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UNRAVELING MATHEMATICAL GIFTEDNESS: CHARACTERISTICS, COGNITIVE PROCESSES AND IDENTIFICATION

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The present doctoral dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy of the University of Cyprus. It is a product of original work of my own, unless otherwise mentioned through references, notes, or any other statements.

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ABSTRACT

The purpose of this study was to develop a theoretical model that describes mathematical giftedness outlining the abilities, cognitive and hypercognitive processes of gifted students in mathematics and to suggest a corresponding identification process that may distinguish mathematically gifted students in 5^{th} and 6^{th} grades of elementary school.

Five hundred and fifty nine students participated in the study. Two instruments were administered. The first test served as a screening instrument and measured students' mathematical abilities related to giftedness in mathematics. Based on their performance, thirty five students were selected to participate in the next stage, during which the second test containing challenging mathematical tasks was administered. During this stage, individual observations took place to investigate, analyse and elucidate the cognitive and hypercognitive processes of gifted students in mathematics employed during problem solving situations.

The results of the study confirmed the multidimensional construct of mathematical giftedness, as an amalgamation of three factors: (a) mathematical abilities, (b) cognitive processes and (c) hypercognitive processes. Each aspect consists of further components related to giftedness in mathematics. Namely, ability in number relations, spatial ability, ability in inclusion relations and creative ability comprise the factor of mathematical abilities. The results of the study showed the existence of three categories of cognitive processes related to giftedness in mathematics, evident through problem solving; articulation of generalizations, flexibility of mental processes and creative thinking. The category of flexibility of mental processes, is further expanded in specific sub-processes; generation of multiple mathematical solutions, reasoning in cycles, control of multiple mathematical relationships at once, fluency for expediency, curtailment of the process of mathematical reasoning and economical thinking and lastly reversibility of mental processes. The third greater category of creative thinking in mathematics involves five processes; namely, construction of mathematical connections, creative spatial ability, holistic and analytic perception of spatial information, focus on product but also on the process and originality in terms of products and processes. The results of this study showed that the cognitive processes are interrelated rather than independent. In addition, the results of the study described the hypercognitive processes of gifted students in mathematics, employed during problem solving situations. These processes are (a) self-regulation and (b) task commitment, perseverance and confidence.

Finally, the findings of the study suggest that it is possible to identify giftedness in mathematics following a two phase domain specific identification process, collecting both quantitative and qualitative evidence and using carefully designed mathematical tasks to reveal gifted students' quality of thinking. Hence, the study formed and empirically assessed an identification process of giftedness in mathematics, aiming to allow the manifestation of mathematical potential in students attending 5th and 6th grade of elementary school. The identification process, allows to examine and describe the nature of the abilities, problem solving reasoning processes that mathematically gifted students use as they engage in solving non-routine mathematical problems, as well as hypercognitive processes employed. It also indicates that observing students during rich problem solving consists of one of the most efficient ways to trace these characteristics and thus capture the manifestation of mathematical giftedness and potential.

ΠΕΡΙΛΗΨΗ

Ο σκοπός της παρούσας εργασίας ήταν η ανάπτυξη ενός θεωρητικού μοντέλου για την περιγραφή της χαρισματικότητας στα μαθηματικά σκιαγραφώντας τις ικανότητες, τις γνωστικές και υπεργνωστικές διεργασίες που εφαρμόζουν οι χαρισματικοί μαθητές στα μαθηματικά και η εισήγηση μίας διαδικασίας αναγνώρισης η οποία να διακρίνει χαρισματικούς μαθητές στα μαθηματικά, στην Ε' και Στ' Δημοτικού.

Στην έρευνα συμμετείχαν 551 μαθητές, στους οποίους χορηγήθηκαν δύο εργαλεία. Το πρώτο εργαλείο μετρούσε τις μαθηματικές ικανότητες που σχετίζονται με τη χαρισματικότητα. Με βάση την επίδοσή τους, τριάντα πέντε μαθητές επιλέγηκαν να λάβουν μέρος στην επόμενη φάση, κατά την οποία χορηγήθηκε το δεύτερο εργαλείο το οποίο περιείχε μαθηματικά έργα που αποτελούσαν πρόκληση για τους μαθητές. Σε αυτή τη φάση, πραγματοποιήθηκε ατομική παρατήρηση για τη διερεύνηση, ανάλυση και διασαφήνιση των γνωστικών και υπεργνωστικών διεργασιών που εφαρμόζουν οι μαθητές σε περιστάσεις επίλυσης προβλήματος.

Τα αποτελέσματα της έρευνας επιβεβαίωσαν την πολυδιάστατη οντότητα της χαρισματικότητας στα μαθηματικά, ως αμάλγαμα τριών παραγόντων: (α) μαθηματικές ικανότητες, (β) γνωστικές διεργασίες και (γ) υπεργνωστικές διεργασίες. Κάθε παράγοντας αποτελείται από περαιτέρω συνιστώσες που σχετίζονται με τη χαρισματικότητα στα μαθηματικά. Συγκεκριμένα, η ικανότητα που αφορά σχέσεις αριθμών, η χωρική ικανότητα, η ικανότητα που αφορά σε σχέσεις εγκλεισμού και η δημιουργική ικανότητα συνιστούν τον παράγοντα των μαθηματικών ικανοτήτων. Τα αποτελέσματα της εργασίας έδειξαν την ύπαρξη τριών κατηγοριών γνωστικών διεργασιών που σχετίζονται με τη μαθηματική χαρισματικότητα, οι οποίες διαφαίνονται κατά τη διάρκεια της επίλυσης προβλήματος. Αυτές αφορούν τη διαμόρφωση γενικεύσεων, την ευελιξία των νοερών διαδικασιών και τη δημιουργική σκέψη. Η κατηγορία της ευελιξίας των νοερών διεργασιών, εκτείνεται περαιτέρω σε συγκεκριμένες διεργασίες, όπως την παραγωγή πολλαπλών μαθηματικών λύσεων, το συλλογισμό σε κύκλους, τον ταυτόχρονο χειρισμό ποικιλίας μαθηματικών σχέσεων, την ευχέρεια με βάση την πρακτικότητα, τη συντόμευση της διαδικασίας του μαθηματικού συλλογισμού και την οικονομία σκέψης και τέλος την αντιστρεψιμότητα των νοερών διαδικασιών. Η τρίτη ευρεία κατηγορία της δημιουργικής σκέψης στα μαθηματικά περιλαμβάνει πέντε διεργασίες: την κατασκευή μαθηματικών διασυνδέσεων, τη δημιουργική χωρική ικανότητα, την ολιστική και αναλυτική αντίληψη των χωρικών πληροφοριών, την επικέντρωση τόσο στο προϊόν όσο και στη διαδικασία και

τη δημιουργικότητα σε σχέση με τα προϊόντα και τις διαδικασίες. Τα αποτελέσματα αυτής της εργασίας έδειξαν ότι οι γνωστικές διεργασίες δεν είναι ανεξάρτητες αλλά σχετίζονται μεταξύ τους. Επίσης, τα αποτελέσματα της εργασίας περιέγραψαν τις υπεργνωστικές διεργασίες των χαρισματικών μαθητών στα μαθηματικά, όπως αυτές εκδηλώνονται κατά την επίλυση προβλήματος. Οι διεργασίες αφορούν την αυτορρύθμιση και τη δέσμευση στο έργο, την επιμονή και την αυτοπεποίθηση.

Τέλος, τα ευρήματα της έρευνας υποδεικνύουν ότι η αναγνώριση της χαρισματικότητας στα μαθηματικά είναι δυνατή μέσα από μια διαδικασία αναγνώρισης δύο φάσεων, επικεντρωμένη στην περιοχή των μαθηματικών, συλλέγοντας ποσοτικά και ποιοτικά δεδομένα και χρησιμοποιώντας προσεκτικά σχεδιασμένα μαθηματικά προβλήματα για την ανάδειξη της ποιότητα της σκέψης των χαρισματικών μαθητών. Κατά συνέπεια, η εργασία ανέπτυξε και αξιολόγησε εμπειρικά μια λεπτομερή διαδικασία αναγνώρισης της χαρισματικότητας στα μαθηματικά, στοχεύοντας να επιτρέψει την εκδήλωση του μαθηματικού δυναμικού μαθητών που φοιτούν στις Ε’ και Στ’ Δημοτικού. Η διαδικασία αναγνώρισης, επιτρέπει την εξέταση και περιγραφή των ικανοτήτων, των γνωστικών διαδικασιών που χρησιμοποιούν οι χαρισματικοί μαθητές στα μαθηματικά κατά την επίλυση ανεξοικείωτων μαθηματικών προβλημάτων, καθώς και των υπεργνωστικών διαδικασιών που χρησιμοποιούν. Η διαδικασία αναγνώρισης υποδεικνύει επίσης ότι η παρατήρηση των μαθητών κατά τη διαδικασία επίλυσης προβλήματος αποτελεί ένα από τους πιο αποτελεσματικούς τρόπους ανίχνευσης των χαρακτηριστικών των μαθητών και επομένως, συλλαμβάνει την εκδήλωση της μαθηματικής χαρισματικότητας και του αντίστοιχου δυναμικού.

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“You don't write because you want to say something;
you write because you've got something to say.”
F. Scott Fitzgerald

To George and my parents
that inspired me to pursue my dreams
and write what I have got to say

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CHAPTER ONE: INTRODUCTION

Introduction

Students with the greatest potential will most likely be the backbone of scientific and technological developments in the future in mathematics (Sheffield, Bennett, Berriozabal, DeArmond, & Wertheimer, 1999). Through the report of the Task Force on Promising Students, Sheffield and her colleagues (1999) have increasingly voiced concern over the vital importance to exploit their power to the maximum, investing in gifted education. Education is critical for the future (Gerver, 2010). In fact, talent in mathematics was proclaimed as one of the most broadly required resources for the 21st century (Office of Science and Technology Policy, 2006), proclaiming the importance of nourishing mathematical giftedness. Still, "gifted children often are neglected, not referred, remain untested, and remain very poorly served by any stretch of the imagination" (Shaughnessy & Persson, 2009, p. 1285). As a result, nations keep losing their most valuable natural resource - human capital.

Investments in gifted education should be based on empirical research findings. However, in recent decades, there has been considerable criticism based on the quality of giftedness research (Heller, 1993; Heller & Schofield, 2000; Ziegler & Raul, 2000). Despite several publications introducing various conceptions of giftedness, only a slight fraction of these definitions have been subjected to adequate empirical examination (Stoeger, 2009). In line with Stoeger's assertion, Friedman-Nimz, O'Brien and Frey (2005) reported that only few of the studies in gifted education add to the body of knowledge through empirically assessed practice. After a comprehensive literature research, through conference proceedings of international conferences on giftedness and articles published in the six most important journals in the area of giftedness over the preceding 12 years, Heller (1993) as well as Heller and Schofield (2000), concluded that less than 20% of them referred to basic research or the empirical examination of models (Stoeger, 2009). Among others, Heller and Schofield (2000) concluded that “this may also suggest that there is still a lack of good quality basic research being undertaken in the area of giftedness per se. Indeed, without such basic research, the nature of giftedness is still open to question” (p. 134). Hence, research should pay more attention on methodological aspects.

In the 21st century, the rapidly expanding globalization, worldwide competition and the great demand for innovation should become the central catalyst of giftedness research. In order to meet the challenges of the new millennium, the field has to expand its perspectives and conceptions, models and approaches should be revised in several regards. Davidson (2009) points to the danger that the fundamental nature of giftedness is becoming obscured. To avoid this, components of models that are not required to describe and predict gifted behavior over time should be removed. As long as the object of giftedness research is not clear, standard definitions are insufficient. In line with Davidson, Stoeger (2009) addresses the future challenge of determining those conceptions of giftedness with the power to survive in the field, therefore subjecting them to empirical clarification.

To this end, the purpose of this study was to describe mathematical giftedness through a theoretical model. At first, the vigilant study of relevant literature both in the field of giftedness and mathematics led to the design and implementation of a two-step identification process for mathematically gifted students, which can be seen as a first step in nurturing students’ talent. Through the observation of mathematically promising students that were carefully selected from the first step of the process, that is an identification instrument, it was possible to delineate specific reasoning, cognitive and metacognitive processes employed whilst engaged in rich problem solving. In this way, an empirically examined and verified comprehensive model describing mathematical giftedness was constructed. This model, was grounded on data collected to respond to the need for empirically grounded research. Furthermore, the model consists of a reply to the necessity for the field of giftedness to expand and revise its conceptions of giftedness, whilst at the same time it focuses in the field of mathematical giftedness. Moreover, this study is expected to guide future research efforts both in identifying and nurturing mathematical giftedness.

The next section presents the issues that initiated this study, organized in two directions; nature of giftedness and the complexity of identification of giftedness.

The Problem

Although giftedness has been extensively studied for centuries, there is still no uniform conception of giftedness (Dai, 2010; Davidson, 2009). The lack of a commonly accepted theoretical foundation of giftedness has caused researchers to vary in perspectives and approaches and to examine factors that affect it in several fields. Even so, the adoption of a comprehensible definition is important for the design of an identification process and subsequent programs to meet the needs of the gifted. At the same time, the complexity of designing and implementing a cohesive identification process has been well documented (VanTassel-Baska, 2005). A discussion on these two tensions surrounding the field of giftedness and serving as a motivation for this study, namely the controversy with respect to the nature of giftedness and the discussion around the complexity of identifying giftedness follows.

Nature of Giftedness

The significance of having a clear image of the components and sources of giftedness has been recognized as one of the key points both for understanding giftedness in a coherent and comprehensive way and fostering the talents of gifted individuals (Davidson, 2009). Despite previous research efforts on studying and identifying giftedness, a number of issues prevent their successfulness. This section addresses these issues, namely the difficulty of reaching a uniform definition of giftedness, the disregard of cognitive processes in theoretical models, the role of innate traits in the development and evolution of giftedness, the arguments on general and specific views of giftedness and the quality of research on giftedness.

Although exceptional abilities have been recognized for centuries (Ziegler & Heller, 2000), there is still no consensus on the definition of giftedness due to the complexity of the construct. Hence, the notion of giftedness has been defined and measured in different ways. For a long time, the field was dominated by narrow conceptualizations of giftedness (Hong & Milgram, 2008) that viewed giftedness in terms of cut off points on certain criteria and equated giftedness with intelligence. In recent decades, there was a notable shift in the research literature, where more liberal definitions and multifaceted views of giftedness are highly favored (e.g., Gagné, 1985, 2004, 2005, 2009; Gardner, 1983; Heller, 2004; Renzulli, 1978, 2002; Sternberg, 1985, 1999). To comply with the trend of multifaceted views of giftedness, contemporary models describing giftedness have been proposed. New types of models include psychological functions (e.g. Gagné, 2009), personality traits (e.g. Milgram & Hong, 2009; Renzulli, 2002), chance (e.g. Gagné, 2009), environmental and social catalysts (e.g. Gagné, 2009; Milgram & Hong, 2009; Renzulli, 2002) in addition to cognitive skills. They embrace a multifaceted view of giftedness and some of them (e.g. Gagné, 2009) take into consideration how potential develops into exceptional domain-specific accomplishments.

Despite the attempts to define giftedness, in the past the field was radically concentrated on traits and neglected findings from the field of cognitive psychology. Most talent models overlooked cognitive processes and therefore neglected their analysis from being incorporated into the models. This tendency to focus on examining traits of gifted individuals was criticized by a number of researchers (Holodnaya, 1993; Shavinina, 1995) and called for a shift in the research paradigm. Throughout the criticism, a number of researchers attempted to investigate the cognitive processes of gifted persons (Holodnaya, 1993; Shore & Kanevsky, 1993; Sternberg, 1986), starting to influence the field. This shift was evident in new conceptions and models of giftedness, such as the Munich Model of giftedness (Heller, 2004) and the Differentiated Model of Giftedness and Talent proposed by Gagné (2004, 2009). The explanatory potential of these new models widely surpassed the conceptions they displaced (Ziegler, 2009).

The role of innate traits on the interpretation of superior performance and achievement has been the subject of numerous studies and is considered to be another controversy in the field. On the surface, the natural abilities' models and the expertise models appear to be on opposing sides (Davidson, 2009; Gagné, 2009; Schneider, 1993). At the one end, a certain degree of inherited characteristics are presumed for the manifestation of giftedness (e.g. Gagné's DMGT). At the other end, supporters of the expert performance approach, propose that superior performance does not depend on innate qualities (Ericsson, Roring, & Nandagopal, 2007); rather on domain-specific training. However, the two approaches can be complementary to one another (Davidson, 2009; Ziegler & Heller, 2000; Ziegler & Stoeger, 2007). According to Feldhusen (2005), nature may help an individual master knowledge in a domain and reach expertise through practice. In sum, previous research findings call for a clarification of the role of innate traits in the development and manifestation of giftedness.

Although general intelligence is important, the level of g required in specific domains for high performance continues to be questionable (Tannenbaum, 1997), while it may depend from the domain (Jensen, 1998). As stated by Csikzentimihalyi (2000) giftedness is field-dependent. However, prior research in the field of giftedness was predominantly concerned with the notion of general giftedness rather than domain-specific giftedness, in our case mathematics. Thus, there was limited focus on theoretical models of mathematical giftedness as well as specially designed identification procedures. VanTassel-Baska (2005) favors domain-specific conceptions of giftedness for promoting talent development. For example, Gardner's (1983) model of multiple intelligences holds aspects of domain-specific views of giftedness. Domain-specific conceptions of giftedness, consider the manifestation of giftedness within specific domains. According to VanTassel-Baska (2005), giftedness is "the manifestation of general intelligence in a specific domain of human functioning at a level significantly beyond the norm such as to show promise for original contributions to a field of endeavor"(p. 359).

Following views on the domain-specificity of giftedness, Song and Porath (2005) suggest that there are cognitive characteristics common to all gifted persons and other cognitive characteristics that are domain-specific. Therefore, domain-specific characteristics would reveal as unique characteristics of individuals gifted in a specific area. However, influential models in the field of giftedness, disagree on common and domain-specific cognitive abilities and the models also lack evidence about the interrelationship between them (Song & Porath, 2005). Therefore, the need for a new model describing giftedness, especially in a specific domain such as mathematics, is evident. In this model, discrimination between general and domain specific characteristics and abilities that contribute to giftedness is required.

The Complexity of the Identification of Giftedness

Conceptions of giftedness should be translated into identification processes and specially designed programs for the students. However, the identification of gifted students has long been a controversial issue (Ziegler, 2009). Since the concept of giftedness has not been clarified yet, the identification of mathematically gifted students is considered to be extremely challenging (Hoeflinger, 1998). Particularly, a number of issues and challenges pertain to the identification of gifted students (VanTassel-Baska, 2005). Underrepresentation of students from certain groups, mismatch between objective, identification and services and underachievement are included.

Research evidence shows that students from racial or ethnic minority groups, economically disadvantaged students, students with limited language proficiency due to their foreign origin and students with physical disabilities are underrepresented in gifted programs (Castellano, 2003, cited in Coleman, 2003). One reason for this phenomenon has been the over reliance on scores from standardized tests for identification purposes (Frasier, Garcia, & Passow, 1995). The underrepresentation of special groups in the identification of gifted students has been also attributed to the subjective information collected during an identification method (Lee, 1999), such as teacher and parent nominations. The evidence on the underrepresentation of students from minority groups call for the design and implementation of inclusive rather than exclusive identification procedures, not biased in any way and accessible for all students that have or show potential.

Not only the decision upon the nature and definition of giftedness as described in the previous section is crucial for the design of identification measures, but also the objective of the identification process should be carefully selected (Heller, 2004). The choice of identification instruments depends on the definition of giftedness adopted and the nature of the gifted program (Nevo, 2008). Therefore, the purpose of identification and the context of identification tools should be carefully considered. In the case of developing an identification system focused in mathematics, special attention should be given to the context of selection instruments in order to ensure that they highlight mathematical strengths, cognitive abilities and patterns of behavior of mathematically gifted students. Following, in order to select gifted students, Lohman (2009) addresses the need of first clarifying the components of expertise in a specific context. As Coleman (2003) argued, a mismatch between identification and consequent nourishment should be eluded. To this end, the purpose and the context of identification tools should be aligned. For instance, the identification process aiming to select gifted students to participate in a mathematics enrichment program should focus on math abilities rather than on general abilities or abilities in another domain. Taking it a step further, Louis, Subotnik, Breland and Lewis (2000) addressed the importance of a suitable identification process for students' welfare. Namely, they addressed the negative consequences when provisions do not match students' needs, such as boredom, frustration, lack of motivation, loss of interest and possible underachievement.

A third issue relates to the lack of ability to identify underachievers, namely the ability not only to identify students that demonstrate giftedness, but also to locate students with potential (VanTassel-Baska, 2005). According to Clark (2008), a gifted student is considered to be an underachiever when despite exceptional performance on some measure of knowledge and skills, that person's performance on school related tasks is significantly lower than the expected average for his age or grade. To shed light to the phenomenon of underachievement, VanTassel-Baska (2005) points to the need to acknowledge that giftedness is also about potential, not only the expressed accomplishments. More specifically, mathematical talent is not given, but should rather be conceptualized as an opportunity or a type of promise (Sheffield, 1999) which might either be actualized or lost under certain conditions (Hong & Milgram, 2008). Giftedness is a developmental construct and as such, it may be exhibited in various aspects and certain times, under certain circumstances and may be expressed in different ways (Reis & Renzulli, 2009). Especially in early ages, the identification should focus in the potential of extraordinary achievement and not only demonstrated extraordinary accomplishments in a specific domain (Hong & Milgram, 2008). In other words, the identification process should allow the recognition of mathematical strengths that may not be immediately apparent (Davis & Rimm, 2004) or as Zollman (2008) points out, "identifying mathematical talent involves identifying not so much that which already exists, as that which might yet come into being". Promise in math is not directly evident from students' achievements on standardized tests, nor is it related to the interest, effort or excitement shown during math teaching (Hoeflinger, 1998).

In other words, researchers point to the need to select both students who demonstrate mathematical giftedness in school and students who are capable of demonstrating it in the future through identification. Therefore, the notion of potential should be incorporated in the theoretical model and also taken under consideration in the identification process. Namely, we have to ensure that students who have the potential to excel but may have not expressed it yet due to several reasons are not excluded during the identification process. This is specifically essential in the proposed study that deals with the sensitive age of 10 to 12 years old.

The aforementioned issues taken together affirm the need for a new identification process that resolves all of the issues noted, namely the underrepresentation of students from certain groups, mismatch between objective, identification and services and underachievement. Taken in mind the aforementioned considerations, this study proposes an equitable two step identification process aiming to recognize mathematical giftedness in students of 10-12 years old.

Purpose and Research Questions of the Study

The purpose of this study is the development of a theoretical model of mathematical giftedness, elucidating and providing detailed examples of the abilities, cognitive and hypercognitive processes of mathematically gifted students and the development of an inclusive identification process for identifying mathematically gifted students in elementary school.

The theoretical model proposed in the study is expected to contribute to deeper understanding of mathematical giftedness, characteristics, reasoning, cognitive and hypercognitive processes of mathematically gifted students and clarify the relationship between mathematical giftedness and general giftedness. The design of the identification process was possible after studying existing frameworks and models of conceptualizing giftedness, talent and mathematical abilities, as well as issues related to the identification of giftedness. The identification process was carefully designed in order to capture mathematical giftedness, allowing all students to demonstrate their diverse characteristics and behaviours in a comprehensive and cohesive manner. The process was focused on children of age 10 to 12 years (5^{th}-6^{th} elementary school grades). Especially the findings of the observation phase of the identification process provide valuable information regarding the reasoning, cognitive and hypercognitive processes used by mathematically promising students. These conclusions may provide guidance to educators and gifted specialists on the aspects to search during the identification of mathematical giftedness, as well as on the organisation of provisions to nurture it in ways that will potentially benefit society.

The following research questions formulate the basis of the research:

(a) Is there a combination of measures that may identify mathematically gifted students in 5^{th}-6^{th} grades of elementary school?

(b) What are the mathematical abilities of mathematically gifted students?

(c) What is the nature of the cognitive processes that mathematically gifted students use as they engage in solving non-routine mathematical problems?

(d) What hypercognitive processes do mathematically gifted students exhibit when engaged in mathematical problem solving?

(e) How can we describe mathematical giftedness through a theoretical model?

Significance and Innovation of the Study

The lack of a consensus in regard to the definition of giftedness, the emphasis to examine general rather than domain specific giftedness, the lack of research studies elucidating mathematical giftedness, methodological issues pertaining to the identification of giftedness as well as the lack of good quality basic research that puts to the test identification processes in real school settings, reveal the importance and need for systematic and high quality research in this field. This study could be considered a response to the considerable criticism that the field has accepted with respect to the quality of giftedness research (Heller, 1993; Heller & Schofield, 2000; Ziegler and Raul, 2000). More specific, this study suggests a new perspective on the contribution and course of giftedness research, addressing some of the unanswered questions in the field of mathematical giftedness. The importance of this study lies on the fact that it aspires to investigate (a) the abilities of mathematically gifted students through a specially designed identification instrument created for this purpose following the latest research considerations, (b) the cognitive and hypercognitive processes of mathematically promising students observed through problem solving with specially selected mathematical tasks that conform to relevant research suggestions, (c) to provide a theoretical model elucidating mathematical giftedness grounded in research data and (d) to suggest and deliver a broad process for the identification of mathematically gifted students 10-12 years old, tested in real school settings.

The originality of the study lies, firstly, on the development of a domain-specific theoretical framework for the description of mathematical giftedness in early ages, grounded in data gathered during the identification process of mathematical giftedness, resulting to an empirically examined and verified model. The suggested theoretical model of mathematical giftedness provides researchers with a new theoretical basis on which to compare existing models and move forward in the field of mathematical giftedness. The proposed theoretical framework takes into account and combines ideas from research theories and models in the domains of giftedness and psychology with special focus in mathematics. More specific, this study consists of an answer to the call of Ziegler (2009) who proposed that the field of giftedness should cooperate with the field of psychology, whilst reforming definitions and concepts of giftedness, by using and examining more domain specific concepts and with less emphasis on traits. The model suggested in this study may contribute to studies on giftedness and mathematical giftedness in that mathematical giftedness may be better understood in terms of interrelationship between domain-specific abilities and processes.

The study provides detailed examples of students' abilities, cognitive and hypercognitive processes and behaviours. The examples provided constitute a rich database for researchers and educators to use as examples on which to base their conclusions in regard to future student responses to the same or similar mathematical tasks. They also illustrate in detail the manifestations of mathematically gifted students cognitive and hypercognitive abilities, as well as reasoning processes and behaviours. This study adds to the research literature by exemplifying how the different processes and behaviours could be manifested. Moreover, this study provides specific examples of the way mathematically gifted abilities think and respond in great detail, whilst past studies may have only provided theoretical assumptions. In addition, this study suggests novel behaviours or reveals relationships amongst them that were not observed before. At the same time it combines several indicators by suggesting different facets of a specific process.

Moreover, this study supplies researchers in the international community with a new identification process that was designed from the start to focus in mathematical giftedness, using recent theoretical frameworks and research designs. The identification instrument allows the recognition of students' strongest abilities and those that need to be developed in order to allow maximal realization of students' potential. In practice, the model is expected to facilitate the identification process of mathematically gifted students and the identification instrument proposed in the study may promote the nourishment of students' mathematical talent. Namely, teachers could identify gifted students through the identification instruments which encompass the components of the theoretical model proposed. The identification process is available for educators to use in school; thus, allowing teachers to identify students who need special treatment at high level. At the same time, the design of the identification process to be appropriate for students in the elementary school (10-12 years old) may help to prevent talent loss, which is considered to be one of the major challenges facing parents and educators nowadays (Hong & Milgram, 2008).

Limitations of the Study

This study has certain limitations that are discussed in this section. The constraints are discussed with reference first to the group administration of the identification test and secondly to the observation of mathematically promising students whilst working with challenging mathematical tasks.

Firstly, in order for students to participate in the study, it was mandatory for parents to give their written consent. Thus, the sampling process was not random, since there was a possibility that parents of students who do not perform well in traditional mathematical assessments at school, may not have given their consent so that their child may participate in the study. In addition, the fact that the test was administered to schools where teachers were willing to administer it, consists another limitation with respect to the sampling process. Moreover, in order to have teachers willing to administer it in their classrooms, the identification test should not take more than three teaching periods for students to complete (two 40-minute periods). Thus, only a limited number of mathematical activities could be included. Furthermore, the fact that the same test was not administered by the researcher but classroom instructors, might have caused for administration conditions to be slightly different from classroom to classroom. However, a detailed administration guide was prepared for all instructors to follow during test administration to minimize this effect.

With regard to the individual administration of the test with challenging mathematical tasks, although the students most probably would have never seen the problems previously, there is a possibility that the results may have been biased by the students' prior knowledge. Their prior mathematical background, knowledge, and problem solving experiences might have affected the manner in which they approached the problems as well as their solution methods. The number of problems presented to the students was limited to some content areas in mathematics due to time limitations so that students would not feel fatigue. In addition, the cognitive and hypercognitive processes observed may have been influenced by the nature of the tasks used. If entirely different tasks were included, slight variations of manifestations would probably be observed. However, this study was grounded on the data collected and interpretations were made according to the specific data.

It was the researcher's belief that the think aloud method would improve students' thinking processes and better assist their approaches compared to the conventional thinking approach. However, participants may have not reported all of their thoughts because they would have to spend additional effort to apply the think aloud method. Finally, we assume that students responded during observation honestly without bias or concern for self-esteem.

Thesis Structure

In this chapter, a description of the problem and an overview of this study, including the purpose and the research questions, were presented. In Chapter Two, a review of the literature related to the contextualisation of giftedness in general is provided, followed by a discussion focused in giftedness in mathematics. Giftedness in mathematics is discussed by commenting on the cognitive, hypercognitive, affective and motivational and environmental characteristics associated with mathematical giftedness. The second chapter also provides information in regard to the identification of giftedness both in general and in mathematics, the process and principles guiding the development of corresponding identification systems and tasks, as well as challenges pertaining identification processes. The methodology used in this study is explained in Chapter Three, introducing the research design and procedure followed, as well as the instrumentation used. Chapter Four presents the results after analysis of data collected during this study, commenting on the structure of mathematical giftedness with regard to mathematical abilities, cognitive and hypercognitive processes employed by gifted learners in mathematics. The discussion of results are presented in Chapter Five positioned in the context of the literature, the research questions and the categories that emerged as a result of data coding and analysis, whereas the proposed model of mathematical giftedness is presented. Consistent with methodologies of grounded theory, various explanations from the literature are applied to explain the research results, whereas innovative aspects that emerged through data analysis are also identified. . In Chapter Six, conclusions of the study, as well as educational implications and suggestions for future research are discussed.

Operational Definitions

Giftedness

Ability or potential for excellence, demonstrated through authentic performance, inferred through aptitude tests and observations of behaviour and performance. It is believed that abilities, motivation, and beliefs can be developed and transformed if students have suitable opportunities and experiences to maximize their potential. Also, cognitive, hypercognitive, affective and motivational domains play an important role, along with environmental and other social factors.

Mathematical giftedness

Demonstration or the potential to demonstrate a combination of high degree of mathematical reasoning, spatial and creative ability and employ great level of cognitive and hypercognitive processes, such as self-monitoring, self-regulation, tasks perseverance and persistence. This is supplemented with unique affective-motivational attributes and environmental-social factors that influence giftedness manifestation and development. It can be observed during problem solving with rich challenging tasks, demonstrating the quality of thinking.

Cognitive processes

Mental processes that reveal the way in which individuals think, reason, perceive and understand the word.

Hypercognitive processes

Processes with supervising and coordinating functions residing in the self-oriented level of the mind, such as self-awareness and self-regulation as well as relevant strategies to accomplish the two. These functions may occur before, concurrently or after cognitions, thus differentiating from metacognition. At a micro-developmental level, the system related to hypercognitive processes controls on line cognitive functioning making decisions about the suitable and efficient use of schemes and cognitive functions for a specific task on hand.

Self-regulation

The ability of individuals to adjust their behavior according to the demands of specific situations since they self-activate and self-direct efforts to acquire knowledge and skills by methodically employing particular hypercognitive strategies.

CHAPTER TWO: LITERATURE

Introduction

This chapter provides a review of the literature that relates to the aims of the study. The goal is, in the words of Creswell (1994), “to present results of similar studies, to relate the present study to the ongoing dialogue in the literature, and to provide a framework for comparing the results of the study with other studies” (p. 37). The review of the literature points to the relation between the research questions of this study to previous research in the field and at the same time pinpoints some of the gaps in the area.

The first section summarizes the problem that served as a motivation for this study. The next section presents and discusses the main concepts, theories and models from the field of general giftedness, proposed over the years. In the following section, consideration is given to the specific context of mathematical giftedness, by discussing this construct with reference in the abilities and characteristics of mathematically gifted persons. It concludes by discussing various identification instruments used to locate giftedness in general as well as mathematical giftedness and it discusses challenges in designing an identification system.

Problem

The modern era in the 21st century is characterized by a plethora of advanced technological tools that require new skills and knowledge, indispensable now but not in previous centuries (Leikin, 2009a). Hence, every citizen should be equipped with all necessary knowledge and skills to cope with the multiple challenges of a rapidly changing world. To do this, only persons characterized by deep and broad knowledge, rich imagination and strong proficiency can make contributions to society and develop further the dynamic changing world (Yerushalmy, 2009; Freiman, 2009). Therefore, to respond to the critical call for developing each country’s resources to the fullest extent, one of the most valuable resources is the ability and creativity of all children (Straker, 1983). Highly creative human capital within STEM (science, technology, engineering, and mathematics) domains is vital for national wealth (National Academies of Science, 2007; Partnership for 21st Century Skills, 2004). More specific, talent in mathematics was asserted as one of the most broadly required resource for the 21st century (Office of Science and Technology Policy, 2006), proclaiming the importance of nourishing mathematical giftedness.

As a result of statements, as the abovementioned one, research on giftedness is growing extensively with a widening international interest. Comprehensive publications such as the International Handbook on Giftedness (Shavinina(Ed.), 2009), the Handbook of Giftedness in Children: Psychoeducational Theory, Research and Best Practices (Pfeiffer (Ed.), 2008) and the International Handbook of Research and Development of Giftedness and Talent (Heller, Monks & Passow (Eds.),1993) reveal the growing attention of the research community to this field in the last decades.

Still, the danger of sacrificing quality over quantity and losing any sort of conceptual outline in the middle of this vast amount of publications on giftedness is evident (Ziegler & Raul, 2000). In support of this argument, VanTassel-Baska (2006) pinpoints that researchers in the field have conducted research in a number of directions "seeking novelty over depth of understanding". Instead, research studies should be related and compared to one another and build a basis for sound generalizations and conclusions in terms of theory and practice. In other words, researchers should focus on conducting meaningful research, to be exact research of the type that will contribute to the growth of the field of gifted education (VanTassel-Baska, 2006). To advance the field by comparing studies and reaching to rich conclusions, the need for reaching to an agreement on terms, theories, models, policies and practices with respect to giftedness is evident. In this way, the needs of gifted children will be accommodated.

Especially in the case of a particular subgroup of gifted persons, in our case mathematically gifted learners, the reformulation and application of specific models, processes and provisions targeted for this specific group is imperative. However, Leikin (2011) illustrated clearly through a review of research publications in mathematics education and in gifted education that "mathematics education is underrepresented in the field of gifted education and, vice versa, the research on giftedness and gifted education is underrepresented in the field of mathematics education" (p.168). Thus, we need a link to bridge the gap between the two disciplines.

At the international level, greater attention has been rewarded to gifted education in mathematics. This awareness is reflected in a number of international forums that lately have focused their work on mathematical creativity and giftedness. Such events were the two Topic Study Groups entitled "Activities and programs for gifted students" (ICME-10, 2004; ICME-11, 2008), organized in the framework of ICME conferences. Since 1999, seven international conferences have been organized entitled “International Conference on Creativity in Mathematics and the Education of Gifted Students”, by a forum formed of educational researchers, mathematicians and mathematics educators with interest in mathematical creativity and mathematics gifted education. To systematize these efforts, in 2010, in the framework of the forums’ conference in Riga, Latvia, the conference participants established the International Group for Mathematical Creativity and Giftedness. A year later, in 2011, the Group was approved as an Affiliate Organization of the International Commission on Mathematical Instruction (ICMI).

Especially during the past decade, as shown by the abovementioned research efforts, planning of conferences and the establishment of a research group, we can observe raising awareness of the importance of the education of mathematically gifted students, partly because of the consciousness of the danger of talent loss (Leikin, 2009a). To the same end, the publications of Genius Denied: How to Stop Wasting our Brightest Young minds (Davidson, Davidson & VanderKam, 2004) as well as Preventing Talent Loss (Hong & Milgram, 2008), brought a heightened awareness of the need for special services for the gifted. Nevertheless, before proceeding to providing special services, gifted students should first be identified among the student population. The identification of gifted students is one of the issues confronting us in the new millennium, since problems surrounding the identification of gifted children have long been debated in the field of giftedness (Van Tassel-Baska, 2005).

To sum it up, prior research in the field of giftedness focused on the examination of general giftedness rather than domain-specific giftedness (Leikin, 2009a). After searching through publications in seven key journals in the field of giftedness and intelligence, Leikin found that in the past decade only a small number of articles have been devoted directly to mathematical giftedness or creativity (Leikin, 2009a, 2011). As a result of the focus in general giftedness, there is limited focus on theoretical models of mathematical giftedness as well as specially designed procedures and instruments for students’ identification. Also, Leikin (2011) addressed the problem of a number of studies on mathematical giftedness that put emphasis primarily on general psychological traits of individuals, whilst they do not investigate the learning and thinking processes of gifted students in mathematics in accordance with contemporary theories of mathematics education.

Before deepening into the research on contextualization of giftedness, it is essential to situate research after commenting on the ontological tension between researchers, with respect to the nature in opposition to nurture issue. Any position taken clearly affects any research attempts in the domain, by shaping beliefs about the nature of giftedness, its manifestations and thus the adoption and employment of identification measures and subsequent provisions.

The Nature in Opposition to Nurture Issue

According to Galton (1869), "nature is all that a man brings with himself into the world; nurture is every influence from without that affects him after his birth" (p.12). The Nature versus Nurture debate stems from the original question of being versus doing / becoming (Delisle, 2003; Subotnik, 2003). In other words, the question is if giftedness is a personal quality that is biologically possessed or if it is an emergent property, achieved through learning, practice and social support, subject to development and change (Dai, 2010). This issue nowadays is manifested in multiple ways, creating a polarization between researchers.

On the one end, there are researchers that believe that giftedness can be explained through differences in brain structures, rapid rate of learning and neurophysiological differences that affect neuronal efficiency (Gagné, 2005; Geake, 2008). It all began after Galton's (1869) research study which reveals one of its main conclusions through its title: "Hereditary Genius". Hence, Galton clearly argued that "nature prevails enormously over nurture" (p.241).

On the other end, there are researchers that believe giftedness is a result of a purposeful effort and practice over an extended time period (Ericsson, 1996; Ericsson, Charness, Feltovich, & Hoffman, 2006) and others that argue that giftedness is situated and heavily dependent of the context (Plucker & Barab, 2005). Ericsson, Roring and Nandagopal (2007) believe that that all actual differences in achievement can be explained by greater practice, instead by innate differences. According to Sternberg (2001) the 'developing expertise view' ''in no way rules out the contribution of genetic factors as a source of individual differences in who will be able to develop a given amount of expertise.'' (p. 161). In other words, as said by Threlfall and Hargreaves (2008) "the expert performance/developing expertise views may be effective as accounts of the manifestation of giftedness in given contexts, but that does not answer the question of whether there are underlying persisting differences between gifted and non-gifted individuals" (p.84).

After a thorough review of the situation with regard to the nature or nurture discussion, at the end, Dai (2010) suggests that there is an "interactive nature of nature and nurture in the developmental process" (p. 48), transforming the discussion into a nature in nurture argument. In addition, he argues that there are "some children who are gifted by natural endowment (being), and others become "gifted" through interests and dedicated efforts (doing/becoming)" (p. 64). Because of the diversity and multitude of phenomena of giftedness, Dai (2010) suggests that the research field should focus more in the development of midrange theories of a specific line of giftedness and talent development, rather than "all purpose grand theories of giftedness" (p.74), since they may better explain the nature and nurture interaction, as well as they may inform and justify specific identification and educational provision practices. Following Dai's suggestions, this study attempts to develop a specific theory of giftedness describing mathematical giftedness to inform identification strategies and provision options.

In the light of the nature and nurture discussion, the next section explores a number of theories of giftedness as they have been articulated to date, in an effort to contextualize giftedness.

Contextualizing Giftedness through Theories and Models

During the last hundred years, a number of complex and cohesive theories and models have been developed in an attempt to capture giftedness. Theory based models are of vital importance, since they consist the foundation on which to base the identification, understanding and examination of giftedness (Davidson, 2009).

The study of intelligence can be considered to be the cradle of research on giftedness, with the first systematic and scientific approach dating in the late 19th century. Intelligence denotes individual differences in a set of cognitive abilities important for learning and problem solving, such as understanding multifaceted ideas, engaging in different reasoning methods and successfully facing real life challenges (Neisser et al., 1996, cited in Dai, 2010). Binet's (1905) and Terman's (1925) efforts to create valid Intelligence Quotient (IQ) tests for measuring intelligence, characterized the field in the first half of the 20th century.

Terman organized the first large-scale study of intelligent children by using massive intelligence testing as an identification method of intellectual giftedness (Dai, 2010). Terman's convictions included the belief that intelligence is a general human ability, in a large extent genetically determined and that it can be measured objectively with an intelligence test, created by Binet and Simon. As a result of this study, one of the first definitions of giftedness, given by Terman in 1925, suggested the use of IQ scores above 140 as an identification criterion, defining giftedness as the possession of high mental power measurable by intelligence tests. This exclusive reliance on one score of an intelligence test as a way of labeling gifted students, promoted an absolutist view of giftedness (Brown, Renzulli, Gubbins, Siegle, Zhang, & Chen, 2005). A year later, Hollingworth (1926) called upon the consideration of all manifestations of giftedness, since a child may express his/her gift in domains such as arts, mechanics or literacy.

The work of Terman represented a milestone in understanding giftedness, but the limited definition adopted, impeded the expansion of the conceptualization of giftedness for many years. In a short time period, the one-dimensional definition provided by Terman influenced and paved the way for the proposition of other psychometric definitions that used a quantitative approach in viewing giftedness in terms of cut off points on certain criteria. For years after Terman, the field was dominated by narrow conceptualizations of giftedness (Hong & Milgram, 2008), equating giftedness with intelligence.

A number of years later, a group of researchers argued that giftedness could not be measured with single IQ tests, realizing that "the gifted and talented come in a tremendous variety of shapes, forms and sizes" (Passow, 1981, p.8). The acknowledgement of the heterogeneity and diversity of abilities of the gifted, provided fertile ground to challenge the Procrustean bed notion of IQ tests was challenged (Dai, 2010). Nowadays, the conceptualization of IQ as giftedness consists of an outdated and flawed concept (Ceci, 1996). Acknowledging that intelligence is not measured only by IQ tests, led to the proposal of novel theories involving more multifaceted and comprehensive approaches to intelligence. Further research findings on brain function and the growth of cognitive and educational psychology have suggested that intelligence is only one aspect in studying giftedness, resulting to two influential much-quoted theories of intelligence in education and psychology, that of Gardner's (theory of multiple intelligences, 1983) and Sternberg's (triarchic theory of intelligence, 1985).

Along the years, there has been a substantial shift in the conceptions about giftedness since the equation of giftedness with high intelligence. Another contribution of Terman's long term study in the advancement of the field was the finding that individuals with high intelligence may not necessarily generate pioneering products. Thus, researchers shifted their research to examine other factors such as creativity and motivation and their role in the construct of giftedness. To this end, Renzulli (1978) proposed the three ring conception of giftedness, forming a new conception of giftedness and talent.

These theories as well as other models are presented in the following sections, commencing with the theory of multiple intelligences (Gardner, 1983, 1993, 1999), continuing with the triarchic theory of intelligence (Sternberg, 1985), the three ring conception of giftedness (Renzulli, 1978, 2002) , the Munich model of giftedness (Heller, 2004), the differentiated model of giftedness and talent with its latest refinements (Gagné, 2009) and at the end, the comprehensive model of giftedness and talent (Hong & Milgram, 2008).

The Theory of Multiple Intelligences

In 1983, Gardner proposed a new theory of multiple intelligences based on neuropsychological analysis of human abilities. The theory was focused on a belief of different and independent skills/abilities ('multiple areas of intelligence'), seven to be exact: (a) linguistic, (b) logical-mathematical, (c) spatial, (d) bodily-kinesthetic, (e) musical, (f) interpersonal, and (g) intrapersonal. Later on, two more were added in 1993 and 1999, 'naturalistic' and 'existentialist' intelligence respectively.

According to Gardner's theory, intelligence is not just a single, unitary construct. Rather, Gardner believes that these abilities are relatively independent of each other. In fact, Gardner considers each ability as a separate intelligence, not as a part of a single whole. Each one represents a relatively autonomous set of problem-solving abilities and each is associated with a specific type of giftedness.

Dimitriadis (2010) provided a description of areas where each intelligence can be experienced according to Gardner's theory:

- Linguistic intelligence, is involved among others in writing, reading, talking and listening;

- Logical-mathematical intelligence in making calculations, solving puzzles and developing proofs;
- Spatial intelligence in moving from one place to another or designating orientation in space;
- Bodily-kinesthetic intelligence in using the body to perform skilled movements (e.g., useful for athletes, dancers, surgeons);
- Musical intelligence in singing, playing music and composing;
- Interpersonal intelligence in understanding other individuals and their relationships (useful for psychologists, teachers and politicians); and
- Intrapersonal intelligence is involved in self-understanding (recognizing one's own thoughts, emotions and actions).
- Naturalist intelligence is involved in understanding the natural world and working successfully within it.
- Existential intelligence, which has yet to be confirmed, will concern a person's ability to raise questions about his/her place in the world. (p. 31)

The Triarchic Theory of Intelligence

According to Sternberg, "intelligence is defined in terms of the ability to achieve success in life in terms of one's personal standards, within one's sociocultural context." (1999, p. 296). In this theory, the conceptualization of intelligence is always situated within a sociocultural context. Hence, a behavior that might be considered gifted in a specific cultural context, or a sign of giftedness, might not considered as such in another culture. The same applies to what constitutes success in life in each culture.

It is Sternberg's belief that giftedness cannot possibly be captured by a single number. Thus, he provided a holistic view on intelligence by describing its analytic, creative and practical components (Sternberg, 1985). Analytic giftedness involves the ability to analyze a problem and understand its parts. An important difference between Sternberg's theory in comparison to other intelligence theories is the way in which the creative and practical dimensions are incorporated into the model. Creative or synthetic giftedness is observed in insightful, intuitive, creative persons or just individuals skilled at coping with relatively novel conditions. Practical giftedness entails the applications of analytical or synthetic ability to everyday realistic situations.

Another important aspect of Sternberg's theory, is that in order to succeed, a person has to capitalize on strengths, correct of and compensate for their weaknesses, adapt to novelty and automatize new skills rapidly. Also, for Sternberg, just as there is no single kind of intelligence; there is no single kind of giftedness. It can manifest in different ways in different situations.

The Three Ring Conception of Giftedness

In an effort to summarize changes in contextualizing giftedness, Renzulli introduced the three ring conception of giftedness (Renzulli, 1978), challenging the traditional view that giftedness is primarily a function of high scores on intelligence tests (Renzulli, 2002). This way, Renzulli made a step towards more liberal or inclusive definitions of giftedness.

According to this model (see Figure 1), above average and not necessarily exceptional ability is a necessary but not sufficient condition for the emergence of giftedness. Necessary qualities are also task commitment and creativity. Therefore, a gifted individual is the person characterized by all three traits. In 1986, Renzulli redefined his model by using the term "gifted behavior" instead of giftedness and added "and/or specific abilities" (p. 73). Recently, Renzulli shifted his emphasis toward other contextual factors, revising the model by including personality and environmental factors (e.g. family, peer-group and school influences). The interactions between the background factors support the development of the three defining qualities of gifted behaviors (Renzulli, 2002). It is important that Renzulli treats giftedness as a dynamic state, thus suggesting that the three qualities suggested "need to come together to create a mesh" (Dai, 2010, p.19)

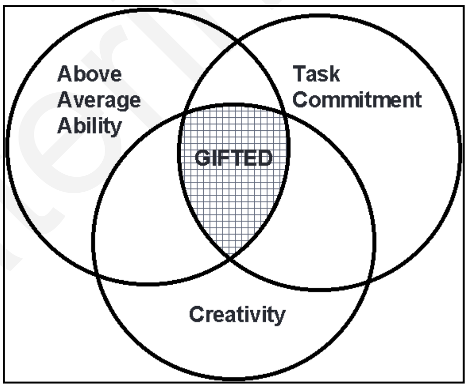


Figure 1. Structure of the three ring conception of giftedness.

Following principles proposed by the positive psychology movement, Renzulli in a project called “Operation Houndstooth”, examined the personal attributes by developing an organizational plan for studying six components this framework. These components are summarized in Figure 2. Their consideration as "co-cognitive factors" by Renzulli (2002) himself, shows their interrelation with the cognitive characteristics, typically related with the development of human abilities.

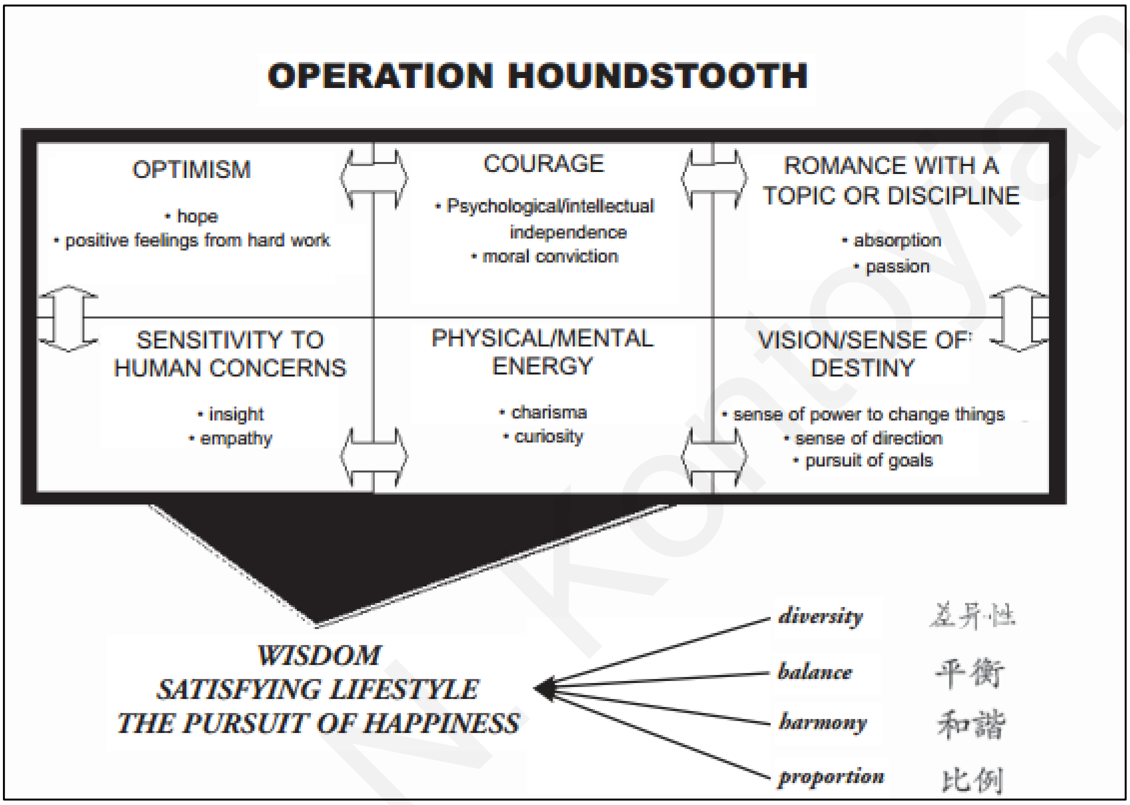


Figure 2. Structure of the “Operation Houndstooth” organizational plan for studying six personal components of the three ring conception of giftedness model. Adapted from “Operation Houndstooth intervention theory: Social capital in today's schools”, by J.S. Renzulli, J. Koehler and E. Fogarty, 2006, *Gifted Child Today, 29*(1), p.17. Copyright 2006 by Sag Publications.

Before Renzulli's decision to add environmental factors and personality traits to his original model, other researchers built up on his first model by incorporating these components too. Such models are the Munich model of giftedness and Gagné's differentiated model of giftedness and talent.

The Munich Model of Giftedness

The Munich model of giftedness, based on multiple factors, is presented in Figure 3 (Heller, 2004). The model is based on four dimensions: talent factors, resulting performance areas, non-cognitive personality factors and environmental factors. In this model, giftedness is considered to be an ability influenced by non-cognitive (e.g. motivation, interests, self-concept, control expectations) and environmental moderators. The moderators are related to the talent/giftedness factors (predictors) and the outstanding performance areas (criteria variables). The moderators control the transition from gifts to revealed performance. In this model, moderators correspond to the term of catalysts as proposed by Gagné (2009) in his model. Heller (2004) acknowledges the fact that although the seven forms of gifts/talents listed in the left of Figure 3 are the most cited in the literature, they do not represent all kinds of giftedness or talent.

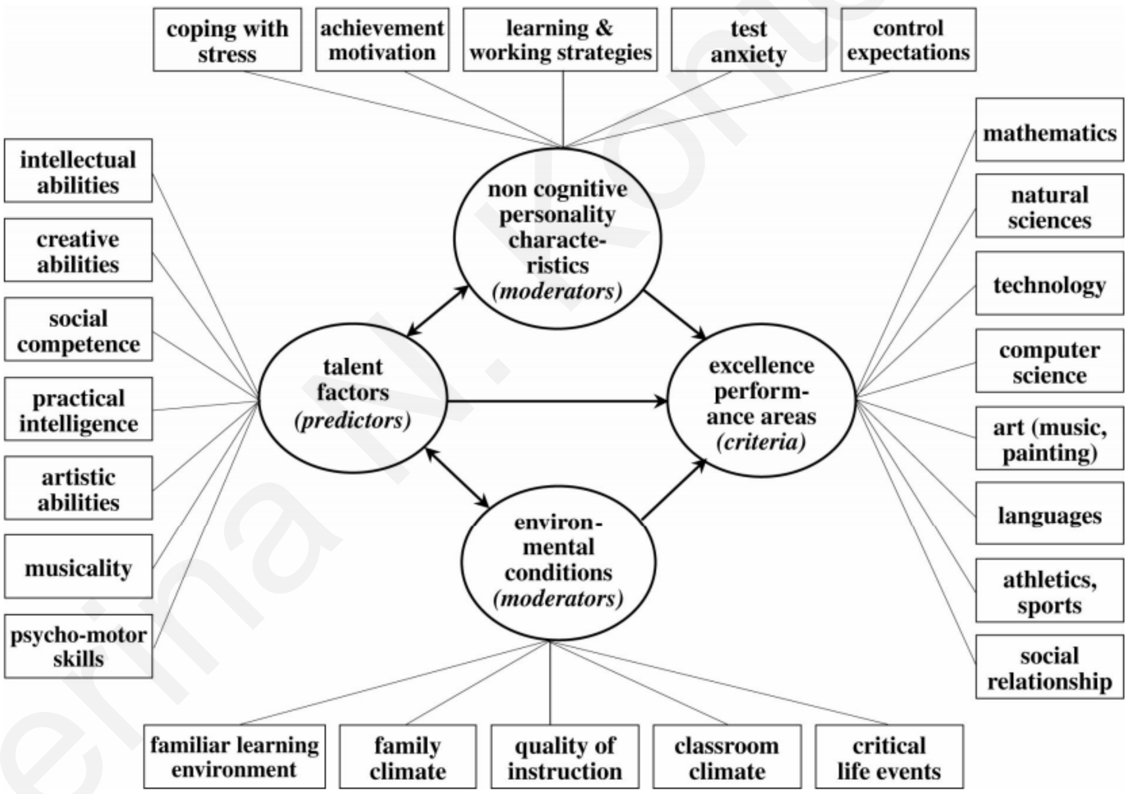


Figure 3. The structure of the Munich model of giftedness. Adapted from "The Munich High Ability Test Battery (MHBT): A multidimensional, multimethod approach" by K.A. Heller and C. Perleth, 2008, *Psychology Science Quarterly, 50*(2), p. 176. Copyright 2008 by Pabst Science Publishers.

On the basis of this model, the Munich High Ability test battery was developed (Heller & Perleth, 2008). Thus, the tests and questionnaires of the Munich High Ability test battery measure various aspects and types of giftedness (predictors) but also several non-cognitive personality and social-environmental learning conditions (moderators). The Munich High Ability test battery contains two dozen tests and standardized questionnaires for the differential assessment of the predictor and moderator variables illustrated in Figure 3.

The Differentiated Model of Giftedness and Talent

While other definitions may refer to giftedness and talent as the same construct, Gagné (2009) points to the fact that researchers do distinguish, implicitly or explicitly, between early emerging and fully developed adult forms of giftedness. Several pairs of terms are used to express this distinction, such as potential/realization, aptitude/achievement, and promise/fulfillment.

Taking advantage of this distinction, Gagné proposed the differentiated model of giftedness and talent, which has evolved and revised several times over the years. Figure 4 presents a revised version of the model, originally proposed in 2008.

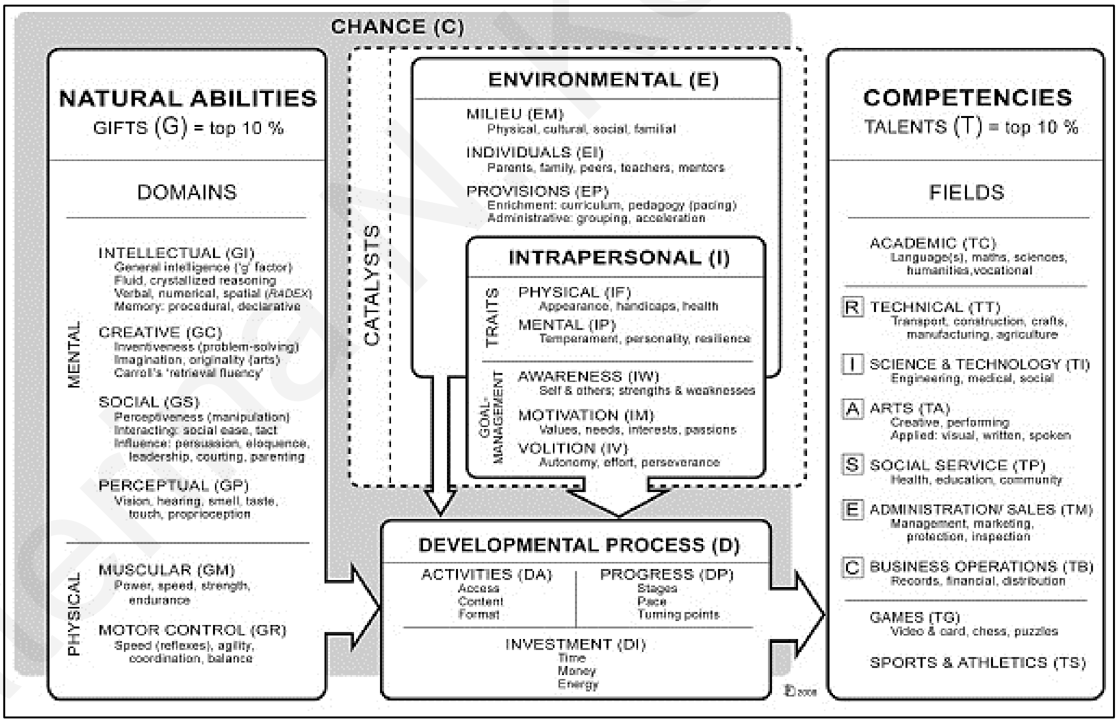


Figure 4. Gagné's differentiated model of giftedness and talent. Adapted from "Academic Talent Development and the Equity Issue in Gifted Education", by F. Gagné, 2011, *Talent Development & Excellence, 3*(1), p. 11. Copyright 2011 by International Research Association for Talent Development and Excellence.

According to this model, giftedness refers to the possession and use of outstanding natural abilities, called aptitudes, in at least one ability domain. Talent denotes the exceptional mastery of systematically developed abilities, competencies in Gagné's terms, in at least one field of human activity. Both giftedness and talent abilities should be developed in a degree that the individual should be placed in the top 10% of his/her age peers in that ability domain or field respectively.

According to Gagné's theory, natural abilities are not innate, but they develop over the years, possibly in a faster pace and larger degree during the early years. Still, natural abilities have indisputable genetic foundations. Regarding talents, the majority of them can be easily evaluated with performance instruments, such as exams and standardized achievement tests. This model does not discriminate against certain occupations, since it emphasizes the presence of talented persons in most professions.

The third basic component of this model is the developmental process, in which gifts are progressively transformed into talents, over a considerable time period and after the systematic pursuit by individuals towards a particular excellence goal. Taking into account this relationship between gifts and talents, we may conclude that a person can be gifted without necessarily being talented (as with the case of underachievers), but not vice versa.

There are two additional components, the intrapersonal catalysts (I), and environmental catalysts (E). As catalysts, they may facilitate or hinder the developmental process by their presence or absence. At the same time, chance has a powerful role in this model, as it places the foundations of a person's talent development possibilities. It designates the degree of control that potentially talented individuals have over natural abilities, environmental influences and intrapersonal characteristics.

Gagné's model with refinements (2004) has received some critique from Feldhusen (2005). In particular, Feldhusen argued that there are terms such as high or natural ability, aptitude or precocity that should be considered instead of insisting on defining giftedness, since, according to Feldhusen, this term is of little use outside of the gifted education area. Also, Feldhusen (2005) pinpoints to the use of old references by Gagné in supporting his arguments about the intrapersonal catalysts.

The Comprehensive Model of Giftedness and Talent

The structure of giftedness model is included among the broader conceptualizations of giftedness, originally proposed by Milgram (1989). This model was updated and revised to result in the comprehensive model of giftedness and talent of Hong and Milgram (2009), shown in Figure 5.

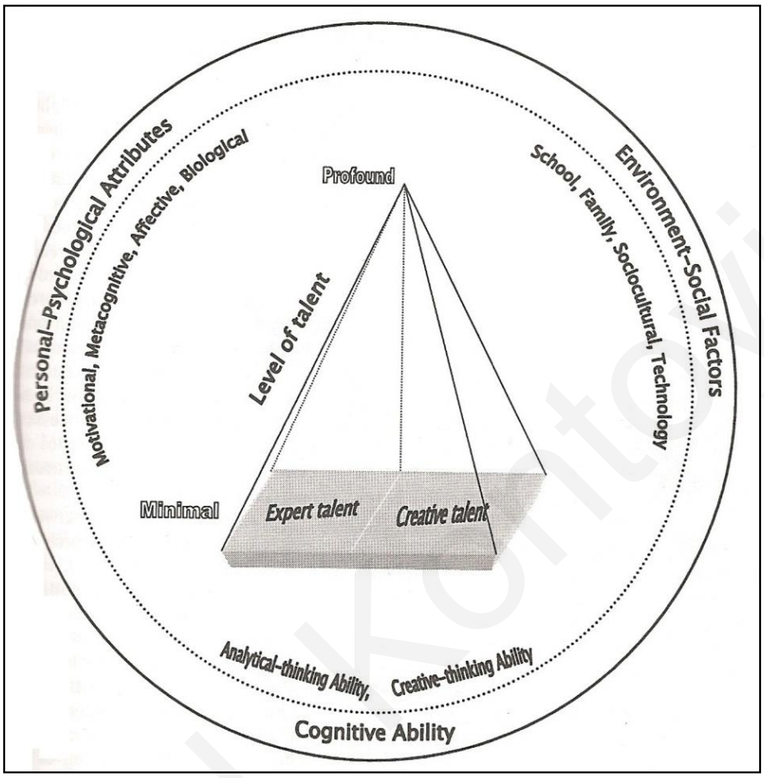


Figure 5. The comprehensive model of giftedness and talent applied in mathematics. Adapted from "Talent loss in mathematics: Causes and solutions", by R. Milgram, and E. Hong, 2009, in R. Leikin, A. Berman & B. Koichu (Eds.), Creativity in mathematics and the education of gifted students, p. 153. Copyright 2009 by Sense Publishers.

The model supports that four cognitive abilities (domain general and specific analytical thinking abilities, domain general and specific creative thinking abilities), personal-psychological and environmental-social variables operate together in each person to determine the type and level of talent.

The aforementioned model emphasizes and distinguishes among expert and creative talent. According to Milgram and Hong (2009), expert talent involves "more analytical or intelligent thinking ability than creative thinking ability. It is logical, systematic ability that utilizes pattern matching based on the massive knowledge base from years of learning and work experiences" (p. 152). In other words, these individuals are experts in their field, and they achieved expertise through education and practice. Creative talents reflects “more creative thinking, that is, the ability to produce ideas that are imaginative, clever, elegant, or surprising, beyond analytical thinking, required as part of the creative process” (p. 152). Creative talent is also developed throughout the years. The authors note that every individual is characterized by both types of talent; the difference is in the level of each talent. Particularly, the model uses a triangle to demonstrate the level of each type of talent found in each individual, indicating that the development of both talents is continuous “ranging from minimal to profound” (Milgram & Hong, 2009, p. 153). The triangular form shows that the number of people with profound expert and creative talents is relatively small.

The major contribution of this model is that it does not distinguish between intelligence and creativity like other theories, rather the authors suggest “a difference of emphasis” (Milgram & Hong, 2009, p. 154). That is, although “the two talents are empirically distinguishable yet highly related” (Milgram & Hong, 2009, p. 154). At the same time, there are domain general and specific abilities, both for analytical and creative thinking, all of which are necessary components of giftedness. Therefore, each person has a unique profile consisting of four types and levels of cognitive abilities, interacting with personal-psychological and environmental-social variables to manifest giftedness and talent.

This section provided an overview of six popular theories in the field of giftedness; the theory of multiple intelligences (Gardner, 1983, 1993, 1999), the triarchic theory of intelligence (Sternberg, 1985), the three ring conception of giftedness (Renzulli, 1978,2002) , the Munich model of giftedness (Heller, 2004), the differentiated model of giftedness and talent with its latest refinements (Gagné, 2009) and at the end, the comprehensive model of giftedness and talent (Hong & Milgram, 2008). A discussion on common grounds between the theories and models presented as well as variations that set them apart follows.

Overlap Between Theories and Models

High ability resulted being a “toothbrush concept” (Ziegler & Raul, 2000). Namely “it seems that everybody has a toothbrush, but nobody wants to use a toothbrush which belongs to somebody else”. Since the conceptualization of giftedness varies from study to study, the results from one study are difficult to be carried over to another. In summary, their review describes a fragmented field whose results are not easily comparable.

Van Tassel-Baska (2005) pointed to the importance of having a coherent and cohesive conception of giftedness. Thus, any proposed model of mathematical giftedness, must be held in high standards (Davidson, 2009), since gifted persons receive services and have opportunities to contribute to society that are not made available to others (Sternberg, 2004). To ensure the quality of a model, the following criteria were proposed by various researchers (Davidson, 1990; Davidson, 2009; Hempel, 1966; Kaplan, 1964):

- Models should exploit previous research knowledge and be empirically supported.
- The components and mechanisms suggested in each model should be precise, internally consistent, and testable as to easily compare it to existing models.
- Models should be economical instead of complicated and understandable, avoiding redundant or vague components.
- Models need to explicate and predict gifted behavior over time periods and across different circumstances.
- Models should be useful. Particularly, they should endow with practical directions on the development and selection of instruments and provisions to identify and promote giftedness.
- Models should allow the production of new empirical research to advance the field of giftedness.

