Perturbation of zeros of entire functions of exponential type

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ΠΕΡΙΛΗΨΗ

Η Διδακτορική Διατριβή χωρίζεται σε δύο ενότητες.

ΕΝΟΤΗΤΑ Ι

Στο πρώτο μέρος μελετάμε την πληρότητα εκθετικών συτημάτων στον χώρο $L^2(-a,a)$.

Έστω $\mu = \{\mu_n, k_n\}_{n=1}^{\infty}$ μία ακολουθία μιγαδικών αριθμών, δηλαδή, τα μ_n είναι διακριτοί μιγαδικοί αριθμοί και το k_n η πολλαπλότητα τους. Με την ακολουθία μ συσχετίζουμε το εκθετικό σύστημα

$$E_\mu = \{t^{k-1}e^{i\mu_n t} : 1 \le k \le k_n\}.$$

Για $1 \le p < \infty$, λέμε ότι το E_μ είναι πλήρες στον $L^p(-a,a)$ εάν $\overline{\text{span}} E_\mu = L^p(-a,a)$. Δηλαδή, για οποιαδήποτε συνάρτηση $f \in L^p(-a,a)$ και τυχαίο $\epsilon > 0$, υπάρχει εκθετικό πολυώνυμο $\sum_{n=1}^{k} P_n(t)e^{i\mu_n t}$ όπου ο συντελεστής $P_n(t)$ είναι πολυώνυμο βαθμού το πολύ $k_n - 1$, έτσι ώστε

$$||f(t) - \sum_{n=1}^{k} P_n(t)e^{i\mu_n t}|| < \epsilon.$$

Εάν ένα σύστημα είναι πλήρες στον $L^p(-a,a)$ αλλά παύει να είναι πλήρες με την αφαίρεση ενός και μόνου όρου, τότε ονομάζεται ακριβές. Συχνά χρησιμοποιούμαι και τον όρο πλεόνασμα $E(\mu; p, a)$. Με αυτόν εννοούμαι το πλήθος των στοιχείων που πρέπει να προστεθούν (αφαιρεθούν) έτσι ώστε το σύστημα να καταστεί ακριβές.

Για να έχει ένα σύστημα πεπερασμένο πλεόνασμα, αναγκαία συνθήκη είναι όπως η ακολουθία μ να ανήκει στην κλάση $\mathcal{B}$. Τα στοιχεία της $\mathcal{B}$ είναι όλες οι ακολουθίες $\mu = \{\mu_n, k_n\}_{-\infty}^{\infty}$ όπου $\Re\mu_n \ge 0$ για $n > 0$ και $\Re\mu_n < 0$ για $n < 0$, οι οποίες έχουν πεπερασμένη άνω πυκνότητα, ο εκθέτης σύγκλισης τους ισούται με το 1, και η σειρά $\sum_{n=-\infty}^{\infty} \frac{|\Im\mu_n|k_n}{|\mu_n|^2}$ συγκλίνει. Ορίζουμε την υποκλάση $\mathcal{B}'$ της $\mathcal{B}$, όπου οι όροι μ_n της $\mu \in \mathcal{B}$ ικανοποιούν την επιπλέον συνθήκη

(*) $|\mu_n - \mu_{n+1}| \le c$ για κάποιο $c > 0, -\infty < n < \infty$.

Ο στόχος μας είναι, για ένα δοσμένο σύστημα E_μ με $\mu \in \mathcal{B}'$, να δώσουμε ένα γενικό τρόπο κατασκευής ενός άλλου συστήματος E_ν με το ίδιο πλεόνασμα στον $L^2(-a,a)$. Η κατασκευή της ακολουθίας ν βασίζεται στην διαμέριση της μ σε τρία υποσύνολα (διαμέριση $\mathcal{P}_{\mu,\delta}$). Δηλαδή γράφουμε

$$\mu = \{\gamma_n\}' \cup \{\lambda_n\}' \cup \{\rho_n\}'.$$

Τότε για μία φραγμένη ακολουθία μιγαδικών αριυμών $\{a_n\}'$ ορίζουμε την νέα ακολουθία ν ως

$$\nu = \{\gamma_n + a_n\}' \cup \{\lambda_n - a_n\}' \cup \{\rho_n\}'. \tag{0.1}$$

Το κύριο μας αποτέλεσμα είναι το ακόλυθο:

ΘΕΩΡΗΜΑ

Έστω $\mu \in \mathcal{B}'$ και για κάποιο $\delta > 0$ έστω $\mathcal{P}_{\mu,\delta}$ μία διαμέριση. Έστω $\{a_n\}'$ μία φραγμένη ακολουθία πραγματικών αριθμών και ν όπως στην (0.1). Τότε ισχύει ότι $E(\nu; 2, a) = E(\mu; 2, a)$. Εάν $\inf \Im\mu_n \ge u \in \mathbb{R}$ μπορούμε να διαλέξουμε τα $\{a_n\}'$ ως ακολουθία μιγαδικών αριθμών αντί για πραγματικών.

ΠΟΡΙΣΜΑ

Για κάθε θετικό ακέραιο q, πραγματικό αριθμό $\alpha \in (0, 1/2\pi)$ και μιγαδικούς αριθμούς

$$\nu_0 = 0, \quad \nu_n = nq + i\alpha \log|nq|, \quad |n| \ge 1,$$

τότε η ακολουθία $\{\nu_n, q\}_{-\infty}^{\infty}$ μας δίνει το ακόλουθο ακριβές σύστημα στον $L^2(-\pi,\pi)$

$$\{t^k e^{it\nu_n} : k = 0, 1, \dots, q-1\}_{n=-\infty}^{\infty}.$$

ΕΝΟΤΗΤΑ ΙΙ

Στο δεύτερο μέρος γενικεύουμε το Θεώρημα Χάσματος των Fabry-Pólya. Ο Pólya απόδειξε ότι εάν $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ είναι μία ακολουθία θετικών αριθμών έτσι ώστε $\lambda_{n+1} - \lambda_n \geq c$ για $c > 0$ και $n/\lambda_n \to D \geq 0$, τότε η σειρά Dirichlet

$$f(z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z}$$

έχει τουλάχιστον ένα ανώμαλο σημείο σε κάθε διάστημα μήκους μεγαλύτερου του $2\pi D$ στην ευθεία της σύγκλισης.

Στόχος είναι να αποδείξουμε παρόμοιου τύπου θεώρημα όπου αντί για αριθμούς $c_n \in \mathbb{C}$ να έχουμε πολυώνυμα.

Ορίζουμε την κλάση $\mathbf{L(c,D)}$ της οποίας τα στοιχεία είναι οι ακολουθίες $\mathbf{A}=\{a_n\}$, $|a_n| \leq |a_{n+1}|$ με τα a_n να ικανοποιούν τις ακόλουθες συνθήκες:

(1) $n/|a_n| \to D \geq 0$.

(2) $|a_n - a_k| \geq c|n-k|$ για $n \neq k$ όπου $c > 0$.

(3) $\sup|\arg a_n| < \pi/2$.

Με δοσμένη την ακολουθία $A \in \mathbf{L(c,D)}$, κατασκευάζουμε ακολουθία $\mathbf{B}$ ως εξής:

ΟΡΙΣΜΟΣ ΚΛΑΣΗΣ $A_{\alpha,\beta}$

Έστω $\mathbf{A} \in \mathbf{L(c,D)}$ και α, β θετικοί αριθμοί έτσι ώστε $\alpha + \beta < 1$. Λέμε ότι η ακολουθία $\mathbf{B}=\{b_n\}_{n=1}^{\infty}$ ανήκει στην κλάση $A_{\alpha,\beta}$ εάν για όλα τα n ισχύει πως

$$b_n \in \{z : |z - a_n| \leq |a_n|^{\alpha}\}$$

και για κάθε $m \neq n$ ισχύει ένα από τα εξής:

(i) $b_m = b_n$.

(ii) $|b_m - b_n| \geq \max\{e^{-|a_m|^{\beta}}, e^{-|a_n|^{\beta}}\}$.

Στην συνέχεια αναδιατάσουμε τα στοιχεία b_n χωρίζοντας τα πρώτα σε ομάδες όρων που έχουν το ίδιο μέτρο, και μετά ως προς το μέγεθος του ορίσματος τους. Έτσι γράφουμε $\{b_n\} = \{\lambda_n, \mu_n\}$ όπου λ_n διακριτοί μιγαδικοί αριθμοί και μ_n η πολλαπλότητα τους. Η μορφή αυτή ονομάζεται η (λ,μ) αναδιάταξη. Σε αυτήν την κλάση, οι όροι πλέον δεν είναι διακριτοί και μπορούν να πλησιάζουν πολύ κοντά.

Το κύριο αποτέλεσμα είναι το ακόλουθο.

ΘΕΩΡΗΜΑ

Έστω ακολουθία $\mathbf{A} \in \mathbf{L(c,D)}$ με θετικούς όρους και $\mathbf{D} > \mathbf{0}$. Έστω ακολουθία $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ έτσι ώστε $\mathbf{B}=\{b_n\}_{n=1}^{\infty}$ με θετικούς όρους και έστω (λ,μ) η αναδιάταξη τους. Τότε κάθε σειρά Taylor-Dirichlet

$$f(z) = \sum_{n=1}^{\infty} \left(\sum_{j=0}^{\mu_n - 1} c_{n_j} z^j \right) e^{-\lambda_n z}, \quad c_{n_{\mu_n - 1}} \neq 0,$$

η οποία ικανοποιεί την συνθήκη

$$\limsup_{n\to\infty} \frac{\log|c_{n_{\mu_n-1}}|}{\lambda_n} = \limsup_{n\to\infty} \frac{\log A_n}{\lambda_n},$$

όπου

$$A_n = \max\{|c_{n_j}| : j = 0, 1, \dots, \mu_n - 1\},$$

έχει τουλάχιστον ένα ανώμαλο σημείο σε κάθε διάστημα μήκους μεγαλύτερου του $2\pi D$ στην ευθεία της σύγκλισης.

Abstract

This thesis is divided into two parts. In the first part, we give new criteria for two complex sequences to have the same *excess* in the sense of Paley and Wiener in $L^2(-a, a)$. As a result, we prove that given any positive integer q, a real number $\alpha \in \left(0, \frac{1}{2\pi}\right)$ and complex numbers

$$\nu_0 = 0, \quad \nu_n = nq + i\alpha \log |nq|, \quad |n| \geq 1,$$

the exponential system $\{t^k e^{it\nu_n} : k = 0, 1, \ldots, q-1\}_{n=-\infty}^{\infty}$ has excess 0 in $L^2(-\pi, \pi)$.

In the second part of the thesis, we give an extension of a theorem of N. Levinson (see Theorems 3.1 and 3.2). As an application, we get a variation of the Fabry Gap Theorem for frequencies with finite upper density (see Theorems 3.3 and 3.4), concerning the location of singularities of Taylor-Dirichlet series, on the boundary of convergence.

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Contents

1 Introduction.

The theory of Nonharmonic Fourier Series in $L^2(-\pi,\pi)$ is concerned with the completeness properties of sets of complex exponentials $\{e^{i\mu_n t}\}_{n=-\infty}^{\infty}$. The study of these series was initiated by Paley and Wiener who showed that the system $\{e^{i\mu_n t}\}_{n=-\infty}^{\infty}$ is a Riesz basis for $L^2(-\pi,\pi)$ whenever each μ_n is real and $|\mu_n - n| \leq L < 1/\pi^2$ for $-\infty < n < \infty$. A system $\{e^{i\mu_n t}\}_{n=-\infty}^{\infty}$ is a Riesz basis in $L^2(-\pi,\pi)$ if it is isomorphic to the basis $\{e^{int}\}_{n=-\infty}^{\infty}$. If this is the case, then each function $f \in L^2(-\pi,\pi)$ has a unique *nonharmonic Fourier* expansion

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{i\mu_n t} \quad (\textit{in the mean})$$

with $\{c_n\}$ in l^2. In general, the sets of complex exponentials $\{e^{i\mu_n t}\}$ which are appropriate, are those which are complete in $L^2(-\pi,\pi)$. In other words, the $\overline{span}\{e^{i\mu_n t}\} = L^2(-\pi,\pi)$, that is, for each $f \in L^2(-\pi,\pi)$ and each $\epsilon > 0$, there is a finite linear combination $\sum_{n=1}^{k} c_n e^{i\mu_n t}$ such that

$$\left|\left| f(t) - \sum_{n=1}^{k} c_n e^{i\mu_n t} \right|\right| < \epsilon.$$

The first part of this thesis is concerned with complete sets of complex exponentials in $L^2(-a,a)$. In fact, we allow for the terms μ_n to have multiplicity greater than 1. Thus we consider the following: let μ=$\{\mu_n, k_n\}_{n=1}^{\infty}$ be a multiplicity sequence, that is, a sequence where $\{\mu_n\}$ are distinct complex numbers satisfying $|\mu_n| \leq |\mu_{n+1}| \mapsto \infty$ as $n \mapsto \infty$, and each μ_n appears $k_n - times$. We associate with this sequence the exponential system

$$E_\mu = \{t^{k-1} e^{i\mu_n t} : 1 \leq k \leq k_n\}. \tag{1.1}$$

For $1 \leq p < \infty$, we say that the system E_μ is complete in $L^p(-a,a)$ if $\overline{span}\, E_\mu = L^p(-a,a)$. By the Hahn-Banach theorem, incompleteness is equivalent to the existence of a non-trivial entire function $F(z)$ which vanishes on μ and which has the integral representation

$$F(z) = \int_{-a}^{a} e^{izt} f(t)dt, \quad f \in L^q(-a,a), \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{1.2}$$

Our goal is to study the stability of a system E_μ under bounded perturbations of the terms of μ. We give conditions under which we obtain equivalent systems in $L^2(-a,a)$, in other words, systems which are simultaneously complete or incomplete in $L^2(-a,a)$. For this, we need to introduce the concept of *excess* in §2 where we state our first main result (Th. 2.10). We remark that the proof of Theorem 2.10 shall occupy most of §4. In addition, some other results will be stated and proved in that section.

The second part of this thesis is concerned with the location of singularities on the boundary of convergence of Taylor-Dirichlet series $f(z)$. Given a multiplicity sequence $\mathbf{B} = \{(\lambda_n, \mu_n)\}_{n=1}^{\infty}$, we let

$$f(z) = \sum_{n=1}^{\infty} p_{\mu_n}(z) e^{-\lambda_n z}, \tag{1.3}$$

where $p_{\mu_n}(z) = \sum_{j=0}^{\mu_n - 1} c_{n_j} z^j$ is a polynomial with $c_{n\mu_n - 1} \neq 0$. Such a series occurs in nature as the solution of an infinite order homogeneous differential equation with constant coefficients. In §3 we state our result (Th. 3.3 and 3.4) which is a strong version of the Fabry Gap Theorem, for frequencies with finite upper density. The result depends on extending a Levinson theorem (Th. 3.1 Th. 3.2) concerning various estiamates of an entire even function. The proof of these results shall occupy §5.

We remark, that our results for both parts of this thesis, depend on the comparison of two entire functions of exponential type, where the zeros of one of the functions are obtained by perturbating the zeros of the other. An entire function $f(z)$ is said to be of exponential type if $|f(z)| \leq Ae^{B|z|}$ for some positive constants A and B. In the following subsection we shall recall various properties of such functions.

1.1 Entire functions of exponential type

An entire function $f(z)$ of exponential type is a function of order 1 and finite type, or a function of order less than 1. The order ρ is defined as

$$\rho = \inf\{\mu : \max_{|z|=r} |f(z)| < e^{r^\mu}, r > r_0(\mu)\}.$$

From this we get that

$$\rho = \limsup_{r\to\infty} \frac{\log\log \max_{|z|=r} |f(z)|}{\log r}.$$

A function of finite order ρ is said to be of type σ if

$$\sigma = \inf\{K : \max_{|z|=r} |f(z)| < e^{Kr^\rho}, r > r_0(K)\}.$$

Then one has that

$$\sigma = \limsup_{r\to\infty} \frac{\log \max_{|z|=r} |f(z)|}{r^\rho}.$$

An entire function $f(z)$ of exponential type which vanishes an infinite number of times, has the following expansion by the Hadamard Factorization Theorem. Let $A = \{a_n\}_{n=1}^{\infty}$ be the set of its zeros so that $|a_n| \le |a_{n+1}| \mapsto \infty$. Then

$$f(z) = kz^m e^{bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}},$$

where k and b are constants, and m denotes the multiplicity at 0. If the order of $f(z)$ is less than 1, then the exponential factors are missing.

The factorization is valid due to the following result of Lindelof.

Theorem 1.1. *Let $A = \{a_n\}$ be the zeros of $f(z)$, an entire function of exponential type. Then the* $\limsup_{t\to\infty} \frac{n_A(t)}{t} < \infty$, *where $n_A(t)$ is the counting function of A, that is,*

$$n_A(t) = \sum_{|a_n|\le t} 1,$$

and the sums $\sum_{|a_n|\le r} \frac{1}{a_n}$ are uniformly bounded with respect to r.

We note that if for a sequence A the relation $\limsup_{t\to\infty} \frac{n_A(t)}{t} < \infty$ holds, we say that it has a finite *upper density*. We say that A has *density* D if the $\lim_{t\to\infty} \frac{n_A(t)}{t}$ exists and is equal to D. In both cases the exponent of convergence of A, κ, satisfies $\kappa \le 1$, where

$$\kappa = \inf\left\{\alpha : \sum_{n=1}^{\infty} \frac{1}{|a_n|^\alpha} < \infty\right\}.$$

The converse is not true. The sequence $a_n = \frac{n}{\log n}$, $n > 2$, is an example where $\kappa = 1$ but the $\limsup_{t\to\infty} \frac{n_A(t)}{t} = \infty$.

Let $f(z)$ be an entire function of exponential type σ. The growth of $f(z)$ in various directions is characterized in terms of its indicator function $h_f(\theta)$. This is defined as

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r}.$$

The indicator function $h_f(\theta)$ is the supporting function of some convex compact set $\mathbf{I_f}$. In other words, there is a compact set $\mathbf{I_f}$, called the *indicator diagram* of $f(z)$, such that

$$h_f(\theta) = \sup_{z\in I_f}\{x\cos\theta + y\sin\theta\} = \sup_{z\in I_f}\{\Re(ze^{-i\theta})\}, \quad \theta \in [0, 2\pi]. \tag{1.4}$$

We note that since $f(z)$ is of type σ, then

$$f(z) = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n, \quad \sigma = \limsup |a_n|^{\frac{1}{n}}. \tag{1.5}$$

To this function there corresponds a function

$$\phi(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^{n+1}}, \tag{1.6}$$

called the *Borel transform* of $f(z)$. The *Borel transform* is a holomorphic function in the domain $|z| > \sigma$. It is possible that $\phi(z)$ can be analytically continued into the disk $|z| < \sigma$. We call the smallest convex compact set containing all the singularities of $\phi(z)$ the *conjugate diagram* of $f(z)$.

We end this section by a beautiful result obtained by George Polya. He established the following remarkable connection between the conjugate diagram and the indicator diagram of an entire function of exponential type.

Theorem 1.2. *The conjugate diagram of an entire function of exponential type is the reflection in the real axis of its indicator diagram.*

2 On the excess of complex exponential systems in $L^2(-a,a)$.

We note that the main sources for this topic are the survey papers of R. Redheffer [34] and of A. Sedletskii [41], as well as the excellent expository account [44] of R. Young.

Let $\mu=\{\mu_n, k_n\}_{n=1}^{\infty}$ be a multiplicity sequence. The completeness in $L^p(-a,a)$ of the system E_μ has lead to the notion of the completeness radius $R(\mu,p)$. This is defined as

$$R(\mu,p) = \sup\{a \geq 0 : E_\mu \textit{ is complete in } L^p(-a,a)\}.$$

The radius $R(\mu,p)$ is the same for all $p \in [1,\infty)$. It is infinite when the series $\sum_{n=1}^{\infty} \frac{|\Im \mu_n| k_n}{|\mu_n|^2}$ diverges and zero when the exponent of convergence is less than 1 (see [34] Theorems 7 and 41). Thus, we are interested for the non-trivial case, that is, when the series converges and the exponent of convergence is equal to 1. We remark, that for some time it was conjectured that $R(\mu,p) = 0$ if the sequence μ is real, with zero density. This was disproved first by Kahane [19].

Theorem 2.1. *There exists a sequence μ with real terms, of zero density, such that $R(\mu,p) = \infty$.*

Furthermore Koosis [25] proved the following.

Theorem 2.2. *There exists a sequence μ with distinct positive integers, of zero density, such that $R(\mu,p) = 2\pi$.*

A very interesting result which compares the radii of two systems, was obtained by Redheffer [37].

Theorem 2.3. *Let $\Lambda = \{\lambda_n\}$ and $\Gamma = \{\gamma_n\}$ be complex sequences converging to infinity, so that*

$$\sum \left| \frac{1}{\lambda_n} - \frac{1}{\gamma_n} \right| < \infty.$$

Then their completeness radii are equal, that is, $R(\Lambda,p) = R(\Gamma,p)$.

One observes, that the result might hold even if $|\mu_n - \lambda_n| \to \infty$. This is the case if, for example, we let $\mu_n = n$ and $\lambda_n = n + \sqrt{n}$.

The situation is quite different as far as the *excess* of two systems is concerned. A. Sedletskii ([39] Th. 4) proved that

Theorem 2.4. *There exist sequences μ and λ with real terms satisfying $|\mu_n - \lambda_n| \to 0$ and yet their excesses are not the same, that is, $E(\mu;p,a) \neq E(\lambda;p,a)$.*

By the term *excess* $E(\mu;p,a)$, we mean the number of terms that have to be removed from (added to) the system E_μ in order for it to become *exact* in $L^p(-a,a)$. The system E_μ is called *exact* if it is complete but becomes incomplete on the removal of a single term. The most classical example is the trigonometric system $\{e^{int}\}_{n=-\infty}^{\infty}$ which is *exact* in $L^p(-\pi,\pi)$ for all $p \in [1,\infty)$ and whose *excess* in $C[-\pi,\pi]$ is equal to -1.

