# ALGEBRAIC COMPLETE INTEGRABILITY OF LOTKA-VOLTERRA EQUATIONS IN THREE AND FOUR DIMENSIONS 

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

AT
UNIVERSITY OF CYPRUS
NICOSIA, CYPRUS
MARCH 2008
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#### Abstract

In this thesis we examine the algebraic complete integrability of Lotka-Volterra equations in three and four dimensions. We restrict our attention to Lotka-Volterra systems that are defined by a skew symmetric matrix. The goal is a complete classification of such systems. The classification is obtained using Painlevé analysis and more specifically by the use of Kowalevski exponents. The imposition of certain integrality conditions on the Kowalevski exponents gives necessary conditions for the algebraic integrability of the corresponding systems.

Therefore the first step is to impose some conditions on the exponents, i.e., we require that all the Kowalevski exponents be integers. This gives a finite number of values of the parameters satisfying such conditions. This step requires some elementary number theoretic techniques as is usual in this type of classification. In the three-dimensional case the general expressions for the Kowalevski exponents are rational and therefore the number theoretic analysis in not very complicated. On the other hand in the four-dimensional case some exponents appear in radical form and therefore the analysis is more involved. The number of cases in the four-dimensional classification is much higher, as expected.

The second step is to check that the leading behavior of the Laurent series solutions agrees with the weights of the corresponding homogeneous vector field defining the dynamical system. In our case the weights are all equal to one and therefore we must exclude the possibility that some of the Laurent series have leading terms with poles of order greater than one. This is a step usually omitted by some authors due to its complexity, but in this thesis we find that it is necessary to check this in detail. To accomplish this step we use old-fashioned Painlevé Analysis, i.e., Laurent series we assume a solution and try to determine the free parameters. In performing Painlevé Analysis we use the fact that the sum of the variables is always a first integral. Surprisingly a Painlevé analysis does not reveal any additional solution besides the ones already found by using the Kowalevski exponents. As we already mentioned, the four-dimensional case is more involved and in fact we do not present all the details in this thesis. In this classification of the algebraic completely integrable Lotka-Volterra systems in three and four dimensions we discover, as expected, some well known integrable systems like the open and periodic Kac-van Moerbeke systems and some systems connected with simple Lie algebras.

To make sure that our conditions are not only necessary but also sufficient we check that the systems obtained are indeed algebraically completely integrable by checking the


number of free parameters. We also have to point out that our classification is up to isomorphism. In other words, if one system is obtained from another by an invertible change of variables, we do not consider them as different. Using this identification we only have six classes of solutions in three dimensions but over one hundred in four dimensions.

The Lotka-Volterra systems are important in population dynamics, Biology, Chemistry, Economics and many other disciplines. They were proposed independently by Alfred J. Lotka in 1925 and Vito Volterra in 1926. The original system was used to describe population evolution in a hierarchical system of competing individuals. This system has close connection with the Toda lattice. The systems were studied by a great number of authors in their various aspects. i.e. complete integrability and chaotic behavior, Poisson and bihamiltonian formulation, stability of solutions and Darboux polynomials.

## ПЕРІАНЧН













 عival $\pi \partial \lambda \dot{\prime} \pi o \lambda \dot{\prime} \pi \lambda o \alpha \eta$. A $\pi o ́ ~ \tau \eta \nu \alpha \alpha \lambda \lambda \eta o ́ \mu \omega \varsigma$, $\sigma \tau \eta \nu \pi \varepsilon p i ́ \pi \tau \omega \sigma \eta \tau \omega \nu \tau \varepsilon \sigma \sigma \alpha ́ \rho \omega \nu \delta \iota \alpha \sigma \tau \alpha ́ \sigma \varepsilon \omega \nu$,

 $\mu \varepsilon \gamma \alpha \lambda \dot{\tau} \tau \varepsilon \rho \circ \varsigma \alpha \pi^{\prime}$ о́ $\sigma о$ $\alpha \nu \alpha \mu \varepsilon \vee$ о́т $\alpha \nu$.