Following the aforementioned criteria when broadening the field by suggesting new models of giftedness is not enough. The way in which a researcher treats the existing research literature when he/she is in front of new research findings plays an important role in the advancement of the field, according to Dai (2010):

> A broadened knowledge base does not mean that the traditional conceptions and theories are all obsolete and should be replaced with new ones. Rather, it means that the old needs to be interpreted and repositioned in the totality of our new understandings. Thus, what I suggest for the field is not a complete overhaul or radical paradigm shift, but a new synthesis that conserves what is useful in the tradition but recasts it in the light of new understandings (p. 35).

Taking into consideration the abovementioned criteria as well as the words of Dai (2010), we may assert that each of the aforementioned models has its strengths and weaknesses, whereas each has been subjected to criticism. Some of the theories are more conservative, relying on a single criterion, whereas other theories are more liberal, employing multiple dimensions, and even emphasize the interactions between components. Some of the models share underlying components whilst others are opposed to each other showing substantial contrasts. Others focus on the hereditary innate abilities, while others point to the important influence of experiences and the environment. Other models rely on demonstrated performance and others emphasize the notion of potential or promise. Each of these theories had a different starting point and different focus e.g., on areas where intelligences are displayed (Gardner, 1983), on psychological processes (Sternberg, 1985), on the aspects that influence the transformation of gifts into talents (Gagné, 1985) or on the definition of gifted behaviors (Renzulli, 1978). Latest models also incorporate the concepts of chance and change. However, the latest theories and models have profoundly changed the way of conceptualizing giftedness but more importantly, they brought to the front a more pluralistic value (Dai, 2010).

In order to make the comparison, the presented theories will be compared along four dimensions; the emphasis placed on innate abilities (genetic factors) and/or experiences and the environment (environmental factors), domain specificity of giftedness, the inclusion of the notion of potential and the role they place on creativity.

The consideration of giftedness as the product of interaction between genetic and environmental factors, leads to the acceptance of the existence of different types of giftedness (Heller & Perleth, 2010). Recently, Renzulli revised his original model of giftedness, by taking into account other contextual factors, such as personality traits and environmental factors (e.g. family, peer-group and school influences). The interactions between the background factors support the development of the three defining qualities of gifted behaviors, according to Renzulli’s theory. This conceptualization is evident also in the Munich model of giftedness and Gagné's differentiated model of giftedness and talent. In the Munich model of giftedness, the non-cognitive factors and social-environmental learning conditions included, have the form of moderators, thus they control the transition from gifts to revealed performance. Similar function one may observe in Gagné's differentiated model of giftedness and talent. More specific, in Gagné’s model, intrapersonal and environmental contributions were noted in the case of the development of gifts into specific talents. Finally, the comprehensive model of giftedness and talent also takes into account personal-psychological and environmental-social variables that operate together in every person to determine the type and level of talent.

With regard to the conceptions of domain specificity of giftedness, the theories share diverse perspectives. Gardner believes that there are distinct types of intelligence, relatively autonomous and independent of each other. Sternberg’s view is compatible with Gardner’s theory of multiple intelligences but differs in being process rather than content-oriented. Giftedness, according to Sternberg, may be manifested in various ways in different circumstances. Sternberg also affirms that the three intelligences are more effectively working when they are all used together, providing a holistic view of intelligence. With regard to Gagné, his model acknowledged the existence of specific domains of natural abilities (gifts), thus allowing their identification. Gagné also distinguished between gifts and talents, in other words between early emerging and fully developed adult forms of giftedness, with gifts progressively transformed into talents, over a considerable time period and after the systematic pursuit by individuals. Renzulli (1986) shifted the emphasis from being gifted ways of developing and observing gifted behaviors in the classroom. He referred to general abilities, and used the term "gifted and talented", interpreting as gifted the generally gifted persons and talented the individuals gifted in a particular area, thus having a specific talent. On a recent perspective, Renzulli (2002) is in accordance with Gardner on the existence of distinct intelligences, each of them related to a specific type of giftedness. In the case of mathematics, Renzulli builds on the view of Krutetskii (1976) who suggested that children with high abilities in mathematics, are characterized by a unique ability, what he refers to as 'mathematical cast of mind'. Milgram and Hong (2009), support that there are domain general and specific abilities, both for analytical and creative thinking, all of which are necessary components of giftedness. Therefore, each person has a unique profile consisting of four types and levels of cognitive abilities.

The notion of potential is not mentioned in every model discussed and if it is, is not given equal emphasis in all models. Renzulli's model was questioned due to the disregard of gifted underachievers, those not highly motivated but with potential to achieve. In the framework of the Munich model of giftedness, the moderators control the transition from gifts to revealed performance. Thus, we may conclude that potential may be in students with gifts, but it depends on the moderators to have demonstrated performance of these gift in a specific area. The notion of potential is stated in a similar way in Gagné's differentiated model of giftedness and talent. Taking into account this relationship between gifts and talents, we can conclude that a person can be gifted without necessarily being talented (as with the case of underachievers), but not vice versa. In order for potential to be realized, for Gagné, a person's abilities should be developed in a degree that the individual should be placed in the top 10% of his/her age peers in that ability domain or field respectively. In the case of the comprehensive model of giftedness and talent, although there is no explicit reference to potential, each person has a unique profile consisting of four types and levels of cognitive abilities, with levels varying from non-gifted to profound.

There are models who acknowledged the important role of creativity in giftedness. However, the models of Renzulli, Sternberg, Gagné, and Milgram and Hong express diverse opinions in regard to the role of creativity and its relationship with giftedness. Namely, Sternberg (1985) considers creativity to be a specific type of giftedness, Renzulli (1978,1986) places creativity in a central position by regarding it as a vital component of giftedness and Gagné considers creativity to be one general ability domain, or gift. Milgram and Hong, assert that they are two independent characteristics of persons (Milgram & Hong, 2009).

In this section, broader conceptions of intelligence and giftedness developed by prominent researchers in the field were presented. After comparing the theories and models discussed, we may reach to a fruitful conclusion. Even though each theory had a different starting point and a different focus, they all lead to a certain conclusion, thus contributing in the same way to gifted education. In particular, all of the aforementioned theories propose that giftedness is a multidimensional construct and as such, identification methods of giftedness now call for a broader approach that will not solely be based on the results from some cognitive ability or intelligence tests, but on more information from various sources and measures.

The studies reported in this chapter, use varying conceptions and definitions. Rather than opposing one model against the other, we should acknowledge their contribution to the advancement of research in the field. Nonetheless, the definition adopted determines and guides the identification of mathematically gifted students; hence, there must be a direct link between definition and identification of giftedness (Callahan, Hunsaker, Adams, Moore, & Bland, 1995). Therefore, it is important to base this study on one concept, since concepts influence identification practices and programming options for the gifted. As shown in the literature, definitions of giftedness also differ in using the terms able, promising, gifted or talented. In this study, it is important to place this research into context by examining the existing literature about concepts and theory driven models of giftedness. As this study focuses in the domain of mathematics, the operational definition of giftedness was drawn on some of the concepts presented in this chapter and is explained in the section of operational definitions. This definition is supported in the next section by a description of traits, abilities and behaviors found to be associated with giftedness in mathematics. Understanding these core abilities and attributes can help prevent talent loss, since our conception of giftedness influences our ways of interpreting it in school for identification and programming (Van Tassel-Baska, 2005).

Giftedness in Mathematics

One of the essential tensions revolving around the concept of giftedness involves whether gifted behaviors should be seen as a fundamentally domain-specific or domain-general phenomenon (Dai, 2010). With studies on gifted students identifying a number of characteristics associated with giftedness (Freeman, 1994; Cross, 1997), these characteristics cannot be generalized to all members of the gifted population since they do not consist a homogeneous group (Clark, 2008). Moreover, the field-dependent character of giftedness was pointed out by Csikszentmihalyi (2000). Combining these two remarks, students may be gifted in a specific area, e.g. mathematics. As such, we may assume that this subgroup may possess different characteristics than other subgroups of the gifted population.

The discrimination of common and domain-specific characteristics of students has implications for identification and nourishment (Song & Porath, 2005); hence they should be considered during identification processes and decisions about special provisions. Namely, students can be early identified and thus an effective educational environment could be established, in order to develop students' domain-specific abilities. Identification methods not focused on the characteristics of the target population, may result to misidentified students. For example, experience gained from the Talent Search Programs has shown that many mathematically gifted students are not actually selected for gifted programs, possibly because mathematically gifted students differ from "all-around" gifted students (Rotigel & Lupkowski-Shoplik, 1999). Benbow and Minor's study (1990) of comparing the test performance of mathematically and verbally gifted students reinforced the arguments for the domain-specificity of giftedness, despite its relatively small sample. Accordingly, this study asserted that mathematical giftedness relates to a "different mix of cognitive abilities, personality traits, and environmental circumstances than verbal talent" (Benbow & Minor, 1990, p. 24).

The next important question for giftedness research to answer is whether differences in abilities between gifted and not gifted individuals are quantitative or qualitative (Sekowski, Siekanska & Klinkosz, 2009; Winner, 2000).

The Nature of Differences

The key question to answer is if gifted individuals are fundamentally different from the rest of the population in regard to the way they perceive, think and learn (Dai, 2010). According to Dai (2010), this is one of the key ontological tensions in the field of giftedness. In other words, the difference of gifted persons lays in a different kind or a different degree of structural and functional organization of the mind compared to non-gifted persons? Shore and Kanevsky (1993) perceive this question as what they call it the ‘‘developmental controversy’’ – whether the superior performance of high-ability students is ‘‘merely precocity’’ or ‘‘does it reflect fundamental differences in thinking processes?’’ (p. 134).

To examine this question, Threlfall and Hargreaves (2008) compared the responses of 9-year-old gifted students with those of average-ability 13-year-old students on the same mathematical problem solving tasks. The research findings showed that 9-year-old gifted and 13-year-old average attaining students in this study performed at a similar level of successful responses, while the answers given and the methods and approaches used had similar profiles across the two groups. Moreover, the fact that no helpful cues were not provided, was not a factor affecting the performance of older non gifted students. These findings suggest that in some degree, a gifted 9 year old approaches mathematical problem solving in a similar way to an average ability 13-year-old student. This study came to the conclusion that the gifted students in the sample were merely precocious, that is, were advanced for their years, rather than having qualitatively different thinking process or possessing some skill that is never present in non-gifted students.

There are other studies that do not point in the direction of giftedness as precociousness, rather as qualitatively different thinking process or organization of the mind. For instance, O’ Boyle (2000) reports that there are morphological and functional characteristics of the mathematically gifted brain in adolescents, concluding that the mathematically gifted adolescents’ brain differs both qualitatively and quantitatively from those of average-math-ability youths. Among others, this study reveals that gifted children in mathematics engage unique parts of the brain not typically employed by children of average math ability. These specific regions have been found to mediate spatial attention and working memory (Mesulam, 2000), and may also be important for deductive reasoning (Knauff, Mulack, Kassubek, Salih, & Greenlee, 2002).

For Dai (2010), a literature review on this thesis and antithesis dilemma, or better put the qualitative-quantitative tension can be answered in a middle ground. More particular, qualitative differences can be observed not only between gifted and non-gifted persons, but also between gifted individuals. The important thing is the bootstrapping process, as Dai (2010) calls it, in which endogenous and exogenous factors interact (cognitive, affective and motivational and environmental), thus causing qualitative differences to emerge, in order for giftedness to develop.

In this framework, a discussion on the cognitive, hypercognitive, affective and motivational and environmental characteristics associated with mathematical giftedness follows in the three next sections.

Cognitive characteristics

In recent decades, the focus of research efforts in giftedness has shifted from a focus on the definition of the gifted to the way the gifted think (Mönks & Mason,1993; Shore, 1986), especially in the early years of development (Robinson, 2000). Even though this shift resulted to more studies investigating differences between the cognitive processes of gifted and non-gifted children, yet cognitive development of the gifted is not fully understood (Steiner, 2006).

In the field of mathematics, several scholars attempted to isolate the cognitive characteristics of gifted learners in mathematics (Greenes, 1981; House, 1987; Krutetskii, 1976, Miller, 1990; Osborne, 1981; Sowell, Zeigler, Berwall, & Cartwright, 1990; Waxman, Robinson, & Mukhopadhyay, 1996). Our discussion will revolve around three major abilities; problem solving, spatial and creative abilities.

Problem solving abilities. Gifted learners differ among other things in problem solving abilities from their average-ability peers, as the review of the empirical research on the identification and portrayal of mathematically gifted students from the 1970s and 1980s revealed (Sowell, Zeigler, Berwall, & Cartwright, 1990). This is in line with Zimmermann who indicated that mathematical giftedness entails "special ways of looking at and attempting to solve mathematical problems" (cited in Wieczerkowski, Cropley, & Prado, 2000, p. 415).

The impact of the Russian psychologist Vadim Krutetskii's seminal work (1976) exploring the nature and structure of mathematical abilities through a twelve year broad research with students is irrefutable, ever since the majority of the research studies on mathematical giftedness students have drawn on this research (Bicknell, 2009). For this reason, the next section will present the findings from the specific study that guided the research field on mathematical giftedness for years.

Mathematical abilities suggested by Krutetskii. Instead of a-priori identifying mathematically gifted students and then studying the differences between these children and non-mathematically gifted children, Krutetskii chose to observe the development of mathematical abilities by comparing students of different ages and abilities. His methodology included the design of a range of rich mathematical tasks (arithmetic, geometric, algebraic and logical) of grade difficulty designed to elicit their abilities and observing students while solving them, following a clinical investigation method. According to Krutetskii (1976), mathematical giftedness is the "unique aggregate of mathematical abilities that open up the possibility of successful performance in mathematical activity" (p. 76). Therefore, his longitudinal research, gave Krutetskii the opportunity to cluster mathematical thinking abilities into three broad cognitive processes, including obtaining mathematical information for the initial orientation to problems, processing mathematical information during problem solving and retaining mathematical information.

His research resulted to a wide list of particular characteristics of mathematically gifted children. With regard to obtaining mathematical information in the beginning of problem solving, his research showed that capable pupils perceive the mathematical material of a problem both analytically and synthetically, grasping the formal structure of the problem as a whole without losing sight of the elements. The term 'analytically' refers to the ability to isolate the elements in the structure of the problem, assess them differently, systematize them and determine their 'hierarchy', while the term 'synthetically', refers to the ability to combine the elements into complexes, looking for mathematical relationships (Krutetskii, 1976).

During problem solving, Krutetskii (1976) observed other differences between the cognitive processes of capable and average students. In particular, he noticed that capable students generalize mathematical content rapidly and broadly 'on the spot'; with a minimal number of exercises. On the contrary, average students, establish generalized links progressively. In addition, capable students show signs of flexibility of mental processes, meaning that they are able to switch rapidly from one operation to another or from one train of thought to another. They are also able to curtail the process of mathematical reasoning by eliminating intermediate steps according to each situation and reverse their mental processes. Finally, they pursue clear, simple and economical methods and solutions in problem solving. All these characteristics, are exhibited in gifted students in ways that are different from average ability students, according to Krutetskii's research.

These qualitative differences in cognitive processes are expressed by the general synthetic component used by Krutetskii, that of “mathematical cast of mind” (p. 302). This term describes the tendency of gifted children to view facts and phenomena through a mathematical prism, have a 'mathematical' perception of the environment and interpretation of the world. In other words, they have a tendency to see mathematics in the ordinary and commonplace (Osborne, 1981) referring to a unique organization of mind that turns the phenomena of the environment into mathematical ones.

The following sections discuss the major cognitive abilities related to mathematics that resulted from empirical research in the field.

Ability to articulate generalizations. Sriraman (2003) defines generalizing in mathematical problem solving as “the process by which one derives or induces from particular cases” (p.152). The ability to form generalizations after engaging higher order cognitive processes has been proved by different researchers to be a distinctive trait of mathematically gifted students.

In his model of multiple intelligences, Gardner (2000) classifies a number of distinct kinds of intelligence. Among them, ‘logical-mathematical intelligence’ is referred. The particular type of intelligence includes both analysis (systematic and logical reasoning) and synthesis (recognizing patterns and articulating generalizations). Thus, synthetic thinking is important in the field of mathematics (Haylock & Thangata, 2007), especially when the solver unites the components of the problem together into a whole. This is evident for instance, when the solver formulates a generalization from a number of specific cases.

Extending the importance of analytic and synthetic thinking in the field of mathematics, Haylock and Thangata (2007) concluded that gifted learners in mathematics excel in higher order cognitive abilities, one of which is synthesis. In his research, Krutetskii (1976) also acknowledged synthetic ability to arrive to generalizations as a trait of mathematically able students. Specifically, he argued that mathematically capable learners perceive the mathematical material of a problem both analytically and synthetically, grasping the formal structure of the problem as a whole without losing sight of the elements. He spoke of their ability to visualize the class of a single problem, recognizing the inferred generality. This generality might be perceived as seemingly unrelated features to other students. Krutetskii (1976) also observed that capable students generalize mathematical content rapidly and broadly with a minimal number of exercises.

Further to Krutetskii's work, House (1987) also asserted that the ability to perceive and generalize about mathematical patterns, structures, relations and operations as well as the ability to reason analytically, deductively, and inductively are included among the indications of mathematical giftedness. Miller (1990) also indicated the ability of abstract thinking using flexible and creative methods and the ability of seeing mathematical patterns and relationships were mentioned as possible indicators of mathematical giftedness. Not only gifted students are able to form generalizations when required, but they are also able to move beyond the answer to a particular problem, arriving to extensions and generalizations (Wolfle, 1986).

Sternberg (1985, 1986), among others, wrote about the "decontextualist" mind of gifted learners. He argued that gifted learners tend to be decontextualists. In this sense, they obtain and store information in long-term memory as a whole, whereas average pupils tend to acquire and store information in small, unrelated pieces. Teachers are the ones to help average learners make connections to eventually see the whole of a concept, whereas this is not required in case of gifted learners.

Flexibility of mental processes. According to Krutetskii (1976), capable students show signs of flexibility of mental processes, in three different ways. First, they are able to switch rapidly from one operation to another or from one train of thought to another, showing evidence of flexible thinking. Second, they are able to curtail the process of mathematical reasoning by eliminating intermediate steps according to each situation and exhibit economy of thought. Third, they possess the ability to reverse their mental processes.

Flexibility in thinking. Krutetskii's name is linked to flexibility since "it may be one of his greatest additions to the discussion of mathematical ability" (Chamberlin, 2012, p. 25). Flexibility in thinking is discussed in two dimensions; generation of multiple mathematical solutions and fluency for expediency.

In line with the first aspect of flexibility in thinking, flexible thinking involves the provision of multiple mathematical solutions (Kinda, 2006, cited in Chamberlin, 2012). According to Chamberlin (2012), advanced mathematical learners are likely to suggest more than one mathematical solving approach to a task, whereas suggesting an additional path seems to be challenging for a typical learner. In fact, the initial path may even hinder the process of providing an alternative solution for typical learners. The provision of multiple solution paths has been widely discussed and is considered a facet of creative ability (Leikin, 2011). Borovik and Gardiner (2007), also report an instinctive inclination of gifted children to approach a problem in different ways. They also reported that even if gifted learners have already solved a particular problem, they have a strong disposition towards finding an alternative solution.

The second aspect of flexibility, refers to fluency for expediency, a term proposed by Chamberlin (2012). According to Chamberlin, typical learners may persist in a specific line of thinking, although it may not be economical or advantageous. This could be observed when students may stick with an insufficient approach, without changing into another method, with the fear of providing a wrong answer. On the contrary, although mathematically gifted learners will start working on a problem with a strategy in mind, they are more likely to switch to a more economical method as soon this is identified. This will result to both a greater level of flexibility by providing different types of strategies to solve a mathematical task and o a greater level of fluency by suggesting a greater number of correct responses.

Curtailment of mathematical reasoning process and economical thinking. Mathematically able pupils are capable of shortening the process of mathematical reasoning by eliminating intermediate steps according to each situation (Krutetskii, 1976), thus abbreviating mathematical reasoning (House, 1987). As an extension of this ability, Krutetskii (1976) asserted that students “strive to find the easiest, clearest and most economical ways to solve problems” (p.223) whereas House (1987) mentioned the effort put by gifted students to find rational and economical solutions.

According to Borovik and Gardiner (2007), the ability to “compress" the reasoning process hinders the risk of a child being misunderstood. This could happen since the child may not provide all the details of the answer, skipping those that he/she thinks are too obvious and/or uninteresting. An example of efficient thinking by curtailing the process and thinking in an economical way that also ended in a learner being misunderstood, is the example of Carl Friedrich Gauss and his teacher. According to the famous story, Gauss’ teacher asked the class to add all integers from 1 to 100 and announce the sum. In about a minute, Gauss came up with the correct answer of 5050. Although Gauss actually found an economical and efficient method to calculate the sum, the reaction of his teacher was to accuse him of cheating. In fact, Gauss created a mathematical model that allowed him to effortlessly calculate the sum, after realizing that the numbers of 1 to 100 could be paired, e.g. 1+100, 99+2. In that way, upon grasping that there will be 50 pairs of 101, Gauss found the product of 50 by 101, which resulted to 5050. Not only he did not cheat, but Gauss systematized his work to arrive to the response. Thus, special attention should be paid to students' thinking process rather than the end product, in order to avoid misunderstandings and overlook of gifted students.

Reversibility of mental processes. Reversibility refers to the ability to mentally go through a series of steps in a problem and then reverse the direction, returning to the starting point. A series of researchers along the years reported this ability as an indicator of mathematical giftedness.

Krutetskii (1976), as well as House (1987) reported that gifted pupils in mathematics are able to switch from a direct to a reverse train of thought exhibiting reversibility of mental processes in mathematical activity. More recently, in a research study of gifted ninth grader' notions of proof, Sriraman (2008) also reported that students revealed their ability to reverse the direction of a mental process to arrive to the required conclusion.

Spatial ability. The relationship between spatial ability and mathematical giftedness has been studied by a number of researchers. The literature review of the empirical research on the identification and description of mathematically gifted students from the 1970s and 1980s revealed that mathematically gifted pupils demonstrate spatial abilities that differ in level and quality in comparison to their average-ability peers (Sowell, Zeigler, Berwall, & Cartwright, 1990). Similar results were found in the study of Benbow and Minor (1990). In another study, Block (1985) found that gifted 4 to 6-year-olds, based on their IQ, varied from average IQ students and chronological age mates with respect to four types of spatial tasks. These tasks required two- and three-dimensional rotations, paper folding, and geometric cross sections.

Krutetskii's work also suggested that spatial ability and visualization of abstract mathematical relationships play an important role in mathematical giftedness (1976), classifying the gifted in mathematics in three types; analytic, geometric or harmonic, differing in the predominance of the verbal logical or visual pictorial system. In particular, Krutetskii referred to the analytic type in which the verbal-logical predominates over the visual-pictorial, so the analytic type tends to think in verbal-logical terms. However, a geometric type is characterized by a stronger visual-pictorial system in comparison to the verbal-logical one. Geometric thinkers attempt to solve a problem using visual means and exhibit a high development of spatial concepts. The harmonic type displays the characteristics of both the analytic and geometric types. In this case, verbal-logical and visual pictorial components are balanced. Yet, students may tend to use or not visual pictorial aids in mental operations. Acknowledging that students differ in their type of mathematical giftedness has subsequent connotations both for identification and provisions. With the heightened emphasis on numbers and arithmetic, there is the potential danger of the gifted students with special abilities in geometric reasoning to be underprivileged with regard to gifted selections.

Diezmann and Watters (1996) pointed to the danger that spatially gifted students may underachieve in school, because of the overemphasis on analytical tasks. In addition, spatially gifted students may find it difficult to verbalize their reasoning (Diezmann & Watters, 1996). Spatially gifted students perform best on spatial mathematical problems and when using spatial methods, such as diagrams and visualisation. Consequently, the identification of these students is dependent on providing such kind of opportunities in order to allow them demonstrate their ability.

Creative ability. Nowadays, creativity shifts from being a background player to holding an important place within the context of giftedness and gifted education (Kaufman, Plucker, & Russell, 2012). Drawing on previous discussion, there are varied views with regard to the relationship between mathematical giftedness and creative abilities. A number of studies consider creativity as a specific type of giftedness (e.g., Sternberg, 1999, 2004), other studies advocate that creativity is an integral part of giftedness (Renzulli, 1978, 1986), whereas others show evidence that they are two independent features of individuals (Milgram & Hong, 2009).

Divergent thinking, originated by Guilford (1950, 1956), is directly linked to flexibility and originality of thinking (Plucker, Runco, & Lim, 2006) and as such it was considered to be synonymous to creativity for many years (Lin & Cho, 2011). Actually, Kim (2008) exemplified a significant correlation of divergent thinking with creative achievement. As a result, creativity was usually measured through divergent thinking instruments (Hong & Milgram, 2010).

Recent studies distinguish between general and specific creativity (Hong & Milgram, 2010; Piirto, 2004; Simonton, 1999). Drawing upon this characterization, they distinguish domain specific and domain general creativity. Specific creativity refers to the clear and distinct ability to create in one domain such as mathematics or arts (Leikin & Lev, 2007). In the domain of mathematics education, Plucker and Zabelina (2009) report a lack of literature win regard to the concept of creativity.

Even though studies on mathematical creativity are not plentiful, the relationship between mathematical giftedness and creativity has been documented through research evidence (e.g. Sriraman, 2005). According to Sriraman (2005), creative students exist in the “fringes” (p. 29) of the set of mathematically gifted students. That is, being mathematically gifted implies that the student is also mathematically creative, whereas being creative does not necessarily refer to a mathematically gifted individual. In line with Sriraman’s claims, Kattou, Kontoyianni, Pitta-Pantazi and Christou’s (2012) research study recently empirically verified that mathematical creativity is a subcomponent of mathematical ability.

A number of studies proceeded to associating mathematically gifted students with creative behaviors. According to Chang (1985), mathematically gifted children make unique links when presented with a demanding mathematical task. In the same direction, Greenes (1981) acknowledged their ability to interpret problem information in original ways. Not only do they interpret information in novel ways, but gifted children proceed in making original inter-subject connections with impressing relative ease (Kanevsky & Geake, 2005; Geake & Dodson, 2005). Miller (1990) referred to gifted students’ ability to exhibit flexible and creative ways when approaching mathematical problems rather than working in a stereotypic mode, leading to the development of unique solutions to ordinary problems (Wolfle, 1986).

In the literature, one may come across many definitions of mathematical creativity, but none of them is universally accepted (Mann, 2006). In an effort to formulate a precise and broadly accepted definition of mathematical creativity, one may find that this is extremely difficult and perhaps a task one may not achieve (Liljedahl & Sriraman, 2006; Sriraman, 2005). Runco (1993) considers creativity to be a construct consisting of “both divergent and convergent thinking, problem finding and problem solving, self-expression, intrinsic motivation, a questioning attitude, and self-confidence” (p. ix). Ervynck (1991) defines mathematical creativity as: “the ability to solve problems or to develop thinking in structures, taking into account of the peculiar logical-deductive nature of the discipline, and of the fitness of the generated concepts to integrate into the core of what is important in mathematics" (p. 47).

After analyzing the research attempting to define mathematical creativity, previous research by Mann (2006) concluded that the lack of a commonly accepted definition for mathematical creativity by the research community hampers subsequent research endeavors. To this respect, Leikin (2009a) suggested a model for the assessment of creativity through the use of multiple solution mathematical tasks. Multiple-solution connecting tasks refer to "tasks that contain an explicit requirement for solving the problem in multiple ways" (Leikin, & Levav-Waynberg, 2008, p.234). Leikin (2009a) provides operational definitions of mathematical creativity and a scoring method for the assessment of creativity, based on fluency, flexibility, and originality, following Torrance (1974). To evaluate originality, Leikin's model employs insight-related levels of creativity as suggested by Ervynk's (1991) in conjunction with conventionality of the solutions provided by students. There are also other researchers that used the concepts of fluency, flexibility and originality to define mathematical creativity in a mathematical context (e.g. Gil, Ben-Zvi, & Apel, 2007). Namely, Gil, Ben-Zvi and Apel (2007) utilized these three components, after defining fluency as the ability of producing many ideas, flexibility as the number of approaches observed in a solution and originality as the possibility of holding extraordinary, new and unique ideas. This definition of the three components of creativity was also used by Ervynck (1991) and Silver (1997). Following this approach in regard to the assessment of creativity, researchers are provided with quantitative measures, thus allowing the comparison between students performing the same task (Pelczer & Rodríguez, 2011).

Apart from cognitive characteristics, there are also hypercognitive characteristics that have been associated to mathematical giftedness.

Hypercognitive characteristics

In his research of mapping the architecture of the mind, Demetriou (2000), coined the term of the hypercognitive system to denote processes, concepts and strategies residing in the self-oriented level of the mind. As such, the hypercognitive system involves abilities such as self-awareness and self-regulation as well as relevant strategies to accomplish the two. The adverb "hyper" in Greek means "on top of", and by joining it to the term "cognitive", designates the supervising and coordinating functions of the processes residing at the self-oriented level (Demetriou et al., 1993). The reason for suggesting a new term instead of metacognition by Demetriou and colleagues, was the belief that functions associated with the hypercognitive system may come before, concurrently or after cognitions. In this regard, metacognition by definition assumes that the functions associated with it come after cognitions. Thus, a new term was required. In addition, Demetriou and Efklides (1994) consider the term "hypercognitive" to be more accurate since it refers to functions applied on other cognitive systems, at two levels. At a macro-developmental level, hypercognition refers to the person's general theory of intellectual functioning (Demetriou & Efklides, 1994). At a micro-developmental level, the hypercognitive system controls on line cognitive functioning (Demetriou, Efklides, & Platsidou, 1993), making decisions about the suitable and efficient use of schemes and cognitive functions for a specific task on hand.

Self-regulation, an integral ability of the hypercognitive system, has been associated with giftedness. The next section discusses the related research findings.

Self-regulation. Self-regulation, is generally defined as the ability of individuals to adjust their behavior according to the demands of specific situations (Block & Block, 1980; Kopp, 1982). The key feature in most definitions of self-regulated learning is the methodical use of metacognitive, motivational and/or behavioral strategies (Malpass, O'Neil, & Hocevar, 1999). In particular, self-regulation is the degree to which individuals are metacognitively, motivationally, and behaviorally proactive participants in their own learning (Zimmerman,1986, 1990) since they self-activate and direct efforts to acquire knowledge and skills by employing particular strategies, rather than passively reacting to teacher instructions (Zimmerman, 1998). In other words, these students set goals, use strategies to achieve them, and closely monitor their acquisition. They are self-confident or self-efficacious about their capabilities to learn and dedicated to their attainment of knowledge and skill (Risemberg & Zimmerman, 1992).

Self-regulation has been linked to giftedness and was present in theories and definitions of giftedness. In his triarchic theory, Sternberg (1986) postulated the importance of metacomponents for intellectual giftedness, including the selection of strategies that will best accomplish anticipated goals, the allocation of resources to better focus attention on the completion of the task and to monitor one's progress so as to establish that goals are being met. Empirical research has also shown the association of self-regulation with high intellectual ability. Previous research concluded that high-IQ children were more advantageous in metacognition (Shore, 2000) compared to average-intelligence children and had greater control over their self-regulatory processes (Calero, García-Martín, Jiménez, Kazén, & Araque, 2007).

In a study of Risemberg and Zimmerman (1992), the authors pointed to differences in regard to self-regulatory abilities amongst the group of gifted students that should not be overlooked. Taken as a whole group in comparison to non-gifted students, gifted employ more self-regulatory strategies, use more cognitively advanced learning strategies, and they complete these strategies in a more effective way. However, since the group of gifted students is not homogenous, there are some gifted students that are relatively weak in self-regulatory abilities. As such, they may not be able to set appropriate goals, or they choose ineffective strategies, or monitor and evaluate their progress. Risemberg and Zimmerman postulate that these students, overlap, to some degree, with the subgroup of gifted underachievers, and the researchers suggest that they may be simply deficient in teachable self-regulatory skills.

The ability to select the appropriate and the most efficient strategies to solve a problematic situation is an integral part of self-regulatory mechanisms. Strategies are, by nature, selected either consciously or unconsciously by the solver to achieve a specific objective (Siegler, 1996; Siegler & Jenkins, 1989).With respect to strategies used during problem solving, gifted children outperform their non-gifted classmates in strategic ability from an early age (Robinson, 2000). In studies of strategy knowledge, gifted children appear to be more advanced in their declarative knowledge of problem-solving strategies (Carr, Alexander, & Schwanenflugel, 1996; Jaušovec, 1991; Montague, 1991). That is, gifted children seem to have a larger and broader repertoire of strategies to use during problem solving. At the same time, when gifted children are in front of a problem solving situation, they often understand better and faster which strategies are suitable for the specific instance (Steiner, 2006). Thus, they select from their broad strategy range only the strategies that have proven effective in the past (Steiner, 2006). In a study of Steiner (2006), using microgenetic methods from cognitive psychology to observe gifted and non-gifted two graders while playing a computer game, the study revealed that gifted and average-ability children differ greatly in their patterns of strategy development and use. Namely, the researcher observed that even though the gifted children showed a certain degree of variability in their strategy choices, they quickly relied more on higher level strategies than the average-ability children that focused mainly on lower level strategies. However, although the gifted group used superior strategies, they failed to complete the task in a less time and significantly fewer trials. In other words, gifted children prefer higher level problem solving strategies even when those strategies are time consuming (Steiner, 2006). This is consistent with other views that gifted children take more time and care when planning problem solving (Shore & Lazar, 1996), resulting to more time required to reach a solution. Davidson and Sternberg (1984) found similar results in their study of insight. In this study, gifted children took longer time to solve higher order word problems, possibly due to their attention to detail and accuracy. In a similar direction, Flavell (1976) advocated that to determine successful problem solving, we should both look at the result and the way the individual monitors and evaluates his or her problem solving process. From this perspective, the gifted children in Steiner's study were more successful in problem solving, since they were much more likely to draw on the information they learned from previous strategies to make future assumptions.

Task perseverance, an important aspect that is related to functions of the hypercognitive system, has been associated with giftedness. The next section discusses the related research findings.

Task perseverance. Similarly to self-regulation, persistence and hard work have been discussed in the theoretical field by being incorporated in models and theories of giftedness. According to Renzulli (1978) and his three-ring theory, giftedness consists of an interaction of intellectual ability, creativity, and task commitment. Task commitment refers to high levels of interest — enthusiasm, hard work, and determination in a particular area, as well as self-confidence and the drive to achieve. Indeed, in a discussion on mathematical abilities and skills, Borovik and Gardiner (2007) reported as one of the traits of mathematically able children, the ability to focus on mathematics for extensive periods without apparent evidence of fatigue. In the theoretical field, in his exertion to deliver a list of characteristics of gifted students, House (1987) denoted the energy and persistence shown in solving mathematics problems.

Empirical research has also reported the relationship between task perseverance and giftedness. For example, Prieto and Castejón (2000, as cited in Calero et al., 2007), reported perseverance in tasks, high self-esteem and metacognitive abilities in high-IQ children. In another study, Bouffard-Bouchard et al. (1991) observed that gifted individuals exhibit sustained effort probably because they acknowledge it value, similarly to the way that those who consistently use a specific strategy acknowledge its significance. This increase in metacognitive understanding shown by gifted students, is likely to lead them to show sustained effort in other tasks as well. To sum up, Bouffard-Bouchard et al. (1991) concluded that gifted students in the study outperformed non gifted classmates not just due to their superior cognitive ability, but primarily because of their more active engagement in the problem-solving process and hard work.

Apart from cognitive factors, models and theories consider the affective and motivational domain as important as other aspects of giftedness. A discussion on the characteristics that fall in this category is presented in the next section.

Affective and Motivational Characteristics. Affective variables have been found by many studies to have an impact to mathematical achievement. For instance, Aiken (1973) showed that affective variables contribute to mathematical ability and mathematical creativity. In another study, Ernest (1985) represented the impact of affect and effort on the achievement in mathematics in his model of the "success cycle". In particular, positive attitudes and beliefs about mathematics typically result to high mathematical self-confidence and self-efficacy. This positive motivation leads to increased effort, persistence, and engagement with challenging problems. When provided with challenging problems, the increased effort is most likely to result to continued success in mathematical problems and mathematics in general. High achievement in mathematics will enhance positive attitudes, creating a cycle, with each component having a positive influence on its successor.

Although it is often assumed that gifted children are predetermined to succeed in life in their domain of giftedness, research evidence show that this is not always the case (Freeman, 1993). Gifted children undeniably possess cognitive potential for success in different achievement areas, but it is the interplay between several internal and external factors that will impact on the realization of this potential (Vlahovic-Stetic, Vizek Vidovic, & Arambasic, 1999).

Models of giftedness acknowledge the importance of affective and motivational factors to the manifestation of giftedness. For example, motivational and affective variables were amongst the personal-psychological attributes suggested by the comprehensive model of giftedness and talent applied in mathematics (Milgram & Hong, 2009) that operate together with cognitive abilities and environmental-social variables in each person to determine the type and level of talent. In addition, the role of motivation in giftedness was shown in the model proposed by Renzulli (1976), with motivation being one of the three necessary traits for a person to be gifted. In the domain of mathematics, the affective domain, including motivation and attitudes, is an often underestimated area in the field of giftedness, despite its importance for the mathematically gifted (Koshy, Ernest, & Casey, 2009).

According to Bandura's (1986) social cognitive theory, students' beliefs about their capabilities to successfully perform school tasks, or self-efficacy beliefs, may strongly predict their capability to carry out such tasks. For this reason, the domain of self-efficacy has been also explored in relation to giftedness in mathematics. Pajares (1996) investigated the role that self-efficacy beliefs play in the mathematical problem solving of middle school gifted students mainstreamed with non-gifted students in algebra classes. In this study, gifted students showed higher math self-efficacy and self-efficacy for self-regulated learning as well as lower math anxiety in comparison to non-gifted students. Although most students were overconfident about their capabilities, gifted students had more accurate self-perceptions and gifted girls were inclined toward under confidence.

There are also studies examining motivational and emotional variables among the gifted population. For instance, Vlahovic-Stetic, Vizek Vidovic and Arambasic (1999), investigated whether motivational-emotional variables could differentiate different groups of mathematically gifted pupils (mathematically gifted achievers and mathematically gifted underachievers) of age 9-10 years and mathematically non gifted pupils. Findings showed that gifted pupils differ from their non-gifted peers in attaining higher levels of intrinsic orientation toward mathematics, lower mathematics anxiety, lower attribution of success to external factors and effort, as well as in lower attribution of failure to external factors and abilities. In addition, gifted achievers have lower attribution of success to effort than gifted underachievers and non-gifted pupils.

Social-emotional and motivational factors are also acknowledged as significant in the design of educational provisions to meet the needs of gifted learners (e.g., Clark, 2008; Janos & Robinson, 1985). Particularly, children's self-concepts (Delisle, 1992; Hoge & Renzulli, 1993) and achievement motivation (McVey & Snow, 1988) are acknowledged as essential supplementary data to information of their cognitive ability and achievement. However, these factors are often overlooked in assessment and subsequent program planning (Porath, 1996). Thus, they should be considered for identification and provision purposes, along with environmental factors that also play a crucial role in the realization of giftedness.

Environmental characteristics

Despite strong genetic influence on intellectual potential, research evidence clearly shows that children's manifestation of talents is also largely determined by environmental characteristics, such as their family lifestyle, values, goals, and other characteristics related to school provisions and factors with respect to more general community climate.

Multidimensional models of giftedness acknowledge these important environmental factors to the development of giftedness. For instance, in the Munich model of giftedness (Heller, 2004), giftedness is considered to be an ability influenced by non-cognitive and environmental moderators. The moderators control the transition from gifts to revealed performance. As such, environmental moderators suggested by Heller (2004) in her model, are the family learning environment, the family climate, the quality of instruction, the classroom climate and critical life events. In this model, moderators correspond to the term of catalysts as proposed by Gagné (2009) in his differentiated model of giftedness and talent. According to Gagné (2009), environmental catalysts may facilitate or hinder the developmental process of transforming gifts to talents by their presence or absence. In this model, Gagné names three groups of environmental influences on the development of talents: milieu, individuals and provisions. Milieu might be translated among others to physical, cultural, social and familial. Influences from individuals might come from parents, teachers, peers, mentors and many others. Finally, environmental catalysts related to provisions may refer but not be limited to programs, activities and services (e.g. enrichment, acceleration, ability grouping). The comprehensive model of giftedness and talent of Hong and Milgram (2008) also pays attention to the role of environmental-social variables to the manifestation of giftedness. These variables are related to home, school and community, culture and socioeconomic status.

The role of family in the development of giftedness in children is undeniable. Reichenberg and Landau (2009) examined the role of family in the development of gifted children, as well as gender differences, and long term outcomes in the gifted as they grow up. In their research, the significant role of the parents involves three dimensions: the identification of giftedness in their children, support of the child's cognitive development and the development of values and goals through family culture (Reichenberg & Landau, 2009). The family's role in the identification of giftedness will be further discussed in the following section about the identification of giftedness. The other two aspects of parental role in gifted development, family support and family culture with regard to development of values and goals, will be briefly discussed below.

Family support is a key factor for fostering a child’s development and growth of gifted potential. From the infant age, parental encouragement and the accessibility to a variety of play materials and learning experiences are of vital importance. Experiences that parents do together with their children strongly influence the child’s understanding: games, chatter, stories, even arguments can be a stimulating way of nurturing a child’s intellectual growth (Landau, 1990; Freeman, 2000). In another study, Radford (1990) followed up 14 children with enriched early language who became “outstanding students in school” in all subject areas. Through a retrospective study design, he found that these adults experienced an enormous amount of verbal, both spoken and written, stimulation when they were children. Radford (1990) furthermore provided evidence that exceptional early achievers often came from homes that were all vigorous, stimulating and usually highly verbal. This was evident even in cases of families of low socioeconomic status.

Other studies showed that when children feel empowered and competent, their motivation increases (Landau, 1990; Freeman, 1992). In contrast, when children consider control of their learning not determined by themselves but located outside themselves, such as the teacher, they will be predisposed to be less involved and motivated to work. Freeman (2000) showed that when poorly motivated children feel empowered and responsible to help others, as when unsuccessful adolescents take on the role of tutors to younger children, their urge to learn may increase.

A rich continuous educational environment is vital in developing intrinsic motivation for curiosity and love of learning (Freeman, 2000). In this study, children from higher socioeconomic status families tended to have higher IQs. According to the research, the parents’ educational accomplishments were responsible for this finding, neither the parents’ professions or intelligence nor the amount of parent–child interaction. “The families of gifted children provided more stimulating activities than did the families of non-gifted children. Moreover, the parents were more involved and apparently more invested in providing their children with a cognitively advantageous home environment” (p. 156). These experiences involved academic and cultural incidents, such as use of library or musical instruments. In this research, the gifted children also influenced and shaped their environments, by demanding more learning experiences and activities in comparison with non-gifted peers (Freeman, 2000).

There are a number of studies that examined the impact of family culture on the development of giftedness, with regard to the development of values and goals, since cultural and family attitudes have an important effect on high achievement. Perkins (1981) found that values and beliefs were a considerable factor for eminent work production by creative people, apart from their talent. In the same direction, Holahan and Sears (1995) studied successful adults and found that their family background and particularly the aspire to success distinguished them rather than intelligence or past school achievement. Both Flynn (1991) and Stevenson (1998) have reached to the conclusion that the culture valuing hard work is probably responsible for the number of Asian students who showed greater school and work success than their higher IQ peers. Indeed, Hess and Azuma (1991) reported that American students required much more assistance and praise than Japanese children in their motivation to learn reflecting a difference in culture the children were brought in.

While cultural values may nurture gifted development, in other cases, they may hamper the achievements of gifted children (Freeman, 2000). For instance, children that do not fit the cultural stereotypes of what considered to be gifted, are less likely to be identified as potentially gifted (Freeman, 2000). Cultural disadvantage because of the family's background, often results to reduced perception and attention, verbal and intellectual abilities, and motivation (Reichenberg & Landau, 2009). There are specific behaviors experienced by parents that narrow the development of gifted children, such as giving orders more frequently than explanations, ignoring or rejecting children's questions, providing scarce opportunities for play and talk.

In another research, Weissler and Landau (1993) examined the family environment of gifted children. Research findings showed differences between parents of gifted children and parents of children with average intellectual abilities in the following aspects; available environmental stimuli, parental academic achievement, and the cognitive interaction between parents and their children. In general, parents of gifted children were more assertive, confident, and liberal.

The literature reviewed in the section of mathematical giftedness suggests that a combination of cognitive, hypercognitive, motivational, affective and emotional characteristics may uniquely contribute to the development of gifted children. Thus, these aspects must be carefully examined and implemented in the identification of giftedness.

Identification of Giftedness

The conceptualization of giftedness provides the theoretical rationale underlying identification (Renzulli, 2002), which in its turn has direct impact on the selection of identification instruments and identification processes. Whilst narrow conceptualizations of giftedness dominated the field for years, they hindered both identification processes as well as efforts to promote manifestation and development of giftedness (Milgram & Hong, 2009). The arguments with regard to the definition of giftedness and the identification of gifted children exist for a century and will certainly continue to exist in the future (Freeman, 1998).

Due to the lack of conceptual clarity as to the nature of giftedness, identification processes have varied widely. Hymer and Michel (2002) provided one possible explanation for the difficulty of designing and agreeing upon a precise identification process for gifted students that is "gifts … elude simple measurement… And while we are usually try to ignore that which we cannot measure, gifts and talents are difficult to ignore – so we persist in our utilitarian search for even more 'accurate' ways of measuring them" (p.10).

In 1981, Passow questioned the two step process of using first an identification process and then proceeding to differentiation. Rather, Passow suggested that planned enrichment educational options could be used as a mean of identification, suggesting a performance and product based criterion of identification of giftedness. Thus, in Passow's (1981) words:

> Identification of the gifted and talented is related not only to systematic observation of and intelligent interpretation of observational data, but to the creation of the right kind of opportunities which facilitate self-identification—identification by performance and product which results in the manifestation of gifted or talented behaviors. (p.10)

Due to the complexity of the identification process, multiple forms of information should be obtained through the identification process for giftedness. The different types of evidence that may be used are presented and discussed in the next section.

Multiple Sources of Evidence

Traditionally, gifted students were identified by scores on intelligence, achievement or aptitude tests. Even alternative identification methods such as teacher or peer nominations and interviews were strongly related to student achievement (Sowell, Zeigler, Berwall, & Cartwright, 1990). Later, a major shift was noted in the research field in relation to the identification of gifted students; namely, IQ was no longer considered as the only and predominant index of giftedness, rather environmental influences were acknowledged (Hartas, Lindsay, & Muijs, 2008). Usually, the original choice of an identification battery depends on the definition adopted, the nature of the gifted program and on the previous experience of the professionals involved in the process (Nevo, 2008). Nevertheless, nowadays there is a lack of valid psychometric instruments to assess the abilities included in modern models of giftedness (Milgram & Hong, 2009).

Identifying gifted individuals raises important issues regarding the types of evidence of giftedness and the validity of assessment processes. Since most definitions include several dimensions to describe giftedness, it is impossible to sample all behaviours using only one test (Salvia & Ysseldyke, 2001). For example, Salvia and Ysseldyke (2001) pointed out that mathematical strength is not directly reflected in students' achievements on standardized tests, nor it is always related to the interest, persistence or enthusiasm shown in the classroom. The concept of giftedness is complex, and as such test results are only part indictors of high ability.

To this end, a combination of valid, reliable, sensitive and objective tools should be used in order to collect information for a student (Coleman, 2003; Davis & Rimm, 2004). This will allow for a range of behaviours to be observed, behaviors otherwise overlooked because they occur outside of the classroom. Whereas not all characteristics of giftedness are easily measurable, thus multiple methods of identification should be employed (Bicknell, 2009), as well as multiple sources of evidence should be collected, including academic performance, intelligence scores as well as information about certain personality characteristics (e.g., persistence, perseverance, resilience), motivation, and interest (Hartas, Lindsay, & Muijs, 2008). Hence, researchers and teachers need to employ a range of methods to identify mathematically promising children (Koshy, Ernest, & Casey, 2009). Charismatic students can be identified using different terms (gifted, talented, promising, etc.) and different instruments (psychological tests, standard assessments, school marks, teachers' observations, etc.) based on several criteria (problem solving behaviour, cognitive abilities, multiple intelligences, personal attitudes, etc.) and (or) their combination (Freiman & Rejali, 2011). The methods most relevant to the identification of mathematically gifted students will be given further attention. These include: intelligence tests, achievement tests, aptitude tests, creativity tests, nominations, and observations.

Intelligence tests. Intelligence tests (IQ tests) were the most commonly used measurement for identifying individuals at the extremes. By considering giftedness as high abstract reasoning (Silverman, 1993), g could be used both as a measure of giftedness and general intelligence. According to Silverman (2009), instruments that view intelligence as abstract reasoning (g) and the richest loadings on general intelligence (g) are the most useful for identification, such as Raven's Progressive Matrices (Raven, Raven, & Court, 2003), the Stanford-Binet scales (Roid, 2003) and the Wechsler scales (Wechsler, 1999).

However, the validity and liability to cultural and social bias of standardized measures such as IQ, has been questioned (Black, 2001). Dai (2010) discussed how four basic assumptions behind intelligence testing have been questioned. Firstly, he argued that traditional intelligence tests are atheoretical by nature. He used the citation of Stephen Jay Gould who, in his book The Mismeasure of Man (1981), supported that research on intelligence measurement is conducting a serious error, that of reification, considering something abstract by nature to have material existence. Due to a certain level of arbitrariness about which and what type of tasks to include in an intelligence test in order to provide a composite score by the performance in various tasks, the measurement is atheoretical (Dai, 2010). The second assumption underlying intelligence testing now being questioned, refers to causal inferences about predictive relationships (Dai, 2010). Traditionally, IQ tests were believed to measure natural aptitude excluding achievement. At the same time, the correlation between natural aptitude and achievement is a cause and effect relationship. These assumptions, have been challenged by many researchers (e.g. Lohman, 2009; Sternberg, 1999).

Milgram and Hong (2009) pointed out that children whose potential differs from the abilities measured by narrow measures used in schools such as IQ tests and school grades may be systematically excluded from gifted programming. The questioning of the use of IQ measures and relevant research findings mentioned, give possible reasons for this exclusion, when this is based on IQ tests. Also, it is of concern that these tests are not able to adequately identify high ability in mathematics. Mathematical talent is a specific aptitude, whereas an IQ score is a summary of many different aptitudes and abilities (Miller, 1990). An individual's IQ consists of several components, only some of which are related to mathematical ability. In a recent study of Kontoyianni, Kattou, Pitta-Pantazi and Christou (2013), through the administration of a mathematical instrument and an intelligence instrument, the researchers illustrated that although intelligence is a predictor of mathematical giftedness, different groups of students are identified by each type of testing. Thus, mathematical testing and intelligence testing do not yield the same results in terms of students identified.

Intelligence tests have also been criticized for the information they provide, when used in the context of an identification process of giftedness. To be more specific, an IQ score does not distinguish between an emergent not yet fully developed ability and a fully developed ability. As for example, having two students of different ages, e.g. a 7-year-old and a 15-year-old, obtaining the same score from an IQ test does not signify any developmental changes from 7 to 15 years old (Dai, 2010). This is due to the fact that IQ tests are based on age norms and are not formed in a way to show a new level of cognitive competence (McCall, 1981). Another concern regarding intelligence testing was raised by Hunt (1999, 2006), providing evidence that intelligence is more differentiated at the very high end of the spectrum. As a result, students with similar high IQ scores may differ in cognitive profiles, thus differ in cognitive strengths and weaknesses. Accordingly, these students may be considered to be equally "gifted", but this IQ score may indicate different things for each student (Dai, 2010). This is specifically concerning, especially in cases of domain specific giftedness, such as in our case mathematical giftedness. A same IQ score for two students, may in fact conceal mathematical giftedness for one student and perhaps artistic giftedness for the other. Thus, other instruments should be used additionally to intelligence testing in order to clarify the type of giftedness that characterizes each student. Moreover, IQ tests solely provide the product of how well a student performed based on age norms, while at the same time no information is obtained on the process of obtaining the particular score (Dai, 2010). Additionally, Lohman and Rocklin (1995) argue that intelligence testing as a selection process does not offer information on how provision should be differentiated to accommodate the needs of identified children. Summing up all concerns, both evaluation of academic performance and cognitive abilities should be used (Naglieri & Ford, 2003), despite their conceptual differences.

Despite the wide spread use of IQ tests, it is particularly concerning that IQ cannot predict success in later life (Stenberg, 1999). Although there is research exemplifying that IQ- tests do correlate .40 to .60 in relation to school success, this interaction is attributed to the similarity of both the IQ test and lesson-learning situations (Hotulainen, 2003). Unfortunately, these school-like situations do not correspond necessarily to the many real life situations after completing school (Renzulli, 1998, [on line], cited in Hotulainen, 2003, p. 14). Thus, even though IQ could at least to a certain extent explain academic school success, this does not necessarily imply anything about success in later years. Equivalent conclusions may be drawn with reference to school achievement (Hotulainen, 2003).

Achievement tests. Achievement tests have been developed in order to measure the effects of instruction (Anastasi, & Urbina, 1997) and therefore measure how much students have learned in a given subject matter content (Ryser, 2004). These tests are often computation oriented and the assumptions about giftedness can be made from the number of items an individual has answered correctly, while little attention is placed on a student's actual mathematical reasoning (Johnson, 1983). For instance, students scoring above the 95th or 97th percentiles in a mathematics achievement test can be considered as having high mathematical ability, but more information is needed to separate high achievers from the truly gifted (Miller, 1990). The wide list of achievement tests includes the Screening Assessment for Gifted Elementary Students (SAGES), the Iowa Tests of Basic Skills, the Stanford Achievement tests, the Metropolitan Achievement Tests, and the California Test of Basic Skills.

Three limitations impede the use of standardized achievement tests as identification instruments of giftedness; the grade equivalent score, low ceiling and the impact of acquired knowledge. Namely, the grade equivalent score (the average score earned by children at a particular grade level on a particular test) is misleading and should be used only as an indication that the child needs special challenge. As for the low ceiling that typical achievement tests have, this makes it difficult to measure high ability and skill levels of the very able children. The effect of the existing knowledge and experience of students taking an achievement test may result to a biased test towards students with more knowledge.

Aptitude tests. Aptitude tests are designed to measure specific abilities that develop over time or the potential for future achievement in specific areas. Mathematics aptitude tests have many common aspects with the results of mathematics achievement tests. Because the aptitude tests place less emphasis on computational skills and more emphasis on mathematical reasoning skills, their results are more useful in identifying mathematically gifted students (Miller, 1990).

Krutetskii (1976) expressed his disagreement with the use of tests that focus only on scores, without studying the process followed by the students. In this case, teachers cannot obtain much information about the mathematical thinking of their students. In the case of multiple choice tests, the results tend to be considered only cumulatively, so their usefulness is limited (Bicknell, 2009). We should be aware that two students may provide the same solution to a mathematical problem, but the solution could be obtained in different ways. If the test was in multiple choice format, the students are limited to select a predetermined response choice, thus not allowing them to show their mathematical thinking processes. Following Krutetskii's remarks, a test that focuses more on reasoning than on content knowledge can also help identify a younger child who has not yet been exposed to curriculum content at higher levels, but can easily master it conceptually (Matthews & Foster, 2005).

The Scholastic Aptitude Test (SAT), a test of verbal and mathematical reasoning ability is the most recognized test for identifying mathematically gifted students in the United States (Bicknell, 2009). The SAT is considered to be appropriate for identifying gifted students between 11-14 years old who top out on individually intelligence tests. This is possible since it was designed for older students and therefore is characterized by high ceilings (Lupkowski-Shoplik & Swiatek, 1999). Other proclaimed advantages of the SAT, was its established validity for predicting the likely ability of young students benefiting from acceleration (Bicknell, 2009). Conversely, Kissane (1986) reported that younger and older students responded to SAT items in qualitatively different ways.

In a response to the general feeling that there was no satisfactory test developed that had been normed for the gifted population, Stanford University developed the Stanford EPGY Mathematical Aptitude Test (SEMAT) (Paek, Holland, & Suppes, 1999). Its objective was to find a solution for low ceilings of tests normed for general populations. These types of tests were designed to measure mathematical ability through challenging but age specific non curriculum based tasks. "These items do not require deep mathematical knowledge to solve. Instead, they require the ability to apply insightfully basic mathematics-related skills" (Paek et al., 1999, p. 339).

One of the two pragmatic movements in the field of gifted education, who, according to Dai (2010), led the way towards the reconceptualization of giftedness, was the study of mathematically precocious youth (SMPY) and subsequently the talent search model (Lubinski & Benbow, 2006; Stanley, 1996). The talent search involved in the study was developed by Stanley and colleagues at the John Hopkins University in the United States aiming to identify middle school students with exceptional performance in mathematics (Stanley, Keating, & Fox, 1974). In the subsequent years, the identification of verbal talent was added in the aims and content of the talent search. The specific talent search bypassed the IQ testing philosophy and by defining giftedness in terms of precocity, follows the idea of ‘‘off level’’ testing (Thomson & Olszewski-Kubilius, 2014) that is tests assess students’ level of prior knowledge and abilities instead of the appropriate level of knowledge according to their chronological age. Thus, ceiling effects are prevented. In the words of Dai (2010) with regard to this approach, ‘epistemological and methodological significance lies in the fact that it represented a more domain-specific approach based on precocious mathematical development rather than assumptions of general intellectual advantage” (p. 19). This approach is both epistemologically and methodological significant. The talent search was found to have good predictive validity (Benbow, 1992). In particular, SAT scores can predict future achievement, such as in 10 years after participation in the talent search.

Above-level tests such as the SAT were used to identify mathematically gifted students (Rotigel & Lupkowski-Shoplik, 1999). Above level tests contained more difficult items. However, Lupkowski-Shoplik, Sayler, and Assouline (1994) revealed third and fourth grade mathematically gifted students performed better on conceptual tasks than computational tasks. These findings might be explained by the fact that gifted students may be bored with repetitive computation tasks. Similar findings were reported by Rotigel (2000) (cited in Assouline & Lupkowski-Shoplik, 2003). Thus, we may conclude that tests with problem solving rather than computational tasks would be a more useful identification tool for mathematical giftedness.

Creativity tests. Everyone has the potential to be creative (Treffinger, 2009). With this perspective, creativity tests can and should be used to provide data whether the student’s creativity “is not yet evident, emerging, expressing, or excelling”, in other words to “understand the richness and breath of creativity” (Treffinger, 2009, p.246). One of the sources of talent loss is the underestimation of the value of creative thinking; thus not emphasizing it in mathematics teaching and identification processes in schools (Milgram & Hong, 2009). In a recent review of the state of the creativity assessment, Kaufman, Plucker and Russell (2012) conclude that, despite the many flaws present in every type of creativity measurement, creativity should be included as part of a gifted assessment battery.

According to Treffinger (2009), with respect to identifying gifted students, there are certain criteria that a creativity test should meet: “(a) yield “hard” data, and preferably a single score or index; (b) is brief and easy to administer and score (preferably objectively); (c) has extensive norms for classifying the students from “gifted” to “average” (or below); (d) is appropriate across ages, genders, and social or cultural differences; and, of course, (e) is cost efficient (or even free)”. (p.245)

For the assessment of creativity, a range of batteries have been developed and used throughout the years, such as divergent thinking tests and rating scales. Comparing all types of batteries with regard to the measurement of creativity, divergent thinking is the backbone of creativity assessment (Kaufman, Plucker, & Russell, 2012). The work of Guilford’s Structure of the Intellect divergent production tests (1967), under the framework of his Structure of the Intellect Model (1950) that placed creativity into a larger context of intelligence, gave a theoretical rational for the use of divergent thinking tests as a mean to assess creativity. Following the Structure of the Intellect Model, the Structure of Intellect Divergent Thinking battery consists of many tests that correspond with the 24 distinct components of divergent thinking, one type for each combination of the four types of content (Figural, Symbolic, Semantic, Behavioral) and six types of product (Units, Classes, Relations, Systems, Transformations, Implications).

Torrance focused on divergent thinking as the basis for creativity and influenced by the assessment of Guilford, made his own tests, the Torrance Tests of Creative Thinking (Torrance, 1974, 2008). As a result of his work, the Torrance Tests of Creative Thinking are the longest running, most carefully, widely studied and used assessment of divergent thinking and creative talent in educational settings (Kaufman, Plucker, & Russell, 2012; Kaufman, Plucker, & Baer, 2008; Sternberg, 2006). The test battery includes of a Figural and Verbal section. The Verbal section assesses the ability to think creatively using words, whereas the Figural tests measure an individual’s ability to think creatively using pictures. The test can be group or individually administered for ages ranging from kindergarten through higher education or beyond (Kim, 2006). The test is standardized and includes detailed norms that were revised accordingly (Torrance, 1974, 2008; Torrance & Ball, 1984). Torrance recommends that the responses should be examined by trained scorers, since inexperienced raters may diverge from the scoring protocol during the assessment of originality, if their personal judgments interfere (Kaufman, Plucker, & Russell, 2012).

In current versions of the Figural Torrance Tests of Creative Thinking (Torrance, 2008), creativity is described through five scores in the following aspects: (a) fluency: the number of responses to a given stimuli), (b) elaboration: the extension of ideas within a specific category of responses to a given stimuli, (c) originality: the uniqueness of responses to a given stimuli, (d) resistance to premature closure: the degree of psychological openness when processing information and (e) abstractedness of titles: the degree to which a title moves beyond a labeling of a picture (Kim, 2006).

Another form of assessing creativity is through checklists, such as the Creativity Checklist (Proctor & Burnett, 2004), the Gifted Rating Scales (Pfeiffer & Jarosewich, 2007) and the Scales for Rating Behavioral Characteristics of Superior Students (Renzulli, Smith, White, Callahan, Hartman, & Westberg, 2004). The Creativity Checklist includes characteristics thought to be typical of a creative individual, that load onto nine scales: fluent thinker, flexible thinker, original thinker, elaborative thinker, intrinsically motivated student, curious/immersed in topic, risk taker, imaginative/intuitive and engages in complex tasks. Yet, there were no norms, no criterion-related or predictive validity established for these checklists (Kaufman, Plucker and Russell, 2012).

The Gifted Rating Scales were designed to be user friendly, involve minimal scoring and interpretation training, they are psychometrically reliable and valid and include a standardized sample (Pfeiffer & Jarosewich, 2007). There are two versions, one suitable for preschool and one for school-aged children. The version for preschoolers is comprised of 60 items that load onto five scales, whereas the version for school-aged students includes six scales with 12 items each (Pfeiffer & Jarosewich, 2007). In a recent study, the version of the Gifted Rating Scales for school aged children was validated with IQ scores on WISC-IV (Pfeiffer & Jarosewich, 2007) and was found successful in identifying intellectual giftedness.

The Scales for Rating Behavioral Characteristics of Superior Students are widely used in the selection of students for gifted and talented programs in Grades K to 12 (Callahan et al., 1995; Hunsaker & Callahan, 1995). Based on a multiple criteria approach to the identification of giftedness, the battery includes 14 scales to identify student abilities in the following areas: learning, motivation, creativity, leadership, art, music, drama, planning, communication (precision), communication (expression), math, reading, science, and technology. The reliability is reported to be significant if the evaluators have been appropriately trained (Center for Creative Learning, 2002). Test-retest and interrater reliability have been found to be excellent for the learning, motivation, and creativity scales (Jarosewich, Pfeiffer, & Morris, 2002).

When using checklists for the assessment of creativity, Kaufman, Plucker and Russell (2012), pinpoint the danger of unintended bias based on global impressions. The researchers emphasize that raters should be encouraged to rate each child individually and not return later to make score comparisons.

According to Kaufman, Plucker and Russell (2012), creativity assessments, such as the ones discussed earlier may have flaws in regard to reliability or validity evidence; others are impractically long to administer or exhaustive to score. However, other forms of identification of giftedness may not illustrate fully an individual's potential. Thus, divergent-thinking or creativity tests may help give a more comprehensive understanding of a person's overall abilities. Similarly, although checklists may not be the most psychometrically sound assessments, they can be used as a form of assessment in combination with other measures to help illustrate the creative abilities of a student (Kaufman, Plucker and Russell, 2012).

Nominations. Nominations by teachers, parents, peers or gifted students themselves are used as a means of identification. Careful examination of nominations should take place before selecting gifted students, due to human judgment and the "weight" given to each nomination during the identification process.

Due to many limitations, nomination by teachers is of limited usefulness (Hoge & Cudmore, 1986) and not the most reliable of identification instruments (Bicknell, 2009; Threlfall & Hargreaves, 2008). Thus, using a variety of identification strategies will minimize the weight of teacher opinion. Due caution is needed to guard against teacher bias and stereotyping (Davis & Rimm, 2004), owed to the tendency of teachers to favor some types of students due to stereotypes, such as students who conform to teachers' guidelines (Hoge & Cudmore, 1986). For instance, Kornhaber (1999) refers to studies according to which ''teachers tend to select compliant students over more challenging students who may have greater potential'' (p. 144). At the same direction, Kissane (1986) also reports that teachers may have overlooked several students during identification.

Lee (1999) further stresses another important limitation of teacher selections for gifted provisions, when these are based on their implicit conceptualizations of giftedness, given that the phenomenon of the underrepresentation of girls has been observed. Furthermore, Montgomery (1996) suggested that checklists can be used to help teachers identify gifted students. Nevertheless, they may be restrictive when teachers put students' performance in categories instead of providing an insight to their profile by describing it across academic and social characteristics (Hartas, Lindsay & Muijs, 2008). There are also other practical considerations that may impede the effectiveness of teacher nominations as an identification method. These considerations rely upon features such as time, teacher expertise, teacher professional development and school policy (Bicknell, 2009).

Benbow pointed to issues of checklist validity, content, and predictive validity in particular with respect to the use of checklists (Benbow, 1992). The Scales for Rating Gifted Students (Ryser & McConnell, 2004) and the Scales for Rating the Behavioral Characteristics of Superior Students (Renzulli, Smith, White, Callahan, Hartman, & Westberg, 2004) are two commonly used rating scales to identify mathematical giftedness, based on published characteristics of mathematically talented students.

Nomination by teachers is one of the most widely used and recommended means of identifying potentially gifted pupils. In a research review, Freeman (1998) showed that teachers would be able to effectively identify giftedness only after they had received adequate training in what to look for. This was complemented by Sheffield (1999), who suggested training and experience for teachers in both mathematical content and in the characteristics and needs of gifted students, prior to assessing them for mathematical giftedness. In line with Sheffield (1999), experienced teachers can identify and nominate gifted students, after commenting upon the specific aspects of students' academic performance (Hartas, Lindsay & Muijs, 2008).

Nomination by parents is suggested after taking into account that no one knows children better than their parents (Davis & Rimm, 2004). Ability in mathematics is often revealed in early childhood years (Straker, 1983). Thus, parents typically detect special abilities in mathematics before their child starts school (Bicknell, 2014; Dimitriadis, 2010). As a result, Koshy (2001) assumes that teachers could consult parents when their child enters school. Koshy (2001) further recommends that the cooperation and communication between parents and teachers should be enhanced by involving parents in the identification of indicators of giftedness. Information from parents can be obtained in several ways, maybe in a form of communication with parents about the aptitudes and abilities of their child, through letters and checklists, even asking parent to describe their child‘s interests on the school admission form, interview the parents or make use of parent questionnaires and surveys (Koshy, 2001).

Gardner (2006) reported a discrepancy between perceptions of parents and teachers of whether a child was considered gifted, possibly due to the fact that parents did not have as many occasions, in comparison to the teachers, to observe a large number of students. Nonetheless, there are also other cases where parents' nominations are often disregarded as biased (Dimitriadis, 2010). There are two possible reasons for this, since there are parents that think that their own child is gifted while this is not the case (Davis &

Rimm, 2004) and others who may underestimate rather than overestimate their children (Chitwood, 1986).

