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ALGEBRAIC THINKING IN SCHOOL MATHEMATICS: UNFOLDING A MULTIFACETED NOTION THROUGH PEDAGOGICAL AND COGNITIVE PERSPECTIVES

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The present Doctoral Dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy of the University of Cyprus. It is a product of original work of my own, unless otherwise mentioned through references, notes, or any other statements.

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ABSTRACT

The notions of algebra and algebraic thinking have been a focus of attention by researchers, policy makers, and curriculum designers during the last decades. Yet, the field of mathematics education has not conceptualized algebra and algebraic thinking in a way that they can explicitly become a part of mathematics teaching and learning in the early grades. This study examines the way algebraic thinking might be conceptualized and the way this conceptualization might inform mathematics instruction at the elementary grades in order to enhance the development of algebraic thinking. The purpose of this study was (a) to empirically test a theoretical model about the core aspects of algebraic thinking, (b) to examine its relation to domain-specific processing abilities, different types of reasoning forms and general cognitive processes of mental action, and (c) to investigate the impact of two concrete instructional approaches on students' algebraic thinking ability.

Six hundred and eighty four students from Grades 4, 5, 6 and 7 participated in the study. Seven different tests were administered: (i) a test measuring algebraic thinking abilities; (ii) a test measuring processes involved in the Specialized Structural Systems (SSSs); (iii) the Naglieri Non-Verbal Ability Test that measures several types of reasoning, (iv) a test measuring deductive reasoning; (v) a test measuring working memory; (vi) a test measuring speed of processing, and (vii) a test measuring control of processing. Two teaching interventionist experiments with 86 fifth graders were also conducted. 42 students participated in the "Semi-structured problem situations" experiment and 44 students participated in the "Structured mathematical investigations" experiment. The purpose of the two interventions was to explore whether it would be possible to engineer instructional interventions that could have positive impact on students' algebraic thinking.

The results of the study verified the theoretical model proposed by Kaput (2008) about the core aspects of algebraic thinking from K-12 grades. According to the model, algebraic thinking can be described using a combination of three distinct but interrelated factors: (i) "Generalized arithmetic", (ii) Functional thinking", and (iii) "Modeling as a domain for expressing and formalizing generalizations". The study describes in detail the three factors through the identification of specific categories of tasks that belong to each factor. Following the suggestions of Kieran

(2004), all tasks were carefully selected in order to satisfy two conditions: (i) the tasks involved processes that are considered to be linked with early algebraic thinking, such as generalization, problem solving, argumentation and justification, prediction and proof, and (ii) the tasks involved verbal expressions, diagrams, drawings or graphs rather than symbols and did not require the use of symbols. The factor “Generalized arithmetic” refers to the use of arithmetic as a domain for expressing and formalizing generalizations. The factor “Functional thinking” refers to the generalization of numerical or geometrical patterns and the exploration and expression of relationships of co-variation and correspondence that are represented in several ways (with table, graphically, diagrammatically, verbally, symbolically). The factor “Modeling as a domain for expressing and formalizing generalizations” involves the construction of models for representing regularities from mathematized situations or phenomena where the regularity itself is secondary to the larger modeling task. Therefore, the findings of the current study verify through empirical data Kaput’s proposed structure of algebraic thinking and also the idea that Kieran (1992) developed about conceptualizing algebraic activity not just as a topic in mathematics curriculum but as a multifaceted activity which encompasses various types of tasks and ways of thinking.

According to the results, the three factors of algebraic thinking remain stable in all the age-groups that the study examined. Nevertheless, students’ algebraic thinking abilities in the fourth grade are different in relation to the students of the other grades. Students’ algebraic thinking abilities in the seventh grade are also different.

Based on the findings regarding the components of algebraic thinking, this study also described four classes of students which reflect broad portraits of students’ skills that can be used to inform our understanding of the way students develop generalization abilities and abilities for representing generalizations and regularities and move from arithmetical to algebraic ways of thinking. Students in the first class hardly solved any type of algebraic tasks, implying the absence of algebraic thinking abilities. Students in the second class had average performance in “Generalized arithmetic” tasks and low performance in the “Functional thinking” and “Modeling” tasks. These students managed to solve with more success the “Generalized arithmetic” tasks by applying arithmetical strategies, implying a level of transition between arithmetical and algebraic ways of thinking. Students in the third class had average performance in the “Generalized arithmetic” and “Functional thinking” tasks and low performance in the “Modeling” tasks. These students managed to solve successfully some of the items that involved correspondence and co-variational relationships as well as finding the nth term in numerical patterns. Students in the fourth class had high abilities in the items of the factors “Generalized arithmetic” and “Functional thinking” and average abilities in the items of the factor “Modeling as a domain for expressing and formalizing generalizations”. These students developed abilities for producing relational reasoning, not only in the context of patterns or co-variational relationships but also in contexts where a regularity is presented through a realistic situation or phenomenon.

This study contributed to theory about the core aspects of algebraic thinking by utilizing research from mathematics education and psychology. Using Demetriou and colleagues’ (2002, 2011, 2015) overarching theory about the architecture and development of the mind as a basis for describing mental action, this study investigated the relationship between algebraic thinking and several cognitive factors and reasoning processes. The results indicated that the relationship between algebraic thinking and cognitive factors changes from age to age. Along the transition of the students from grade to grade, some of the factors appear or disappear in the relationship and some of them remain stable in all age-groups.

The findings of the study indicated that the Causal-Experimental System, Serial Reasoning and Working Memory are significant predictors of students’ algebraic thinking in all age-groups. For this reason, their relationship was further examined. The model that was extracted from quantitative data suggests that the Causal-Experimental System, Serial Reasoning and Working Memory predict algebraic thinking abilities. Thus, this study proposes a model which describes algebraic thinking as a multidimensional factor that is synthesized by the factors of “Generalized Arithmetic”, “Functional Thinking” and “Modeling as a domain for expressing and formalizing generalizations”; this factor is predicted by the operations in the Causal-Experimental system, Serial Reasoning and Working Memory. Serial Reasoning involves the generation of possible relationships and structure between a set of objects. The Causal-Experimental System refers to causal relationships and encloses mental operations such as trial and error, combinatorial hypothesis, systematic experimentation and modeling construction. Working Memory refers to the maximum amount of information and mental acts that the mind can operate concurrently in an efficient way. These three cognitive factors seem to act simultaneously and enable students to extract generalizations when they encounter algebraic problems.

The qualitative analytic system which is responsible for the representation of similarities and differences relationships and the corresponding analogical reasoning which allows the comparison of specific objects of a set with other specifics objects seem to predict students' algebraic thinking in Grade 4. The abilities of the students in Grades 5, 6 and 7 for representing generalities and using symbols seems to be influenced form the Spatial-Imaginal System and the corresponding Spatial Visualization reasoning; these cognitive factors enable the identification of structure in geometrical patterns and the use of visual representations for expressing and formalizing generalizations. Moreover, the Verbal-Propositional System and the reasoning processes by which this system interacts, inductive and deductive reasoning, seem to further enrich students' abilities for generalizing and using symbols. Deductive reasoning appears to be a predictive factor for students' algebraic thinking abilities in Grade 7.

This study explored two approaches that aimed to develop students' algebraic thinking ability; two interventions with ten 80-minutes lessons were examined in respect to their effectiveness, the "Semi-structured problem situations" and the "Structured mathematical investigations". Both interventions involved all of the aspects of algebraic thinking and had similar objectives and characteristics in respect to the quality of instruction. The interventions differed in respect to characteristics of the tasks that were used. In the first intervention, the tasks represented contexts from real life situations; a question was posed and students were encouraged to apply their own strategies for approaching the problem. In the second intervention, the tasks involved investigations that were aided with more assisting questions and scaffolding steps. The results showed that students who received instruction through "Semi-structured problem situations" outperformed students who received instruction through "Structured mathematical investigations" in the algebraic thinking post-test. More detailed results have shown that both experiments had positive impact in the "Generalized arithmetic" component. However, the students involved in the "Semi-structured problem situations" experiment had significantly higher performance in the components of "Functional Thinking" and "Modeling as a domain for expressing and formalizing generalizations". The results of the study showed that students' individual differences in the three cognitive factors that are related to algebraic thinking (Causal-experimental system, Serial Reasoning, Working Memory), and their interactions with the type of instruction had a significant impact on the benefits from the instructional intervention program.

These results provide empirical evidence supporting the arguments from previous literature (e.g. Drijvers, Coddijn & Kindt, 2011; Kaput, 2008; Mason, Graham & Johnston-Wilder, 2005; Radford, 2008) about the multidimensional nature of algebraic thinking and that algebraic thinking 'is not all about literal symbols but rather is about *ways of thinking*' (Kieran, 2011, p.591). The results enlighten the types and features of these ways of thinking by indicating specific cognitive factors and reasoning processes that flow through varying degrees through the three dimensions of algebraic thinking in each age level. Moreover, the results indicate that these ways of thinking are not static and stable but they progressively become more abstract and flexible. The integration of specific features in teachers' practices create viable opportunities in order these ways of thinking to be empowered and formulated through supportive classroom environments.

ΠΕΡΙΛΗΨΗ

Κατά τα τελευταία 20 χρόνια, η άλγεβρα και η αλγεβρική σκέψη αποτέλεσαν αντικείμενο έρευνας για πολλούς ερευνητές, σχεδιαστές εκπαιδευτικών πολιτικών και συγγραφείς αναλυτικών προγραμμάτων. Ωστόσο, στο πεδίο της μαθηματικής παιδείας δεν είχε διευκρινιστεί η έννοια της αλγεβρικής σκέψης με τέτοιο τρόπο ώστε να αποτελέσει αναπόσπαστο κομμάτι της διδασκαλίας και της μάθησης των μαθηματικών στο δημοτικό σχολείο. Η εργασία μελετά την έννοια της αλγεβρικής σκέψης μέσα από παιδαγωγικά και γνωστικά πεδία ανάλυσης, έχοντας ως στόχο την διασαφήνιση της έννοιας και την περιγραφή της σχέσης της με διάφορους γνωστικούς παράγοντες. Επιπλέον, η εργασία στοχεύει στη διαμόρφωση πρακτικών διδασκαλίας και μάθησης στο δημοτικό σχολείο που ενδυναμώνουν την ολόπλευρη ανάπτυξη της αλγεβρικής σκέψης. Ο σκοπός της εργασίας ήταν (α) ο εμπειρικός έλεγχος του θεωρητικού μοντέλου του Kaput (2008) για τα βασικά στοιχεία που συνθέτουν την έννοια της αλγεβρικής σκέψης, (β) η εξέταση της σχέσης μεταξύ της αλγεβρικής σκέψης και διαφόρων γνωστικών παραγόντων και διαδικασιών συλλογισμού και (γ) η εξέταση της επίδρασης δύο διαφορετικών διδακτικών παρεμβάσεων στην ενίσχυση της αλγεβρικής σκέψης των μαθητών δημοτικού σχολείου.

Στην έρευνα συμμετείχαν εξακόσιοι ογδόντα τέσσερις μαθητές από τις Δ’, Ε’ και Στ’ τάξεις του δημοτικού σχολείου και την Α’ γυμνασίου. Στους συμμετέχοντες χορηγήθηκαν συνολικά επτά δοκίμια που αξιολογούσαν: (i) την αλγεβρική σκέψη, (ii) τις ικανότητες στα Εξειδικευμένα Δομικά Συστήματα (ΕΔΟΣ), (iii) διάφορα είδη συλλογισμού (Naglieri Non-Verbal Ability test), (iv) τον επαγωγικό συλλογισμό, (v) την εργαζόμενη μνήμη, (vi) την ταχύτητα επεξεργασίας και (vii) τον έλεγχο επεξεργασίας. Επιπλέον, πραγματοποιήθηκαν δύο διδακτικές παρεμβάσεις στις οποίες συμμετείχαν 86 μαθητές της Ε’ δημοτικού. Στην παρέμβαση «Ημι-δομημένες διερευνήσεις σε ρεαλιστικά προβλήματα» συμμετείχαν 42 μαθητές. Στην παρέμβαση «Δομημένες μαθηματικές διερευνήσεις» συμμετείχαν 44 μαθητές. Ο σκοπός των δύο διδακτικών παρεμβάσεων ήταν η διερεύνηση του ενδεχομένου για ενίσχυση της ανάπτυξης της αλγεβρικής σκέψης των μαθητών δημοτικού σχολείου μέσα από καινοτόμες πρακτικές και διαδικασίες.

Τα αποτελέσματα της έρευνας επιβεβαίωσαν το θεωρητικό μοντέλο που προτάθηκε από τον Kaput (2008) σχετικά με τα βασικά στοιχεία που συνθέτουν την

έννοια της αλγεβρικής σκέψης. Σύμφωνα με το μοντέλο, η αλγεβρική σκέψη αποτελείται από τρεις διακριτούς παράγοντες: (i) τη «Γενικευμένη αριθμητική», (ii) το «Συλλογισμό με μεταβλητές», και (iii) τη «Μοντελοποίηση ως ένα πεδίο για την έκφραση και την επισημοποίηση γενικεύσεων». Η εργασία περιγράφει αναλυτικά τους τρεις παράγοντες μέσα από την αναγνώριση συγκεκριμένων κατηγοριών έργων που ανήκουν σε κάθε παράγοντα. Ακολουθώντας τις εισηγήσεις της Kieran (2004), όλα τα έργα επιλέχθηκαν με προσοχή, ώστε να ικανοποιούν δύο συνθήκες: (i) τα έργα περιλάμβαναν διαδικασίες που συνδέονται με την πρώιμη αλγεβρική σκέψη, όπως γενίκευση, λύση προβλήματος, υπόθεση, επαλήθευση, αιτιολόγηση, πρόβλεψη και απόδειξη και (ii) τα έργα περιλάμβαναν λεκτικές εκφράσεις, διαγράμματα, πίνακες και γραφικές παραστάσεις παρά σύμβολα, ενώ παράλληλα δεν απαιτούσαν τη χρήση συμβόλων. Ο παράγοντας «Γενικευμένη αριθμητική» αναφέρεται στη χρήση της αριθμητικής ως ένα πεδίο για τη διερεύνηση σχέσεων και δομής στους αριθμούς και τις πράξεις και την έκφραση γενικεύσεων. Ο παράγοντας «Συλλογισμός με μεταβλητές» αναφέρεται στη γενίκευση αριθμητικών ή γεωμετρικών μοτίβων και τη διερεύνηση και έκφραση σχέσεων συν-διακύμανσης και συμμεταβολής μέσα από διάφορα είδη αναπαραστάσεων (πίνακας, γραφική παράσταση, διάγραμμα, λεκτικές και συμβολικές εκφράσεις). Ο παράγοντας «Μοντελοποίηση ως ένα πεδίο για την έκφραση και την επισημοποίηση γενικεύσεων» περιλαμβάνει την κατασκευή μοντέλων για αναπαράσταση σχέσεων και γενίκευση κανονικοτήτων από μαθηματικοποιημένες καταστάσεις ή φαινόμενα της καθημερινής ζωής.

Σύμφωνα με τα αποτελέσματα της εργασίας, οι τρεις παράγοντες αλγεβρικής σκέψης παραμένουν σταθεροί σε όλες τις ηλικιακές ομάδες. Ωστόσο, οι ικανότητες αλγεβρικής σκέψης των μαθητών στη Δ’ τάξη διαφοροποιούνται από τις ικανότητες των μαθητών στις τάξεις Ε’, Στ’και Α΄ γυμνασίου. Επιπλέον, οι ικανότητες των μαθητών στην Α’ γυμνασίου διαφοροποιούνται από τις ικανότητες των μαθητών σε όλες τις προηγούμενες τάξεις.

Τα ευρήματα της εργασίας υποδεικνύουν την παρουσία τεσσάρων κατηγοριών που αντανακλούν διαφορετικά επίπεδα ικανοτήτων αλγεβρικής σκέψης ανάμεσα στους μαθητές. Οι μαθητές της πρώτης κατηγορίας αδυνατούσαν να επιλύσουν σχεδόν όλα τα έργα του δοκιμίου αλγεβρικής σκέψης, γεγονός που υποδηλώνει ότι δεν είχαν ακόμα αναπτύξει ικανότητες αλγεβρικής σκέψης. Οι μαθητές στη δεύτερη κατηγορία είχαν μέτρια επίδοση στη «Γενικευμένη αριθμητική» και χαμηλή επίδοση στο «Συλλογισμό με μεταβλητές» και τη «Μοντελοποίηση». Αυτοί οι μαθητές κατάφεραν να επιλύσουν με μεγαλύτερη επιτυχία τα έργα της «Γενικευμένης αριθμητικής» εφαρμόζοντας κυρίως αριθμητικές στρατηγικές. Οι μαθητές στην τρίτη κατηγορία είχαν μέτρια επίδοση στη «Γενικευμένη αριθμητική» και το «Συλλογισμό με μεταβλητές» και χαμηλή επίδοση στη «Μοντελοποίηση». Οι μαθητές αυτοί χρησιμοποιήσαν αριθμητικές στρατηγικές, αλλά κατάφεραν να επιλύσουν με επιτυχία έργα όχι μόνο στον παράγοντα της «Γενικευμένης αριθμητικής», αλλά και στον παράγοντα του «Συλλογισμού με μεταβλητές». Αυτοί οι μαθητές επίσης προσέγγισαν μερικά από τα έργα που περιλάμβαναν σχέσεις συν-διακύμανσης και συμμεταβολής, καθώς και υπολογισμού του νιοστού όρου σε αριθμητικά και γεωμετρικά μοτίβα. Οι μαθητές στην τέταρτη κατηγορία είχαν ψηλές επιδόσεις στη «Γενικευμένη αριθμητική» και το «Συλλογισμό με μεταβλητές» και μέτρια επίδοση στη «Μοντελοποίηση».

Η παρούσα εργασία αξιοποίησε την ολοκληρωμένη θεωρία του Δημητρίου και των συνεργατών του (2002, 2011, 2015) για την αρχιτεκτονική και την ανάπτυξη του νου, με σκοπό να περιγράψει τη σχέση μεταξύ της αλγεβρικής σκέψης και διαφόρων γνωστικών παραγόντων και διαδικασιών συλλογισμού. Τα αποτελέσματα έδειξαν ότι η σχέση αυτή αλλάζει από ηλικία σε ηλικία. Καθώς οι μαθητές μεταβαίνουν από την μια τάξη στην άλλη, ορισμένοι παράγοντες εμφανίζονται η απουσιάζουν από τη σχέση που περιγράφει την αλγεβρική τους σκέψη σε σχέση με διάφορες γνωστικές δομές, ενώ άλλοι παράγοντες δείχνουν να παραμένουν σταθεροί στη σχέση σε όλες τις ηλικιακές ομάδες.

Τα αποτελέσματα της εργασίας έδειξαν ότι το Αιτιώδες-Πειραματικό Σύστημα, ο Σειριακός Συλλογισμός και η Εργαζόμενη Μνήμη εμφανίζονται στη σχέση της αλγεβρικής σκέψης με γνωστικούς παράγοντες σε όλες τις ηλικιακές ομάδες. Για το λόγο αυτό, η σχέση τους εξετάστηκε περαιτέρω μέσα από τη εφαρμογή δομικών μοντέλων στατιστικής ανάλυσης. Το μοντέλο που επιβεβαιώθηκε υποδηλώνει ότι ο Σειριακός Συλλογισμός, το Αιτιώδες-Πειραματικό Σύστημα και η Εργαζόμενη Μνήμη προβλέπουν την αλγεβρική σκέψη των μαθητών. Έτσι, η εργασία αυτή προτείνει ένα μοντέλο που περιγράφει την αλγεβρική σκέψη ως έναν πολυδιάστατο παράγοντα που συντίθεται από τους παράγοντες της «Γενικευμένης Αριθμητικής», του «Συλλογισμού με μεταβλητές» και της «Μοντελοποίησης ως γλώσσας για την έκφραση και την επισημοποίηση γενικεύσεων». Ο παράγοντας αυτός προβλέπεται από τον Σειριακό Συλλογισμό, το Αιτιώδες-Πειραματικό σύστημα και την Εργαζόμενη Μνήμη.

Με βάση τα συμπεράσματα της εργασίας, ο Σειριακός Συλλογισμός επιτρέπει την παρατήρηση της δομής και των σχέσεων με τις οποίες ένα σύνολο στοιχείων συνδέονται μεταξύ τους. Το Αιτιώδες-Πειραματικό Σύστημα επιτρέπει την επεξεργασία αιτιωδών σχέσεων, την κατασκευή υποθέσεων για τον κανόνα που διέπει τις σχέσεις σε ένα σύνολο στοιχείων, τον πειραματισμό για εξέταση της υπόθεσης και την αιτιολόγηση με βάση την αντιστοιχία της υπόθεσης με τα αποτελέσματα του πειράματος. Η Εργαζόμενη Μνήμη συγκρατεί ταυτόχρονα τις πληροφορίες για την παρατηρούμενη σχέση και τις πληροφορίες που αφορούν τη διαδικασία και τα αποτελέσματα του πειράματος. Η παράλληλη λειτουργία αυτών των μηχανισμών επιτρέπει την εξαγωγή γενικεύσεων.

Το Ποιοτικό-Αναλυτικό Σύστημα που είναι υπεύθυνο για την αναπαράσταση σχέσεων ομοιότητας και διαφοράς και ο αντίστοιχος Αναλογικός Συλλογισμός που επιτρέπει τη σύγκριση των επιμέρους στοιχείων ενός συνόλου με επιμέρους στοιχεία, φαίνεται να προβλέπουν τις ικανότητες αλγεβρικής σκέψης των μαθητών στη Δ’ τάξη. Στην Ε’ τάξη, τη Στ΄ τάξη και την Α’ γυμνασίου, η έκφραση γενικεύσεων και η χρήση συμβολισμού φαίνεται να σχετίζεται με τη λειτουργία γνωστικών μηχανισμών όπως, το Οπτικό-Εικονικό Σύστημα και ο χωρικός συλλογισμός που επιτρέπουν την παρατήρηση της δομής και των σχέσεων σε γεωμετρικά μοτίβα και την ευέλικτη διαχείριση οπτικών αναπαραστάσεων για την έκφραση και ερμηνεία γενικεύσεων, όπως τα διαγράμματα και οι γραφικές παραστάσεις, Επιπρόσθετα, το Λεκτικό-Προτασιακό Σύστημα και οι συλλογισμοί με τους οποίους αλληλοεπιδρά, ο επαγωγικός και ο παραγωγικός συλλογισμός, ενισχύουν περεταίρω τις ικανότητες για γενίκευση και χρήση αφηρημένων συμβόλων. Ο επαγωγικός συλλογισμός επιτρέπει την εξέταση ειδικών περιπτώσεων για εξαγωγή γενικεύσεων. Ο παραγωγικός συλλογισμός, που εμφανίζεται ως προβλεπτικός παράγοντας της αλγεβρικής σκέψης στην Α’ γυμνασίου, επιτρέπει την αξιοποίηση γενικεύσεων για την επεξεργασία ειδικών περιπτώσεων,

Η παρούσα εργασία εξέτασε επίσης την επίδραση δύο διδακτικών παρεμβάσεων στην ικανότητα αλγεβρικής σκέψης μαθητών της Ε’ τάξης. Κάθε παρέμβαση περιλάμβανε δέκα ογδοντάλεπτα μαθήματα. Η πρώτη παρέμβαση περιλάμβανε «Ημι-δομημένες διερευνήσεις σε ρεαλιστικά προβλήματα» ενώ η δεύτερη περιλάμβανε «Δομημένες μαθηματικές διερευνήσεις». Και οι δύο διδακτικές παρεμβάσεις αποσκοπούσαν στη ενδυνάμωση όλων των πτυχών της αλγεβρικής σκέψης και είχαν τους ίδιους στόχους και χαρακτηριστικά σε σχέση με την ποιότητα της διδασκαλίας. Οι δύο διδακτικές παρεμβάσεις διέφεραν σε σχέση με τα χαρακτηριστικά των έργων που χρησιμοποιήθηκαν σε καθεμιά από αυτές. Στην πρώτη παρέμβαση, τα έργα παρουσίαζαν ανοικτά αυθεντικά περιβάλλοντα από την καθημερινή ζωή. Στη δεύτερη παρέμβαση, τα έργα αποτελούσαν μαθηματικές διερευνήσεις οι οποίες συνοδεύονταν από βοηθητικές ερωτήσεις και ακολουθούσαν βήματα-σκαλωσιές. Τα αποτελέσματα έδειξαν ότι οι μαθητές που συμμετείχαν στις «Ημι-δομημένες διερευνήσεις σε ρεαλιστικά προβλήματα" είχαν καλύτερα αποτελέσματα σε σύγκριση με τους μαθητές που συμμετείχαν στις "Δομημένες μαθηματικές διερευνήσεις" στο δοκίμιο αλγεβρικής σκέψης που δόθηκε μετά την ολοκλήρωση των παρεμβάσεων. Επιπρόσθετα, τα αποτελέσματα έδειξαν ότι και οι δύο παρεμβάσεις είχαν θετική επίδραση στην επίδοση των μαθητών σε έργα «Γενικευμένης αριθμητικής". Οι μαθητές που συμμετείχαν στις «Ημι-δομημένες διερευνήσεις σε ρεαλιστικά προβλήματα" είχαν στατιστικά σημαντική ψηλότερη επίδοση σε έργα του «Συλλογισμού με μεταβλητές » και της «Μοντελοποίησης».

Τα αποτελέσματα της παρούσας εργασίας έδειξαν επίσης ότι η αλληλεπίδραση των τριών γνωστικών παραγόντων που σχετίζονται με την αλγεβρική σκέψη (Αιτιώδες-Πειραματικό Σύστημα, Σειριακός Συλλογισμός και Εργαζόμενη Μνήμη) με τον τύπο της διδακτικής παρέμβασης είχε σημαντική επίδραση στα οφέλη των μαθητών σε σχέση με τις ικανότητες αλγεβρικής σκέψης μετά την ολοκλήρωση των διδακτικών παρεμβάσεων.

Η εργασία παρέχει εμπειρικά στοιχεία που υποστηρίζουν τις ιδέες που αναπτύχθηκαν από προηγούμενες μελέτες (π.χ. Drijvers, Coddijn & Kindt, 2011· Kaput, 2008· Kieran, 2007·, Mason, Graham & Johnston-Wilder, 2005· Radford, 2008) σχετικά με την πολυδιάστατη φύση της αλγεβρικής σκέψης. Επιπλέον, περιγράφει αναλυτικά τα στοιχεία αυτά μέσα από συγκεκριμένα έργα. Τονίζεται μέσα από τα αποτελέσματα ότι η αλγεβρική σκέψη «δεν είναι όλα όσα σχετίζονται με τα αλγεβρικά σύμβολα, είναι τρόποι σκέψης» (Kieran, 2011, σελ.591). Συγκεκριμένα, τα αποτελέσματα της εργασίας επισημαίνουν τους τύπους και τα χαρακτηριστικά αυτών των τρόπων σκέψης, υποδεικνύοντας συγκεκριμένους γνωστικούς παράγοντες και διαδικασίες συλλογισμού που προβλέπουν τις τρεις διαστάσεις της αλγεβρικό σκέψης σε κάθε ηλικία. Επιπλέον, τα αποτελέσματα δείχνουν ότι αυτοί οι τρόποι σκέψης δεν είναι στατικοί και σταθεροί, αλλά σταδιακά γίνονται πιο αφηρημένοι και ευέλικτοι. Η ενσωμάτωση συγκεκριμένων χαρακτηριστικών στη διδακτική πρακτική δίνει τη δυνατότητα στον εκπαιδευτικό να ενδυναμώσει αυτούς τους τρόπους σκέψης και να τους διαμορφώσει μέσα από υποστηρικτικά περιβάλλοντα μάθησης.

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CHAPTER I

The Problem

Introduction

The notion of algebra historically emerged after arithmetic; reflecting this trajectory, mathematics curricula were usually treated as an assortment of isolated topics, where arithmetic precedes and algebra follows (Carraher & Schliemann, 2007). In recent years, however, there is a growing consensus that algebra is “the getaway to K-12 mathematics reform for the next century” (Kaput, 1998, p.134). Researchers and policy makers seem to agree that students should engage with algebra in a coherent and systematic way throughout their schooling. For this reason, great importance has been given to the development of algebraic thinking, which “moves across the grades” instead of being taught through traditional courses of algebra in the middle school (NCTM, 2000). More specifically, the NCTM’s Principles and Standards for School Mathematics suggested that school mathematics from prekindergarten through grade 12 should empower students’ abilities in:

- exploring patterns, relations, and functions;
- representing the inherent relationships in mathematical situations and structures, by using algebraic notation;
- extracting mathematical models from quantitative relationships;
- analyzing the notion of change within various contexts. (NCTM, 1989; 2000)

The realization of this need stems primarily by the fact that, in the discipline of mathematics, algebra has a fundamental role, serving at least two primary functions. First, algebra is closely linked to the development, establishment and communication of knowledge in all areas of mathematics, including arithmetic, geometry and statistics (NCTM, 2000). Algebraic processes ensure the unity throughout the content of the mathematics curriculum. Second, algebra is considered as an essential language, base and foundation for the participation of the individual in an “economy of knowledge” (Mason & Sutherland, 2002). As it is recommended by the RAND *Mathematics Study Panel Report* (2003), algebra is pivotal not only for exploring most areas of mathematics but also for other disciplines such as science and engineering. Competency in algebra is considered as critical for achieving further study in mathematics and also for accessing professions in science, business and industry (Hatfield, Edwards, Bitter, & Morrow, 2000).

Besides its potential to promote these functions, the focus on algebraic thinking as a widely held goal of K-12 mathematics instruction is supported by the fact that students' abrupt and isolated introduction to algebra in the middle school has led them to experience serious difficulties in understanding core algebraic concepts (Cai & Knuth, 2005). The National Research Council (1998) has characterized the entry-courses to algebra in the United States as an "unmitigated disaster for most students" (p. 1). Previous research highlights, among others, the difficulty that middle and high school students have in understanding the equal sing as bidirectional and using symbols as generalizable numbers or as variables (e.g. Booth, 1984; Vergnaud, 1985). On the one hand, several researchers (e.g. Chazan & Yerushalmy, 2003; Herscovits & Linchevski, 1994; Kuchemann, 1981) have attributed these difficulties to the insufficient cognitive development of the students. On the other, current policy and research discourse is tilted in favor of introducing algebra much earlier, in order to better prepare students for the formal study of algebra in later grades (Cai & Knuth, 2011).

The widespread presence of algebra in school mathematics is considered as important for at least two more reasons. First, it has been suggested that the distinction between arithmetic and algebra deprives meaningful learning of mathematics in the early years (Kieran, 1992). Blanton and Kaput (2005) argued that the mere focus of elementary mathematics on arithmetic and computational fluency dismisses the conceptual development of mathematical ideas. Second, the call for reconceptualizing the nature of school algebra from K-12 grades is underlined by the belief that algebraic thinking is within the conceptual reach of all students. According to Mason, Graham, and Johnston-Wilder (2005),

> Everyone who gets to school has already displayed the powers needed to think algebraically and to make sense of the world mathematically. They have all generalized and expressed generalities to themselves and others.

> What they need is encouragement and permission to develop those powers in a supportive setting (p.ix).

In this context, algebraic thinking is considered as a wide conceptual field which does not coincide with the content of traditional algebra at the secondary grades.

Since mid-1990s, research that investigated the idea of integrating algebraic thinking in the early grades has become intensive. The studies that emerged run through cognitive, curricular and instructional perspectives (Kieran, 2011). Some researchers investigated the nature and components of this kind of thinking. Others attempted to specify the content of mathematical activities that count as algebraic in the elementary school. Another group of researchers reflected on the way by which algebraic thinking might be accessible to younger students and developed instructional approaches that demonstrate routes for developing algebraic thinking. Nevertheless, researchers' efforts to describe algebraic thinking through several perspectives are characterized by diversity (Carraher & Schliemann, 2007). While available studies in the field are considered as 'groundbreaking', still many research questions remain open (Radford, 2012).

The fact that the field of mathematics education has not yet described algebraic thinking in a coherent and systematic way, is a serious threat to the viability of students' learning experiences as well as teachers' knowledge and practices. This study addresses this problem by studying what it might mean to conceptualize algebraic thinking in order to be concretely seen as central in even young students' mathematical instruction rather that as an advanced skill.

The Problem

A number of research studies investigated the transition from arithmetic to algebra. These approaches were based on the fact that algebra starts where arithmetic ends. Filloy and Rojano (1989) provided historical data on the idea of the 'didactic cut' which takes place as the mathematical thinking moves from arithmetic to algebra and students are called to act on unknown terms. Similarly, Herscovics and Linchevsky (1994) referred to the 'cognitive gap' which is inherent between arithmetic and algebra and it becomes obvious through the weakness of students to spontaneously act on the unknown. However, some researchers contended that there is not a clear distinction between these two mathematical domains, suggesting that arithmetic is inherently algebraic and algebra is inherently arithmetical (e.g. Carraher et al., 2006; Lins & Kaput, 2004). For example, pattern activities bring arithmetic and algebra together (Radford, 2014). Furthermore, the structural exploration of expressions that contain only numbers in arithmetical contexts is a prerequisite for the structural exploration of expressions that contain both numbers and letters in algebraic contexts (Watson, 2009). In this perspective, there is a need to comprehensibly describe the relationship between arithmetic and algebra, in order to avoid teaching arithmetic while we are thinking that we teach algebra and vice versa (Radford, 2014).

Some researchers attempted to describe the development of algebraic thinking by analyzing in detail a sequence of advanced transitions from an operational perspective to a structural perspective. (e.g., Mason, 1989; Sfard & Linchevski, 1994; Thomas & Tall, 2001). This body of research suggested that achievement of the fundamental shift from arithmetical to algebraic contexts is cognitively demanding. For example, Sfard and Linchevsky (1994) draw attention to individual learning and understanding by questioning the role of "what one is prepared to notice and able to perceive" (p.192) when confronts algebraic problems. Their model defined two crucial transitions: from the operational conception of a mathematical notion to the structural conception (of an unknown) and then to the functional conception (of a variable). English and Sharry (1996) took the analysis of Sfard and Linchevsky (1994) a step further by suggesting that it is the incorporation of a process of analogical reasoning that constitutes a mental tool for extracting differences and commonalities between mathematical structures and articulating expressions of generality. However, Rivera and Becker (2007) suggested that inductive and abductive reasoning have a pivotal role when students investigate the commonality in a pattern through the prediction of plausible generalizations. The combination of these two reasoning forms prompts the production of conjectures and testable hypotheses in order to construct a sustainable generalization. Other studies also suggested that solving algebraic problems is closely related to reasoning skills such as inductive reasoning (Ellis, 2007; Palla, Potari & Spyrou, 2012) and deductive reasoning

(Pedemonte, 2008). It seems then that research offers various views in respect to the kind of reasoning forms which seem to affect individuals' algebraic thinking ability.

More recently, few studies addressed the need for investigating the correlation of skilled performance in algebra to several cognitive factors, implying that innate constraints might frame the timing and sturdiness of the transition from the operational to the structural understanding of mathematical ideas. For example, Tolar, Lederberg and Fletcher (2009) found that the achievement of college students in algebra depends on a person's computational fluency, 3D spatial ability and working memory. Lee, Ng, Ng and Lim (2004) identified that the skill for pre-algebraic problem solving among 10 years old students is predicted by factors such as the central executive, performance IQ, and literacy. In a recent study, Lee, Ng, Bull, Pe, and Ho (2011) found that high scores of 6-8 years old students in pattern recognition and calculation completely mediate the effects of working memory. Fuchs, Compton, Fuchs, Powell, Schumacher, Hamlett, Vernier and Namkung (2012) pointed to the influence of non-verbal reasoning and oral language to pre-algebraic knowledge.

While these studies provide considerable insights into the relationship of cognitive factors and algebraic thinking, they focused on the investigation of isolated cognitive constructs and they do not employ an overarching model of mental causation. The findings of this body of research can be informed by unified theories of cognitive organization and development, which describe the way by which multiple cognitive constructs, including general cognitive factors of mental action, domain-specialized processing abilities, and reasoning processes, underlie the presence of individual differences in achieving understanding within algebraic tasks. Relative results from psychological research suggested that is important for education to consider comprehensive theories that describe the way the mind causes cognitive behavior, since what teachers have at hand are tools that can influence behavior (Hunt, 2012).

Besides the efforts for illustrating the cognitive dimensions of algebraic thinking, several studies in the mathematics education literature approached the concept of algebraic thinking through epistemological and curricular perspectives. However, few of the existing research studies made an attempt for depicting a thorough picture of the field (Carraher and Schliemann, 2007). On the one hand, NCTM's recommendations regarding the content of algebra in K-12 curriculum and its application in the mathematics classroom have been considered as general and unclear (Howe, 1998). On the other, most of the studies that made an effort to describe the notion of algebraic thinking in the early grades (e.g. Kaput, 1998; Kieran 1996; Kirshner, 2001; Radford, 2000), mix up reasoning processes (e.g. generalization and problem solving) with mathematical topics (e.g. functions and modeling); this is indicative of the fact that the effort for analyzing algebraic thinking is still in its 'infancy' (Carracher and Schliemann, 2007). Moreover, it has not yet been clarified whether early algebraic thinking represents a distinct domain of study or if it is better to be integrated into a more general algebraic terrain that captures the teaching and learning of algebra for both younger and older students (Carraher & Schliemann, 2007).

From an instructional point of view, a number of teaching interventions, indicated the critical role that instruction might play in providing young learners with rich opportunities to develop algebraic concepts from the beginning of their mathematical learning. Mathematics education research gradually offered evidences that as early as in the elementary grades students are able to develop algebraic thinking in appropriate classroom environments (e.g., Blanton & Kaput, 2005; Carpenter & Levi, 2000; Carracher, Brizuela & Schliemann, 2000; Irwin & Britt, 2005; Radford, 2008; Warren & Cooper, 2008). As reported by Watson (2009), these teaching experiments include examples of functional approaches, multi-representational approaches, equation approaches, and generalization approaches. However, Watson (2009) characterized the available research in developing instructional approaches to algebraic thinking as "patchy" because refers on learning in "particular contexts and materials" and not "across contexts and materials" (Watson, 2009, p.24). Similarly, Carracher and Schliemann (2007) emphasized the need for establishing a solid research basis in order the role of mathematics teachers in cultivating algebraic thinking in their classrooms to be clarified. While there are findings that demonstrated elementary school students' capability to think algebraically, it seems that research has not established a thorough understanding of the way in which algebraic thinking is conceptualized and incorporated in early mathematics teaching and learning through appropriate instructional practices.

Aim of the Study

The aim of this study is the development of a better understanding of the notion of algebraic thinking in ways that make sense even in the context of early mathematics instruction. In particular, this study aims to describe an overarching view of the nature and components of algebraic thinking and analyze the relationship between algebraic thinking ability and cognitive factors that affect individuals' behavior. An aim of the current study is also the investigation of promising instructional practices that foster the development of algebraic thinking through mathematics instruction in the elementary grades. The specific purposes of the study are the following:

a) to empirically test a theoretical model about the components and structure of algebraic thinking ability;
b) to identify and describe classes of students that reflect different levels of algebraic thinking ability;
c) to investigate the presence of a possible hierarchical trend in the way the components of algebraic thinking develop;
d) to investigate the relationship among students' algebraic thinking ability and various cognitive factors, such as domain-specific processing abilities, reasoning forms and general cognitive processes of mental action;
e) to investigate the impact of two teaching experiments on students' algebraic thinking ability;
f) to investigate the impact of the interaction between teaching experiments and students' cognitive abilities in their algebraic thinking ability.

Research Questions

This dissertation addresses eight main questions, which define the aim of the study:

1) Which components synthesize 10- to 13-year-old students' algebraic thinking ability and what is the structure of this ability?
2) Is the structure of students' algebraic thinking ability the same or different in relation to age?
3) What are the classes of algebraic thinking ability of 10- to 13-year-old students?
4) What are the characteristics of students' performance in algebraic thinking at different classes of ability?
5) Is there a consistent hierarchical trend in students' algebraic thinking ability?
6) What is the relation of algebraic thinking with domain-specific processes, different types of reasoning forms and general cognitive processes of mental action?
7) What kind of instructional practices nurture algebraic thinking in elementary school mathematics?
8) What is the impact of the interactions between the type of teaching experiment and students' cognitive abilities on their algebraic thinking ability?

Significance and Originality

From a theoretical perspective, this study's significance lies to the fact that aims to set a theoretical framework which informs ways that make sense even in the context of elementary mathematics instruction, in order to enhance the development of algebraic thinking and consequently competence in algebra. This study's special attention to early mathematics instruction is vital. Although the current research and policy discourse requires all levels of mathematics instruction to form a part of an integrated instructional program with the purpose to empower students' algebraic thinking, the elementary grades are obviously disjointed from this program. Moreover, secondary school students seem to face serious problems in achieving higher algebra goals. As evidences from national reports have shown that, about 35% of students fail in completing high school algebra courses and 93% of 17th years-old students fail in solving multistep algebra problems, (U.S Department of Education, 2008). Therefore, the fact that the notion of algebraic thinking has not been conceptualized in a sensible way in order to be easily integrated into younger students' opportunities to learn mathematics constitutes a serious lack in the way mathematics instruction is understood and formulated. This study takes up this problem, considering the significance of offering insight into what algebraic thinking might means, and what it would take to make algebraic thinking central to elementary students' mathematics instruction. Moreover, this dissertation takes into consideration current suggestions for the importance of integrating findings of psychological research and mathematics education research and relating models of information processing abilities rather than neuroscientific evidence to the process of education (Hunt, 2012).

From a practical point of view, the significance of this study lies to the fact that aims to develop a conceptual analytic tool for measuring students' algebraic thinking. This tool can support the detailed description of the components of algebraic thinking and the way students perform in each of these components. Furthermore, this tool might support the design of materials that can be applied in corresponding mathematical activities in the classroom, as well as materials that can be used in teachers' training programs.

This study's originality lies to the fact that aims to parse the notion of algebraic thinking in a viable and integrated way which for the first time crosses both conceptual, cognitive and instructional contexts. This dissertation takes into consideration the suggestions of previous research for not just pushing algebra topics of the secondary school down to the elementary school curriculum. As Mulligan, Cavanagh and Keanan-Brown (2012) suggested, there is a need for reconceptualizing the development of algebraic thinking" (Mulligan, et.al. 2012). The RAND Mathematical Study Panel (2003) also pointed to the need for systematically investigating the topic of algebraic cognition among students of 6 to 12 years old. Similarly, the National Mathematics Advisory Panel [NMAP] (2008) recommended the documentation of factors that predict algebra achievement in order to design interventions that boost the development of basic skills. Hence, this study's originality lies to the fact that seeks to reconceptualize algebraic thinking by examining its association to core mathematical concepts, processes and mental operations in an age span that captures late elementary grades and the first grade of the secondary school. In doing so, algebraic thinking is approached in a way that is responsive to students' effective understanding.

In doing so, this study follows calls for adopting teaching and learning routes that aspire personal conceptual understanding. In 2005, the UK's Department of Education and Skills presented the initiative *Every Child Matters: Change for Children in Schools* for promoting children's learning and development. Among other suggestions, this text posed that schools should contribute to the well-being of children and young people by "helping each pupil to achieve the highest educational standards they possibly can" (p. 2). Concurrently, research has highlighted that teaching practices should correspond to students' needs for learning and pace at which they learn (Tomlinson, 1999). This study's originality is based to the fact that special attention is paid to the role of individual differences in the development of algebraic skilled performance. This study aims to describe the relationship between algebraic thinking and students' cognitive abilities. In addition, concrete instructional practices will be provided for enhancing the development of algebraic thinking and personal conceptual understanding. These practices might influence curricular changes in respect to the teaching and learning of algebra, as well as teachers' education programs.

The conceptualization of algebraic thinking through a theoretical model that is validated by empirical data contributes to theory building in the teaching and learning of algebra in the early grades and more broadly to the K-12 mathematics instruction. Furthermore, the instructional interventions developed by the study can support the design of curriculum materials and provide guidance to elementary school teachers for fostering the development of algebraic thinking in their classrooms.

Limitations

This study aims to describe the notion of algebraic thinking and its relationship to cognitive factors. For this reason, several design decisions were demarcated. The study examines a relatively selected sample of late elementary school students and early secondary school students, in the light of a particular theoretical model. Moreover, this study involves quantitative data and corresponding analysis techniques. The investigation of instructional practices that nurture the development of algebraic thinking involved two particular theoretical frameworks and the approach pursued to collect and analyze data was based on the direct involvement of the researcher in conducting the lessons. All these features of the design and the implementation of the study enforce limitations on the study, with more significant the extent to which its findings might be transferable.

Structure

The present dissertation is structured into six chapters. The first chapter states the problem, the aims and research questions of the intended study. Moreover, the significance and originality of the study are reported, suggesting specific theoretical and practical contributions.

In the second chapter, there is a summary of important literature, which demonstrates understanding of the research issues and identifies gaps that the current research is intended to address. In this chapter, the theoretical framework of the current study is analyzed, with reference to key studies which describe the forms that algebraic thinking might take in the early grades and studies that investigated the relationship between algebraic thinking and cognitive factors. Furthermore, the selection of a specific psychological theory, which was considered as appropriate for the investigation of the relationship between cognitive structures and algebraic abilities, is justified.

The third chapter describes the research design and methodology, including details about the students, the instruments, the design of the two teaching interventions and the research techniques that were employed in the analysis of the data.

The fourth chapter presents the results that emerged from the analysis of the data, with reference to the validation of the proposed model about the components and structure of algebraic thinking, and the presence of different classes of students regarding their algebraic thinking abilities. The results about the relationship of algebraic thinking with domain-specific processing abilities, reasoning processes and general cognitive factors of mental action are also reported. Additionally, this chapter descries the impact of the two teaching interventions on students' algebraic thinking abilities.

The fifth chapter discusses the results of the study and attempts to capture a holistic view of the nature of algebraic thinking. Specifically, a unified model of the quantitative results is presented which informs about the components and structure of the notion of algebraic thinking, its relationship with cognitive characteristics of the individuals and effective instructional practices for empowering algebraic thinking in the elementary grades.

The sixth chapter includes the conclusions of the current study, theoretical, methodological and practical implications, as well as suggestions for future research.

Definition of Concepts

Algebra. Several definitions of algebra can be found in the mathematics education literature. A key aspect, which is involved in almost every description of algebra, is the concept of generalization. Watson (2009) defines algebra as “the way we express generalizations about numbers, quantities, relations and functions” (Watson, 2009, p.3). Nevertheless, generalization seems to be only one of the multiple dimensions of algebra. Kaput (1995) identified five aspects of algebra: generalization and formalization, syntactically guided manipulations, the study of structure, the study of functions, relations and joint variation, and modeling language. NCTM (1998) stated that the content of school algebra could be analyzed in four organizing themes: functions and relations, modeling, structure, and language and representation. Howe (2005) also defined algebra by making references to ideas such as working with variables, representation and modeling of situations, manipulating expressions and equations, uncovering algebraic structure in arithmetical formations. The current study will adopt a definition of algebra which was developed by Mason, Graham, and Johnston-Wilder (2005) and which seems to summarize the many of the possible approaches to the meaning of algebra reflected in literature. This involves four interrelated strands:

i. expressing generality and fostering an awareness of generality,
ii. encountering multiple expressions for the same generality,
iii. experiencing ‘freedom’ when symbols are used as yet-unknowns or as yet-unspecified quantities and ‘constraint’ when these have to be placed in specific equations or inequalities and manipulated through particular methods, and
iv. experiencing structure which leads to express in general both the rules of arithmetic and the rules for manipulating algebraic expressions.

Algebraic thinking. The terms algebraic thinking and algebraic reasoning are used interchangeably within mathematics education literature. However, Kieran (2011) emphasized the need for adopting the term algebraic thinking instead of the term algebraic reasoning; using the term algebraic reasoning points to a confined conceptual field, comparable to other types of mathematical reasoning such as deductive, abductive, inductive and analogical reasoning. Thus, the broader term of algebraic thinking seems to be more appropriate for considering its multifaceted notion and components.

In this study, the term algebraic thinking will be used, in order to reflect the effort for uncovering its complex nature and multiple components. Algebraic thinking is considered as a wide-ranging field of concepts which does not merely coincide with what we already know and teach as school algebra. Specifically, algebraic thinking refers to the presence of 'psychological processes' in the process of solving problems which mathematicians would solve by applying formal algebraic symbolization (Carraher & Schliemann, 2007,). One of the main aims of the current research is to precisely describe which psychological processes are related to the notion of algebraic thinking.

Early algebra. Given the emphasis of this study on the development of algebraic thinking as early as the elementary grades, the term early algebra will be used for referring to the possibility of making algebra and algebraic thinking central to all students' mathematical experiences through the grades. This approach is underlined by the view that algebra and algebraic thinking ought not to follow arithmetic. The development of algebraic thinking should be fostered along all mathematics lessons and not purely at the time that specific aspects of algebra are introduced (Lins & Kaput, 2004). In this context, the term early algebra is distinguished from the term pre-algebra. Pre-algebra approaches "do not question the sequence of arithmetic first, algebra later", (Carraher and Schliemann, 2007, p. 675). This kind of lessons aimed to properly prepare students usually of the ages between 12 to 14 years old on basic concepts, such as equality and variable. Their purpose was to eliminate the difficulties that students could encounter on the formal study of algebra in later grades. In contrast, early algebra approaches suggested the integration of algebra and algebraic thinking across all grades and all topics. Kaput (1998) recommended that school mathematics should be 'algebrafied'. This does not mean that traditional algebra topics should be moved down to the elementary mathematics but requires a coherent conceptualization of the nature and components of algebraic thinking as well as a re-examination of when specific algebraic ideas should be introduced to the students (Knuth, Alibali, McNeil, Weinberg & Stephens, 2005). This view is developed throughout the present study.

Domain-specific processing abilities. Domain-specific processing abilities refer to the processes that underlie problem solving in specific domains of thought. Each of these processes is specialized at the representation, intellectual management, treatment and comprehension of concrete sectors of knowledge of the environment. (Demetriou, Spanoudis and Mouyi, 2011).

Reasoning processes. Reasoning processes are applicable when meanings are transferred from one representation to another. Reasoning by induction, deduction, analogy and abduction are some of the different inferential mechanisms that are used during the transfer of information from an initial representation to a target representation. These types of reasoning are related to each other by common inferential processes which emerge as a separate level in hierarchical models of cognition (Demetriou, Spanoudis and Mouyi, 2011).

General cognitive factors. General cognitive factors are related to the processes that underlie problem solving across different domains. For example the mechanisms of working memory, control of processing and speed of processing are considered as general cognitive factors. This group of cognitive factors includes time parameters regarding the processing of information for solving a problem, speed by which the treatment of stimulus is activated, as well as the examination of the relativity of incoming information to the main objective (Demetriou, Spanoudis and Mouyi, 2011).

CHAPTER II

Literature Review

Introduction

NCTM's Principles and Standards (1989) advocated algebra as the keystone of mathematics reform, suggesting that algebra experiences through K-12 curriculum will empower elementary students' learning with understanding and reduce secondary students' difficulties in learning formal algebra (NCTM, 2012). Since then, research in the field of algebra and early algebra is rapidly evolving. Nevertheless, this field is characterized by diversity, reflecting multiple research perspectives (Kieran, 2011).

The first part of the theoretical framework of this study is structured into three sections, aiming to capture three different perspectives within which previous research has been developed. The first section summarizes literature that is concerned with describing the concepts of algebra, early algebra, and algebraic thinking. The second section refers on the ways by which research, both from the discipline of mathematics education and the discipline of psychology, approached the notion of algebra, early algebra, and algebraic thinking from a cognitive perspective, including developmental aspects and cognitive factors that affect algebraic thinking. In the third section, research which provides insights into pedagogical factors that play a role in supporting the development of students' algebraic thinking will be reported, including teachers' instructional practices, curriculum materials, and technological tools. The second part of the theoretical framework is focused on psychological research and theories of mental causation which might help educators and researchers in better understanding students' mathematical learning and behavior. In particular, the overarching theory of the architecture and development of the mind (Demetriou, Spanoudis & Mouyi, 2011) is thoroughly described, as well as research studies within mathematics education which provide support on the educational implications of this theory. Figure 2.1 presents the structure of the theoretical framework of the study.

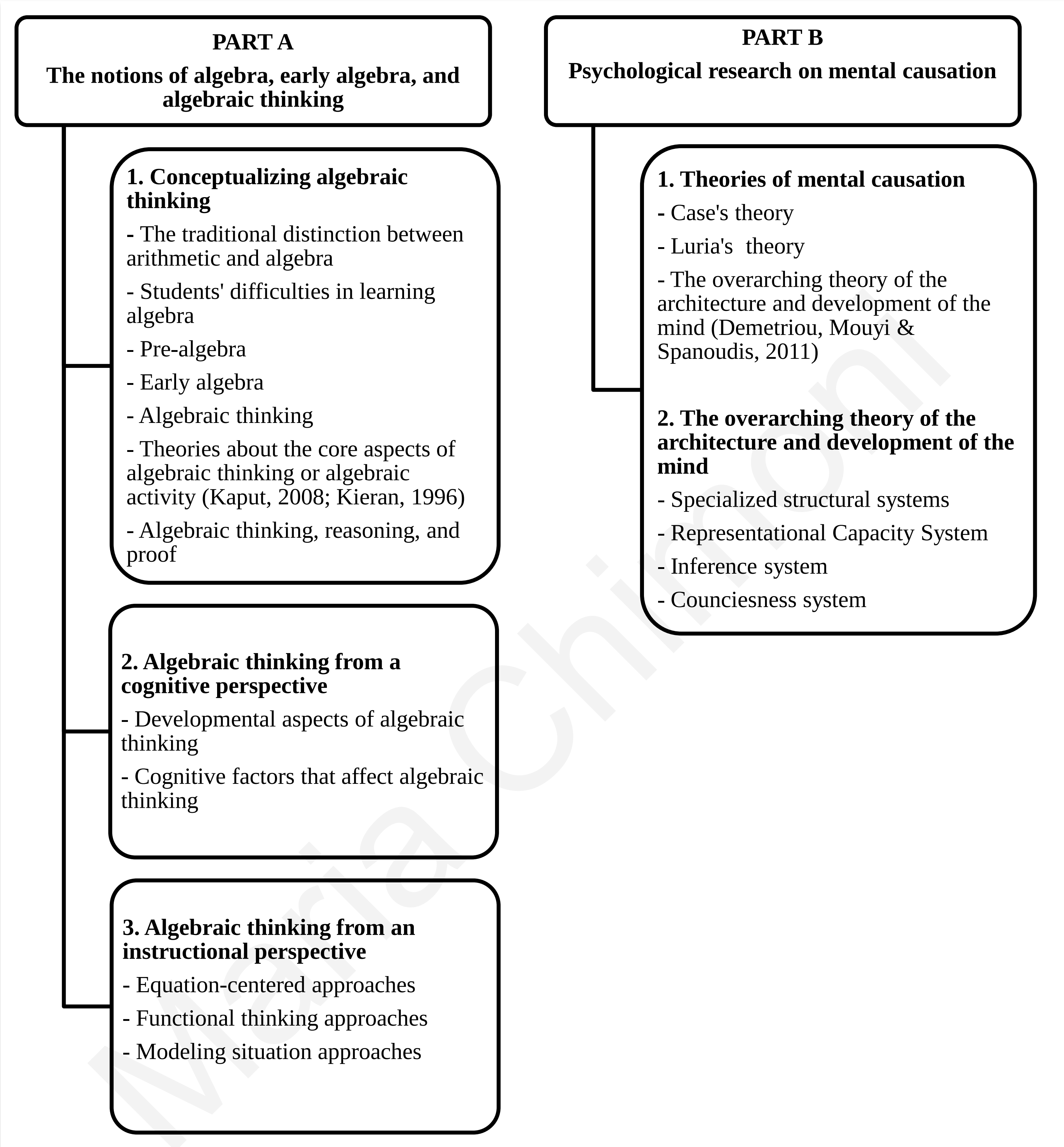


Figure 2.1. The structure of the theoretical framework of the study.

Conceptualizing Algebraic Thinking

The traditional distinction between arithmetic and algebra. The notion of algebra has traditionally been associated with secondary school mathematics. The arithmetic then algebra tradition has been predominant in the mathematics curricula of most countries, indicating the belief that "Historically, algebra grew out of arithmetic and so it ought to grow afresh for each individual" (British Mathematical Association, 1929, p. 219: in Lins & Kaput, 2004). Moreover, this teaching and learning trajectory was justified by the idea that arithmetic is more concrete and easy for students, where algebra is more abstract and difficult (Lins & Kaput, 2004). Following Piaget's developmental trajectory, algebra was associated with higher developmental stages because the abilities required for achieving high performance in algebra were related to formal thinking. For example, Davies (1975, 1984) highlighted the complexity that underlines fundamental algebraic activities such as solving linear equations in relation to performing simple arithmetical operations.

During the decade between 1980s and 1990s, several research studies (e.g., Booth, 1981) made an attempt for defining algebra by introducing the notion of 'generalized arithmetic'. This approach, which refers to the use of letters for expressing general rules of arithmetic, can still be found in algebra research of the present (Kieran, 2006). Kuchemann (1978, 1984) was among the first who attempted to uncover the nature and content of algebra through the investigation of its arithmetical foundations. Furthermore, Kuchemann combined the idea of generalized arithmetic and Piagetian levels of intellectual development, suggesting that the transition of students through levels of using symbols within arithmetical contexts is under the control of cognitive constraints. The standpoint that depicted a cognitive gap between arithmetic and algebra was further strengthen by mathematics education research during 1980s. These studies highlighted the difficulties that middle and high school students face during their algebra courses (e.g. Booth, 1981; Kieran, 1981; Vergnaud, 1985).

Students' difficulties in learning algebra. Various researchers pointed to the difficulty of understanding the meaning of the equals sign. Where in arithmetic the equals sign means that a calculation must be operated, in algebra the equals sign has a bidirectional role. The use of letters as generalized numbers or as variables is also a hurdle for middle school students. For example, students easily perform the operation $6 + 4 = 10$ but not easily understand that $a + b = c$ is a mathematical expression for representing an additive relation. Furthermore, students struggle to manipulate mathematical expressions that contain both letters and numbers [e.g. $4(x + z)$]. In this case, numbers are not involved in calculations but they are treated structurally in the same way as letters. Another difficulty of the students is the application of the commutative and distributive properties (Wagner, 1981). For example, in arithmetical contexts, the expression $4 + b = 7$ can easily be solved by retrieving previous knowledge on the bonds facts. However, $178 + y = 213$ is simpler to be solved by applying the commutative property.

Some researchers attributed the difficulties that middle and high school students face in algebra to the inadequate cognitive development of the students. Filloy and Rojano (1989) referred to the discontinuity among arithmetic and algebra with the term 'didactic cut'. Specifically, they suggested that when students deal with equations of the form $ax + b = c$, where a, b, and c are numbers, by applying the commutative property, they operate through arithmetical perspectives. On the contrary, when students confront equations of the type $ax + b = cx + d$, where they have to manipulate both each side structurally and the concept of the equals sign, they operate through algebraic perspectives. Specifically, in this kind of equations students have to consider the unknown quantities as if they were known (Radford, 2012). According to Filloy and Rojano (1989), these two situations are disjoint due to developmental limitations of the students. Similarly, Herscovits and Linchevski (1994) analyzed the differences between arithmetic and algebra with the term 'cognitive gap'.

Following these ideas, many researchers support that students should not be introduced to algebraic notation before they are developmentally ready (e.g. Linchevski, 2001). Sfard (1992) argues that students will encounter algebra through an operational outlook and then move to more structural conceptions of algebra. For this reason, algebra education should start from an operational perspective instead from a structural perspective. Moreover, most curricula around the world had placed arithmetic before formal algebra teaching. In addition, learning and succeeding in traditional algebra was considered as a privilege of the more skilled students. Students who were not ready were designated to fail in algebra courses. As Kieran (1992) reports, many students experience difficulty in learning algebra. Indicative of this fact is a quotation from NRC (1998) where first year algebra courses in the United States are characterized as “an unmitigated disaster for most students (p.1).

The notion of Pre-algebra. Acknowledging the intrinsic epistemological differences between arithmetic and algebra and the abrupt appearance of formal algebra in the high school, many researchers sought for interventions that would diminish students’ difficulties. Specifically, the aim of these studies was to develop transitional levels of teaching and learning which would assist students to smoothly pass from arithmetical perspectives to algebraic perspectives. Nevertheless, these approaches “do not question the sequence of arithmetic first, algebra later” (Carraher and Schliemann, 2007, p. 675). Their rationale is based on the idea that algebra is associated, almost exclusively, with high school mathematics and that students could better be prepared for these courses if they had the opportunity to investigate the meaning and use of symbols during their mathematical experiences in the middle school.

This kind of studies includes lessons with students of the ages between 12- to 14- years old. Their content is focused on basic concepts, such as equality and variable. For example, Herscovits and Kieran (1980) developed teaching approaches for helping seven and eight grade students in understanding the notion of equality and transforming arithmetical expressions into algebraic equations. Vergnaud (1985), working with eighth and ninth graders, designed activities with two-plate balance scales for exploring the meaning of equality. More recently, Kieran and Saldanha (2005) used a Computer Algebra System, known as CAS, for helping ninth graders in exploring different meanings of the equals’ sign, such as equivalence as a condition which gives equal values for a range of input values of the variables, and equivalence as the condition where expressions are transformations of the same form.

Besides the equation-centered approaches, some studies within the perspective of pre-algebra focused on the notions of generalization, number patterns, and functions. Kieran and Sfard (1999) pointing to the equivalence between forms of functional expressions, used a graphical function approach. In their tasks, students were enabled to observe that equivalent algebraic representations generate the same graphs and hence represent the same relationships between variables. The teaching and learning of variables also seemed to be empowered through instruction that makes use of software environments. For example, Healy, Hoyles, and Sutherland (1990) as well as Ursini (1994, 1997, 2001) used LOGO environments for indicating the idea of generalization within algebraic representations. Sutherland and Rojano (2003) engaged students of 10 years old in spreadsheets activities in order to work with undetermined quantities as if they were known and operate on them. Yerushalmy and Schwartz (1993) and Schwartz (1996) emphasized the important role of software environment which combine multiple forms of representations, such as algebraic notation, graphs, and natural language for enhancing students understanding on the meaning of functions. In particular, Schwartz indicated the value of this kind of software in enabling students to flexibly switch between different kinds of representations.

A more recent study of Knuth et al. (2005) examined middle school students' understanding of two fundamental algebraic concepts, equivalence and variable. Their conclusions highlighted the importance of preparing students at the time they entry middle school grades, by linking their prior arithmetical knowledge to early algebraic thinking. This kind of interventions is considered as pivotal for succeeding when they study formal algebra. Furthermore, understanding equivalence and variable through the participation of the students in pre-algebraic mathematical experiences was found to be related to their performance in solving problems where these two ideas are used.

To recap, research within the context of pre-algebra recognizes the minimal preparation of the students for abstract algebra courses in the high school as problematic. The findings obtained from this kind of studies highlighted the importance of exploring the ideas of equation, equivalence, and variable through various perspectives. Nevertheless, pre-algebra approaches were mainly focused on interventions that take place in the middle school and strongly accepted that arithmetic's place in the mathematics curriculum is prior to algebra's place.

The notion of Early Algebra. Similar to pre-algebra approaches, the strand of early algebra also advocates that many students experience difficulties in learning algebra. Nevertheless, early algebra approaches are tilted in favor of the idea that young students are expected to face difficulties at the time they entry the secondary grades due to the fact that no meaningful learning occurs while they study mathematics in the elementary school. In this perspective, early algebra research focused on ways and content for introducing algebra as early as the first or second grades rather than the sixth or seventh grades (Lins & Kaput, 2004). More recently, studies with kindergarten students demonstrated that early algebra can also become a part of 5-years old students' mathematics education (e.g. Mulligan, English & Mitchelmore, 2008).

At the end of the 1980s, researchers began to sturdily support the idea of reforming the mathematics curriculum in favor of re-examining the content of algebra and identifying ways for introducing core algebraic concepts at the primary grades (Kieran, 2004). Davies (1985, 1989) was among the first who emphasized the need for integrating algebra in the mathematics curriculum for grades 2 or 3. Schoenfeld (1995) also stressed out that algebra should be spread throughout the curriculum instead of being taught at the middle or high school levels (Algebra Initiative Colloquium Working Group, La Campagne, 1995). Kaput (1998) has argued that algebra is the gateway to K-12 mathematics reform for the next century" (p.134) and highlighted the significance of teachers' abilities in enhancing students' opportunities for developing algebraic thinking. Similar to researchers' proposals, the NCTM Standards (2000) underlined the need for introducing activities that empower algebraic thinking abilities of the students from the start of their mathematical learning.

The idea of examining the introduction of students to algebra at a much earlier age instead of restricting its teaching and learning to specific grades or lessons sequences seems to gain ground in the mathematics education discourse. This approach rejects the belief that algebra starts where arithmetic ends only because algebra historically emerged after arithmetic. Moreover, early algebra supporters discard the value of introducing a transitional period in-between arithmetic and algebra courses where pre-algebra lessons will bridge topics of algebra and arithmetic. On the contrary, it is strongly supported that students' failure in understanding algebraic concepts might be rooted to the absence of opportunities for extending mathematics that students are taught during the elementary grades in order to encompass algebraic thinking and especially the concept of generality (Blanton & Kaput, 2011). Specifically, the aim of early algebra is considered to be the introduction of young students to ways of thinking algebraically which progressively become more formal and make use of algebraic symbolism for expressing, establishing and justifying their ideas (Blanton, Levi, Crites & Dougherty, 2011).

A number of researchers analyzed the way by which specific arithmetical contexts might be interwoven with algebraic contexts. For example, it has been suggested that the concept of function could be introduced while students investigate problems of addition (Carraher, Schliemann, Brizuela & Earnest, 2006). The expression $+3$ represents not only an operation on a specific number but also the relationship that connects input and output values (e.g. $f(x) = x + 3$) or mapping notation of the type $x \rightarrow x + 3$. In the context of these possibilities, the objects of arithmetic can be used as both particular and general. Furthermore, algebraic concepts seem to be not just an extra and optional topic but essential in order students to achieve conceptual understanding in mathematics. As Carraher et al. (2006) state "…arithmetic has an inherently algebraic character in that it concerns general cases and structures that can be succinctly captured in algebraic notation" (p. 89).

Despite the current emphasis of research on early algebra, and the corresponding suggestions made by curriculum designers and policy makers, still many questions remain unanswered. For example, the capability of young learners for learning algebraic concepts has not yet been defined (Carraher et al., 2006). The ability of the teachers for teaching algebra in the elementary grades is also a matter of discussion (Carraher and Schliemann, 2008). Still, the content of algebra within early grades have not been coherently defined. Kieran (2011) referred to Subramaniam and Banerjee (2011) who note that arithmetic needs to be viewed with 'algebra eyes' and to Blanton and Kaput (2008) who describe the phenomenon of 'algebrafying' mathematics curriculum as efforts for nurturing classroom norms where the mathematical processes of argumentation, conjecture and justification occur. According to Kieran, combining these two ideas together, a picture of routes for developing algebraic thinking in early grades is depicted; nevertheless, emerging research in the field demonstrates that routes towards early algebra involve much more than these two aspects.

The notion of algebraic thinking. Research in mathematics education assigns considerable importance to the development of algebraic thinking as a way for approaching algebra within the early grades. There is a widespread acceptance of the distinction of algebraic thinking from what we already know and teach as school algebra. Moreover, algebraic thinking is considered to be within the conceptual reach of all students and vital for their participation in society (Mason, Graham & Johnston-Wilder, 2005). For this reason great importance has been given to the development of algebraic thinking across the grades instead on the teaching and learning of algebra through traditional courses in the middle or high school (NCTM, 2000). As it is has been emphasized by Kaput (1999), definitions of algebra that are based on what we were teaching in schools during the last century "is one of simplifying algebraic expressions, solving equations, learning the rules for manipulating symbols, the algebra that almost everyone, it seems, loves to hate" (p.134). Algebraic thinking is unquestionably a broader conceptual field rather than a list of specific tasks.