It is well known, that replacing a **finite** number of terms from a system E_μ by an **equivalent** number of other terms $s_n \notin \mu$, does not change the *excess*. The result is due to N. Levinson ([30] Th. VI).

Theorem 2.5. *The completeness of the exponential system E_μ in $L^p(-a,a)$, $1 \leq p < \infty$, or in $C[-a,a]$ is unaffected if some element of E_μ is replaced by e^{ist}, where $s \notin \mu$.*

Thus, the interesting case is, whether the *excess* of a system is preserved if an infinite number of its terms is replaced. Perhaps, the most celebrated theorem towards this direction is the Alexander-Redheffer theorem [34] (Th. 14).

Theorem 2.6. *For all $p \in [1,\infty)$ the excesses $E(\nu;p,a)$ and $E(\mu;p,a)$ of two exponential systems E_μ and E_ν are equal, assuming that*

$$\sum_{n=1}^{\infty} \frac{|\mu_n - \nu_n|}{1 + |\Im\mu_n| + |\Im\nu_n|} < \infty. \tag{2.1}$$

This theorem is interesting on its own because it does not assume any regularity in the distribution of either of the individual sequences. However, the necessary condition $|\nu_n - \mu_n| \mapsto 0$ for the convergence of the series in (2.1), does not provide a large class of examples.

Remark 2.1. *We note that the Alexander-Redheffer theorem has been generalized for spaces of functions on arcs, other than the interval, and on domains, by Bulat Khabibullin ([21], [22], [23]).*

A positive result in $L^2(-a,a)$, without the condition $|\lambda_n - \gamma_n| \to 0$, but under bounded pure imaginary perturbations instead, was proved by D. Peterson ([34] Th. 17).

Theorem 2.7. *Let $\Lambda = \{\lambda_n\}$ and $\Gamma = \{\gamma_n\}$ be complex sequences converging to infinity, so that*

$$\Re\lambda_n = \Re\gamma_n, \qquad |\Im\lambda_n - \Im\gamma_n| = O(1).$$

Then $E(\Lambda, 2, a) = (\Gamma, 2, a)$.

We remark that the problem remains open for $p \neq 2$. We note however, that the theorem fails for $L^1(-a,a)$ and $C[-a,a]$ ([40] Th. 1).

Another result without the condition $|\lambda_n - \gamma_n| \to 0$, is the following theorem of A. Sedletskii ([38] Th. 2) which covers a case of nonabsolute closeness of the sequences Λ and Γ, preserving the *excess*.

Theorem 2.8. *Let $\Lambda = \{\lambda_n\}$ and $\Gamma = \{\gamma_n\}$ be sequences of* **real** *numbers converging to infinity, so that*

$$\sup_{N\geq 1}\left|\sum_{1}^{N}(\lambda_n - \gamma_n)\right| + \sup_{N\geq 1}\left|\sum_{-1}^{-N}(\lambda_n - \gamma_n)\right| < \infty.$$

Then $E(\Lambda, 2, a) = E(\Gamma, 2, a)$.

At this point we should note that in order for a system E_μ to have a finite *excess*, a **necessary** condition is for μ to belong to (what we shall refer to) the class $\mathcal{B}$. The elements of $\mathcal{B}$ are all the *two − sided* sequences μ, that is, $\mu = \{\mu_n, k_n\}_{-\infty}^{\infty}$ where $\Re\mu_n \geq 0$ for $n > 0$ and $\Re\mu_n < 0$ for $n < 0$, that have a finite upper density, their exponent of convergence is equal to 1 and the series $\sum_{n=-\infty}^{\infty} \frac{|\Im\mu_n| k_n}{|\mu_n|^2}$ converges.

We shall denote by $\mathcal{B}'$ the subclass of $\mathcal{B}$, where in addition the terms μ_n of some sequence $\mu \in \mathcal{B}$ satisfy
(*) $|\mu_n - \mu_{n+1}| \leq c$ for some $c > 0$, $-\infty < n < \infty$.

Our goal is, given some system E_μ with $\mu \in \mathcal{B}'$, to give a general way to generate another system E_ν with the same *excess* in $L^2(-a,a)$. This new sequence ν may have radically different geometric properties. Such an example is provided in Corollary 3.16, where we start with all the terms of μ having multiplicity 1 and construct ν whose terms have multiplicity $q \in \mathbf{N}$. The constuction of ν is based on partitioning μ into at most three sets (see below the $\mathcal{P}_{\mu,\delta}$ partition) and then subjecting two of them to a bounded perturbation.

Our method is particularly useful for systems E_μ having a finite *excess*, even if $\sup|\Im\mu_n| = \infty$. In [40] (Th. 3), A. Sedletskii constructed *exact* systems with unbounded imaginary parts. We state a special form of his result as a theorem.

Theorem 2.9. *If $\mu = \{\mu_n, 1\}_{-\infty}^{\infty}$ with $\mu_0 = 0$ and $\mu_n = n + i\alpha \log|n|$ for $|n| \geq 1$ and $\alpha \in (0, \frac{1}{2\pi})$, then the system E_μ is exact in $L^2(-\pi, \pi)$.*

When A. Sedletskii searches for equivalent systems, he usually imposes the condition that μ is a non-concentrated sequence where the $\sup|\Im\mu_n| < \infty$. A sequence μ is called non-concentrated if $n_\mu(t+1) - n_\mu(t) = O(1)$ where $n_\mu(t)$ is the counting function. But the condition $\sup|\Im\mu_n| < \infty$ is obviously a limitation when one wants to derive other *exact* systems from the one in Theorem 2.9.

Thus, this latter condition and the Alexander-Redheffer theorem with the necessary condition $|\mu_n - \nu_n| \to 0$, are inadequate for what we want to prove. As mentioned before, our method yields equivalent systems E_μ and E_ν, with their sequences μ and ν having different geometric properties, even when their imaginary parts are unbounded.

From now on, when we write a sequence $\{p_n\}'$, a series $\sum'$, or a product $\prod'$, we mean that the index-n is running through all $n \in \mathbf{Z}\backslash\{0\}$.

The $\mathcal{P}_{\mu,\delta}$ partition of some $\mu \in \mathcal{B}'$ and the construction of ν.

Let $\mu \in \mathcal{B}'$. We will partition μ into at most three sets, not necessarily disjoined,

$$\mu = \{\gamma_n\}' \cup \{\lambda_n\}' \cup \{\rho_n\},$$

where $\{\rho_n\}$ might be infinite, finite or empty. This is done as follows: Fix $\delta > 0$ so that $\delta \geq c$ and write μ as $\mu = \{\mu_{n,k} : k = 1, 2, \ldots, k_n\}_{n=-\infty}^{\infty}$. Consider the closed disks $\overline{B}_{n,k} = \overline{B}(\mu_{n,k}, \delta) = \{z : |\mu_{n,k} - z| \leq \delta\}$. Since $\delta \geq c$, then in each $\overline{B}_{n,k}$ there are at least two elements of μ. Thus, we pair $\mu_{n,k}$ with at most one other element of μ which is in $\overline{B}_{n,k}$, and once paired together, they cannot be paired with other ones. Thus, two subsets of μ are constructed, not necessarily disjoined and each containing one of the two elements. We call them $\{\gamma_n\}'$ and $\{\lambda_n\}'$, γ_n is paired with λ_n and satisfy $|\gamma_n - \lambda_n| \leq \delta$. It is not necessary to have $|\gamma_{|n|}| \leq |\gamma_{|n+1|}|$ or $|\lambda_{|n|}| \leq |\lambda_{|n+1|}|$ (see Example 2.1). The remaining (if any) terms of μ we call them $\{\rho_n\}$ and are totally independent, that is, they do not participate in the pairing. We shall refer to such a partition by $\mathcal{P}_{\mu,\delta}$.

Then for some two-sided, bounded sequence of complex numbers $\{a_n\}'$ we define the new sequence ν as

$$\nu = \{\gamma_n + a_n\}' \cup \{\lambda_n - a_n\}' \cup \{\rho_n\}. \tag{2.2}$$

The following example illustrates the above construction.

Example 2.1. *We present a $\mathcal{P}_{\mu,\delta}$ partition when $\mu = \mathbf{Z}$ and $\delta = 4$. From this we construct a new sequence ν. Let $\{\rho_n\} = \{-2, -1, 1, 2\} \cup \{5n\}_{-\infty}^{\infty}$, that is $\{\rho_n\} = \{0, \pm1, \pm2, \pm5, \pm10, \ldots\}$ and let*

$$\begin{aligned} &\gamma_1 = 3,\ \gamma_2 = 4,\ \gamma_3 = 8,\ \gamma_4 = 9,\ \gamma_5 = 13,\ \gamma_6 = 14,\ \ldots, \\ &\lambda_1 = 7,\ \lambda_2 = 6,\ \lambda_3 = 12,\ \lambda_4 = 11,\ \lambda_5 = 17,\ \lambda_6 = 16,\ \ldots. \end{aligned} \tag{2.3}$$

that is, for $n \geq 1$ (similarly for $n \leq -1$) we put

$$\gamma_n = \begin{cases} \frac{5n+1}{2}, & n odd \\ \frac{5n-2}{2}, & n even \end{cases} \qquad \lambda_n = \begin{cases} \frac{5n+9}{2}, & n odd \\ \frac{5n+2}{2}, & n even. \end{cases} \tag{2.4}$$

For $n \geq 1$ take $a_{2n-1} = 2$, $a_{2n} = 1$ and for $n \leq -1$ take $a_{2n+1} = -2$, $a_{2n} = -1$. Then from (2.2) the new sequence ν is $\{0, \pm1, \pm2, \pm5, \pm10, \pm15, \ldots\}$ with all the terms having multiplicity 5, except $\{0, \pm1, \pm2\}$ whose multiplicity is 1.

We now state our main result which is the following:

Theorem 2.10. *Let $\mu \in \mathcal{B}'$ and for some $\delta > 0$ fixed let $\mathcal{P}_{\mu,\delta}$ be a corresponding partition. Let $\{a_n\}'$ be a two-sided bounded sequence of* **real** *numbers and ν as in (2.2). Then the relation $E(\nu; 2, a) = E(\mu; 2, a)$ holds. If $\inf \Im\mu_n \geq u \in \mathbf{R}$ we may choose the $\{a_n\}'$ to be a sequence of* **complex** *numbers instead of real.*

Remark 2.2. *We note that for* **real** *μ and ν, our result follows from Theorem 2.8.*

Corollary 2.1. *Given any positive integer q, a real number $\alpha \in \left(0, \frac{1}{2\pi}\right)$ and complex numbers*

$$\nu_0 = 0, \quad \nu_n = nq + i\alpha \log |nq|, \quad |n| \geq 1,$$

then the sequence $\nu = \{\nu_n, q\}_{-\infty}^{\infty}$ yields the following exact system in $L^2(-\pi, \pi)$

$$\{t^k e^{it\nu_n} : k = 0, 1, \ldots, q-1\}_{n=-\infty}^{\infty}. \tag{2.5}$$

Moreover, we may construct a sequence $\nu = \{\nu_n, k_n\}$ with **different** multiplicities k_n so that for $\alpha \in \left(0, \frac{1}{2\pi}\right)$

Corollary 2.2. *The exponential system*

$$\{t^k e^{it(8n+i\alpha \log 8n)} : k = 0, 1, 2\} \cup \{t^k e^{it[8n-4+i\alpha \log(8n-4)]} : k = 0, 1, 2, 3, 4\} \tag{2.6}$$

is exact in $L^2(-\pi, \pi)$.

We end this section by two standard arguments.

Theorem 2.11. *Let two systems E_μ and E_ν be given. If incompleteness of* **anyone** *of the two systems implies incompleteness of the other, then they have the same excess.*

Proof

Assume that this is not true, say $E(\nu; p, a) < E(\mu; p, a)$. We consider two cases, one with $|E(\nu; p, a) - E(\mu; p, a)| < \infty$ and the other with $|E(\nu; p, a) - E(\mu; p, a)| = \infty$.

Case 1, $|E_\nu - E_\mu| < \infty$: Then $-\infty < E_\nu < \infty$ and $-\infty < E_\mu < \infty$. If necessary, we may add or subtract the same finite number of terms from both systems, in order to get $E_{\nu'}$ and $E_{\mu'}$ respectively, so that $-1 = E_{\nu'} < E_{\mu'}$, thus $E_{\mu'} \geq 0$. Since $E_{\nu'}$ is incomplete, by assumption the same holds for $E_{\mu'}$, thus $E_\mu \leq -1$. Therefore we reach a contradiction.

Case 2, $|E(\nu; p, a) - E(\mu; p, a)| = \infty$: Either $E(\nu; p, a) = -\infty$ and/or $E(\nu; p, a) = +\infty$.

First, assume that $E(\nu; p, a) = -\infty$. Then E_ν is incomplete and by assumption so is E_μ. Thus $-\infty < E(\mu; p, a) < 0$. Add a finite number of terms to both systems in order to get $E_{\nu'}$ and $E_{\mu'}$ respectively, so that $E(\mu'; p, a) \geq 0$. Obviously we have $E(\nu'; p, a) = -\infty$, in other words $E_{\nu'}$ is incomplete. Again by assumption, this implies that $E_{\mu'}$ is incomplete also, thus $E(\mu'; p, a) < 0$. Therefore we reach a contradiction.

Second, assume that $E(\mu; p, a) = +\infty$ and $-\infty < E(\nu; p, a) < \infty$. Once more, add a finite number of terms to both systems in order to get $E_{\nu'}$ and $E_{\mu'}$ respectively, so that $E(\nu'; p, a) < 0$ and $E(\mu'; p, a) = +\infty$. Then, $E_{\nu'}$ is incomplete, and by assumption so is $E_{\mu'}$, thus $E(\mu'; p, a) < 0$. Therefore, once more we reach a contradiction. ♢

Theorem 2.12. *Consider a multiplicity sequence $\mu = \{\mu_n, k_n\}$. Shift all the terms by some amount d to get a multiplicity sequence $\mu' = \{\mu_n + d, k_n\}$. Then their associate systems have the same excess.*

Proof

Assume that E_μ is incomplete in $L^p(-a,a)$. Then there is some $f \in L^q(-a,a)$ so that

$$F(z) = \int_{-a}^{a} e^{izt} f(t)dt \tag{2.7}$$

vanishes on μ. But $F(z)$ can also be written as

$$F(z) = \int_{-a}^{a} e^{i(z+d)t} e^{-idt} f(t)dt, \tag{2.8}$$

with $g(t) = e^{-idt} f(t) \in L^q(-a,a)$. Define now $G(z) = F(z-d)$. Then

$$G(z) = \int_{-a}^{a} e^{izt} g(t)dt, \tag{2.9}$$

vanishes on μ', thus $E_{\mu'}$ is incomplete in $L^p(-a,a)$. Similarly, if one first assumes incompleteness of $E_{\mu'}$ in $L^p(-a,a)$, this yields the same for E_μ. Applying the previous theorem completes the proof. $\diamondsuit$

3 On a theorem of Norman Levinson and a variation of The Fabry Gap Theorem.

The Fabry Gap Theorem (see [30] Th. XXIX) states that if $\mathbf{\Lambda}=\{\lambda_n\}_{n=1}^{\infty}$ is a real positive sequence such that $\lambda_{n+1}-\lambda_n \geq c$ for $c>0$ and $n/\lambda_n \mapsto D \geq 0$ as $n \mapsto \infty$, then the Dirichlet series $f(z)=\sum_{n=1}^{\infty} c_n e^{-\lambda_n z}$ has at least one singularity in every interval of length exceeding $2\pi D$ on the abscissa of convergence.

Pólya proved the theorem relying on certain properties of an entire function which vanishes exactly on $\pm\mathbf{\Lambda}$ (see [30] Th. XXX). Levinson extended the latter result to cover the complex case as well (see [30] Th. XXXI). For a sequence $\mathbf{A}=\{a_n\}_{n=1}^{\infty}$ satisfying for $n \neq k$ the spacing condition $|a_n - a_k| \geq c|n-k|$ for some $c>0$, and the limit relations, $n/a_n \mapsto D \geq 0$ and $\arg a_n \mapsto 0$, he proved that the entire function $F(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{a_n^2}\right)$ satisfies for every $\epsilon>0$ as $r \mapsto \infty$ the following properties:
(1∗) $|F(re^{i\theta})| = O(exp\{\pi r(D|\sin\theta| + \epsilon)\})$.
(2∗) $1/|F(re^{i\theta})| = O(exp\{\pi r(-D|\sin\theta| + \epsilon)\})$ whenever $|re^{i\theta} \pm a_n| \geq c/8$ for all $n \geq 1$.
Furthermore for every $\epsilon>0$ as $n \to \infty$ one has:
(3∗) $1/|F'(a_n)| = O(exp\{\epsilon|a_n|\})$.

We remark that for $D=0$, Vidras [43] dropped the condition $\arg a_n \mapsto 0$ and constructed an entire function of infraexponential type satisfying (2∗) and (3∗).

Our primary goal is to give an extension of Levinsons result. Based on a sequence $\mathbf{A}$ as above, we construct a multiplicity sequence $\mathbf{B}=\{(\lambda_n,\mu_n)\}_{n=1}^{\infty}$. For this sequence $\mathbf{B}$ we prove that the infinite product $G(z)$ which vanishes exactly on $\pm\mathbf{B}$, satisfies $\frac{\mu_n!}{|G^{[\mu_n]}(\lambda_n)|} = O(exp\{\epsilon|\lambda_n|\})$ for every $\epsilon>0$ (see Theorem 3.1). That is, we have a sharp estimate for the μ_n^{th} derivative function of $G(z)$ evaluated on λ_n. Similarly we extend Vidras result (see Theorem 3.2).

These results allow us to get a variation of the Fabry Gap Theorem. For the pre-mentioned constructed multiplicity sequence $\mathbf{B}$, having **real** positive λ_n and density D counting multiplicities, we prove that the Taylor-Dirichlet series in (1.3) that satisfies the relation

$$\limsup_{n\to\infty}\frac{\log|c_{n_{\mu_n-1}}|}{\lambda_n} = \limsup_{n\to\infty}\frac{\log A_n}{\lambda_n}, \tag{3.1}$$

where

$$A_n = \max\{|c_{n_j}| : j=0,1,2,\ldots,\mu_n-1\}, \tag{3.2}$$

has at least one singularity in every interval of length exceeding $2\pi D$ on the abscissa of convergence (see Theorem 3.3). We note that our result holds even if we allow the distinct λ_n terms satisfy the relation $\liminf(\lambda_{n+1}-\lambda_n)=0$ (see Example 3.15). Recall, that even in the simple case when the multiplicity μ_n is equal to 1 for all the λ_n, authors such as Levinson [30] and Mandelbrojt [31] impose the condition $\liminf(\lambda_{n+1}-\lambda_n)>0$.

When $D=0$ the assumption that the λ_n are real can be dropped. We prove that if the λ_n are **complex** numbers such that the $\sup|\arg\lambda_n|<\pi/2$, and if for every $\epsilon>0$ there exists some $n_0 \in \mathbf{N}$ such that

$$\frac{A_n}{|c_{n_{\mu_n-1}}|} \leq e^{\epsilon|\lambda_n|} \qquad \forall n \geq n_0, \tag{3.3}$$

then the boundary of convergence of the Taylor-Dirichlet series in (1.3) is a natural boundary (see Theorem 3.4). We remark that the set of arguments of the λ_n **need not have** a *finite* number of cluster points in the interval $(-\frac{\pi}{2},\frac{\pi}{2})$.

For $D>0$, the desired multiplicity sequence $\mathbf{B}$ shall be constructed from a sequence $\mathbf{A}$ satisfying those properties as stated in Levinsons theorem. For $D=0$, we rely on Vidras result where for $\mathbf{A}$ we replace the limit relation $\arg a_n \to 0$ by the weaker condition $\sup|\arg a_n|<\pi/2$.

Since the condition $\arg a_n \mapsto 0$ is crucial for $D > 0$ but not for $D = 0$, in order to work with anyone of these two cases we denote by $\mathbf{L}(\mathbf{c}, \mathbf{D})$ the class of all sequences $\mathbf{A}=\{a_n\}$, $|a_n| \le |a_{n+1}|$ satisfying the following properties: (1) $n/|a_n| \to D \ge 0$, (2) for $n \ne k$ one has that $|a_n - a_k| \ge c|n-k|$ for some $c > 0$, and (3) the $\sup|\arg a_n| < \pi/2$.

Given some sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$, the construction of the multiplicity sequence $\mathbf{B}$ is as follows:

Definition 3.1. *Let $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and α, β real positive numbers so that $\alpha + \beta < 1$. We say that a sequence $\mathbf{B}=\{b_n\}_{n=1}^{\infty}$ belongs to the class $\mathbf{A}_{\alpha,\beta}$ if for all n we have*

$$b_n \in \{z : |z - a_n| \le |a_n|^{\alpha}\} \tag{3.4}$$

and for all $m \ne n$ one of the following holds:
(i) $b_m = b_n$.
(ii) $|b_m - b_n| \ge \max\{e^{-|a_m|^{\beta}}, e^{-|a_n|^{\beta}}\}$.

One observes that (i) allows for the set $\{\pm\mathbf{B}\}$ to have coinciding terms, thus the entire even function vanishing exactly on $\{\pm\mathbf{B}\}$ has multiple zeros. Also note that (ii) allows for non-coinciding terms to come *very close* to each other, thus, it is possible for the relation $\liminf|\lambda_{n+1} - \lambda_n| = 0$ to hold. We may now rewrite $\mathbf{B}$ in the form of a multiplicity sequence, and this is done as follows: first we split $\{\pm b_n\}$ into groups of terms having the same modulus, and then within each group we order them by the size of their argument, beginning from smaller to larger. The arguments are taken with respect to the principal one, that is, $0 \le \arg b_n < 2\pi$. Thus, we can rewrite the sequence $\mathbf{B}$ as $\{\lambda_n, \mu_n\}_{n=1}^{\infty}$. We shall call this form of $\mathbf{B}$ the (λ, μ) **reordering**.