Regarding peer nomination, Davis and Rimm (2004) mentioned that peers are good at naming gifted and talented classmates, since they spend a lot of time with them each day and it has been proved that they can judge the abilities of their classmates. Koshy (2001) reports that peer nomination can be quite accurate since when the teacher asks children to nominate classmates who they think are mathematically gifted, there are not many discrepancies between student and teacher nominations. In order to find out the perceptions of a student about his classmates, direct questions can be made (e.g. Who is the smartest kid in class? Who is best at math? Who has the most unusual ideas?) or questions can take a game format (e.g. Guess who?). A similar approach was suggested by Jenkins (1979, cited in Gagné, 1989) who suggested playing a game with students. According to this game, students were asked to imagine that they are faced with a difficult situation, e.g. alone in a desert island. Then, students are asked to propose which classmate they would call for help, propose the best leader, fixer, inventor, and entertainer and so on.

Observations. Close and systematic observation of gifted behaviors is a useful and effective way of identification (Bicknell, 2009; Koshy, Ernest, & Casey, 2009), especially in a specific domain, such as mathematics (Karolyi, Ramos-Ford, & Gardner, 2003; Koshy, 2001). Bicknell (2009) highlighted the important consideration that observation could be an effective way of identification, provided that the teacher is aware of the behaviors and signs to look for.

Koshy (2001) argued that observation also includes the creation of suitable opportunities that allow pupils to show their potential as well as systematic monitoring of their work. For instance, Koshy recommends that teachers must provide open ended mathematical tasks, observe the pupils while working on them, listen to them and observe them by keeping written data. It was also Krutetskii's (1976) assertion that the way to identify mathematically gifted students was to monitor them while undertaking specially designed problems to elicit the abilities and characteristics he identified. Unfortunately, according to Wertheimer (1999), Krutetskii's work was not systematically organized since today so that it would be used as an identification instrument in school settings.

It is evident that the variety of existing measures of identifying giftedness implies that the instruments to be included in the identification process should be carefully selected, taking also into account the specific purpose of identification in this study, that of identifying giftedness in mathematics.

Identification of Giftedness in Mathematics

The literature suggests that mathematically gifted students are characterized of unique cognitive, spatial and creative abilities, hypercognitive characteristics and they have qualitatively different mathematical thinking. Hence, identification of mathematical giftedness should incorporate aspects of problem solving, creativity and spatial development (Diezmann & Watters, 1997; Niederer, 2001; Watters & English, 1995) while at the same time it will allow for these abilities along with self-regulatory strategies and hypercognitive processes to be observed. This way, mathematically gifted students with unique ways of thinking about and doing mathematics will be possible to get identified. Numerous of the distinctive behaviors formerly outlined are apparent whilst the student is engaged in problem solving (Niederer & Irwin, 2001). Thus, mathematical problem solving could serve as a means to identify and describe mathematical giftedness.

Identification through the observation of problem solving

Mathematical problem solving is at the heart of any mathematical activity. It has been the focal point of researchers, mathematicians and mathematics educators that seek to better understand the mechanisms of mathematical reasoning process and the development of understanding in mathematics. It is true that mathematically gifted students may be identified through their high levels of reasoning (Sheffield, 1999), given that teachers provide opportunities for them to demonstrate their thinking and cognitive processes, thus distinguishing them from students who are hard workers, but not mathematically gifted students.

Observation of student mathematical problem solving and the surrounding conversation with students may provide useful and priceless information on student's mathematical thinking and abilities. In regard to identification of giftedness, "with younger students, teacher observation of the process, and the surrounding discussion is the most accurate and reliable tool for distinguishing truly gifted students" (Hoeflinger, 1998, p. 245). Firstly, through mathematical problem solving, the researcher may gain valuable insight into a student's mathematical understanding and ability (Bicknell, 2009). If the problems can be solved using a variety of strategies, then the student's response may be quite creative and the selected solution strategy quite sophisticated. In addition, the student may surprise with his/her persistence and determination to reach to a solution. At a subsequent stage, the discussion following the explanation may designate mathematical promise. Thus, as stated by Bicknell (2009) the student should be encouraged to show his/her way of organizing relevant information, elucidate his/her conjecture and justify the provided response. There are empirical findings to confirm Bicknell's proposition, since in a study of Freiman (2006), the analysis of elementary school students' responses to mathematical problem solving tasks in combination to their various solution approaches led to the identification of mathematical talent. Other studies also support this approach in the identification of mathematical giftedness (e.g. Callahan, 2001; Diezmann & English, 2001; Kennard, 2001; Krutetskii, 1976).

Wieczerkowski, Cropley and Prado proposed that mathematical giftedness should be viewed as the unique ways "of looking at and attempting to solve mathematical problems" (2000, p.415). Hence, a more qualitative view of mathematical giftedness is underpinned. Therefore, mathematical giftedness should be identified by collecting student performance data employing multiple assessment approaches, following Cropley (1994) that articulated that a combination of quantitative and qualitative approaches define true mathematical giftedness. This way, different approaches will supplement one another to provide a more comprehensive and detailed representation of a student's performance (vanTassel-Baska, 2014).

Principles for mathematical problem solving task development

Mathematical problem solving entails different types of mathematical problems. For example, problems may differ among others in complexity or level of cognitive demand, reasoning abilities required or assessed, the range of initiative they allow the student to take and the range of solutions to be observed. Albeit the large variety of mathematical problems that exist, not all of them are appropriate to be included in the case of observing mathematical giftedness for several reasons.

The terminology of the mathematical tasks that are suitable to be part on an identification process of giftedness in mathematics differs from one researcher to the other. However, all terms share common ground with each other. In particular, terms proposed are challenging tasks (Greenes, 1997; Leikin, 2007), mathematically rich tasks (Peressini

& Knuth, 2000), non-routine problems (Garofalo, 1993; Pativisan & Niess, 2007; Schoenfeld, Burkhardt, Daro, Ridgway, Schwartz, &Wilcox, 1999) and performance tasks (vanTassel-Baska, 2014). In the following sections, the terms are used interchangeably denoting the tasks developed for identification and observation purposes.

A discussion on the various attributes that should characterize problem solving tasks appropriate for the demonstration of mathematical giftedness follows. It should be noted that some of these criteria were based on relevant principles proposed by vanTassel-Baska (2014) for assessing gifted student learning, enriched with supplementary domain-general and domain-specific criteria to capture mathematical giftedness in a coherent manner.

Rapid and accurate computational ability not a prerequisite. A class of problems are arithmetic word problems, in which the ability to make arithmetic calculations, accuracy and speed are assessed. Often, computational accuracy and compliance to taught procedures are overemphasized in class against reasoning abilities associated with high mathematical ability (Ficici & Siegle, 2008). Moreover, computational proficiency is commonly used as the decisive factor in student selections and not necessarily quality of thought (Johnson, 2000). This could be proven perilous in identification processes, since some mathematically gifted students may be good in understanding mathematical concepts but relatively weak in computations (Lupkowski-Shoplik, & Assouline, 1994). Perhaps surprisingly, lists of the characteristics of mathematically gifted pupils in primary schools do not include prodigious skill in numerical calculations. Krutetskii (1976) found that this was not a necessary component of high mathematical ability. Miserandino, Subotnik and Ou (1995) noted that computational errors made by mathematically gifted children do not imply poor understanding. Sheffield (1994), has also remarked that rapid and accurate computational ability is not a prerequisite or sufficient characteristic of mathematically gifted students. Several mathematically gifted students are impatient. Therefore, instead of spending time for computations, they prefer to proceed to the important elements of the problems. A number of them may be charmed by the rush of completing timed computation tests in the minimum time with the maximum score; others lose their motivation due to the low reasoning demands of this type of tests. Hence, problems that require merely calculations to be solved and arithmetic solutions, in contrast to higher level thinking should be avoided during identification and observation of mathematically gifted students. At the same time, in the context of problem solving, accuracy and speed in performing calculations should not be decisive factors in the identification of giftedness.

Look for promise, rather than demonstrated excellence. There is another important aspect that should be considered in regard to the type of mathematical problem solving to incorporate into the identification process. More specific, it is important to have in mind that gifted students may have not yet realized their potential. Thus, problems should be designed in such a way as to look for promise rather than just demonstrated excellence. Except of students that have not yet manifested giftedness, there also students that are underachievers, that is their performance does not reflect their abilities. These students prefer to hide their abilities for different reasons, to be discussed in a later section. Sowel and colleagues (1990) demonstrated the danger of neglecting underachievers if identification relies solely on test results instead of problem solving abilities which are central to mathematics education and giftedness.

Require higher-level cognitive skills. Due to the depth of gifted learners' cognitive abilities, identification instruments for this population should highlight higher-level cognitive skills. In other words, tasks should move beyond simple recollection of knowledge and call for operating at higher levels of application, such as analysis, synthesis, and evaluation (vanTassel-Baska, 2014). Marzano, Pickering, and McTighe (1993) asserted that tasks for gifted learners should require the use and application of additional processes; comparing, classifying, induction, deduction, constructing support, abstracting, decision making, investigation, problem solving and invention. In addition, research findings denote that economically disadvantaged and minority students perform better on tasks that accentuate fluid compared to crystallized intelligence (Mills & Tissot, 1995). Apart from the level of thinking processes required to employ to solve a mathematical problem, the level of cognitive demands of the tasks needs to be considered before designing the tasks.

Challenge with off level non routine tasks. During identification, there are gifted students that may present several of the characteristics of giftedness spontaneously, and students that may demonstrate their abilities only under special circumstances, such as being presented with challenging mathematical situations (Sheffield, 1994). Sheffield's view is supported by an earlier affirmation made by Krutetskii (1976), that giftedness is a special qualitative combination of abilities, unique for each person (Krutetskii, 1976). So, a student may demonstrate only some of the traits and abilities described in research studies, but still be mathematically gifted. Moreover, Diezmann and Watters (2002b) pointed out that gifted students may demonstrate gifted behaviors and engage in productive mathematical activity employing higher-level cognition, only when the problem solving task becomes sufficiently problematic.

Thus, challenging non routine off level tasks are essential to promote the manifestation of mathematical giftedness (Kell, Lubinski, & Benbow, 2013; Thomson & Olszewski-Kubilius, 2014; Warne, 2014), especially for students that they might otherwise hide their abilities or demonstrate only some of them. Mathematical challenge can be defined as an interesting and motivating mathematical difficulty that a person may overcome (Leikin, 2007). It is essential for the realization of mathematical promise and is also a typical type of activity in which gifted mathematicians are involved. Research revealed that gifted students prefer to solve non routine problems because of the challenge these problems offer (Garofalo, 1993). Given their nature, students are not expected to have solved such type of problems previously, nor have they worked on them in the mathematics curriculum (Pativisan, & Niess, 2007). Non routine problems require thinking flexibility and extension of past knowledge, may involve concepts and techniques that will be explicitly taught at a later stage as well as discovery of connections among mathematical ideas (Schoenfeld, et al., 1999). Challenging mathematical problems suitable for gifted students should entail the formation of generalizations. Reaching to generalizations is of vital importance to analogical reasoning, since they provide a means of categorizing problems with similar mathematical structures (Greenes, 1997). From the evidence provided above, non-routine problems are more likely to stimulate gifted students to demonstrate their high abilities in regard to problem solving.

However, the work of any researcher designing identification instruments becomes challenging as well, since such type of tasks should be designed.

Open-ended format/Multiple reasoning methods. Suitable mathematical problems that promote the manifestation of giftedness should be open to interpretation or solution (Greenes, 1997; Peressini, & Knuth, 2000; vanTassel-Baska, 2014). In an open-ended problem, the answer is neither predetermined nor known in advance. Typically, these problems call for experimentation, data collection, and analysis. Among others, open problems with more than one interpretation, require students to identify possible hypotheses, recognize different problems formed according to their selected assumption and then specify the problem conditions before proceeding to the problem solving process (Greenes, 1997). Problems that allow students to demonstrate their potential in mathematics should also encourage a variety of reasoning and solution methods, allowing students to show evidence of flexibility of thinking (Greenes, 1997; Peressini, & Knuth, 2000). According to Greenes (1997), the processes that students may use during problem solving include analogical, inductive, deductive, spatial, proportional and probabilistic reasoning.

Promote the articulation of thinking processes. Researchers and teachers, working with students at all ability levels, have pointed out another important aspect that should be considered during observation of mathematically gifted learner. During problem solving, students should be expected to justify their explanations (Peressini & Knuth, 2000) and are required to provide some evidence of the thinking processes used in obtaining a solution (vanTassel-Baska, 2014). For example, students may articulate their solution method in words, diagrams or symbols. This way, teachers may gain insight into a gifted student's level of ability in a specific domain. Thus, the role of the researcher in charge of the identification process is critical at this point. More specific, the researcher should employ different instruments and means to record students' verbal explanations, drawings, actions, or even gestures so as to form as much a detailed image of the student's thinking processes at the end of the process. In addition, tasks should emphasize the expression of the thinking processes employed, such as metacognition (vanTassel-Baska, 2014). This will also allow the research to observe possible self-regulatory strategies employed by the learner during problem solving.

Assess spatial ability. Problems involving spatial thinking are an important type of problems to be used to promote the manifestation of mathematical giftedness, as well as otherwise misidentified gifted students. Webb, Lubinski and Benbow (2007) stressed that spatial ability is important for talent identification and should not be neglected as in previous talent identification searches, as for example in the American Talent Search for Intellectual Precocious Youth. According to the same researchers, it is possible to identify a neglected number of mathematically gifted students, if focused on spatial ability.

This perspective was also supported by Shea, Lubinski and Benbow (2001) who highlighted the significance of identifying spatially gifted students, although it can be a challenging process (Olenchak & Reis, 2002). It is of particular concern that spatially gifted children may not be identified by current practices (Shea et al., 2001), as the majority of the commonly used achievement tests as assessments in schools do not include a nonverbal component or do not evaluate spatial ability at all (Mann, 2005). Similarly to Mann, Shea and colleagues acknowledged in their study that verbal and quantitative abilities alone, which are the most frequently assessed areas of intelligence, were inadequate descriptors of gifted children:

> An issue of particular concern is the likelihood that some intellectually promising students are not being identified by current practices, because of the lack of attention given to spatial ability.…there are obviously large numbers of "high-space" (i.e., spatially talented) students who do not meet the minimum math or verbal criteria for participation in talent searches….selecting for the top 3% of verbal-mathematical ability will result in the loss of more than half of the students representing the top 1% of spatial ability! (Shea et al., 2001, p. 612)

Despite the abovementioned assertions, some of the widely used identification instruments, include a spatial thinking component to capture spatially gifted students. For example, the Wechsler Intelligence Scale for Children- Third edition (Wechsler, 1991) contains a block design subtest that may assess spatial ability. However, research has shown that students with spatial strengths that take the WISC tool, present discrepancies between scores on the WISC subtests (Mann, 2005). These discrepancies reveal that the student may have a strength in one area, while at the same time outshined by a weakness in a different area. As a result, the resulting lower Full Scale score on the WISC deprives these students of gifted services they would be otherwise entitled to. To prevent this phenomenon and avoid denying gifted provisions to students with spatial strengths, the revised fourth edition of the WISC tool (Wechsler, 2003) delivers four additional indices in Verbal Comprehension, Perceptual Reasoning, Working Memory, and Processing Speed. Block design is included in the subtests that comprise the Perceptual Reasoning Index in consort with Matrix Reasoning, and Picture Concepts. To avoid such problems, a new domain-specific identification instrument is required, that will incorporate spatial tasks among other to identify mathematical giftedness. Spatial tasks should be carefully assessed in a way so as not to exclude spatially gifted students from subsequent nurturing options.

Assess mathematical creativity. As suggested by the literature, mathematical creativity is another important dimension that should be incorporated into the identification of giftedness in mathematics (Renzulli, 1978). The assessment of creativity in mathematics could be made following one of the two major approaches, as mentioned by Pelczer and Rodríguez (2011): problem solving and problem posing tasks. With respect to mathematical problem solving tasks, Polya (1973), Ervynck (1991) as well as Silver (1997), they all pointed that an expression of creative thinking is reflected in problem solving. The importance of both problem solving and problem posing and its relation to mathematical creativity was stressed by Jensen (1973), who affirmed that in order for a student to be creative in mathematics, posing of mathematical questions, this exploring and extending the original problem, except for the ability to solve problems in multiple ways.

Researchers on the development of mathematical creativity (Leikin, Levav-Waynberg, & Guberman, 2011; Levav-Waynberg, & Leikin, 2009) used their proposed model to evaluate the development of creativity in mathematics using instruction with multiple solution tasks. These studies have shown that due to the instruction using mathematical multiple solution problems, students' flexibility and fluency significantly increased, whereas students' originality decreased non-significantly. As a result, a non-significant decrease in the creativity was reported. The phenomenological paradox of the decrease in originality was explained by the researchers assuming that when students' flexibility increases, more students in the sample generate more solutions and thus, it becomes more difficult to construct a unique answer. Subsequently, since originality appeared to be the strongest component in determining creativity, the researchers assume that from the three components, fluency and flexibility could be considered as dynamic, whereas originality could be considered as a "gift". To conclude, mathematical creative tasks that offer the possibility to examination the originality of ideas and solutions, should be included as a means of identification of gifted students in mathematics.

To sum up, it is our belief that when students are provided with appropriate opportunities to solve challenging mathematical problems, they will be triggered to demonstrate several of the behaviors mentioned and reveal their potential. Several principles should guide the design of the tasks to be part of an identification system of mathematical giftedness. First, rapid and accurate computational ability should not be a decisive factor for gifted learners' selection. Rather, higher level reasoning processes should be valued and fluid intelligence should be accentuated. Tasks should be designed so as to look for mathematical promise, rather than demonstrated excellence. In the context of problem solving, problem conditions should be open to interpretation or solution, with the answer not being predetermined nor known in advance. Problems should also require the use of multiple reasoning methods. At the same time, tasks should emphasize the articulation of the thinking processes employed, as well as self-regulatory strategies. In the context of identification, there should be tasks that assess spatial ability and mathematical creative thinking.

Combined together, the aspects that should characterize tasks for the identification of giftedness, as suggested by previous research will guide this study. Still, a number of problems may impede the identification process of giftedness. These issues are discussed in the following section.

Challenges in the Identification Process

There are many challenges during the design of an equitable and defensive identification system for any type of giftedness, with a number of issues pertaining to the identification of gifted students (Van Tassel-Baska, 2005), such as the underrepresentation of students from different groups, a mismatch between objective, identification and services and underachievement. The following sections present a discussion on the problematic nature of underrepresentation of students from several groups and underachievement followed by a discussion of the danger of experiencing a possible mismatch between objective, identification and services.

Underrepresentation of students from various groups

The issue of underrepresentation is of critical importance to the field of gifted education. Researchers, policy makers and educators should reflect on the possibility that a number of students with needs to differentiated educational opportunities do not receive them, by reason of racism or classism (McBee, 2010). Despite of the vital importance of this issue, it remains poorly understood (McBee, 2010).

According to Frasier (1997), "there is no logical reason to expect that the number of minority students in gifted programs would not be proportional to their representation in the general population" (p. 498). Still, literature on gifted education has identified in many occasions the phenomenon of the numerical underrepresentation of African American, Hispanic, and Native American students in gifted programs (Bernie & Beilke, 2008; Ford, 1998; Naglieri & Ford, 2005; Reid, Romanoff, Algozzine, & Udall, 2000). Yet, only a small number of published studies have adequately addressed the complexity of the topic (McBee, 2010). At the same time, the majority of the literature focused on two aspects that refer to student' individual characteristics. Namely, having a heritage of ethnic minority and a low socioeconomic background were examined.

With regard to the underrepresentation of ethnic minority students, research has investigated the topic of teacher nominations during gifted identification processes. Specifically, since the majority of teachers have White middle-class backgrounds, they may possibly not consistently distinguish the indications of giftedness in students of different cultural backgrounds, thus making nomination procedures unfair (McBee, 2010; McBee, 2006; Moon & Brighton, 2008). In addition, McBee (2006) provided evidence that the nomination phase preceding the testing phase of the gifted identification process may be far more biased than the testing phase. Specifically, in this study McBee found that the probability of nomination for Black students was only 31% as large as the probability for White students, where the pass rate for the testing stage for Black students was 82% as large as the pass rate for White students.

Also, minority students have been reported to significantly underperform on standardized tests in comparison to their peers (e.g. Lewis, Decamp-Fritson, Ramage, McFarland, & Archwamety, 2007; Maker, 1996). Possibly, this phenomenon occurs because this type of tests may be biased in favour of students from the prevailing culture (Ford, Harris, Tyson, & Trotman, 2002). That is, standardized tests might assign minority students lower scores for the same level of ability compared to their peers. From the other hand, there are empirical studies that although purported to identify bias in ability tests, they generally failed (Edwards & Oakland, 2006; Hunter & Schmidt, 2000).

With regard to the underrepresentation of students with low socioeconomic status, the research literature is scarce (McBee, 2010). However, since low socioeconomic background has been linked to affect school performance, one may presume that lower achievement may as well lower the probability of being accepted for gifted provisions (McBee, 2010). Low socioeconomic status has been reported to have a negative effect and thus reduce student achievement in a large degree (Portes, & MacLeod, 1996; Ryan, & French, 1976). Since students from families with low socioeconomic status are likely to underperform, they are also less likely to be identified or nominated for gifted provisions (Stambaugh, 2007; Swanson, 2006). By definition, families with low socioeconomic status lack resources (McBee, 2010). Specifically, Duncan and Brooks-Gunn (2000), point that these families are more likely to live in houses of poor quality, receive reduced medical care, be deficient in healthy foods, live under more stress, and live in high crime areas. Thus, these environmental variables, may affect the manifestation and development of giftedness, as discussed in other sections of the present study.

In a recent study (Peters, & Gentry, 2010), the researchers developed a teacher nomination scale, named Hope Scale, aimed to identify low income elementary school gifted students. It was intended that teachers can use the Hope Scale to rate specific social and academic behaviours of their students. Findings from this study revealed that this instrument was not biased against low-income students as rated by their teachers, thus that the social and academic scales provided similar information with regard to students in both income groups. Still, mean scores for student with low socioeconomic status were lower than their counterparts in both scales. Thus, the researchers still need to norm the Hope Scale on the specific groups for which the scale was designed. This conclusion on the need to norm instruments on the specific groups according to the instruments' intended use, may be generalized to all instruments for gifted identification of minority and low socioeconomic status students.

There are also research studies providing evidence that the gap between the test scores of students with low and average socioeconomic background is larger in case of the verbal subscales on many mental ability tests (Mills, & Tissot, 1995; Tyler-Wood, & Carri, 1993). Thus, there is a tendency nowadays in the field to suggest the use of nonverbal instruments in cases of minority students, students with low socioeconomic status or non-native language speakers (Naglieri & Ford, 2003).

Underachievement

In general, the identification process should have an inclusive rather than an exclusive character. What Birch (1984) refers to as a "narrow identification" should be avoided by ensuring that the identification process is not biased in any way by gender, race, colour, socioeconomic background, physical disability or geographical location. The diversity of the population of gifted students makes it difficult to design an identification process that will not exclude the hard-to-find gifted students, who are underrepresented in programs for gifted students and sometimes are denied services (Coleman, 2003). A part of the population of gifted students consists of the underachievers; that is students whose achievement as shown at school does not reflect their abilities and potential. This group may include culturally different students, students from families with low socioeconomic status or disabled students.

The problem of underachievement is evident in the area of mathematical giftedness as well. Despite the fact that researchers and educators mostly agree that gifted students have special needs, they deserve particular attention and require a different teaching approach, a large number of gifted students in mathematics may not be identified and their needs may be not met in regular classrooms; in fact, they may even experience problems conforming to school routine and, in some cases, become underachievers (Freiman, & Rejali, 2011).

As a result, these neglected students may not be given the opportunity to realize their potential, resulting to talent loss. Leikin (2009a) articulated that the only way to prevent talent loss in mathematics is to apply the equity principle in mathematics education, asserted in 1989 by the National Council of Teachers of Mathematics (NCTM) as the basis for mathematics education. According to this principle, "all students, regardless of their personal characteristics, backgrounds, or physical challenges must have opportunities to study – and the support to learn – mathematics" (Leikin, 2009a, p.387). Parenthetically, Karp (2009) further argues that historically, there has been a shift in the educational system. In particular, the education has changed importantly since it was first available solely for individuals that were members of specific social groups, then shifted to more inclusive professional education, to conclude to education available to all students based on the equity principle. Nonetheless, there was a period when people (mis)interpreted the principle as if it meant that all students should be provided with identical instruction, thus offered the same educational opportunities (Karp, 2009). The adoption of a "one size fits all" approach to education, as if all students were placed on a Procrustean bed was clearly not the right type of provisions for gifted students. Fortunately, later on there were concerns that the equity principle was misinterpreted. This concern was also evident in the field of mathematics education and was expressed through the NCTM (2000) standards. In the standards, it was obvious that there was an effort to highlight the importance of tackling individual student differences in mathematics and making suitable modifications for catering the needs of all students, re-conceptualizing the equity principle: "Equity does not mean that every student should receive identical instructions; instead it demands that reasonable and appropriate accommodations be made to promote access and attainment for all students" (NCTM, 2000, p. 12).

Another facet of the underachievement exists. Namely, the reliance on nontraditional measures reveals the perception that minority students and economically disadvantaged students cannot perform well on traditional instruments or in traditionally school-related areas of performance and therefore we should use other measures (Callahan, 2009). To set new directions, a new commitment of the field of gifted education is required. In Callahan's words, "first, although we must acknowledge that performance on standardized instruments may be lower for particular groups of students, we can begin to examine indicators of success by looking for the highest scorers with a subgroup on traditional assessments and recognizing that such high potential given the social and economic barriers these students face is an indicator of gifted behavior" (p.241).

Findings on the overlooking of underachievers, addresses the need for future research to focus on the identification of mathematically promising students that will go unnoticed unless they are identified.

Mismatch between Objective, Identification and Services

According to Reis (2004), before undertaking any identification process, the first issue to be considered is the purpose of the identification. In this study, the focus is on the identification process for identification of giftedness in mathematics for particular subsequent in-school provisions. Mathematically gifted students can be identified through a variety of methods. As discussed in the previous section, a combination of methods and instruments is the best strategy, considering that the information collected is directly related to the concept of giftedness, and the information is inter-related (Davis & Rimm, 2004).

Coleman (2003) pointed out the possibility of experiencing a mismatch between identification and services, which should be avoided at all times. Moving a step further, the findings of Touron, Reparaz and Peralta (1999) showed the need for the development of test batteries that allow the identification of giftedness in a specific domain and therefore guide the decisions about educational provisions for this group of students. In particular, in case of identifying mathematically gifted students, the identification should focus on math abilities, performance and interest (Coleman, 2003). Conversely, given the range of the individual behaviours and profiles of mathematically gifted students, we should keep in mind that they are not a homogenous group. Although they share a group of similar behaviours, these are not universally shared by all (Bicknell, 2008). In addition, school success in mathematics as reflected in school marks does not entail the presence of mathematical ability, while equally at the same time students that do not succeed in mathematics at school are not necessarily mathematically incapable (Freiman, 2004).

Another controversy in the field of gifted education is the difference between literature and practice. Namely, although the existence of qualitative differences between gifted and non-gifted persons is supported in the literature (e.g. O' Boyle, 2000; O'Boyle, Benbow, & Alexander, 1995), the majority of identification instruments are based on quantitative indicators (Worrell, 2009). Despite contemporary multidimensional definitions of giftedness, narrow identification processes are still employed (Milgram & Hong, 2009). Johnson (1983), Kulm (1990) and Young and Tyre (1992) suggest that instead of just focusing on final, well-formed ability or performance, an appropriate identification method should be designed with the objective to expose the extent, complexity, and functional characteristics of mathematical reasoning. That said, the quality of mathematical reasoning is what sets apart a gifted from a non-gifted child in mathematics (Freiman, 2004). In the same line, Greenes (1981) acknowledges that mathematics programs tend to place larger emphasis on the development of computational skills, thus consequently measuring a students' ability according to successful performance on computations and algorithms, instead of observing students' high order reasoning skills.

Finally, through the instruments selected, the identification process should allow the recognition of mathematical strengths that may not be immediately apparent (Davis, & Rimm, 2004) or as Zollman (2008) points out, "identifying mathematical talent involves identifying not so much that which already exists, as that which might yet come into being", emphasizing the notion of potential for gifted students. Promise in math is not directly evident from students' achievements on standardized tests, nor is it related to the interest, effort or excitement shown during math teaching (Hoeflinger, 1998). For all these reasons, teachers' knowledge regarding the characteristics of mathematically gifted students and the context in which their giftedness might be exhibited is crucial. The teacher's role in the identification process of students gifted in mathematics is crucial. For instance, Kennard (2001) asserts that the provision of challenging material and forms of teacher-pupil interaction is of vital importance for students to demonstrate their abilities.

At the same time, contributions of several researchers (e.g. Gardner, 1983, Renzulli, 1978) suggested a shift from the concept of "being gifted" to the development of gifted behaviours in persons who have the potential to benefit from appropriate educational opportunities. Embracing these thoughts, Brown et al. (2005) suggested a shift from strict identification procedures that result to a permanent selection of students for gifted programs with the belief that they are and they will always be "the gifted", to a more flexible approach.

When we refer to giftedness in children, both Gagné (2004) and Renzulli (2002), emphasize that in order to observe giftedness, there is a need to have a supportive environment that allows children to display their abilities. Such an environment in mathematics is one that allows student to engage in meaningful problem solving challenging tasks. This problem solving context might be utilized as a means of identification, as explained in the previous section presenting the identification of giftedness in mathematics.

Summary

The review of the research on general giftedness, giftedness in mathematics, issues with regard to the identification of both constructs, as well as challenges in the identification process, reveals the need to search for the factors, abilities and processes that constitute mathematical giftedness, which will provide the elements to create an identification process for mathematical giftedness and a theoretical model to clarify the construct.

Due to the focus of research studies in the field of giftedness to examine general giftedness rather than domain-specific giftedness (Leikin, 2009a), there is limited focus on theoretical models of mathematical giftedness as well as specially designed procedures and instruments for students' identification. This study, comes to fill this gap by suggesting a new theoretical model that will contribute to the clarification of the concept of mathematical giftedness and a specially designed identification process accompanied by specially designed identification instruments. Talent in mathematics was asserted as one of the most broadly required resources for the 21st century (Office of Science and Technology Policy, 2006), proclaiming the importance of nourishing mathematical giftedness. Given the widening international interest in research in the field of giftedness (Leikin, 2009, 2011), it is surprising that there is a lack of consensus on the definition of giftedness among the researchers. As a result, the field of research is problematic. In the words of Sternberg (2004), "just as people have bad habits, so can academic fields have bad habits. A bad habit of much of the gifted field is to do research on giftedness or worse, identify children as gifted or not gifted without having a clear conception of what it means to be gifted" (p. xxiii).

Early studies of Terman (1925) and Hollingworth (1926) provided a solid foundation for subsequent research efforts in the field of general giftedness. More recently, Gardner (1985), Sternberg (1985), Renzulli (1978), Heller (1993), Gagné (1985), Milgram and Hong (2008) have made significant research contributions in the area, by proposing theories and models to explain general giftedness as well. However, giftedness in mathematics denotes different abilities than those included in models of general giftedness or models describing domain-specific giftedness other than mathematical giftedness (Benbow & Minor, 1990; Csikszentmihalyi, 2000; Clark, 2008; Rotigel & Lupkowski-Shoplik, 1999). Although the reformulation and application of specific models, processes and provisions targeted for mathematical gifted pupils is required, there is a gap between the field of gifted education and mathematics education. This was clearly illustrated by Leikin (2011) stating that "mathematics education is underrepresented in the field of gifted education and, vice versa, the research on giftedness and gifted education is underrepresented in the field of mathematics education" (p.168). Thus, this study comes to create a link to bridge the gap between the two disciplines.

Hence, subsequent research efforts aiming to define mathematical giftedness such as this study, should isolate and test from these general models, only the aspects that apply to mathematical giftedness as well. For the purposes of this study, we isolated and adopted five aspects: (a) the multidimensionality of the construct of giftedness with subsequent impact on the design of identification processes, (b) the notion of potential or promise as an important dimension of giftedness, (c) the existence of domain-specificity of giftedness, (d) the important role of creativity in giftedness and (e) the use of terms of promising and gifted students interchangeably, as the gifted population included both students whose promise has not yet manifested and also students with their abilities demonstrated.

Research has shown that mathematically gifted students are characterized by a special blend of problem solving (Krutetskii, 1976; Sowell, Zeigler, Berwall & Cartwright, 1990), spatial (Benbow & Minor, 1990; Block, 1985; Sowell, Zeigler, Berwall & Cartwright, 1990) and creative abilities (Geake & Dodson, 2005; Kanevsky & Geake, 2005; Livne & Milgram, 2006; Milgram & Hong, 2009; Renzulli, 1978; Sriraman, 2005; Sternberg, 1999). Except of cognitive abilities, mathematically gifted students' performance depends on hypercognitive (Borovik & Gardiner, 2007; Calero, García-Martín, Jiménez, Kazén, & Araque, 2007; Renzulli, 1978; Sternberg, 1986), affective and motivational characteristics (Koshy, Ernest, & Casey, 2009; Vlahovic-Stetic, Vizek Vidovic, & Arambasic, 1999), as well as environmental ones (Gagné, 2009; Heller, 2004; Hong & Milgram, 2008; Reichenberg & Landau, 2009). Given these findings, the nature of mathematical giftedness should be elucidated, by outlining the abilities of mathematically gifted students as well as the cognitive and hypercognitive processes related to mathematical giftedness. A concern by VanTassel-Baska (2006) identified that researchers in the field have conducted research in a number of directions, whist at this point, research studies should be related and compared to one another, resulting to a foundation for sound generalizations and conclusions in terms of theory and practice. For this reason, this study comes to synthesize research findings in the field of general and mathematical giftedness, design and empirically assess an identification process of giftedness in mathematics, aiming to allow the manifestation of mathematical potential. Especially, this study focuses on cognitive and hypercognitive processes of mathematically gifted students that will be possible to be closely examined through the careful observation of the students during solving challenging tasks. To be able to conduct a thorough analysis of students' work, we had to focus on a limited number of variables. Thus, affective and motivational as well as environmental characteristics will not be investigated in this study.

This could be made possible, by designing and testing a domain specific identification process devoted to mathematical giftedness, assessing abilities and processes. But first, topics that have not received adequate research attention in the past years with regard to identification should be reconsidered. For example, Van Tassel-Baska (2006) raised the issue of a limited number of identification studies that prove the efficacy of identification processes implemented in school settings, not just examining the significance of individual tests. Perhaps this is the reason that Heller and Schofield (2000) express the belief that ''there is still a lack of good quality basic research being undertaken in the area of giftedness'' such that ''the very nature of giftedness is still open to question'' (p. 134).

To capture the nature of giftedness using good quality research, previous studies provide evidence that any identification process of giftedness should be based in multiple sources of evidence. More specific to the use of testing, the review of the research literature points to the use of aptitude tests (Matthews & Foster, 2005; Miller, 1990) and creativity tests (Kaufman, Plucker, & Russell, 2012) as the most effective in identifying giftedness. In addition, research has shown that observing students during rich problem solving consists of one of the most efficient ways to capture the manifestation of mathematical giftedness and potential (Bicknell, 2009; Koshy, 2001; Koshy, Ernest, & Casey, 2009). Thus, the proposed identification process of mathmetical giftedness will follow these guidelines and incorporate mathematical testing and observation of problem solving in challenging tasks. The decision to investigate the thinking processes of gifted students in mathematics, comes also to respond to the problem adressed by Leikin (2011), that a number of studies accentuate predominantly general psychological traits, whilst they do not consider the learning and thinking processes of gifted students in mathematics in accordance with contemporary theories of mathematics education.

To develop taks for the identification of mathmetical giftedness, research suggests several principles, that this study will follow. More specific, rapid and accurate computational ability should not be a determinant factor in excluding students (Sheffield , 1994), whilst tasks should be designed in such a way as to look for promise (Sowell, Zeigler, Berwall, & Cartwright, 1990), rather than demonstrated excellence. Moreover, tasks should require higher-level cognitive skills (vanTassel-Baska, 2014), provide challenge with off level non routine tasks (Diezmann, & Watters, 2002b; Kell, Lubinski, & Benbow, 2013; Sheffield, 1994; Thomson & Olszewski-Kubilius, 2014), allow and promote the use of multiple reasoning methods (Greenes, 1997; Peressini, & Knuth, 2000; vanTassel-Baska, 2014), promote the articulation of thinking processes (Peressini, & Knuth, 2000; vanTassel-Baska, 2014), assess spatial ability (Mann, 2005; Olenchak & Reis, 2002; Shea et al., 2001) and mathematical creativity (Renzulli, 1978).

Given the discussion on the perils pertaining any identification process related to giftedness, we acknowledge the challenge of designing an identification system that will avoid the danger of underrepresenting students from various groups (Coleman, 2003), as well as excluding underachievers (Freiman & Rejali, 2011). A third challenge is to avoid a discrepancy between the objective guiding the identification system, the identification process followed and the content and nature of subsequent services provided (Coleman, 2003; Heller, 2004; Nevo, 2008).

Taken all these issues into consideration, this study aims to design an identification process of mathmetical giftedness that will prove its efficacy implemented in real school settings through good quality basic research, not just examining the significance of individual tests. This effort comes to respond to the relevant concerns of Heller and Schofield (2000) and Van Tassel-Baska (2006). Research in the field suggests to target students of early age, to prevent talent loss which is considered to be one of the major challenges facing parents and educators nowadays (Hong & Milgram, 2008). At the same time, the age group for which the identification process is designed should be considered, since it is awaited to observe the articulation of student thinking processes. To do this, the students should be of an appropriate age to be aware and express their reasoning processes.

CHAPTER THREE: METHODOLOGY

Introduction

The purpose of the present study was to investigate the construct of mathematical giftedness and suggest an instrument suitable for the identification of mathematically gifted students of 5th and 6th grade in elementary school settings. Furthermore, it was the researcher's intent to delineate students' mathematical abilities and investigate their cognitive and hypercognitive processes while engaged in challenging mathematical problem solving tasks.

This chapter presents the research methodology and methods employed for collecting and analyzing the data required for this study. To set the framework under which the research phase was organized, the first section presents the research design and procedure applied in this study. The next section describes the setting and the participants for this study. The third section, presents the instrumentation developed and used in the study. In particular, it presents the test for measuring mathematical problem solving abilities for group administration and the philosophy for the design of the test with mathematical challenging tasks for individual administration as well as the content of the instrument. The chapter also includes an outline of the type of data collected and the methods of statistical analysis employed for answering the research questions of the study.

Research Design and Procedure

For the purpose of this study, a special four phase identification process was designed and implemented for fifth and sixth grade students gifted in mathematics. A structure of the research procedure followed in this study, divided in four phases, is presented below in Figure 6.

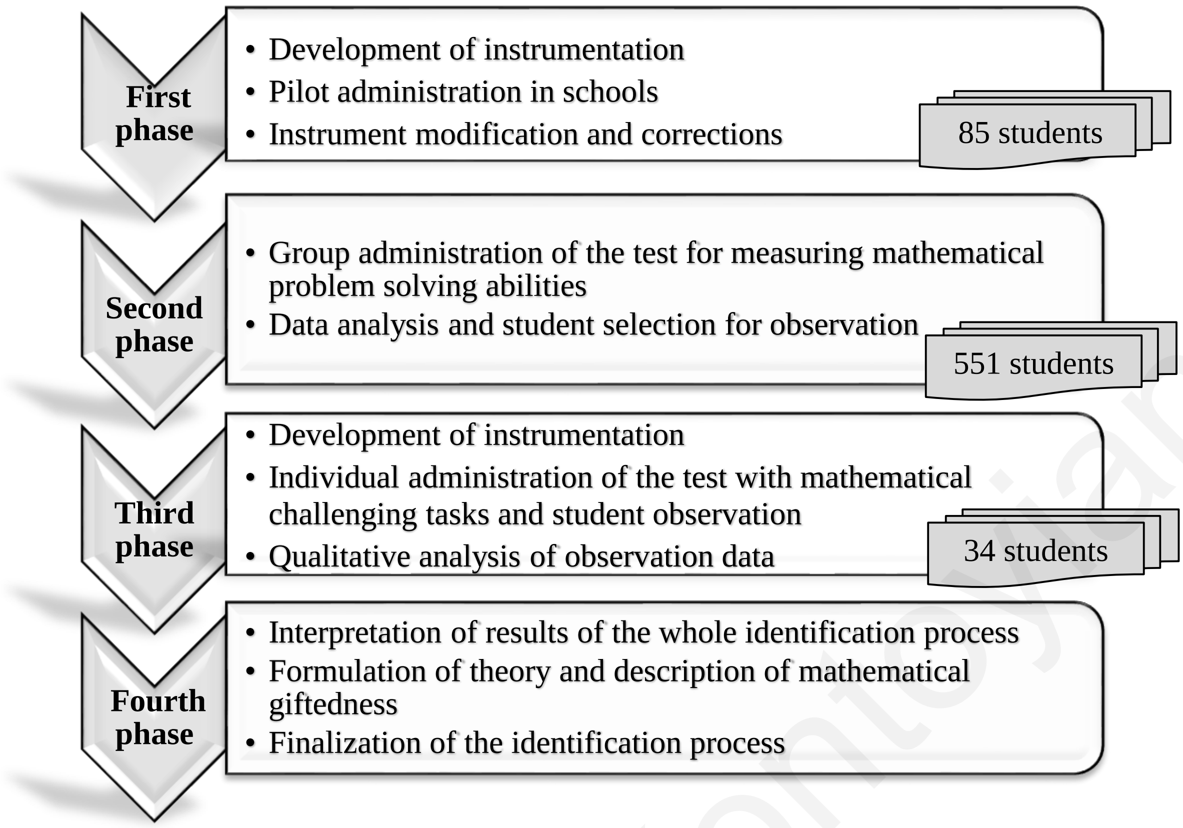


Figure 6. Structure of the research design and procedure followed in the study.

First phase

The first phase included three stages: (a) development of the identification instrument development, (b) pilot administration in schools and (b) instrument modification and corrections. More information about each stage follows.

Initially, the first version of the test for measuring mathematical problem solving abilities was developed to be pilot tested. The content of the instrument as well as the development process will be presented in a later section discussing the instrumentation used. Before proceeding to data collection, the researcher followed the required process to acquire the permission of the Centre of Educational Research and Evaluation (CERE) to conduct the study, which is governed to the Ministry of Education and Culture of Cyprus. At the second stage of this phase, the first version of the mathematical instrument was pilot tested by the researcher in two 5th and two 6th grade classes during December 2011. More specific, from the 85 students that completed the pilot version of the identification instruments, there were 42 fifth graders whereas the remaining 43 students were sixth graders. All participating students obtained the written consent of their parents to participate in the study. The students worked individually to complete the problems included in the test and were allotted 80 minutes (two 40 minute periods) to do this.

At the third stage, the researcher took into consideration student's responses, reaction and comments while solving the items included in the identification instrument during the pilot administration. Accordingly, some modifications to the language and task instructions were made in order to make activities better accessible to students, resulting to the final version of the mathematical instrument.

Second phase

The second phase of the study included two stages: (a) administration of the final version of the mathematical instrument to all participating classes, and (b) analysis of the data collected leading to the selection of a group of students to proceed to the third phase.

During this phase, the final version of the identification instrument was administered to 51 participating classes after obtaining the written consent of parents for their children to participate in the study. More specific, 25 fifth grade and 26 sixth grade classes participated. The test administration took place during February-April 2012. The instrument was administered by the teacher of each class. There were no concerns with respect to objectivity since test administrators, the teachers in this study, were provided with detailed instructions. The test was designed to be suitable for application in groups so that the instructor-testee-interaction is reduced to a minimum. All students completed individually the final version of the identification instrument. Students were allotted 80 minutes (two 40 minute teaching periods) to complete the test.

Next, data collected consisting of students' written responses in their worksheets were coded and analyzed. After measuring the performance of students in the screening instrument, the researcher ranked order the students based on their performance. From the total group of children screened, the children who had attained a score in the top 5% on the screening measure were selected as mathematically gifted. This selection was based on the admission procedures used in many university-affiliated talent searches, such as The Center for Talented Youth at Johns Hopkins University, The Talent Identification Program at Duke University, The Center for Talent Development at Northwestern University, and The Rocky Mountain Talent Search at the University of Denver, were admission standards for entering the talent search require a 95–97 percentile rank score on an achievement test (Lohman, 2005). Since it was the objective of this study to observe striking characteristics of mathematical giftedness, it was our decision to be strict in our selection in order to capture as much intense characteristics as possible, and at the same time allow for inclusive identification. Taking into consideration that Heller (2013) refers to the top 3 to 5% as highly gifted, our decision leaned towards selecting the 5% of the students. Since there were some students with the exact same score, the selected students came up to 6.35% of the total population under examination in order to avoid denying access to the second instrument to students obtaining the exact same score with the last student according to the 5% criterion. The maximum score obtained in the test was 12.76 out of 13. The score obtained by the last student eligible to proceed to the second phase was 9.75 out of 14.

The final sample of students proceeding to the next phase after being identified as mathematically gifted, included 35 students, to be exact 11 5th and 24 6th graders.

Third phase

The third phase of the study included three stages: (a) Development of instrumentation, (b) Individual observation of students by the researcher while engaged in challenging mathematical tasks and discussion and (b) qualitative analysis of the observation data.

At first, the researcher developed the test including mathematical challenging tasks, intended for individual administration with gifted students in mathematics selected from the second phase. The content of the instrument will be presented in a later section discussing the instrumentation used. Two elementary school teachers were requested to review the tasks and comment on their level of difficulty and estimation of time required for high ability 5th and 6th grade students to complete the test. The teachers' comments and suggestions for modifications were considered. The suggestions made were with regard to the phraseology of the tasks.

During the second stage, the researcher communicated with the parents of the 35 students and informed them about the next phase of the study. Thirty four out of 35 students were willing to participate. To this end, meetings were scheduled during July-August 2012. The appointments were scheduled at students' homes in a quiet room, where students felt more comfortable, in a one-to-one setting between the participant and the researcher. Although not following strict duration limits, the researcher informed parents during their phone communication that the whole process was expected to take about two hours.

The same interview process was followed for each of the 34 students. Each participant was required to solve six complex mathematical problems using the think aloud method. At the beginning, the researcher provided each student with general instructions, describing an overview of procedures for the participant to follow including the think aloud protocol. Next, the participants received the test and were told that they could freely move from one activity to the other, allocating their time as they wished. Participants were expected to speak aloud describing their thinking while writing their solutions on the paper. Each interview was intended to elicit the participants' thinking and hypercognitive processes throughout problem solving. For this reason, the researcher had a number of questions for the students, in order to help them exhibit their reasoning methods. Examples of such type of questions were:

- How did you start the problem?
- Did you use a known procedure to solve this problem?
- Can you justify your answer? Please explain your reasoning.
- Is the mathematical relation you identified valid for any number or just for a subset of them?
- How can you be sure that your solution is correct?
- How would you explain your solution to a friend?

The meetings were videotaped and in addition the researcher kept field detailed notes about students' behaviors, processes, actions, even gestures. At the end of each appointment, the researcher reminded the participant that he/she was not to disclose the nature of the problems with other participants that he/she may knew.

At the second stage, the data obtained through the one to one meeting between researcher and students were analyzed qualitatively in an effort to delineate abilities possessed by mathematical promising students as well as specific cognitive and metacognitive processes employed during rich problem solving. The data consisted of students' written responses in their worksheets and transcripts of the conversation between researcher and students, as well as students' explanations to the responses provided.

Fourth phase

The fourth phase of the study included three stages: (a) Interpretation of results of the whole identification process, (b) Formulation of theory and description of mathematical giftedness and (c) Finalization of the identification process.

During the fourth phase, findings acquired both from the group and individual administration of the two mathematical instruments were discussed and interpreted. Exploiting the results and conclusions obtained from data analysis, a theory leading to the contextualization of mathematical giftedness was formulated and research questions of the study were answered. In addition, the identification process as well as the identification instruments were finalized.

Participants

The intent of the research was to find efficient and methods to identify students with mathematical giftedness at the elementary school level and to contextualize mathematical giftedness as it is manifested in this age. The fifth and sixth grade level was selected by the researcher because these students are judged old enough to have some insight into the way they acquire and process information and are able to verbalize their thinking processes.

The sample of this study for the group administration comprised of 551 students from 21 elementary urban and rural schools in Cyprus. In particular, 289 5th and 261 6th grade students participated in the study (age 10 8/12 to 11 8/12 and 11 8/12 to 12 8/12 years old respectively). Thirty five students, of which 11 5th graders and 24 6th graders were identified as students with mathematical promise and were selected to proceed to the third phase of the study, which was the individual observation during rich mathematical problem solving.

Instrumentation

The interest of the study was directed to the identification and manifestation of mathematical giftedness in the elementary mathematics classroom and not general giftedness. Thus, special attention was paid to the context of selection instruments during the development of the proposed identification system in order to ensure that they highlighted mathematical strengths, cognitive abilities and patterns of behavior of mathematically gifted students and may aid in eliciting students' thinking. In addition the methodology used to collect data was critical since the researcher wanted to produce a detailed picture of students' abilities, as well as cognitive and metacognitive processes employed during problem solving.

Two instruments were developed for this study: (a) Test for measuring mathematical problem solving abilities for group administration and (b) Test with mathematical challenging tasks for individual administration.

Test for Measuring Mathematical Problem Solving Abilities for Group Administration

A test for measuring mathematical problem solving abilities was developed. The mathematical problems for this study consisted of non-routine mathematical problems that were selected and modified from a variety of sources, including journals devoted in mathematics education, web pages and mathematical contests (e.g. Kangaroo and Olympiad mathematical contest items). Thus, by drawing upon and extending problems gathered from various sources, the researcher developed a pool of problems to create a new identification mathematical test.

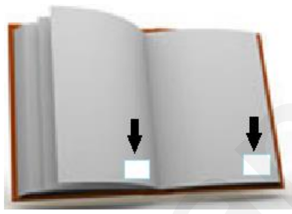
The research field suggests that identification of mathematical giftedness should incorporate aspects of problem solving, spatial development and creativity (Diezmann, & Watters, 1997; Niederer, 2001; Watters & English, 1995). Following these recommendations, the test includes 13 tasks aiming to investigate student abilities in number relations, spatial concepts, and creativity in a mathematical context. Table 1 presents one sample problem from each category. The complete instrument that was used for the purposes of the study can be found in Appendix A.

Table 1

Sample problems from the test for measuring mathematical problem solving abilities

Number Relations

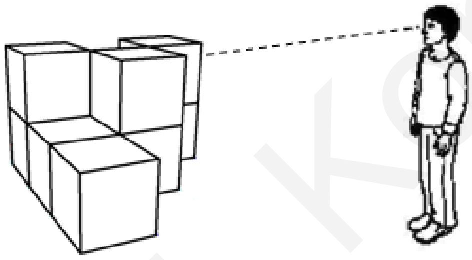
While reading a book, Stella notices that if she multiplies the numbers of the pages in which the book is open in, the unit digit of the result is 6. The book page numbers are two digit numbers. If Stella adds the two page numbers, what is the unit digit of the result?

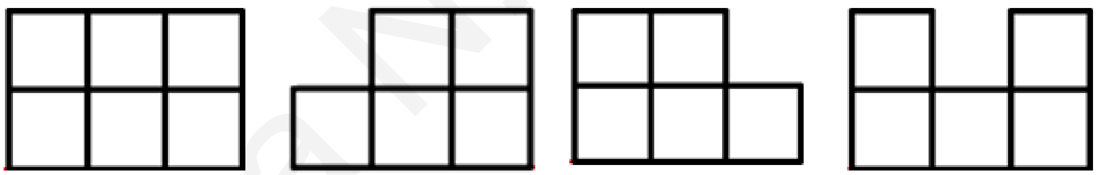


A. 2 B. 4 C. 6 D. 5 E. Other answer

Spatial ability

In this part, a boy is looking towards a solid. The dashed line represents the direction of his sight. The four images below the line present four images; one is the image the boy sees from his position. Circle the right image.

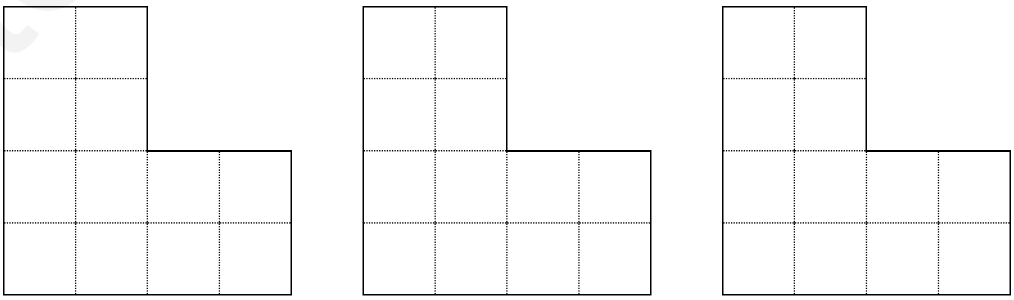




(a) (b) (c) (d)

Mathematical Creativity [a]

Split each boot in four shapes with the same area in as many different ways you can. You may use four colours to show your answer.



[a] Additional figures were provided.

Table 2 presents a categorization of the different types of tasks included in the test.

Table 2

Types of tasks included in the test for measuring mathematical problem solving abilities

Abilities measured	Tasks
Number relations	5, 6, 7, 8, 9, 10, 11, 12
Spatial ability	1, 2, 3, 4
Creativity	13

The items investigating spatial ability were reprinted with permission, since they were used as a part of the instrumentation used in the context of another doctoral dissertation (Pittalis, 2007). The original pool of problems was examined for content validity by four experts with expertise in mathematics and mathematics education. The experts commented on the problems and suggested important feedback for the study.

The majority of the questions, except for the spatial and creative ones, was presented in a ‘‘user-friendly’’ layout, including additional illustrative material, in a five multiple choice format and a box headed ‘‘Show your work’’. The option of presenting the five answer choices in each task was made after careful consideration, since our intent was to draw students’ interest in responding the items of the questionnaire. Since the test level of difficulty was high due to its purpose and given that it would be administered to the general population, we wanted to avoid a large numbers of non-gifted students not approaching the items. Thus the multiple choice format was expected to allow every student to attempt to complete the test.

There are several factors that were taken into consideration in suggesting tasks suitable for measuring mathematical problem solving abilities, to be used to identify mathematical giftedness. These considerations were made following relevant research findings. First, the problems were formed in a way as to challenge students’ thinking processes. Second, the test placed a greater emphasis upon reasoning and learning than on memorization. Third, the type of questions and tasks in the test were selected to be different from ‘‘everyday’’ mathematics questions, since in such a case, the responses of students tend to follow their teaching. If the tasks were similar to those taught in class, then students’ performance would reflect much more on how and how much they have been taught, rather than mathematical reasoning in a novel situation. Thus, the tasks were designed and phrased in a specific way, as questions that are not directly taught in lessons to the specific age group. Although they drew on the elements of the curriculum for the age group, the questions were unlikely to have been rehearsed. In this manner, it was felt that responses to the problems were more likely to be a reflection of qualities in the student rather than the success of teaching. Moreover, it was our objective to identify also promise and underachievers, so these students may be motivated to show their abilities when confronting novel challenging problems out of the level of ordinary teaching, not depending on the taught curriculum. To address underrepresentation of students from different groups, an attempt was made to have short tasks, students with different backgrounds, sociocultural origin, and language or with limited capacity in Greek language would not have problem to deal with. It was not amongst our preferences to turn only to spatial, thus primarily non-verbal tasks to address the issue of underrepresentation, a practice used in intelligence testing, since this choice would limit the range of mathematical abilities to be observed.

To sum up, this type of testing as described previously, was selected for several reasons: it focuses on mathematical reasoning rather than learned mathematical facts from specific curricula and memorization, it is not likely to have a ‘ceiling’ for children at this age, the scoring system allows scores to be compared directly for different test versions or administrations, and it is cost and time effective because it is a group test rather than an individually administered test.

Test with Mathematical Challenging Tasks for Individual Administration

Numerous of the behaviors distinctive of mathematically promising students are apparent whilst the student is engaged in problem solving (Niederer & Irwin, 2001). Thus, mathematical problem solving could serve as a means to identify and describe mathematical giftedness. Mathematically gifted students may be identified through their high levels of reasoning (Sheffield, 1999), given that teachers provide opportunities for them to demonstrate their cognitive processes. This way, mathematically gifted students with unique ways of thinking about and doing mathematics are possible to get identified.

Hence, following recommendations from research literature (e.g. Koshy, 2001; Krutetskii, 1976), it was our intent to give students the opportunity to exhibit their abilities or potential through challenging problems posed to them and observing them through the process. In fact, Hoeflinger (1998) suggested teacher observation of the problem solving process and the surrounding discussion as the most accurate and reliable tool for distinguishing gifted younger students.

To do this, a second instrument was designed for the observation of students by the researcher while engaged in challenging mathematical tasks in a one-to-one setting. Through mathematical problem solving, the researcher may gain valuable insight into a student's mathematical understanding and ability (Bicknell, 2009). Precisely because this stage aimed to record and study student cognitive and hypercognitive processes during problem solving, the selection of mathematical tasks, was important so as to allow a range of behaviors and processes to be observed.

To select the tasks for this instrument, the researcher followed the principles for mathematical problem solving task development, as described in an earlier chapter. More specific, tasks designed for the identification of mathematical giftedness, did not assess rapid and accurate computational ability nor compliance to taught procedures. Rather, tasks extended beyond simple recollection of knowledge and higher level thinking was valued. Thus, gifted learners were required to hypothesize, investigate, apply induction and deduction, make classifications and comparisons, make abstractions and form generalizations, proceed to decision making and apply creative thinking techniques among others to solve the problems.

The tasks also encouraged learners to show their way of organizing relevant information, elucidate their conjecture and justify their answers, in an effort to exhibit their level of mathematical thinking processes. In addition, problems were designed in such a way as to look for promise rather than just demonstrated excellence, to assure the identification of underachievers. This was achieved with the use of non-routine off level tasks. To this end, it was believed that the challenge provided by the tasks and the fact that they differed form everyday problems discussed in class would motivate underachievers to show their abilities. Another parameter for the selection of tasks was the degree to which they were open for interpretation or solution. In such tasks, the answer is neither predetermined nor known in advance. Problems were also selected to encourage a variety of reasoning and solution methods, allowing students to show evidence of flexibility of thinking, a typical characteristic of mathematical giftedness. Moreover, problems were selected to promote the articulation of thinking processes and allow the observation of hypercognitive processes as well. Also, tasks assessing spatial ability were included as well as opportunities to exhibit mathematical creativity.

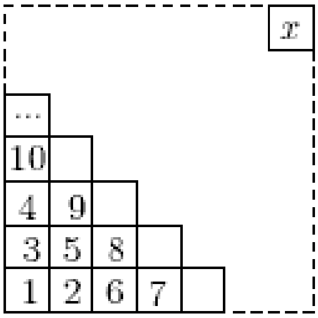
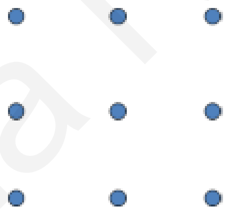
Although a range of properties of appropriate tasks were suggested for identification purposes of mathematical giftedness, another decisive factor for the formation of the test, was the time required for completion. Thus, due to time restrictions, the researcher had to select only a small number of problems for students to work with, whilst at the same time trying to capture mathematical giftedness in a cohesive manner. Accordingly, the researcher had to select a limited number of tasks that in total would satisfy the principles discussed. In the end, after an intensive search of ideas to produce the problems of the test and a vast collection of possible identification tasks, six problems were incorporated in the test for individual administration. A short description of the six activities that were used is presented in Table 3. Table 3 provides the instructions of each of these activities along with a sample of mathematical ideas addressed. The six activities that were used for the purposes of the study can be found in Appendix B.

Table 3

Description of the six activities included in the test with mathematical challenging tasks for individual administration

Activity	Instructions	Sample mathematical ideas addressed
1	Put the following numbers in groups in as many ways as you can. Name each group according to the grouping criterion used. 2, 3, 4, 5, 7, 9, 10, 15, 21, 25, 28, 49 **Restriction verbally explained by the researcher:** Each number should be included only in one group in each grouping attempt. **Modified from:** http://nrich.maths.org/2660	• Number relations and properties (e.g. odd numbers, even numbers, two digit numbers, digit sum, multiples of X) • Inclusion relations • Mathematical creativity

2	Take any two-digit number. Reverse the digits, and add the two numbers. What do you notice? Is it a random number? Can you explain why this happens? **Modified from:** http://nrich.maths.org/7208	• Number relations • Number properties • Generalization • Reasoning and proof • Place value • Multiples of 11
3	Choose any number. Multiply the chosen number by itself. Subtract your starting number. Is the number you're left with odd or even? Can you prove that your result is always true and not just true for the particular number that you chose to start with? **Modified from:** http://nrich.maths.org/8065	• Number relations • Number properties (e.g. square numbers, odd and even numbers) • Generalization • Reasoning and proof
4	Type number 1 000 000 on a calculator. By pressing only the keys 7, +, -, ×, ÷ and = as many times as you wish, reach to number 7. **Source:** http://nrich.maths.org/2649	• Number reasoning • Operations • Reversibility of thinking • Multiples of 7

5	Numbers are placed inside smaller squares, as shown in the picture.  (a)Suggest specific numbers that could be placed in square x and justify your answer. (b) Suggest specific numbers that could NOT be placed in square x and justify your answer. **Modified from:** similar problems included in math contests	• Number relations • Square numbers • Prime and composite Numbers • Geometric reasoning • Area of a square • Properties of a square • Generalization
6	Construct as many different figures of an area of 2 cm^2.The nine dots may guide you in forming line segments. Can you think of figures that would be difficult for some else to think of? [a]  **Modified from:** Haylock (1997)	• Spatial thinking • Geometric reasoning • Area • Measurement • Creative thinking • Fluency, flexibility and originality of ideas

[a] Additional figures were provided and it was verbally clarified that the horizontal and vertical distance between two successive dots is 1 cm.

The purpose of Activity 1 was to find multiple innovative ways to classify a set of 12 integers. Specifically, students had to develop a procedure for grouping the 12 given numbers in different ways, with a specific constraint added by the researcher. That is, a number could be used only once in a specific grouping approach. As such, a particular number should not be eligible to belong to more than one group in each grouping attempt. This restriction increased the difficulty level of the activity and added to the complexity of the proposed solutions. The activity provided opportunities for mathematically promising students to expose the connections they may proceed to in regard to numbers and their relationships. The related mathematical concepts involved in this activity were number relations and number properties such as odd and even numbers, two digit numbers, digit sum, multiples and factors. In addition, this activity was of creative nature and as such students were expected to suggest numerous responses (indicator of fluency), of different categories (indicator of flexibility) and novel ones (indicator of originality). To promote original responses, the researcher triggered students by asking them to think of solutions that another student would not think of. It was of interest to observe the mathematical classification students could come up with and the way in which they would exploit their work in order to produce many different grouping approaches.

The purpose of Activity 2 was to provide an explanation of the properties of the sum of a random two-digit number and the resulting two-digit number after reversing the digits. Specifically, students had to develop a procedure for observing the resulting sums with different starting two-digit numbers and form a generalized argument on the relationship among these sums. In this framework, reasoning and proof were required. Additionally, the activity provided a setting for students to focus and work with the notions of place value and multiples of 11. An important aspect that the researcher was aiming to observe was the process that students would follow in order to arrive to a generalization.

The purpose of Activity 3 was to explain and prove why the difference of a square number with its square root results to an even number. This activity purposely allowed for different levels of response. In fact, this was one of the reasons of including it in the test. In detail, certain students could develop a procedure by experimenting with different numbers and observing the resulting numbers. Other students could immediately think in terms of properties of odd and even numbers to explain the phenomenon without trying out different numbers. The related mathematical concepts appear in this activity were concepts about number properties such as properties involving square numbers, odd and even numbers, whilst the solution required generalization, mathematical reasoning and proof.

The purpose of Activity 4 was to think of and follow a procedure to move from 1 000 000 to number 7 using specific calculator keys. Deliberately the researcher added the constraint of using limited calculator keys, in an attempt to force students to think of a way to organize their thinking and work systematically to suggest a solution. Additionally, the activity provided an excellent situation for students to reveal, among others, evidence of reversibility of mental processes. More specifically, acknowledging that the most effective and time saving approach would be to start from number 7 and move towards 1 000 000 and achieving to find a way to do this, is an indicator of the ability to think reversely. The related mathematical concepts that appear in this activity are numerical reasoning, reversibility of thinking processes and multiples of 7.

The purpose of Activity 5 was to recognize the relationship of square numbers with the area of a square. The presentation of the task purposefully did not make this clear. Specifically, the positioning of succeeding numbers diagonally was aiming to divert students. A number of approaches were possible to be observed. It was interesting to observe whether students would resort to counting the numbers and designing the remaining squares to reach number x, a lower level strategy. A higher level strategy involves taking into account that the number of squares contained in the larger square is basically the area of the square. Thus, rather than counting all the squares a student that conceived this fact is expected to measure the side of the square and calculate the area. Furthermore, a student should relate square numbers with the area of a square forming a generalized conclusion, that will allow for the suggestion of other possible numbers for x. Additionally, a student would then be able to easily suggest numbers that could not be placed in x' s position, in other words any number except of squares. In general, the related mathematical concepts that appear in this activity were numerical reasoning, square numbers, prime and composite numbers, geometric reasoning, area of a square, properties of a square and generalization.

The purpose of Activity 6 was to apply creative mathematical thinking in a spatial context, by constructing as many different figures of an area of 2 cm^2 using a square grid formed by nine dots as guide. Similarly to Activity 1, students were motivated by a direct prompt to think of solutions that another student would not think of, as a way to inspire original responses. This activity was expected to provide many instances of indicators of mathematical giftedness for numerous reasons. First, it was of interest to observe the different figures students could come up. Second, it was our intention to observe the system of figures produced. For example, it would be useful for the research objectives to examine what types of figures would be formed at an earlier stage and whether the subsequent figures would be of a different type. Third, it was of interest to detect the strategy followed by students to keep track of the already formed figures and think of new figures to add to their repertoire of constructed shapes. Fourth, the researcher was keen to examine whether students would prefer to work systematically and would chose to exploit previously formed figures in order to produce many different grouping approaches with minimal effort. A number of related mathematical concepts appear in this activity, such as spatial thinking, creative thinking, fluency, flexibility and originality of ideas, geometric reasoning, area and measurement.

Data Analysis

This section justifies the use of mixed methods of research, combining quantitative and qualitative statistical analysis and how this decision fits the theoretical framework of mathematical giftedness.

Two levels of data were collected for evaluation purposes. The first level of data, to be subjected to quantitative statistical analysis, were the scores from students work in the mathematical abilities instrument. The second tier of performance data came from examining transcripts of video recordings and observation notes taken by the investigator at the third phase of the study, in combination to students' written work on the individual administration of the mathematical tasks and conversation with the researcher.

The statistical techniques for analyzing quantitative data are described first, followed by the description for the statistical techniques for analyzing the qualitative data.

Statistical techniques for analyzing quantitative data

For the analysis of quantitative data, the Mplus statistical package of structural equation modeling was mostly used. In order to answer the question regarding the components and structure of mathematical giftedness, quantitative techniques will be used for analyzing data retrieved from the mathematical test administered to all participants of the study.

To be exact, confirmatory factor analysis (CFA) was conducted in order to investigate the fit of a specific model to the data of the present study, using the statistical modeling program Mplus (Muthén & Muthén, 2010). In confirmatory factor analysis, a proposed model assumes that latent variables corresponding to theoretical construct underlie a set of observed indicators. The relationships that exist within a set of indicators are explained by the covariances between those indicators and the latent variables (Bollen, 1989). Thus, the first step in the CFA procedure starts from a covariance matrix among indicators and decomposes these covariances into the effects of the latent factors upon the observed variables and the random error coefficients (Bollen, 1989).

For testing the fit of the proposed model, the Mplus software uses three fit indices to evaluate the extent to which the data fit the theoretical model under investigation. More specifically, the indices of goodness of fit for the model in which the evaluation of models are based and their optimal values are: (a) the comparative fit index (CFI), the values of which should be equal to or higher than 0.90, (b) the ratio of chi-square to its degree of freedom (x^2/df), which should be less than 1.96 since a significant chi-square indicates lack of satisfactory model fit and (c) the root mean square error of approximation (RMSEA), with acceptable values less than or equal to 0.06 (Muthén & Muthén, 2010).