This idea raised the important issue of which are the aspects of algebraic thinking both in the primary and secondary education. A considerable number of research studies described the kinds of meaning secondary students make when they are engaged with algebraic tasks either through constructivist / cognitive or social / cultural frameworks (Kieren, 2007). More recent research focused on the development of young learners' algebraic thinking (e.g., Irwin & Britt, 2005; Warren & Cooper, 2008; Zaskis & Liljedahl, 2002). While both of these bodies of research provided important advances to the field, it has not yet been clarified whether early algebraic thinking represents a distinct domain of study or if it is better to be integrated into a more general algebraic terrain that captures the teaching and learning of algebra for both younger and older students (Carraher & Schliemann, 2007).

Several researchers made efforts to analyze the nature and content of algebraic thinking, focusing on what individuals do and how their abilities for generalizing and using symbols develop. Lins (1990) declared that algebraic thinking refers to an intended shift from real or mathematical contexts to structure. This process encompasses an emergent competence of the individuals for understanding and using symbols. Kaput (1998) also emphasized the process of symbolization, and the need for using symbolic expressions in order to establish and justify generalizations. Kieran (1996) offered a slightly different view by arguing that algebraic thinking is not only about using symbols in order to express generality; algebraic thinking arises when individuals make use of any kind of representations when they try to manipulate quantitative situations in a relational way (in Kieran, 2011).

Radford (2000) examined more systematically the ways by which generalization might be expressed and highlighted that algebraic thinking is not merely apparent when a precise symbolic language is acquired and applied by the students. Similar to Kieran, Radford (2000) suggested that algebraic thinking entails efforts of the individual to represent generality in certain ways. The identification of a functional relationship constitutes the first step in the process of generalization, where its expression is a further process that does not necessarily involve standardized mathematical symbols; it is a process with semiotic and symbolic nature, where social-linguistic elements of the culture of the individual are inducted to mathematical activities. Consistently, most authors place important role to the natural language as a tool for representing algebraic relations in primitive stages of algebraic thinking development (Carraher & Schliemann, 2007).

Furthermore, Radford (2004) added to the field by clarifying the importance of "semiotic mathematical and non-mathematical" systems in students' production of meaning when they encounter algebraic tasks. In particular, Radford (2004) specified that there are three sources of meaning in algebraic activities; (a) the algebraic structure itself (e.g. the letter-symbolic representations), (b) the problem context (e.g. word problems, modeling activities) and (c) the exterior of the problem context (e.g. social and cultural features, such as language, body movements, and experience). Kieran (2007) reflected on Radford's conceptualization of meaning in algebraic activity, by suggesting that the first source also involves mathematical representations, such as graphs and tables; students could draw on multiple representations in conjunction with letter-symbolic representations for producing meaning in algebraic tasks.

Other definitions of algebraic thinking were focused to one or more aspects of school algebra. Driscoll (1999) contends that algebraic thinking refers to the ability for manipulating quantitative expressions in a way that functional relationships are expressed and justified. This definition is linked to the aspect of manipulating variables and functions. Swafford and Langrall (2000) asserted that algebraic thinking refers to the individuals' ability for manipulating unknown quantities as if they were known. This approach is more closed to the aspect of manipulating and transforming symbolic statements. Likewise, Schmittau (2005) emphasized that a starting point for developing algebraic thinking among young learners is reasoning about relations between undefined quantities, if these can be handled and compared. For example, individuals are able to understand that if a>b and b>c then a>c, even if they don't know the values of a, b, and c.

This kind of demarcations define in a great extend the differences between arithmetical thinking and algebraic thinking. Where arithmetical thinking is dedicated on giving specific quantitative results, algebraic thinking is dedicated to the process and structure of a mathematical operation (Malara & Navara, 2003). Radford (2012) draw on the findings of previous research (e.g., Filloy and Rojano 1989; Filloy, Rojano, and Puig 2007; Kieran1989), for summarizing the main conditions that require the application of algebraic thinking rather than arithmetical thinking: (a) 'indeterminacy': the problem involves unknowns numbers; (b) 'denotation': the unknown numbers involved in the problem have to be symbolized; (c) 'analyticity': the unknown quantities are treated as if they were known numbers. In this context, the students start solving the problem by operating on the unknowns (i.e., applying addition, subtraction, multiplication or division) as if they were known.

Mason and Sutherland (2002) in an attempt for distinguishing algebraic thinking from algebra as appears in school textbooks, offered a description of algebraic thinking which reflects essential abilities for future employees or university students. Specifically, they argue that algebraic thinking involves; (i) formulating, transforming and understanding generalizations, not only in numerical contexts but also in spatial relations, (ii) using symbolic models for predicting and representing mathematical or other situations, and (iii) controlling and using spreadsheets, graphing, programming, and database software.

The previous examples reveal the diversity of approaches through which researchers depicted algebraic thinking. Most of them focus on specific dimensions of this multifaceted notion. Nevertheless, they all seem to agree that algebraic thinking 'is not all about literal symbols but rather is about ways of thinking' (Kieran, 2011, p.591).

Theories about the core aspects of algebraic thinking or algebraic activity. Despite increased emphasis on the notion of algebra in school mathematics, little research has focused on comprehensively clarifying the meaning of algebra in this context, and even less research has attended to this issue in the elementary grades, where algebra has traditionally had a limited role. Bell (1996) offered a particularly useful conceptualization of algebra as a means to express generalizations, relations, and formulas; represent unknowns; and solve equations. Usiskin (1998) analyzed algebra into four conceptions: generalized arithmetic; the set of processes used for solving certain types of problems; the study of relationships among quantities; and the study of structures. Kaput (1995) reported in his early studies five aspects of algebra: generalization and formalization; syntactically guided manipulations; the study of structure; the study of functions, relations and joint variation; and a modeling language.

More recent conceptualizations of algebra have been offered by Mason, Graham, and Johnston-Wilder (2005) and Drijvers, Coddijn, and Kindt (2011). According to Mason et al. (2005), the roots of algebra are found in: expressing generality; using and manipulating multiple expressions for the same generality; using symbols for denoting unknowns or unspecified quantities; and expressing structure as a result of expressing the general rules of arithmetic. Drijvers et al. (2011) described three main components of algebra: making generalizations through the exploration of patterns and formulas; solving equations and in-equalities with reference to specific constraints; investigating functional relationships.

The strands of algebra, as these are reflected through the above studies, provide a guide that informs the design and organization of mathematical lessons. However, all these strands do not merely synthesize a school subject. In an effort to address this misinterpretation, Lee (1997) interviewed a number of mathematicians, teachers, students, and researchers in respect to the question of what is algebra. As he highlighted, one of the themes that appeared to prevail all others was the interpretation of algebra as activity. One of the most influential developments of the past decades in respect to conceptualizing the notion of algebra as an activity is Kieran's (1996) model for synthesizing the activities of school algebra. This model encompasses three types of activities; "generational" activities, "transformational" activities, and "global, meta-level" activities.

i. The *Generational* activities refer to the generation of equations and expressions from various situations. These are considered as objects of algebra. More specifically, the Generational activities involve: (a) exploration of problem situations leading to the formation of equations containing an unknown, (b) exploration of numerical or geometrical patterns leading to the formation of generalizations, (c) exploring numerical relationships leading to the expression of rules. The field of Generational activity is associated with the role of algebra as a linguistic system for expressing meaning or as a *habit of mind* (Kieran, 2007).

ii. The *Transformational* activities refer to the transformation of expressions by applying specific rules. For example, these activities involve collecting like terms, factoring, expanding, substituting one expression for another, adding and multiplying polynomial expressions, exponentiation with polynomials, solving equations and inequalities, simplifying expressions, substituting numerical values into expressions, working with equivalent expressions and equations. Kieran (2007) emphasizes that this kind of activities are not simply skill-based. They are not just a set of techniques but they involve conceptual understanding of algebraic objects.

iii. The *Global/meta level* activities refer to activities which are not strictly algebraic in nature but where algebra is an essential tool for investigating and understanding their meaning. These activities include more general mathematical processes, such as problem solving, modeling, and working with generalizable patterns, justifying and proving, making predictions and conjectures, studying change in functional relationships, identifying structure. These activities do not necessarily involve the representation of relationships in a symbolic way.

Kieran's (1996) model for conceptualizing algebraic activity denotes that algebra is not just a topic in mathematics curriculum. It is rather a multifaceted activity which encompasses various types of tasks and ways of thinking. Algebraic thinking in particular is considered as an approach to quantitative situations which seeks to look for relationships and structure with means that are not strictly letter-symbolic. In this sense, algebraic thinking is a way for introducing students to the more abstract aspects of formal algebra (Kieran, 1996). Nevertheless, as Kieran (2004) pointed out, this model, as well as all of the previous studies reported in this section, was developed in the perspective of understanding the kinds of meaning that secondary students make when they are engaged with algebraic tasks. No direct link is made to the notion of early algebra. In a more recent paper, Kieran (2004) declared that early algebraic thinking is interrelated to the Global meta-level of algebraic activity. According to Kieran (2004), the processes involved in Global-meta level activities are considered as appropriate for the introduction of young learners to algebraic thinking, since they do not require the use of letter-symbolic forms but they provide opportunities for developing algebraic ways of thinking which will be later introduced more formally through *Generational* and *Transformational* activities.

Blanton and Kaput (2005) offered a slightly different perspective from that offered by Kieran (2004) (in Kieran, 2011). While Kieran (2004) argued that younger students could be engaged in *Global/meta-level* activities without the use of letter-symbolic forms, Blanton and Kaput (2005) placed an emphasis on the process of establishing, systematically expressing and justifying generalizations in increasingly more formal forms. Moreover, they highlighted that expressing generalizations with symbols depends on students' age and level. Kaput (2008) further offered a more coherent definition of algebraic thinking by specifying that there are two core aspects of algebraic thinking: (i) making generalizations and expressing those generalizations in increasingly, conventional symbol systems, and (ii) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. In the case of the first aspect, generalizations are produced, justified and expressed in various ways. The second aspect refers to the association of meanings to symbols and to the treatment of symbols independently of their meaning. The second aspect develops after the first aspect since students need first to explore and understand the situations where generalization occurs and then to apply specific associations of symbols. Kaput (2008) asserted that these two aspects of algebraic thinking denote reasoning processes that are considered to flow through varying degrees throughout three strands of algebraic activity: (i) generalized arithmetic, (ii) functional thinking, and (iii) the application of modeling languages for describing generalizations.

This conceptualization breaks down the wide field of algebraic thinking into major components of mathematical activity that can be integrated into teachers' instructional practices. Kaput's (2008) ideas articulated ways in which algebraic activities might be applied both in early algebra and secondary school algebra contexts. Moreover, this breakdown of algebraic thinking into specific strands seems to be helpful in organizing and synthesizing research studies that worked out different dimensions of algebraic thinking and tackling their relation to mathematical thinking and application at a classroom level.

The strand of *Generalized arithmetic* points to the traditional association of algebra with arithmetic. Specifically, it is asserted that understanding arithmetic requires thinking relationally about operations and their properties (Empson, Levi & Carpenter, 2011). Brit and Irwin (2011) also support that investigating operations in a relational way supports understanding in arithmetic, since arithmetical operations and equations are not only viewed as processes for calculation but as relational objects. Unpacking numerical operations and equations by using their properties are considered as fundamental parts of early algebra, where thinking relationally about relationships that are expressed with literal symbols is found at the secondary school level (Kieran, 2011). Hence, it is suggested that mathematics educators can organize instructional activities for helping students to become aware of the structure underneath arithmetic (e.g. Carpenter, Franke, & Levi, 2003). Accordingly, generalized arithmetic as a way for applying algebraic thinking in arithmetical settings involves:

i. using letters for generalizing rules about relations between numbers;

ii. manipulating operations and exploring their properties;

iii. generalizing numerical patterns;

iv. transforming and solving equations ;

v. understanding the equals sign in number relations. (Kaput, 1995; Blanton & Kaput, 2005)

Algebraic thinking as *Functional thinking* refers to the identification and description of functional relationships between independent and dependent variables. This approach focuses on the concepts of change and variation in situations and contexts as well as on the representation of relationships between variables (Kieran, 2011). For example, what makes a patterning activity algebraic in nature is the shift from thinking about particular quantities to extracting and using a rule or calculation method (Radford, 2011). Blanton and Kaput (2005) included in this category the development of symbol sense in order to symbolize quantities for modeling problems and operating on symbolized expressions. Functional approaches also involve:

i. the use of function machines where students give instructions by writing operations and using algebraic symbolization;

ii. comparing multiple representations in order to understand problems about rates of change (graphs, equations, tabular data) (Booth, 1984).

Blanton (2011) suggested that mathematics educators need to consider the capability of elementary schools students to reason about functions and potentially foster the improvement of their performance by organizing appropriate instructional activities. Several studies (e.g. Blanton 2008; Brizuela and Schliemann 2003; Carraher et al. 2008; Kaput and Blanton 2005; Moss et al. 2008) proposed that young students are able to use different forms of representations for being successfully engaged with functional thinking, they can describe relationships of recursion, covariance and correspondence by using symbols and words, and they can apply symbolic language for solving problems with unknown quantities.

The third strand of algebra, *Modeling* is described as the engagement of the learner in the expression and formalization of generalizations from mathematized situations inside or outside mathematics (Blanton & Kaput, 2005). From this perspective, algebraic thinking can be used as a conceptual tool for exploring modeling problems that are derived from complex realistic situations or phenomena.

The conceptualization of modeling encompasses components of algebraic thinking. For example, the representation of a realistic situation involves:

i. using symbols for developing the model;

ii. using isomorphism for illustrating the correspondence between the model and the situation;

iii. manipulating variables either in the model or the situation;

iv. re-translating the transformations between the situation and the model (Watson, 2009b).

Blanton and Kaput (2005) stated that the application of modeling languages for describing generalizations is a form of algebraic thinking that is less common to the students of the elementary grades. Modeling seems to share common features with what Kieren (19960 has described as *Global/meta level* activities. Modeling involves processes, such as problem solving, working with generalizable patterns, justifying and proving, making predictions and conjectures, and identifying structure. These activities do not necessarily involve the representation of relationships in a symbolic way. Moreover, they are not strictly algebraic but they encompass features from other disciplines of mathematics, e.g. statistics and measurement.

Algebraic thinking, reasoning, and proof. Generalization is unquestionably the main route to algebraic thinking. Yet, anticipating, conjecturing, explaining, and justifying also constitute important processes for developing algebraic thinking (Kieran, 2011). According to *Principles and Standards for School Mathematics*, two standards that should be cultivated by curriculum and instruction in all grades are algebra and reasoning and proof (NCTM, 2000). These are naturally related since both of them involve processes of generalization (Lannin, 2003). Blanton and Kaput (2005) noted that algebraic thinking in relation to reasoning and proof can take three forms:

- using generalizations to build other generalizations;
- generalizing mathematical processes or formula;
- testing conjectures, justifying and proving.

These instances of algebraic thinking come to an interplay with all aspects of algebraic thinking. Specifically, Blanton and Kaput (2005) contended that these categories reflect more sophisticated levels of the ability for thinking algebraically as a culture or habit of mind. For example, justification becomes apparent when students construct mathematical arguments to justify general claims for classes of numbers. Although younger students are not able to be engaged into formal proving, they can represent specific numerical expressions by using various representations such as drawings, models, or story contexts and extend models for justifying general claims (Russell, Schifter & Bastable, 2011).

NCTM (2000) suggested that activities such as the investigation of patterns and structures offer opportunities to young students for identifying regularities, produce conjectures about observed regularities, construct and evaluate mathematical arguments. This kind of proving activity reveals the significant role of algebraic thinking. For example, Pedemonte (2007) suggested that generalization is necessary for being engaged into an inductive argumentation. In particular, students of 12th and 13th grades manage to transform an inductive argumentation into a mathematical inductive proof only when they are constructing conjectures by means of a pattern generalization. Pattern generalization focuses on regularity on the results, and it can be visualized as: E1, E2, E3… where E is a property generalized on cases 1, 2, 3, and so on (p. 29). Similarly, Bednarz, Kieran and Lee (1996) supported that generalization, i.e. uncovering and expressing generalities in number patterns, establish a foundation not only for algebraic thinking but also for proving. Bastable and Shifter (1998) argued that tasks of the type "what is the result of adding two even numbers" or "what is the result of adding two odd numbers" can also be used in the classroom discourse in order to trigger students' participation in activities of reasoning about the validity of their ideas. This kind of discussions prompts students to reason about hypothesis and proof, as well as about the quality of their own arguments and the arguments given by others.

Hanna and Janke (1998) argued that "rigorous proof is generally considered as a sequence of formulae within a given system, each formula being either an axiom or derivable from an earlier formula by a rule of the system. This kind of proof clearly reveals the influence of algebra)" (in Pedemonte, 2008, p.386). Taking into account the growing emphasis by recent studies on the important role of algebra for communicating proofs (e.g. Healy & Hoyles, 2000; Pedemonte, 2008), algebraic proof might be considered as a significant feature of algebraic thinking. At the elementary level, the term ‘justification’ is used instead of the term ‘proof’. According to Carpenter et al. (2003) the term justification is more appropriate for considering all the diverse arguments that students may provide when they try to prove their hypotheses. Studies with students of primary and secondary grades have shown that the ability of the students for justification passes through levels where justification is based on external paradigms to levels where justification is based on examples and then to levels where justification is based on mathematical reasoning (e.g. Carpenter et al., 2003; Lannin, 2005; Sowder & Harel, 1998).

Algebraic Thinking from a Cognitive Perspective

Developmental aspects of algebraic thinking. The views of algebraic thinking reported above focused on the establishment of generalizations, taken to mean the detection and expression of structure and a growing understanding of symbolization. Nonetheless, this seems to be an ability of developmental nature. Seeing expressions as structures depends “on the ability to discern details (Piaget, 1969 p. xxv) and application of an intelligence sense of structure (Wertheimer, 1960) and also to know when and how to handle specifics and when to stay with structure” (Watson, 2009, p. 18). Regarding this consideration, learners develop algebraic thinking as they shift from simple calculations to relational thinking.

Kuchemann (1978, 1981), considering the root of algebraic thinking to be the extraction of meaning when letters are used within algebraic tasks, investigated the ways students of the secondary grades treat letters. The results of Kuchemann’s study revealed that students’ understanding of letters diverges through various levels. Seven levels describe the way students tribute meaning to the use of letters within algebraic representations, moving from having no understanding of the meaning of formal symbols to developing deep understanding of the meaning of symbolic representations as variables: (a) letters are evaluated in some way, e.g. $x = 4$, (b) letters are ignored, e.g. 5y taken to be 5, (c) letters are used for the representation of objects, e.g. b=ball, (d) letters are used as specific unknowns that have not yet been defined, (e) letters are used as generalizable numbers, and (f) letters are used as variables.

According to a research from the field of cognitive psychology (Demetriou, 1993), the abilities for relational thinking evolve over seven developmental levels and involve three component abilities: abilities of quantitative specification and representation, abilities of dimensional-directional construction and abilities of dimensional-directional coordination. The quantitative-relational abilities are biased to symbol systems enabling the individual to focus on, represent, and process the quantification-relevant aspects of reality and ignore all irrelevant aspects and properties. Algebraic competence develops at level 3, from 9 to 10 years through early adulthood. In particular, at the age of 9-10 years old, students are able to operate on simple mathematical relations, even those that are symbolically represented. For example, they can identify symbols in equations such as $8 \times b = 5$ and $a + 5 = 8$. At the age of 11-12 years old students are able to coordinate simple structures and operate on undefined structures. For example, the equation $x = y + 3$ is easy to be solved if x is given.

When students are 13-14 years old, they become able to identify complex relationships in relations, such as unbalanced proportions. Furthermore, they can coordinate complex symbolic expressions to determine the value of a variable (e.g., determine the value of x if $x = y + g$ and $x + y + g = 30$). It is also possible to quantify covariance and understand direct ratio. At the ages of 14-16 years old they are able to generalize quantitative dimensions and identify relationships between them. For example, they can understand that the equation $a + b + c = a + x + c$ is true if $b = x$. A dimension can be represented in alternative ways, which can be determined by reference to other representations. They understand the quantification of covariance of inversely proportional quantities. Finally, at the age of 17-18 years old, students search for a variety of relations.

Mason (1989) was among the first who tried to describe the development of algebraic thinking as a process that leads to the detection of structure in mathematical expressions. According to his approach, the development of algebraic abstraction (structural thinking) involves the movement from experience in manipulating objects (whether these be physical, pictorial, symbolic, or mental), to expressing this experience, to articulating the properties of such experience as expressions of generality, and subsequently manipulating such expressions to search for further properties. The actual process of abstraction is considered to lie in the "delicate shift of attention" from seeing the expression as an expression of generality, to seeing it as an object or property that can be manipulated. This does not imply that the latter replaces the former, rather, abstraction entails conceiving mathematical constructs in both ways. Hence, the development of algebraic thinking is ensured as long as the student maintains a "dual awareness of expressions both as entities or objects, and as statements about how a calculation will be performed". Mason (1989) argued that this kind of ability requires effective use of self-monitoring processes.

Sfard and Linchevski (1994) also described the development of algebraic thinking and understanding as a sequence of advanced transitions from an operational perspective to a relational perspective. Moreover, Sfard and Linchevski (1994) draw attention to individual learning by questioning the role of "what one is prepared to notice and able to perceive" (p.192) when confronts algebraic problems. Sfard (1995) and Sfard and Linchevski (1994) have found connections between the historical development of the mathematical knowledge and the development of students' mathematical knowledge. As far as it concerns algebra, it is assumed that historically there is a distinction between the representation of the unknown in equations and the use of letters for representing general solutions. Similarly, students' capability for using symbols passes through these levels. This fact highlights the relation of psychological perspectives to the development and nature of algebraic abstraction (Carracher, Schliemann, Brizuela, & Earnest, 2006).

Specifically, the main question of Sfard and Linchevski (1994) was the extent to which learners are capable of seeing and using the variety of possible interpretations of algebraic objects. According to their model, the learners initially understand algebraic expressions as computational processes. An expression, such as $4 (y + 6) + 2$ represents an arithmetical process. By performing particular operations, the symbol will obtain meaning. At this level, individuals face expressions as means for determining the value of the letter through the application of a prescribed process. The persistence in performing computations in order to take a defined result is called by Collis (1974) as the inability to accept the 'lack of closure'. The expression on the left-hand side is considered by individuals as a process, whereas the expression on the right-hand side is expected to be a product. This idea seems to be interrelated with learners' previous experience with arithmetic where the equals sign '=' triggers the articulation of a result appearing to the left of this sign. For example, students perform differently to problems that are quite similar, such as 'What is x if $2x + 7 = 45$?' and 'If $A = L \times B$ tells us how to work out A, what formula tells us how to work out L?'. In the first case, the final result is a number where in the second the final result is a formula.

In the second level, algebraic expressions are conceptualized as specific entities; they are the products of computations rather than the computation itself. The unknown is considered as a fixed value and the entire expression as one number. Letters are treated as certain unknown numbers and each side of an equation as a concrete series of operations. Nevertheless, at this level students seem to struggle when solving inequalities. An inequality requires testing the values of the component formulae and comparing the results for applying different values of the letter. Hence, in the context of an inequality, the letter plays the role of a variable rather than of a fixed value.

The third level in Sfard and Linchevski's (1994) approach refers to the passage from the algebra of a fixed-value (of an unknown) to the functional algebra (of a variable). In this stage, individuals understand the dual nature of algebraic expressions as both process and product. The symbol represents not a fixed value, but a manageable object. Nevertheless, the functional approach of algebraic representations is not easily accessible even for the more skilled students. The learners usually develop first 'pseudostructural' conceptions. While they may be able to handle a functional relationship, their actions remain instrumental. Students act as if they are handling some kind of object, but their thinking is completely inflexible and structural interpretations are unavailable. For example, many students do not understand the difference between a quadratic inequality and a quadratic equation. In a problem like $z^2 + z + 1 < 1$, students usually apply the formula for the roots mechanically. This kind of behavior demonstrates that students' thinking is not flexible enough and definitely they do not interpret the expression in a structural way.

More recently, Thomas and Tall (2001) offered a slightly more detailed conceptualization of the development of algebraic thinking. Similar to Sfard and

Linchevski (1994), their model refers to the long-term shift between working in contexts of *Simple arithmetic*, *Generalized arithmetic*, *Evaluation algebra*, *Manipulation algebra* and *Axiomatic algebra.* The movement from one level to another is framed against several cognitive difficulties. In brief, students first implement a range of different procedures in order to accomplish a process. In the level of *Simple arithmetic*, the addition of two whole numbers can be done either by step-by-step algorithmic procedures or by more compressed computation procedures. In *Generalized arithmetic*, boxes in equations as referents to the unknowns are replaced by letters. In this level, some children are not able to see algebra expressions as a process/concept because they cannot evaluate them and search for a number as an answer. Another obstacle for many learners seems to be the need for reading expressions in different orders and not only in the left-to-right order. Moving to *Evaluation algebra,* students understand symbols as manageable concepts. According to Thomas and Tall (2001), the use of spreadsheets might play a significant role in achieving this kind of understanding. In such environments, calculations and predictions can be made without the need for manipulating symbols. Furthermore, the potential for representing the same process with different procedures is better understood. For example two equivalent expressions, such as $2n + 6$ and $2 \times (n + 3)$, represent two different procedures of the process of evaluation when n is replaced by a specific value.

In the level of *Manipulation algebra*, equations involve variables that are represented by letters and they have to be approached algebraically. It has been reported that students who succeed in Manipulation algebra have "readily accessible links to alternative procedures and checking mechanisms", as well as "tight links between graphic and symbolic representations" (Crowley, 2000, p. 209: in Thomas and Tall, 2001). The range of procedural techniques leads to the construction of 'procepts' where algebraic expressions have a dual nature; they can be evaluated as a process and manipulated as a concept. On the last developmental stage, students have to reach *Axiomatic algebra* by achieving a cognitive reconstruction. This involves a major discontinuity in development since in axiomatic algebra laws are not built on experiences of the operations in arithmetic but operations have to be seen as 'genuine' laws that lead to the deduction of new properties.

Besides theories and models that describe the development of algebraic thinking through concrete levels, many researchers attempted to provide further explanations of what it means for students to have developed a deep understanding of formal algebraic symbols. For example, Arcavi (1995, 2004) sought a more detailed analysis of the notion of 'symbol sense' among secondary school students. Specifically, Arcavi identifies behaviors that illustrate what is accepted to be examples of symbol sense as soon as algebraic thinking has been developed. According to his investigation, there are six fundamental components of symbol sense. The first one refers to the development of friendliness with symbols. Symbols are readily accessible to students in order to represent relationships, generalizations and proofs. Moreover, there is a feel of when symbols are unnecessary and it is better to make use of other types of representations. The second component is related to the use of syntactic rules for solving equations with meaning and not merely as a mechanical process. Engineering symbolic expressions in order to transform one type of representation to another type is a third component of number sense. For example, students must be able to construct the symbolic expression for a desired graph. The fourth component refers to the capability of individuals for switching between various representations when they try to represent a problem situation until they find the more suitable representation. The fifth component involves the recognition of the need for checking what the symbol means during the implementation of a process and comparing the resulted meaning with those that were expected. At last, the sixth component of symbol sense involves the realization that symbols can have dissimilar roles in different contexts (e.g. in an equation symbols may represent parameters or variables) and the development of an intuitive understanding for those differences.

Lannin (2005) claims that in order to better understand the development of algebraic thinking among young students, it is of great importance to study the justifications given for the generalizations they produce as they explore patterning activities. Student justifications provide a window to view their understanding of the general nature of their rules. Generalization is found to be on the core of algebraic activity, providing a link between numeric situations and symbolic representations. Nevertheless, establishing the validity of a general statement is a challenging task for students. As stated by Lannin, there are four levels of using justifications. At the Level 0 students provide no justification or their responses do not address justification. At Level 1 students' justifications appeal to external authorities. Specifically, they are referred to the correctness stated by other individuals or reference materials. At Level 2, students justify their generalizations by providing empirical evidence. Justification is provided through the correctness of particular examples. At Level 3, students' examples are more generic. Deductive justification is expressed in a particular instance. Lastly, at Level 4 students' justifications have a deductive nature. Validity is given through a deductive argument that is independent of particular instances.

Cognitive factors that affect algebraic thinking. Available research on developmental aspects of algebraic thinking provides support to the assumption that cognitive factors might frame the development of algebraic thinking and in particular, the sequence of advanced transitions from an operational perspective to a structural perspective (e.g., Mason, 1989; Sfard & Linchevski, 1994; Thomas & Tall, 2001). As it is implied by the examples offered by Arcavi (1994, 2005) and Lannin (2005), algebraic skilled-performance depends on the acquisition of multiple capabilities. In the following section, corresponding literature pertaining the relationship of algebra and algebraic thinking with various reasoning processes or cognitive factors is reported.

Reasoning processes. English and Sharry (1996) made an effort to describe the construct that enable individuals to develop algebraic thinking, and mostly relying on Sfard and Linchevski's (1994) model, provided explanations about students' competence for expressing generality. In particular, they showed that analogical reasoning constitutes the mental source of extracting commonalities between relations and constructing mental representations for expressing generalizations. The action of noticing differences and commonalities among things is cognitive in nature and ends up with the formulation of a generalized concept that it does not completely coincide with any of its particular cases. Likewise, Radford (2008) pointed out that, when you verify that "a" is equal to "b" or that "a" is analogous to "b" (as it happens when you verify that two specific trees are equal despite their obvious differences), it means you select certain characteristics of "a" and "b" and you ignore some others.

Radford (2008) took this analysis a step further by developing a definition of the process of generalizing a pattern which encompasses various forms of reasoning:

> Generalizing a pattern algebraically rests on the capability of grasping a commonality noticed on some particulars (say p1, p2, p3,…, pk); extending or generalizing this commonality to all subsequent terms (pk + 1, pk + 2, pk + 3, …), and being able to use the commonality to provide a direct expression of any term of the sequence. (p. 84)

As the quotation suggests, this process first involves the identification of differences and similarities between the parts of the sequence – described as analogical reasoning by English and Sharry (1996). Then the commonality founded is generalized through predicting a plausible generalization as far as it concerns the following terms of the sequence. Rivera and Becker (2007) consider the stage where a plausible generalization is hypothesized to be abductive in nature; according to them it is abductive reasoning that boosts conjecturing and adopting a hypothesis that is considered by the individuals as testable. In the final stage of this process, this commonality becomes the basis for inducing the generalized concept of the sequence. Here, the role of inductive reasoning is considered as pivotal in order for students to come up with the formulation and expression of the nth term of the sequence (Ellis, 2007; Rivera & Becker, 2007). Palla et al. (2012) also suggested that mathematical induction is essential in situations where a geometrical pattern is translated into an algebraic expression.

The emphasis on reasoning forms that enable generalization processes and also personal efforts of the individual for integrating signs and meanings, illustrates the involvement of cognitive systems that facilitate individual to shift from calculating to observing a functional relationship and then expressing it. Moreover, a lot of studies supported that the difficulties that students face in algebra reflect developmental or cognitive obstacles (Carracher et al., 2006). For example, Filloy and Rojano (1989) provided historical data on the idea of the ‘didactic cut’ which takes place as mathematical thinking moves from arithmetic to algebra and students are called to act on unknown terms. Similarly, Herscovics and Linchevsky (1994) referred to the ‘cognitive gap’ which is inherent between arithmetic and algebra and it becomes obvious through the weakness of students to spontaneously act on the unknown. Given the foregoing descriptions on the developmental progression of students' ability for algebraic thinking, it becomes obvious that innate constraints might outline the time and the quality of the transition from operational to structural outlook. Nonetheless, there is a scarcity of research that examines the cognitive framework of algebraic thinking in detail (Tolar et al., 2009).

Domain-specific processing abilities and General cognitive factors of mental action. Some studies from the field of psychology have shown that working memory, three-dimensional (3D) spatial visualization, and computational fluency relate to the general mathematical achievement of adolescents and adults (Engle, Tuholski, Laughlin, & Conway, 1999; Geary, Saults, Liu, & Hoard, 2000; Reuhkala, 2001; in Tolar et.al, 2009). Tolar et al. (2009) examined the way in which these three factors might affect algebra achievement among college students. Their results demonstrated that the successful accomplishment of algebraic tasks depends on a person's computational fluency, where 3D spatial ability and working memory have lower effects. On the other hand, 3D spatial ability mostly affected the Scholastic Assessment in Mathematics (SAT-M) scores. However, computational fluency and 3D spatial ability completely mediated the effect of working memory for both algebra and SAT-M achievement. Lee et al. (2011) also indicated the important role of arithmetic and word problem-solving in setting the basis for engagement with early algebra. In an earlier study, Lee et al. (2004) also showed that the effect of general cognitive factors such as central executive, performance IQ, and literacy was small. Fuchs et al. (2012) indicated that second grade students' pre-algebraic knowledge is indirectly influenced through arithmetical skills by attentive behavior, phonological reasoning, and processing speed.

Algebraic Thinking from an Instructional Perspective

Within the field of mathematics education research, it is important to take into consideration not only the influence of cognitive skills to algebraic thinking but also the exposition of students in environments that foster the development of algebraic thinking (Tolar et al., 2009). The interaction between mental processing and educational experience seems to be reciprocal; therefore, the ways by which cognitive processes are related to performance and the ways by which educational experiences affect cognitive processes need to be specified (Demetriou et al., 2011). In this context, it seems necessary to study possible ways for engaging students to educational experiences that facilitate the emergence of algebraic thinking. In response to calls for improving students' performance in algebra, a number of research studies implemented instructional approaches to algebraic thinking with students of different ages (Rakes, Valentine, McGatha & Ronau, 2010). For example, The 'No Child Left Behind Act' (2002) called for the use of research-based strategies to practically help teachers to choose the most appropriate programs and materials for their particular settings. Moreover, the 70th yearbook of NCTM (Greenes & Rubenstein, 2008) was focused on topics such as the teaching and learning of algebra and suggested practices for improving algebra instruction at the classroom level.

Diverse teaching approaches targeted the learning of one or more concepts and skills that are considered as forms of algebraic thinking (Watson, 2009). Among them, there is an agreement that the development of algebraic thinking affects understanding in higher mathematics (Rakes et al. 2012). Moreover, these studies are considered as interventionist because they do not suggest procedural manipulation of algebraic tasks. Nevertheless, the definition of algebra in this body of literature remains unclear since research takes place in particular aspects, contexts and materials. As Watson (2009) highlighted, research is sporadic and runs throughout specific perspectives. In respect to the issue of early algebra, Canadas, Dooley, Hodgen and Oldenburg (2012) pointed out to the need for re-contextualizing early algebra and clarifying the contribution that each intervention makes to the field as a whole through stronger literature reviews.

This section seeks to provide a valuable synthesis of research on algebra and algebraic thinking instructional practices by investigating what kind of concepts and skills have been studied and how effective have the particular methods been at improving algebra achievement. Three main categories of contexts and materials that aimed to empower students' algebra knowledge and algebraic thinking skills can be found through literature:

- Equation-centered approaches: relationships between expressions are described as equations and sets of techniques for handling, transforming and solving equations in order to find unknown values or represent relationships between variables are introduced.

- Functional thinking approaches: relationships of co-variation and correspondence are analyzed in order to express generalities; functions and their inverses are expressed using multiple representations.
- Modeling situations approaches: variables and their co-variation are identified through the investigation of mathematical problems or real situations.

In the following section, related studies to the above categories of teaching approaches are reported.

Equation-centered approaches. Numerous teaching experiments were focused on traditional problems of representing and finding unknowns. In a teaching experiment with third grade students, Carraher, Brizuela, and Schliemann (2000) used suitable problems, such as Tom is 4 inches taller than Maria; Maria is 6 inches shorter than Leslie, with the aim to introduce the notion of unknown and the need for representing it. In this experiment, although students were puzzled several times, they demonstrated an ability for expressing with a letter a number that is not yet known. Bastable and Shifter (2008) also supported that students become able to construct generalizations about operations and methods when they are provided with appropriate support.

Blanton et al. (2011) suggested that young students are able to generalize properties of operations in supportive classroom environments. This study suggested that students should be encouraged to observe patterns on the way numbers behave when they investigate addition, subtraction, multiplication, and division. For example, students should understand that the series of numbers we add does not have an effect on the final result. At the start of such an investigation, young students are able to express the commutative property of addition through the use of language. This property will become more formal as students grow up and algebraic symbols will be used for expressing the relationship between any two numbers. Therefore, the verbal expressions of students for representing the commutative property of addition will become later on a formal expression of the type $a + b = b + a$.

Blanton and Kaput's (2005) experiment invested on the professional development of teachers for being sensitive to the ways by which they could promote algebraic thinking in their classrooms. In this study, teachers were helped in developing 'algebra eyes and earls' in order to make use of everyday mathematical experiences for drawing students' attention to the algebraic nature of arithmetical activities. Moreover, the mathematics lessons were formulated in order to include tasks that reflected all of the three strands of algebraic thinking as these were described by Kaput. The results of their study showed that they indicated that primary school children were able to invent and solve "missing number" sentences using letters as placeholders, symbolize quantities in patterns, devise and use graphical representations for single variables, and some could write simple relations using letters, codes, "secret messages" or symbols.

In an earlier study, Sutherland and Rojano (1993) involved 10 to 11 years old students in the construction of equivalent expressions using spreadsheets. Their results indicated that spreadsheet technology can assist students to make connections between their informal ideas and the formal algebraic representations. Students seemed that were helped in understanding the meaning of a variable as a quantity that changes when they clicked on a cell that represented a particular case of a generalization. This act supported the construction of a rule which related two or more quantities. In addition, spreadsheets promote flexible and recursive reasoning which allow the emergence of generalization in problem situations.

The CARAPACE study (Kieran, Boileau and Garancon, 1996) also investigated the ways by which 13 years old learners confront graphs and values. The CARAPACE environment involved graphs, data, situations and functions that supported the understanding of equality and equivalence of two functions and the manipulation of equations. It was found that the combination of multiple representations assisted the manipulation of word problems and applications of functions.

Other studies showed that manipulatives can also provide students with rich opportunities for investigating the structure underneath mathematical relationships. For example, rod or diagrams are extensively used in Singapore (Greenes and Rubenstein, 2007) to represent part/whole comparisons, reasoning, and equations.

These manipulatives appear to scaffold students' thinking from actual numbers to structural relationships of addition or repeated addition. Statements in the problem are translated into equalities. These equal lengths are constructed from rods which represent both the actual and the unknown numbers. The rod arrangements or values can then be manipulated to find the value of the unknown pieces. This approach which is used for introducing 11-12 years old students to the notion of equations with variables is similar to the use of Cuisenaire rods in Europe.

The use of technological tools, like graphing calculators, seems to have a critical impact on understanding the notion of equivalence in algebraic expressions and finding the unknown term in secondary school students. For example, Kieran and Saldanha (2005) demonstrated the improvement of a group of 10^{th} graders in considering equations as objects with meaning by using computer algebra systems (CAS). As they reported, graphic representations promoted discussions about the equivalence of expressions not only in the level of purely numerical reasoning. The interpretation of CAS outputs influenced investigations of the concept of equivalence that do not normally occur in traditional algebra classrooms.

Functional thinking approaches. Several research studies suggested that functions must have a prevailing role in algebra instruction (e.g. Blanton, 2011; Schwartz, 1999; Schwartz & Yerushalmy, 1992; Chazan & Yerushalmy, 2003). As described in the previous section, one of the main abilities required for the development of algebraic thinking is the manipulation and understanding of letters as variables rather than as yet-unknowns that need to be calculated. Placing functions at the center of algebra instruction entails the systematic exposition of students to tasks that employ the idea of letters as variables and hence creates opportunities for students to pass from levels of performing calculations to levels of operating with rules for functions (Kaput, 1998). Blanton et al. (2011) suggested that functional thinking requires the establishment of generalizations in respect to the relationships between quantities that continuously change, the expression of such relationships with words, symbols, tables and graphs, and reasoning through these representations about the structure that underlie functions.

A number of studies approached functional thinking through the exploration of sequences of patterns, where students are asked to describe a general term in the sequence. The expectation is that this kind of tasks generates the need for algebraic symbolization. Moss, Beatty and Macnab (2006) worked with 9 year old students in a longitudinal study and found that developing expressions from pattern sequences was an effective introduction to understanding the nature of rules in “guess the rule” problems. Most of the participants in this study seemed to be able for articulating generality. Moss and Mcnab (2011) summarize that pattern activities enhance students’ understanding of functional relationships, act as a basis for moving to more abstract and general mathematical constructs, and empower students’ abilities for hypothesizing and proving.

Cooper and Warren (2007) and Warren and Cooper (2008) also used patterning activities to teach elementary school students ways for expressing generalizations, using various representations, and comparing expressions and structures. This method seemed to have an impact on developing meaning about the use of algebraic symbolization. Specifically, Cooper and Warren’s intervention emphasized the use of algebraic conventions and notations and the underlying operational nature of mathematical expressions. Besides using patterns, they also introduced students to the concept of inverse operations through function machines and a range of mental arithmetic methods.

The findings of Carraher, Martinez and Schliemann (2007) from a one year teaching experiment showed that third grade students are able to make generalizations when they work with variables in arithmetic problems. More specifically, it was shown that instructional environments should support the transition of students from generalizations that are based on term-to-term empirical data to generalizations that are extracted from understanding the mathematical relations between the position of the term and the term which reflects a relation between an independent and a dependent variable. Similarly, Steele (2007) demonstrated ways by which 12 to 13 years old students could manage this transition when they used various forms of data such as pictorial, diagrammatic and numerical. Rivera and Becker (2007) also designed a teaching experiment for studying middle school students’ understanding of sequences of growing figural patterns. They found that the deconstruction of diagrams leads more easily to the identification of a functional formula rather than reasoning inductively from numbers.

Radford (2008) also used geometrical patterns for introducing the concept of generalization. The teacher has the role of drawing students' attention to the structure of the pattern and to the need for identifying a rule that is repeated. According to Radford (20087) two basic processes take place when students explore geometrical patterns. First, previous experiences of the students guide their action to the tasks through a process that is called 'iconicity'. Specifically, students identify similarities and differences between previous experiences and the new situations that are called to investigate. These similarities and differences constitute the basis for articulating a generality. The second process is called 'contraction'. Through this process students are focused on the important parts of the task and their attention is removed from facts that are irrelevant.

Booth (1984) was among the first that showed that lower secondary students working with function machines were capable to construct proper instructions for the machine by writing operations in order and using proper algebraic syntax where necessary. At the end, the students were able to understand the whole expression of the structure underlying the function of the machine. Schliemann, Carraher and Brizuela (2006) also used function tables in order to represent the relationship between the number of items and their price for developing; the aim was to develop third grade students' reasoning about variable quantities and their interrelations. According to their results, students were able to attend the invariant relationship between the values in the first and second column, after they were introduced to a guess-my-rule game. Moreover, the introduction of letters for representing any value of the first variable in a function table seemed to be helpful for emphasizing the existence of a general rule that relates the two variables. Blanton (2008) used function machines for helping young students in searching for the underlined 'secret' rule of the machine. The main purpose of these tasks was to guide students in order to observe and understand the relationship between input and output values.

Likewise, Warren, Cooper, and Lamp (2006) used function machines with 9-10 years-old students. Their results revealed the ability of the students for developing functional thinking and expressing their thoughts through verbal or symbolical representations. Stephens et al. (2012) studied the way 8 to 11 years-old students' functional thinking was influenced by an intervention which focused on early algebra concepts. Their results demonstrated that students who were taught mathematics within the perspective of early algebra improved their performance in tasks such as the development of a functional table, the detection of repeated patterns and understanding linear functions. The use of function machines was also found to be a valuable tool in developing functional thinking.

Goodraw and Schliemann (2003) investigated the impact of graphical representations on students' understanding of functional relationships. Students of 8 years old were able to construct graphical representations and to understand the way coordinates are placed on the grid and form a graphical representation. Moreover, students were able to relate a graphical representation with the corresponding functional relationship and to justify their selection.

Modeling as a domain for expressing and formalizing generalizations. A field that has not yet been investigated in extend is the use of modeling languages for representing mathematical relationships that arise through real situations. Suh and Moyer (2007) examined the learning of algebra in a third grade classroom, by investigating the representation of variables through the use of algebraic models. Their project involved two groups of students that in a course that lasted a week long were engaged to different kinds of algebraic models. More specifically, the students were exposed to virtual and physical manipulative situations and were encouraged to use informal strategies for expressing relational thinking. Their results showed that both kinds of manipulative models were effective in supporting students' algebraic thinking.

An important feature of interventions that aimed to develop students' ability for using symbolic representation as models for representing relationships in algebraic tasks is the application of contexts that have meaning for the students (Blanton & Kaput, 2011). Bodanskii (1991) used problem situations in order to investigate the ability of young students of the first and second grade to use algebraic language. The results of this study indicated that young students performed better than older students of the sixth and seventh grades. The overall conclusion was that students is better to be introduced to symbolic systems for representing equations at the age of six rather than the age of 11. Similarly, Schliemann et al. (2013) suggested that within supportive mathematical activities, young students become capable for translating algebraic problems into formal symbolic representations.

Psychological Research on Mental Causation

Psychological theories. In order to enhance what is currently known in respect to the notion of algebraic thinking, this study aims to investigate its relationship with a set of psychological factors. Such investigations are worthy from an educational perspective. “What skills/thought processes do we need to emphasize” about specific mathematical topics and “What are the mathematical concepts and reasoning processes that prepare and enable students to learn and use algebra?” are questions listed among the research-guiding questions of the National Councils of Teachers of Mathematics’ Research Agenda (NCTM, 2012). Taken together, these two questions imply that mathematics education research calls for investigating in depth the factors that affect the development of students’ algebraic thinking. In this vain, associating mathematics education research and psychological research might better inform the way students’ multiple forms of algebraic thinking unfold under the control of cognitive mechanisms that enable students to apply specific reasoning skills and processes.

In this perspective, this study proposes that algebraic thinking could be investigated in terms of its relation to interrelated systems that represent mental actions for processing information. The findings from previous studies both from the field of mathematics education and psychology suggested the important role of cognitive resources in nurturing the development of algebraic thinking. Although enlightening, these studies examined the way in which algebra performance depends upon specific cognitive constructs, as these were retrieved from other available studies pertaining general mathematical performance. Their research was not designed on the basis of a theoretical model of the mind functioning, which evaluates a range of both general and domain-specific factors as well as different forms of reasoning. Moreover, their algebra test was not based on a theoretical framework which captured all of the different forms of algebraic thinking. For example, Fuchs et al.’s (2012) test of pre-algebraic knowledge included only two types of problems; mathematical equivalence problems with letters standing for missing quantities and function tables. As it was described previously, equivalence problems constitute only one type of algebraic thinking tasks that Blanton and Kaput (2005) included in their description of generalized arithmetic. Similarly, functional thinking is not merely measured by function tables but through multiple representations that tackle the concept of variance and change.

Recognizing both the importance and challenge of better understanding the nature of algebraic thinking from a cognitive perspective, accounts of both general cognitive factors of mental action, reasoning processes and domain-specific processing abilities need to be taken into consideration. On the one hand, analyzing the importance of cognitive factors such as working memory could help teachers in enhancing their students' advancement. For example, Stylianides and Stylianides (2008) suggested that teachers should prevent the unnecessary usage of working memory when students encounter proving tasks; at the same time, they should foster the development of strategies for managing personal working memory capacity. On the other, understanding the individuality of children's information processing styles establishes implementation of instructional practices that guide effective learning (English & Watters, 1995). The importance of information processing in general mathematical achievement is not a new idea. Battista (1994), for example, has demonstrated that high performance in mathematics depends on the ability for interrelating two specific modes for processing information, spatial and verbal processing (Battista, 1994). Brown and Presmeg (1993) have also shown that mathematical achievement in general is related to spatial ability. English and Watters's (1995) study indicated that students with high spatial and verbal-logical ability have better performance in scientific problem solving. In particular, they have shown that inductive reasoning in scientific problem-solving is strongly correlated to integrating information in a holistic or spatial mode. However, there appear to be comparatively few studies that have examined children's information processing modes in relation to algebraic thinking.

To recap, a psychological theory that clarifies mental causation could be useful in setting algebraic thinking into a framework of analysis from a cognitive standpoint. In the section that follows the psychological theories of Luria, Case and

Demetriou and colleagues will be reviewed in order to depict the set of factors that seem to formulate individuals' cognitive skills and educational behavior.

Luria's neuropsychological theory of information processing. One of the most influential developments of the past decades in neuro – psychological research is Luria's model of information processing. This theory (Luria, 1973) provides explanations of individuals' foundations of cognitive functions in overall cognitive processing. In brief, this theory describes three hierarchical functional brain units: the arousal unit, the sensory-input unit and the organization and planning unit. Although the interaction between all of the three units affects any behavior, Luria's model hypothesizes that certain aspects of information processing are associated with each unit.

The first unit comprises the reticular activating system. It is linked with states of consciousness and controls sustained attention. The role of the second unit is the collection, processing and storage of information. The third unit is related to the integration and organization of outputs and includes programming, regulation and verification of information. Luria argues that it is in the second unit that any concrete experience converts into abstract thinking. Specifically, this theory suggests that both verbal and non-verbal information can be processed either simultaneously or successively in the sensory-input unit. In simultaneous processing each piece of information is immediately accessible in relation to another. Successive synthesis refers to information processing in a time dependent sequential mode. Das and Varnhargen (1986), based on the framework provided by Luria, developed a model of information processing that postulates two formats of information synthesis; a simultaneous, quasi-spatial format or a temporally organized format irrespective of the mode of information presentation to the sensory receptor. The way information is processed is influenced by individual's features, the level of attention, the nature of the tasks and their interactions.

Many researchers have adopted Luria's neurophysiological theory in order to explore learning in various content areas. For example, Das and Verhargen (1986) have shown that the capacity to process information in a simultaneous format is correlated with cognitive skills that are important for Piagetian tasks such as conservation, transitive inference and class inclusion. Harris and Wachs (1986) examined the relationship between Luria variables and mathematical achievement. It was found that high scores on the simultaneous processing factor are correlated significantly with success in Scholastic Aptitude Test (SAT) Math scores. Similarly, Watters (1993) indicated that simultaneous synthesis significantly predicts high achievement by 10 years old children in scientific reasoning. Wang, Georgiou and Das (2011) examined children's reading skills, indicating that successive processing predicts reading accuracy and fluency through the effects of phonological awareness whereas simultaneous processing predicts reading accuracy and fluency through the effects of orthographic knowledge. Harris and Wachs (1986) have investigated the relationship between simultaneous and successive synthesis and writing skills. It was found that high successive processing is correlated with fewer sentence errors. High simultaneous processing is also correlated with better performance in indicating relationships between sentences and paragraphs.

Moreover, the psychometric model stemming from Luria's research offers much insight into the individual differences that promote or restrict the expression of spatial ability in tasks that demand high spatial ability. These differences are sought in students' individual information processing styles. Watters and English (1995) have examined the relationship between competencies in scientific problem solving (syllogistic and inductive reasoning) and children's levels of simultaneous and successive synthesis. In this study, subjects were administered a test battery developed by Fitzgerald (1971) which included Matrix Test A and Matrix Test B for measuring simultaneous processing and Number Span, Word String Test and Letter Span Test for measuring successive processing. Furthermore, simultaneous processing was also measured by the Raven's Coloured Progressive Matrices (Raven, Court & Raven, 1986). Subjects were also presented with two tasks which measured reasoning, one for deductive reasoning and one for inductive reasoning. Analysis of the data revealed a significant correlation of deductive and inductive reasoning with both simultaneous and successive synthesis. The strongest correlation was found between simultaneous processing and inductive reasoning. As Watters (1993) claims,

> Simultaneous processing involves information processing in a way that allows linkages to be made independent of temporal limitations. At the perceptual level this is synonymous with spatial processing. At higher cognitive levels this may be the neurological basis for the

> generation of mental models (Johnston-Laird, 1983; Gentner, & Gentner, 1983), of mental capacity (Halford, 1991; diSessa, 1983). (Watters, 1993, p.14).

Given the findings of studies on simultaneous synthesis and successive synthesis which indicate the relationship between individual characteristics and the demands of various learning tasks, an important question arises: which could be the roles of simultaneous (spatial) and successive (verbal-logical) abilities in algebraic thinking. According to Watters and English (1995), there is a need for being aware of the relationships between cognitive characteristics such as information processing styles and scientific problem solving. More specifically, they recommended that reasoning by analogy and relational understanding, as "the establishment of a schematized abstraction of experiences involving a rich network of connections between concepts and incorporating access to problem solving strategies" (p. 11) can be investigated through the lens provided by Luria's psychometric model.

Case's developmental theory of central conceptual structures. Case's theory describes developmental changes in representational and information processing capacity. Development is considered as the progressive construction of higher order control structures and is influenced by the resources of working memory, which also increase as the development moves from a stage to another (Case, 1985). This theory undertakes that conceptual understanding in any domain is actively constructed by the children. In this outline, progress on cognitive development depends on the children's ability for reflecting and rethinking about their own conceptual understanding. Through this procedure, children become able to reorganize their existing structures and then consolidate a new formed structure into a better and more complex coherent structure.

According to Case (1985), there are four stages of development or structures of thinking with three sub-stages within each. However, this theory highlights that developmental stages are influenced by the special characteristics of a particular domain of thought. At the beginning of each stage, a new type of structure is assembled, but it can only be applied in isolation. At the second stage, two such units can be applied in succession, but cannot be integrated into a coherent structure. Finally, at the third stage, two more structures can be applied simultaneously and integrated into a reasonable system. As a result of this integration, the system acquires the general set of properties that Piaget referred to with terms such as 'reversibility' and 'compensation'. Another result is that the system at this stage becomes the base for further progress at the next stage.

As stated by Case, the stages arise at approximately the same ages as Piaget's stages. Nonetheless, these developmental stages are labeled differently. The first stage is the *sensorimotor stage* (0-2 years old), which is characterized by children processing sensory input and thinking in respect to the physical world and the physical impact that they can have on their own environment.

At the second stage, known as the *inter-relational stage* (2-5 years old), children are able to recognize the relationship between two action-reaction units, such as the fact that by pushing a door, the door will open, while by pulling a door, the door will close. Once the child is able to differentiate, coordinate, and consolidate these two action-reaction units, the child becomes capable of what Case signifies as 'inter-relational' thinking. Furthermore, at this stage, children can also understand the effects of adding a door stop to the door. For example, they realize that the door will open and close in a different way due to the presence or absence of the door stop. Another example of capabilities of the children at this stage is the recognition of the effects or outcome of having a heavy weight on one side of a balance beam and a light weight on the other side.

Finally, the children move to the *dimensional stage* (6 to 11 years old). At this stage, children are able to coordinate their conceptual structures for dealing with causation. For example, in balance beam scenarios, the children develop the ability to recognize and anticipate the outcome of having different weights on two sides of the fulcrum of the balance beam. Consequently, in the process of learning to recognize and cognitively manipulate two relationship structures, the child is able to consolidate the inter-relationship functions. At this phase, the children also become able to understand the physics of a balance and the impact of gravity on objects that have different weights. Moreover, children begin to understand the concept of weight as a quantity instead of a characteristic of physical appearance (Case, 1991). This ability becomes obvious when the child focus on the actual value or number of weights on each side of the balance beam, instead of simply arriving at conclusions based on which side “looks” heavier. By the end of the dimensional stage, children can further understand the relationship between two such dimensions, such as the relationship between number and weight. For example, they can relate the number of weights on each side of a balance beam, and the distance of weights placed on each side of the balance beam.

As soon as the differentiation, coordination, and consolidation of two or more dimensions are achieved, the child moves to the fourth developmental stage, known as the *vectorial stage*. Within this final stage, children enter a second sub-stage of vectorial operations, in which they become able to coordinate two dimensional structures. For example, they can coordinate the type of dimensional structure used for the weight-distance effect on the balance beam, and another dimensional structure such as the concepts of fractions and ratios (Case, 1991). Finally, in the third sub-stage of vectorial operations, the child is able to understand abstract systems in which there are no concrete referents to a problem. In the paradigm of the balance beam task, this ability is reflected when the children convert two ratios of weight or distance to two new ratios with a common denominator. The children compare two new abstract terms to draw a conclusion. Therefore, they can successfully predict which side of the balance beam will go down. Throughout all of these specified stages, development proceeds through a recursive process.

Case (1996) argued that a set of *central conceptual structures*, which consist of core semantic units and relations within specific domains or modules of knowledge, is responsible for the manipulation of domain-specific phenomena. These conceptual structures provide the representational units which ensure the function of the control structure that were described in the previous developmental trajectory. As Case (1996) clarifies, processing with the units of a conceptual structure, and the development of conceptual structures, is controlled by the general stage model outlined above.

Central conceptual structures are needed for the functioning of control structures in a particular semantic domain. These structures are domain specific in their semantic development, thus accounting for domain-specific learning and developmental results. Consequently, the general development of a particular semantic domain is constrained by the domain general control structure capacities and constructive possibilities. Forms of information processing and developmental constructions, then, are domain general, while the representational units with respect to which that information processing occurs are domain specific.

Case's model is an information processing model, in which the information processing control structures are hierarchically organized in the stages and sub-stages mentioned above, and the semantic elements which are processed are modularized into central semantic domains. This model can be used in order to investigate levels in students' thinking which are described both in the perspective of time-constraints and explain which are the students' abilities in each stage. Moreover, the role of central conceptual structures in the development of students' thinking can be studied.

Case in collaboration with other researchers has extensively used his theory for investigating the way many mathematical concepts develop, implying that a psychological theory is useful for uncovering features of students' thinking and learning. For example, Griffin and Case (1996) developed a research-based mathematics program, known as *Number Worlds*, for testing this developmental theory. This program investigated the progress of students at risk in developing number sense through their participation in rich activities of making connection and exploring concepts. The application of Case's developmental theory ensured that number sense concepts were introduced to students in an appropriate sequence. The results of this program indicated that students in the control group were experiencing a developmental gap both at the beginning of kindergarten, and at the beginning and at the end of first grade. On the contrary, the students at risk of the experiment group presented better results at all measures of their number sense development. The success of the *Number Worlds* program provided support to Case's psychological theory and the benefits of applying this theory in educational settings. It has been highlighted that teaching number sense through particular instructional principles and set of tools drawn from this theory, seems to be effective and powerful (Bransford, Brown & Cocking, 1999).

Case and Moss (1999) have studied the development of students' understanding of rational numbers. Specifically, this research study used Case's developmental theory for proposing a new curriculum in respect to the teaching and learning of rational numbers. Following the four stages of development of the theory, the first topic of this curriculum was the concept of percent which was introduced in a linear-measurement context. The idea of halving was also emphasized. At the next phase, decimals were introduced. At the last stage the fractional notation as another form for representing decimals was introduced. The researchers compared the results of teaching according to their proposed trajectory in respect to the results of a group of students that followed the traditional curriculum. The results of this study showed that the students in the experiment group demonstrated deeper understanding of rational numbers. These students seemed to be less dependent on whole number strategies when they solved simple problems, and they used more often proportions for justifying their answers. As far as it concerns conventional computation skill, no differences were found among the experiment and the control group.

The overarching theory of the architecture and development of the mind. Demetriou and colleagues (2002; 2011; 2015) introduced the construction of a unified theory of learning, understanding and development, whereby findings and concepts from the psychology of intelligence, the psychology of cognitive development and cognitive psychology are integrated. In brief, they have proposed a model of development and education that is based on the tenet that the construction of computational models could assist researchers' attempts for understanding cognition (Hunt, 2012). According to this theory, the human mind is organized into four systems which function differently during problem solving tasks. However, these systems interact dynamically and changes in one system during development are related to changes in other systems.

(1) The first system is comprised by a set of processing systems, known as *Specialized Structural Systems (SSS)*. These involve information processing mechanisms that specialize in the representation and processing of information coming from different environmental domains. Five domains of thought are identified: categorical, quantitative, causal, spatial and social thought. Each of the SSS involves specific logical processes.

(a) The Qualitative-Analytic system specializes on the representation and processing of similarity and difference relations. Its functioning is based on the specification and disentangling of the properties that may co-define the mathematical objects. The abilities required in the Qualitative-Analytic system contribute to understanding mathematical concepts that are characterized by the inclusion relations connecting the elements of a hierarchy by the horizontal or intersection relations or by the sequential and dimensional structure.

(b) The Causal-Experimental system is specialized on the processing of causal reality structures. The abilities related to this system are combinatorial abilities, hypothesis formation abilities that enable the induction of predictions about possible causal connections on the basis of data patterns, experimentation abilities that enable the conversion of hypotheses to experiments and model construction abilities that enable the mapping of experimentation's results with the original hypothesis in order to reach an acceptable interpretation of the data.

(c) The Spatial-Imaginal system is referred to the visualization of aspects of reality by the "mind's eye" as integral wholes and processed as such. This system involves abilities such as mental rotation, image integration, and image reconstruction. It also directs the activities which are related to location, orientation and experimentation in space.

(d) The Verbal-Propositional system is concerned with the formal relations between mental elements. This system deals more with reasoning, including induction and deduction and involves the abilities of distinguishing between the contextual and formal elements, discarding irrelevant information, and the abilities of meaning construction.

(e) The Quantitative – Relational system is concerned with abilities of construction and reconstruction of the quantitative relations between reality elements varying along one or more dimensions as well as to inter-relate the dimensions themselves. As a representational system, is biased to symbolic representations which enables the thinker to focus on quantitative properties and relations and ignore those properties and relations which are irrelevant to quantitative processing.

The development of each of the five systems runs along three dimensions: complexity, abstraction and flexibility. As individuals grow older, they are able to handle more complex relations among representations of a situation. These relations are further abstracted as new representations. Furthermore the treatment of these relations becomes fluid and flexible since individuals are able to thought either on the specifics of particular representations or on their general form. Hence, "development of the SSS is a continuous process of emergence, differentiation, and integration of new representations" (Demetriou, Spanoudis & Mouyi, 2011, p. 605).

(2) The second system, called the *Representational Capacity System*, refers to information processing potentials which are related to the number of information and mental operations that the mind is able to activate simultaneously. Working memory and speed are basic cognitive resources for the operation of processes in the representational capacity system, such as analyzing and interpreting information.

(3) The third system, known as the *Inference System*, involves reasoning processes for transferring meaning from one representation to another. Reasoning by induction, deduction, analogy and abduction are some of the different inferential mechanisms that are used during the transfer of information from an initial representation to a target representation. These types of reasoning are related to each other by common inferential processes which emerge as a separate level in hierarchical models of cognition; this level interacts with several of the SSS.

(4) The fourth system, called as the *Consciousness System*, includes functions and processes oriented to self-monitoring, self-representation, self-regulation, reflection and recursion. These are core mechanisms of consciousness that are responsible for identifying the goal in a problem solving situation, evaluating the progress of the process and controlling the inconsistencies between the current status of the process and the targeted goal.

The investigation of cognitive capacities that allow individuals to reason mathematically has been a timeless issue. As far as it concerns the concept of Specialized Structural Systems (SSS), these were found to be related to the general mathematical performance of 12 to 18 years old students (Demetriou, Christou & Pitta-Pantazi, 2003). In another study, Pitta-Pantazi et.al (2011), analyzed mathematical ability according to the five SSS, in their effort for identifying mathematically gifted students. Moreover, it was shown that each of the SSS is predisposed in a different way to processes of deduction and induction. The Qualitative-Analytic system is closely related to deductive reasoning rather than inductive reasoning whereas the Causal-Experimental system more frequently makes use of inductive reasoning processes (Kargopoulos & Demetriou, 1998). Induction and deduction processes play an important role while students are engaged to mathematical proof and proving activity (Stylianides & Stylianides, 2008).

Cognitive factors which are included in Demetriou et al.'s (2011) theory were found to be connected to skilled mathematical performance. Mathematical achievement, for example, has been associated to the ability of the individuals to interrelate spatial images and verbal propositions (Krutetskii, 1976; Hermelin & O'Connor, 1986; Bishop, 1989; Battista, 1994). The influence of visual-spatial abilities on children's mathematical achievement seems to be low, when mathematical activities involve computations and problem-solving (Friedman, 1995). However, the effect of spatial ability, when the visual-spatial tasks include mental manipulation of three-dimensional (3D) objects, seems to be higher among adolescents and adults, when assessment involves higher-level mathematical skills rather than arithmetical computation (Friedman, 1995; Casey, Nuttall, & Pezaris, 1997; Reuhkala, 2001). In addition, three-dimensional (3D) spatial processing is associated with working memory (Miyake, Friedman, Rettinger, Shah, & Hegarty, 2001). Individuals with higher working memory capacity may be more proficient at processing spatial information than those with lower working memory capacity, and, as a consequence, perform better on mathematical problems that involve spatial processing.

Furthermore, it has been well documented by literature that components of the representational capacity system, such as working memory and the consciousness system, such as self-monitoring and self-regulation mediate cognitive power (Case, 1992; Pacual-Leone, 1970; Mouyi, 2008). It was also found that these systems communicate with the SSS and the inference system. For example, intelligence depends among others on the efficiency and accuracy of the activated SSS as they deliver their content for representation to the representation system (Demetriou et al., 2011).

Choosing a psychological theory. Demetriou, Spanoudis and Mouyi (2011) provided explanations about the cognitive structures and processes of the human mind, including accounts of the changes in cognitive organization during development. Their theory is considered as appropriate in order to investigate the relation between cognitive factors and algebraic thinking. It has been selected among other psychological theories such as Luria's neuropsychological theory of information processing (Luria, 1973) and Case's developmental theory of central conceptual structures (Case, 1992) for several reasons.

1. The theory describes an overarching paradigm about the architecture of the mind, which integrates influential and widely accepted theories within the field of cognitive, developmental and, differential psychology. More specifically, it combines developmental theories about representational and information processing capacity (e.g. Case, 1985; Luria, 1973) and the experiential structuralism theory (Demetriou, Christou, Spanoudis, & Platsidou, 2002) about the development of thought and consciousness.

2. The theory is strongly supported by extended empirical work, which includes empirical studies with school students (e.g. Demetriou, Kyriakides & Avraamidou, 2003; Demetriou & Kyriakides, 2006; Elia, Gagatsis & Demetriou, 2007; Panaoura, Gagatsis & Demetriou, 2009; Pitta-Pantazi, Christou, Kattou & Kontoyianni, 2011).

3. The theory has recently offered implications for educational applications, suggesting that education plays a crucial role in the development and establishment of cognitive processes and mechanisms (Demetriou, Spanoudis & Mouyi, 2011; Demetriou, 2015). Specifically, the overarching theory of cognitive development and organization denotes that the human mind operates both with general cognitive structures and processes that underlie understanding, problem-solving and learning across different domains and with processes that are domain-specific. Furthermore, it is highlighted that all general mechanisms and all SSS interact variably with the different knowledge domains to be mastered at school (Demetriou et. al., 2011). This suggests that educators can potentially foster the development of all these processes depending on four dimensions:

> (1) developmental time (i.e., the age and developmental condition of the students), (2) educational time (i.e., the grade and prior educational experience and knowledge already attained as specified in the curriculum), (3) the particularities of the subject matter or knowledge domain concerned, and (4) the structural and procedural aspects of education that are crucial for learning, such as the

> education of the teachers, teaching methods, technological support of learning, etc. (Demetriou, et al., 2011, p.628).

The aforementioned recommendation implies that a considerable amount of research work needs to be done in order the tenets of the theory to be integrated into a coherent conception of the instruction of the various curriculum subjects. Nevertheless, as Hunt (2012) points out, this model is based on the description of interrelated systems as mental actions for information processing rather than on neuroscientific evidence that explain the neural bases of cognition. In this way, the model becomes useful for educators and researchers since it provides a framework for generating further, more specific and testable models of specific situations (Hunt, 2012). In the same line of thought, Anderson (2012) supports that Demetriou, Spanoudis and Mouyi's (2011) recommendations laid the ground for the development of more concrete and systematic connections between psychology and teaching and learning.

