as a function of x. Draw the graph of the above function. Find the value of x for which the square A has minimum area". The variable "Pr5a' was used to code teachers' answers to the first question, "Pr5b" was used to code their answer to the second question and "Pr5c" to the third question. A score 1 was assigned if the teachers gave a correct response, 0 if they gave an incorrect answer and 9 for an incomplete task. The strategies employed by the teachers in order to solve the problem were also categorized: "Pr5s1" was used for an algebraic solution while "Pr5s2" was used if the teachers construct a table of values for x and y. Concerning the third problem: "The function $f(x) = ax^2 + bx + 7$ is given (a and b are real numbers). When $x = 0$, $f(x)$ has a maximum value. The $f(x)$ is equal to -5 when $x = 2$. Find how many real solutions the equation $f(x) = ax^2 + bx + 7$ has and explain your answer". The variable "Pr6" was used to code teachers' correct responses. A score 1 was assigned if the teachers gave a correct response, 0 if they gave an incorrect answer and 9 for an incomplete task. The strategies employed by the teachers in order to solve the third problem were also categorized into seven categories: "Pr6s1" was used for a graphical solution (They put the points in the axes and tried to draw the function), "Pr6s2" they made a reference to the positive discriminant (or the Bolzano theorem), "Pr6s3" they gave a correct solution but wrong justification (The function has two solution because is a second degree equation), "Pr6s4" they referred that there is not a solution, "Pr6s5" they referred that there is only one solution, "Pr6s6" they mentioned that there are many solutions and "Pr6s7" they tried to solved it algebraically but this was not possible.

Analysis of the data

Quantitative techniques were used for analyzing the data retrieved from the two tests. Furthermore, qualitative analysis was employed to analyze the data of the nine task-based interviews.

Quantitative analysis

SPSS Software

Descriptive analysis was performed by using SPSS. The descriptive analysis gave valuable information concerning the percentages of correct or wrong responses given by the teachers, the different strategies employed and the categorization of teachers' justifications and answers. The multivariate analysis of variance was also employed in order to determine differences between the groups of teachers (Cyprus and Italy) concerning the various dimensions of the understanding of function. Cluster analysis (Ward's method) was also performed in order to categorize the teachers into groups according to the use of the coordinated or algebraic approach.

Implicative Statistical Analysis and Hierarchical Clustering of Variables

For the analysis of the collected data, the hierarchical clustering of variables and Gras's implicative statistical method were conducted using a computer software called C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive), Version 4.2. (Bodin, Coutourier, & Gras, 2000). These methods of analysis determine the hierarchical similarity connections and the implicative relations of the variables respectively (Gras, 1992). For this research's needs, similarity and implicative diagrams were produced from the application of the analyses to each group (Cyprus and Italy) for both tests (Test A and B).

The hierarchical clustering of variables (Lerman, 1981) is a classification method which aims to identify in a set V of variables, sections of V, less and less subtle, established in an ascending manner. These sections are represented in a hierarchically constructed diagram using a similarity statistical criterion among the variables. The similarity stems from the intersection of the set V of variables with a set E of subjects (or objects). This kind of analysis allows the researcher to study and interpret clusters of variables in terms of typology and decreasing resemblance. The clusters are established in particular levels of the diagram and can be compared with others. This aggregation may be indebted to the conceptual character of every group of variables.

The construction of the hierarchical similarity diagram is based on the following process: Two of the variables that are most similar to each other with respect to the similarity indices of the method are joined together in a group at the highest (first) similarity level. Next, this group may be linked with one variable in a lower similarity level or two other variables that are combined together and establish another group at a lower level, etc. This grouping process goes on until the similarity or the cohesion between the variables or the groups of variables gets very weak. In this research the similarity diagrams will allow the arrangement of the variables, which correspond to subjects' responses in the tasks of the tests, into groups according to their homogeneity.

The implicative statistical analysis (Gras, Peter, Briand, & Philippe, 1997) aims at giving a statistical meaning to expressions like: "if we observe the variable a in a subject, then in general we observe the variable b in the same subject". Thus the underlying principle of the implicative analysis is based on the quasi-implication: "if *a* is true then *b* is more or less true". An implicative diagram represents graphically the network of the quasi-implicative relations among the variables of the set V. In this research the implicative diagrams will contain implicative relations, which will indicate whether success to a specific task implies success to another task related to the former one.

Structural Equation Modelling and CFA

"Structural equation modeling (SEM) is a statistical methodology that takes a hypothesis testing (i.e. confirmatory) approach to the multivariate analysis of a structural theory bearing on some phenomenon" (Byrne, 1994, p. 3). This theory concerns "causal" relations among multiple variables (Bentler, 1988). These relations are represented by structural, namely regression equations, which can be modeled in a pictorial way to allow a better conceptualization of the involved theory.

SEM differs from the more traditional multivariate statistical techniques in at least three dimensions: First, with the use of SEM the analysis of the data is approached in a confirmatory manner rather than in an exploratory way, making hypothesis testing more accessible and easier, compared with other multivariate procedures. Second, whereas SEM gives the estimates of measurement errors, the "conventional" multivariate methods cannot assess or correct for these parameters. Third, SEM involves not only observed but also latent (unobserved) variables, whereas the older techniques incorporate only observed measurements.

Latent and observed variables are two of the most basic concepts of SEM. Latent variables (i.e. factors) represent theoretical or abstract constructs that cannot be observed or measured directly and are rather assumed to lie behind certain observed measures. The measurement of latent variables is obtained indirectly by associating it with other variable/s that is/are observable. The latent variable is based on some behaviour supposed to represent it. This behaviour refers to scores on a particular instrument, like the test of this research, and in turn these measured scores are called observed variables.

Factor analysis is a well known statistical technique for examining associations between observed and latent variables. The covariation among a set of observed variables is investigated to get information on their underlying latent factors. Of primary interest is the strength of the regression paths from the factors to the observed variables. Exploratory Factor Analysis (EFA) and Confirmatory Factor Analysis (CFA) are two basic types of factor analysis. EFA is employed to determine how the observed variables are connected to their underlying constructs in situations where these links are unknown. By contrast, CFA, is used in situations where the researcher aims to test statistically whether a hypothesized linkage pattern between the observed variables and their underlying factors exists. This a priori hypothesis draws on knowledge of related theory and past empirical work in the area of the study. The basic steps that a researcher follows for carrying out CFA are described below: The model is specified based on knowledge of relevant theory and previous empirical research. Using a model-fitting program, such as EQS, the model is analyzed so that the estimates of the model's parameters with the data are derived. Then the tenability of the model is tested based on data that involve all the observed variables of the model. In other words, what is tested is how well the observed data fit the a priori structure. If the hypothesized model is not consistent with the data the model is respecified and the fit of the revised model with the same data is evaluated (Byrne, 1994; Kline, 1998).

The number of levels that the latent factors are away from the observed variables determines whether a factor model is called a first-order, a second-order or a higher order model. Correspondingly, factors one level removed from the observed variables are labeled first-order factors while higher-order factors which are hypothesized to account for the variance and co-variance related to the first-order factors are termed second-order factors. A second or a higher order factor does not have its own set of measured variables.

A structural equation model involves two basic types of components: the variables and the processes or relations among the variables. A schematic representation of a model, which is termed path diagram, provides a visual interpretation of the relations that are hypothesized to hold among the variables under study. The basic notation and schematic presentation of the variables and the relations that are used in path diagrams are described below.

The observed or measured variables, which constitute the actual data of the study, are often designated as Vs and are shown in rectangles. The unmeasured variables, which are hypothetical and represent the structural organization of the phenomenon under study, are of three types: a) the latent factor which is designated as F and represented in the path diagram in ellipses or circles; b) a residual associated with the measurement of each observed variable which is referred to as error and designated as E; and c) a residual or the error associated with the prediction of each factor, which is termed disturbance and designated as D. Residual terms indicate the imperfect measurement of the observed variables and the imperfect prediction of the unobserved factor. Although both kinds of residuals represent errors, the former is termed error, and the latter is referred to as disturbance, to distinguish the error in measurement from error in prediction.

One type of the relations involved in a model is the structural regression coefficients indicating the impact of one variable on another. They are represented by one-way arrows. A second type of processes used in a model is the impact of the errors in the measurement of the observed variables and in the prediction of the latent factors. The impact of random measurement errors on the observed variables and errors in the prediction of factors are represented also as one-way arrows. A third type of processes involved in a model are the covariances or correlations between pairs of variables, which are represented as curved or (sometimes) straight two-way arrows.

The purpose of using CFA in this study was to investigate the structure and the relations between the various dimensions of the understanding of function (problem solving, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic expression, verbal expression), transferring function from one mode of representation to another and coordinated or algebraic approach when dealing with simple function tasks). Bentler's (1995) EQS program was used for testing the CFA model in this study. To test for possible differences between the two groups (Cyprus and Italy) in the structure of their function understanding, multiple group analysis was applied, where the model was fitted separately on the Cypriot and Italian pre-service teachers. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI (Comparative Fit Index) and RMSEA (Root Mean Square Error of Approximation) (Bentler, 1990). The following values of the three indices are needed to support model fit: The observed values for x^2/df should be less than

2.5, the values for CFI should be higher than .9 and the RMSEA values should be lower than .06.

Qualitative analysis

Semi-structured task-based interviews were conducted with nine pre-service teachers in order to triangulate the quantitative data regarding teachers' conceptual understanding of function. The constant comparative method (Denzin & Lincoln, 1998) was used to analyze the qualitative data that emerged from the interviews. The constant comparative method of analyzing qualitative data combines inductive category coding with the simultaneous comparison of all units of meaning obtained (Glaser & Strauss, 1967). As each new unit of meaning is selected for analysis, it is compared to all other units of meaning and subsequently grouped (categorized and coded) with similar units of meaning. If there are no similar units of meaning, a new category is formed (Maykut & Morehouse, 1994). Furthermore, the interviews were analyzed vertically and horizontally. In the case of the vertical analysis we studied separately each participant's responses to the nine tasks of the interview. For this analysis three two-dimensional matrices were constructed. Matrices are the simplest, and probably most used and most useful, way of organizing the data of an interview. In a matrix there is a substantial data reduction which is derived from 8-10 pages of interview notes (Robson, 1993). In this case the rows represent the nine tasks and the columns the nine participants. A horizontal analysis was also performed, in which every task was analyzed separately and teachers' representative answers were emphasized and discussed.

Analysis of the Cypriot curriculum and mathematics books on the concept of function

In order to gain further insight to the way the concept of function is approached to Cypriot middle and high school the mathematics books used for the teaching of the concept were analyzed and the curriculum was explored. In addition, this analysis contributed to the construction of the research's instruments and to the interpretation of the results since many of the difficulties pre-service teachers' face with the concept can be attributed to the way the concept is taught and presented in the textbooks of middle and high school.

Specifically, the teaching of functions starts at the 9th grade and continuous until 12th grade. A big percent (21-30%) of the material of the curriculum deals with this concept.

At 9th grade the students are taught from the algebraic expression of a function to construct a table of values and find the points, these pairs of values represent, on x and y axes, to draw functions that have the form $y = ax$ and $y = ax + b$, to draw the straight lines $y = a$ and $x = b$ and to find the slope of a straight line by using the equation or the graph.

In the 10th grade the students deal with the concept of function more systematically. Particular, they represent a function using multiple representations such as table of values, graph, Venn diagram, algebraic expression and verbal expression and they make conversions from one representation to the other. They recognize and mention examples and non-examples of a function. They find the range and the domain of a function. They draw the graph of a linear function with the use of its equation. They find the equation and draw the graph of a linear function when they are given the slope and one or two points. In the equation $y = ax + b$ they learn the importance and meaning of a and b. They draw the graph of the function $y = \frac{a}{x}$. They draw the graph $y = x^2$ and its translations e.g. $y = ax^2$, $y = ax^2 + d$, $y = a(x-k)^2$. They deal with the function $y = ax^2 + bx + c$.

In the 11th grade the find the domain of several functions. They deal with the equality and operations between functions, and the synthesis of different functions. They define and recognize the one-to-one function. They are taught the limit of a function.

In the 12th grade they deal with functions that are defined parametrically, they are taught the derivative of a function (how to find it and its applications) and the differential of a function.

From the above brief analysis of the material included in the curriculum of mathematics it is asserted that the emphasis on the concept of function is given in 10th grade. The tests of this study were constructed taking into consideration the above material.

Furthermore, we analyzed the modes of representation used in the examples and exercises included in the books of the 9th, 10th, 11th and 12th grade concerning the concept of function. Furthermore, we pinpointed the functions of the different modes of representation on functions in mathematics textbooks. Lesh, Post and Behr (1987b) and

Hitt (1998) proposed ways of analyzing the understanding of a concept that formed a basis for determining the functions of the observed representations.

Table 3.5 presents the percentages of function representation modes in the mathematics textbooks of 9th, 10th 11th and 12th grade. The sum of percentage for each grade in some cases exceeds 100%, because some exercises included more than one category. It is obvious that in all grades the graphical and the symbolic representations are used more than the other modes of representing functions. Following is the verbal expression, the diagrammatic representation and the table with smaller percentages. The fact is noteworthy that the percentages of representations are higher for all the modes of representation in 9th grade. In general, the representations decline as we move to higher grades.

Table 3.5

Percentages of the function representations modes in mathematics textbooks of 9th, 10th, 11th and 12th grade

Grade	Verbal	Symbolic	Diagrammatic	Table	Graphical
9th grade	8.1%	83.5%	1.4%	0.3%	85.8%
10th grade	4.3%	9.6%	0.9%	0.5%	11.3%
11th grade	0.1%	2.8%	0.3%	0%	2.9%
12th grade	0.1%	1.5%	0.1%	0%	2.5%

Tables 3.6 to 3.9 present the percentages of the various function representations' functions from the 9th to 12th grade.

The results in Table 3.6 show that verbal representations used in the textbooks of 9th grade are involved mostly in conversion and treatment tasks. A great percentage of symbolic representations are used in treatment tasks and a smaller percentage in problem solving and recognition tasks. As far as diagrammatic representations are concerned, Table 3.6 shows that are mostly used in problem solving tasks and equally in conversion and treatment tasks. The table appears equally in recognition, treatment and conversion tasks. The graphical representation mostly appears in treatment tasks and less frequently in conversion tasks.

Table 3.6

Percentages of the function representations' functions in 9th grade mathematics textbook

Functions	Verbal	Symbolic	Diagrammatic	Table	Graphical
Recognition	0.5%	7.2%	0.1%	0.1%	4.7%
Treatment	1.6%	63.8%	0.4%	0.1%	73%
Conversion	5%	2.8%	0.4%	0.1%	6%
Problem Solving	1%	10%	0.6%	0%	2.1%

Table 3.7, which addresses the functions exercises in the textbooks of the 10th grade, shows that the verbal representations are involved mostly in conversion and treatment tasks. Symbolic representations are involved primarily in tasks of treatment and less frequently in conversion tasks. Diagrammatic representations are more frequently included in conversion and treatment tasks. The table appears either in conversion tasks or problem solving. The graphical representation is mostly used in treatment and conversion tasks.

Table 3.7

Percentages of the function representations' functions in 10th grade mathematics textbooks

Functions	Verbal	Symbolic	Diagrammatic	Table	Graphical
Recognition	0.3%	0.3%	0.1%	0%	0.8%
Treatment	1.6%	7.2%	0.2%	0%	6.5%
Conversion	2.3%	2.5%	0.5%	0.4%	2.9%
Problem Solving	0.1%	0%	0.1%	0.1%	1.1%

According to Table 3.8, at 11th grade the verbal representations are used exclusively in problem solving. Symbolic representations are involved in conversion tasks and problem solving. Diagrammatic representations are involved in conversion and recognition tasks. The graphical representations are used mainly in conversion tasks and problem solving and less frequently in recognition tasks.

Table 3.8

Percentages of the function representations' functions in 11th grade mathematics textbooks

Functions	Verbal	Symbolic	Diagrammatic	Table	Graphical
Recognition	0%	0%	0.1%	---	0.1%
Treatment	0%	0%	0%	---	0%
Conversion	0%	1.3%	0.2%	---	1.3%
Problem Solving	0.1%	1.5%	0%	---	1.5%

Table 3.9 shows that verbal and symbolic representations used in 12th grade are included exclusively in problem solving. The diagrammatic representations are included in conversion tasks while the graphical representations are involved in conversion tasks and problem solving.

Table 3.9

Percentages of the function representations' functions in 12th grade mathematics textbooks

Functions	Verbal	Symbolic	Diagrammatic	Table	Graphical
Recognition	0%	0%	0%	---	0%
Treatment	0%	0%	0%	---	0%
Conversion	0%	0%	0.1%	---	1%
Problem Solving	0.1%	1.5%	0%	---	1.5%

The findings of this brief analysis of the mathematics textbooks of 9th, 10th, 11th and 12th grades confirm that there is a noteworthy difference between the four grades concerning the representations used. In the 9th grade a plurality of representations is included in the mathematics textbooks. The amount of representations is decreased as we move through the secondary education. In all the grades symbolic and graphical representations are the main modes of representation. In 11th and 12th grade the tabular representation is totally absent. Concerning the function the various representation fulfill, it is observed that in all grades they are mainly involved in conversion and treatment tasks and less frequently in problem solving.

Summary

The present research study was conducted in three phases. The goal of the first phase was to contribute to the understanding of the algebraic and coordinated approaches pre-service teachers develop and use in solving function tasks and to examine which approach is more correlated with their ability in solving complex problems. Furthermore an important goal of this phase was to investigate the stability of these approaches and the stability of their relation. In the first phase four groups of pre-service teachers participated. Particularly, the data of the first group (Group A) were collected in 2005 and the participants were 135 Cypriot pre-service teachers. The data of the second group (Group B) were collected two years later, in 2007 and the participants were 153 Cypriot pre-service teachers. The data of the third group (Group C) were collected in 2008 and the participants were 260 Cypriot pre-service teachers. Finally, the data of the fourth group (Group D) were collected in 2009 and the participants were 200 Italian pre-service teachers. A test consisted of seven tasks was given to the participants.

The goal of the second phase was to explore teachers' display of behavior, cognitive structures and performance in six aspects of the understanding of function: effectiveness in solving complex problems with functions, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic and verbal expression), transferring function from one mode of representation to another and the approach when dealing with simple function tasks. A main concern was also to examine problem solving in relation to the other types of displayed behavior. The participants of the second phase were 279 Cypriot and 206 Italian pre-service teachers. There were given two tests consisted of nine and fourteen tasks, respectively.

The goal of the third phase was to triangulate the quantitative data regarding teachers' understanding of the concept of function and to further investigate pre-service teachers' behavior in the above mentioned aspects of the understanding of function: concept definition, examples of function, the recognition of functions given in various representations, "coordinated" or algebraic approaches when dealing with simple function tasks and effectiveness in problem solving. In the third phase nine of the second phase's participants were chosen for task-based interviews, which were consisted of nine tasks.

Descriptive analysis was performed by using SPSS. The descriptive analysis gave valuable information concerning the percentages of correct or wrong responses given by the teachers, the different strategies employed and the categorization of teachers'

justifications and answers. The multivariate analysis of variance was also employed in order to determine differences between the groups of teachers (Cyprus and Italy) concerning the various dimensions of the understanding of function. Cluster analysis (Ward's method) was also performed in order to categorize the teachers into groups according to the use of the coordinated or algebraic approach. For this research's needs, similarity and implicative diagrams produced-by using a computer software called C.H.I.C.-from the application of the analyses on each group (Cyprus and Italy) for both tests (Test A and B). In addition CFA was performed in order to investigate the structure and the relations between the various dimensions of the understanding of function (problem solving, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic expression, verbal expression), transferring function from one mode of representation to another and coordinated or algebraic approach when dealing with simple function tasks). Furthermore, the constant comparative method (Denzin & Lincoln, 1998) was used to analyze the qualitative data that emerged from the interviews.

CHAPTER IV

RESULTS

Introduction

The results of the data analysis are presented in this chapter. The analysis of the data is focused on answering the research questions presented in chapter one. Descriptive analysis was performed by using SPSS. Furthermore, the similarity statistical method (Lerman, 1981) was conducted using a computer software called C.H.I.C. (Bodin et al., 2000) as well as the confirmatory factor analysis and specifically Bentler's (1995) EQS programme. The results are presented separately for each phase. The goal of the first phase was to contribute to the understanding of the algebraic and coordinated approaches pre-service teachers develop and use in solving function tasks and to examine which approach is more correlated with their ability in solving complex problems. Furthermore, an important goal of this phase was to investigate the stability of these approaches and the stability of their relation. Similarity and implicative diagrams of teachers' responses were constructed in order to indicate and establish the long-lasting relation between the coordinated approach and problem solving. In addition, a structural model is presented that further indicates the interrelations between the two approaches (algebraic and coordinated) and problem solving. The performance of the pre-service teachers in the tasks of the test and in the dimensions of the model are presented. Furthermore, the pre-service teachers are divided into groups according to the approach they used and the performance of each group in problem solving is investigated.

The goal of the second phase was to explore teachers' display of behavior, cognitive structure and performance in six aspects of the understanding of function: effectiveness in solving complex problems with functions, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic expression, verbal expression), transferring function from one mode of representation to another and the approach when dealing with simple function tasks. A main concern was also to examine problem solving in relation to the other types of displayed behavior. Cypriot and Italian pre-service teachers' performance in the tasks involving a definition and examples of the concept of function, recognition of functions given in different representations, conversions from an algebraic representation to a graphical representation of function and vice versa and problem solving is explored. Cypriot and Italian pre-service teachers' conceptions of function on the basis of their concept definitions, of the examples of the notion and of the justifications given in the tasks involving the recognition of function in various representations are discussed. The approach (algebraic or coordinated) Cypriot and Italian pre-service teachers prefer to use when they solve simple function tasks is explored. Cypriot and Italian pre-service teachers' behavior during the solution of tasks involving the definition, the examples, the recognition, the conversions, the approaches and problem solving of function is presented, and differences that exist between them are highlighted. A structural model is presented which determines and documents the importance of multiple representational flexibility and problem solving ability in the conceptual understanding of function. The various dimensions of the multiple representational flexibility and problem solving ability is furthermore examined. Particularly, multiple representational flexibility is a multidimensional concept that involves the concept image which consists of the definition and examples of a concept, the recognition of the concept given in various representations (graphical and diagrammatic representation, symbolic and verbal expression) and the conversions from an algebraic to a graphical representation and vice versa. Furthermore, the role of the coordinated approach and problem solving tasks in the conceptual understanding of function and their relation with the problem solving ability is examined. It is also examined if this structure is the same for the Cypriot and Italian pre-service teachers despite the differences that exist in the curriculum of the two countries. Furthermore, the relation of the coordinated approach with the other dimensions of the conceptual understanding of function is presented.

The goal of the third phase was the triangulation of the quantitative data regarding teachers' understanding of the concept of function and to further investigate pre-service teachers' behavior in several aspects of the understanding of function: concept definition, examples of function, the recognition of functions given in various representations, coordinated or algebraic approaches when dealing with simple function tasks and effectiveness in problem solving. The constant comparative method (Denzin & Lincoln, 1998) was used to analyze the qualitative data that emerged from the interviews. The vertical and horizontal analysis of the nine interviews conducted in this phase is presented and the participants' conceptions, misunderstandings and ideas concerning the concept of function are highlighted.

The results of the first phase

The goal of the first phase was to contribute to the understanding of the algebraic and coordinated approaches pre-service teachers develop and use in solving function tasks and to examine which approach is more correlated with their ability in solving complex problems. Furthermore, an important goal of this phase was to investigate the stability of these approaches and the stability of their relation.

In the first phase four groups of pre-service teachers participated. Particularly, the data of the first group (Group A) were collected in 2005 and the participants were 135 Cypriot pre-service teachers. The data of the second group (Group B) were collected two years later, in 2007 and the participants were 153 Cypriot pre-service teachers. The data of the third group (Group C) were collected in 2008 and the participants were 260 Cypriot pre-service teachers. Finally, the data of the fourth group (Group D) were collected in 2009 and the participants were 200 Italian pre-service teachers. A test (Test A_1) consisting of seven tasks, four simple function tasks and three complex problems, was given.

Descriptive analysis was performed by using SPSS. The descriptive analysis gave valuable information concerning the percentages of correct or wrong responses given by the teachers in the seven tasks of the test as well as the coordinated or algebraic solutions given. The multivariate analysis of variance was also employed in order to determine differences between the four groups of teachers concerning the coordinated, algebraic approaches and problem solving. Cluster analysis (Ward’s method) was also performed in order to categorize the teachers into three groups (coordinated, algebraic and various approaches groups) according to the use of the coordinated or algebraic approach. In order to examine whether there are statistically significant differences between the three groups (coordinated, algebraic and various approaches) concerning their problem solving ability the univariate analysis of variance (ANOVA) and Scheffe test were employed. In addition, four similarity and four implicative diagrams (one for each phase) are presented in order to investigate the relation between the two approaches and problem solving. The interrelations between the coordinated and algebraic approaches and problem solving is further investigated with confirmatory factor analysis.

Descriptive analysis: Pre-service teachers' algebraic and coordinated approaches and performance in problem solving

Table 4.1, shows Cypriot (Groups A, B and C) and Italian (Group D) pre-service teachers' responses to the first four tasks of the test A_1. According to Table 4.1, most of the pre-service teachers, in the four groups, correctly solved Task 1 and 2. Task 1 involved a linear function ($y = 2x$) and Task 2 the simplest form of an equation of a parabola ($y = x^2$). Their achievement radically reduced in tasks involving "complex" quadratic functions (T3 and T4).

Table 4.1

Pre-service teachers' responses to the first four tasks (Groups A, B, C and D)

Tasks (%)		Algebraic approach with correct answer	Coordinated approach with correct answer	Incorrect answer
1	A	54.8	32.5	12.7
	B	56.2	22.2	21.6
	C	55.8	20.4	23.8
	D	33	41.3	25.7
2	A	54.8	31.1	14.1
	B	56.9	25.5	17.6
	C	45.4	24.6	30
	D	21.4	33.5	45.1
3	A	56.3	17.7	26
	B	43.8	15	41.2
	C	44.2	11.2	44.6
	D	20.4	20.9	58.7
4	A	24.4	48.1	27.5
	B	24.8	47.1	28.1
	C	29.2	43.1	27.7
	D	16	33	51

The majority of Cypriot teachers (Groups A, B and C) chose an algebraic approach to solve the first three tasks. In Task 4 most of the Cypriot teachers used a coordinated approach. In this task a coordinated approach seemed easier and more efficient and as a result the percent of Cypriot teachers who used this approach was higher in comparison with the other three tasks. In general, the algebraic solution was predominant in the answers of the Cypriot teachers. Almost a third of the Italian pre-service teachers used an algebraic approach and another third used a coordinated approach. In general, the Italian pre-service teachers gave more incorrect responses than the Cypriot pre-service teachers, used more the coordinated approach and less the algebraic.

In the case of Task 1 ($y = 2x$, $y = 2x + 1$), some pre-service teachers who used an algebraic approach found the points of intersection with x and y axes and constructed the graph. Others constructed a table of values in order to help them construct the graph. The pre-service teachers who used a coordinated approach compared the two equations and mentioned that the slope was the same and the two functions are parallel. Then they referred to the fact that the points of the second function are "one more" than the points of the other. Some of them found a point in order to verify their assertion.

In the case of Tasks 2 ($y = x^2$, $y = x^2 - 1$) and 3 ($y = x^2 + 3x$, $y = x^2 + 3x + 2$), pre-service teachers who used an algebraic approach found the real solutions of the second equation and the minimum point and constructed the graph without using the first graph. In contrast, pre-service teachers who used a coordinated approach first compared the two equations and realized that they are parallel. Then they mentioned that the minimum point in the first case is "one down" and in the second case "two above". Some of them found another point in order to draw the graph more precisely.

In the case of Task 4 ($y = 3x^2 + 2x + 1$, $y = -(3x^2 + 2x + 1)$), the pre-service teachers who used an algebraic approach found the point of intersection with y-axis and the maximum point. The students who used a coordinated approach compared the two equations and mentioned that the two functions are "opposite" and "symmetrical" to the x-axis. In this task, an algebraic approach was more complicated due to the fact that the equation does not have real solutions. Most of the students, after unsuccessful efforts to find the points of intersection with x-axis drew the graph using a coordinated approach.

Table 4.2

Pre-service teachers' responses to the three problems (Groups A, B, C and D)

Problems (%)		Correct answer	Incorrect answer
1a	A	38.5	61.5
	B	22.9	77.1
	C	42.7	57.3
	D	30.6	69.4
1b	A	59.3	40.7
	B	45.8	54.2
	C	35.4	64.6
	D	28.2	71.8
1c	A	70.4	29.6
	B	55.6	44.4
	C	37.7	62.3
	D	26.2	73.8
2a	A	46.7	53.3
	B	35.3	64.7
	C	32.3	67.7
	D	25.2	74.8
2b	A	35.6	64.4
	B	27.5	72.5
	C	28.5	71.5
	D	29.1	70.9
3	A	37	63
	B	20.3	79.7
	C	22.7	77.3
	D	12.6	87.4

Table 4.2 shows pre-service teachers' performance to the three complex problems.

Pre-service teachers' performance was moderate. In Problem 1 only 38.5% of the group A, 22.9% of the group B, 42.7% of the group C and 30.6% of the group D pre-service teachers managed to use the information given in order to give the two equations. Furthermore, higher percentage of the pre-service teachers participating in groups A and B constructed the two graphs correctly (59.3% and 45.8 % respectively) and found their point of intersection (70.4% and 55.6% respectively). Many pre-service teachers participating in these two groups were unable to give the equations but managed to construct the graphs by constructing a table of values for x and y. Some of them did not construct the graphs but found their point of intersection by using the table of values. Concerning the pre-service teachers participating in group C and D smaller percentages managed to construct the graphs (35.4% and 28.2% respectively) and find their point of intersection (37.7% and 26.2% respectively).

In Problem 2 only 46.7% of the group A, 35.3% of the group B, 32.3% of the group C and 25.2% of the group D pre-service teachers managed to construct the graphs. Furthermore, quite small percentage (35.6%, 27.5% , 28.5% and 29.1%) found their point of intersection. In this problem in order to find the point of intersection the pre-service teachers had to solve a second degree equation and that caused difficulties.

Problem 3 was quite difficult for the pre-service teachers participating in the four groups since only 37%, 20.3%, 22.7% and 12.6% respectively managed to solve it correctly.

In general, the Cypriot teachers participating in group A performed better than the teachers of the group B and C in problem solving. This difference is probably due to the fact that the data of group A were collected in 2005 and the participants attended a different kind of high school with different mathematics textbooks and processes for the selection of courses. In general, the Cypriot pre-service teachers performed slightly better than the Italian teachers in problem solving. It is also noteworthy that the participants' performance was radically reduced in problem solving in comparison with the simple function tasks.

Implicative analysis: The relation of the coordinated, algebraic approaches and problem solving

Similarity diagrams

Pre-service teachers'-participating in groups A, B, C and D- correct responses to the tasks and problems are presented in the similarity diagrams in Figure 4.1, 4.2, 4.3 and 4.4 respectively. The similarity analysis (Lerman, 1981) is a classification method which aims to identify in a set V of variables, thicker and thicker partitions of V, established in an ascending manner. These partitions, when fit together are represented in a hierarchically constructed diagram using a similarity statistical criterion among the variables. The similarity is defined by the cross-comparison between a group V of the variables and a group E of the individuals (or objects). This kind of analysis allows for the researcher to study and interpret in terms of typology and decreasing similarity, clusters of variables which are established at particular levels of the diagram and can be opposed to others, in the same levels.

The construction of the similarity diagram is based on the following process: Two of the variables that are the most similar to each other with respect to the similarity indices of the method are joined together in a group at the highest (first) similarity level. Next, this group may be linked with one variable in a lower similarity level or two other variables that are combined together and establish another group at a lower level, etc. This grouping process goes on until the similarity or the cohesion between the variables or the groups of variables gets very weak.

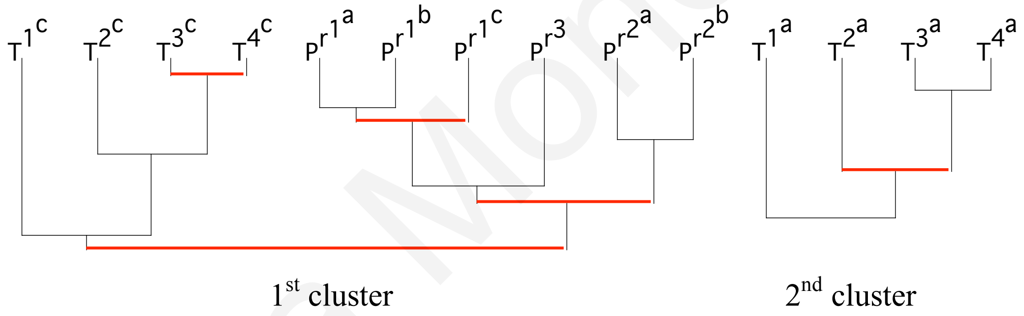


Figure 4.1. Similarity diagram of pre-service teachers' participating in group A responses

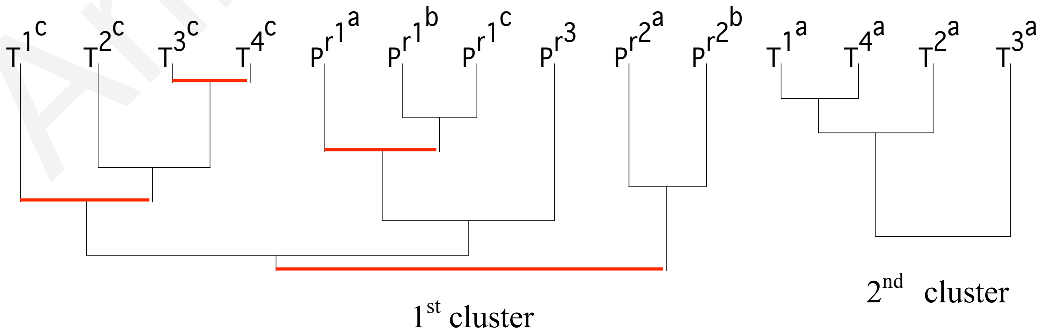


Figure 4.2. Similarity diagram of pre-service teachers' participating in group B responses

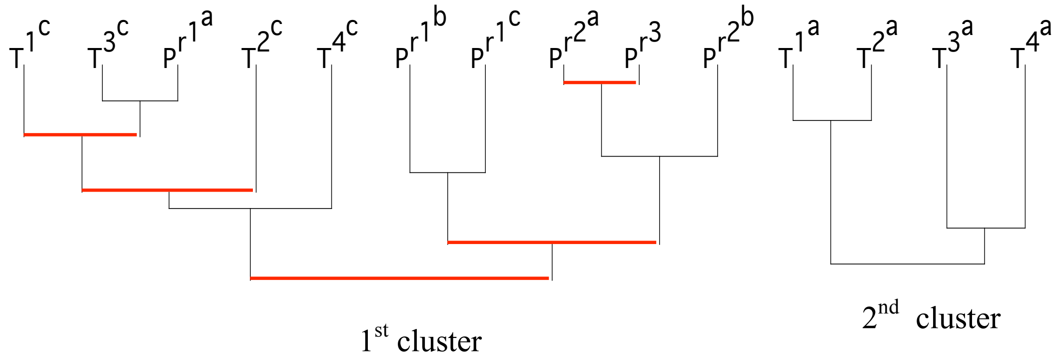


Figure 4.3. Similarity diagram of pre-service teachers' participating in group C responses

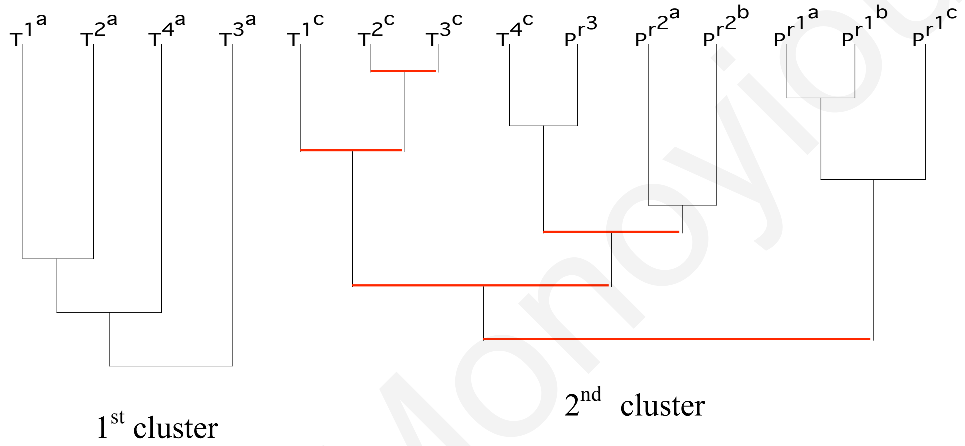


Figure 4.4. Similarity diagram of Italian pre-service teachers' (Group D) responses

The four diagrams are quite similar. More specifically in all the diagrams, two clusters (i.e., groups of variables) can be distinctively identified. The first cluster in Figure 4.1, 4.2 and 4.3 and the second cluster in Figure 4.4 consist of the variables "T1c", T2c", "T3c", "T4c", "Pr1a", "Pr1b", "Pr1c", "Pr2a", "Pr2b" and "Pr3" and refer to the use of the coordinated approach and the problem solving. The second cluster in Figure 4.1, 4.2 and 4.3 and the first cluster in Figure 4.4 consists of the variables "T1a", "T2a", "T3a" and "T4a" which represent the use of algebraic approach. From the similarity diagrams it can be observed that the one cluster includes the variables corresponding to the solution of the complex problems with the variables representing the coordinated approach.

More specifically, students' coordinated approach to simple tasks in functions is closely related with the effectiveness in solving problems. This close connection may indicate that students, who can use effectively different types of representation- in this situation both algebraic and graphical representation- and therefore have a coordinated approach are able to observe the connections and relations in problems, and are more capable in problem solving. It is noteworthy that the similarity clusters presented in the four diagrams are almost the same indicating that the connections and relationships between the approaches and problem solving are very strong and long-lasting.

Implicative diagrams

Figure 4.5 illustrates the implicative diagrams of the pre-service teachers' responses participating in Group A, B, C and D respectively. The implicative statistical analysis (Gras et al., 1997) aims at giving a statistical meaning to expressions like: "if we observe the variable a in a subject of a set E, then in general we observe the variable b in the same subject". Thus, considering that the strict logical implication is rarely fulfilled in natural, human or life sciences situations, the underlying principle of the implicative analysis is based on the following quasi-implication: "if a is true then b is more or less true". A rule is semantically related to this quasi-implication, which is a kind of theorem connecting a premise to a conclusion. An implicative diagram represents graphically the network of the quasi-implicative relations among the variables of a set.

The results of the implicative analysis are in line with the similarity relations explained above. Two separate "chains" of implicative relations among the variables are formed. The two groups of implications correspond to the two similarity clusters of the diagrams presented above. The one group of implicative relations involves the variables concerning the use of algebraic approach (T1a, T2a, T3a, and T4a). The other group of implicative relations refers to variables concerning the use of the coordinated approach and variables concerning solution to the problems (T1c, T2c, T3c, T4c, Pr1a, Pr1b, Pr1c, Pr2a, Pr2b, and Pr3). The first implicative diagram indicates that the pre-service teachers of Group A who used a coordinated approach to solve the Task 1, 2 and 3 and succeeded in those tasks also solved correctly the three problems. The teachers of Group B and D who solved correctly Problem 3 and used a coordinate approach to solve Task 3 also solved correctly the other problems and used the coordinated approach in the other three tasks.

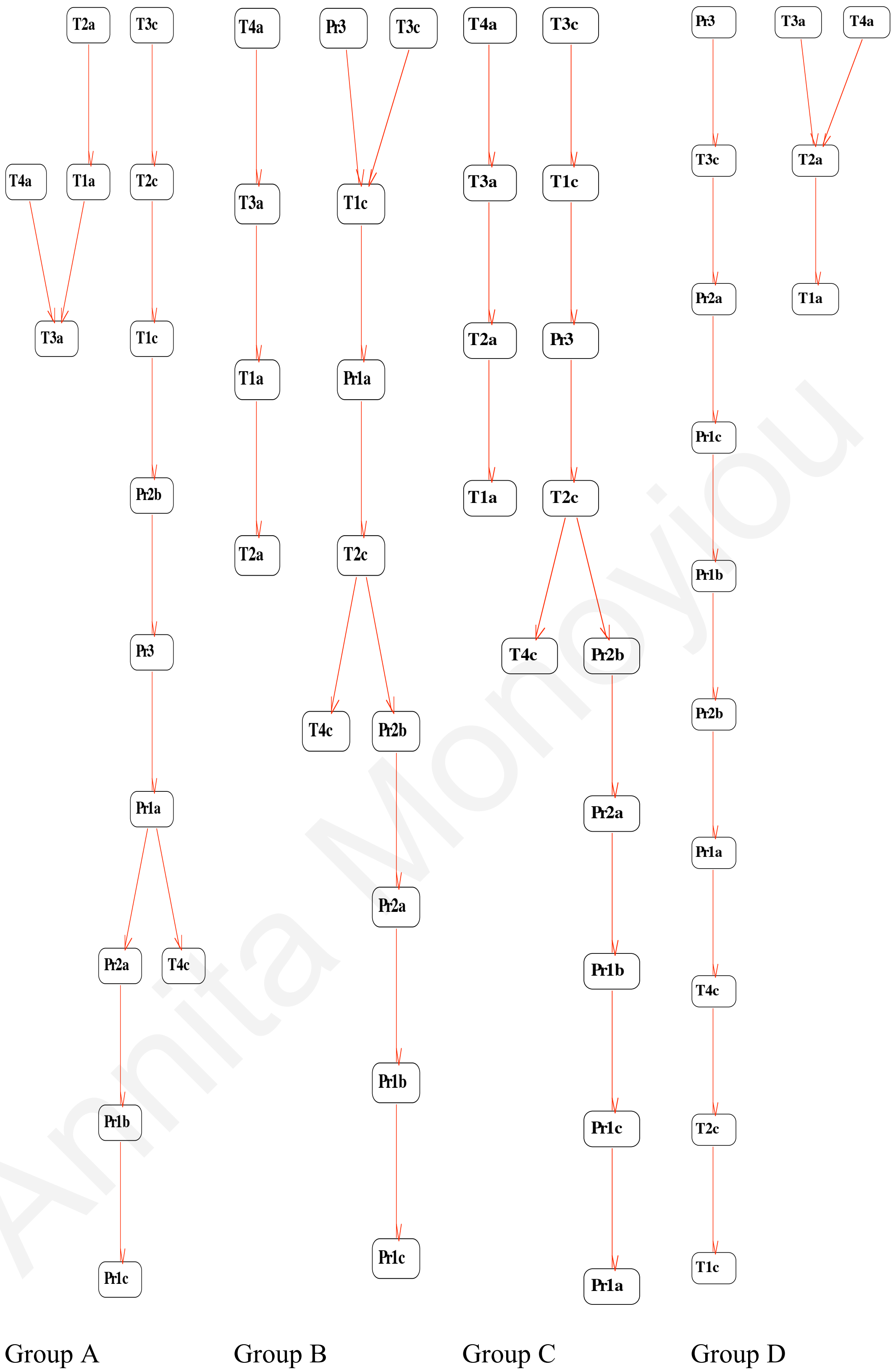


Figure 4.5. Implicative diagrams of the pre-service teachers' responses participating in Group A, B, C and D respectively

Concerning the teachers of Group C those who used a coordinated approach in Task 1 and 3 also solved correctly the three complex problems. According to the above diagrams, teachers who can coordinate two systems of representation and flexibly move from the one to the other, have a solid and coherent understanding of functions and therefore are able to solve complex problems.

Confirmatory factor analysis: A structural model indicating the interrelations between the coordinated, algebraic approaches and problem solving

Confirmatory factor analysis was used to explore the structural organization and iterrelations between the algebraic, coordinated approaches and problem solving. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI (Comparative Fit Index) and RMSEA (Root Mean Square Error of Approximation) (Bentler, 1990).