Remark 3.1. *We point out that the spacing condition* (2) *of a sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ plays a very important role.*

We note that the notation $\mathbf{B}=\{\lambda_n, \mu_n\}_{n=1}^{\infty}$ is not always useful when carrying out various calculations. In such cases we keep the notation $\mathbf{B}=\{b_n\}_{n=1}^{\infty}$. This is more practical since the terms b_n and a_n are related by (3.4). From the latter one also deduces that $n/|b_n| \to D$ as $n \mapsto \infty$. At this point we introduce the following two systems of unions of open disks given some $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$:

$$S_1 = \bigcup_{n=1}^{\infty} B\left(\pm a_n, \frac{e^{-|a_n|^{\beta}}}{3}\right), \tag{3.5}$$

$$S_2 = \bigcup_{n=1}^{\infty} B\left(\pm b_n, \frac{e^{-|a_n|^{\beta}}}{3}\right) \tag{3.6}$$

where as usual

$$B(z_0, r) = \{z : |z - z_0| < r\}.$$

Observe that the disks in S_1 are non-overlapping, whereas in general this is not necessarily true for S_2 since for fixed n we might have $b_n = b_m$ for $m \ne n$. Nevertheless, note that if for fixed n, Γ_n is the set of all integers j so that $b_n = b_j$, that is,

$$\Gamma_n = \{j : b_j = b_n\}, \tag{3.7}$$

then

$$\bigcup_{m\in\Gamma_n} B\left(b_m, \frac{e^{-|a_m|^{\beta}}}{3}\right) = B\left(b_n, \frac{e^{-|a_{l_n}|^{\beta}}}{3}\right), \quad l_n = \min\{m : m \in \Gamma_n\}. \tag{3.8}$$

Relation (3.8) implies that S_2 can be rewritten as an infinite union of non-overlapping disks. Now we are ready to state the extension of Levinsons result.

Theorem 3.1. *Let* $\mathbf{A}=\{a_n\}_{n=1}^{\infty}$ *be a complex sequence satisfying* $\mathbf{A}\in \mathbf{L}(\mathbf{c},\mathbf{D})$ *and* $\arg a_n \mapsto 0$ *as* $n \to \infty$. *Let* $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ *and let* (λ, μ) *be its* **reordering**. *Then the entire function*

$$G(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right)^{\mu_n} \tag{3.9}$$

satisfies for every $\epsilon > 0$ *as* $r \to \infty$ *the following:*

$$|G(re^{i\theta})| = O(exp\{\pi r(D|\sin\theta| + \epsilon)\}) \tag{3.10}$$

and whenever $re^{i\theta} \notin S_2$

$$\frac{1}{|G(re^{i\theta})|} = O(exp\{\pi r(-D|\sin\theta| + \epsilon)\}). \tag{3.11}$$

Furthermore for every $\epsilon > 0$ *as* $n \to \infty$ *one has:*

$$\frac{\mu_n!}{|G^{[\mu_n]}(\lambda_n)|} = O(exp\{\epsilon|\lambda_n|\}). \tag{3.12}$$

If $D = 0$ the previous result holds without the condition $\arg a_n \mapsto 0$.

Theorem 3.2. *Let* $\mathbf{A}=\{a_n\}_{n=1}^{\infty}$ *be a complex sequence so that* $\mathbf{A}\in \mathbf{L}(\mathbf{c},\mathbf{0})$. *Let* $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ *and let* (λ, μ) *be its* **reordering**. *Then* $G(z)$ *as in* (3.9) *satisfies* (3.10), (3.11) *and* (3.12) *with* $D = 0$.

We now recall some basic facts about Taylor-Dirichlet series. Let $\mathbf{B}=\{\lambda_n, \mu_n\}$ be a multiplicity sequence with complex λ_n. Assume the following two properties are satisfied:

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0, \qquad \lim_{n\to\infty} \frac{\mu_n - 1}{\lambda_n} = 0. \tag{3.13}$$

Then according to Valiron [42], the regions of convergence of the Taylor-Dirichlet series $f(z)$ in (1.3) and its two associate series

$$f^*(z) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n z}, \quad f^{**}(z) = \sum_{n=1}^{\infty} A_n z^{\mu_n - 1} e^{-\lambda_n z}, \tag{3.14}$$

are the same. For any point z inside the open convex region, the three series converge absolutely.

If $\mathbf{A} \in \mathbf{L}(\mathbf{c},\mathbf{D})$ is a real positive sequence and $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ is such that $\mathbf{B}=\{b_n\}$ is real positive too, then for the (λ, μ) **reordering** of $\mathbf{B}$ the three series f, f^*, f^{**} as defined in (1.3) and (3.14) have the pre-mentioned properties. Similarly, if instead of a real sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c},\mathbf{D})$ we have a complex sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c},\mathbf{0})$. The claim is proved in Lemma 5.5. We are now ready to present the following strong version of the Fabry Gap Theorem.

Theorem 3.3. *Let* $\mathbf{A} \in \mathbf{L}(\mathbf{c},\mathbf{D})$ *be a real positive sequence for* $\mathbf{D}> 0$. *Let* $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ *so that* $\mathbf{B}=\{b_n\}$ *is real positive too and let* (λ, μ) *be its* **reordering**. *Then any Taylor-Dirichlet series* $f(z)$ *as in* (1.3), *satisfying* (3.1), *has at least one singularity in every interval of length exceeding* $2\pi D$ *on the abscissa of convergence.*

The beauty of this result is evident from the following example

Example 3.1. *Let*

$$\lambda_n = \begin{cases} 98 + \frac{n^2+4n+3}{4}, & n odd \\ 98 + \frac{n^2+2n}{4} + \frac{2}{n}, & n even \end{cases} \qquad \mu_n = \begin{cases} \frac{n+3}{2}, & n odd \\ \frac{n}{2}, & n even. \end{cases} \tag{3.15}$$

Then any Taylor-Dirichlet series $f(z)$ *as in* (1.3), *satisfying* (3.1), *has at least one singularity in every interval of length exceeding* 2π *on the abscissa of convergence.*

Proof : Consider the real positive sequence $\mathbf{A}$=$\{a_n\}_{n=1}^{\infty}$ where $a_n = 98 + n$, and note that $\mathbf{A} \in \mathbf{L}(\mathbf{1}, \mathbf{1})$. Rewrite it as $\bigcup_{n=1}^{\infty}\{a_{n^2+k}\}_{k=0}^{n} \cup \{a_{n^2+k}\}_{k=n+1}^{2n}$.

For any fixed $n \in \mathbf{N}^+$ and all $k = 0, 1, 2, \ldots, n$, define $b_{n^2+k} = a_{n^2+n}$, and for all $j = n+1, n+2, \ldots, 2n$, define $b_{n^2+j} = a_{n^2+n} + 1/n$. Then let $\mathbf{B}$=$\{b_n\}_1^{\infty}$ and note that these terms are not distinct. Furthermore, the non-coinciding terms come *very close* to each other. We claim that $\mathbf{B}$ is an $\mathbf{A}_{\alpha,\beta}$ sequence. Indeed, for $k = 0, 1, 2, \ldots, n$ one gets

$$|b_{n^2+k} - a_{n^2+k}| = |k - n| < (98 + n^2 + k)^{\frac{1}{2}} = (a_{n^2+k})^{\frac{1}{2}}. \tag{3.16}$$

Similarly for $j = n+1, n+2, \ldots, 2n$. Thus (3.4) is satisfied for $\alpha = 1/2$. Also note that for any $k = 0, 1, 2, \ldots, n$ and any $j = n+1, n+2, \ldots, 2n$, one has

$$|b_{n^2+k} - b_{n^2+j}| = \frac{1}{n} > \max\{e^{-|a_{n^2+k}|^{\frac{1}{4}}}, e^{-|a_{n^2+j}|^{\frac{1}{4}}}\}. \tag{3.17}$$

In other words these non-coinciding terms of $\mathbf{B}$ satisfy condition (ii) of Definition (3.4) for $\beta = 1/4$. From these two relations it follows that $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ for $\alpha = 1/2$ and $\beta = 1/4$.

By simple calculations we find that the (λ, μ) reordering of $\mathbf{B}$ in (3.15) is valid. Theorem 3.3 completes the proof. $\diamondsuit$

When $D = 0$ the assumption that the λ_n are real can be dropped. In fact, the set of arguments of the λ_n **need not have** a *finite* number of cluster points in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. We prove the following result:

Theorem 3.4. *Let* $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{0})$ *be a complex sequence so that* $\sup|\arg a_n| \leq \tau < \pi/2$. *Let* $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ *and* (λ, μ) *its* **reordering**. *Then any Taylor-Dirichlet series* $f(z)$ *as in* (1.3)*, satisfying* (3.3)*, has its boundary of convergence as a natural boundary.*

We note that other results concerning the location of singularities of Taylor-Dirichlet series have been derived by Blambert, Parvatham, and Berland (see [9], [10], [11]). A key role in their work is played by the set of zeros of the polynomials $p_{\mu_n}(z)$. In our case this has been avoided.

4 On the excess of complex exponential systems in $L^2(-a,a)$. Proof of the results.

This section is mainly devoted to the proof of Theorem 2.10. But first, we prove its Corollary 2.1 and then present some further results.

Proof of Corollary 2.1 : We consider the case when $q=5$. For other values of q the proof is similar.

Change the five terms μ_n for $-2 \le n \le 2$ into five zeros. By Theorem (2.5) the *excess* is not altered. Next, for every n so that $|n| \ge 1$, keep the terms μ_{5n} fixed and shift vertically the terms $\mu_{5n+\beta}$ for $\beta \in \{-2,-1,1,2\}$, so that their new imaginary part is equal to $\Im \mu_{5n} = \alpha \log|5n|$. Observe that this vertical shifting is bounded, thus from [34] (Th. 17) the *excess* does not change. Write these new terms as $\{5n+\beta+i\alpha\log|5n| : \beta \in \{-2,-1,0,1,2\}\}_{|n|\ge 1}$. We then proceed with a partition as in relations (2.3) and (2.4), that is, for $|n| \ge 1$ keep $5n+i\alpha\log|5n|$ fixed and pair $5n-2+i\alpha\log|5n|$ with $5n+2+i\alpha\log|5n|$ and $5n-1+i\alpha\log|5n|$ with $5n+1+i\alpha\log|5n|$. Then carry out the same shifting as in Example 2.1 to get $5n+i\alpha\log|5n|$ with multiplicity 5. By Theorem 2.10 the result is valid. $\diamondsuit$

Similarly one proves Corollary 2.2.

4.1 Some additional results.

When $\mu = \mathbf{Z}$ the set of integers, a more general result holds, when compared to Theorem 2.10. Although this result might be known to some people, nevertheless, since we could not trace it in the literature we state it here.

Theorem 4.1. *Let $\mathcal{P}_{\mathbf{Z},\delta}$ be a partition for some $\delta > 0$ fixed. Let $\{a_n\}'$ be a bounded two-sided sequence of complex numbers and define ν as in (2.2). Then $E(Z;p,\pi) = E(\nu;p,\pi)$ for all $p \in (1,\infty)$.*

Remark 4.1. *The theorem fails for $L^1(-\pi,\pi)$ and $C[-\pi,\pi]$.*

Two further results will be proved where the condition $\sup|\Im\mu_n| < \infty$ is once more not essential. The first one generalizes the following recent result from [36]:

Theorem 4.2. *Let $\{\mu_n\}_{-\infty}^{\infty}$ be a sequence satisfying $|\mu_n - n| \le c$ for some $c > 0$. Let $\lambda_0 = \mu_0$,*

$$\lambda_n = \mu_n + \alpha, \quad \lambda_{-n} = \mu_{-n} - \beta, \quad n > 0, \tag{4.1}$$

where $\alpha \ge 0$ and $\beta \ge 0$. Then $E(\lambda;2,\pi) \le E(\mu;2,\pi)$.

The authors of [36] asked whether their result remains true, assuming that $|\Im\mu_n| \le c$ and $|\mu_n + \mu_{-n}| \le 2c$, instead of $|\mu_n - n| \le c$. The answer is affirmative, and in fact the assumption that the imaginary parts are bounded is not required. Our result is as follows:

Theorem 4.3. *Let $\mu \in \mathcal{B}$ and assume that for some $c > 0$ the condition $|\mu_n + \mu_{-n}| \le c$ is satisfied for every $n \ge 1$. Assume also the condition $|\Re\mu_n| \ge (\Im\mu_n)^2$ holds for $|n| \ge 1$, and suppose that $E(\mu;2,a)$ is finite for some $a > 0$. Let $\lambda_0 = \mu_0$,*

$$\lambda_n = \mu_n + \alpha, \quad \lambda_{-n} = \mu_{-n} - \beta, \quad n > 0 \tag{4.2}$$

where $\alpha \ge 0$ and $\beta \ge 0$. Then $E(\lambda;2,a) \le E(\mu;2,a)$.

We note that another generalization of Theorem 4.2 with unbounded imaginary parts, is due to A. Boivin and H. Zhong [14].

If we now combine Theorems 2.10 and 4.3, another more interesting result is obtained. The constants α and β may be replaced by a bounded two-sided real sequence $\{\epsilon_n\}'$ subject to the condition

$$\epsilon_{2n-1} + \epsilon_{2n} = \delta_1 \geq 0, \ n \geq 1, \quad \epsilon_{2n+1} + \epsilon_{2n} = \delta_2 \geq 0, \ n \leq -1. \tag{4.3}$$

Theorem 4.4. *Let μ as in Theorem 4.3, and assume that $\mu \in \mathcal{B}'$ as well. Let $\mathcal{P}_{\mu,\delta}$ be a partition with the set $\{\rho_n\}$ finite, and let $\{\epsilon_n\}'$ be a bounded two-sided real sequence satisfying (4.3). Then construct the sequence*

$$\nu = \{\gamma_n + \epsilon_{2n-1}\}_1^{\infty} \cup \{\lambda_n + \epsilon_{2n}\}_1^{\infty} \cup \{\gamma_n - \epsilon_{2n+1}\}_{-1}^{-\infty} \cup \{\lambda_n - \epsilon_{2n}\}_{-1}^{-\infty} \cup \{\rho_n\}.$$

Assuming that $E(\mu; 2, a)$ is finite for some $a > 0$, the relation $E(\nu; 2, a) \leq E(\mu; 2, a)$ holds.

Proof: As usual write $\mu = \{\gamma_n\}' \cup \{\lambda_n\}' \cup \{\rho_n\}$. Then construct a new sequence $\tau = \{\tau_n\}' \cup \{\rho_n\}$ so that

$$\tau_{2n-1} = \gamma_n + \frac{\epsilon_{2n-1} - \epsilon_{2n}}{2}, \quad \tau_{2n} = \lambda_n - \frac{\epsilon_{2n-1} - \epsilon_{2n}}{2}, \quad n \geq 1,$$

$$\tau_{2n+1} = \gamma_n - \frac{\epsilon_{2n+1} - \epsilon_{2n}}{2}, \quad \tau_{2n} = \lambda_n + \frac{\epsilon_{2n+1} - \epsilon_{2n}}{2}, \quad n \leq -1.$$

Since $\{\epsilon_n\}'$ is bounded, the fractions are uniformly bounded also. It follows from Theorem 2.10 that $E(\tau; 2, a) = E(\mu; 2, a)$.

Next, observe that one obtains ν by shifting to the right (left) all the terms of $\{\tau_n\}'$ with positive (negative) index-n, by the same amount δ_1 (δ_2). This holds since $\gamma_n + \epsilon_{2n-1} = \tau_{2n-1} + \delta_1/2$ and $\lambda_n + \epsilon_{2n} = \tau_{2n} + \delta_1/2$ for $n \geq 1$. Similarly $\gamma_n - \epsilon_{2n+1} = \tau_{2n+1} - \delta_2/2$ and $\lambda_n - \epsilon_{2n} = \tau_{2n} - \delta_2/2$ for $n \leq -1$. Then from Theorem 4.3 one has that $E(\nu; 2, a) \leq E(\tau; 2, a)$. The relation $E(\nu; 2, a) \leq E(\mu; 2, a)$ is now obvious. $\diamondsuit$

As a special case of Theorem 4.4, let $\epsilon_{2n-1} = \delta_1$ and $\epsilon_{2n} = 0$. Then only **half** of the terms are shifted and the inequality still holds. We also note that similar results with inequalities, but with **real** sequences, are found in [39] (Th. 1).

The rest of this section is divided into three subsections. Our main result, Theorem 2.10, is proved in subsection 4.3. For its proof a crucial role is played by a meromomorphic function whose properties are discussed in subsection 4.2. In subsection 4.3 we also prove Theorem 4.1, and Theorem 4.3 is proved in subsection 4.4.

4.2 Constructing a meromorphic function, that replaces frequencies.

Throughout this subsection, we assume that $\mu \in \mathcal{B}'$ with $\Im\mu_n \geq 0$ for all $n \in \mathbf{Z}$. For $\delta > 0$ fixed, $\mathcal{P}_{\mu,\delta}$ is the partition of μ, $\mu = \{\gamma_n\}' \cup \{\lambda_n\}' \cup \{\rho_n\}$. For the two-sided bounded sequence of complex numbers $\{a_n\}'$ we construct the sequence ν as in (2.2).

A well known theorem of Plancherel-Polya [44] (Theorem 16, p. 79) states that if a function $F(z)$ of exponential type belongs to $L^p(-\infty, \infty)$, then $F(x - it) \in L^p(-\infty, \infty)$ for any $t \in \mathbf{R}$. Motivated by this, we define for every $t \in (0, \infty)$ the function

$$\prod{}' \frac{\left(1 - \frac{z}{\gamma_n + a_n}\right)\left(1 - \frac{z}{\lambda_n - a_n}\right) e^{\frac{z}{\gamma_n + a_n} + \frac{z}{\lambda_n - a_n}}}{\left(1 - \frac{z - it}{\gamma_n}\right)\left(1 - \frac{z - it}{\lambda_n}\right) e^{\frac{z - it}{\gamma_n} + \frac{z - it}{\lambda_n}}}. \tag{4.4}$$

Standard calculations show that (4.4) defines a meromorphic function of z in the complex plane with poles at $\{\gamma_n + it\} \cup \{\lambda_n + it\}$. Note also that since the exponent of convergence for μ is 1 and $\{a_n\}'$ is bounded, then the series

$$\sum{}' \left(\frac{1}{\gamma_n} - \frac{1}{\gamma_n + a_n} + \frac{1}{\lambda_n} - \frac{1}{\lambda_n - a_n} \right)$$

converges to some $\omega \in \mathbf{C}$. Thus, multiplication of $e^{\omega z}$ with the function in (4.4) for fixed $t \in (0, \infty)$ yields the meromorphic function of z

$$\prod{}' \frac{\left(1 - \frac{z}{\gamma_n + a_n}\right)\left(1 - \frac{z}{\lambda_n - a_n}\right)}{\left(1 - \frac{z - it}{\gamma_n}\right)\left(1 - \frac{z - it}{\lambda_n}\right)} e^{\frac{it}{\gamma_n} + \frac{it}{\lambda_n}}. \tag{4.5}$$

We denote this function by $M(z,t)$ and remark that for some $t = t_0$, $M(z,t_0)$ has a certain upper bound on the real line (see Prop. 4.1) which is very crucial for proving Theorem 2.10. The key to all these is the following:

Lemma 4.1. *There exists a positive t_0 so that for any $n \in \mathbf{Z} \setminus \{\mathbf{0}\}$ and all $x \in \mathbf{R}$ one has*

$$\left| \frac{(\gamma_n + a_n - x)(\lambda_n - a_n - x)}{(\gamma_n - x + it_0)(\lambda_n - x + it_0)} \right| \leq 1. \tag{4.6}$$

Proof: When the $\{a_n'\}$ are imaginary numbers, the proof is rather easy. Thus, we will prove it for the real case, and as a result the complex case follows as well. Let

$$(I) = |(\gamma_n - x + it)(\lambda_n - x + it)|^2, \quad (II) = |(\gamma_n + a_n - x)(\lambda_n - a_n - x)|^2.$$

Denote the quantity $(I) - (II)$ by $g_n(x,t)$. Observe that relation (4.6) is proved as soon as we show that there is some $t = t_0 > 0$, independent of n and x, so that $g_n(x,t_0) \geq 0$ for any $n \in \mathbf{Z} \setminus \{\mathbf{0}\}$ and all $x \in \mathbf{R}$.