This dissertation aspires a more coherent conceptualization of a specific aspect in mathematics performance. This purpose corresponds to the clarification of the third and fourth dimension reported above for specifying the particularities of the subject matter or knowledge domain concerned – in this case of algebraic thinking - and the structural and procedural aspects of education that are crucial for learning.

Tolar et al. (2009) proposed that there are several theoretical reasons for hypothesizing the connection of a variety cognitive factors to the development of algebraic thinking. As they contended, working memory could be related to algebraic thinking, since when individuals solve algebraic problems, they need to maintain multiple conceptions of mathematical expressions and to retrieve previous algorithmic knowledge. As far as it concerns, spatial ability, Tolar et al. (2009) argued that algebraic tasks that include functional relations need to be manipulated mentally through the incorporation of spatial representations. Moreover, patterns need to be represented graphically. Their study, indeed verified the relationship of these cognitive factors to algebra achievement. Nonetheless their research was based to these hypotheses and was constrained to the mere investigation of these two factors in combination with computational fluency. This dissertation hypothesizes that insights into the relationship between algebraic thinking and a range of cognitive mechanisms can be gleaned from the theory of Demetriou et al. (2011). This hypothesis is in alignment with Demetriou et al. (2011) who suggested through their theory that "the possibilities afforded to an individual by a particular profile of processing and inferential possibilities must be invested in readily available domain-specific, socially and culturally relevant skills and knowledge" (p. 628).

Summary. In this chapter, a review of theoretical and research studies on students' ability for algebraic thinking has been presented. Specifically, the literature review focused on theories and research studies that aimed to describe the concepts of algebra and algebraic thinking. The traditional distinction between algebra and arithmetic, and the notions of pre-algebra, early algebra, and algebraic thinking were analyzed. Furthermore, the theories of Kieran (1996) and Kaput (2008) which describe algebraic activity and algebraic thinking in extend were presented. Kaput's conceptualization of algebraic thinking from K-12 grades, reflects the multifaceted nature and content of algebraic thinking on which this dissertation is focused. Relying on Kaput's theoretical framework, this study aims to investigate whether different types of algebraic tasks could be used to explore the core aspects of algebraic thinking, and the extent to which different aged-groups of students reflect these aspects.

This chapter was also referred on theories and studies both from the discipline of mathematics education and the discipline of psychology, which highlighted the cognitive framework that possibly frames the development of algebraic thinking among students, including developmental stages and cognitive factors that affect algebraic thinking. Recognizing both the importance and challenge of better understanding the way students' algebraic thinking unfolds under the control of cognitive factors such as domain-specific processing abilities, reasoning forms and general cognitive factors, this study aims to provide further explanations about students' algebraic thinking abilities by focusing on accounts of a range of cognitive processes.

The literature review also included research studies that foster ways for introducing algebraic concepts and algebraic thinking in the secondary and primary school level. The findings of these studies indicated that specific instructional practices, curriculum materials, and technological tools might be used for supporting the emergence of algebraic thinking as early as the primary grades. Nevertheless, the translation of these findings into concrete instructional practices, that cross contexts and materials and that teachers can easily use to cultivate the various aspects and abilities of algebraic thinking, requires additional investigation.

The last part of this chapter provided descriptions of the tenets of a corpus of psychological theories on students' development of conceptual understanding. These theories provide a broad portrait of students' emerging cognitive skills that can be used to inform mathematics education researchers' understanding of the approximate ages at which students may be able to acquire different abilities of meaning construction that are essential to their engagement with mathematics. The theories of Luria, Case, and Demetriou and colleagues designate that educators seem to be benefited by findings of psychological research which describe when and how students achieve conceptual understanding in various semantic domains of though and knowledge. In particular, the overarching theory of the architecture and development of the mind (Demetriou, Spanoudis & Mouyi, 2011) was thoroughly described, as well as research studies within mathematics education which provide support on the educational implications of this theory. This overview suggests that student' abilities for algebraic thinking might be better described through providing explanations about the way a corpus of cognitive factors are related to algebraic thinking. The four systems that Demetriou et al's (2011) theory describes as core features of mental action will be integrated in a way that they will thoroughly describe algebraic thinking abilities.

CHAPTER III

Methodology

Introduction

The third chapter describes the research design and methodology of the current study. The first section provides information about the participants of the study. The second section presents the data collection instruments that were developed and used in the study. The third section refers to the data collection procedures and the way the data instruments were marked. The fourth section provides information about the design and procedure of instructional interventions. The final section outlines the data analysis procedures.

The Participants of the Study

The participants of the study were 684 students from 10 different schools (3 urban and 7 rural) and 42 different classrooms in Cyprus. In order to investigate algebraic thinking ability across an age-span, the participants represented four age-groups; 170 were fourth-graders (10 years old), 164 were fifth-graders (11 years old), 184 were sixth-graders (12 years old), and 166 were seventh-graders (13 years old). The participants were selected by convenience due to the fact that the sample had to be large. There were approximately equal numbers of males and females in the population of the study. The elementary school participants (10-12 years old) were students of five schools from Nicosia district, one from Limassol district, and two from Larnaca district. The middle school (13 years old) participants were students of one middle school from Nicosia district and one middle school from Larnaca district. In order to examine instructional practices that cultivate algebraic thinking in elementary school mathematics, a sample of 96 fifth-graders from two different elementary schools and four different classrooms, attended a teaching intervention program. These students were also selected by convenience.

Taking into consideration the fact that the data collection instruments would be the same for all of the participants of the study, no younger or older groups of students were selected. On the one hand, third grade students would not be able to understand or manipulate the tasks of the test, probably due to developmental reasons and absence of experience with such tasks.

The mathematics curriculum in effect at the school year that the study was conducted did not include algebra as a distinct domain of mathematics education or precise objectives regarding algebra teaching and learning. Elementary school students were introduced to algebraic problems such as balance scale tasks in the fourth grade. Students at Grades 4 and 5 had occasionally the experience of pattern activities and activities involving the interpretation of linear graphs. In Grade 5, the properties of operations (associative property of addition and multiplication, distributive property of multiplication, properties of 0, and order of operations) were introduced. Symbols, as a way for expressing unknowns in equations, were not introduced before Grade 6. The formal integration of algebra within mathematics education appeared in Grade 7. The corresponding lessons in 7th graders books emphasized the use and meaning of symbols as a tool for representing unknown quantities as well as the investigation of algebraic rules for solving equations.

Eighth grade students were considered as more skillful in solving algebraic tasks due to their more intensive involvement in algebra courses. Hence, in order to include eight grade students in the study, the test should have been adapted to their previous knowledge and experience, meaning to include in the test complex algebraic tasks (e.g. equations with variables and equivalent expressions) that would be abstract for the students of younger ages.

Data Collection Instruments

In order to address the multiple research questions set by the current study, various data collection instruments were required. Specifically, the participants were tested with seven different tests: (i) a test including items that examine algebraic thinking abilities; (ii) a test that addresses processes involved in the Specialized Structural Systems (SSSs); (iii) the Naglieri non-verbal ability test that addresses reasoning processes in the inferential system, (iv) a deductive reasoning test; (v) a working memory test; (vi) a speed of processing test, and (vii) a control of processing test. The construction of the test on algebraic thinking was mainly based on Kaput's (2008) theoretical framework and the recommendations from other related literature, curricula, and textbooks, national and international studies on mathematics achievement. For capturing the cognitive component, as this is described by the overarching theory of the architecture and development of the mind (Demetriou, Spanoudis & Mouyi, 2011), tasks from previous studies were also selected and adapted. Specifically, four different tests that were used in previous studies of Demetriou and colleagues were selected and adapted to capture the Hypercognitive System and the Specialized Structural Systems. Moreover, the Naglieri Nonverbal Ability Test (NNAT) and a deductive reasoning test were used for measuring aspects of the inferential system. Table 3.1 presents the title of each test and the parameters that aims to measure.

Table 3.1

Data collection instruments of the study

	Tests	Ability/ System to measure	Components
1	Algebraic thinking Test	Algebraic thinking abilities	• Generalized Arithmetic • Functional Thinking • Modelling
2	Specialized Structural Systems Test	Specialized Structural Systems	• Spatial-Imaginal • Causal-Experimental • Qualitative-Analytic • Verbal-Propositional
3	Naglieri Non-Verbal Ability Test	Inferential System	• Serial Reasoning • Spatial Visualization • Reasoning by Analogy • Pattern Completion
4	Deductive Reasoning Test	Inferential System	• Deductive Reasoning
5	Working Memory Test	Hypercognitive System	• Working Memory
6	Speed of Processing Test	Hypercognitive System	• Speed of Processing
7	Control of Processing Test	Hypercognitive System	• Control of Processing

Test on algebraic thinking abilities. As it has already been mentioned, algebraic thinking is not only about learning the mathematical language for representing algebraic expressions. What is emphasized in literature is the relation of algebraic thinking to the ability for expressing and justifying generalizations in several problem solving contexts by following processes of symbolization (Kaput & Blanton, 2005). At the elementary school level, algebraic thinking is embedded in activities as generalized laws about numbers and patterns, expressing change and relationships through the manipulation of variables, manipulating symbolic expressions, and expressing structure in modeling situations, within and outside mathematics (Kaput, 2008).

The test on algebraic thinking was constructed on the basis of the descriptions of algebraic thinking from existing theory and research. The identification of appropriate tasks in order to measure algebraic thinking was mainly based on the theoretical models of Kaput (2008) and Kieran (1996; 2007). Moreover, a content analysis of several curricula was conducted, in order to identify all the possible kinds of mathematical activities that are considered by curriculum designers as algebraic in nature. An additional requirement for designing the test was a content analysis of the curriculum (Cyprus Ministry of Education and Culture, 2005) and textbooks (Cyprus Ministry of Education and Culture, 1998; 1999; 2001) employed in Cypriot schools at the time the research was conducted. This review revealed that students were introduced to topics of algebra occasionally during the fourth grade, at the middle of the fifth grade and then at the sixth grade. These concepts mostly referred to the notion of balance scale problems, symbolization for representing unknowns and interpreting graphical representations. As soon as they entered the seventh grade, students studied algebra more systematically, focusing on the concepts of equations and equivalence.

The test consisted of tasks that were adapted from previous research studies related to the notions of algebra and algebraic thinking or algebraic proof. In particular, the design and content of the test was based on previous studies of Blanton and Kaput (2005), Drijvers et al. (2011), Kieran (1997; 2004; 2007; 2011), Mason et al, (2005) and Stylianides and Stylianides (2008). Furthermore, the test involved tasks that were included in past research studies that evaluated students' mathematical achievement in international or national level (e.g. TIMSS - Grade 4, 2011; NAEP – Grade 4, 2011; MCAS – Grade 4, 2012). The test included 21 tasks which denoted three forms of algebraic activity (see Appendix 1). In particular, assuming that these 21 tasks required different aspects of algebraic thinking to be processed, they were accordingly categorized into three groups which reflected the three strands of algebra as these were described by Kaput's (2008) theoretical framework: *Generalized arithmetic*, *Functional thinking*, and *Modeling as a domain for expressing and* *formalizing generalizations*. There were also tasks that took into consideration the relationship of algebraic thinking with roving procedures. The tasks included both open-ended questions and multiple choice questions.

A variety of tasks in the test intended to capture the strand of *generalized arithmetic*. In accord with previous studies, the items of the generalized arithmetic strand examined students' performance on identifying generalizations and structure within arithmetical contexts. Tasks from several research studies guided the construction of the items in this strand of algebraic thinking abilities, such as Blanton and Kaput (2005), Kieran (2004), Mason et al. (2005), MCAS (2012), NAEP (2011), Stylianides & Stylianides (2008) and TIMSS (2011). The test involved items that addressed properties of numbers and number theory, such as determining if the sum of two numbers is an odd or even number (item ga1, see Appendix I) and determining if the sum of two multi-digit numbers is an odd or even number (item ga7, see Table 3.2 or Appendix I). There were also items that examined operations and their properties, such as analyzing whole numbers into possible sums and examining the structure of those sums (item ga2, see Table 3.2 or Appendix I), relating place-value properties to operations' algorithms (item ga3, see Table 3.2 or Appendix I) and identify the structure of mathematical operations using the hundredths table (item ga4, see Table 3.2 or Appendix I). In addition, there were items that addressed the concepts of equality (items ga6 and ga8, see Appendix I) and inequality (item ga5, see Table 3.2 or Appendix I). The manipulation of this kind of activities required an ability of attending the structure of a mathematical expression relying on the fact that numbers are placeholders rather than on the computation of specific numbers.

The problems of the *Functional thinking* strand were categorized into three types of activities: (i) representing and interpreting data graphically, and (iii) identifying and expressing functional relationships (correspondence and co-variation relationships), and (iv) identifying and describing numerical and geometric patterns. These items were adapted from items that were included in relative research studies that were undertaken by Blanton and Kaput (2005), MCAS (2012), NAEP (2011), TIMSS (2011), Radford (2008), and Rivera (2007). The first type of items examined students' ability for encoding and decoding information graphically in order to analyze a functional relationship (item ft1, see Appendix I, and item ft6, see Table 3.2 or Appendix I). One item asked students to specify the correspondence among a correspondence relationship and generalize a rule that describes this relationship in a verbal form (item ft3, see Table 3.2 or Appendix I). The remaining items in this strand required finding the nth term in arithmetical and geometrical patterns and expressing this generalization in a verbal, symbolic or any other form (items ft2, ft5 see Appendix I and item ft2, see Table 3.2 or Appendix I).

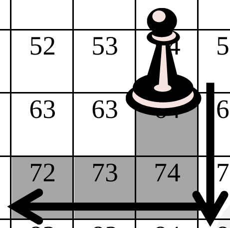
The next strand of algebraic thinking activities was associated with *Modeling* activities that required the expression and formalization of generalizations. In particular, these items (see Table 3.2 or Appendix I) engaged the participants with the analysis of information that were presented verbally, symbolically or in a table. The construction of these items was guided by items used in the studies of Mason et al. (2005) and school textbooks (e.g. California Mathematics– Grade 6, 2008). This group of items included the following: Modeling with a symbolic expression the relationship between Celsius and Fahrenheit degrees (item mod1, see Table 3.2 or Appendix I), modeling with a symbolic expression the process for calculating the area of a square (item mod2, see Appendix I), modeling with symbolic or verbal expressions three sale offers (item mod3, see Appendix I), modeling with symbolic or verbal expressions two offers for taking computer lessons (item mod4, see Table 3.2 or Appendix I), modeling with a table two offers for downloading songs (item mod5, see Table 3.2 or Appendix I), modeling a figural pattern (item mod6, see Appendix I) and modeling with a symbolic expression a function machine (item mod7, see Appendix I). All of these tasks required establishing relationships between the variables involved in a phenomenon or situation and associating meanings extracted from the modeling situation to symbols or other forms of representations, such as verbal expressions, diagrams, graphs or table).

Table 3.2

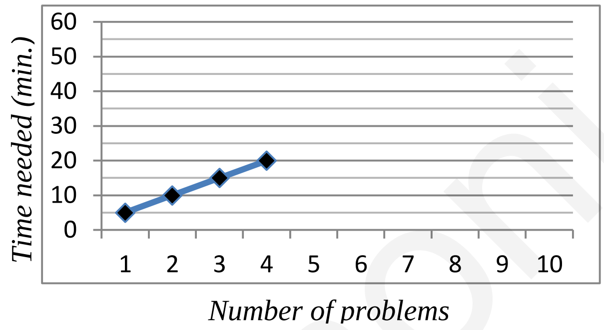
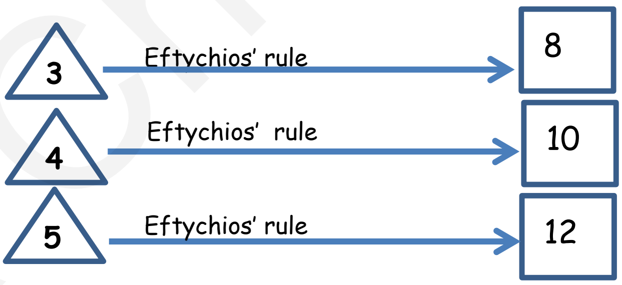
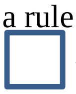
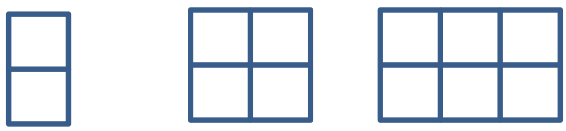
Examples of tasks included in the test on algebraic thinking abilities

1. Algebraic Thinking as Generalized Arithmetic		
A. Exploring properties of numbers	Determining if the sum of two multi-digit numbers is an odd or even number (ga7)	Is the sum of the following addition an even or an odd number? Explain your thinking. 1245676 + 4535731
B. Exploring properties of numbers and operations	Analyzing whole numbers into possible sums and examined the structure of those sums (ga2)	Nikiforos found the sum 80 + 50 with the following way: ***80 + 20 + 30 = 100 + 30*** ***= 130*** Use the procedure that Nikiforos applied in order to calculate the sum 70 + 50.
	Relating place-value properties to the multiplication algorithm (ga3)	Vasiliki calculated the following multiplication. ***35*** ***X <u>22</u>*** ***70*** ***+ <u>70</u>*** ***140*** Is Vasilikis' solution right? Explain your answer.

(continued)

<table>
<tr><td></td><td>Representing addition in the hundredths table (ga4)</td><td>A pawn was moved from number 64 to number 72.
<table>
<tr><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td><td>16</td><td>17</td><td>18</td><td>19</td><td>20</td></tr>
<tr><td>21</td><td>22</td><td>23</td><td>24</td><td>25</td><td>26</td><td>27</td><td>28</td><td>29</td><td>30</td></tr>
<tr><td>31</td><td>32</td><td>33</td><td>34</td><td>35</td><td>36</td><td>37</td><td>38</td><td>39</td><td>40</td></tr>
<tr><td>41</td><td>42</td><td>43</td><td>44</td><td>45</td><td>46</td><td>47</td><td>48</td><td>49</td><td>50</td></tr>
<tr><td>51</td><td>52</td><td>53</td><td>[illegible]</td><td>55</td><td>56</td><td>57</td><td>58</td><td>59</td><td>60</td></tr>
<tr><td>61</td><td>63</td><td>63</td><td>[illegible]</td><td>65</td><td>66</td><td>67</td><td>68</td><td>69</td><td>70</td></tr>
<tr><td>71</td><td>72</td><td>73</td><td>74</td><td>75</td><td>76</td><td>77</td><td>78</td><td>79</td><td>80</td></tr>
<tr><td>81</td><td>82</td><td>83</td><td>84</td><td>85</td><td>86</td><td>87</td><td>88</td><td>89</td><td>90</td></tr>
<tr><td>91</td><td>92</td><td>93</td><td>94</td><td>95</td><td>96</td><td>97</td><td>98</td><td>99</td><td>100</td></tr>
</table>

Which operation represents its movement?
(a) 64 + 8
(b) 64 + 10 - 2
(c) 64 + 6 + 2
(d) 64 + 10 + 2</td></tr>
<tr><td>C. Exploring equality / inequality as expressing a relationship between quantities</td><td>Solve an inequality (ga5)</td><td>For which value of β the following inequality is right?
12 > 3 X β
(a) 2
(b) 3
(c) 4
(d) 5</td></tr>
</table>

(continued)

2. Algebraic thinking as functional thinking		
A. Representing and interpreting data graphically	Interpreting a graph (ft6)	The graphical representation presents the time that Stavriani needs for solving some mathematical problems.  How much time does Stavriani need for solving 3 problems?
B. Finding functional relationships	Choosing the appropriate verbal expression for representing a recursive relationship (ft3)	 Eftychios applied a rule to the  in order to get the number in the  . Which was his rule? Express it in words or any other form.
C. Identifying and describing numerical and geometric patterns	Calculating the nth term in the geometrical pattern of even numbers (ft4)	Vasilis is arranging squares in the following way:  Figure 1 Figure 2 Figure 3 How many squares there will be in the 16th figure?

(continued)

<table>
<tr><td colspan="3">3. Modeling as a domain of expressing and formalizing generalizations</td></tr>
<tr><td>A. Generalizing regularities from mathematized situations</td><td>Modeling with symbolic or verbal expressions two offers (mod4)</td><td>Joanna will take computers lessons twice a week. Which is the best offer?

OFFER A
€8 for each lesson

OFFER B
€50 for the first 5 lessons of the month and then €4 for every additional lesson</td></tr>
<tr><td>B. Symbolizing quantities and operating with symbolized expressions</td><td>Modeling with a symbolic expression the relationship between Celsius and Fahrenheir degrees (mod1)</td><td>The temperature can be measured in Celsius (C) degrees or in Fahrenheit (F) degrees. Some children have converted the degrees in Celsius to Fahrenheit. Choose the symbolic expression that describes the relationship between these two measurement units?

<table>
<tr><th>Celsius (C)</th><th>Fahrenheit (F)</th></tr>
<tr><td>25</td><td>47</td></tr>
<tr><td>30</td><td>86</td></tr>
<tr><td>40</td><td>104</td></tr>
</table>

$9 \times \mathbf{25} = (\mathbf{47} - 32) \times 5$
$9 \times \mathbf{30} = (\mathbf{86} - 32) \times 5$
$9 \times \mathbf{40} = (\mathbf{104} - 32) \times 5$

(α) $9 \times C = (F-32) \times 5$
(β) $9 \times F = (C-32) \times 5$
(γ) $9 \times C = (47 - F) \times 5$
(δ) $9 \times F = (32 - C) \times 5$</td></tr>
</table>

Test on Domain-Specific Processing Abilities (Specialized Structural Systems). The items involved in the test about the SSS were adopted from previous tests that were designed from Demetriou and colleagues (e.g. Demetriou & Kyriakides, 2006). In particular, tasks from the WISC-R intelligence test were used, since previous literature has indicated that they can measure all of the domains of thought that are included in the SSS (Case, Demetriou, Plastidou & Kazi, 2001). The test measured students' abilities in four types of domain-specific processing abilities: the Spatial-Imaginal system, the Causal-Experimental system, the Qualitative-Analytic system, and the Verbal-Propositional system.

The Spatial – Imaginal items involved spatial visualization. Specifically, these tasks referred to mental rotation around vertical, horizontal and diagonal axes. These items required the ability to manipulate complex spatial information when several stages are needed to produce the correct solution.

The Causal – Experimental items involved cause-effect relations. Specifically, these items involved the representation of causal relations between objects and persons and operations related to causality. In this perspective, the production of inductive inferences was expected. Students were requested to perform systematic experimentation by isolating variables and model construction for explaining an experimental result. For example, students had to examine various combinations between different materials, such as soda and milk and the way these combinations affect the preparation of a cake. This kind of tasks assessed students' understanding of basic types of causal relationships. The causal relations represented in these tasks were as follows: necessary and non-sufficient, non-necessary and non-sufficient, and incompatible relative to an effect.

The Qualitative – Analytic items addressed categorical though. The corresponding items asked students to identify similarity and difference relations and make inductive inferences. Specifically, these items involved classification and forming hierarchies of interrelated concepts about class relationships, specification of the semantic and logical relations between properties, transformation of properties into mental objects and construction of conceptual systems. The first items in this battery included Raven-like questions (Raven test, 2000) that were made up of a series of drawings or patterns with one missing piece. The individuals were asked to fill the missing piece. The Raven-like matrices appeared in a series of increasing complexity. Students were asked to integrate two dimensions of a pattern (e.g. shape or background). To integrate two aspects of background in order to construct a complex layout and to integrate various patterns through a complex transformation.

The second part in this battery included verbal analogies. In the first analogy, students were given a, b, and c components and were asked to specify d component by choosing one of three alternatives. The second analogy presented a pair of familiar terms and relations and students were asked to construct the second pair. The third analogy required the specification of the d term but involved abstract relations. The fourth analogy required the students to understand the two higher (second order) relations and specify one of them.

Finally, the last part of Qualitative-Analytic items included items of class inclusion. Students were asked to manipulate objects/animals in reference to the relationship between the classes involved. The corresponding items requested students to compare classes with clear class inclusion relations, by specifying the combination of characteristics of the objects/animals involved.

The Verbal – Propositional items included questions of inferring a conclusion from two premises based on logical rules - implication and transitivity. Each question involved two premises and one conclusion and participants were asked to judge the validity of the conclusion. These items required students to reason inductively and deductively. The abilities associated with these activities included abstraction of information in goal-relevant ways, differentiation of the contextual from the formal elements, elimination of biases from inferential processes, and securing validity of inference.

Two aspects of verbal – propositional reasoning were included in the test, the propositional reasoning and reasoning in pragmatic contexts. The degree of difficulty in each task depended on the type of the logical relations involved, the validity of the argument, and the intuitiveness of the premises. Some logical relations (e.g. transitivity, disjunction, and modus ponens) are found to be more easily constructed than other relations (e.g. implication, modus tollens). The propositional reasoning items involved two premises and a conclusion, and student's task was to indicate if the conclusion was right, wrong, or undecidable. In particular, this item addressed modus tollens (i.e. p or q; not p; therefore p). The reasoning in pragmatic contexts items addressed modus ponens (i.e. if p then q, p, therefore q).

The pragmatic reasoning items addressed the ability to draw conclusions from a pragmatic context. More specifically, a series of dialogues are presented and the students are requested to draw the logically valid conclusion suggested by the dialogues by integrating the premises (Item F3, see appendix II). The relations involved here are similar to the relations involved in the propositional reasoning items.

Tests on processes in the inferential system. Two tests that are considered to measure inferential processes that underlie reasoning and problem solving were used. The first test was the Naglieri Non-Verbal Ability Test (NNAT) which assesses students' non-verbal syllogism and abilities of problem solving. The second test was developed by English and Watters (1995) and assessed students' deductive reasoning ability.

The Naglieri Nonverbal Ability Test (NNAT). The Naglieri Nonverbal Ability Test (NNAT) is a nonverbal test of general ability. Specifically, the NNAT measures cognitive ability independently of linguistic and cultural background (Naglieri, 2003). The tasks of the test are all multiple choice tasks. The NNAT's administration requires a time period of approximately 30 minutes. There are seven different levels of the test corresponding to different age-groups of students. In accord with the level, the NNAT may be administered to K–12 school children. The administration may be processed on an individual or group basis. The NNAT test has been administered to more than 100 000 students around the world and was found to have very good indices of validity. Usually this test is used as a means for identifying potentially gifted and talented students. If the student achieves high scores in the NNAT test, the student might be placed in a gifted and talented program.

The Naglieri Non-Verbal Ability test is a matrix reasoning type of exam that contains diagrams and shapes that form patterns and shapes. It contains four different types of questions: (i) Pattern Completion, (ii) Reasoning by Analogy, (iii) Serial

Reasoning and (iv) Spatial Visualization. Examples for each type of questions are presented in Table 3.3.

The Pattern Completion questions (Item 4, see Table 3.3) involved series of patterns with one missing part. The students were asked to choose among five possible answers the missing part of the pattern by focusing on aspects of orientation and special characteristics.

The Reasoning by Analogy questions (Item 2, see Table 3.3) required the students to identify relationships between a series of drawings by focusing on their special characteristics. Specifically, students were asked to identify similarities and differences between the drawings.

The Serial Reasoning questions (Item 5, see Table 3.3) requested students to conceive the way a set of drawings are placed in a series, horizontally and vertically, and then identify the changes on the way the drawings are placed in another series. Serial reasoning shares common features with Inductive Reasoning, which indicates the progression from particular/individual instances to broader generalizations. In the case of Serial Reasoning items, students had to study the specific instances and generalize the rule that guided their placement.

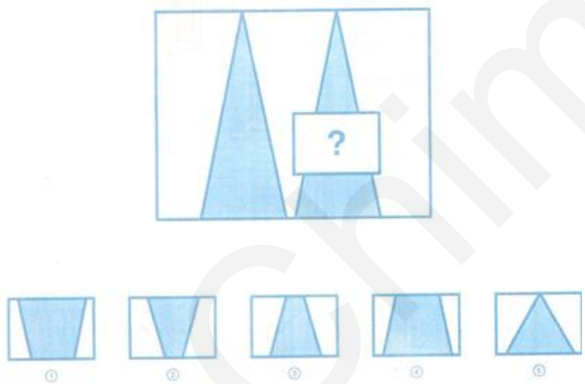
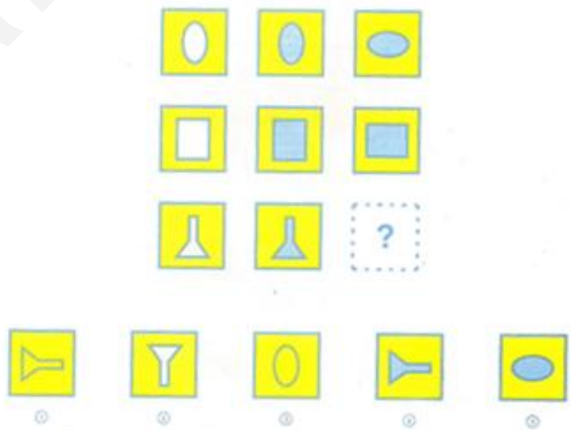
The Spatial Visualization (see Table 3.3) questions requested students to visualize the way a shape will look after a transformation or the way two shapes will look after they are combined.

This study made use of the NNAT Level D for the students of grade 4, the Level E for the students of grades 5 and 6, and the Level F for the students of grade 7. The Level D included 6 pattern completion tasks, 10 reasoning by analogy tasks, 8 serial reasoning tasks, and 14 spatial visualization tasks. The Level E included 5 pattern completion tasks, 6 reasoning by analogy tasks, 8 serial reasoning tasks, and 19 spatial visualization tasks. The Level F included 6 pattern completion tasks, 10 reasoning by analogy tasks, 8 serial reasoning tasks, and 14 spatial visualization tasks.

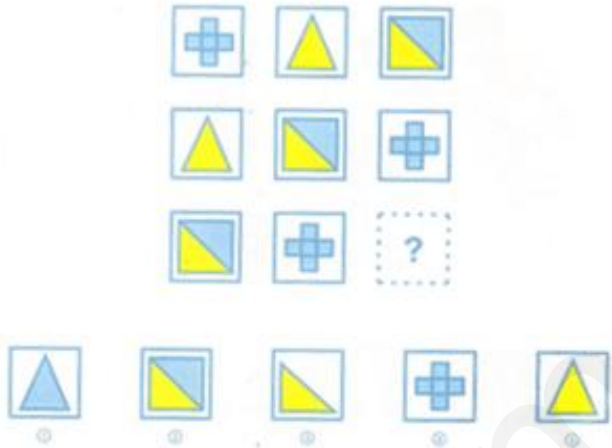
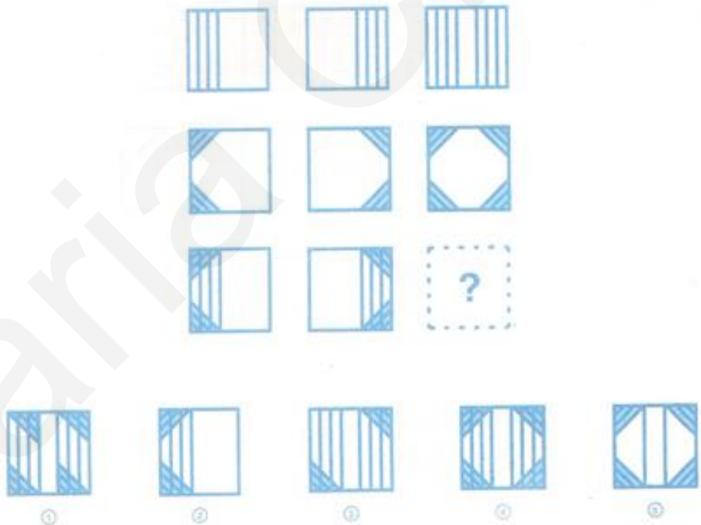
The NNAT was selected among other rival tests that are used extensively for assessing students cognitive ability, such as the Raven Progressive Matrices test (Raven, 2000), due to the fact that NNAT includes different categories of questions which reflect different types of reasoning processes. Specifically, these categories request students to use in great extend inductive reasoning (diSessa, 1983, 1993; in English and Watters, 1995) which is associated to scientific explanation. This process entails observation, isolation of phenomena, consideration, reflection combination, and at the end generalization of abstractions and theories.

Table 3.3

Examples of types of questions in the Naglieri Nonverbal Ability Test

Pattern Completion

Reasoning by analogy


(continued)

Serial reasoning

Spatial visualization


Test on Deductive Reasoning. A test on syllogism that requires deductive reasoning was used, following methodological practices suggested by existing theory and research on deductive reasoning. In particular, the test on deductive reasoning was adapted from a test that was used by Watters and English (1995). This test was considered as appropriate due to the fact that it was used for measuring deductive reasoning among students that were approximately of the same age as the participants in the current study.

The items in this test represented 10 syllogisms which requested the students to reason deductively. This process included the analysis of premises that describe formal truth relationships, without reference to the empirical or practical truth value of the premises. After this kind of analysis is conducted, a logical fact, result or consequence is derived. According to Nickerson (1986; in English and Watters, 1995) deductive reasoning involves the abilities of developing arguments, evaluating the validity of hypotheses and the plausibility of assertions, deciding possible paths for action and reflect on the possible consequences of the decisions taken.

The premise of the five first items involved fantasy or make-believe animals called Bongos and Wobbles (Item A, see Table 3.4). Students were encouraged to believe that they visited a planet where this kind of creatures lived. The following five items contained premises involving real entities (Item Z, see Table 3.4). Nevertheless, both sets of items described phenomena or behaviors that did not correspond to real-life experiences.

Table 3.4

Examples of types of questions in the Deductive Reasoning Test

Syllogisms including creatures of fantasy
A. Bangos have big eyes. Animals with big eyes like the sun. Ten is a Bango. Does Ten like the sun? **YES or NO**
Syllogisms including real-entities
Z. All dogs drink milk. The animals that drink milk are meow. Do dogs meow? **YES or NO**

Tests on processes in the representational capacity system. The test on the representational capacity system involved two computer-based activities that were designed in the context of a former study of Demetriou, Mouyi, and Spanoudis (2008) which investigated the structure and development of intelligence. Specifically, the first activity (Figure 3.1) measured the working memory of the students by evaluating their ability to remember a figure which appears on the computer screen and to distinguish it from other comparable figures.

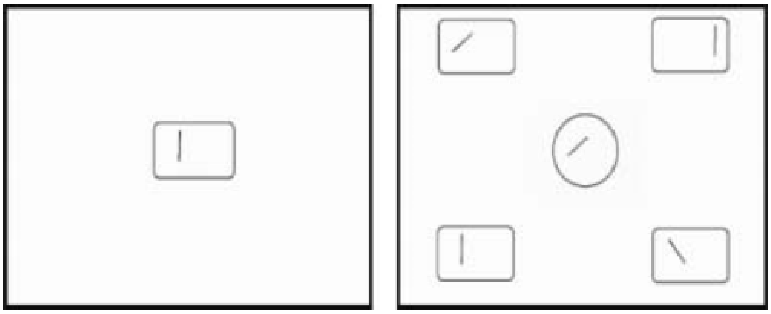


Figure 3.1. Example of tasks included in Working Memory test.

The second activity (Figure 3.2) evaluated speed and control of processing. The computer screen is divided into two parts. The individuals had to focus on the form of a stimuli presented (green circle or blue square) and press the right or left keyboard arrow according to the form of the shape (left arrow for the green circle and right arrow for the blue square). Moreover, they had to ignore the position (left or right) where the figure appears on the screen and press the left arrow if the green circle appeared and the right arrow if the blue square appeared. For half of the items of this test the green circle appeared in the same direction as the keyboard button that had to be pressed (left side of the screen, left arrow). These items measured speed of processing. For the other half of the items, the green circle appeared on the right side of the screen and the student had to press the left arrow (right side of the screen, left arrow). These items, for which the keyboard arrow to be pressed is inconsistent with the part of the screen that the shape appears, measured control of processing.

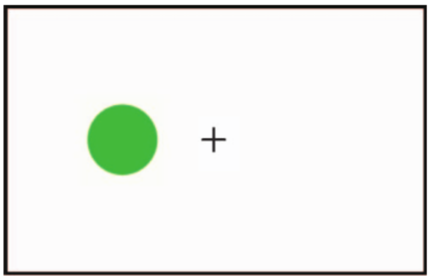
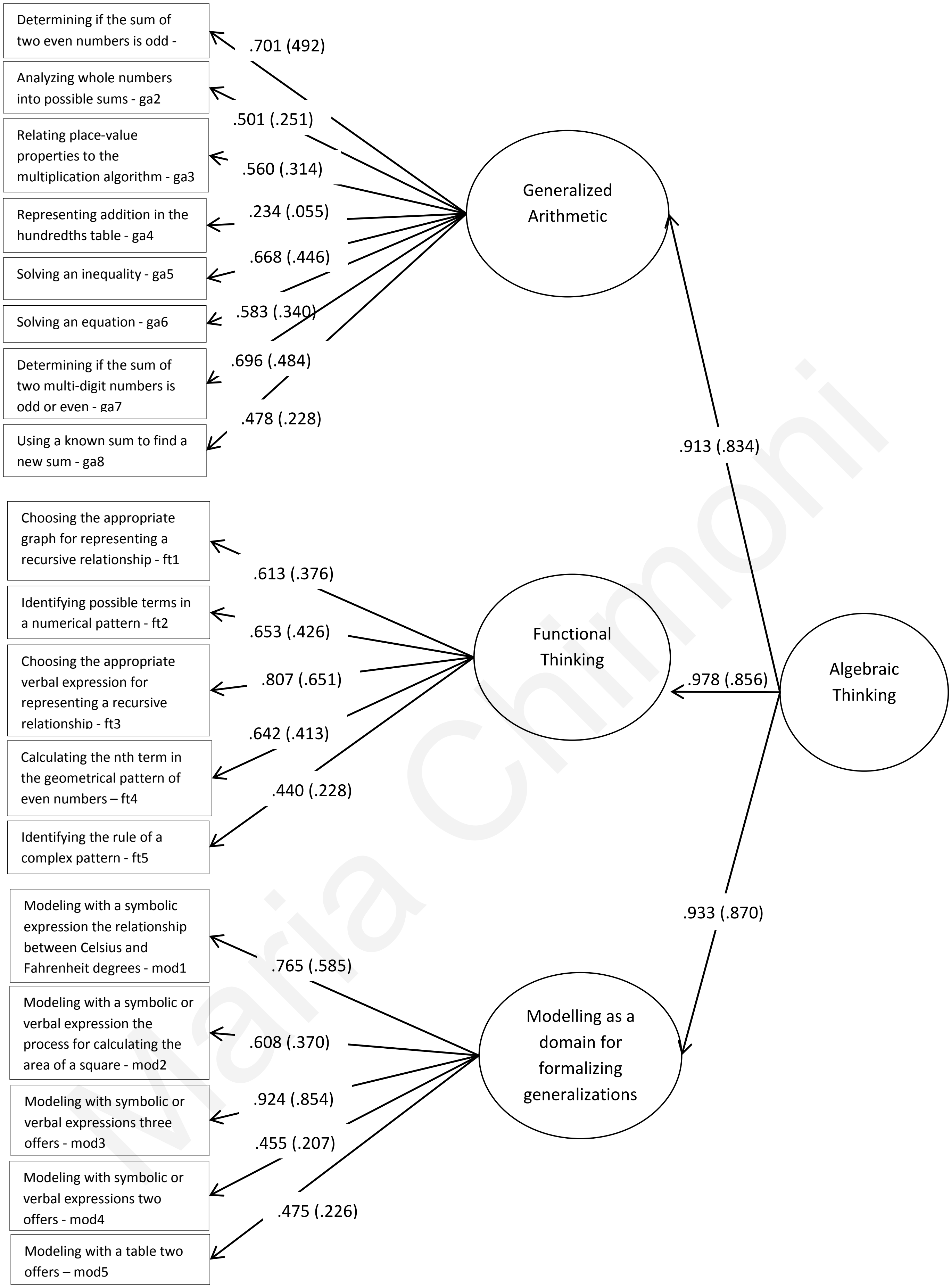


Figure 3.2. Example of tasks included in Speed and Control of Processing test.

Validity of Instruments

The items of the algebraic thinking test were developed for the aims of the current research, based on related literature, national exams and textbooks. The algebraic thinking test was structured in three parts, each measuring one of the three components of algebraic thinking: Generalized arithmetic, Functional thinking and Modeling as a domain for expressing and formalizing generalizations. For this reason, it was necessary to examine the construct validity of the items to measure the three categories. In order to evaluate the construct validity of the test, a pilot administration of the test was conducted. The results of this analysis (see Figure 3.3) indicated that the fitting of the model to the data gathered from the pilot study were satisfactory (CFI=.969, TLI=.979, x2=141.647, df =88, x2/df =1.61, p<.01, RMSEA=.036).The pilot study also guided the modification of some items. For example, item ga5 had a low factor loading and for this reason it was modified – from an open-ended question became a multiple choice question. Some factors were enriched with more items and corrections were made in the wording of some items.

The internal consistency of scores measured by Cronbach's alpha was satisfactory for the algebraic thinking test (a=0.87). The internal consistencies of scores measured by Cronbach's alpha were also satisfactory for the NNAT test (a=0.84), the Deductive Reasoning test (a=.79), and the Hypercognitive tests (a=.96).



Note. The first number indicates factor loading and the number in the parenthesis indicates the corresponding interpreted dispersion (r^2)

Figure 3.3. Model of construct validity of the algebraic thinking test.

The Proposed Model

It has been well documented that algebraic thinking is a wide conceptual field which does not merely coincide with what we know as traditional algebra (e.g. Blanton & Kaput, 2005; Kaput, 2008; Kieran, 1992; Mason, Graham & Johnston-Wilder, 2005). This idea raised the important issue of which are the aspects of algebraic thinking both in the primary and secondary education. Kaput, for many years, sought ways for defining algebraic thinking (Carraher & Schliemann, 2007). Blanton and Kaput (2005) described algebraic thinking as the process of establishing, systematically expressing and justifying generalizations in increasingly formal ways. They highlighted that expressing generalizations with symbols depends on students' age and level. Kaput (2008) further specified that there are two core aspects of algebraic thinking: (i) making generalizations and expressing those generalizations in increasingly systematic, conventional symbol systems, and (ii) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. The first aspect refers to the way, generalizations are produced, justified and expressed in various ways. The second aspect refers to the association of meanings to symbols and to the treatment of symbols independently of their meaning. Kaput (2008) asserted that these two aspects of algebraic thinking denote reasoning processes that are considered to flow in varying degrees throughout three strands of algebraic activity: (i) generalized arithmetic, (ii) functional thinking, and (iii) the application of modeling languages for describing generalizations.

Relying on Kaput's theoretical framework, this study investigated whether different types of algebraic tasks could describe the basic components of algebraic thinking, and the extent to which different age-groups of students reflect these aspects. The proposed model consisted of three first-order factors: Generalized Arithmetic, Functional Thinking and Modeling as a domain for expressing and formalizing generalizations.

Generalized arithmetic was related to items of "Exploring properties of numbers" (see items ga1, ga7 in Table 3.5), "Exploring properties of numbers and operations" (see items ga2, ga3, ga4 and ga7 in Table 3.5) and "Exploring equality / inequality as expressing a relationship between quantities" (see items ga5, ga6 and ga8 in Table 3.5).

Functional thinking items involved “Representing and interpreting data graphically” (see items ft1 and ft6 in Table 3.5), “Finding functional relationships” (see item ft3 in Table 3.5) and “Identifying and describing numerical and geometric patterns” (see items ft2, ft4 and ft5 in Table 3.5).

Modeling items involved “Generalizing regularities from mathematized situations” (see items mod3, mod4 and mod5 in Table 3.5) and “Symbolizing quantities and operating with symbolized expressions” (see items mod1, mod6 and mod7 in Table 3.5).

Table 3. 5

Description of Factors and Items of Factors of the Algebraic Thinking Ability

1st order Factor		Description of Items
Generalized Arithmetic	Exploring properties of numbers	Ga1 Determining if the sum of two even numbers is an odd or an even number
		Ga7 Determining if the sum of two multi-digit numbers is an odd or an even number
	Exploring properties of numbers and operations”	Ga2 Analyzing whole numbers into possible sums
		Ga3 Relating place-value properties of numbers to the multiplication algorithm
		Ga4 Representing addition in the hundredths table

(continued)

	Exploring equality / inequality as expressing a relationship between quantities	Ga5 Solving and inequality Ga6 Solving an equation with one unknown Ge8 Using a known sum to find a new sum.
Functional thinking	Representing and interpreting data graphically	Ft1 Choosing the appropriate graph for representing a recursive relationship Ft6 Interpreting a graph
	Finding functional relationships (correspondence , co-variation)	Ft3 Choosing the appropriate verbal expression for representing a recursive relationship
	Identifying and describing numerical and geometric patterns	Ft2 Identify possible in terms in a numerical pattern Ft4 calculating the nth term in the geometrical pattern of even numbers Ft5 developing the rule of a complex pattern

(continued)

	Generalizing regularities from mathematized situations	Mod3 Modeling with symbolic or verbal expressions the process three sales offers
		Mod4 Modeling with symbolic or verbal expressions two offers for computer lessons
Modeling as a domain of expressing and formalizing generalizations through algebraic symbols		Mod5 Modeling with a table two offers for downloading songs
	Symbolizing quantities and operating with symbolized expressions	Mod1 Modeling with a symbolic expression the relationship between Celsius degrees and Fahrenheit degrees
		Mod2 Modeling with a verbal or a symbolic expression the process for calculating the area of a square
		Mod6 Modeling a figural pattern
		Mod7 Modeling with a symbolic expression a function table

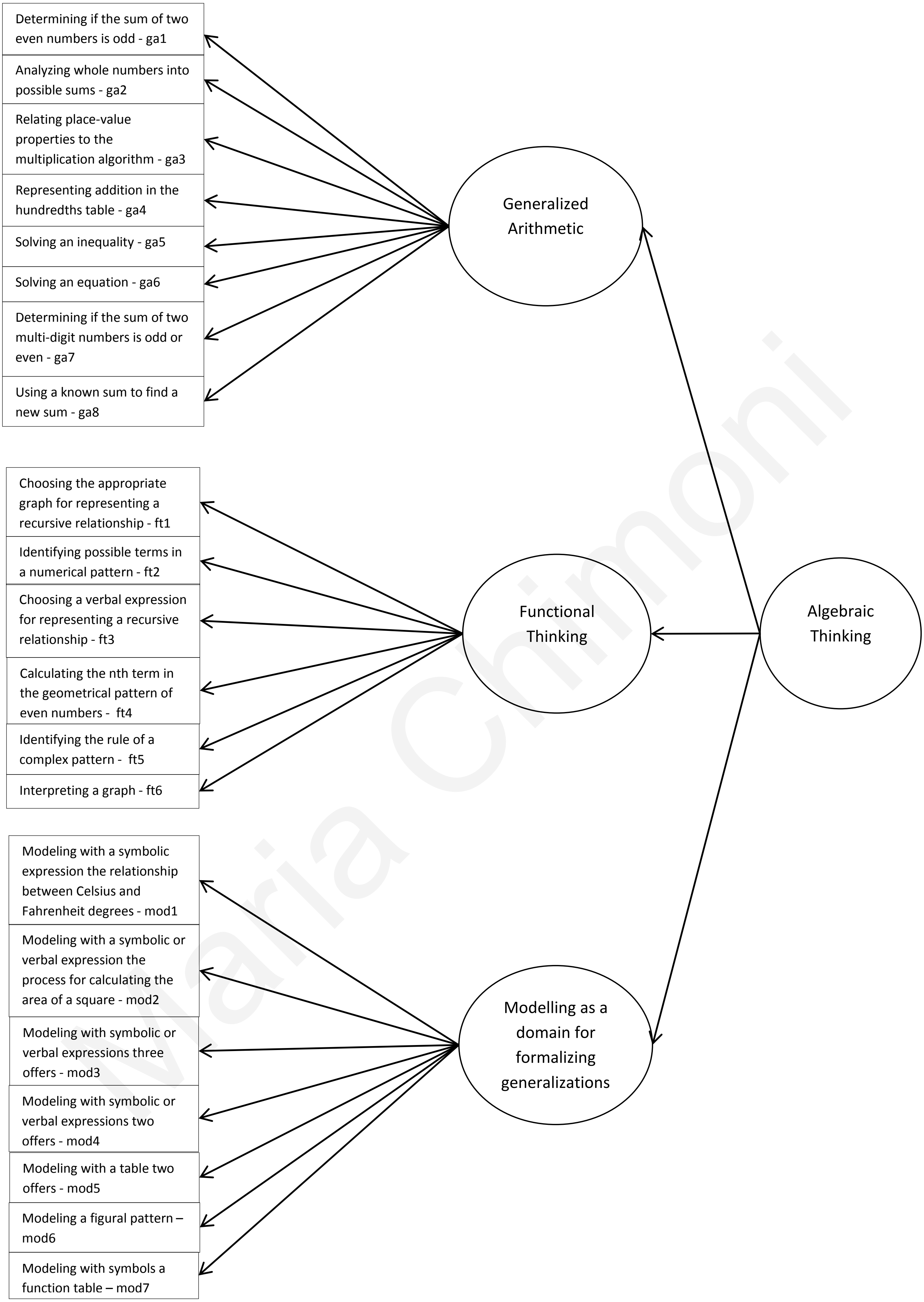


Figure 3.4. The proposed model for the structure of algebraic thinking.

Teaching Experiments

The design experiment methodology. The final part of the study focuses on the effect of two teaching experiments on students' algebraic thinking. In particular, the aim of the intervention was to assist students to make progress with respect to the three components of algebraic thinking.

The design of the intervention is grounded in the suggestions of available literature pertaining teaching experiments. In particular, Cobb, Confrey, diSessa, Lehrer and Schauble (2003) described five features which are involved in preparing for and carrying out a teaching experiment and in conducting a retrospective analysis of data and results produced during the experiment. First, the design experiment should explain the process of learning and describe the means that will provide support for that learning. For example, the design experiment should be based on a psychological model of the process by which students are expected to develop deep understanding of the particular mathematical topic of foci, together with the type of tasks and teacher practices that can support that learning. Second, it should be ensured that the experiment will be highly interventionist; in order to make certain the interventionist nature of the experiment and to investigate the possibilities of educational improvement, the methodology of the experiment should draw on the theoretical and empirical results of prior research. Third, the design experiment should be both prospective and reflective in nature. While the design is implemented in accordance to a hypothesized learning process, an equally important objective is to generate and test alternative conjectures when a prospective conjecture is refuted. A fourth characteristic should be the iterative design. A design experiment is consisted of cycles of application and revision. Finally, the theoretical products of a design experiment should precisely inform domain specific learning processes and speak directly to the types of problems that practitioners address in the course of their work.

In line with the suggestions made by Cobb et al. (2003), two highly interventionist teaching experiments were conducted. These kinds of experiments were selected for several reasons. First, their formulation reflects features of larger projects that were found in related literature and were exclusively designed for promoting algebraic thinking among students of elementary grades. Additionally, these projects supported investigative learning. Also, both projects were found to be effective. However, the tasks used in each project seemed to be different in respect to the extent of providing students with scaffolding steps for accomplishing mathematical investigations.

Pre- and post-tests were administered to the students in order to measure their algebraic thinking and cognitive abilities. The results of the tests were analyzed in order to compare the effectiveness of the two teaching approaches.

Clarifying the process of learning and the means for supporting it. According to the first suggestion of Cobb et al. (2003), the design experiments were based on the conceptual and cognitive analysis of the notion of algebraic thinking in order to clarify specific content, cognitive processes that affect learning and the means that will support that learning. Situated in this context, the experiments were also guided by the results of the quantitative analysis of the data, about the relationship between algebraic thinking and various types of cognitive factors.

Teaching experiment I: Semi-structured Problem Situations. The first teaching experiment used semi-structured problems which reflected situations and phenomena from the real life. These tasks addressed a general question and students were given time to get engaged to the problem situation, analyze and combine information and apply their own strategies and pathways for solving the task. Moreover, these tasks employed some features of modeling like tasks. For example, modeling activities engage children to authentic situations that need to be interpreted and described in mathematical ways (Lesh, 2003). Key mathematical constructs are embedded within the problem context and are formulated by the students as they investigate the problem. An important feature of modeling activities is that they require students to figure out ways for solving the problem (English, 2004). Moreover, they might need to use various technological tools for examining all of the given information and organizing and interpreting their data (e.g. spreadsheets).

This kind of tasks is cognitively demanding and aim to generate understanding of relative mathematical concepts. In the same line of thought, the first teaching experiment included intellectually stimulating activities. These activities provided students with opportunities for generating algebraic thinking as a meaningful mathematical construct. Specifically, modeling like tasks are considered as appropriate for enhancing the development of algebraic thinking because models involve the description and interpretation of complex systems of information through the application of processes such as, constructing, explaining, justifying, predicting, generalizing, conjecturing, and representing (English, 2011; English & Watters, 2005). Most of these processes coincide with what Kieran (2011) has described as important aspects that research has addressed pertaining algebraic thinking in the early grades. These processes are also integrated in most of the cognitive factors that were examined by the current research and are considered by psychologists to affect students' educational behavior, such as processes of the causal-experimental system, spatial-imaginal system, inductive and deductive reasoning. Moreover, mathematical models focus on structural characteristics of phenomena (e.g. patterns, interactions, and relationships among elements) (English & Watters, 2005). In this context, modeling like activities provide the learners with opportunities for developing and applying algebraic thinking as a powerful tool for understanding and predicting the behavior of problem situations.

Teaching experiment II: Structured Mathematical Investigations. The second teaching experiment included tasks that aimed to assist students in investigating all important aspects of algebraic thinking, through scaffolded activities that directed students to develop relational thinking, through identifying and understanding structure in mathematical concepts. Specifically, the tasks involved several assisting steps and pathways which guided students' investigation to the extraction of an explicit conclusion. This kind of activities is considered as relevant and important for enhancing algebraic thinking since they apply fundamental processes, such as pattern building, generalization, formulation and expression of relationships and progressive symbolization.

This experiment draw upon the findings from two large evaluation studies that aimed to enhance young children's early algebraic thinking and their awareness of the structural development of mathematics, the GEAAR project (Kaput & Blanton, 1999) and the PASMAP program (Mulligan, 2009; Mulligan, English & Mitchelmore, 2011).

The “Generalizing to Extend Arithmetic to Algebraic Reasoning” (GEAAR) project was designed to develop teachers’ abilities to identify and strategically build upon students’ attempts to reason algebraically and to use appropriate classroom instructional activities to support this (Kaput & Blanton, 1999). The strategy of GEAAR increased teachers’ capacity to transform instructional materials in order to shift the focus of their practice from arithmetic to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical facts and relationships. This project grouped teachers across grade levels and engaged them in solving authentic mathematical tasks and reflecting on the algebraic character of these tasks and how they might play out mathematically and pedagogically in their classroom. Teachers then adapted these tasks to their particular grade levels and implemented them in their own classrooms, focusing on students’ emergent algebraic thinking.

Mulligan and colleagues developed a “Pattern and Structure Mathematics Awareness Program” (PASMAP) that focused explicitly on raising primary school students’ awareness of mathematical pattern and structure through a variety of well-connected pattern-eliciting experiences (Mulligan, 2009; Mulligan, English & Mitchelmore, 2011). In this program, learning experiences are scaffolded in order children to look for and represent patterns and structure across a variety of concepts and then transfer this structure to other concepts. The PASMAP promoted simple or ‘emergent generalization’ in young children’s mathematical thinking across a range of concepts. This study documented astonishing changes in children’s structural awareness and development of mathematical concepts that were above the expected for their age level. PASMAP also had an effect on their scores on independent mathematics assessments.

The interventionist nature of the experiments. According to the second feature reported by Cobb et al. (2003), both of the experiments will draw on the results of the literature presented above, in order to ensure their interventionist nature. The studies reported above lend strong support to the hypothesis that cultivating algebraic thinking in young students should lead to a general improvement in the quality of their mathematical understanding. However, most of the studies did not have sufficiently large or representative sample and did not track and describe in depth the effects on algebraic thinking. Moreover, most studies lacked a comparison group which also had the opportunity to engage in algebraic thinking within practices and materials that are different from those that are expected to be used in a traditional classroom.

To overcome these gaps, both of the interventions adopted characteristics for utilizing conceptual understanding and reaching advanced forms of reasoning, such as 'promoting the active development of knowledge rather than the acquisition of static knowledge' and creating highly cognitive demanding learning activities (English, 2011). In order to approach this goal, the design of this intervention was based on the results of a systematic review and meta-analysis of algebra instruction improvement strategies conducted by Rakes et al. (2010). The recommendation of this systematic review was that the use of coherent curricula, instructional strategies, manipulatives, and technology to develop conceptual understanding has significantly positive effects on improving learning in algebra. Coherent curricula refer to the design and organization of curricula that comprehensively serve the principles and standards set by policy makers in respect to learning. Instructional strategies involve teaching methods such as cooperative learning, multiple representations, and assessment strategies. Manipulatives refer to the use of objects that help understanding a particular concept, e.g. rectangular tiles for developing polynomial multiplication skills. Technology involves tools such as graphic calculators, computer programs, and java applets. In this context, the two teaching experiments were designed in order to adapt all of these effective strategies with the aim to attend a possible positive effect on students' algebraic thinking.

The current intervention set a "coherent curriculum" that cultivated students' algebraic skills in robust ways through their mathematical experiences. The aim was to design of a broad range of activities which encompassed the multiple forms of algebraic thinking as these were described through the literature review and are enclosed in the proposed model of algebraic thinking. In this context, each intervention was developed through ten lessons of 80-minutes duration (see Table 3.6. or Appendix II). Lessons 3-4 were focused on "Generalized arithmetic", lessons 1,2,6,7 on "Functional thinking", lessons 5,8,9,10 on "Modeling as a domain for expressing and formalizing generalizations" and algebraic proof. In particular, the instruction was arranged around mathematical activities that promoted algebraic thinking through the use of appropriate tasks and classroom interactions. The goal was to build habits of mind whereby students naturally were engaged in algebraic tasks and used instructional strategies (e.g. cooperative learning) and tools (technological tools, objects, structures and processes) that supported the implementation of the tasks.

Table 3.6

Structure of Instructional Interventions and Objectives for each Lesson

LESSONS	Topic	Objectives: To develop students' ability to…
3-4	Generalized arithmetic	• Apply properties and relationships of whole numbers • Apply properties of operations on whole numbers • Treat equalities as objects that express quantitative relationships • Treat numbers by attending structure rather than computations • Solve missing numbers sentences
1, 2, 6, 7	Functional thinking	• Symbolize quantities and operate on symbolized expressions • Encode information graphically for analyzing a functional relationship • Identify correspondence among quantities or co-variation relationships and develop a rule that describes the relationship

(continued)

		• Identify and describe numerical and geometrical patterns
5, 8, 9, 10	Modeling / Algebraic proof	• Generalize regularities from mathematized situations • Use generalizations to build other generalizations • Describe, justify, and test generalizations • Generalize a mathematical process or formula

In this perspective, the interventionist nature of both experiments lay on the fact that they applied the same theoretical framework and similar characteristics as far as it concerns the content, the quality of instruction, the investigative nature of learning, and the cognitive demand of the learning activities. In both teaching experiments the goal was the teaching and learning of the same content. Nevertheless, they were elaborated through different types of tasks. Hence, the two teaching experiments were compared in relation to the type of the tasks through which algebraic thinking was expected to be emerged. One group was involved in semi-structured problem situations that moved beyond typical problem solving and involved authentic contexts. Students were expected to work collaboratively in order to uncover the mathematics enclosed in the situation, organize schemes of work by themselves, formulate strategies and build proper mathematical models for representing the situation. The other group was involved in scaffolded and structured mathematical investigations. Students were expected to come to specific viable conclusions while following the assisting steps and questions in each investigation. In both experiments, technology had a pivotal role, since digital work was encouraged through the use of online applets.

Figure 3.5 summarizes the similarities and differences among the two teaching experiments.

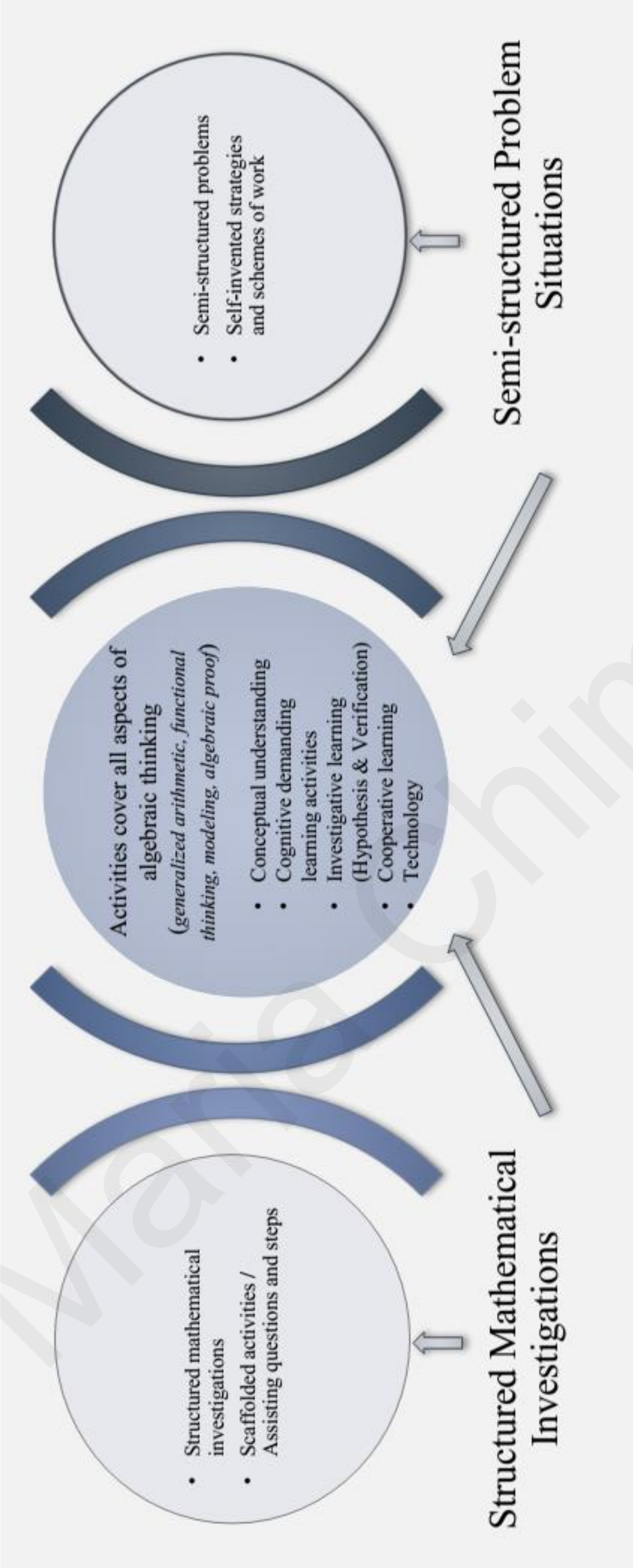


Figure 3.5. Similarities and differences between the two teaching experiments.

Table 3.7 presents an example of the activities in one of the lessons focused on generalized arithmetic. The topic of the lesson is examining properties of numbers. Table 3.8 presents an example of the activities in one of the lessons focused on functional thinking. The topic of this lesson was generalizing the nth term in pattern tasks. Appendix II includes all the lessons that were conducted in both teaching interventions.

Table 3.7

Example of tasks in generalized arithmetic lessons

Teaching experiment 1: Semi structured problem situation

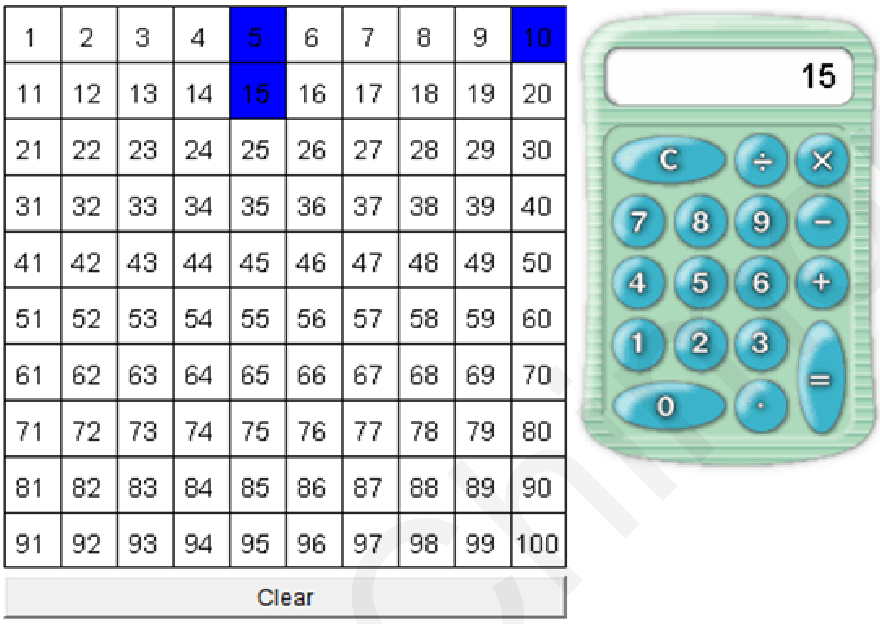
1. Constantinos noticed that last year the school holidays of the 25th of March and 1st of April were on the same day of the week. Both of them were on Tuesday. He is wondering if this happens every year. Check in other years' calendars if the schools holidays of the 25th of March and 1st of April were on the same day of the week. Explain your thinking and your answer.

(continued)

Teaching experiment 2: Structured mathematical investigation

Use the applet:

http://www.nctm.org/java/eexamples/4.5/standalone1.asp



1. What is the instruction that you have to write on the calculator in order to color:

a) the multiples of 5

____

b) the multiples of 7

____

c) the pattern 1, 12, 23, 34, …

____

Table 3.8

Examples of tasks in functional thinking lessons

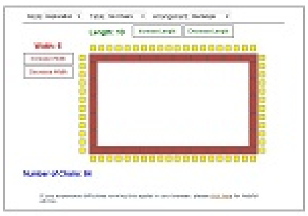
Teaching experiment 1: Semi-structured problem situation

Fanis is a waiter. He is preparing the tables for the lunch reservations.

(a) The restaurant has square tables. How many square tables Fanis needs to place side by side for Mr. Charalambous' reservation? Justify your answer.

You can use the applet:

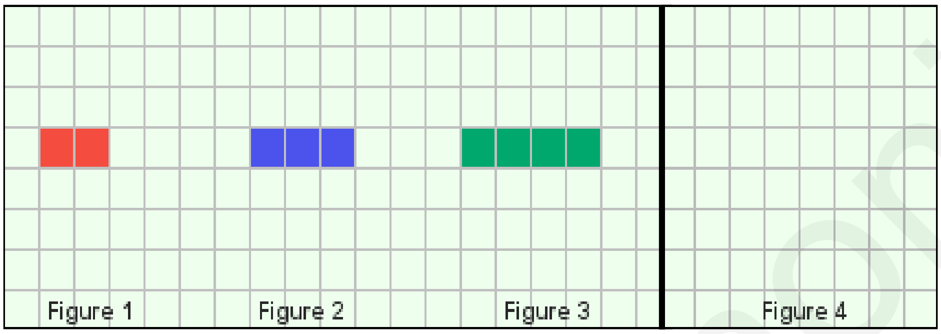
http://illuminations.nctm.org/Activity.aspx?id=3542



(continued)

Teaching experiment 2: Structured mathematical investigation

1. Investigate the following pattern.



(a) How many squares are colored in each figure?

Figure 1: __________

Figure 2: __________

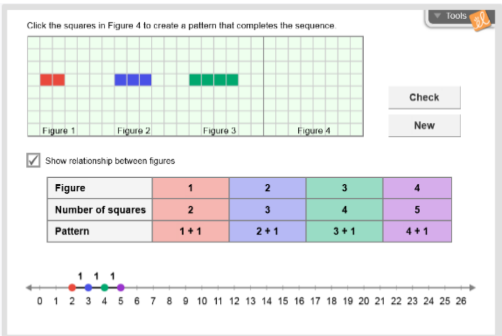
Figure 3: __________

(b) How many colored squares will the 4th Figure of the pattern have?