Statistical techniques for analyzing qualitative data

For a better description and investigation of students' cognitive and hypercognitive processes when engaged in rich mathematical problem solving, qualitative methodology and data analysis was employed. Qualitative information may be used to elucidate quantitatively derived findings (Strauss & Corbin, 1998). While a large source of data collection in this research involved quantitative data retrieved from the mathematical abilities test, it was supported with qualitative data collected in the field in an effort to develop an understanding of the cognitive and hypercognitive reasoning processes behind student responses during the subsequent one-to-one meetings between the gifted students in mathematics and the researcher. Using quantitative and qualitative data "as supplements, as mutual verification, and most important for use, as different forms of data on the same subject" (Glaser & Strauss, 1967, p. 18) in many instances is necessary to obtain evidence pertinent to specific research.

The grounded theory approach (Glaser & Strauss, 1967) was applied to code and analyze the qualitative data, which consisted of students' responses and reactions during the individual interview with researcher and observation. In grounded theory approach (Glaser & Strauss, 1967) the theory emerges inductively from the data and data is systematically collected and analyzed throughout the research process. The basic rationale behind the grounded theory approach emphasizes the generation of theory through the inductive examination of qualitative information, in contrast to other approaches that use information to verify existing theory. As such, data are not forced to create a predetermined theory. Stated simply by Charmaz (2006), "grounded theory methods consist of systematic, yet flexible guidelines for collecting and analyzing qualitative data to construct theories "grounded" in the data themselves" (p.2). Grounded theory approach is consistent with modified analytic induction qualitative research methodology where data is gathered and analyzed in an attempt to develop a descriptive model of the phenomena studied (Bogdan & Biklen, 1998). Thus, students' records help to produce a continuous trail of documentation and they are used to reflect on the nature of the students' mathematical thinking and understanding, with regard to cognitive and hypercognitive processes.

Data Coding

This section describes the procedures followed for data coding, to satisfy the objectives of the study and organize data in such a way as to answer the research questions posed. The procedures for coding quantitative data will be described first, followed by the description for the procedures for analyzing the qualitative data.

Procedures for coding quantitative data

For the coding of the test measuring mathematical abilities, three different procedures were followed according to the type of the task. All six tasks measuring ability in number relations, the two tasks measuring ability in inclusion relations and three out of four tasks measuring spatial ability were multiple choice, with four or five alternative responses. In these tasks, one mark was given to each correct response and zero marks were given to each incorrect response. For the coding of the remaining task measuring spatial ability (Problem 2, see Appendix A), partial credit was given. Specifically, the task was graded as follows: 0 marks for incorrect response, 0.25 for identifying one correct edge, 0.50 for identifying two correct edges, 0.75 for identifying three correct edges and 1 for identifying four correct edges.

Finally, the assessment of students' creativity measured through Problem 13 was based on the fluency, flexibility and originality of their solutions (Torrance, 1974). For fluency, the number of correct solutions was counted. For the fluency score we calculated the ratio: number of the correct mathematical solutions that the student provided, to the maximum number of correct mathematical solutions provided by a student in the population under investigation.

For flexibility, the number of different types or categories of solutions was measured. For the flexibility score we calculated the ratio: number of different types of correct solutions (depending on the type of four same area figures constructed—e.g.

figures consisting of four squares, using figures that include half squares divided diagonally, using figures that include half squares divided horizontally, using figures that contain half squares divided vertically, a mixed combination) that the student provided, to the maximum number of different types of solutions provided by a student in the population under investigation.

Originality was calculated by comparing a student's solutions with the solutions provided by all students and the rarest correct solution received the highest score. For the originality score, we calculated the frequency of each solution's appearance, in relation to the sample under investigation. A student was given the score 1 for originality if one or more of his/her answers appeared in 1 % of the sample's answers. Correspondingly, a student was given a score of 0.8 if the frequency of one or more of his/her answers appeared in between 1 and 5 %, 0.6 if the frequency of one or more of his/her answers appeared in between 6 and 10 %, 0.4 if the frequency of one or more of his/her answers appeared in between 11 and 20 %, 0.2 if one or more of his/her answers appeared in more than 20 % of the sample's answers.

Three different numbers (fluency, flexibility and originality scores), each up to 1, were calculated for each student. Then the sum of the three indicators divided by three was calculated, to provide a score up to 1, indicating the score in the mathematical creativity task.

Procedures for coding qualitative data

Data collected during the individual administration of challenging mathematical tasks and observation consisted of the transcripts of video recordings and observation notes taken by the investigator. The transcribed data was coded and analyzed using techniques from grounded theory (Glaser & Strauss, 1967). The objective of coding was to outline the processes and build greater categories (Strauss & Corbin, 1998). The constant comparative method was applied to compare the behaviors and processes of all students observed and to isolate the similarities of their cognitive and hypercognitive processes as found in the data.

Three types of coding were applied in regard to qualitative data; open, axial, and selective coding (Strauss & Corbin, 1998). Open coding refers to the process concerned with identifying, naming, categorizing and describing phenomena found in the text. In other words, it is the part of analysis of generating initial concepts from data. To obtain the initial concepts, the researcher reads through data several times and then starts to create tentative labels for chunks of data that summarize what is happening, based on the meaning that emerges from the data, rather than existing theory. In this phase, examples of participants' words can be recorded and properties of each code are established. At a second level, axial coding refers to the development and linking of concepts into conceptual families. Codes, including both categories and properties, emerged from open coding are now related to each other, via a combination of inductive and deductive thinking. In other words, axial coding consists of identifying relationships among the open codes. The third level of coding is selective, where the relationships are formalized into theoretical frameworks. To achieve this, one category is identified as a core category, and all other categories are related to that category. In other words, a single storyline is developed around which all everything else is draped.

For this study, coding began by reading the transcripts line by line and spontaneously creating memos using words that described cognitive and hypercognitive processes of gifted students observed. Through reading and re-reading of transcripts and field notes, the researcher labeled a plethora of variables, including concepts and properties. After finishing with identifying processes and their properties, a second coding cycle began. At the axial coding level, the researcher related codes and properties identified earlier, forming greater categories of cognitive and hypercognitive processes and establishing relationships amongst them. During the third level of selective coding, the researcher identified the core category and all other categories revolved around it in order to create a unified theoretical framework describing the cognitive and hypercognitive process of gifted students in mathematics.

There is an important note to make about the place of literature in a grounded theory approach. In an emergent study, as this is, Glaser (1978) recommends reading widely while avoiding the literature most closely related to the research subject, pinpointing the danger of reading constraining coding and memoing. To achieve theoretical sensitivity, that is perceiving variables and relationships, the researcher must begin with as few predetermined ideas, particularly hypotheses, as possible so he or she can be as sensitive to the data as possible (Strauss & Corbin, 1990). This does not mean that the researcher must start with a tabula rasa, as is often assumed. As Dey (1999) best described it, having an open mind is essential, not an empty head. In grounded theory approaches, it is the way prior knowledge is used to inform our analysis rather than to direct it that makes the difference. In addition, literature can be used as 'data' and constantly compared with the emerging categories to be integrated in the theory (Glaser, 1992). For the purposes of this study, the researcher followed this recommendation, by keeping an open-mind to avoid having existing literature on the field constraining coding and memoing. The researcher was sensitive to the data collected, from which categories emerged to form the suggested theoretical model of giftedness in mathematics.

Results based on analysis of quantitative data as well as the categories that emerged as a result of qualitative data coding and analysis are presented in the next chapter.

CHAPTER FOUR: DATA ANALYSIS AND RESULTS

Introduction

This chapter presents the results after analysis of data collected during this study. The first section illustrates the empirically tested model for describing the components of mathematical giftedness in regard to mathematical abilities as occurred by the screening instrument used for identification purposes. The largest part of this chapter is devoted to the second section, which comments on the findings in regard to cognitive and hypercognitive processes of mathematically gifted students based on student's observation working with rich mathematical tasks.

Structure of Mathematical Giftedness with Regard to Abilities

The theoretical model according to the theoretical background of the study and the relevant literature in regard to mathematical abilities of gifted students in mathematics is presented in Figure 7. According to the suggested model, it was hypothesized that mathematical giftedness can be described across three dimensions; ability in number relations, spatial ability, and creative ability.

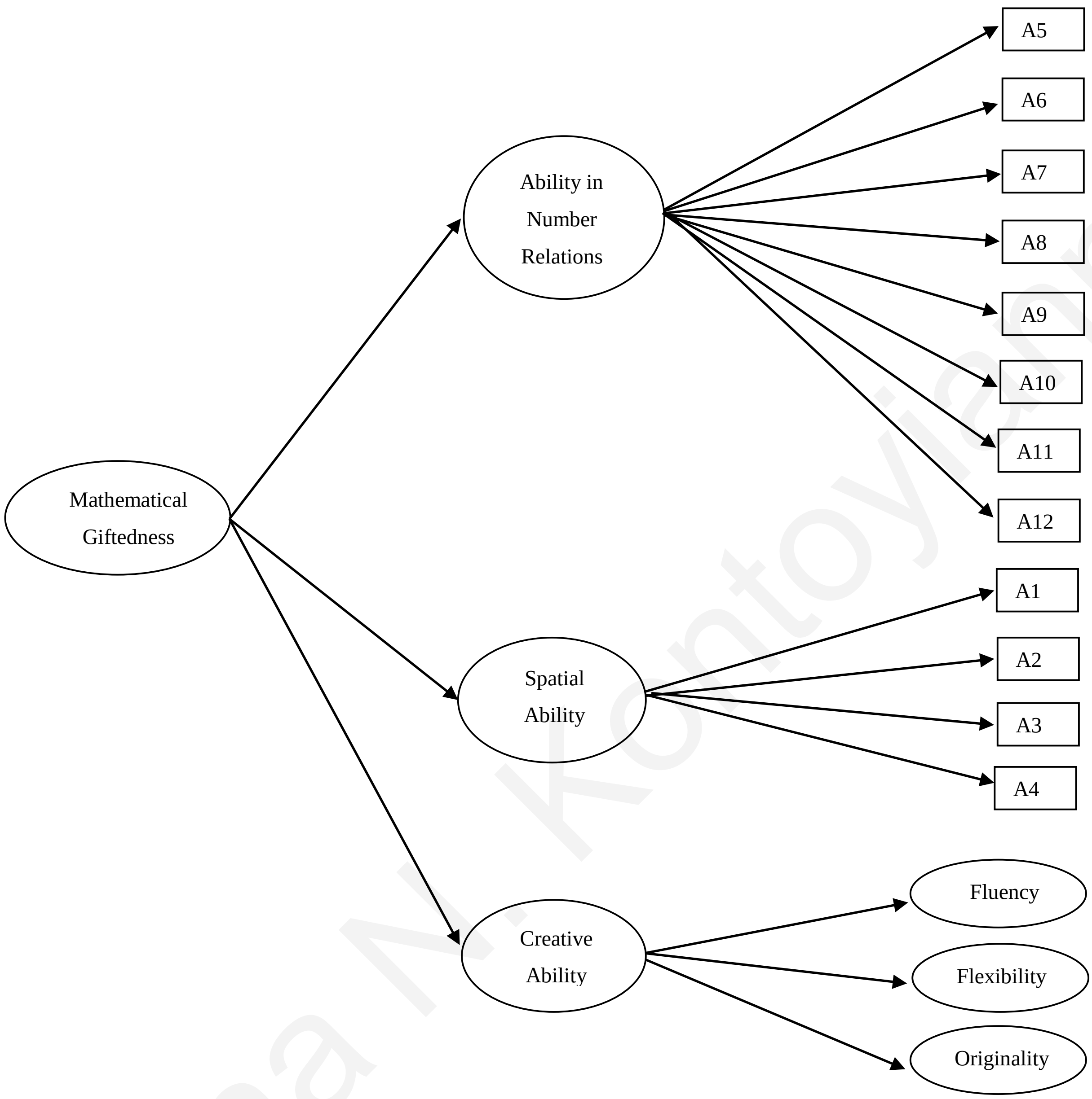
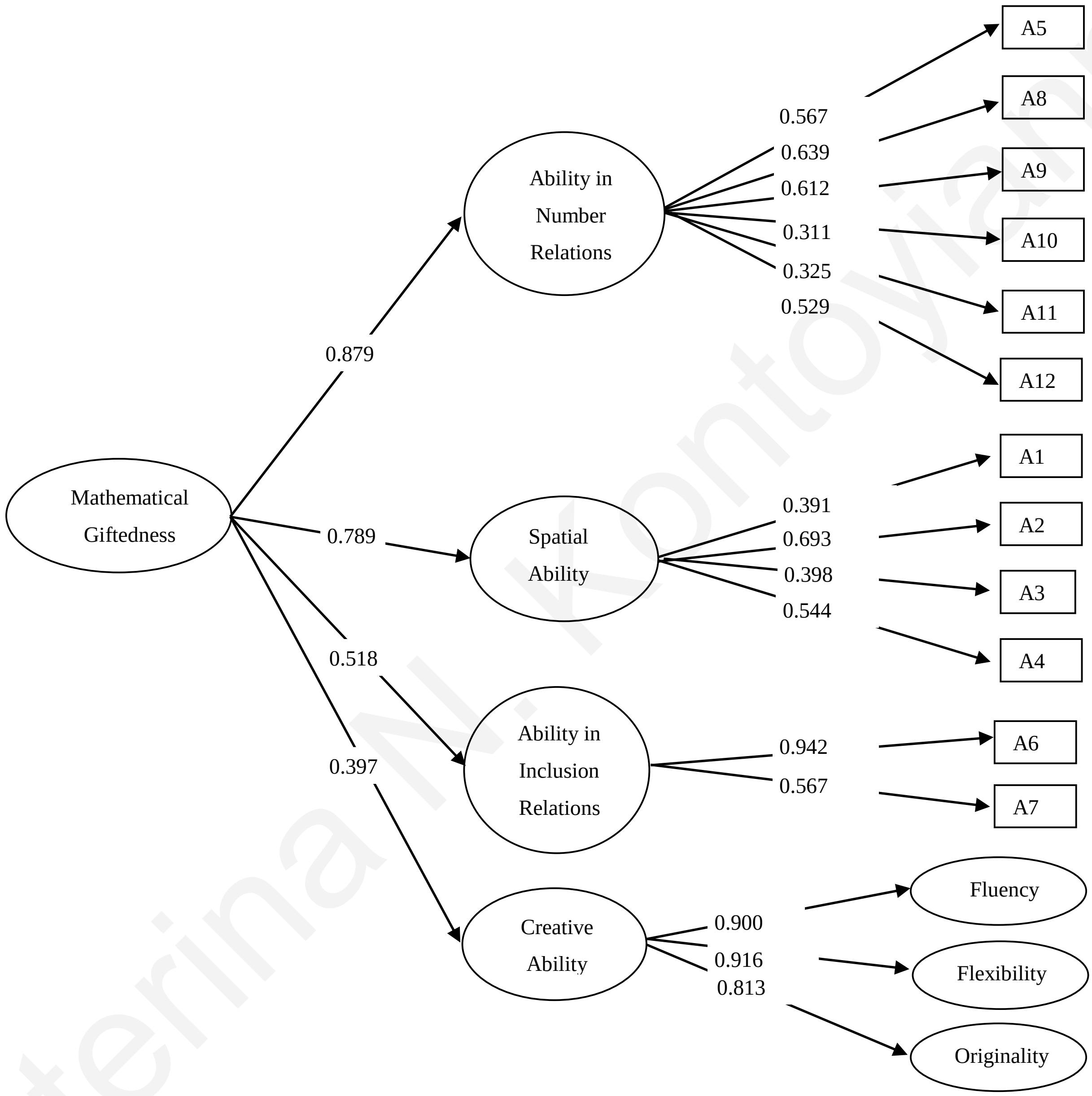


Figure 7. The suggested model of abilities related to mathematical giftedness.

In order to investigate the structure of giftedness with regard to mathematical abilities, the construct validity of the theoretical model was tested, employing a confirmatory factor analysis (CFA). The results of the confirmatory factor analysis showed that the hypothesized model did not fit the data of the study in a satisfactory level.

To be more specific, a subsequent CFA showed that a more satisfactory model was produced when activities 6 and 7 formed an additional distinct factor of mathematical ability. The results of the confirmatory factor analysis showed that the observed (students' responses to each task) and theoretical factor structures (the components of the theoretical model) matched the data set of the present study and determined the "goodness of fit" of the factor model (CFI=0.958, x^2=111.669, df=86, x^2/df= 1.298, RMSEA=0.023). Figure 8, presents the structural equation model with the latent and observed variables and their indicators.



Note: The number indicates factor loading.

Figure 8. The model of abilities related to mathematical giftedness.

Hence, the theoretical model of four first-order factors and one second-order factor can describe mathematical giftedness in regard to mathematical abilities. The factor loadings of all the items to their corresponding factors are statistically significant. The fitting of data to a new structure of the theoretical model shows that the items used were in fact suitable instruments for measuring mathematical giftedness in four salient factors and that the model could represent distinct factors across which students' mathematical abilities related to giftedness can be organized and should be considered during the identification of mathematical giftedness. Thus, students' abilities in (a) number relations, (b) inclusion relations, (c) spatial tasks and (e) creative tasks are important for the identification of mathematical giftedness.

Mathematical giftedness entails four types of abilities with statistically significant loadings; ability in number relations ($r=.879$, $p<.05$), spatial ability ($r=.789$, $p<.05$), ability in inclusion relations ($r=.518$, $p<.05$) and creative ability ($r=.397$, $p<.05$). The data suggest that according to students' responses, ability in number relations and spatial ability contribute more than ability in inclusion relations and creative ability. In addition, the loadings of fluency ($r=.900$, $p<.05$), flexibility ($r=.916$, $p<.05$) and originality ($r=.813$, $p<.05$) suggest that they constitute the factor of mathematical creativity.

Given the structure of mathematical abilities related to mathematical giftedness, Table 3.4 presents a revised categorization of the tasks included in the test for measuring mathematical problem solving abilities.

Table 3.4

Revised types of tasks included in the test for measuring mathematical problem solving abilities

Abilities measured	Tasks
Number relations	5, 8, 9, 10, 11, 12
Inclusion relations	6, 7
Spatial ability	1, 2, 3, 4
Creativity	13

Cognitive and Hypercognitive Processes based on Students' Observation

One of the goals of this study was to reveal the cognitive and hypercognitive processes employed when dealing with rich mathematical problems. After closely examining the recordings of students' observation, the following sections present and analyze such processes, as experienced during the process. To this end, specific excerpts derived of the reasoning and argumentation provided by the students are presented to help extract these behaviors and build up a synthesized image of their abilities and processes employed while problem solving.

Before proceeding into presenting and commenting on the findings, it is vital to note that the characteristics observed were influenced firstly by the tasks selected to be used for observation purposes and also the methodology used. To be more exact, the observation tasks included tasks that required students to experiment, observe or/and discover mathematical relations, reach to conclusions, form generalized arguments and apply creative thinking among others. Therefore, it was expected that the particular type of tasks were inevitable to reveal if the students observed possess such type of mathematical abilities and expose to what extent these abilities are developed. It should however be clarified that although the nature of the activities promoted the demonstration of students' potential, the behaviors that were observed revealed traits that characterized the students, not the tasks.

The same applies to the methodology used. Namely, the researcher was observing students, recording down their thoughts, words and noting down possible expressions and gestures. The researcher intervened to clarify something or give an additional hint to the student when jammed. This approach gave the opportunity to students to reveal their abilities, express their reasoning process and expose their capabilities. The fact that everything were documented in multiple ways, allowed the comparison of multiple sources of data. Hence, it was possible to avoid omitting any important aspect or behavior on behalf of the students.

It is also important to underline that some characteristics have been found to be age-oriented/dependent. In particular, fifth graders had more difficulties to approach certain tasks in comparison to sixth graders. This was due to the difference in the amount of mathematical experiences fifth graders were exposed to, as well as to the lack of specific mathematical knowledge that sixth graders may have been taught already. This resulted to different strategies used in some cases. For example, in Activity 6, where the students were required to draw figures with an area of 2 cm^2, some sixth graders exploited their knowledge of the formula to calculate the area of a triangle, among others, to produce a variety of figures. This was not the case for fifth graders, since they were not taught the formula yet at school. As a consequence, they relied on other strategies to formulate different figures in the specific activity.

The following sections illustrate and analyze the processes witnessed during student's observation. In particular, the findings comment upon the processes of arriving at generalizations, flexibility of mental processes, creative thinking in mathematics and hypercognitive processes.

Arriving at Generalizations

Mathematically capable children were able to arrive at generalizations with ease, as shown in many instances during observation. This could have been achieved in various ways. Firstly, there were students that were able to generalize "on the spot" from the start of the problem, thus grasping the structure of it at once. The following excerpt is distinctive of this thinking process, as recorded in Problem 3:

> R: Can you convince me that the result will always be even, regardless of your selection of starting number?
> S9: Because square numbers are always even, no, they are not always even, how about 25? Wait, there are two cases. 12 times 12 is 144. It is even and it results to an even number.
> R: You mean that its square number is even?
> S6: Yes. So, if this is even then the result is also even. Even minus even, equals even number.
> R: Nice, so you convinced me about this case.
> S6: For odd numbers, odd times odd equals odd, but odd minus odd equals to an even number.

Notice that at the start, S16 mistakenly said that square numbers are always even. However, the student quickly changed his argument and split the task into two cases, one with odd as a starting number and the other with even. Instead of trying out random numbers to examine the task in hand, this student was able to form a generalized argument from the start and thus provide a response to the problem easily.

Generalizations were also made about other relationships observed in the same task. In particular, another student also provided a justification for always ending up with an even number through generalization at the same task. This student was also able to perceive that the product results from the multiplication of any number and his preceding number. From this realization, the specific student also took advantage of his knowledge about multiples of odd numbers. This awareness saved him from doing any calculations on paper.

> S1: If you try number 7 which is an odd number, and multiply it with itself it will be 49. In odd numbers the multiple is once odd and once even. So, if here it is odd, then the previous multiple is even. In the case of even numbers, multiples are always even numbers.

In the same task, another student, (S13), this time a fifth grader, was also able to reach to a generalized argument, after trying out specific numbers. The succeeding part of her thinking illustrates the process followed to arrive to the generalized argument.

> S13: Thinking about it, it is a multiple [meaning that the result is a multiple of the starting number], because 3 times 3 equals 9, minus 3, 6, and 6 is a multiple of 3. Or 10 times 10, 100, minus 10, 90, it is again related to 10.
> R: Correct. How can you be sure that the number is even? For example, from 6 you ended in 30. You said it is a multiple of the starting number.
> S13: Five times 6. Always one less.
> R: Does this happen also in the case of 10?
> S13: 10 times 10 minus 10, 90. 9 times 10. One time less.
> R: Can you explain why the result is always one time less?
> S13: Because you subtract a 6. Five times 6.
> R: How can you be so sure that this number will be number 6? If you had started with number 7, what product will be your result?
> S13: 42. It could matter that it is multiplied with an even number… even if you start with an even number it will be multiplied with an odd number. If you start with an odd number, you will multiply with an even. You always have the product of an even and an odd number.
> R: What does this say to you?
> S13: The product of an odd with an even number is always an even number.

Signs of the ability to generalize were also evident in other activities. In another example in Activity 5, a number of students arrived to the solution without having to draw or count the missing squares. Below is the relevant excerpt from the discussion with S16.

> R: Which number could be in the position of x?
> S16: There are 7 squares here [mentally counts the squares in the lower row] so 49.
> R: How did you end up in 49 without drawing all the squares?
> S16: 7 times 7 equals 49.
> R: Why did you use 7 times 7?
> S16: Area.
> R: So, what is declared by number x?
> S16: Area.

R: Can you suggest more numbers that could be placed in the position of x in another square?
S16: 4, 9, 16, 25, 36, 49, 64.
R: Well done. If I have a square with an area of 64 square units, what is the length of the side?
S16: 8.
R: Very good. Now, can you tell me some numbers that could not be placed in the position of x?
S16: 5,8,20.
R: If I have a shape with area of 5 square units, what shape could it be?
S16: A rectangle.

This student was able to comprehend the properties of numbers placed in position X, by correlating the number with area. After acknowledging this, the student was also able to suggest that shapes with an area different that this suggested by numbers in X position, could take the shape of a rectangle.

Another student (S18) also arrived at a generalized argument at the same task, by making a distinct correlation of square numbers and the area of a square figure. More specific the student concluded that all square numbers could be positioned in x's position, since the shape provided is a square.

R: Can you tell me of another number that could be placed in x's position, instead of 49?
S18: Any number I can think of? 4.
R: How did you think of 4?
S18: Any square number.
R: Why is that?
S18: Take for example 100, 10*10. The numbers should have two identical factors.
R: Why do we need to use square numbers in this task?
S18: Because we have a square shape.

These behaviors show that mathematically gifted students have the ability to arrive at generalized arguments with ease. Indeed, the abovementioned excerpts illustrate that mathematically gifted students may observe the underlying structure and components of a posed problem and identify similarities and differences amongst the components. This allows them to acknowledge the conditions under which specific properties tend to apply for a certain set of numbers and formulate a generalized argument to express these relationships. For some students this may occur without trials with specific numbers or experimentation, while others may require a minimal number of experimentation to grasp the connections among the components of the problem situation.

Students also demonstrated their ability to use generalizations as a means to produce many different responses. Through the provision of a generalized model, a diversity of cases was implied without the need to point the specific cases. This happened in the case of several students in Activity 6, which required spatial thinking, sense of space and mathematical creativity. In his effort to produce many different and novel figures with an area of 2 cm^2, as advised in the Activity, a student resorted to a generalized argument. Firstly, S22 drew the figures shown in Figure 9.

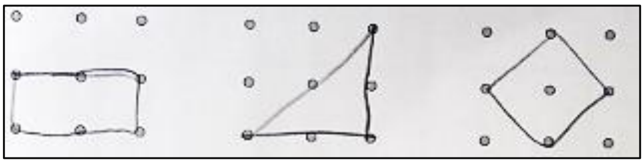


Figure 9. Figures suggested by S26 in Activity 6 during observation.

When the student was asked to explain the way he used to calculate the area of the end figure in order to be sure that the shape had the required area of 2 cm^2, the student provided an unexpected answer. To be exact, the student asserted that the total area is 4 cm^2, referring to a mental square area formed by the exterior eight dots. Thus, he concluded that the requirement of the task was to mark half of the whole area, in other words 2 cm^2. In conclusion, the student processed the data provided in this activity and was able to extract an important judgment. The activity was thus proven very simple for this student after making this generalization, which allowed him to produce a lot of figures with minimal effort.

The arrival at a generalized argument contributed to a range of creative responses in the case of another student. Namely, S12 formed the four figures illustrated in Figure 10 in Activity 6.

Figure 10. Generalization as a means to produce creative responses (S12, Activity 6).

After drawing the illustrated figures, the student declared that a large number of similar shapes could be made by sliding one of the two squares to different positions to the left or to the right. S12, in this case, was able to take advantage of the generalization he reached at, and after forming the first figures, he refrained from doing anymore shapes using this technique. This was no longer considered necessary, since after phrasing his generalized conclusion, all possible similar figures using the sliding approach had already been proposed through the generalization.

In the same activity, S22 suggested a different generalized approach to form many figures with the same area. Firstly, the student drew a scalene triangle with a base length and altitude of 2 cm, as illustrated in Figure 11.

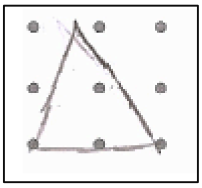


Figure 11. Generalization as a means to produce creative responses (S22, Activity 6).

Then, S22 suggested a strategy to produce many similar triangles to the first one by arriving at a generalization. To be exact, the student claimed "There are many ways to create many triangles with the same area. Just move right or left the upper vertex of the triangle while keeping the same altitude and the same base length". Similarly to S22's aforementioned behavior, this student was able to visualize the possible alternate triangles with the same area and he did not proceed into designing any other figures following the same strategy. Hence, afterwards, S22 proceeded to a different type of thinking to form other figures.

In the last two cases of S7 and S12, the expression of a generalized strategy to form many similar but not identical shapes, thus increasing the number of responses provided, allowed students to save time and effort and therefore they were able to proceed on thinking of other strategies to draw new figures with the same area. To sum up, the aforementioned cases illustrate in general that mathematically gifted students employ processes that aid them to arrive at generalized arguments, form a mathematical proof, create a class of similar responses easily and arrive to conclusions and extensions.

Flexibility of Mental Processes

During observation, one of the most striking abilities displaye by mathmetically gifted students was that of flexibility of mental processes. Flexibility of mental processes was manifested in different ways. Results will be presented along these facets; flexible thinking through the generation of multiple mathematical solutions, reasoning in cycles, handling multiple mathematical relationships at once, fluency for expediency, curtailment of the process of mathematical reasoning and economical thinking and reversibility of mental processes.

Generation of multiple mathematical solutions. An activity that triggered the production of many mathematical solutions was Activity 1. Namely, students had to come up with many different and innovative ways to classify 12 given numbers in groups. A popular way to start grouping the 12 numbers was to create four groups with multiples of 2, 3, 5 and 7. However, soon the students discovered that by creating these four groups, some of the numbers were eligible to be part of many groups. For instance, number 15 is both a multiple of 3 and 5, while number 28 is a multiple of 2 and 7. Thus, students had to figure a way to create groups in that way to avoid having the same number in more than one group. This obstacle led to a variability of different suggestions.

For instance, a specific student (S13) chose to exclude number 15 from multiples of 5, by rephrasing the group's name into "multiples of 5 except of numbers with 1 as a tens digit". The same student, chose to avoid having number 28 in multiples of 7, by renaming the group into "multiples of 7 except of numbers with 2 as a tens digit".

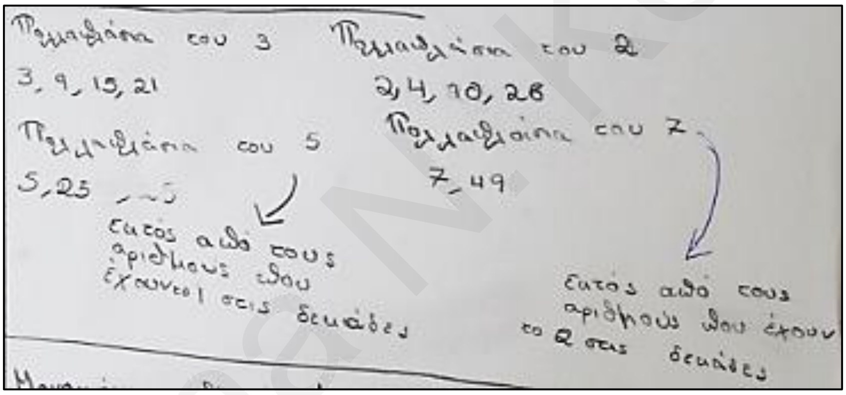


Figure 12. One of the grouping attempts suggested by S13 (Activity 1).

Another student (S7) chose different numbers to exclude. At first, the student created a group of multiples of 5. Then, S7 created a group of multiples of 7. Afterwards, he formed another group consisted of multiples of 2, but wanted to exclude number 10 since it was already included in multiples of 5 and number 28 since it was previously added to multiples of 7. To overcome this obstacle, the student chose to rename the group into "multiples of 2 up to 8". At the end, the student formed a fourth group named as "multiples of 3 up to 9" in order to group 3 and 9, but not numbers 15 and 21.

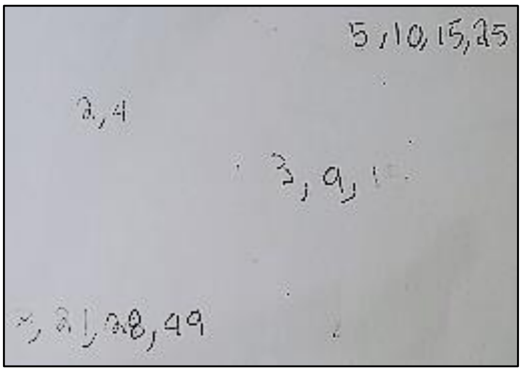


Figure 13. One of the grouping attempts suggested by S7 (Activity 1).

A similar approach with multiples was proposed by S20. The first group generated, was that of multiples of 5, consisting of 5, 10, 15 and 25. Afterwards, a second group was formed consisting of multiples of 7, including 7, 21, 28 and 49. Then the student chose to form a group with multiples of 3, but 21 and 15 were already been grouped previously. At that point, the student chose to remove 21 from multiples of 7, place it in the group of multiples of 3. Then, the student renamed both groups to comply with the numbers included into "multiples of 7 that are not multiples of 3" and "multiples of 3 that are not multiples of 5". Left at the end with 2 and 4, the student preferred to group them as "two numbers whose sum is 6".

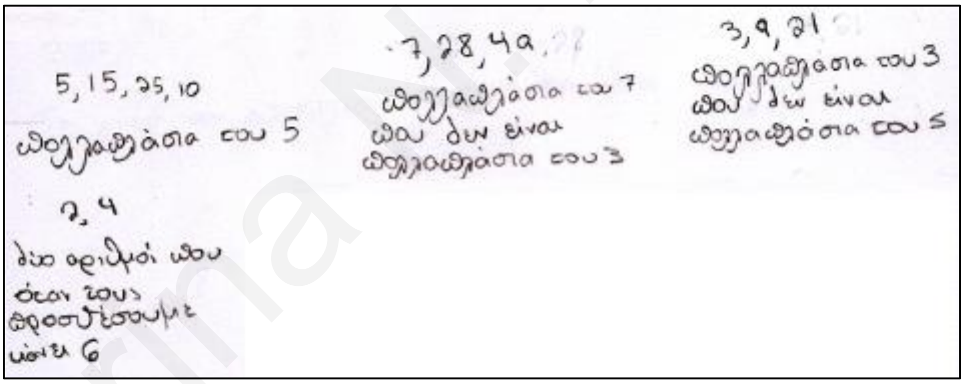


Figure 14. One of the grouping attempts suggested by S20 (Activity 1).

Another grouping attempt that suggested the use of multiples as a grouping criterion, was proposed by S32. In fact the student initially grouped multiples of 7 and then multiples of 5. When trying to create a third group of multiples, he suggested to group numbers divided by 2, writing down numbers 2 and 4. Then with a prompt from the researcher to check for numbers eligible to be part of more than one group, he observed that this is the case of number 21, being a multiple of 3 and 7. He quickly thought of a solution to avoid this.

S32: Multiples of 3 until 20.
R: Well done! Is there any other number who may be grouped in two groups?
S32: Number 5. It belongs in the groups divided by 3 and 5.
R: It should be a part of only one group.
S32: Numbers divided by 5, except of 11, up to 20 [renaming the group].
R: Nice! In the group of multiples of numbers divided by 2, you could also put numbers 10 and 28. What can you do now that they are grouped in other groups?
S32: Up to 9 [referring to numbers divided by 2, extending the group name to include numbers divided by 2, up to number 9].

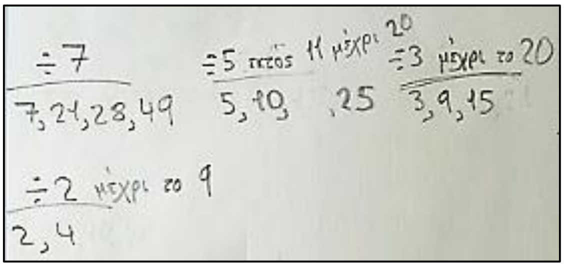


Figure 15. One of the grouping attempts suggested by S32 (Activity 1).

Another student (S31) focused in a larger extent on multiples and divisors and produced three unique ways to group the 12 given numbers incorporating the mathematical relations of 2, 3, 5 and 7, paying attention to alter the name in order to avoid having numbers eligible to belong in two groups. Specifically, the numbers were grouped in odd multiples of 7 that are not multiples of 3, multiples of 2 that are not multiples of 5, multiples of 3 that are not multiples of 5 and multiples of 5.

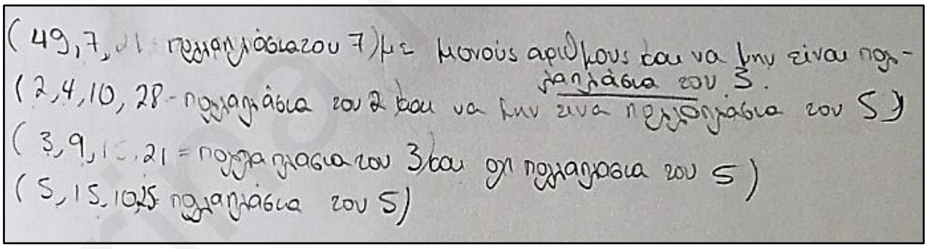


Figure 16. First grouping attempt by S31 (Activity 1).

In a second attempt, the same student (S31) chose to divide the numbers into two groups, by exploiting the groupings used in his first attempt. To be specific, the student selected to unite the third and fourth groups proposed previously, and create a new group named by the composition of the two groups, as “multiples of 3 or 5”. Then, the student united the first two groups proposed in the previous attempt, now entitled as “multiples of

7 or 2 that are not multiples of 3".

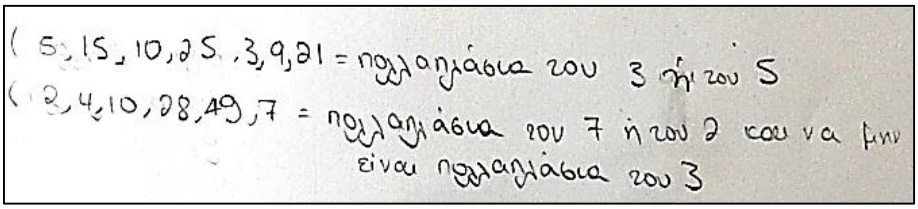


Figure 17. Second grouping attempt by S31 (Activity 1).

This is an ingenious decision on behalf of the student, since is a simple and quick way to make a novel grouping without spending much effort. Rather, the student chose to modify the previous grouping attempt. This behavior shows that this student is able to flexibly handle mathematical relations and easily alter a type of organization to produce a new one, by exploiting a previous proposed solution. Furthermore, the student seems to acknowledge that different ways of grouping, can be based in the same mathematical conditions, and can be adjusted to produce many different ways. In this case, the student based his strategy upon multiples. It is also important to note that the student shows signs of handling easily conditions and operations of sets, such as the union of sets that was witnessed in the specific extract.

As a third grouping attempt, the student provided four groups. The first group included numbers 5, 15 and 25 and was named as odd multiples of 5. The second group was the group of even numbers, consisting of 2, 4, 10 and 28. The third group consisted of 7 and 49 and was entitled "not even multiples of 7". Numbers 3 and 9 were part of the fourth group, under the name "Divisors of 9." This attempt, has some similarities with the previous grouping attempts. In fact, multiples of 5 are renamed into odd multiples of 5 to exclude number 10. This way, number 10 could be part of the group of even numbers. At the same time, a group of odd multiples of 7 was formed, but with no other constraints, compared to the one proposed in the second attempt, where multiples of 3 were excluded, to be specific number 21. The previously used set of multiples of 3 was renamed into divisors of 9, to disregard numbers like 21 and 15 that could be part of a possible set of multiples of 3.

In the third attempt, the student showed that he recognized the mathematical connection between multiples and divisors and he was able to switch flexibly into the concept that best served the purpose of the particular grouping he wished to use. In general, once more, this student displayed his ability to think creatively in mathematics, while relying in a common ground in all three attempts, that of multiples and divisors of numbers. He easily combined sets to form new groups, modified previously used groups to create different ones, and handled multiples mathematical relationships in parallel.

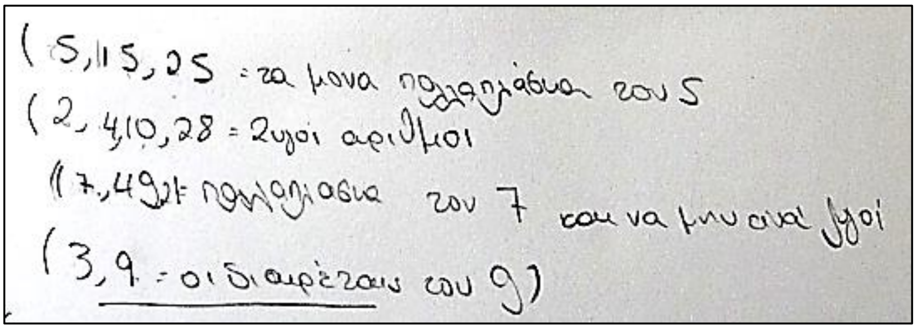


Figure 18. Third grouping attempt by S31 in Activity 1 during observation.

A different student (S1) resulted in seven different ways to group the given numbers in Activity 1, varying in level of difficulty of mathematical relationships, as shown in the following excerpt. The notable ease with which this student renames the criteria by which the numbers are grouped in order to allow for numbers to be excluded from specific groups is apparent. This shows his flexibility to switch his thinking and reorganize the groups created to respond to the task's restrictions.

> S1: I am making a group with multiples of 5 [noting 5, 10, 15 and 25]. I group multiples of 7 [noting 7, 21, 28 and 49 and crossing out "used" numbers]. Then prime numbers [noting 2 and 3].
> R: What about 5 and 7? Aren't they entitled to get into the group of prime numbers?
> S1: Yes, but they cannot be placed in both groups [as a task restriction].
> R: How can you change the title of the group in order to remove 5 and 7?
> S1: Prime numbers up to 3.
> R: Well done.
> S1: I am left with these two [numbers 4 and 9]… I do not know.
> R: They are related somehow.
> S1: Oh, they are square numbers. [noting square numbers 4 and 9]
> R: Are there any more square numbers?
> S1: Yes, 25 and 49.
> R: So, how are you going to keep them out of this group?
> S1: Square numbers up to 9.

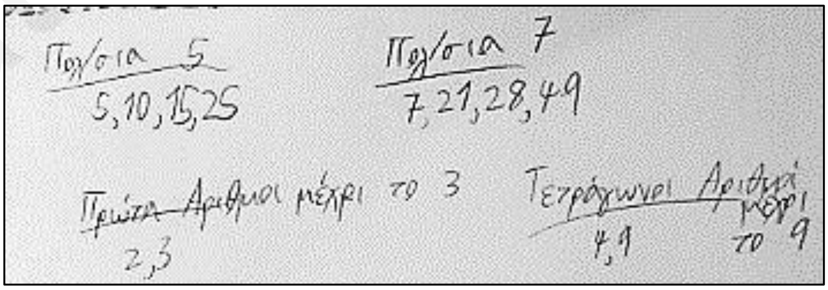


Figure 19. First grouping attempt by S1 (Activity 1).

In his first grouping, S1 created a group of multiples of 5 and a second one with multiples of 7. Proceeding to the formation of a third group of prime numbers, he acknowledged with the help of the researcher that numbers 5 and 7 are also prime numbers. In front of this realization, the student chose to rename the group of prime numbers into “Prime numbers up to 3”. Left with numbers 4 and 9, the student grouped them under “Square numbers”. Acknowledging that the previously grouped numbers 25 and 49 are also square numbers, he extended the name of the fourth group, by naming it into “Square numbers up to 9”.

In the second attempt, S1 distinguished among odd and even numbers, whilst in the third attempt, he differentiated between numbers larger or less than 20. For his fourth grouping, the student chose to form two groups, after one-digit and two-digit numbers.

Next, the student attempted a grouping with more than two groups, returning to more complex mathematical relations.

> S1: … [thinking]
> E: What are you thinking?
> S1: If I can… I am thinking of numbers who are divisible by 2, numbers divisible by 7… I will put numbers divisible by 2 [noting 2, 4, 10 and 28] I will have numbers divisible by 3 and have 15, 9 and 21 in this group. I can also make a group as numbers divisible by 5 over 15.
> R: To avoid having 15 in this group.
> S1: 10 and 15.
> R: Ok.
> S1: [noting this group on paper]. I‘ve only got 25…. Here I forgot to put number 3 [adding 3 in the group of number divisible by 3] and I still have left numbers 5, 7 and 49… Oh, the biggest square number is 49 and numbers who divide 35 evenly, no. I will name the group numbers that when multiplied result to product 35, and these are 5 and 7.

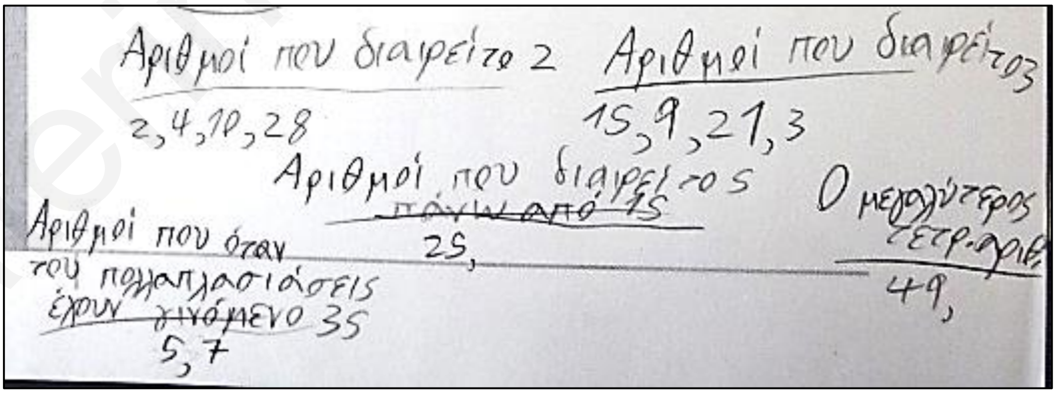


Figure 20. Fifth grouping attempt by S1 (Activity 1).

The student followed the underlying idea of numbers divided by 2, 3 and 5, followed by specific limitations to comply with the task limitations of having each number being a part of only one group. However, he differentiated from using the group of numbers divided by 7 again. Instead, the student formed two more groups, one consisting of the largest square number and another with factors of 35. This is a very clever variation on behalf of the student. It shows that although he bases his response into former responses, he is able to make the necessary modifications in order to produce a new original response.

In the sixth grouping effort, the student added another property into his repertoire of grouping criteria, by furthermore observing the unit digit of numbers.

> S1: I will group numbers that end in 5 and these are 5, 15 and 25 [marking already grouped numbers by putting them in a circle]. Numbers that ends in 0, that is 10. I will put numbers that divide 28 evenly, 28, 7, 2 and 4…. I will group numbers that end in 9, that is 9 and 49 and we are left with 3 and 21. 3 divides 21 but where to put it…. I will name numbers who divide 21 except of 7 and 1 [noting the group down].

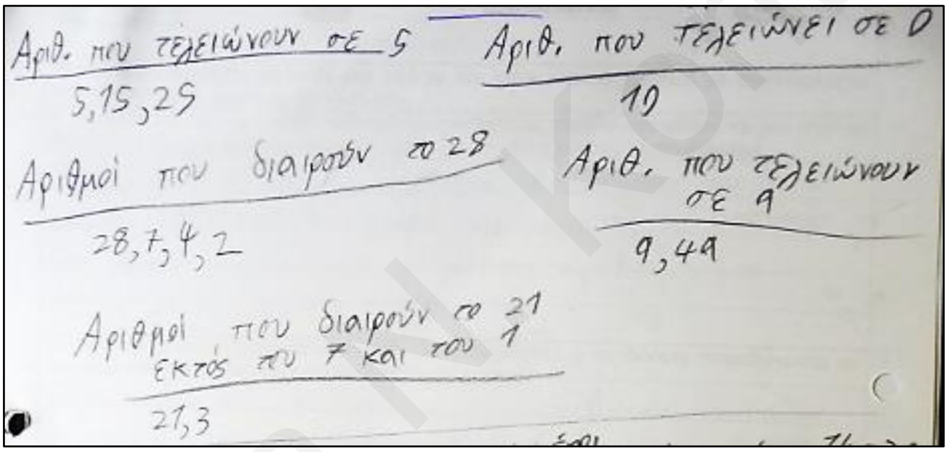


Figure 21. Sixth grouping attempt by S1 (Activity 1).

The student proceeded with remarkable ease to the sixth grouping attempt. In this case, the student focused in the unit digit of numbers to form the first two groups. Then, he created a group with divisors of 28 and then another group with numbers ending in 9. Left with 3 and 21, he was troubled with which grouping criterion to use. After some thinking, he decided to form a fourth group, named “Divisors of 21 except of 7 and 1”. After gaining experience through previous attempts, the student was able to form criteria with exclusive titles effortlessly. The change in regard to the ease of forming the grouping rules is apparent. In his last attempt, the student selected to create groups according to a specific range.

> S1: … I will group numbers up to 5, numbers up to 10, numbers up to 25 and numbers over 25.

R: Ok.
S1:In group “numbers up to 5” we put 2, 3, 4 and 5, in group “number up to 15” we put 7, 9, 10 and 15, “up to 25”, numbers 21 and 25 and “over 25”, 28 and 49.
R: Nice. If someone looks at your group, this group here is named “Numbers up to 15”. Thus, one can also put 2, 3, 4 and 5 in this group. How can we avoid this?
S1: Numbers from 6 to 15, and from 16 to 25.

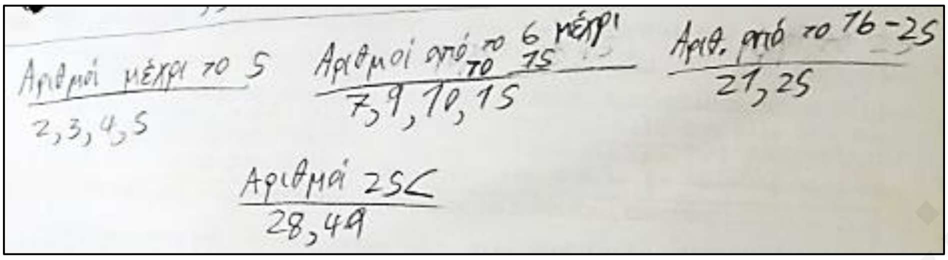


Figure 22. Seventh grouping attempt by S1 (Activity 1).

With a small remark from the researcher in regard to the precision of the group names, the student was able to think of other names for two of the four groups in order to be more accurate and clearly reflect the numbers included in the specific set, without allowing other numbers to invade into the groups.

A fifth grader (S13) did not understand at first that a specific limitation accompanied the task. That is, numbers should be organized in groups in such a way that each number could belong in only one group.

S13: I first thought of odds and evens [creates the two groups and listed the numbers]. Then multiples of 3, 4, 5 and 7 [lists multiples of 3 and 4]
S13: Now I have 2 left out.
R: Also, 21 is not also a multiple of 7?
S13: I already put it in the group with multiples of 3.
R: When a group is named “Multiples of 7” then 21 is entitled to be a part of the group. Since the task asks for numbers to belong to only one group, you should change the group names to have 21 belong to only one group. Can you do this?
S13: … [Thinking]

It is important to observe how this fifth grader attempted to overcome this difficulty and the way in which she created precise name groups to comply with the problem’s restrictions. This student needed some assistance in defining the group names to exclude numbers from belonging to two or more groups. In this case, the researcher consciously tried to help the student in her Zone of Proximal Development as necessary, and tapered off this aid as it became unnecessary.

R: For example, how can you change the name of the group named “Multiples of 7” so that 21 will not belong to this group? [at this point the group includes numbers 7 and 49]
S13: 7 times 7 equals 49, so something like this…Multiple of 3 times… Multiples of 7 except of… From 2 to 6? [her intention is to exclude multiples of 7 from 7 x 2 to 7 x 6]
R: Can you explain this?
S13: If we want 7 in this group, it can be multiplied only with 1 and itself…
R: Yes, but as I understand, number 21 is the one you want to exclude from the group. What type of numbers do you want to exclude from multiples of 7 so that 21 will be excluded?
S13: If we say multiples with the factor except of 7 [being] odd, then we will exclude 21 which uses the number 3, but then also 49 is excluded because 7 is used … and 28 is a multiple of 7 also [number 28 has already been grouped in multiples of 4]
R: So you want to take out 21 and 28.
S13: Numbers from 20 to 30, numbers with 2 as a tens digit.
R: So, what is the final rule you came up with?
S13: Multiples of 7 except of those with 2 as a tens digit.
R: Check your groups once more [number 2 remains ungrouped]…
S13: No, number 2 is missing again… unless it goes… but the others are also evens… If I say “multiples of 2” [instead of the group she now has formed of “multiples of 4”] 4 and 28 will stay [in the group], but then, not multiplied by 5, but again there is 10 [number 10 is already grouped in the group of multiples of 5]…
R: Oh, so 10 is bothering you. Can you remove it somehow from that group?
S13: Oh, over 15. Oh, I have it in two groups [number 15 is also found in multiples of 3].
R: So, we also have a problem with 15…
S13: So I am going to remove it [from the group], except from the numbers with one as a tens digit [noting the group name “multiple of 5 except from numbers with 1 as a tens digit].

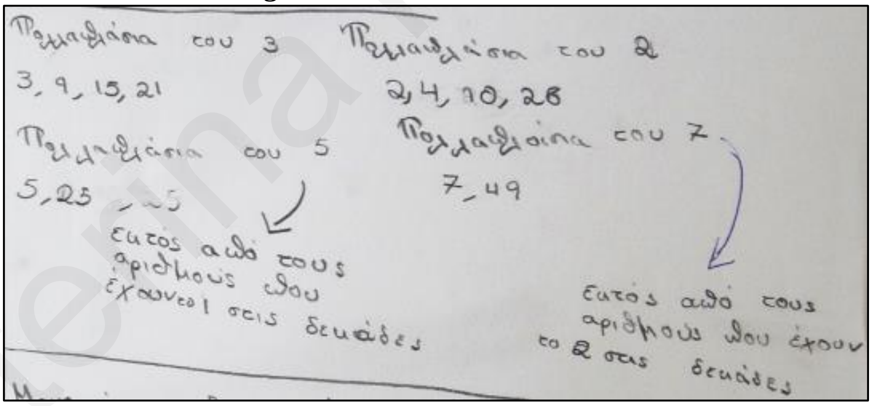


Figure 23. Second grouping attempt by S13 (Activity 1).

R: Can you think of another way? There are many ways.
S13: Numbers that have 1, 2 as a tens digit but what happens with those that do not have a tens digit?
R: How are the numbers with just one digit called?
S13: Single digit numbers. [Creates three groups, single digit numbers, numbers with 1 as a tens digit and numbers with 2 as a tens digit]. Now 49 is left alone.
R: What can you do now?

S13: Tens digit over 1 [Renames the group with numbers with 2 as a tens digit and inserts 49 apart from 21, 25 and 28].

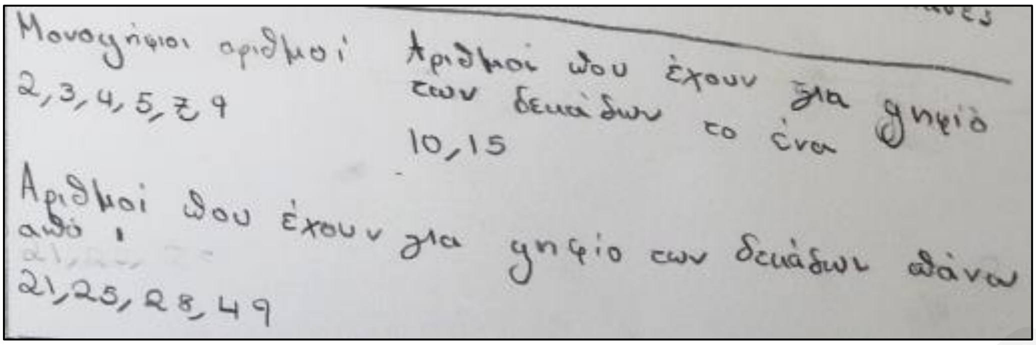


Figure 24. Third grouping attempt by S13 (Activity 1).

R: Well done... so far you found three different ways to group numbers. Can you think of a fourth one?
S13: I look at numbers that start from 10 and onwards.
R: So what are you thinking?
S13: If they can all form a group… for example 2 times 2 is 4, 3 times 3 is 9, 7 times 7 is 49, 25… the rest…
R: What are these numbers called, the ones that are a product of a number with itself?
S13: Square numbers. So, numbers and their squares. [noting a group with numbers 2, 4, 3, 9, 5, 25, 7 and 49]. Now I have left with 10, 15, 21 and 28… I am trying to think but what I come up with is related to things I've already done … Two of them are related to 5 and the other two are related to 7.
R: What is the relationship of 10 and 15?
S13: Multiples of 5.
R: Then 5 and 25 are entitled to enter this group.
S13: So I will have numbers with 1 as a tens digit [noting a group with numbers 10 and 15]. I will have in the other team multiples of 7 with 2 as a tens digit [noting a group with numbers 21 and 28].

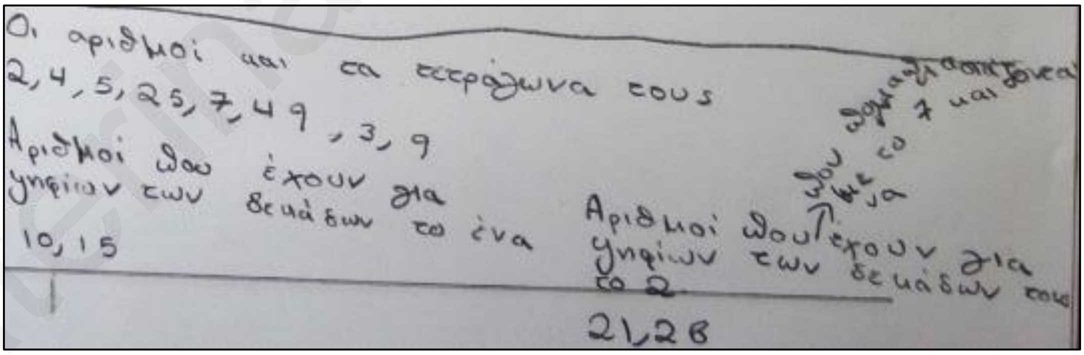


Figure 25. Fourth grouping attempt by S13 (Activity 1).

In addition to all other ways of organizing numbers in group in Task 1, a different student (S16) chose to turn to exponents, as shown in the excerpt that follows, when his previous group formation allowed for numbers to be grouped in more than one group.

Thus, exponents seemed like a good choice to overcome this impediment. At first, S16 formed four groups and noted ÷ 2, ÷ 3, ÷ 5 and ÷ 7 on paper, as titles for the groups. He then placed numbers 2, 4, 10 and 28 in numbers divisible by 2. He also placed numbers 3, 9, 15 and 21 in numbers divisible by 3. Afterwards, he placed numbers 5, 10, 15 and 25 in numbers divisible by 5 and lastly he put numbers 7, 21 and 49 in the group of numbers divisible by 7.

R: Correct thinking. But in this manner. We have some numbers…
S16: Used in more than one group.
R: How could you change a group's name to avoid this?
S16: … [erasing number 15].
R: Tell me what you thought…
S16: [Changing the title of the group previously named as numbers divisible by 5 into powers of 5 retaining the numbers 5, 10 and 25. Note that number 10 is mistakenly now position in this group].
R: Can you remind me the meaning of exponents?
S16: Number 5 in the second power means 25. Oh, 10 should not stay here [erasing number 10]. I'll do it also in the case of number 7 [referring to the group with numbers divisible by 7 that contains 7, 21, 49 while acknowledging that 21 is positioned in two groups], since only 7 and 49 should remain in this group.

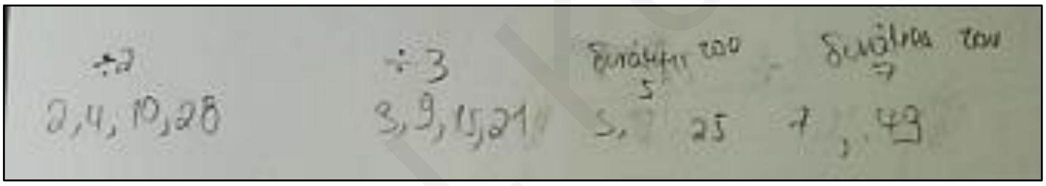


Figure 26. One of the grouping attempts by S16 (Activity 1).

It is worthy to comment that this was the only student that used the concept of power in groupings. Although when coming up with the idea of changing the group's name from "number divisible by 5" into "power of 5" the student mistakenly kept number 10 in the group, when asked by the researcher to clarify the meaning of exponents, the student showed that he has not mistaken the term and immediately acknowledged and corrected his error.

Another student, not native Cypriot but with Russian origin, thus not being fluent in Greek, was eager and drawn to the activity. As shown in the extract mentioned below, in some cases S12 did not know the exact Greek word for a mathematical concept, although he knew the concept. For this reason, he first grouped the numbers and afterwards he asked the researcher for the exact Greek word for that concept.

S12: How are the numbers that end in 0, 2, 3, 4, 6, and 8 called?
R: Even.

S12: Yes [creating two columns and dividing numbers into odds and evens]. How do you call numbers 3, 5…?
R: Odds.

The student had in mind the specific classification, but did not remember the Greek word for odds and evens. In a test only measuring the outcome, the student's answer would be considered incomplete. Next, the student proceeded into a second grouping attempt by grouping in four columns, numbers up to 10, numbers up to 30 and numbers from 30 to 50. Immediately after finishing, the student started a new grouping attempt, showing signs that he already had thought of specific mathematical associations with the given numbers.

S12: How do you call the numbers with two numbers?
R: Do you refer to double-digits?
S12: [without responding, the student created a column with double-digit numbers]… and single-digit numbers? [the student required confirmation for the appropriate Greek word].
R: Bravo.

The student seemed to be so focused in his thoughts, one may say he seemed "lost" in his world. He needed the researcher just to aid with the suitable Greek mathematical terms, otherwise he was self-absorbed deep in his thoughts. Straightaway, the student initiated a new grouping attempt, forming new groups with the following names: "Number 1 +Number 2, evens", "Number 1 + Number 2, odds", "Number1 x Number 2 = evens", "Number1 x Number2 = odds", "Number x Number = Evens", "Number x Number = Odds" and he put numbers in each group.

A more detailed idea about this student's work on this task is provided (see Figure 27).

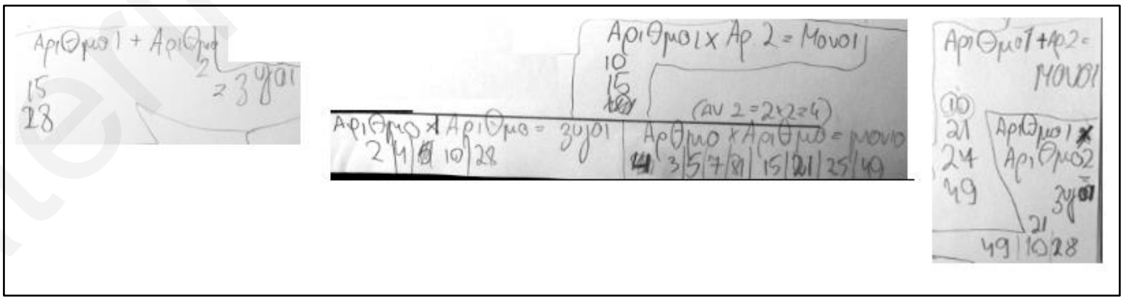


Figure 27. Part of the grouping attempts suggested by S12 (Activity 1).

The researcher let the student work on the task without any distractions. So, as soon as the student created the aforementioned 6 groups, some clarifications were needed about the meaning of the group names.

R: I would like you to explain me the rationale behind some group names. For example, what do you mean here? [pointing to the group name “Number1 x Number2 = Odd”]
S12: Number times second, if it is odd [not correct Greek] if it’s even [he circles the tens digit and then the units digit as he explains].
R: Oh, so when you multiply the tens digit and the units digit you get an odd number.
S12: Yes.

Although the suggested groupings shown in Figure 27 do not comply with the tasks’ instructions, that is to provide a grouping for the 12 given numbers before proceeding to a new grouping attempt, the researcher did not indicate this fact to the student. Although this was not intended by the task, this student used original ways to group the given numbers. It is noteworthy that no other student from the sample used this type of categorization, by noticing the sum or product of the number digits. Except of being extremely focused in his assignment, in addition, all the time that the student was working at the specific task, he had a constant smile on his face, giving the impression that he was aware of his abilities.

Another original way to group the 12 given numbers in Activity 1, was proposed by S26 (see Figure 28). In this case, the student mentally calculated the sum of all numbers and then decided to split the numbers into two groups, after making the following argument:

S26: Since the sum (of all numbers) is 178, 178 /2 equals 89, I will make two groups with sum of 89 in each group.

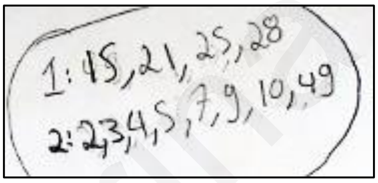


Figure 28. One of the grouping attempts suggested by S26 (Activity 1).

This student’s grouping attempt was completely unique since it was not suggested by any of the other students. It was also impressive that the student chose to make all calculations mentally, and managed to select the correct numbers to achieve the same sum in both groups.

From the excerpts and figures discussed in this section, the ability of the students to produce a variety of mathematical solutions is obvious. It was very interesting to observe the different approaches the students came up with to refrain from including the same number in more than two groups. The specific limitation was one of the reasons of the students in total producing that many different responses. Throughout each student's work, it became clear that student's thinking evolved through practice and experience accumulated as grouping attempts increased. In fact, the researcher soon observed signs of flexible thinking in terms of reasoning in cycles. Namely, it was shown that students were able to flexibly handle mathematical relations and easily alter a type of organization to produce a new one, by exploiting a previous proposed solution. Furthermore, students seemed to acknowledge that different ways of grouping, can be based in the same mathematical conditions, and can be adjusted to produce many different ways. This characteristic behavior of thinking in cycles is further discussed and supported by relevant data collected in the next section.

Reasoning in cycles. Certain tasks used during observation, especially the creative task requiring students to provide different and original ways to group 12 given numbers, resulted to the production of a vast variety of distinct groupings among the students and allowed the observation of an important behavior. Namely, the majority of students showed evidence of flexible reasoning, in terms of thinking in cycles. This type of thinking was exhibited merely due to the limitation that the researcher expressed orally to students during the task execution. Specifically, the researcher added a constraint to the task; each number should be included only in one group in each grouping attempt, thus requiring students to comply with this restriction. For instance, students had to rephrase group names in order to exclude a number from a specific group after being part of a different group, completely reorganize their groups or combine different mathematical relationships to the group names to conform to the task limitation. Specific examples of this characteristic behavior of thinking in cycles to improve the proposed solution to meet the demands of the task follow, by tracking the students' steps from the start till the end of their engagement with the specific task.

For example, S18 initially proposed to organize numbers into two groups, into odds and evens. His second grouping started by creating a group of numbers divided by 3 (noting down the numbers 3,9,15 and 21) and then suggesting to form a second group combined of numbers divided by 5. At this point, the researcher drawn the student's attention to the task's limitation, that of a number being a member of only one group in each grouping attempt.

> S18: Then 15 would belong to both groups.
> R: If you wished to create a group with multiples of 5, what would you do to avoid having 15 in this group?
> S18: … I would write numbers divided by 5, smaller or larger than 15. [noting down numbers 5, 25 and 10]….Another way is to have prime numbers and in another group I will place numbers that can be divided by 7.

Although the student started a new second grouping attempt incorporating multiples of 3 and multiples of 5 excluding 15 to avoid 15 being a port of both groups, he came up with a third way to group numbers. At this point, the student decided to write down the group named as prime numbers consisting of 2,3,5,7 in case he forgot it later. Then, he continued with his second categorization attempt. This time, the student formed another group with numbers divided by 2 (noting down the numbers 2, 4 and 28) and numbers divided by 7 (noting down the numbers 7 and 49). Note that the student mistakenly did not place number 28 and 21 into numbers divided by 7 and also number 10 in the group of multiples of 2. These errors were made due to the fact that the student used to strikethrough the numbers already included in groups. Unfortunately, this strategy prevented him from seeing that with the proposed new group names there would be numbers belonging in more than one group. When the researcher brought this fact to the attention of the student, the student very quickly found a way to alter the group names in order to resolve this issue.