One has

$$\begin{aligned}
(I) &= [(\Re\gamma_n - x)^2 + (\Im\gamma_n + t)^2][(\Re\lambda_n - x)^2 + (\Im\lambda_n + t)^2] \\
&= (\Re\gamma_n - x)^2(\Re\lambda_n - x)^2 + (\Im\gamma_n + t)^2(\Im\lambda_n + t)^2 \\
&+ (\Im\gamma_n + t)^2(\Re\lambda_n - x)^2 + (\Im\lambda_n + t)^2(\Re\gamma_n - x)^2 \\
&= (\Re\gamma_n - x)^2(\Re\lambda_n - x)^2 + \omega_n(t) + \tau_n(t)(\Re\gamma_n - x)^2 \\
&+ \sigma_n(t)(\Re\lambda_n - x)^2,
\end{aligned} \tag{4.7}$$

where

$$\omega_n(t) = (\Im\gamma_n + t)^2(\Im\lambda_n + t)^2, \quad \tau_n(t) = (\Im\lambda_n + t)^2, \quad \sigma_n(t) = (\Im\gamma_n + t)^2. \tag{4.8}$$

Similarly

$$\begin{aligned}
(II) &= [(\Re\gamma_n - x + a_n)^2 + (\Im\gamma_n)^2][(\Re\lambda_n - x - a_n)^2 + (\Im\lambda_n)^2] \\
&= [(\Re\gamma_n - x)^2 + 2a_n(\Re\gamma_n - x) + p_n][(\Re\lambda_n - x)^2 - 2a_n(\Re\lambda_n - x) + q_n]
\end{aligned}$$

where

$$q_n = a_n^2 + (\Im\lambda_n)^2, \qquad p_n = a_n^2 + (\Im\gamma_n)^2. \tag{4.9}$$

If we expand the terms we get:

$$\begin{aligned}
(II) &= (\Re\gamma_n - x)^2(\Re\lambda_n - x)^2 - 2a_n(\Re\lambda_n - x)(\Re\gamma_n - x)^2 + q_n(\Re\gamma_n - x)^2 \\
&+ 2a_n(\Re\gamma_n - x)(\Re\lambda_n - x)^2 - 4a_n^2(\Re\gamma_n - x)(\Re\lambda_n - x) \\
&+ 2a_nq_n(\Re\gamma_n - x) + p_n(\Re\lambda_n - x)^2 - 2a_np_n(\Re\lambda_n - x) + p_nq_n \\
&= (\Re\gamma_n - x)^2(\Re\lambda_n - x)^2 + q_n(\Re\gamma_n - x)^2 + p_n(\Re\lambda_n - x)^2 \\
&+ \xi_n(\Re\gamma_n - x)(\Re\lambda_n - x) + 2a_nq_n(\Re\gamma_n - x) - 2a_np_n(\Re\lambda_n - x) \\
&+ p_nq_n
\end{aligned} \tag{4.10}$$

where

$$\xi_n = 2a_n(\Re\lambda_n - \Re\gamma_n) - 4a_n^2.$$

Since $a_n = O(1)$ and $|\lambda_n - \gamma_n| = O(1)$ then the $\sup|\xi_n| < \infty$. From now on we let $t \gg \sup|\xi_n|$.

Since $g_n(x,t) = (I) - (II)$ then from (4.7) and (4.10) one gets

$$g_n(x,t) = [\tau_n(t) - q_n](\Re\gamma_n - x)^2 + [\sigma_n(t) - p_n](\Re\lambda_n - x)^2 + \omega_n(t) + \Upsilon_n(x), \tag{4.11}$$

where

$$\Upsilon_n(x) = -\xi_n(\Re\gamma_n - x)(\Re\lambda_n - x) - 2a_n q_n(\Re\gamma_n - x) + 2a_n p_n(\Re\lambda_n - x) - p_n q_n. \tag{4.12}$$

Observe now that since $\{a_n\}'$ is bounded, then for large t fixed we have

$$\tau_n(t) - q_n \approx t^2 + 2t\Im\lambda_n, \quad \sigma_n(t) - p_n \approx t^2 + 2t\Im\gamma_n. \tag{4.13}$$

Since $t \gg \sup|\xi_n|$, both quantities above are bigger than the $\sup|\xi_n|$, and this implies that the coefficient of x^2 in (4.11) is positive. Thus for t fixed, large enough, $g_n(x,t)$ has a minimum. Our goal is to prove that for t fixed $g_n(x,t)$ is non-negative there, thus everywhere else as well. This suffices to complete the proof.

We differentiate $g_n(x,t)$ with respect to x to get

$$\begin{aligned} g_n'(x,t) &= 2(x - \Re\gamma_n)(t^2 + 2t\Im\lambda_n) + 2(x - \Re\lambda_n)(t^2 + 2t\Im\gamma_n) \\ &- \xi_n(x - \Re\gamma_n) - \xi_n(x - \Re\lambda_n) + 2a_n q_n - 2a_n p_n \\ &= 2(x - \Re\gamma_n)[t^2 + 2t\Im\lambda_n - \xi_n] + 2(x - \Re\lambda_n)(t^2 + 2t\Im\gamma_n - \xi_n) \\ &+ 2a_n q_n - 2a_n p_n. \end{aligned} \tag{4.14}$$

It follows that $g_n'(x,t) = 0$ when

$$\begin{aligned} x &= \frac{a_n(p_n - q_n)}{2t^2 - 2\xi_n + 2t\Im\gamma_n + 2t\Im\lambda_n} + \frac{\Re\gamma_n}{2}\left(\frac{t^2 - \xi_n + 2t\Im\lambda_n}{t^2 - \xi_n + t\Im\gamma_n + t\Im\lambda_n}\right) \\ &+ \frac{\Re\lambda_n}{2}\left(\frac{t^2 - \xi_n + 2t\Im\gamma_n}{t^2 - \xi_n + t\Im\gamma_n + t\Im\lambda_n}\right). \end{aligned} \tag{4.15}$$

Consider now the first fraction. From (4.9) one has

$$\frac{a_n(p_n - q_n)}{2t^2 - 2\xi_n + 2t\Im\gamma_n + 2t\Im\lambda_n} = \frac{a_n(\Im\gamma_n - \Im\lambda_n)(\Im\gamma_n + \Im\lambda_n)}{2t^2 - 2\xi_n + 2t(\Im\gamma_n + \Im\lambda_n)}.$$

Since $a_n = O(1)$, $|\gamma_n - \lambda_n| = O(1)$ and $\xi_n = O(1)$, it follows that for large t the fraction is very *small*. Call this fixed $t > 0$, t_0. Since $t_0 \gg \sup|\xi_n|$, then ξ_n has no effect in the other two fractions of (4.15). All these imply that $g_n'(x,t_0)$ takes its minimum value at $x = x_0$ where

$$x_0 \approx \frac{\Re\gamma_n}{2}\left(\frac{t_0 + 2\Im\lambda_n}{t_0 + \Im\gamma_n + \Im\lambda_n}\right) + \frac{\Re\lambda_n}{2}\left(\frac{t_0 + 2\Im\gamma_n}{t_0 + \Im\gamma_n + \Im\lambda_n}\right). \tag{4.16}$$

This implies that

$$\Re\gamma_n - x_0 \approx \left(\frac{\Re\gamma_n - \Re\lambda_n}{2}\right)\left(\frac{t_0 + 2\Im\gamma_n}{t_0 + \Im\gamma_n + \Im\lambda_n}\right) \tag{4.17}$$

and

$$\Re\lambda_n - x_0 \approx \left(\frac{\Re\lambda_n - \Re\gamma_n}{2}\right)\left(\frac{t_0 + 2\Im\lambda_n}{t_0 + \Im\gamma_n + \Im\lambda_n}\right). \tag{4.18}$$

If $\Im\gamma_n \le t_0$ and $\Im\lambda_n \le t_0$, then one gets

$$\frac{1}{3} \le \frac{t_0 + 2\Im\gamma_n}{t_0 + \Im\gamma_n + \Im\lambda_n} \le 3, \quad \frac{1}{3} \le \frac{t_0 + 2\Im\lambda_n}{t_0 + \Im\gamma_n + \Im\lambda_n} \le 3.$$

If $\Im\gamma_n \geq t_0$ then the relation $\frac{\Im\gamma_n}{2} < \Im\lambda_n < 2\Im\gamma_n$ holds since $\{\gamma_n - \lambda_n\}'$ is bounded and t_0 is large. Similarly if $\Im\lambda_n \geq t_0$. Then one gets

$$\frac{1}{4} < \frac{t_0 + 2\Im\gamma_n}{t_0 + \Im\gamma_n + \Im\lambda_n} < 3, \quad \frac{1}{4} < \frac{t_0 + 2\Im\lambda_n}{t_0 + \Im\gamma_n + \Im\lambda_n} < 5.$$

Either way, substitution in (4.17) and (4.18) yields that $|\Re\gamma_n - x_0| < 3\delta$ and $|\Re\lambda_n - x_0| < 3\delta$ since $|\gamma_n - \lambda_n| \leq \delta$. Then from (4.9) and the boundedness of $\{a_n\}'$ we deduce that there is some positive constant κ so that $\Upsilon_n(x)$ in (4.12) satisfies

$$|\Upsilon_n(x_0)| \leq \kappa + \kappa(\Im\gamma_n)^2 + \kappa(\Im\lambda_n)^2 + (\Im\gamma_n)^2(\Im\lambda_n)^2. \tag{4.19}$$

We now go back to relation (4.11). Observe that

$$g_n(x_0, t_0) \geq \omega_n(t_0) + \Upsilon_n(x_0), \tag{4.20}$$

and from (4.8) one has

$$\omega_n(t_0) \geq (\Im\gamma_n)^2(\Im\lambda_n)^2 + t_0^4 + t_0^2(\Im\gamma_n)^2 + t_0^2(\Im\lambda_n)^2. \tag{4.21}$$

Since t_0 is large, it follows from (4.19) and (4.21) that $\omega_n(t_0) + \Upsilon_n(x_0) > 0$. Thus $g_n(x_0, t_0) > 0$ and this completes the proof. $\diamondsuit$

Proposition 4.1. *There exist constants $A > 0$, $C > 0$ so that $\forall x \in \mathbf{R}$ the meromorphic function $M(z, t_0)$ in* (4.5) *where t_0 is as in Lemma* 4.1, *satisfies*

$$|M(x, t_0)| \leq Ae^{Ct_0}. \tag{4.22}$$

Proof: Let us write $|M(x, t_0)|$ as

$$|M(x, t_0)| = \left| {\prod}' \frac{\gamma_n\lambda_n}{(\gamma_n + a_n)(\lambda_n - a_n)} e^{\frac{it_0}{\gamma_n} + \frac{it_0}{\lambda_n}} \frac{(\gamma_n + a_n - x)(\lambda_n - a_n - x)}{(\gamma_n - x + it_0)(\lambda_n - x + it_0)} \right|. \tag{4.23}$$

Since $\{a_n\}'$ is bounded, $|\gamma_n - \lambda_n| \leq \delta$, and the exponent of convergence for $\{\gamma_n\}'$ and $\{\lambda_n\}'$ is less than or equal to 1, then one deduces that the series

$${\sum}' \frac{a_n^2 + a_n(\gamma_n - \lambda_n)}{(\gamma_n + a_n)(\lambda_n - a_n)}$$

converges absolutely. It follows that the infinite product

$${\prod}' \left| \frac{\gamma_n\lambda_n}{(\gamma_n + a_n)(\lambda_n - a_n)} \right|$$

converges and is bounded above by some positive A. Also

$${\prod}' \left| e^{\left(\frac{it_0}{\gamma_n} + \frac{it_0}{\lambda_n}\right)} \right| = {\prod}' e^{\Re\left(\frac{it_0}{\gamma_n} + \frac{it_0}{\lambda_n}\right)} = e^{t_0 \sum' \frac{\Im\gamma_n}{|\gamma_n|^2} + \frac{\Im\lambda_n}{|\lambda_n|^2}} = e^{Ct_0}$$

for some $C > 0$ since by definition the series converges. Applying Lemma 4.1 gives the upper bound Ae^{Ct_0} for the product in (4.23). $\diamondsuit$

4.3 Proof of Theorems 2.10 and 4.1.

Proof of Theorem 2.10: By Theorem 2.11, in order to derive equivalent systems it suffices to prove that incompleteness of anyone of the two systems implies incompleteness of the other. To achieve this, we need to have *symmetric* conditions with respect to their associated sequences, and in our case this holds since the terms a_n of the sequence causing the perturbations have no $pre-assigned\ argument$. We compare this with Theorem 4.3 where due to the lack of such conditions (α and β are *positive*), we cannot deduce equivalence.

We assume that E_μ is incomplete. This implies the existence of a non-trivial entire function F of exponential type $\sigma \leq a$, which vanishes on some sequence $\tau \supset \mu$ with the properties:
(i) $F \in L^2(-\infty, \infty)$ and so does $F(x - it)$ for all $t \in \mathbf{R}$.
(ii) $F(z) = \int_{-a}^{a} f(t)e^{izt}dt$ for some $f \in L^2(-a, a)$.
(iii) The conjugate diagram of F is a vertical line segment of length 2σ, thus its indicator function satisfies $h_F(\pi/2) + h_F(-\pi/2) = 2\sigma$.
(iv) $\sum \frac{|\Im \tau_n|}{|\tau_n|^2} < \infty$.
(v) $\lim_{r\to\infty} \frac{n_+(r,\phi)}{r} = \lim_{r\to\infty} \frac{n_-(r,\phi)}{r} = \sigma/\pi$ where $n_+(r, \phi)$ and $n_-(r, \phi)$ are the numbers of zeros of F in the sectors $\{z : |z| \leq r,\ |\arg z| \leq \phi\}$ and $\{z : |z| \leq r,\ |\pi - \arg z| \leq \phi\}$ respectively, for $\phi \in (0, \pi)$.

Our goal is to show that there is some function G vanishing on ν with similar properties as F. This will prove incompleteness of E_ν.

For some $d \in \mathbf{C}$ we can write F as

$$F(z) = e^{dz} \prod \left(1 - \frac{z}{w_n}\right) e^{\frac{z}{w_n}} \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{\mu_n}\right)^{k_n} e^{\frac{zk_n}{\mu_n}}, \tag{4.24}$$

where k_n is the multiplicity of μ_n and $\{w_n\} = \tau \setminus \mu$. Note that the set $\{w_n\}$ might be infinite, finite or the empty set. We can also assume that $\Im \tau_n \geq 0$ for all $n \in \mathbf{Z}$. For if $\{\omega_n\} \subset \tau$ and $\Im \omega_n < 0$, then multiplication of F by a Blaschke product which vanishes on $\{\overline{\omega}_n\}$ and has poles on $\{\omega_n\}$ yields a function in $L^2(-\infty, \infty)$ whose zeros are all in the upper half-plane.

Then proceed with the $\mathcal{P}_{\mu,\delta}$ partition and construct ν as in (2.2). Let $M(z, 0)$ be the meromorphic function as in (4.5) for $t = 0$. Consider then $t_0 > 0$ as in Lemma 4.1 and denote by $G(z)$ the function $e^{-idt_0}F(z)M(z, 0)$. Then based on the partition of μ one expresses $G(z)$ as

$$e^{d(z-it_0)} \prod \left(1 - \frac{z}{w_n}\right) e^{\frac{z}{w_n}} \ {\prod}' \left(1 - \frac{z}{\gamma_n + a_n}\right)\left(1 - \frac{z}{\lambda_n - a_n}\right) e^{\frac{z}{\gamma_n} + \frac{z}{\lambda_n}},$$

where the $\{\rho_n\}$ terms have been included in $\{w_n\}$.

Note that μ is replaced by ν and this due to the bounded sequence $\{a_n\}'$. It follows that G is of exponential type as well. For the same reason properties (iv) and (v) do not change which implies the same for (iii). Then we can assume, without loss of generality, that $h_G(\pm\pi/2) = \sigma$. To complete the proof, we have to show that $G \in L^2(-\infty, \infty)$.

From (4.24) and the partition of μ, we may write $\frac{F(x-it_0)}{e^{d(x-it_0)}}$ as

$$\prod \left(1 - \frac{x - it_0}{w_n}\right) e^{\frac{x-it_0}{w_n}} \ {\prod}' \left(1 - \frac{x - it_0}{\gamma_n}\right)\left(1 - \frac{x - it_0}{\lambda_n}\right) e^{\frac{x-it_0}{\gamma_n}} e^{\frac{x-it_0}{\lambda_n}}.$$

Then one gets

$$\left|\frac{G(x)}{F(x - it_0)}\right| = \left|\prod \left(\frac{1 - \frac{x}{w_n}}{1 - \frac{x-it_0}{w_n}}\right) e^{\frac{it_0}{w_n}} {\prod}' \frac{\left(1 - \frac{x}{\gamma_n + a_n}\right)\left(1 - \frac{x}{\lambda_n - a_n}\right)}{\left(1 - \frac{x-it_0}{\gamma_n}\right)\left(1 - \frac{x-it_0}{\lambda_n}\right)} e^{\frac{it_0}{\gamma_n} + \frac{it_0}{\lambda_n}}\right|.$$

But the $\prod'$ function is the meromorphic function $M(x, t_0)$. Thus, from Proposition 4.1 there are positive constants A and C so that

$$\left|\frac{G(x)}{F(x-it_0)}\right| \leq Ae^{Ct_0}\prod\left|\frac{w_n - x}{w_n - x + it_0}\right|\left|e^{\frac{it_0}{w_n}}\right|,$$

for every $x \in \mathbf{R}$. Since $\Im w_n \geq 0$ and $t_0 > 0$, we also have $|w_n - x| < |w_n - x + it_0|$. Combining this with the convergence of the series $\sum \frac{\Im w_n}{|w_n|^2}$, we deduce that

$$|G(x)| \leq \phi(t_0)|F(x - it_0)| \quad \forall x \in \mathbf{R}, \tag{4.25}$$

where ϕ depends only on t_0. This relation implies that $G \in L^2(-\infty, \infty)$. Then by the Paley-Wiener theorem, G admits the integral representation

$$G(z) = \int_{-a}^{a} g(t)e^{izt}dt, \quad g \in L^2(-a, a). \tag{4.26}$$

Since G vanishes on ν, this implies that E_ν in incomplete in $L^2(-a, a)$. ♢

Proof of Theorem 4.1: Let $\mu = \mathbf{Z}$ and $P_{\mathbf{Z},\delta}$ its partition with the term $0 \in \{\rho_n\}$. Let ν be the new sequence and $\nu' = \nu \setminus \{0\}$. Since $\{e^{int}\}_{-\infty}^{\infty}$ is *exact* in $L^2(-\pi, \pi)$, then from Theorem 2.10 one has $E(\nu; 2, \pi) = 0$ as well. But the *excess* is a decreasing function of p and changes at most by 1 (see [44] p. 98 problems 1, 2). Thus
(A) $E(\nu; p, \pi)$ is either 0 or 1 for any $p \in (1, 2)$.
(B) $E(\nu; p, \pi)$ is either 0 or -1 for any $p \in (2, \infty)$.
We will show that in both cases E_ν is *exact*.

Case $1 < p < 2$: Consider the function $F(z) = \frac{\sin \pi z}{z}$. Then $F(x) \in L^p(-\infty, \infty)$ for all $p > 1$ and vanishes exactly on $\mathbf{Z} \setminus \{\mathbf{0}\}$. Let $M(z, 0)$ be the usual meromorphic function and define $G(z)$ as before. Then $G(z)$ is an entire function of exponential type not exceeding π, and vanishes exactly on ν'. As in (4.25) one has that $G(x) \in L^p(-\infty, \infty)$ for all $p > 1$. Consider now any $1 < p_0 < 2$. Then, from [13] (Th. 6.4) G admits the integral representation

$$G(z) = \int_{-a}^{a} g(t)e^{izt}dt, \quad g \in L^{q_0}(-a, a), \quad p_0^{-1} + q_0^{-1} = 1. \tag{4.27}$$

This implies that $E_{\nu'}$ is incomplete in $L^{p_0}(-\pi, \pi)$, thus $E(\nu'; p_0, \pi) \leq -1$. It follows that $E(\nu; p_0, \pi) \leq 0$. Combining this with (A), shows that $E(\nu; p_0, \pi) = 0$.

Case $2 < p < \infty$: Assume $E(\nu; p_0, \pi) = -1$ for some $p_0 \in (2, \infty)$. Thus, there exists a non-trivial $f \in L^{q_0}(-\pi, \pi)$, $p_0^{-1} + q_0^{-1} = 1$, so that

$$H(z) = \int_{-\pi}^{\pi} f(t)e^{izt}dt$$

is an entire function which vanishes **exactly** on ν. The latter holds since if $H(u) = 0$ for some $u \notin \nu$, then $E_\nu \cup \{e^{iut}\}$ is incomplete contradicting the fact that $E(\nu; p_0, \pi) = -1$. Since $q_0 \in (1, 2)$, from [13] (Th. 6.5) one has that H is of exponential type π and $H \in L^{p_0}(-\infty, \infty)$. Thus $H(z) = ke^{cz}z\prod_{\nu_n \in \nu'}(1 - z/\nu_n)e^{\frac{z}{\nu_n}}$ for some constants $k, c \in \mathbf{C}$.

We then consider the usual meromorphic function $M(z, 0)$ but this time with $\{\nu_n + it\}$ as its poles and $\mathbf{Z}$ as its zeros. Define analogously $G(z) = H(z)M(z, 0)$. Then G is an entire function of exponential type, vanishes exactly on $\mathbf{Z}$, and as in (4.25) $G(x) \in L^{p_0}(-\infty, \infty)$. But this implies that $\sin \pi x \in L^{p_0}(-\infty, \infty)$ as well, which is false. Therefore $E(\nu; p_0, \pi) \neq -1$, thus E_ν is *exact*. ♢

The theorem fails for $L^1(-\pi, \pi)$ and $C[-\pi, \pi]$: Consider the system

$$E_\nu = \{e^{int}\}_{-\infty}^{0} \cup \{e^{it(n+ih(-1)^n)}\}_{1}^{\infty}, \quad h > 0,$$

and compare it with the system $\{e^{int}\}_{-\infty}^{\infty}$ which is *exact* for all $1 \leq p < \infty$ and whose *excess* equals -1 in $C[-\pi, \pi]$. From what we have already proved, it follows that the *excess* is unaltered for $1 < p < \infty$. However, in [40] A. Sedletskii proved that the excess of E_ν in $L^1(-\pi, \pi)$ is 1, and in the space $C[-\pi, \pi]$ it is 0. ♢

4.4 Proof of Theorem 4.3.

We will follow very closely the steps of the proof of Theorem 4.2, as given by its authors.

First, we need to make the following simplifications to enable us prove the result:

(a) By Theorem 2.12 replacing the sequence $\mu = \{\mu_n\}$ with $\{\mu_n + d\}$ for any $d \in \mathbf{R}$, does not change the completeness properties of the new system. Thus we replace β with 0 and α with $\alpha_0 = \alpha + \beta$. However, in what follows we treat α as a variable where $0 \leq \alpha \leq \alpha_0$.