____

(c) Use the applet: http://www.explorelearning.com/index.cfm?method=cResource.dspDetail&ResourceID=219

Sketch Figure 4 and then check your answer by clicking on «check».



(d) How many colored squares will the 20th figure of the pattern have? Explain your answer.

Analysis

Statistical techniques for analyzing quantitative data. To address the research questions of the study, the MPLUS statistical package of structural equation analysis (Muthén & Muthén, 1998) was used in order to perform Confirmatory Factor Analysis (CFA) and Latent Class Analysis (LCA). Moreover, the SPSS statistical package was used in order to perform descriptive statistics analyses.

The Confirmatory Factor Analysis (CFA) was used for two reasons (i) to test the construct validity of the algebraic thinking instrument and (ii) to investigate whether the theoretical assumptions of the model about the core aspects of algebraic thinking fit the data of the study. The Confirmatory Factor Analysis was applied at the whole population that participated in the study. Moreover, the Multiple Group Confirmatory Factor Analysis was applied in order to examine whether the model fits the data of each age-group separately (Grade 4, Grade 5, Grade 6 and Grade 7). Goodness of fit of the data is evaluated by using three indices: (i) chi-square to its degree of freedom ratio (x^2/df), (ii) Comparative Fit index (CFI), and (iii) Root Mean-Square Error of Approximation (RMSEA). The observed values of x^2/df should be less than 2, the values for CFI should be higher than 0.9, and the RMSEA values should be close to zero (Marcoulides & Schumacker, 1996).

The Latent Class Analysis (LCA) was used to test whether there are different classes of students that are at different levels of algebraic thinking ability. Specifically, this analysis detected groups of students with similar behavior (Marcoulides & Schumacher, 1996). More than one fit indices are used to evaluate the possibility of grouping students into different groups: (i) the Entropy index, which needs to be the highest possible, (ii) the AIC index which needs to be the lowest possible, and (iii) the BIC which needs to be the lowest possible.

Regressions analysis and Structural Equation Model (SEM) analysis was used to investigate the relation between algebraic thinking abilities and the cognitive factors. Specifically, for each age-group in separation, several regression models were investigated in order to define the cognitive factor that influence algebraic thinking. Then SEM analyses were used with the aim to test (i) the validity of a model where the abilities of the students in the different cognitive systems (SSS, Inferential System, Hypercognitive System) predict algebraic thinking ability, and (ii) the validity of a model where algebraic thinking ability and the abilities of the students in the different cognitive systems (SSS, Inferential System, Hypercognitive System) are sub-factors of a more general ability, namely "Generalization abilities".

Using the SPSS statistical package, the means and the standard deviations of the answers of the students in the items of all of the tests were measured. Moreover, analysis of variance (ANOVA) and multivariate analysis of variance (MANOVA) was conducted. First it was investigated whether there are statistically significant differences between the performance of different age-groups in the components of algebraic thinking ability, in their overall performance in the algebraic thinking ability, and their performance in the different cognitive abilities of the human mind. Second, it was investigated whether there are statistically significant differences between the different classes of students as far as it concerns the algebraic thinking ability (as these were extracted from the latent class analysis) in their performance in the different components of algebraic thinking and in their overall performance in the algebraic thinking ability, and their performance in the different cognitive abilities of the human mind.

Statistical techniques for analyzing the impact of the instructional intervention. In order to test the means and standard deviations in the algebraic thinking test and the cognitive tests of the two groups that participated in the instructional intervention program, the SPSS statistical package was used. Paired-sample t-test was performed in order to measure the differences in the performance of students of the same group in the pre-tests and the post-tests. Moreover, multivariate analysis of covariance (MANCOVA) was used to examine the impact of the two teaching interventions (structured mathematical investigations and semi-structured problem situations) on the participants' algebraic thinking abilities. The type of intervention was the independent variable. Students' performance in algebraic thinking pre-test was considered as the covariate, and the performance differences between the pre- and post- tests as dependent variables.

A multivariate analysis of covariance (MANCOVA) was also used in order to investigate the impact of the interactions between the type of teaching experiment (structured mathematical investigations and semi-structured problem situations) and students' individual characteristics. The MANCOVA analysis was applied with the purpose of evaluating the moderation effects of the intervention and students' cognitive abilities, in respect to students' benefits in algebraic thinking, while adjusting for covariates in the students' abilities prior to the intervention program. Modeling as a domain for expressing and formalizing generalizations). In the analysis, the dependent variables were the benefits in students' overall algebraic thinking abilities, in generalized arithmetic concepts, functional thinking concepts and modeling concepts. The fixed factor was the intervention type (structured mathematical investigations and semi-structured problem situations). The covariates were the cognitive factors.

Scoring of the Data Collection Instruments

For the data coding of the algebraic thinking test, different procedures were followed according to the type of the task. Considering that students had adequate time to complete the test during administration, items with no response were graded with 0 marks.

For the multiple choice tasks which had four alternative responses, one mark was given to each correct response and zero marks were given to each incorrect response (items ga5, ga6, ga7, ft1, ft2, ft3, mod1, mod6 and mod7).

For the coding of the tasks that had two sub-questions, partial credit was given, considering the maximum sum of the marks of the sub-questions to be equal to 1 (items ft5 and mod 5). Specifically, in the item "Developing the rule of a complex pattern" (ft5) the scoring was as follows: 0 mark for incorrect responses to all sub-questions, 0.50 for answering correct the first sub-question and 0.50 for answering correct the second sub-question. In the item "Modeling with a table two offers for downloading songs" (mod5), the scoring was as follows: 0 mark for incorrect responses to all sub-questions, 0.33 for completing correct the first row of the table, 0.33 for completing correct the second row of the table and 0.33 for answering correct the last question.

In the items where students had to justify their answers (ga1, ga3 and ga7), the scoring was as follows: 0 mark for fully incorrect responses, 0.50 for giving a correct answer without justifying the answer or giving a wrong justification and 1 mark for giving a correct answer and a proper justification.

In the remaining items (ga2, ga8, mod1, mod2, ft4, mod3 and ft6), 0 mark was given for an incorrect response and 1 mark was given for a fully correct response.

The Specialized Structural Systems test, the Naglieri Non-Verbal Ability test and the Deductive Reasoning test involved only multiple choice items, with four alternative responses each. In these tests, 1 mark was given to each correct response and 0 marks were given to each incorrect response. Items in these tests with no response were also graded with zero, since students had adequate time to complete all of the tests during administration.

For the measurement of students' working memory, speed of processing, and control of processing, computer based tests were used which calculated the total time needed for completing the tasks and the number of correct responses that each student gave. The final score of the students in these tests was their reaction times. These were calculated by dividing the total time needed for the tasks by the number of correct responses of each student. As a result, three reaction times were measured for each student.

Procedure

The procedure of the study included six phases. The first phase involved the literature review which guided the establishment of the theoretical framework of the study and the construction and selection of the instruments. The second phase included the administration of the instruments in a pilot study and their modification. The third phase involved the administration of the final instruments to the students. The fourth phase included the analysis of the data collected. The fifth phase involved the design and conduction of the teaching interventions in two fifth grade classes. Finally, the sixth phase included the analysis of all the data collected and the extraction of conclusions.

At the first phase research findings and theoretical frameworks that examine the notions of algebra and algebraic thinking were synthesized. Moreover, psychological studies that were used in the context of education and offered insights about mental causation were reviewed. This analysis guided the design, construction, and selection of instruments. Specifically, several curricula, textbooks, national and international studies on mathematics achievement, and research studies were reviewed in order to select and adapt tasks that were appropriate for measuring students' abilities on algebraic thinking. Following Kaput's (2008) theoretical framework and the recommendations from related literature (e.g. Drijvers et al, 2011; Kieran, 2004; Mason et al, 2005; Radford, 2008), the tasks were grouped into three categories: generalized arithmetic, functional thinking and modeling as a domain for expressing and formalizing generalizations. For measuring the parameters of the cognitive component, as these were extracted from the overarching theory of the architecture and development of the mind (Demetriou, Spanoudis & Mouyi, 2011), tests from previous studies were selected and adapted. Specifically, four different tests that were used in previous studies of Demetriou and colleagues were selected and adapted to capture the Representational Capacity System, the Specialized Structural Systems. Moreover, the Naglieri Nonverbal Ability Test (NNAT) and a deductive reasoning that was developed by Watters and English (1996) were used for measuring aspects of the inferential system.

The second phase included the evaluation of the tests in a pilot study. In particular, the tests were administered to three fourth grade classes (58 students), three fifth grade classes (66 students), three sixth grade classes (82 students), and four seventh grade classes (95 students). The administration of the tests in the pilot study aimed to examine the level of difficulty in the tasks, the difficulties of the participants in understanding the instructions and the time needed for completing the tests.

Following the pilot study and based on its results, in the third phase the tests were modified and administered to the whole sample. Some items were modified, in order to be better clarified and improve the wording of instructions. Also, in the final test some tasks were removed due to time restrictions. Then, in the fourth phase, the analysis of the quantitative data collected through the tests was conducted.

The fifth phase included the teaching interventions in four fifth grade classes. The students of these classes were divided into two different teaching intervention experiments, both aiming to enhance their algebraic thinking abilities. The duration of each program was six weeks long, with two eighty-minute period lessons each week.

All lessons took place in students' classroom or in the school computer laboratory and were conducted by the researcher.

The final phase of the study included the analysis of all the collected data, the discussion and the conclusions based on the theoretical framework that was used as a guide for designing the research.

CHAPTER IV

Data Analysis and Results

Introduction

This chapter presents the study results pertaining to the research questions, which aimed at a thorough exploration of the nature and content of algebraic thinking from pedagogical and cognitive perspectives. In particular, this chapter presents the quantitative results of this study, as they were extracted from the analysis of data collected from tests that measured algebraic thinking and various types of cognitive factors. Additionally, the results that arose from the conduction of the instructional intervention program are described. These refer to the comparison of two different teaching experiments for developing students' algebraic thinking.

The chapter consists of four sections. The first section addresses the first two research questions of the study about the structure and components of algebraic thinking. Kaput's model about the aspects that synthesize algebraic thinking is empirically tested. The stability of this model across the four age-groups that were included in the sample (Grade 4, Grade 5, Grade 6 and Grade 7) is also examined.

In the second section, questions third, fourth and fifth are addressed. Specifically, the results from evaluating the possibility of grouping students into different groups considering the level of their algebraic thinking ability are presented. Then, the characteristics and differences between these groups are described in detail. Moreover, the presence of a hierarchical route that prescribes levels of algebraic thinking ability is investigated. Both sections one and two of this chapter contribute to better understand the concept of algebraic thinking from a pedagogical perspective.

In the third section of this chapter, the sixth research question is approached. This refers to the extent to which there is an association between algebraic thinking and different types of cognitive factors. In particular, the third section presents a detailed description of the relationship among students' algebraic thinking and (i) domain-specific processes of mental action, (ii) different types of reasoning forms, and (iii) general cognitive processes of mental action. This analysis is conducted separately for each of the four age-groups that are involved in the study. Finally, this section outlines which of these factors are significantly related to algebraic thinking. The aim is to define the nature of the relationship between different types of cognitive factors and algebraic thinking. As significant factors in this relationship are considered to be only the factors that appear to be common and significant in all of the four age-groups. By identifying factors that affect students' performance in the algebraic thinking test, this section contributes in further defining algebraic thinking from a cognitive perspective.

The fourth section of this chapter explores the possibility of enhancing students' algebraic thinking using innovative teaching approaches. Questions seventh and eighth are examined pertaining the impact of a particular intervention program. Specifically, the two teaching experiments are compared in respect to their effect in fifth grade students' algebraic thinking abilities. The interventionist nature of both experiments lies to the fact that they consider the results reported in the previous two sections for articulating the goals and objectives of a series of lessons in several algebraic concepts. Similar characteristics are applied in both experiments as far as it concerns the quality and content of instruction and the cognitive demand of the learning activities. Nevertheless, the tasks and questions used in each experiment have different characteristics. One group of students is involved in semi-structured realistic problem situations that move beyond typical problem solving and the other group is involved in more scaffolded and structured mathematical investigations. Algebraic thinking is integrated in viable ways in the activities of both experiments. This chapter provides further evidence regarding the aim under consideration for unpacking the concept of algebraic thinking from a pedagogical perspective and offers insight into the way the components of algebraic thinking can be manifested in teachers' practices.

The Structure and Components of Algebraic Thinking

This section reports the results concerning the nature and content of algebraic thinking from an epistemological perspective. This section is organized along the first two questions of this study:

1) Which components synthesize 10- to 13-year-old students' algebraic thinking ability and what is the structure of this ability?

2) Is the structure of students' algebraic thinking ability the same or different in relation to age?

In order to answer these questions, the data collected from the algebraic thinking test are analyzed from a quantitative standpoint. First, the results of descriptive statistics analyses are considered. Then, the verification of the theoretical model about the structure and components of algebraic thinking is examined through confirmatory factor analyses. This section concludes considering the stability of the proposed model across the four age-groups of the participants.

Descriptive statistics of the algebraic thinking ability test. Table 4.1 presents the results of descriptive statistics analysis for each of the 21 items that were included in the algebraic thinking test. Depending on their content and structure, these items were categorized in three distinct groups, which reflected the three first-order factors in the proposed model for the algebraic thinking ability. The first three categories of Table 4.1 correspond to the means, standard deviations and range of the algebraic thinking measures; the next three categories represent the information concerning the distribution of scores on categorical and continuous variables. As shown in Table 4.1, the highest mean of the students' performance was in the item "Solving an equality" which belongs in the component of Generalized Arithmetic (M=.806). The second higher mean of the students' performance was and in the item "Analyzing whole numbers into possible sums" which also belongs to the component of Generalized arithmetic. The third and fourth highest means of subjects' performance belong to the component of Functional thinking. Specifically, in the items "Choosing the appropriate verbal expression for representing a recursive relationship" and "Calculating the nth term in the geometrical pattern of even numbers" the students appear to have high means of performance (M=.569 and M=.559 respectively). In all of the items in the component of Modeling the students appear to have means that are lower than .400. The highest mean in this component was in the item "Modeling with a symbolic expression the relationship between Celsius and Fahrenheit degrees" (M= .383). The lowest mean of the students in this component was in the item "Modeling with a table two offers for downloading songs" (M=.243). In the other two modeling items which both required the comparison of offers and decision making about the best offer ("Modeling with a symbolic or verbal expressions three sales offers that represent proportional relationships" and "Modeling with symbolic or verbal expressions two offers for attending computer lessons"), the students' performance was also very low (M=.249 and M=.269 respectively).

The maximum value of performance in all of the categories of items was 1 and the minimum was 0. The range of the students' performance was 1, showing that there were students that responded correctly to the item, as well as students that did not respond correctly. The skewness values provide indication of the symmetry of the distribution. Kurtosis provides information about the peakedness of the distribution. If the distribution is perfectly normal it means that skewness and kurtosis value of 0. In the current study, the values of these indices were smaller than 2 and larger than -2. This result suggests that the variables of the students' performance for the three categories of items in the algebraic thinking test follow a normal distribution.

The reliability coefficient of the algebraic thinking test items was Cronbach's Alpha = .875 which is considered as very good. The reliability indices for the three groups of items were at satisfactory levels (a_{GA}=.688, a_{FT}=.739, a_{MOD} =.734, GA=Generalized arithmetic, FT=Functional thinking and MOD=Modeling as a domain for expressing and formalizing generalizations).

Table 4.1

Descriptive Results of the Students' Performance in Algebraic Thinking Components

Test	Mean	Standard Deviation	Range	Skewness	Kyrtosis
A. Generalized arithmetic					
Determining if the sum of two even numbers is either odd or even number (ga1)	.531	.499	1.00	-.123	-1.991
Analyzing whole numbers into possible sums (ga2)	.648	.478	1.00	-.620	-1.621
Relating place-value properties to the multiplication algorithm (ga3)	.322	.467	1.00	.765	-1.418
Representing addition using the hundredths table (ga4)	.427	.495	1.00	.296	-1.918
Solving an inequality (ga5)	.528	.500	1.00	-.112	-1.993
Solving an equation (ga6)	.806	.396	1.00	-1.547	.396
Determining if the sum of two multi-digit numbers is either odd or even number (ga7)	.504	.500	1.00	-.018	-2.006
Calculating a sum by using a known sum (ga8)	.474	.499	1.00	.106	-1.995
B. Functional thinking					
Choosing the appropriating graph for representing a recursive relationship (ft1)	.462	.498	1.00	.153	-1.982
Identifying possible numbers in a numerical pattern (ft2)	.437	.496	1.00	.254	-1.941

(continued)

Choosing the appropriate verbal expression for representing a recursive relationship (ft3)	.569	.495	1.00	-.278	-1.928
Calculating the nth term in the geometrical pattern of even numbers (ft4)	.559	.496	1.00	-.236	-1.950
Developing the rule of a complex geometrical pattern (ft5)	.333	.401	1.00	.677	-1.117
Interpreting a graph (ft6)	.431	.497	1.00	.278	-1.928
C. Modeling					
Modeling with a symbolic expression the relationship between Celsius and Fahrenheit degrees (mod1)	.383	.487	1.00	.482	-1.773
Modeling with a symbolic or verbal expressions the process for calculating the area of a square (mod2)	.366	.482	1.00	.560	-1.692
Modeling with a symbolic or verbal expressions three sales offers that represent proportional relationships (mod3)	.249	.432	1.00	1.166	-.642
Modeling with symbolic or verbal expressions two offers for attending computer lessons (mod4)	.269	.444	1.00	1.044	-.913

(continued)

Modeling with a table two offers for downloading songs (mod5)	.243	.405	1.00	1.192	-.414
Modeling with a symbolic or verbal expression a figural pattern (mod6)	.319	.466	1.00	.780	-1.396
Modeling with a symbolic expression a machine (mod7)	.292	.455	1.00	.915	-1.167

Table 4.2 presents the correlations between all the items of the test measuring the algebraic thinking ability. The variables correspond to the 21 items of the algebraic thinking test. The first immediate observation from this table is that all correlations between the same types of items for each component of the algebraic thinking ability (items that belong to the same first-order factor of the proposed model) are statistically significant and the correlation is significant at the 0.01 level. For the "Generalized arithmetic" factor, the highest correlation appears to be between items ga1 and ga7 ($r=.363$, $p<.01$). These items examine the ability for generalizing properties and relationships of numbers. Item ga1 is also highly correlated with item ga3 which examines the identification of properties of operations ($r=.328$, $p<.01$). A high correlation appears between the items ga7-ga5 and ga7-ga8 ($r=.328$, $p<.01$ and $r=.284$, $p<0.1$ respectively). Both items ga5 and ga8 correspond to the group of items that examine the ability for treating equations and finding the unknown.

For the "Functional thinking" factor, the highest correlation appears between items ft5 and ft6 ($r=.441$, $p<.01$). The item ft5 requires the translation of a pattern that is represented geometrically to its numerical representation. The item ft6 requires the translation of a functional relationship that is represented graphically to its numerical expression. The item ft5 is also highly correlated with item ft3 ($r=.432$, $p<.01$). The item ft3 requires the translation of a functional rule that is represented diagrammatically to its verbal expression. Another high correlation that is observed within the component of "Functional thinking" is between the items ft3 and ft6 ($r=.371$, $p<.01$). The items ft1 and ft2 also appear to have a high correlation ($r=.344$, p<.01). The item ft1 represents a functional relationship through a graphical representation where ft2 represents a numerical pattern.

For the “Modeling as a domain of expressing and formalizing generalizations” factor, the results show that the items mod1 and mod3 have the highest correlation (r=.493, p<.01). The item mod1 refers to the interpretation of a procedure that some children follow in order to construct a model for easily converting Fahrenheit degrees to Celsius degrees. The item mod3 requires the construction of a model in order to compare offers for buying shampoos in three different supermarkets. The items mod5 and mod6 are also highly correlated (r=.484, p<.01). The item mod5 refers to the extraction of a model from data that are represented within a table. The aim is to easily compare two offers for downloading songs from the internet. The item mod6 refers to the extraction of a model from data that are represented within a figural pattern involving of different pictures of balls. The aim is to construct a model through a numerical expression that describes the way the balls are set in the sequence. The items mod3 and mod5 that both correspond to the group of items that require the comparison of offers and decision making about the best offer, also seem to be highly correlated (r=.364, p<.01).

The descriptive statistics just reported provide some first insights into the inquiry under exploration. Specifically, they indicate that students under the age-span being explored by the study exhibit different performance in different types of algebraic tasks. Students seem to perform better on the “Generalized arithmetic” tasks rather than on the “Functional thinking” tasks or the “Modeling” tasks; they exhibited higher scores in “Generalized arithmetic” tasks comparing to the “Functional thinking” tasks, while their performance is quite low in the “Modeling” tasks.

In sum, the findings reported above show that in overall there are different mean scores in different categories of algebraic thinking tasks and high correlations are observed between items that belong to the same group. For example, all of the items that were categorized in the “Generalized arithmetic” factor are highly correlated. Similar correlations were found between the performance in the items of “Functional thinking” and “Modeling as a domain for expressing and formalizing generalizations”. At the same time all items present high correlations either at the level of significance 0.01 or 0.05. No significant correlations appear between the item ga4 and items ft5, ft6, mod5 and mod6. This result might be attributed to the fact that the item ga4 requires students to generalize an arithmetical rule about the way an operation can be represented in the hundredths table where items ft5, ft6, mod5 and mod6 involve functional rules that are based on the observation of patterns.

Table 4.2

Correlations between Subjects' Performance in the Items of the Algebraic Thinking Test

	ga1	ga2	ga3	ga4	ga5	ga6	ga7	ga8	ft1	ft2	ft3	ft4	ft5	ft6
ga1	1													
ga2	.226**	1												
ga3	.328**	.167**	1											
ga4	.107**	.185**	.171**	1										
ga5	.313**	.203**	.194**	.130**	1									
ga6	.189**	.202**	.149**	.185**	.275**	1								
ga7	.363**	.260**	.244**	.128**	.328**	.274**	1							
ga8	.264**	.130**	.231**	.081*	.252**	.185**	.284**	1						
ft1	.307**	.180**	.235**	.131**	.260**	.181**	.221**	.219**	1					
ft2	.303**	.237**	.169**	.127**	.302**	.202**	.272**	.250**	.342**	1				
ft3	.352**	.254**	.239**	.143**	.329**	.303**	.347**	.229**	.333**	.339**	1			
ft4	.254**	.192**	.304**	.187**	.318**	.164**	.214**	.163**	.292**	.277**	.344**	1		
ft5	.291**	.195**	.215**	.071	.221**	.270**	.261**	.136**	.251**	.309**	.432**	.323**	1	
ft6	.210**	.191**	.178**	.066	.221**	.234**	.214**	.173**	.247**	.215**	.371**	.316**	.441**	1
.mod1	.259**	.254**	.236**	.208**	.324**	.205**	.270**	.204**	.301**	.361**	.298**	.352**	.271**	.273**
mod2	.227**	.198**	.186**	.094*	.225**	.189**	.188**	.156**	.277**	.310**	.299**	.287**	.241**	.252**
mod3	.283**	.198**	.249**	.208**	.368**	.223**	.387**	.240**	.356**	.393**	.357**	.435**	.368**	.346**
mod4	.214**	.095*	.203**	.123**	.164**	.156**	.127**	.091*	.145**	.177**	.162**	.225**	.254**	.277**
mod5	.283**	.152**	.297**	.120**	.239**	.145**	.264**	.221**	.276**	.271**	.290**	.314**	.298**	.420**
mod6	.379**	.163**	.301**	.057	.289**	.154**	.276**	.256**	.298**	.283**	.330**	.345**	.381**	.355**
mod7	.267**	.144**	.314**	.108**	.170**	.153**	.226**	.266**	.223**	.263**	.287**	.329**	.319**	.362**

**p<.01, **p<.05

(continued)

Table 4.2

Correlations between Subjects' Performance in the Items of the Algebraic Thinking Test

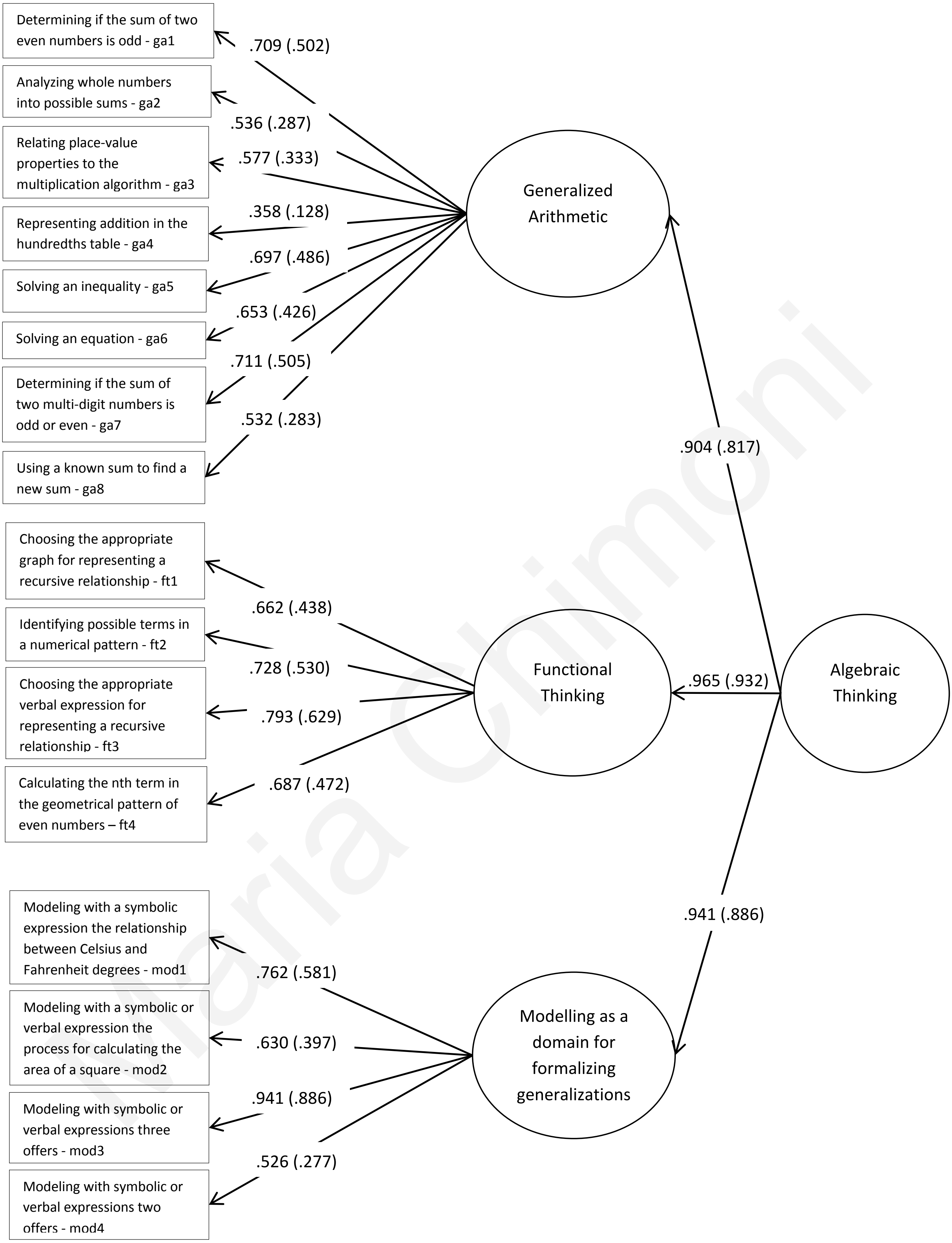
	mod1	mod2	mod3	mod4	mod5	mod6	mod7
mod1	1						
mod2	.320**	1					
mod3	.493**	.350**	1				
mod4	.193**	.197**	.231**	1			
mod5	.358**	.278**	.364**	.335**	1		
mod6	.307**	.211**	.412**	.243**	.484**	1	
mod7	.267**	.220**	.207**	.248**	.455**	.450**	1

**p<.01, **p<.05

The structure of algebraic thinking ability. This part is organized in two subdivisions. In the first subdivision, the results of the confirmatory factor analysis for testing the validity of Kaput' model about the structure and components of algebraic thinking are presented. Specifically, Confirmatory Factor Analysis (CFA) is used to test whether the set of measures in the algebraic thinking test has three-factorial dimensionality reflecting that algebraic thinking ability is synthesized by three different factors (Generalized arithmetic, Functional thinking and Modeling as a domain for expressing and formalizing generalizations). In the second subdivision, the results of exploring the extent to which this model remains stable across the four age-groups of the participants are described.

The results of the confirmatory factor analysis showed that the data of the research fitted the theoretical model at a satisfactory level (CFI=.990, TLI=.988, x^2=144.427, df =101, x^2/df =1.43, p<.01, $RMSEA$=.025,). Hence, the theoretical model of three first order factors and one second order factor can describe algebraic thinking ability. The factor loadings of all of the items to their corresponding factors are statistically significant, as shown in Figure 4.1. The distinct nature of the three factors of the model is confirmed by the fact that all observable variables load only on one first-order factor. The fitting of the data to the structure of the theoretical model confirms that the items in the algebraic thinking test measure three distinct dimensions of algebraic thinking ability.

The results of the analysis indicated that the interpreted dispersion of the items that were finally included in the model was relatively high (see Figure 4.1.). Therefore the items interpret the dispersion of the factors of the model. The factor loadings of all the first-order factors to the corresponding higher order factor were statistically significant and very high. The factors of performance in the items of "Functional thinking" , "Modeling as a domain for expressing and formalizing generalizations" and "Generalized arithmetic" had a high ability for predicting the second order factor (r^2=.932, p<.01, r^2=.886, p<.01 and r^2=.817, p<.01 respectively).



Note. The first number indicates factor loading and the number in the parenthesis indicates the corresponding interpreted dispersion (r^2)

Figure 4.1. The model of algebraic thinking ability.

Students' ability in algebraic thinking components. Table 4.3 presents the results of descriptive statistics in respect to the students' performance in the three factors of algebraic thinking ability. In order to describe the performance of the students as high, average and low, the sum of the mean of students' overall performance in the algebraic thinking test plus the standard deviation was used as indices of reference (mean + standard deviation). Specifically, the mean of the overall performance in the algebraic thinking test was .44 and the standard deviation was .25. Their sum is .69. In the light of this consideration, the performance of a group is considered to be high when is equal or higher than .69, average when is lower than .69 and higher or equal to .44 and low when is lower than .44.

The first three categories of table 4.3 correspond to the means, standard deviations and range of the algebraic thinking measures; the next three categories represent the information concerning the distribution of scores on continuous variables. As shown in Table 4.3, the highest mean of the students was in the "Generalized Arithmetic" items (M=.530). The second higher mean of the students was and in the "Functional Thinking" items (M=.507). The lowest mean of the students was in the "Modelling for expressing and formalizing generalizations" items (M=.302). According to these results, students appear to have an average performance in the items of "Generalized Arithmetic" and "Functional thinking". Their performance is low in the items of "Modeling".

The maximum value of performance in all of the categories of items was 1 and the minimum was 0. The range of the students' performance was 1, showing that there were subjects that responded correctly to all of the items of a specific category, as well as subjects that did not respond correctly to any item of a specific category. The values of skewness and kurtosis were smaller than 2 and larger than -2. This result suggests that the variables of the students' performance for the three categories of items in the algebraic thinking test follow a normal distribution.

Table 4.3

Descriptive Results of the Students' Performance in Algebraic Thinking Components

Items of the test	Mean	Standard Deviation	Range	Skewness	Kyrtosis
Generalized Arithmetic	.530	.230	1	-.172	-.798
Functional Thinking	.507	.352	1	-.044	-1.275
Modelling	.302	.302	1	.605	-.375

Correlations between the three components of algebraic thinking ability. Table 4.4 presents the correlations between the performances of the subjects in the three categories of items in the algebraic thinking test which represent the three first order factors of the proposed model for the structure and components of the algebraic thinking. The highest correlation appears between the factors "Functional thinking" and "Modelling as a domain for expressing and formalizing generalizations" ($r=.601$, $p<.01$). As far as it concerns the rest of the correlations, "Generalized arithmetic" and "Functional thinking" are highly correlated ($r=.587$, $p<.01$). Also, the factor "Generalized arithmetic" and "Modelling as a domain for formalizing generalizations" have a high correlation ($r=.564$, $p<.01$). The fact that all correlations between the subjects' performance in the three categories of items in the algebraic thinking test were statistically significant suggests that they measure the same ability. This finding resonates with the theoretical framework that was verified with the Confirmatory Factor Analysis, which posits that different categories of items constitute algebraic thinking, as a higher order factor.

Table 4.4

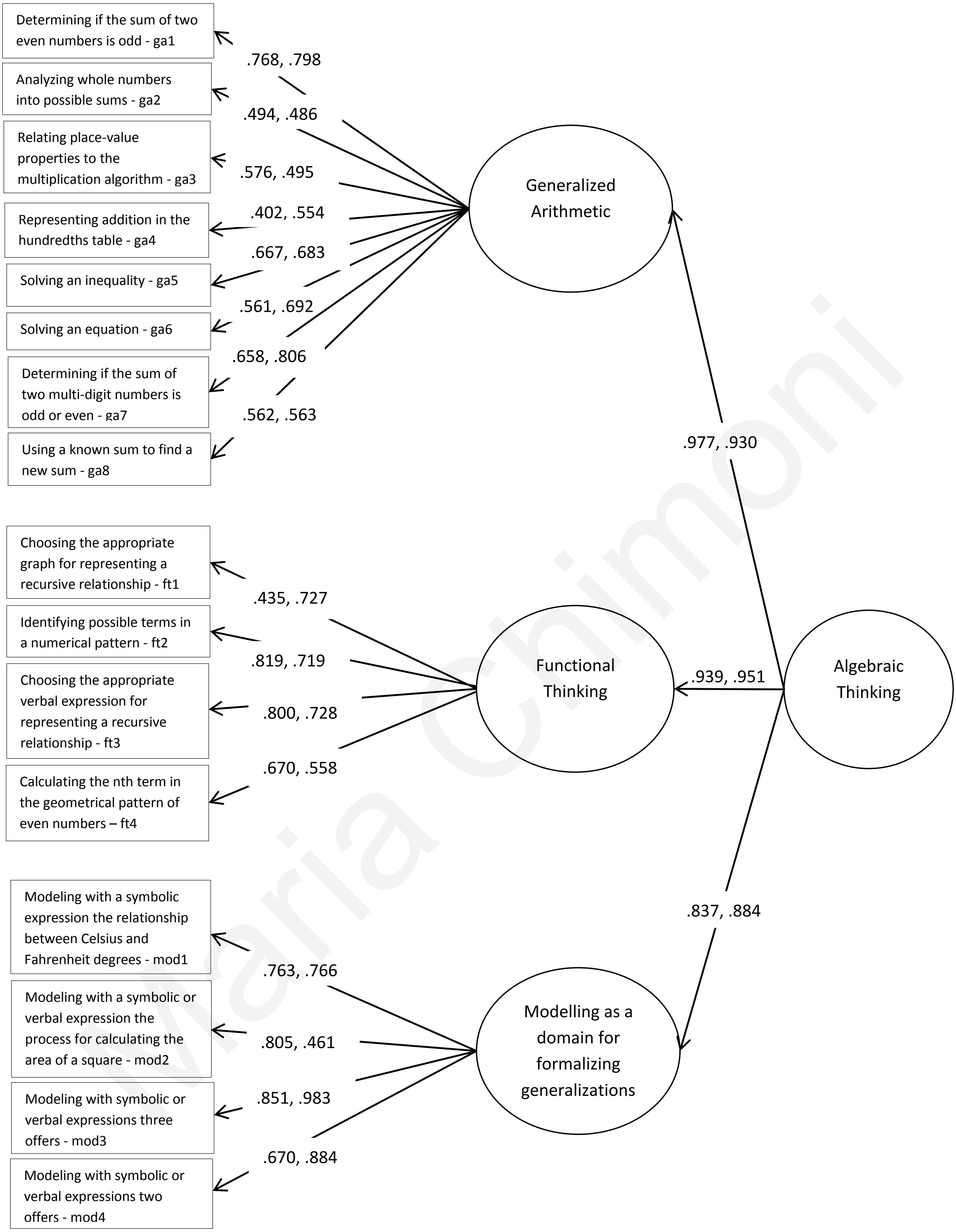
Correlations Between the First Order Factors of Algebraic Thinking Ability

	GA	FT	MOD
GA	1		
FT	,587**	1	
MOD	,564**	**,601****	1

Note. Code GA corresponds to the factor "Generalized Arithmetic, FT to the factor "Functional Thinking", MOD to the factor "Modeling as a domain for expressing and formalizing generalizations"
**p<.01

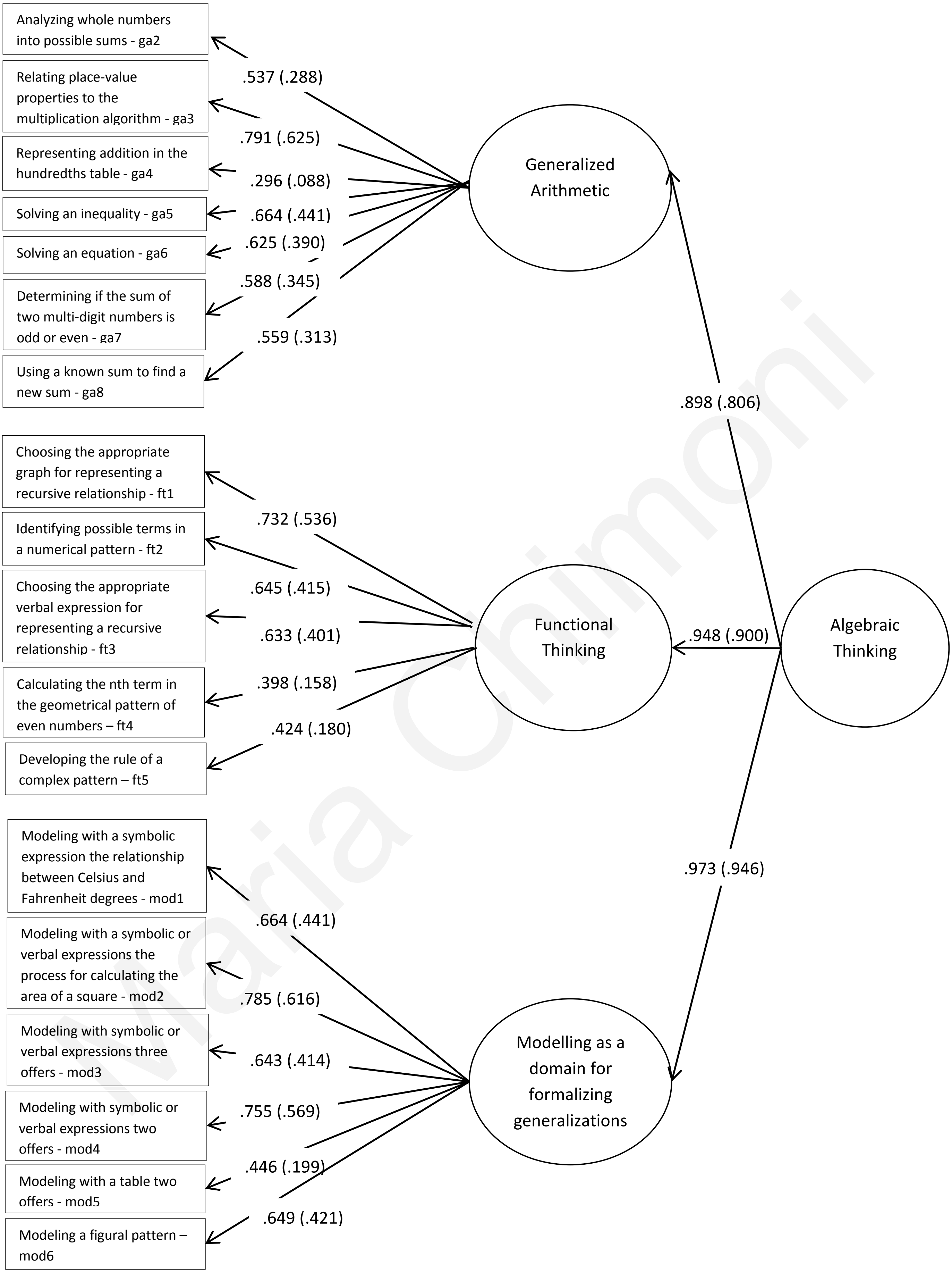
Examination of the stability of the model for algebraic thinking. To examine the stability of the structure of the proposed model for algebraic thinking ability, the validity of the model was tested considering the presence of four different age-groups in the sample. The results of this analysis confirmed its stability when only two out of the four age-groups of the participants were included in the analysis. Specifically, the results of the confirmatory factor analysis suggested that the model remains stable only for Grades 5 and 6 (see Figure 4.2).

Considering this result, the validity of the model was tested in Grade 4 and Grade 7 separately. The results confirm that in these groups, three distinct first order factors compose a second-order factor. Nevertheless, the items of the best fitting model that seem to have statistically significant factor loadings to the corresponding first-order factors were not exactly the same. This result suggests that the model remains stable from grade to grade in respect to the structure. However, the items that interpret the dispersion of each factor are slightly differentiated, reflecting different abilities of students of different ages for solving specific algebraic tasks. In each age-group, the factor loadings of all the first-order factors to the corresponding higher-order factor were statistically significant and very high. Figures 4.3, 4.4 and 4.5 present analytically the results for Grade 5 and 6, Grade 4 and Grade 7.



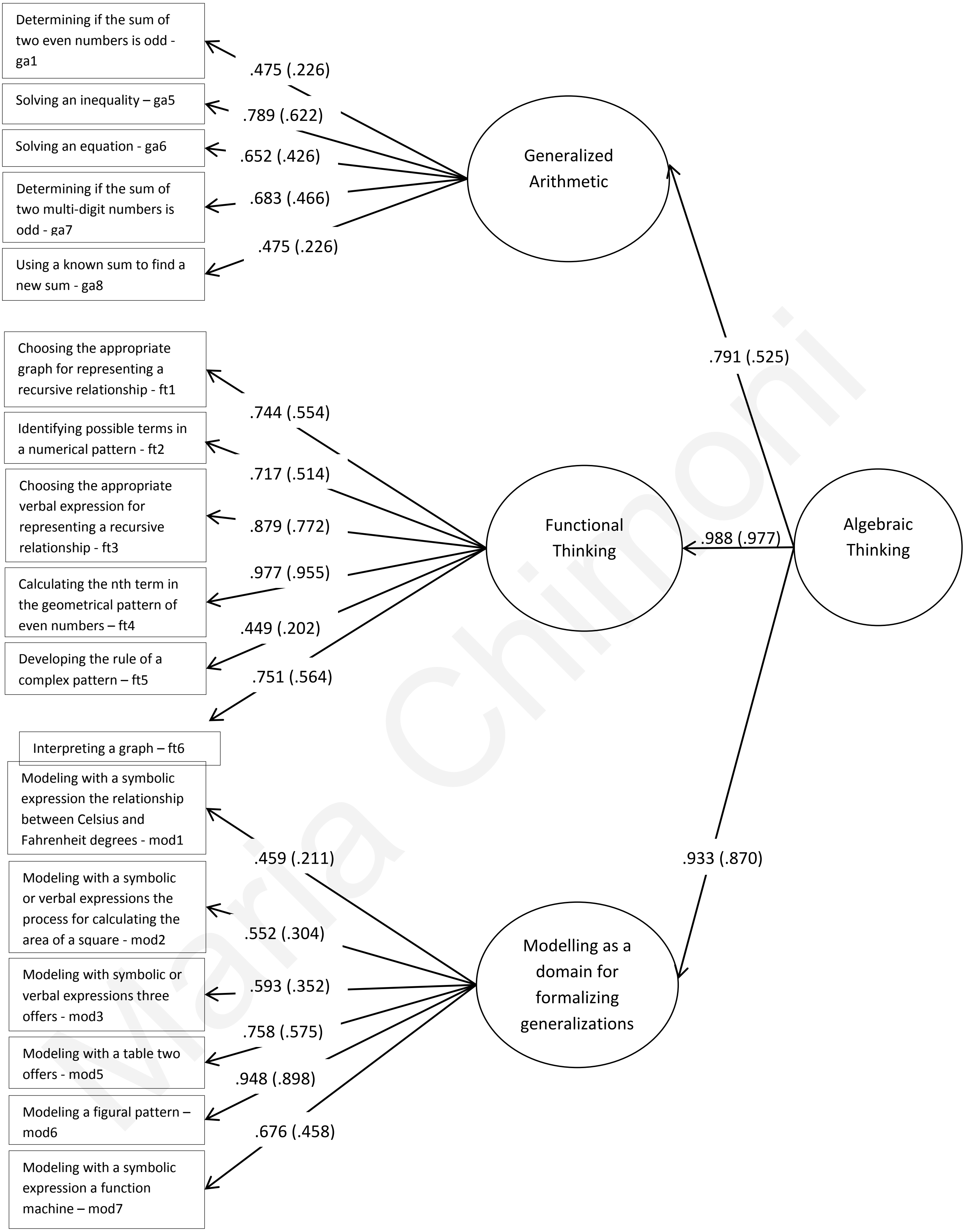
Note. The first number indicates factor loading for Grade 5 and the second for Grade 6.
CFI=.951, TLI=.949, x2=305.730, df=212, x2/df=1.44, p<.05, RMSEA=.051

Figure 4.2. The model of algebraic thinking ability for Grades 5 and 6.



CFI=.956, TLI=.949, x2=166.211, df=132, x2/df =1.26, p<.05, RMSEA=.039

Figure 4.3. The model of algebraic thinking ability for Grade 4.



CFI=.961, *TLI*=.955, x^2=182.534, df =116, x^2/df =1.57, p<.01, *RMSEA*=.059

Figure 4.4. The model of algebraic thinking ability for Grade 7.

Students' ability in algebraic thinking components by grade level. After providing evidence for the stability of the model of algebraic thinking ability across the four age-groups, in the next parts further details about the performance of the students at each age-group are explored. Table 4.5 presents descriptive results (means and the standard deviations) regarding students' performance in the algebraic thinking test at each grade level. The results of the analysis showed that the mean increases from grade to grade. The performance of the students in Grade 4 is considered as low. The performance of the students in Grades 5, 6 and 7 is considered as average.

Table 4.5

Means and Standard Deviations of the Students' Performance in Algebraic Thinking by Grade Level

Factor	Grade 4		Grade 5		Grade 6		Grade 7	
	M	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
Algebraic Thinking	.404	.332	.508	.246	.582	.265	.606	.195

To further investigate whether there are any differences in the performance of the students from different grade levels in algebraic thinking ability components, an analysis of variance (ANOVA) was performed. In this analysis, the dependent variable was the performance in the algebraic thinking test and the independent variable was students' grade level. The results of the analysis of variance (ANOVA), as reported in Table 4.6, showed that there are statistically significant differences between the students of the four grades that participated in the study (Pillai's $F=20.798$, $p<.01$).

Table 4.6

Results of the Analysis of Variance for Algebraic Thinking Ability by Grade

	Sum of Squares	df	Mean Square	F	Significance
Between Groups	927.430	3	309.143	20.798	.000
Within Groups	9349.652	629	14.864		

Table 4.7 presents the results of post-hoc analyses (Bonferroni criterion) which were performed in order to further investigate statistically significant differences in algebraic thinking ability between the different grade levels.

Table 4.7

Comparisons of the Students' Performance in Algebraic Thinking Ability at the four Grade Levels

Dependent Variable	School Grade X	School Grade Y	Post-hoc significance
Algebraic Thinking	Grade 4	Grade 5	.000
		Grade 6	.000
		Grade 7	.000
	Grade 5	Grade 4	.000
		Grade 6	1.000
		Grade 7	.001
	Grade 6	Grade 4	.000
		Grade 5	1.000
		Grade 7	.036
	Grade 7	Grade 4	.000
		Grade 5	.001
		Grade 6	.036

As it is shown, there are statistically significant differences between Grade 4 and all other grade levels in their algebraic thinking ability. There are also statistically significant differences between Grade 7 and all other grades. The results indicate that there is not a statistical difference between Grade 5 and Grade 6 regarding their algebraic thinking ability.

Table 4.8 presents the means and the standard deviations of the students at each grade level, regarding their performance in the three components of algebraic thinking ability. The results of the analysis showed that the mean scores increase from grade to grade.

Table 4.8

Means and Standard Deviations of the Subjects' Performance in Algebraic Thinking Components by Grade Level

Factor	Grade 4		Grade 5		Grade 6		Grade 7	
	M	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
Generalized Arithmetic	.500	.262	.556	.264	.582	.265	.669	.209
Functional Thinking	.373	.323	.514	.337	.604	.325	.609	.342
Modelling as a domain for formalizing generalizations	.254	.258	.364	.320	.407	.255	.513	.257

In order to investigate whether there are any differences in the performances of the students from different grade levels in the three components of algebraic thinking, a multivariate analysis of variance (MANOVA) was performed. In this analysis, there were three dependent variables: the performance in the items of "Generalized arithmetic", the performance in the items of "Functional thinking" and the performance in the items of “Modeling”. The independent variable was students’ grade level. The results of the analysis showed that there are statistically significant differences between the students of the four grade levels that participated in the study (Pillai’s F=1148.548, p<.01). As presented in Table 4.9, there are statistically significant differences between the students of the four grades in their abilities in the three components of algebraic thinking ability.

Table 4.9

Results of the Multiple Analysis of Variance for the Components of Algebraic Thinking Ability by Grade

	Sum of Squares	Degrees of Freedom	Mean Square	F	Significance
Generalized Arithmetic	140.406	3	46.802	11.417	.000
Functional Thinking	94.024	3	31.341	17.856	.000
Modeling as a Domain for Formalizing generalizations	129.960	3	43.320	23.009	.000

Post-hoc analyses were also performed to reveal statistically significant differences in ability in the components of algebraic thinking between the grade levels. The data of this analysis are presented in Table 4.10. As it is shown, in the “Generalized arithmetic” component indicate that there are not statistically significant differences between Grade 4 and Grade 5 and between Grade 5 and Grade 6. There are statistically significant differences between Grade 7 and all other Grade levels.

Table 4.10

Comparisons of the Students' Performance in the Components of Algebraic Thinking Ability at the four Grade Levels

Dependent Variable	School Grade X	School Grade Y	Post-hoc significance
Generalized Arithmetic	Grade 4	Grade 5	.253
		Grade 6	.018
		Grade 7	.000
	Grade 5	Grade 4	.253
		Grade 6	1.000
		Grade 7	.001
	Grade 6	Grade 4	.018
		Grade 5	1.000
		Grade 7	.019
	Grade 7	Grade 4	.000
		Grade 5	.001
		Grade 6	.019
Functional Thinking	Grade 4	Grade 5	.001
		Grade 6	.000
		Grade 7	.000
	Grade 5	Grade 4	.001
		Grade 6	.082
		Grade 7	.084
	Grade 6	Grade 4	.000
		Grade 5	.082
		Grade 7	1.000
	Grade 7	Grade 4	.000
		Grade 5	.084
		Grade 6	1.000
Modeling as a Domain for Formalizing Generalizations	Grade 4	Grade 5	.002
		Grade 6	.000
		Grade 7	.000
	Grade 5	Grade 4	.002
		Grade 6	.920
		Grade 7	.000
	Grade 6	Grade 4	.000
		Grade 5	.920
		Grade 7	.006
	Grade 7	Grade 4	.000
		Grade 5	.000
		Grade 6	.006

As far as it concerns the component of "Functional thinking", the analysis indicates that there are statistically significant differences between Grade 4 and all other Grade levels. There are not any significant differences between Grade 5 and Grade 6 and between Grade 6 and Grade 7, in respect to the abilities of the students in the "Functional thinking" factor.

As it is shown in Table 4.10, in the "Modeling" component, there are statistically significant differences between there are not statistically significant differences between Grade 4 and all other Grade levels. There are also statistically significant differences between Grade 7 and all other Grade levels. Grade 5 and Grade 6 appear not to have any statistically significant differences regarding the abilities of the students in solving the modeling tasks.

Classes of Ability in the Components of Algebraic Thinking

In this section, as part of the investigation of students' algebraic thinking from a pedagogical perspective, the possibility for tracing classes of algebraic thinking ability is examined. Understanding the extent to which students in the sample varied according to their level of ability will be based on their performance in the algebraic thinking test.

Specifically, this section presents the results concerning the third aim of the study by taking up the third, fourth and fifth research questions:

(3) What are the classes of algebraic thinking ability of 10- to 13-year-old students?

(4) What are the characteristics of students' performance in algebraic thinking at different groups of ability?

(5) Is there a consisted hierarchical trend of students' algebraic thinking ability?

This section is organized in three parts. The first part addresses the first question about the possibility of identifying groups of students which reflect different levels of algebraic thinking ability; namely, it presents the results of the quantitative analyses pertaining to analyze students ‘performance in the items of the algebraic thinking test. Statistical procedures for classifying students based on their ability in the components of algebraic thinking are described. Building on the results of the first part, the second part looks across the identified classes of students and describes in detail their quantitative features in the three factors of algebraic thinking ability. These results indicate which specific items in the algebraic thinking test were able to solve the students in each class. Finally, in the third part, the existence of a specific hierarchical trend across the three factors of algebraic thinking is considered and tested.

Classes of students in the components of algebraic thinking. In order to investigate whether there are subgroups of students with similar behavior in respect to their ability in algebraic thinking components, students’ performance in the items of the algebraic thinking test was used. Specifically, the statistical method of latent class analysis (LCA) was applied which is part of mixture growth analysis (Muthen & Muthen, 1998). This method can be used for finding sub-types of related cases (latent classes) from multivariate data and for classifying cases into their most likely latent class. Given a sample of subjects measured on several variables, LCA can be used for examining whether there is a small number of basic groups into which cases fall. Once the latent class model is estimated, subjects can be classified into their most likely latent classes by means of recruitment probabilities. A recruitment probability is the probability that, for a randomly selected member of a given class, a given response pattern will be observed (Muthen & Muthen, 1998).

The validity of four consecutive models was tested, according to which the subjects of the study could be divided into two, three, four, or five groups of similar behavior to the algebraic thinking test. The model with five groups of students was not taken into consideration due to the fact that the average class probability of the subjects to be classified in a specific group was not satisfactory. The results concerning the assumption that there are two, three or four classes of subjects are presented in Table 4.11. The best fitting model with the smallest AIC (88.431) and BIC (169.934) indices and the model where the analyses indicated recruitment (at least 10 times) of the best loglikelihood was the one involving four classes. The entropy statistic is a summary measure which assesses the quality of the classification. The entropy value is closed to 1, indicating high classification accuracy.

Table 4.11

Fit Indices for Models with Different Number of Classes

Indices	Entropy	AIC	BIC	Adjusted BIC
2 Classes model	.813	240.623	285.903	254.151
3 Classes model	.832	109.089	172.481	128.029
4 Classes model	.809	88.431	169.934	112.782

Taking into consideration the average class probabilities as shown in Table 4.12, it can be concluded that classes were quite distinct, indicating that each class has its own characteristics.

Table 4.12

Average Latent Class Probabilities

Latent class probabilities	Class 1	Class 2	Class 3	Class 4
Class 1 Subjects	.904	.000	.096	.000
Class 2 Subjects	.000	.854	.081	.066
Class 3 Subjects	.037	.094	.869	.000
Class 4 Subjects	.000	.079	.000	.921

Descriptive results of the four classes of students in algebraic thinking ability. According to the latent class analysis the percentages of the students that are classified within the four Classes are: 31.4% was within Class 1, 23.1% was within Class 2, 19.3% was within Class 3 and 26.2% was within Class 4.

Table 4.13 presents the way that the percentage of students that are classified within each class varies according to grade level. 37.2% of Grade 4 students are classified within Class 1, 26.6% are classified within Class 2, 21.2% are classified within Class 3 and 11.2% are classified within Class 4. As far as it concerns Grade 5, 22.8% of the students are classified within Class 1, 22.8% are classified within Class 2, 23.5% are classified within Class 3 and the 26.8% are classified within Class 4. The percentages of Grade 6 students were 21.4% for Class 1, 25.3% for Class 2, 31.8% for Class 3, and 31.3% for Class 4. The percentages of Grade 7 students were 18.6% for Class 1, 25.3% for Class 2, 23.5% for Class 3 and 30.7% for Class 4.

Table 4.13

Percentages of Students in the Four Classes of Algebraic Thinking Ability

	Class 1	Class 2	Class 3	Class 4
Grade 4	37.2 %	26.6%	21.2%	11.2%
Grade 5	22.8%	22.8.%	23.5%	26.8%
Grade 6	21.4%	25.3%	31.8%	31.3%
Grade 7	18.6%	25.3%	23.5%	30.7%
Sum	31.4%	23.1%	19.3%	26.2%

The majority of Grade 4 students are classified within Class 1 (37.2%). Similar percentages of Grade 5 students are classified within Class 1, Class 2 and Class 3 (22.8%, 22.8% and 23.5% respectively). The majority of the students in Grade 5 are classified within Class 4 (26.8%). Similar percentages of Class 6 students are classified within Class 3 and Class 4 (31.8% and 31.3% respectively). The majority of

Grade 7 students are classified within Class 4 (30.7%). The percentage of the students that belong in Class 4 seems to be increased from Grade 4 to Grade 5. Similar percentages of students from Grade 6 and Grade 7 seem to belong to Class 4 (31.3% and 30.7% respectively).

In order to investigate whether there are statistically significant differences between the subjects of the four classes concerning their overall performance in the algebraic thinking test, analysis of variance was performed (ANOVA). The results of this analysis suggest that there are statistically important differences regarding the general algebraic thinking ability of the students in each class (Pillai's F=400.621, $p<.01$). The means and standard deviations of each class regarding the overall performance in the algebraic thinking test are reported in Table 4.14.

Table 4.14

Means and Standard Deviations of Students' Performance in Overall Algebraic Thinking Ability in Each Class

	Class 1		Class 2		Class 3		Class 4	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Overall Algebraic Thinking Ability	.202	.128	.394	.130	.521	.161	.709	.173

Additionally, a multiple analysis of variance (MANOVA) was performed in order to investigate whether there are statistically significant differences in students' ability in the three components of algebraic thinking (Generalized arithmetic, Functional thinking and Modeling as a domain for expressing and formalizing generalizations). This analysis showed that there are statistically significant differences between the groups in the three algebraic thinking components (Pillai's F= 82.133, $p<.01$). Table 4.15 presents the means and standard deviations of the students in each class in the three components of algebraic thinking. The mean performance of each class in the three components of algebraic thinking was significantly higher than the corresponding mean of the previous class. All the differences are statistically significant at the <.01 level. In order to describe the performance of the students in each class as high, average and low, the sum of the mean of students' overall performance in the algebraic thinking test plus the standard deviation was used (mean + standard deviation). As reported in the previous section of this chapter, the mean was .44 and the standard deviation was .25. An indice for considering the performance of the students as high, average and low could be their sum (.44 + .25 = .69). In the light of this consideration, the performance of a group is considered to be high when is equal or higher than .69, average when is lower than .69 and higher or equal to .44 and low when is lower than .44.

Table 4.15

Means and Standard Deviations of Performance in the Components of Algebraic Thinking for Each Class

	Class 1		Class 2		Class 3		Class 4	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Generalized arithmetic	.322	.222	.502	.224	.608	.231	.747	.175
Functional thinking	.097	.129	.427	.124	.549	.234	.717	.273
Modeling	.094	.158	.196	.040	.361	.277	.642	.268

Table 4.15 shows that students in Class 4 outperformed students in Class 1, Class 2 and Class 3 in the tasks of all of the three components of algebraic thinking. The means of students in Class 1 in all tasks was below .44, showing that these students had difficulties in conceptualizing algebraic thinking ideas. Class 2 students had difficulties in the tasks of "Functional thinking" and especially in the tasks of

“Modeling as a domain for expressing and formalizing generalizations”, since their mean of performance was below .44. These students were more successful in the tasks of “Generalized arithmetic” ($\bar{x}$=.502). Class 3 students had difficulties in the tasks of “Modeling as a domain for expressing and formalizing generalizations” since their mean of performance was below .44. These students were more successful in solving “Generalized arithmetic” tasks and “Functional thinking” tasks ($\bar{x}$=.608 and $\bar{x}$=.549 respectively). Finally, class 4 students not only seemed to have a high performance in the “Generalized arithmetic” tasks and “Functional thinking” tasks ($\bar{x}$=.747 and $\bar{x}$ =.717 respectively) but also to have an ability to solve the “Modeling as a domain for expressing and formalizing generalizations” tasks ($\bar{x}$=.642).

Table 4.17 summarizes the characteristics of the four classes of the students regarding the means of the classes in the three components of algebraic thinking. It seems that the students of Class 1 had low performance in all of the components of algebraic thinking. The students of Class 2 had an average performance in the factor “Generalized Arithmetic” and low performance in the factors “Functional Thinking” and “Modeling as a domain for expressing and formalizing generalizations”. The students of Class 3 had average performance in the factors “Generalized arithmetic” and “Functional thinking” and low performance in the factor “Modeling as a domain for expressing and formalizing generalizations”. The students of Class 4 had high performance in the factors “Generalized arithmetic” and “Functional thinking” and average performance in the factor “Modeling as a domain for expressing and formalizing generalizations”.

Table 4.16

Characteristics of the four Classes in the Components of Algebraic Thinking

	Class1	Class 2	Class 3	Class 4
High Performance (M≥.69)				GA, FT
Average Performance (.69>M≥.44)		GA	GA, FT	MOD
Low performance (M<.44)	GA, FT, MOD	FT, MOD	MOD	

Note. The code GA corresponds to the factor "Generalized arithmetic", FT corresponds to the factor "Functional thinking" and MOD to the factor "Modeling as a domain for expressing and formalizing generalizations".

Characteristics of the four classes in the factors of algebraic thinking. In this section, the characteristics of the four classes of students regarding their performance in algebraic thinking are presented. This report is organized along the three distinct factors of algebraic thinking. First, the factor of generalized arithmetic is considered. In particular, the performance of each class of students in the corresponding tasks of generalized arithmetic is presented. This description is repeated for the factors of functional thinking and generalized arithmetic.

Students' characteristics of performance in the tasks of the factor "Generalized arithmetic". Table 4.18 presents the overall performance of the four groups of ability in the tasks of the factor "Generalized arithmetic". Students of Class 1 had low performance in the items ga1, ga2, ga3, ga4, ga5, ga7, and ga8. Specifically less than 50% of the students of Class1 responded correctly in these items (25.1%, 44.2%, 13%, 30.7%, 27%, 27% and 28.4%) respectively. Students of Class 1 had average performance only in the item ga6 where they had to select the possible value of an unknown in an equation (62.3% responded correctly).

Students of Class 2 had low performance in items ga3, ga4, and ga7, with 29.1% of the students responding correctly in item ga3, 41.8% in item ga4 and 42.4% in item ga7. Students in Class 2 had average performance in items ga1, ga2, ga5, and ga8 (47.5%, 64.6%, 47.5%, and 46.2% respectively responded correctly). Students in Class 2 had high performance in the item ga6 (82.3% responded correctly).

Students of Class 3 had low performance only in the item ga3 where the percentage of success was 26.4%. In items ga1, ga4, ga5, ga7, and ga8 students in Class 3 had average performance (65.9%, 50%, 59.8%, 58.3%, and 52.3% respectively responded correctly).Similar to students in Class 2, students in Class 3 also had a high performance in item ga6 (87.9%).

Students of Class 4 did not have low performance in any item of the "Generalized arithmetic" component. They had average performance in items ga3, ga4, and ga8 (54.7%, 52.5%, and 67.6% was the percentage of success in each item respectively). The percentages of students' success in items ga1, ga2, ga5, ga6 and ga7 were 82.1%, 81.6%, 83.2%, 95.5% and 79.9%), indicating high performance in these items.

Table 4.17

Characteristics of the four Classes in the Factor Generalized Arithmetic

	Class1	Class 2	Class 3	Class 4
High Performance (M≥.69)		ga6	ga2, ga6	ga1, ga2, ga5, ga6, ga7
Average Performance (.69>M≥.44)	ga6	ga1, ga2, ga5, ga8	ga1, ga4, ga5, ga7, ga8	ga3, ga4, ga8
Low performance (M<.44)	ga1, ga2, ga3, ga4, ga5, ga7, ga8	ga3, ga4, ga7	ga3	

Note. **p<.01, ga1: Determining if the sum of two even numbers is odd, ga2: Analyzing whole numbers into possible sums, ga3: Relating place-value properties to the multiplication algorithm, ga4: Representing addition in the hundredths table, ga5: Solving an inequality, ga6: Solving an equation, ga7: Determining if the sum of two multi-digit numbers is odd or even, ga8: Using a known sum to find a new sum

Students' characteristics of performance in the tasks of the factor "Functional thinking". Table 4.19 presents the overall performance of the students in the four groups of ability in the items of the factor "Functional thinking". Students of Class 1 had low performance in all of the items in this factor. Specifically, the percentages of correct responses in the items f1, f2, f3, and f4 were very low (11%, 6.5%, 7.9%, and 13% respectively). Item f4, which had the highest percentage of success among students in Class 1, represented a geometrical pattern with the rule L=2n. The second higher percentage of success was in the item f1 which requested students to choose the appropriate graph for representing a recursive relationship.

Students of Class 2 had low performance in items f1 and f2, with 34.2% and 29.1% of the students responding correctly to these items, respectively. More than

50% of the students had an average performance in items f3 and f4 (55.1% and 52.5% of the students responded correctly respectively).

Students in Class 3 had low performance only in the item f1, where 42.4% of the students were able to give a correct answer. In the items f2 and f4, the percentages of success were 49.2% and 55.3% respectively. Students in Class 3 appeared to have a high percentage of success in the item f3 (72.7%), indicating an ability in linking a a recursive relationship with its corresponding verbal expression.

Students in Class 4 did not have low performance in any of the items of the "Functional thinking" factor. Moreover, they had average performance only in the item f1 (66.5% responded correctly). In the items f2, f3, and f4 students appeared to have high percentage of success (73.2%, 73.7%, and 73.2% respectively).

Table 4.18

Characteristics of the four Classes in the Factor Functional Thinking

	Class1	Class 2	Class 3	Class 4
High Performance (M≥.69)			f3	f2, f3, f4
Average Performance (.69>M≥.44)		f3, f4	f2, f4	f1
Low performance (M<.44)	f1, f2, f3, f4	f1, f2	f1	

Note. **p<.01, f1: Choosing the appropriate graph for representing a recursive relationship, f2: Identifying possible terms in a numerical pattern, f3: Choosing the appropriate verbal expression for representing a recursive relationship, f4: Calculating the nth term in the geometrical pattern of even numbers

Students' characteristics of performance in the tasks of the factor "Modeling as a domain for expressing and formalizing generalizations". Table 4.20 presents the overall performance of the students of the four groups of ability in the items of the factor "Modeling as a domain for expressing and formalizing generalizations". Students of Class 1 had low performance in all of the items of this factor. In the items mod1, mod2, and mod4 students' percentages of success were slightly above 10% (12.6%, 14.9%, and 11.2% respectively). In the items mod3 and mod 5, the percentages of success were very low. For the item mod3 3.3% of the answers were correct. For the item mod5 3.7% of the students' answers were correct and 3.3% were partially correct.

Similar to students of Class1, students of Class2 had low performance in all of the items of the items of the factor "Modeling as a domain for expressing and formalizing generalizations". Nevertheless, their percentages of success in some items were much larger comparing to the percentages of the students of Class 1. These were: mod1 - 28.5%, mod2 - 25.3% and mod4 - 20.9%. In the item mod 5 the percentage of the students that answered correctly was 13.3% where the percentage of the students that gave a partially correct answer was 8.2%. In the item mod3, the percentage of success was very low (5.7%).