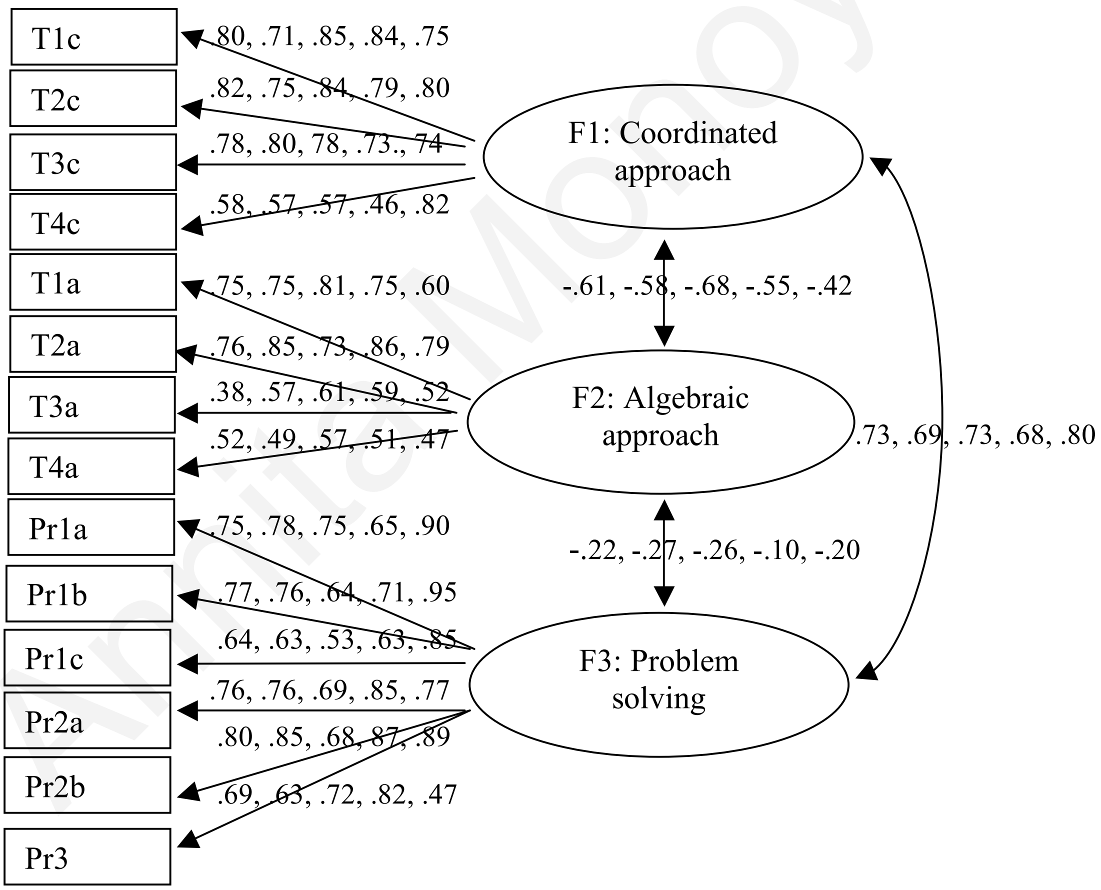


Figure 4.6. The confirmatory factor analysis model accounting for performance on the tasks by the whole sample and groups A, B, C and D separately

Note: The first, second, third, fourth and fifth coefficients of each factor stand for the application of the model on the performance of the whole sample, pre-service teachers of the group A, B, C and D respectively.

The following values of the three indices are needed to hold true for supporting an adequate fit of the model: the observed values for x^2/df should be less than 2.5, the values for CFI should be higher than 0.9 and the RMSEA values should be lower than 0.06.

A series of models were tested and compared. Specifically, the first model involved three first-order factors representing the coordinated approach, the algebraic approach and problem solving and one second order factor on which all of the first-order factors were regressed. The fit of this model was not satisfactory [x^2 (56) =195.17; CFI=0.98; RMSEA=0.057, 90% confidence interval for RMSEA=0.049-0.066].

The second model (see Figure 4.6) involved three first-order factors that are intercorrelated. The fit of this model was very good [x^2 (52) =126.4; CFI=0.99; RMSEA=0.044, 90% confidence interval for RMSEA=0.034-0.053]. Therefore, it is suggested that the two approaches and problem solving are intercorrelated.

To test for possible differences between the four groups in the structure described above, multiple group analysis was applied, where the model was fitted separately on the participants of each group. The model was first tested under the assumption that the relations of the observed variables to the first-order factors would be equal across the four groups. Although the fit of this model was acceptable, [x^2 (290) =487.94; CFI=0.97; RMSEA=0.060, 90% confidence interval for RMSEA=0.051-0.069], some of the equality constraints were found not to hold.

As a result, these constraints were released. Releasing the constraints resulted in an improvement of the model fit [x^2 (272) =382.97; CFI=0.98; RMSEA=0.047, 90% confidence interval for RMSEA=0.035-0.057].

It is noteworthy that the relation of the first-order factor standing for the coordinated approach (F1) with the first-order factor standing for problem solving (F3) is very strong. Attention is also drawn to the fact that the first-order factor standing for the algebraic approach (F2) is negatively related with the coordinated approach (F1) and problem solving (F3).

In order to examine whether there are statistically significant differences between the teachers of groups A, B, C and D concerning the approach they used and their problem solving ability, a multivariate analysis of variance (MANOVA) was performed. Furthermore, the Scheffe test was employed. Overall, the effects of teachers' phase were significant (Pillai's $F_{(9, 2250)} = 10.43$, $p<0.001$).

Table 4.3 presents the mean and standard deviation of the algebraic, coordinated approach and problem solving of the four groups. Particularly, there were significant differences between the four groups concerning the effectiveness in the algebraic approach ($F_{(3, 748)} = 20.6$, $p<0.001$) and problem solving ($F_{(3, 748)} = 10.3$, $p<0.001$).

Specifically concerning the algebraic approach, significant differences were found between the first and the fourth group ($\overline{X}_1-\overline{X}_4=.25$, $p<0.001$), the second and the fourth group ($\overline{X}_2-\overline{X}_4=.22$, $p<0.001$) and the third and fourth group ($\overline{X}_3-\overline{X}_4=.21$, $p<0.001$). Concerning problem solving significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.13$, $p<0.05$), the first and third group ($\overline{X}_1-\overline{X}_3=.15$, $p<0.01$) and the first and fourth group ($\overline{X}_1-\overline{X}_4=.23$, $p<0.001$).

There were not statistically significant differences between the teachers of group A, B, C and D concerning the coordinated approach ($F_{(3, 748)} = 2.3$, $p=0.08$). As far as the coordinated approach is concerned the teachers of group A ($\overline{X}=0.32$, SD=0.35) and D ($\overline{X}=0.32$, SD=0.39) used it more often than the teachers of group B ($\overline{X}=0.27$, SD=0.35) and C ($\overline{X}=0.25$, SD=0.32), although this difference was not statistically significant.

Table 4.3

The mean and standard deviation of the algebraic, coordinated approach and problem solving for the four groups (Groups A, B, C and D)

Groups	Algebraic approach		Coordinated approach		Problem solving	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
A	0.48	0.36	0.32	0.35	0.48	0.37
B	0.45	0.37	0.27	0.35	0.35	0.34
C	0.44	0.38	0.25	0.32	0.33	0.38
D	0.23	0.29	0.32	0.39	0.25	0.37

It is noteworthy that overall the Cypriot pre-service teachers used more often the algebraic approach than the coordinated in order to solve the simple function tasks. In contrast, the Italian pre-service teachers used more often the coordinated approach even though this difference was not statistically significant.

The relation of the coordinated approach with problem solving

In order to examine whether the pre-service teachers of the four groups who used a coordinated approach to solve the simple functions tasks performed better in problem solving, Ward's method of hierarchical cluster analysis was used. The teachers were clustered into three distinct groups. In the first group 159 teachers were clustered who used systematically the coordinated approach ($\overline{X}_{coordinated}$=0.90, $SD_{coordinated}$=0.12; $\overline{X}_{algebraic}$=0.07, $SD_{algebraic}$=0.11). In the second group 241 teachers were clustered who used extensively the algebraic approach ($\overline{X}_{coordinated}$=0.08, $SD_{coordinated}$=0.12; $\overline{X}_{algebraic}$=0.86, $SD_{algebraic}$=0.12). In the third group 354 teachers were clustered who used other approaches or used equally the algebraic and coordinated approach ($\overline{X}_{coordinated}$=0.15, $SD_{coordinated}$=0.19; $\overline{X}_{algebraic}$=0.22, $SD_{algebraic}$=0.21).

In order to examine whether there are statistically significant differences between the three groups (coordinated, algebraic and various approaches) concerning their problem solving ability the univariate analysis of variance (ANOVA) and Scheffe test were performed.

Table 4.4

The mean and standard deviation of the problem solving for the three groups

Groups	Problem solving	
	$\overline{X}$	SD
1: Coordinated approach	0.74	0.31
2: Algebraic approach	0.32	0.32
3: Various approaches	0.17	0.29

Overall, the effects of teachers' group were significant (Pillai's $F_{(2, 751)}$ = 188.56, $p<0.001$). Table 4.4 presents the mean and standard deviation of problem solving for the three groups of teachers. Specifically, significant differences were found between the first and second group ($\overline{X}_1$-$\overline{X}_2$=.42, $p<0.001$), the first and the third group ($\overline{X}_1$-$\overline{X}_3$=.57, $p<0.001$) and the second and third group ($\overline{X}_2$-$\overline{X}_3$=.15, $p<0.001$). In general the pre-service teachers who used the coordinated approach had better performance than the algebraic and various approaches groups in problem solving.

The results of the second phase

The goal of the second phase was to explore teachers' display of behavior, cognitive structure and performance in six aspects of the understanding of function: effectiveness in solving complex problems with functions, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic expression, verbal expression), transferring functions from one mode of representation to another and the algebraic or coordinated approaches when dealing with simple function tasks. A main concern was also to examine problem solving in relation to the other types of displayed behavior.

The participants of the second phase were 279 Cypriot and 206 Italian pre-service teachers. Two tests (Test A_2 and Test B) were given. The first test consisted of nine tasks: a definition task, a task requiring examples of the concept, four simple functions tasks and three complex problems. The second test consisted of fourteen tasks: three definition tasks, four tasks involving the recognition of functions given in various representations, four tasks involving conversions from a graphical to an algebraic representation of function and vice versa and three complex problems.

Descriptive analysis was performed by using SPSS. The descriptive analysis gave valuable information concerning the percentages of correct or wrong responses given by the teachers, the algebraic and coordinated solutions given, the different strategies employed and the categorization of teachers' justifications and answers. The multivariate analysis of variance was also employed in order to determine differences between the Cypriot and Italian pre-service teachers concerning the various dimensions of the understanding of function. Cluster analysis (Ward's method) was also performed in order to categorize Cypriot and Italian teachers into three groups (coordinated, algebraic and various approaches groups) according to the use of the coordinated or algebraic approach. In order to examine whether there are statistically significant differences between the three groups (coordinated, algebraic and various approaches) concerning their performance in the other dimensions of the understanding of function the multivariate analysis of variance (MANOVA) and Scheffe test were employed. In addition, similarity and implicative diagrams were constructed in order to investigate Cypriot and Italian pre-service teachers' behaviour in the various dimensions of the understanding of function. The interrelations between the several aspects investigated in this study and their relation with the conceptual understanding of function is further explored with the confirmatory factor analysis.

Descriptive analysis: Cypriot and Italian pre-service teachers' performance in the various aspects of the understanding of function

Table 4.5, shows Cypriot and Italian pre-service teachers' responses to the nine tasks of the first test.

Table 4.5

Cypriot and Italian pre-service teachers' responses to the nine tasks of the test A_2

Tasks	Correct responses (%) Cyprus	Correct responses (%) Italy
D1	33.1	33.5
Ex1	40.4	28.2
Ex2	28.8	27.2
Ap1	76.5	74.3
Ap2	70.8	54.4
Ap3	53.8	40.8
Ap4	71.2	49
Pr1a	42.7	32
Pr1b	35.4	29.1
Pr1c	37.3	26.2
Pr2a	32.3	31.1
Pr2b	28.5	29.1
Pr3	22.7	12.6

According to Table 4.5, only a third of the Cypriot and Italian pre-service teachers (33.1%, 33.5% respectively) managed to give a correct definition of the concept of function. Quite low percentages of Cypriot (40.4%, 28.2%) and Italian (28.8%, 27.2%) pre-service teachers managed to give two correct examples from the applications of functions in everyday life. Concerning the first simple function task, most of the Cypriot and Italian pre-service teachers (76.5%, 74.3%) gave a correct response since this task involved the transformation of a simple linear function ($y = 2x$). Concerning the second simple function task that involved the transformation of the simplest form of an equation of a parabola ($y = x^2$) their achievement was also high (70.8%, 54.4% respectively). Their achievement radically reduced in tasks involving the transformations of "complex" quadratic functions (Ap3). Particularly, only 53.8% of the Cypriot and 40.8% of the Italian pre-service teachers managed to solve the third simple function task. Concerning the fourth simple function task although it involved the transformation of a "complex" quadratic function larger percentage of Cypriot and Italian pre-service teachers (71.2%, 49%) managed to solve it correctly since they used a coordinated approach that was easy and efficient in this task.

Pre-service teachers' performance in problem solving was moderate. In Problem 1 only 42.7% of the Cypriot and 32% of the Italian pre-service teachers managed to use the information given in order to give the two equations. Smaller percentages of the Cypriot and Italian pre-service teachers managed to construct the graphs (35.4% and 29.1% respectively) and find their point of intersection (37.3% and 26.2% respectively).

In Problem 2 only 32.3% of the Cypriot and 31.1% of the Italian pre-service teachers managed to construct the graphs. Furthermore, quite a small percentage (28.5%, 29.1% respectively) found their point of intersection. In this problem in order to find the point of intersection the pre-service teachers had to solve a second degree equation and that caused difficulties.

Problem 3 was quite difficult for the Cypriot and Italian pre-service teachers since only 22.7% and 12.6% respectively managed to solve it correctly. In general Cypriot pre-service teachers' performance was higher in all of the tasks of the first test. In addition, it is noteworthy that pre-service teachers' performance was radically decreased in the problem solving tasks.

Table 4.6, shows Cypriot and Italian pre-service teachers' responses to the first task of the test.

Table 4.6

Cypriot and Italian pre-service teachers' responses to the first task of the first test involving a definition of function

Definition Categories	Cypriot teachers' (%)	Italian teachers' (%)
D1c1	18.5	0
D1c2	6.2	26.7
D1c3	8.5	5.8
D1c4	42.3	32
D1c5	15.4	14.6

In this task we categorized teachers' definitions in five categories. In the first category (D1c1) we included all the answers involving an accurate definition of function. Only a small percentage of Cypriot teachers (18.5%) managed to give an accurate definition of the concept. Answers like "Function consists of an ordered triple of sets, which may be written as (X, Y, F). X is the domain of the function, Y is the co-domain, and F is a set of ordered pairs. In each of these ordered pairs (a, b), the first element a is from the domain, the second element b is from the co-domain, and every element in the domain is the first element in one and only one ordered pair" were accounted in the first category.

In the second category (D1c2) we included all the answers involving definitions of a special kind of function (e.g. real function, function one-to-one, continuous function). Only 6.2% of the Cypriot and 26.7% of the Italian pre-service teachers gave a definition of a special kind of function. Answers like "Function is a relation between two variables so that one value of x (or the independent variable) corresponds to one value of y (or the dependent variable) and vice versa" were accounted in the second category.

In the third category (D1c3) we involved all the answers that made a correct reference to the relation between variables but without the definition of the domain and range. Only 8.5% and 5.8% of the Cypriot and Italian pre-service teachers respectively gave this kind of definition. Answers like "Function is a relation between two variables so that one value of x (or the independent variable) corresponds to one value of y (or the dependent variable)" were accounted in the third category.

In the fourth category (D1c4) we included all answers that made a reference to an ambiguous relation (answers that made reference to a relation between variables or elements of sets, or a verbal or symbolic example were included in this group). Higher percentage of Cypriot and Italian pre-service teachers gave this kind of answer (42.3% and 32% respectively). Answers like "Function is an equation with two dependent variables", "Function is a relation in which an element x is linked with another element y" or even "Function is a mathematical relation connecting two quantities" were coded as D1c4.

The fifth category (D1c5) involved other answers pre-service teachers gave (this type of answers made reference to sets but without relation, or reference to relation without sets or elements of sets). Small percentages of Cypriot and Italian pre-service teachers' responses were included in this category (15.4% and 14.6% respectively). Answers like "Function is a relation" or "Function is a mathematical concept that is influenced by two variables" or "Function is the identification of parts of a set" were included in this category.

Table 4.7, shows Cypriot and Italian pre-service teachers' responses to the second task of the test. In this task we categorized teachers' examples from the applications of functions in everyday life in six categories.

Table 4.7

Cypriot and Italian pre-service teachers' responses to the second task of the first test involving examples of function

Example 1	Cypriot teachers' (%)	Italian teachers' (%)	Example 2	Cypriot teachers' (%)	Italian teachers' (%)
Ex1c1	5	4.4	Ex2c1	5	3.9
Ex1c2	31.5	17	Ex2c2	20.8	17.5
Ex1c3	3.1	5.3	Ex2c3	1.9	3.9
Ex1c4	10	0.5	Ex2c4	6.9	0
Ex1c5	5.4	2.4	Ex2c5	3.5	2.9
Ex1c6	15.8	9.2	Ex2c6	14.2	4.9

The first category (Ex1c1-Ex2c1) included all the answers referring to an example of a function with the use of discrete elements of sets. Small percentage of Cypriot (5%, 5%) and Italian pre-service teachers (4.4%, 3.9%) gave this kind of example. Examples of the first kind were "Each person corresponds to the size of his shoes", "Each student corresponds to his/her mark on the test" made use of sets with discrete elements.

The second category (Ex1c2-Ex2c2) involved examples of a continuous function from physics. Quite high percentage of Cypriot (31.5%, 20.8%) and Italian (17%, 17.5%) pre-service teachers gave this kind of answers. Answers such as "The height of trees is a function of time", "Atmospheric pressure is a function of altitude" were included in this category.

The third category (Ex1c3-Ex2c3) involved examples of one-to-one function. Small percentage of Cypriot (3.1%, 1.9%) and Italian (5.3%, 3.9%) pre-service teachers gave an example of one-to-one function. Such answers were "Every citizen has his own identity number", "Every graduate has his own different degree" and "Every country corresponds to its own unique name".

The fourth category (Ex1c4-Ex2c4) involved examples presenting an ambiguous relation between elements of sets. Small percentage of Cypriot (10%, 6.9%) and Italian (0.5%, 0%) pre-service teachers gave examples included in this category. Such answers were "There is a relation between students and their books", "The prices of vegetables depends on the production", "We correspond the marks of girls in a classroom to those of boys".

The fifth category (Ex1c5-Ex2c5) involved examples of an equation instead of a function. Small percentage of Cypriot (5.4%, 3.5%) and Italian (2.4%, 2.9%) pre-service teachers' responses were included in this category. Examples such as "There are 2x boys and 3y girls in a classroom and all the children are 60. If the boys are 15 we can calculate the number of girls", "Kostas has x number of toffees and John has double that number. How many toffees do the two friends have?" were included in this category.

Finally, in the sixth category (Ex1c6-Ex2c6) included answers, which were ambiguous and in addition they did not define any variables or sets, and referred to general transformations of real world. Higher percentages of Cypriot (15.8%, 14.2%) and Italian (9.2%, 4.9%) pre-service teachers' responses were included in this category. Such answers were "Health depends on smoking", "Success in a test depends on the hours of studying", "In the relation of children and parents, the children are the dependent variable and parents the independent variable".

In general most of the Cypriot and Italian pre-service teachers gave either an example of a continuous function from physics or an example that referred to general transformations of the real world.

Table 4.8, shows Cypriot and Italian pre-service teachers' approaches to the four simple function tasks.

We categorized pre-service teachers' responses as algebraic or coordinated. A solution was coded as "algebraic" if teachers did not use the information provided by the graph of the first function and they proceeded constructing the graph of the second function by finding pairs of values for x and y. On the contrary, a solution was coded as coordinated if teachers observed and used the relation between the two functions in constructing the graph of the second function. In this case teachers used and coordinated two systems of representation, the algebraic and the graphical one. They noticed the relationship between the two equations given and they interpreted this relationship graphically by manipulating the function as an entity.

The majority of Cypriot teachers (55.4%, 45%, 43.8%) chose an algebraic approach to solve the first three tasks. In Task 4, 28.8% of Cypriot teachers used an algebraic approach and 43.1% used a coordinated approach. In this task a coordinated approach seemed easier and more efficient and as a result the percent of teachers who used this approach was higher in comparison with the other three tasks. In general, the algebraic solution was predominant in the answers of the Cypriot teachers.

The majority of the Italian pre-service teachers used a coordinated approach (41.3%, 33.5%, 20.9%, 33%) in order to solve the four simple function tasks. In general, the Italian pre-service teachers gave more incorrect responses than the Cypriot pre-service teachers, used more the coordinated approach and less the algebraic.

Table 4.8

Cypriot and Italian pre-service teachers' algebraic or coordinated approaches to the four simple function tasks of the first test

Tasks	Correct responses (%) Cyprus	Correct responses (%) Italy
Ap1ral	55.4	33
Ap1rco	20.4	41.3
Ap2ral	45	21.4
Ap2rco	24.6	33.5
Ap3ral	43.8	20.4
Ap3rco	11.2	20.9
Ap4ral	28.8	16
Ap4rco	43.1	33

Pre-service teachers' algebraic and coordinated solutions were furthermore categorized into six categories. Particularly, in the first category (c1) teachers' responses who found the points of intersection with the axes were included, in the second category (c2) teachers' responses who found two or more random points were included, in the third category (c3) all the responses that involved a table of values for x and y were included, in the fourth category (c4) all the responses that involved only a verbal explanation were included , in the fifth category (c5) the responses of the teachers who gave a verbal explanation but they also found one or more points were included and finally the sixth category (c6) included other answers.

The first three categories correspond to an algebraic approach while the fourth and fifth categories to a coordinated approach. Table 4.9 shows that most of the Cypriot and Italian pre-service teachers who used an algebraic approach in order to reach a solution made a table of values. Most of the Cypriot teachers who used a coordinated approach gave only a verbal explanation. The Italian pre-service teachers' gave a verbal explanation but also tried to find points in order to verify their answer.

Table 4.9

Categorization of the Cypriot and Italian pre-service teachers' responses to the four simple function tasks of the first test

Categories	1 Cyprus (%)	1 Italy (%)	2 Cyprus (%)	2 Italy (%)	3 Cyprus (%)	3 Italy (%)	4 Cyprus (%)	4 Italy (%)
c1	18.8	4.4	20.4	4.4	20	12.6	10.8	3.4
c2	26.9	7.3	19.2	1.5	18.1	1	13.1	1.5
c3	23.1	26.7	23.8	30.1	31.9	24.3	15.8	23.3
c4	16.2	14.6	23.5	13.1	10.4	7.8	41.2	18
c5	9.2	16	6.2	12.6	8.5	13.6	5.4	8.3
c6	1.9	12.6	3.8	15	2.7	9.2	3.1	11.7

Table 4.10, shows Cypriot and Italian pre-service teachers' strategies employed in order to solve the three complex problems of the first test.

The strategies employed by the teachers in order to solve the first and second problem were categorized into two groups. The first category (Pr1s1, Pr2s1) was used for an algebraic solution while the second category (Pr1s2, Pr2s2) was used if the teachers constructed a table of values for x and y. Most of the Cypriot and Italian pre-service teachers solved both problems algebraically. Smaller was the percentage of teachers who constructed a table of values in order to solve the two problems.

The strategies employed by the teachers in order to solve the third problem were also categorized into seven categories: "Pr3s1" was used for a graphical solution (They put the points in the axes and tried to draw the function), "Pr3s2" they made a reference to the positive discriminant (or to the Bolzano theorem), "Pr3s3" they gave a correct solution but wrong justification (The function has two solution because is a second degree equation), "Pr3s4" they referred that there is not a solution, "Pr3s5" they referred that there is only one solution, "Pr3s6" they mentioned that there are infinity solutions, "Pr3s7" they tried to solved it algebraically but this was not possible.

Table 4.10

Cypriot and Italian pre-service teachers' strategies employed for the solution of the three complex problems of the first test

Strategies	Cypriot teachers (%)	Italian teachers (%)
Pr1s1	43.1	26.2
Pr1s2	18.1	3.4
Pr2s1	40.4	32
Pr2s2	11.5	5.8
Pr3s1	15.4	7.3
Pr3s2	6.9	4.9
Pr3s3	1.9	0.5
Pr3s4	2.7	0
Pr3s5	2.3	0.5
Pr3s6	0	7.3
Pr3s7	31.9	14.6

Most of the Cypriot and Italian pre-service teachers (31.9%, 14.6%) tried to solve the problem algebraically but this was not possible. The Cypriot and Italian teachers who gave a correct solution to this problem mainly gave a graphical solution (15.4%, 7.3%).

Table 4.11, shows Cypriot and Italian pre-service teachers' correct responses to the fourteen tasks of the second test.

According to Table 4.11, Cypriot pre-service teachers' performance in the tasks requiring a definition of the concept of function (D2, D3, D4) was high (62%, 74.2%, 70.1%). In contrast Italian teachers' performance was lower (27.7%, 51%, 33%). Cypriot teachers' performance in the tasks requiring a conversion from an algebraic to a graphical representation of the concept (Coag1, Coag2, Coag3) was also very high (52.8%, 79.3%, 83.4%). Italian teachers' performance was lower (34.5%, 44.2%, 35%). A large percentage of Cypriot teachers' (79%, 77.5%, 65.7%) managed also to make the conversion from a graphical to an algebraic representation of function (Coga1, Coga2, Coga3). Italian teachers' performance in these tasks was lower (42.2%, 43.2%, 32.5%). Concerning the recognition of functions given in different representations –diagram (Red1-Red5), graph (Reg1-Reg6), symbolic expression (Res1-Res6) and verbal expression (Rev1-Rev4) -

overall Cypriot teachers' performance was high. In contrast Italian teachers' performance was lower.

Table 4.11

Cypriot and Italian pre-service teachers' correct responses to the fourteen tasks of the second test

Tasks	Correct responses (%) Cyprus	Correct responses (%) Italy	Tasks	Correct responses (%) Cyprus	Correct responses (%) Italy
D2	62	27.7	Reg6	69.7	37.4
D3	74.2	51	Res1	78.2	45.1
D4	70.1	33	Res2	96.3	54.9
Coag1	52.8	34.5	Res3	65.3	29.1
Coag2	79.3	44.2	Res4	66.1	34.5
Coag3	83.4	35	Res5	79.3	44.7
Coga1	79	42.2	Res6	61.3	35.9
Coga2	77.5	43.2	Rev1	81.9	47.6
Coga3	65.7	32.5	Rev2	94.1	52.4
Red1	86.7	48.1	Rev3	29.2	19.4
Red2	63.5	22.3	Rev4	80.8	44.2
Red3	90.4	42.2	Pr4a	52.8	24.8
Red4	64.9	19.4	Pr4b	55	26.2
Red5	78.2	34.5	Pr4c	52.4	26.7
Reg1	91.9	40.8	Pr4d	48	20.4
Reg2	73.8	31.6	Pr5a	46.9	24.8
Reg3	86.7	43.2	Pr5b	35.4	18.9
Reg4	63.5	29.6	Pr5c	34.3	21.8
Reg5	69.7	34.5	Pr6	40.6	19.4

Cypriot and Italian pre-service teachers' achievement reduced radically in solving complex problems on functions.

In Problem 4 only 52.8% of the Cypriot and 24.8% of the Italian pre-service teachers managed to use the information given in order to give the equation. A bigger percentage of Cypriot pre-service teachers and a smaller percentage of Italian pre-service teachers managed to construct the graph (55% and 26.2% respectively). Some of the teachers managed to construct the graph without using the equation by using a table of values for x and y. Smaller was the percentage of Cypriot and Italian teachers who managed to use the graph or equation correctly in order to find particular values. Particularly, 52.4% of the Cypriot teachers and 26.7% of the Italian teachers managed to give a correct answer to Pr4c. Similarly, only 48% of the Cypriot pre-service teachers and 20.4% of the Italian teachers managed to answer correctly Problem 4d.

In Problem 5 only 46.9% of the Cypriot and 24.8% of the Italian pre-service teachers managed to give the equation. Furthermore, quite small percentages (35.4%, 18.9% respectively) were able to draw the graph. Quite small was also the percentage of Cypriot and Italian pre-service teachers who managed to find the minimum point of the graph (34.3%, 21.8% respectively). This problem involved a second degree equation and that caused difficulties for the teachers.

Problem 6 was quite difficult for the Cypriot and Italian pre-service teachers since only 40.6% and 19.4% respectively managed to solve it correctly.

In general Cypriot pre-service teachers' performance was higher in all of the tasks of the second test in comparison with the Italian pre-service teachers. Furthermore, it is noteworthy that teachers' performance in the definition, conversion and recognition tasks was higher than their performance in problem solving. Concerning the tasks of the second test requiring a definition of function teachers' performance was higher in comparison with the task involving a definition of function in the first test. Their performance was higher since in these tasks they had to use the definition in order to give an answer and not to write a formal definition of the concept.

Table 4.12, shows Cypriot and Italian pre-service teachers' responses to the first task of the second test.

Table 4.12

Cypriot and Italian pre-service teachers' responses to the first task of the second test involving a definition of function

Definition Categories	Cypriot teachers' (%)	Italian teachers' (%)
D2c1	32.1	7.8
D2c2	17.7	2.9
D2c3	33.2	6.8
D2c4	8.1	72.3

In this task teachers were asked if there exists a function all of whose values are equal to each other. Although, many of the Cypriot teachers (62%) gave a correct response to this question only half of them gave also a correct justification. Similarly, while 27.7% of the Italian teachers gave a correct answer only 7.8% gave also a right justification. A big percentage of Cypriot teachers (17.7%) gave a wrong justification mentioning that a function all of whose values are equal to each other is $y = x$. This is a function whose values are equal to their arguments and not to each other. The percentage of Italian pre-service teachers who gave this answer was lower (2.9%). A big percentage of the Cypriot pre-service teachers' gave other incorrect justifications (33.2%) while a big percentage of Italian pre-service teachers did not justify their answer (72.3%).

Table 4.13 shows Cypriot and Italian pre-service teachers' justifications for the fourth task of the second test involving recognition of functions given in a diagrammatic representation.

According to the table a big percentage of Cypriot pre-service teachers managed to justify their answer correctly. Quite low was the percentage of Italian teachers who justified their answer correctly. The majority of Italian pre-service teachers did not give justifications for their answers.

From teachers' justifications a number of misunderstandings concerning the concept of function emerged. An idea held by the teachers was that a function is necessarily an injective (one-to-one) correspondence. This was noticeable in the justifications of many teachers who stated that "There are many x for some y, so this is not a function" (Red1c2, Red2c2, Red4c2). Some teachers have the misunderstanding that a function must be surjective. This was evident in the cases they stated "This is not a function because there is a y without an x" (Red2c3, Red4c3). The teachers were also very much distracted by the arrow diagrams, which, were presented on incompact frames (Red2, Red4), thus expressing the idea that in a graph of a function domain and range should be compact sets. Negative answers for the diagrams were that "It is not a function because the domain or range splits into two sub domains".

Table 4.13

Cypriot and Italian pre-service teachers' justifications for the fourth task of the second test involving recognition of functions given in a diagrammatic representation

Tasks	Categories	Cypriot teachers' (%)	Italian teachers' (%)
Red1	c1: Right justification: For every x there is a y	79.7	30.6
	c2: There are many x for some y	8.1	2.4
	c3: Other justifications	2.2	3.4
	c4: No justification	9.2	52.9
Red2	c1: Right justification: For every x there is a y	57.6	15.5
	c2: There are many x for some y	5.5	5.8
	c3: There is a y without an x	12.5	0.5
	c4: The domain splits into two sub-domains	8.1	1.5
	c5: Other justifications	1.8	3.4
	c6: No justification	13.3	64.6
Red3	c1: Right justification: There is an x with two y	82.7	28.2
	c2: For every x there is a y	4.1	3.4
	c3: Other justifications	1.8	1.5
	c4: No justification	9.6	53.9
Red4	c1: Right justification: For every x there is a y	55.7	11.2
	c2: There are many x for some y	7.7	7.8
	c3: There is a y without an x	9.6	0.5
	c4: The range splits into two sets	5.5	1.5
	c5: Other justifications	1.8	3.4
	c6: No justification	17.7	68
Red5	c1: Right justification: There is an x without y	72	19.9
	c2: For every x there is a y	11.4	4.9
	c3: Other justifications	2.6	4.9
	c4: No justification	11.8	58.3

Table 4.14 shows Cypriot and Italian pre-service teachers' justifications to the fifth task of the second test involving recognition of functions given in a graphical representation.

Table 4.14

Cypriot and Italian pre-service teachers' justifications for the fifth task of the second test involving recognition of functions given in a graphical representation

Tasks	Categories	Cypriot teachers' (%)	Italian teachers' (%)
Reg1	c1: Right justification: For every x there are two y	72.7	22.8
	c2: It is a function because it is a parabola	2.2	5.3
	c3: Other justifications	9.6	4.9
	c4: No justification	14	57.3
Reg2	c1: Right justification: For every x there are two y	69	18.9
	c2: It is a function because it is a hyperbola	7.4	6.3
	c3: Other justifications	6.3	5.3
	c4: No justification	17	57.8
Reg3	c1: Right justification: For every x there is a y	66.8	15
	c2: It is a function because it is a straight line	4.1	9.2
	c3: It is not a function because there is not an x in the equation	2.2	1.9
	c4: Other justifications	7.4	4.9
	c5: No justification	18.1	58.3
Reg4	c1: Right justification: For every x there is a y	54.2	16
	c2: It is not continuous	16.2	1.9
	c3: Other justifications	6.3	2.4
	c4: No justification	19.2	69.9
Reg5	c1: Right justification: There are many y for an x	58.3	13.6
	c2: It is not a function because there is not a y in the equation	2.2	10.2
	c3: It is a function because it is a straight line	7	3.9
	c4: Other justifications	8.5	4.9
	c5: No justification	22.9	58.7
Reg6	c1: Right justification: For every x there is a y	58.3	14.1
	c2: It is not continuous	10.3	2.4
	c3: Other justifications	5.9	4.9
	c4: No justification	21	67.5

According to the table a big percentage of Cypriot pre-service teachers managed to justify their answer correctly. Quite low was the percentage of Italian teachers who justified their answer correctly. The majority of Italian pre-service teachers did not give justifications for their answers.

From teachers' justifications a number of misunderstandings concerning the concept of function emerged. Teachers held the idea that a parabola, a hyperbola or a straight line is always a function (Reg1c2, Reg2c2, Reg3c2, Reg5c3). They also stated that a graph of a function should be continuous (Reg4c2, Reg6c2). Teachers justified their answer stating explicitly that "the graph is not continuous, and therefore, cannot represent a function". Another idea that was observed among the teachers was that a function must essentially contain two variables or unknowns (Reg3c3, Reg5c2). The answers that the graphs $y = \frac{4}{3}$ and $x = -\frac{3}{2}$ that represent a straight line parallel to the x- or the y- axis cannot define functions, were justified with "x (or y) is constant and therefore it is not an unknown and a function must contain two unknowns".

Table 4.15 shows Cypriot and Italian pre-service teachers' justifications for the sixth task of the second test involving recognition of functions given in a symbolic expression.

According to the table a big percentage of Cypriot pre-service teachers managed to justify their answer correctly. Quite low was the percentage of Italian teachers who justified their answer correctly. The majority of Italian pre-service teachers did not give justifications for their answers. From teachers' justifications a number of misunderstandings concerning the concept of function emerged. Teachers held the idea that a straight line is always a function (Res1c3, Res3c3). They also expressed the idea that any symbolic expression that contains an x and a y is a function (Res2c2, Res4c2, Res5c2). For example the relation $x^2 + y^2 = 25$ was considered a function, since it included x and y. Furthermore the answers that the expressions $5x + 3 = 0$ and $4y + 1 = 0$ cannot define functions, were justified with "x (or y) do not appear in the expression, therefore a function cannot be defined" (Res1c2, Res3c2). Also some students, while trying to explain their wrong decision that the algebraic expression in $f(x) = x$ for $x \geq 0$ and $f(x) = -x$ for $x \leq 0$ does not represent a function, stated that "there is something weird with this function. It is not continuous" (Res6c2). Some of them also stated that "two different values of x correspond to the same value of $f(x)$ and therefore the expression is not a function".

Table 4.15

Cypriot and Italian pre-service teachers' justifications for the sixth task of the second test involving recognition of functions given in a symbolic expression

Tasks	Categories	Cypriot teachers' (%)	Italian teachers' (%)
Res1	c1: Right justification: For every x there are many y	43.2	21.8
	c2: It is not a function because there is not a y in the expression	24	5.8
	c3: It is a function because is a straight line	7.4	2.4
	c4: Other justifications	12.9	8.3
	c5: No justification	12.2	52.4
Res2	c1: Right justification: For every x there is a y	63.1	26.2
	c2: It is a function because there is an x and a y	12.9	1.5
	c3: Other justifications	7	1.9
	c4: No justification	14.8	58.3
Res3	c1: Right justification: For every x there is a y	47.2	20.4
	c2: It is not a function because there is not an x in the expression	20.7	5.3
	c3: It is a function because is a straight line	3.3	1.9
	c4: Other justifications	10.7	4.4
	c5: No justification	15.9	58.3
Res4	c1: Right justification: For every x there are two y	57.2	18.9
	c2: It is a function because there is an x and a y	11.1	1.5
	c3: Other justifications	11.8	11.2
	c4: No justification	16.6	58.7
Res5	c1: Right justification: For every x there is a y	52.8	20.9
	c2: It is a function because there is an x and a y	7	0.5
	c3: Other justifications	11.4	3.9
	c4: No justification	22.5	64.6
Res6	c1: Right justification: For every x there is a y	42.1	15
	c2: It is not continuous	8.9	1
	c3: Other justifications	16.2	3.9
	c4: No justification	24	69.9

Table 4.16 shows Cypriot and Italian pre-service teachers' justifications for the seventh task of the second test involving recognition of functions given in a verbal expression.

Table 4.16

Cypriot and Italian pre-service teachers' justifications for the seventh task of the second test involving recognition of functions given in a verbal expression

Tasks	Categories	Cypriot teachers' (%)	Italian teachers' (%)
Rev1	c1: Right justification: For every x there are many y	77.9	32
	c2: It is a function because there is a relationship between two variables	7.4	1.5
	c3: Other justifications	6.6	3.9
	c4: No justification	7.7	52.9
Rev2	c1: Right justification: For every x there is a y	87.5	29.6
	c2: For one y there are many x	0.7	1.5
	c3: Other justifications	1.8	3.4
	c4: No justification	8.9	55.3
Rev3	c1: Right justification: For every x there is a y	20.7	11.2
	c2: It is not a function because one x has two y	59.4	15.5
	c3: Other justifications	4.1	3.9
	c4: No justification	14.4	58.7
Rev4	c1: Right justification: For every x there are many y	74.9	23.3
	c2: It is a function because there is a relationship between two variables	6.3	3.4
	c3: Other justifications	4.8	2.4
	c4: No justification	11.4	61.7

According to the table a big percentage of Cypriot pre-service teachers managed to justify their answer correctly. Quite low was the percentage of Italian teachers who justified their answer correctly. The majority of Italian pre-service teachers did not give justifications for their answers. From teachers' justifications a number of misunderstandings concerning the concept of function emerged. Many teachers held the idea that every time we have an expression that describes the relation between two variables we have a function (Rev1c2, Rev4c2). Another idea held by the teachers was that a function is necessarily an injective (one-to-one) correspondence. This was noticeable in the explanation given in Rev2 asking whether the correspondence between every football game and the score achieved defines a function. Negative answers were justified with the fact that "two football games may have the same score". Another idea emerged in Rev3 asking whether the correspondence of the scripts to the couple of marks given by the first and the second examiner is a function. Negative answers justified with "there are two y for one x". In this case the subjects do not realize that the two marks form a pair of values. The elements of the range must be particular values and not pairs of values.

Table 4.17, shows Cypriot and Italian pre-service teachers' strategies employed in order to solve the three complex problems of the second test. The strategies employed by the teachers in order to solve the fourth and fifth problem were categorized into two groups. The first category (Pr4s1, Pr4s1) was used for an algebraic solution while the second category (Pr4s2, Pr4s2) was used if the teachers constructed a table of values for x and y.

Table 4.17

Cypriot and Italian pre-service teachers' strategies involved in solving the three complex problems of the second test

Strategies	Cypriot teachers (%)	Italian teachers (%)
Pr4s1	59	33.5
Pr4s2	8.5	0.5
Pr5s1	46.5	27.7
Pr5s2	8.5	0.5
Pr6s1	31	9.2
Pr6s2	8.8	6.8
Pr6s3	0.7	4.9
Pr6s4	0.4	0
Pr6s5	1.1	0.5
Pr6s6	0	1.5
Pr6s7	11.8	1

Most of the Cypriot and Italian pre-service teachers solved both problems algebraically. Smaller was the percentage of teachers who constructed a table of values in order to solve the two problems.

The strategies employed by the teachers in order to solve the third problem were also categorized into seven categories: “Pr6s1” was used for a graphical solution (They put the points in the axes and tried to draw the function), “Pr6s2” they made a reference to the positive discriminant (or to the Bolzano theorem), “Pr6s3” they gave a correct solution but wrong justification (The function has two solution because is a second degree equation), “Pr6s4” they claimed that there is not a solution, “Pr6s5” they claimed that there is only one solution, “Pr6s6” they mentioned that there are infinitely many solutions and “Pr6s7” they tried to solved it algebraically but this was not possible.

The Cypriot and Italian teachers who gave a correct solution to this problem mainly gave a graphical solution (31%, 9.2%). Some of the Cypriot and Italian pre-service teachers solved this problem by making a reference to the positive discriminant (7%, 6.8%). A quite high percentage of Cypriot teachers tried to give an algebraic solution to the problem (11.8%) although this was not possible.

Implicative analysis: Cypriot and Italian pre-service teachers’ behavior in the various dimensions of the understanding of function

Similarity diagrams of the first test

Cypriot and Italian pre-service teachers’ correct responses to the tasks and problems are presented in the similarity diagrams in Figure 4.7 and 4.8 respectively. The similarity analysis (Lerman, 1981) is a classification method which aims to identify in a set V of variables, thicker and thicker partitions of V, established in an ascending manner. These partitions, when fit together are represented in a hierarchically constructed diagram (tree) using a similarity statistical criterion among the variables. The similarity is defined by the cross-comparison between a group V of the variables and a group E of the individuals (or objects). This kind of analysis allows for the researcher to study and interpret in terms of typology and decreasing similarity, clusters of variables which are established at particular levels of the diagram and can be opposed to others, in the same levels.

The construction of the similarity diagram is based on the following process: Two of the variables that are the most similar to each other with respect to the similarity indices of the method are joined together in a group at the highest (first) similarity level. Next, this group may be linked with one variable in a lower similarity level or two other variables that are combined together and establish another group at a lower level, etc. This grouping process goes on until the similarity or the cohesion between the variables or the groups of variables gets very weak.

More specifically, in figure 4.7 three clusters (i.e., groups of variables) can be distinctively identified. The first cluster consists of the variables "D1r", "Ex1r", "Ex2r", "Ap1rco", "Ap2rco", "Ap3rco", "Ap4rco", "Pr1a", "Pr1b", "Pr1c", Pr2a", "Pr2b", "Pr3" and "Pr3s1" which represent the right definition and examples of the concept of function, the use of the coordinated approach, problem solving and the use of the graphical strategy in the third problem. The second cluster consists of the variables "D1w", "Ex1w", "Ex2w", "Pr3s7" which represent the wrong definition and examples of the concept and the algebraic approach in the third problem. The third cluster involves the variables "Ap1ral", "Ap2ral", "Ap3ral" and "Ap4ral" which represent the algebraic approach to the four simple function tasks.

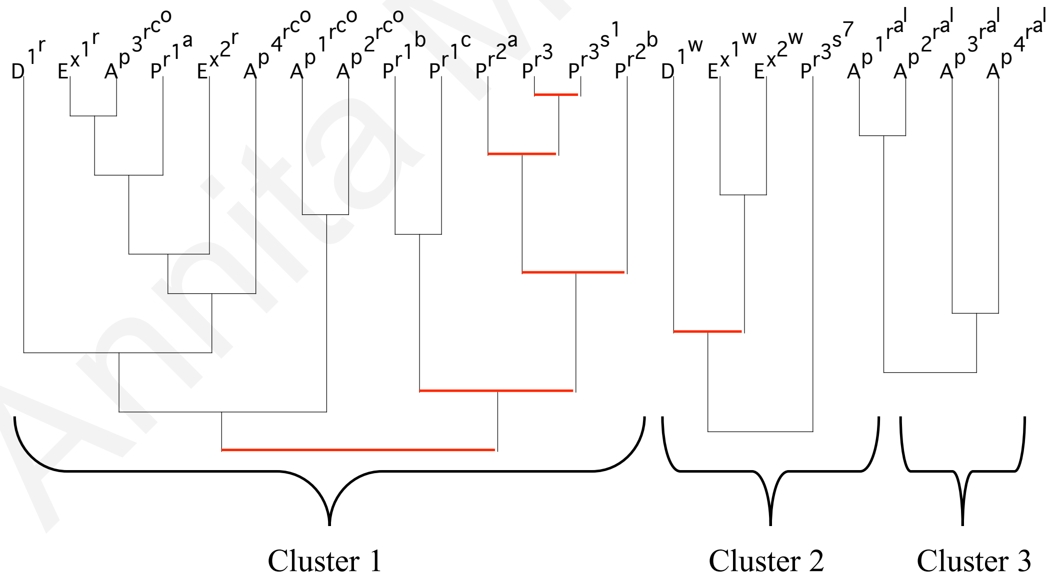


Figure 4.7. Similarity diagram of the Cypriot pre-service teachers' responses in the first test

In the second similarity diagram (Figure 4.8) four clusters (i.e., groups of variables) can be distinctively identified. The first cluster consists of the variables "D1r", "Ap1rco", "Ap2rco", "Ap3rco", "Ap4rco", "Ap3ral", "Pr1a", "Pr1b", "Pr1c", Pr2a", "Pr2b", "Pr3"and "Pr3s1" which represent the right definition of the concept of function, the use of the coordinated approach, problem solving and the use of the graphical strategy in the third problem. The second cluster consists of the variables "D1w", "Ex1w", "Ex2w" and "Pr3s7" which represent the wrong definition and examples of the concept and the algebraic approach in the third problem. The third cluster involves the variables "Ex1r", "Ex2r" which represent the right examples of the concept of function. The fourth cluster involves the variables "Ap1ral", "Ap2ral", "Ap4ral" which represent the algebraic approach to the simple function tasks.