 $\varepsilon \chi \vartheta \varepsilon ́ \tau \varepsilon \varsigma ~ K o w a l e v s k i . ~ ' O \pi \omega \varsigma ~ \chi \alpha \iota ~ \sigma \tau о ~ \pi р \omega ́ \tau о ~ \beta \eta ́ \mu \alpha, ~ \eta ~ \pi \varepsilon р i ́ \pi \tau \omega \sigma \eta ~ \tau \omega \nu ~ \tau \varepsilon \sigma \sigma \alpha ́ p \omega \nu ~ \delta เ \alpha \sigma \tau \alpha ́ \sigma \varepsilon \omega \nu$










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 $\tau \omega \nu \lambda ט ́ \sigma \varepsilon \omega \nu$ 久 $\alpha \iota \tau \alpha \pi o \lambda \cup \omega ́ \nu \cup \mu \alpha$ Darboux.

## Acknowledgement

This thesis was done under the supervision of Pantelis Damianou. I thank him for introducing me in the theory of the Hamiltonian systems and for the help and the useful suggestions he gave to me. I thank Pol Vanhaecke for his help in a significant point of this thesis. I also thank all the members of the Committee for their corrections which improved my thesis very much.

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## 1 Introduction

In this thesis we consider Lotka-Volterra systems. The most general form of the equations is

$$
\dot{x}_{i}=\varepsilon_{i} x_{i}+\sum_{j=1}^{n} a_{i j} x_{i} x_{j}, \quad i=1,2, \ldots, n
$$

following [22] and [33].
More precisely we examine the algebraic complete integrability of the three- and fourdimensional cases of Lotka-Volterra equations without linear terms $\left(\varepsilon_{i}=0\right)$; we also assume that the matrix $A=\left(a_{i j}\right)$ is skew-symmetric. The basic tools for the required classification are the use of Painlevé analysis, the examination of the eigenvalues of the Kowalevski matrix and other standard Lax pair and Poisson techniques.

The associated Poisson Bracket for the Lotka-Volterra systems is defined by

$$
\left\{x_{i}, x_{j}\right\}=a_{i j} x_{i} x_{j}, \quad i, j=1,2, \ldots, n
$$

One can write the system in the Hamiltonian form $\dot{x}_{i}=\left\{x_{i}, H\right\}$, where $H=\sum x_{i}$. The complete integrability in the three-dimensional case can be easily proved. In addition to the Hamiltonian function $H$, there exists a Casimir function, $F$, since $\operatorname{rank}(\pi)=2$, where $\pi$ is the Poisson matrix, i.e. $\pi_{i, j}=\left\{x_{i}, x_{j}\right\}$. The formula for this Casimir function $F$ is given afterwards. In the four-dimensional case this does not hold generally, but it does under some conditions that are discussed. In this thesis we are interested in the algebraically complete integrability of these systems. The definition of an algebraic completely integrable system, due to Adler and van Moerbeke [4], is given later. We have to point out that algebraic integrability does not imply integrable and vice versa.

The application of Painlevé analysis and especially of the ARS algorithm (see [1], [2], [10], [11], [5]) is useful in calculating the Laurent solution of a system and check if there are $(n-1)$ free parameters. This fact is necessary in proving the algebraic integrability. Some examples from [10] are in detail in order to see how the ARS algorithm can be applied.

Another important tool of this work is the Kowalevski exponents and their properties. The definition of Kowalevski exponents and some relevant results are from the book [4] and [17]. We compute the Kowalevski exponents of some Hamiltonian systems that are related to simple Lie Algebras. In the review article of Goriely [23] one can find many properties of these exponents and some relations of the exponents of the algebraic completely integrable systems. These are necessary conditions for the algebraic integrability of the systems and
so they have an important role for our work. Using the result of a particular Proposition in [4] we see that some number-theoretic techniques are also needed.

We can see that there is a connection between the two basic tools of our work. Actually in [4] we can see an application of the ARS algorithm in weight homogeneous systems that contains the calculation of the Kowalevski exponents.

In order to prove the algebraic complete integrability of a system $(n-1)$ free parameters are needed in the Laurent expansion of its solution. In the cases that Kowalevski exponents can be defined we can find a remark in [4] that is very important to our work because since this assures us that the free parameters appear in a finite number of steps of calculation. So we find the Lotka-Volterra systems that satisfy some necessary conditions of algebraic complete integrability and then using this result we can check which of these systems are algebraic completely integrable by a finite number of steps of calculation and this is how we end up classifying the algebraic completely integrable Lotka-Volterra equations in three and four dimensions.