Students of Class 3 also had low performance in items mod1, mod3, mod4 and mod5. The percentages of success in the items mod1, mod3, and mod4 were 43.2%, 23.5%, 38.6% respectively. In the item mod5 9.1% of the students' answers were partially correct and 23.5% were correct. Students in Class 3 appeared to have average performance in item mod2 where 47% of the students responded correctly.

Students of Class 4 did not have low performance in any of the items of the factor "Modeling as a domain for expressing and formalizing generalizations". Specifically, these students had the percentages of success in the items mod2, mod3, and mod4, were 64.8%, 68.7%, and 63.1%, respectively, indicating an average performance. In the item mod 5, 44.1% of the students answered correctly where 12.8% of the students gave a partially correct answer. Furthermore, these students appeared to have high performance in the item mod1. A percentage of 74.3% answered correctly to the task, which requested students to model with a symbolic expression the relationship between Celsius and Fahrenheit degrees.

Table 4.19

Characteristics of the four Classes in the Factor Modeling as domain for expressing and formalizing generalizations

	Class1	Class 2	Class 3	Class 4
High Performance (M≥.69)				mod1
Average Performance (.69>M≥.44)			mod2	mod2, mod3, mod4, mod5
Low performance (M<.44)	mod1, mod2, mod3, mod4, mod5	mod1, mod2, mod3, mod4, mod5	mod1, mod3, mod4, mod5	

Note. **p<.01, mod1: Modeling with a symbolic expression the relationship between Celsius and Fahrenheit degrees, Mp2: Examining offers by modeling them through algebraic symbols, mod2: Modeling with a symbolic or verbal expression the process for calculating the area of a square, mod3: Modeling with symbolic or verbal expressions three offers, mod4: Modeling with symbolic or verbal expressions two offers, mod5: Modeling with a table two offers.

Summary of the characteristics of the four groups of algebraic thinking ability. This section presents a summary of the characteristics of the four groups of algebraic thinking. Table 4.21 presents a summary of the results. As it is extracted from the results described in the previous section, the students in Class 1 had average performance only in the item of solving an equation (ga5), where it was requested to identify the missing value of an unknown symbolized by a letter in an additive relationship. This task does not request the identification of the structure underneath the operation. On the contrary, this task relies on the calculative skills of the subjects.

Students in Class 2 appeared to have high performance in the item of solving equations. These students also had average performance in functional thinking tasks, such as the identification of the pattern of even numbers. Despite the fact that the performance of the students in Class 2 in the items of “Modeling as a domain for expressing and formalizing generalizations” was characterized as low, their percentages of success in these items were much higher than those of the students in Class 1. This result shows that students in Class 2 outperformed in respect to students in Class1, in the modeling tasks. Their highest percentage of success was found in the item were they had to model the relationship of Fahrenheit and Celsius degrees by choosing the appropriate expression.

Students in Class 3 were very successful in in the items of analyzing whole numbers into possible sums and solving an equation. The item in which they appeared to have more difficulties was the item were they had to identify and explain the error in a multiplication algorithm by focusing on place-value properties. Also these students had average or high performance in most of the items of the “Functional thinking” factor. Their highest percentage of success was in the item of linking a recursive relationship that it was represented through a diagram with an appropriate verbal expression. Additionally, these students were very successful in solving the modeling task where they had to relate the Fahrenheit and Celsius degrees in a symbolic way. In the items, such as the development of a tabular representation for examining the offers of two internet companies for downloading songs, students in Class 3 seemed to have a low performance.

The results indicated that students in Class 4 had high performance in three more items of the “Generalized arithmetic” factor, comparing to students in Class 3. Specifically, students in Class 4 were able to gain high scores in the items of determining if the sum of two even numbers is odd or even, solving an inequality, and determining if the sum of two multi-digit numbers is odd or even. These items required the application of properties of numbers and a conceptual understanding of the equality/inequality symbols. Moreover, the students of Class 4 had high performance in three out of the four items of the “Functional thinking” factor. Students within this Class solved successfully the modeling problems. Similar to the students of Class 3, their highest percentage of success was in the item that involved the relationship between Fahrenheit and Celsius degrees. Their percentage of success in items that required the comparison of offers and decision making about the most advantageous offer, students in Class 4 had satisfactory results.

Table 4.20

Characteristics of the four Classes in the three factors

	Class1	Class 2	Class 3	Class 4
High Performance (M≥.69)		ga6	ga3, ga6, f1	ga1, ga2, ga5, ga6, ga7, f2, f3, f4, mod1
Average Performance (.69>M≥.44)	ga6	ga1, ga2, ga5, ga8, f3, f4	ga1, ga4, ga5, ga7, ga8, f2, f4, mod2	ga3, ga4, ga8, f1, mod2, mod3, mod4, mod5
Low performance (M<.44)	ga1, ga2, ga3, ga4, ga5, ga7, ga8, f1, f2, f3, f4, mod1, mod2, mod3, mod4, mod5	ga3, ga4, ga7, f1, f2, mod1, mod2, mod3, mod4, mod5	ga3, f1, mod1, mod3, mod4, mod5	

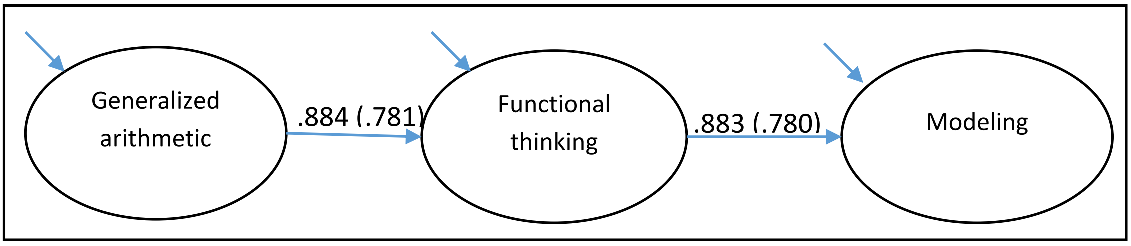
Note. **p<.01, ga1: Determining if the sum of two even numbers is odd, ga2: Analyzing whole numbers into possible sums, ga3: Relating place-value properties to the multiplication algorithm, ga4: Representing addition in the hundredths table, ga5: Solving an inequality, ga6: Solving an equation, ga7: Determining if the sum of two multi-digit numbers is odd or even, ga8: Using a known sum to find a new sum , f1: Choosing the appropriate graph for representing a recursive relationship, f2: Identifying possible terms in a numerical pattern, f3: Choosing the appropriate verbal expression for representing a recursive relationship, f4: Calculating the nth term in the geometrical pattern of even numbers, mod1: Modeling with a symbolic expression the relationship between Celsius and Fahrenheit degrees, Mp2: Examining offers by modeling them through algebraic symbols, mod2: Modeling with a symbolic or verbal expression the process for calculating the area of a square, mod3: Modeling with symbolic or verbal expressions three offers, mod4: Modeling with symbolic or verbal expressions two offers, mod5: Modeling with a table two offers.

Hierarchy of Algebraic Thinking Components

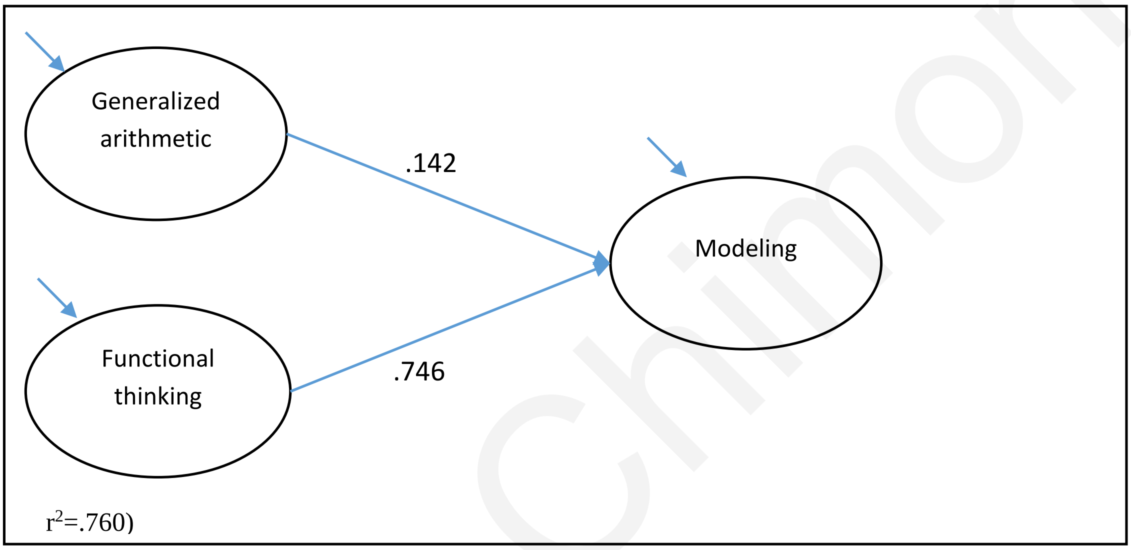
The presence of a consistent trend in level of difficulty across the three factors of algebraic thinking supports the hypothesis of the existence of a specific hierarchical trend. The results of the latent class analysis implied that there are four classes of students. Class 1 had low performance in all the components of algebraic thinking. Class 2 had an average performance in the “Generalized arithmetic” tasks. Class 3 had average performance in the “Generalized arithmetic” tasks and “Functional thinking” tasks. Class 4 had high performance in the “Generalized arithmetic” tasks and the “Functional thinking” tasks. Additionally, Class 4 had average performance in the tasks of the component “Modeling as a domain for expressing and formalizing generalizations”. This result denotes that, students grasp generalized arithmetic concepts first and then they grasp functional thinking concepts. The concepts of modeling are grasped only after generalized arithmetic and functional thinking have been conceptualized.

To further examine this sequence, three models were tested for specifying the nature of the hierarchical trend of students’ understanding of the algebraic thinking concepts. The first model, which results from the data of the previous analyses, assumes that students first understand the generalized arithmetic concepts and then are able to understand the concepts of functional thinking and modeling (see Figure 4.6). The second model assumes that students first grasp both generalized arithmetic and functional thinking concepts. Since, only students in Class 4 were able to have an average performance in the modeling concepts, it was assumed that in order to understand and develop these concepts students might need to simultaneously have developed the generalized arithmetic and functional thinking concepts. The third model assumes that modeling concepts are understood only after students first understand the concepts of generalized arithmetic. After grasping the modeling concepts, students become able to understand the concepts of functional thinking. This model was assumed based on the results of the confirmatory factor analysis and particularly on the fact that “Functional thinking” had the highest loading to the second order factor in the model of algebraic thinking ability. Latent Path analysis was used to examine the model that best fits the empirical data.

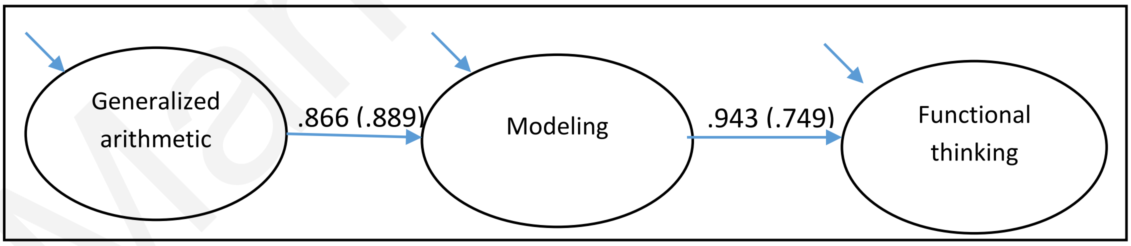
Model 1



Model 2



Model 3



Note. **p<.01, The first number indicates the regression coefficient and the number in parenthesis indicate the proportion of variability that can be explained (r^2)

Figure 4.5. The comparison of the three models of the hierarchy of algebraic thinking components.

Table 4.21

Fitting Indices of Model 1, Model 2 and Model 3

	CFI	*TLI*	x^2/df	RMSEA
Model 1	.959	.953	1.73	.033
Model 2	.959	.952	1.73	.033
Model 3	.952	.944	1.87	.036

From Figure 4.6 and Table 4.17, we can deduce that the best fitting model is model 1, since it has the best fitting indices and high regression coefficients for each of the algebraic thinking components. Specifically, the fitting indices are adequate to provide evidence that supports the structure implied in it (CFI=.959, TLI=.953, x^2=201.853, df =117, x^2/df=1.73, RMSEA=.033). While these fit indices seem to be the same also for model 2, Figure 4.6 shows that in model 2 the regression coefficient of modeling concepts on generalized arithmetic concepts is very low (.179). For this reason, Model 2 cannot be considered as the best model for describing the developmental trend among the algebraic thinking components.

These results reaffirm the developmental trend as described above and indicate that students are first more fluent in doing the generalized arithmetic tasks and then in doing the functional thinking tasks and then the modeling tasks.

Algebraic Thinking Ability and its Relation to Cognitive factors

The preceding analyses largely resonated with the existence of three distinct algebraic thinking components, as three first-order factors were yielded from the quantitative analysis. These factors appear to synthesize a second-order factor which reflects the concept of algebraic thinking. Moreover, the results presented in the previous section indicated that there are four quite distinct classes of students regarding their algebraic thinking ability; each class has its own characteristics. Further examinations have shown that there is a consistent trend in level of difficulty across the three components of algebraic thinking; the data implied that students grasp generalized arithmetic concepts first and then move to grasp functional thinking concepts. Modeling concepts are grasped only after generalized arithmetic and functional thinking have been conceptualized. This part of the study moves a step further, in order to explore this concept within a cognitive perspective.

The development of this section follows the fifth, sixth and seventh aim of the current study pertaining the investigation of the relationship between students' algebraic thinking and three categories of cognitive factors: (i) domain-specific information processing abilities, (ii) reasoning processes and (iii) general cognitive structures. Specifically the question under examination is the following:

(6) What is the relation of algebraic thinking with domain-specific processes, different types of reasoning forms and general cognitive processes of mental action?

An understanding of the association between algebraic thinking and various types of cognitive factors will be built upon data gathered from six sources:

(i) the Specialized Structural Systems test
(ii) the Naglieri Non-Verbal Ability test
(iii) the Deductive Reasoning test
(iv) the Working Memory test
(v) the Control of Processing test
(vi) the Speed of processing test

In what follows, each of the above aims is extensively elaborated. In particular, this part is organized in three sections. Each section provides results in respect to the relationship between algebraic thinking and the cognitive factors measured by the aforementioned tests. Results are reported for each age-group (Grade 4, Grade 5, Grade 6 and Grade7). Specifically, regression analyses were applied in each age-group separately in order to identify predictive relationships between different cognitive factors and algebraic thinking. Then, conclusions about which of these factors are related to algebraic thinking ability were assisted by triangulation between the different data sources and by observing similarities and differences in the results of each grade level.

Algebraic thinking ability and its relation to domain-specific information processing abilities. This section presents the results concerning the investigation of the relation between students' ability in algebraic thinking and specific information processing abilities. In order to measure students' abilities in domain-specific processes, a test on Specialized Structural Systems was used. The test measured students' abilities in four cognitive constructs: the Spatial-Imaginal System, the Causal-Experimental System, the Qualitative-Analytic System, and the Verbal-Propositional System. Descriptive information in respect to the test on Specialized Structural Systems is described first. Secondly, the results of correlation analyses between all the factors of algebraic thinking and the four types of Specialized Structural Systems are reported. Finally, the relationship between algebraic thinking and Specialized Structural Systems for each age-group (Grade 4, Grade 5, Grade 6 and Grade7) is described. Regression analyses were applied in each age-group separately. The purpose was to generate for each age group an equation to describe the statistical relationship between the four types of Specialized Structural Systems (predictor variables) and algebraic thinking (response variable).

Descriptive results of the test on Specialized Structural Systems. Table 4.23 presents the results of descriptive statistics analysis in each of the four categories of the items in the Specialized Structural Systems test. The first three categories of this table correspond to the means, standard deviations and range of the algebraic thinking measures; the next three categories represent the information concerning the distribution of scores on continuous variables. As the figures in Table 4.23 show, the highest mean of the students was in the items of the “Qualitative-Analytic System” (M=.624). The second higher mean of the subjects was and in the items of the “Causal-Experimental System” (M=.603). The lowest means of the subjects were in the items of “Spatial-Imaginal System” (M=.548) and “Verbal-Propositional System” (M=.358). The maximum value of performance in all of the categories of items was 1 and the minimum was 0. The range of thestudents’ performance was 1, showing that there were students that responded correctly to all of the items of a specific category, as well as students that did not respond correctly to any item of a specific category. The Skewness and Kyrtosis values suggest that the variables of the students’ performance for the items of the four systems in the test follow a normal distribution.

Table 4.22

Descriptive Results of the Specialized Structural Systems Test According to the Category of the Item

Items of the test	Mean	Standard Deviation	Range	Skewness	Kyrtosis
Spatial-Imaginal System	.548	.304	1	-.089	-.826
Causal-Experimental System	.603	.353	1	-.410	-1.068
Qualitative-Analytic System	.624	.231	1	-.429	.095
Verbal-Propositional System	.358	.815	1	.463	-.491

Relation between the factors of algebraic thinking and the Specialized Structural Systems. Table 4.24 shows the correlations between the three factors of algebraic thinking and the Specialized Structural Systems. There were significant correlations between all the factors of algebraic thinking and the four categories of the Specialized Structural Systems.

Table 4.23

Correlations between the Performance of the Subjects in the Algebraic Thinking Factors and the Specialized Structural Systems

System / Factor	Spatial-Imaginal	Causal-Experimental	Qualitative-Analytic	Verbal-Propositional	SSSs
Generalized arithmetic	.089*	.302**	.240**	.251**	.242**
Functional thinking	.132**	.334**	.313**	.292**	.295**
Modeling	.105*	.350**	.305**	.317**	.268**
Algebraic Thinking	.125**	.384**	.330**	.333**	.310**

*p<.05, **p<.01

Students' performance in all of the algebraic thinking factors is positively related to the abilities involved in Specialized Structural Systems ($r_{gen.arithmetic}$=.242, $r_{funct.thinking}$=.295, $r_{modeling}$=.268, p<.01). The overall performance in the algebraic thinking test is also positively related to the abilities involved in Specialized Structural Systems ($r_{alg.thinking}$=.310, p<.01). Moreover, performances in the three factors and in the overall performance in the algebraic thinking test appear to have the highest correlation with the "Causal Experimental" system.

Relation between algebraic thinking ability and Specialized Structural Systems in Grade 4. Table 4.25 presents the results of Multiple Regression Analysis, where the performance of the fourth graders in the algebraic thinking test is explained by their performance in the four types of tasks in the Specialized Structural Systems test.

Table 4.24

Regression Analysis of the Performance in each of the four Specialized Structural Systems with Dependent Variable the Performance in Algebraic Thinking in Grade 4

Algebraic thinking	B	SE	Beta
Spatial-Imaginal	.083	.052	.118
Causal-Experimental	.143	.048	.225*
Qualitative-Analytic	.203	.069	.218*
Verbal-Propositional	.092	.063	.108
R^2=.193			
*p<.05			

According to the model, two out of the four Specialized Structural Systems exert a significant influence on the prediction of individuals' performance in algebraic thinking. The Causal-Experimental and the Qualitative-Analytic systems seem to have a positive effect on the dependent variable, which means that the higher these abilities are the higher is the performance of fourth graders in algebraic thinking tasks. The data show that the factor with the greatest effect on the prediction of achievement in algebraic thinking tasks is the Causal-Experimental system (β=.225). The Qualitative-Analytic system also explains a respectable proportion of variance in the fourth graders' performance in the algebraic thinking test (β=.218). The Spatial-Imaginal and the Verbal-Propositional systems do not seem to be significant predictors of students' algebraic thinking at this age group.

On the basis of the results reported in Table 4.24, the model of the regression equation was extracted. Figure 4.7 presents the coefficients of the multiple regression model. The overall performance of fourth graders in the algebraic thinking test (AT) is the criterion (depended variables) and the four types of Specialized Structural Systems, the Spatial-Imaginal (SI), the Causal Experimental (CE), the Qualitative-Analytic (QA) and the Verbal-Propositional (VP), are the predictors (independent variables).

$$AT_{4thG} = .225(CE) + .218(QA) + .118(SI) + .108(VP) + 8.504$$

Note. AT4thG: Algebraic Thinking-Grade 4, SI: Spatial-Imaginal, CE: Causal Experimental, QA: Qualitative-Analytic, VP: Verbal-Propositional

Figure 4.6. The regression model for the relation of algebraic thinking and Specialized Structural Systems in Grade 4.

Relation between algebraic thinking ability and Specialized Structural Systems in Grade 5. Table 4.26 presents the results of Multiple Regression Analysis, where the performance of the fifth graders in the algebraic thinking test is explained by their performance in the four types of tasks in the Specialized Structural Systems test.

Table 4.25

Regression Analysis of the Performance in each of the Four Specialized Structural Systems with Dependent Variable the Performance in Algebraic Thinking in Grade 5

Algebraic thinking	B	SE	Beta
Spatial-Imaginal	.165	.045	.222**
Causal-Experimental	.405	.045	.588**
Qualitative-Analytic	.031	.052	.034
Verbal-Propositional	.102	.041	.135*

R^2=.612
**p<.01, *p<.05

Three out of the four Specialized Structural Systems appear to exert a significant influence on the prediction of individuals' performance in algebraic thinking, as shown in Table 4.26. The Causal-Experimental, the Spatial-Imaginal and the Verbal-Propositional systems seem to have a significant positive effect on the dependent variable. This result indicates that the higher these abilities are the higher is the performance of fifth graders in the items of the algebraic thinking test. Similar to the results of the fourth graders, the factor with the greatest effect on the prediction of performance in algebraic thinking tasks is the Causal-Experimental system (β=.588). In fifth grade, the effect of this system on algebraic thinking becomes larger. The Spatial-Imaginal also explains an important proportion of variance in the performance in the algebraic thinking test (β=.222). The Verbal-Propositional system appears to have a positive effect on fifth graders' algebraic thinking (β=.135). In contrast to fourth graders' results, the Qualitative-Analytic system does not seem to be a significant predictor of students' algebraic thinking at this age group.

Figure 4.8 illustrates the corresponding to the above results regression equation. The overall performance of fifth graders in the algebraic thinking test (AT) is the criterion (depended variables) and the four types of Specialized Structural Systems, the Spatial-Imaginal (SI), the Causal Experimental (CE), the Qualitative-Analytic (QA) and the Verbal-Propositional (VP), are the predictors (independent variables).

$$AT_{5thG} = .588(CE) + .222(SI). + .135(VP) + .34(QA) + 21.652$$

Note. AT5thG: Algebraic Thinking-Grade 5, SI: Spatial-Imaginal, CE: Causal Experimental, QA: Qualitative-Analytic, VP: Verbal-Propositional

Figure 4.7. The regression model for the relation of algebraic thinking and Specialized Structural Systems in Grade 5.

Relation between algebraic thinking ability and Specialized Structural Systems in Grade 6. In order to examine the way sixth grades' performance in the algebraic thinking test is explained by their performance in the four types of tasks in the Specialized Structural Systems test, Multiple Regression Analysis was conducted for this age group. The results are presented in Table 4.27.

As can be seen in Table 4.27, three out of the four Specialized Structural Systems exert a significant influence on the prediction of individuals' performance in algebraic thinking. The Causal-Experimental, the Verbal-Propositional and the Spatial-Imaginal systems seem to have a positive effect on the dependent variable (β=.413, β=.210 and β=.170 respectively). Similar to the corresponding results in Grade 5, the Causal-Experimental system appears to have great effect on the prediction of achievement in algebraic thinking tasks.

Table 4.26

Regression Analysis of the performance in each of the four Specialized Structural Systems with dependent variable the performance in algebraic thinking in Grade 6

Algebraic thinking	B	SE	Beta
Spatial-Imaginal	.014	.006	.170*
Causal-Experimental	.313	.056	.413**
Qualitative-Analytic	.115	.082	.150
Verbal-Propositional	.166	.056	.210*

R^2=.354
**p<.01, *p<.05

Based on the results of the regression analysis, a regression model was extracted for the performance of sixth graders in algebraic thinking (see Figure 4.9). The overall performance of sixth graders in the algebraic thinking test (AT) is the criterion (depended variables) and the four types of Specialized Structural Systems, the Spatial-Imaginal (SI), the Causal Experimental (CE), the Qualitative-Analytic (QA) and the Verbal-Propositional (VP), are the predictors (independent variables).

$$AT_{6thG} = .413(CE) + .210(VP) + .170(SI) + .150(QA) + 5.326$$

Note. AT6thG: Algebraic Thinking-Grade 6, SI: Spatial-Imaginal, CE: Causal Experimental, QA: Qualitative-Analytic, VP: Verbal-Propositional

Figure 4.8. The regression model for the relation of algebraic thinking and Specialized Structural Systems in Grade 6.

Relation between algebraic thinking ability and Specialized Structural Systems in Grade 7. The results of Multiple Regression Analysis that examined whether there is a predictive relationship between seventh grades' performance in the four types of tasks in the Specialized Structural Systems test and their algebraic thinking ability test are reported in Table 4.27.

Table 4.27

Regression Analysis of the Performance in each of the four Specialized Structural Systems with Dependent Variable the Performance in Algebraic Thinking in Grade 7

Algebraic thinking	B	SE	Beta
Spatial-Imaginal	.144	.061	.220*
Causal-Experimental	.124	.054	.217*
Qualitative-Analytic	.069	.061	.109
Verbal-Propositional	.101	.055	.153 *

R^2=.352
*p<.05

According to the model, three out of the four Specialized Structural Systems exert a significant influence on the prediction of individuals' performance in algebraic thinking. Like Grade 5, the Causal-Experimental, the Spatial-Imaginal and the Verbal-Propositional systems seem to have a positive effect on the dependent variable, which means that the higher these abilities are the higher is the performance of seventh graders in algebraic thinking tasks. In contrast to the results regarding Grades 5 and 6, in Grade 7 the factor with the greatest effect on the prediction of performance in algebraic thinking tasks is the Spatial-Imaginal (β=.220). The Causal-Experimental explains a respectable proportion of variance in the performance in the algebraic thinking test (β=.217). The Verbal-Propositional system also seems to have a statistically significant effect on fifth graders' algebraic thinking. The Qualitative-Analytic system does not seem to be a significant predictor of students' algebraic thinking at this age group.

Figure 4.10 presents the regression equation that was extracted from the analysis described above. The overall performance of sixth graders in the algebraic thinking test (AT) is the criterion (depended variables) and the four types of Specialized Structural Systems, the Spatial-Imaginal (SI), the Causal Experimental (CE), the Qualitative-Analytic (QA) and the Verbal-Propositional (VP), are the predictors (independent variables).

$$AT_{7thG} = .220(SI) + .217(CE) + .153(VP) + .109(QA) + 27.546$$

Note. AT7thG: Algebraic Thinking-Grade 7, SI: Spatial-Imaginal, CE: Causal Experimental, QA: Qualitative-Analytic, VP: Verbal-Propositional

Figure 4.9. The regression model for the relation of algebraic thinking and Specialized Structural Systems in Grade 7.

Algebraic thinking ability and its relation to reasoning processes. This section presents the results regarding the relation between students' ability in algebraic thinking and different types of reasoning processes. In order to measure students' abilities in several types of reasoning processes two different tests were used. The first one measured students' deductive reasoning and the second one was the Naglieri Non-Verbal Ability Test (NNAT) which measures overall cognitive ability. Nevertheless, the score of the students in the NNAT test can be split into four types of reasoning processes: (i) Serial reasoning, (ii) Spatial Visualization, (iii) Reasoning by Analogy and (iv) Pattern Completion. The descriptive information of the deductive reasoning tests and the NNAT test are presented first. Then, the results of correlation analyses between all the different types of reasoning processes and algebraic thinking components are presented. Finally, the association between algebraic thinking and the different types of reasoning processes for each age-group separately (Grade 4, Grade 5, Grade 6 and Grade7) is defined. Similar to the previous part, where the association of algebraic thinking and the Specialized Structural Systems was described, this part aspires the generation of an equation for each age group, in order to describe the statistical relationship between the different types of reasoning processes (predictor variables) and algebraic thinking (response variable).

Descriptive results of the test on Deductive Reasoning test. Table 4.29 presents the results of descriptive statistics analysis in the overall performance of the students in the deductive reasoning test. The first three categories of this table correspond to the means, standard deviations and range of the algebraic thinking measures; the next three categories represent the information concerning the distribution of scores on continuous variables.

As the figures in Table 4.29 show, students had an average performance in this test (*M*=.512). The maximum value of performance in all of the categories of items was 1 and the minimum was 0. The range of the students' performance was 1, showing that there were students that responded correctly to all of the items of a specific category, as well as students that did not respond correctly to any item of a specific category. The Skewness and Kyrtosis values suggest that the variables of the students' performance for the items of the four systems in the test follow a normal distribution.

Table 4.28

Descriptive Results of the Deductive Reasoning Test

	Mean	Standard Deviation	Range	Skewness	Kyrtosis
Deductive Reasoning	.512	.251	1	.369	-.158

Relation between the factors of algebraic thinking and deductive reasoning. Table 4.30 presents the correlations between the three factors of algebraic thinking and deductive reasoning. According to the results of this analysis, deductive reasoning appears to be significantly related with all the factors of algebraic thinking ability.

Students' performance in all of the algebraic thinking factors is positively related to the abilities involved in deductive reasoning ($r_{gen.arithmetic}$=.242, $r_{funct.thinking}$=.275, $r_{modeling}$=.278, $p<.01$). The overall performance in the algebraic thinking test is also positively related to the abilities involved in deductive reasoning ($r_{alg.thinking}$=.308, $p<.01$). Moreover, students' performance in the factor of "Functional thinking" appears to have the highest correlation with the abilities involved in deductive reasoning.

Table 4.29

Correlations between the Performance of the Students in the Algebraic Thinking Factors and Deductive Reasoning

System / Factor	Deductive Reasoning
Generalized arithmetic	.242**
Functional thinking	.275**
Modeling	.278**
Algebraic Thinking	.308**

**p<.01

Relation between algebraic thinking ability and deductive reasoning in Grade 4. Table 4.31 presents the results of Regression Analysis, where the performance of the fourth graders in the algebraic thinking test is explained by their performance in the deductive reasoning test.

Table 4.30

Regression Analysis of the Performance in the Deductive Reasoning test with Dependent Variable the Performance in Algebraic Thinking in Grade 4

Algebraic thinking	B	SE	Beta
Deductive Reasoning	2.996	.695	.318**
R^2=.101 **p<.01			

According to the model, deductive reasoning exerts a significant influence on the prediction of fourth graders' performance in algebraic thinking (β=.318). Nevertheless, the value of R^2 is .101, which tells us that can account for 10.1% of the variation in algebraic thinking. This means that almost 90% of the variation in algebraic thinking cannot be explained by deductive reasoning. Figure 4.11 presents the coefficients of the multiple regression model. The overall performance of fourth graders in the algebraic thinking test (AT) is the criterion (depended variables) and deductive reasoning (DR) is the predictors (independent variable).

$$AT_{4thG} = .318(DR) + 23.015$$

Note. AT4thG: Algebraic Thinking-Grade 4, DR=Deductive Reasoning

Figure 4.10. The regression model for the relation of algebraic thinking and Deductive Reasoning in Grade 4.

Relation between algebraic thinking ability and deductive reasoning in Grade 5. Table 4.32 illustrates the relationship between the performance of the fifth graders in the algebraic thinking test and their performance in deductive reasoning test, after the conduction of Regression analyses.

Table 4.31

Regression Analysis of the Performance in the Deductive Reasoning Test with Dependent Variable the Performance in Algebraic Thinking in Grade 5

Algebraic thinking	B	SE	Beta
Deductive Reasoning	1.923	.682	.227*

R^2=.052
*p<.05

As set by the figures in Table 4.32, deductive reasoning exerts a significant influence on the prediction of fourth graders' performance in algebraic thinking (β=.227). Nevertheless, as in the case of fourth graders, the R Square of the model is very low. Figure 4.12 presents the coefficients of the multiple regression model. The overall performance of fifth graders in the algebraic thinking test (AT) is the criterion (depended variables) and deductive reasoning (DR) is the predictors (independent variable).

$$AT_{5thG} = .227(DR) + 39.101$$

Note. AT5thG: Algebraic Thinking-Grade 5, DR=Deductive Reasoning

Figure 4.11. The regression model for the relation of algebraic thinking and Deductive Reasoning in Grade 5.

Relation between algebraic thinking ability and deductive reasoning in Grade 6. Table 4.33 presents the results of Regression Analysis, where the performance of the sixth graders in the algebraic thinking test is explained by their performance in the deductive reasoning test.

Table 4.32

Regression Analysis of the Performance in the Deductive Reasoning Test with Dependent Variable the Performance in Algebraic Thinking in Grade 6

Algebraic thinking	B	SE	Beta
Deductive Reasoning	2.928	.796	.294**
R^2=.086 **p<.01			

According to the results, deductive reasoning exerts a significant influence on the prediction of sixth graders' performance in algebraic thinking (β=.294). However, the value of R Square is very low, indicating that a large proportion of the variation in algebraic thinking cannot be explained by deductive reasoning. Figure 4.13 presents the coefficients of the multiple regression model. The overall performance of sixth graders in the algebraic thinking test (AT) is the criterion (depended variables) and deductive reasoning (DR) is the predictors (independent variable).

$$AT_{6thG} = .294(DR) + 33.761$$

Note. AT6thG: Algebraic Thinking-Grade 6, DR=Deductive Reasoning

Figure 4.12. The regression model for the relation of algebraic thinking and Deductive Reasoning in Grade 6.

Relation between algebraic thinking ability and deductive reasoning in Grade 7. Table 4.34 presents the corresponding results of conducting Regression Analysis, where the dependent variable is the performance of the seventh graders in the algebraic thinking test and the independent variable is their performance in the deductive reasoning test.

Table 4.33

Regression Analysis of the Performance in the Deductive Reasoning test with Dependent Variable the Performance in Algebraic Thinking in Grade 7

Algebraic thinking	B	SE	Beta
Deductive Reasoning	3.336	.689	.402**
R^2=.161			
**p<.01			

According to the model, deductive reasoning exerts a significant influence on the prediction of sixth graders' performance in algebraic thinking (β=.402). Figure 4.14 presents the coefficients of the multiple regression model. The overall performance of seventh graders in the algebraic thinking test (AT) is the criterion (depended variables) and deductive reasoning (DR) is the predictors (independent variable).

$$AT_{7thG} = .402(DR) + 31.001$$

Note. AT7thG: Algebraic Thinking-Grade 7, DR=Deductive Reasoning

Figure 4.13. The regression model for the relation of algebraic thinking and Deductive Reasoning in Grade 7.

Descriptive results of the Naglieri Non-Verbal Ability test. Table 4.35 presents the results of descriptive statistics analysis in each of the four categories of the items in the Naglieri Non-Verbal Ability Test. The first three categories of this table correspond to the means, standard deviations and range of the algebraic thinking measures; the next three categories represent the information concerning the distribution of scores on continuous variables. As the figures in Table 4.34 illustrate, the highest mean of the subjects was in the items of "Spatial Visualization" (M=.637). The second higher mean of the subjects was and in the items of "Serial reasoning" (M=.427). The lowest means of the subjects were in the items of "Pattern Completion" (M=.397) and "Reasoning by Analogy" (M=.376). The maximum value of performance in all of the categories of items was 1 and the minimum was 0. The range of the subjects' performance was 1, showing that there were subjects that responded correctly to all of the items of a specific category, as well as subjects that did not respond correctly to any item of a specific category. The Skewness and Kyrtosis values suggest that the variables of the subjects' performance for the items of the four systems in the test follow a normal distribution.

Table 4.34

Descriptive Results of the Naglieri Non-Verbal Ability Test

Items of the test	Mean	Standard Deviation	Range	Skewness	Kyrtosis
Reasoning by Analogy	.376	.161	1	.979	.483
Pattern Completion	.397	.164	1	-.158	-.270
Spatial Visualization	.637	.332	1	.357	-.149
Serial Reasoning	.427	.213	1	.274	.480

Relation between the factors of algebraic thinking and reasoning processes in the Naglieri Non-Verbal Ability test. Table 4.36 presents the correlations between the three factors of algebraic thinking and reasoning processes in the Naglieri Non-Verbal Ability test (NNAT). There were significant correlations between all the factors of algebraic thinking and the four types of reasoning processes involved in the NNAT.

Table 4.35

Correlations between the Performance of the Subjects in the Algebraic Thinking Factors and the NNAT Abilities

System / Factor	Reasoning by Analogy	Pattern Completion	Spatial Visualization	Serial Reasoning	Overall Cognitive Ability
Generalized arithmetic	.435**	.388**	.451**	.526**	.453**
Functional thinking	.375**	.358**	.395**	.464**	.422**
Modeling	.376**	.335**	.455**	.449**	.422**
Algebraic Thinking	.470**	.427**	.512**	.569**	.510**

**p<.01

Students' performance in all of the algebraic thinking factors is positively related to the overall cognitive abilities involved in the NNAT ($r_{gen.arithmetic}$=.423, $r_{funct.thinking}$=.460, $r_{modeling}$=.409, $p<.01$). Students' performance in the factor of "Functional thinking" appears to have the highest correlation with the overall cognitive ability. The overall performance in the algebraic thinking test is also positively related to the overall cognitive ability ($r_{alg.thinking}$=.472, $p<.01$). Moreover, the data in Table 4.35 show that students' overall performance in the algebraic thinking test has the highest correlations with the abilities involved in "Serial reasoning" and "Spatial Visualization".

Relation between algebraic thinking ability and NNAT abilities in Grade 4. In order to examine the relationship between the performance of students in Grade 4 with their performance in the NNAT, Multiple Regression Analysis was conducted. Specifically, the performance of the students in the algebraic thinking test was considered as the dependent variable where their performance in the four types of processes included in the NNAT were considered as the independent variables. The results of this analysis are reported in Table 4.36.

Table 4.36

Regression Analysis of the Performance in Each of the Four Types of Tasks in the NNAT with Dependent Variable the Performance in Algebraic Thinking in Grade 4

Algebraic thinking	B	SE	Beta
Reasoning by Analogy	.225	.074	.252*
Pattern Completion	.098	.075	.129
Spatial Visualization	.141	.099	.152
Serial Reasoning	.220	.089	.232*

R^2=.426
*$p<.05$

According to the model, two out of the four types of abilities involved in the NNAT exert a significant influence on the prediction of individuals' performance in algebraic thinking. The Reasoning by Analogy and the Serial Reasoning processes seem to have a positive effect on the dependent variable, which means that the higher these abilities are the higher is the performance of fourth graders in algebraic thinking (β=.252 and β=.232 respectively). The Pattern Completion and the Spatial Visualization processes do not seem to be significant predictors of students' algebraic thinking at this age group.

The results reported above, provide information for extracting the model of the regression equation. Figure 4.15 presents the coefficients of the multiple regression model. The overall performance of fourth graders in the algebraic thinking test (AT) is the criterion (depended variables) and the four types of cognitive abilities, the Reasoning by Analogy (RA), the Pattern Completion (PC), the Spatial Visualization (SV) and the Serial Reasoning (SR) are the predictors (independent variables). The value of the constant was not included in the following equation since its level of significance was higher than .05.

$$AT_{4thG} = .252(RA) + 232(SR) +.152(SV) + .129(PC) + .386$$

Note. AT4thG: Algebraic Thinking-Grade 4, RA=Reasoning by Analogy, SR=Serial Reasoning, SV=Spatial Visualization and PC=Pattern Completion

Figure 4.14. The regression model for the relation of algebraic thinking and NNAT abilities in Grade 4.

Relation between algebraic thinking ability and NNAT abilities in Grade 5. Multiple Regression analysis was conducted for examining the relationship between the performance in the algebraic thinking test and the NNAT for the students in Grade 5. Similar to the previous part, the aim was to describe the way by which the performance in the algebraic thinking test is explained by the performance in the four types of tasks in the NNAT test. As shown in Table 4.37, this relationship in Grade 5 does not follow the same pattern as in Grade 5. In particular, the Reasoning by Analogy does not appear to be a significant predictor of the fifth grades in the algebraic thinking test.

According to the model, two out of the four types of abilities involved in the NNAT exert a significant influence on the prediction of individuals' performance in algebraic thinking. The Serial Reasoning and the Spatial Visualization processes seem to have a positive effect on the dependent variable, which means that the higher these abilities are the higher is the performance of fifth graders in algebraic thinking (β=.369 and β=.184 respectively). The Pattern Completion and the Reasoning by Analogy processes do not seem to be a significant predictor of students' algebraic thinking at this age group.

Table 4.37

Regression Analysis of the Performance in Each of the Four Types of Tasks in the NNAT with Dependent Variable the Performance in Algebraic Thinking in Grade 5

Algebraic thinking	B	SE	Beta
Reasoning by Analogy	.075	.061	.093
Pattern Completion	.013	.093	.012
Spatial Visualization	.181	.084	.184*
Serial Reasoning	.358	.085	.369**

R^2=.305

*p<.05 , **p<.01

On the basis of the results reported in Table 4.38, the model of the regression equation was extracted. Figure 4.16 presents the coefficients of the multiple regression model. The overall performance of fifth graders in the algebraic thinking test (AT) is the criterion (depended variables) and the four types of of cognitive abilities, the Reasoning by Analogy (RA), the Pattern Completion (PC), the Spatial Visualization (SV) and the Serial Reasoning (SR), are the predictors (independent variables).

$$AT_{5thG} = .369(SR) + .184(SV) + .093(PC) + .012(RA) + 20.661$$

Note. AT5thG: Algebraic Thinking-Grade 5, RA=Reasoning by Analogy, SR=Serial Reasoning, SV=Spatial Visualization and PC=Pattern Completion

Figure 4.15. The regression model for the relation of algebraic thinking and NNAT abilities in Grade 5.

Relation between algebraic thinking ability and NNAT abilities in Grade 6. Table 4.39 presents the results of Multiple Regression Analysis, pertaining the relationship between algebraic thinking and the four types of processes involved in the NNAT test. Similar to the results described in the previous section regarding fifth graders, the Spatial Visualization and the Serial Reasoning processes seem to have a positive effect on the dependent variable.

As the Table 4.38 shows, the Spatial Visualization and the Serial Reasoning processes exert a significant influence on the prediction of individuals' performance in algebraic thinking, which means that the higher these abilities are the higher is the performance of sixth graders in algebraic thinking (β=.337 and β=.283 respectively). The Pattern Completion and the Reasoning by Analogy processes do not seem to be significant predictors of students' algebraic thinking at this age group.

Table 4.38

Regression Analysis of the Performance in each of the Four Types of Tasks in the NNAT with Dependent Variable the Performance in Algebraic Thinking in Grade 6

Algebraic thinking	B	SE	Beta
Reasoning by Analogy	.039	.060	.048
Pattern Completion	.074	.086	.066
Spatial Visualization	.371	.096	.337**
Serial Reasoning	.284	.298	.283*

R^2=.421

*p<.05, **p<.01

On the basis of the results reported in Table 4.39, the model of the regression equation was extracted. Figure 4.17 presents the coefficients of the multiple regression model. The overall performance of sixth graders in the algebraic thinking test (AT) is the criterion (depended variables) and the four types of cognitive abilities, the Reasoning by Analogy (RA), the Pattern Completion (PC), the Spatial

Visualization (SV) and the Serial Reasoning (SR), are the predictors (independent variables).

$$AT_{6thG} = .337(SV) + .298(SR) + .066(PC) + .048(RA) + 14.807$$

Note. AT6thG: Algebraic Thinking-Grade 6, RA=Reasoning by Analogy, SR=Serial Reasoning, SV=Spatial Visualization and PC=Pattern Completion

Figure 4.16. The regression model for the relation of algebraic thinking and NNAT abilities in Grade 6.

Relation between algebraic thinking ability and NNAT abilities in Grade 7. Table 4.40 presents the results of Multiple Regression Analysis, where the performance of the seventh graders in the algebraic thinking test is explained by their performance in the four types of tasks in the NNAT test.

According to the model, two out of the four types of abilities involved in the NNAT exert a significant influence on the prediction of individuals' performance in algebraic thinking. The Spatial Visualization and the Serial Reasoning processes seem to have a positive effect on the dependent variable, which means that the higher these abilities are the higher is the performance of seventh graders in algebraic thinking (β=.447 and β=.312 respectively). The Pattern Completion and the Reasoning by Analogy processes do not seem to be significant predictors of students' algebraic thinking at this age group.

Table 4.39

Regression Analysis of the Performance in each of the Four Types of Tasks in the NNAT with Dependent Variable the Performance in Algebraic Thinking in Grade 7

Algebraic thinking	B	SE	Beta
Reasoning by Analogy	.074	.091	.065
Pattern Completion	.034	.066	.036
Spatial Visualization	.515	.050	.447**
Serial Reasoning	.314	.081	.312**

R^2=.544

**p<.01

On the basis of the results reported in Table 4.40, the model of the regression equation was extracted. Figure 4.18 presents the coefficients of the multiple regression model. The overall performance of seventh graders in the algebraic thinking test (AT) is the criterion (depended variables) and the four types of cognitive abilities, the Reasoning by Analogy (RA), the Pattern Completion (PC), the Spatial Visualization (SV) and the Serial Reasoning (SR), are the predictors (independent variables).

$$AT_{7thG} = .447(SV) + .312(SR) + .065(RA) + .036(PC) + 6.726$$

Note. AT7thG: Algebraic Thinking-Grade 7, RA=Reasoning by Analogy, SR=Serial Reasoning, SV=Spatial Visualization and PC=Pattern Completion

Figure 4.17. The regression model for the relation of algebraic thinking and NNAT abilities in Grade 7.

Algebraic thinking ability and its relation to General Cognitive Processes of Mental Action. This section presents the results regarding the relation between students' ability in algebraic thinking and general cognitive structures. In order to measure students' abilities in general cognitive processes of mental action three different tests were used. The first one measured working memory, the second one measured Control of Processing and the third one measured Speed of Processing. All of the tests were computer-based.

This section is organized along three subdivisions concerning the three general cognitive structures measured in the current study. In each subdivision, the descriptive information of each test is described first, followed by the results of correlation analyses between all the factors of algebraic thinking and the general cognitive structure. Finally, the relationship between algebraic thinking and each of the general cognitive structures is described through the conduction of Regression Analysis. Specifically, regression analyses were applied in each age-group separately. In the vein of the previous parts regarding Specialized Structural Systems and Reasoning Processes, the purpose was to generate for each age group an equation to describe the statistical relationship between the general cognitive processes (predictor variable) and algebraic thinking (response variable).

Descriptive results of the test on Working Memory. Table 4.41 presents the results of descriptive statistics analysis in the overall performance of the students in the Working Memory test. The first three categories of this table correspond to the means, standard deviations and range of the Working Memory measures; the next three categories represent the information concerning the distribution of scores on continuous variables.

As the figures in Table 4.41 set out, students had an average performance in this test (*M*=.597). The maximum value of performance in all of the categories of items was 1 and the minimum was 0. The range of the subjects' performance was 1, showing that there were subjects that responded correctly to all of the items of a specific category, as well as subjects that did not respond correctly to any item of a specific category. The Skewness and Kyrtosis values were higher than -2 and lower than 2, suggesting that the variables of the subjects' performance for the items of the four systems in the test follow a normal distribution.

Table 4.40

Descriptive Results of the Working Memory Test

	Mean	Standard Deviation	Range	Skewness	Kyrtosis
Working Memory	.597	.236	1	-.925	.361

Relation between the factors of algebraic thinking and Working Memory. Table 4.42 presents the correlations between the three factors of algebraic thinking and Working Memory. According to the results of this analysis, Working Memory appears to be significantly related with all the factors of algebraic thinking ability.

Table 4.41

Correlations between the Performance of the Subjects in the Algebraic Thinking Factors and Working Memory

System / Factor	Working Memory
Generalized arithmetic	.133**
Functional thinking	.249**
Modeling	.166**
Algebraic Thinking	.204**

**p<.01

Students' performance in all of the algebraic thinking factors is positively related to the abilities involved in deductive reasoning. The correlation between the performance of the students in the items of "Generalized arithmetic" and their Working Memory appears to be significant at the .05 level $r_{gen.arithmetic}$=.248, p<.01). "Functional thinking" and "Modeling as a domain for expressing and formalizing generalizations" are also significantly correlated with Working Memory ($r_{funct.thinking}$=.248, $r_{modeling}$=.194, p<.01). The overall performance in the algebraic thinking test is also positively related to students' Working Memory ($r_{alg.thinking}$=.285, p<.01). Moreover, students' performance in the factor of "Functional thinking" appears to have the highest correlation with their Working Memory.

Relation between algebraic thinking ability and Working Memory in Grade 4. Table 4.43 presents the results of Regression Analysis, where the overall performance of the fourth graders in the algebraic thinking test is explained by their performance in the Working Memory test.

Table 4.42

Regression Analysis of the Performance in the Working Memory Test with Dependent Variable the Performance in Algebraic Thinking in Grade 4

Algebraic thinking	B	SE	Beta
Working Memory	.496	.069	.520**
R^2=.270 **p<.01			

According to the model, Working Memory exerts a significant influence on the prediction of fourth graders' performance in algebraic thinking (β=.520). Figure 4.19 presents the coefficients of the multiple regression model. The overall performance of fourth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Working Memory (WM) is the predictor (independent variable).

$$AT_{4thG} = .520(WM) + 14.530$$

Note. AT4thG: Algebraic Thinking-Grade 4, WM=Working Memory

Figure 4.18. The regression model for the relation of algebraic thinking and Working Memory in Grade 4.

Relation between algebraic thinking ability and Working Memory in Grade 5. Table 4.44 illustrates the relationship between the performance of the fifth graders in the algebraic thinking test and their performance in the Working Memory test, after the conduction of Regression analyses.

Table 4.43

Regression Analysis of the Performance in the Working Memory Test with Dependent Variable the Performance in Algebraic Thinking in Grade 5

Algebraic thinking	B	SE	Beta
Working Memory	.957	.047	.890**
R^2=.791 *p<.01			

As set by the figures in Table 4.44, Working Memory exerts a significant influence on the prediction of fifth graders' performance in algebraic thinking (β=.890). Figure 4.20 presents the coefficients of the multiple regression model. The overall performance of fifth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Working Memory (WM) is the predictor (independent variable).

$$AT_{5thG} = .890(WM) - .662$$

Note. AT5thG: Algebraic Thinking-Grade 5, WM=Working Memory

Figure 4.19. The regression model for the relation of algebraic thinking and Working Memory in Grade 5.

Relation between algebraic thinking ability and Working Memory in Grade 6. Table 4.45 presents the results of Regression Analysis, where the performance of the sixth graders in the algebraic thinking test is explained by their performance in the Working Memory test.

Table 4.44

Regression Analysis of the Performance in the Working Memory Test with Dependent Variable the Performance in Algebraic Thinking in Grade 6

Algebraic thinking	B	SE	Beta
Deductive Reasoning	.963	.058	.838*
R^2=.703			
*p<.05			

Similar to Grades 4 and 5, Working Memory is a significant predictor of sixth graders' performance in algebraic thinking (β=.838). Figure 4.21 presents the coefficients of the multiple regression model. The overall performance of sixth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Working Memory (WM) is the predictor (independent variable).

$$AT_{6thG} = .838(WM) - .056$$

Note. AT6thG: Algebraic Thinking-Grade 6, WM=Working Memory

Figure 4.20. The regression model for the relation of algebraic thinking and Working Memory in Grade 6.

Relation between algebraic thinking ability and Working Memory in Grade 7. Table 4.46 presents the corresponding results of conducting Regression Analysis, where the dependent variable is the performance of seventh graders in the algebraic thinking test and the independent variable is their performance in the Working Memory test.

Table 4.45

Regression Analysis of the Performance in the Working Memory Test with Dependent Variable the Performance in Algebraic Thinking in Grade 7

Algebraic thinking	B	SE	Beta
Working Memory	.406	.099	.392**
R^2=.154 **p<.01			

According to the model, Working Memroy is a significant predictor of sixth graders' performance in algebraic thinking (β=.392). Figure 4.22 presents the coefficients of the multiple regression model. The overall performance of seventh graders in the algebraic thinking test (AT) is the criterion (depended variables) and Working Memory (WM) is the predictor (independent variable).

$$AT_{7thG} = .392(WM) + 23.989$$

Note. AT7thG: Algebraic Thinking-Grade 7, WM=Working Memory

Figure 4.21. The regression model for the relation of algebraic thinking and Working Memory in Grade 7.

Descriptive results of the test on Control of Processing. Table 4.47 presents the results of descriptive statistics analysis in the overall performance of the students in the Control of Processing test. The first three categories of this table correspond to the means, standard deviations and range of the Control of Processing measures; the next three categories represent the information concerning the distribution of scores on continuous variables.

As the figures in Table 4.47 set out, students had a high performance in this test (*M*=.690). The maximum value of performance in all of the categories of items was 1 and the minimum was 0. The range of the students' performance was 1, showing that there were students that responded correctly to all of the items of a specific category, as well as subjects that did not respond correctly to any item of a specific category. The Skewness and Kyrtosis values were higher than -2 and lower than 2, suggesting that the variables of the students' performance for the items of the four systems in the test follow a normal distribution.

Table 4.46

Descriptive Results of the Control of Processing Test

	Mean	Standard Deviation	Range	Skewness	Kyrtosis
Control of Processing	.690	.236	1	-.925	.361

Relation between the factors of algebraic thinking and Control of Processing. Table 4.48 presents the correlations between the three factors of algebraic thinking and Control of Processing. According to the results of this analysis, Control of Processing appears to be significantly related with all the factors of algebraic thinking ability.

Table 4.47

Correlations between the Performance of the Subjects in the Algebraic Thinking Factors and Control of Processing

System / Factor	Control of Processing
Generalized arithmetic	.126*
Functional thinking	.242**
Modeling	.157**
Algebraic Thinking	.195**

*p<.05, **p<.01

Students' performance in all of the factors is positively related to the cognitive factor of Control of Processing. In contrast to the corresponding results that involved "Generalized arithmetic" and Working Memory, this analysis shows that "Generalized arithmetic" is significantly correlated with Control of Processing ($r_{gen.arithmetic}$=.126, p<.05). "Functional thinking" and "Modeling as a domain for expressing and formalizing generalizations" are also significantly correlated with Control of Processing ($r_{funct.thinking}$=.242 and $r_{modeling}$=.157, p<.01). The overall performance in the algebraic thinking test is positively related to students' Control of Processing ($r_{alg.thinking}$=.195, p<.01). The highest correlation appears between Control of Processing and "Functional thinking".

Relation between algebraic thinking ability and Control of Processing in Grade 4. Table 4.49 presents the results of Regression Analysis, where the overall performance of the fourth graders in the algebraic thinking test is explained by their performance in the Control of Processing test.

Table 4.48

Regression Analysis of the Performance in the Control of Processing Test with Dependent Variable the Performance in Algebraic Thinking in Grade 4

Algebraic thinking	B	SE	Beta
Control of Processing	.190	.066	.225**
R^2=.050 **p<.01			

According to the model, Control of Processing exerts a significant influence on the prediction of fourth graders' performance in algebraic thinking (β=.225).Though, the value of R Square of the model is very low. Figure 4.23 presents the coefficients of the multiple regression model. The overall performance of fourth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Control of Processing (CP) is the predictor (independent variable).

$$AT_{4thG} = .225(CP) + 28.011$$

Note. AT4thG: Algebraic Thinking-Grade 4, CP=Control of Processing

Figure 4.22. The regression model for the relation of algebraic thinking and Control of Processing in Grade 4.

Relation between algebraic thinking ability and Control of Processing in Grade 5. Table 4.50 illustrates the relationship between the performance of fifth graders in the algebraic thinking test and their performance in the Control of Processing test, after the conduction of Regression analyses.

Table 4.49

Regression Analysis of the Performance in the Control of Processing Test with Dependent Variable the Performance in Algebraic Thinking in Grade 5

Algebraic thinking	B	SE	Beta
Control of Processing	.103	.059	.136
R^2=.018			

As illustrated by the figures in Table 4.50, Control of Processing does not influence fifth graders' performance in algebraic thinking since the correlation is not significant at the .05 level. Figure 4.24 presents the coefficients of the multiple regression model. The overall performance of fifth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Control of Processing (CP) is the predictor (independent variable).

$$AT_{5thG} = .136(CP) + 45.003$$

Note. AT5thG: Algebraic Thinking-Grade 5, CP=Control of Processing

Figure 4.23. The regression model for the relation of algebraic thinking and Control of Processing in Grade 5.

Relation between algebraic thinking ability and Control of Processing in Grade 6. Table 4.51 presents the results of Regression Analysis, where the performance of the sixth graders in the algebraic thinking test is explained by their performance in the Control of Processing test.

Table 4.50

Regression Analysis of the Performance in the Control of Processing Test with Dependent Variable the Performance in Algebraic Thinking in Grade 6

Algebraic thinking	B	SE	Beta
Control of Processing	.122	.059	.159*
R^2=.025 *p<.05			

Similar to Grades 4, Control of Processing appears to influence sixth graders' performance in algebraic thinking (β=.159). Like Grade 4, the value of R Square of the model is very low. Figure 4.25 presents the coefficients of the multiple regression model. The overall performance of sixth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Control of Processing (CP) is the predictor (independent variable).

$$AT_{6thG} = .159(CP) + 45.148$$

Note. AT6thG: Algebraic Thinking-Grade 6, CP=Control of Processing

Figure 4.24. The regression model for the relation of algebraic thinking and Control of Processing in Grade 6.

Relation between algebraic thinking ability and Control of Processing in Grade 7. Table 4.52 presents the corresponding results of conducting Regression Analysis, where the dependent variable is the performance of seventh graders in the algebraic thinking test and the independent variable is their performance in the Control of Processing test.

Table 4.51

Regression Analysis of the Performance in the Control of Processing test with Dependent Variable the Performance in Algebraic Thinking in Grade 7

Algebraic thinking	B	SE	Beta
Control of Processing	.131	.047	.213**
R^2=.045			
**p<.01			

According to the model, Control of Processing is a significant predictor of seventh graders' performance in algebraic thinking (*β*=.213). Nevertheless, the value of R Square is very low. Figure 4.26 presents the coefficients of the multiple regression model. The overall performance of seventh graders in the algebraic thinking test (AT) is the criterion (depended variables) and Control of Processing (CP) is the predictor (independent variable).

$$AT_{7thG} = .213(CP) + 41.539$$

Note. AT7thG: Algebraic Thinking-Grade 7, CP=Control of Processing

Figure 4.25. The regression model for the relation of algebraic thinking and Control of Processing in Grade 7.

Descriptive results of the test on Speed of Processing. Table 4.53 presents the results of descriptive statistics analysis in the overall performance of the students in the Speed of Processing test. The first three categories of this table correspond to the means, standard deviations and range of the Speed of Processing measures; the next three categories represent the information concerning the distribution of scores on continuous variables.

As the figures in Table 4.53 set out, students had a high performance in this test (*M*=.723). The maximum value of performance in all of the categories of items was 1 and the minimum was 0. The range of the students' performance was 1, showing that there were students that responded correctly to all of the items of a specific category, as well as students that did not respond correctly to any item of a specific category. The Skewness and Kyrtosis values were higher than -2 and lower than 2, suggesting that the variables of the students' performance for the items of the four systems in the test follow a normal distribution.

Table 4.52

Descriptive Results of the Speed of Processing Test

	Mean	Standard Deviation	Range	Skewness	Kyrtosis
Speed of Processing	.723	.162	1	-.925	.361

Relation between the factors of algebraic thinking and Speed of Processing. Table 4.54 presents the correlations between the three factors of algebraic thinking and Speed of Processing. According to the results of this analysis, Speed of Processing appears to be significantly related with all the factors of algebraic thinking ability.

Table 4.53

Correlations between the Performance of the Students in the Algebraic Thinking Factors and Speed of Processing

System / Factor	Speed of Processing
Generalized arithmetic	.108*
Functional thinking	.248**
Modeling	.194**
Algebraic Thinking	.285**

*p<.05, **p<.01

Students' performance in all of the factors is positively related to the cognitive factor of Speed of Processing. Similar to the corresponding results that involved the correlation of algebraic thinking and Control of Processing, "Generalized arithmetic", "Functional thinking" and "Modeling as a domain for expressing and formalizing generalizations" are significantly correlated with Speed of Processing ($r_{gen.arithmetic}$=.108, p<.05, $r_{funct.thinking}$=.248 and $r_{modeling}$=.194, p<.01). The overall performance in the algebraic thinking test is positively related to students' Control of Processing ($r_{alg.thinking}$=.285, p<.01). The highest correlation appears between Control of Processing and "Functional thinking".

Relation between algebraic thinking ability and Speed of Processing in Grade 4. Table 4.55 presents the results of Regression Analysis, where the overall performance of the fourth graders in the algebraic thinking test is explained by their performance in the Speed of Processing test.

Table 4.54

Regression Analysis of the Performance in the Speed of Processing Test with Dependent Variable the Performance in Algebraic Thinking in Grade 4

Algebraic thinking	B	SE	Beta
Speed of Processing	.172	.066	.204*
R^2=.042 *p<.05			

According to the model, Speed of Processing is a significant predictor of fourth graders' performance in algebraic thinking (β=.204).Despite the fact that the value of Beta is statistically significant, the value of R Square of the model is very low. Figure 4.27 presents the coefficients of the multiple regression model. The overall performance of fourth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Speed of Processing (SP) is the predictor (independent variable).

$$AT_{4thG} = .204(CP) + 27.063$$

Note. AT4thG: Algebraic Thinking-Grade 4, SP=Speed of Processing

Figure 4.26. The regression model for the relation of algebraic thinking and Speed of Processing in Grade 4.

Relation between algebraic thinking ability and Speed of Processing in Grade 5. Table 4.56 illustrates the relationship between the performance of fifth graders in the algebraic thinking test and their performance in the Speed of Processing test, after the conduction of Regression analyses.

Table 4.55

Regression Analysis of the Performance in the Speed of Processing Test with Dependent Variable the Performance in Algebraic Thinking in Grade 5

Algebraic thinking	B	SE	Beta
Speed of Processing	.064	.055	.092
R^2=.009			

As set by the figures in Table 4.56, Speed of Processing does not influence fifth graders' performance in algebraic thinking since the correlation is not significant at the .05 level. Figure 4.28 presents the coefficients of the multiple regression model. The overall performance of fifth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Speed of Processing (SP) is the predictor (independent variable).

$$AT_{5thG} = .092(SP) + 46.006$$

Note. AT5thG: Algebraic Thinking-Grade 5, SP=Speed of Processing

Figure 4.27. The regression model for the relation of algebraic thinking and Speed of Processing in Grade 5.

Relation between algebraic thinking ability and Speed of Processing in Grade 6. Table 4.57 presents the results of Regression Analysis, where the performance of the sixth graders in the algebraic thinking test is explained by their performance in the Speed of Processing test.

Table 4.56

Regression Analysis of the Performance in the Speed of Processing Test with Dependent Variable the Performance in Algebraic Thinking in Grade 6

Algebraic thinking	B	SE	Beta
Speed of Processing	.106	.054	.150
R^2=.023			

As set by the figures in Table 4.57, Speed of Processing does not influence sixth graders' performance in algebraic thinking since the correlation is not significant at the .05 level. Figure 4.29 presents the coefficients of the multiple regression model. The overall performance of fifth graders in the algebraic thinking test (AT) is the criterion (depended variables) and Speed of Processing (SP) is the predictor (independent variable).

$$AT_{6thG} = .150(SP) + 45.086$$

Note. AT6thG: Algebraic Thinking-Grade 6, SP=Speed of Processing

Figure 4.28. The regression model for the relation of algebraic thinking and Speed of Processing in Grade 6.

Relation between algebraic thinking ability and Speed of Processing in Grade 7. Table 4.58 presents the corresponding results of conducting Regression Analysis, where the dependent variable is the performance of seventh graders in the algebraic thinking test and the independent variable is their performance in the Speed of Processing test.

Table 4.57

Regression Analysis of the Performance in the Speed of Processing Test with Dependent Variable the Performance in Algebraic Thinking in Grade 7

Algebraic thinking	B	SE	Beta
Speed of Processing	.119	.044	.207**
R^2=.043			
**p<.01			

According to the model, Speed of Processing is a significant predictor of seventh graders' performance in algebraic thinking (β=.207). Though, the value of R Square of the model is very low. Figure 4.30 presents the coefficients of the multiple regression model. The overall performance of seventh graders in the algebraic thinking test (AT) is the criterion (depended variables) and Speed of Processing (SP) is the predictor (independent variable).

$$AT_{7thG} = .207(SP) + 41.509$$

Note. AT7thG: Algebraic Thinking-Grade 7, SP=Speed of Processing

Figure 4.29. The regression model for the relation of algebraic thinking and Speed of Processing in Grade 7.

Summary of the results regarding the relation of algebraic thinking to Specialized Structural Systems, Reasoning Processes and General Cognitive Structures of Mental Action. As presented in the previous parts, multiple regression analyses were applied with the purpose of investigating the relationship between algebraic thinking and different types of cognitive factors. Specifically, regression analyses were conducted in each age-group separately (Grade 4, Grade 5, Grade 6 and Grade7). Based on these results, several model equations were generated in order to describe statistical associations between the different types of cognitive factors (predictor variables) and algebraic thinking (response variable) in each age-group.

The quantitative analysis showed positive high correlations with some of the Specialized Structural Systems and some of the processes involved in the Naglieri Non-Verbal Ability Test (NNAT). Positive moderate correlations were observed between algebraic thinking and deductive reasoning. Also positive moderate correlations were found between algebraic thinking and some of the processes involved in the tests of the Hypercognitive System.

Table 4.59 summarizes the results for each age-group. An important observation made from this table is that the four age-groups present differentiations in respect to the cognitive factors found to be significant predictors of algebraic thinking ability. Nevertheless, the four age groups also share similarities. Table 4.63 indicates which cognitive factors appear to be common in respect to their response in predicting students' performance in algebraic thinking in all of the four age groups.

According to the results presented in Table 4.63 the abilities that seem to be common at all of the four age groups and predict individuals' performance in algebraic thinking are:

i. **Causal-Experimental Ability** (measured by the Specialized Structural Systems Test)
ii. **Serial Reasoning** (measured by the NNAT)
iii. **Working Memory** (measured by the Working Memory test)

Table 4.58

Summary of the Cognitive Factors Predicting Individual's Performance in Algebraic Thinking in Each Age Group

	Measurement tool	Type of Cognitive process	Grade 4	Grade 5	Grade 6	Grade 7
Domain - Specific Information Processing Abilities	Specialized Structural Systems Test	Spatial-Imaginal		✓	✓	✓
		Causal-Experimental	✓	✓	✓	✓
		Qualitative-Analytic	✓			
		Verbal-Propositional		✓	✓	✓
Reasoning Processes	Deductive reasoning Test	Deductive Reasoning				✓
	Naglieri Non-Verbal Ability Test	Reasoning by Analogy	✓			
		Pattern Completion				
		Spatial Visualization		✓	✓	✓
		Serial Reasoning	✓	✓	✓	✓
General Cognitive Processes of Mental Action	Hyper-cognitive System Tests	Working Memory	✓	✓	✓	✓
		Control of Processing				
		Speed of Processing				

The Qualitative-Analytic System appears as a predictor of the algebraic thinking ability only in Grades 4. Reasoning by Analogy also appears to be a predictor of algebraic thinking ability only in Grade 4. Deductive Reasoning, Pattern Completion, Control of Processing and Speed of Processing do not seem to significantly predict students' performance in algebraic thinking in any age-group.

Relation between algebraic thinking and the cognitive factors found to predict algebraic thinking ability in all age-groups. This section presents the results of Structural Equation Modelling analysis (SEM) which was conducted with the aim of further examining the mediating effect of cognitive factors on algebraic thinking. Specifically, the results obtained in the previous section informed the construction of a theoretical model which expands the concept of algebraic thinking so as to describe its association with specific cognitive factors.