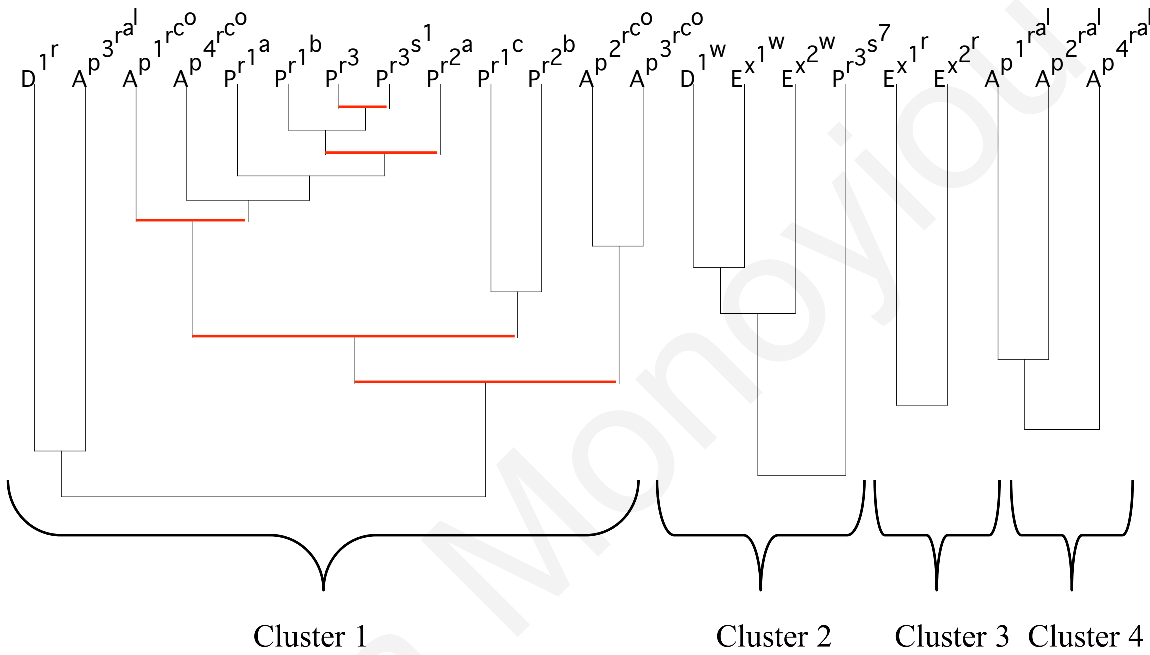


Figure4.8. Similarity diagram of the Italian pre-service teachers' responses in the first test

From both similarity diagrams it can observed that the first cluster includes the variables corresponding to the solution of the complex problems with the variables representing the coordinated approach, the graphical solution of problem 3 and the right definition of the concept of function. Concerning the similarity diagram of the Cypriot pre-service teachers the first cluster also includes the right examples of the concept. More specifically, students' coordinated approach to simple tasks in functions is closely related with effectiveness in solving problems, the graphical solution of a problem and the right definition of the concept. This close connection may indicate that students, who can use effectively different types of representation- in this situation both algebraic and graphical representations- , who can give a right definition and examples of a concept, are able to observe the connections and relations in problems, and are more capable in problem solving. The second cluster in both diagrams involves the wrong definition and examples of the concept with the algebraic solution of problem 3. The third cluster of the first diagram and the fourth cluster of the second diagram involve the algebraic solution of the simple function tasks.

Implicative diagrams of the first test

Figures 4.9 and 4.10 illustrate the implicative diagrams of the Cypriot and Italian pre-service teachers' responses. The implicative statistical analysis (Gras et al., 1997) aims at giving a statistical meaning to expressions like: "if we observe the variable a in a subject of a set E, then in general we observe the variable b in the same subject". Thus, considering that the strict logical implication is rarely fulfilled in natural, human or life sciences situations, the underlying principle of the implicative analysis is based on the following quasi-implication: "if a is true then b is more or less true". A rule is semantically related to this quasi-implication, which is a kind of theorem connecting a premise to a conclusion. An implicative diagram represents graphically the network of the quasi-implicative relations among the variables of a set.

The results of the implicative analysis are in line with the similarity relations explained above. In Figure 4.9, three separate "chains" of implicative relations among the variables are formed.

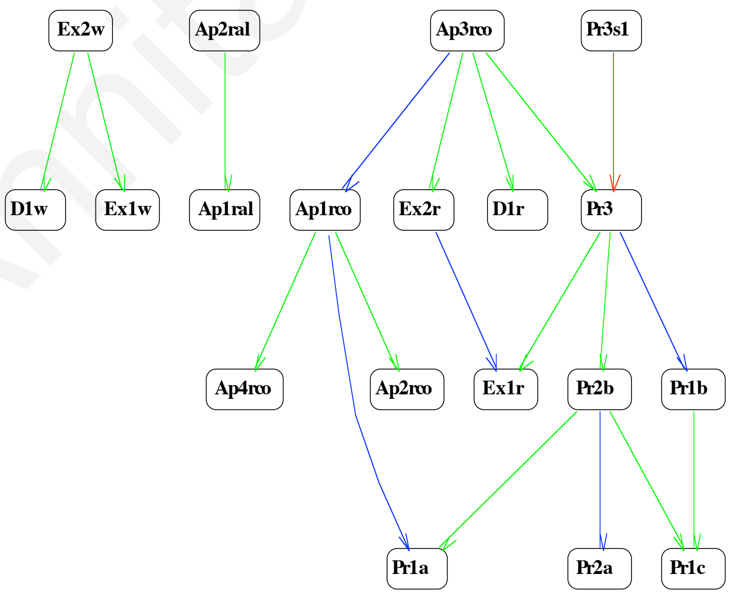


Figure 4.9. Implicative diagram of the Cypriot pre-service teachers' responses in the first test

The first chain involves the variables corresponding to the wrong definition and wrong examples of the concept (D1w, Ex1w, Ex2w). The teachers who gave a wrong example of the concept also gave a wrong definition. The second chain involves the variables representing the algebraic solution to the simple function tasks (Ap2ral, Ap1ral). The third chain includes the variables corresponding to the coordinated approach to the simple function tasks, the problem solving, the right definition and examples of the concept and the graphical solution of the third problem (Ap1rco, Ap2rco, Ap3rco, Ap4rco, Pr1a, Pr1b, Pr1c, Pr2a, Pr2b, Pr3, D1r, Ex1r, Ex2r, Pr3s1). The third chain indicates that teachers who used a coordinated approach to solve Task 3 and a graphical approach to solve problem 3 and succeeded in those tasks also solved correctly the three problems and gave a right definition and examples of the concept.

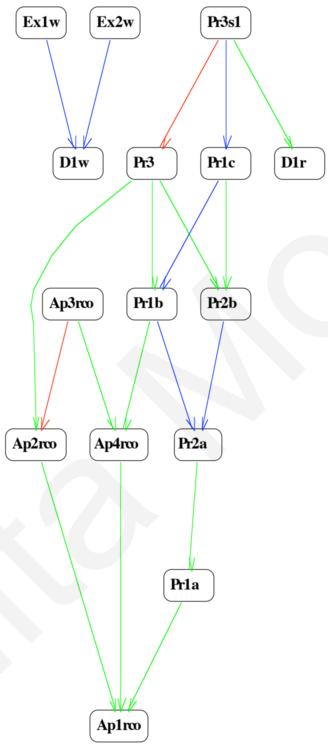


Figure 4.10. Implicative diagram of the Italian pre-service teachers' responses in the first test

In Figure 4.10, two separate "chains" of implicative relations among the variables are formed. The first chain is similar to the first chain of the above diagram and involves the variables corresponding to the wrong definition and wrong examples of the concept (D1w, Ex1w, Ex2w). The Italian teachers who gave wrong examples of the concept also gave a wrong definition. The second chain includes the variables corresponding to the coordinated approach to the simple function tasks, the problem solving, the right definition of the concept and the graphical solution of the third problem (Ap1rco, Ap2rco, Ap3rco, Ap4rco, Pr1a, Pr1b, Pr1c, Pr2a, Pr2b, Pr3, D1r, Pr3s1). The third chain indicates that teachers who used a graphical approach to solve problem 3 also solved correctly the three problems, used a coordinated approach to solve the three simple function tasks and gave a right definition of the concept.

According to the above diagrams, students who can coordinate two systems of representation and flexibly move from the one to the other, have a solid and coherent understanding of functions and therefore are able to solve complex problems, to give a right definition and examples of the concept.

Similarity diagrams of the second test

Cypriot and Italian pre-service teachers' correct responses to the tasks and problems of the second test are presented in the similarity diagrams in Figure 4.11 and 4.12 respectively.

More specifically, in the first figure (Figure 4.11) two clusters (i.e., groups of variables) can be distinctively identified. The first cluster is divided into two groups. The first group involves the variables referring to the definition of a function (D2, D4), the recognition of functions given in graphical, symbolic and diagrammatic representation (Reg3, Reg5, Res3, Res4, Res5, Red5), a task that involves a conversion from a graphical to an algebraic representation of a function (Coga1) and the problem solving tasks (Pr4a, Pr5a, Pr5b, Pr5c). The second group consists of a task requiring a definition of a function (D3), the recognition of functions given in a graphical representation (Reg2, Reg4, Reg6), tasks that involve the conversion from a graphical to an algebraic representation of a function and vice versa (Coga2, Coag2) and problem solving (Pr4b, Pr4c, Pr4d, Pr6). It is noteworthy that the closer relations are observed between problem solving and conversions tasks. Particularly, problem 5 (Pr5a, Pr5b, Pr5c) is closely linked with a task involving a conversion from a graphical to an algebraic representation of function (Coga1). Similarly, problem 4 (Pr4b, Pr4c, Pr4d) is closely linked with a task involving a conversion from an algebraic to a graphical representation of function (Coag2).

Concerning the second cluster, it is divided into two groups of variables. The first group consists of tasks that involve the recognition of functions given in verbal, graphical and symbolic representation (Rev1, Reg1, Res6) and tasks that involve conversions from a graphical to an algebraic representation of a function and vice versa (Coag1, Coag3, Coga3). The second group involves recognition tasks where the functions are given in a diagrammatic representation, a verbal and a symbolic expression (Red1, Red2, Red3, Red4, Rev2, Rev3, Rev4, Res1, Res2). Thus, pre-service teachers seem to handle in a consistent way the tasks involving the recognition of functions presented in the form of a diagram, a verbal and a symbolic expression and the tasks involving conversions from an algebraic to a graphical representation of function and vice versa.

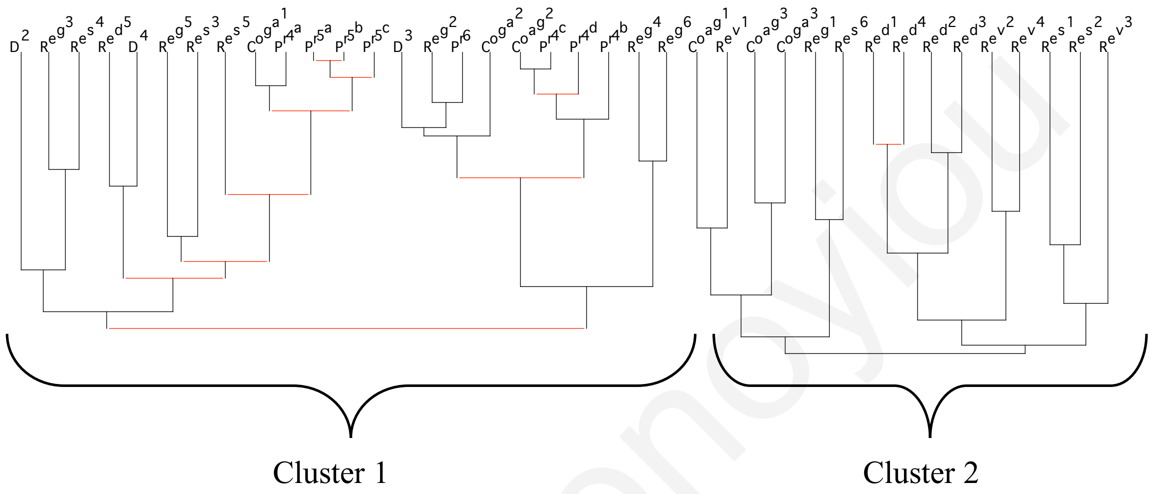


Figure 4.11. Similarity diagram of the Cypriot pre-service teachers' responses in the tasks of the second test

To sum up, the above connections in the similarity diagram and more specifically the fact that the second cluster included only tasks involving the recognition of functions presented in the form of a diagram, a verbal and a symbolic expression and the tasks involving conversions from an algebraic to a graphical representation of function and vice versa indicates that the teachers handled these tasks in a distinct way. In addition, problem solving tasks were related only with few conversions tasks a fact that indicates teachers' inconsistent behaviour. Teachers' inconsistent behaviour in dealing with tasks of different cognitive features was also uncovered by the formation of the two similarity clusters. Teachers' constructed problem solving ability were found not to be consistent with the ability to recognize the concept in different forms or the definition of the concept, even though carrying out the particular tasks required the use of these conceptions. Thus, evidence is provided for the existence of the phenomenon of compartmentalization in some extent in teachers' behaviour when confronting different types of tasks as regards the cognitive abilities they require, even though their content is similar.

In the second figure (Figure 4.12) two clusters (i.e., groups of variables) can be distinctively identified. The first cluster is divided into two groups. The first group involves the variables referring to the definition of a function (D2, D4) with the variables represent the recognition of functions given in a verbal expression (Rev1, Rev3). The second group consists of a task requiring a definition of a function (D3), the recognition of a function given in a graphical representation (Reg4) and tasks that involve the conversion from an algebraic to a graphical representation of a function (Coag1, Coag3). Pre-service teachers seem to handle in a consistent way the tasks involving a definition of the concept, the recognition of functions presented in the form of a graph and a verbal expression and the tasks involving conversions from an algebraic to a graphical representation of function.

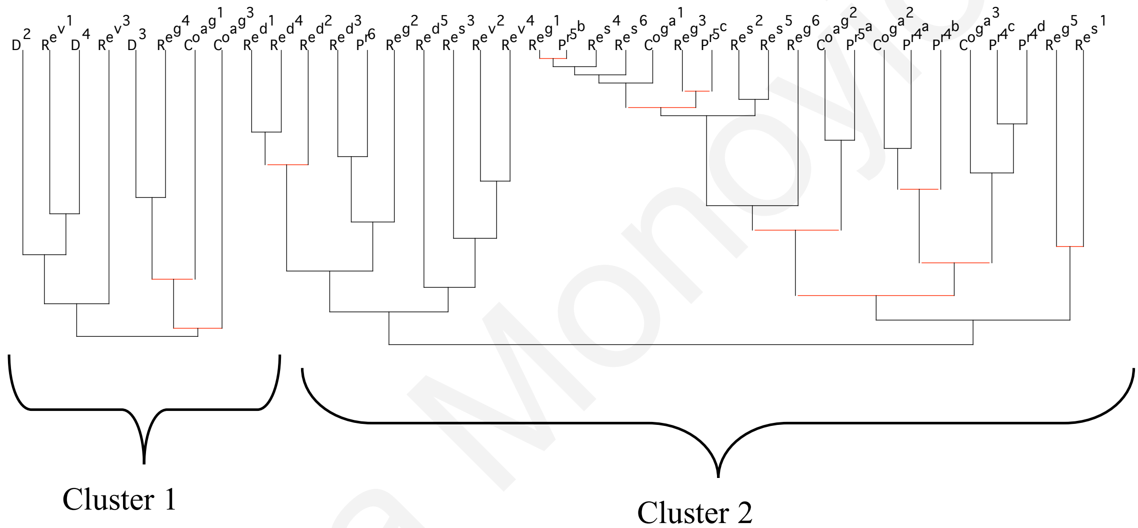


Figure 4.12. Similarity diagram of the Italian pre-service teachers' responses in the tasks of the second test

Concerning the second cluster it is divided into two groups of variables. The first group consists of tasks that involve the recognition of functions given in diagrammatic, verbal, graphical and symbolic representation (Red1, Red2, Red3, Red4, Red5, Rev2, Rev4, Reg2, Res3) and problem 6. The second group involves recognition tasks where the functions are given in a graphical representation and a symbolic expression (Reg1, Reg3, Reg5, Reg6, Res1, Res2, Res4, Res5, Res6), tasks that involve the conversion from a graphical to an algebraic representation of a function and vice versa (Coga1, Coga2, Coga3, Coag1) and problem solving tasks (Pr4a, Pr4b, Pr4c, Pr4d, Pr5a, Pr5b, Pr5c). The closer relations are observed between problem solving and recognition tasks given in a graphical representation (Reg1-Pr5b, Reg3-Pr5c). Close are also the relations between problem solving tasks and some conversions tasks (Coag2-Pr5a, Coga2-Pr4a-Pr4b, Coga3-Pr4c-Pr4d).

To sum up, the above connections in the similarity diagram and more specifically the fact that the first cluster included only tasks involving the definition of function, the recognition of functions presented in the form of a graph and a verbal expression and the tasks involving conversions from an algebraic to a graphical representation of function indicates that the teachers handled these tasks in a distinct way. Furthermore, teachers' inconsistent behaviour was evident by the fact that problem solving was related only with recognition tasks given in a diagrammatical and graphical representation and conversion tasks. Teachers' inconsistent behaviour in dealing with tasks of different cognitive features was also uncovered by the formation of the two similarity clusters. Teachers' problem solving ability was found not to be consistent with the ability to give a definition of the concept, even though carrying out the particular tasks required the use of the definition. Thus, evidence is provided for the existence of the phenomenon of compartmentalization in some extent in teachers' behaviour when confronting different types of tasks as regards the cognitive abilities they require, even though their content is similar.

Implicative diagrams of the second test

Figures 4.13 and 4.14 illustrate the implicative diagrams of the Cypriot and Italian pre-service teachers' responses.

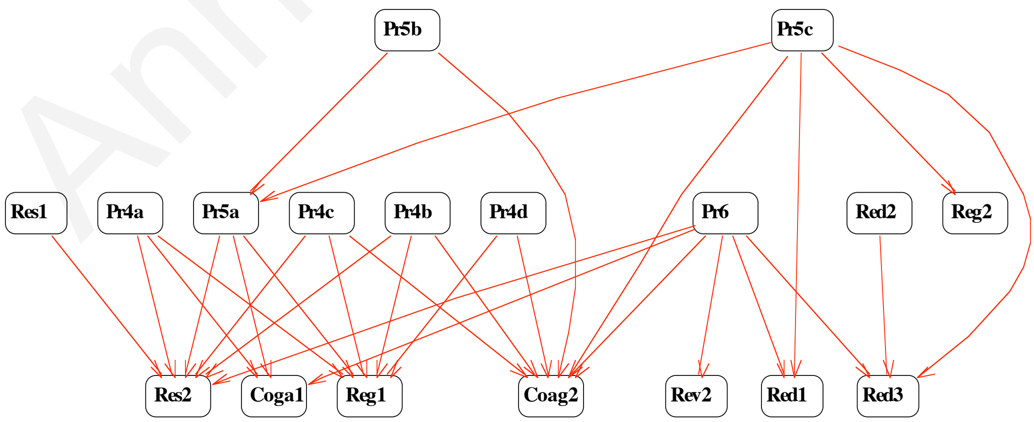


Figure 4.13. Implicative diagram of the Cypriot pre-service teachers' responses in the second test

In Figure 4.13, a group of implicative relations among the variables are formed. The group involves the variables corresponding to problem solving (Pr4a, Pr4b, Pr4c, Pr4d, Pr5a, Pr5b, Pr5c, Pr6), the tasks involving the recognition of function given in symbolic, graphical, verbal and diagrammatic representation (Res1, Res2, Reg1, Reg2, Rev2, Red1, Red2, Red3) and the tasks requiring conversions from an algebraic to a graphical representation of function and vice versa (Coga1, Coag2). The implicative relations among the variables indicate that the Cypriot teachers who managed to give correct solutions to the problems were also successful in the recognition of functions given in different representations and to the conversions. It is noteworthy that from the implicative diagram are totally absent implicative relations between problem solving and definition tasks.

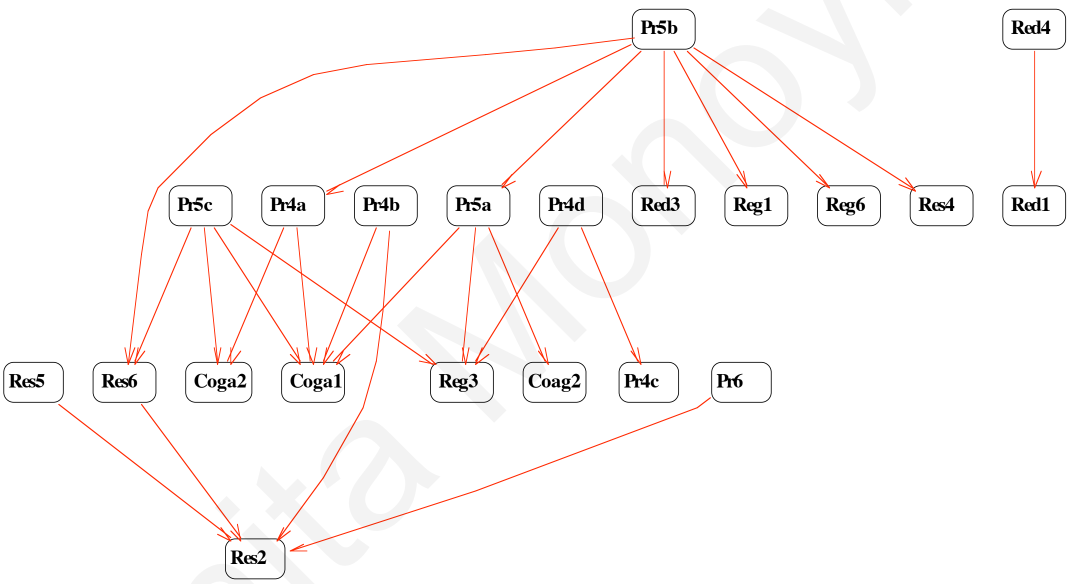


Figure 4.14 Implicative diagram of the Italian pre-service teachers' responses in the second test

In Figure 4.14, two separate groups of implicative relations among the variables are formed. The first group is similar to the group of the above diagram and involves the variables corresponding to problem solving (Pr4a, Pr4b, Pr4c, Pr4d, Pr5a, Pr5b, Pr5c, Pr6), the tasks involving the recognition of function given in symbolic, graphical, verbal and diagrammatic representation (Res2, Res4, Res5, Res6, Reg1, Reg3, Reg6, Red3) and the tasks requiring a conversion from an algebraic to a graphical representation of function (Coga1, Coga2, Coag2). This indicates that Italian teachers' proficiency in tackling problems entails success in recognition and conversion tasks involving different types of representations of functions. The second group involves only two variables (Red4, Red1) that refer to the recognition of functions given in a diagrammatic representation. Again there are not implicative relations between the definition of the concept and problem solving tasks.

Similarity diagrams of both tests

Cypriot and Italian pre-service teachers' correct responses to the tasks and problems of both tests are presented in the similarity diagrams in Figure 4.15 and 4.16 respectively.

More specifically, in the first figure (Figure 4.15) four clusters (i.e., groups of variables) can be distinctively identified. The first cluster is divided into two groups. The first group involves a variable referring to the definition of a function (D1) and the recognition of functions given in a verbal expression (Rev1, Rev2, Rev4). The second group consists of tasks that require the recognition of functions given in a symbolic and verbal expression (Res1, Res2, Rev3).

The second cluster is also divided into two groups. The first group consists of the examples of a function (Ex1, Ex2), the coordinated approach to problem 3 (Ap3rco), the recognition of a function given in a graphical representation (Reg5) and problem solving (Pr1a). The second group consists of a variable referring to the definition of a function (D4), the coordinated approach to task 2 (Ap2rco) and the recognition of functions given in a graphical representation (Reg4, Reg6).

The third cluster is also divided into two groups. The first group involves the variables representing a coordinated approach in tasks 1 and 4 (Ap1rco, Ap4rco) and tasks that involve conversions from an algebraic to a graphical representation of a function (Coag1, Coag3). The second group consists of tasks that require the recognition of functions given in a diagrammatic representation (Red1, Red2, Red3, Red4).

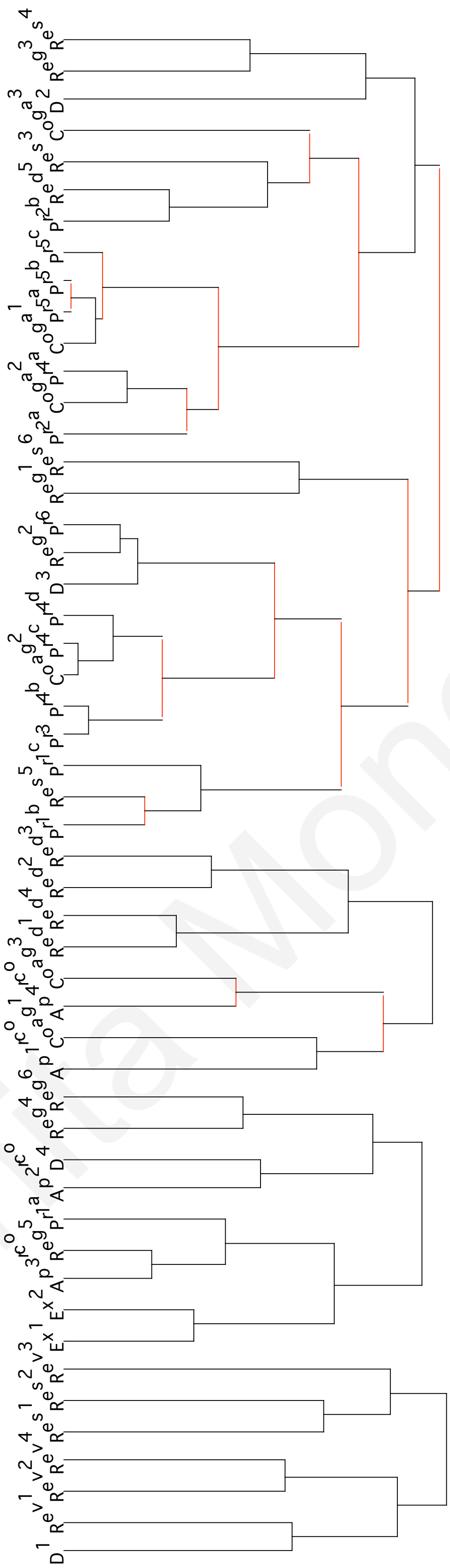


Figure 4.15. Similarity diagram of the Cypriot pre-service teachers' responses in the tasks of the first and second test

The fourth cluster is divided into two groups. The first group consists of a variable that demands a definition of the concept (D3), problem solving (Pr1b, Pr1c, Pr3, Pr4b, Pr4c, Pr4d, Pr6), a task that requires a conversion from an algebraic to a graphical representation of a function (Coag2) and recognition of function given in a symbolic expression and graphical representation (Res5, Res6, Reg1, Reg2). The second group consists of the variables that involve the conversion from a graphical to an algebraic representation of functions (Coga1, Coga2, Coga3), the definition of function (D2), the recognition of functions given in diagrammatic, symbolic and graphical representation (Red5, Res3, Res4, Reg3) and problem solving (Pr2a, Pr2b, Pr4a, Pr5a, Pr5b, Pr5c).

From the above diagram it is noteworthy that the strongest relation exists between the variables representing problem 5 (Pr5a, Pr5b, Pr5c) and the conversion from a graphical to an algebraic representation of function (Coga1). Also a strong relation exists between problem solving (Pr4b, Pr4c, Pr4d, Pr3) and a conversion from an algebraic to a graphical representation of function (Coag2). Furthermore, a strong relation exists between problem solving (Pr2a, Pr4a) and a conversion from a graphical to an algebraic representation of function (Coga2). These strong relations indicate that the teachers who successfully made conversions between algebraic and graphical representations of function and vice versa were also very successful in problem solving.

To sum up, the above connections in the similarity diagram and the fact that four distinct clusters were constructed indicates that the teachers handled these tasks in a distinct and inconsistent way. Teachers' problem solving ability was found not to be consistent with the ability to recognize the concept in different forms, examples and definitions of the concept even though carrying out the particular task required the use of these conceptions. Thus, evidence is provided for the existence of the phenomenon of compartmentalization.

In the second figure (Figure 4.16) four clusters (i.e., groups of variables) can be distinctively identified. The first cluster involves a variable referring to the definition of a function (D1) and the examples of the concept (Ex1, Ex2).

The second cluster is divided into two groups. The first group consists of the coordinated approach to the four simple function tasks (Ap1rco, Ap2rco, Ap3rco, Ap4rco), the conversion from an algebraic to a graphical representation of function (Coag3) and problem solving (Pr2a, Pr2b, Pr3). The second group consists of the variables representing problem solving (Pr1a, Pr1b, Pr1c).

The third cluster is also divided into two groups. The first group consists of tasks that require the recognition of functions given in a verbal expression (Rev1, Rev3) and tasks that demand a definition of the concept (D2, D4). The second group involves the recognition of function given in a graphical representation (Reg4), the definition of the concept (D3) and a task that involves a conversion from an algebraic to a graphical representation of a function (Coag1).

The fourth cluster is divided into two groups. The first group consists of tasks that require the recognition of functions given in a diagrammatic and graphical representation, a symbolic and verbal expression (Red1, Red2, Red3, Red4, Red5, Reg2, Rev2, Rev4, Res3) and the solution of problem 6 (Pr6). The second group consists of the variables that involve the conversion from a graphical to an algebraic representation of functions and vice versa (Coga1, Coga2, Coga3, Coag2), the recognition of functions given in a symbolic and graphical representation (Reg1, Reg3, Reg5, Reg6, Res1, Res2, Res4, Res5, Res6) and problem solving (Pr4a, Pr4b, Pr4c, Pr4d, Pr5a, Pr5b, Pr5c).

From the above diagram it is noteworthy that strong relations exists between problem 5 (Pr5b, Pr5c) and the recognition of functions given in a diagrammatic representation (Reg1, Reg3). These strong relations indicate that the teachers who successfully recognised a function given in a diagrammatic representation were also very successful in problem solving.

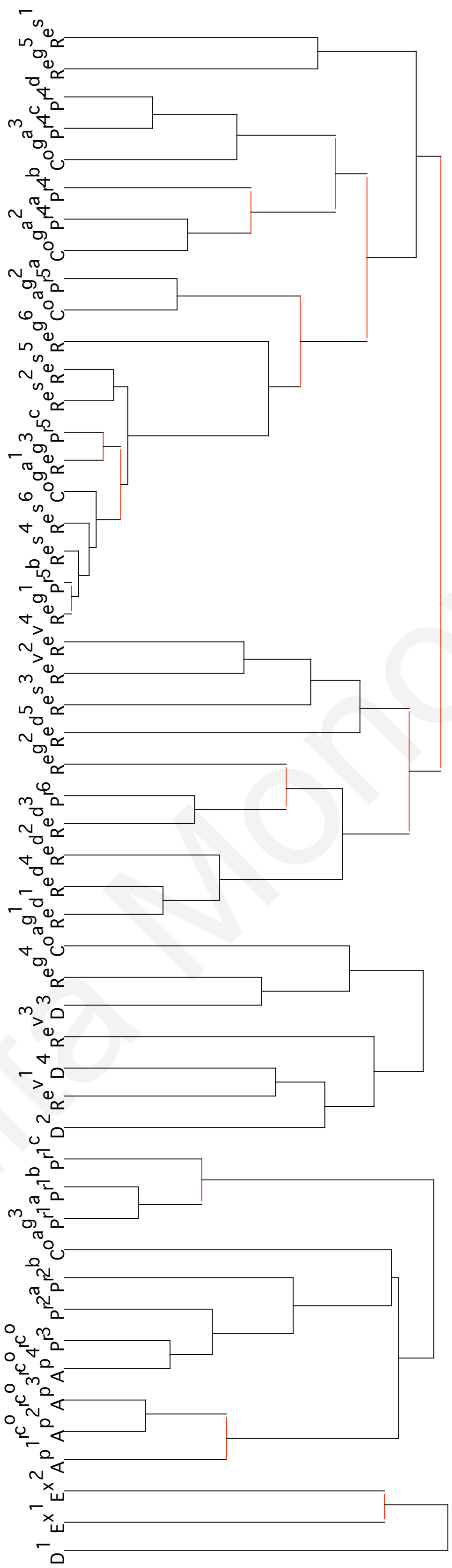


Figure 4.16. Similarity diagram of the Italian pre-service teachers' responses in the tasks of the first and second test

To sum up, the above connections in the similarity diagram and the fact that four distinct clusters were constructed indicates that the teachers handled these tasks in a distinct and inconsistent way. Teachers' problem solving ability was found not to be consistent with the ability to give examples and definition of the concept even though carrying out the particular task required the use of these conceptions. Thus, evidence is provided for the existence of the phenomenon of compartmentalization to some extent in teachers' behaviour when confronting different types of tasks as regards the cognitive abilities they require, even though their content is similar.

Implicative diagrams of both tests

Figures 4.17 and 4.18 illustrate the implicative diagrams of the Cypriot and Italian pre-service teachers' responses to the tasks of both tests.

In Figure 4.17, a group of implicative relations among the variables are formed. The group involves the variables corresponding to problem solving (Pr1b, Pr1c, Pr2a, Pr2b, Pr3, Pr4a, Pr4b, Pr4c, Pr4d, Pr5a, Pr5b, Pr5c, Pr6), the tasks involving the recognition of function given in symbolic, graphical, verbal and diagrammatic representation (Res1, Res2, Reg1, Reg2, Reg3, Reg5, Rev2, Red1, Red2, Red3), the tasks requiring a conversion from an algebraic to a graphical representation of function and vice versa (Coga1, Coga2, Coag2) and the coordinated approach to tasks 1 and 3 (Ap1rco, Ap3rco).

The implicative relations among the variables indicate that the Cypriot teachers who managed to give correct solutions to the problems also were successful in the recognition of functions given in different representations and in the conversions. Furthermore, the Cypriot pre-service teachers who gave a coordinated approach to tasks 1 and 3 managed to recognize functions given in different representations and to make conversions from a graphical to an algebraic representation of a function. It is noteworthy that there are not implicative relations between the definition, examples of the concept and problem solving.

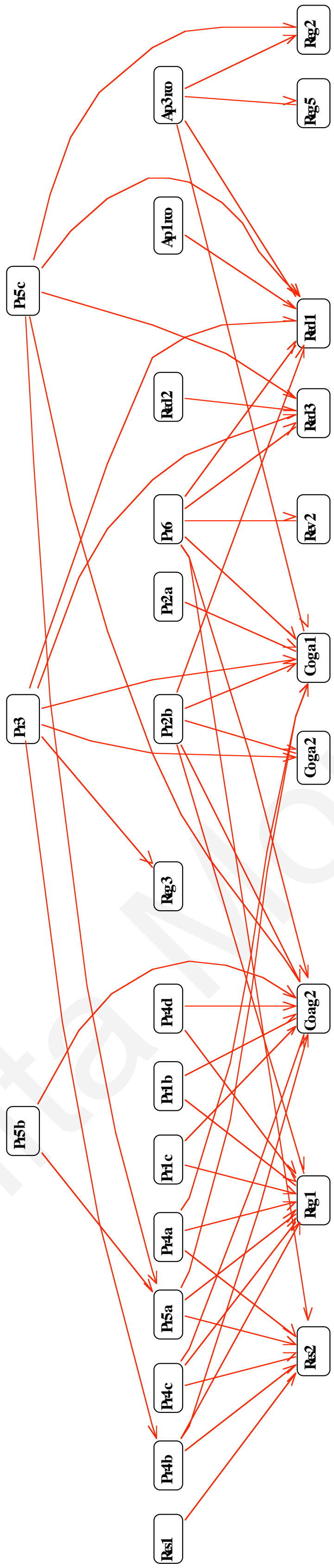


Figure 4.17. Implicative diagram of the Cypriot pre-service teachers' responses in the tasks of the first and second test

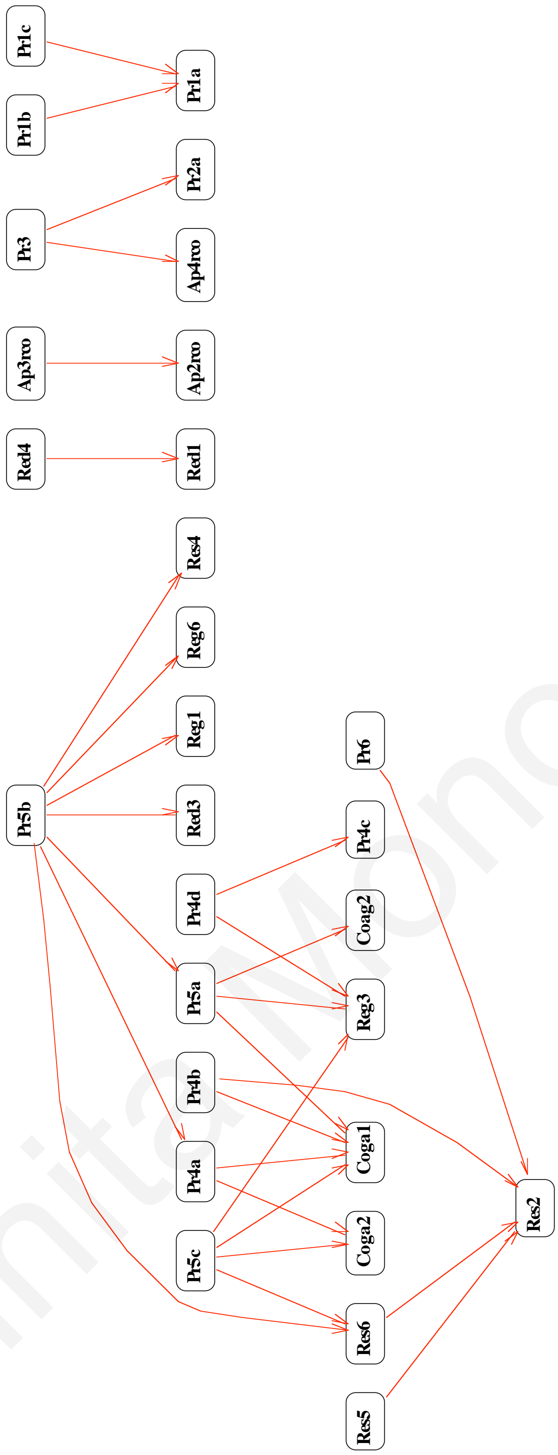


Figure 4.18. Implicative diagram of the Italian pre-service teachers' responses in the tasks of the first and second test

In Figure 4.18, five separate groups of implicative relations among the variables are formed. The first group is similar to the group of the above diagram and involves the variables corresponding to problem solving (Pr4a, Pr4b, Pr4c, Pr4d, Pr5a, Pr5b, Pr5c, Pr6), the tasks involving the recognition of function given in symbolic, graphical and diagrammatic representation (Res2, Res4, Res5, Res6, Reg1, Reg3, Reg6, Red3) and the tasks requiring a conversion from an algebraic to a graphical representation of function and vice versa (Coga1, Coga2, Coag2). This indicates that Italian teachers' proficiency in tackling problems entails success in recognition and conversion tasks involving different types of representation of functions. The second group involves only two variables (Red4, Red1) that refer to the recognition of functions given in a diagrammatic representation. The third group involves two variables referring to the coordinated approach to tasks 3 and 2 (Ap3rco, Ap2rco). The fourth group involves problem solving and the coordinated approach to task 4 (Ap4rco, Pr3, Pr2a). It indicates that a successful solution to problem 3 leads also to a successful solution to problem 2a and to a coordinated approach to task 4. The fifth group involves the variables concerning the solution of problem 1 (Pr1a, Pr1b, Pr1c).

From both implicative diagrams it emerged that pre-service teachers' problem solving ability is related only with recognition and conversions tasks. Furthermore, problems are at the beginning of the implicative diagrams, a fact that indicates that problem solving is a complex process. This is in line with the fact that pre-service teachers' performance in problem solving was low in comparison with the other dimensions of the understanding of function.

Confirmatory factor analysis: The interrelations between the concept image, the coordinated and algebraic approaches and problem solving

Confirmatory factor analysis was used to explore the structural organization of the various dimensions of the understanding of function involved in Test A_2: the concept image, the coordinated and algebraic approaches and problem solving. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: x^2, CFI (Comparative Fit Index) and RMSEA (Root Mean Square Error of Approximation) (Bentler, 1990). The following values of the three indices are needed to hold true for supporting an adequate fit of the model: the observed values for x^2/df should be less than 2.5, the values for CFI should be higher than 0.9 and the RMSEA values should be lower than 0.06.

A series of models were tested and compared. Specifically, the first model involved four first-order factors representing the concept image, the coordinated approach, the algebraic approach and problem solving and one second-order factor on which all of the first-order factors were regressed. The fit of this model was not satisfactory [x^2 (94) =239.70; CFI=0.97; RMSEA=0.058, 90% confidence interval for RMSEA=0.049-0.067].

The second model (see Figure 4.19) involves four first-order factors that are intercorrelated. The fit of this model was very good [x^2 (99) =204.87; CFI=0.98; RMSEA=0.048, 90% confidence interval for RMSEA=0.039-0.057]. Therefore, it is suggested that the two approaches, namely, coordinated and algebraic, the concept image and problem solving are intercorrelated.

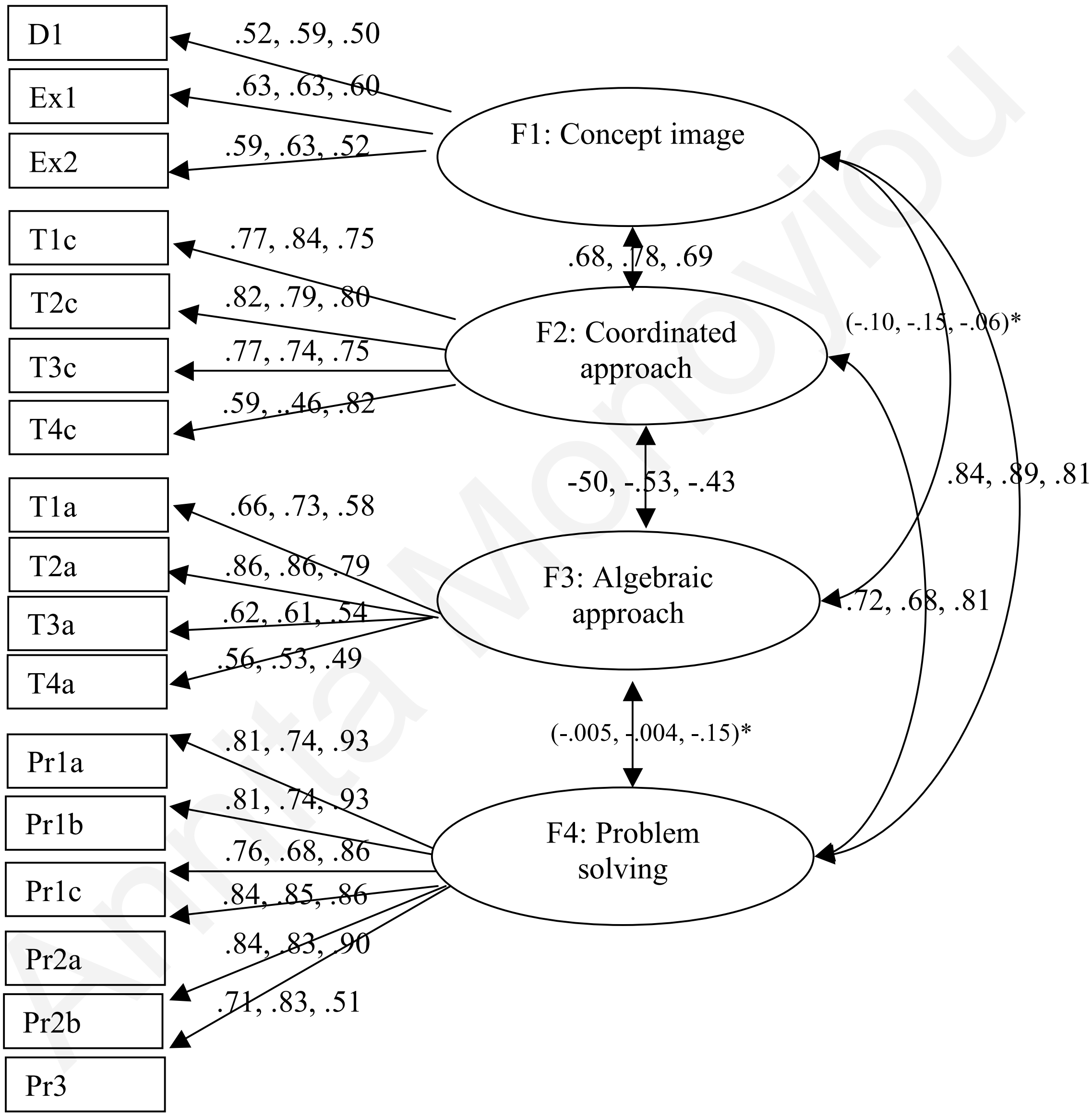


Figure 4.19. The confirmatory factor analysis model accounting for performance on the tasks of the first test by the whole sample, the Cypriot and Italian pre-service teachers separately. Note: The first, second and third coefficients of each factor stand for the application of the model on the performance of the whole sample, Cypriot and Italian pre-service teachers respectively.