S18: I will have numbers divided by 2 except of numbers ending in zero. And here where I have 21 and 28 in the group [the group with numbers divided by 7], I will rephrase the name into numbers divided by 7 except of two digit numbers whose tens digit is 2.

In this grouping attempt, the student showed his ability to think in cycles. First, the student, created some of the groups and then realized that there are specific numbers that are allowed to be included in two groups. The interesting fact is that the student strived to find a different group name in order to accommodate the numbers he originally chose for each group, rather than erasing and removing numbers and placing them in another existing group. This specific choice that the student made, led him to combine two mathematical conditions for the same grouping. Figure 29 presents the final groupings as it occurred after the abovementioned discussion. As shown, the student was able to combine two mathematical criteria as group names in three out of four groups proposed in this attempt. This kind of thinking is far more difficult and complex than using only one mathematical relation per grouping.

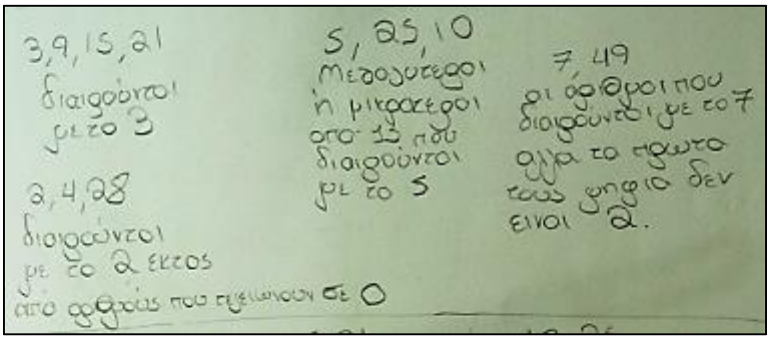


Figure 29. Second grouping attempt suggested by S18 (Activity 1).

As soon as this second grouping was achieved, the student proceeded to a third grouping attempt. This attempt started earlier with the student noting down a group of prime numbers (including numbers 2, 3, 5 and 7) so that he would not forget it while working at the same time on the second grouping attempt. In addition to the group of prime numbers, S18 added four more groups with the following names; “numbers divided by 3” (noting numbers 9, 15 and 21), “numbers divided by 5” (noting numbers 10 and 25), “numbers divided by 4” (noting numbers 4 and 28) and “numbers whose digit sum is larger than 12” (noting number 49). Again, there were numbers that should belong in more than one group, but the strategy of mark with a strikethrough the already grouped numbers prevented the student from having them also in mind when creating new groups.

> S18: Number 3 should also be placed in this group [pointing to the group with multiples of 3].
> R: So, what are you going to do?
> S18: Multiples of 3 larger than 8 [adding this into the name of the group]... Numbers 5 and 15 should also be in this category [pointing to the group with numbers divided by 5] 15 and 5 are allowed to be in here… and “numbers whose sum is larger than 6 or less than 3” [referring to the digit sum of the numbers included in the group of numbers divided by 5, which is instantly added in the name of the group].

Once more, the student exhibited evidence of flexible thinking. At first, the student formed the five groups involving different mathematical relationships such as multiples, prime numbers and observing the sum of digits. In a second look, the student realized that there were numbers that belong also to other groups. From this point, a second thinking cycle began, with the student preferring to alter the existing group names by narrowing down the range of numbers to be inserted. This way, only the numbers the student originally chose for each group were eligible to be in the group, and there was no need to erase and remove numbers and place them in another group. Figure 30 presents the final grouping as it occurred after the abovementioned discussion. Once more, the ability of the student to think, combine and handle multiple mathematical relationships was evident. In this specific attempt, the student combined multiples within a specific range and multiples with a specific range of digits' sum.

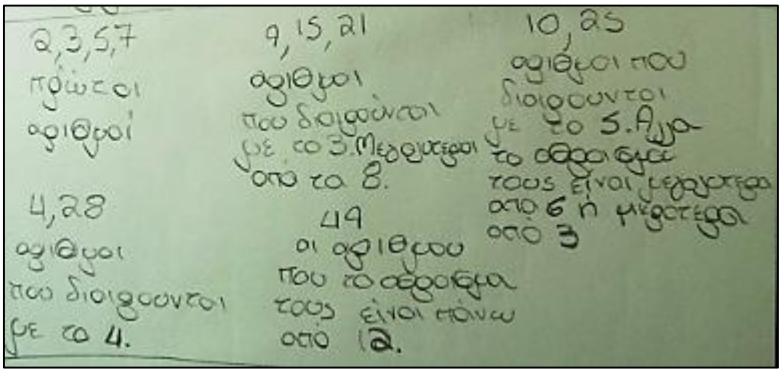


Figure 30. Third grouping attempt suggested by S18 (Activity 1).

In another attempt, the same student started by creating a group with multiples of 5 (writing down the numbers 5, 10, 15 and 25). For the next groups, the student argued on the numbers to include and the names of the groups:

> S18: Numbers whose square root is larger than 8, no, numbers divided by 2 [writes down 2, 4 and 28], that way 10 is also eligible for this group, except of 10 [this exception is noted to the name of the group]. Numbers whose square root is larger than 2 and smaller than 8, oh, we also have 25, so... numbers that can be divided by 3 or 7 [noting down numbers 3, 7, 21 and 49]. Now, we have another problem, since 28 is also divided by 7 and 15 is also divided by 3, so we should change the name into "odd numbers divided by 3 or 7".
> R: What type of number is 15?
> S18: Odd.
> R: What about 28?
> S18: Even …Oh, numbers that can be divided by 3 or 7 and the sum of their digits is not 10 or 6. This grouping attempt is correct [after checking the groups].
> R: There is one number left out of the groups.
> S18: Number 9… Oh, it goes here [adding number 9 into the group of numbers that can be divided by 3 or 7 and the sum of their digits is not 10 or 6].

Figure 31 presents the final grouping as it occurred after the abovementioned discussion. In this attempt, after the experience of the previous grouping attempts, the student seemed to be able to form more easily the complex combined mathematical criteria. For example, in the case of forming the second group of multiples of 2, the student quickly realized that 10 was previously added into the first group of multiples of 5. Simultaneously, he suggested to release it from the group proposing the revised name of the group. The student also seemed to feel more comfortable to use multiple mathematical relationships in the same group. As a result, he suggested a mathematical categorization involving multiples of one of two numbers, at the same time excluding numbers with specific digits' sum.

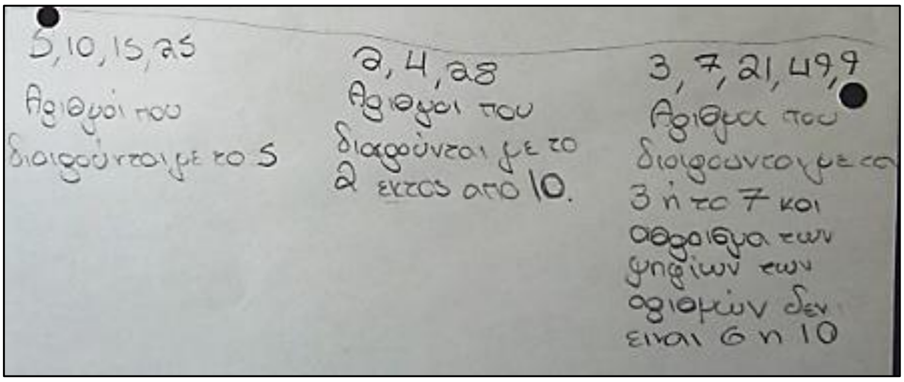


Figure 31. Fourth grouping attempt suggested by S18 (Activity 1).

In the following grouping attempt, the student created four groups. The groups referred to "square numbers" (writing down the numbers 9, 25 and 49), "numbers that can be divided by 2" (writing down the numbers 2, 4, 10 and 28), "numbers that can be divided by 3 except of 9" (writing down the numbers 3, 15 and 21), "numbers smaller than 8 and larger than 4 (writing down the numbers 5 and 7). He then checked the groups in order to see if all numbers had been grouped or if there were numbers that were eligible to be part of two groups.

> S18: Here [pointing to the first group of square numbers], number 4 should also be included.
> R: What are you thinking to do?
> S18: Numbers that are not evens, hey, that are odds in fact [adding this extension into the name of the first group].

Figure 32 presents the final grouping as it occurred after the abovementioned discussion. In this attempt, the student created the four groups from nothing, using two mathematical criteria in three out of the four groups. Later, he checked the groups and noticed that according to his formerly expressed mathematical criteria, number four was allowed to be part of two groups. The student easily thought of a solution to avoid number four from being a member of the group of square numbers by incorporating an additional mathematical criterion, that of being an odd square number.

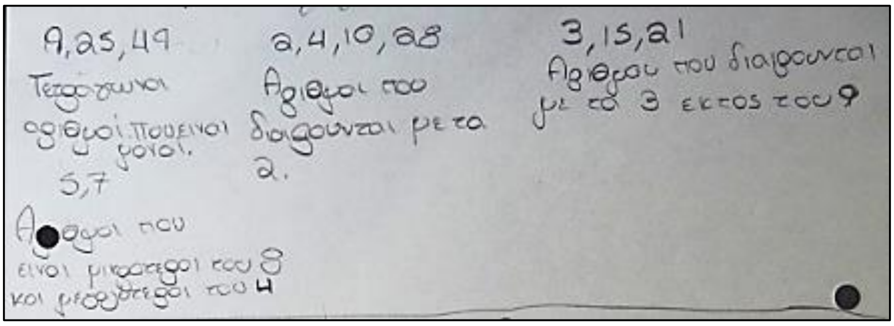


Figure 32. Fifth grouping attempt suggested by S18 (Activity 1).

Next, the student noted down numbers 5, 10, 15 and 25 under the name of multiples of 5. He then continued to create more groups.

> S18: Numbers divided by 2, but the sum of their digits to be larger than 1, so that 10 will not be part of this group [noting down numbers 2, 4 and 28 and naming the group as “numbers divided by 2 and the sum of their digits is larger than 1]. Numbers that are divided by 7, except of 28 [noting down numbers 7 and 49 creating a third group].

In comparison with previous grouping attempts, one may observe a change in student’s working style. In this attempt, the student was able to form the second and third groups in order to exclude a specific number from the start in each group. Based on the experience accumulated from the previous attempts, the student improved his technique by thinking ahead and thus incorporating at once the exceptions into the rules of the groups.

R: Are all numbers grouped?
S18: 3, 9 and 21 have not been placed in a group yet…Numbers divided by 3, except of 21 and 28.

The student wanted to create a new fourth group with two of the three remaining ungrouped numbers. As before, he showed his ability to simultaneously form the group name with the constraint to exclude specific numbers. By excluding number 21, the student acknowledged that number 21 is also eligible to be part of the previously created group, the group consisted of numbers divided by number 7, although he had forgotten to include it in the group before.

R: Where should the number 21 be placed?
S18: In numbers divided by 7, or numbers divided by 3.
R: It is your choice where to put the number.
S18: I will change the name here [pointing to the group of multiples of 3 excluding number 28] in to “numbers divided by 3 except of 28 and 21” in order to place this [number 21] in the other group. Except of 15 [renaming the group of numbers divided by 3 into “numbers divided by 3 except of 21” and adding number 21 into the group].

Once more, the student thinks in cycles, in a more elaborated way in comparison to previous grouping attempts. Here, the student attempted to state from the start the refined names of the groups, including the restraints to limit specific numbers from entering the groups. When despite his efforts to do this in parallel, he was confronted with a number eligible to be included in two groups, he exhibited great ease in restating the categories in order to accommodate the numbers according to his wish. Figure 33 presents the final grouping as it occurred after the abovementioned discussion.

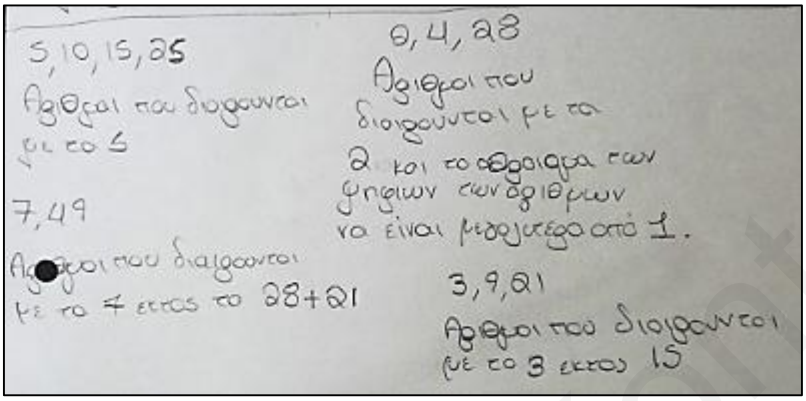


Figure 33. Sixth grouping attempt suggested by S18 (Activity 1).

In a new grouping attempt, the student created and named the four groups all at once. There was no need to proceed into a new cycle, since all numbers were grouped and each number was eligible to be part of only one group.

> S18: Factors of 20 [writing down the numbers 2, 10, 4 and 5]…factors of 21 [writing down the numbers 3, 7 and 21], odd numbers larger than 8 except of number 21 [writing down the numbers 9, 15, 25 and 49] … numbers with 10 as a digits’ sum [noting number 28].

Figure 34 presents the final grouping as it occurred after the abovementioned discussion. It is impressive that the student was able to formulate all groups at once, without the need to rephrase some of the grouping criteria, as in the case of previous grouping attempts. One may assume that through the experience and knowledge gained through previous grouping attempts, the student was now ready to suggest a new grouping method that followed all the task’s constraints simultaneously. Furthermore, the level of student’s thinking is evident in the complexity of the criteria he used to form the third group. More specific, the categorization of odd numbers larger than 8 except of 21 involves paying attention into three relations all at once; odd numbers, a specific range of odd numbers, and the exclusion of a specific number.

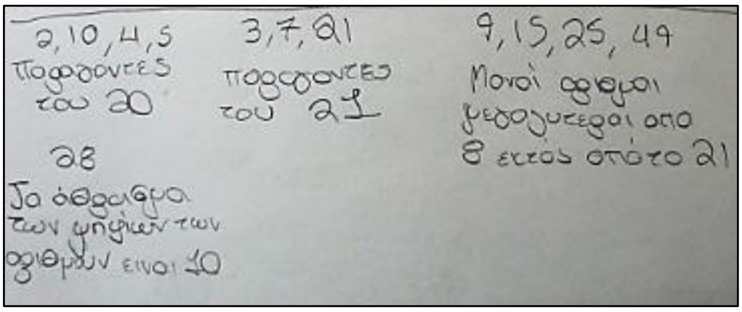


Figure 34. Seventh grouping attempt suggested by S18 (Activity 1).

In an eighth grouping attempt, the student suggested a new mathematical relation that he did not use before, that of square root.

> S18: Numbers with 2 as one of their digits [writing down the numbers 2, 21, 25 and 28] …factors of 15 [writing down the numbers 3, 5 and 15]…factors of 63 [writing down the numbers 7 and 9] except of 3… factors of 40 [writing down the numbers 4 and 10]. Square numbers with a square root, larger than 6 [writing down number 49].
> R: Are there any other numbers that are factors of 40?
> S18: 2 and 5. Except of 5 and 2 [adding this constraint to the name of the group of factors of number 40]…

Figure 35 presents the final grouping as it occurred after the abovementioned discussion. In this attempt, the student, was able to consider as a grouping method, the use of multiples of a specific number excluding at the same numbers that have already been included in previous groups. However, in the case of factors of 40, the student forgot that 2 and 5, although already grouped in other groups are also factors of 40. Taking the lead from the question of the researcher, the student clearly showed once more his ability to rephrase the name of the groups to accommodate the numbers accordingly. Left with number 49 at the end, the student quickly thought of creating a group with square numbers. At the same time, to ensure that 9 and 25 would not be entitled to be part of this group, the student chose to limit the eligible numbers, by stating that the group included numbers whose square root is larger than 6.

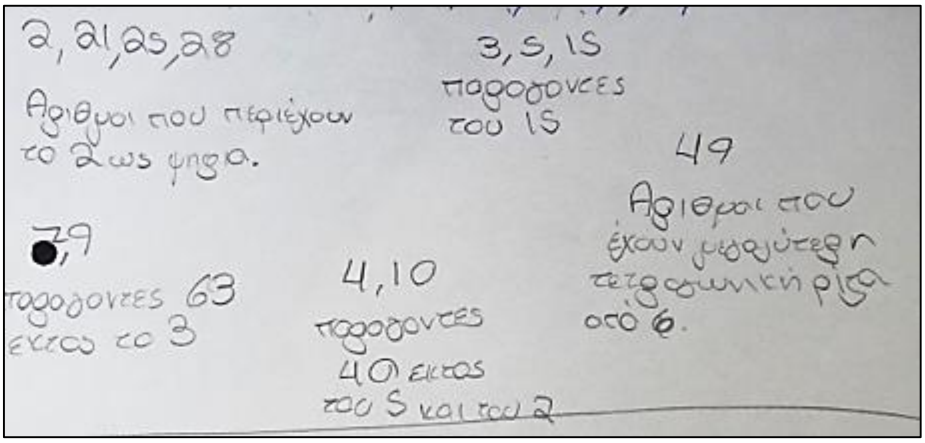


Figure 35. Eighth grouping attempt suggested by S18 (Activity 1).

In the next categorization of the twelve numbers, the student had a clear vision.

S18: Numbers having an even number as the sum of their digits, let's say 15, 1 plus 5, 6 [writing down the numbers 2, 4, 15 and 28]… even numbers [writing down the numbers 3, 5, 7, 9, 21, 25 and 49], numbers that end in zero [writing down number 10].
R: Is there any odd number left out?
S18: [immediately] Number 15, so except of 15.

Figure 36 presents the final grouping as it occurred after the abovementioned discussion. In this categorization, the student involved a variety of mathematical relations, such as having an even digits' sum, the set of odd numbers excluding a specific numbers and the ending digit being zero.

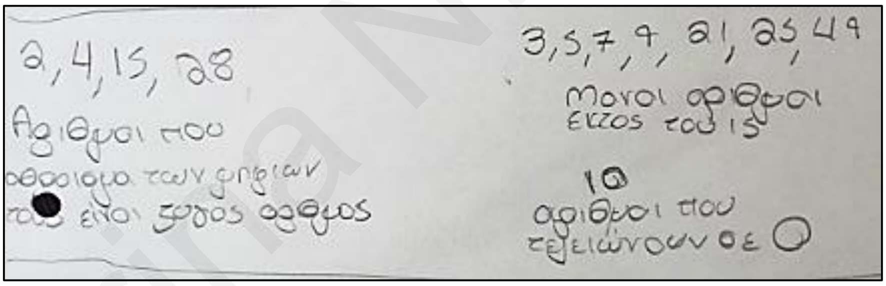


Figure 36. Ninth grouping attempt suggested by S18 (Activity 1).

In conclusion, this specific student provided nine distinct successful categorizations of the given set of 12 numbers, with ease. The student seemed ready to provide more groupings, but there was a need to move to other tasks too due to the amount of time allotted to the specific activity and the limited time available. Eight out of nine proposed grouping methods were entirely original, since they were not suggested by any of the other students. The only exemption was the grouping involving odd and even numbers. Throughout his work in this task, the ability of the student to think, combine and handle multiple mathematical relationships was evident. More specific, multiples within a specific range, multiples with a specific range of digits' sum, multiples of one of two numbers, while at the same time excluding numbers with specific digits' sum, numbers containing a specific digit, a specific range of square numbers, multiples of a number, factors of a number, numbers with a specific range of square root, were some of the mathematical relations mentioned by the student.

The student's working strategy evolved as time passed. At first, the first attempt focused on two simple groups; that is, even and odd numbers. Then the student tried to group the numbers in more than two groups in each grouping attempt. In the second grouping attempt, the student showed his ability to think in cycles. First, the student, created some of the groups and then realized that there were specific numbers that were allowed to be included in two groups. The interesting fact was that the student strived to find a different group name in order to accommodate the numbers he originally chose for each group, rather than erasing and removing numbers and placing them in another existing group. That specific choice that the student made, led him to combine two mathematical conditions for the same grouping. To be exact, the student was able to combine two mathematical criteria as group names in three out of four groups proposed in this attempt. This kind of thinking is far more difficult and complex than using only one mathematical relation per grouping. After the experience of the first two grouping attempts, the student seemed to be able to form more easily the complex combined mathematical criteria. When faced with a number eligible to be a member of two groups, the student came up with a solution very easily, in order to exclude that number from one of the two groups by incorporating an additional mathematical criterion. However, this working style did not remain the same throughout all attempts. In fact, during the sixth grouping attempt, a change in student's working style was observed. The student attempted to form the second and third groups in order to exclude a specific number from the start in each group, instead of thinking in cycles and revising the groups. In general, based on the experience accumulated from the previous attempts, the student improved his technique by trying to think ahead and incorporate at once the exceptions into the rules of the groups.

Another example of thinking in cycles was the work of S14. S14's way of thinking was not straight-forward. Namely, the student first thought of a way to group numbers, then when experiencing difficulties, he went back, rephrased some of the groups, even delete some of them and created new ones. Therefore, it is not that in one moment the whole solution is there and the student just writes it down. It involves a lot of thinking, reorganizing one's thoughts and checking if all parameters are satisfied. There is another interesting aspect of the way S14 handled a problematic situation and communicating this to others. Although S14was talkative when expressing his thoughts, still, there were several things he preferred to do in his mind. For this reason, although he knew what he wanted to say, people who listened to him might not understand his thoughts, because he didn't form complete sentences as he was at the same time thinking. Notice also that in the attempt described below, three completely seemingly unrelated sets of numbers manage to provide the required groupings; square, prime and two-digit triangular numbers.

> S14: 4, 9, 25, 49 [writing down "square numbers"]. Prime numbers [writing down the numbers 2, 3, 5 and7] … [thinking] … I should group, how do we call them, that are divided, that have a divisor... more than two divisors... [writing down "prime numbers" and the numbers 2, 3, 5 and 7]. Place triangular numbers noting "triangular numbers" and the numbers 3 and 10]. Three times four, 12, six, four times five, 20, 10, ok, five times six, 30, 15, ok [adding number 15 to the group of triangular numbers], 21 next [writing down numbers 21 and 28]. Ok, all numbers have been placed.
> R: Here [pointing to the groups of prime and triangular numbers], you have number 3 in two groups. So, you should remove it from one of the two groups.
> S14: Only two digit numbers.
> R: What about 25 and 49?
> S14: Oh yes… Two digit triangular numbers.

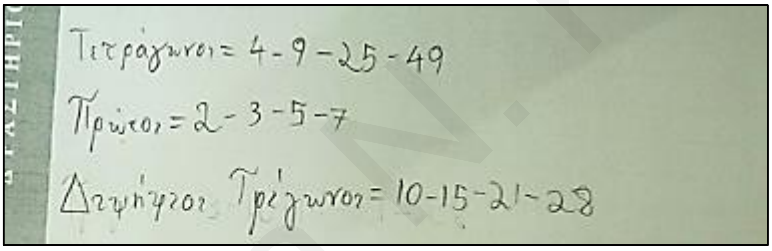


Figure 37. First grouping attempt suggested by S14 (Activity 1).

As a second grouping attempt, the student easily sorted the numbers into odds and evens. For a start in a third grouping way, S14 focused on consecutive numbers, forming a group entitled "consecutive numbers" and inserting the numbers 2, 3, 4, 5 and 9, 10].

> S14: Numbers left out are unrelated …
>
> R: You mean they cannot be placed in one group… They can be placed in more than one groups if you wish.
> S14: Numbers 7 and 49, square number and square root. Also numbers 5 and 25 but I've already placed them in a group… [forming a third group entitled "divisors of 7" and inserting numbers 7, 21, 28 and 49. After realizing that numbers 7 and 49 belong to two groups, he erases the group of square number and square root]. We are left with numbers 25 and 15 ungrouped.… [thinking, seemed to be stuck].
> R: What do numbers 15 and 25 have in common?
> S14: Number 5, but…

R: What do you mean?
S14: Five [pointing to the unit digit, getting ready to note it down]. No ... we also have number 5 here [pointing to number 5 already included in the group of consecutive numbers].
R: What do you mean?
S14: I can’t comment on the units’ digit since we also have five in the first group.
R: How can you change your group name to avoid having number 5 in the group? You need numbers 15 and 25 but not number 5.
S14: Two digit numbers.
R: Well done!
S14: Two digit numbers with a common number… no … let’s put … two digit numbers with common unit digit of 5.

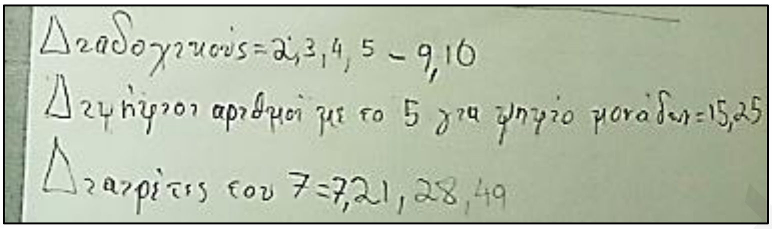


Figure 38. Third grouping attempt suggested by S14 (Activity 1).

As a fourth grouping attempt, the student very easily sorted the numbers into single-digit and double-digit numbers. For a start in a fifth grouping way, S14 focused again on consecutive numbers like he did before in the third grouping attempt. However, quickly he turned into other relationships. More specific, influenced by the previous grouping attempt into single-digit and double digit numbers, the student ended up using single and two digit numbers, further categorizing them into odds and evens, creating four groups.

S14: What if we say each number has one divisor or more? Let’s say 2 and 28 but there is also 4 and 7... But 9 doesn’t ... Can we have consecutive prime numbers? Not prime numbers, odd consecutive, 3, 5, 7, 9… What if we have pairs? 3 with 5, 7 with 9, but 15…
R: If you put 3,5,7,9 according to your plan, what would the group’s name be?
S14: Odd single digit numbers [noting the numbers 3, 5, 7 and 9]…. Even single digit numbers 2 and 4 … we should have odd double digit numbers [noting the numbers 15, 21, 25 and 49 and a fourth group entitled “even double digit numbers”, noting down the numbers 16 and 28].

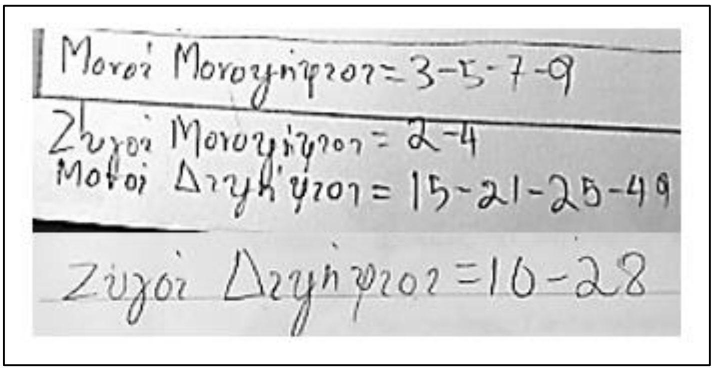


Figure 39. Fourth grouping attempt suggested by S14 (Activity 1).

This student, similarly to other students mentioned previously, showed his ability to handle multiple mathematical relationships simultaneously, as illustrated in his first, third and fifth grouping attempts.

In total, the excerpts shown in this section reveal that mathematically promising students are able to provide complex and highly original solutions, showing great ease in manipulating the data of the task in hand and also handling multiple mathematical relations simultaneously. While problem solving, an effective strategy used is that of thinking in cycles. This was mainly triggered by the need to come up with new grouping ways compared to the previously suggested ones. In some other cases, whilst in the process of a new grouping attempt, students may have found out that they were repeating a previously suggested grouping attempt or that there were numbers belonging to other groups as well. Two possible reactions to this realization were observed. Either to come up with a different group name in order to accommodate the originally chosen numbers for each group and differentiate from a previous attempt, or to erase and remove some of the numbers and placing them in a new group. Especially in the case of being confronted with a number eligible to be included in two groups, students exhibited great ease in restating the categories in order to accommodate the numbers according to their wish. The need to alter their original grouping approach, gave the opportunity to students to combine more than one mathematical criterion for the same grouping. As time passed by, students felt more comfortable to use multiple mathematical relationships in the same group. Sometimes, a revolution in students' working style occurred. Usually it was during the first attempts that students realized that they had numbers entitled to be part of more than one group and were put in the process of thinking how to overcome this obstacle. Later, based on the experience accumulated from the previous attempts, there were several students that improved their working approach by thinking ahead and incorporate at once the exceptions into the rules of the groups from the start. Thus, with a more strategic approach applied, there was no need to proceed in a new thinking cycle anymore. This ability of course was able to be developed and observed thanks to the creative nature of the specific task, that allowed the student to attempt to approach the problem in many different ways, thus forcing them to repeat in innovative ways the thinking process by employing different mathematical grouping criteria. Hence, we may deduce that creative tasks are important to be included in any assessment of mathematical giftedness, since they provide the opportunity to a potentially gifted person to show his/her abilities, knowledge and skills, that could otherwise be overlooked of tasks were of a certain type.

Control of multiple mathematical relationships at once. It was remarkable how the vast majority of students observed was able to handle multiple mathematical relationships simultaneously. This was evident in most examples described in previous sections. In this section, it is the intent of the researcher to refer to occasions were students exhibited their ability to combine simultaneously a number of mathematical relationships without repeating examples discussed previously. So, in order to avoid repeating data, this section analyses instances where this ability was documented, other than those aforementioned to display other cognitive processes. Obviously, all occasions discussed before where students, among others, showed evidence of their ability to make multiple mathematical associations concurrently, enhance and complement our findings discussed here that this is an ability possessed by mathematically promising students.

An occasion where this ability was manifested was in an approach suggested by S26 in Activity 1 and it is illustrated in Figure 40. In this approach, the student verbally explained that he tried to create "as many groups as possible, whose sum is an odd number".

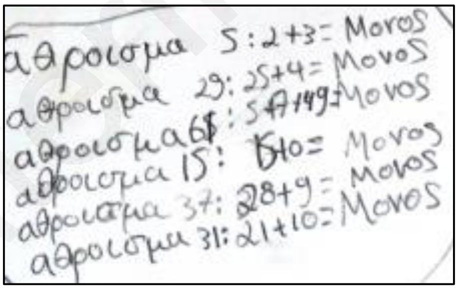


Figure 40. A grouping attempt suggested by S26 (Activity 1).

Hence, the student chose to focus on two relationships at the same time, specifically getting an odd sum and also obtaining a different sum in each group to differentiate amongst groups. This reasoning is complex, since it requires to combine two mathematical criteria. The student had to make the necessary combinations to obtain suitable sets of numbers to produce six different odd sums. This approach was also unique and original, since it was not proposed by any other student observed for the purposes of this study.

The management of multiple mathematical criteria was also represented in unique ways. For instance, S34 was finding it difficult to verbalize his thinking and express the way in which he chose to group the selected numbers. Therefore, he chose to exemplify his grouping approach through a representation he clearly felt comfortable with and reflected his thinking; a Venn diagram (Figure 41).

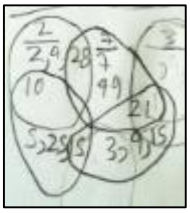


Figure 41. A grouping attempt suggested by S34 (Activity 1).

In his attempt, the student chose to illustrate relations of union and intersection in the way he felt more comfortable with, that is a pictorial informative representation. In this way, the student managed to organize the numbers in such a way that multiple mathematical relations may be observed. For instance, once may comment on the numbers that are multiples of 2 and 7, while at the same time are not multiples of 5. Another remark could be to group together numbers that are multiples of 3 and 5, but not multiples of 2. There are so many complex mathematical relationships to be made from this single diagram, and the student provided this representation with great ease, as it was the way he had already visualized the numbers in his mind.

To sum up, mathematical promising students are capable of handling multiple mathematical relationships simultaneously. Just to mention some of the relationships observed in students' solutions, that include multiples within a specific range, multiples with a specific range of digits' sum, multiples of one of two numbers, while at the same time excluding numbers with specific digits' sum, numbers containing a specific digit, a specific range of square numbers, multiples of a number, factors of a number, numbers with a specific range of square root. Also, mathematically gifted students are able to represent them in a way it allows them to observe the associations needed.

Fluency for expediency. During observation, the researcher observed a striking working style of mathematical promising students; they identified an economical method and they stuck with it to achieve greater level of fluency and flexibility.

For most of the students, Activity 6 was perceived as an anchoring task. In other words, the students used the first figures they made as a foundation on which to base their subsequent figures, by exploiting to the fullest the existing figures in order to come up with new figures, as many as possible. This behavior was manifested in three different approaches: sliding pieces, rearranging pieces and "complete what is missing" technique.

An indicative example of this ability is shown in Figure 42, using the sliding approach.

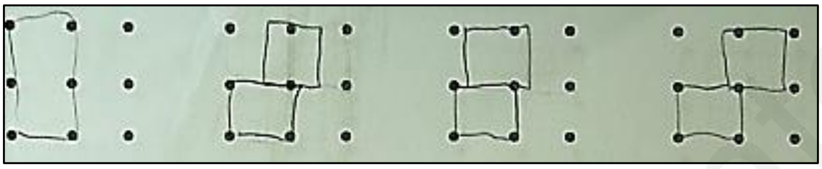


Figure 42. The sliding approach employed by S12 (Activity 6).

S12, in this case, was able to take advantage of the first constructed figure of a rectangle to produce as many similar shapes possible with the least load on his behalf. What he did to produce the other three figures was to slide slightly the upper square to the left or to the right, making sure it is adjacent to the bottom square to avoid splitting the figure. This action, implies that the student acknowledged that all he had to do is keep the area of two squares the same, but with minimal effort and sliding one of the two parts, he could produce other shapes too. The student also proceeded to a generalized statement, declaring that a large number of similar shapes could be made by sliding one of the two squares to different positions to the left or to the right. His technique seemed to be not only evolving as figures are added, but it was also dynamic, since the student was able to produce new figures by modifying slightly the original figure by just sliding one square piece and not reshaping the figure.

A great number of students also attempted to produce new figures using their original figures us a starting point, using the rearrangement approach. However, their approach was different in comparison to the aforementioned method of sliding pieces. Following this method, students assembled different pieces to create an initial figure with the requisite area of 2 cm^2. Then, instead of sliding fragments to form new figures with the same area, this group of students chose to reorganize the same fragments into different positions and orientations, transforming the existing figure into a new one (Figure 43).

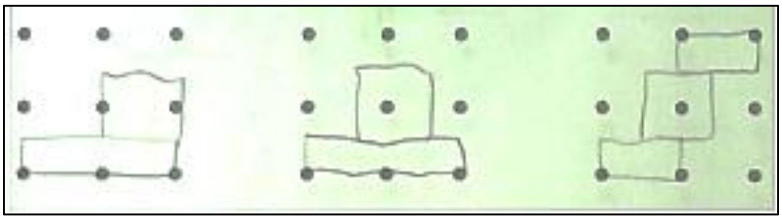


Figure 43. The strategy of rearrangement of shapes employed by S12 (Activity 6).

As presented in Figure 43, S12 used the “sliding method” to produce the second figure based on the initial figure, while afterwards he chose to alter his approach. More specific, he selected to rearrange the three pieces to create a third figure. Later on, the student moved on to break more the fragments used to incorporate fragments of an area of ¼ cm^2. The approach to rearrange the pieces so as to come up with new figures with minimal effort was proven effective once more for this student (see Figure 44).

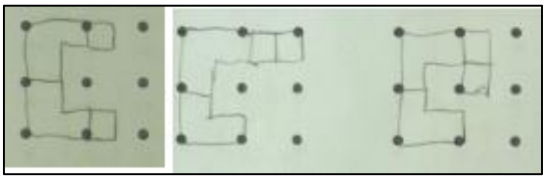


Figure 44. The strategy of rearrangement of shapes employed by S12 (Activity 6).

In Figure 44, the student first formed the first shape with two pieces of ¾ cm^2 each and two pieces of ¼ cm^2 each. Then, he chose to retain intact the two parts of ¾ cm^2 and repositioned the remaining two pieces of ¼ cm^2 each. While minimal effort was exerted, two novel figures obtained.

S26 applied the rearrangement approach more than once during his work. Figure 45 presents some of the figures that were produced.

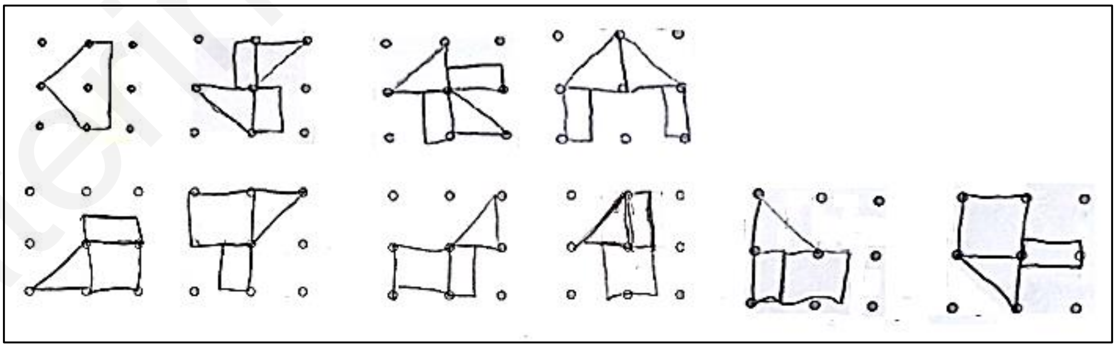


Figure 45. The strategy of rearrangement of shapes employed by S26 (Activity 6).

In the first row, the student first made a figure composed of two ½ cm^2 triangles and two ½ cm^2 rectangles. Then, the student relocated the four pieces to form three more figures with the same area. In the second row, the student first formed the first shape with one square piece of 1 cm^2 and two pieces of ½ cm^2 each, one being a rectangle and the other a triangle. Then, he chose to reposition the three pieces. This decision resulted to five more new figures. Once more, with two starting figures and using the rearrangement technique, eight novel figures were obtained with the least cognitive load on behalf of the student. Notice that the student shows that he feels comfortable with this approach, since it is applied more than once an each time different type of shapes are composed to synthesize the end figures.

S9 also found the rearrangement approach accommodating that he used twice during his work. Figure 46, presents some of the figures that were produced.

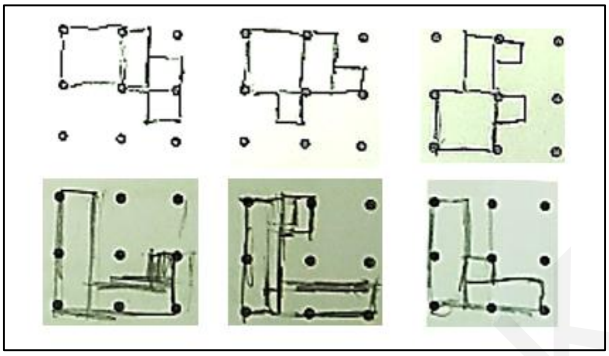


Figure 46. The strategy of rearrangement of shapes employed by S9 (Activity 6).

In the first row, the student first made a figure consisting of four components; one square piece of 1 cm^2, one ½ cm^2 rectangle and two ¼ cm^2 squares. Then, the student rearranged the four pieces, producing two more figures, retaining the area constant. In the second row, the student first formed the first shape with one rectangular piece of 1 cm^2, a rectangular piece of ¾ cm^2, and a square pieces of ¼ cm^2. The subsequent two figures were produced after repositioning the three fragments. Similarly to the previous student, this student feels competent to use this approach. This is why it is used more than once, each time with different types of shapes combined.

S10 applied the rearrangement approach in his work using pieces of 1 cm^2 and ¼ cm^2. These figures are presented in Figure 47.

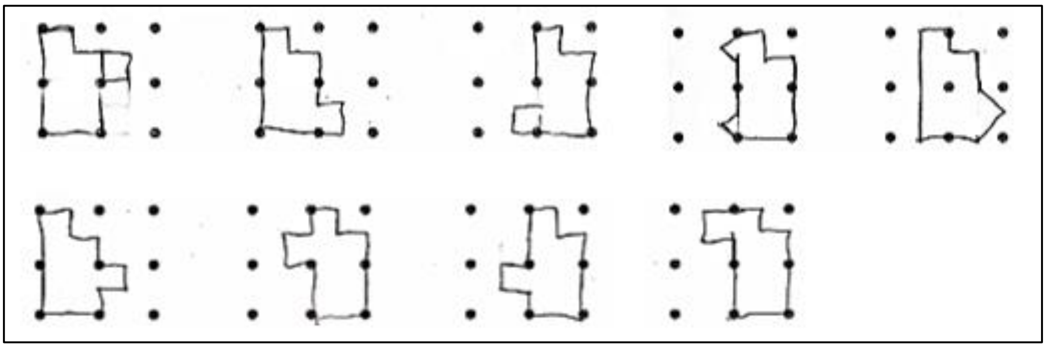


Figure 47. The strategy of rearrangement of shapes employed by S10 (Activity 6).

The first figure was constructed out of a large piece of 1 ¾ cm^2, and a smaller piece of ¼ cm^2. In order to make new figures, the student decided to keep the large piece intact, although in some examples this piece just slid to the right. In the first three figures, what really changed from one figure to the next was the position of the smaller piece of ¼ cm^2. In the fourth and fifth figures, to provide some differentiation, the student replaced the square piece of ¼ cm^2 with two triangular pieces of 1/8 cm^2, placing them in different positions once more. In the second row, the student returned to the use of a square piece of ¼ cm^2. Again, the larger piece remained the same and the student experimented with the position of the square piece of ¼ cm^2.

S7 worked in a similar way to S10, but his figures came out more elaborate due to the use of triangular pieces of ½ and ¼ cm^2 (Figure 48).

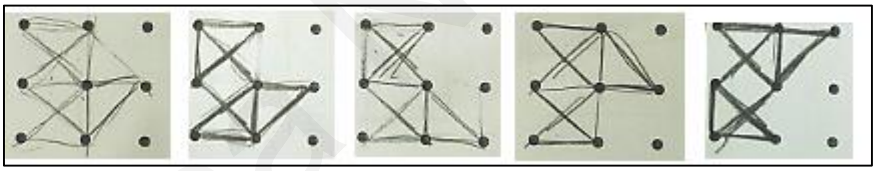


Figure 48. The strategy of rearrangement of shapes employed by S7 (Activity 6).

In order to provide many similar but not identical shapes effortessly, S7 chose to keep one part of the figure the same and experiment with the remaining parts. Namely, the left part of the figures, consisting of four triangular pieces of ¼ cm^2, resulting to two hourglasses sitting on top of each other, presented itself in all five figures. The student used this part as the backbone on which to build the end figures by adding the remaining pieces to reach the requested area of 2 cm^2. The remaining 1 cm^2 was completed in different variations. Notice the similarity between the ways the remaining 1 cm^2 was added in the first two figures. The difference between them is just the one triangular piece of ¼ cm^2 that is placed at the right part of the top hourglass in the first figure, whereas in the second figure the same part is placed at the left part of the top hourglass. In order to create the third figure, the student repeated the second figure, with the difference of changing the orientation of the triangle of $\frac{1}{2}$ cm^2. For the fourth figure, the student chose to relocate the same triangle, though the orientation remained the same. To achieve the fifth figure, the student retained the triangle of $\frac{1}{2}$ cm^2 at the upper right part, altering its orientation in comparison to the previous figure.

The third approach to produce many similar but not identical figures while conserving on time and cognitive load, was the “complete what is missing” technique. As shown from the work of the students who used it, this approach was suitable and convenient to use in cases of using pieces of $\frac{3}{4}$ cm^2 as components of the requested figures. One of the students who applied this method to produce various figures was S29, with a part of her work being illustrated in Figure 49.

*Figure 49.*The “complete what is missing” technique employed by S29 (Activity 6).

S29 was mentally visualizing the area produced by two squares of 1 cm^2. Then, she drew pieces of $\frac{3}{4}$ while simultaneously thinking the parts that were missing to complete the area of 2 cm^2. At that point, she placed the missing parts in another position, adjacent to the already placed figure. For example, in the first figure, the student first drew two pieces of $\frac{3}{4}$ cm^2 and at the same time she was able to attach two more pieces of $\frac{1}{4}$ cm^2 each, to compensate for what was missing to reach to an area of 2 cm^2. Similarly, in the second figure, the student again drew two pieces of $\frac{3}{4}$ cm^2 and then added a triangular piece of $\frac{1}{2}$ cm^2. For the last two figures, the strategy slightly changed. Instead of starting by drawing two pieces of $\frac{3}{4}$ cm^2 each, S26 favored to draw a square of 1 cm^2 accompanied by a piece of $\frac{3}{4}$ cm^2. Then, she continued to complete the figures by adding a triangular overhang of the required $\frac{1}{4}$ cm^2.

The same philosophy was applied in the case of the work of S35 (Figure 50). Similarly to S29, the student started by drawing pieces of $\frac{3}{4}$ cm^2 while simultaneously thinking about the parts that were missing to complete the area of 2 cm^2. To reach the requested area, he placed the missing parts in another position, adjacent to the already placed figure.

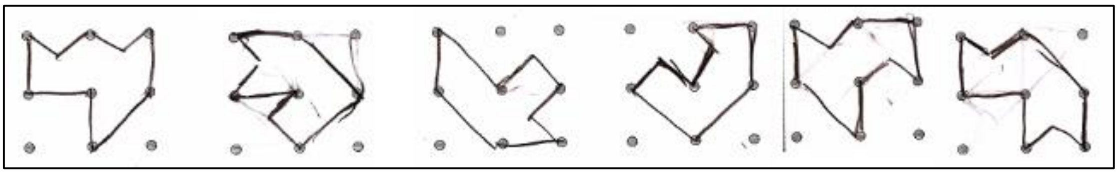


Figure 50. The “complete what is missing” technique employed by S35 (Activity 6).

More specific, the student started by drawing two pieces of ¾ cm^2 and then added a triangular piece of ½ cm^2. For the second figure, the student used again two pieces of ¾ cm^2, this time positioned differently and also a triangular piece of ½ cm^2. For the third figure, the student started by drawing the piece of ¾ cm^2 at the bottom right part of the figure while simultaneously added the triangular piece of ¼ cm^2 on top of it. Afterwards, she completed what was missing with the left part of the figure. For the fourth figure, the student started by drawing the piece of ¾ cm^2 at the upper right part of the figure while simultaneously added the triangular piece of ¼ cm^2 at the upper left part. Afterwards, she completed what was missing with the two triangular shapes of ½ cm^2 at the bottom half of the figure. For the last two figures, the student first placed two pieces of ¾ cm^2 and then completed the figure with a triangular piece of ½ cm^2 adjacent to the existing figure.

In this section, the ability to exploit to the fullest a specific strategy in order to achieve a greater level both of fluency and flexibility was exemplified through numerous descriptions of students’ work. This ability required students to perceive the problematic situation posed to them as an anchoring task and was manifested in three different approaches. Thus, students managed to show once more their vast repertoire of strategies and their ability to select the one they find most appropriate each time that also would benefit them more in the problematic situation they were found in.

Curtailment of the process of mathematical reasoning and economical thinking. At times, students observed displayed signs of their ability to curtail the process of mathematical reasoning by eliminating intermediate steps or even chose to work in an economical way of thinking.

During observation, there were times where students pursued an economical way to reach to a solution. For example, in case of Problem 3, a student chose to find the square of a smaller number in comparison to a larger number he had first thought of.

> S9: Let me try with number 32. No, it's too large number to multiply with itself [He erases the number]. I will try with three. 9, 9 minus 3, 6, even again. The result is always even.

The student's decision of using a smaller number to help him arrive to a conclusion, made his calculations easier. His words show that he acknowledged that number 32 was too large to multiply with itself and that he also recognizes that regardless of the size of the number, he may reach to the same generalization. Thus, he chose to perform calculations with a smaller number, showing the ability to choose the most economical way to solve a problematic situation.

Similarly, the manner in which students behaved in the context of Activity 6, revealed their economical way of thinking. For example, students chose to proceed to minor modifications to the first figures they formed, in order to produce many different and original figures with the least possible effort. This was obtained using three approaches; (a) sliding the composing pieces of the figures slightly to another direction without splitting the figures, (b) rearranging the composing pieces by placing them in a different way while keeping the area constant and (d) looking to compensate for the area that is missing to a formed figure by adding an additional piece.

In the same activity, other techniques used by students also provided evidence that they tend to function and think in an economical way. For example, instead of working randomly and producing diverse figures alternating the way of working to form each figure, the majority of the students chose to focus in a specific way to produce figures and evolve the specific technique. Only when the capacity to produce new figures using the same underlying idea was exhausted, students moved on to think of a different way to create new figures. Hence, students worked in an economical manner by focusing their efforts following a specific philosophy each time. For instance, a student chose to compose figures by using eight square pieces of ¼ cm^2. In order to work economically, the student chose firstly to follow the technique of reorganizing the eight square pieces to make the production of new figures easier, as shown in Figure 51.

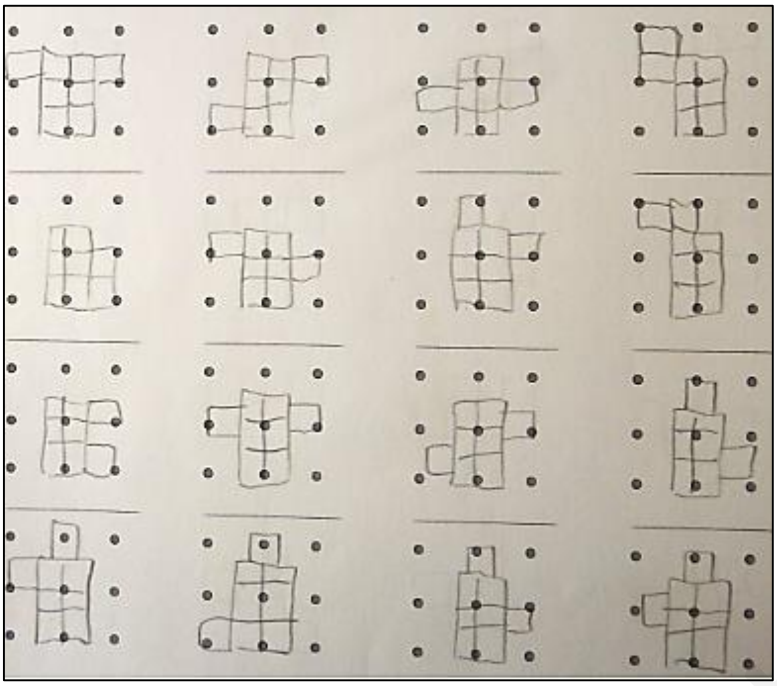


Figure 51. Economy of thought as evidenced in a small segment of S12’s work (Activity 6).

The figure illustrates only a small fraction of this student’s work, since the student provided many more figures by rearranging the eight square pieces. Since a large number of figures was going to be produced by rearranging the eight pieces of ¼ cm^2, the student came up with a way to organize his actions in order to avoid repeat previously formed figures and making sure that he did not miss any possible version of the figures using the same idea. For the specific example, the student decided to have in the same position in all figures the six of the eight square pieces (two columns with three pieces in each column in the bottom central designated dot area) and reorganize the remaining two pieces, Therefore, the researcher selected to present in Figure 51, only the figures produced when remaining dedicated to the specific technique of leaving intact the six square pieces of ¼ cm^2. Another important part of economical way of thinking, was that the student chose to exhaust this strategy until no more figures could be formed with the same way. Thus, after drawing the figures shown, the student proceed into forming a different backbone of six square pieces on which to base new figures and started again to construct novel figures by rearranging the two other pieces. This was once more continued until no more shapes could be formed with the same backbone of six pieces.

These behaviors indicate that mathematically capable students are able to select shortest route and the most economical way to work under complex situations. They have the capability to simplify their work, as in selecting simpler numbers to work with, if this is does not affect the outcome. In addition, they have shown that they can exploit a task's conditions into their own benefit; thus, they were able to produce many correct responses with minimal effort and minor modifications. Furthermore, they may evolve and persist on a certain working strategy while afterwards exhausting a certain way of working until no more responses could be produced by working in the same way, Also, they may develop a specific system to keep track of their work and make sure no duplicate responses are provided, in case of creative tasks where fluency is an important parameter.

Reversibility of mental processes. Student's work in a particular problem, namely Problem 4, made possible the appearance of another distinctive behavior during problem solving. This was the ability to reverse a thinking process, in other words, thinking backwards from the end to the start to solve a problem. This was made possible through the nature of the particular task.

More specific, students were asked to type number 1 000 000 on a calculator. Then, by pressing only the keys 7, +, -, ×, ÷ and = as many times as they wish, they should get to number 7 as a result. Of all the problems posed to the students during observation, this was the one they found most difficult. It was a type of problem that they had never faced before and although it posed certain restrictions, there was an unknown field to explore and an unknown route to discover.

As such, the first thing students did, was to start trying out different options, to reach to number 7. The most common approach followed was to start subtracting numbers formed out of digits of 7 in order to reach to the desired result. Although a series of random subtractions at first, sometimes this approach led to the required outcome. This was the case for example of S16, whose attempt was proven successful. The student used a series of subsequent subtractions with subtrahends made of 7s digits, ranging from 7777777 to 7, to reach to number 1 and then to number 7 by multiplication. The last six operations used to arrive to number 7 on screen, are shown below.

106-77=29

29-7=22

22-7=15

15-7=8

8-7=1

1x7=7

After this productive attempt, the researcher pointed out to the student that although this was a possible path to lead to the desired outcome, it was in a degree based to random subtractions. The researcher gave a hint to the student to see if he could work backwards, starting from number 7 and trying to reach to 1 000 000.

> R: If you were to think backwards, starting from number 7, what could you do to reach to 1000 000?
> S16: In the previous way, I ended up to 8 and then to 1.
> R: So, if you could find an easy way to reach to number 1? What could you have on the screen to make one operation and get to number 1?
> S16: Something divided by 77 or 777 or 7777...
> R: What do all of these numbers you proposed have in common?
> S16: They are all divided by 7 and they all have the same digit, 7.
> R: If you could reach to a number composed of sevens, what would you do next?
> S16: Divide it with itself. [typing 1 000 000 x 7 = 7 000 000, then 7 000 000 + 777 777 = 7 777 777, next 7 777 777 / 7 777 777 =1 and finally 1x7 = 7]
> R: Great job! Except of using 1 times 7 to reach to 7, is there any other number with which you could result to 7?
> S16: Yes, by going to number 0 and then add 7.
> R: And what could be in the previous screen on the calculator to construct 0?
> R: 7 777 777 minus 7 777 777, equals 0.

After the aid provided by the researcher, the student was able to work reversely. Not only the student managed to reach to number 7 by forming a number composed out of sevens and resulting to number 1, but easily he suggested a second approach by getting to number 0. More importantly, the student suggested the second approach without writing anything on paper or typing on the calculator. This suggests that he already conceived the underlying principles of the problem and he could easily think of an alternative solution to end up to the result, based on his previous attempt.

In the same activity, S4 approached it in an interesting manner, trying to work in an organized way. This student realized that random subtractions are not only time consuming, but they might not yield the desired outcome. At first, S4 typed 1 000 000 divided by 7, resulting to a quotient equal to 142857.14. Next, the student cleared the calculator screen and then he typed 1 000 000 divided by 142 000, resulting to a quotient of 7.0422535. At this point, the student seemed to be troubled.

> R: So, the problem is that it does not divide with 7 without leaving a remainder?
> S4: Yes. [typing 1 000 000/7=142857.14, writing the result on the paper, typing 7× 142858=1 000 006]. It doesn't exactly fit… [typing 7 000 000/7=1 000 000]. Ah, nice, I think [typing 1 000 0007×7=7 000 000, then typing 7 000 000-77-77-77=6 999 769]. If I could subtract sevens all the time from 1 000 000 until reaching to 7, but that needs a lot of time… [typing 7 000 000- 777 777= 6222223]… [erasing everything from the calculator screen and typing again the number 7 000 000].

The student first tried to divide one million by 7 and realized that the division produces a remainder. By multiplying reversely 7 times the remainder rounded up, he realizes that 1 000 006 can be divided with number 7 without leaving a remainder. It was fascinating to observe if this student would exploit this gained knowledge.

> R: What are you thinking?
> S4: I'm thinking of dividing or subtracting something… to reach to a smaller number from here, or subtracting to result to a smaller number, or dividing and afterwards subtracting… I have to turn this number to a smaller one [typing 1 000 000…whispering, erasing everything on paper and writing down again 142857, whispering again]…I am thinking if I can reach to a number who if divided by 7 will give me a 7…
> R: Ok…
> S4: …Because I can press only 7… I think [typing 1 000 000], no… 13, 20, 27...
> R: What are you thinking?
> S4: Cause 1 000 006 is a multiple of 7… and if I was going to add a number [referring to number seven], that number would be one more time seven is multiplied…

The student supposed at that point that division, subtraction or a combination of the two operations were his only options to reach to a smaller number. Moments later, he tried to use the newly found multiple of 7, to be exact number 1 000 006. Immediately, he changed his mind and made a statement with enthusiasm.

> S4: Or if I could get to a number made only of sevens and divide it by itself so it gets to 1 and then multiple by 7 and result to 7! Or subtract itself, get to 0, add 7 and reach to 7.
> R: Can I see it?
> S4: 1 000 000, I have to get only to 7s… [whispering, giving the impression of making mental calculations]. Aha! [typing 1 000 000 × 7= 7 000 000, next 7 000 000 + 777 777= 7 777 777]. This number minus itself, equals 0, plus 7, equals 7. The same but divided by itself, equals 1, multiplied by 7, equals 7.

This student was also able to reverse his thinking process. The student was able to suggest both possible approaches to 7, one from number 1 and the other from number 0. Moreover, just like the previous student whose work was described earlier, S4 here verbalized his thinking process explicitly for both approaches. The fact that both problem solving methods were proposed at the same time, suggests that the student was aware that they share common elements, like getting there from a number whose digits are all seven.

In addition both students whose work was described here show that the ability to reverse the thinking process has to be triggered by a specific situation that requires it in order to achieve the desired outcome. Although it may not come naturally at once, a person may show this ability, when needed.

In the case of a fifth grader (S7), the problem seems more difficult in comparison to the way sixth graders worked. Although able to reverse his thinking by the start of working on the specific problem, he would not find the solution of this problem, without the hints provided by the researcher. However, when aided, the student was able to come up with a successful approach to reach to the desired number 7.

> S7: We can try 1 000 000 divided by 777.
> R: How did you choose number 777 instead of another number?
> S7: I just thought about it [pressing 1 000 000 / 777= 1 287,001].

Afterwards, the student started making a number of trials with different operations and numbers composed of sevens that resulted to non-integers. The last number on the screen was 39,594. The researcher wanted to check if the student had made any thought that could provide evidence of thinking backwards.

> R: Before reaching to number 7, did you think about the number that could take you right after to number 7?
> S7: Yes, 49.
> R: So, you are trying to reach to number 49. Is there anything that makes this difficult for you?
> S7: 39 plus 7, equals 46… but the decimals are an obstacle.

Here, it is obvious that the student had already thought reversely from the start of the problem. His aim had transformed from reaching to 7 into reaching number 49. By reaching to number 39, even it was not integer but it also contained decimals, the student was thinking of how to reach to 49. At that point, the researcher decided to provide a hint to see if the student was able to take advantage of it and use it in order to solve the problem.

> R: Except of 49, is there another number that with an operation could give you a result of 7?
> S7: Multiples of 7.
> R: For example?
> S7: If you have 28, you will subtract 7 and 7 again and 7 again and you will get 7 as a result.

The student managed to acknowledge that any multiple of 7 on the screen could lead the way to reach to the desired number 7. This was an important aspect and also an indication of the ability to generalize on behalf of the student. Aided by from the researcher when stuck, this fifth grader was also able to reach to the desired result. However, in comparison to sixth graders, he experienced more difficulties specifically in this activity.

In general, behaviors described in this section validate that mathematically capable students possess the ability to reverse a thinking process. In other words, they can think backwards from the end to the start to solve a problem, if this is suitable. In this activity, the two possible approaches of reaching first to number one and the other to zero, shared many common elements. The fact that students were able to propose the second approach easily after the first was proposed, suggests that they already conceived the underlying principles of the problem and could easily think of an alternative solution to end up to the result, based on previous attempts. Furthermore, it was shown that the ability to reverse the thinking process has to be generated as a precondition need to yield the desired outcome. Even though it may not originate naturally at once, a person may show this ability, as soon as it is required.

Creative Thinking in Mathematics

Through the tasks posed to students during observation, it was made possible to observe their creative abilities in mathematics. Results are discussed in three subsections; construction of mathematical connections, creative spatial ability and unconventionality.

Construction of mathematical connections. It was of great importance to the research study to observe students' ability to interrelate different mathematical associations. The first activity that the students engaged with during observations provided an excellent medium for students to expose this ability. In this task, students were asked to group a set of numbers in as many possible ways they could think of. This task provided unlimited possibilities to students, allowing them to focus on number relations, properties about numbers, various number sets, associations based on the relation between digits and many more. The different solutions that the students proposed are presented in Appendix C accompanied by the percentage of students proposing each type of categorization.

As shown in Appendix C, 59 different types of responses were observed in Activity 1. A careful examination of the responses reveals the variety of mathematical criteria involved and the different perspectives by which the data information was perceived upon.

An important dimension of students' responses was the number of mathematical relations involved in a specific group name. For example, a student may have used only one mathematical criterion as a grouping name, such as "odd numbers" or "numbers larger than 18". Another student, may have been able to come up with a group name relating two mathematical criteria in a specific group name, such as "odd number larger than 8" or "multiples of 3 that are not multiples of 5". In some cases, there were students capable of employing three mathematical criteria in a single group name, such as “odd numbers larger than 18, except of 21” or “number divided by 3 or 7 and their digits’ sum is not 6 or 10”. In the first case, the student focused in (a) the subset of odd numbers, (b) selected a specific range over 18 and (c) excluded number 21 from the group. In the second case, the student focused in (a) the subset of numbers divided by 3, (c) the subset of numbers divided by 7 and (c) excluded digits’ sum of 6 or 10. Table 4 provides a more detailed picture of students’ ability to use multiple mathematical relations in a specific group name during Activity 1.

Table 4

Students’ ability to employ multiple mathematical relations into a specific group name during Activity 1.

	Maximum number of mathematical relations employed in a specific group name		
	One	Two	Three
Number of students (Percentage)	19 (0.56)	12 (0.35)	3 (0.09)

As shown in the Table 4, 19 students (56%) were able to reach to maximum one mathematical relation used in the same group name in any of their attempts in Activity 1. 12 students (35%) managed to combine two mathematical relations in a specific group name. More importantly, there were three students (9%) in the sample that attained to use three mathematical relations in a particular group name. Note that this may have happened more than once. However, data included in Table 4 shows the maximum number of mathematical relations employed in a specific group name at a specific instance during the activity.

Table 5 provides important information in regard to students’ ability to employ multiple mathematical relations into a specific group name during Activity 1 in comparison to the number of correct responses provided, in terms of fluency.

Table 5

Students' ability to employ multiple mathematical relations into a specific group name during Activity 1 in comparison to the number of correct responses provided (fluency).

		Maximum number of mathematical relations employed in a specific group name			
		Up to one	Up to two	Up to three	Total
	1-3	14	5		19 (0.56)
Fluency	4-6	3	5	2	10 (0.29)
	7-10	2	2	1	5 (0.15)
		19 (0.56)	12 (0.35)	3 (0.09)	34

As shown from Table 5, students that provided fewer correct responses were less likely to combine more than one mathematical relation in a particular group name. In particular, 14 of the 19 students that were able to reach only to one mathematical criterion, also provided maximum of three correct grouping attempts. Three of 19 students provided maximum of six responses and two of them managed to produce a maximum of 10 responses. From a total of 12 students that provided a combination of two mathematical criteria in a group name, five students provided a maximum of three correct responses, six of them provided a maximum of six correct responses, whilst two of them were capable of producing a maximum of 10 correct responses. As fluency increased, it was more likely to see student being able to associate two or even three mathematical criteria in a single group name. More specific, three mathematical associations in a group name were observed in cases where more than four correct grouping attempts were suggested. Two out of three students that suggested three mathematical relations in a specific group name provided a maximum of six correct grouping approaches. There was also one student that was able to provide more than seven correct ways to group the 12 given numbers and also suggest group names that associated three mathematical relations. In conclusion, fluency in terms of the number of correct responses seems to be related to the ability to combine multiple mathematical criteria.

Except of fluency, originality is also an important aspect of any creative task. It was the intent of the researcher to investigate the frequency in which a totally original response was reported, that is a response that only one student out of the whole sample thought of.

23 out of 34 (67%) students provided at least one completely original response than no other student suggested. It was interesting to see the relation of the provision of a completely novel response to the maximum number of mathematical relations employed in a particular group name. From the 23 (67%) students that suggested at least one completely original response than no other student suggested, eight (23%) students managed to proposed up to one mathematical relation in a specific group name, 12 (35%) students were capable of combining up to two mathematical relations in a specific group name, whereas there were three (9%) students that connected three mathematical criteria in a group name.

Hence, mathematical promising students, are capable of interrelating mathematical concepts in numerous ways, exhibiting fluency. They also seem to be competent to combine multiple mathematical criteria together to achieve a required outcome and handle multiple data in parallel. Moreover, they may provide original responses to differentiate both among their previously suggested answers and responses provided by others.

Creative spatial ability. The observation of creative ability in the context of a spatial task was important to the researcher to examine the way mathematical promising students would deal with it to produce creative products. As such, Activity 6 required students to reveal their creative ability in a task that also involved spatial thinking and sense of space. The activity was spread across one sheet of paper, providing space for 20 figures to be formed. For each figure, nine dots with horizontal and vertical distance between two succeeding dots being 1 cm to facilitate the formation of figures were provided (see Figure 52).

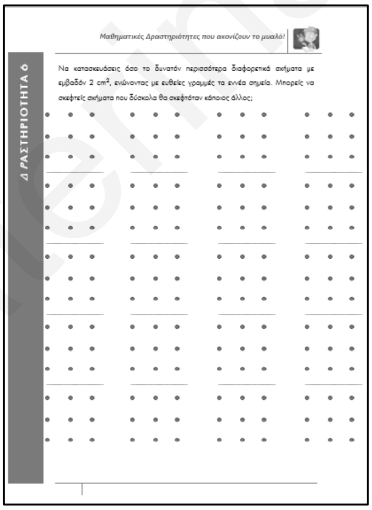


Figure 52. The layout of Activity 6.

The role of the researcher was critical in the specific task, since she was motivating the students to form as many figures possible. In addition, when the students were blocked, the researcher encouraged them to think of figures that another student would not think of, in an effort to trigger original responses. This behavior on behalf of the researcher was shown to help students keep trying and thinking of new figures. Thus, a type of Zone of Proximal Development was established, that had a great impact on students' creative performance.

In the following sections, findings related to creative spatial ability as occurred during Activity 6 will be presented and discussed. This will be made by analyzing the behaviors, thinking processes and approaches used by students, in an attempt to pinpoint specific students' characteristics that denote their potential in mathematics. First, a brief discussion will be made on the types of shapes that were used to produce the end figures, to serve as a background for the subsequent analysis of students' characteristics.

For this activity, a number of shapes served as the foundations which combined all together, they formed the requested figures of an area of 2 cm^2. Amongst the most popular shapes to be put together, where squares of 1 and ¼ cm^2, triangles of 1, ½, ¼ and 1/8 cm^2, rectangles of 1 and ½ cm^2. At first, students exhibited a preference to combine alike shapes of larger size (e.g. two squares of 1 cm^2, two triangles of 1 cm^2, four triangles of ½ cm^2), as shown in Figure 53.

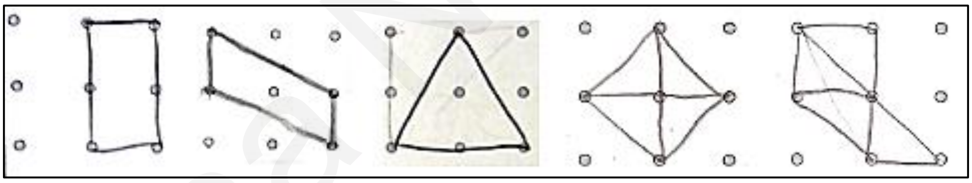


Figure 53. Combination of similar shapes of larger size, as a starting strategy (Activity 6).