The three-dimensional case leads to few non isomorphic algebraic integrable systems. The corresponding number in the four-dimensional case becomes bigger. All the known integrable cases appear including the open and periodic Kac-van Moerbeke (KM) systems.

The special case of the KM-system has been used as a model for predator-prey evolution systems in [44]. The Hamiltonian description of this system appeared in the book of Fadeev and Takhtajan [19] and was investigated later by Damianou [14] who looked for the relation between Volterra model and the Toda lattices. The integrability of the KMsystems was established in [28] and [35]. The Volterra's realization of this system can be found in [7] and in a more general form in [6].

Finally we illustrate in this thesis the method of Kowalevski exponents on some integrable systems related to simple Lie algebras. In this thesis we also consider the completely integrable Hamiltonian systems that are defined by a set of the roots of some simple Lie Algebras. Bogoyavlensky in [8] and in [9] constructed integrable Hamiltonian systems connected with simple Lie algebras, generalizing the KM-system. In [15] Damianou and Kouzaris had found a relation between Birkhoff integrable systems starting from Volterra systems. A Birkhoff system is discussed in Subsection 4.2.1.

In the next few subsections we give a brief historical review of Kowalevski exponents and Painleve analysis.

### 1.1 History of Kowalevski exponents

The theory of linear differential equations in the complex plane was first developed by Lazarus Fuchs in 1866. He succeeded in characterizing those differential equations the solutions of which have no essential singurality in the extended complex plane.

Consider the system

$$
\begin{equation*}
\dot{z}=A(t) z, \quad z \in \mathbf{C}^{n} . \tag{1}
\end{equation*}
$$

Fuchs proved that in the case of a regular singular point located at $t=t_{0}$, i.e.

$$
A(t)=\frac{H(t)}{t-t_{0}}
$$

where $H(t)$ is analytic in $t \in \mathbf{C}$, the fundamental solution of (1) can be expressed as a convergent Laurent series if the solutions of the indicial equation

$$
\operatorname{det}(H(t)-\lambda I)=0
$$

are distinct and do not differ by an integer. Later, Frobenius considered the case in which the roots of the indicial equation differ by an integer.

The idea of using Laurent expansion solutions for investigating differential equations motivated two students of Weierstrass, Paul Hoyer and Sophie Kowalevski, to investigate this area. Paul Hoyer [26] studied the system

$$
\frac{d}{d t}\left(\begin{array}{l}
x_{1}  \tag{2}\\
x_{2} \\
x_{3}
\end{array}\right)=\left(\begin{array}{lll}
a_{1} & a_{2} & a_{3} \\
b_{1} & b_{2} & b_{3} \\
c_{1} & c_{2} & c_{3}
\end{array}\right)\left(\begin{array}{c}
x_{2} x_{3} \\
x_{1} x_{3} \\
x_{1} x_{2}
\end{array}\right)
$$

His thesis project was to find the required condition on the parameters so that the Kowalevski exponents (in modern terminology) are integers.
S. Kowalevski [30], before her work on rigid bodies, also considered a similar system, a 3-dimensional Lotka-Volterra system of the form

$$
\frac{d}{d t}\left(\begin{array}{l}
x_{1}  \tag{3}\\
x_{2} \\
x_{3}
\end{array}\right)=\left(\begin{array}{lll}
a_{1} x_{1} & a_{2} x_{1} & a_{3} x_{1} \\
b_{1} x_{2} & b_{2} x_{2} & b_{3} x_{2} \\
c_{1} x_{3} & c_{2} x_{3} & c_{3} x_{3}
\end{array}\right)\left(\begin{array}{c}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right) .
$$

In the present thesis we consider Lotka-Volterra systems defined by a skew-symmetric matrix in three and four dimensions.

### 1.2 Kowalevski's work

The work of Kowalevski in rigid bodies (Bordin prize 1898) is important for two reasons. One is the discovery of a new case which helps constructing a general solution in terms of theta functions of two variables. The other is that she proved that there are no other case for which the solution is single-valued [30, 31].