* These relations are not statistically significant.

To test for possible differences between the two groups in the structure described above, multiple group analysis was applied, where the model was fitted separately on the Cypriot and Italian pre-service teachers. The model was first tested under the assumption that the relations of the observed variables to the first-order factors would be equal across the two groups. Although the fit of this model was acceptable, [x^2 (210) =387.39; CFI=0.96; RMSEA=0.060, 90% confidence interval for RMSEA=0.051-0.070], some of the equality constraints were found not to hold.

As a result, these constraints were released. Releasing the constraints resulted in an improvement of the model fit [x^2 (207) =322.59; CFI=0.98; RMSEA=0.049, 90% confidence interval for RMSEA=0.038-0.059].

It is noteworthy that the relations of the first-order factor standing for the coordinated approach (F2) with the first-order factors standing for problem solving (F4) and concept image (F1) are very strong. Furthermore, quite strong is the relation between concept image (F1) and problem solving (F4). Attention is also drawn to the fact that the first-order factor standing for the algebraic approach (F3) is negatively related with the coordinated approach (F2). There are also negative relations between the algebraic approach (F3) and problem solving (F4) and the algebraic approach (F3) and concept image (F1) although these relations are not statistically significant.

In order to examine whether there are statistically significant differences between the Cypriot and Italian pre-service teachers concerning the concept image, the approach they used and their problem solving ability, a multivariate analysis of variance (MANOVA) was performed. Overall, the effects of teachers' nationality were significant (Pillai's $F_{(4, 461)} = 11.62$, $p<0.001$).

Table 4.18 presents the mean and standard deviation of concept image, the coordinated, algebraic approach and problem solving for the two groups.

Table 4.18

The mean and standard deviation of the concept image, the coordinated, the algebraic approaches and problem solving for the Cypriot and Italian pre-service teachers

Groups	Concept image		Coordinated approach		Algebraic approach		Problem solving	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
Cyprus	0.34	0.38	0.25	0.32	0.43	0.38	0.33	0.38
Italy	0.30	0.34	0.32	0.39	0.23	0.29	0.28	0.38

Particularly, there were significant differences between the two groups concerning the coordinated ($F_{(4, 461)} = 4.95$, $p<0.05$) and algebraic approaches ($F_{(4, 461)} = 41.4$, $p<0.01$). There were not statistically significant differences concerning the concept image ($F_{(4, 461)} = 1.75$, $p=0.18$) and problem solving ($F_{(4, 461)} = 3.31$, $p=0.07$).

The Italian pre-service teachers ($\overline{X}$=0.32, SD=0.39) used the coordinated approach more than the Cypriot pre-service teachers ($\overline{X}$=0.25, SD=0.32). Concerning the algebraic approach, the Italian pre-service teachers ($\overline{X}$=0.23, SD=0.29) used this approach less than the Cypriot pre-service teachers ($\overline{X}$=0.43, SD=0.38). The Cypriot pre-service teachers ($\overline{X}$=0.33, SD=0.38) had better results in problem solving than the Italian pre-service teachers ($\overline{X}$=0.28, SD=0.38). They also had better results in concept definition ($\overline{X}$=0.34, SD=0.38, $\overline{X}$=0.30, SD=0.34 respectively). These differences were not statistically significant.

It is noteworthy that overall the Cypriot pre-service teachers used more often the algebraic approach than the coordinated in order to solve the simple function tasks. In contrast, the Italian pre-service teachers used more often the coordinated approach. The Cypriot pre-service teachers had better results in concept image and problem solving.

Confirmatory factor analysis: The interrelations of the concept definition, recognition, conversions and problem solving

Confirmatory factor analysis was used to explore the structural organization of the various dimensions concerning the understanding of function involved in Test B: definition of function, recognition of the concept in diagrammatic, graphical, symbolic and verbal representations, conversion from an algebraic to a graphical representation of function and vice versa and problem solving. Bentler's (1995) EQS programme was used for the analysis.

A series of models were tested and compared. Specifically, the first model involved four first-order factors representing the concept definition, the recognition of functions given in different representation, the conversions from a graphical to an algebraic representation of function and vice versa and problem solving and one second-order factor on which all of the first-order factors were regressed. The fit of this model was quite good [x^2 (90) =173.71; CFI=0.98; RMSEA=0.044, 90% confidence interval for RMSEA=0.034-0.054].

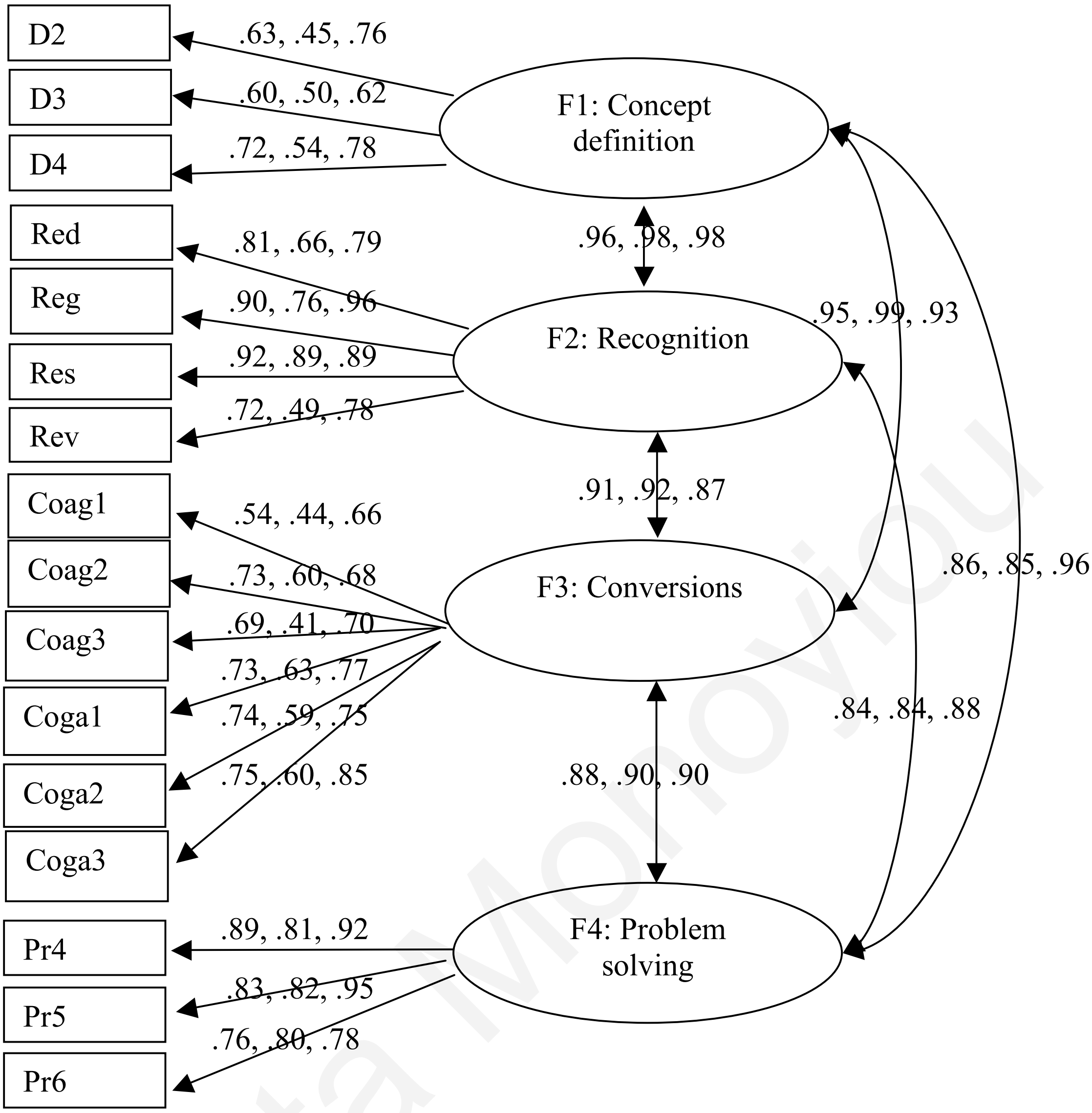


Figure 4.20. The confirmatory factor analysis model accounting for performance on the tasks of the second test by the whole sample, the Cypriot and Italian pre-service teachers separately Note: The first, second and third coefficients of each factor stand for the application of the model on the performance of the whole sample, Cypriot and Italian pre-service teachers respectively.

The second model (see Figure 4.20) involves four first-order factors that are intercorrelated. The fit of this model was very good and in contrast with the previous model the indices were slightly better [x^2 (88) =166.49; CFI=0.98; RMSEA=0.043, 90% confidence interval for RMSEA=0.033-0.053]. Therefore, it is suggested that the concept definition, the recognition of functions given in different representations, the conversions of functions from a graphical to an algebraic representation and vice versa and problem solving are intercorrelated.

To test for possible differences between the two groups in the structure described above, multiple group analysis was applied, where the model was fitted separately on the Cypriot and Italian pre-service teachers. The model was first tested under the assumption that the relations of the observed variables to the first-order factors would be equal across the two groups. Although the fit of this model was acceptable, [x^2 (189) =312.23; CFI=0.97; RMSEA=0.052, 90% confidence interval for RMSEA=0.042-0.062], some of the equality constraints were found not to hold.

As a result, these constraints were released. Releasing the constraints resulted in an improvement of the model fit [x^2 (186) =276.19; CFI=0.98; RMSEA=0.045, 90% confidence interval for RMSEA=0.033-0.056].

It is noteworthy that the relations between the four first-order factors are very strong indicating the fact that the four components are strongly related. Furthermore, the higher loadings are observed between the three first-order factors representing the definition, the recognition and the conversions indicating that the strong relation exists between these dimensions. In addition it is noteworthy that the loadings in all cases are higher for the Italian pre-service teachers indicating that this structure is stronger for this group.

In order to examine whether there are statistically significant differences between the Cypriot and Italian pre-service teachers concerning the concept definition, the recognition of the concept given in various representations, the conversion from an algebraic to a graphical representation of the concept and vice versa and their problem solving ability, a multivariate analysis of variance (MANOVA) was performed. Overall, the effects of teachers' nationality were significant (Pillai's $F_{(4, 472)} = 68.54$, $p<0.001$). Table 4.19 presents the mean and standard deviation of concept definition, the recognition of the concept, the conversions and problem solving of the two groups.

Concerning the concept definition Cypriot pre-service teachers ($\overline{X}$=0.69, SD=0.32) performed better than the Italian pre-service teachers ($\overline{X}$=0.37, SD=0.39) and this difference was statistically significant ($F_{(4, 472)} = 93$, $p<0.001$). The Cypriot pre-service teachers ($\overline{X}$=0.75, SD=0.20) had better results in the recognition of functions given in various representations than the Italian pre-service teachers ($\overline{X}$=0.38, SD=0.34). They also had better results in the conversions of functions from a graphical to an algebraic representation and vice versa ($\overline{X}$=0.73, SD=0.28, $\overline{X}$=0.39, SD=0.38 respectively). These differences were statistically significant ($F_{(4, 472)} = 219.47$, $p<0.001$, $F_{(4, 472)} = 127.48$, $p<0.001$ respectively). Statistically significant was also the difference observed in their problem solving ability ($F_{(4, 472)} = 36.70$, $p<0.01$). Cypriot pre-service teachers ($\overline{X}$=0.44, SD=0.41) performed better than the Italian pre-service teachers ($\overline{X}$=0.22, SD=0.36).

Table 4.19

The mean and standard deviation of the concept definition, the recognition, the conversion and problem solving for the Cypriot and Italian pre-service teachers

Groups	Concept Definition		Recognition of functions		Conversions of functions		Problem solving	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
Cyprus	0.69	0.32	0.75	0.20	0.73	0.28	0.44	0.41
Italy	0.37	0.39	0.38	0.34	0.39	0.38	0.22	0.36

It is noteworthy that overall the Cypriot pre-service teachers performed better than the Italian pre-service teachers in all the tasks. Furthermore, Cypriot and Italian teachers' performance in problem solving was lower than in the other tasks requiring a definition, a recognition and conversions of the concept.

Confirmatory factor analysis: A model for the conceptual understanding of function

Confirmatory factor analysis was used to explore the structural organization of the various dimensions concerning the understanding of function involved in both tests (Test A_2 and B): concept image, recognition of the concept in diagrammatic, graphical, symbolic and verbal representations, conversion from an algebraic to a graphical representation of function and vice versa, coordinated approach to simple function tasks and problem solving. Bentler's (1995) EQS programme was used for the analysis.

A series of models were tested and compared. Specifically, the first model involved only one first-order factor associated with all of the tasks. The fit of this model was poor [x^2 (276) =2133.13; CFI=.74; RMSEA=.12, 90% confidence interval for RMSEA=0.116-0.126], indicating that a single common factor is not sufficient to describe the solution of all the tasks in the test.

The second model involved five first-order factors and one second-order factor on which all of the first-order factors were regressed. The first-order factors stand for the concept image, the recognition of functions given in a diagrammatic, a graphical, a symbolic and a verbal expression, the conversions from a graphical to an algebraic representation and vice versa, the coordinated approach and problem solving. The second-order factor stands for the conceptual understanding of functions. The fit of this model was also poor [x^2 (270) =1203.01; CFI=.87; RMSEA=0.087, 90% confidence interval for RMSEA=0.082-0.092].

Table 4.20

Goodness of Fit Indices of the Models

Models	x^2	df	x^2/df	CFI	RMSEA
General model with one first order factor	2133.13 p=0.001	276	7.72	0.74	0.12
General model with five first order factors and a second order factor	1203.01 p=0.001	270	4.45	0.87	0.087
General model with five first order factors, two second order factors and a third order factor	360.92 p=0.001	178	2	0.97	0.047
Multiple group model for Cypriot and Italian pre-service teachers with equality constraints	548.29 p=0.001	369	1.48	0.97	0.046
Multiple group model for Cypriot and Italian pre-service teachers	508.17 p=0.001	366	1.38	0.98	0.041

The third model (see Figure 4.21) involved five first-order factors, two second-order factors and a third-order factor. The fit of this model was very good [x^2 (178) =360.92; CFI=0.97; RMSEA=0.047, 90% confidence interval for RMSEA=0.040-0.054]. The third-order model which is considered appropriate for interpreting the conceptual understanding of function, involved five first-order factors, two second-order factors and one third-order factor. The two second-order factors that correspond to the multiple representational flexibility and problem solving ability, respectively, regressed on a third-order factor that stands for the conceptual understanding of function. On the second-order factor that stands for the multiple representational flexibility three first-order factors are regressed. The first first-order factor referred to the concept image which consists of definition and examples tasks, the second first-order factor to the recognition of functions given in various representations and the third first-order factor to conversion tasks from a graphical to an algebraic representation of function and vice versa.

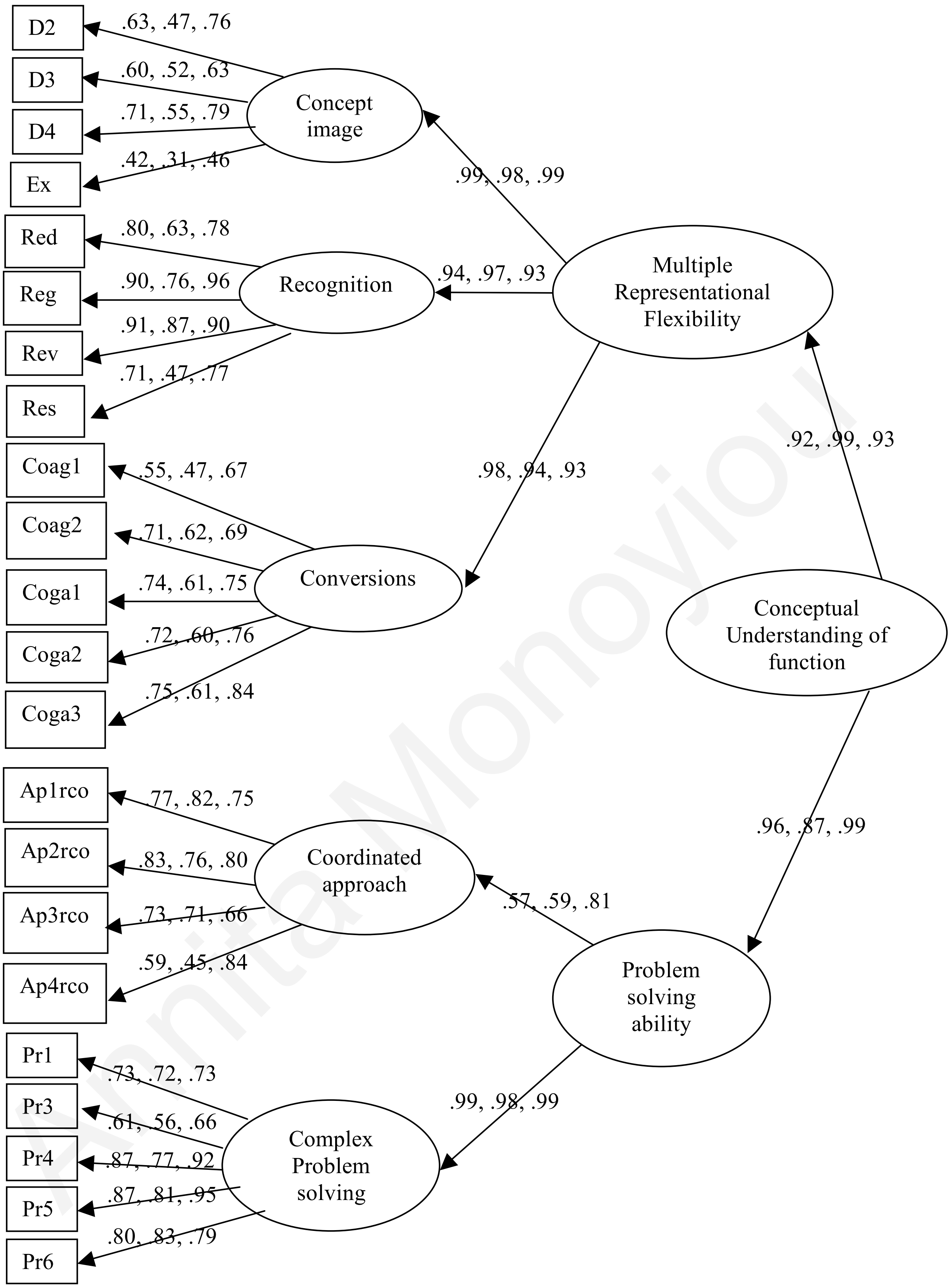


Figure 4.21. The confirmatory factor analysis model accounting for performance on the tasks of both tests by the whole sample, the Cypriot and Italian pre-service teachers separately

Thus, the findings revealed that concept image, the recognition of the concept given in various representations and the conversions from a graphical to an algebraic representation of the concept have a differential effect on the multiple representational flexibility concerning the concept of function. On the second-order factor that corresponds to problem solving ability two first-order factors were regressed. The first first-order factor involved the coordinated approach to simple function tasks and the second first-order factor consisted of the complex problems. Therefore the results indicated that coordinated approach to simple function tasks and the complex problems have an effect on problem solving ability.

To test for possible differences between the two countries in the structure described above, multiple-group analysis was applied, where the higher order model was fitted separately on each group. The model was first tested under the assumption that the relations of the observed variables to the five first-order factors would be equal across the two groups. The fit of this model was quite good [x^2 (369) = 548.29; CFI=.97; RMSEA=.046, 90% confidence interval for RMSEA=0.038-0.054]. In order to achieve an improvement of the model some of the equality constraints were released.

Releasing some of the constraints resulted in a considerable improvement of the model fit [x^2 (366) = 508.17; CFI =0.98; RMSEA = 0.041, 90% confidence interval for RMSEA=0.032-0.050]. The fit indices of all the models tested are illustrated in Table 4.20. Although the same structure holds for the two groups, in the multiple group model (see Figure 4.21), many of the factor loadings are stronger in the group of the Italian pre-service teachers. This finding indicated that the dependence of the conceptual understanding of function varies across the two groups.

It is noteworthy that the relations between the three first-order factors representing the concept image, the recognition and the conversion with the second-order factor representing the multiple representational flexibility are very strong indicating the fact that these three components are strongly related with representational flexibility. Furthermore, multiple representational flexibility and problem solving are strongly related with a conceptual understanding of function.

In order to examine whether there are statistically significant differences between the Cypriot and Italian pre-service teachers concerning the concept image, the recognition of the concept given in various representations, the conversion from an algebraic to a graphical representation of the concept and vice versa, their coordinated approach and their problem solving ability, a multivariate analysis of variance (MANOVA) was performed. Overall, the effects of teachers' nationality were significant (Pillai's $F_{(5, 452)} = 57.55$,

$p<0.001$). Table 4.21 presents the mean and standard deviation of concept definition, the recognition of the concept, the conversions, the coordinated approach and problem solving for the two groups.

Concerning the concept image Cypriot pre-service teachers ($\overline{X}$=0.60, SD=0.29) performed better than the Italian pre-service teachers ($\overline{X}$=0.35, SD=0.34) and this difference was statistically significant ($F_{(5, 452)} = 72.77$, $p<0.001$). The Cypriot pre-service teachers ($\overline{X}$=0.74, SD=0.20) had better results in the recognition of functions given in various representations than the Italian pre-service teachers ($\overline{X}$=0.38, SD=0.34). They also had better results in the conversions of functions from a graphical to an algebraic representation and vice versa ($\overline{X}$=0.70, SD=0.39, $\overline{X}$=0.39, SD=0.39 respectively). These differences were statistically significant ($F_{(5, 452)} = 196.86$, $p<0.001$, $F_{(5, 452)} = 91.18$, $p<0.001$ respectively). Statistically significant was also the difference observed in their coordinated approach to the simple function tasks ($F_{(5, 452)} = 4.47$, $p<0.05$). Cypriot pre-service teachers ($\overline{X}$=0.25, SD=0.32) used less the coordinated approach than the Italian pre-service teachers ($\overline{X}$=0.32, SD=0.39). Statistically significant was also the difference observed in their problem solving ability ($F_{(5, 452)} = 26.69$, $p<0.001$). Cypriot pre-service teachers ($\overline{X}$=0.38, SD=0.36) performed better than the Italian pre-service teachers ($\overline{X}$=0.21, SD=0.33).

Table 4.21

The mean and standard deviation of the concept image, the recognition, the conversions, the coordinated approach and problem solving for the Cypriot and Italian pre-service teachers

Groups	Concept image		Recognition of functions		Conversions of functions		Coordinated approach		Problem solving	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
Cyprus	0.60	0.29	0.74	0.20	0.70	0.30	0.25	0.32	0.38	0.36
Italy	0.35	0.34	0.38	0.34	0.39	0.39	0.32	0.39	0.21	0.33

It is noteworthy that overall the Cypriot pre-service teachers performed better than the Italian pre-service teachers in all the tasks except the coordinated approach. Furthermore, Cypriot and Italian teachers' performance in problem solving was lower than in the other tasks requiring a definition, a recognition and conversions of the concept.

The relation of the coordinated approach with the other dimensions of the understanding of function (concept image, recognition, conversions, problem solving)

In order to examine whether the Cypriot and Italian teachers who used a coordinated approach to solve the simple functions tasks performed better in the other dimensions of the understanding of function (concept image, recognition, conversions, problem solving) the Ward's method of hierarchical cluster analysis was used. The teachers were clustered into three distinct groups.

Concerning the Cypriot teachers, in the first group 49 teachers were clustered who used systematically the coordinated approach ($\overline{X}_{coordinated}$=0.83, $SD_{coordinated}$=0.19; $\overline{X}_{algebraic}$=0.08, $SD_{algebraic}$=0.12). In the second group 108 teachers were clustered who used extensively the algebraic approach ($\overline{X}_{coordinated}$=0.06, $SD_{coordinated}$=0.11; $\overline{X}_{algebraic}$=0.82, $SD_{algebraic}$=0.18). In the third group 95 teachers were clustered who used other approaches or used equally the algebraic and coordinated approach ($\overline{X}_{coordinated}$=0.17, $SD_{coordinated}$=0.16; $\overline{X}_{algebraic}$=0.18, $SD_{algebraic}$=0.20).

Concerning the Italian teachers, in the first group 66 teachers were clustered who used systematically the coordinated approach ($\overline{X}_{coordinated}$=0.84, $SD_{coordinated}$=0.19; $\overline{X}_{algebraic}$=0.05, $SD_{algebraic}$=0.09). In the second group 43 teachers were clustered who used extensively the algebraic approach ($\overline{X}_{coordinated}$=0.03, $SD_{coordinated}$=0.09; $\overline{X}_{algebraic}$=0.72, $SD_{algebraic}$=0.18). In the third group 97 teachers were clustered who used other approaches or used equally the algebraic and coordinated approach ($\overline{X}_{coordinated}$=0.09, $SD_{coordinated}$=0.17; $\overline{X}_{algebraic}$=0.13, $SD_{algebraic}$=0.17).

In order to examine whether there are statistically significant differences between the three groups (coordinated, algebraic and various approaches groups) concerning their concept image, recognition, conversions and problem solving ability, two multivariate analyses of variance (MANOVA), one for the Cypriot and one for the Italian pre-service teachers, were performed. Furthermore, the Scheffe test was employed.

Overall, the effects of teachers' group were significant for both the Cypriot (Pillai's $F_{(8, 494)}$ = 983.66, $p<0.001$) and the Italian (Pillai's $F_{(8, 402)}$ = 85.30, $p<0.001$) pre-service teachers.

Particularly, concerning the Cypriot teachers there were significant differences between the three groups concerning the concept image ($F_{(4, 246)} = 18.41$, $p<0.001$), the recognition of functions given in various representations ($F_{(4, 246)} = 14.67$, $p<0.001$), the conversions ($F_{(4, 246)} = 25.82$, $p<0.001$) and problem solving ($F_{(4, 246)} = 54.55$, $p<0.001$). Table 4.22 presents the mean and standard deviation of concept image, recognition, conversions and problem solving of the three groups for Cypriot teachers.

Specifically concerning the concept definition, significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.25$, $p<0.001$) and the first and third group ($\overline{X}_1-\overline{X}_3=.28$, $p<0.001$). Similarly concerning the recognition of functions significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.12$, $p<0.01$) and the first and third group ($\overline{X}_1-\overline{X}_3=.19$, $p<0.001$). As far as the conversions is concerned significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.13$, $p<0.05$), the first and third group ($\overline{X}_1-\overline{X}_3=.33$, $p<0.001$) and the second and third group ($\overline{X}_1-\overline{X}_2=.20$, $p<0.001$). Lastly concerning problem solving significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.38$, $p<0.001$), the first and third group ($\overline{X}_1-\overline{X}_3=.55$, $p<0.001$) and the second and third group ($\overline{X}_1-\overline{X}_2=.17$, $p<0.001$).

Table 4.22

The mean and standard deviation of the concept image, the recognition, the conversions and problem solving for the three groups of Cypriot pre-service teachers

Groups	Concept image		Recognition of functions		Conversions of functions		Problem solving	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
1:Coordinated	0.81	0.25	0.86	0.18	0.88	0.20	0.74	0.35
2:Algebraic	0.56	0.28	0.74	0.19	0.75	0.28	0.36	0.30
3:Other approaches	0.53	0.27	0.67	0.20	0.55	0.30	0.19	0.27

Concerning the Italian teachers there were also significant differences between the three groups concerning the concept image ($F_{(4, 200)} = 26.29$, $p<0.001$), the recognition of functions given in various representations ($F_{(4, 200)} = 20.74$, $p<0.001$), the conversions ($F_{(4, 200)} = 17.82$, $p<0.001$) and problem solving ($F_{(4, 200)} = 59.54$, $p<0.001$).

Table 4.23 presents the mean and standard deviation of concept image, recognition, conversions and problem solving of the three groups for Italian teachers.

Table 4.23

The mean and standard deviation of the concept image, the recognition, the conversions and problem solving for the three groups of Italian pre-service teachers

Groups	Concept image		Recognition of functions		Conversions of functions		Problem solving	
	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD	$\overline{X}$	SD
Coordinated	0.57	0.36	0.58	0.37	0.60	0.44	0.52	0.35
Algebraic	0.30	0.30	0.33	0.28	0.40	0.34	0.08	0.18
Other approaches	0.22	0.26	0.26	0.27	0.25	0.31	0.09	0.24

In all cases significant differences were found only between the coordinated and the algebraic group and the coordinated and the various approaches group. The differences between the algebraic and the various approaches group were not significant. Specifically concerning the concept image, significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.27$, $p<0.001$) and the first and third group ($\overline{X}_1-\overline{X}_3=.35$, $p<0.001$). Similarly concerning the recognition of functions significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.25$, $p<0.001$) and the first and third group ($\overline{X}_1-\overline{X}_3=.32$, $p<0.001$). As far as the conversions are concerned significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.20$, $p<0.05$) and the first and third group ($\overline{X}_1-\overline{X}_3=.35$, $p<0.001$). Lastly concerning problem solving significant differences were found between the first and second group ($\overline{X}_1-\overline{X}_2=.44$, $p<0.001$) and the first and third group ($\overline{X}_1-\overline{X}_3=.43$, $p<0.001$).

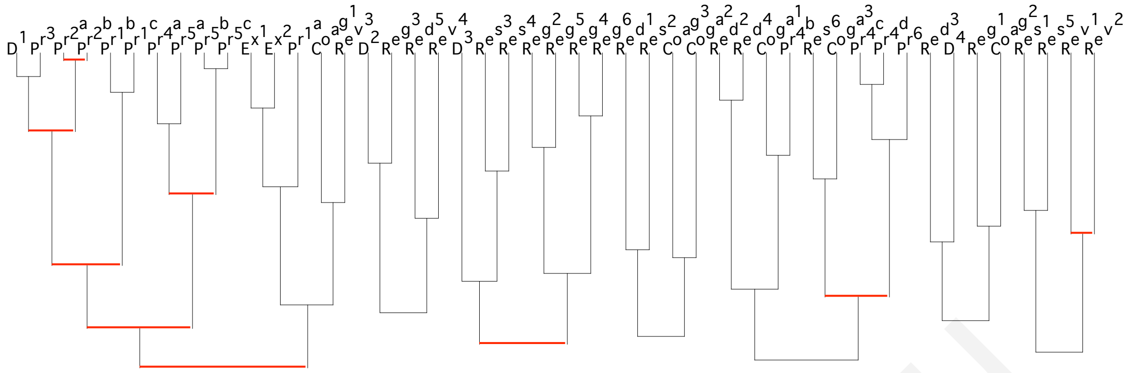
In general both the Cypriot and the Italian coordinated approach groups performed better than the other groups in all the dimensions of the understanding of function.

In order to examine whether there were differences in the way Cypriot and Italian teachers who used a coordinated approach behaved during the solution of the definition, examples, recognition, conversion tasks and problem solving three similarity diagrams for Cypriot and three for Italian pre-service teachers were employed, one for each group.

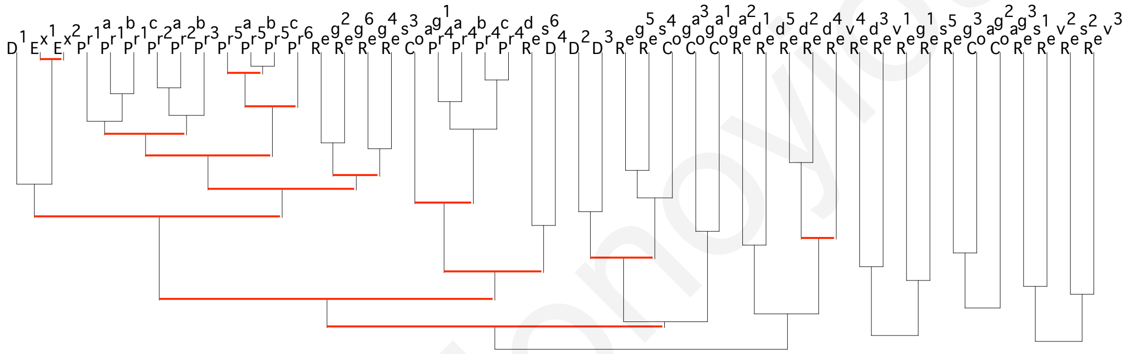
Figure 4.22 and figure 4.23 show the similarity diagrams of the Cypriot and Italian teachers' responses participating in the coordinated, algebraic and various approaches group respectively. In the case of the Cypriot pre-service teachers' coordinated group, problem solving is related in the first and fifth cluster with all the other dimensions of the understanding of function the definition and examples of the concept, the conversions from an algebraic to a graphical representation of the concept and vice versa and the recognition of the concept given in various representations. In the case of the Cypriot teachers' algebraic group, the problem solving tasks are "isolated" from the other dimensions of the understanding of function. Specifically, "Pr1a", "Pr1b", "Pr1c", "Pr2a", "Pr2b" and "Pr3" that are the problem solving tasks of the first test are strongly related between them and with the problem solving tasks "Pr5a", "Pr5b", "Pr5c" and "Pr6" that are involved in the second test. The problem solving tasks "Pr4a", "Pr4b", "Pr4c' and "Pr4d" are also connected in a separate cluster. In the similarity diagram of the Cypriot teachers' various approaches group, the problem solving is even more isolated forming a separate cluster. Only the problem solving task "Pr1a" is related with the examples of the concept "Ex1" and "Ex2".

Similar results are obtained from the three similarity diagrams of Italian pre-service teachers. In the case of the Italian pre-service teachers' coordinated group, problem solving is related in the first and fourth cluster with all the other dimensions of the understanding of function the definition and examples of the concept, the conversions from an algebraic to a graphical representation of the concept and vice versa and the recognition of the concept given in various representations. In the case of the Italian teachers' algebraic group, the problem solving tasks are "isolated" from the other dimensions of the understanding of function. Specifically, "Pr1a", "Pr1b", "Pr1c", "Pr2a", "Pr2b", "Pr3", "Pr4a", 'Pr4b", "Pr4c", "Pr4d", "Pr5b" and "Pr5c" are strongly connected in a separate cluster. The only exceptions are "Pr5a" which is connected with a conversion task and "Pr6" which is connected with a recognition task. In the similarity diagram of the Italian teachers' various approaches group, the problem solving is even more isolated forming a separate cluster. Only the problem task "Pr6" is related with the definition of the concept "D3" but this connection is rather weak.

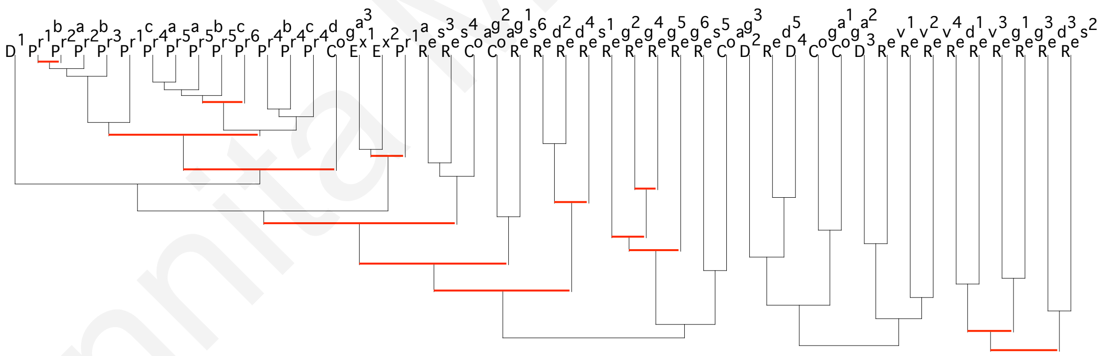
In general from the similarity diagrams of both the Cypriot and the Italian pre-service teachers it was noteworthy that in the case of the algebraic and the various approaches groups similarity groups by variables corresponding to the same cognitive type of tasks, i.e. function definition, examples, recognition of functions, conversions of functions and function problem solving, were formed revealing that pre-service teachers responded in tasks involving the same kind of mathematical thinking in a consistent and coherent manner. However, lack of connections between different types of tasks, i.e., providing a correct definition of function, recognizing the concept in different contexts, conversions from an algebraic to a graphical representation of function and function problem solving, indicated pupils' incoherent behaviour in dealing with tasks of different cognitive features referring to the same concept.



1: Coordinated group

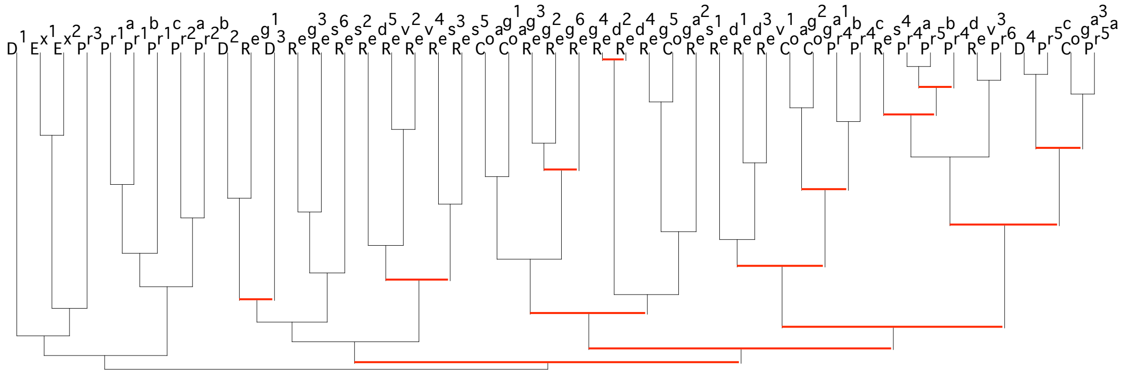


2: Algebraic group

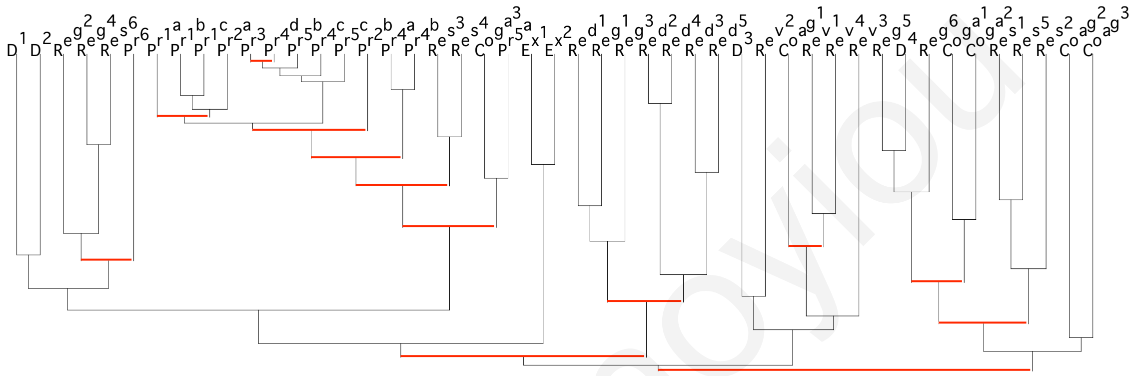


3: Various approaches group

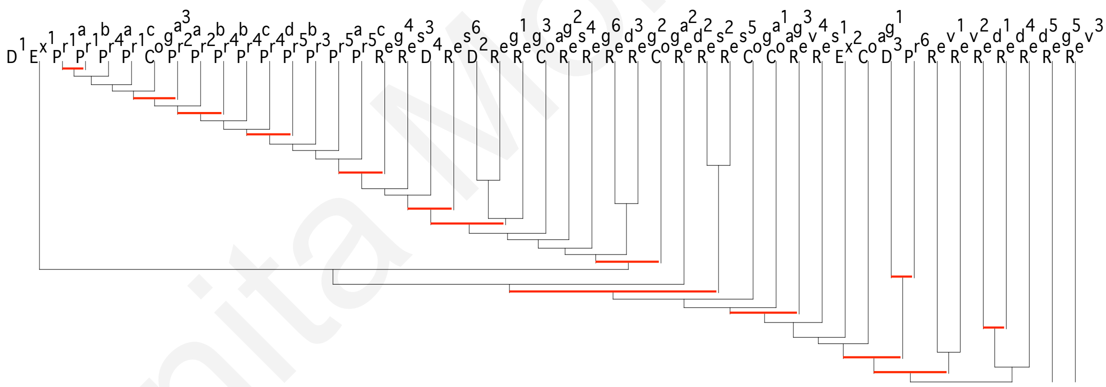
Figure 4.22. Similarity diagrams of the Cypriot pre-service teachers' responses participating in the coordinated, algebraic and various approaches groups respectively



Coordinated group



Algebraic group



Various approaches group

Figure 4.23. Similarity diagrams of the Italian pre-service teachers' responses participating in the coordinated, algebraic and various approaches groups respectively

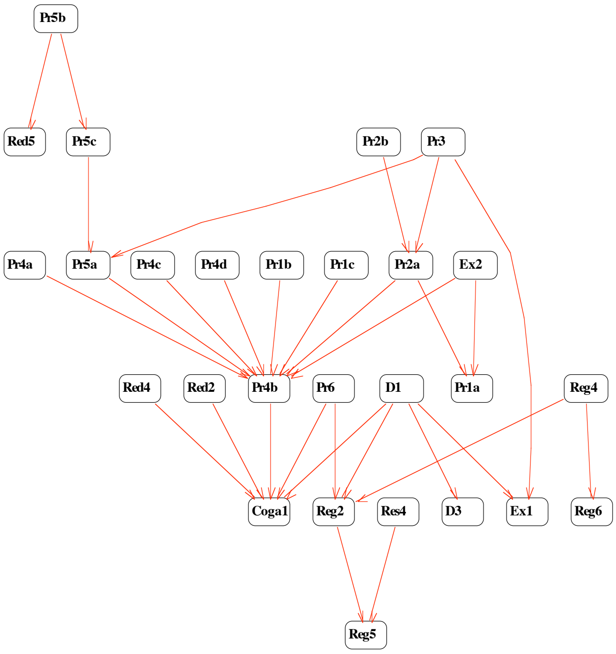
Thus, the phenomenon of compartmentalization holds for these two groups of pre-service teachers (algebraic and various approaches group) in the case of tasks involving different mathematical reasoning and use of this concept. In the case of the coordinated approach group a de-compartmentalized way of thinking was observed since problem solving was found to be related to all the other dimensions of the conceptual understanding of function.

The findings of the implicative diagrams (Figure 4.24 and 4.25) are in line with the similarity connections described and discussed above. A first observation is that in the case of the algebraic groups for both the Cypriot and Italian pre-service teachers the implicative diagrams involve a limited number of connections between the tasks. Furthermore, in both diagrams different implicative chains are formed. In the case of Cypriot pre-service teachers, four different chains are formed with the first chain involving only recognition tasks and the fourth only problem solving tasks. The second and third chains involve different tasks (definition, examples, conversion, recognition and problem solving) but the implicative relations among them are limited. As far as the Italian pre-service teachers are concerned three implicative chains are formed. The first involves recognition tasks given in a symbolic representation, the second one involves a problem solving task and some recognition tasks and the third chain involves two problem solving tasks and some recognition tasks. In the case of the Italian algebraic group implicative relations exist only between problem solving and the recognition of the concept.

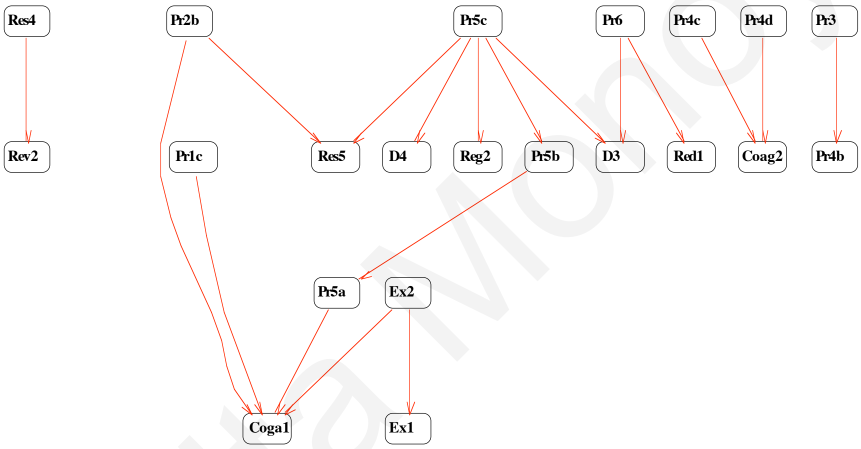
In the case of the Cypriot and Italian coordinated group the implicative diagrams are quite rich in implicative relations. It is noteworthy that in these diagrams implicative relations exist between all the different tasks of the understanding of function. Concerning the various approaches groups the problem solving tasks are mainly related to the recognition and conversion tasks. In the case of the Cypriot teachers three chains are formed with the first two involving only recognition tasks given in a symbolic and a diagrammatic expression respectively.