Later on, to produce more and diverse shapes, they tended to break their starting units into smaller units with the same area to keep the requested total area unchanging. Thus, figures started to appear where a combination of bigger and smaller pieces was made. The result was the construction of unusual and bizarre shapes that inspired multiple novel responses. Figure 54 illustrates a small sample of such type of responses produced by students in Activity 6.

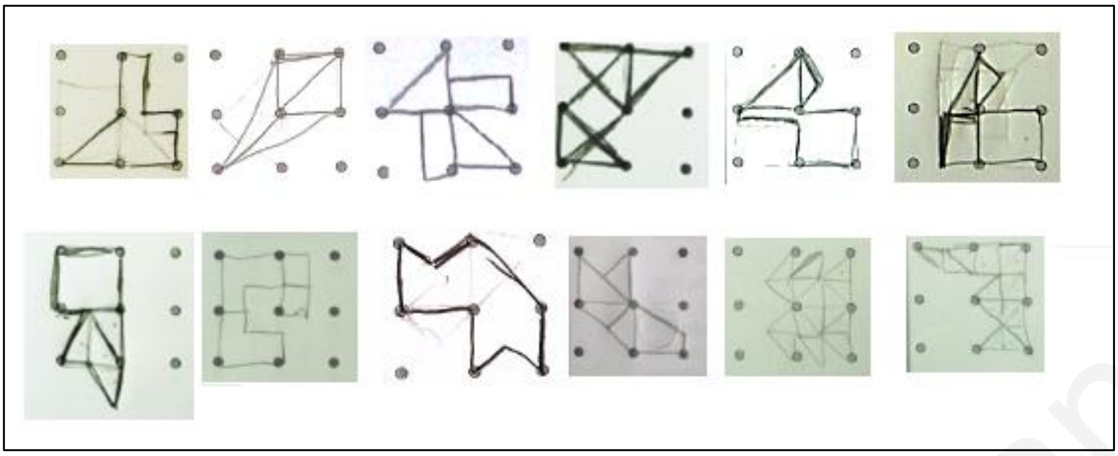


Figure 54. Figures produced after a combination of bigger and smaller size shapes (Activity 6).

In the following sections, specific students' characteristics that denote their potential in mathematics are presented. The delineation of these traits was obtained through the study of the behaviors, thinking processes and approaches used by students in Activity 6. Although these traits were observed during their involvement with the specific activity, they characterize the students and not the task.

Holistic and analytic perception of spatial information. For this activity, amongst the first figures the students made, were the rectangle with length 2 cm and width 1 cm, the right triangle with legs of 2 cm and the rhombus/square with diagonals of 2 cm (see Figure 55).

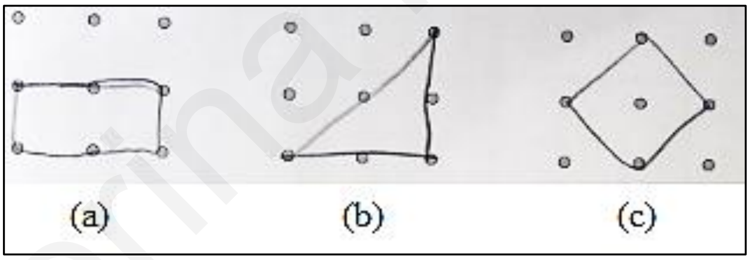


Figure 55. The most common figures suggested by students (Activity 6).

More specifically, the majority of the students had in mind to create a closed area of 2 cm^2, thinking in terms of combining squares of 1 cm^2 (see Shape (a), Figure 55). When students were asked to explain the way they calculated the area of the end figure in order to be sure that the shaped had the required area of 2 cm^2, their answers varied. This revealed that although the end product might be the same for two students, this does not necessary imply that the same thinking process was followed to form the figure. For instance, the vast majority of students combined a square of 1 cm^2 and two triangles of ½ cm^2 forming a right triangle (see Shape (b), Figure 55). When asking one of the students to explain how he calculated the area, the student answered that since all the area covering the nine dots is 4 cm^2, he alleged that all the task required was to select half of the whole area, in other words 2 cm^2. This student showed his ability to read the spatial information provided in the task in two different ways. The student was able to conceive the information holistically by perceiving the available area of nine dots as an area of 4 cm^2 that has to be split in order to comply with the requirements of the activity. Furthermore, the same student was also able to perceive the spatial information provided also analytically, by producing subsequent figures after combining smaller area parts that add up to an area of 2 cm^2. In addition, the student was able to rearrange the same parts used before to produce new figures.

The ability to perceive spatial information holistically was proven to be accommodating in providing the opportunity to visualize and suggest multiple figures based on the prototype figure, as in the case of S22. Among the many figures the student proposed, was a scalene triangle, demonstrated in Figure 56.

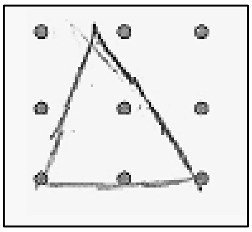


Figure 56. The use of a triangle as a basis to produce many different triangles by S22 (Activity 6).

After forming the triangle, S22 made an argument about possible extensions of the specific figure. In particular, the student stated "There are many ways to create many triangles with the same area. Just move right or left the upper vertex of the triangle while keeping the same altitude and the same base length". This student was able to conceive the information holistically by perceiving the figure as a single triangle, treating it as one shape without dissecting it into smaller shapes. Aware of the formula to calculate the area of a triangle, this student was able to visualize the possible alternate triangles with the same area and verbally express how this could be obtained. The student felt no need to proceed into designing any other figure according to the extension he suggested. Rather, he felt that his explanation was enough to denote his ability to produce many shapes with the same underlying philosophy. Moreover, the student was also able to perceive the spatial information provided analytically, by producing subsequent figures after combining smaller area parts that add up to an area of 2 cm^2. Thus, afterwards, S22 proceeded to a different type of thinking to form other figures.

Focus on product but also on the process. Students did not feel the need to make a typical figure shape, a figure that is known or a figure that is symmetrical. Rather, they were more focused in complying with the requirements of the activity and were merely drawn by a fixation to create as many different and original figures possible with the necessary area. This tendency to provide original shapes led to interesting peculiar products. This was largely evident in cases where smaller area pieces were used, such as pieces with an area of ¼ and 1/8 cm^2 (see Figure 57).

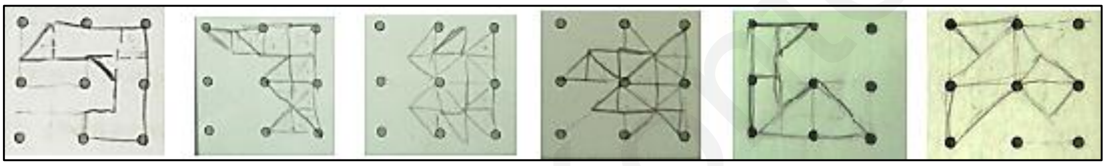


Figure 57. Sample of atypical figures (Activity 6).

In general, the observation of students while working in this spatial creative task, allowed the researcher to gain valuable insight into their thinking processes, techniques and approaches used during process solving. The findings reveal that mathematically promising students are able to perceive and manipulate mathematical information both holistically as well as analytically. They also have a strictly structured idea about what they want to do and the way they want to work. This idea is being processed while time passes, experience is accumulated and more figures are added. As a result, their technique evolves through practice, figures become more elaborate, and approaches become more sophisticated. Moreover, they are able to control their thinking by staying focused in their target and do not decline of their course. They may also devise systems to keep track of the way they work in order to avoid repeating the same figures. Also, students were so focused in the process of achieving as many different and original products that ended up in doing many peculiar figures.

Originality. Originality is considered to be the pinnacle of creative thinking. Hence, in the framework of the study, the researcher was interested to examine whether originality would characterize students' work. Indeed, in several cases, students resulted in original products or employed original ways to approach a particular problematic situation. For that reason, the presentation of results with regard to originality will be discussed along these two parameters. To be specific, first, findings related to the establishment of original products will be discussed and second, original processes used to reach to a conclusion will be analyzed.

Original products. Original assumptions as well as products that were not anticipated, were witnessed during students' work.

For example, a fifth grade student (S7) arrived to an unanticipated and original conclusion when working with Problem 5. In the specific activity, the researcher attempted to extract examples of numbers that could be placed in x's position, but also more importantly numbers that could not be placed in x's position. Thus, the researcher aimed to observe if the students conceptually understood the importance of x's position, that of being a square number and that any number except of a square number could not be placed in that position. The observer, at the same time, wanted to examine if the students could reach to a generalization about the properties of numbers in x's position, except of providing specific numbers. At first, the student concluded that square numbers are the set of numbers that could be found in x's position. Then, the researcher tried to extract a generalization of the set of numbers that could not be placed in x's position.

R: Apart from numbers 4 and 9, can you tell me another number?
S7: Four plus five… let me try, 16 [without drawing]. 4,9,16…
R: Why do these numbers fit into x's position? How did you propose 16 without counting?
S7: Four times four.
R: For number nine;
S7: Three times three.
E: How about number 49;
S7: Seven times seven.
R: So do these numbers have a relation in math? How can I calculate them?
S7: If I divide them with one side of the square…
R: What they will result to?
S7: The other side!
E: Correct!
R: So, I need two numbers who cannot be placed in x's position.
S7: Prime numbers.
R: Why?
S7: Since prime numbers can only be divided by themselves and 1!

As a fifth grader, this student was not taught square numbers in school, so the researcher was not expecting this terminology or concept to come up during the discussion with a fifth grader. Although the researcher anticipated responses for specific numbers who cannot be placed in x's position, which included any number except of square numbers, this fifth grader was able to focus on prime numbers as a special set of numbers. This fifth grader was able to acknowledge that numbers who are a product of an integer with itself cannot by definition be also prime numbers. Except from an unexpected response, this argument also showed the student's ability to conceive the underlying properties of the numbers involved and focus on a special set of numbers that is prime numbers that do not meet the conditions of being candidate numbers for the specific problem.

The same fifth grader also made an unanticipated observation during working in Problem 2. In this case, the student strived to determine if the result of a process of reversing a randomly selected two-digit number and adding the two two-digit numbers together is random and if not, to clarify and explain the mathematical relationship behind this process. Namely, S7 first noted easily that all sums are multiples of 11. In an effort to find an explanation for this phenomenon, the student made an important observation.

S7: …. I noticed that if you make them like this, x [pointing to the diagonals]. If you look at them like this, they are 11 and 22. Twenty-two is two times 11.
R: So if 22 is 2 times 11, how many times 11 we have at the end?
S7: 3.

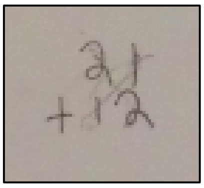


Figure 58. Figure drawn by S7 (Activity 2).

After researcher's prompts, the student realized that by rearranging the two numbers added, the result is a multiple of 11. However, S7 was not able to verbally form a generalized argument in regard to the sum of multiples of 11, namely that adding multiples of 11 always result to a multiple of 11 too. Still, the researcher believes that the student might have intuitively understood the connection but was unable to formally express this thought.

Another attempt to Activity 2 by a different student, this time a sixth grader, confirmed that mathematically promising students thinking may result to original findings. At the same time, student's action also portrayed gifted students' ability to look for mathematical associations everywhere and furthermore being fixated with finding the answer they are looking for. A description of this student's work follows.

At first, S18 calculated the sum of 10 and 01, 32 and 23. Then, in order to test his assumption that the result would always be a number with all digits the same, he calculated the sum of 54 and 55. The conclusion in which he reached to is the following:

> S18: Basically, both numbers should be under nine [referring to the sum of the digits of the starting two-digit number], because then the result is above 100 and the result is no longer a two-digit number.

This student acknowledged that the same digit result can no longer be obtained after the sum of the digits of the starting number is over 9. After prompted by the researcher to look for a similarity in results, he focused in another interesting relationship, a relationship that involved the difference of multiples of the starting numbers and a multiple of 9. He also noted that all results (multiples of 11) had a common characteristic, since the sum of their digits is always an even number. In order to arrive to an elaborate argument about the relationship with multiples of 9, a process of experimentation with different possible relations was observed. The next excerpt from this student's work illustrates his strive to find a relation in the results and justify his answer.

> R: Do you notice any similarity in the results?
> S18: … Well, I think, here for example, 32 when multiplied by 2 results to 64, minus 9 equals 55. Here, 54, 2 times results to 108, minus 9 equals 99. 86, times 2, 172, it does not match. Here, 63 times 2, 126, minus 63, does not match again. So…..
> R: Good work, nice relationship you got there even regarding the numbers that fit. If you focus on the results, what these numbers have in common?
> S18: … Well, if we add the digits, the result is always an even number.
> R: Ok, what other numbers could you have as a result of the same process?
> S18: 66, 77, 88, 44, 33, 22…
> R: Do they have anything else in common?
> S18: They are all divided by 11.
> R: Very well.
> S18: So 154 should be divided by 11 [proceeds to vertical division to confirm this]. 14 times.
> R: Ok. So, is it a coincidence that they are all multiples of 11? Why number 154 is equal to the product of 14 times 11?
> S18: … Here it's the same with what I've noticed before. 43 times 2, 86, minus 9, 77 again. But I do not know why, I do not understand… Aha! The previous is also plus 9, times 2 plus 9 because they have a difference of 9 between them [he refers to the two numbers with reversed digits] so one number has to be plus 9 and the other minus 9 in order to balance.
> R: What happens with 53?
> S18: [noting down 35+53=88]… 53 plus 53 equals 106, so here it has to be 106 minus 18 that equals 88. Oh, here the digits of the number…
> R: Why do we have this difference? Why in one case do we have minus 9 and in another case minus18?

S18: It is larger by 10 [43 turned into 53], but the units have also changed [34 turned into 35]). [writing down 36 + 63 = 99] 63 times two, 126 minus… aaaaaah, it will be 27. 9, 18, 27 ….. aaah, [noting after each addition the subtraction that could be calculated to obtain the same sum]. Here, when we have 4 as a unit digit, minus 9, when we have 5, minus 18, 6, minus 27, let's try another one [noting 37 + 73 = 110]. So, here it should be minus 36, so 73 times 2 equals 146 minus 36, oh! Correct! So, when the units digit is 4, minus 9, 5,minus 18, 6,minus 27,7,minus 36,8,minus 45,9,minus 54 and when is 0, minus 0!
R: Why is it 0?
S18: They are the same, so we only have to do 0 times 2. As it increases … let's say from 33 to 34 , minus 9, each time that the units digit increases, the number we subtract from the starting number after doubling it also increases.
R: Why do we subtract 9 and not any other number?
S18: …Oh, we multiply by 2 and we subtract the difference of the two numbers... Now I get that you multiply the largest number by 2 and then you subtract the difference between the two numbers [also writing down the process with 32 and 31 as starting numbers]. Here, you can also add 9 [pointing to 32 + 23].
R: Can you explain this?
S18: If I multiplied 23 times 2, then it equals 46, plus 9 equals 54.
R: Well done.
S18: Basically here [pointing to the previously noted examples], I used to double the second numbers in the addition which were the largest, but I could use the smallest and then add instead of subtract [turning page to proceed to the next task].

Although the researcher tried to help the student explain the relationship of results with multiples of 11, the student was focused on another relationship, one he had previously observed with multiples of 9 and could not let go without finding an explanation. The student was determined to find why this relationship is valid for only some of the results obtained. The realization of the mathematical relationship came after an aha moment, a sudden insight into the problem. This illumination stage was accompanied with emotional relief and excitement in behalf of the student, as it came after a period of time when no progress had been made and the student was struggling to associate his findings. The conclusion in which the student resulted to finally, was an original finding that was not observed or suggested by any other student in our sample. The reasoning process followed until to reach to the conclusion was notable, as well as the persistence of the student to arrive at a conclusion.

Not only the student was able to find the mathematical relationship he strived for, but he was also able to provide multiple possible ways to calculate the result of the process suggested by the problem. His behavior suggests that he is able to flexibly manipulate the given problem, by providing alternative ways to reach to a specific outcome. For example, he was able to suggest doubling the largest of the two digit numbers and then subtract the difference of the two numbers which is a multiple of 9. Moreover, he also concluded that the same result can be obtained by doubling the smaller of the two digit numbers and then add the difference of the two numbers which is again a multiple of 9, based on the difference of the digits of the starting number.

As mentioned in a previous section, Activity 1 triggered a variety of responses from mathematically gifted students. By nature as a task that allowed and explicitly asked for the maximum number of different types of responses, as well as original ones, the students acted accordingly. In regard to original products, 23 out of 34 students (67%) managed to provide a minimum of one completely original response than no other student suggested. This could be considered a high percentage, considering the great variety of responses that were produced. Namely, this Activity resulted to 59 different types of responses produced by the 34 students in total. In other words, 23 out of 59 (39 %) different types of responses were completely original. That is, there was no duplicate answer of each of these 23 responses suggested by any other student but one in the sample.

The subsequent section presents specific examples of students' work where original processes were employed.

Original processes. A number of students employed original processes for a number of problems posed. To accomplish this, students exhibited risk taking that is necessary to achieve innovation, by breaking the boundaries of the posed problematic situations. In addition, they also strived to find mathematical relationships in all places, providing multiple ways to reach to a specific conclusion.

An original approach was employed by S9 while working on Problem 4. In particular, the student made an important comment on relations of divisibility, taking a novel approach to solve the problem.

> S9: … First I have to reach to a multiple of 7 and then [typing 1 000 000 x 7 = 7 000 000, next 7 000 000 + 7 = 7 000 007 and afterwards 7 000 007 – 7 / 7 = 7 000 006]. … [subtracting seven from 1 000 000 each time until reaching to 999 979]. There is no…
> R: What do you mean?
> S9: There is no divisibility criterion for 7, it must be difficult.
> R: How can you reach to a 7?
> S9: By turning all digits to 7 [subtracting multiple sevens from 999 979 at the same time, where he was before].
> R: So, in which number are you planning to go?
> S9: 777.
> R: And how are you going to reach it?
> S9: By subtracting 7 many times.
> R: Isn't this very time consuming?

> S9: Yes.
> R: Can you think of a shortest route?
> S9: It won't work. Let's check this, that it doesn't work [typing 1 000 000/7=142 857,142]. No. Let's try something different. I have to do it on paper to find the remainder. The calculator does not show the remainder, only the decimal points [proceeding to vertical division of 1 000 000 divided by 7 with quotient 142857 and 1 as a remainder]. Number 1 as a remainder. If this was smaller by 1, it would be much easier. 1 less is a multiple of 7. We found the magic number [circling number 1]. We have to think how to decrease the number by 1…

The results obtained from each strategy selected and decision taken, had an impact on the next steps the student decided to take. Although S16 first realized that in order to reach in number 7 he had to turn all digits to 7, he then changed his strategy and attempted to divide 1 000 000 by seven which resulted to a non-whole number. In front of this finding, he decided to perform the division on paper in order to find the remainder and use it to find the closest multiple of 7 to 1 000 000. Although not the actual strategy the student used in the end, it shows his deep understanding of the relations of divisibility.

Not only students were able to proceed in making unique observations, there were also students that were capable of providing multiple solution paths. In the example described below, S7 was working on Problem 3 and tried to explain why the result was always an even number. His first approach to the problem consisted of an original way to justify that the result always ends up to be an even number, since the student noted a pattern between consecutive results.

> S7: The result with number 2 and the result with number 3 has a difference of 4. 4 times 4, which is 16, minus 4, 12. Two plus 4, 6 plus 6, 12. For every next number you try…
> R: Why do we always end up with an even number?
> S7: … [noting on paper until starting number 5]. Every time we add 2 in order to get the result [referring to the difference between two consecutive results, namely 2, 6, 12 and 20 which increases by 2 each time].

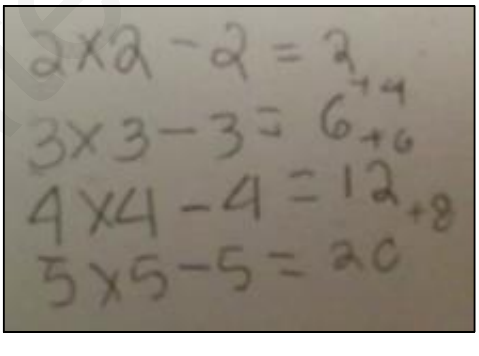


Figure 59. Calculations noted by S7 (Activity 3).

R: So, can you predict the next result if you started with number 6?
S7: Yes, 30 and then 42.
R: Well done. How can we explain why it always results to an even number?
S7: Because the number you add each time to find the difference with the next number is always even.
R: Explain this to me please.
S7: For example, the difference between 2 and 6, is 4, that is even. 6 is an even number too.

During his attempt, the student came upon a particular mathematical relationship, a pattern who allowed him to calculate the next number in the pattern of results. This conclusion allowed him to explain why the resulting number is always even, focusing on a different mathematical relationship than other students. Later on, he also concluded to the properties of even and odd numbers and provided another proof that the resulting number will always be even.

R: Ok. If you observe each line separately instead of focusing on the results, can you find of another explanation why you always end up with an even number?
S7: Two times 2, 4, it's the same you add to find the next number but it doesn't match at the next one… If you multiply 2 times 1, you get 2. If you multiply 3 times 2, you get 6.
R: And afterwards?
S7: Four times 3, equals 12.
R: Well done. Why 4 times 4 and then minus 4 equals 4 times 3?
S7: Because instead of multiplying 4 times 4 and then subtract 4, it is better to calculate in one step 4 times 3.

Thus, this example shows the close relationship of fluency, flexibility and originality as aspects of creative thinking. A student that provides multiple ways to solve a problem, is more likely to suggest original approaches, displaying different categories of responses. Although both approaches justify the conclusion of the result being always an even number, they differ in practicality. Observing the difference of two consecutive results allows to calculate only the next result, whereas observing the relation between the starting number and the end result permits the calculation of any result. Thus, in a similar situation that this student may encounter in his life, we may assume that he will be able to provide different solution paths and choose which strategy serves his purpose better.

To achieve innovation, students had to take risks and think out of the box. Indeed, Activity 6 provided an excellent means for students to break the boundaries and restrictions of the specific activity, allowing to think differently with an open mind. Activity 6 required students to provide as many different figures of 2 cm^2 as possible. For this reason, students were provided with nine dots placed in a square grid format to facilitate calculations of length. One may assume that the nine dots provided by the researcher to facilitate the students into forming the figures by using them as guides in terms of distances (lengths) would cause students to feel restrained in regard to the possible shapes to occur. All students based a significant percentage of the figures drawn by overlapping some of the dots. At the same time, their figures were placed inside the square grid of nine dots.

However, this was not the case for all of the students. Several students did not by any means feel restricted by the dots provided as guidelines and this was evident in their creations.

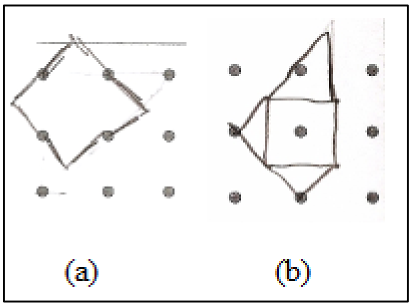


Figure 60. Breaking the boundaries of the designated nine dots (Activity 6).

Figure 60 presents two indicative figures produced by different students. In Figure (a), the student created a rhombus by combining a square piece in the center with an area of 1 cm^2 and four surrounding triangular fragments of an area of 1/4 cm^2 each. Due to the placement of the central square piece at the upper left corner of the nine dots' area from the start, the surrounding triangle had to be placed outside of the dots. While another student might be intimidated by this fact and even erase the figure, this student surely was not a conformist. Similarly, in Figure (b), the student combined a square piece in the center with an area of 1 cm^2, two surrounding triangular fragments of an area of 1/4 cm^2 each and finally one triangular piece of an area of 1/2 cm^2 on top of the square. Likewise, the last piece had to extend beyond the dots and this student had no trouble in placing it outside of the boundaries.

Although these two students were daring enough to step out of the boundaries, some others were also bold enough to draw their own dots and extend the region to work with, as shown below in Figure 61.

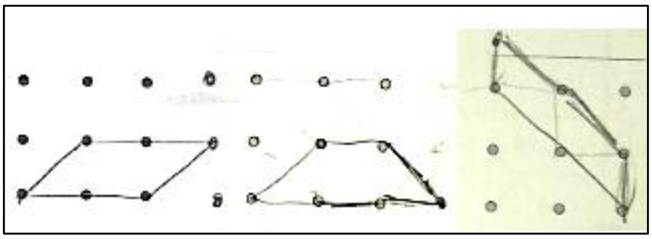


Figure 61. Breaking the boundaries of the designated nine dots, by drawing new ones (Activity 6).

In general, originality seems to be associated with a willingness to step out of one self's comfort zone, take risks and think outside of the box. From the data collected, one may assume that the students observed are characterized by a sense of freedom. They dare to extend beyond the boundaries of the specific activity with no attention paid on conventions that would normally intimidate others. They are capable of thinking differently than others with an open mind and they are not afraid to take risks. In this way, they appear willing to take new perspectives toward day-to-day work. The driving force behind original approaches and products seems to be frustration. They don't feel that "enough" is being done, and that the "normal" way of thinking just isn't getting it done. Thus, they feel compelled to find new ideas and approaches to handle a problematic situation.

In regard to approaches taken, mathematically gifted students tend to approach problems from different perspectives in comparison to other students. Furthermore, the behaviors described in this section disclose that not every mathematically gifted student deals a problem with the same way, nor that mathematically gifted students focus in the same mathematical relationships. Instead, the abovementioned excerpts illustrate that mathematically gifted students may provide original solutions to a problem and moreover they may suggest different paths to reach to a solution.

This section shed light to another behavior linked to mathematically gifted students in mathematics. As shown, students did not only observe mathematical relationships necessary for reaching to the required outcome during the problem solving process. Rather, they also discovered and provided reasoning in regard to other mathematical relations, thus extending what was already posed by the problematic condition itself. These extensions to the original problems were not anticipated to be observed. Indeed, these incidents were a pleasant surprise for the research study.

Furthermore, our findings indicate that originality does not occur only in cases of working on divergent thinking tasks. Rather, thinking in original ways is a more general characteristic that describes students' creative thinking processes and may be employed in other types of problematic situations as well. Thus, original processes and products should not be mistakenly expected to be observed exclusively in multiple solution tasks. Indeed, creative thinking is sometimes viewed as thinking triggered only by tasks that require multiple solutions.

Hypercognitive Processes

The ability of individuals to adjust their behavior according to the demands of specific situations including the selection of strategies that will best accomplish anticipated goals, the allocation of resources to better focus attention on the completion of the task and to monitor one's progress so as to establish that goals are being met, was one of the abilities observed in the students of the study. In addition, task commitment, sustained effort and perseverance were also recorded in students' work.

To give a more detailed image and gain insight into the hypercognitive processes, the work of specific students will now be presented and discussed across two dimensions; (a) self-regulation (b) task commitment, perseverance and confidence.

Self-regulation. The selection of particular strategies, the switch to alternative strategies according to their efficiency, and the control of progress being made was among the self-regulatory processes observed in mathematically promising students' work.

The first student whose work is to be discussed to expose these self-regulatory processes is S12, a sixth grader. S12 filled four sheets of paper with his creations, in other words 80 figures were suggested. Throughout his work, it was evident that there is great creative potential in this student. Figure 62 presents the first 20 figures formed by the students as shown in the 1st sheet of paper filled.

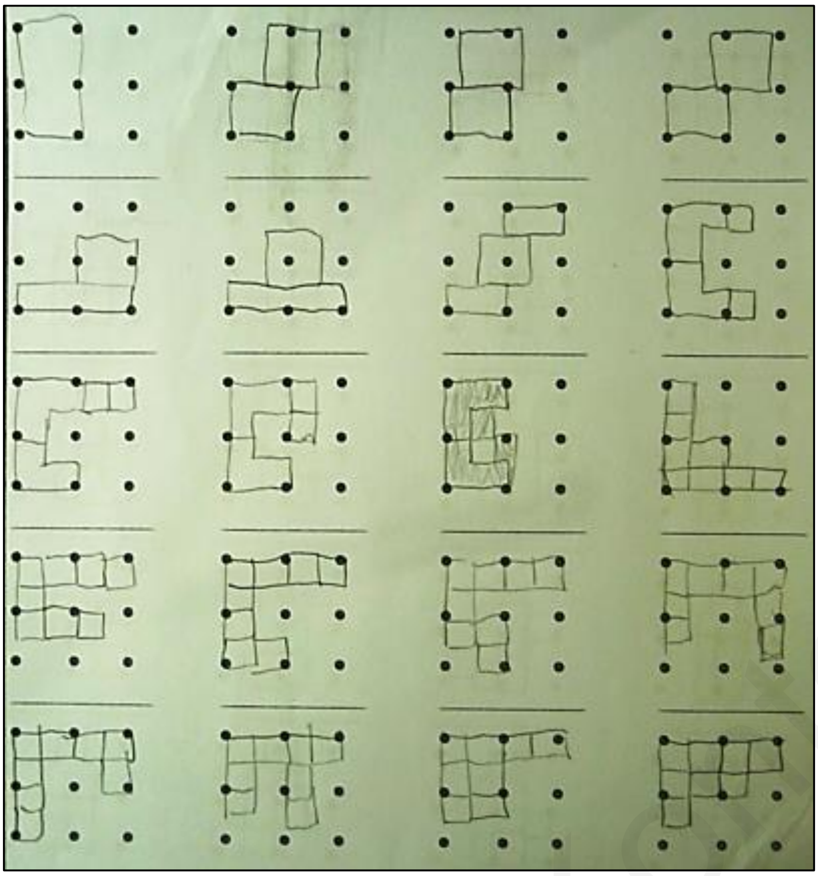


Figure 62. Figures 1-20 (first sheet of paper) by S12 (Activity 6).

With regard to strategy selection, the student applied the sliding technique with two squares of 1 cm^2 to obtain the first four figures. This technique as well as this student's work was analyzed in a relevant section before. For shapes 5 to 7, the student used one square of 1 cm^2 and two rectangular pieces of ½ cm^2 each. Exploiting the rearrangement approach, he easily managed to produce the three figures. To obtain figures 8 to 11, the student used two pieces of ¾ cm^2 and two square pieces of ¼ pieces each. Once more, the rearrangement approach was used and four figures were added to this student's repertoire with minimal effort.

Notice the evolution of this student's technique as time passed. At first, he began working with larger units and fewer pieces. To be able to produce more and different shapes with the same technique, thus less effort, the student moved to work with smaller size units, combining pieces of various sizes. At that moment, the student formed the 12^{th} shape, a figure made of eight square pieces of ¼ cm^2. However, as new figures were added, the student realized that he had to devise a system to keep track of the formed figures in order to avoid repeating them. This system was made obvious in the next creations of the student. In order to fill the first sheet of paper, the student decided to keep the four of the eight pieces of ¼ cm^2 permanent in the top of the designated area, and rearrange the four remaining square pieces of ¼ cm^2. With the same thinking strategy, not only the first sheet of paper was filled, but also four more figures were produced (Figures 21 to 24) in a second sheet of paper, as shown in Figure 63. It is important to note that the researcher provided one sheet of paper to each student for this activity. It was also however explained that additional sheets were available, if students wanted to use them to produce more figures.

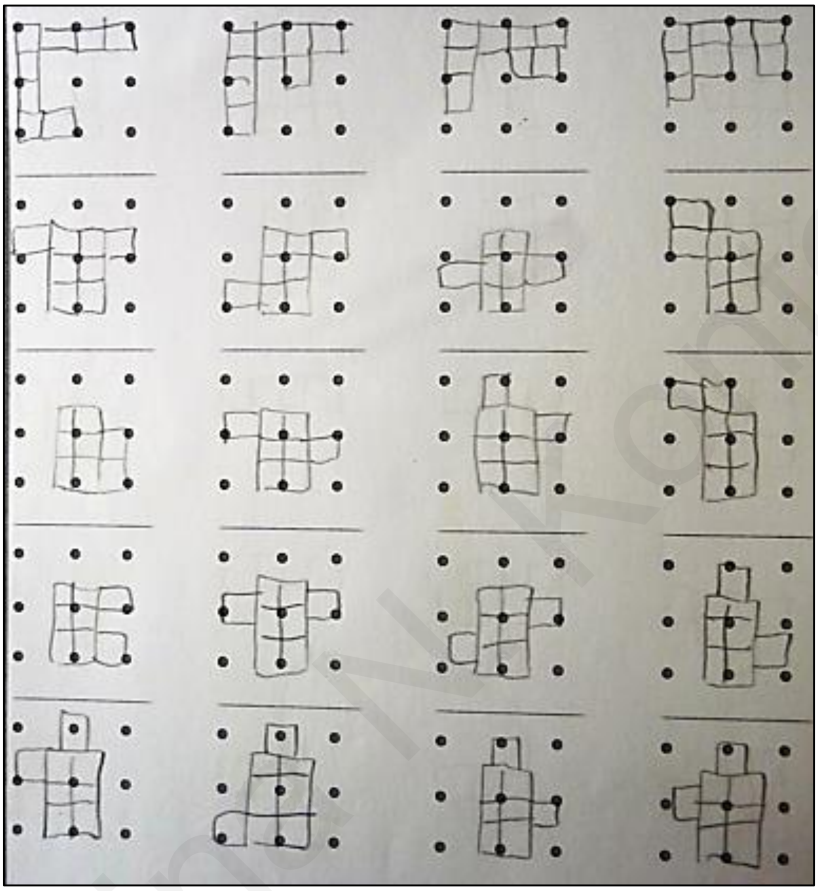


Figure 63. Figures 21-40 (second sheet of paper) by S12 (Activity 6).

After producing the first 24 shapes, the student continued to apply the rearrangement approach in a slightly different way. Now, six square pieces of ¼ cm^2 remained in the same position (two columns of three pieces each) and the two residual square pieces of ¼ cm^2 were repositioned in each figure. This was the case for figures 25 to 40, ending in two sheets of paper full of different figures. Also, notice figures 34 to 40 where the student placed one or both of the two repositioned squares in such places where the dots did not osculate any of the squares' vertices. This behavior provides evidence that the student felt free to work and create figures, rather than restrained from the dots that were provided simply to help as guidelines.

Figure 64 illustrates Figures 41 to 60, created by S12 using the third sheet of paper he requested.

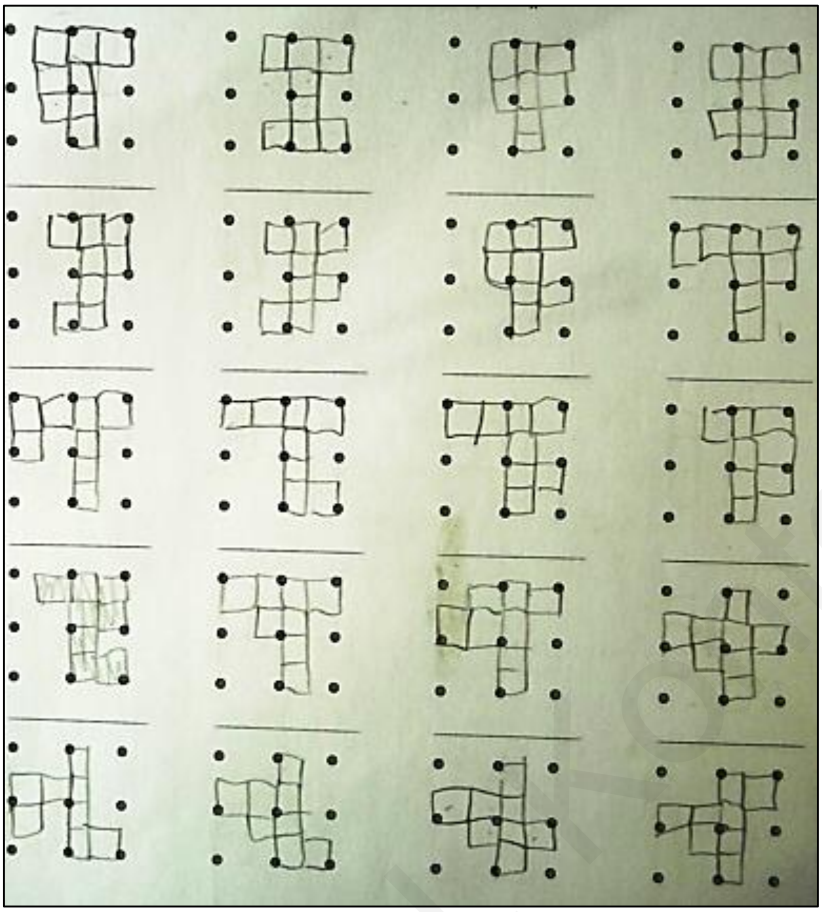


Figure 64. Figures 41-60 (third sheet of paper) by S12 (Activity 6).

Since the rearrangement strategy proved to be productive previously, the student did not have any hesitations in continuing to apply it in the subsequent figures. Once more, six square pieces of ¼ cm^2 remained in the same position and the two residual square pieces of ¼ cm^2 were repositioned in each figure. The difference with the only just constructed figures was the shape of the six immovable pieces. From that point on, a T shape figure, made of a column of four square pieces of ¼ cm^2 in the center of the designated area and two square pieces on the upper left and right side each remained in the same position. To complete the necessary area as requested by the activity's directions, the two residual square pieces of ¼ cm^2 were repositioned in each figure. This was the underlying idea behind Figures 41 through 55. Figures 56 through 73 were obtained with a slight modification of the previous strategy that produced Figures 41 through 55. Since the next group of figures is partially arranged in both third and fourth sheets of paper, Figure 65 illustrates Figures 61 to 80, created by S12 using the fourth sheet of paper he requested.

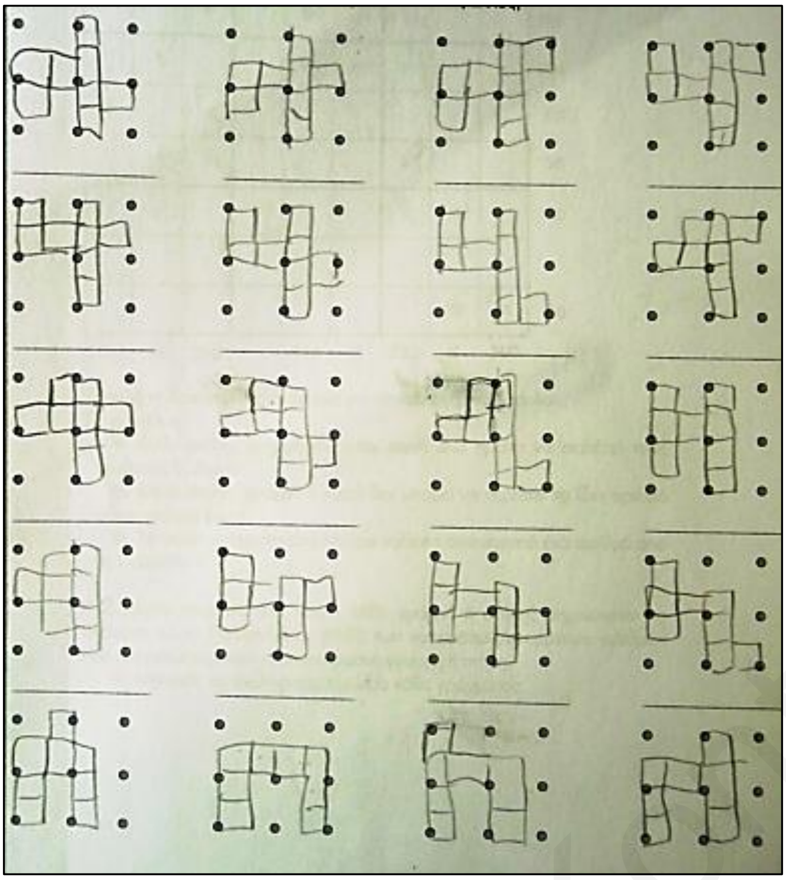


Figure 65. Figures 61-80 (fourth sheet of paper) by S12 (Activity 6).

To obtain Figures 56 through 73, the student maintained the same fixed column of four square pieces of ¼ cm^2 in the center of the designated area as before.

This time, two square pieces were fixed to the left of the column, at the second row. To fulfill the purpose of the task, the two remaining square pieces of ¼ cm^2 were repositioned in each subsequent figure.

There was also another truly remarkable finding in regard to the way this student worked to create new novel figures and was observed by the researcher during this student's work. Not only, six pieces were kept in the same place, but the student was moreover able to control another condition.To obtain Figures 61 through 67 with the minimal effort possible, the student was moving only one square piece in comparison to the previously formed figure. In this way, he also kept track of the previously constructed figures and was kept assured that he was not repeating any shape. Proceeding to Figure 68 and later to fill the third row (Figures 69 to 71 and 73), the student came up with a different way to make new figures. This time, the student selected to move only one square piece in comparison to the figure previously formed directly above it, in the former row. To construct shapes 74 through 80, S12 maintained a core backbone shape on which to base the subsequent figures. This backbone shape was made after removing one square piece from the former basic shape that was used to create Figures 60-73. More particular, the core shape was now similar to a seven-like shape and consisted of a row of three square pieces of ¼ cm^2 immediately followed by a column of two square pieces of ¼ cm^2. The remaining three pieces revolved around them in different positions, resulting to eight new figures.

After constructing 80 figures, all having the same area, and filling four sheets of paper, the student did not request an additional piece of paper. He stopped making new figures because it was time consuming rather than he could not suggest more figures. Besides, his work did not leave any doubts about his great control of thinking his self-regulatory abilities as well as his creative ability in mathematics and spatial sense.

In general, this student's work has been a great sample of a progressive evolving technique, starting from larger and moving on to smaller pieces to work with. He also demonstrated his persistence on following the rearrangement approach, which was proved to be a reliable strategy to produce numerous figures with little effort and cognitive load. He also devised and thought of a genius system to keep track of previously formed figures, thus avoided repetitions and kept being focused in his target. The key here seems to be the progression of thinking and the evolution of the strategy used, in order to produce a variety and a great numbers of figures, without great effort. Moreover, the researcher observed and noted down the great speed and convenience in which the student kept forming innovative figures. The researcher also recorded that the student did not take even a close look at his end products, since all he cared about was the process. So, after completing one figure, he rushed on to the next one, without even taking a glimpse into his accomplishments.

Another example of a self-regulatory technique used to focus attention on task demands and ensure that the objectives are met follows. During Activity 1, S1 made up a marking system to keep track of the numbers he already had placed in a group in a specific grouping attempt. The system changed as grouping attempts increased. This marking strategy is shown in Figure 66.

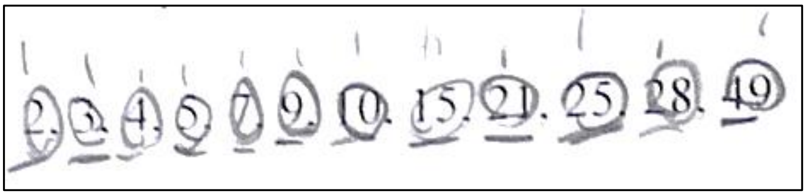


Figure 66. Marking system of grouped numbers used by S1 as a self-regulatory technique (Activity 1).

At first, the student used to place a line segment over already used numbers to avoid repeating them in a new formed group, in the first grouping attempt. Since the proposition of group formation in second, third and fourth attempts made it relatively easy to keep track of already grouped numbers, the student did not proceed to any markings of the original numbers proposed by the activity. However, when working on the fifth attempt which proposed more than two groups and at the same time suggested more complex mathematical relations, the student proceeded to underlining already grouped numbers, to distinguish from the numbers already circled. The need for marking previously grouped numbers was evident in the sixth attempt. In this case, the student chose to circle each grouped number to distinguish the remaining numbers to group.

To sum up, several signs of self-regulatory processes were observed during mathematically promising students' work. Namely, gifted students were able to select specific strategies according to task demands. They were also capable of switching to alternative strategies according to their efficiency or remain loyal to a certain strategy if it was proven to be effective previously. Another important ability that was observed was the ability to control the progress being made and act accordingly to fulfill the task's objectives. As shown, an effective way to achieve this was through a method to remember their previous actions in order to avoid repetitions, keep progressing their thinking and accomplish the task successfully.

Task commitment, perseverance and confidence. Students' interest and willingness to work on certain tasks was apparent during observation. At the same time, the sustained effort put into their work and persistence to achieve the desired outcome especially in difficult tasks, was remarkable. Moreover, there were students that were confident and assured about their abilities.

More specific, there were students who were really drawn by the creative mathematical activities (Activities 1 and 6). Although spending a significant amount of time on the specific tasks, the majority of the students were eager to continue to work, striving to provide different solutions and original ones. This behavior was also encouraged by the researcher, who motivated students throughout their work in these two tasks, to try and find more ways and also ways to solve the activities that another student would not think of. Thus, it was intended by the researcher to provoke students' creative responses.

It is worthy to note another interesting fact. Students and their parents were informed prior to our session that the whole observation process was planned to last about 2-2 ½ hours long to organize their family schedule. The students were by no means pressured by the researcher at any point of the observation to rush on to activities to meet the time frame. Thus, although aware of the duration the researcher foresaid to the family, several students devoted 40 minutes just for the first creative activity, without stressing about it nor rushing to other activities. Rather, they spend their time into thinking of new creative and original ways. Namely, the largest amount of creative responses in Activity 1 was 9 groupings!

The same enthusiasm to work was exhibited in other activities as well. For example, S18 was rolling over pages to find which activity to do next after finishing Activity 2. Suddenly, he spotted Activity Number 6 and proclaimed with great enthusiasm:

S18: Wow! Should we do that?

The student was fascinated by the fact that the activity was one including spatial thinking and it was presented in a page filled up by dots to help draw the figures. He retained the same enthusiasm during his work in this activity, providing 16 different figures, some of which were truly unique.

In the same activity persistence was also shown. S12 kept asking for more sheets of paper when there was no empty space to write. Eventually, this student filled four sheets of paper trying to form as many different figures with an area of 2 cm^2. As a result, this student was able to provide 80 different figures (see Appendices D1-D4). This student was able to create a remarkable system to keep track of the previously formed figures in order not to repeat them later. At the same time, the system he came up with, allowed him to transform the previously formed figures into new ones, whilst at the same time avoiding to redo a recently formed figure, as discussed in previous sections. He also felt confident about himself and his abilities. This was evident throughout the activity, when he was constantly smiling. There were times were he checked on the researcher to see her reaction, in other words, if she was impressed by his work.

In total, students were really drawn into the mathematical activities posed to them and were eager to work with them. In many occasions, they spend a considerable amount of time on a specific activity, either to provide as many solutions possible, or to accomplish a task if it was difficult and was troubling them for some time. It was rare to observe a student that would leave a task unfinished to proceed to other tasks. Rather, they preferred to persist on a problem, struggling to find a solution. During observation, students also demonstrated their self-confidence, revealing that they are aware of their above average abilities. Indeed, a specific student kept looking at the researcher to see if she was impressed by his work. We may assume from the findings that when proper challenging mathematical activities that stimulate gifted students are posed to them, they are greatly motivated which in turn impacts on their level of engagement and subsequently their performance.

CHAPTER FIVE: DISCUSSION

Introduction

Talent in mathematics was declared as the most broadly required resources for the 21st century (Office of Science and Technology Policy, 2006), asserting the significance of nurturing giftedness in mathematics. Despite such assertions and the awareness of the importance of providing for gifted students in mathematics, still, “gifted children often are neglected, not referred, remain untested, and remain very poorly served by any stretch of the imagination” (Shaughnessy & Persson, 2009, p. 1285). Consequently nations keep losing their most valuable natural resource - human capital, resulting to talent loss (Davidson, Davidson & VanderKam, 2004; Hong & Milgram, 2008; Leikin, 2009a).

In recent decades, the field of giftedness research has subjected to substantial criticism in regard to the quality of relevant research (Heller, 1993; Heller & Schofield, 2000; Ziegler & Raul, 2000). More specific, only a slight fraction of the plethora of conceptions of giftedness that were introduced in the research literature, were a product of adequate empirical examination (Stoeger, 2009). Friedman-Nimz and colleagues (2005) came to the conclusion that few of the studies in gifted education add to the body of knowledge through empirically assessed practice. In the same direction, Heller and Schofield noted “...without such basic research, the nature of giftedness is still open to question” (Heller & Schofield, 2000, p. 134). Therefore, in the future, research should pay more attention on methodological aspects and subject proposed conceptions and approaches to empirical examination.

In addition, the challenges of the new millennium call for a shift in the field of giftedness. The field has to expand its perspectives and conceptions, whereas models and approaches should be revised to keep up with recent discoveries. In fact, Stoeger (2009) pinpoints the necessity to subject conceptions of giftedness to empirical clarification, in an effort to preserve only the ones with the power to survive in the field. Researchers should focus on conducting meaningful research, research that will contribute to the growth of the field of gifted education (VanTassel-Baska, 2006). This was not the case since a common phenomenon in the field was to conduct research in a number of directions "seeking novelty over depth of understanding" (VanTassel-Baska, 2006). Instead, research studies should be related and compared in order to allow for sound generalizations and conclusions. This is especially important for domain-specific giftedness that calls for the general models, conceptualizations, approaches and provisions to be modified to correspond to the type of giftedness, such as giftedness in mathematics that is addressed in this study.

Taking in consideration the abovementioned topics, this study addresses the concerns raised by researchers in the field and comes to resolve a number of issues. The aim of the study was to develop a theoretical model of mathematical giftedness, provide a descriptive profile of the abilities, as well as cognitive and hypercognitive processes of gifted students in mathematics and to develop an inclusive identification process for identifying mathematically gifted students in elementary school. Addressing the need for good quality research, the conceptions, abilities, processes and instruments proposed were subjected to empirical examination, past research findings were compared and a solid foundation for generalizations and conclusions in terms of theory and practice was built. Moreover, research products and suggestions are domain-specific and allow for the advancement of the field of mathematical giftedness and gifted education.

The specific purposes of the study were the following: (a) To identify and create a combination of measures that may accurately recognize mathematically gifted students in 5th-6th grades of elementary school, (b) to outline the abilities of mathematically gifted students, (c) to investigate the nature of the cognitive processes that mathematically gifted students use as they engage in solving non-routine challenging mathematical problems, (d) to investigate the hypercognitive processes that mathematically gifted students exhibit when engaged in mathematical problem solving and (e) to describe mathematical giftedness through a theoretical model.

This chapter presents the discussion of the results positioned in the context of the literature, the research questions and the categories that emerged as a result of data coding and analysis. Consistent with methodologies of grounded theory, various explanations from the literature are applied to interpret the research results, whereas innovative aspects that emerged through data analysis are also identified.

Identification of mathematical giftedness in the upper elementary grades

The results of this study showed that is possible to identify mathematical giftedness in early ages, if the suitable identification process is followed.

A concern by VanTassel-Baska (2006) identified that researchers in the field have conducted research in a number of directions, whilst at this point, research studies should be related and compared to one another, resulting to a foundation for sound generalizations and conclusions in terms of theory and practice. For this reason, this study comes to synthesize research findings in the field of general and mathematical giftedness, form and empirically assess an identification process of giftedness in mathematics, aiming to allow the manifestation of mathematical potential. To develop an inclusive domain specific identification process to capture manifestations of mathematical giftedness, this study acknowledged and exploited aspects from general models of giftedness that apply to mathematical giftedness as well. These aspects had a subsequent impact on and guided the design of identification processes. For the purposes of this study, five aspects were adopted: (a) the multidimensionality of the construct of giftedness (Gagné, 2008; Gardner, 1983, 1993, 1999; Heller, 2004; Hong & Milgram, 2008; Renzulli, 1978, 2002; Sternberg, 1985), (b) the notion of potential or promise as an important dimension of giftedness (Gagné, 2009), (c) the existence of domain-specificity of giftedness (Csikzentimihalyi 2000; Song & Porath, 2005; VanTassel-Baska, 2005), (d) the important role of creativity in giftedness (Kaufman, Plucker, & Russell, 2012) and (e) the use of terms of promising and gifted students interchangeably, as the gifted population included both students whose promise has not yet manifested and also students with their abilities demonstrated.

This study designed an identification process of mathmetical giftedness that proved its efficacy by being implemented in real school settings through good quality basic research, not just examining the significance of individual tests. The study came as a response to the relevant concerns of Heller and Schofield (2000) and Van Tassel-Baska (2006). Research in the field suggested to target students of early age in an attempt to prevent talent loss, which is considered to be one of the major challenges facing parents and educators nowadays (Hong & Milgram, 2008). That is why the study targeted students at the elementary level. In addition, the age group was important, since it was awaited to observe the articulation of student thinking processes. Thus, students had to be of an appropriate age to be aware and express their reasoning processes. Therefore, the study focused and therefore suggests the proposed identification process to be applied to students attending 5th and 6th grade of elementary school. The identification process suggested was created from the beginning to focus on the identification of mathematical giftedness rather than general or any other specific type of giftedness. This is consistent with concerns that a discrepancy between the objective guiding the identification system, the identification process followed and the content and nature of subsequent services provided should be avoided (Coleman, 2003; Heller, 2004; Nevo, 2008).The suggested identification process includes two stages, and collects multiple forms of evidence, both qualitative and quantitative, to form a detailed profile of students. This is compliant to previous studies that provide evidence that any identification process of giftedness should be based in multiple sources of evidence to capture the nature of giftedness using good quality research (Freiman & Rejali, 2011; Hartas, Lindsay, & Muijs, 2008; Koshy, Ernest, & Casey, 2009; Salvia & Ysseldyke, 2001).

To be exact, the first stage concerns the administration of a group administered test to 5th and 6th grade students. The nature of the tasks included are similar to the one used in aptitude testing, since the review of the research literature points to the use of aptitude tests (Matthews & Foster, 2005; Miller, 1990) as the most effective in identifying giftedness. With regard to the content of the screening instrument, our focus was on mathematical abilities that have been found in the research field to be associated with giftedness in mathematics. For this reason, the proposed identification screening instrument was based in problem solving and measures number reasoning, spatial reasoning and creative reasoning. The three factors are consistent with earlier research recommendations. More specific, research has shown that mathematically gifted students are characterized by a special blend of problem solving (Krutetskii, 1976; Sowell, Zeigler, Berwall, & Cartwright, 1990), spatial (Benbow & Minor, 1990; Block, 1985; Sowell, Zeigler, Berwall, & Cartwright, 1990) and creative abilities (Geake & Dodson, 2005; Kanevsky & Geake, 2005; Livne & Milgram, 2006; Milgram & Hong, 2009; Renzulli, 1978; Sriraman, 2005; Sternberg, 1999). Thus, this study put to the test and synthesized the mathematical abilities that distinguish gifted students in mathematics focusing on mathematical reasoning rather than learned mathematical facts from specific curricula and memorization. The screening instrument included non-routine tasks, and by default, it is not likely to have a ‘ceiling’ for children at this age. Moreover the scoring system allowed scores to be compared directly for different test versions or administrations, and it is cost and time effective because it is a group test rather than an individually administered test.

According to their performance, students considered to be mathematical promising were identified and were eligible to take part to the second stage of the identification process. Given the wealth of information collected on the nature of cognitive and hypercognitive processes employed by students gifted in mathematics, other confirming previous research findings and others proposing new facets and directions, we may assume that the screening instrument has efficiently served its purpose. Moreover, in our discussion with the parents of the students that were selected to participate in the second stage, we were informed that several of them were medalists of Mathematical Olympiads or Kangaroo Contests organized in Cyprus. In addition, in an informal discussion with parents, we discovered that many of the selected students shared interests and characteristics often described in literature as typical characteristics of gifted children. Among these, parents mentioned the ability to speak or read from an early age. Two striking examples are mentioned. First, there was a fifth grade student, among the students identifies as mathematically promising, with a high level of spatial sense and a passion for LEGO constructs. More specific, the student, had a storage room in his house devoted in his passion for constructing figures made of LEGO pieces. In fact, the room was filled up to ceiling with helicopters, castles, animals, structures and vehicles constructed out of LEGO, placed in carefully constructed shelves. In fact, the mother of this student mentioned that a trip to London was devoted to a visit to a Lego Exhibition at a Museum and a visit at a large LEGO Store to get supplies for his next LEGO projects. Second, there was a sixth grade student that learned by himself how to repair mobile phones, when his father mobile phone accidentally fell into the water and volunteered to fix it. To do this, he searched the web himself and managed to repair the phone. Since then, he managed to gather some money by helping his dad in the family business, and spent the money to buy equipment to repair electrical devices. At the time of observation, the researcher visited this boy's room filled with special equipment and accessories and was told that he fixes broken electronic devices, that friend and relatives send to him, being aware of his ability. With the money he makes each time a device is repaired, this student buys more equipment and accessories to continue his hobby.

For the purposes of the second stage, the study proposes a second instrument, one that yields qualitative information. This complies with researchers that pointed out the importance of collecting qualitative evidence during the identification of giftedness (Salvia & Ysseldyke, 2001).

Except of cognitive abilities and processes, mathematically gifted students' performance depends on hypercognitive attributes (Borovik & Gardiner, 2007; Calero, García-Martín, Jiménez, Kazén, & Araque, 2007; Renzulli, 1978; Sternberg, 1986). To conduct a thorough analysis of students' work, the second stage of the identification process focuses on the investigation of cognitive and hypercognitive processes of mathematically gifted students that will be possible to be closely examined through the careful observation of the students during solving challenging tasks.

The choice to incorporate mathematical testing and observation of problem solving in challenging tasks is compliant to reseach findings, with research showing that observing students during rich problem solving consists of one of the most efficient ways to capture the manifestation of mathematical giftedness and potential (Bicknell, 2009; Koshy, 2001; Koshy, Ernest, & Casey, 2009). The decision to investigate the thinking processes of gifted students in mathematics, comes also to respond to the problem adressed by Leikin (2011), that a number of studies accentuate predominantly general psychological traits, whilst they do not consider the learning and thinking processes of gifted students in mathematics in accordance with contemporary theories of mathematics education.

To develop taks for the identification of mathmetical giftedness, research suggests several principles, that this study also followed. More specific, rapid and accurate computational ability should not be a determinant factor in excluding students (Sheffield (1994), whilst tasks should be designed in such a way as to look for promise (Sowell, Zeigler, Berwall, & Cartwright, 1990), rather than demonstrated excellence. Moreover, tasks should require higher-level cognitive skills (vanTassel-Baska, 2014), provide challenge with off level non routine tasks (Diezmann & Watters, 2002b; Kell, Lubinski, & Benbow, 2013; Sheffield, 1994; Thomson & Olszewski-Kubilius, 2014; Warne, 2014), allow and promote the use of multiple reasoning methods (Greenes, 1997; Peressini & Knuth, 2000; vanTassel-Baska, 2014), promote the articulation of thinking processes (Peressini & Knuth, 2000; vanTassel-Baska, 2014), assess spatial ability (Mann, 2005; Olenchak & Reis, 2002; Shea et al., 2001) and mathematical creativity (Renzulli, 1978).

There are several factors that were taken into consideration in suggesting tasks suitable for measuring mathematical problem solving abilities, to be used to identify mathematical giftedness. These considerations were made following relevant research findings. First, the problems were formed in a way as to challenge students' thinking processes. Second, the test placed a greater emphasis upon reasoning and learning than on memorization. Third, the type of questions and tasks in the test were selected to be different from ''everyday'' mathematics questions, since in such a case, the responses of students tend to follow their teaching. If the tasks were similar to those taught in class, then students' performance would reflect much more on how and how much they have been taught, rather than mathematical reasoning in a novel situation. Thus, the tasks were designed and phrased in a specific way, as questions that are not directly taught in lessons to the specific age group. Although they drew on the elements of the curriculum for the age group, the questions were unlikely to have been rehearsed. In this manner, it was felt that responses to the problems were more likely to be a reflection of qualities in the student rather than the success of teaching.

The suggested identification process identifies promise and underachievers. The danger of neglecting underachievers (Freiman & Rejali, 2011) if the identification process is not designed properly, was stated in research studies. To do this, challenging problems out of the level of ordinary teaching were included, that do not depend on the taught curriculum. This was a consciously made decision, since it was believed that these students would be motivated to show their abilities when confronting novel mathematical problems. This was achieved and acknowledged in our communication with teachers after the data analysis of the screening instrument to ask for the telephone numbers of mathematically promising students' parents to ask permission for their children to participate in the second phase. More specific, in some cases, teachers were surprised by the names of the students labeled as mathematically promising, since they did not acknowledge the capabilities of quite a few students. In fact, teachers showed that some of them did not show enthusiasm or interest in classroom, typical characteristics of underachievers (Salvia & Ysseldyke, 2001).

To address underrepresentation of students from different groups, an attempt was made to have short tasks, students with different backgrounds, sociocultural origin, and language or with limited capacity in Greek language would not have problem to deal with. It was not amongst our preferences to turn only to spatial, thus primarily non-verbal tasks to address the issue of underrepresentation, a practice used in intelligence testing, since this choice would limit the range of mathematical abilities and processes to be observed. Our consideration to undress underrepresentation was supported by relevant research concerns (e.g. Coleman, 2003). Our identification process addressed the problem efficiently, since one of the students screened from the first instrument was of Russian origin, at the time of observation a Cypriot resident. In fact, he experienced difficulties in the use of the Greek language, and during observation, he felt comfortable to communicate in English. Although not fluent in Greek, this student was amongst the students who outperformed on both instruments, possessing high level mathmetical ability, self-regulation, self-confidence, ability to perceive mathmetical relations easily and comprehension of the mathmetical structure of problems.

Abilities of gifted students in mathematics

The quantitative analysis of the data collected from the screening instrument, allowed to outline students' mathematical abilities related to mathematical giftedness. Namely, mathematical giftedness entails four types of abilities; ability in number relations, spatial ability, ability in inclusion relations and creative ability. The data suggest that according to students' responses, ability in number relations and spatial ability contribute more than ability in inclusion relations and creative ability. Ability in number relations, spatial ability and creative ability were anticipated to be related to mathematical giftedness, since these findings confirm previous related research discoveries. For example, Krutetskii (1976) illustrated through his comprehensive research the abilities of mathematically capable students in mathematical reasoning and number relations. In addition, Webb, Lubinski and Benbow (2007) were amongst the researchers that argued on the spatial abilities of gifted students in mathematics. Also, creative ability was related to mathematical giftedness, as for example in studies of Sriraman (2005). On the other hand, the finding that reasoning associated with relationships of inclusion is one of the four mathematical abilities that can be used to measure mathematical giftedness consists a contribution of this study to the research literature. To date, there is no study that we are aware of that investigated or precisely referred to this ability as indicator of mathematical giftedness. Hence, it is intriguing to further explore the association of this ability to giftedness in subsequent research studies.

The fitting of data to the model and confirmatory factor analysis in general confirms that the items used were in fact suitable instruments for measuring mathematical giftedness in four salient factors and that the model could represent distinct factors across which students' mathematical abilities related to giftedness can be organised and should be considered during the identification of mathematical giftedness. Thus, students' abilities in (a) number relations, (b) spatial concepts, (c) creative tasks and (e) inclusion relations are important for the identification of mathematical giftedness.

Description of the cognitive processes of mathematically gifted students during problem solving

Through the observation of mathematically promising students working with challenging mathematical tasks, the nature of cognitive processes related to mathematical giftedness was elucidated. The characteristics observed were influenced firstly by the tasks selected to be used for observation purposes and also the methodology used. To be more exact, the observation tasks included tasks that required students to experiment, observe or/and discover mathematical relations, reach to conclusions, form generalized arguments and apply creative thinking among others. Therefore, it was expected that the particular type of tasks were inevitable to reveal if the students observed possess such type of mathematical abilities and expose to what extent these abilities are developed. It should however be clarified that although the nature of the activities promoted the demonstration of students' potential, the behaviors that were observed revealed traits that characterized the students, not the tasks.

Generalization

Firstly, behaviors described in Chapter Four showed that mathematically gifted students were able to arrive at generalized arguments with relative ease. Indeed, mathematically gifted students were able to observe the underlying structure and components of a posed problem and identified similarities and differences amongst the components. These findings are consistent with Krutetskii's conclusions (1976), that mathematically capable children perceive the mathematical material of a problem both analytically and synthetically, grasping the formal structure of the problem as a whole without losing sight of the elements, being able to visualize the class of a single problem, recognizing the inferred generality, which might be perceived as seemingly unrelated features to other students.

In this study, this ability allowed students to acknowledge the conditions under which specific properties tended to apply for a certain set of numbers and thus, they formulated a generalized argument to express these relationships. For some students, this occurred without trials with specific numbers or experimentation, while others required a minimal number of experimentation to grasp the connections among the components of the problem situation. This finding is also consistent with similar findings of Krutetskii (1976), who also observed that capable students generalize mathematical content rapidly and broadly with a minimal number of exercises. However, the findings of study also offer a slightly different perspective, by pinpointing that researchers and teachers in the process of observing and identifying giftedness in mathematics, should not expect that all students will necessarily grasp the problem's components and generalize 'on the spot'. Rather, they may need to develop a process to arrive to the generalized argument and this particular process will reveal among other their higher order cognitive processes and unique abilities.

In general, the findings of this study regarding to generalization are consistent with the findings of other research studies who found that synthetic thinking is important in the field of mathematics (Haylock & Thangata, 2007) and intelligence (Gardner, 2000). In fact, 'logical-mathematical intelligence', as proposed by Gardner (2000) includes synthesis, by recognizing patterns and articulating generalizations. In their turn, Haylock and Thangata (2007), pointed to the essential mathematical ability to unite the components of the problem together into a whole during a problem solving situation, such as for instance, when the solver formulates a generalization from a number of specific cases.

During observation, students also demonstrated their ability to use generalizations as a means to produce many different responses. Through the provision of a generalized model, a diversity of cases was implied without the need to point the specific cases. The ability to form generalizations was proven important in creative tasks that required spatial thinking, too. More specific, the expression of a generalized strategy to form many similar but not identical shapes, thus increasing the number of responses provided, allowed students to save time and effort and therefore they were able to proceed on thinking of other strategies to draw new figures with the same area. Thus, the findings of this study add to the literature, by commenting on the revealed relationship of the ability to generalize to creative ability. As shown, grasping the underlying structure and forming a generalized idea about the demands of a creative task, allows gifted students to form in turn a generalized strategy. As a result, students are more productive and generate more responses with less effort.

Flexibility of Mental Processes

A greater category of interrelated cognitive processes that emerged through students' observation was that of flexibility of mental processes. This was a striking ability of students gifted in mathmetics and was manifested in different ways. This general category was also suggested by Krutetskii (1976), in his study of the structure of abilities of mathematically able children. In fact, his name is linked to flexibility since "it may be one of his greatest additions to the discussion of mathematical ability" (Chamberlin, 2012, p.25). This study confirms Krutetskii's findings whilst at the same time adds supplementary dimensions of flexibility to the literature. Discussion of results will be organized across these dimensions; flexible thinking through the generation of multiple mathematical solutions, reasoning in cycles, handling multiple mathematical relationships at once, fluency for expediency, curtailment of the process of mathematical reasoning and economical thinking and reversibility of mental processes.

Flexible thinking through the generation of multiple mathematical solutions. From the excerpts and figures discussed in the relevant section of results, the ability of the students to think flexibly through the production of a range of mathematical solutions was obvious. This finding is in line with similar research findings (Chamberlin, 2012) who suggested this to be a discriminating behavior amongst gifted and non-gifted learners in mathematics. According to Chamberlin (2012), advanced mathematical learners are likely to suggest more than one mathematical solving approach to a task, whereas suggesting an additional path, seems to be challenging for a typical learner. In fact, the initial path may even hinder the process of providing an alternative solution for typical learners. Consistent with Chamberlin's findings, it was clear in our findings that students' initial solutions provided an excellent foundation for constructing subsequent solution paths based on knowledge accumulated.