(b) Since $\Re\mu_n \mapsto \infty$ as $n \mapsto \infty$ we assume that $\Re\mu_n \geq 1$ for all $n \geq 1$. Then the condition $(\Re\mu_n) \geq (\Im\mu_n)^2$ yields $(\Re\mu_n)^2 \geq (\Im\mu_n)^2$ for all $n \geq 1$ as well.

(c) Since $E(\mu; 2, a)$ is finite, we can assume that E_μ is an *exact* system in $L^2(-a, a)$ by adding or removing a finite number of terms. By Theorem (2.5), we can also assume that $\mu_n \neq 0$ for all $n \in \mathbf{N}$. We then claim that for $\alpha = 0$ the entire function

$$F(z, \alpha) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\mu_n + \alpha}\right)\left(1 - \frac{z}{\mu_{-n}}\right)$$

belongs to $L^2(-\infty, \infty)$. This would not have been true if we had retained the factor corresponding to μ_0. We remark that the convergence of the product is justified since $|\mu_n + \mu_{-n}| = O(1)$.

Let us justify the claim. Since E_μ is *exact*, then

$$\lim_{r\to\infty} \prod_{|\mu_n|\leq r} \left(1 - \frac{z}{\mu_n}\right) \tag{4.28}$$

converges uniformly on compact sets of $\mathbf{C}$ and $\frac{G(z)}{z-\mu_n} \in L^2(-\infty, \infty)$ for any μ_n (see [29]). It is also known that the $\lim_{r\to\infty} \sum_{|\mu_n|\leq r} \frac{1}{\mu_n}$ exists and is finite, say τ. We also note that due to the condition $|\mu_n + \mu_{-n}| = O(1)$, one deduces that the $\lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{\mu_n} + \frac{1}{\mu_{-n}}\right)$ exists. Furthermore, by simple calculations one deduces that the relation $\tau = \lim_{r\to\infty} \sum_{|\mu_n|\leq r} \frac{1}{\mu_n} = \frac{1}{\mu_0} + \sum_{n=1}^{\infty} \left(\frac{1}{\mu_n} + \frac{1}{\mu_{-n}}\right)$ is valid. Thus, the product in (4.28) is equal to

$$e^{-\tau z} \lim_{r\to\infty} \prod_{|\mu_n|\leq r} \left(1 - \frac{z}{\mu_n}\right) e^{\frac{z}{\mu_n}}, \tag{4.29}$$

and also equal to

$$e^{-\tau z} \left(1 - \frac{z}{\mu_0}\right) e^{\frac{z}{\mu_0}} \prod_{n=1}^{\infty} \left(1 - \frac{z}{\mu_n}\right) e^{\frac{z}{\mu_n}} \left(1 - \frac{z}{\mu_{-n}}\right) e^{\frac{z}{\mu_{-n}}}. \tag{4.30}$$

Finally, the relation $\tau = \frac{1}{\mu_0} + \sum_{n=1}^{\infty} \left(\frac{1}{\mu_n} + \frac{1}{\mu_{-n}}\right)$ shows that our claim holds.

Theorem 4.3 follows as soon as we show that $F(x, \alpha_0) \in L^2(-\infty, \infty)$ as well. In other words, the integral $\int_{-R}^{R} |F(x, \alpha_0)|^2 dx$ denoted by $S(R, \alpha_0)$ must converge to a real number as $R \mapsto \infty$. Thus we decompose $S(R, \alpha_0)$ into the form

$$S(R, \alpha_0) = \int_{-R}^{0} |F(x, \alpha_0)|^2 dx + \int_{0}^{\alpha_0+c+1} |F(x, \alpha_0)|^2 dx + \int_{\alpha_0+c+1}^{R} |F(x, \alpha_0)|^2 dx,$$

and observe that the middle integral is finite and independent of R. In order to complete the proof we have to show that the first and third integrals denoted by $I(R, \alpha_0)$ and $III(R, \alpha_0)$ respectively, converge to a real number as $R \mapsto \infty$. Comparison is made with respect to $I(R, 0)$ and $III(R, 0)$ which converge since $F(x, 0) \in L^2(-\infty, \infty)$. As already mentioned we treat α as a variable, with $\alpha \in [0, \alpha_0]$.

We give the proof for $III(R,\alpha_0)$. After the substitution $u = x-\alpha$, we have

$$\begin{aligned} III(R,\alpha) &= \int_{c+1}^{R-\alpha} \prod_{n=1}^{\infty} \left|1-\frac{u+\alpha}{\mu_n+\alpha}\right|^2 \left|1-\frac{u+\alpha}{\mu_{-n}}\right|^2 du \\ &= \int_{c+1}^{R-\alpha} \prod_{n=1}^{\infty} \left|\frac{\mu_n-u}{\mu_{-n}}\right|^2 \left|\frac{\mu_{-n}-u-\alpha}{\mu_n+\alpha}\right|^2 du. \end{aligned} \tag{4.31}$$

Observe that

$$\left|\frac{\mu_{-n}-u-\alpha}{\mu_n+\alpha}\right|^2 = \frac{(\Re\mu_{-n}-u-\alpha)^2+(\Im\mu_{-n})^2}{(\Re\mu_n+\alpha)^2+(\Im\mu_{-n})^2}.$$

Denote the whole fraction by $L_n(u,\alpha)$ and the denominator by $U_n(\alpha)$. For fixed $u \geq c+1$, differentiating $L_n(u,\alpha)$ with respect to α gives

$$\begin{aligned} & U_n^2(\alpha)L_n'(u,\alpha) = -2(\Re\mu_{-n}-u-\alpha)[(\Re\mu_n+\alpha)^2+(\Im\mu_{-n})^2] \\ - \quad & [(\Re\mu_{-n}-u-\alpha)^2+(\Im\mu_{-n})^2](\Re\mu_n+\alpha) = -2(\Re\mu_{-n}-u-\alpha)[(\Re\mu_n+\alpha)^2 \\ + \quad & (\Im\mu_{-n})^2+(\Re\mu_{-n}-u-\alpha)(\Re\mu_n+\alpha)] - 2(\Im\mu_{-n})^2(\Re\mu_n+\alpha) \\ \leq \quad & -2(\Re\mu_{-n}-u-\alpha)[(\Re\mu_n+\alpha)(\Re\mu_n+\Re\mu_{-n}-u)+(\Im\mu_n)^2]. \end{aligned} \tag{4.32}$$

We show now that (4.32) is negative. Since $|\Re\mu_n+\Re\mu_{-n}| \leq c$ and $u \geq c+1$ then $(\Re\mu_n+\alpha)(\Re\mu_n+\Re\mu_{-n}-u) \leq -\Re\mu_n$. Therefore $(\Re\mu_n+\alpha)(\Re\mu_n+\Re\mu_{-n}-u)+(\Im\mu_n)^2 \leq -\Re\mu_n+(\Im\mu_n)^2 \leq 0$, since $\Re\mu_n \geq (\Im\mu_n)^2$. But $\Re\mu_{-n}-u-\alpha < 0$ as well since $u > 0$. Thus (4.32) is negative and the same is true for $L_n'(u,\alpha)$. This implies that for fixed $u \geq c+1$, $L_n(u,\alpha)$ is a decreasing function of α. Thus $L_n(u,\alpha_0) \leq L_n(u,0)$ for all $u \geq c+1$. It then follows that $III(R,\alpha_0)$ converges to a real number as $R \mapsto \infty$.

Similarly we prove it for $I(R,\alpha_0)$ using the conditions $x \leq 0$ and $(\Re\mu_n)^2 \geq (\Im\mu_n)^2$. $\diamondsuit$

5 On a theorem of Norman Levinson and a variation of The Fabry Gap Theorem. Proof of the results.

This section is devoted to the proofs of Theorems 3.1, 3.2, 3.3 and 3.4. The first two are given in subsection 5.3, and the others follow in subsection 5.4. But first, we state and prove some auxiliary results. One of them, Lemma 5.4, is very crucial and its proof is given separately in subsection 5.2.

5.1 Some auxiliary results

Lemma 5.1. *Let* $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ *and* $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$. *Then*

$$\lim_{r\to\infty} \frac{n_B(r)}{r} = D \tag{5.1}$$

where $n_B(r) = \sum_{|b_n|\le r} 1$ *is the counting function of* $\mathbf{B}$.

Proof : Assume that this is not true. Then there exists some $\epsilon > 0$ and a sequence of positive numbers $\{r_k\}_{n=1}^{\infty}$ so that $|n_B(r_k) - Dr_k| > \epsilon r_k$. Assume $n_B(r_k) > (D+\epsilon)r_k$ (similarly for $n_B(r_k) < (D-\epsilon)r_k$), that is, $\sum_{|b_j|\le r_k} 1 > (D+\epsilon)r_k$. On the other hand, $j < (D+\epsilon)|b_j|$ for $j > j_0$ since $n/|b_n| \to D$. Thus, if $|b_j| \le r_k$ then $j < (D+\epsilon)r_k$, in other words, one has $\sum_{|b_j|\le r_k} 1 < (D+\epsilon)r_k$. Therefore we reach a contradiction. $\diamondsuit$

Let Γ_n be as in (3.7). One deduces that if $j \in \Gamma_n$ then $\Gamma_j = \Gamma_n$. We also define $m(n)$ to be the number of terms of Γ_n and we shall refer to $m(n)$ as the pseudo-multiplicity of b_n. In the lemma that follows, we get an upper bound for $m(n)$ with respect to b_n. This bound is used in the proof of Theorem 3.1.

Lemma 5.2. *There exist positive constants* ψ *and* χ *so that for any* n *one has* $m(n) \le \psi|a_n|^{\alpha} \le \chi|b_n|^{\alpha}$.

Proof : First note that the relation $|a_n|/2 \le |b_n| \le 2|a_n|$ holds for all $n > n_0$ since $|a_n - b_n| \le |a_n|^{\alpha}$. Consider now any $j \in \Gamma_n$. Then

$$\begin{aligned} |a_j| &= |(a_j - b_j) + (b_j - b_n) + b_n| \le |a_j|^{\alpha} + |b_n| \\ &\le \frac{|a_j|}{2} + 2|a_n|. \end{aligned}$$

It follows that $|a_j| \le 4|a_n|$. Then one also gets

$$\begin{aligned} |a_n - a_j| &= |(a_n - b_n) + (b_n - b_j) + (b_j - a_j)| \le |a_n|^{\alpha} + |a_j|^{\alpha} \\ &\le 5|a_n|^{\alpha}. \end{aligned}$$

Finally, the spacing condition $|a_n - a_k| \ge c|n-k|$ yields that for any $j \in \Gamma_n$ one has

$$c|j-n| \le 5|a_n|^{\alpha}. \tag{5.2}$$

Since $m(n)$ is the number of terms of Γ_n, then $m(n) \le 2\max\{|j-n| : j \in \Gamma_n\}$. From (5.2) it follows that there exists a positive ψ so that $m(n) \le \psi|a_n|^{\alpha}$. Finally, the relation $|a_n| \le 2|b_n|$ yields a positive χ so that $m(n) \le \chi|b_n|^{\alpha}$. $\diamondsuit$

Similarly, we get an upper bound for the multiplicity μ_n of the point λ_n in the (λ, μ) reordering of $\mathbf{B}$. This bound is used in the proof of the variation of the Fabry Gap Theorem.

Lemma 5.3. *For any* n *one has* $\mu_n \le \chi|\lambda_n|^{\alpha}$.

Proof : Let $\lambda_n = b_k$ for some $k \in N^+$. From the previous lemma we know that $m(k) \le \chi|b_k|^\alpha$ for some $\chi > 0$. But the pseudo-multiplicity $m(k)$ of b_k is the multiplicity μ_n of λ_n. Thus, one obtains the relation $\mu_n \le \chi|\lambda_n|^\alpha$. $\diamondsuit$

We introduce now the meromorphic function $M(z)$ through which we replace an $\mathbf{L}(\mathbf{c}, \mathbf{D})$ sequence by a multiplicity sequence.

Given sequences $\mathbf{A}$=$\{a_n\}$ and $\mathbf{B}$=$\{b_n\}$, where $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$, we denote by $M(z)$ the infinite product

$$M(z) = \prod_{n=1}^{\infty} \left(\frac{1 - \frac{z^2}{b_n^2}}{1 - \frac{z^2}{a_n^2}} \right). \tag{5.3}$$

Standard calculations show that $M(z)$ converges absolutely and uniformly on compact subsets of $\mathbf{C}\backslash\{\pm a_n\}$. Thus it defines a meromorphic function in the complex plane. The possibility of several terms of $\{\pm\mathbf{B}\}$ to coincide with a particular a_k is not excluded. Thus, this a_k is not a pole. However, it makes no harm to keep $M(z)$ in the form as in (5.3).

Our goal is to find upper and lower bounds of $|M(z)|$ outside disks whose centers are the elements of $\{\pm\mathbf{A}\}$ and $\{\pm\mathbf{B}\}$ respectively. This is done in the following lemma which is very crucial for proving Theorems 3.1 and 3.2. The proof of this lemma will occupy subsection 5.2.

Lemma 5.4. *Let $M(z)$ be the meromorphic function as in* (5.3) *and let S_1, S_2 be the two systems defined in* (3.5) *and* (3.6)*, respectively. Then for every $\epsilon > 0$ as $r \to \infty$ one has*

$$|M(re^{i\theta})| = O(e^{\epsilon r}) \text{ whenever } re^{i\theta} \notin S_1, \tag{5.4}$$

$$\frac{1}{|M(re^{i\theta})|} = O(e^{\epsilon r}) \text{ whenever } re^{i\theta} \notin S_2. \tag{5.5}$$

Another important lemma, cited already in §3, is the following

Lemma 5.5. *Let $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ be a real positive sequence and let $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ so that $\mathbf{B}$=$\{b_n\}$ is real positive too, with (λ, μ) its* **reordering***. Then the regions of convergence of the three series f, f^*, f^{**} as defined in* (1.3) *and* (3.14) *are the same. For any point z inside the open convex region, the three series converge absolutely. Similarly, if instead of a real sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ we have a complex sequence $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{0})$.*

Proof : We have to show that (3.13) is satisfied. First, note that from Lemma 5.3 one deduces that the right limit of (3.13) is valid. Thus, it remains to verify the left limit.

We claim that

$$|\lambda_n| \ge |a_n|/2, \quad n \ge 1. \tag{5.6}$$

This implies that

$$\frac{\log n}{|\lambda_n|} = \frac{\log n}{n} \frac{n}{|a_n|} \frac{|a_n|}{|\lambda_n|} \le \frac{\log n}{n} \frac{n}{|a_n|} \frac{1}{2} \to 0, \quad n \to \infty, \tag{5.7}$$

since $n/|a_n| \to D$, and we are done.

Let us justify our claim. It is obvious that $|\lambda_1| \ge |a_1|/2$. Assume that $|\lambda_k| \ge |a_k|/2$ for some k. We will prove that $|\lambda_{k+1}| \ge |a_{k+1}|/2$ as well.

Note that there is at least one b_n so that $\lambda_{k+1} = b_n$. If $|\lambda_{k+1}| < |a_{k+1}|/2$ then $|a_n|/2 \le |b_n| = |\lambda_{k+1}| < |a_{k+1}|/2$. Since $|a_n|/2 < |a_{k+1}|$, this implies that $n \le k$ since $|a_n| \le |a_{n+1}|$ for all $n \in \mathbf{N}$, therefore $\lambda_{k+1} \in \{b_m\}_{m=1}^k$. This means that $\{(\lambda_m, \mu_m)\}_{m=1}^k \ne \{b_m\}_{m=1}^k$, thus there is some $b_j \in \{(\lambda_m, \mu_n)\}_{m=1}^k$ with $j \ge k+1$. It follows that

$$\frac{|a_{k+1}|}{2} > |\lambda_{k+1}| \ge |\lambda_k| \ge |b_j| \ge \frac{|a_j|}{2} \ge \frac{|a_{k+1}|}{2}, \tag{5.8}$$

that is, $|a_{k+1}| > |a_{k+1}|$ which is false. Thus $|\lambda_{k+1}| \ge |a_{k+1}|/2$ and this completes the proof. $\diamondsuit$

The last lemma in this section is needed for Theorem 3.4.

Lemma 5.6. *Let* $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{0})$ *be a complex sequence and let* $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$ *with* (λ, μ) *its* **reordering**. *If the series* $f^*(z)$ *in* (1.3) *converges absolutely at* z_0*, then it converges absolutely for any point* z *which lies inside the region*

$$\Omega = \left\{ z : \frac{|\Im(z - z_0)|}{\Re(z - z_0)} \leq \frac{1}{\tan\tau},\ \Re(z - z_0) > 0 \right\}, \tag{5.9}$$

where $\tau = \sup|\arg\lambda_n|$.

Proof : If $z \in \Omega$ then one has $-\frac{\Re(z-z_0)}{\tan\tau} \leq \Im(z - z_0) \leq \frac{\Re(z-z_0)}{\tan\tau}$, and this implies that $-\frac{\Re(z-z_0)}{|\tan\theta_n|} \leq \Im(z-z_0) \leq \frac{\Re(z-z_0)}{|\tan\theta_n|}$, $\theta_n = \arg\lambda_n$, since $|\tan\theta_n| \leq \tan\tau$ for all $n \in \mathbf{N}$. Thus, we have $-\frac{\Re(z-z_0)\Re\lambda_n}{|\Im\lambda_n|} \leq \Im(z - z_0) \leq \frac{\Re(z-z_0)\Re\lambda_n}{|\Im\lambda_n|}$ and this implies that $-\Re\lambda_n\Re(z - z_0) + \Im\lambda_n\Im(z - z_0) \leq 0$ for all $n \in \mathbf{N}$. Therefore, if $z \in \Omega$ then $|A_n e^{-\lambda_n z}| \leq |A_n e^{-\lambda_n z_0}|$ for all $n \in \mathbf{N}$ and the result follows. $\diamondsuit$

5.2 On the lower and upper bounds of the infinite product $M(z)$

First, we need to factorize the meromorphic function $M(z)$ into a product of six factors. This factorization is based on an involved partitioning and counting of the terms $\{a_n, b_n\}$ in subdomains of $\mathbf{C}$.

Let $\epsilon > 0$ fixed. Since $|a_n - b_n| \leq |a_n|^\alpha$, $n/|a_n| \to D$ and $|a_n|^{\alpha-1} \to 0$, then there exists a positive integer $n(\epsilon)$ so that for all $n \geq n(\epsilon)$ we have

$$1 - \epsilon \leq \frac{|a_n|}{|b_n|} \leq 1 + \epsilon, \qquad D - \epsilon \leq \frac{n}{|a_n|} \leq D + \epsilon, \qquad |a_n|^{\alpha-1} < \frac{\epsilon^2}{1-\epsilon}. \tag{5.10}$$

Define the function

$$\xi(r, \epsilon) = \inf\{n : |a_n| \geq (1 + \epsilon)r\}, \tag{5.11}$$

in variable r. For $\epsilon > 0$ fixed, $\xi(r, \epsilon)$ is an increasing function of r. Fix r_0 large enough so that $\xi(r_0, \epsilon) > n(\epsilon)$. Then if $\rho > r_0$ one has $\xi(\rho, \epsilon) \geq \xi(r_0, \epsilon) \geq n(\epsilon)$. Recall also that $n/|a_n| \mapsto D$ implies that $n_A(r)/r \mapsto D$ where $n_A(r)$ denotes as usual the counting function of the sequence $\mathbf{A}$. Then for any $r > r_1(\epsilon) = \frac{r_0}{1-\epsilon}$ one has

$$(D - \epsilon)r < n_A(r) < (D + \epsilon)r, \tag{5.12}$$

$$(D - \epsilon)(1 + \epsilon)r < n_A\left((1 + \epsilon)r\right) < (D + \epsilon)(1 + \epsilon)r, \tag{5.13}$$

$$(D - \epsilon)(1 - \epsilon)r < n_A\left((1 - \epsilon)r\right) < (D + \epsilon)(1 - \epsilon)r. \tag{5.14}$$

For sufficiently large z, $|z| > r_1(\epsilon)$, break up sequences $\mathbf{A}$ and $\mathbf{B}$ into five disjoint sets which depend simultaneously on ϵ and z. This partition is as follows:

$$L_1(\epsilon,\, z) = \{(a_n,\, b_n) : n \leq n(\epsilon)\},$$

$$L_2(\epsilon,\, z) = \{(a_n,\, b_n) : |a_n| \leq (1 - \epsilon)|z|\} \setminus L_1,$$

$$L_3(\epsilon,\, z) = \{(a_n,\, b_n) : |a_n - z| < 2|a_n|^\alpha\} \cup \{(a_n,\, b_n) : |a_n + z| < 2|a_n|^\alpha\},$$

$$L_4(\epsilon,\, z) = \{(a_n,\, b_n) : (1 - \epsilon)|z| < |a_n| < (1 + \epsilon)|z|\} \setminus L_3,$$

$$L_5(\epsilon,\, z) = \{(a_n,\, b_n) : |a_n| \geq (1 + \epsilon)|z|\}.$$

We note that for $|z|$ sufficiently large, L_3 is a proper subset of the annulus $A_\epsilon(z) = \{w : (1 - \epsilon)|z| < |w| < (1+\epsilon)|z|\}$. The reason is as follows: let $a_n \in L_3$ and assume that $|a_n - z| < 2|a_n|^\alpha$

(similarly if $|a_n + z| < 2|a_n|^\alpha$). This implies that $|a_n| < 2|z|$ thus $|a_n - z| < 2|z|^\alpha < \epsilon|z|$ for $|z|$ sufficiently large. Therefore $a_n \in A_\epsilon(z)$ and thus L_4 is well defined.