The motion of a rigid body around a fixed point is ruled by the system,

$$
\begin{array}{ll}
A \dot{\omega}_{1}+(C-B) \omega_{2} \omega_{3}+\left(x_{3} k_{2}-x_{2} k_{3}\right)=0, & \dot{k}_{1}=\omega_{3} k_{2}+\omega_{2} k_{3} \\
B \dot{\omega}_{2}+(A-C) \omega_{3} \omega_{1}+\left(x_{1} k_{3}-x_{3} k_{1}\right)=0, & \dot{k}_{2}=\omega_{1} k_{3}+\omega_{3} k_{1}  \tag{4}\\
C \dot{\omega}_{3}+(B-A) \omega_{1} \omega_{2}+\left(x_{2} k_{1}-x_{1} k_{2}\right)=0, & \dot{k}_{3}=\omega_{2} k_{1}+\omega_{1} k_{2},
\end{array}
$$

depending on six parameters that are the positive components $A, B$ and $C$ of the diagonal momentum $I$ and the real components $x_{1}, x_{2}$ and $x_{3}$ of the vector $\overrightarrow{O M}$ starting from a fixed point $O$ to the mass center $M$. We have three integrals

$$
\begin{align*}
K_{1} & =(I \vec{\Omega}) \cdot \vec{\Omega}-2 \overrightarrow{O M} \cdot \vec{k} \\
& =A \omega_{1}^{2}+B \omega_{2}^{2}+C \omega_{3}^{2}-2\left(x_{1} k_{1}+x_{2} k_{2}+x_{3} k_{3}\right), \\
K_{2} & =(I \vec{\Omega} \cdot \vec{k})=A \omega_{1} k_{1}+B \omega_{2} k_{2}+C \omega_{3} k_{3},  \tag{5}\\
K_{3} & =\vec{k} \cdot \vec{k}=k_{1}^{2}+k_{2}^{2}+k_{3}^{2} .
\end{align*}
$$

A fourth first integral, independent of time, is needed as a sufficient condition to prove integrability. As we mentioned above, Kowalevski's work there were three known such cases:

- The isotropic case with $A=B=C$, for which

$$
\begin{equation*}
K_{4}=\overrightarrow{O M} \cdot \vec{\Omega}=x_{1} \omega_{1}+x_{2} \omega_{2}+x_{3} \omega_{3} . \tag{6}
\end{equation*}
$$

- The Euler - Poinsot case with the fixed point $O$ being the origin $(0,0,0)$ and $M=O$ for which

$$
\begin{equation*}
K_{4}=|I \vec{\Omega}|^{2}=A^{2} \omega_{1}^{2}+B^{2} \omega_{2}^{2}+C^{2} \omega_{3}^{2} \tag{7}
\end{equation*}
$$

- The Lagrange - Poisson case with $A=B$ and $x_{1}=x_{2}=0$ for which

$$
\begin{equation*}
K_{4}=\omega_{3} . \tag{8}
\end{equation*}
$$

For these three cases the general solution is meromorphic [25].
Let now

$$
\begin{equation*}
\omega_{i}=t^{n} \sum_{j=0}^{\infty} \omega_{i j} t^{j}, k_{i}=t^{m} \sum_{j=0}^{\infty} k_{i j} t^{m_{i}+j} \quad \text { with } \prod_{i=1}^{3} \omega_{i 0} k_{i 0} \neq 0 \tag{9}
\end{equation*}
$$

and $\omega_{i j}, k_{i j} \in \mathbf{C}$ for $i=1,2,3$.
The case in which $n=-1$ and $m=-2$ was defined in [30] and the $\omega_{i 0}$ and $k_{i 0}$ are solutions of the simultaneous equations