A common observation in all the implicative diagrams is that in the beginning of the implicative chains are the problem solving tasks. This indicates that Cypriot and Italian pre-service teachers' proficiency in tackling complex problems involving the concept of function entails success in recognition and conversion tasks involving different types of representation of functions. It also implies competency in providing an accurate definition and an appropriate examples of function.

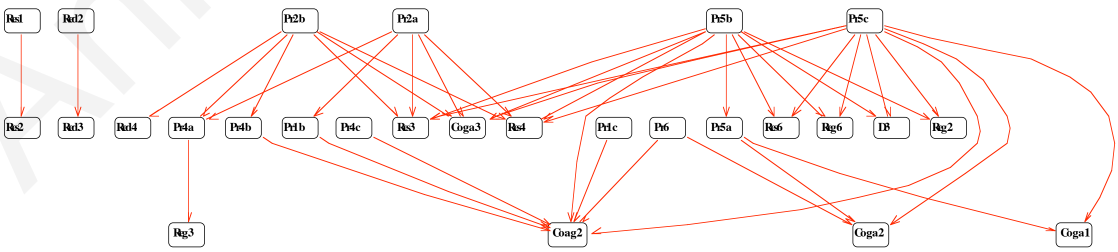
These findings are in line with the fact that all the pre-service teachers who used a coordinated approach, were very successful in problem solving, in definition and examples of the concept and furthermore succeeded at the use of the various representations in the recognition and conversion tasks of the test.



Coordinated group

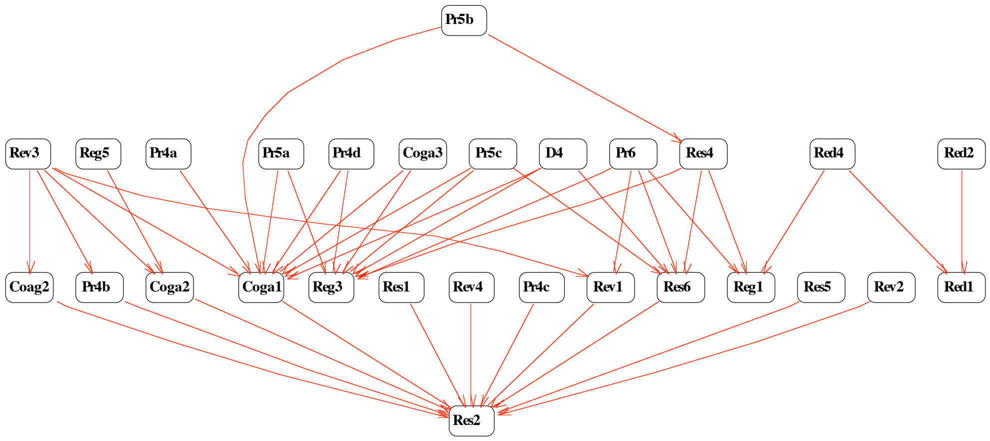


Algebraic group

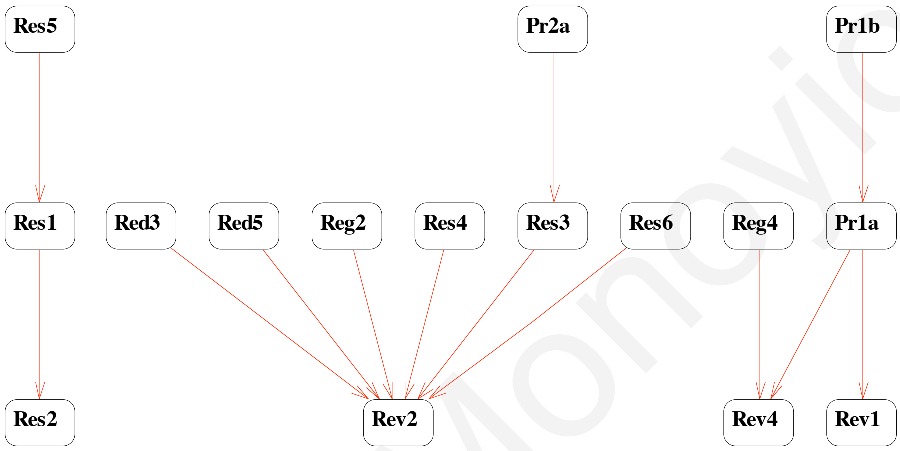


Various approaches group

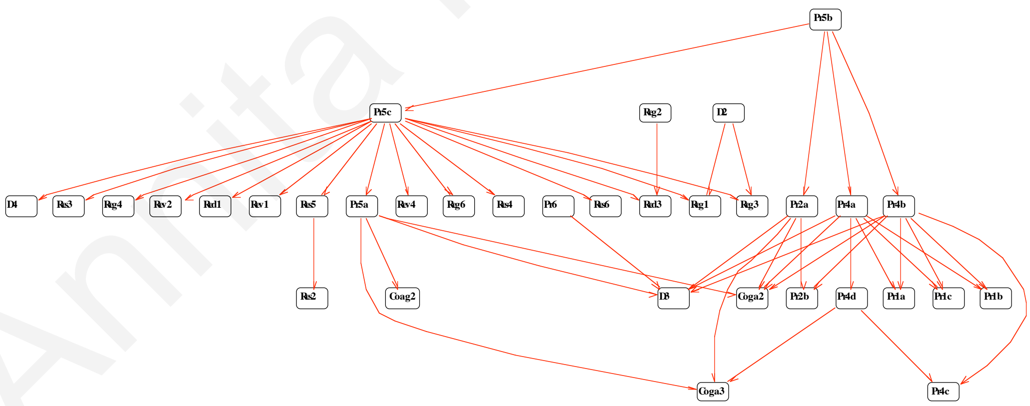
Figure 4.24. Implicative diagrams of the Cypriot pre-service teachers' responses participating in the coordinated, algebraic and various approaches groups respectively



Coordinated group



Algebraic group



Various approaches group

Figure 4.25. Implicative diagrams of the Italian pre-service teachers' responses participating in the coordinated, algebraic and various approaches groups respectively

The results of the third phase

The goal of the third phase was the triangulation of the quantitative data regarding teachers understanding of the concept of function and the further investigation of pre-service teachers' behaviour in the above mentioned five aspects of the understanding of function: concept definition, examples of function, the recognition of functions given in various representations, "coordinated" or algebraic approaches when dealing with simple function tasks and effectiveness in problem solving. Nine task-based interviews were conducted. Task-based interviews were used because they are powerful means to focus on "subjects' purposes of addressing mathematical tasks, rather than just on patterns of correct and incorrect answers in the results they produce" (Goldin, 2000).

The participants were nine of the second phase's Cypriot participants. They were chosen according to their performance and their approach (algebraic or coordinated) in the two tests of the second phase. The interview consisted of nine tasks. In the first task the teachers were asked to give a definition of the concept of function and a simple example from the applications of functions in everyday life. In the second task they were asked if $f(-2) = f(3) = 4$ is a function or not. In this task the teachers had to employ a right definition of the concept in order to give a correct answer. In the third task they were asked to give an example of a function f such that for any real numbers x, y in the domain of f the following equation holds: $f(x+y) = f(x) + f(y)$ (Sajka, 2003). In the fourth task they were asked to recognize the concept given in different representations (arrow diagrams, graphs, symbolic, verbal). The fifth and sixth tasks were vertical and horizontal transformation tasks. In the fifth task the function $y = x^2 + x$ was given. The teachers were asked to draw the function $y = x^2 + x + 1$. In the seventh task the function $\sin(x)$ was given. The teachers were asked to draw the function $\sin(x+1)$. The other three tasks were complex problems with functions. The first problem involved a linear function, the second a quadratic function and the third an exponential relationship. The problems examined teachers' ability to use various representations, their ability to translate one representation to the other and their conceptual understanding of the concept.

The interviews were analysed vertically and horizontally. In the case of the vertical analysis we studied separately each participant's responses to the nine tasks of the interview. For this analysis three two-dimensional matrices were constructed. Matrices are the simplest, and probably most used and most useful, way of organising the data of an interview. In a matrix there is a substantial data reduction which is derived from 8-10 pages of interview notes (Robson, 1993). In this case the rows represent the nine tasks and the columns the nine participants.

In the first matrix (Table 4.24) the responses of the high performance participants to the nine tasks of the interview are presented. The two participants gave an accurate definition for the concept of function while the third gave a definition of a special kind of function (one-to-one function). They all gave a correct example from the applications of functions in everyday life by using discrete elements of sets. The third participant although she gave a correct example hers example was of a one-to-one function. The two participants gave also correct responses to the tasks involving a definition of function (Task 2) and an example of the concept (Task 3). The third participant gave a wrong response to the task 2 influenced by the one-to-one definition of function she gave. She also did not manage to solve task 3 involving an example of function. All the participants recognized successfully the functions given to them in various representations diagrammatic, graphical, symbolical and verbal. They all used a coordinated approach to solve the two simple tasks. In the case of Task 5 they all gave correct responses. In the case of Task 6, which was quite difficult, since it involved a horizontal transformation only one of them managed to give a correct response. The other two gave a response the first one with a minor mistake while the other with a major mistake. Their performance in problem solving was high. Particularly, they all solved correctly the problems 7 and 8. Problem 7 involved a linear function while problem 8 involved a quadratic function. In problem 9 which involved an exponential relation, they all recognized this relation and the two of them managed also to give the equation. In general, the high performance teachers were quite successful in all the aspects of the understanding of function.

In the second matrix (Table 4.25) the responses of the medium performance participants to the nine tasks of the interview are presented. The first participant gave an accurate definition for the concept of function, the second gave a definition of a special kind of function (one-to-one function) while the third participant made a reference to an ambiguous relation. They all gave a correct example from the applications of functions in everyday life. Two of them gave an example by using discrete elements of sets while the third participant gave an example of a continuous function. All the participants gave wrong responses to the tasks involving a definition of function (Task 2) and an example of the concept (Task 3). The two participants recognized successfully the functions given to them in various representations diagrammatic, graphical, symbolical and verbal. The third participant did not manage to recognize the function given to him/her in a symbolic and verbal expression. They all used a coordinated approach to solve the two simple tasks. In the case of Task 5 they all gave correct responses. In the case of Task 6, which was quite difficult, they all gave responses with major mistakes. Their performance in problem solving was moderate. Particularly, in problem 7 which involved a linear function the second participant did not manage to find the particular value and the third one did not manage to find the equation. In problem 8, they all found the equation and drew the graph of the function successfully but they did not manage to find the maximum point. In problem 9 which involved an exponential relation, they all recognized this relation as linear. In general, the medium performance teachers showed limited understanding of the concept of function.

In the third matrix (Table 4.26) the responses of the low performance participants to the nine tasks of the interview are presented. All the participants gave wrong definitions and made a reference to an ambiguous relation. They also all gave wrong examples from the applications of functions in everyday life. Two of them gave an example of an equation in a symbolic form and the other gave an example presenting an uncertain transformation of the real world. All the participants gave wrong responses to the tasks involving a definition of function (Task 2) and an example of the concept (Task 3). All the participants recognized successfully the function given to them in a diagrammatic representation. Two of them managed also to recognize correctly the function given to them in a graphical representation. The participants did not manage to recognize the functions given to them in a symbolic and verbal expression. They all used an algebraic approach to solve Task 5 and gave correct response. In the case of Task 6, which was quite difficult, they all used a coordinated approach since they were not able to use an algebraic approach but they gave responses with major mistakes. Their performance in problem solving was low. Particularly, in problem 7 which involved a linear function only two of them managed to find the equation and draw the graph correctly while only one of them managed to find the particular value. In problem 8, they all found the equation, two of them drew the graph of the function successfully but they did not manage to find the maximum point. In problem 9 which involved an exponential relation, they all recognized this relation as linear. In general, the low performance teachers displayed ambiguous or limited ideas of the function concept.

Table 4.24

Pre-service teachers' -who had a high performance on the two tests of the second phase-responses to the nine tasks of the task-based interview

	High Performance		
	1	2	3
1a. Definition	Accurate	Accurate	One-to-one
1b. Example of the applications in everyday life	Correct Discrete elements	Correct Discrete elements	Correct Discrete elements (one-to-one function)
2. Definition	Correct	Correct	Wrong
3. Example	Correct	Correct	Wrong
4a. Recognition (Diagram)	Correct	Correct	Correct
4b. Recognition (Graph)	Correct	Correct	Correct
4c. Recognition (Symbolic)	Correct	Correct	Correct
4d. Recognition (Verbal)	Correct	Correct	Correct
5. Approach	Coordinated Correct	Coordinated Correct	Coordinated Correct
6. Approach	Coordinated Correct	Coordinated Major mistake	Coordinated Minor mistake
7a. Linear-Problem (Find Equation)	Correct	Correct	Correct
7b. Problem (Draw Graph)	Correct	Correct	Correct
7c. Problem (Find a value)	Correct	Correct	Correct
8a. Quadratic-Problem (Find Equation)	Correct	Correct	Correct
8b. Problem (Draw a graph)	Correct	Correct	Correct
8c.Problem (Find the maximum)	Correct	Correct	Correct
9. Exponential-Problem (Find equation and graph)	Recognize the exponential relation without giving an equation	Recognize the exponential relation and gives the equation.	Recognize the exponential relation and gives the equation.

Table 4.25

Pre-service teachers' -who had a medium performance on the two tests of the second phase- responses to the nine tasks of the task-based interview

	Medium Performance		
	4	5	6
1a. Definition	Accurate	One-to-one Function	Ambiguous relation
1b. Example of the applications in everyday life	Correct Discrete elements	Correct Discrete elements	Correct Continuous function
2. Definition	Wrong	Wrong	Wrong
3. Example	Wrong	Wrong	Wrong
4a. Recognition (Diagram)	Correct	Correct	Correct
4b. Recognition (Graph)	Correct	Correct	Correct
4c. Recognition (Symbolic)	Correct	Correct	Wrong
4d. Recognition (Verbal)	Correct	Correct	Wrong
5. Approach	Coordinated Correct	Coordinated Correct	Coordinated Correct
6. Approach	Coordinated Major mistake	Coordinated Major mistake	Coordinated Major mistake
7a. Linear-Problem (Find Equation)	Correct	Correct	Wrong
7b. Problem (Draw Graph)	Correct	Correct	Correct
7c. Problem (Find a value)	Correct	Wrong	Correct
8a. Quadratic-Problem (Find Equation)	Correct	Correct	Correct
8b. Problem (Draw a graph)	Correct	Correct	Correct
8c.Problem (Find the maximum)	Wrong	Wrong	Wrong
9. Exponential-Problem (Find equation and graph)	Linear function	Linear function	Linear function

Table 4.26

Pre-service teachers' -who performed low on the two tests of the second phase- responses to the nine tasks of the task-based interview

	Low Performance		
	7	8	9
1a. Definition	Ambiguous relation	Ambiguous relation	Ambiguous relation
1b. Example of the applications in everyday life	Wrong Equation	Wrong Equation	Wrong Description of a situation
2. Definition	Wrong	Wrong	Wrong
3. Example	Wrong	Wrong	Wrong
4a. Recognition (Diagram)	Correct	Correct	Correct
4b. Recognition (Graph)	Correct	Correct	Wrong
4c. Recognition (Symbolic)	Wrong	Wrong	Wrong
4d. Recognition (Verbal)	Wrong	Wrong	Wrong
5. Approach	Algebraic Correct	Algebraic Correct	Algebraic Correct
6. Approach	Coordinated Major mistake	Coordinated Major mistake	Coordinated Major mistake
7a. Linear-Problem (Find Equation)	Correct	Correct	Wrong
7b. Problem (Draw Graph)	Correct	Wrong	Correct
7c. Problem (Find a value)	Wrong	Wrong	Correct
8a. Quadratic-Problem (Find Equation)	Correct	Correct	Correct
8b. Problem (Draw a graph)	Wrong	Correct	Correct
8c.Problem (Find the maximum)	Wrong	Wrong	Wrong
9. Exponential-Problem (Find equation and graph)	Linear function	Linear function	Linear function

In order to gain further information concerning teachers' conceptual understanding of the concept of function we also performed a horizontal analysis. Every task was analyzed separately and teachers' representative answers were emphasized and discussed.

Task 1:

1a. What is a function? (Give a simple definition).

1b. Can you give a simple example from the applications of functions in everyday life?

Table 4.27

Pre-service teachers' representative answers to task 1a involving the definition of function

Participant	Emphasized aspects	Representative protocols
1	Accurate definition	"Function is for example one set of values. Every element of the first set (A) has a corresponding element in the second set (B). For every x there is a y. We cannot have for one x, two y. Two values of x can lead to a y."
3	Special kind of function (one-to-one function)	"Function is a one to one correspondence between a domain and a range. An example is the mother with her child."
6	Ambiguous definition	"The function has two variables that are related the one to the other."
7	Ambiguous definition	"P7: I cannot tell you exactly what a function is. The only think I can tell you is that it is an equation for example $y = 2x + 1$ … I: So every equation is a function? For example $y^2 = x$ is a function? P7: No, it is not a function. It must be $y = ax + b$."

Table 4.28

Pre-service teachers' representative answers to task 1b involving an example of the concept

Participant	Emphasized aspects	Representative protocols
2	Correct example-Discrete elements	"P2: Let's say that we have two teams, boys and girls. A girl will dance with the boys. I: If a girl dances with more than one boy? P2: No, this is not a function. Every girl must dance only with one boy."
3	Correct example-Discrete elements (one-to-one function)	"P3: An example is the mother with her child. I: If a mother has more than one child? P3: Then it is not a function. Every x has only one y."
4	Correct example-Discrete elements	"The grades. A student will have one grade. But some students may have the same grade."
6	Correct example – Continuous function	"An example is the speed with time."
7	Wrong example-equation	"If we have $y = ax + b$ we put values for x and take a y. So x is an independent variable and y is a dependent variable."
8	Wrong example-equation	"I: Can you give me an example of a function? P8: You mean you want me to tell you an equation? I: Whatever you like… P8: For example f. Function is usually f … $f(x)$ equals something."
9	Wrong example-Description of a situation	"The car and the speed is an example."

In Task 1, pre-service teachers were asked to give a simple definition of function and an example from the applications of functions in everyday life. Table 4.27 shows pre-service teachers' representative answers to task 1a involving the definition of function. Three teachers managed to give an accurate definition for the concept of function defining the domain and range and making a reference to the relation between the two variables (par. 1). Two teachers gave a definition of a special kind of function particularly they made reference to an injective function (one-to-one correspondence) (par. 3). The other four teachers made a reference to an ambiguous relation. Particularly, three of them made a reference to a relation between variables or elements of sets (par. 6). The other gave a symbolic example. Participant 7 has the belief that a function is an equation that has the form $y = ax + b$, so all functions are linear.

Table 4.28 shows pre-service teachers' representative answers to task 1b involving an example of the applications of functions in everyday life. Five teachers gave a correct example of a function with the use of discrete elements of sets although one of them gave an example of an injective function (par. 3). Two of them gave the characteristic example with the students and their grades (par. 4). A pre-service teacher gave an example of a continuous function from physics. Particularly he/she made a reference to the speed of a car as a function of time (par. 6). Since this is a quite popular example another teacher tried to give this example but was restricted to the description of a real world situation (par.9). Two teachers gave examples presenting an equation instead of a function (par. 7 and 8).

In Task 2 pre-service teachers were asked if $f(-2) = f(3) = 4$ is a function or not. Table 4.29 shows pre-service teachers' representative answers to task 2. Only two teachers gave correct answers to this task indicating that it is a function since two values of x have the same y but not the opposite (par.1). The other seven teachers claimed that this is not a function since for two values of x we have the same y (par. 8). These teachers may have the misunderstanding that a function must be a one-to-one correspondence.

Task 2:

If $f(-2) = f(3) = 4$*, then* f *is a function. This statement is right or wrong and why?*

Table 4.29

Pre-service teachers' representative answers to task 2 involving a definition of the concept

	Emphasized aspects	Representative protocols
1	Correct answer	"P1: It is a function because for two values of x we have one y. The opposite is not possible. I: What do you mean? P1: One value of x cannot lead to two values of y."
8	Wrong answer – The function must be an injective relation	"P8: No it is not a function. I: Why? P8: Because $x = -2$ and $x = 3$ and we cannot have for both $y = 4$."

In Task 3 pre-service teachers were asked to give an example of a particular function. Table 4.30 shows pre-service teachers' representative answers to this task. This task was quite difficult for the participants since only two of them managed to give a correct answer. The two participants who gave a correct example managed this after several tries (par. 2). The other seven participants did not manage to give an example of such function and characterized this task as very difficult (par. 5).

Task 3:

Give an example of a function f such that for any real numbers x, y in the domain of f the following equation holds: $f(x+y) = f(x) + f(y)$. *(Sajka, 2003)*

Table 4.30

Pre-service teachers' representative answers to task 3 involving an example of the concept

	Emphasized aspects	Representative protocols
2	Correct example	"P2: For example $f(x) = x + 3$, for $x = 2$ then $y = 5$ and for $x = 1$ then $y = 4$. Then if we put $1 + 2 = 3$ for x then $y = 6$ and not 9. This is not the function. I: Can you think another example? P2: $y = ax$ … for example $y = 3x$. For $x = 0$ then $y = 0$. For $x = 1$ then $y = 3$. So for $f(1) = 0 + 3$.. That's correct."
5	Wrong example	"P5: You mean that if I put 8 at $x + y$ it will be the same with $f(5) + f(3)$ I: Yes… P5: I understand this but I am not sure if I can give an example."

In Task 4 pre-service teachers were asked to recognize functions given in various representations, namely, diagrammatic, graphical, symbolic and verbal expressions. Table 4.31 shows pre-service teachers' representative answers to task 4a involving the recognition of a function given in a diagram. This task was very easy since all the participants stated that this is not a function and gave a right justification.

Task 4:

Which of the following correspondences are functions? Explain your answers.

Task 4a:

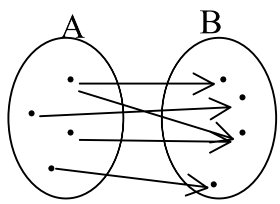


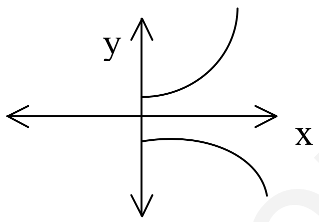
Table 4.31

Pre-service teachers' representative answers to task 4a involving the recognition of a function given in a diagrammatic representation

	Emphasized aspects	Representative protocols
2	Correct answer	"P2: This is not a function. For an x we have two y."

Table 4.32 shows pre-service teachers' representative answers to task 4b involving the recognition of a function given in a graph.

Task 4b:



Again this task was quite easy for the participants since eight of them gave a correct answer. Specifically, three of them gave a correct answer and justified their solution graphically. Particularly, they mentioned that if we draw a vertical line we can see that for one x we have two y (par. 1, see figure 4.26). The other five teachers gave a correct answer and justified it be saying that for one x we have two y. One of them before giving a justification she asked if the graph is one or two separate graphs. The teacher who gave a wrong response held the idea that every graph is a function.

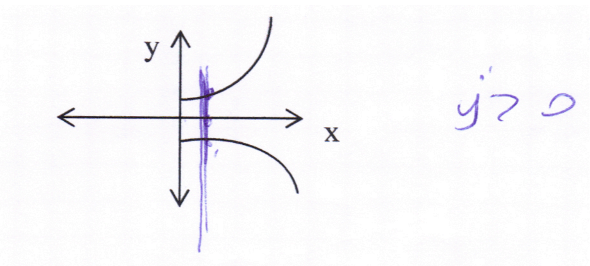


Figure 4.26. Participant's 1 inscriptions for task 4b

Table 4.32

Pre-service teachers' representative answers to task 4b involving the recognition of a function given in a graphical representation

	Emphasized aspects	Representative protocols
1	Correct answer – Graphical solution (Vertical line)	"I: What about this graph? Is it a function or not? P1: If I draw a vertical line, I can see that for one x we have two y. It is not a function. I: Under which circumstances can this graph be a function? P1: For example for y>0 or for y<0."(see figure 4.26)
6	Correct answer	"P6: It is not a function. Is this one graph or two different graphs? I: It is one graph. P6: It is not a function. For the same x we have two y."
9	Wrong answer	"It is a function. It is a graph."

Table 4.33 shows pre-service teachers' representative answers to task 4c involving the recognition of a function given in a symbolic expression.

Task 4c:

$x^2 + y^2 = 2 \quad , \ x \in R$

Table 4.33

Pre-service teachers' representative answers to task 4c involving the recognition of a function given in a symbolic representation

	Emphasized aspects	Representative protocols
1	Correct answer	"I: What about this equation? Is it a function or not? P1: Sure it is not a function. Is it the equation of a circle? I am not sure about it….. $y^2 = 2 - x^2$ That means that for one x there are two y. I: Under which circumstances can this equation be a function? P1: We have a root. We can take either the minus or plus."
3	Correct answer	"It is not a function. This is the equation of a circle. For one x we have two y."
7	Wrong answer	"It is a function. It's an equation so it is a function."
8	Wrong answer	"We have to solve this equation and make it $y =$ …. Can we do that? I am not sure. If $y^2 = x^2 - 2$. So this is a function because we have y equals something…"

Five teachers gave a correct response to task 4c (par. 1 and par. 3). Three of them mentioned that this is the equation of a circle so it is not a function. The other two stated that for every x there are two y so it is not a function. Four of the pre-service teachers stated that this is an equation with x and y so it is a function (par. 7 and par. 8). They held the idea that every equation is a function.

Table 4.34 shows pre-service teachers' representative answers to task 4d involving the recognition of a function given in a verbal expression.

Task 4d:

We correspond every human with his/her nationality.

Table 4.34

Pre-service teachers' representative answers to task 4d involving the recognition of a function given in a verbal representation

	Emphasized aspects	Representative protocols
4	Correct answer	"I: What about this verbal expression? Is it a function or not? P4: If a person has only one nationality it is a function. If not, it isn't a function. Can a person have two nationalities? I: Yes maybe. P4: Ok, then it is not a function."
7	Wrong answer	"We have two sets. In the first set we have the names of the people and in the second set we have their nationalities. We correspond every human with her/his nationality so it is a function."

Five teachers gave a correct response to this task indicating that this expression is a function if a person has only one nationality (par. 4). Furthermore, they emphasized the fact that if a person has more than one nationality then it is not a function. The other four teachers answered that it is a function since we correspond every human with his/her nationality (par. 7).

Tasks 5 and 6 were simple function tasks involving a vertical and a horizontal transformation respectively. Particularly, in each task, there were two functions in algebraic form and one of them was also in a graphical representation. Pre-service teachers were asked to interpret graphically the second function. In these tasks we also characterized teachers' answers as algebraic or coordinated. A solution was coded as "algebraic" if teachers did not use the information provided by the graph of the first function and they proceeded constructing the graph of the second function by finding pairs of values for x and y. On the contrary, a solution was coded as coordinated if students observed and used the relation between the two functions in constructing the graph of the second function.Table 4.35 shows pre-service teachers' representative answers to task 5.

Task 5:

In the following diagram $y = x^2 + x$ *is given. Draw the function* $y = x^2 + x + 1$.

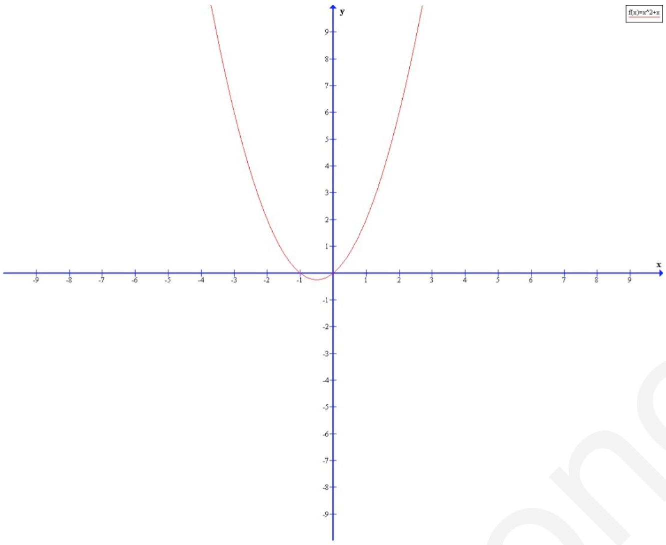


Table 4.35

Pre-service teachers' representative answers to task 5 involving the coordinated or algebraic approaches to simple function tasks

	Emphasized aspects	Representative protocols
4	Correct answer – Coordinated approach	"P4: Ok, for $x = 0$ then $y = 1$. So, I found a point. It will be the same function put it will intersect the axis here. I: Why it will be the same? P4: Because is plus 1. So it will alter plus 1."
7	Correct answer – Algebraic approach	"I will find values for x and y. For example for $x = 0$ then $y = 1$, for $x = 1$ then $y = 3$ and for $x = 2$ then $y = 4 + 2 + 1 = 7$. I have three points and now I can draw the function."

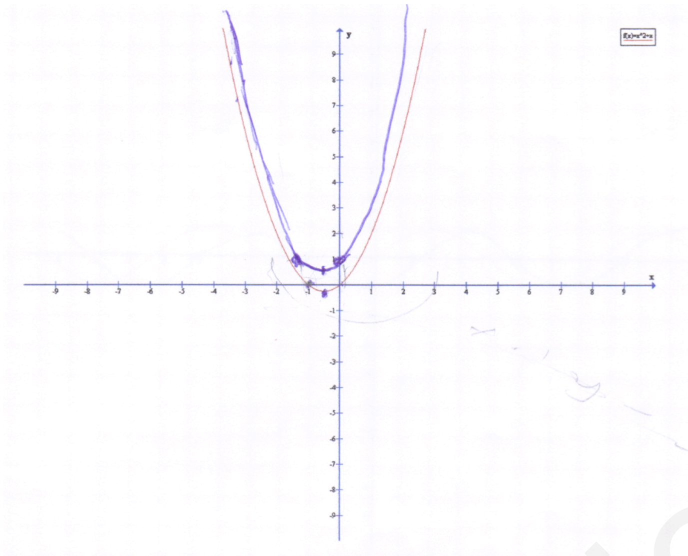


Figure 4.27. Participant 1's inscriptions for task 5

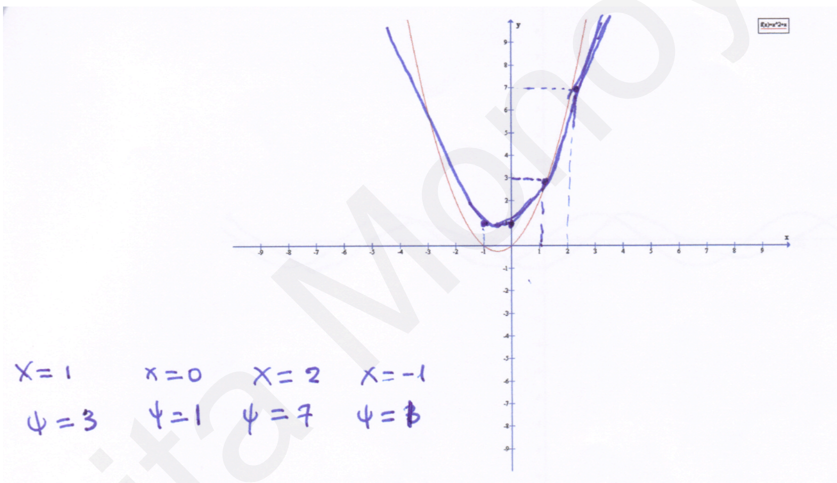


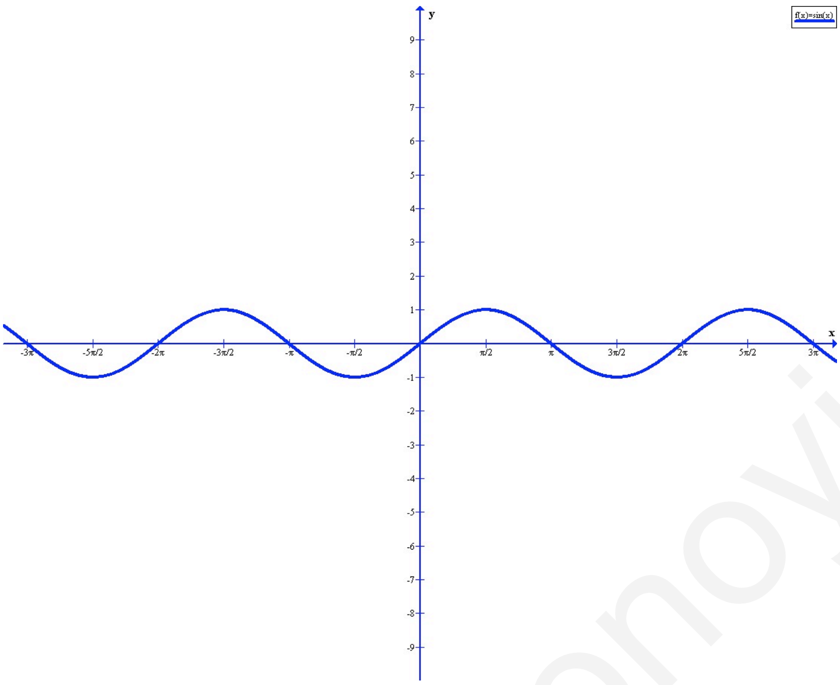
Figure 4.28. Participant 8's inscriptions for task 5

Six teachers gave a correct response by using a coordinated approach. They claimed that the points of the second function are "one more" than the points of the first. They drew the graph quite easily (see figure 4.27). Four of them also found a point in order to confirm their thoughts. The other three used an algebraic approach in order to reach a correct response. Particularly they constructed a table of values in order to help them construct the graph (see figure 4.28). They found three points and then they drew the graph.

Table 4.36 shows pre-service teachers' representative answers to task 6.

Task 6:

In the following diagram $\sin(x)$ *is given. Draw the function* $\sin(x+1)$.



In the task 6 all the teachers used a coordinated approach to reach a solution since as they mentioned they could not find the values of y in this case. Only one of the teachers managed to give a correct response noticing that the value that changes is x and not y and indicating that the graph will move to the left and not to the right even though it is plus one (par. 1). One of the teachers made a minor mistake mentioning that the variable that changes is x but claiming that the graph will move to the right because is plus one (par.3, see figure 4.29). The other seven teachers gave wrong responses to this task. Six of them mentioned that the graph will move one up (par. 7, see figure 4.30). One teacher answered that the graph will move to the right and up. Most of the teachers who gave wrong responses mentioned that they were influenced by the previous task. They also mentioned that they were not sure about their answers but they did not know how to find the values of y since the equation involved the “sin”.

Table 4.36

Pre-service teachers' representative answers to task 6 involving the coordinated or algebraic approaches to simple function tasks

	Emphasized aspects	Representative protocols
1	Correct response - Coordinated approach	"P1: It will move one to the left. $f(x)$ $f(x+1)$. I: Why to the left? P1: If we have $\sin 180^{o}$ then 0. In the second case we must have less than 180 in order to have 0. So it will move to the left."
3	Response with a minor mistake – Coordinated approach	"P3: Ok, that is the same like before…..Oh, just a minute. In this case the x gets larger and not y. It is different from the previous example. So the graph will move in the x axis right or left. I: What do you think? P3: It will move to the right one. Because it is plus one. It will be like this…." (see figure 4.29)
4	Response with a major mistake – Coordinated approach	"P4: Is this a function? I: Yes, it is. Why? P4: It is not like the others.. P4: It will intersect the axis here instead of here. It will be like this? I: Why did you draw it like this? P4: Because is plus one. It will go one up."
5	Response with a major mistake – Coordinated approach	"I don't remember how we can find the sin of a number. Here we can see that the one that changes is x. It changes one. It will move to the right one and up…"
7	Response with a major mistake – Coordinated approach	"I cannot put values. I don't know how to find sin…. Maybe it will move one up." (see figure 4.30)

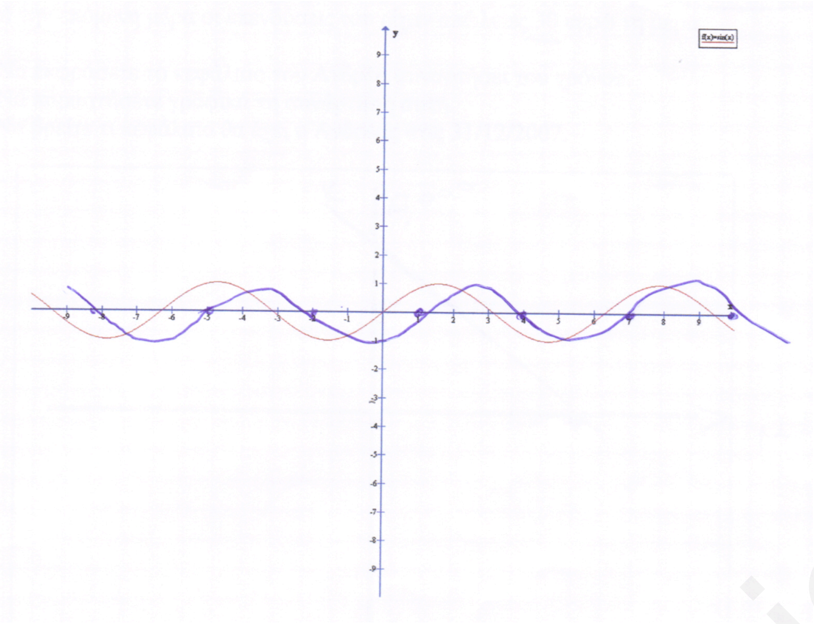


Figure 4.29. Participant 3's inscriptions for task 6

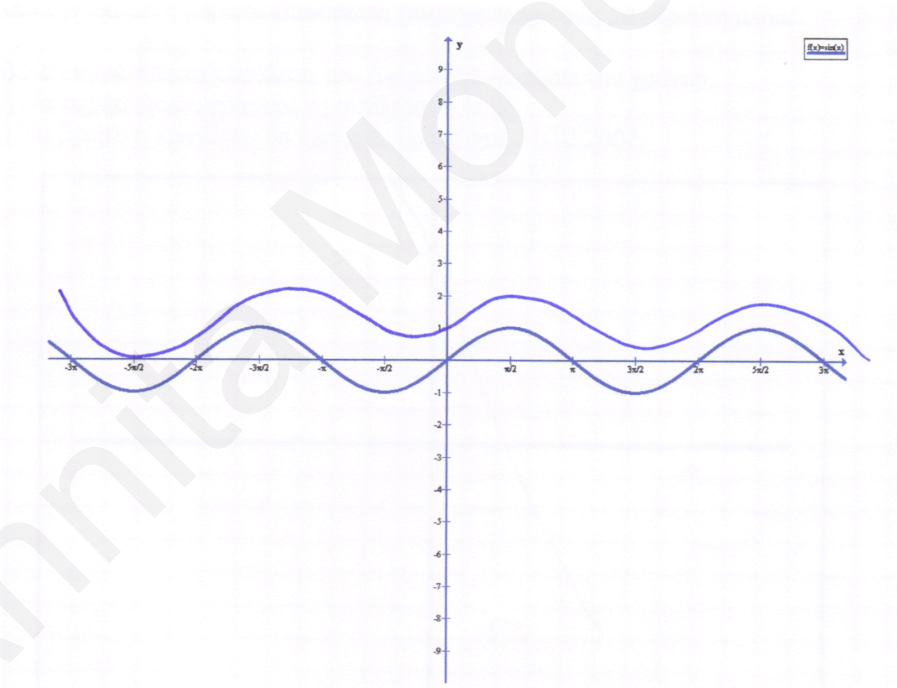


Figure 4.30. Participant 7's inscriptions for task 6

Tasks 7, 8 and 9 were complex problems with functions. The first involved a linear function, the second a quadratic and the third an exponential one. Table 4.37 shows teachers' representative answers to problem 7 involving a linear function.

Task 7:

Mr. Nick invested in the stock market a fund of 13,000 euro on 31/1/2007. Starting from the next day his investments had losses 30 euro per day.

a) Express Mr. Nick's investments (fund) as a function of time.

b) Draw the graph of the above function.

c) Find Mr. Nick's fund on 31/12/2007.

Eight of the participants managed to draw correctly the graph of the linear function. Five of them made a table of values in order to help them to draw this graph successfully. The other four found the points of intersection with the axes, mentioned that the slope is negative and managed to draw the graph easily. One of them, although she found the values of x and y correctly she did not draw a straight line but a curve indicating limited understanding for the concept of function (par. 8, see figure 4.31). Concerning the equation seven of the teachers gave the correct equation. One teacher gave a wrong equation (par. 6, see figure 4.32) while another could not give the equation (par. 9). Both teachers who gave wrong equations managed to draw the graph correctly by finding pairs of values and answered correctly the last question. Three teachers gave wrong responses to the last question. One of them found the fund for twelve months instead of eleven (par. 7). The others subtracted the number of days instead of Nick's losses (par. 8, see figure 4.33).

Table 4.37

Pre-service teachers' representative answers to problem 7 involving a linear function

	Emphasized aspects	Representative protocols
4	Correct answers to all the questions	P4: $y = 13000 - 30t$ I: Can you draw this function? P4: I will put t (time) in the x axis. So for $t = 0$ the $y = 13000$. I: Ok. P4: Now for $y = 0$, $13000 = 30t$, $t = 433$. I will find the point of intersection with x axis. The graph will be like this (a straight line).

		I: Can you find Mr. Nick's fund on 31/12/2007? P4: After twelve months. No, after eleven months. So, $t = 11$. I: It is days. P4: Ok, so every month has 30 days. I: Some months have 31 days. N: Yes, that's right. A year has 365 days minus one month (30 days). I: 31 days. It is the January. P4: Ok. So $y = 13000 - 30(365 - 31)$
6	Wrong equation Correct graph and value	P6: $x = y - 30$. It is the equation. I: Can you draw this function? P6: For the first day is minus 30. So $13000 - 30 = 12970$. I will then say for 10 days. Minus 300. So it is 12700. It will be like this…(She draws the function correctly). I: What about the third question? P6: For eleven months. So $365 - 31 = 334$, $334.30 = 10020$, $13000 - 10020 = 2980$. (see figure 4.32)
7	Correct equation and graph Wrong value	P7: $y = 13000 - 30x$ I: Can you draw this function? P7: Yes, it is a straight line. Like this (She draws the function correctly). P7: For 12 months. $12.30 = 360$ so $y = 13000 - 30.360$
8	Correct equation Wrong graph and value	P8: The equation is $f(x) = 13000 - 30y$ I: Can you draw this equation?

P8: Yes. I will find values for x and y.

if $y = 3$ then $f(x) = 13000 - 90 = 12010$

if $y = 0$ then $f(x) = 13000$

if $y = 5$ then $f(x) = 13000 - 150 = 12850$

I have three points. I can draw this (She draws a curve).

I: What about the third question?

P8: $f(x) = 13000 - 360 = 12640$

I: 360?

P8: It is the number of the days. (see figure 4.31)

9 No equation

Correct graph and value

P9: I will make a table.

1	13000	30
2	12970	60
3	12940	90
4	12910	120

I cannot give the equation. He loses 30 euro every day.

I: Can you draw the function?

P9: Yes it will be like this. (She draws the graph correctly)

I: Can you tell me the equation now?

P9: No it is difficult to give the equation…

I: What about the third question?