In order to investigate this relationship, the structure of two theoretical models was examined. Model 1 assumes that the three cognitive factors that were extracted from the analyses in the previous section (Serial Reasoning, the Causal-Experimental System and Working Memory) can predict performance in algebraic thinking, as a multidimensional factor synthesized by the factors of "Generalized Arithmetic", "Functional Thinking" and "Modeling as a domain for expressing and formalizing generalizations". Model 2 assumes that algebraic thinking and the abilities involved in the three cognitive factors are sub-factors of a more general ability, namely "Generalization abilities".

The results of the Structural Equation Modeling Analysis showed that the best model to describe the relation between algebraic thinking and the three cognitive factors was Model 1 (see Table 4.60). According to these results the three cognitive factors can predict algebraic thinking.

Table 4.59

Fit Indices of Models for the Relation between Algebraic Thinking Ability and the Five Cognitive Factors

	CFI	*TLI*	χ^2	*df*	χ^2/df	*p*	*RMSEA*
Model 1	.959	.953	227.007	129	1.760	.000	.033
Model 2	.919	.907	335.791	133	2.525	.000	.047

As it is shown in Figure 4.31, the five cognitive factors (Spatial Visualization, Serial Reasoning, Deductive Reasoning, the Causal-Experimental Structural System and Working Memory) have high ability for predicting students' algebraic thinking ability.

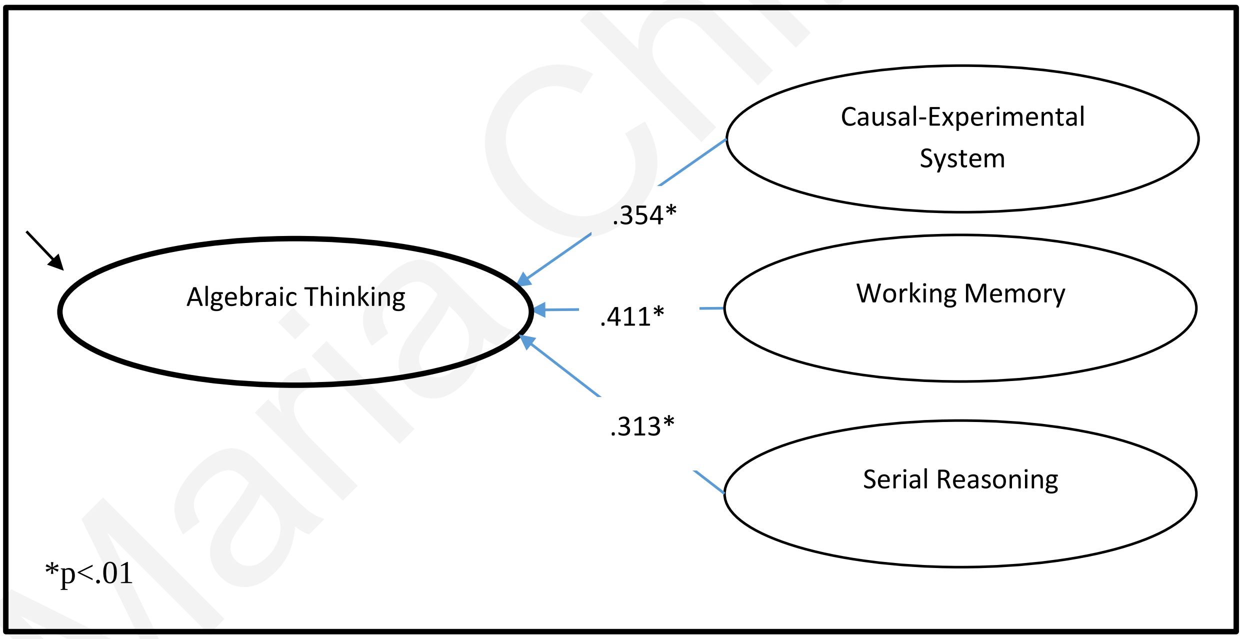


Figure 4.30. The model for the relation between algebraic thinking and the five cognitive factors.

The Impact of two Teaching Experiments on Algebraic Thinking Abilities

The final section of this chapter presents the results related to the sixth aim of the research. Specifically, this section reports on the impact of two different interventionist teaching experiments on enhancing fifth grade students' algebraic thinking abilities. The research questions that are answered are the following:

(7) What kind of instructional practices nurture algebraic thinking in elementary school mathematics?

(8) What is the impact of the interactions between the type of teaching experiment and students' cognitive abilities on their algebraic thinking ability?

In order to examine the impact of two different teaching experiments, one based in scaffolded and structured mathematical investigations (Group 1) and one based on semi-structured problem situations (Group 2), two groups of fifth grade students with equal abilities were formed. First, the descriptive results of the abilities of the two groups and their mean comparisons are presented. Second, the impact of the two teaching interventions is presented.

Group mean comparisons prior to the intervention. Table 4.61 presents the means and standard deviations for the two groups, regarding their abilities in algebraic thinking and abilities involved in the Causal-Experimental System, Spatial Visualization, Serial Reasoning, Deductive Reasoning and Working memory. In order to compare the abilities of the two groups and their personal traits prior to the intervention, a MANOVA analysis was conducted. The results suggest that the two groups did not have any statistically significant differences in their algebraic thinking abilities or in their cognitive abilities ($F=.576$, $p>.05$).

Table 4.60

Mean Comparisons of the Two Groups Prior to the Intervention

Performance	Group 1		Group 2			
	M_1	*SD*	M_2	*SD*	*F*	*p*
Overall Algebraic Thinking	.337	.195	.368	.151	.576	.449
Generalized arithmetic	.467	.326	.473	.235	.009	.897
Functional thinking	.302	.263	.404	.228	2.998	.107
Modeling	.223	.241	.183	.202	.565	.454
Causal Experimental System	.322	.260	.267	.253	.804	.298
Spatial Visualization	.364	.217	.299	.244	1.370	.204
Serial Reasoning	.457	.192	.419	.227	.574	.454
Deductive Reasoning	.460	.183	.470	.288	.088	.794
Working Memory	.540	.246	.503	.196	.404	.493

The impact of the two teaching experiments. In order to compare the impact of the two teaching interventions on the groups' performance in the algebraic thinking post-test, controlling for their pre-test scores, multivariate analysis of covariance (MANCOVA) was applied. Table 4.62 presents the results. The analysis indicated significant overall intervention effects, controlling for pre-test scores in the algebraic thinking test (Pillai's F=9.586, p<.05). As shown in Table 4.69, the students the semi-structured problem situations group had a significantly higher overall performance in algebraic thinking rather than the students in the structured mathematical investigations group. The effect size indices for the overall algebraic thinking ability (partial n2=.088) suggest that the effect of the semi-structured problem situations intervention over the structured mathematical investigations intervention were moderate. The students in the semi-structured problem situations group had significantly higher performance in the functional thinking component (Pillai's F=26.845, p<.01) and in the modeling component (Pillai's F=9.804, p<.05) I comparison to the students in the structured mathematical investigations group. The effect size indices for the functional thinking component (partial n2=.286) and the modeling component (partial n2=.128) suggest that the effect of the semi-structured problem situations intervention over the structured mathematical investigations intervention was moderate. The performance of the students in the semi-structured problem situations group in the generalized arithmetic component did not have any significant difference in relation to the performance of the students in the structured mathematical investigations group (Pillai's F=.081, p>.05).

Table 4.61

Results of the Multiple Covariance Analysis Between the Two Intervention Groups Post-test Performance in Algebraic Thinking

	Structured Mathematical Investigation Group		Authentic Problem Situations Group					
Ability	Mean[1]	*SE*	Mean[1]	*SE*	*df*	*F*	*p*	n_p^2
Overall Algebraic Thinking	.452	.206	.570	.179	1	6.452	.013*	.088
Generalized arithmetic	.663	.213	.647	.246	1	.081	.777	.001
Functional Thinking	.369	.225	.547	.270	1	26.845	.000**	.286
Modeling	.291	.291	.509	.319	1	9.804	.003*	.128

[1] Estimated Marginal Means

*p<.05, **p<.01

In order to compare the differences within the groups' pre-test and post-test scores in the overall algebraic thinking ability and in the components of "Generalized arithmetic", "Functional thinking" and "Modeling as a domain for formalizing and expressing generalizations", paired-samples t-test scores were performed. Table 4.63 presents the means and the standard deviations of the pre-tests and the post-tests of overall algebraic thinking ability, "Generalized arithmetic", "Functional thinking" and "Modeling as a domain for formalizing and expressing generalizations" for the structured mathematical investigations group. The results of paired-samples t-tests showed statistically significant differences in the mean difference between the pre and post-tests means of performance of the structured mathematical investigations group. Students in this group had a significant increase in their overall algebraic thinking ability and in the "Generalized arithmetic" component. The results also show that no statistically significant differences exist between pre- and post-tests means of performance in the "Functional thinking" component and in the "Modeling as a domain for formalizing and expressing generalizations" component.

Table 4.62

T-test Comparisons between Pre-test and Post-test Performance of the Structured Mathematical Investigations Group Subjects in Overall Algebraic Thinking Ability and in Algebraic Thinking Components

	Pre-test		Post-test			
Ability	M	SD	M	SD	T(df)	p
Overall Algebraic Thinking	.337	.195	.452	.206	-5.519(33)	.000**
Generalized arithmetic	.467	.326	.663	.213	-4.112(33)	.000**
Functional Thinking	.302	.263	.369	.225	-2.774(33)	.09
Modeling	.223	.241	.291	.291	-1.231(33)	.227

**p<.01

Table 4.64 presents the means and the standard deviations of the pre-tests and the post-tests of overall algebraic thinking ability, “Generalized arithmetic”, “Functional thinking” and “Modeling as a domain for formalizing and expressing generalizations” for the semi-structured problem situations group. The results of paired-samples t-tests showed statistically significant differences in the mean difference between the pre and post-tests means of performance of the authentic problem situations group. Students in this group had a significant increase in their overall algebraic thinking ability and in all of the components of algebraic thinking.

Table 4.63

T-test Comparisons between Pre-test and Post-test Performance of the Semi-structured Problem Situations Group Subjects in Overall Algebraic Thinking Ability and in Algebraic Thinking Components

	Pre-test		Post-test			
Ability	M	SD	M	SD	T(df)	p
Overall Algebraic Thinking	.368	.151	.570	.179	-10.147(34)	.000**
Generalized arithmetic	.473	.235	.647	.246	-4.818(34	.000**
Functional Thinking	.404	.228	.547	.270	-5.663(34)	.000**
Modeling	.183	.202	.509	.319	-9.926(34)	.000**

**p<.01

Interactions between teaching interventions and students' individual differences. A multivariate analysis of covariance (MANCOVA) was used in order to investigate the impact of the interactions between the type of teaching experiment (structured mathematical investigations and semi-structured problem situations) and students' individual characteristics. Taking into consideration the results described in previous sections of this Chapter, students' individual characteristics were defined on the basis of their scores in the pre-test regarding the Causal-Experimental system, Serial reasoning and Working memory. Specifically, the three factors that were found to be important predictors of students' algebraic thinking were used as indicators for students' cognitive abilities.

The MANCOVA analysis was applied with the purpose of evaluating the moderation effects of the intervention and students' cognitive abilities, in respect to students' benefits in algebraic-thinking, while adjusting for covariates in the students' abilities prior to the intervention program. Students' benefits were calculated as the difference between their post-test and pre-test scores for their overall algebraic thinking ability and for the three factors of algebraic thinking (Generalized arithmetic, Functional thinking, Modeling as a domain for expressing and formalizing generalizations).

In the analysis, the dependent variables were the benefits in students' overall algebraic thinking abilities, in generalized arithmetic concepts, functional thinking concepts and modeling concepts. The fixed factor was the intervention type (structured mathematical investigations and semi-structured problem situations). The covariates were the three cognitive factors (causal-experimental system, serial reasoning and working memory). Table 4.65 presents the results of the analysis.

Table 4.64

Results of the Analysis for the Effects of Interactions Between Teaching Experiment Type and Students' Cognitive Abilities on Students' Benefits in Overall Algebraic Thinking Ability and Algebraic Thinking Factors

	df	*MS*	*F*	*p*	n_p^2
Benefits in algebraic thinking ability					
Main effects					
Teaching Experiment type × Causal-Experimental System × Serial Reasoning × Working Memory	2	49.340	4.218	.019*	.115
Benefits in generalized arithmetic					
Main effects					
Teaching Experiment type × Causal-Experimental System × Serial Reasoning × Working Memory	2	3.471	.676	.512	.020
Benefits in functional thinking					
Main effects					
Teaching Experiment type × Causal-Experimental System × Serial Reasoning × Working Memory	2	6.630	3.490	.036*	.097
Benefits in modeling					
Main effects					
Teaching Experiment type × Causal-Experimental System × Serial Reasoning × Working Memory	2	2.236	1.351	266	.040

*p<.05

According to the results the main effect of the interaction between the teaching experiment type and the three cognitive factors on the benefits of the students regarding their overall algebraic thinking ability was significant while adjusting for the covariates (F=4.120, $p<.05$, $n_p^2=.115$). The data also indicate that the main effect of the interaction between the teaching experiment type and the three cognitive factors on the benefits of the students regarding functional thinking concepts was significant adjusting for the covariates (F=2.236, $p<.05$, $n_p^2=.040$).

These results eliminate the covariates' effects on the relationship between the type of the experiment and the benefits of the students. In this way, more variability in the model between the independent and dependent variables is explained. Specifically, the data in Table 4.65 illustrate that the three cognitive factors (Causal-Experimental System, Serial Reasoning and Working Memory) moderate the impact of the independent factor (Intervention type) on two out of the four dependent variables (Benefits in algebraic thinking ability and Benefits in Functional thinking).

Associating these results to the results reported in the previous sections of this chapter, illustrates the way algebraic thinking as a multifaceted concept that is comprised by three distinct components and is affected by specific cognitive factors. Moreover, the results indicate that the relationship between cognitive characteristics of the individuals and the type of mathematics instruction is dynamic and interactive.

CHAPTER V

Discussion

The introductory section of Chapter I outlined a critical problem in mathematical teaching and learning. Although calls from policy makers and curriculum designers during the last twenty years highlighted that algebraic thinking should become central to all students' mathematical experiences across K-12 grades, available research has not conceptualized algebraic thinking, at least explicitly. The field of mathematics education remained unclear about: (i) the nature and components of algebraic thinking in the early grades, (ii) the similarities and differences of algebraic thinking between elementary and secondary grades, (iii) the relationship between algebra and arithmetic, (iv) the reasoning processes and other types of cognitive factors that assist algebra learning, and (v) the key instructional practices for promoting algebraic thinking in elementary school mathematics.

This study took up these issues of concern, by focusing on three interrelated goals. The first goal centers on the development of a thorough understanding of the structure and components of algebraic thinking in the context of elementary mathematics by empirically testing the theoretical model proposed by Kaput (2008). The second goal extends the first, by involving the relation of algebraic thinking to specific reasoning processes and cognitive constructs, in order to develop a thorough understanding of factors that facilitate the development of algebraic thinking. The third goal examines concrete instructional practices that consider a sensible conceptualization of algebraic thinking in order to foster its development in the elementary mathematics classroom.

This study is significant for at least the following reasons: (a) algebraic thinking has a pivotal role in students' learning because it is closely linked to the development, establishment and communication of knowledge in all areas of mathematics, including arithmetic, geometry and, statistics; it is not possible to create viable opportunities for students to learn mathematics without having a comprehensive understanding of algebraic thinking, (b) prior experiences have shown that students' abrupt and isolated introduction to algebra in the middle school had led them to experience serious difficulties in understanding core algebraic concepts; there is a need for defining algebraic thinking in a way that considers students' pass from elementary grades to secondary grades of schooling, (c) the emphasis of early mathematics instruction on arithmetic and computational fluency is considered as a preventer for the development of conceptual understanding among young learners; it is important to describe the relationship between arithmetic and algebra and enable the presence of both in young students' mathematical experiences.

This chapter is organized in terms of the three foregoing goals, which correspond to the aims and research questions of the study. For each question, the main findings of the study are presented and discussed.

Which Components Synthesize 10- to 13-year-old Students' Algebraic Thinking Ability and What is the Structure of this Ability?

The components and structure of algebraic thinking. Kaput's (1995; 2005; 2008) theoretical model about the core aspects of algebraic thinking from K-12 grades has been used in a great extent within the field of mathematics education. In particular, it is considered as one of the most influential developments of the past decades for many reasons. First, Kaput conceptualized the notion of algebraic thinking as multidimensional; many research studies were based on this idea for developing further research on the field and offering explicit details of the characteristics of algebraic thinking. Secondly, this conceptualization broke down the wide field of algebraic thinking into three major and clearly distinguished components that can be easily integrated into teachers' instructional practices. Thirdly, Kaput's model articulated ways in which algebraic activities might be designed and applied both in early and secondary school algebra contexts, considering that algebraic thinking involves (i) the construction of generalizations and the expression of those generalizations in increasingly, conventional symbol systems, and (ii) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms.

However, this model has not been verified before with empirical data. Moreover, the components of the model were not explicitly defined through specific tasks. Noting this gap and with the aim to describe the components and structure of algebraic thinking, this study used appropriate quantitative methods to empirically test a model of algebraic thinking based on the theoretical foundations set by Kaput. The results of the study confirm that the concept of algebraic thinking is synthesized by three distinct but interrelated factors: (a) Generalized arithmetic, (b) Functional thinking and (c) Modeling as a domain for expressing and formalizing generalizations. In order to measure and verify these three factors, this study developed and used a corresponding algebraic thinking test.

The factor "Generalized arithmetic" refers to the use of arithmetic as a domain for expressing and formalizing generalizations. This study, with the aim to examine students' abilities in generalized arithmetic concepts, involved the following categories of tasks: (i) exploring properties and relationships of whole numbers (e.g. odd and even numbers), (ii) exploring properties of operations on whole numbers (e.g. distributive property of multiplication, associative property of addition), (iii) exploring equality and inequality as expressing a relationship between quantities and understanding the equals sign in number relations (e.g. solving and manipulating equations), and (iv) treating numbers as placeholders and attending the structure of numbers rather than relying on computations (e.g. determining if the sum of two multi-digit numbers is odd or even without performing calculations).

The factor "Functional thinking" refers to the generalization of numerical or geometrical patterns and the exploration and expression of relationships of co-variation and correspondence that are represented in several ways (with table, graphically, diagrammatically, verbally, symbolically). In this study, the items of "Functional thinking" involved the following categories of tasks: (i) finding variation within a sequence of values (recursive patterning), (ii) co-variational thinking based on analyzing how two quantities vary simultaneously and in keeping that change as an explicit, dynamic part of a function's description (e.g., "as *x* increases by one, *y* increases by three"), (iii) correspondence relationships based on identifying a correlation between variables (e.g., "*y* is 3 times *x* plus 2"), and (iv) comparing multiple representations in order to understand problems about rates of change (e.g. graphs, equations, tabular data).

The factor “Modeling as a domain for expressing and formalizing generalizations” involves the construction of models for generalizing regularities from mathematized situations or phenomena where the regularity itself is secondary to the larger modeling task. In order to measure this factor, the test of the study included tasks that required the identification and expression of regularities in real life problems, such as decision making about the best sales offer or in mathematical tasks that students needed to construct a model in order to represent an algebraic relationship (table, diagram, symbolic expression). Moreover, the test included tasks that required the expression and use of a regularity developed by repeated reasoning, such as a mathematical process or formula that address broad concepts of mathematics (e.g. the formula of area) or a figural pattern.

The results of the study indicated that the aforementioned tasks loaded only on one first-order factor. This fact confirms the high interpreted dispersion of the tasks and the distinct nature of the three factors of the model. Therefore, this study confirms that, “Generalized arithmetic”, “Functional thinking” and “Modeling as a domain for expressing and formalizing generalizations” represent three first-order factors and algebraic thinking represents a second, higher-order factor. Therefore, the findings of the current study verify through empirical data Kaput’s proposed structure of algebraic thinking and also the idea that Kieran (1992) developed first about conceptualizing algebraic activity not just as a topic in mathematics curriculum but as a multifaceted activity which encompasses various types of tasks and ways of thinking.

All tasks were carefully selected in order to satisfy two conditions: (i) the tasks involved processes that are considered to be linked with early algebraic thinking, such as generalization, problem solving, argumentation and justification, prediction and proof, and (ii) the tasks involved multiple forms of representation, such as verbal expressions, diagrams, drawings or graphs rather than symbols and did not require the use of symbols. Thus, the results lend support to the argument of several researchers (e.g. Carraher & Schliemann, 2014; Kieran, 2004, Radford, 2000) that algebraic thinking in the early grades can take place in the absence of algebraic notation.

Students' performance in "Generalized arithmetic". The results of the study showed that the factor "Generalized arithmetic" had the highest mean of performance among the students in the sample. Students' means of performance in the factors of "Functional thinking" and "Modeling as a domain for expressing and formalizing generalizations" were lower. Not surprising, students' higher mean of performance was in the item "Solving an equation" which belongs to the factor "Generalized arithmetic". The second higher mean of students' performance was in the item "Decomposition of whole numbers into possible sums" which also belongs to the factor "Generalized arithmetic". Both items represent the way thinking relationally is integrated in arithmetical settings. According to Kieran (2014) the ability to describe relations and solve procedures in a general way remains a timeless characterization of algebraic thinking from the years that Freudenthal (1977) made an effort to describe algebra until today.

This result illustrates the answer to the common question among researchers about the relationship between arithmetic and algebra. According to the results, "Generalized arithmetic" represents the module where arithmetic and algebra co-exist and interact; this interaction enables students to smoothly pass from arithmetical settings to algebraic settings. There is not a clear cut between them. As soon as students start to think about relationships and understand structure within numbers and properties of operations, their thinking moves from arithmetic to algebra. This result, based on empirical data, reflects Sfard and Linchevski's (1994) argument that learners initially understand algebraic expressions as computational processes. An expression, such as $4 (y + 6) + 2$ represents an arithmetical process. By performing particular operations, the symbol will obtain meaning. At this level, individuals face expressions as means for determining the value of the letter through the application of a prescribed process. Nevertheless, this level enables them to start detecting structure in mathematical expressions. Consequently, "Generalized arithmetic" is considered as an essential component of elementary mathematics instruction which will set the space for students in order to achieve what Mason (1989) have called, a "delicate shift of attention" from seeing the expression as an expression of calculations to seeing it as an expression of generality and then to seeing it as an object or property that can be manipulated.

Students' performance in "Functional thinking". The descriptive results showed that students had high means of performance in two of the items in the factor "Functional thinking". The majority of the students was able to solve the item "Choosing the appropriate verbal expression for representing a recursive relationship". This item involved a simple correspondence relationship based on the relation of variables (y is x plus 7). This relation was represented diagrammatically and students were called to choose the appropriate verbal expression. Additionally, students were able to solve the item "Identifying the pattern of even numbers when they are generated geometrically". The identification of simple rules in numerical patterns (such as "multiplying by 2" or "adding 7") represent cases of examining topics that are arithmetical in nature through functional approaches. This result is in accordance to the results found by Blanton and Kaput (2005) regarding the ability of students as young as 3rd graders for representing additive and multiplicative relationships transitioning from iconic to natural language. Consistently, Carracher and Schliemann (2014) offered the example of "multiplication by 3" for presenting the way early mathematics might integrate an algebraic character; "Multiplication by 3 is viewed as a subset of the integer function, 3n, that maps a set of input values to unique output values, thus preparing the ground for the continuous function, f(x) = 3x" (p.195).

This study indicated that the factor of "Functional thinking" is a distinct component of algebraic thinking. Hence, this study confirms that recursion, co-variance and correspondence represent basic concepts of algebraic thinking not only in the secondary grades but also in the elementary grades. This study conceptualizes functional thinking in such a way that it could be seen not as an advanced form of algebraic thinking that can be cultivated only in the upper grades of schooling but as central in even young students' learning of mathematics. This result confirms the suggestions of previous studies (e.g. Blanton, 2011; Brizuela and Schliemann 2003; Carraher et al. 2008; Kaput and Blanton 2005; Moss et al. 2008) about the capability of elementary school students to reason about functions and use different forms of representations for being successfully engaged with functional thinking. Consequently, this result specifies that the absence of functional thinking from elementary mathematics instruction, could become a serious threat to students' opportunities to develop algebraic thinking. Students should be provided with rich opportunities to develop functional thinking skills from the start of their mathematical study.

Students' performance in "Modeling as a domain for expressing and formalizing generalizations". The means of performance among students concerning the items in the factor "Modeling as a domain for expressing and formalizing generalizations" were low. This factor included items that in an extent share similarities with the so-called "algebra problem solving", which, as reported by Kieran (2014), is an area of algebra learning that challenges many students. Nowadays, "algebra problem solving" has been broadened to include not merely word problems but also non-routine tasks that are purely symbolic and do not connect to real world (Kieran, 2014). In this study, an effort was made to include in the factor "Modeling as a domain for expressing and formalizing generalizations" mostly items that present situations of the real world and only the items "Modeling a figural pattern" and "Modeling with a symbolic expression a function table" was disconnected from realistic settings.

All of the items in this factor hindered regularities which were secondary to the general modeling task. This kind of items provides opportunities to students to express and formalize generalizations from mathematized situations inside or outside mathematics (Blanton & Kaput, 2005). From this perspective, algebraic thinking is used as a conceptual tool for exploring modeling problems that are derived from complex realistic situations or phenomena. Therefore, the engagement of students in modeling activities is considered as crucial in order to develop both ways of thinking that are algebraic, such as working with generalizable patterns and identifying structure and ways of thinking that are not exclusively algebraic, such as problem solving, justifying and proving, making predictions and conjectures.

A possible reason for students' low performance in the modeling items may be the fact that they were cognitively demanding. Modeling tasks required students to synthesize sets of information in order to produce and test a hypothesis through the use of appropriate models. In this perspective, such activities might require the integration of a more complex spectrum of cognitive constructs and reasoning processes. According to Kieran (2007), modeling is involved in the global-meta level of algebraic activity and addresses not only algebraic thinking but also more general mathematical processes. The following sections of this Chapter will further enlighten the role of reasoning processes and other types of cognitive factors in students' performance in the algebraic thinking test.

Another possible explanation of the results reported above might be students' mathematical experiences. Specifically, the mathematics curriculum in effect at the school year that the study was conducted did not include algebra as a specific domain of mathematics education or precise objectives regarding algebra teaching and learning. Students at Grades 4 and 5 had occasionally the experience of pattern activities and activities involving the interpretation of linear graphs. In Grade 5, the properties of operations (associative property of addition and multiplication, distributive property of multiplication, properties of 0, and order of operations) were introduced. Symbols, as a way for expressing unknowns in equations, were not introduced before Grade 6. The formal integration of algebra within mathematics education appeared in Grade 7. The corresponding lessons in 7th graders books emphasized the use and meaning of symbols as a tool for representing unknown quantities as well as the investigation of algebraic rules for solving equations.

Correlations between the three components of algebraic thinking. The results yielded from the study showed that the correlations between the three first-order factors were statistically significant. The highest correlation appears between the factors "Functional thinking" and "Modeling as a domain for expressing and formalizing generalizations". Intertwined with this result might be the fact that modeling activities share common features with functional thinking activities. As it was reported in Chapter II, modeling encompasses components of algebraic thinking, such as the manipulation of symbols, the illustration of the correspondence between the situation and the model and re-translations between the situation and the model (Watson, 2009b). What differentiates the two factors is that the modeling task embodies a mathematized situation either inside or outside mathematics. The object in the modeling items is the identification of the algebraic components that are enclosed in the task as well as the development of a modeling language for analyzing the situation and finding a logical answer. Moreover, this kind of tasks involve in a great degree the use of generalizations for supporting intuitive arguments and discussion about the rationality or incorrectness of a decision on the basis of reasoning about the relations among quantities.

Is the Structure of Students' Algebraic Thinking Ability the Same or Different in Relation to Age?

The structure of the model across Grades 4, 5, 6 and 7. The results pertaining the stability of the verified model for algebraic thinking across the four age-groups showed that the model remains stable only for Grades 5 and 6. Considering this result, the validity of the model was tested in Grade 4 and Grade 7 separately. The results confirm that in these groups, three distinct factors compose a higher-order factor. Nevertheless, the items of the best fitting model that seem to have statistically significant factor loadings to the corresponding factors were not exactly the same. This result suggests that the model remains stable from grade to grade in respect to the structure. However, in each factor the items that interpret the dispersion of the factor are slightly differentiated, reflecting different abilities of students of different age-levels for solving specific algebraic tasks.

This finding signifies that algebraic thinking ability in the higher primary grades and in the beginning of secondary education is synthesized by three distinct factors, as these were described above. Besides, Kaput's argumentation was that the model of algebraic thinking refers to a broader conceptualization that captures both primary and secondary education. Even though the level of ability in algebraic thinking varies between the students of different grade levels, the dimensions of this ability are the same for all students, with respect to their educational level and cognitive abilities. The fact that the content and the extent of the instructional time devoted to algebraic concepts from grade to grade are different does not appear to differentiate the structure of ability in algebraic thinking.

For the students of Grades 4 and 7, the model remains stable as far as it concerns its three-dimensionality but the items that load to the three first order factors are slightly different. A possible reason for this differentiation might be the fact that according to several psychological theories (e.g. Piaget, 1970; Pascual-Leone, 1970;

Case, 1985), the corresponding ages of the students in these grades (9-10 years old and 12-13 years old) represent transitional stages in students' development in respect to their cognitive abilities and educational behavior. According to Demetriou, Spanoudis and Mouyi (2008), students that are 9-10 years old are only able to construct simple math relations of the type a + 5 = 8 ; students' of 12 years old develop proportional reasoning and they are able to co-ordinate symbolic structures; students' of 13 years old develop algebraic reasoning that is based on mutually specified symbols systems. These descriptions as well as others reporting differences across developmental cycles in students' abilities regarding specialized domains, working memory and processing efficiency and inferential skills (e.g. Demetriou et al., 2002, Demetriou, Spanoudis & Mouyi, 2010; Demetriou, Spanoudis & Shayer, 2015) might provide viable explanations about the different behavior of the students in Grade 4 and Grade 7.

In order to measure algebraic thinking in four different grades of school education, the test was constructed on the basis of Kaput's model. On the one hand, the test included items that corresponded to the three factors as reported above; on the other, the items were carefully selected considering the majority of them not to involve algebraic notation. In the items of the factor "Functional thinking" all variables were mainly represented either through verbal expressions, diagrams, drawings or graphs. Moreover, these tasks did not require the representation of an extracted generalization with symbols but in one of the aforementioned forms. As Carraher and Schliemann (2014) stated, algebraic thinking in the early grades can take place in the absence of algebraic notation. Also, Radford (2014) argued that what allowed researchers to discuss about the possibility of developing algebraic thinking in the early grades is the rejection of the idea that notations are a manifestation of algebraic thinking. Moreover, the use of letters in algebra is not a sufficient evidence for thinking algebraically. For example, students could solve the equation "2x+2=10+x" by replacing notations with particular numbers (e.g., x=1 or x=2, etc.) and applying trial and error strategies (Radford, 20014, p.4). Similarly, Kieran (2004) suggested the following definition for algebraic thinking in the early grades:

> Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all,

> such as, analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting (p.149).

Radford (2006) suggested that the alphanumeric algebraic symbolism is a modern invention, which appeared during the Renaissance years as a need for representing quantities and relationships between quantities. Radford (2006) also refered to paradigms from history that indicate that notations are not necessarily manifestations of algebraic thinking: (i) Euclid used in his Elements letters without the aim for expressing algebraic ideas; (ii) Ancient Chinese mathematicians solved systems of equations without using notations (iii) Babylonian scribes used geometric diagrams to think algebraically. Hence, the use of letters in algebra is neither an essential nor an adequate condition for demonstrating that a student is thinking algebraically.

The results of the study also show that the distinction between arithmetic and algebra cannot be cast in terms of notations, as it has often been supported. Based on the epistemological analyses of the past along with the theoretical contribution of researchers like Kieran (1996; 2004) and Kaput (2000; 2008), the test on algebraic thinking was constructed by items that required students to think algebraically and satisfied the following conditions that according to Radford (2014) characterize algebraic thinking: (i) the tasks involved unspecified numbers (fixed unknowns and variables), (ii) the unspecified numbers had to be named or symbolized (alphanumeric symbols, natural language, unconventional signs, a mixture of these), and (iii) the unspecified numbers had to be treated as if they were known and students had to operate on them (e.g. perform additions, subtractions, multiplications and divisions). As the results of the current study show, students' algebraic thinking from Grades 4 to 7 can be characterized by three distinct factors measured by items that satisfy the conditions described above. Moreover this result, supports Carracher and Schliemann's (2014) claim that the incorporation of early algebra in young learners' education is dependent on teachers' fluidity to handle algebraic representations, especially those that are expressed through natural language, diagrams, tables and graphs.

Students' performance in the algebraic thinking test across grades. The results of the study showed that there are significant differences in the algebraic thinking abilities of the students in the four grades (Grade 4, Grade 5, Grade 6 and Grade 7). The means of the overall performance of the students in the algebraic thinking test increases from grade to grade. Moreover, the means in the performance of the students in each of the three factors of algebraic thinking also increase from grade to grade.

This result is in accordance with the content of students' mathematics education related to algebraic concepts, as this was described in the previous part of this chapter. Moreover, this result supports the hypothesis of this study about the relations of algebraic thinking with significant cognitive factors that are age-dependent and influenced by developmental constraints. As Kaput asserted, there are two core types of reasoning processes which flow through varying degrees throughout the three factors of algebraic thinking; (i) making generalizations and expressing those generalizations in increasingly, conventional symbol systems, and (ii) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. In the case of the first reasoning process, generalizations are produced, justified and expressed in various ways. The second reasoning process refers to the association of meanings to symbols and to the treatment of symbols independently of their meaning. These reasoning processes might be related to different types of cognitive factors and accordingly be responsible for the significant differences among the performance of students of different ages. The part of the current study that discusses the findings about the relation of algebraic thinking with various cognitive factors, purposes to enlighten the reasons underneath the differentiations in the abilities of the students from grade to grade.

What are the Classes of Algebraic Thinking Ability of 10- to 13-year-old Students?

This study indicates that there are four different classes-groups of students which can describe their algebraic thinking abilities. Class 1 students had low performance in all of the three factors of algebraic thinking. Class 2 students had average performance in the factor "Generalized arithmetic" and low performance in the factors “Functional thinking” and “Modelling as a domain for expressing and formalizing generalizations”. Class 3 students had average performance in the factors “Generalized arithmetic” and “Functional thinking” and low performance only in the factor “Modelling as a domain for expressing generalizations”. Class 4 students did not have low performance in any of the three factors of algebraic thinking. These students had high performance in the factors “Generalized arithmetic” and “Functional thinking” and average performance only in the factor “Modelling as a domain for expressing generalizations”.

Nearly 1/2 of the students in Grade 4 were classified within Class 1. Around 1/3 of the students in Grade 5 and around 1/4 of the students in Grades 6 and 7 were classified within Class 1. Class 2 had almost equal percentages of students from the four grades; around 1/5 of the students in each grade were classified within Class 2. Class 3 had also almost equal percentages of students from the four grades; around 1/4 of the students in each grade were classified within Class 3. Class 4 was mostly represented by students in the seventh, sixth and fifth grade. Furthermore, the percentage of the students that belong in Class 4 seems to be increased from Grade 4 to Grade 7.

These results indicate that fourth grade students’ level of algebraic thinking abilities is different in comparison to the fifth, sixth and seventh grade students. Students in Grade 4 are able to solve some tasks from the generalized arithmetic factor and simple patterning tasks from the functional thinking factor. Students in Grade 5 and 6 do not seem to have important differences in respect to their algebraic thinking abilities, since they are both able to solve the tasks from the generalized arithmetic factor and most of the tasks in the functional thinking factor. A transition seems to occur as soon as students enter secondary school, since the majority of the students in this grade were classified within Classes 3 and 4. These students have more abilities in solving not only the generalized arithmetic and functional thinking tasks but also the modeling tasks. Therefore, it can be argued that there are three broad transitional stages regarding the abilities of the students in algebraic thinking over the ages 10 to 13 years old. The first stage includes students who are at 9 to 10 years old. The second stage includes the students who are at 10 to 12 years old. The third stage includes the students who are at 12 to 13 years old. These results support the hypothesis that differences in the cognitive characteristics of the students pertaining the involvement of specific reasoning forms or other cognitive factors, affect their individual algebraic thinking abilities. As reported in the previous section of this chapter, these results are in accordance with the developmental stages described by Piaget (Piaget & Inhelder, 1967), concerning advances in the cognitive behavior of the students.

What are the Characteristics of Students' Performance in Algebraic Thinking at Different Classes of Ability?

The results of the study indicated that students in each of the four classes have different abilities. In the following section, the four classes are described based on the abilities of the students in the three factors of algebraic thinking. A brief report of some typical characteristics observed while scoring students' answers in the test of algebraic thinking, is also included for each class separately. These characteristics concern students' nature of approaches to specific tasks of the algebraic thinking test.

Characteristics of the first class of algebraic thinking ability. The results of the study have shown that the students of the first class had low abilities in all of the three factors of algebraic thinking. Regarding the factor "Generalized arithmetic", the majority of the students in this class was unable to respond correctly to most of the items included in this factor. The only item that they appear to solve successfully was the item "Solving missing number sentences", which required finding the unknown in an equation representing an additive relationship. Specifically, this task was mainly solved through trial and error strategies where students tested possible numbers before giving an answer for the value of the unknown that was involved in the equality. As far as it concerns the factor "Functional thinking", the highest percentage of correct responses was found to be in the item "Choosing the appropriate verbal expression for representing a recursive relationship". This relationship was represented through a diagram and student should translate it into a verbal representation. In the factor "Modeling as a domain for expressing and formalizing generalizations" less than 30% of the students within Class 1 were able to respond correctly to any of its items.

Generally, the students that according to the quantitative analysis probably belong in Class 1 had many difficulties regarding the ability to generalize and manipulate expressions as generalizable objects that represent relationships between quantities rather than calculating procedures. The main problem of these students seem to be their failure in developing strategies for operating on the unknown. According to the studies reported previously and other analyses that aimed to describe what is algebra (e.g., Filloy, Rojano & Puig, 2007; Kieran 1989, 1990; Radford & Puig, 2007; Serfati, 1999), the way of thinking that distinguishes arithmetic from algebra is the consideration of indeterminate quantities as if they were something known. The students in Class 1 seemed to fail operating on the unknowns as if they were specific numbers, thus failing in gaining respectable mean scores in any of the three types of tasks involved in the algebraic thinking test.

Characteristics of the second class of algebraic thinking ability. The students of the second class had average abilities in the tasks of "Generalized arithmetic" and low abilities in the tasks of 'Functional thinking" and "Modeling as a domain for expressing and formalizing generalizations". What was perceived while scoring the tests of the students, was that the strategies they followed in order to solve the tasks in the Factor "Generalized arithmetic" were arithmetical in nature. For example, in the item "Relating place-value properties to the multiplication algorithm", most students in this class needed to do the multiplication from the start, to find that the result was false rather than relying on the fact the in a two-digit multiplication the second digit by which we multiply is at the tens place. Others stated that the answer was false since no place was left under the units place, indicating a procedural understanding of the algorithm rather than a conceptual understanding. In the item "Determining if the sum of two multi-digit numbers is either odd or even", the majority of the students performed the addition in order to calculate the sum of the two multi-digit numbers instead of attending the structure of the numbers and confront them as placeholders. According to Blanton and Kaput (2005), this kind of behavior is indicative of students that have developed the ideas of arithmetic within a procedural perspective rather than a conceptual one. Their mathematical knowledge is constrained to the field of arithmetic and computational procedures.

Drawing on the above observations, these students seem to use their arithmetical knowledge, in order to express and formalize generalizations. The most common strategies used by these students were “Guess and check” and “Working backwards”. Several researchers consider these strategies as arithmetical rather than as algebraic (e.g. Bednarz & Janvier, 1996; Davydov, 1990; Dougherty, 2001). Nonetheless, Sfard and Linchevski (1994) argue that these strategies signify the beginning of thinking algebraically and a smooth transition between arithmetic and algebra. Hence, the current study addresses once again the problem of defining the relationship between arithmetic and algebra. Students in Class 2 seem to make an effort for better understanding the algebraic tasks by using as tools the knowledge and abilities they have developed through the strand of arithmetic. These tools provide them the basis for smoothly passing from arithmetic and algebra.

Nonetheless, due to the limited prospects that these strategies provide, students in Class 2 seem to fail in solving tasks of the factor “Functional thinking. In these cases, such as the translation of a co-variation relationship that is represented verbally to its graphical representation, arithmetic methods prove not to be sufficient. Algebraic thinking has to be applied by the analytic manner where students deal with indeterminate numbers. According to Vergnaud (1998), this kind of activities, which include concepts such as functions, variables and parameters reflect the new objects that students have to treat as soon as they move from the field of arithmetic to the field of algebra.

Characteristics of the third class of algebraic thinking ability. The students that according to the results belong in Class 3, had average abilities in the factors of “Generalized arithmetic” and “Functional thinking”. These students had low performance means in the tasks of the factor “Modeling as a domain for expressing and formalizing generalizations”. The higher abilities of the students of Class 3 can be explained by the fact that they were more able in treating numerical expressions as relationships rather than as directions for performing calculations. Moreover, the students of this class managed to solve more successfully some of the items that involved correspondence and co-variational relationships as well as finding the nth term in numerical patterns. Fujii και Stephens (2008) argue that the identification of repeated patterns is one of the more important indicators of students’ ability for thinking relationally rather than thinking with specific numbers.

Nonetheless, the task “Identifying possible numbers in a numerical pattern” which had the highest mean of performance among the items of the factor “Functional thinking” (62%) was mainly approached with arithmetical strategies. Specifically, in order to extend either a numerical or geometric sequence, students need to notice the relationship between the numbers or the way the figures in the pattern are placed structurally (Mulligan and Mitchelmore 2009); especially in the geometrical patterns, students need to uncover the regularity that involves the association between the spatial structure and the corresponding numerical structure. Contrary to this, most of the students within Class 3 were merely focused on the numerical aspect of the terms. Their activity relied to counting or “Guess and check” strategies.

Consistently with Howe (2005), producing a formula within patterning activities might merely be based on guessing the formula and trying it; nonetheless, this strategy is based on arithmetic concepts. This might be a possible reason for the low means of performance of students in Class 3 in more complex patterns, such as the item “Developing the rule of a complex geometrical pattern” (43.5%). In this kind of task the counting or try-and-error methods are not sufficient. Usually, the students start counting the number of hexagons and squares in Figures 1, 2, and 3, and realize that the number of hexagons and squares increase by the same number each time. However, as the students quickly notice this recursive relationship between consecutive figures, this is not a practical way to answer the question about Figure 100. Students in Class 3 also seem to rely on their arithmetical knowledge and abilities for untangling the algebraic tasks of the test. Comparing to students in Class 2, students in Class 3 seem to have attained a higher level of abilities in the strand of algebra.

Characteristics of the fourth class of algebraic thinking ability. According to the results of the study, the students of the fourth Class had high abilities in the items of the factors “Generalized arithmetic” and “Functional thinking” and average abilities in the items of the factor “Modeling as a domain for expressing and formalizing generalizations”. The majority of the students that belong in this class comes from Grade 7. This result is consistent with Blanton and Kaput (2005) who argue that the factor of modeling is mostly apparent within students of secondary education. A significant observation is that the students in Class 4 demonstrated an ability for producing formulas while solving patterning tasks rather than applying trial and error. Thus their success in the items of the factor “Functional thinking” was based in an extent to their ability for attending generalizing processes where general functional relationships were first identified (e.g., the number of hexagons and squares in the item “Developing the rule of a complex geometrical pattern”) and then simplified in order to be expresses as the rule for finding any term of the sequence.

In the factor “Modeling as a domain for expressing and formalizing generalizations”, students in Class 4 were able to respond more successfully comparing to the students that belong in Classes 1, 2 and 3. Specifically, they demonstrated an ability for responding correctly mainly to the tasks that involved phenomena or situations from real life settings. A possible reason for this result might be the fact that these students had developed in a great extend the abilities required for solving the tasks of the factors “Generalized arithmetic” and “Functional thinking”.

Students in Class 4 seem to have developed abilities for producing relational reasoning, not only in the context of patterns or co-variational relationships but also in contexts where a regularity is presented through a realistic situation or phenomenon. An ability that seems to be central when students confront modeling tasks is the ability for translating representations of one form to another as well as the ability for choosing the appropriate model (e.g. symbolic expression, table, and graph). As past research studies have shown, flexibility in managing multiple representations and constructing appropriate representations for treating “mathematically” a situation remains difficult for the majority of students. (e.g. Lamon, 1998).

Summarizing the results in this section, it seems that there are four classes of students reflecting different levels of algebraic thinking abilities. Students in Class 1 seem not to have successfully develop algebraic thinking abilities, since they are not able to solve with success any tasks of the algebraic thinking test. Students in Class 2 and 3 seem to be in an intermediate stage where some arithmetical strategies help them to attend algebraic tasks both in the “Generalized arithmetic” factor and the “Functional thinking” factor. Students in Class 4 seem to have achieved the passing from arithmetical modes of thinking to algebraic ones, demonstrating relational reasoning when solving algebraic tasks and applying strategies that enable them to construct viable generalizations.

Is there a Consistent Hierarchical Trend of Students' Algebraic Thinking Ability?

The data indicated that there is a hierarchical relationship in the development of ability in algebraic thinking. Specifically, the results of the study specified that the factor "Modeling as a domain for expressing and formalizing generalizations" is grasped only after the factors "Generalized arithmetic" and "Functional thinking" have been conceptualized by the students. Specifically, it seemed that this hierarchy follows a progression of (i) "Generalized arithmetic, (ii) "Functional thinking" and (iii) "Modeling as a domain for expressing and formalizing generalizations". The analysis indicated that this hierarchy is confirmed. Students are more successful first in undertaking the generalized arithmetic tasks. Later on, they are successful in undertaking the functional thinking tasks. It seems that the modeling tasks can be understood once students have mastered the previous two aspects of algebraic thinking. Thus, it can be argued that, for students in Grade 4 to 7, understanding of the basic algebraic ideas progresses from "Generalized arithmetic", to "Functional thinking", to "Modeling as a domain for expressing and formalizing generalizations".

This finding is important, since it implies that the factors of algebraic thinking do not develop simultaneously. A possible reason might be the emphasis in the mathematics classroom and especially in the primary schools, which does not focus on algebra itself as a topic of interest. Furthermore, modeling tasks are absent from the school textbooks (the ones used by the time the study was conducted. Considering the differences that psychological theories describe in students' abilities in specialized domains (e,g, Demetriou, Spanoudis & Mouyi, 2010), working memory and processing efficiency (e.g. Demetriou et al., 2002) and reasoning (e.g. Demetriou, Spanoudis & Shayer, 2015), another possible reason for this result may be the extent to which specific cognitive factors influence students' abilities in each of the three components.

Besides, available theoretical descriptions on developmental aspects of algebraic thinking, as these were reported in Chapter 2 of the current study, imply that the development of algebraic thinking is a process that evolves from thinking with the specifics to thinking abstractly, from thinking with fix-values (unknowns) to thinking about quantities that vary (variables) (e.g. Kuchemann, 1981; Mason, 1989; Sfard & Linchevski, 1994; Thomas & Tall, 2001). Along this developmental process, the meaning and use of symbols also alters, from understanding and using symbols as a way for representing relationships to applying syntactic rules for solving equations and then understanding the different role that the same symbol might take in an equation (Arcavi, 2005).

What is the Relation of Algebraic Thinking with Domain-Specific Processes, Different Types of Reasoning Forms and General Cognitive Processes of Mental Action?

The study's special attention to the relationship between algebraic thinking and several cognitive constructs is crucial. As it was reported above students' algebraic thinking abilities are differentiated in between the grades of schooling as well as within a single grade. A serious possible reason for observing this differentiation was attributed to individual differences among the students pertaining their cognitive characteristics. This study confirms that that there is a strong relationship between algebraic thinking and several cognitive factors. Specifically, the following categories of cognitive factors were measured for the purposes of this study:

(vii) Specialized Structural Systems (Spatial-Imaginal System, Causal-Experimental System, Qualitative-Analytic System, and Verbal-Propositional System)

(viii) Reasoning processes (Deductive Reasoning and the processes measured by the Naglieri Non-Verbal Ability test - Reasoning by Analogy, Pattern Completion, Spatial Visualization and Serial Reasoning)

(ix) General cognitive structures of mental action (Working Memory, Control of Processing and Speed of processing)

The predictive relationship between various cognitive factors and algebraic thinking across Grades 4, 5, 6 and 7. The analysis of the data produced several model equations that described the statistical associations between the above factors (predictor variables) and algebraic thinking (response variables) in each age group separately (Grades 4, 5, 6 and 7). The results indicated that the predictive relationship between algebraic thinking and different cognitive factors changes from age to age. Along the transition of the students from grade to grade, some of these factors appear or disappear in the relationship and some of them remain stable in all age-groups. Hence, the results of this part of the study explain the reasons for observing differentiation in respect to the tasks in the algebraic thinking test that students in each grade level where able to solve as well as in the algebraic thinking abilities of the students.

The findings of the study suggest that the algebraic thinking ability of the students in Grade 4 is in a great extent subtle to the effect of different types of cognitive factors. In particular, it was found that fourth graders' algebraic thinking is influenced by the Causal-Experimental System and the Qualitative-Analytic System. As far as it concerns the reasoning processes, in Grade 4 it seems that Reasoning by Analogy and Serial Reasoning play an important role. Working memory appears to exert a significant influence on fourth graders algebraic thinking.

The results of the current study have shown that the relationship of different cognitive factors and algebraic thinking changes as students move from Grade 4 to Grade 5. Fifth graders' algebraic thinking appears to be influenced by the Spatial-Imaginal System, the Causal-Experimental System and the Verbal-Propositional System. Spatial Visualization and Serial reasoning were also identified as significant predictors of algebraic thinking in Grade 5. From the general cognitive structures of mental action, fifth graders' algebraic thinking seems to be influenced by Working Memory. Summarizing the results pertaining the changes in the relationship of different cognitive factors and algebraic thinking for Grades 4 and 5, it seems that the Qualitative-Analytic System and Reasoning by Analogy do not continue to be significant predictors of algebraic thinking as students move from Grade 4 to Grade 5.

Moreover, in Grade 5, Spatial-Imaginal System and the corresponding Spatial Visualization process are added in the relationship.

The findings of the regression analyses in Grade 6 showed that there are no differences between the predictive relationship of different cognitive factors and algebraic thinking between Grade 5 and Grade 6. The factors that influence algebraic thinking in Grade 5, as these were reported in the previous paragraph, remain the same as students move to Grade 6.

The results regarding Grade 7 showed that all the factors that influence fifth and sixth graders' algebraic thinking continue to be significant in Grade 7. Nevertheless, there is an important difference between Grades 5 and 6 and Grade 7. The factor of Deductive Reasoning appears for the first time in Grade 7 as a significant predictor of students' algebraic thinking.

These findings are in line with past and recent research in cognitive psychology which aims to explain the causal role of various cognitive factors in different types of cognitive activity and also the educational behavior of individuals. Demetriou, Spanoudis and Shayer (2015) emphasize that all important theories describe four major levels of intellectual development, reflecting transitions between $1\frac{1}{2}$ – 2, 6-7, and 11-12 years old (e.g. Piaget, 1970; Pascual-Leone, 1970; Case, 1985). These transitions are related to changes in the way individuals operate with representations, moving from concrete to increasingly more abstract representations.

In particular, Demetriou and colleagues described through their research the way several cognitive factors are developed and enriched in respect to the age of the individuals. Specifically, the Qualitative-Analytic System enables students from the age of 9-10 years old to transfer experiences and knowledge from familiar contexts to unfamiliar contexts based on the identification of similarities and differences. Later on, at the age of 11-12 years this ability becomes more flexible (Demetriou, Spanoudis & Mouyi, 2008).

The Causal-Experimental System also starts by the age of 9-10 years old to facilitate the testing of theories about causal relations between objects and persons, while at the age of 11-12 years old this system enables individuals to isolate variables in causal relationships (Demetriou, Spanoudis & Mouyi, 2008).

The Verbal-Propositional System facilitates interaction between persons, direct action, and organization of inference across different domains and situations; it is strongly interrelated with the processes of inductive and deductive reasoning (Demetriou, Spanoudis & Mouyi, 2007). According to Demetriou et al. (2002), the verbal propositional system does not indicate noticeable and stable improvement between the ages of 8 to 10 years old; it is rather at the age of 12 that students begin to develop Verbal-Propositional abilities.

The Spatial-Imaginal System enables students of 9-10 years old to represent familiar persons and objects; this system is improved by the age of 11-12 years old where students become able to imagine non real objects (Demetriou, Spanoudis & Mouyi, 2008). The Spatial-Imaginal System appears to improve faster than the verbal-propositional system, as by the age of 10-11 years old students become able to solve complex tasks, such as mental rotation (Demetriou at al., 2002).

Inference is initially based on perceptual similarity (Demetriou, Spanoudis & Mouyi, 2008). Similarly, analogical reasoning facilitates individuals to transfer meaning on the basis of similarity relationships between objects and concepts (Demetriou, Spanoudis & Mouyi, 2008). As far as it concerns the processes of inductive and deductive reasoning, Demetriou, Spanoudis and Mouyi (2007) have shown that both of them are improved through the ages of 6 to 12 years old. However, the performance of the students in inductive reasoning tasks is consistently higher than deductive reasoning tasks. The development of both inductive and deductive reasoning is influenced by processing efficiency, working memory and information integration. Therefore, deductive reasoning seems to require more support from the various processes. Inductive reasoning develops from the age of 6 to the age of 12 years and proceeds from abilities for identifying patterns and formulating generalizations on the basis of a single dimension or relation to handling hidden or implied relations that require the combination of information available to the senses with information stored in long-term memory. At the last level, inductive reasoning is based on theoretical supposition where multiple parameters and relations can be simultaneously considered and manipulated and generalizations can be extracted. Deductive reasoning is associated with awareness of cognitive processes and cognitive control which enable individuals to search systematically for the relations suggested by premises of an argument and their relations.

Working memory is the cognitive construct related to the maintenance of information in an active state while that information or other information is being processed (Tolar rt al., 2009). As Demetriou at al. (2002) suggested, the more demanding the operations to be performed on information are, the less capacity available for storing information. For this reason, working memroy seems to have a pivotal role when students are engaged to problem solving activities. Working memory abilities improve as students get older and as a consequence problem-solving abilities are imporved systematically with age. However, the exact capacity available at succesive phases of development cannot be specified; what matters is the nature of information and the operational demands of the problem associated with what can be stored (Demetriou at al., 2002).

The results of the current study have shown that fourth graders were based on their abilities regarding the Qualitative-Analytic System and Analogical Reasoning in order to construct inductive inferences guided by similarity-difference relations. For example, this kind of abilities assisted students in identifying the similarities between the terms of a numerical pattern in order to figure out its next term. Inductive reasoning abilities of the first level also might have facilitated the identification of patterns. In a similar vein, Radford (2008) argued that generalization processes in the early grades encompass grasping commonalities among particular elements in a pattern structure.

Students in higher grades appear to rely in their improved Verbal-Propositional abilities for better organizing their inferences, probably in the modeling tasks where they had to interpret the verbal problem, identify its mathematical content and construct plausible inferences. Moreover, older students have improved abilities in the Spatial-Imaginal System and the corresponding Spatial Visualization process. This result is in accordance with previous studies that showed the relation of mathematical achievement to students' spatial ability (e.g. Battista, 1994; Brown and Presmeg, 1993). In the case of algebra, the identification and analysis of figural patterns, requires noticing the relationship between the way the figures in the pattern are placed structurally and discovering the association between the spatial structure and the corresponding numerical structure of each figure (Mulligan and Mitchelmore, 2009). In addition, algebraic thinking involves the ability to represent functional relations graphically and to manipulate visual-spatial representations mentally (Tolar et al., 2009). Consistently, the results of the study index spatial visualization abilities as an important factor for solving algebraic tasks.

Deductive Reasoning appears as a significant predictor of algebraic thinking only in Grade 7. One reasonable explanation for this result might be the fact that deductive reasoning is strongly associated to the notion of proof (e.g. Stylianides & Stylianides, 2008). According to Blanton and Kaput (2005) activities that are included in all of the three factors of algebraic thinking and mainly in the factor of “Modeling” involve proving procedures such as using generalizations to build other generalizations, generalizing mathematical processes, testing conjectures, and justifying. Hence, students’ abilities in Grade 7 for solving the modeling tasks is attributed to the fact that at this age, deductive reasoning has more been developed comparing to the other age groups. For example, the seventh graders were able to successfully solve the item “Determining if the sum of two odd numbers is an even or an odd number” which entailed proving procedures.

The Causal-Experimental System appears to remain stable regarding its effect in algebraic thinking along the four age-groups. A possible reason for this result might be the fact that the Causal-Experimental system refers to overt and covert causal relationships and encloses mental operations such as trial and error, combinatorial hypothesis, systematic experimentation and modeling construction (Demetriou, Spanoudis & Shayer, 2015). These mental operations are in alignment with the requirements of the tasks on all the factors of algebraic thinking (Generalized arithmetic, Functional thinking, Modeling as a domain for expressing and formalizing generalizations) as well as the strategies that students applied in order to be able to complete these tasks in each of the four classes of algebraic thinking abilities that were described in the previous section.

This study confirms that Serial Reasoning is an important predictor of algebraic thinking in all age groups. Serial reasoning shares common features with inductive reasoning. This ability has been considered by related literature as crucial for the engagement in activities of determining pattern rules, recognizing the part that is repeated, and finding not observable terms (e.g. Warren & Cooper, 2008). Palla et al. (2012) also suggest that mathematical induction is applied in situations where a geometrical pattern is translated into an algebraic expression. Ayalon and Even (2013) emphasized the role of inductive reasoning when students investigate algebraic expressions. Martinez and Pedemonte (2014) have shown that a prerequisite for linking inductive argumentation in arithmetic and deductive proof in algebra is the co-existence of arithmetic and algebra for supporting the arguments developed within an argumentation. All these literature reports indicate the important role of inductive reasoning for completing successfully tasks that belong to all of the three factors of algebraic thinking. The results of the current study seem to be aligned with the recommendations of available literature.

The fact that, in the current study, Working Memory seems to stimulate students' algebraic thinking in all grades might be explained by the fact that, as reported above, individual differences in learning at this age-span are sensitive to differences regarding processing and representational efficiency. Specifically, in algebraic problems students are called to handle multiple forms of mathematical expressions, such as, objects with features or a set of procedures, and switch between them as appropriate; this process requires students to retrieve algorithms and facts from long-term memory (Tolar et al., 2009).

The relationship between significant cognitive factors and algebraic thinking in all grades. The findings of the study indicated that there are three important abilities that seem to be common at all of the four age groups and are associated to individuals' performance in algebraic thinking: (i) the Causal-Experimental System (ii) Serial Reasoning and (iii) Working Memory. In order to better describe the nature of this relationship, this study used Structural Equation Modeling Analysis. The results of the analysis suggest that the three cognitive factors predict algebraic thinking. An important contribution of the study is the extraction of a thoroughgoing model where algebraic thinking is a multidimensional factor that is synthesized by the factors of "Generalized Arithmetic", "Functional Thinking" and "Modeling as a domain for expressing and formalizing generalizations". This factor is predicted by the Causal-Experimental system, Serial Reasoning and Working memory.

Serial Reasoning involves the generation of possible relationships and structure between a set of objects. Based on this reasoning process, a general rule is extracted and used for identifying any object of the set. The Causal-Experimental System refers to overt and covert causal relationships and encloses mental operations such as trial and error, combinatorial hypothesis, systematic experimentation, and model construction. Working Memory refers to the maximum amount of information and mental acts that the mind can operate concurrently in an efficient way. These three cognitive factors seem to act simultaneously and enable students to extract generalizations when they encounter algebraic problems.

According to the results, students' abilities for generalization is predicted by their abilities for observing relationships in series of objects, and then making plausible hypotheses about the rules that guide these relationships. Emerging processes in the Causal-Experimental System permit students to test their hypotheses through experimentation and examine the correspondence among the results of their experiment and their initial hypothesis. The construction of a model among the initial hypothesis and the experiment results leads to the establishment of a final generalization about the observed relationship and its justification.

Working memory plays an important role in enabling the Serial Reasoning and Causal-Experimental procedures to act simultaneously. As suggested from studies in the field of psychology, working memory' role is pivotal for determining the complexity of the relations that the mind examines and the problem-solving tasks that may be implemented; working memory involves not only the storage of information but also the orientation to the currently active mental goal and the integration of information across different models of representation (Demetriou, Spanoudis & Shayer, 2015).

What Kind of Instructional Practices Nurture Algebraic Thinking in Elementary School Mathematics?

This study focused on the question of whether it would be possible to engineer an instructional intervention that could have positive impact on the three factors of algebraic thinking. Answering this question is important since prior interventions in this area tended to (i) focus on one aspect of algebraic thinking thereby producing knowledge that could not alone promote the overall development and enrichment of students' algebraic thinking, (ii) extract conclusions through the comparison of one type of interventionist series of lessons to the traditional series lessons. This study followed a different methodology. First the lessons that were designed involved all of the aspects of algebraic thinking. Second, two types of interventionist teaching experiments were designed and conducted, the “Semi-structured problem situations” experiment and the “Structured mathematical investigations” experiment. Both teaching experiments had similar objectives and characteristics in respect to the quality of instruction which was investigative in nature. Though, the tasks in each experiment had different characteristics. In the “Semi-structured problem situations”, the tasks represented authentic contexts from the real life and the questions used were exploratory. In the “Structured mathematical investigations” the tasks involved investigations that were aided with more assisting questions and scaffolding steps.

The results of the study showed that the instruction with “Semi-structured problem situations” had better learning outcomes compared to instruction with “Structured mathematical investigations”, while controlling for preliminary differences regarding students’ algebraic thinking ability and cognitive characteristics. Specifically, the students who received instruction that was developed through “Semi-structured problem situations” outperformed students who received instruction that was developed through “Structured mathematical investigations” in the algebraic thinking post-test. Nevertheless, more detailed results regarding the effect of the two types of teaching experiments have shown that both experiments had positive impact in the “Generalized arithmetic” component. What seems to have affected the overall outcome of the comparison between the two teaching experiments seem to be the fact that the students involved in the “Semi-structured problem situations” had significantly higher performance in the components of “Functional Thinking” and “Modeling as a domain for expressing and formalizing generalizations” in respect to the students that were involved in the “Structured mathematical investigations”.

The tasks that were selected for both teaching experiments involved characteristics that according to Drijvers, Goddijn and Kindt (2011) signify whether an activity is algebraic in nature or not. Specifically, the tasks involved implicit or explicit generalization, patterns of relationships between numbers, logical reasoning with unknown or yet-unknown quantities, mathematical operations with variables, tables and graphs that represent formulas, formulas and expressions that describe situations in which units and quantities play a role, and processes for solving problems contain steps that are based on calculation rules, but that do not necessarily have any meaning in the context of the problem.

Despite the fact that these characteristics were established for the tasks included in both experiments, the results have shown that the experiments had different effects on students' algebraic thinking. A possible explanation regarding this result might be the fact that the two teaching experiments involved different tasks in respect to the way algebraic thinking was expected to be emerged. The "Semi-structured problem situations" involved complex thinking to solve tasks in which there was not a predictable, well-tested approach explicitly suggested by the task. The tasks involved in the "Structured mathematical investigations" used procedures in a manner that deep levels of understanding of mathematical concepts and ideas were maintained and developed. Although students followed suggested pathways through the problems, the pathways tended to be broad, general procedures that had closed connections to the underlying conceptual ideas. While both types of tasks had high cognitive demands, it appears that the first type of tasks facilitated in a greater extend the students to achieve advanced gains in the components of "Functional thinking" and "Modeling as a domain for expressing and formalizing generalizations.

Specifically, the activities that were included in the "Semi-structured problem situations" share common features with modelling approaches to mathematical problem solving. As English and Sriraman (2013) describe, modeling tasks offer richer learning experiences than the typical word problem activities where students have to record problem information with arithmetic quantities and operations through a one- or two-step process. These word problems are usually constrained to problem-solving contexts that are artificially constructed in order to point to the relevant concept (Hamilton, 2007; in English & Sriraman, 2013). In contrast, modelling tasks provide opportunities for children to elicit their own mathematical models as they analyze the problem. That is, the problems require to make sense of the situation so that they can mathematize it themselves in ways that are meaningful to them. This involves a cyclic process of interpreting information, selecting relevant quantities, identifying operations that may lead to new quantities, and creating meaningful representations (Lesh & Doerr, 2003; in Enlish & Sriraman, 2013).

The above features are reported by many researchers as important processes that should be integrated in young children' algebraic thinking. As it was reported in the first section of this chapter, algebraic thinking in the early grades should involve activities where students are engaged in analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. Modeling tasks seem to offer a supportive learning environment for involving all of the above processes.

As far as it concerns the "Structured mathematical investigations", the type of tasks that were included in this experiment also seem to have helped the students to gain some advances to their algebraic thinking controlling for their initial abilities. The "Structured mathematical investigations" experiment aimed in pointing students' attention to the structural relationships that are involved in algebraic activities. This purpose was approached through more scaffolded learning experiences that promote emergent generalization across a range of concepts. While this kind of teaching experiment was found to have an equal effect to children's abilities in "Generalized arithmetic" concepts comparing to "Semi-structured problem situations" experiment, the gains of the students regarding "Functional thinking" and "Modeling as a domain for expressing and formalizing generalization" were lower.

Another interrelated reason for the aforementioned results might be the way cognitive skills were addressed in the teaching experiments. As reported already the tasks in both interventions were cognitively demanding. Both "Semi-structured problem situations" and "Structured mathematical investigations" involved problem-solving processes and skills. Also, the tasks practiced core operators in the SSS such as processing and construction of associations between meaning and representations (Qualitative-Analytic system), hypothesizing and testing causal relationships (Causal-Experimental System) and spatial processing that extends from basic information processing mechanisms related to speed and control of processing to spatial working memory and spatial reasoning (Spatial-Imaginal system).

The fact that "Semi structured problem situations" had higher benefits in the components of "Functional thinking" and "Modeling" might be attributed to the fact that the corresponding activities triggered operators of the Verbal-Propositional System, such as logical exploration and coherence and the Inference System in general, as in the authentic problem situations students were more intensively involved in testing and arguing.

Summarizing the results from the teaching intervention program, “Structured mathematical investigations” tasks might be more appropriate for students of younger ages that need to develop an awareness of mathematical patterns and structure and smoothly pass from arithmetic to algebra through scaffolded learning experiences that are mostly focused on topics of the “Generalized arithmetic” factor and some aspects of the “Functional thinking” factor, such as pattern-eliciting activities. “Semi-structured problem situations” might be more appropriate as students need to further apply their developed awareness of mathematical pattern and structure to other aspects that are related to the factors of “Functional thinking” and “Modeling as a domain for expressing and formalizing generalizations”.

This suggestion is in alignment with the findings regarding the relationship between algebraic thinking and cognitive factors as students move from grade to grade. On the one hand, in the tasks of the “Structured mathematical investigations” experiment, the students had to look for and represent patterns across a variety of concepts, by mainly identifying commonalities and differences. This requirement might be more facilitated by the Reasoning by Analogy factor which was found to be an important predictor of algebraic thinking in fourth graders. On the other, the “Semi-structured problem situations” experiment involved tasks that required students to uncover the mathematical relationships in realistic situations. In most cases students had to spatially analyze and represent the situation and then deductively or inductively come to a conclusion. Thus, it seems that this kind of tasks are in a greater degree facilitated by the spatial abilities of the students and deductive reasoning.