$$
\begin{gathered}
A \omega_{1,0}+(B-C) \omega_{2,0} \omega_{3,0}+x_{2} k_{3,0}-x_{3} k_{2,0}=0, \\
B \omega_{2,0}+(C-A) \omega_{3,0} \omega_{1,0}+x_{3} k_{1,0}-x_{1} k_{3,0}=0, \\
C \omega_{3,0}+(A-B) \omega_{1,0} \omega_{2,0}+x_{1} k_{2,0}-x_{2} k_{1,0}=0, \\
2 k_{1,0}+\omega_{3,0} k_{2,0}-\omega_{2,0} k_{3,0}=0, \\
2 k_{2,0}+\omega_{1,0} k_{3,0}-\omega_{3,0} k_{1,0}=0, \\
2 k_{3,0}+\omega_{2,0} k_{1,0}-\omega_{1,0} k_{2,0}=0
\end{gathered}
$$

and the linear system for $j \geq 1$ is

$$
\left(\begin{array}{cccccc}
(j-1) A & (C-B) \omega_{3,0} & (C-B) \omega_{2,0} & 0 & x_{3} & -x_{2} \\
(A-C) \omega_{3,0} & (j-1) B & (A-C) \omega_{1,0} & -x_{3} & 0 & x_{1} \\
(B-A) \omega_{2,0} & (B-A) \omega_{1,0} & (j-1) C & x_{2} & -x_{1} & 0 \\
0 & k_{3,0} & -k_{2,0} & j-2 & -\omega_{3,0} & \omega_{2,0} \\
-k_{3,0} & 0 & k_{1,0} & \omega_{3,0} & j-2 & -\omega_{1,0} \\
k_{2,0} & -k_{1,0} & 0 & -\omega_{2,0} & \omega_{1,0} & j-2
\end{array}\right)\left(\begin{array}{l}
\omega_{1, j} \\
\omega_{2, j} \\
\omega_{3, j} \\
k_{1, j} \\
k_{2, j} \\
k_{3, j}
\end{array}\right)+\mathbf{Q}_{j}=0 .
$$

The determinant, $\operatorname{det} \mathbf{P}$, must have five positive zeroes.
If $A, B$ and $C$ are all different and $M \neq O$, there exists a unique solution to the simultaneous equation depending on an arbitrary parameter and the root of an equation of eighth degree, given in [31],

$$
\begin{equation*}
\operatorname{det} \mathbf{P}=A B C(j+1)(j-2)(j-4)\left(j^{2}-j-\mu\right), \tag{10}
\end{equation*}
$$

where $\mu$ is an expression of $A, B, C, x_{1}, x_{2}$ and $x_{3}$.

Kowalevski found the subcase

$$
\begin{equation*}
A=B, \quad\left(x_{1}, x_{2}\right) \neq(0,0), \quad \omega_{1,0}^{2}+\omega_{2,0}^{2}=0, \tag{11}
\end{equation*}
$$

for which the unique solution is

$$
\begin{gathered}
\omega_{1,0}=-\frac{i C}{2\left(x_{1}+i x_{2}\right) \lambda}, \omega_{2,0}=i \omega_{1,0}, \omega_{3,0}=2 i, \quad i^{2}=-1, \\
k_{1,0}=-\frac{2 C}{x_{1}+i x_{2}}, k_{2,0}=i k_{1,0}, \quad k_{3,0}=0,
\end{gathered}
$$

$\operatorname{det} \mathbf{P}=A B C(j+1)(j-2)(j-4)(j+1-2 C / A)(j-2+2 C / A)$,
where $\lambda$ is the solution of the equation

$$
\begin{equation*}
2 C-A-4 \lambda x_{3}=0, \quad \lambda \neq 0 . \tag{12}
\end{equation*}
$$

There exist five positive integer indices if and only if $A=2 C, x_{3}=0$ and the first integral

$$
\begin{equation*}
A=B=2 C, x_{3}=0 \Longrightarrow K_{4}=\left|C\left(\omega_{1}+i \omega_{2}\right)^{2}+\left(x_{1}+i x_{2}\right)\left(k_{1}+i k_{2}\right)\right|^{2} \tag{13}
\end{equation*}
$$

This gives a complete proof of reducibility to quadratures.
In the next section we present the work Painlevé which is one of the two basic tools for the desired classification.

## 2 The work of Painlevé

Paul Painlevé considered the problem of looking at all second-order differential equations which possess the Painlevé property (simply called P-property). The following definition of this property is for first-order differential equations.