P9: 11 months… $365 - 31 = 334$ .It will be $13000 - 334.30 = 13000 - 10020 = 2980$

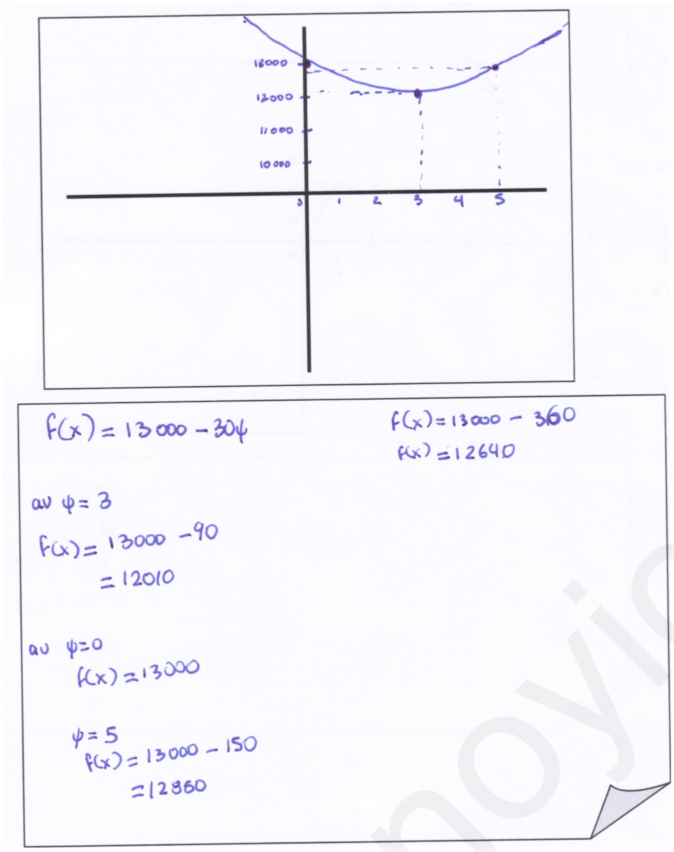


Figure 4.31. Participant 8's inscriptions for task 7

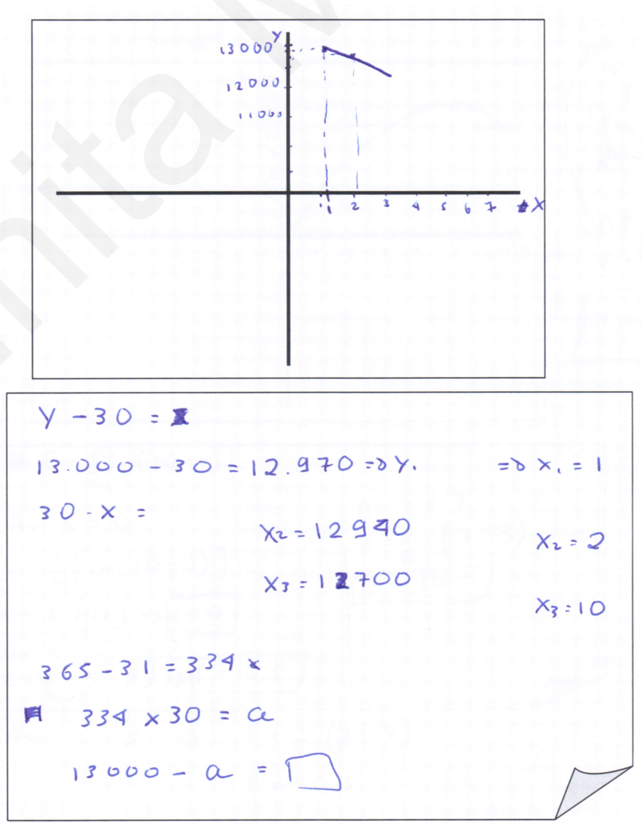


Figure 4.32. Participant 6's inscriptions for task 7

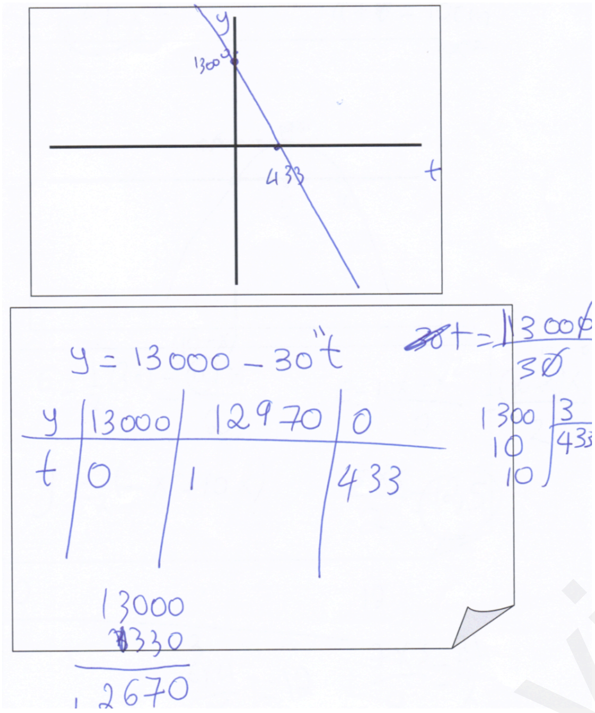


Figure 4.33. Participant 5's inscriptions for task 7

Table 4.38 shows teachers' representative answers to problem 8 involving a quadratic function. All the participants managed to find correctly the equation of the area of the right triangle as a function of its side AB=x. Eight participants drew correctly the graph of the function. Only one participant drew the graph wrongly (par. 7, see figure 4.34). Six of the participants in order to draw the graph of the function gave values to x and y. The other three participants found the points of intersection with the x axis and the maximum point in order to draw the graph. Concerning the question 8c only three teachers managed to give a formal and correct proof. The other five teachers claimed that since the area is smaller for $x = 6$ (A=12cm^2), $x = 4$ (A=12cm^2) or $x = 7$ (A=10.5cm^2) then for $x = 5$ (A=12.5cm^2) the area is the maximum. One teacher did not prove it at all that for $x = 5$ the area is maximum.

Task 8:

The sum of the two legs of a right triangle ABC (A= 90^O) is 10cm.

(a) Find the area of the right triangle as a function of its side AB=x.

(b) Draw the graph of the above function.

(c) Prove that the right triangle has maximum area, when it is also isosceles.

Table 4.38

Pre-service teachers' representative answers to problem 8 involving a quadratic function

	Emphasized aspects	Representative protocols
1	Correct equation, graph and proof	P1: What do you mean "as a function of its side $AB = x$"? I: The side AB is x. P1: Ok, now I understand. $AB + AC = 10$, $x + AC = 10$, $AC = 10 - x$, $\frac{x(10-x)}{2} = A$, $A = \frac{10x - x^2}{2}$ $A = 5x - \frac{1}{2}x^2$ I: Can you draw this function? P1: Yes. x^2 is negative so we have a maximum point. I will also find an intersection with the axes. For $y = 0$, $5x - \frac{1}{2}x^2 = 0$, $x^2 = 10x$, $x^2 - 10x = 0$ $x(x-10) = 0$, $x = 0$, $x = 10$ I will find the maximum point. $f'(x) = 5 - x$, $5 - x = 0$, $x = 5$ We have a maximum point for $x = 5$, $y = 25 - \frac{25}{2}$, $y = 12.5$ I: Can you prove that the right triangle has maximum area, when it is also isosceles? P1: We have maximum when x equals 5. The sum of the two vertical lines is 10. So the triangle has maximum area, when it is also isosceles.
5	Correct equation and graph. Empirical proof	P5: Ok. The area of a triangle is $\frac{1}{2}$ multiply by base

	(For $x = 6$ and $x = 4$, the area is 12cm^2. For $x = 3$ and $x = 7$, the area is 10.5cm^2.So for $x = 5$ the area is maximum)	by height. $A = \frac{1}{2}bh = \frac{ax}{2}$ I: Can you replace a? A: Yes. $a + b = 10$ so $a = 10 - x$. $y = \frac{1}{2}(-x^2 + 10x) = 10x - \frac{x^2}{2} = \frac{(10-x)x}{2}$ I: Can you draw this function? P5: I will give values for x. $x = 0$ $A = 0$, $x = 10$ $A = 0$. It will be something like this. It will be a curve. I have to find the maximum also….I do not remember the type. I: Can you prove that the right triangle has maximum area, when it is also isosceles? P5: The sum of the two sides is 10cm. Each side will be 5cm. So the area will be $\frac{5.5}{2} = 12.5$ cm^2. I have to prove that this area is the maximum. I: Yes. A: If the one side was 6 and the other 4, then the area will by $\frac{6.4}{2} = 12$ cm^2. If the one side was 7 and the other 3 it will be $\frac{3.7}{2} = \frac{21}{2} = 10.5$ cm^2. So the 12.5 cm^2 is the maximum.
6	Correct equation and graph. Empirical proof (For $x = 3$ and $x = 7$, the area is 10.5cm^2.So for $x = 5$ the area is maximum)	P6: $x + AC = 10$, $x = 10 - AC$, $AC = 10 - x$, $A = \frac{x(10-x)}{2}$ That's the area. I: Can you draw this function? D: I will give values to x… $x = 0$, $A = 0$. $x = 5$, $A = \frac{5.5}{2} = \frac{25}{2} = 12.5$. $x = 3$, $A = \frac{3.7}{2} = 10.5$. $x = 7$,

		$A=\frac{3.7}{2}=10.5$. It will be like this. A curve. I: What about the third question now? P6: We can see this from the graph…12.5 is the maximum area. (see figure 4.35)
7	Correct equation. Wrong graph. No proof.	P7: The area is $A=\frac{ab}{2}$, $A=\frac{ax}{2}$ so $y=\frac{ax}{2}$ I: Can you give the area as a function of x only? P7: How? Aaa… $a+x=10$, $a=x-10$. So $y=\frac{(10-x)x}{2}=\frac{10x-x^2}{2}$. That's the area. I: Can you draw this function? P7: I will find pairs of values for x and y. For $x=1$ then $y=\frac{10-1}{2}=\frac{9}{2}=4.5$. For $x=0$ $y=0$. For $x=-1$ $y=\frac{-10-1}{2}=\frac{-11}{2}=-5.5$. I have three points so I can draw the graph now. It is like this. I: Can you prove that the right triangle has maximum area, when it is also isosceles? P7: Isosceles. That means that both sides will be x. So $2x=10$, $x=\frac{10}{2}$, $A=\frac{\frac{10}{2}\frac{10}{2}}{2}=\frac{\frac{100}{4}}{2}=\frac{100}{8}=12.5$. That's the maximum area. I: How do we know this? That this is the maximum area? P7: We just prove it. (see figure 4.34)
8	Correct equation and graph.	P8: $AB+AC=10$, $A=\frac{AB.AC}{2}$, $A=\frac{x(10-x)}{2}$

	Empirical proof (For $x = 6$ and $x = 4$, the area is $12cm^2$. So for $x = 5$ the area is maximum)	I: Can you draw this function? P8: Yes, it will be for $x = 1$, $y = 4.5$. For $x = 0$, $y = 0$. For $x = 2$, $y = 16$. For $x = 6$, $y = 12$. Now I can draw it. I: What about the third question? P8: Isosceles. That means that the two sides are 5cm. The area it will be $12.5cm^2$. For $x = 6$ we found before that the area will be 12. So for $x = 5$ we have maximum area.

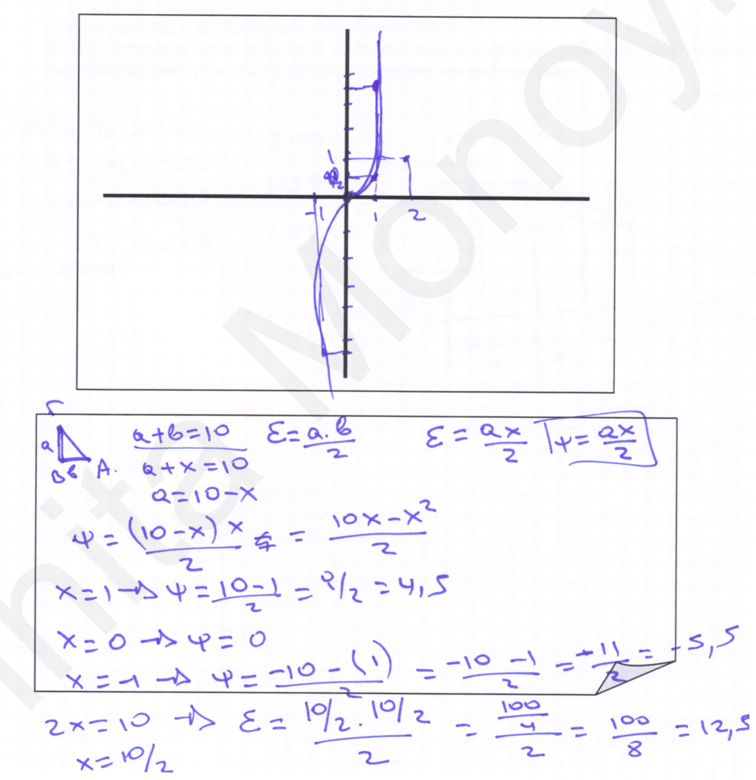


Figure 4.34. Participant 7's inscriptions for task 8

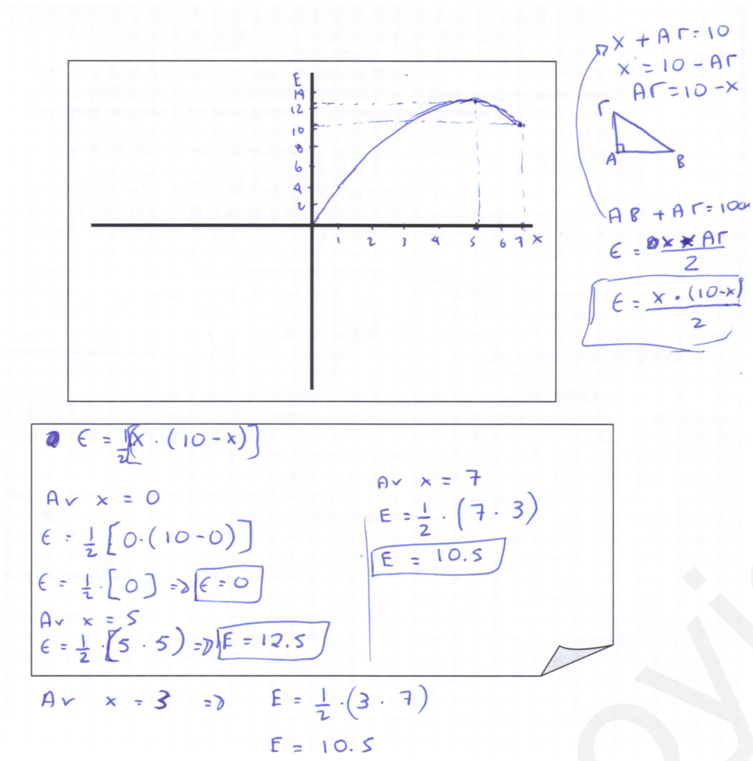


Figure 4.35 Participant 6's inscriptions for task 8

Table 4.39 shows teachers' representative answers to problem 9 involving an exponential relation. Only three teachers recognize the exponential relation involved in this problem (par.1 and par. 2, see figure 4.36). Furthermore, two of them managed also to give the equation. The other six teachers considered the relation to be linear and drew a linear function indicating limited understanding for the concept of function (par. 8 and par 5, see figures 4.37 and 4.38).

Task 9:

Several species of whales have been declared endangered. When the population of a particular whale species falls dangerously low, biologists encourage governments to agree to a ban on hunting the species. Suppose that, in 1994, there were only 5000 whales of a particular species and the number of whales in the next two years were 4500 and 4050 respectively. Given the population was predicted to decline in this manner,

(a) What will be the whale population in 2001? (Presmeg & Nenduradu, 2005)

Table 4.39

Pre-service teachers' representative answers to problem 9 involving an exponential relation

<table>
<tr><th></th><th>Emphasized aspects</th><th>Representative protocols</th></tr>
<tr><td>1</td><td>Recognize the exponential relation without giving an equation</td><td>P1: It is a pattern.

I will make a table.
<table><tr><td>1994</td><td>5000</td></tr><tr><td>1995</td><td>4500</td></tr><tr><td>1996</td><td>4050</td></tr></table>
From the first to the second we have 500. From the second to the third 450. From the third to the fourth 400? The second difference is constant? 50?

I: Can you give the equation?

P1: Yes, $W = 500 -$ ….No. Is it an exponential function?

I: Can you give the function?

P1: No, it is difficult. I don't remember.</td></tr>
<tr><td>2</td><td>Recognize the exponential relation and gives the equation.</td><td>P2: I will make a table of value.
<table><tr><td>1994</td><td>5000</td></tr><tr><td>1995</td><td>4500</td></tr><tr><td>1996</td><td>4050</td></tr></table>
S: From the first to the second 500, from the second to third 450….Lets say that y=the number of whales and x=years.

I: Can you give the equation?

N: Yes, $Y = 5000 - x450$. No, minus 500. $y = 5000 - x500$. For $x = 1$ it is ok. But for $x = 2$ it is not ok. This equation does not work. No, it is not like this. It is not a straight line.</td></tr>
</table>

The values are not analogous. It is a curve. It decreases. For $x = 0$ y will be 5000. For $x = 1$ y will be 4500. So we have to put minus or a fraction in order to have a decrease. It is an exponential function. $y = 5000.(a)^x$ .We find $a = \frac{4500}{5000} = \frac{9}{10}$, so $y = 5000.(\frac{9}{10})^x$. That's ok. (see figure 4.36)

5 | Linear equation | P5: I will make a table.

1994	5000
1995	4500
1996	4050

From 1994 to 1995 we have minus 500. Then we have minus 450. Then we will have minus 400. Then minus 350. Then minus 300. It will reduce 50 every year. The second difference is 50. In the first year minus 50. In the second minus 100.

I: Can you give the equation?

P5: Yes, $y = 5000 - 50x$. No it is not like this. $y = 5000 - (500 - 50x)$.

I: Can you draw this function?

N: It is a straight line. The slope it is negative. It is like this… (see figure 4.38)

8 | Linear equation | P8: I will make a table.

1994	5000
1995	4500
1996	4050

From 1994 to 1995 we have minus 500. Then we have minus 450.

I: Can you give the equation?

P8: Yes, $y = 5000 - 50x$, x is the years

I: Can you draw this function?

P8: It is a linear function. It will be like this.

(see figure 4.37)

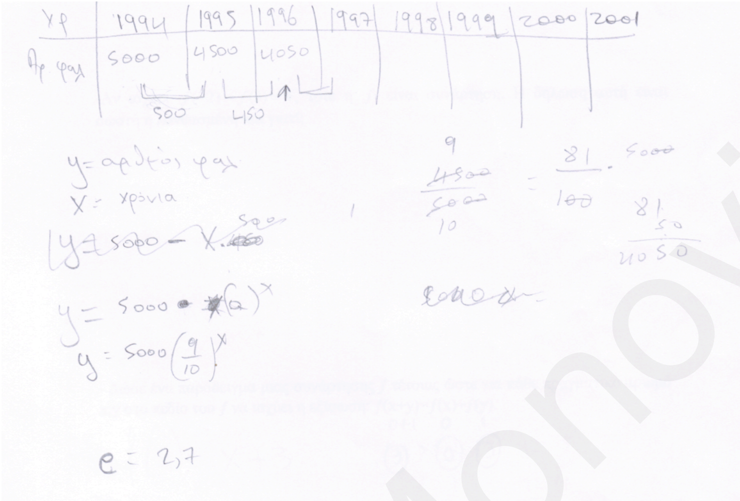


Figure 4.36. Participant 2's inscriptions for task 9

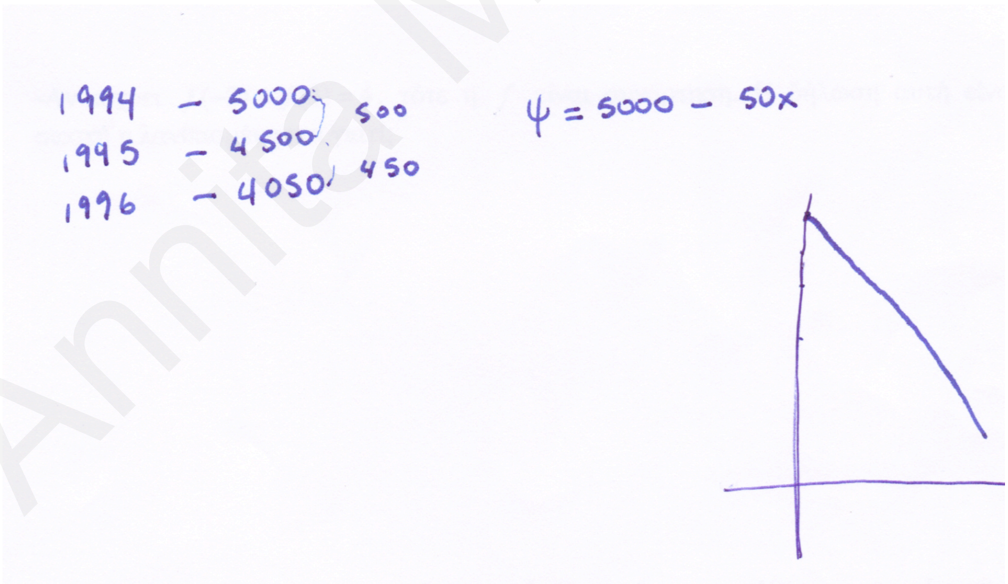


Figure 4.37. Participant 8's inscriptions for task 9

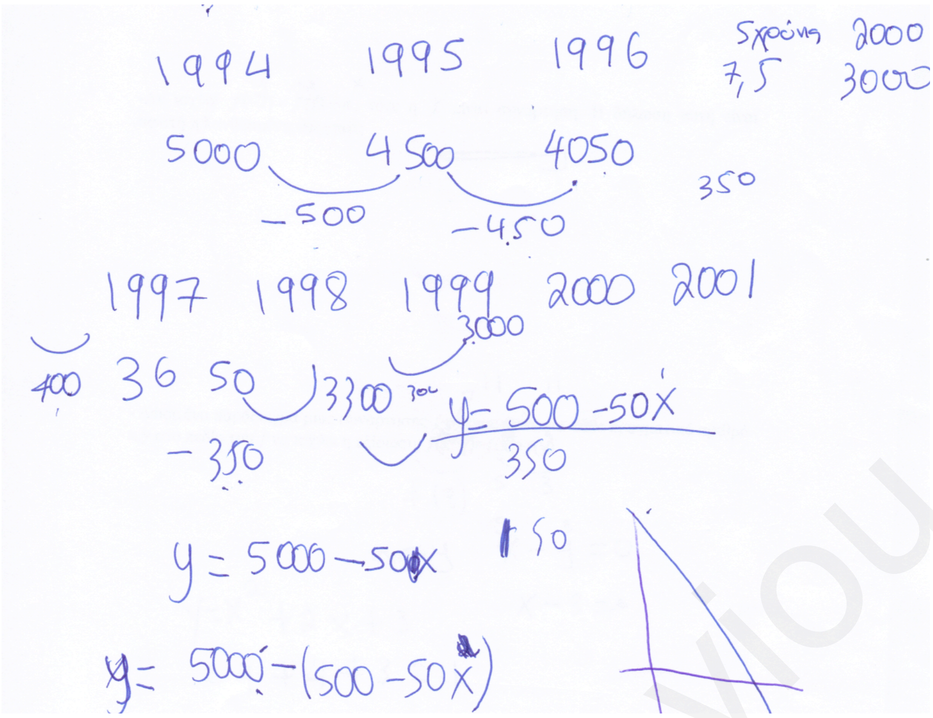


Figure 4.38. Participant's 5 inscriptions for task 9

Summary

The general aim of this research study was to explore Cypriot and Italian pre-service teachers' display of behavior, cognitive structures and performance in different aspects of the understanding of function. Basic dimensions explored were the multiple representational flexibility and problem solving.

From the first phase of this research study the stability and long-lasting character of the algebraic and coordinated approaches emerged. Furthermore, the strong relation of the coordinated approach with problem solving was evident from the similarity, implicative diagrams and the structural model.

From the second phase a structural model emerged which underlined the importance of multiple representational flexibility and problem solving ability in the conceptual understanding of function. From the structural model it was evident that the multiple representational flexibility is a multidimensional concept that involves the concept image which consists of the concept definition and examples of the concept, the recognition of the concept given in various representations (graphical and diagrammatic representation, symbolic and verbal expression) and the conversions from an algebraic to a graphical representation and vice versa. As far as the problem solving ability is concerned it was found that the coordinated approach to simple function tasks and complex problems with functions are its basic components. Although Cypriot teachers performed better than the Italian teachers in all the dimensions of the conceptual understanding of function the above structure remained invariant for both groups. Furthermore, Cypriot and Italian pre-service teachers were divided into three groups according to the approach they follow and used in order to examine the relation of the coordinated approach with the other dimensions of the understanding of function (concept image, recognition, conversions, problem solving). It was found that the coordinated approach group outperformed the other two groups in all the dimensions of the understanding of function.

Cypriot and Italian pre-service teachers' behavior when dealing with tasks involving the various dimensions of the conceptual understanding of function was also examined. The phenomenon of compartmentalization was evident to some extent in both the Cypriot and Italian pre-service teachers' behaviour when confronting different types of tasks as regards the cognitive abilities they require, even though their content was similar.

In the third phase the structure of the conceptual understanding of function was further validated with qualitative data. Furthermore, several misunderstandings and ideas held by the pre-service teachers for the concept of function were uncovered.

CHAPTER V

DISCUSSION

Introduction

In this chapter the conclusions of the present research study are discussed. The main results of the research are presented, with basic aim their interpretation and comparison with previous research. Conclusions are drawn based on the discussion of the results, suggestions for future research are made and a number of implications for mathematics teaching and learning are presented.

This chapter is organized in three parts according to the three phases and presents the results separately for each phase.

In the first part the stability and long-lasting character of the coordinated approach is discussed as well as the strong relation of this approach with problem solving ability. Explanations concerning teachers' preference for the algebraic approach are given and their low performance in problem solving is explained.

In the second part the structural model which underlined the importance of multiple representational flexibility and problem solving ability in the conceptual understanding of function is discussed. The various dimensions of the multiple representational flexibility are referred to and discussed namely, the concept image which consists of the concept definition and examples of the concept, the recognition of the concept given in various representations (graphical and diagrammatic representation, symbolic and verbal expression) and the conversions from an algebraic to a graphical representation and vice versa. As far as the problem solving ability is concerned its basic components are presented and discussed. The invariance of the above structure for the Cypriot and Italian pre-service teachers is also discussed. Cypriots and Italians pre-service teachers' performance and behavior when dealing with tasks involving the various dimensions of the conceptual understanding of function it is also discussed. The phenomenon of compartmentalization which exists to some extent in both the Cypriot and Italian pre-service teachers' behaviour when confronting different types of tasks as regards the cognitive abilities they require, even though their content was similar is explained.

Furthermore, Cypriot and Italian pre-service teachers were divided into three groups according to the approach they followed and used in order to examine the relation of the coordinated approach with the other dimensions of the understanding of function (concept image, recognition, conversions, problem solving). The power of the coordinated approach and its role as a predictive factor for the performance in all the others dimensions of the understanding of function is pinpointed.

In the third part the results that emerged for the interviews are discussed and particularly the several misunderstandings and ideas held by the pre-service teachers for the concept of function.

Discussing the results of the first phase: The stability and long lasting character of the algebraic and coordinated approaches and their relation with problem solving

The aim of the first phase was twofold. Firstly to contribute to the understanding of the algebraic and coordinated approaches pre-service teachers develop and use in solving function tasks and to examine which approach is more correlated with their ability in solving complex problems, and secondly to investigate the stability of these approaches and the stability of their relation. Thus, the following research questions were examined:

- What approach (algebraic or coordinated) do pre-service teachers prefer to use when they solve simple function tasks?
- How able are pre-service teachers to solve complex function problems?
- Which approach (algebraic or coordinated) is more correlated with pre-service teachers' ability in solving complex problems?
- How stable and long lasting are these approaches and their relation with problem solving?

The basic aim of this phase was to identify and examine the approach pre-service teachers -participating in the four groups- used in order to solve simple function tasks that involved vertical transformations of functions. It is important to know whether teachers are flexible in using algebraic and graphical representations in these functions tasks.

Most of the Cypriot pre-service teachers, participating in groups A, B and C, used an algebraic approach in order to solve the simple function tasks. A coordinated approach is fundamental in solving problems even though many teachers, as it emerged, have not mastered even the fundamentals of this approach. This finding is in line with the results of previous studies that suggest that many students deal with functions pointwise (Mousoulides & Gagatsis, 2004; Knuth, 200; Even 1998; Bell & Janvier, 1981) and cannot use effectively the geometric-coordinated approach, which originates within the object perspective. Ponte (1984) (as cited in Schwarz et al., 1990) has shown that students take a discrete, static approach to functions. Particularly, they do not conceive of them as objects but as consisting of points, and thus fundamentally discrete in nature. Students can plot and read these points, but cannot think of a function as it behaves over intervals or in a global way. These studies also indicate that a global-coordinated approach to functions is more powerful than a pointwise approach. Students who can easily and freely use a global approach have a better and more powerful understanding of the relationships between graphic and algebraic representations and are more successful in problem solving.

Students' preference for the algebraic solution is probably due to the curricular and instructional emphasis dominated by a focus on algebraic representations and their manipulation (Dugdale, 1993). Kaldrimidou and Ikonomou (1998) showed that teachers and students pay much more attention to algebraic symbols and problems than to pictures and graphs. Furthermore, as Eisenberg and Dreyfus (1991) pointed out, the way knowledge is constructed in schools favours mostly the analytic elaboration of the notion which deteriorates the approach of function from the graphical point of view. In their textbooks, students are usually asked to construct graphs from given equations using pairs of values. As a result, students fail to connect algebraic and graphical representations and therefore fail to develop a "global-coordinated" approach. In addition the transformation of functions is a topic that has been taught with strong emphasis on algebraic symbolism and in relative isolation from the visual transformational topics (Borba & Confrey, 1996). Furthermore, as it emerged from the analysis of the Cypriot textbooks, the symbolic and algebraic representations of functions were predominant.

In contrast, the Italian pre-service teachers (Group D) used less the algebraic approach than the Cypriot pre-service teachers and more the coordinated approach even though this difference was not statistically significant. This difference is probably the result of the differences that exist between the two educational systems. Furthermore, from a closer look at the tests of the Italian pre-service teachers (Group D), some of them used the coordinated approach rather occasionally and superficially. They probably were affected by the didactical contract indicating that all the data given in a problem or exercise must be used in order to reach an answer. The didactical contract rules notion was first introduced by Brousseau (1983) as a set of partly explicit and mainly implicit set of rules that defines the relationship between the teacher, the pupil and the mathematical knowledge. Learning cannot be obtained under the conditions of the didactical contract, but under the breaches of it (as cited in Gagatsis, 1992). Most pupils' behavior when solving word problems is explained through their reaction in line with didactical contract rules. For instance, pupils are obliged to give an answer to every problem presented to them, which according to them is always correct and sensible, so they must combine all the data of the problem to arrive at an answer.

Similarly with Baker et al. (2001b) and Lage and Gaismain-Trigueros (2006) who have investigated the transformations of various functions from a perspective of Action-Process-Object-Schema (APOS theory) we can claim that the pre-service teachers who used an algebraic approach act at an action level since they can perform operations on functions and variables step by step. In contrast the pre-service teachers who have a coordinated approach act at an object level since they are able to think functions as a whole, in any representational context, working flexibly in different representational contexts as well. Furthermore, Lage and Gaisman-Trigueros (2006) showed that students who were classified at an action level showed a weak understanding of the concept of function. A similar result emerged in this research study since pre-service teachers who used an algebraic approach had a low performance in problem solving indicating limited understanding of the concept of function.

Pre-service teachers' performance in problem solving was moderate. Teachers participating in group A performed slightly better than teachers of groups B, C and D. This result is due to the fact that the participants of the first group graduated from a slightly different type of high school with different textbooks and different procedures for the selection of lessons. Although problems used in this study are among those taught at school, subjects had difficulties. This finding suggests that in order to give a correct solution to a complex function problem the students must be able to handle different representations of function flexibly and move easily from one representation to the other. This is in accord with the results of previous studies that underline that problem solving is a complex process that involves various abilities such as the ability to handle flexibly various representations of a concept and make conversions between them (Mousoulides & Gagatsis, 2004; Even; 1998; Knuth, 2000; Niemi, 1996; Stenning et al., 1995; Monaghan et al., 1999). Students who have a better and more powerful understanding of the relationships between different kinds of representations are more successful in problem solving (Gagatsis & Shiakalli, 2004).

Another important finding of this study is the relation between the coordinated approach and problem solving. In the four similarity diagrams –one for each group- two clusters of variables emerged. The one cluster included the variables corresponding to the solution of the complex problems with the variables representing the coordinated approach, indicating the strong relation that exists between them and the other the variables representing the algebraic approach. This finding was further verified by the results of the implicative diagrams and the confirmatory factor analysis. Particularly, the results of the implicative analysis were in line with the similarity relations explained above. Two separate "chains" of implicative relations among the variables were formed. The one group of implicative relations involved the variables concerning the use of the algebraic approach. The other group of implicative relations involved the variables representing the use of the coordinated approach with the variables concerning the solution of the problems. Furthermore, from the implicative relations it was inferred that the pre-service teachers who used a coordinated approach to solve the simple function tasks and succeeded in those tasks also solved correctly the three problems.

These results were further reinforced by the results of the confirmatory factor analysis. Specifically a model with three first-order factors that were interrelated emerged. The three first-order factors of the model represented the coordinated, the algebraic approach and problem solving. It is noteworthy that the relation of the first-order factor standing for the coordinated approach with the first-order factor standing for problem solving was very strong for all the groups. Attention was also drawn to the fact that the first-order factor standing for the algebraic approach was negatively related with the coordinated approach and problem solving.

Furthermore, the pre-service teachers were clustered into three categories according to the approach they used in order to solve the four simple function tasks. The first cluster involved the teachers who used a coordinated approach systematically, the second cluster included the teachers who used an algebraic approach with consistency and the third cluster involved the teachers who used other approaches or used equally the algebraic and coordinated approach. It was noteworthy that the teachers who used a coordinated approach outperformed the other groups in problem solving indicating that the coordinated approach is more powerful and can lead to successful problem solving.

The data used in the similarity diagrams, the implicative relations, the confirmatory factor analysis and the cluster analysis suggest that the pre-service teachers participating in the four groups who have a coherent understanding of the concept of function (coordinated approach) can easily understand the relationships between symbolic and graphical representations and, therefore, are able to provide successful solutions to complex problems.

The above results are in line with the results of previous studies indicating that the geometric approach (coordinated) enables students to manipulate functions as an entity, and thus students are capable of finding connections and relations between the different representations involved in problems (Knuth, 2000; Moschkovich et al., 1993).

It is noteworthy that the association between the coordinated approach and problem solving ability is strong and stable. The algebraic and the geometrical-coordinated approaches were identified also in other research studies as was mentioned before (Even, 1998; Mousoulides & Gagatsis, 2004), but the stability of these approaches was not evident. In this research study a very important result was the stability of the two approaches and the long-lasting relation of the coordinated approach with problem solving. The data of the first group were collected in 2005, of the second group in 2007 and of the third group in 2008. In the meantime major changes have happened in the Cypriot educational system concerning the educational program, the examination material and the textbooks. The data of the fourth group were collected from Italy in 2009 where the educational system has several differences in comparison with the Cypriot educational system. Taking into account the above it was very important and noteworthy that the teachers' approaches were the same in all the groups and a strong relationship between the coordinated approach and problem solving ability existed.

Discussing the results of the second phase: A model for the conceptual understanding of function

The aim of the second phase was to explore pre-service teachers' display of behavior in six aspects of the understanding of function: effectiveness in solving a word problem, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic and verbal expression), transferring function from one mode of representation to another and the algebraic or coordinated approaches when dealing with simple function tasks. A main concern was to examine problem solving in relation to the other types of displayed behavior.

Thus, the following research questions were examined:

- How able are Cypriot and Italian pre-service teachers to give a right definition, to give correct examples of the concept of function, to recognize functions given in different representations, to make conversions from an algebraic to a graphical representation of function and vice versa and to solve complex problems involving the concept?
- What approach (algebraic or coordinated) do Cypriot and Italian pre-service teachers prefer to use when they solve simple function tasks?
- What conceptions do Cypriot and Italian pre-service teachers have of function on the basis of their concept definitions, examples of the notion and justifications given in the tasks involving the recognition of function given in various representations?
- How do Cypriot and Italian pre-service teachers behave during the solution of tasks involving the definition of the concept, examples of function, the recognition of functions given in various representations, the conversions from an algebraic to a graphical representation and vice versa, the approach when dealing with simple function tasks and problem solving?
- What are the structure and the relationships between the concept image, the examples, the recognition, the conversions, the approaches and problem solving?
- What are the the similarities between Cypriot and Italian pre-service teachers in regard with the structure of their function understanding?
- What differences exist in the behavior of the coordinated, algebraic and various approaches groups of pre-service teachers during the solution of tasks involving the definition, the examples, the recognition, the conversions and the problem solving of function?

Only a third of the Cypriot and Italian pre-service teachers managed to give a correct definition of the concept of function. Concerning the Cypriot teachers half of them gave an accurate definition of the concept. In contrast, none of the Italian pre-service teachers gave an accurate definition but a large percentage gave a definition of a special kind of function, particularly they gave a definition of a one-to-one function. This finding is in accord with the results of previous studies indicating that one-valuedness is a dominating criterion that students use for deciding whether a given correspondence is a function or not (Vinner & Dreyfus, 1989). This idea is also associated with the process of enumeration, which involves one-to-one correspondence as a matter of routine for the students. A large percentage of Cypriot and Italian pre-service teachers made a reference to an ambiguous relation. Particularly they made reference to a relation between variables or elements of sets, or a verbal or symbolic example. The large percentage of teachers who gave an ambiguous definition indicates that it is quite difficult to provide an appropriate definition of the concept, probably because the formal definition is not discussed so systematically and in an explicit manner in school mathematics. Students' constructed image of the function concept may deviate from the formal definition of the concept that is introduced to high school students (Vinner, 1983). This is what the concept of "concept image" was intended to alert researchers and teachers to. The difficulty in giving an appropriate definition of function is in line with the results of Elia and Spyrou (2006), which showed that the majority of university students (prospective teachers) did not give a correct definition, but made reference to an ambiguous relation. Furthermore, students generally appear to hold to the requirement that the function is reasonable and describable by a formula (Graham & Ferrini-Mundy, 1990).

Concerning the tasks involved in the second test that required for their solution a correct definition of the concept of function, Cypriot pre-service teachers' performance was quite good. Specifically, more than half of the Cypriot teachers' gave correct responses to these tasks. In these tasks their performance was higher in comparison to the task of the first test requiring writing a simple definition of function since in these tasks they have to use the definition in order to give an answer and not to state a definition. Tall and Vinner (1981) claimed that in many cases we may learn to recognize concepts from experience, and use them appropriately in different contexts without knowing a precise definition for them at all. Similarly, in this case the absence of a formal definition did not impede the development of a rich and flexible concept (Tall & Vinner, 1981) since the pre-service teachers used the definition and gave correct responses to these tasks. Italian pre-service teachers' performance in these tasks was lower than the Cypriot teachers' performance. Thus, Italian pre-service teachers had difficulties not only in stating a correct definition of function but also in using it to reach a correct solution to tasks.

Small were the percentages of Cypriot and Italian pre-service teachers who managed to give correct examples for the concept of function. Most of them gave examples of a continuous function from physics. High percentages of Cypriot and Italian pre-service teachers gave examples that were ambiguous and in addition they did not define any variables or sets, and made a reference to general transformations of the real world. Similarly, Hazzan and Zazkis (1997) in a research study found that being asked to generate examples is a relatively difficult task for students. Even though it is a difficult task many researchers claimed that examples generated by learners serve as powerful pedagogical tool for enhancing the learning of mathematics at a variety of levels (Watson & Mason, 2005; Zazkis & Leikin, 2008). Zazkis and Leikin (2007) also suggested that the generation of examples may serve also as a research tool since by the examination of the examples generated by learners, researchers may draw inferences about their knowledge about a concept. In this case the examples generated by pre-service teachers were used in order to access participants' image for the concept of function.

A large percentage of Cypriot teachers managed to make conversions from an algebraic to a graphical representation of functions and vice versa and to recognize functions given in different representations (diagrammatic, graphical, symbolic expression, verbal expression). In contrast, Italian pre-service teachers' performance was lower in these tasks. From the brief analysis of the Cypriot textbooks it was evident that the functions in all grades are mainly involved in conversions and treatments tasks, and that explains students' high performance in these tasks.

Concerning the recognition of functions given in various representations a big percentage of Cypriot pre-service teachers managed to justify their answer correctly. Quite low was the percentage of Italian teachers who justified their answer correctly. The majority of Italian pre-service teachers did not give justifications for their answers. From teachers' justifications a number of misunderstandings concerning the concept of function emerged.

An idea held by the teachers was that a function is necessarily an injective (one-to-one) correspondence, an idea that also emerged from the definition task. This finding is in accord with the results of previous studies indicating that many students or even pre-service teachers held the idea that a function is one-to-one correspondence (Vinner & Dreyfus, 1989; Elia & Spyrou, 2006; Evangelidou et al., 2004). The idea of uniqueness is particularly condensed and leads to identification of function as one-to-one function. Although this idea works for a wide range of situations and problems involving functions, it becomes a strong obstacle for the understanding of function as a wider concept.

Furthermore, some teachers have the misunderstanding that a function must always be surjective. In addition, the teachers were also very much distracted by the arrow diagrams, which, were presented on incompact frames thus expressing the idea that in a graph of a function domain and range should be compact sets. The split domain emerged also in other studies and played a crucial role in the justifications given by the respondents and in their decision for rejecting or not a relation as a function (Vinner & Dreyfus, 1989; Elia & Spyrou, 2006).

Teachers held the idea that a parabola, a hyperbola or a straight line are always functions. Another idea that was observed among the teachers was that a function must essentially contain two variables or unknowns. They furthermore expressed the idea that any symbolic expression that contains an x and a y is a function or every time we have a verbal expression that describes the relation between two variables we have a function.

They also stated that a graph of a function should be continuous. The discontinuity was also mentioned in other studies as a reason for rejecting given relations as functions (Vinner & Dreyfus, 1989; Elia & Spyrou, 2006). The pre-service teachers held also the misunderstanding that the elements of the range must be particular values and not pairs of values.

Most of the Cypriot teachers used an algebraic approach in order to solve the simple function tasks. In contrast, a smaller percentage of Italian pre-service teachers used an algebraic approach while a larger percentage used a coordinated approach. A quite large percentage of Italian pre-service teachers gave incorrect responses to the four simple function tasks. The algebraic solution was predominant in the answers of the Cypriot teachers. This is in line with the results of previous studies indicating that even if a coordinated approach is fundamental in problem solving many teachers have not mastered even the fundamentals of this approach and deal with functions pointwise using an algebraic approach (Mousoulides & Gagatsis, 2004; Knuth, 200; Even 1998; Bell & Janvier, 1981). Cypriot teachers' preference for the algebraic solution as was mentioned in the previous section is probably the curricular and instructional emphasis dominated by a focus on algebraic representations and their manipulation (Dugdale, 1993). In general, the Italian pre-service teachers gave more incorrect responses than the Cypriot pre-service teachers, used more the coordinated approach and less the algebraic. From a closer look in the tests of the Italian pre-service teachers' some of them used the coordinated approach rather occasionally and superficially. Furthermore, they probably were affected by the didactical contract indicating that all the data given in a problem or exercise must be used in order to reach an answer.

Cypriot and Italian pre-service teachers' achievement reduced radically in solving a complex problem on functions. Particularly, teachers' performance in problem solving was moderate. Cypriot teachers performed slightly better than Italian teachers. This finding is consistent with the results of many research studies (Mousoulides & Gagatsis, 2004; Gagatsis & Shiakalli, 2004; Elia et al., 2007; Elia et al., 2008; Elia & Spyrou, 2006; Even, 1998; Hitt, 1998), which identified students' university and pre-service teachers' great difficulties in providing a correct solution to problems on function. This is due to the fact that problem solving is a complex process that involves various abilities and in this case probably skills referring to the other aspects of the understanding of function. For instance, the solution of the particular problems of the test required among other abilities the coordination of various representations of function, i.e., verbal, algebraic and graphic, as well as acquisition of what a function is (definition) and of different types of functions (examples). The above findings strongly suggest that problem solving is at the core of the understanding of the concept of function, supporting the view of Stanic and Kilpatrick (1988) that a problem situation is central to mathematics learning. In addition it is noteworthy that even Cypriot pre-service teachers who had a high performance in conversion and recognition tasks had a moderate performance in problem solving. This is in line with the results of a research study conducted by Presmeg and Nenduradu (2005). Their results emerged from the case study of a pre-service teacher who while he could flexibly use various representations (tabular, algebraic and graphical representation) in order to solve an algebraic problem, could not recognize the exponential relation involved in it and gave a linear interpretation showing lack of conceptual understanding concerning the concept of function.

Another important finding of this study is the relation between the definition of function, the examples of the concept, the algebraic approach, the coordinated approach and problem solving that emerged from the analysis of the first test. In the similarity diagram of the Cypriot pre-service teachers' responses, problem solving was strongly connected with the correct definition of the concept, the correct examples and the coordinated approach. The variables representing the algebraic approach formed a distinct cluster while the wrong definition and examples of the concept formed another cluster. In the similarity diagram of the Italian pre-service teachers' responses the correct definition of the concept, the coordinated approach and problem solving were strongly connected. Again the variables representing the algebraic approach and the variables representing the wrong definition and examples were clustered separately.

The findings that emerged from the similarity diagrams were further verified by the results of the implicative diagrams. Particularly, the results of the implicative analysis were in line with the similarity relations explained above. In the implicative diagram of the Cypriot teachers' responses the strongest chain was formed among the variables representing the correct concept definition, the correct examples of the concept, the coordinated approach and problem solving. In the implicative diagram of the Italian pre-service teachers' responses the strongest chain was formed among the variables representing the correct concept definition, the coordinated approach and problem solving. From the implicative diagrams it was also evident that teachers who used a coordinated approach were also very successful in problem solving, gave a correct definition and examples of the concept. Similarly, many researchers pointed out that students' problem solving effectiveness had a predictive role in whether they would successfully employ the concept of function in various forms of representation, in giving a definition and examples of the concept (Elia et al., 2007; Elia et al., 2008).