It should be noted that students in this study provided multiple mathematical solutions, both in cases that this was explicitly required, as in creative tasks, but in other tasks as well, revealing their instinctive and strong inclination of gifted children to approach a problem in different ways, regardless of its nature. This finding is consistent with relevant findings of Borovik and Gardiner (2007). Moreover, students in this study were in a constant search for possible additional relations amongst the components of the problem. This distinctive behavior reminded the researcher of mathematically gifted students' strive to look for mathematics everywhere, mentioned by Osborne (1981) as a trait of gifted learners in mathematics. This tendency for continuously looking for possible mathematical argumentations and conclusions for the same task, in some cases, even resulted in observing and verbally expressing non anticipated relations. These unique relationships revealed the wealth of mathematical connections that gifted students in mathematics are able to make and the interconnectedness of ideas in their mind.

Reasoning in cycles. The majority of students showed evidence of flexible reasoning, in terms of thinking in cycles. This characteristic behavior of thinking in cycles became evident when students were confronted with the need to improve a proposed solution to meet the demands of a task, as in the case of Activity 1 and the limitation added by the researcher, or when they were dealing with a creative tasks and had to think of a way to produce a different outcome. In the case of Activity 1 with numbers to be grouped, this process was mainly triggered by the need to come up with new grouping ways compared to the previously suggested ones. In some other cases, whilst in the mid of the process of a new grouping attempt, students may have found out that they were repeating a previously suggested grouping attempt or that there were numbers belonging to other groups as well.

Namely, it was shown that students were able to flexibly handle mathematical relations and easily alter a type of organization to produce a new one, by exploiting a previous proposed solution. Two possible reactions to this realization were observed. Either to come up with a different group name in order to accommodate the originally chosen numbers for each group and differentiate from a previous attempt, or to erase and remove some of the numbers and placing them in a new group. Especially in the case of being confronted with a number eligible to be included in two groups, students exhibited great ease in restating the categories in order to accommodate the numbers according to their wish. The need to alter their original grouping approach, gave the opportunity to students to combine more than one mathematical criterion for the same grouping.

Throughout each student's work, it became clear that student's thinking evolved through practice and experience accumulated as correct responses increased. As time passed by, students felt more comfortable to use multiple mathematical relationships in the same group. Sometimes, a revolution in students' working style occurred. Usually it was during the first attempts that students realized that they had numbers allowed to be part of more than one group and were put in the process of thinking how to overcome this obstacle. Later, based on the experience accumulated from the previous attempts, there were several students that improved their working approach by thinking ahead and incorporate at once the exceptions into the rules of the groups from the start. Thus, with a more strategic approach applied, there was no need to proceed in a new thinking cycle anymore. Furthermore, students seemed to acknowledge that different responses, can be based in the same mathematical conditions, and can be adjusted to produce many different ways.

This process was able to be witnessed thanks to the creative nature of the specific task that allowed students to attempt to approach the problem in many different ways, thus forcing them to repeat in innovative ways the thinking process by employing different mathematical grouping criteria. Hence, we may deduce that creative tasks are important to be included in any assessment of mathematical giftedness, since they provide the opportunity to a potentially gifted person to show his/her abilities, knowledge and skills, that could otherwise be overlooked if tasks were of a certain type.

Although the general ability of flexibility of mental processes was suggested by Krutetskii (1976), the specific behavior of reasoning in cycles is a contribution of this study to the existing literature in regard to the field of giftedness. Namely, this study suggests and provides a plethora of descriptive examples and empirical evidence that reasoning in cycles is one of the characteristic cognitive processes that gifted learners in mathematics exploit when dealing with mathematical problems, to refine their solutions or produce many alternative solutions.

Control of multiple mathematical relationships at once. It was remarkable how the vast majority of students observed was able to handle multiple mathematical relationships simultaneously. Also, they were able to represent them in a way it allowed them to observe the associations needed.

Fluency for expediency. During observation, another striking cognitive process used by students was the identification of an economical method and the persistence in using it to achieve greater level of fluency and flexibility. This finding is consistent with the cognitive process suggested by Chamberlin (2012) that first introduced the term of fluency for expediency, and was an inspiration for this study to accordingly name the cognitive process observed. According to Chamberlin, typical learners may persist in a specific line of thinking, although it may not be economical or advantageous. This could be observed when students may stick with an insufficient approach, without changing into another method, with the fear of providing a wrong answer. On the contrary, although mathematically gifted individuals will start working on a problem with a strategy in mind, they are more likely to switch to a more economical method as soon this is identified. This will result to both a greater level of flexibility by providing different types of strategies to solve a mathematical task, and also a greater level of fluency, by suggesting a greater number of correct responses.

Indeed, our findings support Chamberlin's (2012) description of mathematically gifted learners using strategies to their own benefit. Indeed, there were numerous instances during students' work where they exploited to the fullest a specific strategy in order to achieve a greater level both of fluency and flexibility. Especially in Activity 6 where this process was observed, students perceived the problematic situation as an anchoring task. This process was manifested in three different approaches; sliding pieces, rearranging pieces and "complete what is missing" technique.

In regard to the technique of sliding pieces, students were able to produce many correct outcomes by slightly modifying the original figure by just sliding one square piece and not reshaping the whole figure. Acknowledging that the only requirement was to keep the area constant, students were able to produce many shapes deriving from the initial figure with minimal effort and sliding one of the pieces. There was one student that proceeded to a generalized statement, declaring that a great number of similar shapes could be made by sliding one of the two squares to different positions to the left or to the right.

In regard to the approach of rearranging pieces, after assembling different pieces to form an initial figure with the requisite area of 2 cm^2, there were students that preferred to reorganize the same fragments into different positions and orientations, transforming the existing figure into a new one. The third approach to produce many similar but not identical figures while conserving on time and cognitive load, was the "complete what is missing" technique. This approach was suitable and convenient to use in cases of using pieces of ¾ cm^2 as components of the requested figures. More specific, student started by drawing pieces of ¾ cm^2 while simultaneously thinking about the parts that were missing to complete the area of 2 cm^2. To reach the requested area, they placed the missing parts in another position, adjacent to the already placed figure. Therefore, students managed to show once more their vast repertoire of strategies and their ability to select the one they find most appropriate each time that also would benefit them more in the problematic situation they were found in.

Similarly to the case of reasoning in cycles, although the general ability of flexibility of mental processes was suggested by Krutetskii (1976), the specific behavior of fluency for expediency is a contribution of this study to the existing literature in regard to the field of giftedness. Although Chamberlin suggested the term quite recently in 2012, he did not provide any empirical evidence to support his assumption. On the contrary, this study validates Chamberlin's affirms and specifically suggests and provides a plethora of descriptive examples and empirical evidence that fluency for expediency consists one of the distinctive cognitive processes that gifted learners in mathematics exploit during problem solving.

Curtailment of the process of mathematical reasoning and economical thinking. At times, students observed displayed signs of their ability to curtail the process of mathematical reasoning by eliminating intermediate steps or even chose to work in an economical way of thinking. For example, a student chose to use a smaller number instead of a larger number in an activity of experimentation and generalization. This decision helped him arrive to a conclusion by quickly grasping the necessary relationship and made his calculations easier.

In another instance in the case of Activity 6, students chose to proceed to minor modifications to the first figures they formed, in order to produce many different and original figures with the least possible effort. For example, instead of working randomly and producing diverse figures alternating the way of working to form each figure, the majority of the students chose to focus in a specific way to produce figures and evolve the selected technique. Only when the capacity to produce new figures using the same underlying idea was exhausted, students moved on to think of a different way to create new figures. Hence, students worked in an economical manner by focusing their efforts following a specific philosophy each time.

Our findings support the findings of House (1987) and Krutetskii (1976), who referred to mathematically able pupils' ability of abbreviating mathematical reasoning (House, 1987) and striving to find rational, easy, clear and the most economical ways to solve problems". In general, the findings of this study suggest that mathematically gifted learners are able to select the shortest route and the most economical way to work under complex situations. They have the capability to simplify their work, as in selecting simpler numbers to work with, if this is does not affect the outcome. In addition, they have shown that they can exploit a task's conditions into their own benefit; thus, they are able to produce many correct responses with minimal effort and minor modifications. Furthermore, they may evolve and persist on a certain working strategy while afterwards exhausting a certain way of working until no more responses could be produced by working in the same way, Also, they may develop a specific system to keep track of their work and make sure no duplicate responses are provided, in case of creative tasks where fluency is an important parameter.

Although this process is helpful for gifted students, Borovik and Gardiner (2007) alerted that the ability to “compress" the reasoning process hinders the risk of a child being misunderstood. This could happen since the child may not provide all the details of the answer, skipping those that he/she thinks are too obvious and/or uninteresting. Thus, our findings in combination of Borovik and Gardiner’s (2007) warnings, validate our decisions on paying attention to students’ thinking processes rather than just the end product in the framework of our identification process, in order to avoid misunderstandings and overlook of gifted students. Our findings also suggest that educators should accordingly pay particular attention to students’ argumentation and reasoning process and accept curtailed approaches in classroom.

Reversibility of mental processes. Students’ work made possible the appearance of another distinctive process during problem solving. This was the ability to reverse a thinking process, in other words, thinking backwards from the end to the start to solve a problem. This was made possible through the nature of the particular task.

In general, students’ efforts validate that mathematically capable students are able to reverse a thinking process. In other words, they can think backwards from the end to the start to solve a problem, if this fits the situation. The results of this study are consistent with the suggestion of a series of researchers along the years that reported this ability as an indicator of mathematical giftedness (House, 1987; Krutetskii, 1976). Namely, both researchers reported that gifted pupils in mathematics are able to switch from a direct to a reverse train of thought exhibiting reversibility of mental processes in mathematical activity. Findings are also consistent with a more recent study conducted by Sriraman (2008) that also reported that students revealed their ability to reverse the direction of a mental process to arrive to the required conclusion.

Furthermore, it was shown that the ability to reverse the thinking process has to be generated as a precondition needed to yield the desired outcome. Even though it may not originate naturally at once, a person may show this ability, as soon as it is required. This finding adds a new perspective, suggesting that the ability to reverse mental processes has to be triggered by the problematic situation, in order to be manifested. In fact, if there is no significant gain from using it, it may not appear.

Creative Thinking in Mathematics

Through the tasks posed to students during observation, it was made possible to observe their creative abilities in mathematics. Conclusions based on the results are discussed in three subsections; construction of mathematical connections, creative spatial ability and originality.

Construction of mathematical connections. It was of great importance to the research study to observe students' ability to interrelate different mathematical associations.

As shown in Chapter Four, students were able to connect mathematical relations in many different ways, exhibiting great fluency. The findings of this study are consistent with the literature, according to which fluency is considered to be a vital component of creativity (Gil, Ben-Zvi, & Apel, 2007; Ervynck, 1991; Silver, 1997; Leikin, 2009a; Pelczer & Rodríguez, 2011; Torrance, 1974). Moreover, findings of this study support other relevant research findings (Kanevsky & Geake, 2005; Geake & Dodson, 2005), according to which gifted children are characterized by high creativity, while making original inter-subject connections with impressing relative ease. Indeed, students' responses revealed the wealth of associations they can make between mathematical concepts, relating notions involving place value, odd and even numbers, digit sum, range of numbers, multiples, factors, exponents, square numbers and square roots. These findings are consistent with other equivalent findings, according to which, mathematically gifted children make unique links when presented with a demanding mathematical task (Chang, 1985).

Except of fluency, mathematically gifted students showed signs of flexibility, since there were different perspectives by which the data information was perceived upon, resulting to different categories of responses, differing in complexity. Namely, a significant percentage of students, managed to combine two mathematical relations in a specific group name, whereas some of them even attained to use three mathematical relations in a particular group name, at a specific instance during the activity. Other researchers have similarly discussed flexibility in terms of different categories of responses, in other words, different approaches, as an integral aspect of creative ability (Gil, Ben-Zvi, &Apel, 2007; Ervynck, 1991; Silver, 1997; Leikin, 2009a; Pelczer & Rodríguez, 2011; Torrance, 1974).

The findings of this study indicate that fluency in terms of the number of correct responses seems to be related to the ability to combine multiple mathematical criteria. This finding was based on the observation that students that provided fewer correct responses were less likely to combine more than one mathematical relation in a particular group name. As fluency increased, it was more likely to see student being able to associate two or even three mathematical criteria in a single group name.

Except of the relation between fluency and the complexity of mathematical relations proposed, data of this study also report a relation of originality in terms of the novelty of a response to the ability to combine multiple mathematical criteria. It was of interest to the researcher to investigate the frequency in which a totally original response was reported, that is a response that only one student out of the whole sample thought of was investigated. Results showed that a significant number of students (67%) provided at least one completely original response than no other student suggested. From this group of students, almost half of them were capable of combining up to two or up to three mathematical relations in a specific group name. The results of our study support Miller's findings (1990) that denoted the ability of gifted students in mathematics to work with mathematical problems in flexible and creative ways rather than in a stereotypic mode.

In general, our findings suggest that mathematical promising students, are capable of interrelating mathematical concepts in numerous ways, exhibiting fluency and the ability to make connections among mathematical concepts. They are also competent to combine multiple mathematical criteria together to achieve a required outcome and handle multiple data in parallel. Furthermore, they may provide original responses to differentiate both among previously suggested answers and responses provided by others.

Creative spatial ability. The observation of creative ability in the context of a spatial task, as in the case of Activity 6, provided interesting findings. Although spatial ability has awarded its significance in the field of mathematical giftedness (Benbow and Minor, 1990; Block, 1985; Krutetskii, 1976; Sowell, Zeigler, Berwall, & Cartwright, 1990), there is a scarcity of research in regard to the relation of spatial thinking skills and creativity (Mann, 2004). Amongst the few studies that investigated the two constructs, the results of Baum (1984) and Silverman (2002), suggest that students with spatial strengths excel when engaged in tasks that require higher order thinking skills and creative problem solving, two dimensions that this study took into account. In addition, in a recently published research study, Mann (2014), provided evidence to support that students with high spatial abilities have a preference for innovation, providing support for the argument on the existence of a direct relationship between spatial abilities and creativity. In this study, Mann observed that “innovation is a preference for these children as they develop their own problem-solving strategies and a deep understanding of how seemingly unrelated concepts can be integrated” (p. 67). Another recent study of Kell, Lubinski, Benbow, & Steiger (2013), acknowledged that spatial ability has a unique role in the development of creativity. This study comes to fill this gap, due to the lack of relevant research, by providing evidence on how creative thinking is manifested in tasks requiring spatial thinking skills, and especially adds to the literature by commenting on how this relationship is manifested by gifted learners in mathematics. Thus, this study provides evidence of a close relationship between creative spatial ability and giftedness.

The discussion of the findings in regard to creative spatial ability is organized in two sections; holistic and analytic perception of spatial information, focus on product but also on the process.

Holistic and analytic perception of spatial information. The results of this study suggest that mathematically gifted students are able to read the spatial information provided in the task in two different ways. Namely, they are able to conceive the information both holistically and analytically, thus following corresponding strategies according to the way spatial information is perceived at a specific instance. This difference in the way of perceiving spatial information became evident in cases where although two students might have produced the same end product, the same thinking process was not necessarily followed by both learners, in order to form the figure.

Particularly, the ability to perceive spatial information holistically was proven to be accommodating in providing the opportunity to visualize and suggest multiple figures based on the prototype figure, thus increasing fluency. Conceiving spatial information holistically, allowed students to visualize other same area figures and verbalize their thinking processes. Our findings also suggest that there are gifted learners that they may generalize by describing a method that will provide many alternative creative products. According to this study’s results, these students believe that the verbal expression of a method to produce many creative products with the same underlying philosophy is enough to denote their ability to produce a plethora of shapes; hence, they do not proceed in designing more shapes using the same strategy. The results of this study provide a new perspective into the contextualization on mathematical giftedness, by proposing the cognitive process of perceiving spatial information both holistically and analytically as an indicator of mathematical giftedness. At the same time, this cognitive process has been shown to improve creativity in mathematics, by facilitating fluency and flexibility.

Focus on product but also on the process. The findings of this study suggest that gifted students in mathematics are fascinated by creative challenging spatial tasks. Indeed, they are so drawn into them that they appear to be fixated to provide as many different and original products. They are so overwhelmed by the task, that they do not feel the need to provide a typical figure. This concentration in the process and tendency to provide original responses leads to interesting peculiar products. In general, the findings reveal that gifted students in mathematics have a strictly structured idea about what they want to do and the way they want to work. This idea is being processed while time passes, experience is accumulated and more figures are added. As a result, their technique evolves through practice, figures become more elaborate, and approaches become more sophisticated. This cognitive process consists of a contribution of this study to the research field, since focusing both on product and process should be considered as an indicator of creative spatial ability that in turn is related to giftedness in mathematics.

Originality. The results of this study suggest that originality consists of one of the most striking manifestations of creativity in mathematics. Although not clear in this study, other studies (Leikin, 2009b; Leikin & Lev, 2013) advocate that originality is the strongest component in determining creativity, amongst the fluency, flexibility, and originality triad. Leikin's as well as Leikin's and Lev's findings are consistent with definitions of creativity at an absolute level: being creative means being original (e.g. Liljedahl & Sriraman, 2006). This study acknowledges the power of originality in determining creativity and mathematical giftedness, but further research is needed to support Leikin's confirmations. Following her findings, Leikin (2009b) further hypothesizes that fluency and flexibility are of a dynamic nature and as such they can be developed, whereas originality is of the "gift" type. If this is the case, we may assume, based on Leikin's affirms that originality is one of the most valuable indicators of mathematical giftedness.

Recently, Kajander, Manuel and Sriraman (2013) provided an alternative view of creativity; rather than considering creativity as an adjective to describe the nature of a problem, a different perspective may distinguish among creative products or processes. This study further expands this classification. Transferring it to the creative component of originality, the results of the study suggest that originality exhibited by gifted students in mathematics can be described across original products and original processes. Hence, the discussion on the relevant results with regard to originality is organized across these two dimensions.

Original products. The results of this study reveal that gifted students in mathematics tend to provide original products to differentiate their work and provide innovative, often original assumptions, unique observations as well as unanticipated products. This study, suggests that original products may result from gifted students' strive to look for mathematical associations everywhere and furthermore being fixated with finding the answer they are looking for. This tendency to see mathematics everywhere was referred to as 'mathematical cast of mind' by Krutetskii (1976). Namely, the findings of this study reveal that gifted students in mathematics may provide reasoning in regard to other mathematical relations additionally to the ones required to reach to the desired outcome, thus extending what was already posed by the problematic condition itself. In fact, extending the problematic situation is an important finding of the study.

In addition, the results indicate that the endeavor of gifted students in mathematics to find mathematical relationships in all places, leads to the provision of multiple solution paths based in different associations and argumentation paths. Thus, this example shows the close relationship of flexibility and originality as aspects of creative thinking. A student that provides multiple types of solving a problem, is more likely to suggest original approaches, displaying different categories of responses. This finding is consistent with prior research findings that consider flexibility as a necessary condition for originality when solving multiple solution problems (Leikin & Lev, 2013).

Considering that originality is relative to the work of other individuals of the reference group proposing responses to the same undertaking (Leikin & Lev, 2013), the fact that the majority of the 34 students observed in this study managed to provide a minimum of one completely original response than no other student suggested, is a remarkable achievement. This is truly impressive, considering that the sample consisted completely of gifted learners in mathematics, that as shown all exhibit enhanced creative abilities and were able to provide truly unique products amongst a great variety of responses. Findings of this study support Wolfle's (1986) findings that mathematically gifted students develop unique solutions to common problems.

Originality is not only exhibited in original products, but also originality is expressed in terms of the processes mathematically gifted students use during mathematical problem solving. The next section discusses the related findings.

Original processes. The results of this study suggest that gifted students in mathematics employ original processes during problem solving, not just end in creating original products and providing original responses. This study also reports that the use of original processes requires gifted students in mathematics to take risks by breaking the boundaries of the posed problematic situations. Originality in terms of using novel problem solving strategies, as exhibited in this study validates recent findings of Mann (2014), who noted that innovation is a preference for gifted children in mathematics as they formulate their own problem-solving strategies and a profound understanding of how seemingly dissimilar notions can be combined.

In regard to approaches taken, the results of this study suggest that mathematically gifted students tend to approach problems from different perspectives in comparison to other students. Hence, findings disclose that not every mathematically gifted student deals a problem with the same way, nor that mathematically gifted students focus in the same mathematical relationships. These findings support Greenes's prior findings (1981) commenting on the ability to interpret problem information in original ways. To do this and achieve innovation, empirical evidence from students' work suggest that they should exhibit a willingness to step out of one self's comfort zone, take risks and think outside of the box. In addition, the results of the study suggest that gifted students in mathematics are characterized by a sense of freedom. They dare to extend beyond the boundaries of a specific activity with no attention paid on conventions that would normally intimidate others and are capable of thinking differently than others with an open mind.

Furthermore, our findings indicate that originality does not occur only in cases of working on divergent thinking tasks. Rather, thinking in original ways is a more general characteristic that describes students' creative thinking processes and may be employed in other types of problematic situations as well. Thus, this study suggests a new perspective, noting that original processes and products should not be mistakenly expected to be observed exclusively in multiple solution tasks, whereas creative thinking should not perceived as thinking triggered only by tasks that require multiple solutions.

Description of the Hypercognitive Processes of Mathematically Gifted Students during Problem Solving

The results of the study showed that two basic hypercognitive processes, were employed by mathematically promising students; (a) self-regulation (b) task commitment, perseverance and confidence.

Self-regulation

The results of this study report several signs of self-regulatory processes during mathematically promising students' work. Firstly, the results of this study showed the great strategic ability of gifted students in mathematics. This finding is supported in literature, where gifted learners have been reported to have a larger and broader repertoire of strategies to use during problem solving compared to their non-gifted peers (Carr, Alexander, & Schwanenflugel, 1996; Jaušovec, 1991; Montague, 1991). Namely, although gifted students possess a great repertoire of strategies, they are quickly able to regulate their thinking processes and select specific strategies according to task demands. The results of this study are consistent with Steiner's (2006) research findings, that report that in front of a problem solving situation, gifted students often understand better and faster which strategies are suitable for the specific instance, in comparison to non-gifted students (Steiner, 2006). The results of this study also suggest that gifted students in mathematics are capable of switching to alternative strategies according to their efficiency or remain loyal to a certain strategy if it was proven to be effective previously. Findings of this study support prior similar findings of Steiner (2006) that gifted students may select from their broad strategy range only the strategies that have proven effective in the past.

Furthermore, the results of this study suggest that another self-regulatory process of gifted students in mathematics is to monitor, control and evaluate the progress being made, staying focused in their target and not declining off course. To achieve this, gifted students in mathematics may design systems to keep track of their working attempts, so as to remember their previous actions in order to avoid repetitions, keep progressing their thinking and accomplish the task successfully. These findings validate previous findings reporting that gifted students self-activate and direct efforts to acquire knowledge by employing particular strategies (Zimmerman, 1998; Sternberg, 1986), such as setting goals, using strategies to achieve them, and closely monitoring their acquisition.

Task Commitment, Perseverance and Confidence

The results of this study reveal mathematically gifted students' task commitment, interest and willingness to work on challenging tasks. In addition, mathematically gifted students have been shown to put sustained effort into their work and remarkable persistence to achieve the desired outcome especially in demanding mathematical tasks.

In total, our findings suggest that mathematically gifted learners are really drawn by challenging non routine mathematical activities and are eager to work with them, providing evidence of great commitment to the task. In fact, in many occasions throughout this study, gifted learners in mathematics devoted a considerable amount of time on a specific activity, either to provide as many solutions possible, or to accomplish a demanding task that kept troubling them for quite some time. It was rare to observe a student that would leave a task unfinished to proceed to other tasks. Rather, students preferred to persist on a problem, struggling to find a solution. The results of this study are consistent with prior research findings, that gifted students are dedicated to their attainment of knowledge and skill (Risemberg & Zimmerman, 1992) and the outperformance of gifted students over their non-gifted peers can also be explained by their more active engagement in the problem-solving process and hard work, further to superior cognitive ability (Bouffard-Bouchard et al., 1991). This study further validates the inclusion of task commitment, as a cluster of giftedness, in Renzulli's (1978) three ring-theory of giftedness. In fact, Renzulli (1978) added task commitment in his model, referring to high levels of interest, enthusiasm, hard work, and determination in a particular area, as well as self-confidence and the drive to achieve. Since the studies of Risemberg and Zimmerman (1992) and Bouffard-Bouchard et al. (1991) as well as the three-ring theory of giftedness (Renzulli, 1978) refer to the group of gifted students in general, the results of this study further validate prior findings by additionally pointing task commitment as a behavior associated with giftedness in a specific domain, in this case mathematical giftedness. In the research field of mathematical giftedness research, Borovik and Gardiner (2007) reported as one of the traits of mathematically able children, the ability to focus on mathematics for extensive periods without apparent evidence of fatigue. Hence, this study comes to validate Borovik and Gardiner's domain-specific findings of task commitment and perseverance observed in gifted students in mathematics.

Moreover, our findings suggest that gifted learners in mathematics are confident and assured about their above average abilities. The findings of this study are compliant with findings of studies of Risemberg and Zimmerman (1992) that reported gifted students being confident about their capabilities to learn. We may assume from the findings that when exposed to proper challenging mathematical activities, gifted students in mathematics are greatly stimulated which in turn impacts on their level of engagement, perseverance and subsequently their performance.

Constructing a Theoretical Model describing Giftedness in Mathematics

Taken into account results obtained both from the quantitative and qualitative data collected and the valuable information gained, the results of the study propose an empirically validated model (see Figure 67) aiming to deliver a descriptive profile of giftedness by unfolding the abilities, cognitive and hypercognitive processes of gifted students in mathematics.

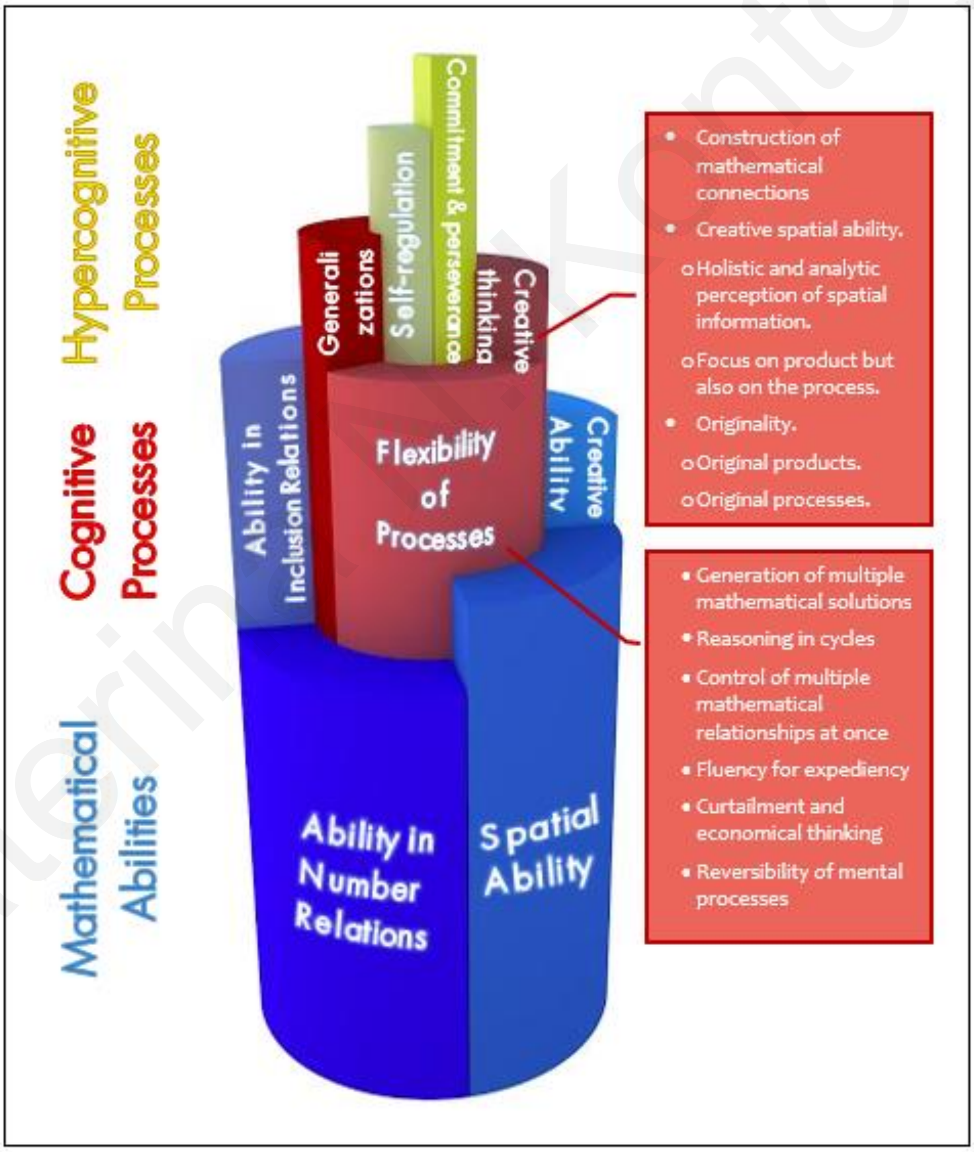


Figure 67. The theoretical model of mathematical giftedness.

The theoretical model, validates the multidimensional nature of the construct of mathematical giftedness. The model of this study confirms that giftedness in mathematics can be described using an amalgamation of three primary aspects, namely mathematical abilities, cognitive processes and hypercognitive processes, manifested during rich and challenging problem solving mathematical experiences. The model has three layers, resulting to a solid cylinder-like figure in the center and two hollow cylinder-like fragmented components as the intermediate and exterior layers. The interior cylinder in yellow color tones represents the hypercognitive processes and is further divided into two parts, representing the two types of hypercognitive processes observed throughout students' work; (a) self-regulation and (b) commitment and perseverance. The intermediate layer in red color tones represents the cognitive processes employed by mathematically gifted students and is further distributed in three principal processes; flexibility of mental processes, creative thinking and generalizations. The first two cognitive processes are further described by additional sub-processes, as shown in the model figure. The exterior layer in blue color tones represents four types of mathematical abilities; (a) ability in number relations, (b) spatial ability, (c) ability in inclusion relations and (d) creativity.

Each of the three dimensions of mathematical giftedness is deliberately represented in the model in a different color and in a different layer composed of attached fragments. The model represents the interrelationship of mathematical processes, an aspect that was evident during the observation of students engaging in rich problem solving situations. Instead of being independent processes, they coexist and are manifested according to the challenges a student faces. For example, fluency for expediency is related to the provision of multiple mathematical solutions and it is likely to manifest in an effort to provide creative responses. At the same time, the provision of multiple solutions may become easier and more productive if the student is able to grasp the underlying structure of the problematic situation and develop a generalized strategy. Thus, the types of cognitive processes are not isolated from each other and may be manifested simultaneously. In fact, it is in the watchful eye of the observer to identify and distinguish the facets of these cognitive processes during a student's work.

A deliberate selection was also made in regard to the representation of the three central dimensions of the model in three different but attached layers. The exterior layer was intentionally reserved for mathematical abilities, to indicate that they are the first thing an observer, either a parent, a teacher or an external observer in general, may notice in gifted students in mathematics. Indeed, their extraordinary level of abilities is the first thing to draw someone's attention to further investigate these students. However, abilities are just the tip of the iceberg when we refer to giftedness in mathematics. This study aims to raise awareness to the fact that mathematically gifted students employ cognitive processes to interpret and deal with mathematical information in a unique way and level, through generalizing, thinking creatively and flexibly manipulating their cognitive processes. Cognitive processes provide additional information for students' potential in mathematics in comparison to the mere assessment of abilities through a test and especially, observing just the final score. A person has to look even deeper in order to observe and focus on student's hypercognitive processes. This is why hypercognitive processes were appointed the inner part of the figure. For instance, self-regulatory techniques may be employed by a gifted learner in mathematics in order to control his/her thinking and adjust to the problem's requirements. However, an observer has to know the dimensions to look for. As such, parents are not likely to identify hypercognitive dimensions of mathematical giftedness in their children if they are not educated to do so.

CHAPTER SIX: CONCLUSIONS

Introduction

The aim and innovative aspect of the study was to describe mathematical giftedness through a theoretical model outlining the abilities, cognitive and hypercognitive processes of gifted students in mathematics based in empirical evidence and to suggest a corresponding identification process that may accurately recognize mathematically gifted students in 5th-6th grades of elementary school. This chapter presents the statement of the conclusions, as they emerge from the discussion of the findings of the study. This chapter also acknowledges the education application of these findings in the field of education, thus provides relevant education applications and ends with suggestions for further research.

Conclusions of the Study

This study, adds to the body of knowledge in terms of giftedness contextualization and identification through emprically based research, as a response to relevant alarming findings regarding the small percentage of publications in giftedness providing empirical data to support and validate affirmations (Freidman-Nimz et al., 2005), even in studies proposing models of giftedness (Heller, 1993). In fact, Stoeger (2009) underlines the necessity to subject conceptions of giftedness to empirical clarification, in an effort to preserve only the ones with the power to survive in the field. Hence, Dai (2010) suggested a synthesis that conserves the useful existing research findings, now reorganizing them in the light of new understandings. This is significant for domain-specific giftedness that calls for the general models, conceptualizations, approaches and provisions to be modified to correspond to the type of giftedness, such as giftedness in mathematics that is addressed in this study.

In this context, the results of the study propose an empirically validated model aiming to deliver a descriptive profile of giftedness by unfolding the abilities, cognitive and hypercognitive processes of gifted students in mathematics, with data collected from 5^{th} and 6^{th} graders. Namely, the model of this study validates the multidimensionality of the construct of giftedness and suggests that giftedness in mathematics can be described using a combination of three principal aspects; mathematical abilities, cognitive processes and hypercognitive processes, demonstrated through rich and challenging problem solving mathematical experiences. For the formation of the model, aspects from theories of general giftedness were taken into account, as well as findings from the field of psychology. Moreover, research findings allow for the advancement of the field of mathematical giftedness and gifted education, retaining the old aspects of giftedness appearing in this study, but recasting them in the light of new knowledge and findings.

The proposed theoretical model provides important information for researchers, educators and parents in regard to the dimensions of mathematical giftedness and how it is manifested in early ages. It further points to the aspects that an observer should take into consideration in an effort to capture giftedness in mathematics. In this way, this model may guide identification processes specifically aiming to recognize giftedness in mathematics. For example, the choice to represent mathematical abilities in the model as the exterior layer, indicates that although abilities are the first dimension a parent, a teacher, a researcher or an external observer in general, may notice in gifted students in mathematics, giftedness is by no means limited to extraordinary mathematical abilities. For this reason, the model also includes two major categories of components related to mathematical giftedness. Firstly, this study aims to raise awareness to the fact that mathematically gifted students employ cognitive processes to interpret and deal with mathematical information in a unique way and level, through generalizing, thinking creatively and flexibly manipulating their cognitive processes. Secondly, hypercognitive processes are more difficult to distinguish and this is shown in the model by appointing the inner part of the model to hypercognitive processes. Hence, the model suggests that cognitive and hypercognitive processes are less likely to be identified by not educated persons, such as the parents of gifted children.

The suggested model adds to the literature and allows for the advancement of the field of mathematical giftedness and gifted education, retaining long-standing aspects of giftedness appearing in this study, but recasting them in the light of new knowledge and findings. For instance, flexibility of processes has been suggested by other researchers as well (e.g. Krutetskii, 1976) as indicators of mathematical giftedness. Still, this study, describes flexibility of processes by suggesting additional sub-processes, such as fluency for expediency that expand the concept and aid in capturing giftedness in mathematics in a comprehensive manner.

The newly established model consists of this study's contribution to the research field, since it does not only account for domain specific abilities, cognitive and hypercognitive processes relevant to mathematical giftedness but also their interrelationships. The components of the model include both aspects previously mentioned in the literature, as well as new aspects that emerged during the observation of gifted students' endeavor with challenging tasks. What is important is to provide the necessary conditions for mathematical giftedness to manifest and allow students to articulate their reasoning and hypercognitive processes.

In addition, the model proposed in this study provides evidence on how to translate the model of mathematical giftedness into a cohesive identification process capable of sufficiently capture mathematical giftedness.

Therefore, in order to prevent the present-day challenge of talent loss (Milgram & Hong, 2008; Leikin, 2009a), the study suggests the proposed identification process to be applied to students attending 5th and 6th grade of elementary school. The identification process suggested was created from the beginning to focus on the identification of mathematical giftedness rather than general or any other specific type of giftedness, in order to ensure consistency between the objective guiding the identification system, the identification process followed and the content and nature of subsequent services provided.

The suggested identification process includes two stages, and collects multiple forms of evidence, both qualitative and quantitative, to form a detailed profile of students. With regard to the content of the first screening instrument the proposed identification screening instrument was based in mathematical abilities that have been found in the research field to be associated with giftedness in mathematics. Based on prior research findings, the second stage of the identification process focuses on the investigation of cognitive (Matthews & Foster, 2005) and hypercognitive (Borovik & Gardiner, 2007; Calero, García-Martín, Jiménez, Kazén, & Araque, 2007; Renzulli, 1978; Sternberg, 1986) processes of mathematically gifted students that will be possible to be closely examined through the careful observation of the students during solving challenging tasks.

In contrast to studies that accentuate predominantly general psychological traits, whilst they do not consider the learning and thinking processes of gifted students in mathematics in accordance with contemporary theories of mathematics education, as noted by Leikin (2011), this study selected to investigate the thinking processes of gifted students in mathematics through the observation of problem solving in challenging tasks. Indeed, observing students during rich problem solving consists of one of the most efficient ways to capture the manifestation of mathematical giftedness and potential (Bicknell, 2009; Koshy, 2001; Koshy, Ernest & Casey, 2009). Following observations that the quality of mathematical reasoning is what sets apart a gifted from a non-gifted child in mathematics (Freiman, 2004), cognitive and hypercognitive processes provide additional information for students' potential in mathematics in comparison to the mere assessment of abilities through a test and especially, observing only the final score. To conclude, this study suggests that the power of the gifted children results from the unique merge and combination of all three components (abilities, cognitive and hypercognitive processes); thus the process of thinking should be allowed to be exposed to reveal their potential.

The suggested identification process identifies promise and underachievers, responding to relevant concerns (Freiman & Rejali, 2011), by including challenging situations different from those included in everyday teaching, in an attempt to motivate underachievers to show their abilities. In addition to address the challenge of underrepresentation of students from different groups (Coleman, 2003), an attempt was made to have short tasks, students with different backgrounds, sociocultural origin, and language or with limited capacity in Greek language would not have problem to deal with.

Through the detailed observation of mathematically gifted students, a substantial amount of information in regard to the reasoning processes, behaviours and hyper-cognitive processes used by mathematically promising students was collected. Following methodologies from grounded theory (Glaser & Strauss, 1967), this study adds to the literature, by gaining understanding that was constantly verified by the data collected, refining the emergent categories based on constant compararison for category development. Cognitive and hypercognitive processes observed may have been influenced by the nature of the tasks used. If completely different tasks were included, slight variations of manifestations would probably be observed. However, this study was grounded on the data collected and interpretations were made according to the specific data. In fact, one of the greatest strengths of this study is the plethora of descriptive examples from authentic extracts during observation, that illustrate in a coherent way the cognitive and hypercognitive processes associated with mathematical giftedness.

This study validated previous research findings with regard to processes employed by gifted students in mathematics, whilst at the same time verified specific dimensions and enriched research literature with new aspects of the way mathmetical giftedness is manifested. More specific, this study adds to the literature by proposing that the cognitive processes of mathematically gifted students unveiled through problem solving situations, can be described across three dimensions; articulation of generalizations, flexibility of mental processes and creative thinking. The category of flexibility of mental processes, is further expanded in more specific processes; generation of multiple mathematical solutions, reasoning in cycles, control of multiple mathematical relationships at once, fluency for expediency, curtailment of the process of mathematical reasoning and economical thinking and lastly reversibility of mental processes. The third greater category of creative thinking in mathematics involves five processes; namely, construction of mathematical connections, creative spatial ability, holistic and analytic perception of spatial information, focus on product but also on the process and originality in terms of products and processes.

In regard to the articulation of generalizations, although this study confirms that gifted learners are able to formulate a generalized argument, the findings of the study also offer a slightly different perspective, by pinpointing that researchers and teachers in the process of observing and identifying giftedness in mathematics, should not expect that all students will necessarily grasp the problem's components and generalize 'on the spot'. Rather, they may need to develop a process to arrive to the generalized argument and this particular process will reveal among other their higher order cognitive processes and unique abilities.

In respect to flexibility of processes, this study confirms prior findings and furthermore adds supplementary dimensions of flexibility to the literature, whilst at the same time indicates interrelationships amongst the components of flexibility. More specific, the behavior of reasoning in cycles is a contribution of this study to the existing literature. Namely, this study suggests and provides a plethora of descriptive examples and empirical evidence that reasoning in cycles is one of the characteristic cognitive processes that gifted learners in mathematics exploit. The same can be said for controlling multiple mathmetical relationships in parallel, and combining them to observe the associations needed. The process of exhibiting fluency for expediency is another contribution of this study to the existing literature in regard to the field of giftedness. Although Chamberlin suggested the term quite recently in 2012, he did not provide any empirical evidence to support his assumption. On the contrary, this study validates Chamberlin's (2012) affirms and specifically suggests and provides indicative instances of this process being employed by gifted learners in mathematics during problem solving. In regard to the reversibility of processes, this study comes to add a new perspective by suggesting that it has to be has to be triggered by the problematic situation, in order to be manifested. In fact, if there is no significant gain from using it, it may not appear. Although flexibility has been associated with mathematical creativity (Leikin, Levav-Waynberg, & Guberman, 2011), in terms of the categories of responses provided in a specific creative situation, this study deliberately selects to have a distinct category of cognitive processes under the concept of flexibility of mental processes, denoting a more general process. This process of flexibly altering between processes, reasoning in cycles to refine solutions or provide a range of them, controlling multiple relations in parallel, manipulating mental processes to maximize results, going back and forth reversing the mental process according to what is beneficial to the specific situation, may also occur in creative tasks, but it is also a process that may apparent in any type of rich mathmetical problem solving task. Thus, the association of flexible thinking with creativity is not neglected in this study. Rather, its importance is acknowledged.

In regard to creative thinking, this study comes to fill the research gap due to the lack of relevant research (Mann, 2004), by providing evidence of a close relationship between creative spatial ability and giftedness and especially adds to the literature by commenting on how this relationship is manifested by gifted learners in mathematics. The study also provides a new perspective into the contextualization on mathematical giftedness, by proposing the cognitive processes of perceiving spatial information both holistically and analytically and focusing both on product and process as additional indicators of mathematical giftedness. In addition, this study suggests that originality is one of the most valuable indicators of mathematical giftedness and can be described across original products and original processes. Furthermore, the study suggests a new perspective, noting that original processes and products should not be mistakenly expected to be observed exclusively in multiple solution tasks, whereas creative thinking should not perceived as thinking triggered only by tasks that require multiple solutions.

In addition, the study provides evidence on the hypercognitive processes of gifted students in mathematics, employed during problem solving situations. These processes are self-regulation and task commitment, perseverance and confidence. This study pinpoints the importance of taking also into account hypercognitive processes to identify giftedness.

Instructional Implications

The educational system should encourage, nurture and support gifted students, not only for the benefit of the individual student, but also for the benefit of society. This study provides awareness to educators on the level of abilities of gifted students in mathematics. An understanding of children gifted in mathematics may enable educators to better comprehend how to teach them in the classroom, ensuring that the next generation of gifted students in mathematics will be more eagerly identified. Moreover, understanding gifted children's thinking may not only help to create better assessment procedures and curriculum options for gifted children but for all children (Friedman & Shore, 2000). As a result, a better and deeper understanding of ways to aid mathematically gifted students to develop higher and meaningful mathematical thinking would lead to the elaboration of efficient teaching methods for all students.

The model of this study acknowledges to mathematics teachers the important dimensions of abilities in number relations, inclusion relations, spatial and creative abilities, and emphasizes the importance of providing students with a variety of activities that refer to the different dimensions identified to facilitate its development, manifestation and nourishment. In addition, being aware of the abilities, as well as the cognitive, hypercognitive and problem solving processes that mathematically gifted students employ, educators may better plan appropriate educational activities as to promote and assess these abilities, and also to fulfil their needs by providing the appropriate level of challenge. Many elementary textbooks may not have a satisfactory number of authentically challenging tasks for gifted students in mathematics, although some exceptions appear to exist (Stylianides & Stylianides, 2008). For this reason, this study alerts teachers of mathematically gifted students that they may need to provide enriching opportunities better suited to their students' needs, in order to genuinely challenge them.

The results of the study provide additional suggestions on important principles to consider during teaching. Namely, the study stresses the importance of allow students time to articulate their thinking processes in class rather than just emphasizing the correct responses. At the same time, rapid computational ability should not be a determinant factor of mathematical ability or giftedness. For this reason, mathematics teachers should emphasize more on mathematical reasoning and argumentation. In addition, spatial and creative thinking should not be neglected.

With reference to specific processes found to be indicators of mathematical giftedness, the striking finding of flexibility of mental processes as a behaviour typical of gifted learners in mathematics, should empower educators to promote this behaviour in classroom. More specific, teachers should value and accept different solution methods, discuss on their effectiveness according to a particular problem-solving situation, promote the refinement of students' solution methods by triggering them to look for alternative solution paths, and use activities that require reversibility of mental processes. With regard to the curtailment of reasoning processes, educators should pay particular attention to students' reasoning processes, accept curtailed approaches during problem solving, so that no misunderstandings will occur that may suppress their enthusiasm for mathematics and end up being underachievers. Additional findings that creativity is strongly related to mathematical giftedness and also that originality is one of the strongest components of giftedness, should inform teachers to promote creative responses and multiple solution tasks in their teaching, valuing especially original products and processes.

Being aware of the different characteristics that may indicate mathematical giftedness, educators should be alert to observe any of these signs in their classrooms and further investigate individual cases. The rich database of student examples as described in this study provides an explanatory framework for a teacher to base his conclusions about a specific student in his classroom.

Concluding, this study connects empirical findings of indicators of mathematical giftedness, with theoretical models of giftedness. Hence, this study contributes to the credibility of these abilities and processes, both from the perspective of teachers and of researchers. The insights can therefore be used for educational purposes and pave the way for a deepened discussion about mathematical giftedness among different groups involved in the nurturing of giftedness and development of gifted education.

Suggestions for Future Research

This study investigated the structure of abilities, cognitive and hypercognitive processes of gifted students in mathematics. It has confirmed that giftedness is synthesized by four abilities; number, spatial and creative reasoning as well as reasoning on inclusion relations. The specific finding that reasoning associated with relationships of inclusion is one of the four mathematical abilities that can be used to measure mathematical giftedness consists a contribution of this study to the research literature. To date, there is no study that we are aware of that investigated or precisely referred to this ability as indicator of mathematical giftedness. Hence, it is intriguing to further explore the association of this ability to giftedness in subsequent research studies.

Moreover, the study has confirmed that students employ cognitive and hypercognitive processes in a unique way and each category is synthesized by a number of sub processes. Future research studies could also consider the inclusion of other possible dimensions, which were not investigated in the framework of this study. Such dimensions could be affective, motivational and environmental processes to isolate the variables that affect mathematical giftedness in early ages. These findings could complement and added to the proposed theoretical model of mathematical giftedness. In addition, it would be worthy to extend the findings of this study and investigate the relationship of mathematical giftedness and general giftedness, incorporating measures of general giftedness and intelligence.

This study also designed and implemented an identification process for giftedness in mathematics in the two upper grades of elementary school. Similar studies could be conducted to design corresponding identification processes for other ages, even lower grades. It would be interesting to find a way to capture mathematical giftedness in a younger age than the one investigated in this study, with subsequent impact on the mathematical content of the instruments. It would equally be of interest to the research field to examine if cognitive and hypercognitive processes in a younger age differ from those described in this study; for example, whether they are also manifested but in a level appropriate to the participants' age and abilities.

The findings of this study provide substantial evidence for a relationship between spatial ability and creativity in giftedness in mathematics. Given the lack of studies clarifying this relationship in the literature, future studies could fill this research gap and investigate the relationship between spatial ability and mathematical creativity, illuminating the way in which creative spatial ability is manifested in mathematics. In the future, inclusion of spatial tasks in the assessment of mathematical giftedness will result to the identification of spatially gifted learners and subsequently their inclusion in gifted provisions.

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APPENDIX

Appendix A

The test for measuring mathematical problem solving abilities for group administration

Όνομα:………………………………………………………………………… Τάξη:………………………

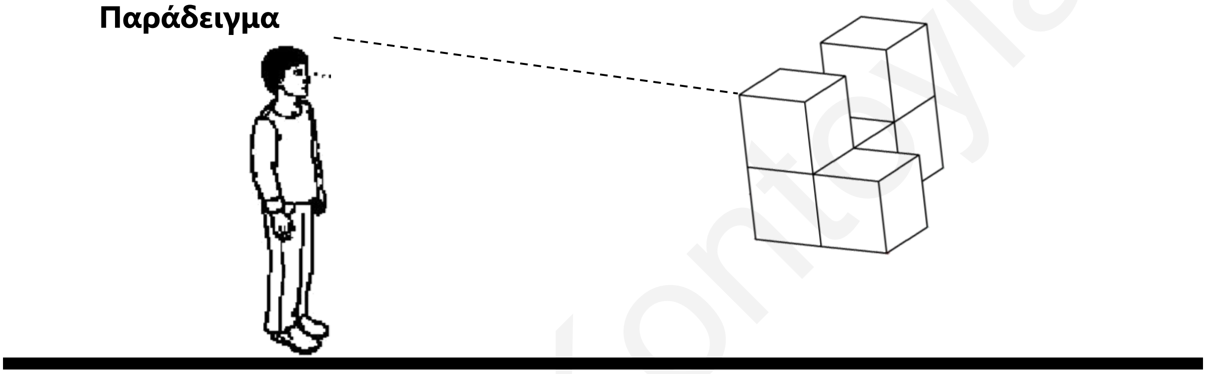
Σχολείο:…………………………………………………………… Ημερομηνία:………………………

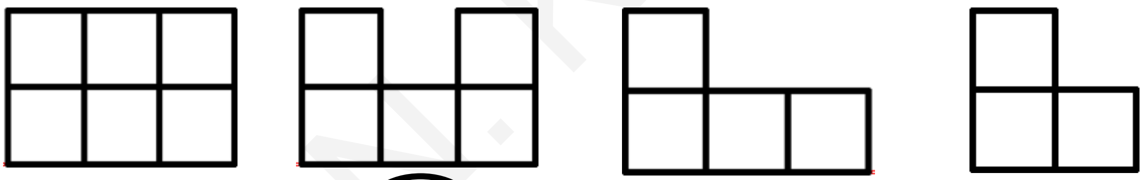
Να διαβάσεις προσεκτικά και να εργαστείς στις επόμενες δραστηριότητες. Καλή επιτυχία!

ΜΕΡΟΣ Α

Στο μέρος αυτό παρουσιάζεται ένας άνθρωπος να κοιτάζει προς ένα στερεό. Η διακεκομμένη γραμμή δείχνει την κατεύθυνση του βλέμματός του. Οι τέσσερις εικόνες που υπάρχουν κάτω από τη γραμμή δείχνουν ποια εικόνα θα μπορούσε να έχει μπροστά του ο άνθρωπος από τη θέση που βρίσκεται. Να βάλεις σε κύκλο τη σωστή.

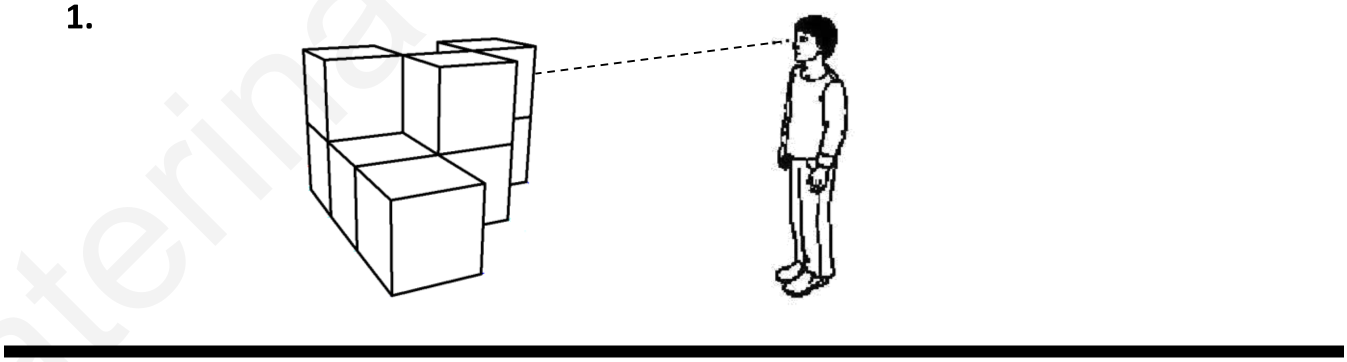
Παράδειγμα

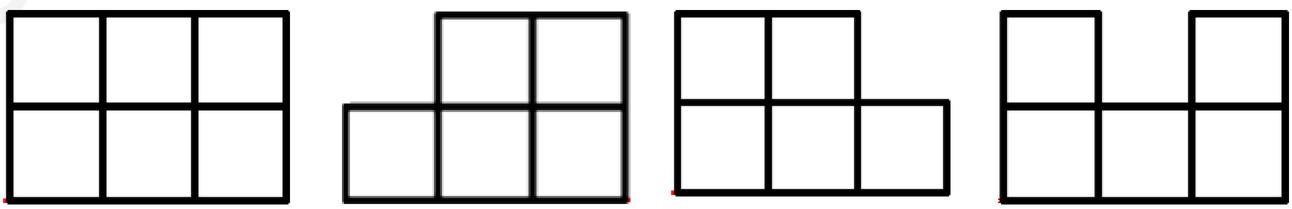




(α) (β) (γ) (δ)

1.

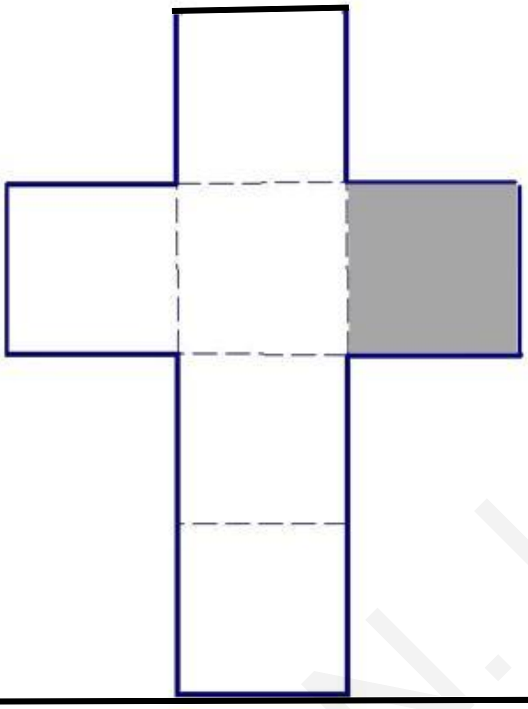


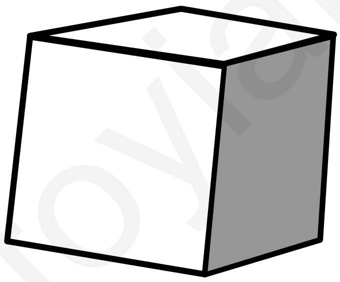


(α) (β) (γ) (δ)

Στο μέρος αυτό παρουσιάζεται στα αριστερά ένα κομμάτι χαρτόνι το οποίο θα διπλωθεί κατά μήκος των ΔΙΑΚΕΚΟΜΜΕΝΩΝ γραμμών για να σχηματιστεί το στερεό που υπάρχει στα δεξιά. Στον πίνακα που υπάρχει μετά τα σχήματα να συμπληρώσεις σε ποια ΑΚΜΗ του στερεού αντιστοιχεί η κάθε αριθμημένη πλευρά του χαρτονιού.

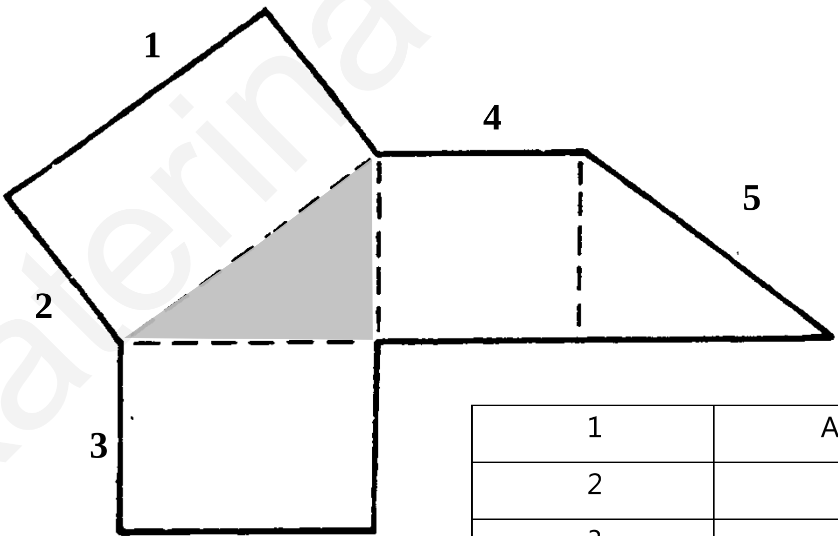
Παράδειγμα:

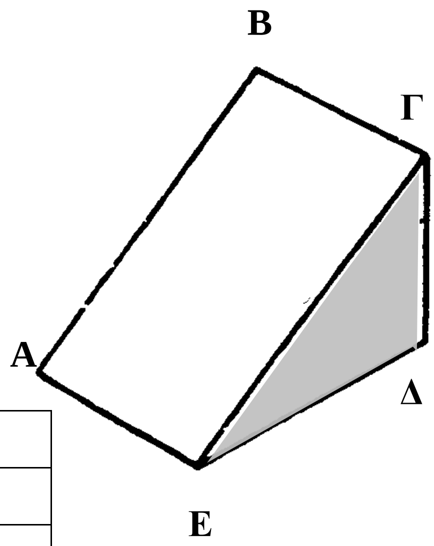




1	ΒΓ
2	ΓΗ
3	ΑΔ
4	ΔΓ
5	ΓΗ

2.

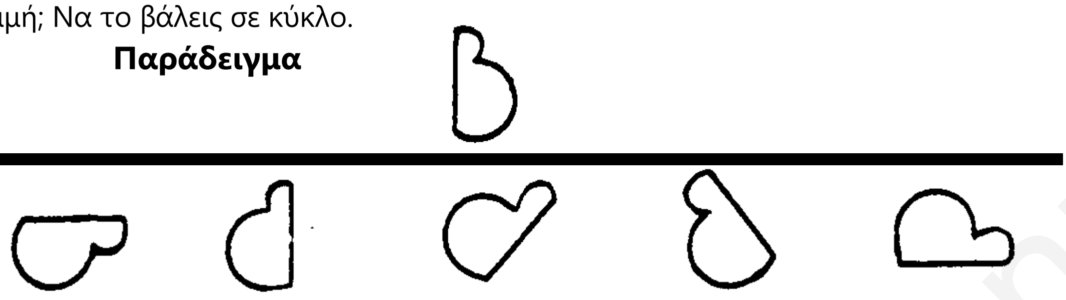




1	ΑΒ
2	
3	
4	
5	

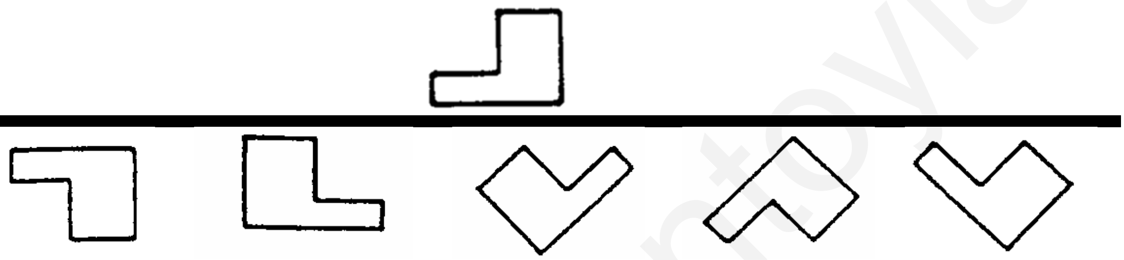
ΜΕΡΟΣ Β Στο μέρος αυτό δίνεται ένα σχήμα πάνω από τη γραμμή και πέντε σχήματα κάτω από τη γραμμή. Ποιο από τα σχήματα που βρίσκονται κάτω από τη γραμμή μπορεί να προκύψει από περιστροφή του σχήματος που βρίσκεται πάνω από τη γραμμή; Να το βάλεις σε κύκλο.

Παράδειγμα



(α) (β) (γ) (δ) (ε)

3.

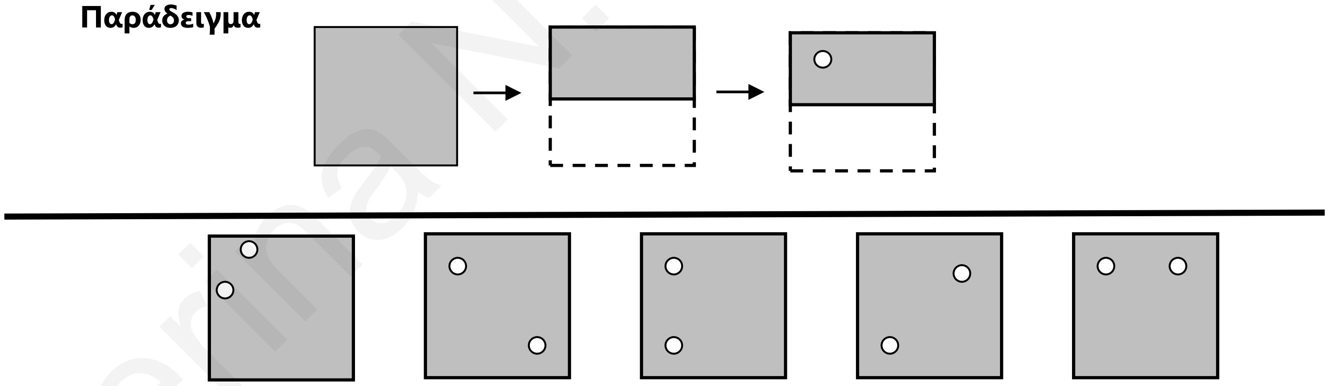


(α) (β) (γ) (δ) (ε)

ΜΕΡΟΣ Γ

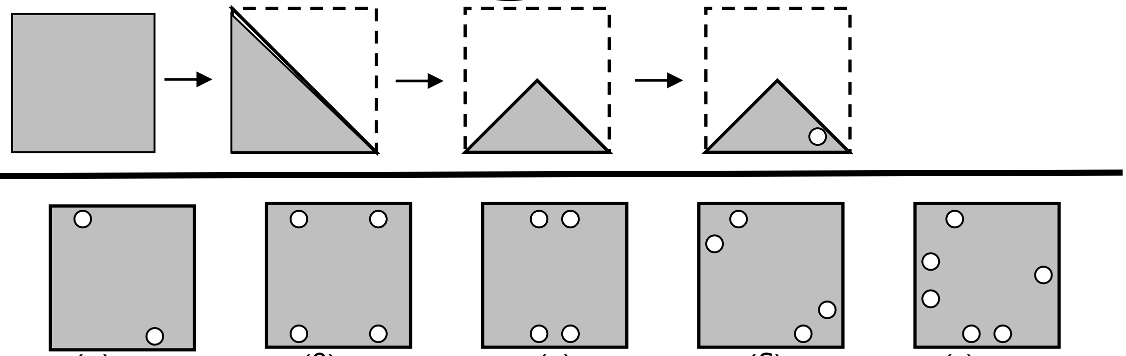
Στο μέρος αυτό παρουσιάζεται πάνω από τη γραμμή ο τρόπος με τον οποίο διπλώνεται ένα τετράγωνο γκρίζο χαρτόνι και η θέση στην οποία ανοίγουμε μια τρύπα όταν το χαρτόνι είναι διπλωμένο. Να βάλεις σε κύκλο το σχήμα κάτω από τη γραμμή που δείχνει πώς θα φαίνεται το χαρτόνι όταν ανοιχθεί.

Παράδειγμα



(α) (β) (γ) (δ) (ε)

4.



(α) (β) (γ) (δ) (ε)

ΜΕΡΟΣ Δ

Να διαβάσεις προσεκτικά και να εργαστείς στις επόμενες δραστηριότητες. Στο κουτί με την ένδειξη «Χώρος για σημειώσεις», <u>προσπάθησε να γράψεις αυτά που σκέφτεσαι, είτε αυτό είναι κάποιο σχέδιο, κάποια πράξη ή κάποια δοκιμή.</u>

Καλή επιτυχία!

5. Η Ελένη έφτιαξε μία τάρτα για τους φίλους που θα έρθουν για καφέ. Δε θυμάται όμως πόσοι τελικά θα έρθουν, 3, 5 ή 6 φίλοι. Επειδή θέλει να είναι σίγουρη ότι κάθε φίλος της θα πάρει την ίδια ποσότητα από την τάρτα, σε πόσα κομμάτια πρέπει να κόψει την τάρτα, ώστε να μπορεί να μοιράσει στα ίσα την τάρτα και στους 3, και στους 5, αλλά και στους 6 φίλους;

Α. 12 Β. 15 Γ. 18 Δ. 24 Ε. 30

6. Σε ένα εστιατόριο, ορισμένοι πελάτες παράγγειλαν γλυκό. 12 άτομα έφαγαν σοκολατίνα και 8 άτομα έφαγαν μηλόπιτα. <u>Από αυτούς</u>, πέντε άτομα έφαγαν και σοκολατίνα και μηλόπιτα. Πόσα άτομα έφαγαν γλυκό;

Α. 5 Β. 20 Γ. 15 Δ. 25 Ε. 13

Χώρος για σημειώσεις

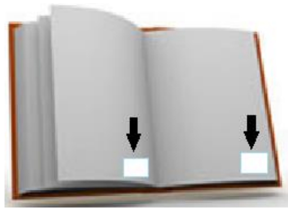
7. Ο κ. Θανάσης έφτιαξε 14 κομμάτια τούρτας με κρέμα. Πέντε κομμάτια έχουν πάνω φράουλες, τέσσερα κομμάτια έχουν κεράσια, δύο κομμάτια έχουν και τα δύο. Πόσα

κομμάτια είναι χωρίς φρούτα;

Α. 7 Β.11 Γ. 8 Δ. 5 Ε. 3

Χώρος για σημειώσεις

8. Διαβάζοντας ένα βιβλίο, η Στέλλα παρατηρεί ότι αν πολλαπλασιάσει τους αριθμούς των δύο σελίδων στις οποίες είναι ανοικτό το βιβλίο, το ψηφίο των μονάδων του αποτελέσματος είναι 6. Οι δύο αριθμοί είναι διψήφιοι. Αν προσθέσω τους αριθμούς των δύο σελίδων, ποιο είναι το ψηφίο των μονάδων του αποτελέσματος;

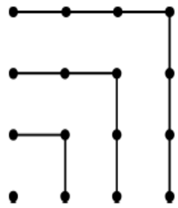


Α. 2 Β. 4 Γ. 6 Δ. 5 Ε. Άλλη απάντηση

Χώρος για σημειώσεις

9. Όπως φαίνεται στη διπλανή εικόνα, 1+3+5+7=4x4.

Ποια είναι η τιμή του 1+3+5+7+…+17+19+21;



Α. 10x10 Β. 11x11 Γ. 12x12 Δ. 13x13 Ε. 14x14

Χώρος για σημειώσεις

10. Τρία τετράδια και δύο κασετίνες στοιχίζουν €32. Τέσσερα τετράδια και τρεις κασετίνες στοιχίζουν €44. Πόσα στοιχίζουν δύο τετράδια και μία κασετίνα;

Α. €3 Β. €20 Γ. €15 Δ. €12 Ε. €76

Χώρος για σημειώσεις

11. Μια ομάδα από φίλους θέλουν να αγοράσουν ένα δώρο για το φίλο τους Ηλία που έχει γενέθλια. Έχουν βρει το δώρο που τους αρέσει και προσπαθούν να υπολογίσουν το ποσό που θα πρέπει να πληρώσει ο καθένας:

Κυριάκος: Αν πληρώσει €3 ο καθένας, θα υπολείπονται €5.
Μάριος: Αν πληρώσει €5 ο καθένας, θα περισσεύουν €5.