### 2.1 The P-property

## Definition 1

Consider the ODE (Ordinary Differential Equation)

$$
\frac{d x}{d t}=f(x, t), \quad x \in \mathbf{C}
$$

where $f$ is rational in $x$ and analytic in $t$. This ODE possesses the Painlevé property (or the $P$-property) if its general solution has no movable singularities other than poles.

This property can be also defined for second-order differential equations for which the function $f$ in the equation

$$
\frac{d^{2} x}{d t^{2}}=f\left(\frac{d x}{d t}, x, t\right)
$$

is rational in $d x / d t$, algebraic in $x$ and analytic in $t$.
The following example points out how we can calculate the Laurent series of a differential equation.

Example 1 We find the solution of

$$
\begin{equation*}
\frac{d x}{d t}=x-x^{2} \tag{14}
\end{equation*}
$$

near a singularity $t_{*}$. Let $\tau=t-t_{*}$. We firstly seek to determine the leading behavior of the solution.

$$
x(t)=\frac{A}{\tau^{p}}, \quad A \neq 0
$$

The leading behavior of each term is

$$
\begin{align*}
x(t) & \sim A \tau^{-p} \\
\dot{x}(t) & \sim-A p \tau^{-p-1}  \tag{15}\\
-x^{2}(t) & \sim-A^{2} \tau^{-2 p} .
\end{align*}
$$

Equating the leading behavior of each side of (14) we have

$$
-2 p=-p-1 \Rightarrow p=1 .
$$

Equating also the leading coefficients (15) of each side we have that

$$
-A p=-A^{2} \Longrightarrow A(A-p)=0 \Longrightarrow A=p=1, \text { since } A \neq 0 \text {. }
$$

Therefore we have that

$$
\begin{aligned}
\Rightarrow x(t) & =\frac{1}{\tau}+a_{0}+a_{1} \tau+a_{2} \tau^{2}+\ldots \\
\dot{x} & =-\frac{1}{\tau^{2}}+a_{1}+2 a_{2} \tau+\ldots \\
x^{2} & =\frac{1}{\tau^{2}}+\frac{2 a_{0}}{\tau}+a_{0}^{2}+2 a_{1}+2 a_{0} a_{1} \tau+2 a_{2} \tau+\ldots
\end{aligned}
$$

Substituting into (14) and equating the coefficients of $\tau^{k}$ of each side ( $k \in \mathbf{N} \cup\{0\}$ ) we have that

$$
\begin{gathered}
\Rightarrow a_{0}=\frac{1}{2}, \quad a_{1}=\frac{1}{12}, \ldots \\
\Rightarrow x(t)=\frac{1}{\tau}+\frac{1}{2}+\frac{\tau}{12}+\ldots \\
\Longrightarrow x(t)=\frac{1}{\tau}+\sum_{n=0}^{\infty} a_{n} \tau^{n} .
\end{gathered}
$$

Painlevé was well aware of Kowalevski's work, ([30,31]) but he realized that in order to test this property the existence of the Laurent series is just a necessary condition.

### 2.2 Classification of ODEs possessing the P-property

There has been a classification of first- and second-order ODEs that possess the $P$-property.

### 2.2.1 First-order ODEs

Paul Painlevé proved in 1900 the following theorem, in which we can see that only one type of first-order ODEs possesses this $P$-property and all those that can be transformed into it.

## Theorem 1

Only one ODE possesses the $P$-property and this is the Riccati equation:

$$
\begin{equation*}
\frac{d x}{d t}=a(t) x^{2}+b(t) x+c(t) \tag{16}
\end{equation*}
$$

The transformation

$$
\begin{equation*}
x(t)=-\frac{\dot{\psi}}{a(t) \psi}, \quad \text { where } \dot{\psi}=\frac{d \psi}{d t} \tag{17}
\end{equation*}
$$

transforms the (16) into a linear second-order ODE.

It is implied that the equations that can be transformed into a Riccati equation also possess this property, like the equation

$$
\frac{d x}{d t}=x-x^{n}
$$

that possesses the P-property, since it can be transformed into (16)

$$
\dot{\psi}=(n-1) \psi-(n-1) \psi^{2},
$$

using the Bernoulli transformation

$$
x=\psi^{\frac{1}{n-1}} .
$$

Painlevé proved that there is a finite set of second-order ODEs that possess the $P$-property. Some of them are displayed in the next theorem.