We shall also denote by $L_6(\epsilon, z)$ the set

$$L_2(\epsilon, z) \cup L_3(\epsilon, z) \cup L_4(\epsilon, z).$$

One should note that the set which makes the crucial difference is the set L_3. It allows to make more precise counting for the terms of the factor $P_3(z)$, defined in (5.17).

Define pointwise, the six product factors

$$M_i(z) = \prod_{(a_n,\, b_n) \in L_i} \left(\frac{1 - \frac{z^2}{b_n^2}}{1 - \frac{z^2}{a_n^2}} \right) \quad i = 1, 5, \tag{5.15}$$

$$U(z) = \prod_{(a_n, b_n) \in L_6} \left(\frac{a_n^2}{b_n^2} \right), \tag{5.16}$$

$$P_i(z) = \prod_{(a_n,\, b_n) \in L_i} \left(\frac{b_n^2 - z^2}{a_n^2 - z^2} \right) \quad i = 2, 3, 4, \tag{5.17}$$

and note that

$$\begin{aligned} M(z) &= \prod_{(a_n,\, b_n) \in L_1} \left(\frac{1 - \frac{z^2}{b_n^2}}{1 - \frac{z^2}{a_n^2}} \right) \prod_{(a_n,\, b_n) \in L_5} \left(\frac{1 - \frac{z^2}{b_n^2}}{1 - \frac{z^2}{a_n^2}} \right) \prod_{(a_n,\, b_n) \in L_6} \left(\frac{1 - \frac{z^2}{b_n^2}}{1 - \frac{z^2}{a_n^2}} \right) \\ &= M_1(z) M_5(z) \prod_{(a_n,\, b_n) \in L_6} \left(\frac{a_n^2}{b_n^2} \right) \prod_{(a_n,\, b_n) \in L_6} \left(\frac{b_n^2 - z^2}{a_n^2 - z^2} \right) \\ &= M_1(z) M_5(z) U(z) P_2(z) P_3(z) P_4(z). \end{aligned} \tag{5.18}$$

We remark that (5.18) is the desired factorization of $M(z)$ into a product of six factors.

In order to prove Lemma 5.4, we will obtain upper and lower bounds for each one of these six functions, outside the systems S_1 and S_2 respectively as defined in (3.5) and (3.6). First we prove the following result:

Lemma 5.7. *Let $\mathbf{A} \in \mathbf{L}(\mathbf{c}, \mathbf{D})$ and $\mathbf{B} \in \mathbf{A}_{\alpha,\beta}$. Let $\epsilon > 0$ fixed and consider the function $\xi(r, \epsilon)$ as in* (5.11). *Then for all $\theta \in [0, 2\pi]$ we have*

$$\lim_{r \to \infty} \frac{1}{r} \log \prod_{n=\xi(r,\epsilon)}^{\infty} \left| \left(\frac{1 - \frac{re^{i\theta}}{b_n}}{1 - \frac{re^{i\theta}}{a_n}} \right) \right| = 0. \tag{5.19}$$

The limit is uniform with respect to θ.

Proof : Since $|a_n|^{\alpha-1} \mapsto 0$ and $|b_n - a_n| \le |a_n|^\alpha$, then for some r' fixed we have

$$|b_n| \ge \frac{|a_n|}{2} \quad \textit{and} \quad |a_n|^{\alpha-1} < \frac{\epsilon}{4},$$

for every $n \ge \xi(r', \epsilon)$. Observe that for any r we get

$$|a_n| - r \ge \frac{\epsilon |a_n|}{1+\epsilon},$$

for every $n \ge \xi(r, \epsilon)$. Then for any $r > r'$ the previous inequalities yield

$$\left| \left(\frac{\frac{re^{i\theta}}{a_n} - \frac{re^{i\theta}}{b_n}}{1 - \frac{re^{i\theta}}{a_n}} \right) \right| = \frac{r|b_n - a_n|}{|b_n||a_n - re^{i\theta}|} \le \frac{2r(1+\epsilon)|a_n|^{\alpha-2}}{\epsilon} \le \frac{2|a_n|^{\alpha-1}}{\epsilon} < \frac{1}{2}, \tag{5.20}$$

for every $n \geq \xi(r,\epsilon)$. Applying the inequality $|\log(1+w)| \leq \frac{3|w|}{2}$, which holds when $|w| < \frac{1}{2}$, together with (3.11) yields for any $r > r'$ the following :

$$\left|\log\left(\frac{1-\frac{re^{i\theta}}{b_n}}{1-\frac{re^{i\theta}}{a_n}}\right)\right| \leq \frac{3r(1+\epsilon)|a_n|^{\alpha-2}}{\epsilon}, \tag{5.21}$$

for every $n \geq \xi(r,\epsilon)$. Then, since $\log\left|\left(\frac{1-\frac{re^{i\theta}}{b_n}}{1-\frac{re^{i\theta}}{a_n}}\right)\right| \leq \left|\log\left(\frac{1-\frac{re^{i\theta}}{b_n}}{1-\frac{re^{i\theta}}{a_n}}\right)\right|$, one deduces

$$\begin{aligned} \frac{1}{r}\left|\log\prod_{n=\xi(r)}^{\infty}\left|\left(\frac{1-\frac{re^{i\theta}}{b_n}}{1-\frac{re^{i\theta}}{a_n}}\right)\right|\right| &\leq \frac{1}{r}\sum_{n=\xi(r)}^{\infty}\left|\log\left(\frac{1-\frac{re^{i\theta}}{b_n}}{1-\frac{re^{i\theta}}{a_n}}\right)\right| \\ \leq \quad \frac{1}{r}\sum_{n=\xi(r)}^{\infty}\frac{3r(1+\epsilon)|a_n|^{\alpha-2}}{\epsilon} &= \frac{3(1+\epsilon)}{\epsilon}\sum_{n=\xi(r)}^{\infty}|a_n|^{\alpha-2}. \end{aligned} \tag{5.22}$$

Since $n/|a_n| \mapsto D$ as $n \mapsto \infty$, one gets $\sum_{n=1}^{\infty}|a_n|^{\alpha-2} < \infty$, thus the last term in (5.22) converges to 0 as $r \mapsto \infty$. This implies that the limit in equation (5.19) is valid and one can also see that it is independent of θ. $\diamondsuit$

Proof of Lemma 5.4. Let $0 < \epsilon < 1/2$ fixed. Recall that the factorization of $M(z)$ into the product of six factors holds for all $z : |z| \geq r_1(\epsilon)$ for some r_1. We will get upper and lower bounds for each one of them outside the systems S_1 and S_2 respectively. However, we remark that these systems are crucial only for the factor $P_3(z)$, not for the other ones.

Estimates for $M_1(z)$

Since the number of terms in L_1 is independend of z it follows that

$$\lim_{|z|\to\infty}\frac{1}{|z|}\log\prod_{n=1}^{n(\epsilon)}\left|\left(\frac{1-\frac{z^2}{b_n^2}}{1-\frac{z^2}{a_n^2}}\right)\right| = 0.$$

Thus there exists an $r_2 > r_1$ such that

$$e^{-\epsilon|z|} \leq |M_1(z)| \leq e^{\epsilon|z|} \quad \forall z : |z| \geq r_2. \tag{5.23}$$

Estimates for $M_5(z)$

From Lemma (5.7) we deduce that there exists an $r_3 > r_1$ such that

$$e^{-\epsilon|z|} \leq |M_5(z)| \leq e^{\epsilon|z|} \quad \forall z : |z| \geq r_3. \tag{5.24}$$

Estimates for $U(z)$

Relation (5.13) gives the upper bound $n_A\left((1+\epsilon)|z|\right) < (D+\epsilon)(1+\epsilon)|z|$ for the number of terms of the set L_6 in the open disk $B(0,(1+\epsilon)|z|)$. This and relation (5.10) give for all $z : |z| \geq r_1$

$$(1-\epsilon)^{2(D+\epsilon)(1+\epsilon)|z|} \leq \prod_{(a_n,b_n)\in L_6}\left|\left(\frac{a_n^2}{b_n^2}\right)\right| \leq (1+\epsilon)^{2(D+\epsilon)(1+\epsilon)|z|}. \tag{5.25}$$

Application of the inequalities $1+\epsilon \leq e^{\epsilon}$ and $1-\epsilon \geq e^{-2\epsilon}$ which hold if $\epsilon \in (0,1/2)$, implies for all $z : |z| \geq r_1$

$$e^{-\epsilon|z|4(D+\epsilon)(1+\epsilon)} \leq |U(z)| \leq e^{\epsilon|z|2(D+\epsilon)(1+\epsilon)}. \tag{5.26}$$

Estimates for $P_2(z)$

Relation (5.14) gives the upper bound $n_A((1-\epsilon)|z|) < (D+\epsilon)(1-\epsilon)|z|$ for the number of terms of the set L_2 in the open disk $B(0,(1-\epsilon)|z|)$. Notice also that if $(a_n, b_n) \in L_2$ then the inequality $|a_n| \le (1-\epsilon)|z|$ implies that $|a_n \pm z| > \epsilon|z| \ge \epsilon|a_n|/(1-\epsilon)$. Then one has

$$\left|\frac{b_n - a_n}{a_n \pm z}\right| \le \frac{|a_n|^\alpha}{\epsilon|z|} \le \frac{|a_n|^\alpha}{\frac{\epsilon|a_n|}{1-\epsilon}} \le \epsilon,$$

with relation (5.10) yielding the last inequality, and this gives

$$1-\epsilon \le \left|\frac{b_n \pm z}{a_n \pm z}\right| \le 1+\epsilon.$$

Then for all $z : |z| \ge r_1$ we have the estimate

$$(1-\epsilon)^{2(D+\epsilon)(1-\epsilon)|z|} \le \prod_{(a_n,b_n)\in L_2} \left|\left(\frac{b_n^2 - z^2}{a_n^2 - z^2}\right)\right| \le (1+\epsilon)^{2(D+\epsilon)(1-\epsilon)|z|}, \tag{5.27}$$

and as before we deduce that

$$e^{-\epsilon|z|4(D+\epsilon)(1-\epsilon)} \le |P_2(z)| \le e^{\epsilon|z|2(D+\epsilon)(1-\epsilon)}. \tag{5.28}$$

Estimates for $P_4(z)$

Relations (5.13) and (5.14) yield the upper bound $(2\epsilon D+2\epsilon)|z|$ for the number of terms in L_4. Notice also that if $(a_n, b_n) \in L_4$ then by definition $(a_n, b_n) \notin L_3$. Thus we have $|a_n \pm z| \ge 2|a_n|^\alpha$. Then

$$\left|\frac{b_n - a_n}{a_n \pm z}\right| \le \frac{|a_n|^\alpha}{2|a_n|^\alpha} = \frac{1}{2},$$

and this gives

$$\frac{1}{2} \le \left|\frac{b_n \pm z}{a_n \pm z}\right| \le \frac{3}{2}.$$

Then for all $z : |z| \ge r_1$ we have the estimate

$$\left(\frac{1}{4}\right)^{|z|(2\epsilon D+2\epsilon)} \le \prod_{(a_n,b_n)\in L_4} \left|\left(\frac{b_n^2 - z^2}{a_n^2 - z^2}\right)\right| \le \left(\frac{9}{4}\right)^{|z|(2\epsilon D+2\epsilon)}, \tag{5.29}$$

in other words

$$e^{-\epsilon|z|(\log 4)(2D+2)} \le |P_4| \le e^{\epsilon|z|(\log \frac{9}{4})(2D+2)}. \tag{5.30}$$

Estimates for $P_3(z)$

This time the systems of disks are crucial. We find an upper bound outside S_1 and a lower bound outside S_2.

Without loss of generality, we assume that for any z outside the two systems of disks, if $(a_n, b_n) \in L_3(\epsilon, z)$ then it is relation $|a_n - z| < 2|a_n|^\alpha$ which holds and not $|a_n + z| < 2|a_n|^\alpha$. This implies that $|a_n + z| > |a_n|$. The relation $|a_n - z| < 2|a_n|^\alpha$ also implies that $|a_n| < 2|z|$. Then we have the following:

For z outside $\bigcup_{n=1}^\infty B(\pm a_n, e^{-|a_n|^\beta}/3)$ and for $(a_n, b_n) \in L_3(\epsilon, z)$ we get

$$\left|\frac{b_n - z}{a_n - z}\right| = \left|1 + \frac{b_n - a_n}{a_n - z}\right| \le 1 + \left|\frac{b_n - a_n}{a_n - z}\right| \le 1 + \frac{3|a_n|^\alpha}{e^{-|a_n|^\beta}} < e^{3|z|^\beta}$$

and

$$\left|\frac{b_n + z}{a_n + z}\right| = \left|1 + \frac{b_n - a_n}{a_n + z}\right| \le 1 + \left|\frac{b_n - a_n}{a_n + z}\right| \le 1 + \frac{|a_n|^\alpha}{|a_n|} < 2.$$

For z outside $\bigcup_{n=1}^{\infty} B(\pm b_n, e^{-|a_n|^\beta}/3)$ and for $(a_n, b_n) \in L_3(\epsilon, z)$ we get

$$\left|\frac{b_n - z}{a_n - z}\right| \geq \frac{e^{-|a_n|^\beta}/3}{2|a_n|^\alpha} > \frac{e^{-2|z|^\beta}}{12|z|^\alpha} > e^{-3|z|^\beta}$$

and

$$\left|\frac{b_n + z}{a_n + z}\right| = \left|1 + \frac{b_n - a_n}{a_n + z}\right| \geq \left|1 - \left|\frac{b_n - a_n}{a_n + z}\right|\right| \geq 1 - \frac{|a_n|^\alpha}{|a_n|} > \frac{1}{2}.$$

One also needs to obtain an upper bound for the number of terms in $L_3(\epsilon, z)$. Notice that if $(a_n,\, b_n) \in L_3(\epsilon, z)$ then $|a_n - z| \leq |a_n|^\alpha < 2|z|^\alpha$. In fact, we obtain an upper bound for the number of terms of $\{a_n\}$ in the disk $B(z,\, 2|z|^\alpha)$. We claim that this is $|z|^\gamma$ for some $\gamma \in (\alpha,\, 1)$. Indeed, suppose that for some integer $k > 0$ we have $a_k \in B(z,\, 2|z|^\alpha)$ and $a_n \notin B(z,\, 2|z|^\alpha)$ for every $n < k$. Assume there are integers $k_m \geq 0$ so that $a_{k+k_m} \in B(z,\, 2|z|^\alpha)$. Recalling that $|a_k - a_n| \geq c|k - n|$, the following estimates hold

$$4|z|^\alpha \geq |a_k - a_{k+k_m}| \geq ck_m \geq cm.$$

Thus, $m \leq 4|z|^\alpha/c \leq |z|^\gamma$ for some $\gamma \in (\alpha,\, 1)$. We can take γ close to α so that $\gamma + \beta < 1$ as well. This last result and the above inequalities show that $|P_3(z)| \leq (2e^{3|z|^\beta})^{|z|^\gamma}$ and $|P_3(z)| \geq (\frac{1}{2}e^{-3|z|^\beta})^{|z|^\gamma}$ whenever $z \notin S_1$ and $z \notin S_2$ respectively. Then observe that the relation $(2e^{3|z|^\beta})^{|z|^\gamma} = 2^{|z|^\gamma} e^{3|z|^{\beta+\gamma}} \leq e^{\epsilon|z|}$ holds as $|z| \mapsto \infty$ since $\gamma + \beta < 1$. Thus as $|z| \mapsto \infty$ we get

$$|P_3(z)| \leq e^{\epsilon|z|} \textit{ and } |P_3(z)| \geq e^{-\epsilon|z|} \tag{5.31}$$

whenever $z \notin S_1$ or $z \notin S_2$ respectively.

By a suitable renormalisation of all the estimates we have obtained from the various steps, we conclude the proof of Lemma 5.4. $\diamondsuit$

5.3 Proof of Theorems 3.1 and 3.2

Proof of Theorem 3.1 : Let $F(z)$ be the entire even function vanishing exactly at $\{\pm\mathbf{A}\}$ and let $M(z)$ be the meromorphic function as in relation (5.3). Define $G(z) = F(z)M(z)$. Thus

$$G(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{b_n^2}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right)^{\mu_n}, \tag{5.32}$$

with the second equality valid using the (λ, μ) reordering of $\mathbf{B}$.

From the properties $(1*)$, $(2*)$ of $F(z)$ and Lemma 5.4, we deduce the following:
If $re^{i\theta} \notin \bigcup_{n=1}^{\infty} B(\pm a_n, e^{-|a_n|^\beta}/3)$ one has for every $\epsilon > 0$ as $r \mapsto \infty$

$$G(re^{i\theta}) = O(exp\{\pi r(D|\sin\theta| + \epsilon)\}). \tag{5.33}$$

If $re^{i\theta} \notin \bigcup_{n=1}^{\infty} B(\pm b_n, e^{-|a_n|^\beta}/3)$ and $|re^{i\theta} \pm a_n| \geq c/8\ \forall n \in \mathbf{N}^+$ then one has for every $\epsilon > 0$ as $r \mapsto \infty$

$$\frac{1}{|G(re^{i\theta})|} = O(exp\{\pi r(-D|\sin\theta| + \epsilon)\}). \tag{5.34}$$

From (5.33), (5.34), we will deduce (3.10) and (3.11) respectively. Then we also get (3.12).
Proof of (3.10):
Since $G(z)$ is an entire function, its maximum value over any closed disk $\overline{B}(a_n, e^{-|a_n|^\beta}/3)$ is taken on the boundary. But for all $z \in \partial B(a_n, e^{-|a_n|^\beta}/3)$, relation (5.33) is satisfied, implying the same for the interior.
Proof of (3.11):
Notice that from (5.34) we have to remove the condition $|re^{i\theta} \pm a_n| \geq c/8\ \forall n \in \mathbf{N}^+$. Therefore,

suppose that $\{z_k = r_k e^{i\theta_k}\}$ is a sequence of complex numbers so that for any $k \in \mathbf{N}^+$ we have $|z_k - a_k| < c/8$ and $|z_k \pm b_n| \geq e^{-|a_n|^\beta}/3$ for all $n \in \mathbf{N}^+$. Our goal is to show that (5.34) holds for $1/|G(r_k e^{i\theta_k})|$. This makes the condition $|re^{i\theta} \pm a_n| \geq c/8\ \forall n \in \mathbf{N}^+$ redundant, thus proving (3.11).

Take any $k \in \mathbf{N}^+$. Observe that for an arbitrary $z \in \partial B(a_k, c/8)$ the relation $|z - b_n| \geq e^{-|a_n|^\beta}/3$ might not hold for all $n \in \mathbf{N}^+$. Thus, we consider the larger closed disk $\overline{B}(a_k, c/6)$ and claim that there is a constant $\tau \in (c/8, c/6)$ so that for all $z \in \partial B(a_k, \tau)$, one has $|z - b_n| \geq e^{-|a_n|^\beta}/3$ for all $n \in \mathbf{N}^+$. In other words, every point of this circle satisfies this spacing condition. Needless to say that the condition $|z - a_n| \geq c/8$ for all $n \in \mathbf{N}^+$ is also well satisfied for any $z \in \partial B(a_k, \tau)$.

Let us justify our claim. Note that it suffices to prove it for the set $\{b_n :\ b_n \in \overline{B}(a_k, c/6)\}$.

First, we get an upper bound for the number of zeros of $G(z)$ inside this closed disk. For any such zero there is an integer k_m, not necessarily positive, so that $|a_k - b_{k+k_m}| \leq c/6$. This relation and the usual conditions $|a_n - a_k| \geq c|n-k|$, $|b_{k+k_m} - a_{k+k_m}| \leq |a_{k+k_m}|^\alpha$ and $|a_n| \leq 2|b_n|$, yield

$$\begin{aligned} c|m| \leq c|k_m| \leq |a_k - a_{k+k_m}| &= |(a_k - b_{k+k_m}) + (b_{k+k_m} - a_{k+k_m})| \\ &\leq c/6 + |a_{k+k_m}|^\alpha \\ &\leq 2|b_{k+k_m}|^\alpha \approx 2|a_k|^\alpha. \end{aligned} \tag{5.35}$$

Thus, there are at *most* $4|a_k|^\alpha/c$ zeros inside $\overline{B}(a_k; c/6)$.

Next, consider all the annuli $C_\nu(a_k) \subset \overline{B}(a_k, c/6)$ for $\nu = 0, 1, 2, \ldots, l(a_k)$,

$$C_\nu(a_k) = \left\{ z :\ \frac{c}{8} + (\nu - 1)e^{-|a_k|^{\frac{\beta}{2}}} < |z - a_k| \leq \frac{c}{8} + (\nu + 1)e^{-|a_k|^{\frac{\beta}{2}}} \right\}.$$

In fact, $C_\nu(a_k) \subset C(a_k)$ where $C(a_k)$ is the annulus

$$C(a_k) = \left\{ z :\ c/8 - e^{-|a_k|^{\frac{\beta}{2}}} \leq |z - a_k| \leq c/6 \right\}.$$

Assume that every annulus $C_\nu(a_k) \subset C(a_k)$ contains at least one b_n. Since the width of each $C_\nu(a_k)$ is $2e^{-|a_k|^{\frac{\beta}{2}}}$, then the number of such annuli in $C(a_k)$ is of magnitude $\frac{ce^{|a_k|^{\frac{\beta}{2}}}}{48}$. By assumption, there should be at *least* as many b_n terms in the disk $\bar{B}(a_k; c/6)$. This contradicts (5.35) since $|a_k|^\alpha < e^{|a_k|^{\frac{\beta}{2}}}$ when k is sufficiently large. Thus, there exists some ν_0 so that the intersection of the annulus $C_{\nu_0}(a_k)$ and the sequence $\mathbf{B}$ is the empty set. Then, taking $\tau = c/8 + \nu_0 e^{-|a_k|^{\frac{\beta}{2}}}$, shows that any $z \in \partial B(a_k, \tau)$ satisfies $|z - b_n| \geq e^{-|a_k|^{\frac{\beta}{2}}}$ for all $n \in \mathbf{N}^+$.