As Radford (2004) argued, “semiotic mathematical and non-mathematical” systems in students’ production of meaning when they encounter algebraic tasks are very important. In particular, Radford (2004) specified that there are three sources of meaning in algebraic activities; (a) the algebraic structure itself (e.g. the letter-symbolic representations), (b) the problem context (e.g. word problems, modeling activities) and (c) the exterior of the problem context (e.g. social and cultural features, such as language, body movements, and experience). The structured mathematical investigation tasks seemed to have involved in a greater extent the first source where semi-structured problem situation tasks involved more the second source. Nevertheless, as Radford declares, both sources are important in algebraic activities.

Accordingly, the results of this study provide support to the argument that considering all types of meaning sources along with specific cognitive skills in the design of educational environments and not merely focusing on specific aspects of mathematical knowledge, might powerfully facilitate students' gains in algebraic thinking ability. Thus we may say that the power of the intervention derives from the synergetic effect of the entire set of design and implementation features of the tasks which combined the core aspects of algebraic thinking and core practices that require the emergence of concrete cognitive constructs. Concluding, the types of tasks that are used in algebraic thinking lessons might be differentiated regarding their "semi-structured" or "structured" character depending on the age of the individuals and the corresponding specific topics of learning.

What is the Impact of the Interactions between the Type of Teaching Experiment and Students' Cognitive Abilities on their Algebraic Thinking Ability?

The results of the study showed that students' individual differences in the three cognitive factors that are related to algebraic thinking, and their interactions with the type of instruction have a significant impact on the benefits from the instructional intervention program. This result supports several claims that were raised in literature regarding the impact of cognitive skills on the teaching and learning outcomes. The findings of this study suggest that various cognitive constructs interact with the type of intervention and this interaction affects the overall benefit of the students. Hence, it seems that the benefits of the students in both teaching experiments and especially in the "Semi-structured problem situations" experiment where the overall benefits were larger, can be accounted to the fact that they allowed the interaction between the content and type of the tasks with specific cognitive constructs, such as the Causal-Experimental System, Serial Reasoning and Working Memory. This result enhances the arguments made in the previous section pertaining the promising effect of teaching experiments which promote the involvement of specific cognitive constructs in the learning experience.

Summary of the Results: Conceptualizing Algebraic Thinking as "Ways of Thinking"

Figure 5.1 summarizes the results of the current study pertaining the conceptualization of algebraic thinking through the identification of (i) its main components and (ii) its ineraction with fundamental cognitive factors and reasoning processes. The connection of these two parts of the study and related findings aims to illustrate a concrete conceptualization of algebraic thinking as "ways of thinking".

In the right hand side of the diagram (see Figure 5.1), the gradation of shading in the three factors of algebraic thinking (from light grey to black) denotes students' level of abilities in each age-level. Next to the three factors, information about the abilities of the students along two core axes are reported: (i) the abilities for formulating and expressing generalizations and the nature of strategies used, and (ii) the type of notation used for expressing relationships and generalizations. On the left hand side of the diagram, the cognitive factors that were found to predict algebraic thinking in each age-level are presented. With bold are marked the factors that were found to be common in all age-levels. With italics are marked the cognitive factors that appear only in one age-level.

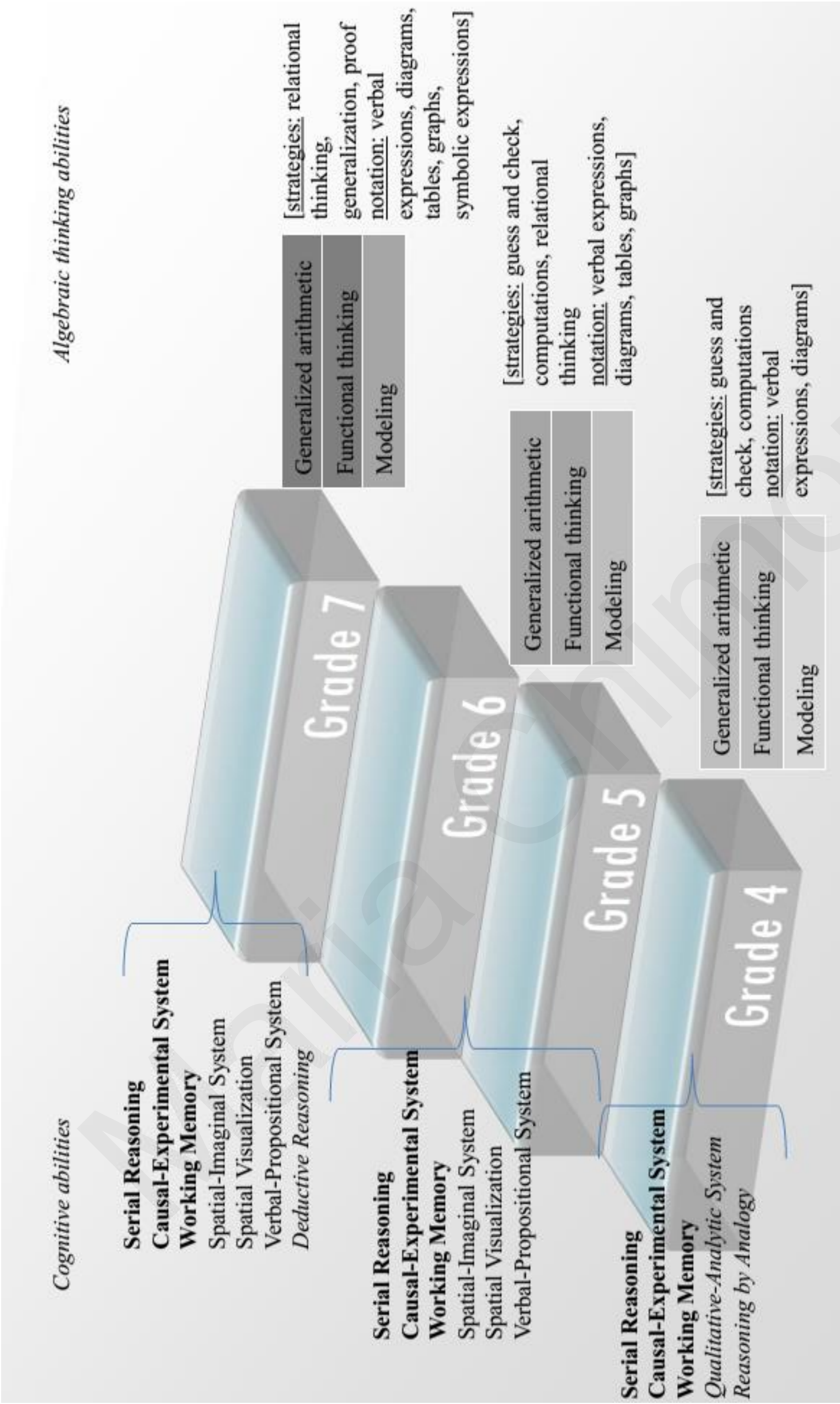


Figure 5.1. The components of algebraic thinking and its relation to cognitive factors.

Interpreting the diagram vertically. As the right hand side of the diagram illustrates, students' algebraic thinking between the ages of 10 to 13 years old is comprised by three distinct factors. Nevertheless, students' abilities along the two axes of strategies and notation are differentiated from level to level. Specifically, at the lower grades, students' strategies seem to be based on arithmetical knowledge and computation procedures. Moving upwards, these strategies become more algebraic in nature, involving relational reasoning. At the upper grades, these strategies are enriched and enable the manipulation of generalizations as independent objects that are not strictly related to particular numbers or situations. Hence, these generalizations can be symbolized through the use of formal algebraic notation.

As the left hand side of the diagram shows, there are three main cognitive factors that predict algebraic thinking in all age-levels; Serial Reasoning, the Causal-Experimental system and Working Memory.

Serial reasoning, which involves features of inductive reasoning, allows students to generate possible hypotheses about the relationships and structure between a set of objects. Based on this reasoning process, students make an effort for extracting a general rule that can be used for identifying any object of the set.

The Causal-Experimental System refers to overt and covert causal relationships and encloses mental operations such as trial and error, combinatorial hypothesis, systematic experimentation, and model construction. These kinds of processes permit students to test their initial hypotheses about the observed relationship in a set of objects, through experimentation and examine the correspondence among the results of their experiment and their initial hypothesis. The construction of a model among the initial hypothesis and the experiment results leads to the establishment of a final generalization about the observed relationship and its justification.

Working Memory refers to the maximum amount of information and mental acts that the mind can operate concurrently in an efficient way. Hence, Working Memory plays an important role in enabling the Serial Reasoning and Causal-Experimental procedures to act simultaneously. As suggested from studies in the field of psychology, working memory's role is pivotal for determining the complexity of the relations that the mind examines and the problem-solving tasks that may be implemented; working memory involves not only the storage of information but also the orientation to the currently active mental goal and the integration of information across different models of representation (Demetriou, Spanoudis & Shayer, 2015).

The interaction between Serial Reasoning, Causal-Experimental System and Working Memory in all grades seems to predict students' abilities for solving generalization tasks. Furthermore, this interaction provides support to the argument of Blanton and Kaput (2008) who described the phenomenon of 'algebrafying' mathematics curriculum as an effort for nurturing classroom norms where the mathematical processes of argumentation, conjecture and justification occur.

While abilities in Serial Reasoning, the Causal-Experimental System and Working Memory appear as crucial for developing algebraic thinking abilities, there are some cognitive factors that do not remain stable in all age-levels. A horizontal correspondence between students' abilities in algebraic thinking and their cognitive abilities in each age-level might provide further insights into the reasons for observing differentiations in students' performance in algebraic thinking from level to level.

Interpreting the diagram horizontally. Starting from the lower levels, the right hand side of the diagram illustrates that fourth graders' algebraic thinking is comprised by three distinct factors. Nevertheless, for the majority of the students, their performance in the corresponding tasks was low, indicating low abilities in each factor. Looking at the left hand side of the diagram, the factors that seem to predict algebraic thinking at this age are the Causal-Experimental System, the Qualitative-Analytic System, Reasoning by Analogy, Serial Reasoning and Working Memory. These students were able to solve some generalized arithmetic tasks (e.g. solving equations) and some functional thinking tasks (e.g. simple numerical patterns that required the identification of the next term of the pattern and not the general term). The Qualitative-Analytic System and Analogical Reasoning which appear only in Grade 4 as significant predictors of algebraic thinking, seem to have facilitated students to construct inductive inferences guided by similarity-difference relations and to use strategies that they are mainly arithmetical in nature. Specifically, analogical reasoning is a specific type of inductive reasoning which is applied in relations as such (e.g. London is to the UK what Paris is to France) (Demetriou, 2015).

According to Sfard and Linchevsky (1994) these strategies signify the beginning of thinking algebraically and a smooth transition between arithmetic and algebra. As it was reported in Chapter II, Sfard and Linchevsky (1994) described the development of algebraic thinking and understanding as a sequence of advanced transitions from an operational perspective to a relational perspective. English and Sharry (1996), reflecting on Sfard and Linchevsky's study, argued that the mental source that triggers this transition is analogical reasoning. It seems that in the case of fourth graders, their strategies and manipulation of notation reflect this transitional stage, where Reasoning by Analogy and Qualitative-Analytic play a central role.

Moving to the second age-level, the diagram illustrates that students in Grades 5 and 6 had average performance in the generalized arithmetic and functional thinking tasks and low performance in the modeling tasks. Similar to Grade 4, the Causal-Experimental System, Serial reasoning and Working Memory predict algebraic thinking for both Grades 5 and 6. However, in these age-levels, the Spatial-Imaginal System and the corresponding Spatial Visualization process, and the Verbal-Propositional System, seem to be empowered and added as significant predictors of algebraic thinking. As students get older, their abilities in the Spatial-Imaginal System are improved, resulting to a higher performance in functional thinking tasks, such as the identification and analysis of figural patterns which require noticing the relationship between the ways the figures in the pattern are placed structurally and discovering the association between the spatial structure and the corresponding numerical structure of each figure. Spatial Visualization abilities also seem to enable students in manipulating visual-spatial representations mentally, such as graphs (Tolar et al., 2009). This result also aligns with Mason and Sutherland's (2002) argument about important features of algebraic thinking; specifically, their first feature referred to algebraic thinking as formulating, transforming and understanding generalizations, not only in numerical contexts but also in spatial relations.

Students in Grade 5 and 6 also appeared to rely in their improved Verbal-Propositional abilities for better organizing and extracting inferences, probably in the functional thinking tasks where they managed to form verbal expressions for interpreting or representing a functional relationship. The Verbal-Propositional abilities are also important in the modeling tasks where students had to interpret the verbal problem, identify its mathematical content, and construct plausible inferences. According to Demetriou, Spanoudis and Mouyi (2008), verbal-propositional abilities encompass the identification of truth in information, abstraction of information in goal-relevant ways, differentiation of the contextual from the formal elements, elimination of biases from inferential process, and establishing validity of inference. This system, enables students to move from the particular to the general and from the general to the particular in flexible ways, and work out concepts that are not any more dependent on particular numbers but on abstract objects.

Students in Grades 5 and 6 seemed to be more able in treating numerical expressions as relationships rather than as directions for performing calculations. Moreover, these students solved more successfully some of the items that involved correspondence and co-variational relationships as well as finding the nth term in numerical patterns, indicating an advanced ability for thinking relationally rather than thinking with specific numbers. However, arithmetical strategies were also apparent. Some students focused on the numerical aspect of the terms and their strategies relied to counting or "Guess and check". All of these features in the behavior of fifth and sixth grade students reflect a more advanced stage in the process of moving form operational to structural perspectives of thinking, where generalization is expressed through various ways and, as Kieran (2011) and Blanton and Kaput (2005) have highlighted, the mathematical processes of argumentation, conjecture and justification begin to have a pivotal role when students manipulate and solve algebraic tasks.

Students in Grade 7 seem to have higher abilities in all factors of algebraic thinking. Their strategies involved producing relational reasoning, not only in the context of patterns or co-variational relationships but also in contexts where a regularity is presented through a realistic situation or phenomenon. Students at this age-group also seemed to have developed a sense of the meaning of symbols as they managed to solve the modeling task in the algebraic thinking test that required the development of a symbolic expression for modeling a function table. Deductive Reasoning appears for the first time as a significant predictor of algebraic thinking in Grade 7. Deductive reasoning is the type of inference where students transfer meaning from general premises to specific premises (Demetriou, 2015) and enables individuals to search systematically for and envision the relations suggested by the premises of an argument and their relations. According to Demetriou, Spanoudis, and Mouyi (2011), deductive reasoning as such does not appear before representations are differentiated from each other and expressed into natural language. In the perspective of the current study, deductive reasoning seems to have facilitated students in following proving procedures such as using generalizations to build other generalizations, generalizing mathematical processes, testing conjectures, and justifying, and also in expressing their inferences with multiple representations, involving formal algebraic notation. Hence, students' abilities in Grade 7 for solving the modeling tasks can be attributed to the fact that at this age group, the factor of deductive reasoning has been developed in a larger extent compared to the other age groups.

Implications. These results provide empirical evidence supporting the arguments from previous literature (e.g. Drijvers, Coddijn & Kindt, 2011; Kaput, 2008; Mason, Graham & Johnston-Wilder, 2005; Radford, 2008) about the multidimensional nature of algebraic thinking and that algebraic thinking 'is not all about literal symbols but rather is about ways of thinking' (Kieran, 2011, p.591). The results enlighten the types and features of these ways of thinking by indicating specific cognitive factors and reasoning processes that flow through varying degrees through the three dimensions of algebraic thinking in each age level. Moreover, the results indicate that these ways of thinking are not static and stable but they progressively become more abstract and flexible.

This study also indicates that in lower age-levels, algebraic thinking is not apparent through the use of precise symbolic language. As Radford (2000) suggested, algebraic thinking entails efforts of the individual to represent generality in certain ways; the expression of generalizations is a process with semiotic and symbolic nature, where social-linguistic elements of the culture of the individual are inducted to mathematical activities. Kieran (2007) also argued that students are facilitated through a variety of mathematical representations to search for and identify structures, such as graphs and tables. The results of the current study reflect this idea, since they provide evidence of the way students move from formulating and expressing generalizations with means that are not strictly formal to the use of letter-symbolic representations

CHAPTER VI

Conclusions

Prior research that has been successfully enlighten the notion of algebraic thinking focused primarily on secondary school students' algebraic thinking or most often on one of its multiple aspects. This fact hindered the building of foundations for effectively integrating algebraic thinking in earlier levels of schooling. This study explored a different approach to the issue of describing in detail the notion of algebraic thinking in the elementary grades. In particular, the theoretical model proposed by Kaput (2008) about the core aspects of algebraic thinking was for the first time empirically tested. This study confirms, based on empirical data, that algebraic thinking can be described using a combination of three distinct but interrelated factors: (i) "Generalized arithmetic", (ii) Functional thinking", and (iii) "Modeling as a domain for expressing and formalizing generalizations". The findings which derived from applying multiple methods of statistical analysis in data gathered from fourth, fifth, sixth and seventh graders, offered good evidence for the presence of the three basic components in students' algebraic thinking ability, thus creating the basis for examining the potential applicability of this model in other age groups, either younger or older.

The tasks in the algebraic thinking test study were carefully selected so as to unfold each of the three factors in the model and to measure algebraic thinking in four different grades of school education. The wide use of algebraic notation in the tasks was avoided in purpose, since Kaput highlighted that algebraic thinking involves making generalizations and expressing those generalizations in forms that are not necessarily symbolic but increasingly become more conventional. Carraher and Schliemann (2014) also stated that algebraic thinking in the early grades can take place in the absence of algebraic notation. Similarly, Radford (2014) argued that what allowed researchers to discuss about the possibility of developing algebraic thinking in the early grades is the rejection of the idea that notations are a manifestation of algebraic thinking. Therefore, the majority of the tasks involved verbal expressions, diagrams, drawings or graphs rather than symbols. Moreover, these tasks did not require the representation of an extracted generalization with symbols but in one of the aforementioned forms.

Based on the findings regarding the components of algebraic thinking, this study also described four classes of students which reflect broad portraits of students' abilities and skills that can be used to inform our understanding of the way students develop generalization abilities and move from arithmetical to algebraic ways of thinking. In contrast to existing theoretical approaches which focused on designating border lines in between algebra and arithmetic, this study supports that there are not explicit compounds between them since the factor of "Generalized arithmetic" offers students the opportunity to see arithmetic in algebra and algebra in arithmetic. Particularly, the results indicated that some students attended the "Generalized arithmetic" tasks and even some of the "Functional thinking" tasks through the application of arithmetical strategies such as counting and guess and check. Nonetheless, students illustrated an awareness of the structure in the tasks. This is exactly the level where arithmetic and algebra co-exist; arithmetical and algebraic modes of thinking come into an interplay and assist students to progressively move to more abstract and advanced forms of thinking. This level reflects Subramaniam and Banerjee's (2011) emphasis on the need for viewing arithmetic with 'algebra eyes'.

Specifically, this study describes four classes of students with different levels of algebraic thinking abilities. Students in Class 1 hardly solved any type of algebraic tasks, implying a primitive level of algebraic thinking ability. Students in Class 2 seem to be in an intermediate stage since they managed to solve the "Generalized arithmetic" tasks by applying arithmetical strategies. As Sfard and Linchevski (1994) argued, these strategies signify the beginning of thinking algebraically and a smooth transition between arithmetic and algebra. Students in Class 2 seem to make an effort for solving algebraic tasks by using as tools the knowledge and abilities they have developed through the strand of arithmetic.

Students in Class 3 also used arithmetical strategies but they managed to attend algebraic tasks not only in the factor of "Generalized arithmetic" but also in the factor of "Functional thinking". These students were more able in treating numerical expressions as relationships rather than as directions for performing calculations.

Moreover, the students managed to solve successfully some of the items that involved correspondence and co-variational relationships as well as finding the nth term in numerical patterns. As Fujii και Stephens (2008) argued, the identification of repeated patterns is one of the most important indicators of students' ability for reasoning relationally rather than thinking with specific numbers. Mulligan and Mitchelmore (2009) also supported that when students are able to extend either a numerical or geometric sequence, they indicate a structural awareness of the way the numbers or the figures in the pattern are placed.

Students in Class 4 had high abilities in the items of the factors "Generalized arithmetic" and "Functional thinking" and average abilities in the items of the factor "Modeling as a domain for expressing and formalizing generalizations". These students seem to have developed abilities for producing relational reasoning, not only in the context of patterns or co-variational relationships but also in contexts where a regularity is presented through a realistic situation or phenomenon. Moreover, these students were able to manipulate and /or translate representations of one form to another as well as to select the appropriate model for representing a complex problem (e.g. symbolic expression, table, and graph).

The results of the study show that the majority of the students in Class 1 came from Grade 4. The majority of fifth and sixth graders were categorized in either Classes 2 or 3. The majority of the students in Class 4 came from Grade 7. These results indicate that the abilities of the students in each class are not age independent. Consequently, students' algebraic thinking cannot be independent regarding their learning experiences or the cognitive skills that characterize each age-group.

Analyses pertaining the existence of a hierarchical trend in the way the components of algebraic thinking develop indicated that students are more successful first in doing the generalized arithmetic tasks and later on in doing the functional thinking tasks. Students were only able to deal with the modeling tasks once they have been successful in the generalized arithmetic and functional thinking tasks. Thus, this study indicates that, for students from Grades 4, 5, 6 and 7, the development of algebraic thinking progresses from generalized arithmetic, to functional thinking, to modeling. This analysis specifies previous descriptions from available literature which supported that the development of algebraic thinking is a process that evolves from thinking with the specifics to thinking abstractly, from thinking with fix-values (unknowns) to thinking about quantities that vary (variables) (e.g. Kuchemann, 1981; Mason, 1989; Sfard & Linchevski, 1994; Thomas & Tall, 2001). Along this developmental process, the meaning and use of notation also alters, from understanding and using notation through verbal expression, diagrams or graphs to the use of formal symbols for representing relationships. According to Arcavi (2005), this ability is extended in the middle school in order students to be able to apply syntactic rules for solving equations and then understanding the different role that the same symbol might take in an equation.

This study contributed to theory about the core aspects of algebraic thinking by utilizing research from mathematics education and psychology. In particular, this study proposes and exemplifies a model that aims to cast light on the cognitive constructs of the individuals that affect their algebraic thinking ability. Using Demetriou and colleagues' (2002, 2011, 2015) overarching theory about the architecture and development of the mind as a basis for describing mental action, this study investigated the relationship between algebraic thinking and several cognitive factors and reasoning processes. The results indicated that the relationship between algebraic thinking and cognitive factors changes from age to age. Along the transition of the students from grade to grade, some of the factors appear or disappear in the relationship and some of them remain stable in all age-groups. This analysis can be used to inform our understanding of when students are expected to overcome innate constraints related to their algebraic thinking ability.

The findings of the study suggest that the algebraic thinking ability of the students in Grade 4 is predicted by the Causal-Experimental System, the Qualitative-Analytic System, Reasoning by Analogy, Serial Reasoning and Working Memory. The algebraic thinking ability of students in both Grades 5 and 6 is predicted by the Spatial-Imaginal System, the Causal-Experimental System, the Verbal-Propositional System, Spatial Visualization, Serial reasoning and Working memory. The results regarding Grade 7 showed that all the factors that predict fifth and sixth graders' algebraic thinking continue to be significant in Grade 7. Nevertheless, the factor of Deductive Reasoning also appears in Grade 7 as a significant predictor of students' algebraic thinking. These findings are in line with past and recent research in cognitive psychology which aims to explain the important role of various cognitive factors in different types of cognitive activity and also the educational behavior of individuals. As in cognitive psychology, the relationship of algebraic thinking with cognitive factors changes as soon as students move from Grade 4 to Grade 5 and then from Grade 6 to Grade 7 (e.g. Piaget, 1970; Pascual-Leone, 1970; Case, 1985). These transitions denote changes in the way individuals operate with representations, moving from concrete to increasingly more abstract representations.

The fact that processes in the Qualitative-Analytic System and Reasoning by analogy appear to influence fourth graders algebraic thinking might explain the abilities of these students to manipulate and successfully solve many generalized arithmetic tasks and simple patterns, as these cognitive factors enable the identification of similarities and differences between several structures in a set.

In a similar vein with the results of psychological studies (e.g. Demetriou, Spanoudis and Mouyi, 2008), students in Grades 5 and 6 appeared to have improved Verbal-Propositional and Spatial-Imaginal skills; these abilities might explain their successful manipulation of tasks that involved proving procedures, such as the items involving operations with odd and even numbers, and tasks that involved the analysis of spatial structures, such as geometrical patterns and co-variation relationships that were represented with graphs.

The fact that deductive reasoning appears as a significant factor of seventh graders' algebraic thinking is also in alignment with previous evidence from psychological studies (e.g. Demetriou et al, 2002), which indicated that this process is enriched as students get older. The advanced skills of seventh graders in deductive reasoning might have facilitated their higher performance in algebraic tasks that required the application of general methods in order to reach a viable conclusion in a specific situation (e.g., the comparison of sales offers).

The findings of the study indicated that the Causal-Experimental System, Serial reasoning and Working Memory appear in the relationship of algebraic thinking with cognitive factors in all age-groups. For this reason, their relationship was further examined through Structural Equation Modeling analyses. The model that was extracted from these analyses suggests that the Causal-Experimental System, Serial Reasoning and Working Memory predict algebraic thinking abilities. Thus, the proposed model of algebraic thinking offers insights into the prior-unclear relationship between algebraic thinking and specific cognitive constructs. According to the model, algebraic thinking is a multidimensional concept that is synthesized by the factors of "Generalized Arithmetic", "Functional Thinking" and "Modeling as a domain for expressing and formalizing generalizations". This factor is associated with the operations of the Causal-Experimental system which belongs to the Specialized Structural Systems, the Serial Reasoning which belongs to the Inference System and Working Memory which belongs to the Representational Capacity System.

Serial reasoning shares common features with inductive reasoning. At initial stages, this ability enables students to identify patterns and formulate generalizations on the basis of a single dimension or relation. In more advanced levels, this ability facilitates the formulations of generalizations in more complex patterns. Similarly, this ability has been considered by related literature as crucial for the engagement of the students in activities for determining pattern rules, recognizing the part that is repeated, and finding not observable terms (e.g. Ellis, 2007; Rivera & Becker, 2008; Warren & Cooper, 2008).

The Causal-Experimental System seems to predict algebraic thinking since it refers to overt and covert causal relationships and encloses mental operations such as trial and error, combinatorial hypothesis, systematic experimentation and modeling construction (Demetriou, Spanoudis & Shayer, 2015). According to the results, students' abilities for generalization is predicted by their abilities for observing relationships in series of objects through Serial Reasoning processes, and then making plausible hypotheses about the rules that guide these relationships. Emerging processes in the Causal-Experimental System permit students to test their hypotheses through experimentation and examine the correspondence among the results of their experiment and their initial hypothesis. The construction of a model among the initial hypothesis and the experiment results leads to the establishment of a final generalization about the observed relationship and its justification.

Working memory plays an important role in enabling the Serial Reasoning and Causal-Experimental procedures to act simultaneously. This cognitive construct is related to the maintenance of information in an active state while that information or other information is being processed (Tolar et al., 2009). As Demetriou at al. (2002) suggested, the more demanding the operations to be performed on information are, the less capacity available for storing information. For this reason, working memory seems to have a pivotal role when students are engaged to problem solving activities. Specifically, in algebraic problems students are called to handle multiple forms of mathematical expressions, such as, objects with features or a set of procedures, and switch between them accordingly (Tolar et al., 2009).

Based on the above results, this study explored two approaches to the issue of impacting positively on fifth grade students' algebraic thinking ability; two interventions with ten 80-minutes lessons, the "Semi-structured problem situations" and the "Structured mathematical investigations", were examined in respect to their effectiveness. Both interventions involved all of the aspects of algebraic thinking and had similar objectives and characteristics in respect to the quality of instruction. The interventions differed in respect to characteristics of the tasks that were used. In the first intervention, the tasks represented contexts from real life and the questions used were more exploratory. In the second intervention, the tasks involved mathematical investigations that were aided with more assisting questions and scaffolding steps. The findings, which derived from the analysis of pre-test and post-test data, offered good evidence for the positive impact of both interventions on students' algebraic thinking in the particular context of the classes . However, the results showed that the instruction with "Semi-structured problem situations" had better learning outcomes compared to instruction with "Structured mathematical investigations", while controlling for preliminary differences regarding students' algebraic thinking ability and cognitive characteristics. Specifically, the students who received instruction through "Semi-structured problem situations" outperformed students who received instruction through "Structured mathematical investigations" in the algebraic thinking post-test. More detailed results have shown that both experiments had equal positive impact in the "Generalized arithmetic" component. The students involved in the "Semi-structured problem situations" experiment had significantly higher performance in the components of "Functional Thinking" and "Modeling as a domain for expressing and formalizing generalizations" comparing to the students that were involved in the "Structured mathematical investigations" experiment.

As it appears, the activities that were included in the "Semi-structured problem situations", which share common features with modeling approaches to mathematical problem solving provided students with opportunities to elicit their own mathematical models as they analyzed real-life problem situations. These problems required to interpret the problem information, select relevant quantities, identify operations that may lead to new quantities, and create meaningful representations. Thus, this kind of tasks seem to offer a supportive learning environment for involving important algebraic thinking aspects, such as analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. These processes remind of Blanton and Kaput's (2008) description of the phenomenon of 'algebrafying' mathematics curriculum as efforts for nurturing classroom norms where the mathematical processes of argumentation, conjecture and justification occur.

The "Structured mathematical investigations" seem to have helped the students to gain some advances to their algebraic thinking controlling for their initial abilities. The "Structured mathematical investigations" experiment aimed in pointing students' attention to the structural relationships that are involved in algebraic activities. This purpose was approached through more scaffolded learning experiences that promote emergent generalization across a range of concepts. While this kind of teaching experiment was found to have an equal effect to children's abilities in "Generalized arithmetic" concepts comparing to "Semi-structured problem situations" experiment, the gains of the students regarding "Functional thinking" and 'Modeling as a domain for expressing and formalizing generalization" were lower.

Taking into consideration the results of both interventions, the "Structured mathematical investigations" tasks might be more appropriate for students of younger ages that need to develop an awareness of mathematical patterns and structure and start to develop algebraic thinking through scaffolded learning experiences that are mostly focused on topics of the "Generalized arithmetic" factor and some aspects of the "Functional thinking" factor, such as pattern-eliciting activities. "Semi-structured problem situations" might be more appropriate as students need to further apply their developed awareness of mathematical pattern and structure to other aspects that are related to the factors of "Functional thinking" and "Modeling as a domain for expressing and formalizing generalizations". Concluding, the types of tasks that are used in algebraic thinking lessons might be differentiated regarding their "semi-structured" or "structured" character depending on the age of the individuals, the corresponding specific topics of learning or even the way that matches students' style of learning.

The results of the study showed that students' individual differences in the three cognitive factors that are related to algebraic thinking (Causal-experimental system, Serial Reasoning, Working Memory), and their interactions with the type of instruction had a significant impact on the benefits from the instructional intervention program. This result supports the argument that teaching experiments which promote the involvement of specific cognitive constructs in the learning experience might have positive effect on students' algebraic thinking. Further, the results demonstrate that algebraic thinking competence is associated with approaches that also enable the development of competence in specific cognitive constructs.

Concluding, a major contribution of this study is the description of a broad portrait of students' emerging algebraic thinking abilities which dynamically interact with students' emerging cognitive skills as these are defined by cognitive factors that reflect three systems of mental action - the Specialized Structural Systems, the Inferential System and the Hypercognitive System. Therefore, this study conceptualizes algebraic thinking, through concrete paradigms and descriptions extracted form empirical data as "ways of thinking". This conceptualization can be used in order to inform educators' understanding of the approximate ages at which students may be able to master different forms of algebraic thinking and cognitive skills that are essential to their engagement with algebraic tasks. In addition, the two teaching experiments developed in the perspective of these results, offer useful insights into the way different forms of algebraic thinking and cognitive requirements necessary for the development of algebraic thinking might be cultivated in school mathematics classroom.

Theoretical, Methodological and Practical Contributions of the Study

This study offers insights into the notion of algebraic thinking by extending previous theoretical frameworks in order to describe not only the core components of algebraic thinking but also their relationship to specific cognitive constructs. For the first time, a study approached Kaput's theoretical framework from an empirical perspective and verified its content and structure. Moreover, this analysis refers to the higher grades of elementary school and first grade of secondary school, establishing a basis for understanding in what ways algebraic thinking changes as students move from the elementary to the secondary school. From a theoretical standpoint, this study described in detail the theoretical foundations of algebraic thinking in respect to its nature and content and its relation to various cognitive factors and reasoning processes, for every age group in separation. This analysis reflects upon a timeless enquiry in mathematics education pertaining the relationship between arithmetic and algebra and also declares the role of algebraic notation in the process of developing students' algebraic thinking.

From a methodological standpoint, this study offers a tool for measuring students' algebraic thinking for Grades 4, 5, 6 and 7. The algebraic thinking test that was designed and used in this study clarifies the tasks which can be used in order to describe students' abilities in the three factors of algebraic thinking. Taking into consideration the fact that the tool refers to a range of ages, the tasks were carefully selected in order to avoid the need for involving algebraic notation and requiring the use of algebraic notation for representing relationships, equations and formulas. In contrast, the test makes use of a variety of representations, such as verbal expressions, tables, figural representations, graphs and diagrams. The test includes both open-ended questions and multiple choice tasks.

From a practical standpoint, the theoretical model of the study acknowledges researchers, educators and policy makers about the basic components of algebraic thinking, thus serving as a tool for integrating algebraic thinking in a viable way in mathematics curricula and instructional programs. Moreover, the model of algebraic thinking and the test of algebraic thinking represent a comprehensive set of tasks that can be used as a guide for further designing and implementing teaching-related algebraic tasks and explore further their effectiveness through experimental studies that investigate teachers' training and preparation programs.

This study also offered insights into the often-problematic relationship between theory on fundamental mathematical concepts and the practical work of mathematics teaching and learning. Specifically, it contributes to understanding of the way theoretical ideas on the mathematics content knowledge can be used to design appropriate instructional programs. In this case, the results pertaining the structure and components of algebraic thinking were used in order to design an instructional program that covers all aspects of algebraic thinking for students in Grade 5. In addition, the program took into consideration the need for designing mathematical activities that promise to promote the involvement of specific cognitive skills of the students during mathematics instruction. Specifically, the instructional program proposes conditions that increase the need for using specific cognitive skills and reasoning processes, such as causal-experimental contexts, spatial visualization and inductive reasoning, in order to empower the developmental progression in students' algebraic thinking.

Following this methodology, as well as the results of the current research, similar instructional programs can be designed with the purpose of further enhancing students algebraic thinking either in Grades 4, 6 or 7. The model of algebraic thinking as well as the cognitive factors that appear to predict students' abilities in each grade, can potentially guide the development of appropriate tasks and corresponding lessons plans. In this perspective, algebraic thinking can be integrated in students' mathematical experiences in a viable way which corresponds to their specific needs and capabilities, depending on their age.

Limitations of the Study

To facilitate the exploration of the notion of algebraic thinking through identifying its main components and its association with cognitive factors, research was based on quantitative data and analyses. The fact that no qualitative data were collected resulted in findings that are not supported by detailed information about students' common errors and thinking while solving algebraic tasks.

To explore the notion of algebraic thinking, a test was constructed which involved multiple-choice questions and open-ended questions. Moreover, the Specialized Structural Systems test, the Deductive Reasoning test and The Naglieri Non-Verbal Ability test were all multiple-choice tests. The scoring and analysis undertaken highlighted some of the limitations of using multiple-choice questions, since this format assist students to correctly answer a question despite the fact that they might not have acquired the knowledge that the question is designed to capture.

Moreover, with this format is not possible to trace students' reasoning in answering these questions.

To investigate the possibility of empowering students' algebraic thinking through promising instructional practices, two teaching experiments were conducted and compared. The teaching experiments followed specific theoretical backgrounds regarding their characteristics and the design of the tasks involved. Inevitably, this fact imposes constraints to the study, the most important that other significant theories and models in respect to mathematics instruction were not taken into consideration. For this reason, the outcomes of the intervention program are strictly interpreted on the basis of the specific design decisions that were taken.

Implications for Further Research

This study proposes a model which clarifies the basic components of algebraic thinking and important cognitive factors that interact with algebraic thinking. According to the model, algebraic thinking is a multidimensional factor that is synthesized by the factors of "Generalized Arithmetic", "Functional Thinking" and "Modeling as a domain for expressing and formalizing generalizations". This factor is associated with the Causal-Experimental system, Serial Reasoning and Working Memory. The sample that the study used was limited to students from Grades 4, 5, 6 and 7. For this reason, it is important to conduct similar research studies that investigate students' algebraic thinking with younger and older students. Specifically, it is important to investigate the way the tenets of this model apply to other algebraic tasks that capture students' algebraic thinking abilities in either Grades 1-3 or Grades 7-11. Moreover, another direction of research can be the investigation of the way students' engagement with algebraic tasks at different levels of schooling can be organized based on the forms of algebraic thinking that they can master and their corresponding cognitive skills.

This study identified four classes which reflect groups of students with different abilities in algebraic thinking. The classes exemplify the tasks that students in each class are able to solve as well as some strategies that students applied while solving the algebraic thinking test. Complementary studies can be designed in order to investigate students' abilities, conceptions, strategies, common errors and difficulties in each class by using qualitative methods of research. All of the results in this study were based on quantitative data and corresponding analyses. For this reason, the conduction of qualitative studies will further cast light on students' algebraic thinking abilities. Similar studies can also be conducted to validate the existence of these classes of abilities and investigate the possibility of additional classes in higher levels of education.

Moreover, in the light of the proposed model of algebraic thinking, the development of students' symbol sense can further be investigated and described. Specifically, it would be significant to depict through a longitudinal, qualitative study, the way different forms of notation are interpreted and used in each of the three factors of algebraic thinking along different ages.

The two interventions that were designed and implemented in the context of the current study have shown that the corresponding lessons indeed helped fifth graders to develop further their algebraic thinking. In particular, the "Semi-structured problem situations" experiment appeared to have a more positive effect on students' algebraic thinking comparing to the "Structured mathematical investigations" experiment. Consequently, it remains open to further examine whether this program will be equally effective with students that belong in other age groups, either younger or older. In addition, future studies can investigate the effects of other characteristics of the intervention, such as the mixture and use of both semi-structured problem situations and structured mathematical investigations in the same experiment.

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APPENDICES

APPENDIX I: ALGEBRAIC THINKING TEST

ΔΟΚΙΜΙΟ ΑΛΓΕΒΡΙΚΗΣ ΣΚΕΨΗΣ

1. Συμφωνείς με την πιο κάτω δήλωση; Να δικαιολογήσεις την απάντηση σου.

Το άθροισμα δύο ζυγών αριθμών είναι πάντα μονός αριθμός.

____

____

____

(ga1)

2. Ο Νικηφόρος υπολόγισε το άθροισμα 80 + 50 με τον ακόλουθο τρόπο:

80 + 20 + 30 = 100 + 30
= 130

Να χρησιμοποιήσεις τον τρόπο του Νικηφόρου, για να υπολογίσεις το άθροισμα 70 + 50.

(ga2)

3. Να παρατηρήσεις τον τρόπο με τον οποίο η Βασιλική εκτέλεσε τον πολλαπλασιασμό 35 Χ 22.

```
   35
X  22
   70
+  70
  140
```

Είναι ορθή η απάντηση της Βασιλικής; Να δικαιολογήσεις την απάντηση σου.

____

____

(ga3)

4. Ένα πιόνι μετακινήθηκε από τον αριθμό 64 στον αριθμό 72.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	63	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Ποια πράξη παρουσιάζει την κίνηση του;

(α) 64 + 8

(β) 64 + 10 - 2

(γ) 64 + 6 + 2

(δ) 64 + 10 + 2

(ga4)

5. Για ποια τιμή του β ισχύει η ανισότητα;

12 < 3 X β

(α) 2

(β) 3

(γ) 4

(δ) 5

(ga5)

6. Να υπολογίσεις την αξία του Ν, ώστε να ισχύει η πιο κάτω ισότητα.

Ν + 4 = 12

(ga6)

7. Το άθροισμα της μαθηματικής πρότασης ***1245676 + 4535731*** είναι ζυγός ή μονός αριθμός; Να επεξηγήσεις.

____

____

____

(ga7)

8. Αν γνωρίσεις ότι ☆ + ☆ = 4 τότε,

☆ + ☆ + 6 = ;

(ga8)

9. Η θερμοκρασία μπορεί να μετρηθεί τόσο σε βαθμούς Κελσίου (C°) όσο και σε βαθμούς Φαρενάιτ (F°). Τα παιδιά έχουν μετατρέψει τους βαθμούς Κελσίου σε Φαρενάιτ και συμπλήρωσαν τον πίνακα. Ποια σχέση συνδέει τα δύο μεγέθη;

*9 X **25** = (**47** - 32) X 5*

*9 X **30** = (**86** - 32) X 5*

*9 X **40** = (**104** - 32) X 5*

Κελσίου (C°)	Φαρενάιτ (F°)
25	47
30	86
40	104

(α) 9 X C = (F-32) X 5

(β) 9 X F = (C-32) X 5

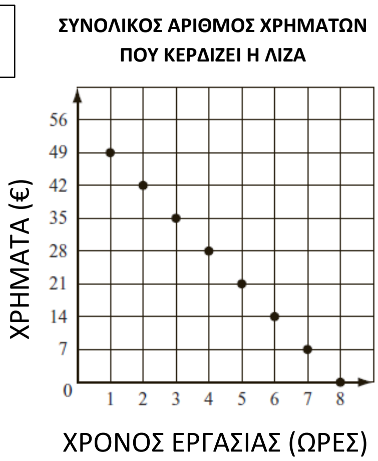
(γ) 9 X C = (47 - F) X 5

(δ) 9 X F = (32 - C) X 5

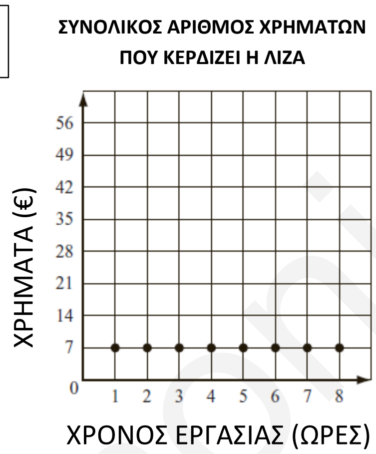
(mod1)

10. Η Λίζα κερδίζει €7 για κάθε ώρα που εργάζεται. Ποια από τις πιο κάτω γραφικές παραστάσεις παρουσιάζει τον συνολικό αριθμό χρημάτων που κερδίζει η Λίζα σε σχέση με τις ώρες εργασίας της;

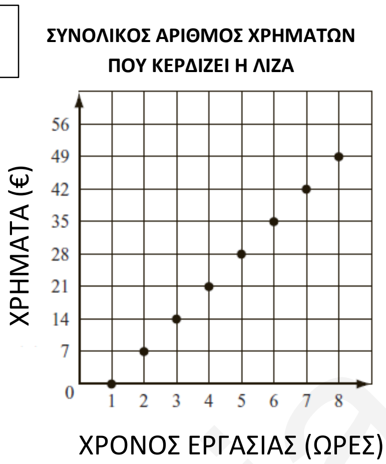
Α.



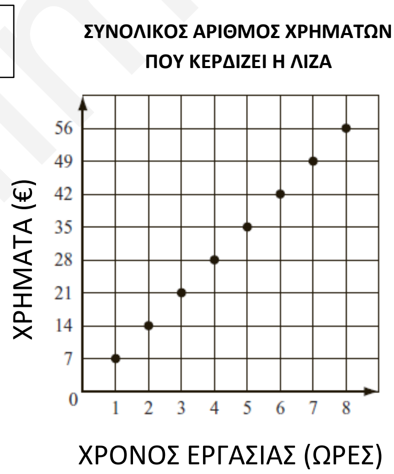
Β.



Γ.



Δ.



(ft1)

11. Αν το μοτίβο 3, 6, 9, 12 συνεχιστεί, ποιος από τους αριθμούς θα μπορούσε να είναι ένας από τους αριθμούς του μοτίβου;

(α) 26

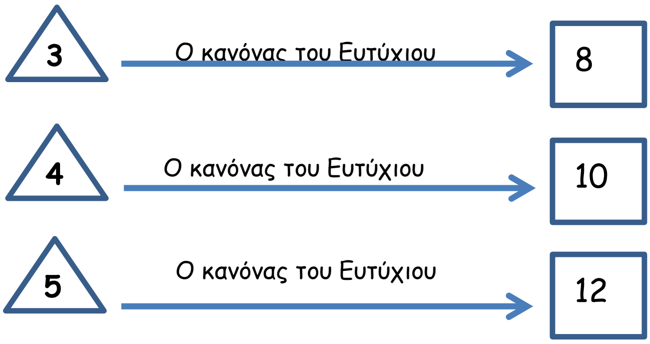
(β) 27

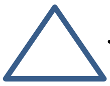
(γ) 28

(δ) 29

(ft2)

12.



Ο Ευτύχιος εφάρμοσε ένα κανόνα για να πάρει τον αριθμό στο  από τον αριθμό στο  . Ποιος ήταν ο κανόνας;

(α) Πολλαπλασιάζω επί 1 και προσθέτω 5.

(β) Πολλαπλασιάζω επί 2 και προσθέτω 2.

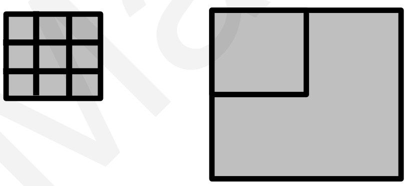
(γ) Πολλαπλασιάζω επί 3 και αφαιρώ 1.

(δ) Πολλαπλασιάζω επί 4 και αφαιρώ 4.

(ft3)

13. Να βρεις με διαφορετικούς τρόπους πόσα μικρά τετράγωνα χωράνε στο μεγάλο τετράγωνο.

Ποιος είναι ο πιο σύντομος τρόπος; Να επεξηγήσεις.



(mod2)

14. Να βάλεις σε σειρά τις πιο κάτω προσφορές αρχίζοντας από αυτήν που προσφέρει τη μεγαλύτερη έκπτωση:

- *Δύο στην τιμή του ενός.*
- *Αγόρασε δύο και πάρε ένα δωρεάν.*
- Τρία στην τιμή του ενός.

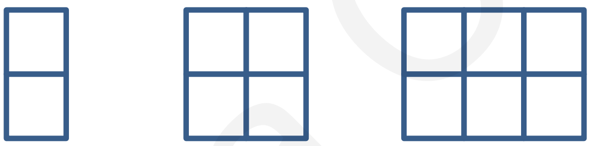
1. ______________________________

2. ______________________________

3. ______________________________

(mod3)

15. Ο Βασίλης τοποθετεί τετράγωνα με τον ακόλουθο τρόπο:

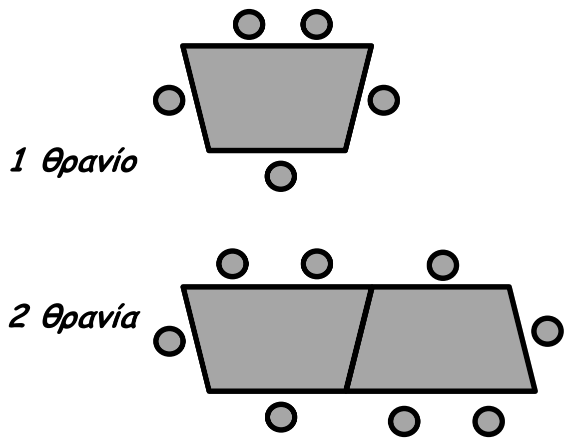


Σχήμα 1　Σχήμα 2　Σχήμα 3

Πόσα τετράγωνα θα υπάρχουν στο 16^{o} σχήμα;

(ft4)

16. Σε ένα θρανίο με σχήμα τραπέζιο μπορούν να καθίσουν 5 παιδιά. Αν ενωθούν δύο τέτοια θρανία μπορούν να καθίσουν 8 παιδιά.



(α) Πόσα παιδιά μπορούν να καθίσουν σε 3 θρανία;

(β) Πόσα παιδιά μπορούν να καθίσουν σε 10 θρανία;

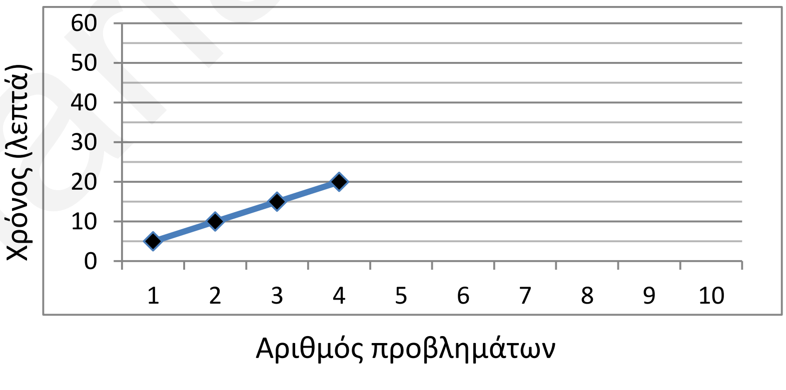
(ft5)

17. Η Ιωάννα θα κάνει μάθημα ηλεκτρονικών υπολογιστών δύο φορές τη βδομάδα. Ποια προσφορά θα της σύστηνε να επιλέξει, ώστε να πληρώσει όσο το δυνατόν λιγότερα χρήματα στο τέλος του μήνα;

ΠΡΟΣΦΟΡΑ Α	**ΠΡΟΣΦΟΡΑ Β**
€8 για κάθε μάθημα	€50 για τα 5 πρώτα μαθήματα του μήνα και €4 για κάθε επιπλέον μάθημα.

(mod4)

18. Η γραφική παράσταση δείχνει το χρόνο που πήρε στη Σταυριανή για να λύσει προβλήματα.



Πόσο χρόνο πήρε στη Σταυριανή να λύσει 3 προβλήματα;

(ft6)

19. Πιο κάτω φαίνονται οι προσφορές δύο μουσικών εταιριών στο διαδίκτυο.
Η κάθε προσφορά περιλαμβάνει 10 δωρεάν τραγούδια κάθε μήνα.

ΜΟΝΟ ΕΠΙΤΥΧΙΕΣ:	€20 το μήνα, €3 για κάθε επιπλέον τραγούδι.
ΤΑ ΚΑΛΥΤΕΡΑ ΤΡΑΓΟΥΔΙΑ:	€30 το μήνα, €2 για κάθε επιπλέον τραγούδι.

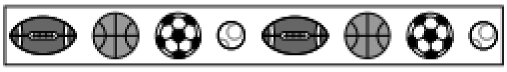
(α) Να συμπληρώσεις τον πίνακα.

Αριθμός επιπλέον τραγουδιών κάθε μήνα	**3**	**6**	**9**	**12**	**15**
ΜΟΝΟ ΕΠΙΤΥΧΙΕΣ / ΣΥΝΟΛΙΚΟ ΚΟΣΤΟΣ	€29				
ΤΑ ΚΑΛΥΤΕΡΑ ΤΡΑΓΟΥΔΙΑ/ ΣΥΝΟΛΙΚΟ ΚΟΣΤΟΣ	€36				

(β) Ποια εταιρία θα σύστηνες σε μια φίλη σου αν σκοπεύει να αγοράζει 9 επιπλέον τραγούδια το μήνα;

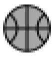
(mod5)

20. Η Λίζα κατασκευάζει το πιο κάτω μοτίβο με αυτοκόλλητα. Ποια θα είναι η $15^{η}$ μπάλα;



Start

A. 

B. 

C. 

D. 

(mod6)

21. Να επιλέξεις την εξίσωση που περιγράφει το μοτίβο του πίνακα;

Εισερχόμενα (χ)	16	19	22	25	28	31
Εξερχόμενα (ψ)	9	12	15			

(α) 16 – 7 = ψ

(β) χ – 7 = ψ

(γ) χ + 6 = ω

(δ) ψ + 6 = χ

(mod7)

APPENDIX II: TEACHING EXPERIMENTS

Άλγεβρα & Αλγεβρική Σκέψη

Ημι-Δομημένες Διερευνήσεις σε Ρεαλιστικά Προβλήματα

Περιεχόμενα

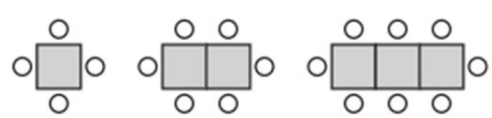
ΜΑΘΗΜΑ 1

Ο Φάνης εργάζεται σε ένα εστιατόριο. Ετοιμάζει τα τραπέζια, για να υποδεχθεί τις κρατήσεις που έχει το εστιατόριο για το μεσημέρι.

Λίστα Κρατήσεων

Όνομα	Αριθμός ατόμων
Γεωργίου	4
Δημητρίου	6
Στεφάνου	8
Χαραλάμπους	16
Κυριάκου	22
Βασιλείου	24

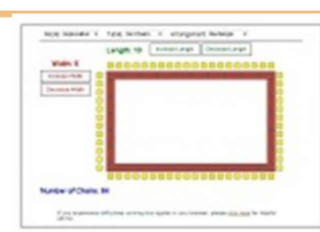
(α) Το εστιατόριο διαθέτει τετράγωνα τραπέζια. Να υπολογίσεις τον αριθμό των τετράγωνων τραπεζιών που θα ενώσει ο Φάνης για την κράτηση των 16 ατόμων.



(β) Το εστιατόριο διαθέτει και τραπέζια που έχουν σχήμα εξάγωνο. Ο Φάνης θα χρησιμοποιήσει αυτά τα τραπέζια για την κράτηση των 22 ατόμων.

(γ) Ο Φάνης σκέφτεται ότι για την κράτηση των 24 ατόμων θα ήταν καλύτερα να χρησιμοποιήσει τα τραπέζια που έχουν σχήμα εξάγωνο.
Συμφωνείς με τον Φάνη; Να αιτιολογήσεις την απάντησή σου.

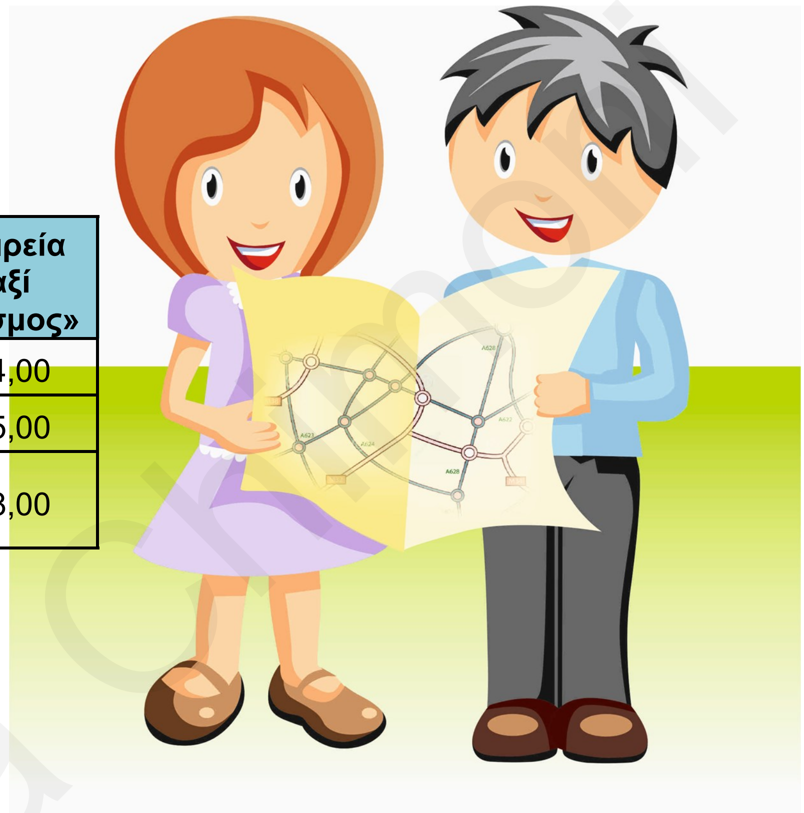
Προτεινόμενο εφαρμογίδιο:
http://illuminations.nctm.org/Activity.aspx?id=3542



ΜΑΘΗΜΑ 2

Ο Γιάννης και η Νίκη βρίσκονται για διακοπές σε μια ευρωπαϊκή πόλη. Παρατηρούν τις τιμές δύο εταιρειών ταξί για τη μετάβασή τους από το ξενοδοχείο σε διάφορα σημεία της πόλης.

Προορισμός	**Εταιρεία ταξί «Άλφα»**	**Εταιρεία ταξί «Κόσμος»**
Μουσείο (3 Km)	€9,00	€14,00
Στάδιο (4 Km)	€12,00	€15,00
Αεροδρόμιο (7 Km)	€21,00	€18,00



(α) Θέλουν να επισκεφθούν το ενυδρείο που βρίσκεται σε απόσταση 6 Km από το ξενοδοχείο. Ποια από τις δύο εταιρείες νομίζεις ότι θα πρέπει να επιλέξουν; Να αιτιολογήσεις την απάντησή σου.

(β) Ο Γιάννης και η Νίκη είχαν επιλέξει την προηγούμενες μέρες την εταιρεία ταξί «Άλφα» για τη μετάβασή τους σε τρία σημεία της πόλης. Πιο κάτω, φαίνεται το αντίστοιχο ποσό που πλήρωσαν για κάθε διαδρομή.

Ξενοδοχείο – Ζωολογικός κήπος
€15

Ξενοδοχείο – Εμπορικό κέντρο
€27

Ξενοδοχείο – Θέατρο
€6

Νομίζεις ότι ήταν ορθή η επιλογή της συγκεκριμένης εταιρείας και για τις τρεις διαδρομές; Να αιτιολογήσεις την απάντησή σου.

ΜΑΘΗΜΑ 3

Ο Κωνσταντίνος παρατήρησε ότι πέρυσι οι σχολικές αργίες της 25ης Μαρτίου και της 1ης Απριλίου ήταν και οι δύο ημέρα Τρίτη.

(α) Να ελέγξεις σε ημερολόγια άλλων ετών κατά πόσο κάθε χρόνο οι αργίες της 25ης Μαρτίου και της 1ης Απριλίου είναι η ίδια μέρα της βδομάδας.

Να εξηγήσεις τη σκέψη σου.

(β) Η Βασιλική μελέτησε το ημερολόγιο του 2014 και έκανε κάποιες παρατηρήσεις.

Η 1η Μαρτίου και η 1η Φεβρουαρίου ήταν και οι δύο ημέρα Σάββατο. Κάθε χρόνο η 1η Μαρτίου και η 1η Φεβρουαρίου θα είναι η ίδια μέρα της εβδομάδας.

Είναι ορθή η υπόθεση της Βασιλικής; Να επεξηγήσεις.

ΜΑΘΗΜΑ 4

Ο κύριος Γιάννης θέλει να υπολογίσει τη μάζα ενός σακουλιού που περιέχει μήλα. Τοποθέτησε το σακούλι στη ζυγαριά, μαζί με μερικά βαρίδια των 100 g.

Με ποιο τρόπο μπορεί να βρει τη μάζα του σακουλιού; Να επεξηγήσεις.

(β) Να αναπαραστήσεις την εξίσωση, σχεδιάζοντας βαρίδια στη ζυγαριά. Στη συνέχεια, να λύσεις την εξίσωση. Κάθε βαρίδιο είναι ίσο με 100 g.

χ + 1 = 4

ψ + 3 = 7

(γ) Να λύσεις τις πιο κάτω εξισώσεις.

Χ - 8= 12 12 + ψ = 27

(δ) Να γράψεις μια δική σου εξίσωση με λύση το 5

ΜΑΘΗΜΑ 5

Η εταιρεία επιδιόρθωσης ηλεκτρικών συσκευών «Το γρήγορο εργαλείο» χρεώνει €30 για κάθε επίσκεψη και €45 για κάθε ώρα που διαρκεί η επιδιόρθωση.

(α) Το πλυντήριο ρούχων της κυρίας Φλώριας έχει χαλάσει. Καλεί την πιο πάνω εταιρεία να έρθει να το επιδιορθώσει αλλά έχει στη διάθεσή της μόνο €120.

Ποσό χρόνο είναι δυνατόν να διαρκέσει η επιδιόρθωση ώστε το συνολικό κόστος να μην υπερβεί το ποσό αυτό;
Να αιτιολογήσεις την απάντησή σου.

(β)

«Οι πολυμήχανοι μάστορες»

Επιδιορθώνονται ηλεκτρικές συσκευές

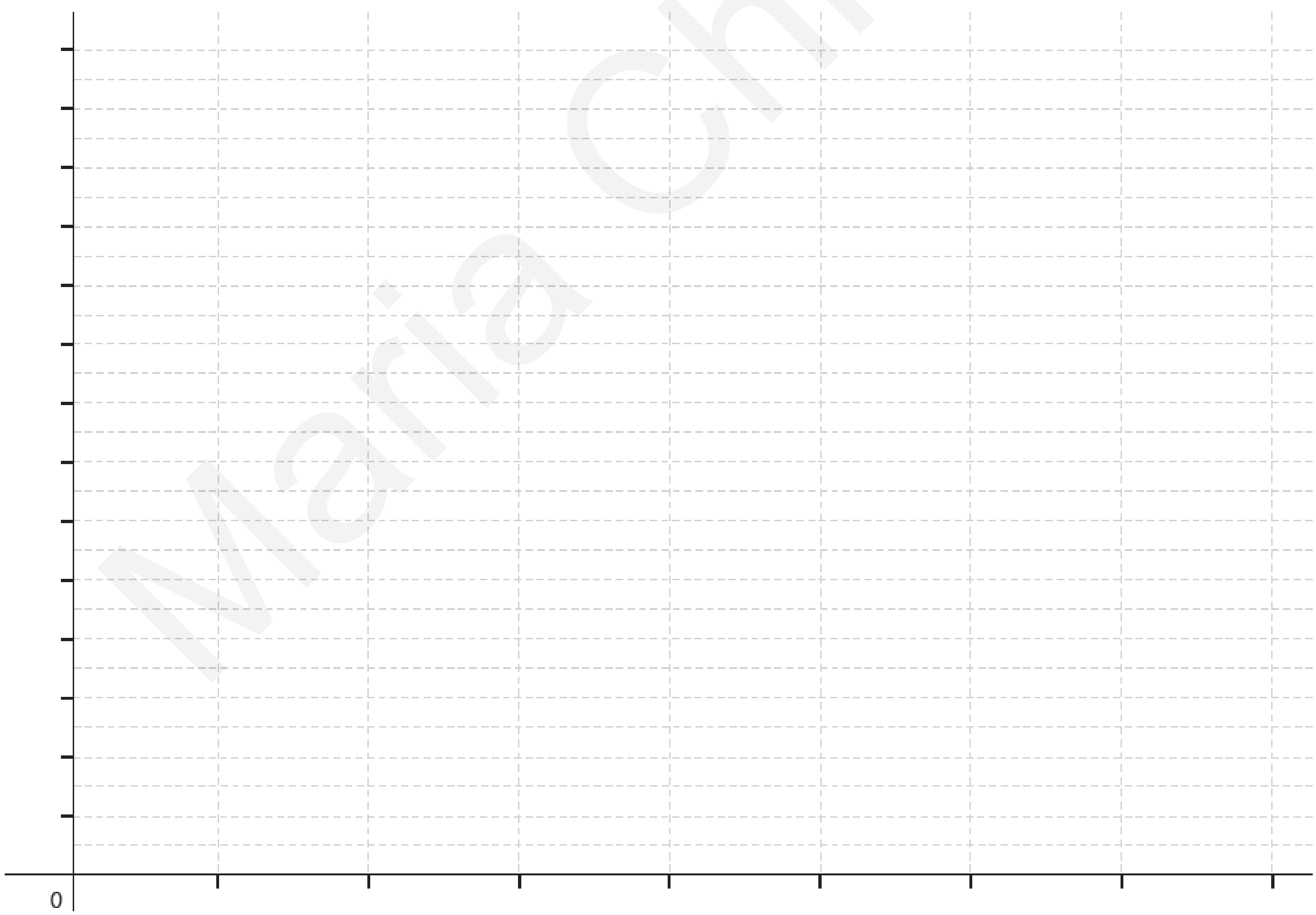
€45 για κάθε επίσκεψη και €30 / ώρα.
Τηλ. 9000007788

Επιδιόρθωση ηλεκτρικών συσκευών
«Το γρήγορο εργαλείο»

€30 για κάθε επίσκεψη και €45 / ώρα.

Να μελετήσεις τις πιο πάνω προσφορές για επιδιορθώσεις ηλεκτρικών συσκευών.

Να δείξεις στη γραφική παράσταση το συνολικό κόστος επιδιόρθωσης μιας ηλεκτρικής συσκευής για 1, 2, 3, 4 και 5 ώρες διάρκειας της επιδιόρθωσης.



(β) Να εκφράσεις με λόγια ή σύμβολα τον τρόπο υπολογισμού του κόστους της επιδιόρθωσης για οποιοδήποτε αριθμό ωρών, για κάθε εταιρεία.

Οι πολυμήχανοι μάστορες:

………………………………………………………………………………………

………………………………………………………………………………………

Το γρήγορο εργαλείο:

………………………………………………………………………………………

………………………………………………………………………………………

(γ) Ποια από τις δύο εταιρείες, «Το γρήγορο εργαλείο» ή «Οι πολυμήχανοι μάστορες», είναι φθηνότερη για επιδιορθώσεις με μεγάλη διάρκεια και ποια είναι φθηνότερη για επιδιορθώσεις μικρής διάρκειας; Να επεξηγήσεις.

ΜΑΘΗΜΑ 6

(α) Ο Μάριος έτρεξε απόσταση 100 m σε 10 δευτερόλεπτα. Ο Νίκος έτρεξε απόσταση 60 μέτρα σε 5 δευτερόλεπτα.

- Ποιος δρομέας έτρεξε τη μεγαλύτερη απόσταση;

____

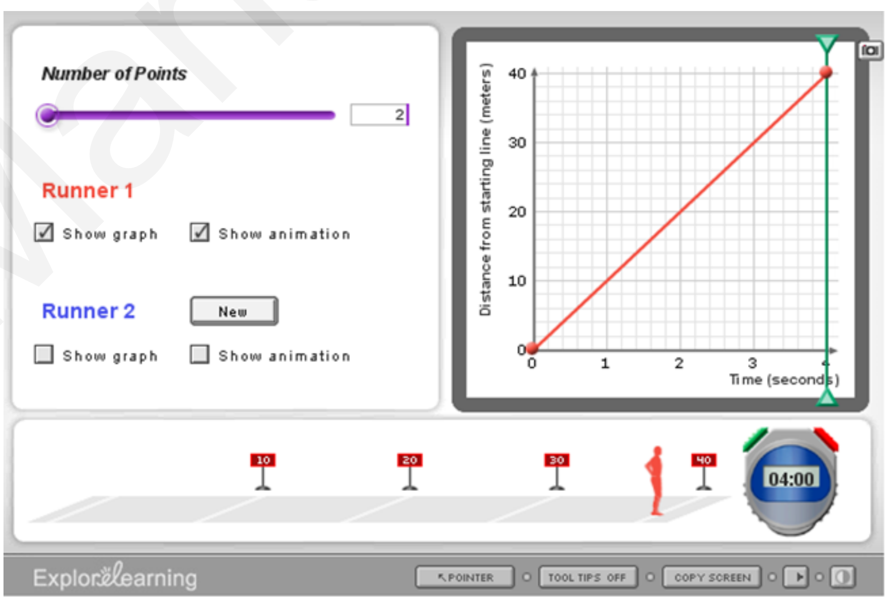
____

- Ποιος δρομέας είναι ο πιο γρήγορος; Να επεξηγήσεις.

____

____

β) Να χρησιμοποιήσεις το εφαρμογίδιο:
http://www.explorelearning.com/index.cfm?method=cResource.dspDetail&ResourceID=625



(α) Να πατήσεις το πράσινο κουμπί (Έναρξη). Τι παρατηρείς να συμβαίνει;

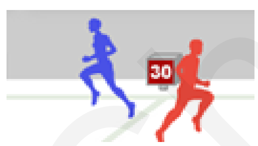
____

(β) Να μετακινήσεις την πράσινη ράβδο στο σημείο όπου ο χρόνος είναι ίσος με 1 δευτερόλεπτο.