### 2.2.2 Second-order ODEs

In the case of second-order ODEs we can see that there are many more cases possessing this $P$-property than in the case of first-order ODEs.

## Theorem 2

There are exactly 50 ODEs of the form

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}=f\left(x, \frac{d x}{d t}, t\right), \tag{18}
\end{equation*}
$$

where $f$ is rational in $d x / d t$, algebraic in $x$ and analytic in $t$, the solutions have no movable singularities other than poles. The first 44 were already known. We display the first three of the remaining six new ones.

$$
\begin{aligned}
\text { (PI) } & \frac{d^{2} x}{d t^{2}}=6 x^{2}+t \\
\text { (PII) } & \frac{d^{2} x}{d t^{2}}=2 x^{3}+t x+a \\
\text { (PIII) } & \frac{d^{2} x}{d t^{2}}=\frac{1}{x}\left(\frac{d x}{d t}\right)^{2}-\frac{1}{t} \frac{d x}{d t}+\frac{1}{t}\left(a x^{2}+b\right)+c x^{2}+\frac{d}{x} .
\end{aligned}
$$

### 2.3 ARS Algorithm

In 1981 Ablowitz, Ramani and Segur conjectured that all reductions of integrable Partial Differential Equations (PDEs) possess the Painlevé property. They constructed a procedure (ARS algorithm [1], [2]) in the spirit of Kowalevski's work in order to check the Painlevé Property and then the integrability of the system. Another equivalent method is that of Yoshida (1983). The ARS algorithm has three steps. In the first one we find
the leading behavior of the solution. In the second one, the terms that have arbitrary coefficients are determined. We always know that the term $t^{-1-p}$ always appears, where $p$ is the leading behavior of the solution found in the first step. In the last compatibility step one has to prove that indeed the coefficients of the terms found in the second step are arbitrary constants.

Example 2 We apply this algorithm to the simplest equation of Theorem 2

$$
\begin{equation*}
\text { (PI) } \frac{d^{2} x}{d t^{2}}=6 x^{2}+t \tag{19}
\end{equation*}
$$

Step 1.
We firstly seek to determine the leading behavior of the solution

$$
\begin{equation*}
x(t)=c \tau^{p}+\ldots, \quad p<0, \text { where } \tau=t-t_{*} . \tag{20}
\end{equation*}
$$

From (19) we have that

$$
\begin{aligned}
p(p-1) c \tau^{p-2}=6 c^{2} \tau^{2 p} & \Rightarrow 2 p=p-2 \Rightarrow p=-2 \\
& \Rightarrow 6 c=6 c^{2} \Rightarrow c=1,
\end{aligned}
$$

because $c \neq 0$. Otherwise $c \tau^{p}$ would not be the leading behavior of the solution. Now we want to find another arbitrary constant $\alpha$ so that

$$
x(t)=\frac{1}{\tau^{2}}+\ldots+\alpha \tau^{r-2}+\ldots
$$

Step 2.
We determine the value of $r$ such that the coefficient of the term $\tau^{r-2}$ is an arbitrary constant.

$$
\begin{gathered}
\ddot{x} \sim(r-2)(r-3) \alpha \tau^{r-4} \\
6 x^{2}=6\left(\frac{1}{\tau^{2}}+\ldots+\alpha \tau^{r-2}\right)^{2} \sim 12 \frac{1}{\tau^{2}} \alpha \tau^{r-2}=12 \alpha \tau^{r-4} .
\end{gathered}
$$

We equate the leading behaviors of the solution and we have

$$
\begin{align*}
(r-2)(r-3) \alpha \tau^{r-4} & =12 \alpha \tau^{r-4}  \tag{21}\\
{[(r-2)(r-3)-12] \alpha \tau^{r-4} } & =\text { other terms of } \tau^{r-4}
\end{align*}
$$

We want $\alpha$ to be an arbitrary constant. This occurs if and only if

$$
\begin{aligned}
(r-2)(r-3)-12=0 & \Rightarrow r^{2}-5 r+6-12=0 \\
\Rightarrow r^{2}-5 r-6=0 & \Rightarrow(r+1)(r-6)=0
\end{aligned}
$$

Since the term $(r+1)$ always appears, the value of $r$ in order to have another arbitrary constant is the number 6. Therefore the arbitrary constant is expected to be the coefficient of $\tau^{4}$.