Concerning the similarity and implicative diagrams of the second test the fact that problem solving was related only to some recognition and conversion tasks while some clusters involved for example only recognition tasks indicated that the teachers handled these tasks in a distinct way. Furthermore, the fact that two similarity clusters were formed in both the Cypriot and Italian pre-service teachers' similarity diagrams indicated teachers' inconsistent behaviour in dealing with tasks of different cognitive features. Thus, evidence is provided for the existence of the phenomenon of compartmentalization to some extent in teachers' behaviour when confronting different types of tasks as regards the cognitive abilities they require, even though their content is similar. Even though pre-service teachers could make conversions or recognize functions given in different representations they were not essentially in a position to resolve function problems involving conversions from one mode of representation to another. This was evidenced also by the significant proportions of teachers who made successful conversions or recognized functions represented in graphic form, symbolic form, verbal expression or arrow diagrams, but did not provide a correct solution for the problems. The inconsistent way students behave during the solution of tasks with different cognitive features is also revealed in other studies (Duval, 2002; Gagatsis et al., 2002; Elia et al., 2007). The phenomenon of compartmentalization was also evident in the similarity and implicative diagrams of both tests where four clusters were formed and teachers' problem solving ability was found not to be consistent with the ability to recognize the concept in different forms, examples and definitions of the concept even though carrying out the particular task required the use of these conceptions.

In addition, from the implicative relations we can assert that Cypriot and Italian teachers' proficiency in tackling problems entails success in recognition and conversion tasks involving different types of representation of functions. So in order to give a correct solution to a problem the teachers must be able to recognize the concept given in various representations and make conversions between different representations of the concept.

The results of the similarity and implicative relations were further reinforced by the results of the confirmatory factor analysis. Specifically from the analysis of the data of the first test a model with four first-order factors that were interrelated emerged. The four first-order factors of the model represented the definition of the concept, the coordinated, the algebraic approach and problem solving. It was noteworthy that the relation of the first-order factor standing for the coordinated approach with the first-order factor standing for problem solving was very strong for all the pre-service teachers. Very strong was also the relations between the concept definition and problem solving as well as the concept definition and coordinated approach. Attention was also drawn to the fact that the first-order factor standing for the algebraic approach is negatively related with the coordinated approach, problem solving and concept definition.

From the analysis of the data of the second test a model with four first-order factors that were interrelated emerged. The four first-order factors of the model represented the definition of the concept, the recognition of the concept given in various representations, the conversions from an algebraic to a graphical representation and vice versa and problem solving. It was noteworthy that the relations between the four first-order factors were very strong, indicating the fact that the four components were strongly related. The interrelations between some aspects of the understanding of function were also apparent in other studies (Elia et al., 2007; Elia et al., 2008) but in the present research study a more composite model emerged. The two models were verified for both the Cypriot and Italian pre-service teachers. This fact indicates that despite the differences that exist in their performance the interrelations of the various aspects of the understanding of function are the same for all the teachers.

It was noteworthy that the two models that emerged from the analysis of the two tests were not sufficient to interpret the conceptual understanding of function. They just gave a strong indication for the interrelations that exist between the various aspects examined.

Thus, a third-order model that emerged from the analysis of all the data was considered appropriate for interpreting the conceptual understanding of function. The model involved five first-order factors, two second-order factors and one third-order factor. The two second-order factors corresponded to the multiple representational flexibility and problem solving ability and the third-order factor to the conceptual understanding of function. This finding is in line with the results of previous studies that underline the important role of multiple representations (Ainsworth, 1999; Cheng, 1999; Even, 1998; Lesh et al., 1987b) and problem solving (Lambdin, 2003; Reys et al., 1989; Schoenfeld, 1992) in the understanding of mathematical concepts. Furthermore, the important relation between the representational flexibility and problem solving is highlighted, verifying the results of previous studies (Gagatsis et al., 2010; Gagatsis et al., 2009; Elia et al., 2007; Gagatsis & Shiakalli, 2004; Hitt, 1998; Monoyiou & Gagatsis, 2008a; Niemi, 1996). Particularly, Niemi (1996) showed that the ability to recognize fractions given in various diagrammatic representations can predict the performance in problem solving tasks involving the particular concept. Additionally, Gagatsis and Shiakalli (2004) and Hitt (1998) claimed that the ability to translate from one mode of representation to another is closely related with function problem solving.

On the second-order factor that stands for the multiple representational flexibility the first-order factors referring to the concept image, the recognition of the concept given in various representations and the conversions from an algebraic to a graphical representation of the concept and vice versa are regressed. The important role of recognition, treatment and conversion in representational flexibility was also highlighted in other studies (Gagatsis et al., 2010; Duval, 2006; Even, 1998). In this study a new dimension of the multiple representational flexibility emerged, that is the concept image, which consists of the concept definition and examples of the concept.

On the second-order factor that stands for problem solving the first-order factors referring to the solution of complex problems and to the coordinated approach to simple function tasks involving transformations of functions were regressed. Thus, the strong relation of the coordinated approach with problem solving emerged (Monoyiou & Gagatsis, 2008b).

The results of the confirmatory factor analysis are in line with the distinction between an exercise and a mathematical problem that is discussed by a lot of researchers in the field of mathematics education (D' Amore & Zan, 1996; Jonassen, 2000; Philippou & Christou, 1995; Polya, 1945). Even though giving a definition or an example of the concept or the solution of the recognition and conversion tasks is a difficult process, the way of solution and the processes are already known. In the case of problem solving the teachers must discover a novel way to reach a solution since the solution is not already known (Schoenfeld, 1983). They must discover a strategy or even combine different strategies in order to understand the situation of the problem and reach a solution (English, 1996).

Furthermore, the pre-service teachers were clustered into three categories according to the approach they used in order to solve the four simple function tasks. The first cluster involved the teachers who used a coordinated approach systematically, the second cluster included the teachers who used an algebraic approach with consistency and the third cluster involved the teachers who used other approaches or equally the algebraic and coordinated approach. It was noteworthy that the teachers who used a coordinated approach outperformed the other groups in problem solving, in the definition and examples of the concept, in the recognition of the concept given in various representations and in the conversions from an algebraic to a graphical representation and vice versa, indicating that the coordinated approach is more powerful and can lead to successful problem solving (Even, 1998; Mousoulides & Gagatsis, 2004; Knuth, 2000).

The results of the third phase: Pre-service teachers' misunderstandings and ideas held for the concept of function

In order to triangulate the quantitative data regarding teachers' understanding of the concept of function and to further investigate pre-service teachers' behavior in the above mentioned five aspects of the understanding of function namely, concept definition, examples of function, the recognition of functions given in various representations, "coordinated" or algebraic approach when dealing with simple function tasks and effectiveness in problem solving, nine task-based interviews were conducted. The following research questions were examined with the task-based interviews:

- How able are the nine participants to give a right definition, to give examples of the concept, to recognize functions given in different representations and to solve problems?
- What approach do the participants use (algebraic or coordinated) in order to solve simple function tasks?
- What differences exist between the three groups (high, medium and low performance) concerning their performance in the nine tasks of the interview?
- What conceptions, misunderstandings and ideas do pre-service teachers have of function?

The high performance participants were quite successful in all the tasks of the interview. They managed to give an accurate definition of function, they gave correct examples of the concept, they recognized the concept given in various representations and they solved quite easily the problems involving a linear and a quadratic function. Concerning the problem involving an exponential relation they all managed to recognize the exponential relation and two of them managed also to give the equation. They all chose a coordinated approach to solve the two function tasks. In the first task they all managed to reach a correct solution while in the second only one of them managed to give a correct solution. In the second task, which involved a horizontal transformation, in order to reach a correct solution the participants should use a coordinated approach but not superficially. A superficial use of the coordinated approach and the lack of real conceptual understanding of the concept led the participants to wrong answers.

Only one of the medium performance participants gave an accurate definition of function while the others gave a definition of a one-to-one function and an ambiguous definition. They all gave correct examples of the applications of function in everyday life. They gave wrong responses in the case of the more complicated tasks requiring the application of the definition and examples of function. Two of them managed to recognize correctly the functions given to them in different representations. The others made some mistakes in the case of the symbolic and verbal expressions. They all used a coordinated approach to solve the two tasks. In the case of the first task they gave correct responses but in the case of the second task the superficial use of the coordinated approach led to major mistakes. In the case of the problems involving a linear and a quadratic function their performance was quite good. They made mistakes in some parts of the problems that were more demanding and required a conceptual understanding of function. Concerning the problem involving an exponential relation they all interpreted the relation as linear and drew the graph as a straight line.

The low performance participants gave ambiguous definitions and wrong examples for the concept of function. They recognized the function given to them in a diagram and two of them managed to recognize correctly the function given to them in a graph. In contrast, they all gave incorrect responses in the case of the recognition of functions given in a symbolic and a verbal expression. In the case of the two simple function tasks in the first one they used an algebraic approach that led them to a correct response while in the case of the second task since they could not use an algebraic approach they turned to a coordinated approach but since they used it rather superficial they gave wrong responses. Their performance in the solution of the problems involving a linear and a quadratic function was moderate. In some cases they had difficulties in giving the equation or in drawing the graph of the function. Concerning the problem involving an exponential relation they all gave wrong responses by giving the equation of a linear function and by drawing a straight line.

From the above results it emerged that the different aspects of the concept of function are interrelated. However, every aspect describes students' acquisition of the complex concept of function in a unique way. By this we mean that it is not sufficient to make general inferences such as "students have an understanding of the concept of function" in the sense that they are reasonably successful in giving a definition of the concept or providing examples or even recognizing functions in different forms of representation, separately. In many cases students can make conversions or recognize the concept given in various representations but they cannot solve complex problems involving the concept and thus they show ambiguous ideas or limited understanding about a particular concept (Presmeg & Nenduradu, 2005). It is necessary to rely on a composite model like the one that emerged in this study in order to characterize students' understanding of the concept.

From the task-based interviews a lot of misunderstandings and ideas teachers held concerning the concept of function emerged. An idea held by many teachers concerning functions is that it is a one-to-one correspondence (Vinner & Dreyfus, 1989; Elia & Spyrou, 2006). Particularly, two teachers gave a definition of a special kind of function, making a reference to an injective function (one-to-one correspondence) and one of them also gave an example of an injective function. Additionally in the task in which they were asked if $f(-2) = f(3) = 4$ is a function or not seven teachers claimed that this is not a function since for two values of x we have the same y. Some participants expressed also the idea that every graph and every equation is a function. Furthermore, they stated that if the equation has x and y it is a function.

From the interviews it is worth mentioning and giving emphasis to the two tasks with which the participants had difficulties and in which only few of them reached a correct solution. In the first task $\sin(x)$ was given. The teachers were asked to draw the function $\sin(x+1)$. In this task all the teachers used a coordinated approach to reach a solution since as the mentioned they could not find the values of y in this case. The teachers could not ignore the "trigonometric nature" of this task and therefore did not consider the problem in a general way (Even, 1998). Only one of the teachers managed to give a correct response noticing that the value that changes is x and not y and indicating that the graph will move to the left and not to the right even though it is plus one. This teacher showed a conceptual understanding of function. Eisenberg and Dreyfus (1994) conducted an extensive exploration of students' understanding of function transformations and they acknowledged the difficulty in visualizing a horizontal translation in comparison to a vertical one. Furthermore, Baker et al. (2001b) investigated the understanding of transformations of various functions from a perspective of Action-Process-Object-Schema (APOS) theory. They confirmed the observation that vertical transformations appear easier for the students than horizontal transformations and explained this based on their theoretical perspective, claiming that "vertical transformations are actions performed directly on the basic functions, while horizontal transformations consist of actions that are performed on the independent variable of the function and further action is needed on the object resulting from the first action to get the result of the transformation" (Baker et al., 2001b, p. 47). Furthermore, students' difficulty with function transformation was attributed, at least in part, to their incomplete understanding of the concept of function. Baker et al. agree with Eisenberg and Dreyfus in their observation that an object conception of function may be a prerequisite to the effective understanding of transformations of functions.

The second task that presented difficulties for teachers was the problem involving an exponential relation. Only three teachers recognize the exponential relation involved in this problem. Furthermore, two of them managed also to give the equation. The other six teachers consider the relation to be linear and drew a linear function indicating limited understanding for the concept of function. This problem was also used by Presmeg and Nenduradu (2005) in a research study that aimed to identify and characterize different representations that pre-service teachers use- and how they use them-in solving algebraic problems involving exponential relationships. The results of this study showed that the use of different inscriptions (representations) and the conversion between registers is insufficient goal of instruction and does not necessarily imply relational understanding (Skemp, 1987). The results of the present study are in line with the findings of Presmeg and Nenduradu (2005) and move a step forward indicating that in order to achieve relational understanding of a concept multiple factors are involved.

Considering the inconsistencies demonstrated by a number of teachers among the different aspects of the understanding of function examined in this study, all variables investigated here, i.e. definitions, examples, recognition, conversions, coordinated or algebraic approach and problem solving, seemed to contribute to the understanding of the concept and describe in their own unique way different aspects of students' acquisition of this complex concept. Thus, evidence is provided for the necessity of using a composite model, constituted (at least) by these types of behaviour, for examining and explaining how the function concept is understood. Coherent understanding of the concept may be indicated primarily by successful problem solving, which is built on the basis of correct definition and examples, flexibility in dealing with multiple representations in recognition and conversion tasks and a coordinate approach. Limited and ambiguous aspects of the function concept may be revealed by students' deficits in dealing with at least one of the aspects of the understanding of function investigated here.

Implications for teaching and suggestions for further research

The aim of this research study was to explore teachers' display of behavior in six aspects of the understanding of function: effectiveness in solving a word problem, concept definition, examples of function, recognizing functions given in different representations (diagrammatic, graphical, symbolic expression, verbal expression), transferring function from one mode of representation to another and the algebraic or coordinated approaches when dealing with simple function tasks. A main concern was also to examine problem solving in relation to the other types of displayed behavior.

The results of this study have direct implications for teaching and assessment. One must remember that in order to teach functions, it is important to include the different dimensions of studying function in his instruction and assessment: definition, examples, recognition, conversion, coordinated approach and problem solving. To employ effectively this model it is also important for the teachers to have in mind and make appropriate use of the connections among its components. By using this model in students' assessment, teachers can identify in which of these domains students have difficulties as regards the understanding of function. On the basis of the assessment results, teaching must develop mathematical understanding in a way that builds on students' constructed knowledge and abilities. In other words, strong emphasis should be given to the domains that are less familiar or known in some aspects and on their connection to the domains or aspects of a domain in which students are more capable. For example, students who are able to give an appropriate definition and examples of function applications, can be helped to elaborate their knowledge at first by using a familiar representation system and a diversity of other representations to represent their definition and examples; next, by recognizing whether a given mathematical relation in different systems of representation is a function or not in terms of their definition, by identifying the same types of function in various representations and carrying out a conversion of a function from one system of representation to another in different directions. These didactical implications are in line with Steinbring' s (1997) idea that mathematical meaning is developed in the interplay between a reference context and sign systems of the mathematical concept in question.

Nevertheless, further research is needed to investigate at a practical level the effectiveness of such didactical processes for teaching the complex concept of function addressing prospective teachers. It could be interesting to examine whether designing and implementing didactic activities that are not restricted in limited and separately taught aspects, but interconnected with each other on the basis of the above forms of understanding of the notion, may contribute to the development of a global and coherent understanding of function and successful problem solving. The results of such a research study would be enlightening for mathematics educators about the importance of using a composite model constituted by these types of behaviour as a means not only to examine and explain how the function concept is understood by students, but to teach functions at secondary school.

It seems that there is a need for further investigation into the subject. The participants of this study were pre-service teachers. In the future, it is interesting to conduct the same research with students attending middle and high school and examine whether the model for the conceptual understanding of function proposed here applies and remains invariant for these students. Besides, longitudinal performance investigation of the conceptual understanding of function as they move from middle to high school should be carried out.

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APPENDIX 1

Test A_1 (included the tasks 3-9)

Test A_2 (included all the tasks)

Name:…………………………………………………………………

Lesson:……………………………………………………………….

Year in University:…………………………………………………..

Type of High School:……………………………………………….

Specialization in High School:…………………………………….

Specialization in University:…………………………………………

1. According to you what is a function? Give a simple definition.

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2. Give two simple examples from the applications of functions in everyday life.

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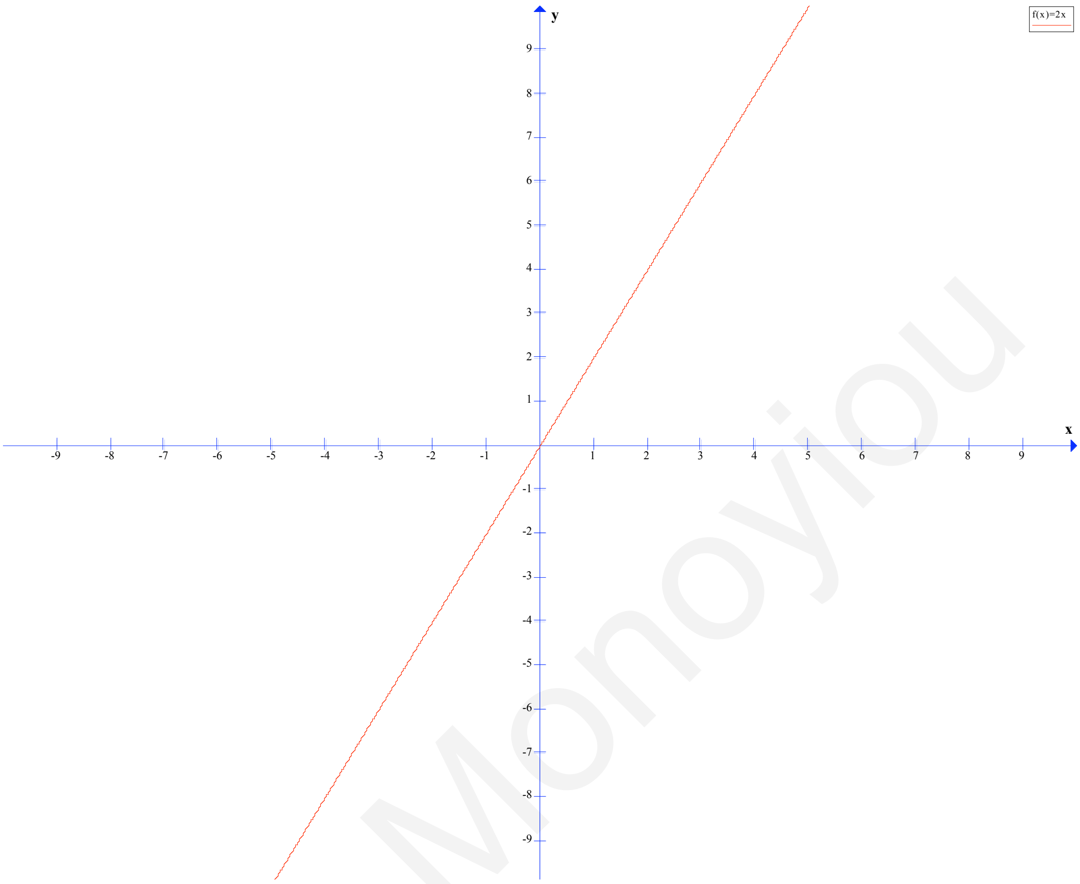
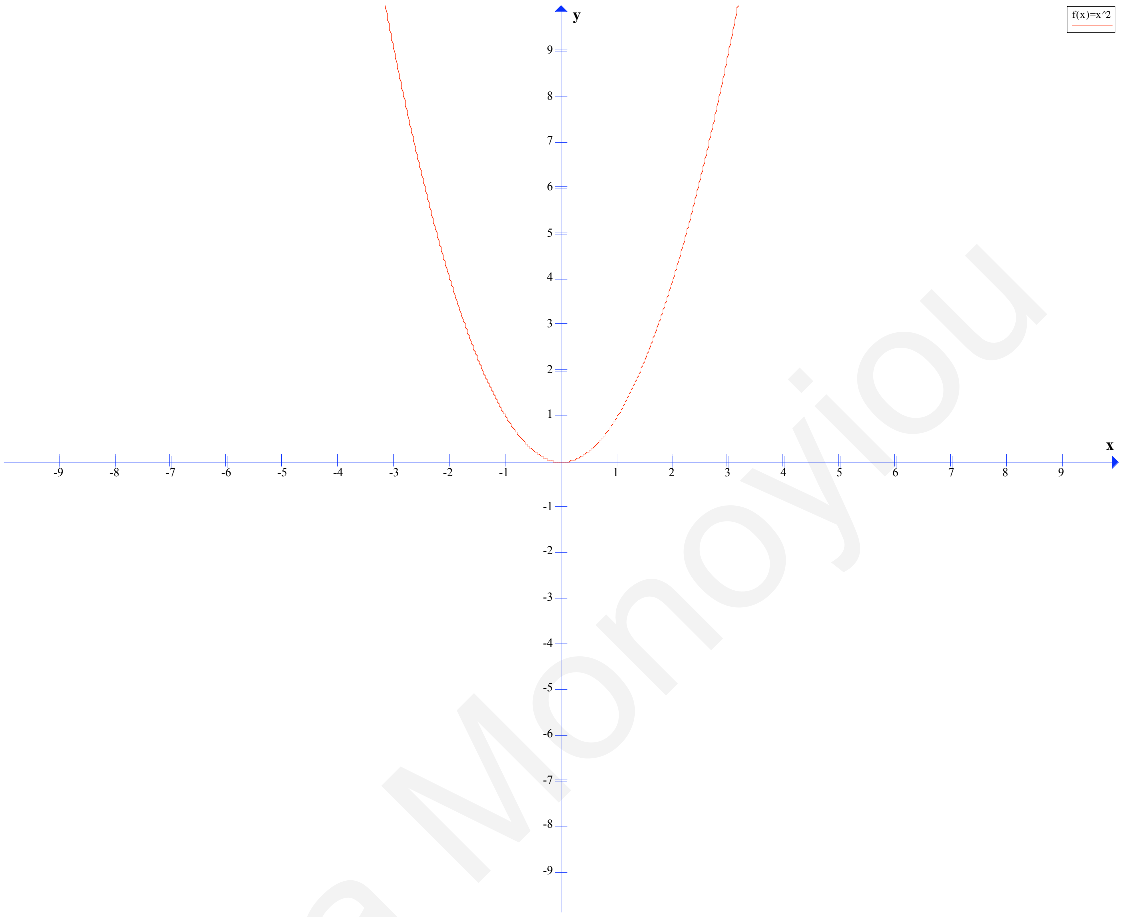
3. In the following diagram **y=2x** is given. Draw the function **y=2x+1**.

(You can use the space below to make calculations or write your thoughts)



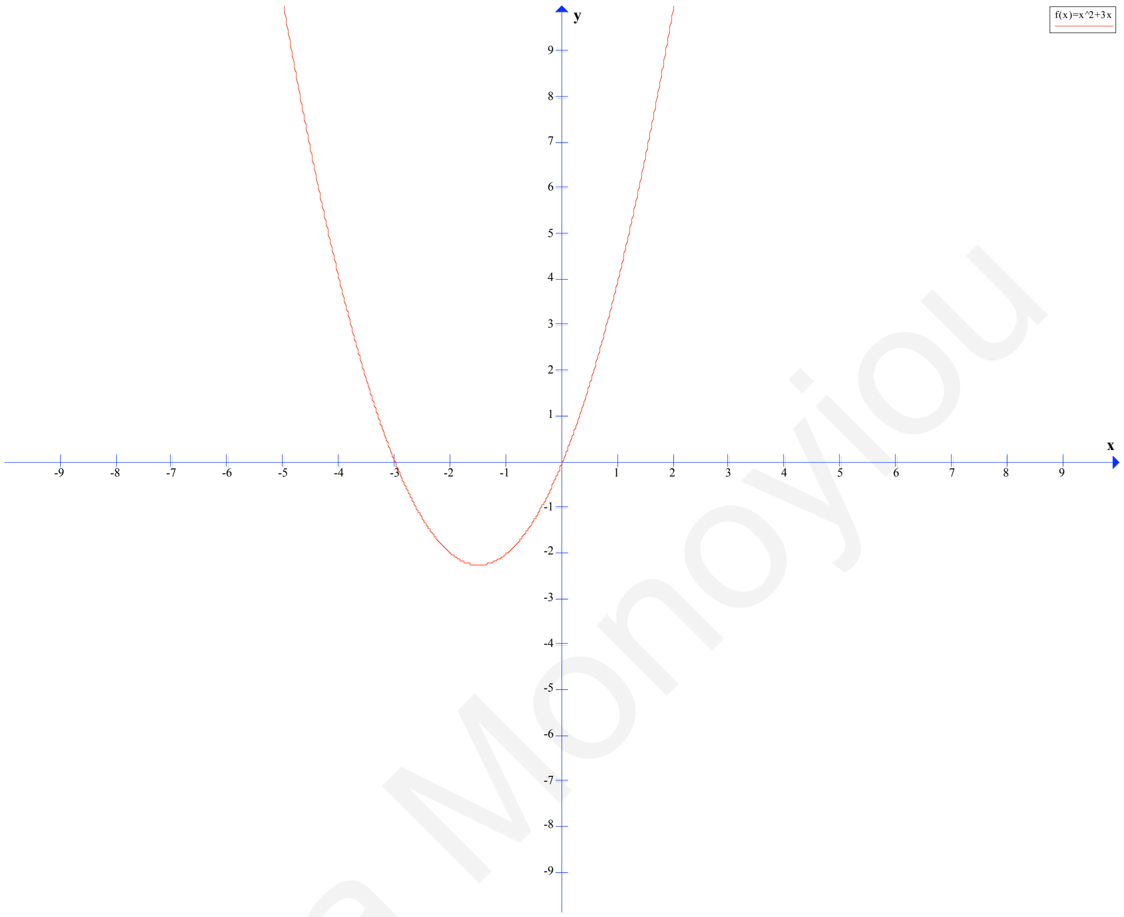
4. In the following diagram $\mathbf{y=x^2}$ is given. Draw the function $\mathbf{y=x^2-1}$.

(You can use the space below to make calculations or write your thoughts)



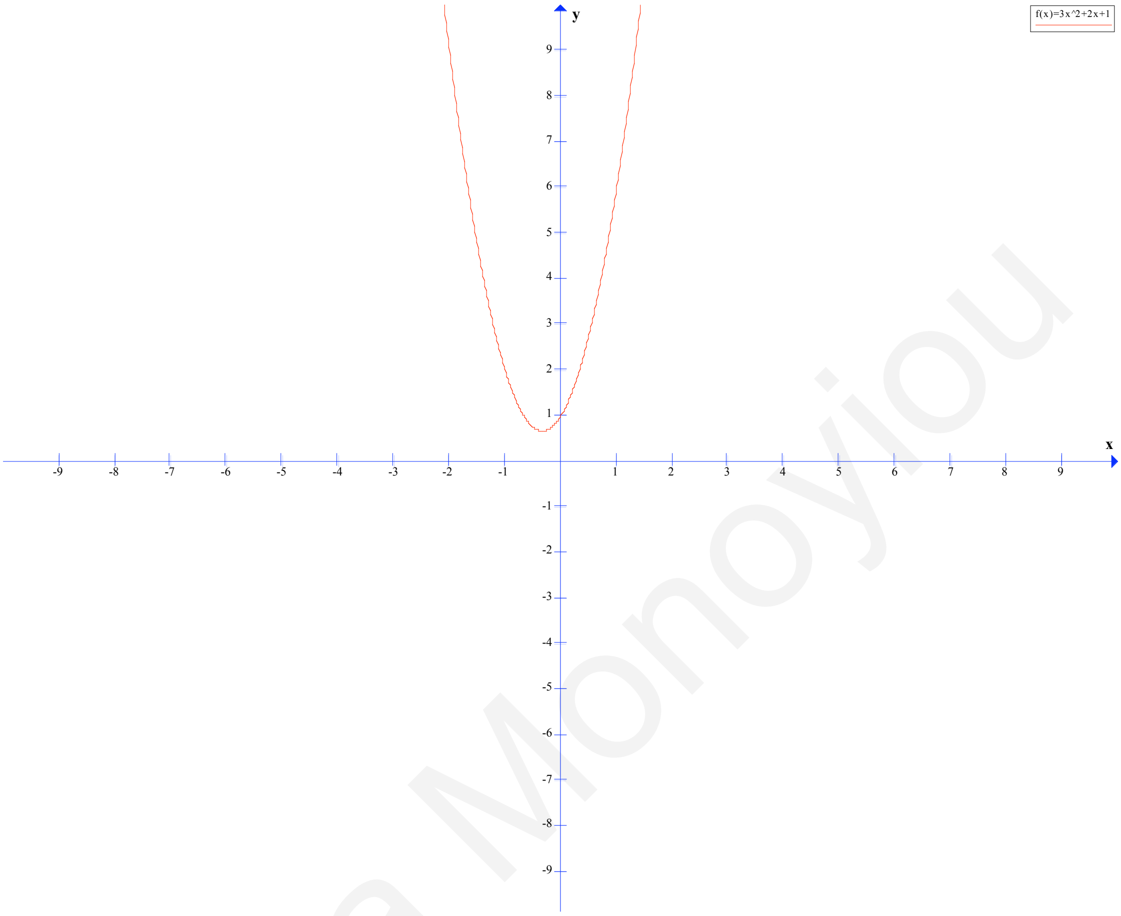
5. In the following diagram $\mathbf{y=x^2+3x}$ is given. Draw the function $\mathbf{y=x^2+3x+2}$.

(You can use the space below to make calculations or write your thoughts)

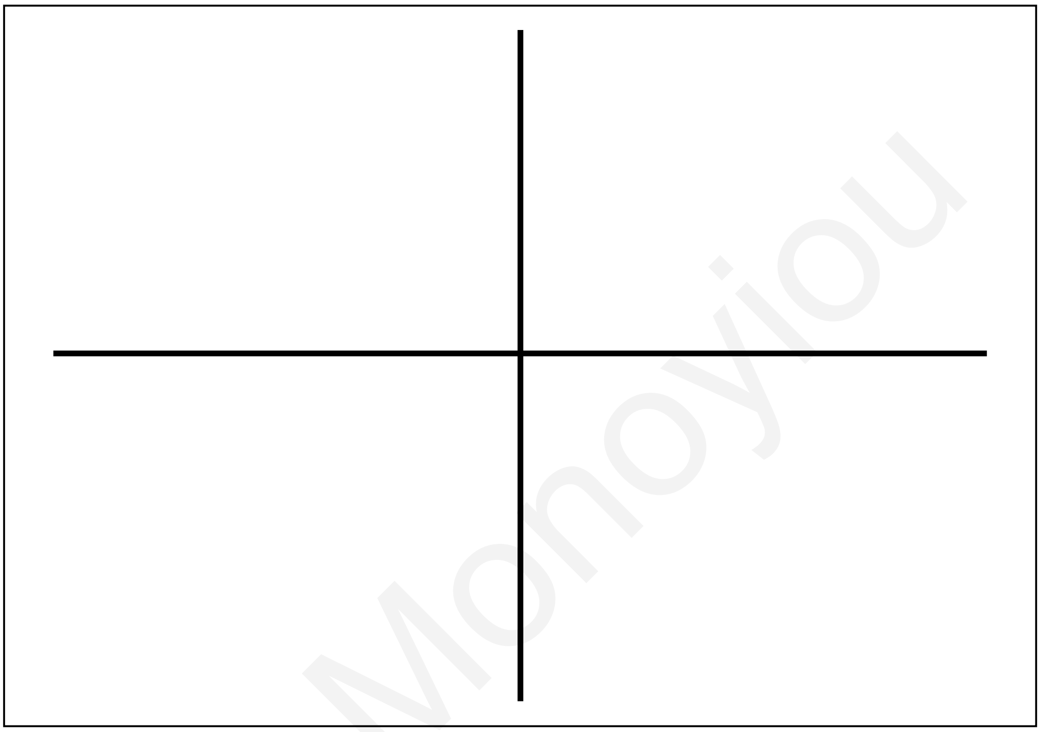


6. In the following diagram **$y=3x^2+2x+1$** is given. Draw the function **$y= -(3x^2+2x+1)$.**

(You can use the space below to make calculations or write your thoughts)



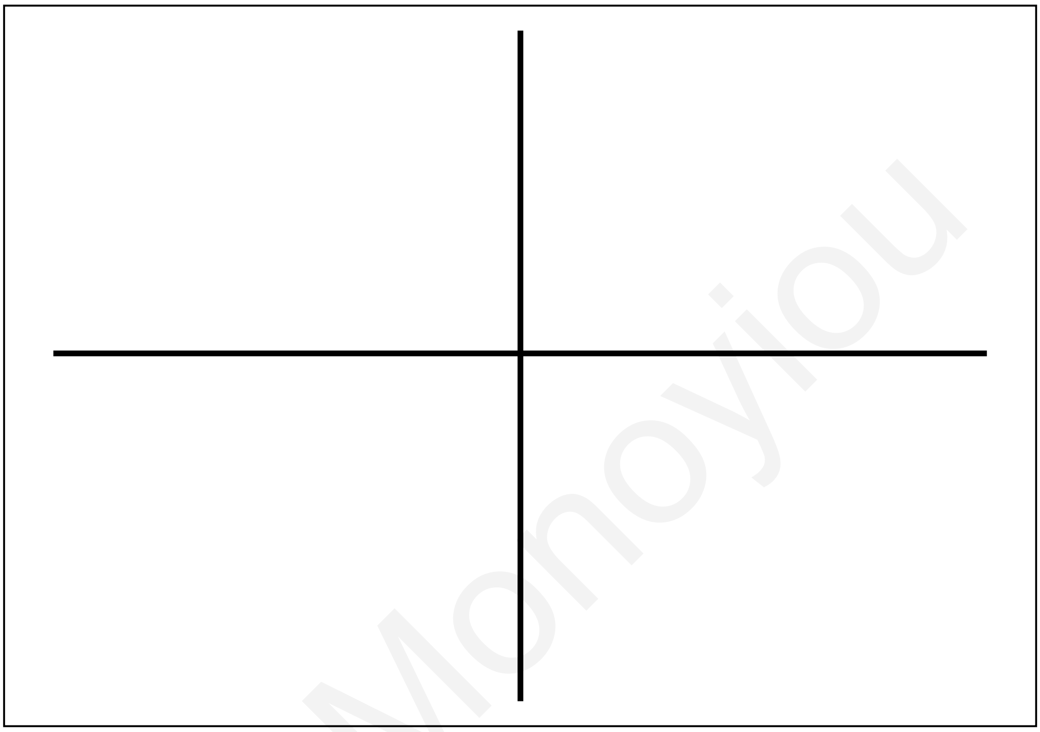
7. A tank contains 600 L of petrol (initial amount). A tank car is filling the tank with petrol. The tank car contains 2000 L of petrol and the rate of filling is 100 L per minute.
    a. Use the information in order to give the two equations.
    b. Draw the two graphs (the volume of the petrol in the tank as a function of time t and the volume of the petrol in the tank car as a function of time t).
    c. Find when the amounts of petrol in the tank and in the car would be equal.



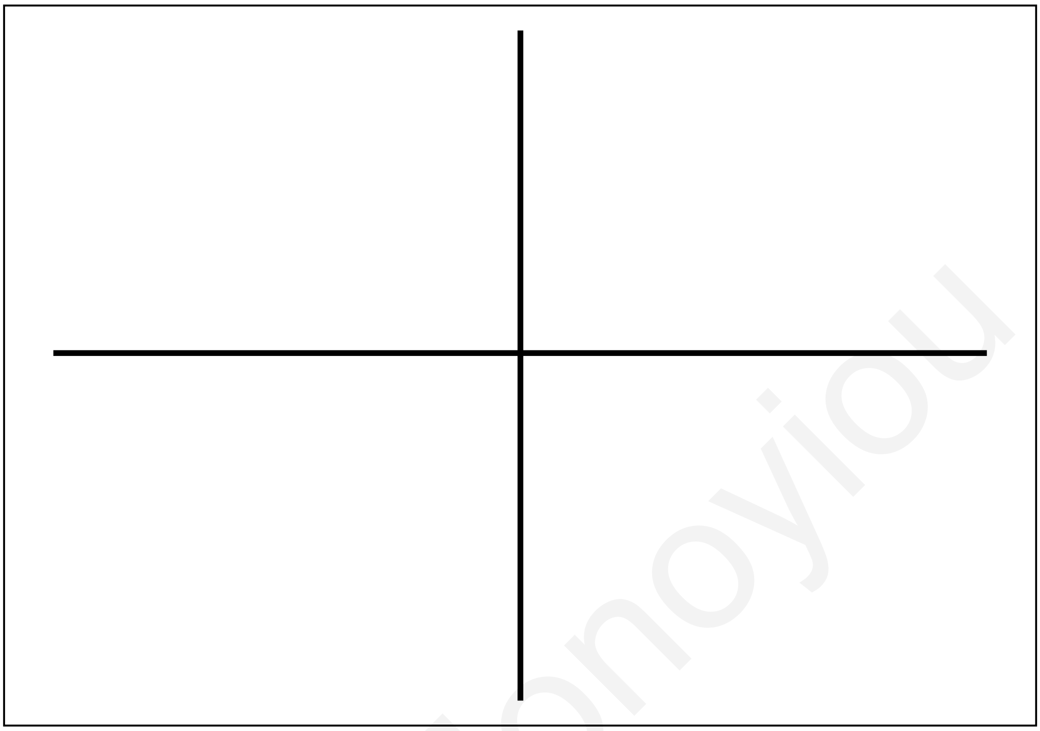
8. In an ant colony the number of ants (A) increases according to the function: $A=t^2+1000$ (t= the number of days). The amount of seeds, the ants save in the colony, increases according to the function $S=3t+3000$ (t= the number of days).

a. Use the information in order to draw the graphs.

b. Find when the number of ants in the colony and the number of seeds would be equal.



9. The function $f(x)= ax^2+bx+c$ is given. Numbers a, b and c are real numbers and the f(x) is equal to 4 when x=2 and f(x) is equal to -6 when x=7. Find how many real solutions the equation $\mathbf{ax^2+bx+c}$ has and explain your answer.



Test A_2

Name:…………………………………………. Lesson:……………………………….

Year in University:………… Specialization in University:……………………………

Type of High School:……………………………………………….

Specialization in High School:……………………………………….

1. Does there exist a function all of whose values are equal to each other?
 Explain your answer.

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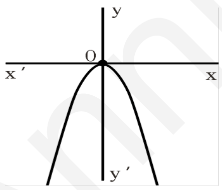
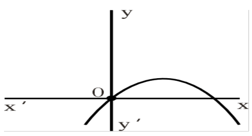
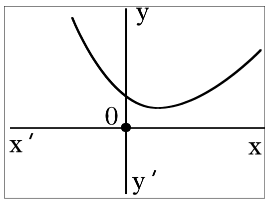
2. Can f be a function, if $f(-2) = 3$ and $f(-2) = 0$?

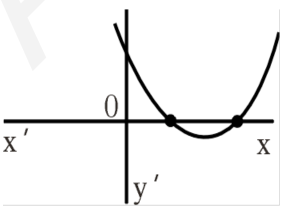
 Yes or No

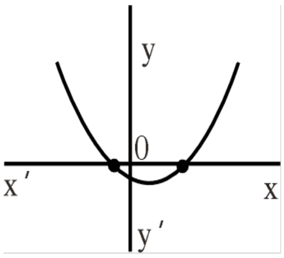
3. The function $ax^2 + bx + c$ is given. For this function $a \neq 0$ and $a.c < 0$.
Which of the following graphs represents the above function?

A. 

B. 

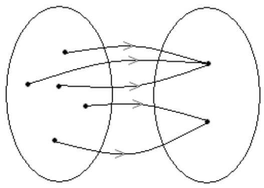
C. 

D. 

E. 

4. Examine which of the following correspondences presented in the form of Venn diagrams are functions. Circle the right answer and give an explanation.

(a)



Yes or No

Explanation:

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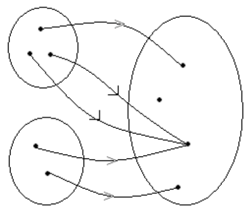
(b)



Yes or No

Explanation:

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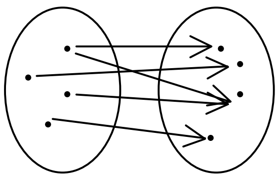
(c)



Yes or No

Explanation:

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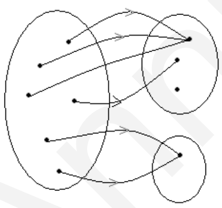
(d)



Yes or No

Explanation:

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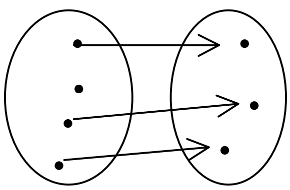
(e)



Yes or No

Explanation:

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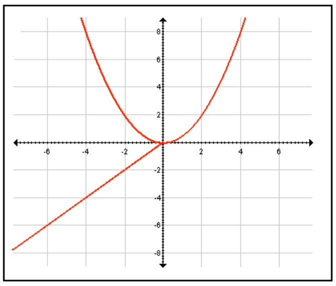
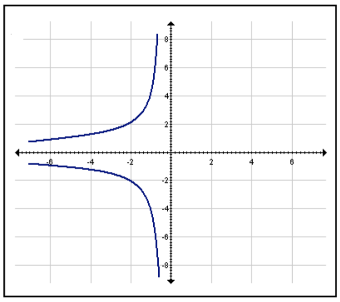
5. Examine which of the following graphs represent functions. Circle the right answer and give an explanation.

(a)



(b)



Yes or No

Explanation:

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Yes or No

Explanation:

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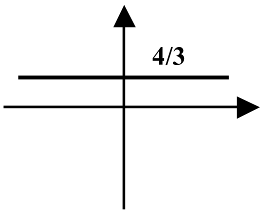
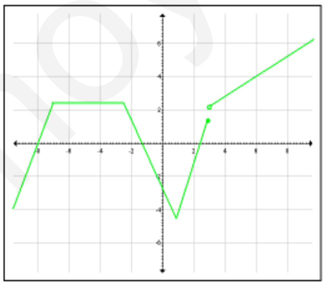
(c)



(d)



Yes or No

Explanation:

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Yes or No

Explanation:

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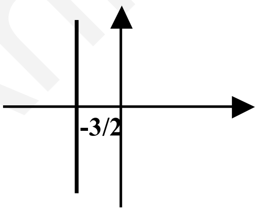
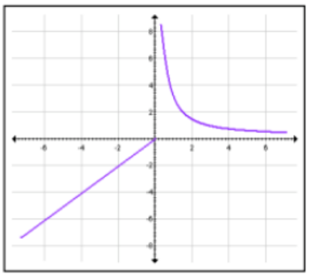
(e)



(f)



Yes or No

Explanation:

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Yes or No

Explanation:

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6. Examine whether the following symbolic expressions may define functions and justify your answer.

(a) $5x+3=0$

Yes or No

Explanation:

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(b) $2x+y=0$

Yes or No

Explanation:

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(c) $4y+1=0$

Yes or No

Explanation:

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(d) $x^2+y^2=25$

Yes or No

Explanation:

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(e) $x^3-y=0$

Yes or No

Explanation:

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(f) $f(x) \begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$

Yes or No

Explanation:

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7. Explain whether we define a function when:

a) In the set of the girls of a class, we correspond a girl with different classmates of hers (George, Homer, Jason, Thanasis, etc.) with whom she will probably dance at a party.

Yes or No

Explanation:

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b) In a football championship, we correspond every football game to the score achieved.

Yes or No

Explanation:

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c) In the set of the scripts at the university entrance examinations, we correspond every script to the couple of marks given by the first and the second examiner.

Yes or No

Explanation:

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d) In the set of the candidates for employment in different work positions in a big organization, we correspond every candidate with the posts for which she/he applies for work in the organization (candidates may apply for more than one post).

Yes or No

Explanation:

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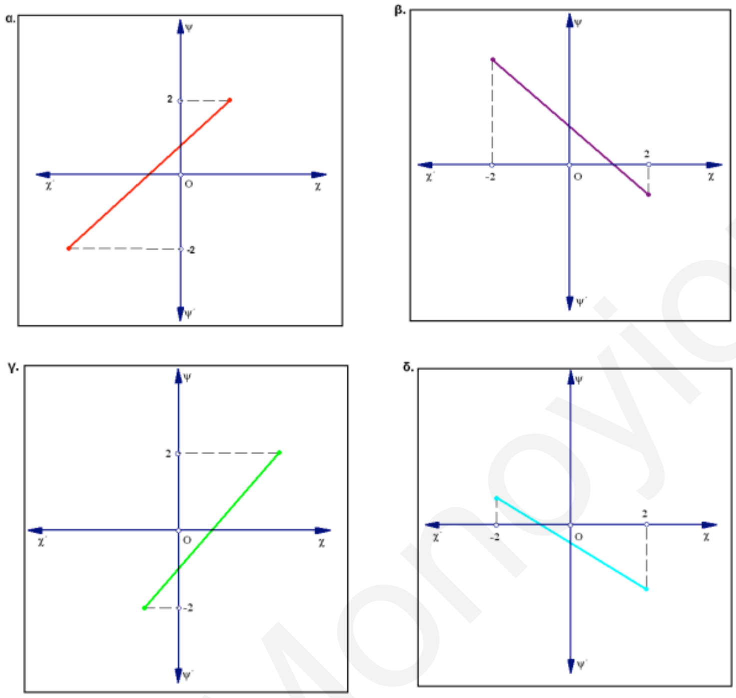
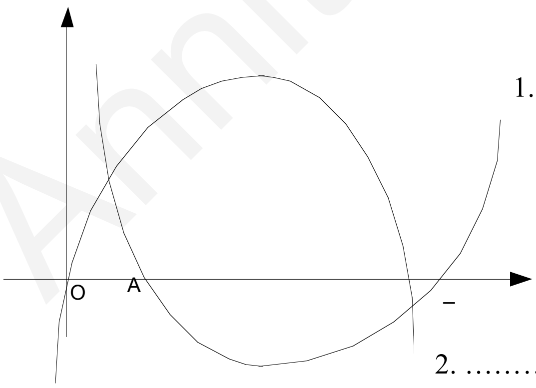
8. Can f be a function, if $f(-2) = f(3) = 4$?