Finally note, that for every $b_n \in B(a_k, \tau)$ one gets $|a_n| \geq |a_k|/2$ from (5.35). Thus $e^{-|a_k|^{\frac{\beta}{2}}} \geq e^{-|2a_n|^{\frac{\beta}{2}}} \geq e^{-|a_n|^\beta}/3$. This yields for all $z \in \partial B(a_k, \tau)$ the relation $|z - b_n| \geq e^{-|a_n|^\beta}/3$ for all $n \in N^+$. Our claim is now fully justified.

Assume now that $G(z)$ has zeros in the closed disk $\overline{B}(a_k, \tau)$, and define $Y_k = \{n :\ b_n \in \overline{B}(a_k, \tau)\}$. Then write

$$G(z) = \prod_{n \in Y_k} \left(1 - \frac{z^2}{b_n^2}\right) \prod_{n \notin Y_k} \left(1 - \frac{z^2}{b_n^2}\right) \tag{5.36}$$

For all $n \in Y_k$ since $|z_k - a_k| \leq \tau$ and $|b_n - a_k| \leq \tau$, then one has that $|z_k| \approx |a_k| \approx |b_n|$. Also from (5.35) one has that $|a_n| \leq 2|a_k|$ for all $n \in Y_k$. Combining all these with $|z_k \pm b_n| \geq e^{-|a_n|^\beta}/3$, yields for all $n \in Y_k$

$$\left| \left(1 - \frac{z_k^2}{b_n^2}\right) \right| = \left| \frac{(b_n - z_k)(b_n + z_k)}{b_n^2} \right| \geq \frac{e^{-2|a_n|^\beta}}{9|b_n|^2} \geq \frac{e^{-8|a_k|^\beta}}{9|b_n|^2} \approx \frac{e^{-8|z_k|^\beta}}{9|z_k|^2} \geq e^{-10|z_k|^\beta}.$$

The last inequality holds since $0 < \beta < 1$. Combination with (5.35), yields

$$\frac{1}{\left|\prod_{n\in Y_k}\left(1-\frac{z_k^2}{b_n^2}\right)\right|} \leq e^{\frac{40}{c} r_k^{\alpha+\beta}} = O(e^{\epsilon r_k}). \tag{5.37}$$

Next, note that for all $n \in Y_k$ and all $z \in \partial B(a_k, \tau)$ one gets $|1 - z^2/b_n^2| \leq 1$. This implies that for all $z \in \partial B(a_k, \tau)$ one has:

$$|G(z)| = \left|\prod_{n\in Y_k}\left(1-\frac{z^2}{b_n^2}\right)\right|\left|\prod_{n\notin Y_k}\left(1-\frac{z^2}{b_n^2}\right)\right| \leq \left|\prod_{n\notin Y_k}\left(1-\frac{z^2}{b_n^2}\right)\right|. \tag{5.38}$$

Observe that the right product in (5.36) has no zeros in the closed disk $\overline{B}(a_k, \tau)$. Thus it takes its minimum value on the boundary. This and the previous relation yield

$$\left|\prod_{n\notin Y_k}\left(1-\frac{z_k^2}{b_n^2}\right)\right| \geq \min_{z\in\partial B(a_k,\tau)}\left|\prod_{n\notin Y_k}\left(1-\frac{z^2}{b_n^2}\right)\right| \geq \min_{z\in\partial B(a_k,\tau)} |G(z)|. \tag{5.39}$$

But for all $z \in \partial B(a_k, \tau)$ relation (5.34) holds. Thus, for every $\epsilon > 0$ as $k \mapsto \infty$ we get

$$\frac{1}{\left|\prod_{n\notin Y_k}\left(1-\frac{z_k^2}{b_n^2}\right)\right|} = O(exp\{\pi r_k(-D|\sin\theta_k| + \epsilon)\}). \tag{5.40}$$

This relation and (5.37) yield that $1/|G(r_k e^{i\theta_k})|$ satisfies (5.34).
Proof of (3.12):
Write

$$G(z) = \left(1-\frac{z^2}{\lambda_n^2}\right)^{\mu_n}\prod_{k\neq n}\left(1-\frac{z^2}{\lambda_k^2}\right)^{\mu_k}. \tag{5.41}$$

Then one deduces that

$$\frac{G^{[\mu_n]}(\lambda_n)}{\mu_n!} = \frac{(-2)^{\mu_n}}{\lambda_n^{\mu_n}}\prod_{k\neq n}\left(1-\frac{\lambda_n^2}{\lambda_k^2}\right)^{\mu_k}. \tag{5.42}$$

Let us now obtain an upper bound for $|\lambda_n|^{\mu_n}$. From Lemma (5.2) we get

$$|\lambda_n|^{\mu_n} \leq |\lambda_n|^{\chi|\lambda_n|^{\alpha}} \leq e^{\epsilon|\lambda_n|} \tag{5.43}$$

since $\chi|\lambda_n|^{\alpha}\log|\lambda_n| \leq \epsilon|\lambda_n|$.

Next we get a lower bound for the infinite product in (5.42). We note that for any λ_n there is some b_j so that $\lambda_n = b_j$. Thus, consider the closed disk $\overline{B}(b_j, e^{-|a_{l_j}|^{\beta}}/3)$ where l_j is as in (3.8). We claim that for all z on the boundary one has $z \notin S_2$, in other words we have $|z \pm b_k| \geq e^{-|a_k|^{\beta}}/3$ for all $k \in \mathbf{N}^+$.

Indeed, if $b_k \in \Gamma_j$ the relation is trivial. Assume the opposite and consider the case when $|a_k| \leq |a_{l_j}|$. Then from *(ii)* in Definition 3.4 one has $|b_{l_j} - b_k| \geq e^{-|a_k|^{\beta}}$, thus $|z - b_k| = |(z - b_{l_j}) + (b_{l_j} - b_k)| \geq 2e^{-|a_k|^{\beta}}/3$. Similarly we treat the case $|a_k| > |a_{l_j}|$, and our claim is justified.

Then observe that for all z on the boundary we get that $|(1 - z^2/b_j^2)^{m(j)}| < 1$. Therefore for any such z one has

$$\frac{1}{\left|\prod_{k\neq n}\left(1-\frac{z^2}{\lambda_k^2}\right)^{\mu_k}\right|} < \frac{1}{\left|\prod_{k=1}^{\infty}\left(1-\frac{z^2}{\lambda_k^2}\right)^{\mu_k}\right|} = O(e^{\epsilon|z|}) = O(e^{\epsilon|\lambda_n|}). \tag{5.44}$$

The first equality is valid since every z on the boundary satisfies $z \notin S_2$, thus we apply (3.11). The second holds since $|\lambda_n| \approx |z|$ for all z on the boundary.

Next, observe that the right product in (5.41) has no zeros in the closed disk $\overline{B}(b_j, e^{-|a_{l_j}|^\beta}/3)$. Thus it takes its minimum value on the boundary. Combining this with (5.44) gives

$$\frac{1}{\left|\prod_{k\neq n}\left(1-\frac{\lambda_n^2}{\lambda_k^2}\right)^{\mu_k}\right|} = O(e^{\epsilon|\lambda_n|}). \tag{5.45}$$

Substitution of (5.45) and (5.43) into (5.42), gives (3.12).

This concludes the proof of Theorem 3.1. $\diamondsuit$

Proof of Theorem 3.2 : Similar to the previous one.

5.4 Proof of Theorems 3.3 and 3.4

Proof of Theorem 3.3: We follow the lines of the proof of Theorem XXIX in [30].

Let $f(z)$, $f^*(z)$, $f^{**}(z)$ and A_n as defined in (1.3), (3.14) and (3.2). From Lemma 5.5, the regions of convergence of the three series are the same. Since the λ_n are real positive numbers, we consider the non-trivial case, that is when the three series converge in identical half-planes of the form $\Re z > x_0$, $x_0 \in \mathbf{R}$. With no loss of generality we assume that the abscissa of convergence (ordinary and absolute) is the line $x = 0$. In other words the relation

$$\limsup_{n\to\infty} \frac{\log A_n}{\lambda_n} = 0 \tag{5.46}$$

holds. Thus, all three series converge absolutely and uniformly in any half-plane $x \geq \tau > 0$. One also notes that from (3.1) we have

$$\limsup_{n\to\infty} \frac{\log|c_{n_{\mu_n - 1}}|}{\lambda_n} = 0. \tag{5.47}$$

Suppose now that there exists an interval of length greater than $2\pi D$ on the line $x = 0$ on which $f(z)$ has no singularity. Then with no loss of generality we can also assume that this interval is $-B \leq y \leq B$ where $B > \pi D$. This implies the existence of some $a > 0$ such that $f(z)$ is analytic for $x \geq -a$, $|y| \leq B$. We put

$$\gamma = \arctan \frac{a}{4\pi D}. \tag{5.48}$$

so that $0 < \gamma < \pi/2$ and let

$$b = \frac{(B - \pi D)\tan\gamma}{2}. \tag{5.49}$$

The rest of the proof is broken into three steps. The first two steps are rather straightforward generalizations of parts of the proof of the original Fabry Gap theorem to be found in [30] and are given here for the sake of completeness. Only the third step requires considerable new effort. To facilitate the reader we give an outline for each one of the steps of the proof.

Step 1

Since $f(z)$ is regular in the semi-strip ($x \geq -a$, $|y| \leq B$), then for all $w \in \mathbf{C}$ so that $\Re w < 0$, we define

$$\begin{aligned} H(R, w) &= \int_{-a}^{-a+iB} f(z)e^{wz}dz + \int_{-a+iB}^{b+iB} f(z)e^{wz}dz + \int_{b+iB}^{R+iB} f(z)e^{wz}dz \\ &+ \int_{R+iB}^{R} f(z)e^{wz}dz \end{aligned} \tag{5.50}$$

where $R > b$ and the paths of the integrals are the segments joining the various points. Then we prove that $H(R, w)$ converges as $R \mapsto \infty$, and if we denote this limit by $H(w)$, one has

$$\begin{aligned} H(w) &= \int_{-a}^{-a+iB} f(z)e^{wz}dz + \int_{-a+iB}^{b+iB} f(z)e^{wz}dz \\ &+ \sum_{n=1}^{\infty} e^{(b+iB)(w-\lambda_n)} \sum_{j=0}^{\mu_n-1} c_{n_j} \sum_{l=0}^{j} \frac{(-1)^{l+1}(b+iB)^{j-l}}{(w-\lambda_n)^{l+1}} \frac{j!}{(j-l)!}. \end{aligned} \tag{5.51}$$

But now $H(w)$ is well defined for all $w \in \mathbf{C}\backslash\{\lambda_n\}$. In fact it is analytic in $\mathbf{C}\backslash\{\lambda_n\}$.

Next, we define $J(w) = H(w)G(w)$ where $G(w)$ is the entire even function defined in Theorem 3.1. Then $J(w)$ is an entire function in the complex plane.

Step 2

We prove that for some $\delta > 0$ the relation $|J(\rho e^{\pm i\gamma})| = O\left(e^{-\delta\rho\cos\gamma}\right)$ holds, thus $|e^{\delta w}J(w)| = O(1)$ for $\arg w = \pm\gamma$.

Step 3

We show that $e^{\delta w}J(w)$ is a function of exponential type bounded in the angle $|\arg w| \leq \gamma$. In particular, for real w this implies that $J(w) = O(e^{-\delta w})$, thus $J(\lambda_m) = O(e^{-\delta\lambda_m})$. This eventually yields the relation $|c_{m_{\mu_m-1}}| = O(e^{-\frac{\delta}{2}\lambda_m})$ which contradicts relation (5.47). This will complete the proof of our theorem.

All three steps make use of the convergence of

$$\sum_{n=1}^{\infty} A_n |b+iB|^{\mu_n-1} |e^{-\lambda_n(b+iB)}|, \tag{5.52}$$

due to $f^{**}(b+iB)$. We also need the following result:

Lemma 5.8. *Let $\{L_n\}_{n=1}^{\infty}$ be anyone of the following sequences: $\{\lambda_n\}$, $\{\frac{\lambda_n}{3}\}$, $\{\lambda_n \sin\gamma\}$ or $\{\lambda_n^{\alpha+\eta}\}$ where $\eta > 0$ so that $\alpha + \beta + \eta < 1$. Then*

$$\left| \sum_{j=0}^{\mu_n-1} c_{n_j} \sum_{l=0}^{j} \frac{j!}{(j-l)!} \left(\frac{1}{L_n}\right)^{l+1} (b+iB)^{j-l} \right| < A_n |b+iB|^{\mu_n-1}. \tag{5.53}$$

Proof. Observe that it is enough to prove it for $\{L_n\} = \{\lambda_n^{\alpha+\eta}\}$. For $j = 0, 1, \ldots, \mu_n - 1$, one gets

$$\begin{aligned} & \sum_{l=0}^{j} \frac{j!}{(j-l)!} \left(\frac{1}{\lambda_n^{\alpha+\eta}}\right)^{l+1} < \frac{1}{\lambda_n^{\alpha+\eta}} \sum_{l=0}^{j} \left(\frac{j}{\lambda_n^{\alpha+\eta}}\right)^{l} < \frac{1}{\lambda_n^{\alpha+\eta}} \sum_{l=0}^{j} \left(\frac{\mu_n}{\lambda_n^{\alpha+\eta}}\right)^{l} \\ &= \left(\frac{1}{\lambda_n^{\alpha+\eta}}\right) \left(\frac{\lambda_n^{\alpha+\eta}}{\lambda_n^{\alpha+\eta} - \mu_n}\right) \left[1 - \left(\frac{\mu_n}{\lambda_n^{\alpha+\eta}}\right)^{j+1}\right] < \left(\frac{2}{\lambda_n^{\alpha+\eta}}\right), \end{aligned}$$

since $\mu_n \leq \chi\lambda_n^{\alpha}$ from Lemma 5.3. Then (5.53) follows easily. ♢

We proceed now with the proof of the various steps.

Proof of Step 1: Observe that the first two integrals in (5.50) are independent of R, thus we deal with the other two. We will prove that (5.51) holds as $R \mapsto \infty$.

The absolute convergence of $f(z)$ in the interval $[b+iB, R+iB]$, justifies integrating it term by term to get the following:

$$\begin{aligned}&\int_{b+iB}^{R+iB} f(z)e^{wz}dz = \\ &- \sum_{n=1}^{\infty} e^{(R+iB)(w-\lambda_n)} \sum_{j=0}^{\mu_n-1} c_{n_j} \sum_{l=0}^{j} \frac{(-1)^{l+1}(R+iB)^{j-l}}{(w-\lambda_n)^{l+1}} \frac{j!}{(j-l)!} \\ &+ \sum_{n=1}^{\infty} e^{(b+iB)(w-\lambda_n)} \sum_{j=0}^{\mu_n-1} c_{n_j} \sum_{l=0}^{j} \frac{(-1)^{l+1}(b+iB)^{j-l}}{(w-\lambda_n)^{l+1}} \frac{j!}{(j-l)!}\end{aligned}$$

Denote by $I(R,w)$ the infinite series which depends on R. We will show that $I(R,w) \mapsto 0$ as $R \mapsto \infty$.

Since $\Re w < 0$ then $|w - \lambda_n| > \lambda_n$. It follows from Lemma (5.8) that

$$\left| \sum_{j=0}^{\mu_n-1} c_{n_j} \sum_{l=0}^{j} \frac{(-1)^{l+1}(R+iB)^{j-l}}{(w-\lambda_n)^{l+1}} \frac{j!}{(j-l)!} \right| \leq A_n |R+iB|^{\mu_n-1}.$$

Hence

$$\begin{aligned}|I(R,w)| &\leq \sum_{n=1}^{\infty} A_n |R+iB|^{\mu_n-1} |e^{(R+iB)(w-\lambda_n)}| \\ &\leq e^{-B\Im w} \sum_{n=1}^{\infty} A_n (2R)^{\mu_n-1} e^{-R\lambda_n}.\end{aligned}$$

On the other hand, since $\mu_n/\lambda_n \mapsto 0$ one has $(2R)^{\mu_n-1} < e^{\frac{\lambda_n R}{4}}$ and therefore

$$|I(R,w)| < e^{-B\Im w} \sum_{n=1}^{\infty} 2A_n e^{-\frac{3R}{4}\lambda_n} \leq e^{-B\Im w} e^{-\frac{R}{4}\lambda_1} \sum_{n=1}^{\infty} A_n e^{-R\frac{\lambda_n}{2}}. \tag{5.54}$$

From relation (5.46) one gets that $\limsup_{n\to\infty} (\log A_n)/(\lambda_n/2) = 0$, and this implies that the Dirichlet series $f^{***}(z) = \sum_{n=1}^{\infty} A_n e^{-\frac{\lambda_n}{2}z}$ converges absolutely for any z if $\Re z > 0$. Thus, the series

$$f^{***}(R) = \sum_{n=1}^{\infty} A_n e^{-\frac{\lambda_n}{2}R}$$

is defined for all $R > 0$ and is a positive decreasing function. Therefore there exists some $M > 0$ so that for all $R > 1$ one has $f^{***}(R) < M$. Combining this with (5.54), shows that $I(R,w) \mapsto 0$ as $R \mapsto \infty$.

Similarly one deduces that

$$\lim_{R\to\infty} \int_{R+iB}^{R} f(z)e^{wz}dz = 0. \tag{5.55}$$

Therefore for all w with $\Re w < 0$ one has that the $\lim_{R\to\infty} H(R,w)$ exists. If we denote this by $H(w)$ then $H(w)$ has the form as in (5.51).

Next we prove that $H(w)$ is well defined for all $w \in \mathbf{C}\backslash\{\lambda_n\}$. In fact, we prove that $H(w)$ is analytic in $\mathbf{C}\backslash\{\lambda_n\}$.

Note that the two integrals in (5.51) define analytic functions of w in the whole complex plane. Thus, it remains to prove that the infinite series converges uniformly on any compact subset $K \in \mathbf{C}$ such that $K \cap \{\lambda_n\}_{n=1}^{\infty} = \emptyset$.

Consider such a compact K. Then there exists an $n_0 \in \mathbf{N}$ so that for all $w \in K$ one has $|w - \lambda_n| \geq \lambda_n/2$ for all $n \geq n_0$. Let $q = \max\{|e^{(b+iB)w}| : w \in K\}$. For all $w \in K$ define

$$I_{n_0}(w) = \sum_{n=n_0}^{\infty} e^{(b+iB)(w-\lambda_n)} \sum_{j=0}^{\mu_n - 1} c_{n_j} \sum_{l=0}^{j} \frac{(-1)^{l+1}(b+iB)^{j-l}}{(w-\lambda_n)^{l+1}} \frac{j!}{(j-l)!}.$$

Then for all $w \in K$, it follows from Lemma 5.8 and (5.52) that

$$|I_{n_0}(w)| \leq q \sum_{n=n_0}^{\infty} A_n |b+iB|^{\mu_n - 1} |e^{-\lambda_n(b+iB)}| \mapsto 0, \quad n_0 \mapsto \infty.$$

This implies uniform convergence on K.

<u>Proof of Step 2:</u> Let γ as in (5.48). Then

$$\begin{aligned} H(\rho e^{i\gamma}) &= \int_{-a}^{-a+iB} f(z) e^{z\rho e^{i\gamma}} dz + \int_{-a+iB}^{b+iB} f(z) e^{z\rho e^{i\gamma}} dz \\ &+ \sum_{n=1}^{\infty} e^{(b+iB)(\rho e^{i\gamma} - \lambda_n)} \sum_{j=1}^{\mu_n} c_{n_j} \sum_{l=0}^{\mu_n - 1} \frac{(-1)^{l+1}(b+iB)^{j-l}}{(\rho e^{i\gamma} - \lambda_n)^{l+1}} \frac{j!}{(j-l)!}. \end{aligned}$$

Denote the infinite series by $T(\rho e^{i\gamma})$ and note that $|\rho e^{i\gamma} - \lambda_n| \geq \lambda_n \sin\gamma$. Then from Lemma 5.8 and (5.52), it follows that

$$\left|T(\rho e^{i\gamma})\right| = O(e^{b\rho\cos\gamma - B\rho\sin\gamma}). \tag{5.56}$$

One also notes that

$$\left| \int_{-a}^{-a+iB} + \int_{-a+iB}^{b+iB} \right| = O\left(e^{-a\rho\cos\gamma} + e^{b\rho\cos\gamma - B\rho\sin\gamma}\right). \tag{5.57}$$

Then by choosing the path of integration as the reflection in the real axis of that used in (5.50), we get that (5.56) and (5.57) hold for $\rho e^{-i\gamma}$ as well. Thus

$$H(\rho e^{\pm i\gamma}) = O\left(e^{-a\rho\cos\gamma} + e^{b\rho\cos\gamma - B\rho\sin\gamma}\right). \tag{5.58}$$

From the definition of $J(w)$ above, one deduces that

$$J(w) = G(w) \int_{-a}^{-a+iB} f(z) e^{wz} dz + G(w) \int_{-a+iB}^{b+iB} f(z) e^{wz} dz + Q(w), \tag{5.59}$$

where $Q(w)$ is the entire function defined as

$$G(w) \sum_{n=1}^{\infty} e^{(b+iB)(w-\lambda_n)} \sum_{j=0}^{\mu_n - 1} c_{n_j} \sum_{l=0}^{j} \frac{(-1)^{l+1}(b+iB)^{j-l}}{(w-\lambda_n)^{l+1}} \frac{j!}{(j-l)!}. \tag{5.60}$$

Note that $Q(w)$ is also written as

$$\sum_{n=1}^{\infty} e^{(b+iB)(w-\lambda_n)} \sum_{j=0}^{\mu_n - 1} c_{n_j} \sum_{l=0}^{j} \frac{(-1)^{\mu_n + l + 1}(b+iB)^{j-l} j!}{(j-l)!} (\lambda_n - w)^{\mu_n - l - 1} G_n(w), \tag{5.61}$$

where

$$G_n(w) = \frac{(\lambda_n + w)^{\mu_n}}{\lambda_n^{2\mu_n}} \prod_{k \neq n} \left(1 - \frac{w^2}{\lambda_k^2}\right)^{\mu_k}. \tag{5.62}$$

One observes that combining (3.10) and (5.58), gives for every $\epsilon > 0$

$$|J(\rho e^{\pm i\gamma})| = O\left(e^{-a\rho\cos\gamma+\pi\rho D\sin\gamma+\epsilon\rho} + e^{b\rho\cos\gamma+(\pi D-B)\rho\sin\gamma+\epsilon\rho}\right).$$

From (5.49) one also deduces that $B - \pi D = 2b\cot\gamma$, and since ϵ is arbitrarily small this yields

$$|J(\rho e^{\pm i\gamma})| = O\left(e^{-\frac{a}{2}\rho\cos\gamma+\pi\rho D\sin\gamma} + e^{-\frac{b}{2}\rho\cos\gamma}\right).$$

Relation (5.48) implies that $\pi D\sin\gamma = \frac{1}{4}a\cos\gamma$, thus for $\delta = \frac{1}{4}\min(a,b)$ we have

$$|J(\rho e^{\pm i\gamma})| = O\left(e^{-\delta\rho\cos\gamma}\right).$$

Therefore

$$|e^{\delta w}J(w)| = O(1), \quad \arg w = \pm\gamma. \tag{5.63}$$

Proof of Step 3: We will show that (5.63) holds in the angle $|\Im w| \leq \gamma$. In order to do this, first we prove that $J(w)$ is an entire function of exponential type. From (5.59) observe that it suffices to work with the function $Q(w)$.