- Πόση απόσταση είχε καλύψει ο δρομέας τη συγκεκριμένη χρονική στιγμή;

  ____

- Ποιες είναι οι συντεταγμένες του σημείου στη γραφική παράσταση που δείχνει τη θέση και το χρόνο του δρομέα; ____________________



(γ) Μια γραφική παράσταση που δείχνει τη σχέση της απόστασης με το χρόνο, περιέχει το σημείο (4,15). Τι σημαίνει αυτό για τον δρομέα;

____

____

(δ) Να ρυθμίσεις τα σημεία της γραφικής παράστασης ώστε να είναι 3. Η γραφική παράσταση να περιέχει τα σημεία (0,0), (2,10) και (4,40).

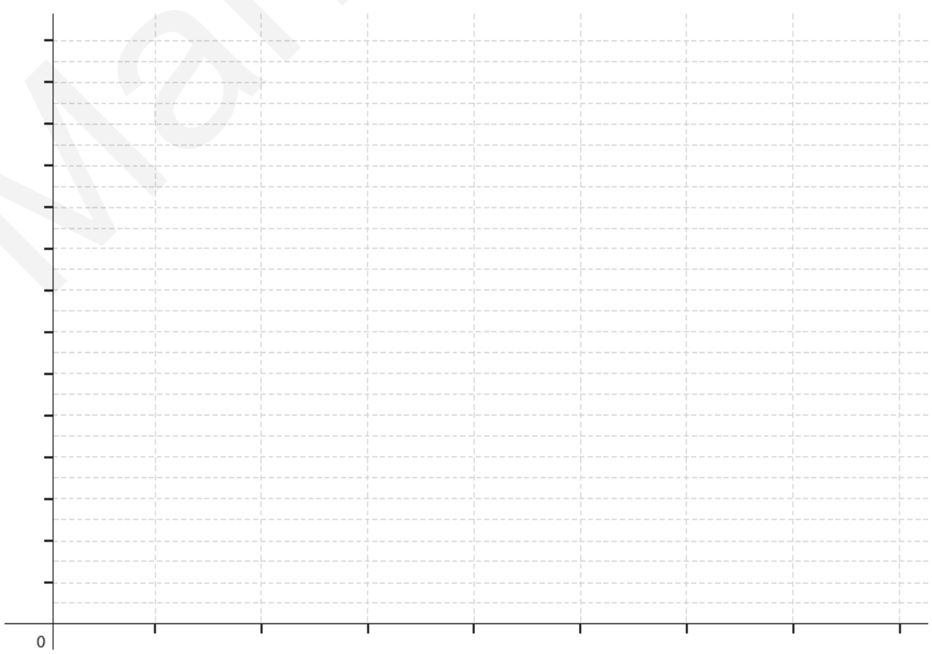
- Από ποιο σημείο αρχίζει ο δρομέας να τρέχει; ________________
- Που θα βρίσκεται μετά από 2 δευτερόλεπτα; ________________
- Που θα βρίσκεται μετά από 4 δευτερόλεπτα; ________________
- Να βάλεις σε κύκλο τη χρονική περίοδο κατά την οποία ο δρομέας έτρεχε πιο γρήγορα.

0 – 2 δευτερόλεπτα   2 – 4 δευτερόλεπτα

(ε) Δύο δρομείς θα τρέξουν μια διαδρομή με μήκος 40 m.

- Να δείξεις στη γραφική παράσταση με ποιο τρόπο ο Δρομέας 1 θα μπορέσει να έρθει πρώτος.

(στ) Να δείξεις στη γραφική παράσταση με ποιο τρόπο ο Δρομέας 2 θα είναι στην αρχή πίσω από τον Δρομέα 1, αλλά στη συνέχεια θα τον προσπεράσει και θα έρθει πρώτος.



ΜΑΘΗΜΑ 7

Να διαβάσεις το πιο κάτω άρθρο.

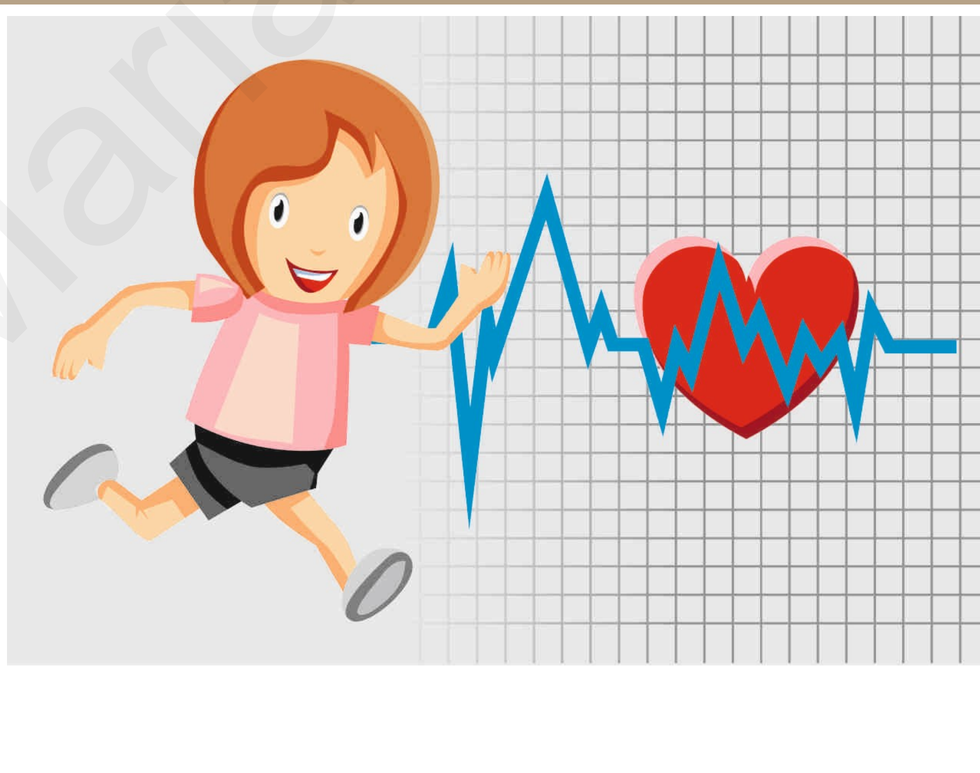
«Οι άνθρωποι πρέπει να περιορίσουν τις προσπάθειες στην άθληση τους, ώστε να μην υπερβαίνουν ένα συγκεκριμένο αριθμό καρδιακών παλμών. Για χρόνια, η σχέση μεταξύ του μέγιστου επιτρεπτού αριθμού καρδιακών παλμών ενός ατόμου και της ηλικίας του περιγραφόταν με τον ακόλουθο τύπο:

Μέγιστος επιτρεπτός αριθμός καρδιακών παλμών = 220 – ηλικία

Πρόσφατη έρευνα έδειξε ότι αυτός ο τύπος πρέπει να τροποποιηθεί. Ο νέος τύπος είναι ο ακόλουθος:

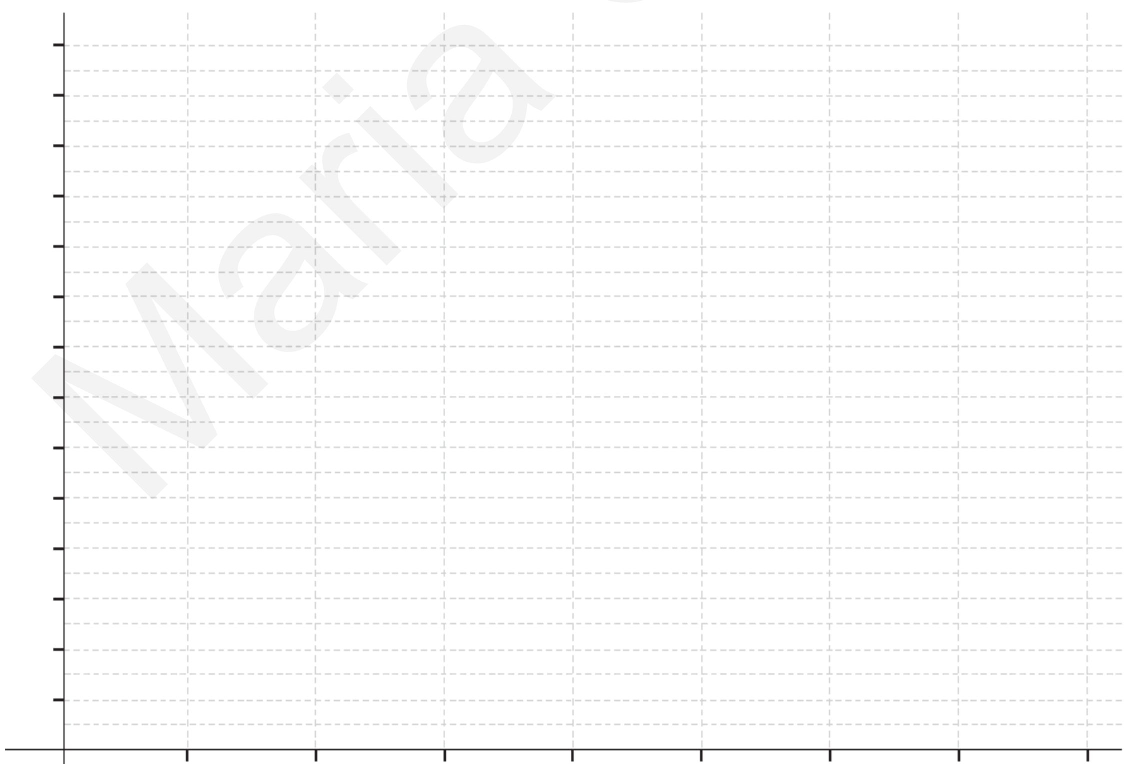
Μέγιστος επιτρεπτός αριθμός καρδιακών παλμών = 208 – (0.7 x ηλικία)

Η χρήση του νέου τύπου αντί του παλιού, έχει ως αποτέλεσμα ο μέγιστος επιτρεπτός αριθμός παλμών της καρδιάς ανά λεπτό να είναι ελαφρώς μειωμένος για τα νέα άτομα και ελαφρώς αυξημένος για τους πιο ηλικιωμένους.»



Από ποια ηλικία και μετά ο μέγιστος επιτρεπτός αριθμός καρδιακών παλμών αυξάνεται ως αποτέλεσμα της εισαγωγής του νέου τύπου; Να αιτιολογήσεις την απάντησή

Μπορείς να κατασκευάσεις γραφική παράσταση, για να παρουσιάσεις την απάντησή σου.

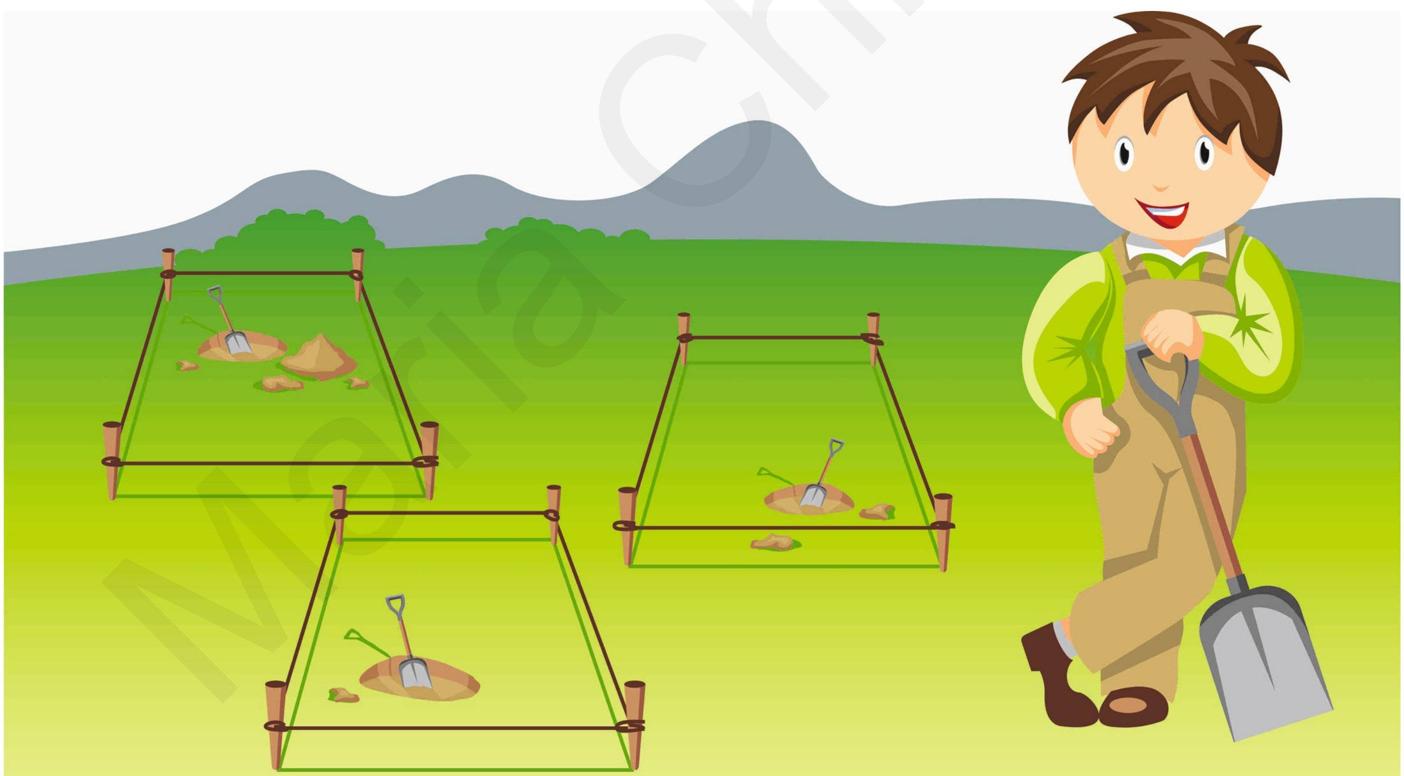


ΜΑΘΗΜΑ 8

Κατά τη διάρκεια του 19ου αιώνα πολλοί ερευνητές ταξίδευαν στη Βόρεια Αμερική για να βρουν χρυσό.

Ο Dan Jackson ήταν ιδιοκτήτης γης στην οποία είχε ανευρεθεί χρυσός και ενοικίαζε κομμάτια γης σε διάφορους ερευνητές.

Ο Dan έδωσε σε κάθε ερευνητή 4 ξύλα και σχοινί με μήκος 100 m. Κάθε ερευνητής έπρεπε να χρησιμοποιήσει τα ξύλα και το σχοινί για να οριοθετήσει ένα ορθογώνιο κομμάτι γης μέσα στο οποίο μπορούσε να σκάψει.



(β) Με ποιο τρόπο θα πρέπει να τοποθετήσει ένας ερευνητής τα ξύλα ώστε να οροθετήσει το μεγαλύτερο κομμάτι γης που μπορεί; Να αιτιολογήσεις την απάντησή σου.

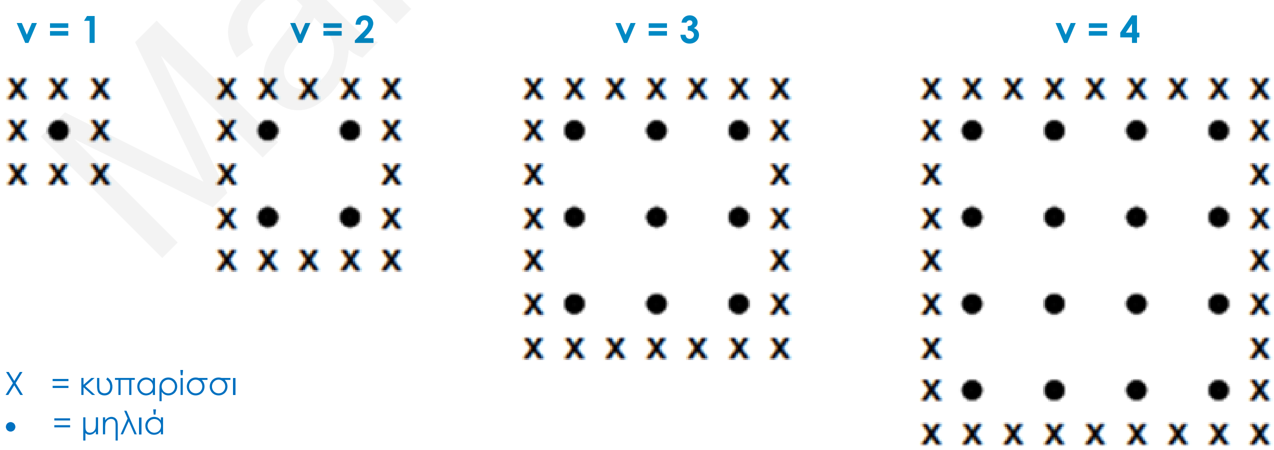
(γ) Ένας ερευνητής είχε μια ιδέα:

«Να δέσουμε όλα τα σχοινιά μαζί! Έτσι, μπορούμε να οριοθετήσουμε μεγαλύτερο κομμάτι γης και να δουλέψουμε μαζί. Θα μοιράσουμε το χρυσό που θα βρούμε.»

Έχει δίκαιο ο ερευνητής, αν δουλέψουν όλοι ερευνητές μαζί και χρησιμοποιήσουν 4 μόνο ξύλα;

ΜΑΘΗΜΑ 9

Ένας γεωργός φυτεύει μηλιές με τρόπο που να σχηματίζεται ένα μοτίβο με τετράγωνα. Για να προστατεύει τις μηλιές από τον άνεμο, φυτεύει γύρω από το κάθε τετράγωνο κυπαρίσσια.



Χ = κυπαρίσσι
• = μηλιά

(β) Ο γεωργός θέλει να δημιουργήσει ένα πολύ μεγάλο χώρο με πολλές σειρές δέντρων.

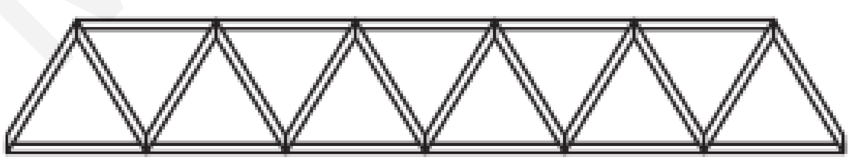
Όσο ο γεωργός κάνει το χώρο μεγαλύτερο, ποια δέντρα θα αυξηθούν περισσότερο, οι μηλιές ή τα κυπαρίσσια;

Να αιτιολογήσεις την απάντησή σου.

ΜΑΘΗΜΑ 10

Ένα εργοστάσιο κατασκευάζει μεταλλικά δικτυώματα από ράβδους, όπως αυτά που φαίνονται στην πιο κάτω εικόνα.

Το μήκος κάθε δικτυώματος είναι ίσο με τον αριθμό των ράβδων στο κάτω μέρος του δικτυώματος. Για παράδειγμα, το πιο κάτω δικτύωμα έχει μήκος 6 m.



$L = 6\ m$

(β) Το εργοστάσιο έλαβε μια παραγγελία μέσω φαξ για την κατασκευή ενός δικτυώματος. Όμως στο φαξ δεν φαινόταν καθαρά ο συνολικός αριθμός των ράβδων που χρειάζονταν. Κάποιοι έλεγαν ότι ίσως είναι 47 και κάποιοι 48.

(α) Ποιος νομίζεις ότι είναι ο ορθός αριθμός;

(β) Ποιο είναι το μήκος του δικτυώματος;

Άλγεβρα & Αλγεβρική Σκέψη

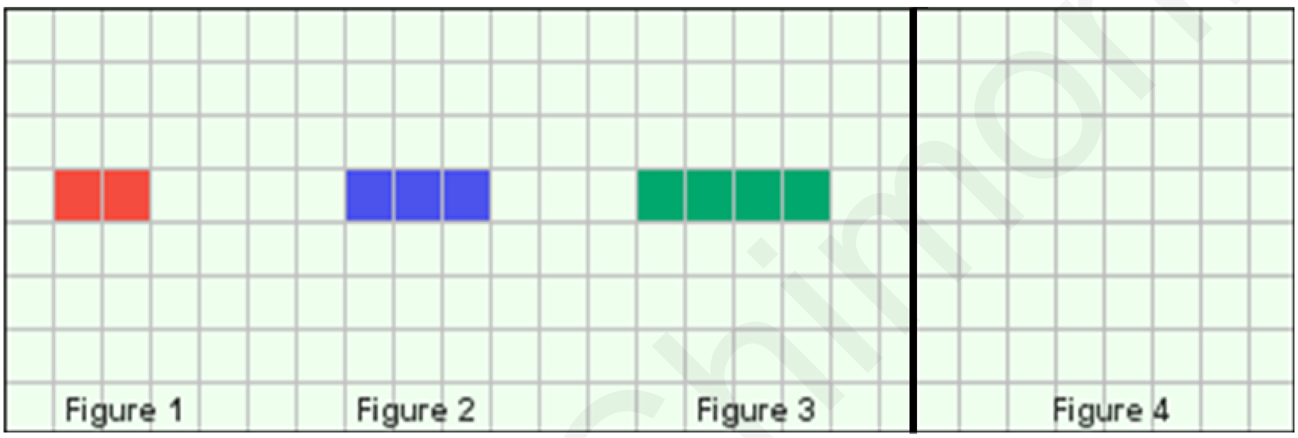
Δομημένες Μαθηματικές
Διερευνήσεις

Περιεχόμενα

ΜΑΘΗΜΑ 1

Να χρησιμοποιήσεις το εφαρμογίδιο:
http://www.explorelearning.com/index.cfm?

(α) Να παρατηρήσεις το μοτίβο της πιο κάτω εικόνας.



(α) Πόσα τετράγωνα είναι χρωματισμένα σε κάθε εικόνα;

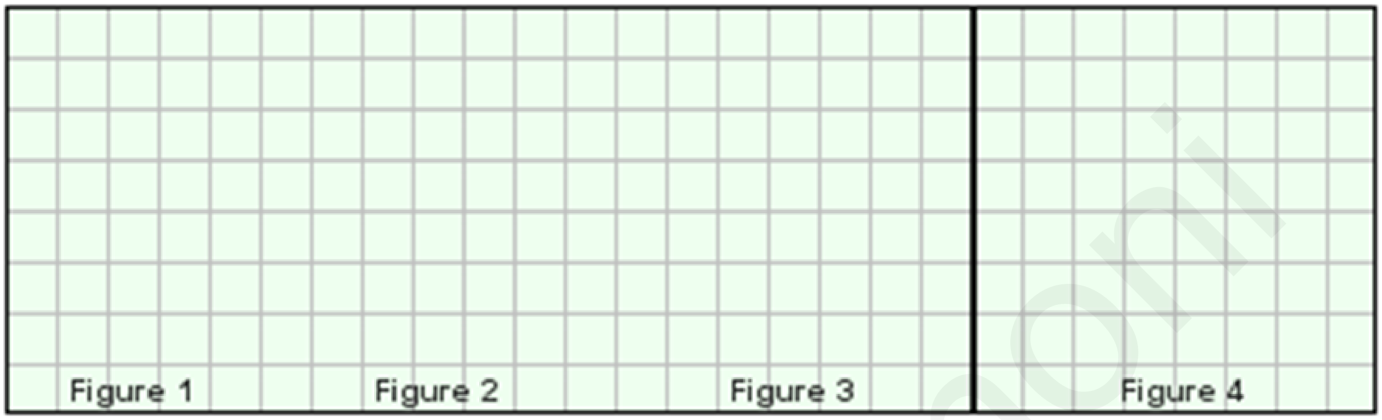
Εικόνα 1: __________
Εικόνα 2: __________
Εικόνα 3: __________

- Πόσα τετράγωνα θα έχει η Εικόνα 4 του μοτίβου; ____________
- Να σχεδιάσεις την Εικόνα 4 και στη συνέχεια, να ελέγξεις την απάντησή σου, πατώντας «check».
- Πόσα χρωματισμένα τετράγωνα θα έχει η 10η εικόνα του μοτίβου; Να εξηγήσεις.

____

____

(β) Να πατήσεις new για να εμφανιστεί ένα καινούριο μοτίβο. Στο πιο κάτω πλαίσιο, να σχεδιάσεις το μοτίβο που βλέπεις. Κάτω από κάθε εικόνα να γράψεις τον αριθμό των χρωματισμένων τετραγώνων.



(γ)

- Πόσα τετράγωνα θα έχει η εικόνα 4; Να αιτιολογήσεις την απάντησή σου.

  ____

- Να σχεδιάσεις την Εικόνα 4 και στη συνέχεια, να ελέγξεις την απάντησή σου, πατώντας «check».

- Να γράψεις τον κανόνα του μοτίβου και στη συνέχεια, να ελέγξεις την απάντησή σου, πατώντας «Show relationship between figures».

  ____

- Να παρατηρήσεις την αριθμητική γραμμή κάτω από τον πίνακα. Με ποιο τρόπο η αριθμητική γραμμή παρουσιάζει πόσα τετράγωνα είναι χρωματισμένα σε κάθε εικόνα;

  ____

  ____

- Πόσα χρωματισμένα τετράγωνα θα έχει η 10η εικόνα του μοτίβου; Να εξηγήσεις.

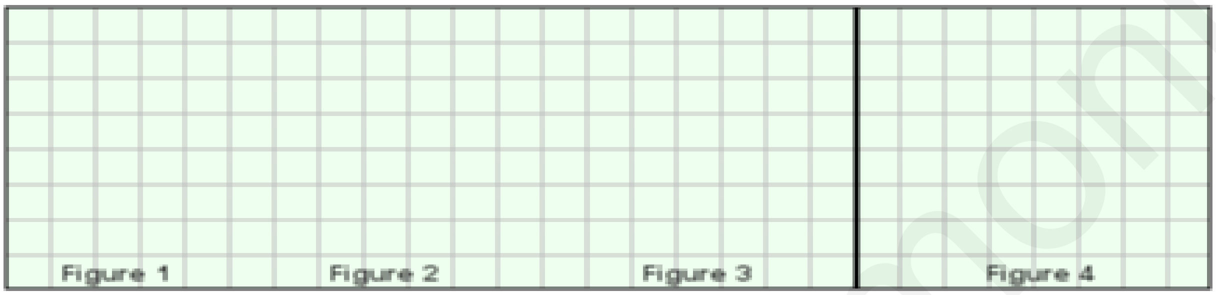
  ____

  ____

(δ) Να πατήσεις «new» για να εμφανιστούν δύο νέα μοτίβα. Για κάθε μοτίβο:

- Να σχεδιάσεις τις τρεις πρώτες εικόνες και να γράψεις τον αριθμό των χρωματισμένων τετραγώνων κάτω από κάθε εικόνα.
- Να υπολογίσεις των αριθμό των χρωματισμένων τετραγώνων στην Εικόνα 4. Να σχεδιάσεις την Εικόνα 4 και στη συνέχεια, να πατήσεις «check» για να ελέγξεις την απάντησή σου.

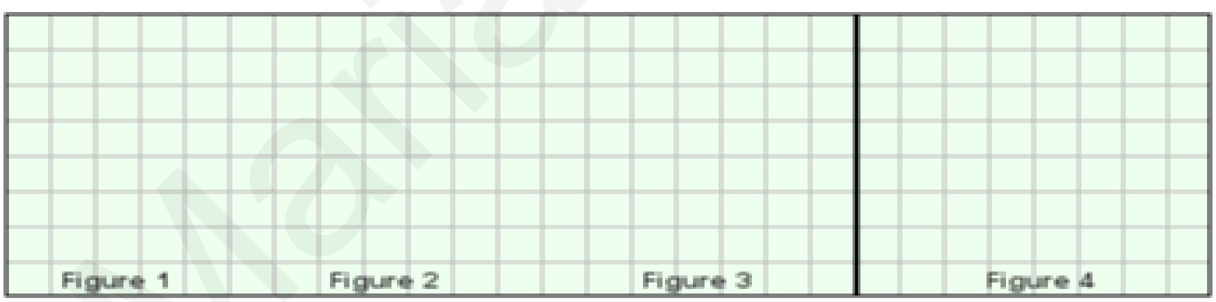
ΜΟΤΙΒΟ 1



ΚΑΝΟΝΑΣ: ____

Αριθμός χρωματισμένων τετραγώνων στην Εικόνα 10:

ΜΟΤΙΒΟ 2

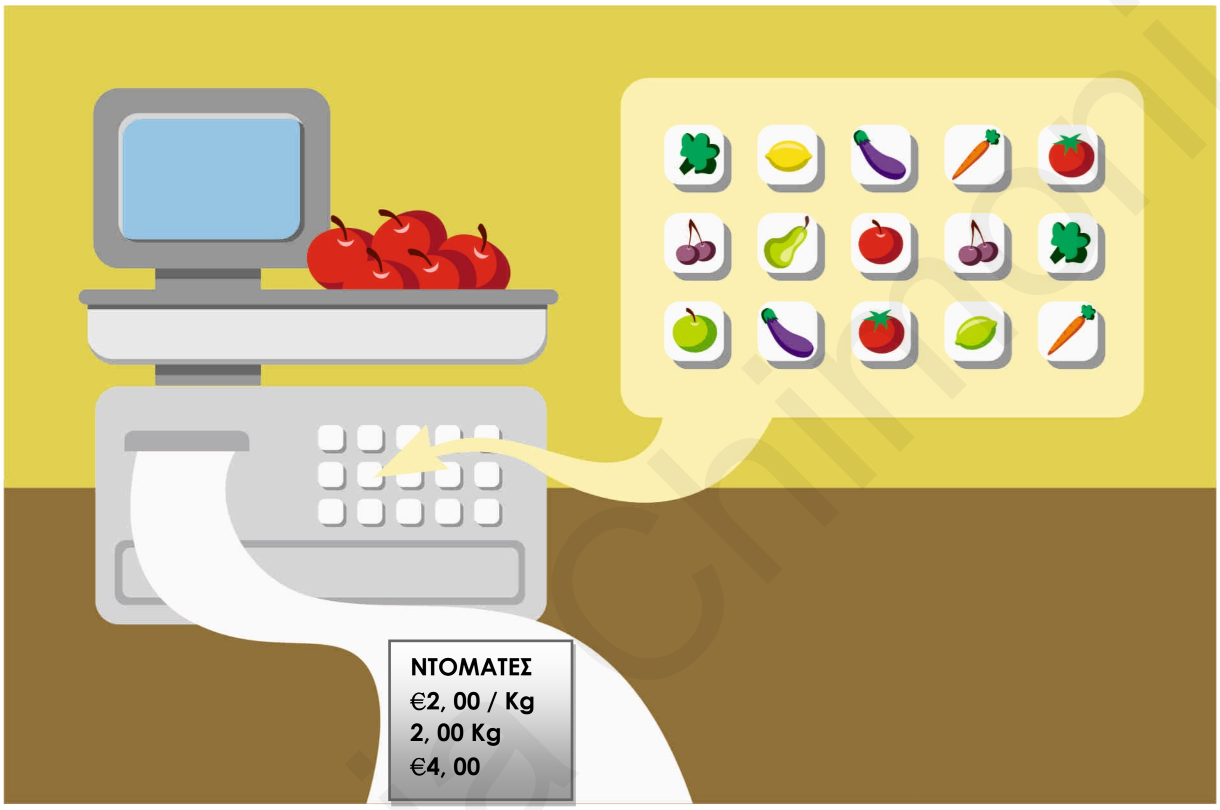


ΚΑΝΟΝΑΣ: ____

Αριθμός χρωματισμένων τετραγώνων στην Εικόνα 10:

ΜΑΘΗΜΑ 2

Στη φρουταρία «Το φρέσκο φρούτο», οι πελάτες μπορούν να ζυγίσουν μόνοι τους τα φρούτα ή τα λαχανικά που θα αγοράσουν, επιλέγοντας το αντίστοιχο κουμπί.



Η μηχανή τυπώνει μια σημείωση με το είδος του φρούτου ή του λαχανικού, την τιμή του ανά κιλό, τη μάζα και το συνολικό κόστος.

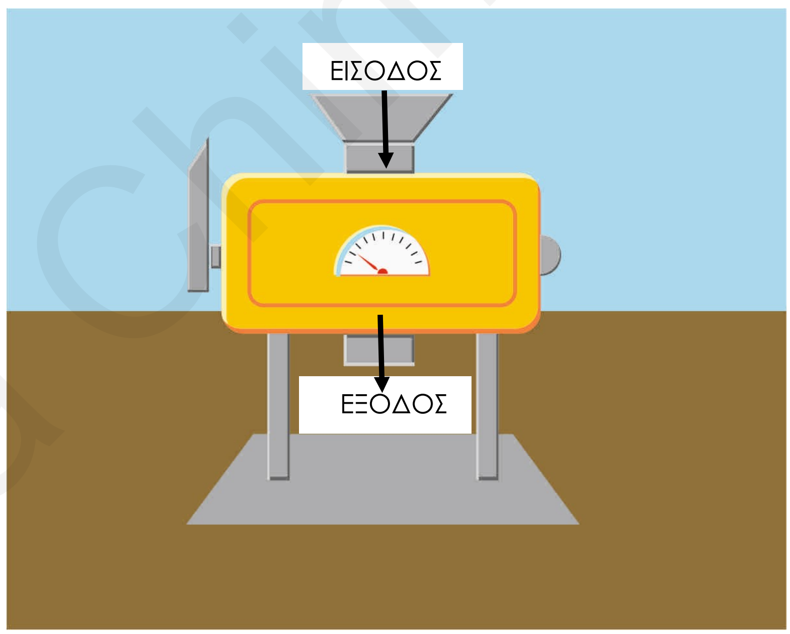
(α) Να συμπληρώσεις τον πίνακα.

Ντομάτες Μάζα (Kg)	Τιμή ανά κιλό	Συνολικό κόστος
4	€2	
1	€2	
3	€2	
5	€2	

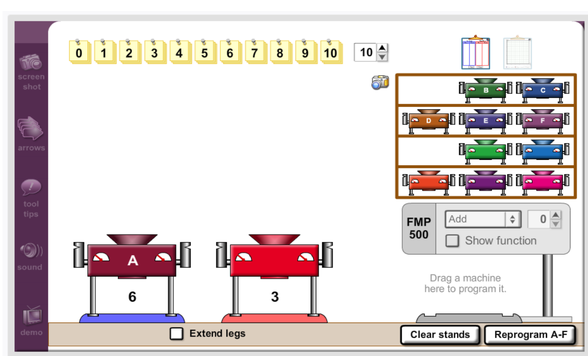
(β) Οι μπανάνες κοστίζουν €2,50 το κιλό. Πόσο θα κοστίσουν 3 Kg μπανάνες;

(γ) Αν η κυρία Νίκη αγόρασε 4 Kg μήλα και πλήρωσε €10,00, να υπολογίσεις την τιμή των μήλων ανά κιλό.

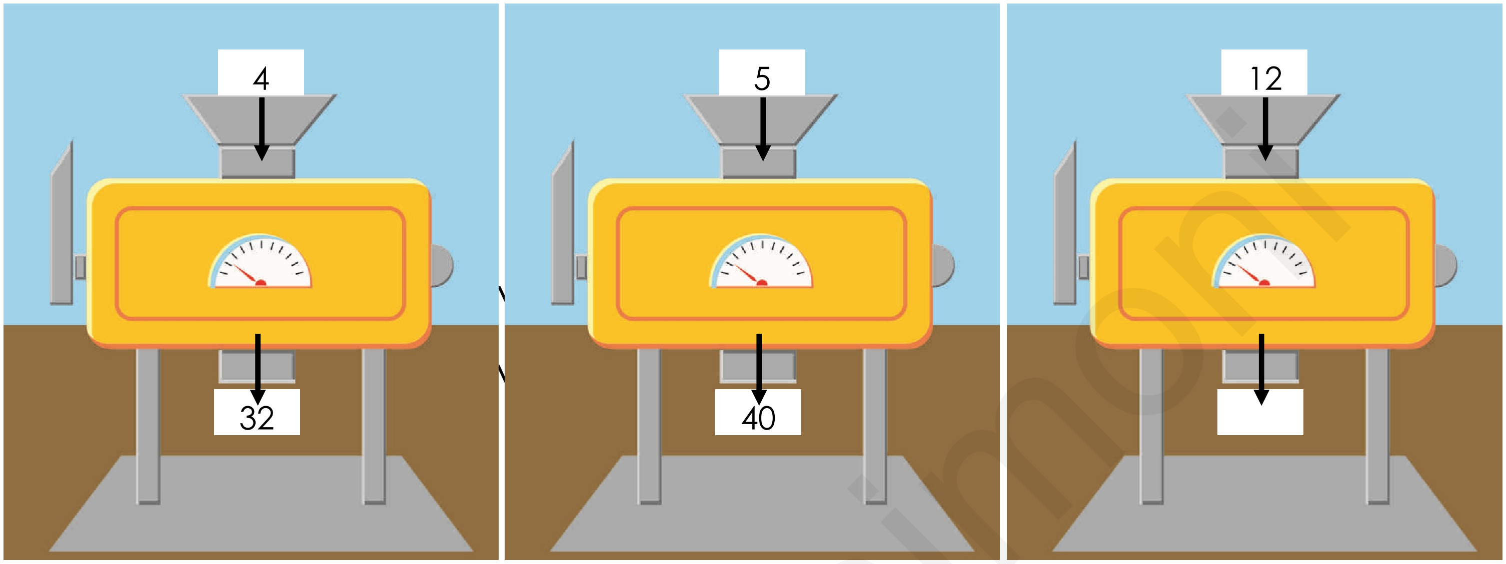
(δ) Η ζυγαριά μοιάζει με μια μηχανή που στην είσοδο της εισέρχεται η μάζα των φρούτων ή των λαχανικών και στην έξοδο εξέρχεται η συνολική τους τιμή.



Μπορείς να χρησιμοποιήσεις το εφαρμογίδιο https://www.explorelearning.com/index.cfm?method=cResource.dspDetail&resourceID=1035 , για να μελετήσεις τον τρόπο λειτουργίας διαφόρων μηχανών.

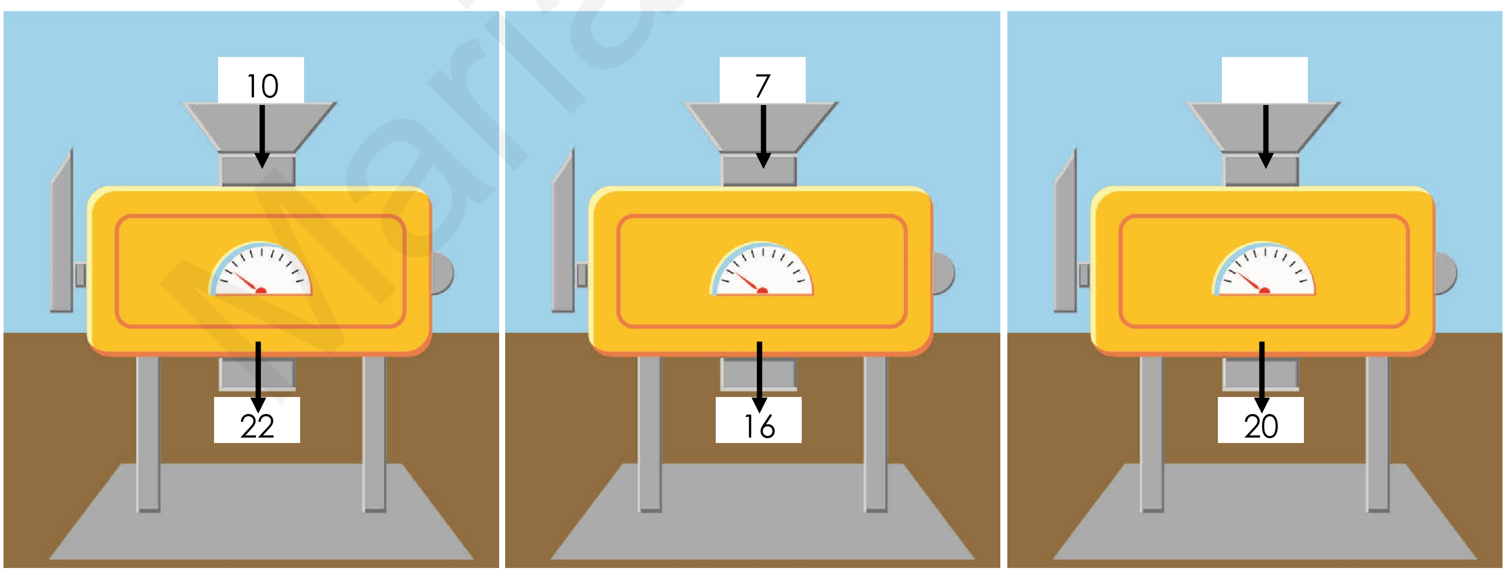


Να βρεις τον κανόνα λειτουργίας της πιο κάτω μηχανής και να συμπληρώσεις.



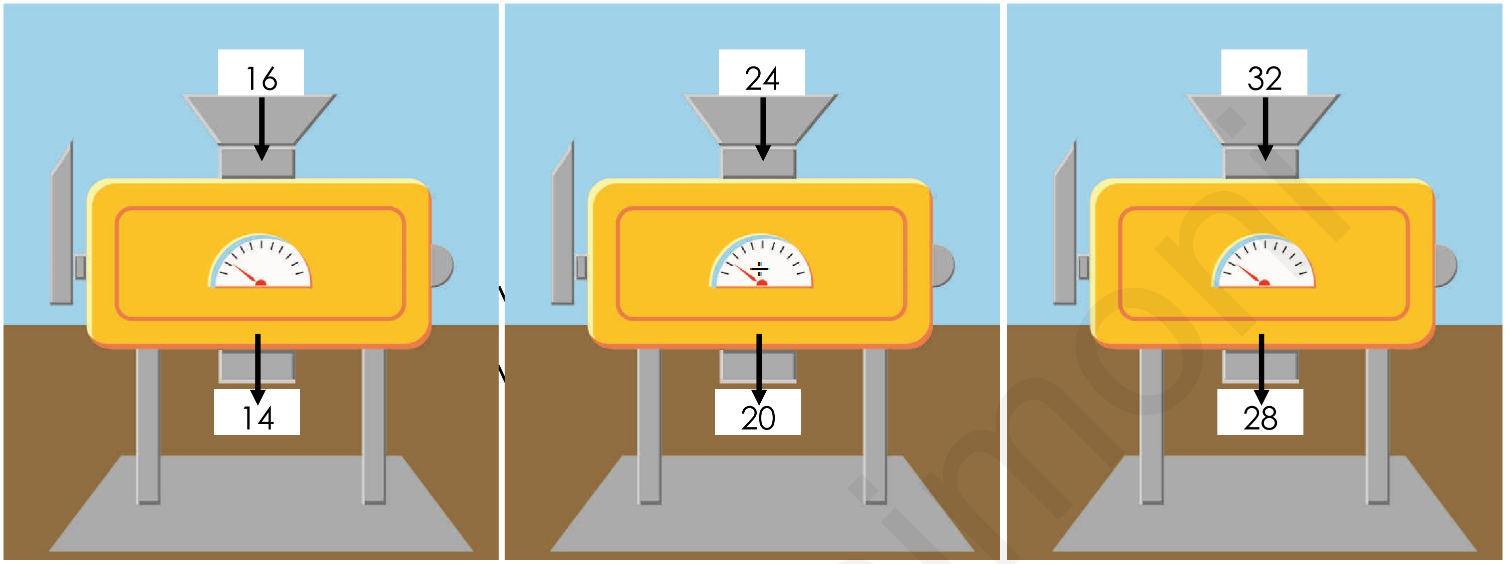
Κανόνας: ____

(ε) Να βρεις τον κανόνα λειτουργίας της πιο κάτω μηχανής και να συμπληρώσεις.



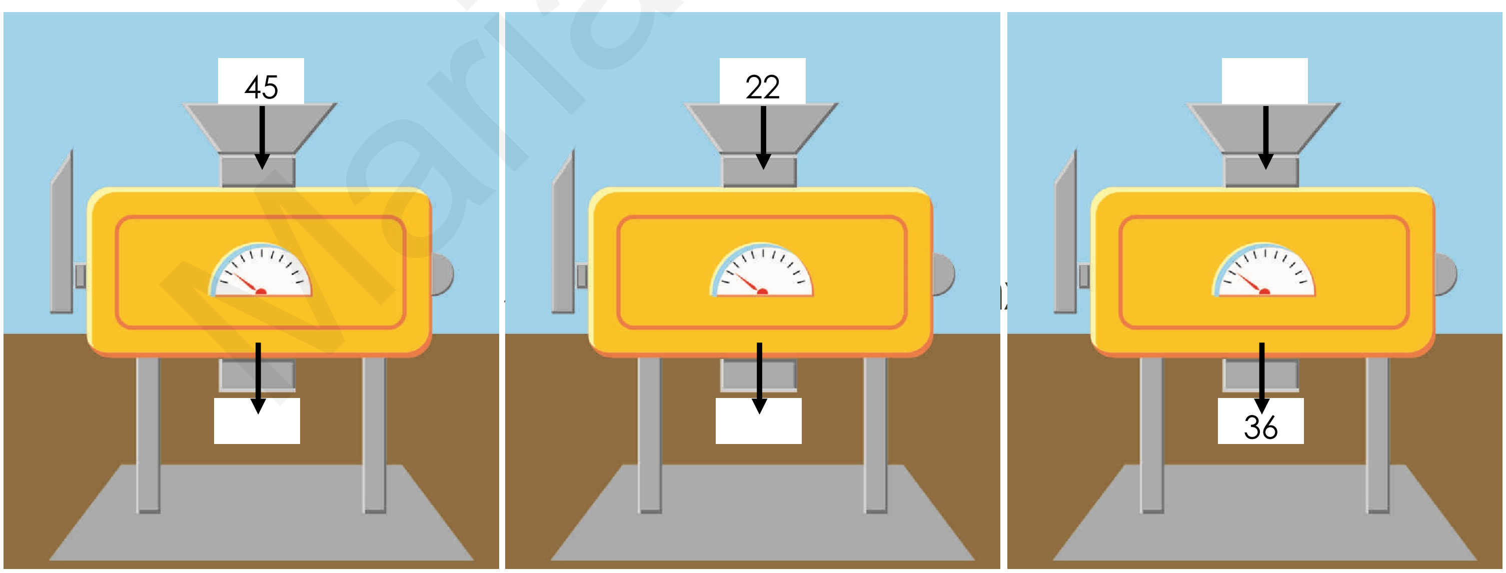
Κανόνας: ____

(στ) Να επιλέξεις την μαθηματική πρόταση που περιγράφει τον κανόνα λειτουργίας της πιο κάτω μηχανής.



i. χ ÷ 4 = ψ   ii. χ – 4 = ψ   iii. ψ + 8 = χ   iv. 16 – 4 = ψ

(ζ) Να συμπληρώσεις τις τιμές στην πιο κάτω μηχανή, αν ο κανόνας που ακολουθεί είναι (χ + 5) - 4 = ψ.

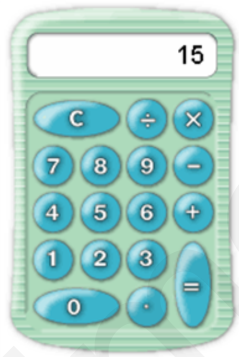


ΜΑΘΗΜΑ 3

Να χρησιμοποιήσεις το εφαρμογίδιο:
http://www.nctm.org/standards/content.aspx?id=25013

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Clear



Ποια εντολή πρέπει να γράψεις στην υπολογιστική μηχανή ώστε να χρωματιστούν:

(α) τα πολλαπλάσια του 5

____

____

(β) το μοτίβο 1, 12, 23, 34, …

____

____

(γ) τα πολλαπλάσια του 10

____

____

(δ) οι αριθμοί 75 και 86

____

____

(ε) οι ζυγοί αριθμοί

____

____

(στ) οι μονοί αριθμοί

____

____

(ζ) ένα μοτίβο με αριθμούς που περιλαμβάνει και τον αριθμό 100

____

____

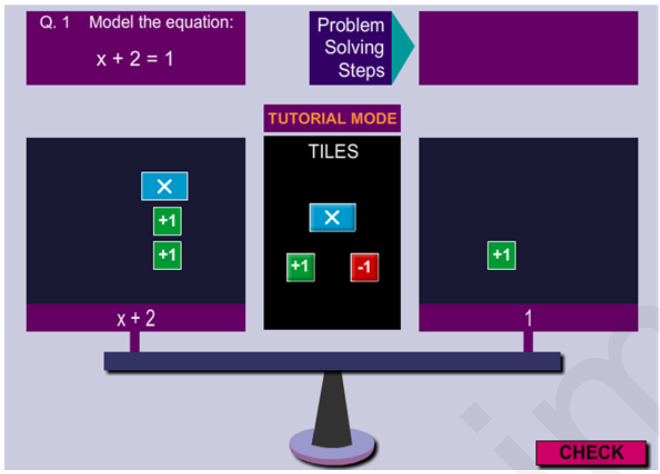
(η) ένα μοτίβο με αριθμούς που δεν περιλαμβάνει τον αριθμό 100

____

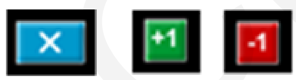
____

ΜΑΘΗΜΑ 4

Να χρησιμοποιήσεις το εφαρμογίδιο¨
http://www.mathplayground.com/AlgebraEquations.html



(α) Να μελετήσεις το εφαρμογίδιο και να αναπαραστήσεις την εξίσωση που εμφανίζεται στο δεξί πάνω μέρος της οθόνης, χρησιμοποιώντας τα εικονίδια:



(β) Να λύσεις την εξίσωση και να ελέγξεις την απάντησή σου.

(γ) Να ακολουθήσεις την ίδια διαδικασία για τις επόμενες δύο εξισώσεις που θα εμφανιστούν στο εφαρμογίδιο.

(δ) Να λύσεις τις εξισώσεις.

ω + 7 = 10	χ + 3 = 5	6 + γ = 10	α - 5 = 9

ΜΑΘΗΜΑ 5

(α) Να αντιστοιχίσεις τις εκφράσεις με την κατάλληλη αλγεβρική αναπαράσταση.

Διπλασίασε έναν αριθμό και πρόσθεσε τρία.	(2 × κ) + 3
Αφαίρεσε πέντε από έναν αριθμό και μετά πολλαπλασίασε επί δύο.	(μ : 7) + 6
Διαίρεσε με το εφτά έναν αριθμό και μετά πρόσθεσε έξι.	(λ - 5) × 2
Ένας αριθμός αφαιρείται από το εικοσιένα.	π - 21
Το εικοσιένα αφαιρείται από έναν αριθμό.	21 - τ

(β) Να γράψεις μια συμβολική αναπαράσταση για κάθε έκφραση.

Τέσσερα περισσότερα από ένα αριθμό ………………………………..

Ένας αριθμός προστίθεται στο τέσσερα ………………………………..

Τέσσερα λιγότερα από ένα αριθμό ………………………………..

Ένας αριθμός αφαιρείται από το τέσσερα ………………………………..

(γ) Ο Χάρης εργάζεται στο πρατήριο βενζίνης. Κερδίζει €7 την ώρα τις καθημερινές και €9 την ώρα το Σαββατοκύριακο.

i. Να υπολογίσεις πόσα χρήματα κέρδισε συνολικά ο Χάρης αν:

- Εργάστηκε 8 ώρες τις καθημερινές και 12 ώρες το Σαββατοκύριακο.

- Εργάστηκε 4 ώρες τις καθημερινές και 5 ώρες το Σαββατοκύριακο.

ii. Να γράψεις μια συμβολική αναπαράσταση, για να δείξεις με ποιο τρόπο μπορεί να υπολογίζει ο Χάρης το μισθό του κάθε βδομάδα, για οποιοδήποτε αριθμό ωρών.

iii. Ο Χάρης χρειάζεται €115 για να αγοράσει ένα κινητό τηλέφωνο. Εργάστηκε 5 ώρες το Σαββατοκύριακο. Πόσες ώρες πρέπει να εργαστεί τις καθημερινές για να πάρει τα χρήματα που χρειάζεται;

ΜΑΘΗΜΑ 6

(α) Να συμπληρώσεις τον πίνακα εισόδου και εξόδου για κάθε έκφραση.

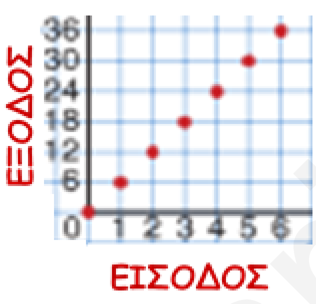
4 × K

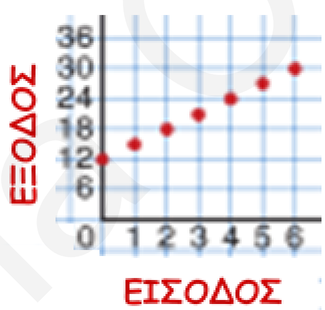
ΕΙΣΟΔΟΣ K	ΕΞΟΔΟΣ
1	
2	
3	
4	
5	

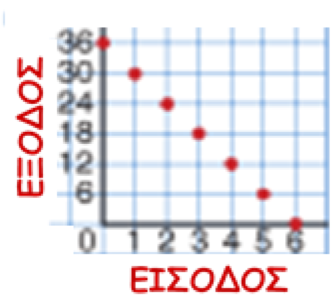
P + 3

ΕΙΣΟΔΟΣ P	ΕΞΟΔΟΣ
1	
2	
3	
4	
5	

β) Να αντιστοιχίσεις κάθε γραφική παράσταση με την κατάσταση που παρουσιάζει.





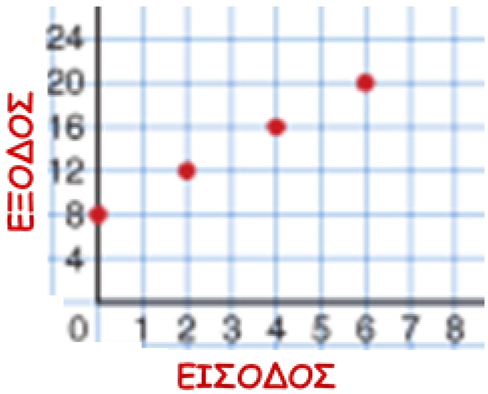


Ο δρομέας καλύπτει 6 μέτρα κάθε 1 δευτερόλεπτο.

Στην έδρα υπήρχαν 36 βιβλία. Κάθε παιδί παίρνει από 6 βιβλία.

Η εγγραφή στο κατάστημα ενοικίασης ταινιών είναι €12. Κάθε ταινία που ενοικιάζεται χρεώνεται με €3.

(γ) Να παρατηρήσεις τη γραφική παράσταση και να συμπληρώσεις τον πίνακα.



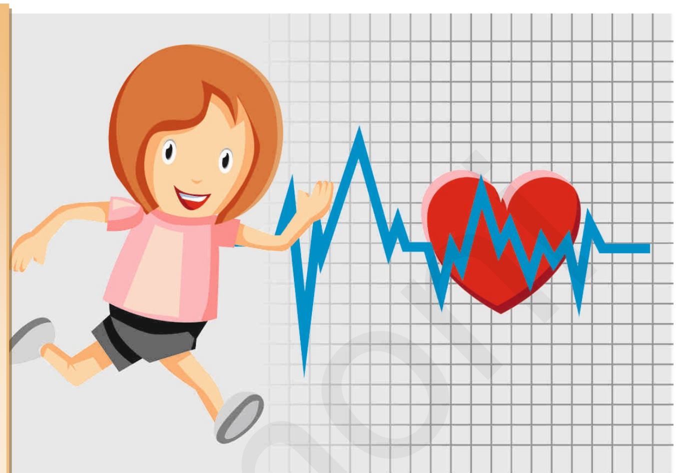
ΕΙΣΟΔΟΣ	ΕΞΟΔΟΣ

(δ) Ποιος αριθμός θα είναι στην έξοδο, αν στην είσοδο είναι ο αριθμός 8. Να επεκτείνεις τη γραφική παράσταση.

(δ) Να περιγράψεις μια κατάσταση που μπορεί να παρουσιάζει η πιο πάνω γραφική παράσταση.

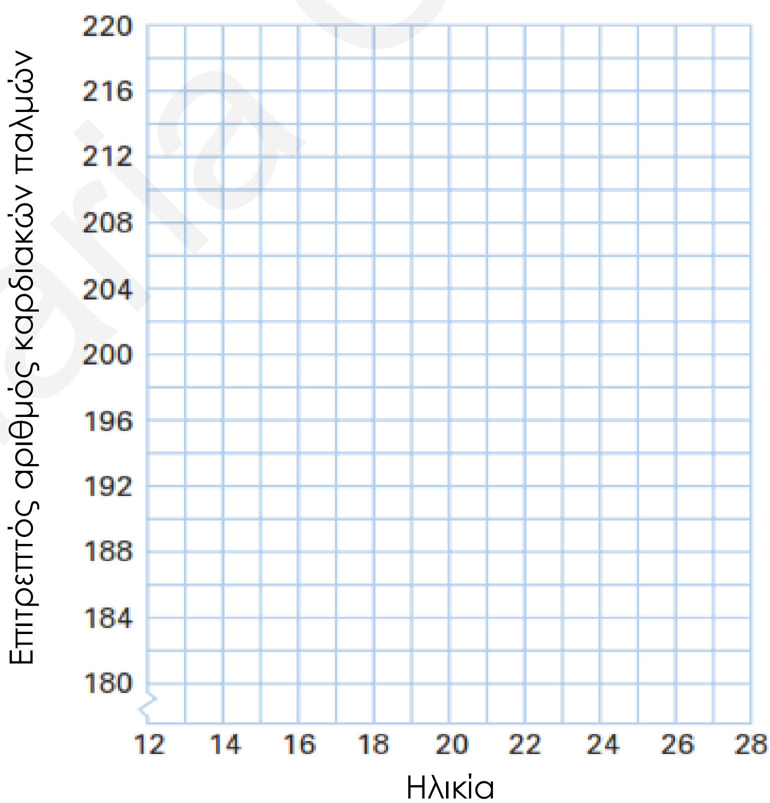
ΜΑΘΗΜΑ 7

Οι άνθρωποι πρέπει να περιορίσουν τις προσπάθειες στην άθληση τους, ώστε να μην υπερβαίνουν ένα συγκεκριμένο αριθμό καρδιακών παλμών. Ο μέγιστος επιτρεπτός αριθμός καρδιακών παλμών του ανθρώπου όταν βρίσκεται σε κατάσταση ηρεμίας υπολογίζεται με τον εξής τύπο: Αφαιρείς την ηλικία σου (Α) σε χρόνια από τον αριθμό 220 για να βρεις τον μέγιστο επιτρεπτό καρδιακό παλμό (Μ).



(α) Να γράψεις μια συμβολική αναπαράσταση που αναπαριστά την πιο πάνω σχέση.

(β) Να χρησιμοποιήσεις τον πιο πάνω τύπο για να συμπληρώσεις τη γραφική



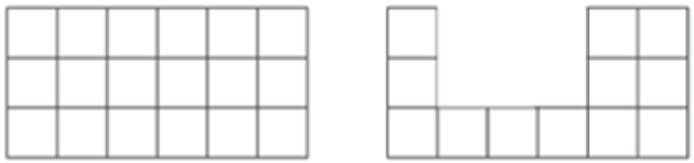
(γ) Ποιος έχει υψηλότερο επιτρεπτό αριθμό καρδιακών παλμών, εσύ ή ο/η δάσκαλος/α σου; Να εξηγήσεις.

(δ) Ο Δημήτρης είναι προπονητής της κολύμβησης. Δίνει οδηγίες στους αθλητές του, ώστε κατά τη διάρκεια της προπόνησής τους να φτάνουν στο 75% του επιτρεπτό αριθμού καρδιακών παλμών.

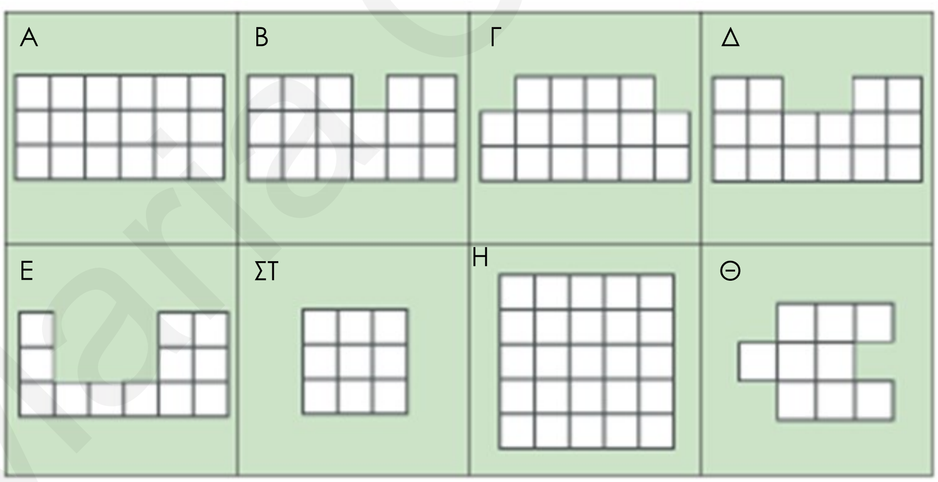
Να γράψεις ένα καινούριο μαθηματικό τύπο με τον οποίο οι αθλητές μπορούν να υπολογίσουν τον επιτρεπτό αριθμό καρδιακών παλμών κατά την προπόνησή τους (Π) σε σχέση με τον επιτρεπτό αριθμό καρδια-

ΜΑΘΗΜΑ 8

(α) Να συγκρίνεις την περίμετρο και το εμβαδόν των πιο κάτω σχημάτων.

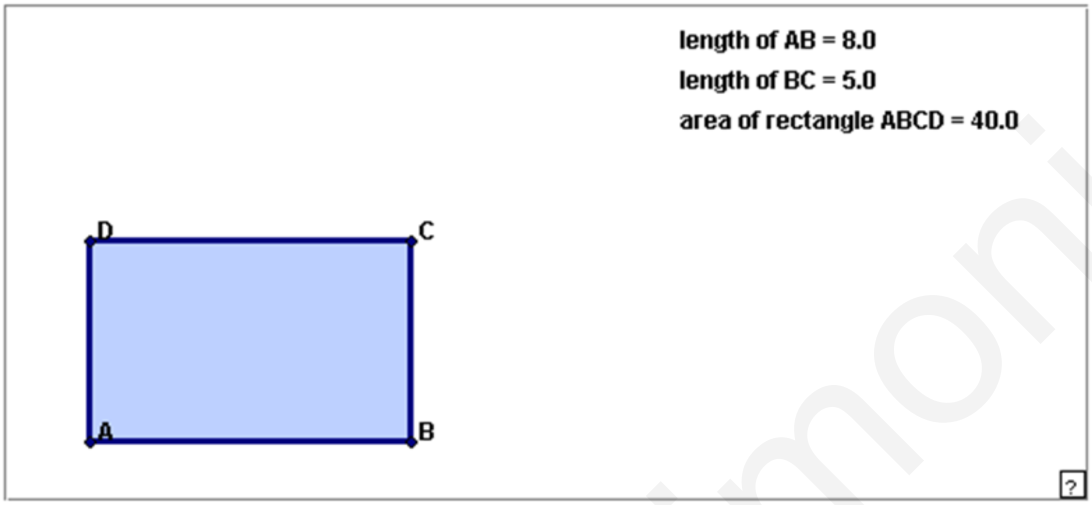


(β) Να συγκρίνεις την περίμετρο και το εμβαδόν των πιο κάτω σχημάτων.



(β) Να χρησιμοποιήσεις το εφαρμογίδιο:

http://illuminations.nctm.org/Activity.aspx?id=4159



(γ) Ποιο είναι το μεγαλύτερο εμβαδόν που μπορεί να έχει ένα ορθογώνιο με περίμετρο 100. Να αιτιολογήσεις την απάντησή σου.

ΜΑΘΗΜΑ 9

Ένας τετράγωνος αριθμός μπορεί να αναπαρασταθεί με τελείες που δημιουργούν ένα τετράγωνο.

1ος 2ος 3ος 4ος

Ένας τρίγωνος αριθμός μπορεί να αναπαρασταθεί με τελείες που σχηματίζουν ένα τρίγωνο.

1ος 2ος 3ος 4ος

(α) Να βρεις τους επόμενους πέντε τετράγωνους και τρίγωνους αριθμούς.

(β) Με ποιο τρόπο μπορείς να βρεις έναν οποιονδήποτε τετράγωνο ή τρίγωνο αριθμό;

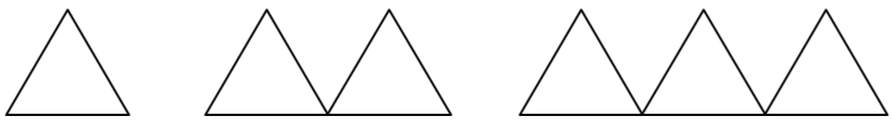
(γ) Να χρησιμοποιήσεις σειρές με τελείες για να αναπαραστήσεις τις πιο κάτω σχέσεις.

Το άθροισμα δύο διαδοχικών τρίγωνων αριθμών είναι τετράγωνος αριθμός.

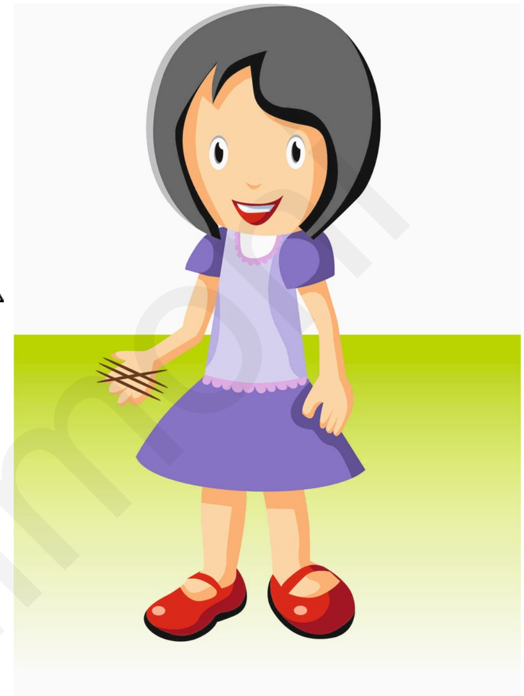
Το αποτέλεσμα της πράξης 8 φορές έναν τρίγωνο αριθμό συν 1 είναι ένας τετράγωνος αριθμός.

ΜΑΘΗΜΑ 10

Η Χριστίνα φτιάχνει το πιο κάτω μοτίβο, χρησιμοποιώντας οδοντογλυφίδες.



ΣΧΗΜΑ 1 ΣΧΗΜΑ 2 ΣΧΗΜΑ 3



(α) Να σχεδιάσεις το επόμενο σχήμα στο μοτίβο της Χριστίνας.

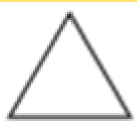
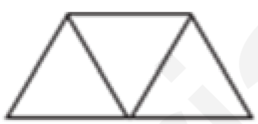
(β) Πόσες οδοντογλυφίδες θα χρειαστούν για να κατασκευάσει το 10ο σχήμα του μοτίβου. Να αιτιολογήσεις την απάντησή σου.

(γ) Η Χριστίνα κατασκεύασε ένα νέο μοτίβο με τις οδοντογλυφίδες. .

ΣΧΗΜΑ 1 ***ΣΧΗΜΑ 2*** ***ΣΧΗΜΑ 3***

i. Να σχεδιάσεις το επόμενο σχήμα του νέου μοτίβου.

ii. Να συμπληρώσεις τον πίνακα.

	Μήκος σχήματος (Μ)	Αριθμός οδοντογλυφίδων (Τ)
	1	
	2	
	3	
	4	

(γ) Πόσες οδοντογλυφίδες θα χρειαστεί για να φτιάξει το 10ο σχήμα;
Να αιτιολογήσεις την απάντησή σου.