Yes or No

9. Which of the following graphs corresponds to the algebraic expression $y = -x + 1$, $-2 \leq x \leq 2$?



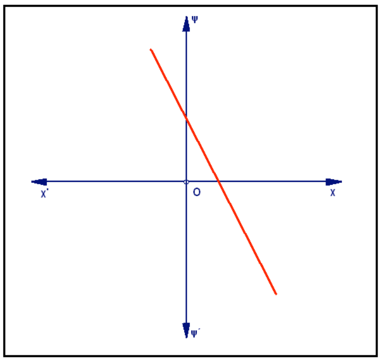
10. In the following figure two functions y=x (2-x) and y=(x-1) (x-3), are given.
Which of the following graphs corresponds to each of the algebraic expressions given?



1. …………….

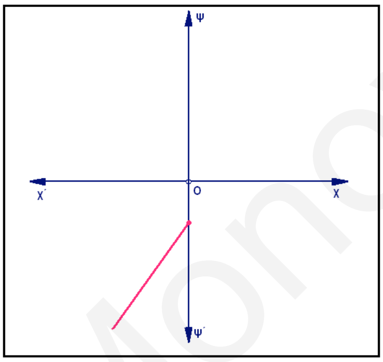
2. …………….

11. Choose the function that corresponds to each graph.



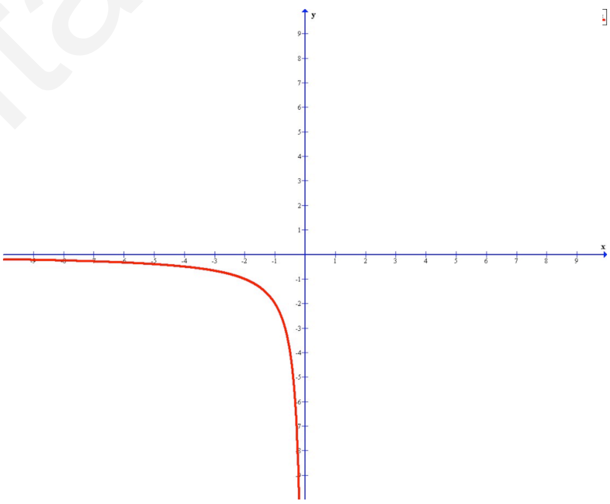
a) y + 5x = 0 c) y + 3 = 2x

b) y = -5x-2 d) y + 3x = 1



a) y = -3x, x>0 c) y + 3x = -1, x ≥ 0

b) y -3x = 1, x<0 d) y – 3x = -1, x ≤ 0

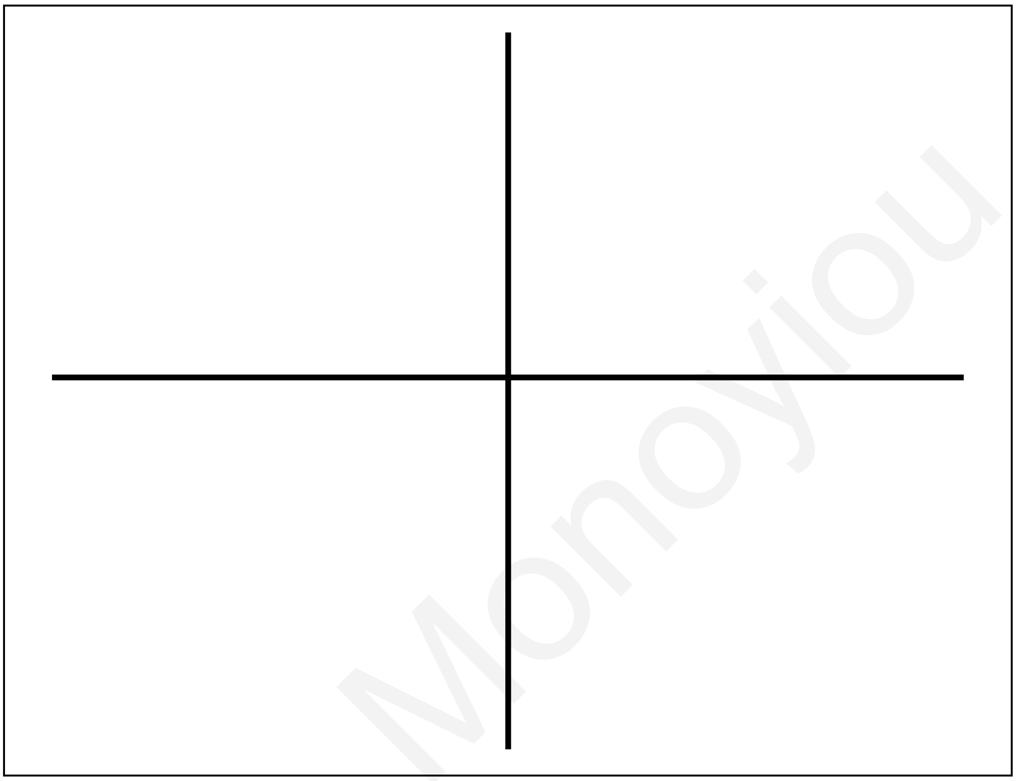


a) y = -2/x, x<0 c) xy=2, x<0

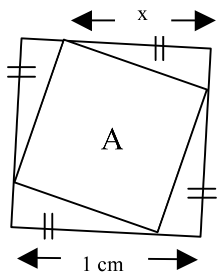
b) xy=-2 d) y=2/x

12. A parachutist jumps from an airplane which is in 3000 m height (above the earth). The parachutist falls with stable speed 30 m/s.

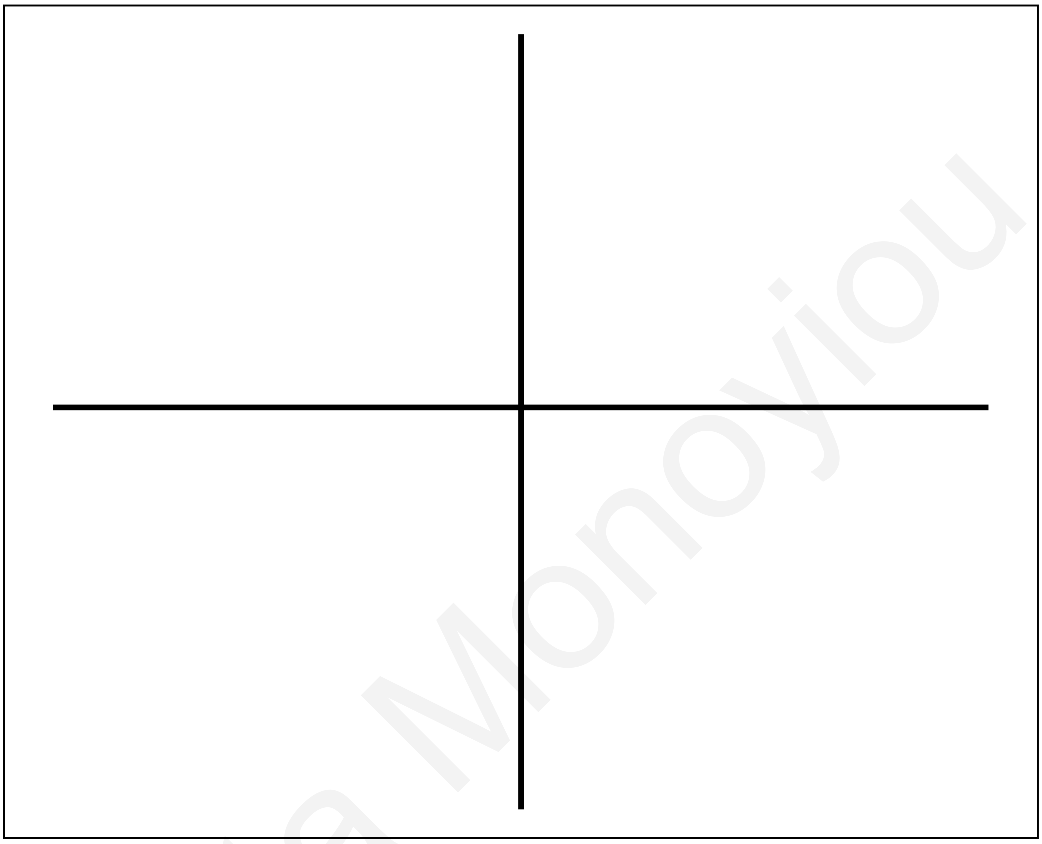
a) Express the parachutist's height as function of time.
b) Draw the graph of the above function.
c) Find the parachutist's height (from earth) 1 minute after his/her fall.
d) In what height the parachutist will be 20 minutes after his/her fall? (Give an explanation)



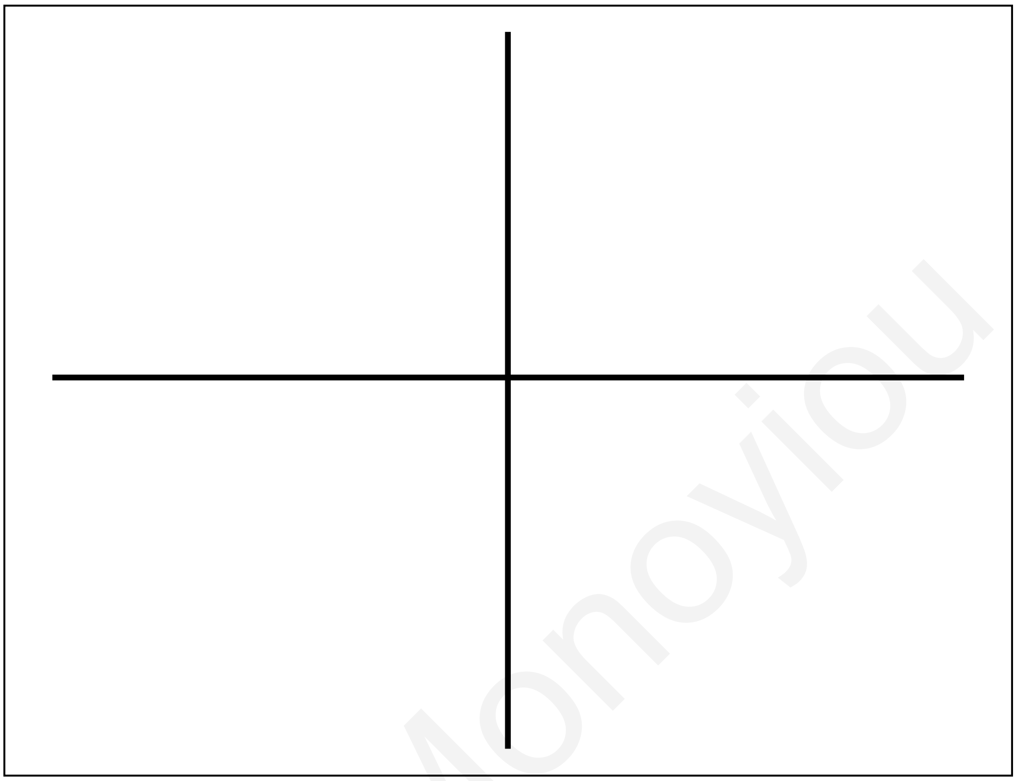
13. In the next figure a square with side 1 cm is given.
Inside the square we draw another square A.
The marked segments have the same length x.



a) Express the area of the square A as a function of x.
b) Draw the graph of the above function.
c) Find the value of x for which the square A has minimum area.



14. The function $f(x) = ax^2 + bx + 7$ is given (a and b are real numbers). When x=0, $f(x)$ has a maximum value. The $f(x)$ is equal to -5 when x=2. Find how many real solutions the equation $\mathbf{ax^2 + bx + 7}$ has and explain your answer.



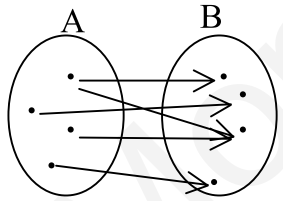
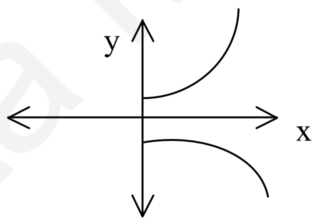
Interviews

Task 1: What is a function? (Give a definition). Can you give a simple example from the applications of functions in everyday life?

Task 2: If $f(-2) = f(3) = 4$, then f is a function. This statement is right or wrong and why?

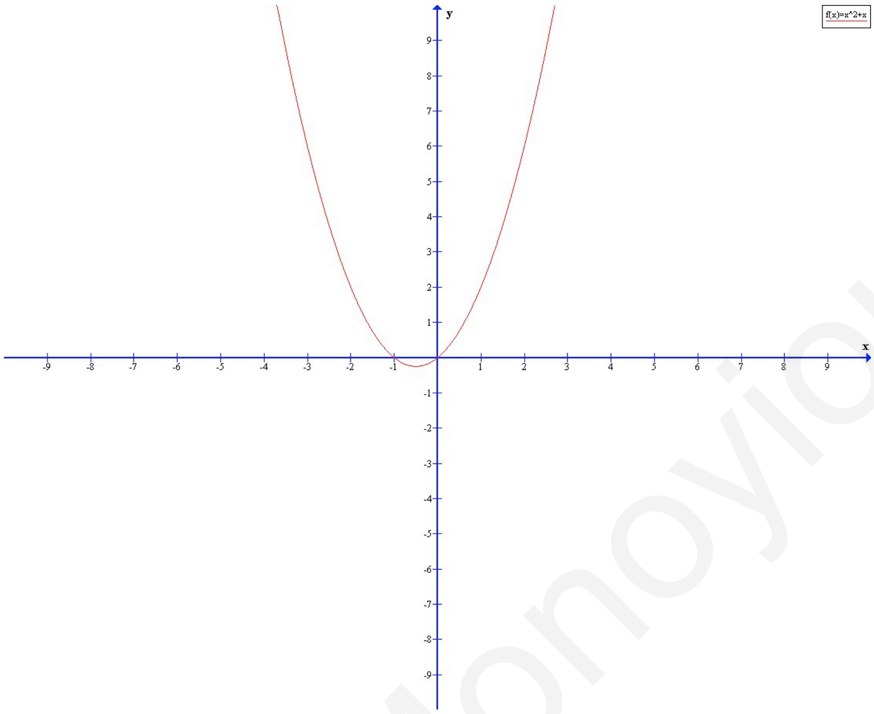
Task 3: Give an example of a function f such that for any real numbers x, y in the domain of f the following equation holds: $f(x+y) = f(x) + f(y)$.

Task 4: Which of the following correspondences are functions? Explain your answers.

Task 4a: 
Task 4b: 
Task 4c: $x^2 + y^2 = 2 \ , \quad x \in R$
Task 4d: We correspond every human with his/her nationality.

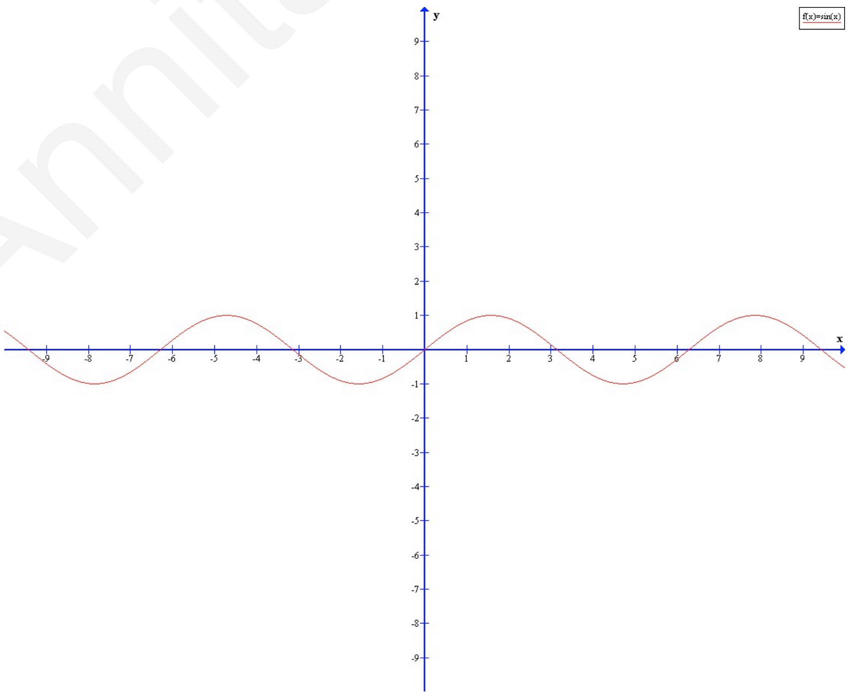
Task 5:

In the following diagram $y=x^2+x$ is given. Draw the function $y=x^2+x+1$.



Task 6:

In the following diagram **sin(x)** is given. Draw the function **sin(x+1)**.



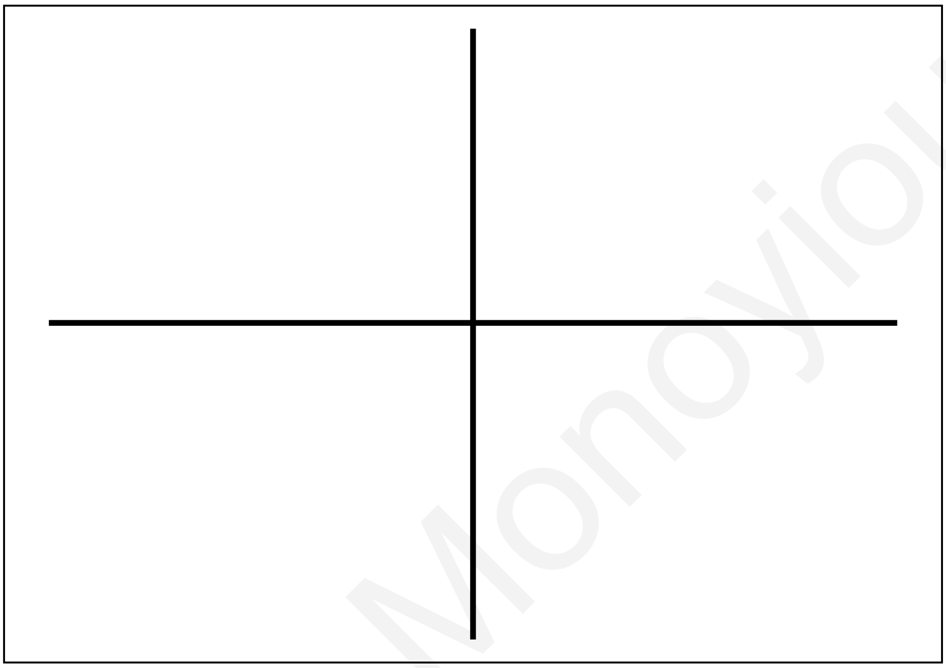
Task 7:

Mr. Nick invested in the stock market a fund of 13,000 euro on 31/1/2007. Starting from the next day his investments had losses 30 euro per day.

a) Express Mr. Nick’s investments (fund) as a function of time.

b) Draw the graph of the above function.

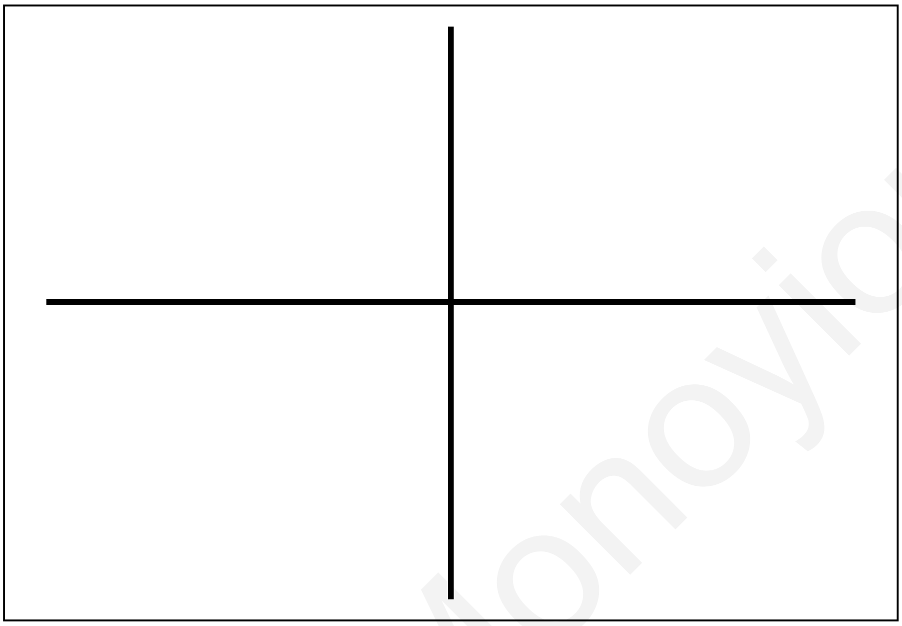
c) Find Mr. Nick’s fund on 31/12/2007.



Task 8:

The sum of the two vertical sides of a right triangle ABC ($A= 90^O$) is 10cm.

(a) Find the area of the right triangle as a function of its side AB=x.
(b) Draw the graph of the above function.
(c) Prove that the right triangle has maximum area, when it is also isosceles.



Task 9:

Several species of whales have been declared endangered. When the population of a particular whale species falls dangerously low, biologists encourage governments to agree to a ban on hunting the species. Suppose that, in 1994, there were only 5000 whales of a particular species and the number of whales in the next two years were 4500 and 4050 respectively. Given the population was predicted to decline in this manner, what will be the whale population in 2001?

APPENDIX 2

Covariance matrix of the variables involved in the structural model that emerged in Phase A (Whole sample)

	T1c	T1a	T2c	T2a	T3c	T3a	T4c	T4a	Pr1a	Pr1b	Pr1c	Pr2a	Pr2b	Pr3
T1c	.205													
T1a	-.142	.250												
T2c	.138	-.099	.204											
T2a	-.086	.148	-.122	.245										
T3c	.094	-.066	.103	-.060	.133									
T3a	-.032	.102	-.036	.125	-.063	.240								
T4c	.096	-.054	.098	-.040	.080	-.008	.244							
T4a	-0.34	.074	-047	.085	-.032	.078	-.100	.182						
Pr1a	.099	-.037	.092	-.020	.069	.032	.083	-.004	.227					
Pr1b	.092	-.024	.091	-.006	.069	.035	.090	-.011	.138	.240				
Pr1c	.072	-.015	.077	.008	.057	.036	.075	-.006	.122	.179	.247			
Pr2a	.092	-.024	.088	.002	.066	.048	.081	.001	.128	.140	.116	.223		
Pr2b	.106	-.045	.094	-.019	.077	.022	.085	-.007	.131	.129	.113	.155	.209	
Pr3	.079	-.033	.074	-.015	.057	.020	.079	-.013	.094	.108	.089	.115	.106	.172

Covariance matrix of the variables involved in the structural model that emerged in Phase A (Group A)

	T1c	T1a	T2c	T2a	T3c	T3a	T4c	T4a	Pr1a	Pr1b	Pr1c	Pr2a	Pr2b	Pr3
T1c	.221													
T1a	-.180	.250												
T2c	.137	-.105	.216											
T2a	-.098	.145	-.172	.250										
T3c	.076	-.061	.108	-.083	.147									
T3a	-.021	.092	-.057	.129	-.101	.248								
T4c	.073	-.020	.073	-.012	.093	.003	.252							
T4a	-.021	.052	-.032	.059	-.044	.093	-.119	.186						
Pr1a	.090	-.041	.081	-.034	.080	.005	.104	-.020	.239					
Pr1b	.082	-.029	.098	-.044	.058	.022	.101	-.019	.143	.243				
Pr1c	.053	-.030	.070	-.030	.038	.011	.054	-.017	.100	.155	.210			
Pr2a	.078	-.019	.070	-.011	.058	.034	.109	-.108	.155	.139	.080	.251		
Pr2b	.107	-.062	.090	-.047	.086	-.015	.096	-.020	.131	.124	.084	.154	.231	
Pr3	.095	-.048	.070	-.025	.046	.029	.082	-.002	.117	.122	.073	.109	.121	.235

Covariance matrix of the variables involved in the structural model that emerged in Phase A (Group B)

	T1c	T1a	T2c	T2a	T3c	T3a	T4c	T4a	Pr1a	Pr1b	Pr1c	Pr2a	Pr2b	Pr3
T1c	.174													
T1a	-.126	.248												
T2c	.134	-.098	.191											
T2a	-.101	.145	-.146	.247										
T3c	.098	-.078	.100	-.073	.129									
T3a	-.052	.121	-.053	.105	-.066	.248								
T4c	.099	-.102	.103	-.098	.080	-.030	.251							
T4a	-.049	.103	-.057	.088	-.038	.081	-.118	.188						
Pr1a	.087	-.050	.099	-.065	.057	.018	.063	-.018	.178					
Pr1b	.075	-.009	.087	-.032	.069	.009	.040	-.003	.105	.250				
Pr1c	.047	-.005	.055	-.022	.041	.005	.039	-.007	.083	.172	.249			
Pr2a	.086	-.022	.074	-.018	.058	.035	.050	.004	.110	.107	.059	.230		
Pr2b	.090	-.043	.081	-.045	.064	.017	.061	-.009	.095	.084	.050	.139	.200	
Pr3	.080	-.042	.080	-.050	.061	-.010	.086	-.024	.092	.091	.071	.092	.095	.163

Covariance matrix of the variables involved in the structural model that emerged in Phase A (Group C)

	T1c	T1a	T2c	T2a	T3c	T3a	T4c	T4a	Pr1a	Pr1b	Pr1c	Pr2a	Pr2b	Pr3
T1c	.163													
T1a	-.114	.248												
T2c	.120	-.099	.186											
T2a	-.077	.159	-.112	.249										
T3c	.078	-.055	.073	-.043	.099									
T3a	-.029	.100	-.040	.134	-.050	.248								
T4c	.078	-.044	.075	-.050	.052	-.037	.246							
T4a	-.037	.080	-.053	.098	-.025	.090	-.126	.208						
Pr1a	.094	-.027	.072	.006	.056	.046	.043	.018	.246					
Pr1b	.086	-.028	.075	.005	.057	.036	.059	-.007	.119	.230				
Pr1c	.073	-.014	.069	.014	.054	.030	.046	.009	.124	.160	.236			
Pr2a	.088	-.019	.082	.011	.064	.053	.042	.021	.116	.136	.133	.220		
Pr2b	.088	-.024	.076	.009	.065	.024	.055	.009	.121	.127	.128	.163	.204	
Pr3	.089	-.038	.087	-.018	.063	.015	.072	-.016	.104	.120	.103	.139	.132	.176

Covariance matrix of the variables involved in the structural model that emerged in Phase A (Group D)

	T1c	T1a	T2c	T2a	T3c	T3a	T4c	T4a	Pr1a	Pr1b	Pr1c	Pr2a	Pr2b	Pr3
T1c	.244													
T1a	-.137	.222												
T2c	.154	-.087	.224											
T2a	-.045	.090	-.072	.169										
T3c	.113	-.064	.135	-.045	.166									
T3a	-.002	.054	.005	.068	-.043	.163								
T4c	.146	-.075	.147	-.032	.111	.010	.222							
T4a	-.013	.044	-.034	0.63	-.024	.031	-.053	.135						
Pr1a	.127	-.048	.121	.012	.092	.030	.133	-.020	.213					
Pr1b	.123	-.054	.115	-.021	.092	.020	.141	-.026	.187	.203				
Pr1c	.116	-.053	.112	-.022	.086	.019	.123	-.028	.168	.165	.194			
Pr2a	.120	-.054	.120	-.015	.084	.031	.121	-.021	.137	.143	.124	.190		
Pr2b	.138	-.062	.126	-.019	.095	.043	.133	-.018	.169	.161	.153	.155	.207	
Pr3	.065	-.032	0.60	-.008	.056	.008	.080	-.015	.059	.067	.055	.085	.065	.111

Covariance matrix of the variables involved in the structural model that emerged in Phase B for Test A_2 (Whole sample)

	D1	Ex1	Ex2	T1a	T1c	T2a	T2c	T3a	T3c	T4a	T4c	Pr1a	Pr1b	Pr1c	Pr2a	Pr2b	Pr3
D1	.222																
Ex1	.077	.228															
Ex2	.065	.127	.203														
T1a	-.016	-.020	-.029	.249													
T1c	.071	.073	.071	-.135	.209												
T2a	-.016	-.005	-.016	.141	-.075	.227											
T2c	.066	.059	.055	-.098	.139	-.099	.204										
T3a	.030	.025	.009	.093	-.028	.119	-.025	.223									
T3c	.058	.053	.049	-.064	.098	-.049	.102	-.052	.131								
T4a	.009	.003	-.003	.071	-.032	.090	-.047	.071	-.027	.178							
T4c	.050	.067	.044	-.051	.103	-.035	.105	-.009	.076	-.090	.238						
Pr1a	.073	.114	.087	-.031	.107	.006	.094	.047	.072	.006	.087	.236					
Pr1b	.076	.099	.080	-.037	.103	-.001	.092	.035	.072	-.011	.097	.153	.220				
Pr1c	.064	.095	.081	-.025	.087	.004	.086	.031	.066	-.004	.083	.148	.163	.220			
Pr2a	.092	.102	.085	-.035	.110	.002	.107	.053	.076	.008	.086	.141	.150	.138	.217		
Pr2b	.085	.093	.089	-.041	.110	-.003	.098	.033	.078	-.002	.089	.146	.145	.139	.169	.205	
Pr3	.070	.078	.073	-.029	.073	-.007	.073	.018	.058	-.012	.078	.090	.100	.085	.114	.102	.149

Covariance matrix of the variables involved in the structural model that emerged in Phase B for Test A_2 (Cypriot pre-service teachers)

	D1	Ex1	Ex2	T1a	T1c	T2a	T2c	T3a	T3c	T4a	T4c	Pr1a	Pr1b	Pr1c	Pr2a	Pr2b	Pr3
D1	.222																
Ex1	.082	.242															
Ex2	.082	.146	.206														
T1a	-.033	-.031	-.045	.248													
T1c	.083	.080	.080	-.113	.163												
T2a	-.037	-.024	-.018	.159	-.077	.248											
T2c	.065	.066	.068	-.098	.120	-.111	.186										
T3a	.020	.008	.008	.100	-.028	.134	-.039	.247									
T3c	.059	.059	.053	-.054	.078	-.043	.073	-.049	.099								
T4a	.016	.-005	-.002	.079	-.036	.097	-.052	.089	-.025	.206							
T4c	.035	.042	.037	-.043	.078	-.048	.075	-.035	.052	-.125	.246						
Pr1a	.067	.120	.100	-.025	.094	.008	.072	.048	.056	.019	.043	.246					
Pr1b	.083	.088	.087	-.027	.086	.006	.075	.037	.057	-.006	.059	.119	.230				
Pr1c	.065	.092	.093	-.014	.074	.013	.070	.029	.055	.008	.047	.126	.161	.235			
Pr2a	.109	.108	.107	-.017	.088	.012	.082	.055	.064	.022	.042	.116	.136	.134	.220		
Pr2b	.083	.089	.099	-.023	.088	.010	.076	.025	.065	.010	.055	.121	.127	.129	.163	.204	
Pr3	.091	.093	.096	-.037	.089	-.018	.087	.016	.063	-.016	.072	.104	.120	.104	.139	.132	.176

Covariance matrix of the variables involved in the structural model that emerged in Phase B for Test A_2 (Italian pre-service teachers)

	D1	Ex1	Ex2	T1a	T1c	T2a	T2c	T3a	T3c	T4a	T4c	Pr1a	Pr1b	Pr1c	Pr2a	Pr2b	Pr3
D1	.224																
Ex1	.071	.203															
Ex2	.045	.104	.199														
T1a	.006	-.020	-.012	.222													
T1c	.056	.078	.063	-.137	.244												
T2a	.011	.003	-.014	.090	-.045	.169											
T2c	.068	.056	.040	-.087	.154	-.072	.224										
T3a	.044	.030	.008	.054	-.002	.068	.005	.163									
T3c	.057	.053	.045	-.064	.113	-.045	.135	-.043	.166								
T4a	.000	.003	-.005	.044	-.013	.063	-.034	.031	-.024	.135							
T4c	.069	.092	.051	-.075	.146	-.032	.147	.010	.111	-.053	.222						
Pr1a	.082	.100	.069	-.053	.135	-.010	.126	.032	.099	-.017	.138	.219					
Pr1b	.068	.108	.072	-.058	.133	-.019	.117	.023	.095	-.022	.142	.194	.207				
Pr1c	.063	.092	.065	-.053	.116	-.022	.112	.019	.086	-.028	.123	.169	.162	.194			
Pr2a	.071	.093	.057	-.059	.139	-.013	.139	.049	.091	-.011	.141	.173	.168	.143	.215		
Pr2b	.087	.098	.077	-.062	.138	-.019	.126	.043	.095	-.018	.133	.179	.168	.153	.177	.207	
Pr3	.045	.052	.044	-.032	.065	-.008	.060	.008	.056	-.015	.080	.067	.070	.055	.083	.065	.111

Covariance matrix of the variables involved in the structural model that emerged in Phase B for Test B (Whole sample)

	D2	D3	D4	Red	Reg	Res	Rev	Coag1	Coag2	Coag3	Coga1	Coga2	Coga3	Pr4	Pr5	Pr6
D2	.250															
D3	.094	.230														
D4	.104	.110	.249													
Red	.101	.087	.110	.149												
Reg	.121	.099	.119	.112	.151											
Res	.098	.088	.112	.102	.115	.128										
Rev	.076	.080	.094	.087	.082	.079	.111									
Coag1	.072	.086	.089	.077	.084	.087	.069	.248								
Coag2	.098	.098	.125	.094	.113	.104	.078	.086	.230							
Coag3	.108	.065	.115	.093	.106	.102	.075	.089	.130	.235						
Coga1	.101	.084	.139	.106	.115	.103	.085	.086	.124	.122	.233					
Coga2	.105	.097	.120	.097	.123	.105	.077	.090	.129	.122	.156	.234				
Coga3	.119	.107	.125	.097	.110	.112	.081	.099	.130	.124	.131	.135	.250			
Pr4	.108	.093	.119	.099	.115	.114	.072	.118	.126	.112	.132	.129	.141	.207		
Pr5	.103	.087	.120	.088	.107	.102	.064	.093	.104	.092	.114	.107	.124	.147	.189	
Pr6	.112	.100	.115	.097	.109	.101	.067	.096	.100	.093	.104	.105	.120	.142	.144	.216

Covariance matrix of the variables involved in the structural model that emerged in Phase B for Test B (Cypriot pre-service teachers)

	D2	D3	D4	Red	Reg	Res	Rev	Coag1	Coag2	Coag3	Coga1	Coga2	Coga3	Pr4	Pr5	Pr6
D2	.236															
D3	.057	.192														
D4	.034	.063	.210													
Red	.037	.046	.050	.081												
Reg	.054	.053	.049	.043	.084											
Res	.039	.040	.050	.042	.047	.063										
Rev	.021	.033	.033	.023	.018	.015	.045									
Coag1	.024	.048	.051	.050	.038	.048	.030	.250								
Coag2	.047	.061	.068	.038	.052	.053	.025	.043	.165							
Coag3	.029	.005	.039	.021	.022	.034	.010	.040	.073	.139						
Coga1	.038	.027	.074	.049	.047	.042	.022	.048	.060	.032	.167					
Coga2	.047	.042	.055	.039	.058	.048	.017	.045	.068	.044	.086	.175				
Coga3	.069	.081	.060	.041	.045	.062	.023	.052	.084	.050	.068	.071	.226			
Pr4	.054	.063	.070	.058	.065	.077	.025	.106	.102	.059	.093	.093	.105	.212		
Pr5	.063	.066	.084	.056	.067	.072	.028	.077	.073	.045	.082	.075	.092	.137	.206	
Pr6	.081	.094	.092	.070	.075	.074	.035	.089	.080	.053	.078	.081	.099	.143	.152	.242

Covariance matrix of the variables involved in the structural model that emerged in Phase B for Test B (Italian pre-service teachers)

	D2	D3	D4	Red	Reg	Res	Rev	Coag1	Coag2	Coag3	Coga1	Coga2	Coga3	Pr4	Pr5	Pr6
D2	.201															
D3	.097	.251														
D4	.123	.124	.222													
Red	.100	.085	.099	.131												
Reg	.132	.108	.129	.105	.150											
Res	.111	.107	.123	.098	.128	.150										
Rev	.089	.103	.111	.097	.097	.104	.144									
Coag1	.099	.111	.100	.067	.104	.103	.089	.227								
Coag2	.097	.101	.127	.081	.113	.105	.087	.106	.248							
Coag3	.117	.080	.113	.067	.109	.098	.076	.103	.108	.228						
Coga1	.112	.111	.148	.090	.122	.113	.103	.098	.134	.139	.245					
Coga2	.114	.125	.135	.089	.131	.115	.096	.114	.140	.131	.178	.247				
Coga3	.119	.097	.141	.089	.120	.116	.099	.126	.124	.130	.145	.156	.221			
Pr4	.126	.098	.125	.085	.120	.110	.086	.105	.103	.107	.126	.123	.137	.157		
Pr5	.123	.093	.131	.088	.121	.110	.082	.097	.111	.107	.122	.117	.135	.134	.151	
Pr6	.112	.081	.101	.080	.107	.096	.073	.084	.085	.088	.093	.096	.107	.109	.115	.157

Covariance matrix of the variables involved in the structural model that emerged in Phase B for both tests (Test A_2 and B) (Whole sample)

	D2	D3	D4	Ex	Red	Reg	Res	Rev	Coag 1	Coag 2	Coga 1	Coga 2	Coga 3	Ap1 rco	Ap2 rco	Ap3 rco	Ap4 rco	Pr1	Pr3	Pr4	Pr5	Pr6
D2	.249																					
D3	.093	.233																				
D4	.104	.108	.250																			
Ex	.054	.039	.050	.173																		
Red	.077	.087	.088	.062	.247																	
Reg	.100	.098	.123	.037	.084	.233																
Res	.100	.083	.140	.055	.087	.124	.236															
Rev	.105	.096	.120	.051	.091	.129	.156	.237														
Coag1	.121	.106	.123	.048	.098	.130	.132	.136	.251													
Coag2	.103	.087	.109	.035	.076	.093	.106	.096	.097	.148												
Coga1	.123	.099	.118	.049	.084	.112	.115	.123	.110	.111	.151											
Coga2	.099	.087	.110	.044	.087	.104	.103	.105	.112	.101	.114	.128										
Coga3	.079	.081	.095	.035	.069	.079	.086	.078	.081	.088	.082	.079	.113									
Ap1rco	.055	.034	.039	.073	.067	.034	.033	.034	.047	.021	.033	.034	.013	.210								
Ap2rco	.067	.047	.055	.058	.065	.041	.053	.054	.051	.034	.052	.044	.027	.139	.206							
Ap3rco	.041	.025	.022	.052	.040	.021	.029	.023	.034	.018	.027	.025	.015	.099	.103	.133						
Ap4rco	.078	.058	.078	.056	.068	.077	.053	.068	.084	.059	.070	.070	.043	.103	.105	.077	.238					
Pr1	.084	.066	.090	.094	.082	.088	.090	.092	.088	.064	.081	.076	.049	.097	.090	0.70	0.90	.180				
Pr3	.062	.049	.064	.076	.073	.060	.064	.061	.063	.049	.065	.061	.038	.073	.073	.058	.079	.091	.151			
Pr4	.108	.091	.116	.069	.121	.126	.132	.128	.140	.098	.115	.113	.072	.066	.078	.052	.088	.123	.091	.205		
Pr5	.103	.087	.119	.067	.097	.104	.114	.107	.124	.087	.107	.102	.064	.077	.080	.055	.088	.113	.096	.146	.186	
Pr6	.114	.100	.114	.061	.101	.100	.103	.105	.121	.097	.110	.101	.067	.068	.082	.055	.092	.092	.092	.141	.142	.213

Covariance matrix of the variables involved in the structural model that emerged in Phase B for both tests (Test A_2 and B) (Cypriot pre-service teachers)

	D2	D3	D4	Ex	Red	Reg	Res	Rev	Coag 1	Coag 2	Coga 1	Coga 2	Coga 3	Ap1 rco	Ap2 rco	Ap3 rco	Ap4 rco	Pr1	Pr3	Pr4	Pr5	Pr6
D2	.237																					
D3	.056	.198																				
D4	.033	.060	.215																			
Ex	.037	.026	.030	.188																		
Red	.031	.049	.050	.067	.251																	
Reg	.050	.061	.065	.030	.040	.169																
Res	.036	.025	.076	.038	.050	.061	.171															
Rev	.046	.040	.055	.039	.046	.069	.084	.180														
Coag1	.071	.082	.056	.041	.049	.085	.068	.070	.227													
Coag2	.041	.047	.049	.018	.051	.037	.051	.040	.042	.083												
Coga1	.057	.054	.047	.032	.037	.052	.048	.059	.046	.043	.086											
Coga2	.040	.039	.047	.031	.049	.053	.042	.049	.062	.042	.047	.064										
Coga3	.024	.034	.033	.019	.030	.026	.024	.018	.022	.024	.018	.015	.047									
Ap1rco	.027	.036	.036	.082	.064	.036	.025	.033	.040	.021	.023	.030	.017	.164								
Ap2rco	.044	.036	.050	.069	.069	.034	.047	.047	.031	.023	.036	.023	.026	.120	.188							
Ap3rco	.024	.023	.016	.057	.044	.021	.025	.019	.028	.019	.022	.020	.016	.080	.075	.102						
Ap4rco	.026	.030	.047	.040	.041	.057	.011	.034	.054	.022	.024	.037	.016	.078	.075	.054	.246					
Pr1	.042	.042	.069	.097	.071	.069	.063	.069	.062	.036	.048	.053	.022	.086	.072	.057	.051	.171				
Pr3	.038	.040	.049	.096	.089	.046	.051	.051	.045	.033	.048	.046	.021	.091	.089	.065	.074	.111	.180			
Pr4	.053	.060	.066	.068	.113	.104	.094	.093	.105	.059	.065	.077	.025	.062	.063	.045	.047	.114	.094	.213		
Pr5	.062	.067	.084	.068	.084	.073	.083	.076	.093	.057	.068	.072	.028	.069	.066	.049	.058	.111	.104	.137	.204	
Pr6	.084	.096	.091	.057	.100	.081	.079	.081	.102	.072	.077	.075	.035	.073	.084	.058	.079	.090	.102	.143	.149	.240

Covariance matrix of the variables involved in the structural model that emerged in Phase B for both tests (Test A_2 and B) (Italian pre-service teachers)

	D2	D3	D4	Ex	Red	Reg	Res	Rev	Coag 1	Coag 2	Coga 1	Coga 2	Coga 3	Ap1 rco	Ap2 rco	Ap3 rco	Ap4 rco	Pr1	Pr3	Pr4	Pr5	Pr6
D2	.201																					
D3	.097	.251																				
D4	.123	.124	.222																			
Ex	.060	.046	.059	.152																		
Red	.099	.111	.100	.048	.227																	
Reg	.097	.101	.127	.031	.106	.248																
Res	.112	.111	.148	.061	.098	.134	.245															
Rev	.114	.125	.135	.051	.114	.140	.178	.247														
Coag1	.119	.097	.141	.044	.126	.124	.145	.156	.221													
Coag2	.100	.085	.099	.039	.067	.081	.090	.089	.089	.131												
Coga1	.132	.108	.129	.055	.104	.113	.122	.131	.120	.105	.150											
Coga2	.111	.107	.123	.047	.103	.105	.113	.115	.116	.098	.128	.150										
Coga3	.089	.103	.111	.042	.089	.087	.103	.096	.099	.097	.097	.104	.144									
Ap1rco	.129	.057	.083	.071	.091	.070	.083	.075	.094	.069	.091	.077	.043	.244								
Ap2rco	.112	.072	.079	.048	.069	.066	.077	.079	.091	.068	.090	.086	.043	.154	.224							
Ap3rco	.079	.039	.048	.049	.045	.039	.053	.046	.059	.038	.052	.049	.030	.113	.135	.166						
Ap4rco	.123	.080	.095	.072	.091	.083	.084	.091	.102	.080	.104	.092	.059	.146	.147	.111	.222					
Pr1	.114	.082	.094	.086	.087	.090	.102	.101	.100	.074	.098	.085	.063	.124	.117	.092	.133	.185				
Pr3	.072	.048	.061	.048	.044	.056	.059	.053	.066	.045	.062	.060	.042	.065	.060	.056	.080	.061	.111			
Pr4	.126	.098	.125	.059	.105	.103	.126	.123	.137	.085	.120	.110	.086	.102	.109	.075	.124	.120	.073	.157		
Pr5	.123	.093	.131	.060	.097	.111	.122	.117	.135	.088	.121	.110	.082	.105	.105	.071	.116	.107	.078	.134	.151	
Pr6	.112	.081	.101	.058	.084	.085	.093	.096	.107	.080	.107	.096	.073	.085	.091	.062	.097	.083	.068	.109	.115	.157