Consider some $\eta > 0$ so that $\alpha + \beta + \eta < 1$. For every $w \in \mathbf{C}$ so that $w \notin S_2$ where S_2 is the system defined in (3.6), we partition the sequence $\{\lambda_n\}$ into two sets as follows:

$$A(w) = \{\lambda_n : |w - \lambda_n| > \lambda_n^{\alpha+\eta}\}$$

and

$$B(w) = \{\lambda_n : |w - \lambda_n| \leq \lambda_n^{\alpha+\eta}\}.$$

Then we write

$$Q(w) = Q_A(w) + Q_B(w)$$

where $Q_A(w)$ is defined as

$$Q_A(w) = G(w)\sum_{\lambda_n\in A(w)} e^{(b+iB)(w-\lambda_n)}\sum_{j=0}^{\mu_n-1} c_{n_j}\sum_{l=0}^{j}\frac{(-1)^{l+1}(b+iB)^{j-l}}{(w-\lambda_n)^{l+1}}\frac{j!}{(j-l)!}. \tag{5.64}$$

Similarly one defines $Q_B(w)$.

Consider now $Q_A(w)$. We remark that in this case the condition $w \notin S_2$ plays no role. Note that from Lemma 5.8, we deduce for any $\lambda_n \in A(w)$ that

$$\left|\sum_{j=0}^{\mu_n-1} c_{n_j}\sum_{l=0}^{j}\frac{(-1)^{l+1}(b+iB)^{j-l}}{(w-\lambda_n)^{l+1}}\frac{j!}{(j-l)!}\right| \leq A_n|b+iB|^{\mu_n-1}.$$

Thus

$$|Q_A(w)| \leq |G(w)||e^{(b+iB)w}|\sum_{\lambda_n\in A(w)} A_n|b+iB|^{\mu_n-1}|e^{-\lambda_n(b+iB)}|,$$

and observe that the series is bounded above by the one in (5.52). This implies that $|Q_A(w)| = O(e^{\kappa|w|})$ for some $\kappa > 0$.

Next, we consider $Q_B(w)$. We can also write it as

$$\sum_{\lambda_n\in B(w)} e^{(b+iB)(w-\lambda_n)}\left[\sum_{j=0}^{\mu_n-1} c_{n_j}\sum_{l=0}^{j}\frac{((b+iB)^{j-l}j!}{(-1)^{l+1}(j-l)!}(\lambda_n-w)^{\mu_n-l-1}\right]G_n(w), \tag{5.65}$$

where $G_n(w)$ is defined in (5.62). Note that for any λ_n there is a b_j so that $\lambda_n = b_j$, thus $\mu_n = m(j)$ the pseudo-multiplicity of b_j. Since $w \notin S_2$, then one gets

$$|G_n(w)| = \frac{|G(w)|}{|\lambda_n - w|^{\mu_n}} = \frac{|G(w)|}{|b_j - w|^{m(j)}} \le |G(w)| \left(3e^{|a_j|^\beta}\right)^{m(j)}. \tag{5.66}$$

Fix some $\epsilon > 0$. Then from Lemma 5.2 we get

$$\left(3e^{|a_j|^\beta}\right)^{m(j)} \le \left(3e^{|a_j|^\beta}\right)^{\psi|a_j|^\alpha} = 3e^{\psi|a_j|^{\alpha+\beta}} \le e^{\epsilon|a_j|},$$

with the last inequality valid since $\alpha + \beta < 1$. One also observes that $|a_j| \le 2|b_j| = 2\lambda_n \le 4|w|$ since $\lambda_n \in B(w)$. Thus for all $\lambda_n \in B(w)$ one has $|G_n(w)| \le |G(w)|e^{4\epsilon|w|}$. This implies that there are constants $A' > 0$ and $A'' > 0$, so that for any $w \in \mathbf{C}\backslash S_2$ and all $\lambda_n \in B(w)$ one has

$$|G_n(w)| \le A'e^{A''|w|}. \tag{5.67}$$

Next, observe that for any $\lambda_n \in B(w)$, we have $|\lambda_n - w|^{\mu_n - l - 1} \le (\lambda_n^{\alpha+\eta})^{\mu_n - l - 1}$. Combining this with (5.67) shows that $|Q_B(w)|$ is bounded above by

$$A'e^{A''|w|} \sum_{\lambda_n \in B(w)} \left|e^{(b+iB)(w-\lambda_n)}\right| \left(\lambda_n^{\alpha+\eta}\right)^{\mu_n} \sum_{j=0}^{\mu_n - 1} |c_{n_j}| \sum_{l=0}^{j} \frac{|b+iB|^{j-l} j!}{(j-l)! \left(\lambda_n^{\alpha+\eta}\right)^{l+1}}.$$

Then from Lemma 5.8 we get that

$$|Q_B(w)| \le A'e^{A''|w|} \left|e^{(b+iB)w}\right| \sum_{\lambda_n \in B(w)} \left(\lambda_n^{\alpha+\eta}\right)^{\mu_n} A_n |b+iB|^{\mu_n - 1} \left|e^{-\lambda_n(b+iB)}\right|. \tag{5.68}$$

Note also that from Lemma 5.3 one gets

$$\left(\lambda_n^{\alpha+\eta}\right)^{\mu_n} \le \left(\lambda_n^{\alpha+\eta}\right)^{\chi\lambda_n^\alpha} = \lambda_n^{\chi(\alpha+\eta)\lambda_n^\alpha}| \le e^{\epsilon\lambda_n} \le e^{2\epsilon|w|},$$

and combining this with (5.68) gives

$$|Q_B(w)| \le A'e^{A''|w|} \left|e^{(b+iB)w}\right| e^{2\epsilon|w|} \sum_{\lambda_n \in B(w)} A_n |b+iB|^{\mu_n - 1} \left|e^{-\lambda_n(b+iB)}\right|,$$

with the series bounded above by the one in (5.52). This implies that $|Q_B(w)| = O(e^{\sigma|w|})$ for some $\sigma > 0$, provided $w \notin S_2$.

Since $Q(w) = Q_A(w) + Q_B(w)$ it follows that $|Q(w)| = O(e^{\upsilon|w|})$ for some $\upsilon > 0$, provided $w \notin S_2$. But according to (3.8), S_2 is the union of non-overlapping disks whose radius tends to zero. Since $Q(w)$ is an entire function, its maximum value over any such closed disk is obtained on the boundary. All these imply that $|Q(w)| = O(e^{\upsilon|w|})$ for all w. It then follows that $J(w)$ is an entire function of exponential type. Combining this result with relation (5.63) and a Phragmen-Lindelof theorem [30] (Th. C, p. 243), it yields

$$|e^{\delta w} J(w)| = O(1), \quad |\arg w| \le \gamma. \tag{5.69}$$

In particular, for real w this implies that $J(w) = O(e^{-\delta w})$, thus

$$J(\lambda_m) = O(e^{-\delta\lambda_m}). \tag{5.70}$$

Note also that from (5.59) and (5.60), one deduces that

$$J(\lambda_m) = \frac{c_{m_{\mu_m - 1}} 2^{\mu_m} (\mu_m - 1)!}{\lambda_m^{\mu_m}} \prod_{k \ne m} \left(1 - \frac{\lambda_m^2}{\lambda_k^2}\right)^{\mu_k}. \tag{5.71}$$

Then from (5.42) we can write

$$c_{m_{\mu_m-1}} = J(\lambda_m)\left[\frac{\mu_m!}{G^{[\mu_m]}(\lambda_m)}\right]\left[\frac{1}{(-1)^{\mu_m}(\mu_m-1)!}\right]. \tag{5.72}$$

If we now apply (3.12) and (5.70) to (5.72), it yields for every $\epsilon > 0$

$$|c_{m_{\mu_m-1}}| = O(e^{(-\delta+\epsilon)\lambda_m}). \tag{5.73}$$

Since ϵ is arbitrary we get that

$$|c_{m_{\mu_m-1}}| = O(e^{-\frac{\delta}{2}\lambda_m}). \tag{5.74}$$

But this contradicts relation (5.47), and this completes the proof of our theorem. ♢

Proof of Theorem 3.4: Let us assume that the boundary of convergence is not a natural boundary. That is, for some point z_0 on the boundary the series can be continued analytically in a disk $B(z_0, r)$ for some $r > 0$. For convenience, we replace $f(z)$ by $g(z)$ where

$$g(z) = \sum_{n=1}^{\infty} p_{\mu_n}(z)e^{-\lambda_n z_0}e^{-\lambda_n z}$$

thus $z = 0$ replaces z_0. Similarly we replace $f^*(z)$ by $g^*(z)$ such that $g^*(z) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n z_0}e^{-\lambda_n z}$.

Since $z = 0$ is a boundary point, then for every $r' < r$ the disk $B(0, r')$ contains points z so that $g(z)$ converges absolutely. From Lemma 5.6 this implies that $g(z)$ converges absolutely on every point of the arc $S = \{z : |z| = r, |\arg z| < \frac{1}{\tan\tau}\}$. The same holds for any point z in the region $W = \{z : |z| \geq r, |\arg z| < \frac{1}{\tan\tau}\}$. Then, choose some point $s \in S$ so that $\Im s = \Delta > 0$ and fix positive constants a and b so that $a < r < b$. We remark that the rest of the proof is similar to the previous proof of Theorem 3.3. To facilitate the reader we have kept the same notation.

We break the proof into three steps as done previously.

For Step 1, let $R > b$ and $\Re w < 0$. Then define

$$\begin{aligned} H(R, w) &= \int_{-a}^{-a+i\Delta} g(z)e^{wz}dz + \int_{-a+i\Delta}^{b+i\Delta} g(z)e^{wz}dz + \int_{b+i\Delta}^{R+i\Delta} g(z)e^{wz}dz \\ &+ \int_{R+i\Delta}^{R} g(z)e^{wz}dz \end{aligned} \tag{5.75}$$

where the paths of the integrals are the segments joining the various points.

The absolute convergence of $g(z)$ in the intervals $[b + i\Delta, R + i\Delta]$ and $[R + i\Delta, R]$, justifies integration term by term. Letting $R \mapsto \infty$ and using the relation $|w - \lambda_n| \geq |\lambda_n|\cos\tau$ which holds since $\Re w < 0$, we apply Lemma 5.8 which is valid for $L_n = |\lambda_n|\cos\tau$. This yields that

$$\begin{aligned} H(w) &= \int_{-a}^{-a+i\Delta} g(z)e^{wz}dz + \int_{-a+i\Delta}^{b+i\Delta} g(z)e^{wz}dz \\ &+ \sum_{n=1}^{\infty} e^{(b+i\Delta)(w-\lambda_n)} \sum_{j=0}^{\mu_n-1} c_{n_j}e^{-\lambda_n z_0} \sum_{l=0}^{j} \frac{(-1)^{l+1}(b+i\Delta)^{j-l}}{(w-\lambda_n)^{l+1}} \frac{j!}{(j-l)!}, \end{aligned}$$

is an analytic function in $\mathbf{C}\backslash\{\lambda_n\}$. If we compare this relation with (5.51) we note that this time we have the term $c_{n_j}e^{-\lambda_n z_0}$ instead of just c_{n_j}.

For Step 2, since $\sup|\arg\lambda_n| \leq \tau < \pi/2$ we can choose a constant γ so that $\tau < \gamma < \pi/2$ and $\gamma > \arctan\frac{2b}{\Delta}$. Then we get the following estimates on the ray $\arg w = \gamma$:

$$\left|\int_{-a}^{-a+i\Delta} g(z)e^{\rho e^{i\gamma}z}dz + \int_{-a+i\Delta}^{b+i\Delta} g(z)e^{\rho e^{i\gamma}z}dz\right| = O\left(e^{-a\rho\cos\gamma} + e^{b\rho\cos\gamma - \Delta\rho\sin\gamma}\right) \tag{5.76}$$

and

$$\left|T(\rho e^{i\gamma})\right| = O(e^{b\rho\cos\gamma-\Delta\rho\sin\gamma}), \tag{5.77}$$

where $T(w)$ is the infinite series in the expression of $H(w)$. The estimate for (5.77) holds since $|\rho e^{i\gamma}-\lambda_n| > |\lambda_n|\sin(\gamma-\tau)$ and Lemma 5.8 is valid for $L_n = |\lambda_n|\sin(\gamma-\tau)$. Then by choosing the path of integration as the reflection in the real axis of that used in (5.75), we get that (5.76) and (5.77) hold for $\rho e^{-i\gamma}$ as well. Thus

$$H(\rho e^{\pm i\gamma}) = O\left(e^{-a\rho\cos\gamma} + e^{b\rho\cos\gamma-\Delta\rho\sin\gamma}\right). \tag{5.78}$$

Next, define $J(w) = G(w)H(w)$ where $G(w)$ is the entire function of infraexponential type as in Theorem 3.2. Then for every $\epsilon > 0$ one has

$$|J(re^{\pm i\gamma}| = O\left(e^{-a\rho\cos\gamma+\epsilon\rho} + e^{b\rho\cos\gamma-\Delta\rho\sin\gamma+\epsilon\rho}\right). \tag{5.79}$$

Since $\gamma > \arctan\frac{2b}{\Delta}$ andf ϵ is arbitrarily small, one has that

$$|J(re^{\pm i\gamma}| = O\left(e^{-\delta\rho\cos\gamma}\right), \tag{5.80}$$

for some $\delta > 0$, that is, $|e^{\delta w}J(w)| = O(1)$ for $\arg w = \pm\gamma$.

For Step 3, we get again that $J(w)$ is a function of exponential type, thus by the Phragmen-Lindelof theorem [30] (Th. C, p. 243) one has $|e^{\delta w}J(w)| = O(1)$ for $|\arg w| \leq \gamma$. This implies that $J(\lambda_n) = O(e^{-\delta\Re\lambda_n})$ and since $\Re\lambda_n \geq |\lambda_n|\cos\tau$ then $J(\lambda_n) = O\left(e^{-\delta\cos\tau|\lambda_n|}\right)$. Then, since the term $c_{n_{\mu_n-1}}e^{-\lambda_n z_0}$ is equal to the right-hand side of (5.72), an application of Theorem 3.2 yields

$$\left|c_{n_{\mu_n-1}}e^{-\lambda_n z_0}\right| = O\left(e^{-\frac{\delta\cos\tau}{2}|\lambda_n|}\right). \tag{5.81}$$

Then from (3.3) we get

$$\left|A_n e^{-\lambda_n z_0}\right| = O\left(e^{-\frac{\delta\cos\tau}{4}|\lambda_n|}\right). \tag{5.82}$$

This relation implies that for some $x < 0$ the series $g^*(x)$ converges absolutely. By Lemma 5.6 the same holds for all z in the region $\Omega = \{z : \frac{|\Im(z-x)|}{\Re(z-x)} \leq \frac{1}{\tan\tau},\ \Re(z-x) > 0\}$. But then the point $z = 0$ is an interior point of Ω, a contradiction, since $z = 0$ is a boundary point of the region of convergence of $g(z)$ and $g^*(z)$. $\diamondsuit$

6 Future projects.

In this last section we discuss some possible future projects. The first one deals once more with complete exponential systems in $L^p(-a,a)$.

6.1 Complete exponential systems in $L^p(-a,a)$

In this subsection we discuss briefly two open problems for complete exponential systems in $L^p(-a,a)$. The first one was already mentioned in §2. We know that if the terms of a sequence μ are subjected to bounded pure imaginary perturbations, then the *excess* is preserved in $L^2(-a,a)$ and that the theorem fails for $L^1(-\pi,\pi)$ and $C[-\pi,\pi]$. The problem is still **open** for $p \notin \{1,2\}$.

The next problem deals with non-concentrated sequences $\mu = \{\mu_n, k_n\}$. As we know, A. Sedlestkii usually imposses this condition. The question that we pose is the following: Does μ have to be a non-concentrated sequence in order for its system E_μ to have a finite *excess* in $L^p(-a,a)$? An affirmitive answer to the question, would imply that the multiplicities k_n are necessarily uniformly bounded, a result which is valid at least in the case of Riesz bases, a particular example of *exact* systems.

In the following subsection we discuss the Carleman formulas in complex analysis, whose aim is to restore a function holomorphic in a domain $\mathcal{D}$ by its values on a part M of the boundary $\partial\mathcal{D}$, provided that M is of positive Lebesgue measure.

6.2 Carleman formulas in complex analysis

In the theory of boundary values of holomorphic functions of one complex variable a question was raised about the description of the class of holomorphic in a domain $\mathcal{D}$ functions which are represented using their boundary values by the Cauchy integral formula. The answer was very clear and was obtained for the case of the disk by F. and M. Riesz (1916) and for other domains by V. Smirnov (1932). Their result states that this class of functions coincides with the Hardy class $\mathcal{H}^1(\mathcal{D})$, for the disk, and with the class $E^1(\mathcal{D})$ for other domains.

Definition 6.1. *A function $f(z)$ holomorphic in a domain $\mathcal{D}$ is said to belong to the class $E^p(\mathcal{D})$, $p>0$, if there exists a sequence of curves γ_m in $\mathcal{D}$ converging to $\partial\mathcal{D}$ such that*

$$\int_{\gamma_m} |f(z)|^p |dz| \leq C_1,$$

whereC_1 is independent of m.

During the last years there was a number of research papers devoted to the Carleman formulas for holomorphic functions of one and several variables (their survey can be found in [1]). These formulas solve the problem of the reconstruction of holomorphic functions in the interior points of a domain $\mathcal{D}$ from their values on a subset $M \subset \partial\mathcal{D}$ of positive measure. In order to do this, one needs to construct a *quenching* function ϕ, that is a function which is holomorphic and bounded in $\mathcal{D}$, satisfying two conditions:
(1) $|\phi(z)| = 1$ almost everywhere on $\partial\mathcal{D} \setminus \mathcal{M}$,
(2) $|\phi(z)| > 1$ in $\mathcal{D}$.

The idea of this function was first introduced by T. Carleman and then developed by Goluzin-Krylov. They proved the following:

Theorem 6.1. *If $f \in E^1(\mathcal{D})$ and the set $M \subset \partial\mathcal{D}$ has positive Lebesgue measure, then for any point $z \in \mathcal{D}$ the Carleman formula*

$$f(z) = \lim_{m\to\infty} \frac{1}{2\pi i} \int_M f(\zeta) \left[\frac{\phi(\zeta)}{\phi(z)}\right]^m \frac{d\zeta}{\zeta - z} \tag{6.1}$$

is valid. The convergence in (6.1) *is uniform on compact subsets in $\mathcal{D}$.*

Therefore, naturally arises the following problem:
Can we describe the class of holomorphic functions that are represented by Carleman formulas as in (6.1)?

L. Aizenberg, A. Tumanov and A. Vidras (see [2]) conjectured that a necessary and sufficient condition for a holomorphic function f to be representable by Carleman formulas over the set M is that f must belong to the class $\mathcal{H}^1$ near the set M. In other words, we must have $f \in E^1(W_n)$, where $\{W_n\}$ is an Ahflors-regular exhaustion of $\mathcal{D}$, that is, $\overline{M_n} = \overline{\partial W_n \cap M} \subset M$.

We note that positive results were obtained in [2] for the bisected disk, that is a simply connected domain $\Omega \subset \mathcal{D}$, $0 \neq \Omega$, whose boundary consists of an arc of the unit circle and of an Ahflors-regular curve M in the unit disk joining two points on $|z| = 1$. Similar results were obtained in [3] for a simply connected domain in the right-half plane whose boundary consisted of a vertical segment on the imaginary axis and an Ahflors-regular curve joining the endpoints of the segment and the corresponding Fok-Kuni integral representation formula. We also note that other results were derived in [4] and [15].

At this point we remark that the Carleman integral representation formulas **are not** preserved under conformal mappings. Thus, for different domains we are obliged to attack the problem with a different *quenching* function. This will be the case for the following problem that we shall study: For a special class of bounded, simply connected domains $V \subset C$ with piecewise smooth boundary, for example regular polygons, we shall derive a Carleman formula representing all those holomorphic functions $f \in \mathcal{H}(\mathcal{V})$ from their boundary values (if they exist) on the arc $M \subset \partial V$, whose length satisfies $l(m) < l(\partial V)$ which belong to the Hardy class $\mathcal{H}^1$ near the arc M.

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