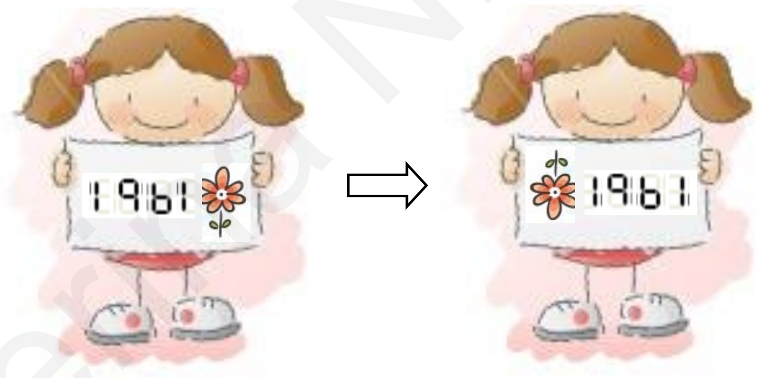
Πόσα θα πρέπει να πληρώσει ο καθένας για να μαζέψουν το ακριβές ποσό για το δώρο;

Α. €4,50 Β. €4,00 Γ. €4,80 Δ. €4,75 Ε. €10

Χώρος για σημειώσεις

12. Η Κυριακή έγραψε σε ένα χαρτί τα ψηφία 0123456789 ,όπως ακριβώς τα βλέπεις.

Στη συνέχεια, διάλεξε ορισμένα από αυτά και έγραψε τον αριθμό 1961 σε ένα χαρτί. Στη συνέχεια, γύρισε ανάποδα το χαρτί και παρατήρησε ότι ο αριθμός φαίνεται ο ίδιος, αν τον γυρίσουμε ανάποδα.

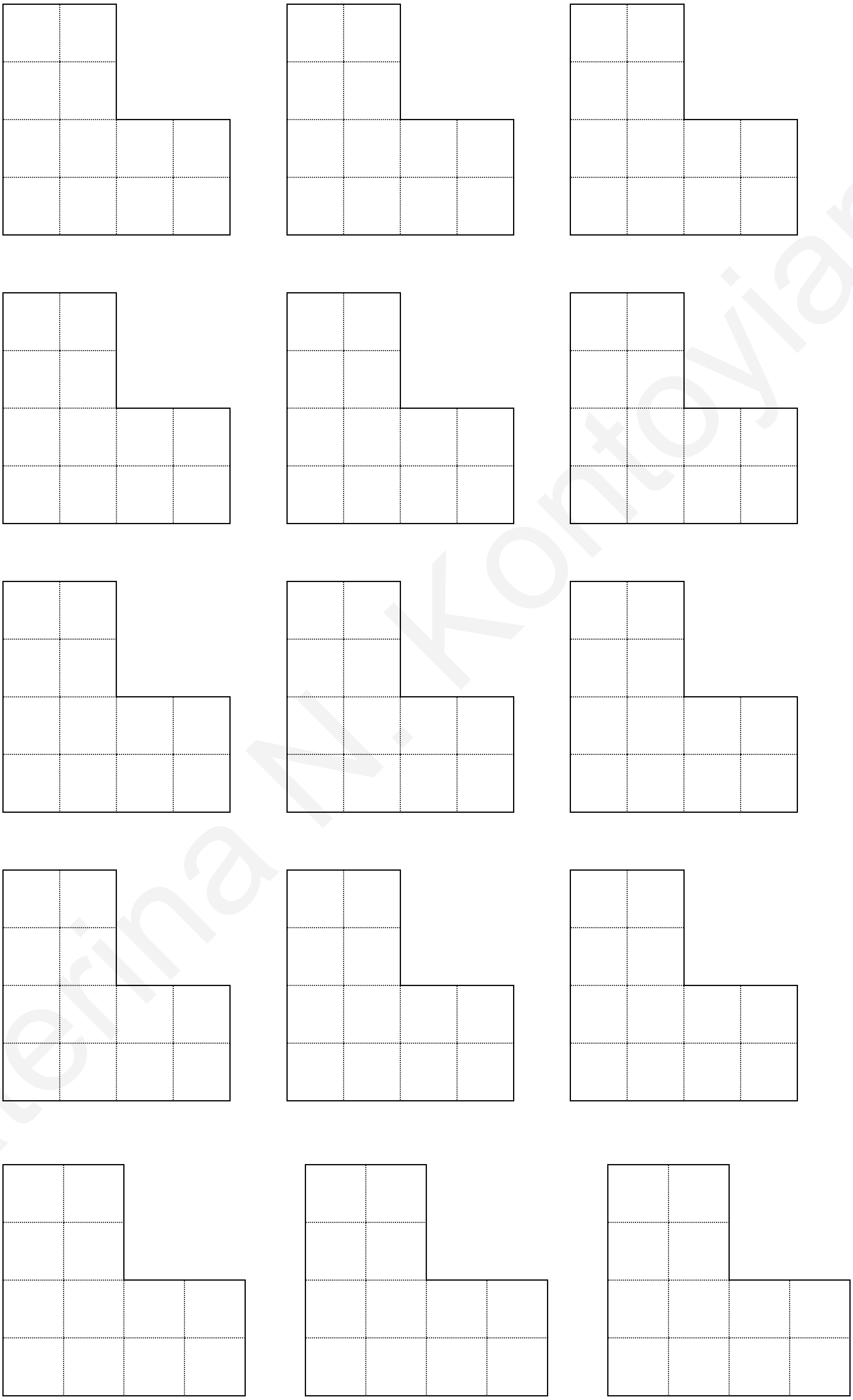


Ποιος είναι ο <u>επόμενος αριθμός μετά το 1961</u> που μπορεί να δείχνει το ίδιο αν τον γυρίσουμε ανάποδα; Μπορείς να διαλέξεις όποια ψηφία θέλεις από τα ψηφία που έγραψε αρχικά η Κυριακή.

Χώρος για σημειώσεις

Απάντηση: ....

13. Να χωρίσεις κάθε μπότα σε τέσσερα σχήματα με ίσο εμβαδόν με όσο το δυνατόν περισσότερους και διαφορετικούς τρόπους. Μπορείς να χρησιμοποιήσεις τέσσερα χρώματα.



Appendix B

The test with mathematical challenging tasks for individual administration

Να χρησιμοποιήσεις τους πιο κάτω αριθμούς για να τους χωρίσεις σε ομάδες με όσο το δυνατόν περισσότερους τρόπους. Να δώσεις ένα όνομα σε κάθε ομάδα με βάση το κριτήριο ομαδοποίησης που χρησιμοποίησες.

2, 3, 4, 5, 7, 9, 10, 15, 21, 25, 28, 49

ΔΡΑΣΤΗΡΙΟΤΗΤΑ 2

- Να επιλέξεις ένα διψήφιο αριθμό.
- Να αντιστρέψεις τα ψηφία και στη συνέχεια να προσθέσεις τους δύο διψήφιους αριθμούς.
- Τι παρατηρείς για το αποτέλεσμα; Είναι τυχαίος αριθμός;
- Μπορείς να εξηγήσεις γιατί συμβαίνει αυτό;

ΔΡΑΣΤΗΡΙΟΤΗΤΑ 3

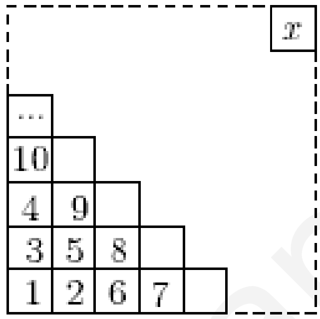
- Να επιλέξεις ένα τυχαίο αριθμό.
- Να πολλαπλασιάσεις αυτό τον αριθμό με τον εαυτό του.
- Να αφαιρέσεις τον αρχικό σου αριθμό.
- Το αποτέλεσμά σου είναι άρτιος ή περιττός αριθμός;
- Μπορείς να αποδείξεις ότι το αποτέλεσμά σου είναι πάντα αληθές και όχι αληθές μόνο για το συγκεκριμένο αριθμό που επέλεξες να χρησιμοποιήσεις στην αρχή;

ΔΡΑΣΤΗΡΙΟΤΗΤΑ 4

Να αναγράψεις τον αριθμό 1 000 000 στην οθόνη της υπολογιστικής μηχανής. Πατώντας μόνο τα πλήκτρα 7 και +, -, ×, ÷ και = όσες φορές θέλεις, να φτάσεις στο αποτέλεσμα 7.

ΔΡΑΣΤΗΡΙΟΤΗΤΑ 5

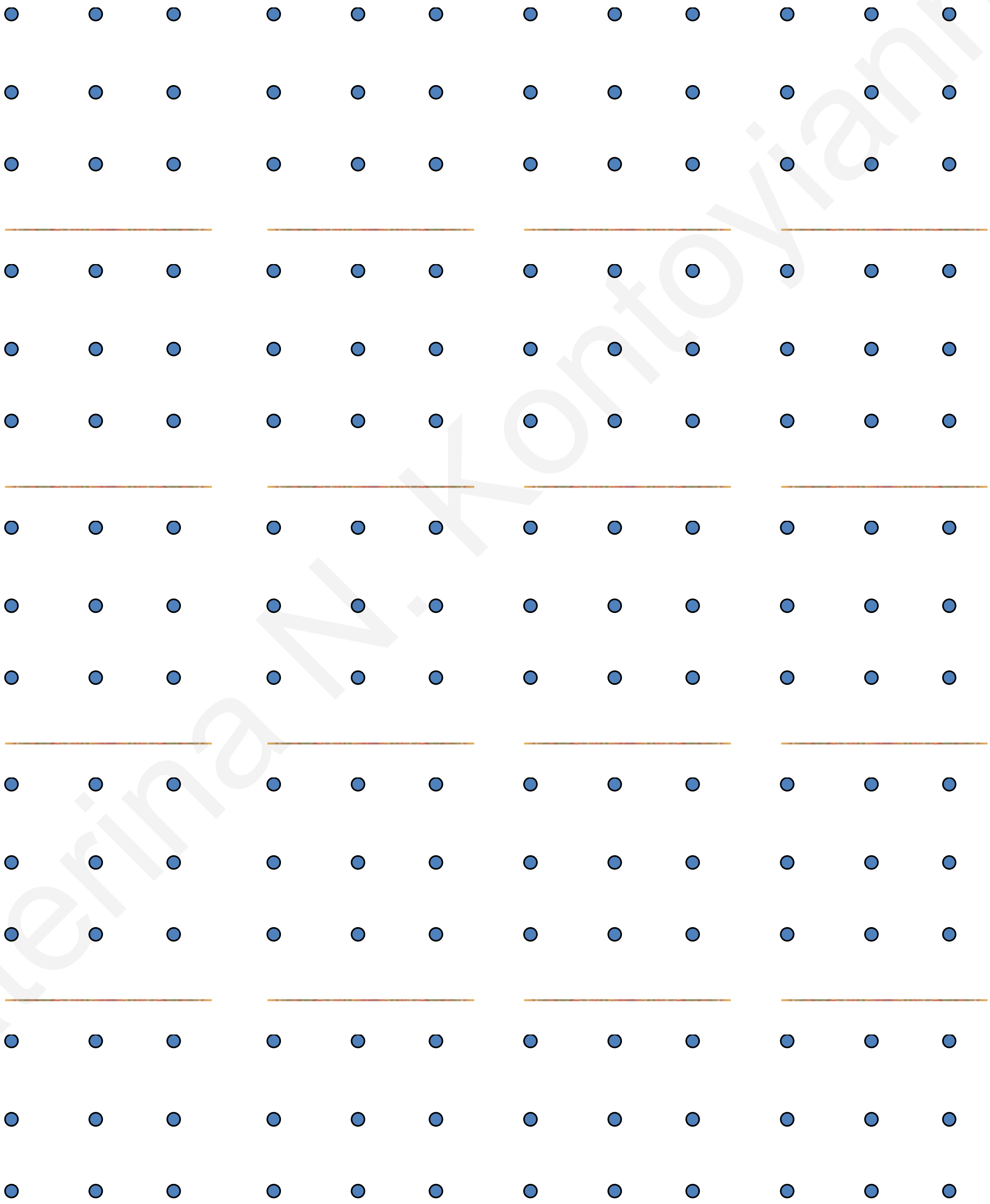
- Μέσα στα μικρά τετράγωνα έχουν τοποθετηθεί αριθμοί όπως φαίνεται στην εικόνα.



- Να προτείνεις ορισμένους αριθμούς που θα μπορούσαν να τοποθετηθούν μέσα στο τετράγωνο x και να δικαιολογήσεις την απάντησή σου.
- Να προτείνεις ορισμένους αριθμούς που ΔΕΝ θα μπορούσαν να τοποθετηθούν μέσα στο τετράγωνο x και να δικαιολογήσεις την απάντησή σου.

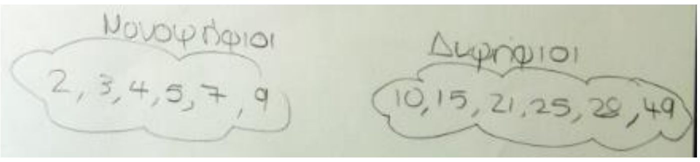
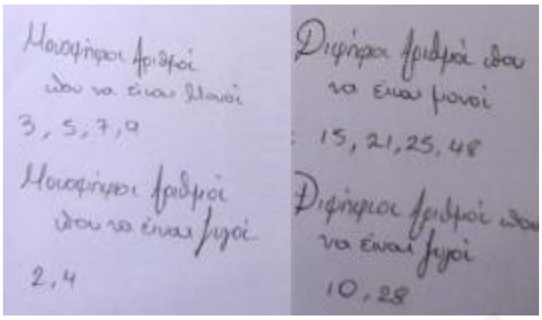
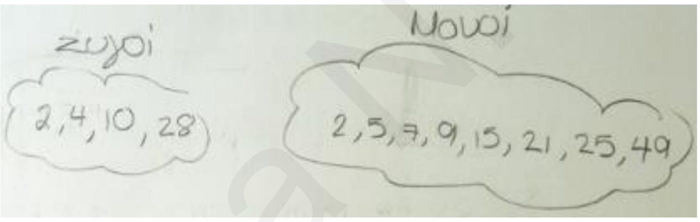
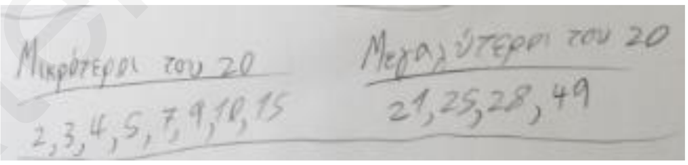
ΔΡΑΣΤΗΡΙΟΤΗΤΑ 6

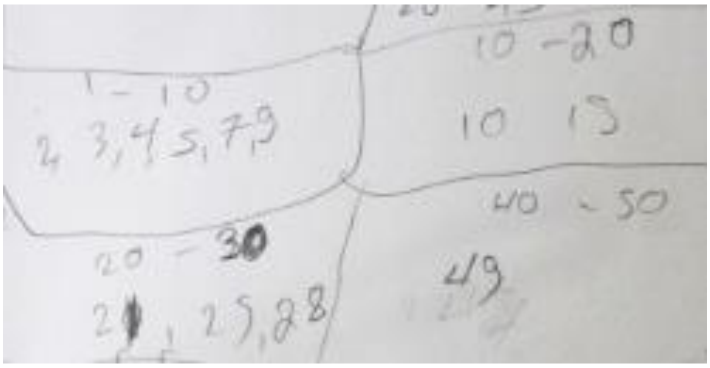
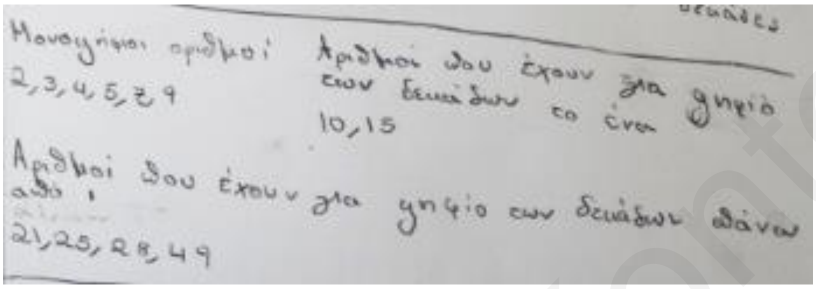
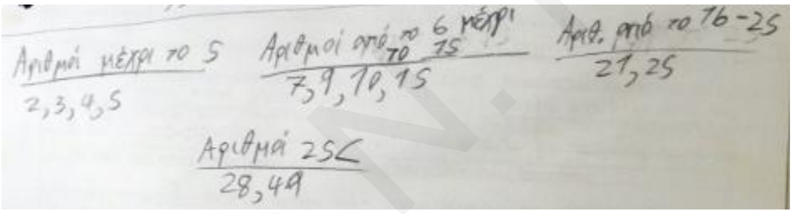
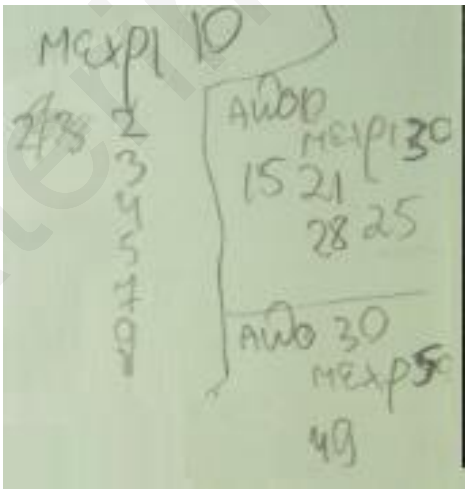
Να κατασκευάσεις όσο το δυνατόν περισσότερα διαφορετικά σχήματα με εμβαδόν 2 cm^2, ενώνοντας με ευθείες γραμμές τα εννέα σημεία. Μπορείς να σκεφτείς σχήματα που δύσκολα θα σκεφτόταν κάποιος άλλος;

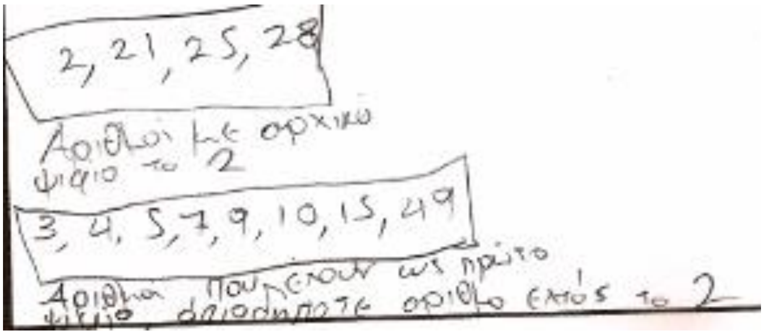


Appendix C

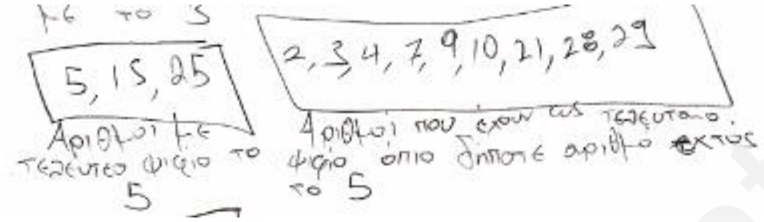
Grouping responses proposed by students in Activity 1 of the test with mathematical challenging tasks for individual administration

Suggested grouping method [a]	Percentage of students suggesting the specific response
	0.38
	0.06
	0.91
	0.06

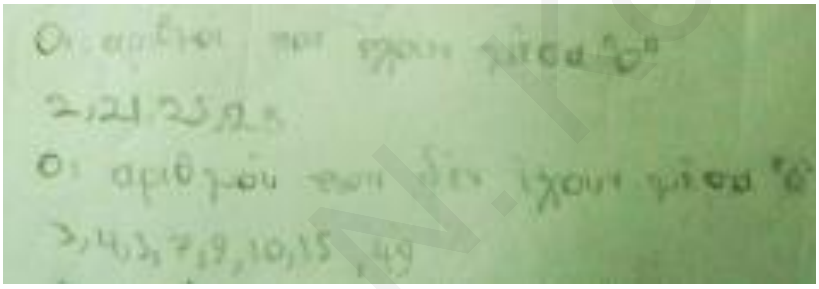
	0.06
	0.03
	0.03
	0.03


2, 21, 25, 28
Αριθμοί με αρχικό ψηφίο το 2
3, 4, 5, 7, 9, 10, 15, 49
Αριθμοί που έχουν ως πρώτο ψηφίο οποιοδήποτε αριθμό εκτός το 2

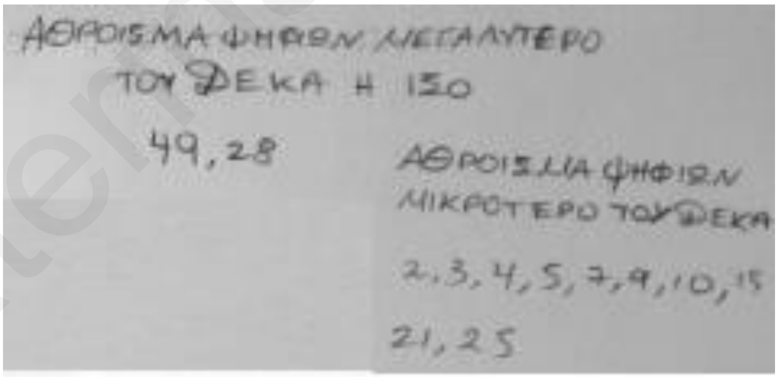
0.03


με το 5
5, 15, 25
Αριθμοί με τελευταίο ψηφίο το 5
2, 3, 4, 7, 9, 10, 21, 28, 29
Αριθμοί που έχουν ως τελευταίο ψηφίο οποιοδήποτε αριθμό εκτός το 5

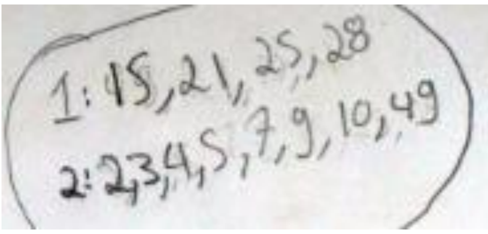
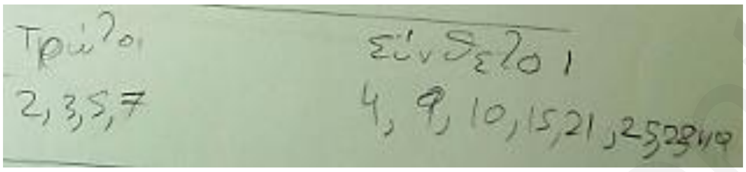
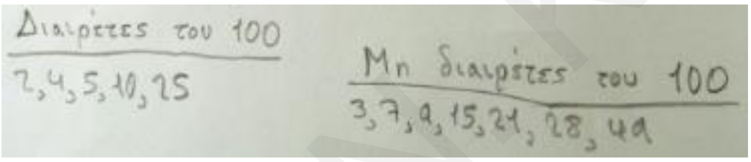
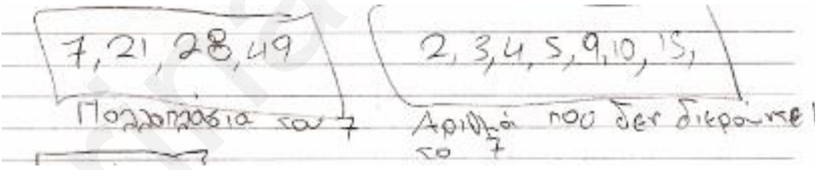
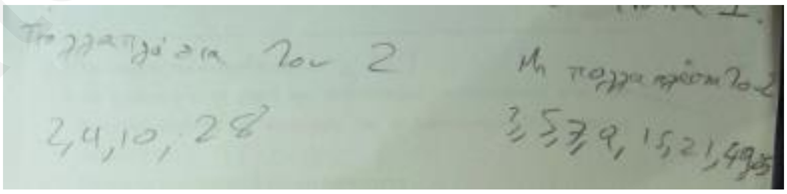
0.03

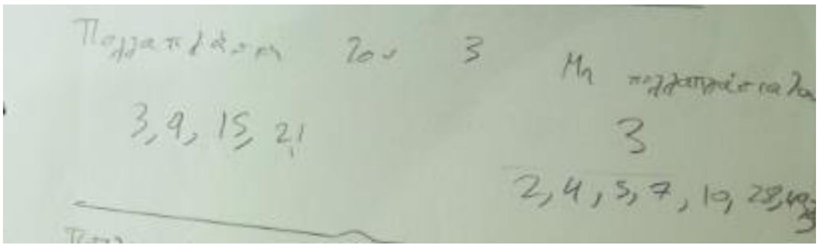

Οι αριθμοί που έχουν ψηφίο "2"
2, 21, 25, 28
Οι αριθμοί που δεν έχουν ψηφίο "2"
3, 4, 5, 7, 9, 10, 15, 49

0.03


ΑΘΡΟΙΣΜΑ ΨΗΦΙΩΝ ΜΕΓΑΛΥΤΕΡΟ ΤΟΥ ΔΕΚΑ Η ΙΣΟ
49, 28
ΑΘΡΟΙΣΜΑ ΨΗΦΙΩΝ ΜΙΚΡΟΤΕΡΟ ΤΟΥ ΔΕΚΑ
2, 3, 4, 5, 7, 9, 10, 15
21, 25

0.03

 Rational verbally expressed: Since the sum is 178, 178/2 equals 89, I will make two groups with sum of 89 in each group.	0.03
	0.38
	0.06
	0.06
	0.03



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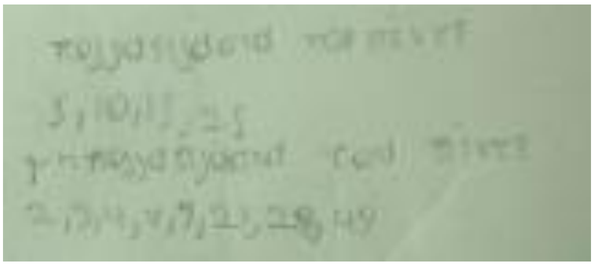

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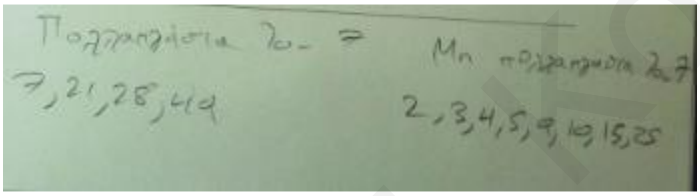

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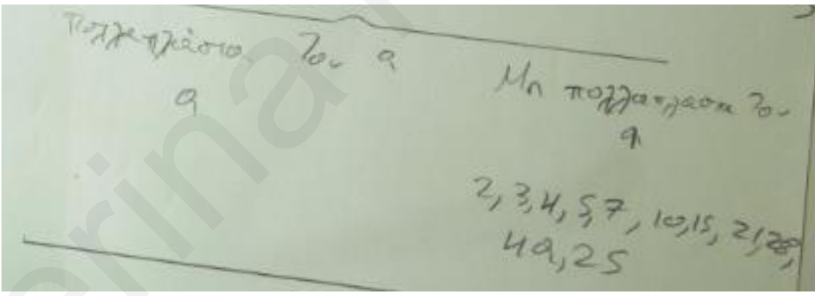

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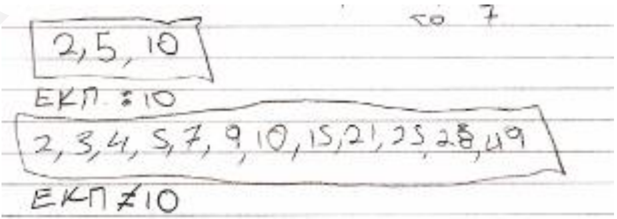

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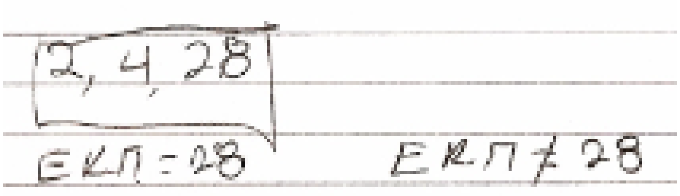

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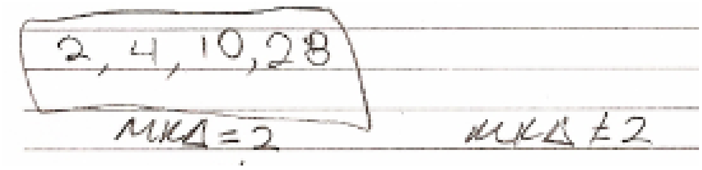

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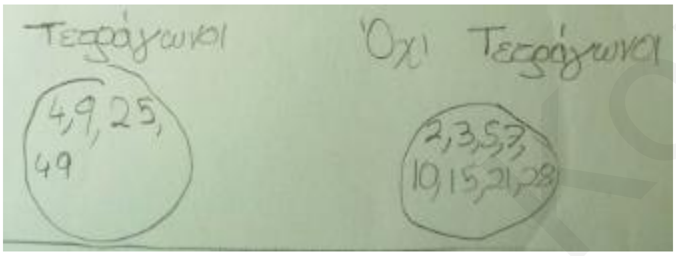
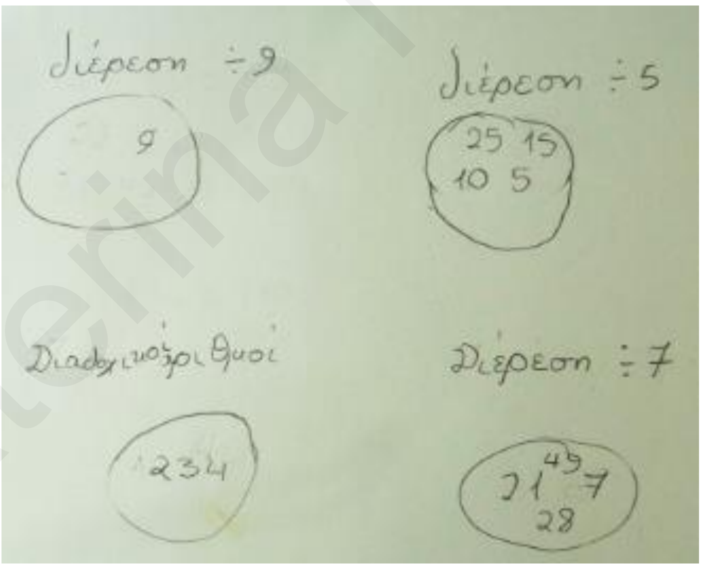


0.15



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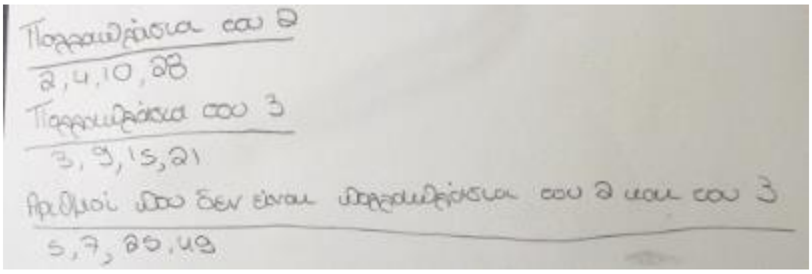

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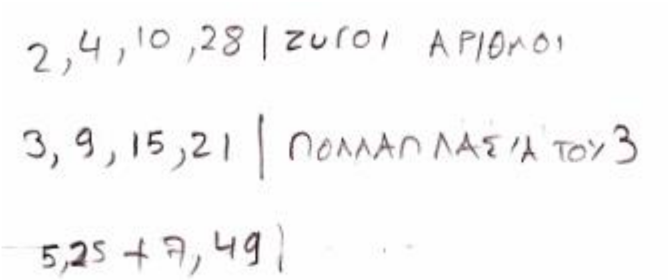
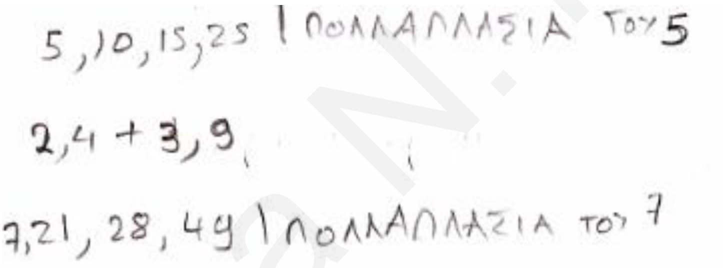


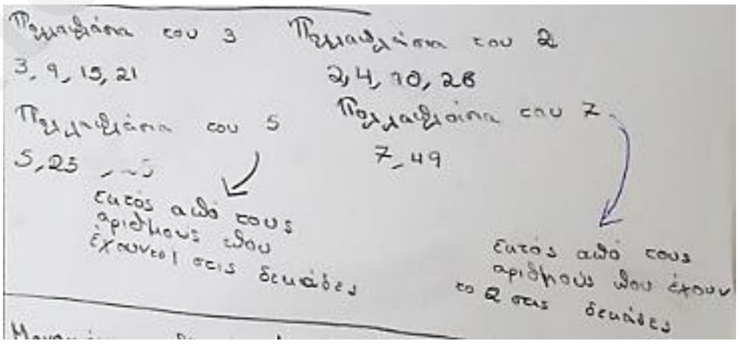
Last group: Pairs of numbers and their squares

0.03



Second group: Pairs of numbers and their squares

0.03



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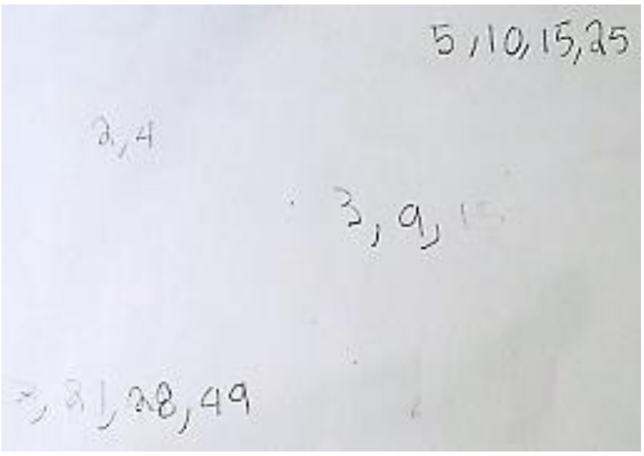


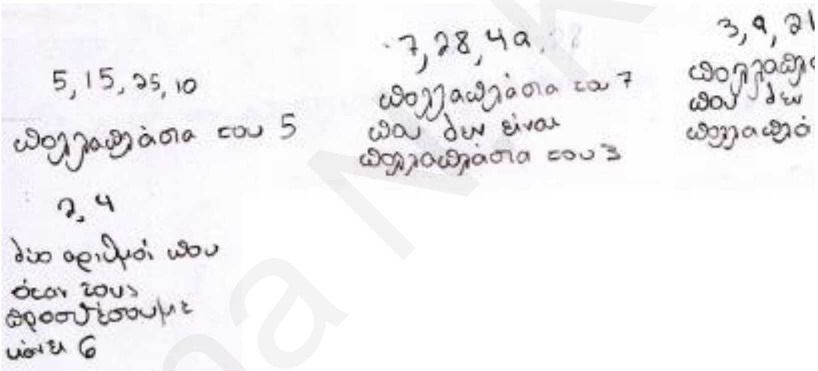
0.03

Numbers 2 and 4: Multiples of 2 up to 8

Numbers 5,10,15 and 25: Multiples of 5

Numbers 3 and 9: Multiples of 3 up to 9

Numbers 7,21,28 and 49: Multiples of 7



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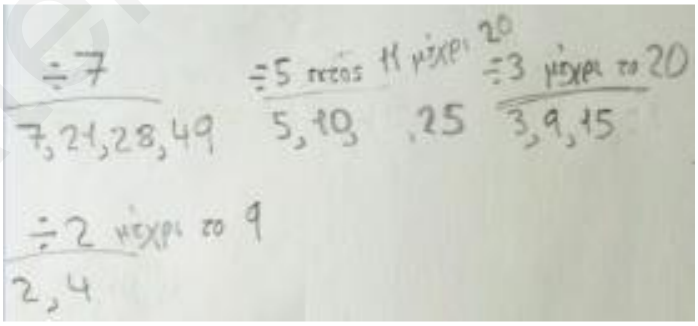

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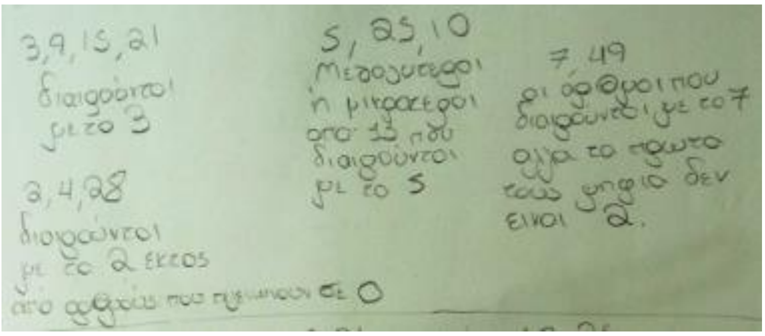
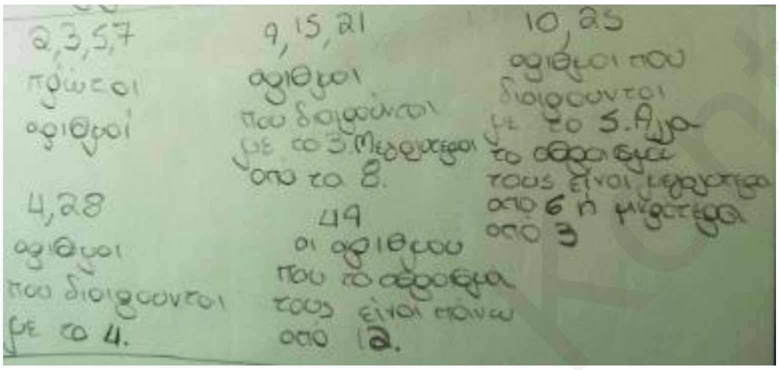
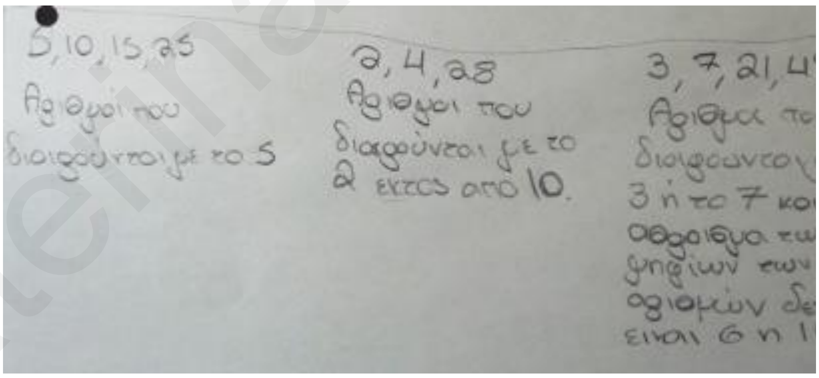

3,9,15,21
διαιρούνται με το 3
5, 25, 10
Μεγαλύτεροι ή μικρότεροι από 13 που διαιρούνται με το 5
7, 49
οι αριθμοί που διαιρούνται με το 7 αλλά το πρώτο τους ψηφίο δεν είναι 2.
2, 4, 28
διαιρούνται με το 2 εκτός από αριθμούς που τελειώνουν σε 0

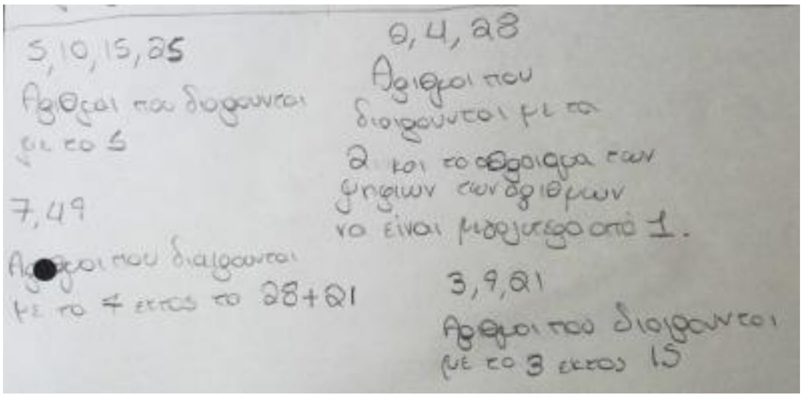
0.03


2,3,5,7
πρώτοι αριθμοί
9, 15, 21
αριθμοί που διαιρούνται με το 3. Μεγαλύτεροι από το 8.
10, 25
αριθμοί που διαιρούνται με το 5. Αλλά το άθροισμα τους είναι μεγαλύτερο από 6 ή μικρότερο από 3
4, 28
αριθμοί που διαιρούνται με το 4.
49
οι αριθμοί που το άθροισμα τους είναι πάνω από 12.

0.03


5, 10, 15, 25
Αριθμοί που διαιρούνται με το 5
2, 4, 28
Αριθμοί που διαιρούνται με το 2 εκτός από 10.
3, 7, 21, 4
είναι 6 ή 1

0.03



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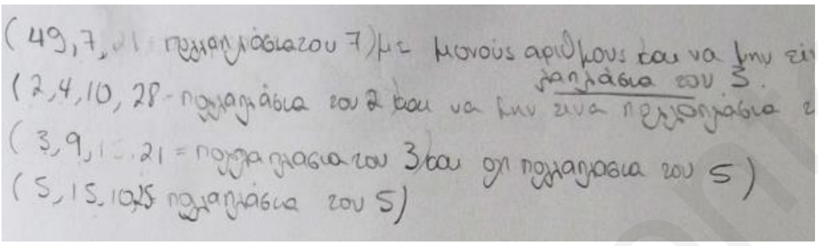

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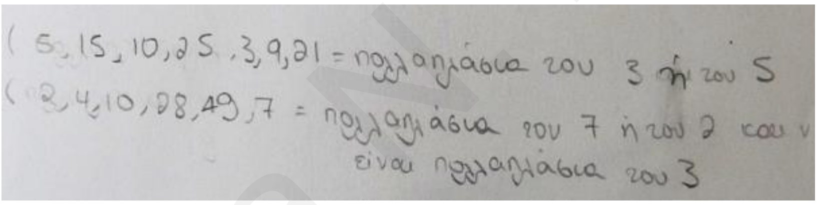

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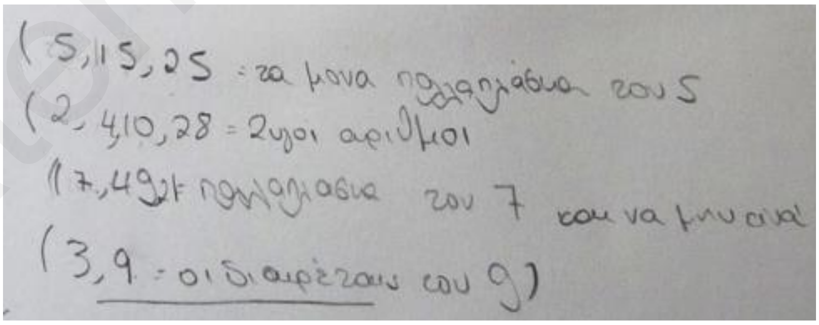

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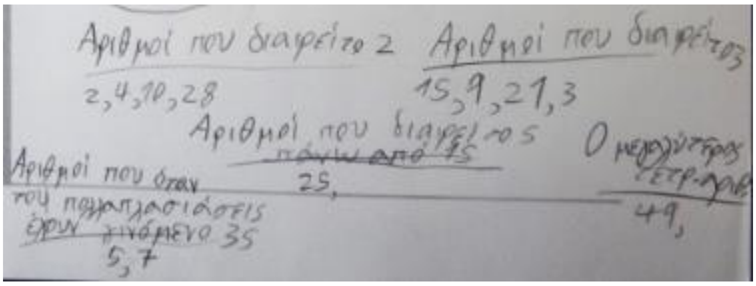

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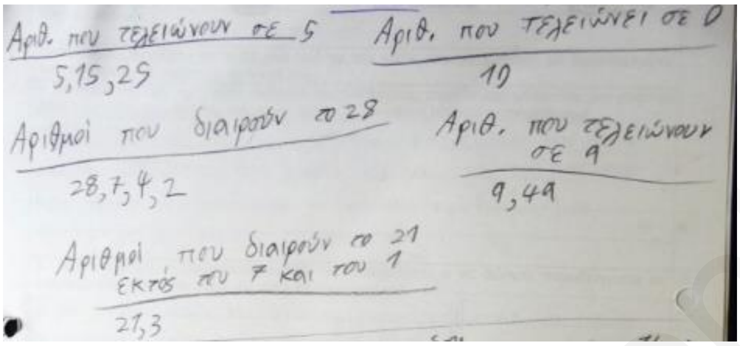

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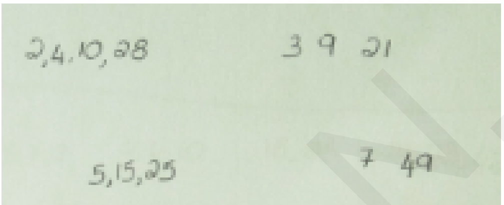


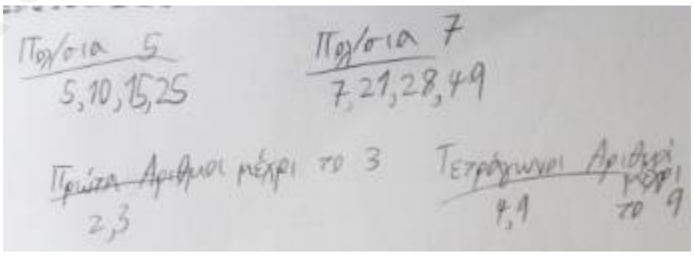
0.03

Numbers 2,4,10 and 28: Numbers that can be divided by 2

Numbers 3,9 and 21: Numbers divided by 3 except 15

Numbers 5,15 and 25: Numbers divided by 5 except 10

Numbers 7 and 49: Numbers divided by 7 except 21,28



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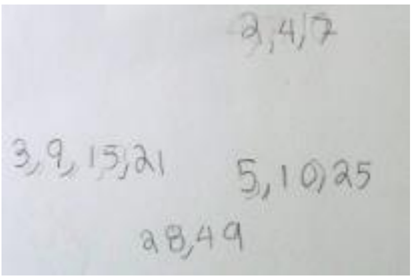


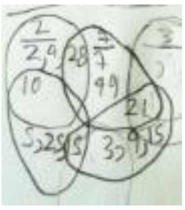
0.03

Numbers 2,4 and 7: Numbers with increased difference (2+2=4, 4+3=7)

Numbers 3,9,15 and 21: Numbers with difference of 6 from one to the other

Numbers 5,10 and 25: Multiples of 5 except of 15

Numbers 28 and 49: Multiples of 7 larger than 21



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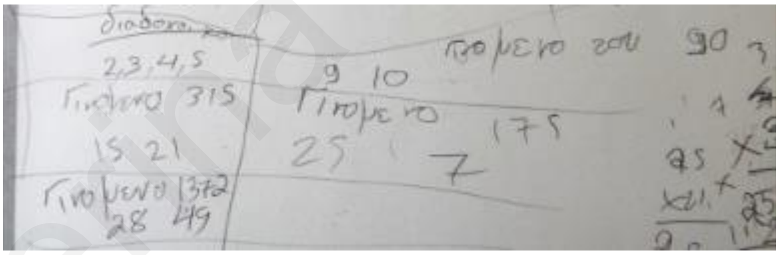

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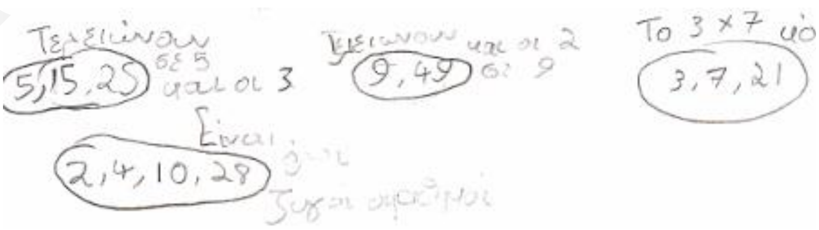

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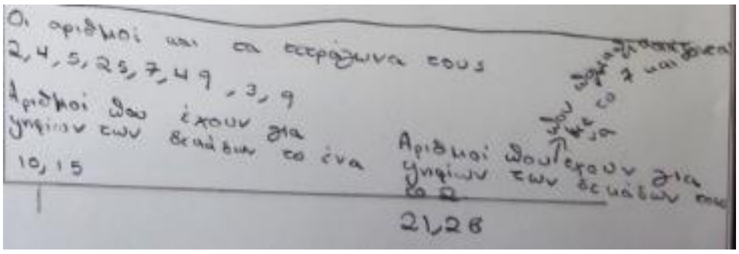

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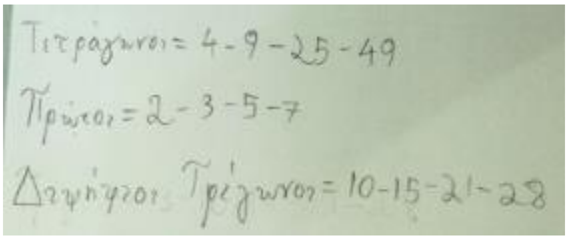

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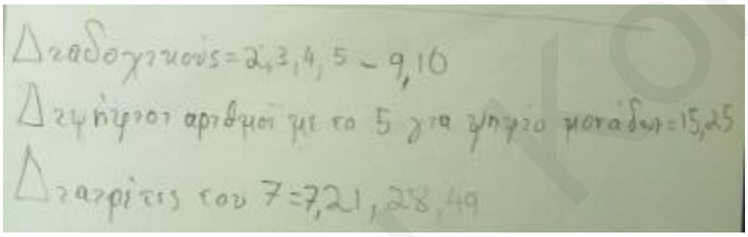

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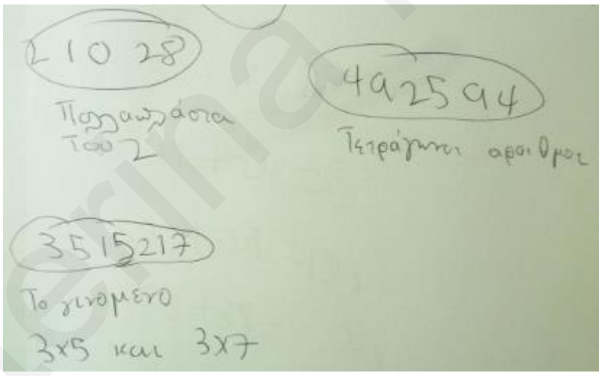

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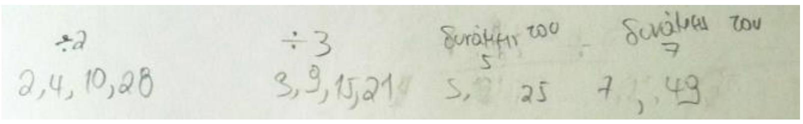

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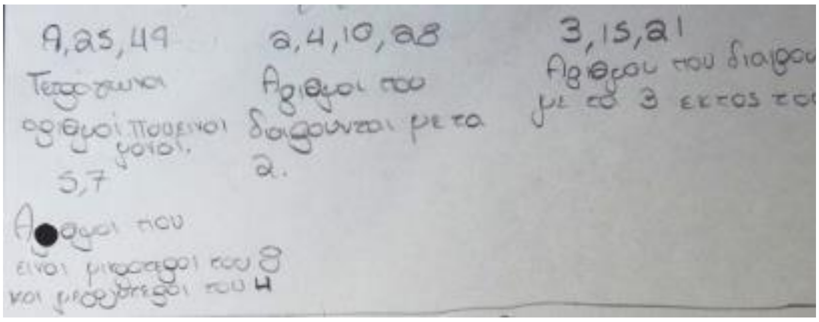

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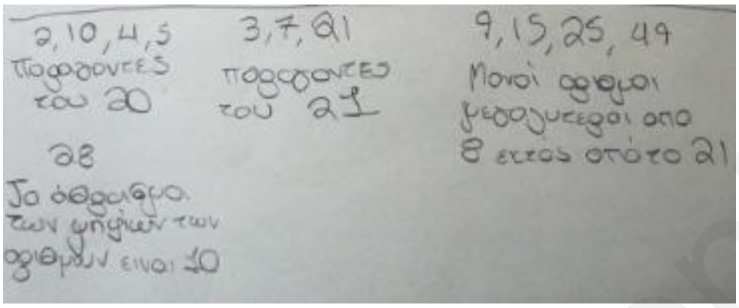

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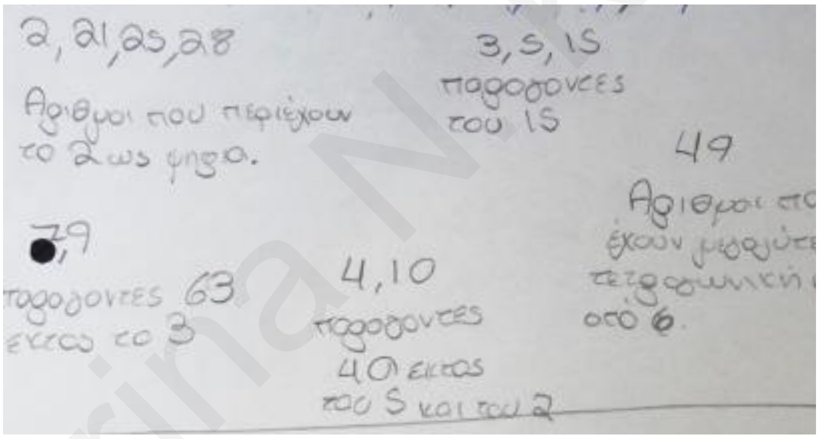

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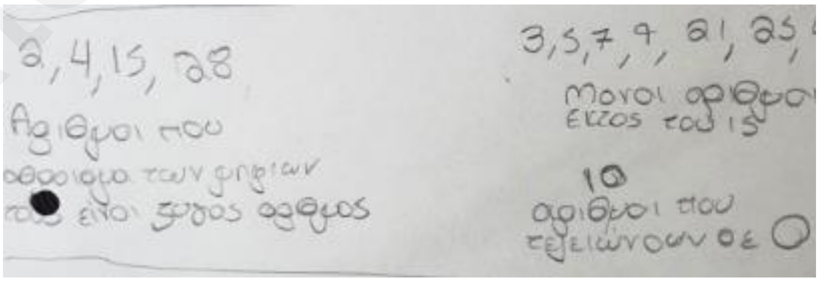
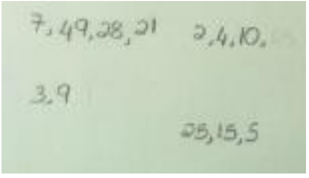
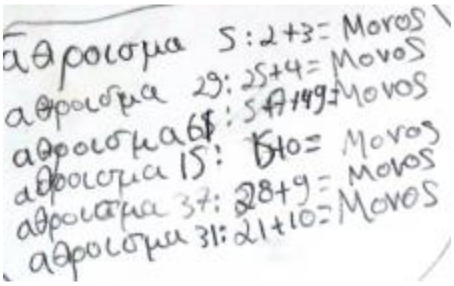


0.03



0.03

Numbers 7,49,28 and 21: Multiples of 7
Numbers 2,4 and 10: Multiples of 2 up to 10
Numbers 3 and 9: Divided by 3 up to 9
Numbers 5, 15 and 25: Multiples of 5 except 10



0.03

Rational verbally expressed: As many groups as possible, whose sum is an odd number

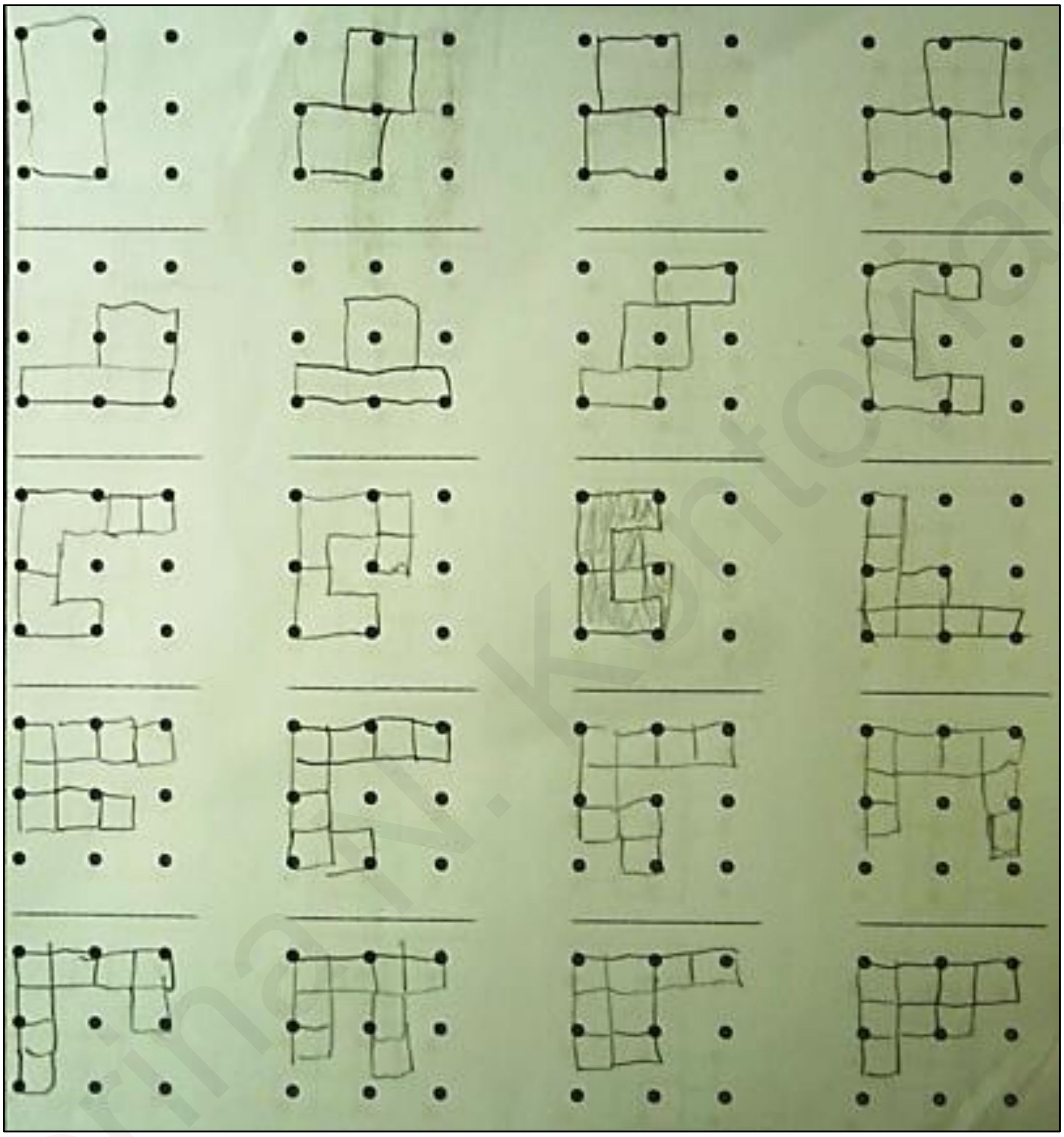
[a] In cases where students expressed the group names verbally, group names are provided under the corresponding approach.

Appendix D

Figures created by S12 in Activity 6 of the test with challenging tasks for individual administration

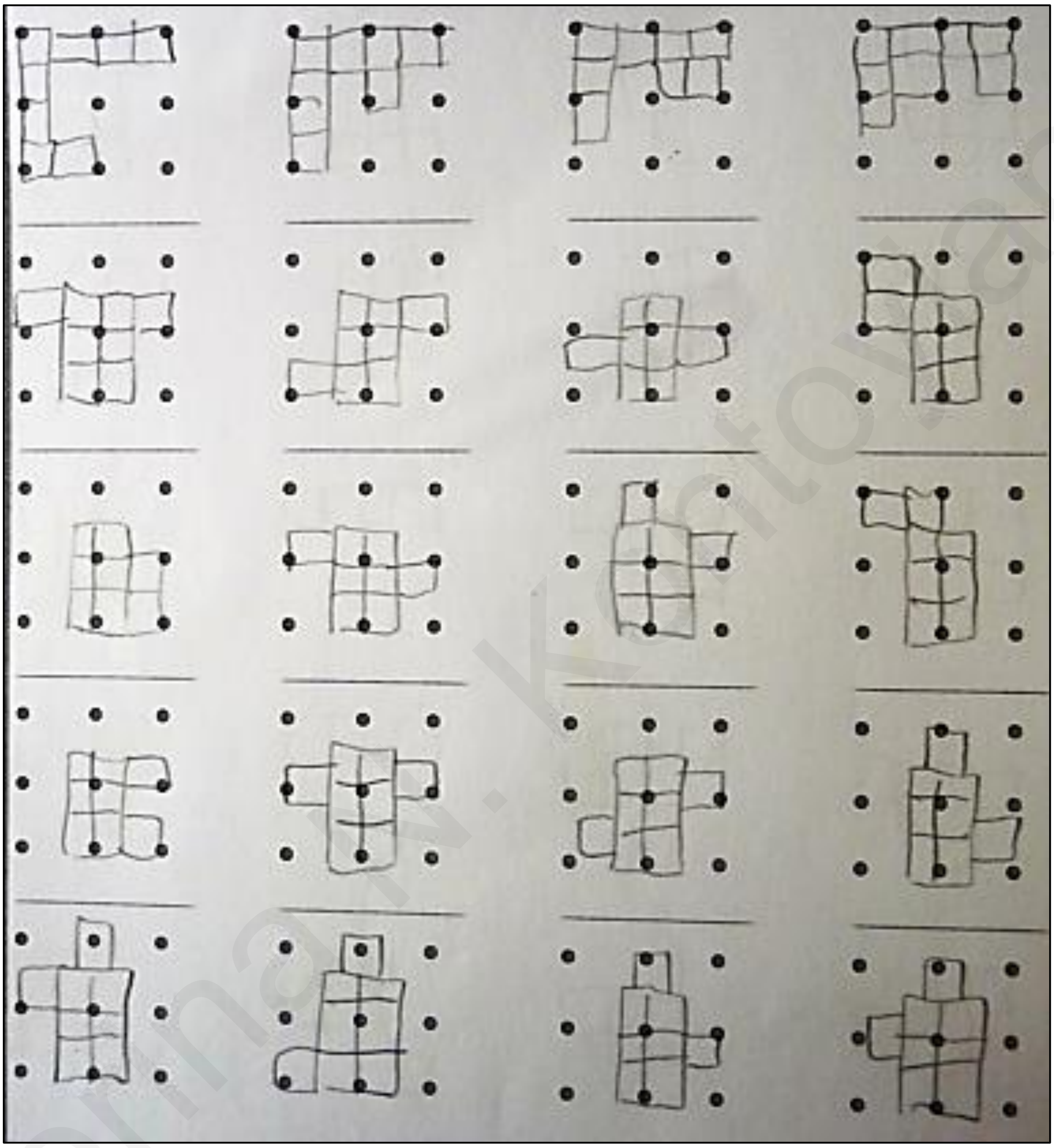
Appendix D1

Figures created by S12 in Activity 6 of the test with challenging tasks for individual administration (Page 1 of 4)



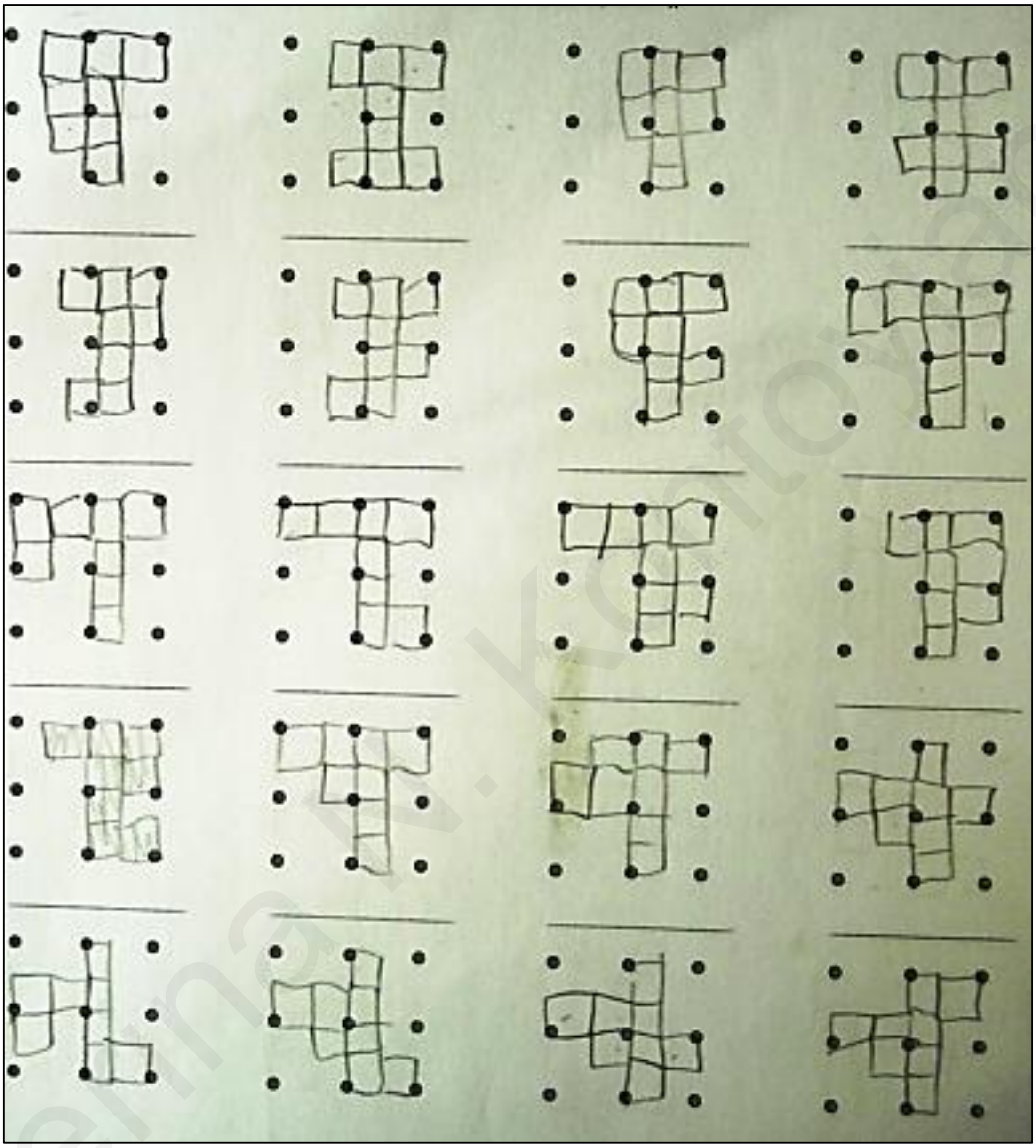
Appendix D2

Figures created by S12 in Activity 6 of the test with challenging tasks for individual administration (Page 2 of 4)



Appendix D3

Figures created by S12 in Activity 6 of the test with challenging tasks for individual administration (Page 3 of 4)



Appendix D4

Figures created by S12 in Activity 6 of the test with challenging tasks for individual administration (Page 4 of 4)