Step 3.
Now we find the coefficients of the powers of $\tau$ and we test the compatibility condition. That means that we have to prove that the coefficient of $\tau^{4}$ is an arbitrary constant as found in the previous step of this algorithm.

$$
\begin{aligned}
x(t) & =\frac{1}{\tau^{2}}+\frac{a_{0}}{\tau}+a_{1}+a_{2} \tau+a_{3} \tau^{2}+a_{4} \tau^{3}+a_{5} \tau^{4}+\ldots \\
\ddot{x}(t) & =\frac{6}{\tau^{4}}+\frac{2 a_{0}}{\tau^{3}}+2 a_{3}+6 a_{4} \tau+12 \tau^{2}+\ldots \\
6 x^{2}(t) & =\ldots=\frac{6}{\tau^{4}}+\frac{12 a_{0}}{\tau^{3}} \frac{12 a_{1}+6 a_{0}^{2}}{\tau^{2}}+\frac{12 a_{0} a_{1}+12 a_{2}}{\tau} \\
& +\left(6 a_{1}^{2}+12 a_{3}+12 a_{0} a_{2}\right)+\left(12 a_{4}+12 a_{0} a_{3}+12 a_{1} a_{2}\right) \tau \\
& +\left(12 a_{5}+12 a_{0} a_{4}+12 a_{1} a_{3}+a_{2}\right) \tau^{2}+\ldots \\
t & =\tau+t_{*} .
\end{aligned}
$$

With respect to the ODE (19) equating the coefficients of all the powers of $\tau$ leads to

$$
\begin{gathered}
\frac{1}{\tau^{4}}: 6=6 \\
\frac{1}{\tau^{3}}: 2 a_{0}=12 a_{0} \Rightarrow a_{0}=0 \\
\frac{1}{\tau^{2}}: 0=12 a_{1}+6 a_{0}^{2} \Rightarrow a_{1}=0 \\
\frac{1}{\tau}: 0=12 a_{0} a_{1}+12 a_{2} \Rightarrow a_{2}=0 \\
\text { Constant terms }: \quad 2 a_{3}=6 a_{1}^{2}+12 a_{3}+12 a_{0} a_{2}-t_{*} \Rightarrow a_{3}=\frac{t_{*}}{10} \\
\tau: 6 a_{4}=12 a_{4}+12 a_{0} a_{3}+12 a_{1} a_{2}+1 \Rightarrow a_{4}=-\frac{1}{6} \\
\tau^{2}: \quad 12 a_{5}=12 a_{5}+12 a_{0} a_{4}+12 a_{1} a_{3}+6 a_{2}^{2} \Rightarrow 12 a_{5}=12 a_{5}
\end{gathered}
$$

Therefore $a_{5}$, the coefficient of $\tau^{4}$, is the arbitrary constant as it was expected and so the necessary conditions are satisfied.

This algorithm can be applied to systems of ODEs, as well.
It is now time to define the Kowalevski exponents and how these quantities are related to the algebraic integrability or algebraic non integrability of a weight-homogeneous vector fields. The exposition follows the book [4].

## 3 Background

### 3.1 Basic definitions

The first thing to do is to define what is a (weight) homogeneous polynomial of weight $k$.

## Definition 2

A polynomial $f \in \mathbf{C}\left[x_{1}, x_{2}, \ldots, x_{n}\right]$ is called $a$ weight-homogeneous polynomial of weight $k$ with respect to a vector $v=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ if

$$
f\left(t^{v_{1}} x_{1}, \ldots, t^{v_{n}} x_{n}\right)=t^{k} f\left(x_{1}, x_{2}, \ldots, x_{n}\right)
$$

and the vector $v$ is called weight vector. The weight $k$ is denoted by $\varpi(f)$.

This definition is given in order to see a special kind of vector field in which we interested for the classification below.

## Definition 3

A polynomial vector field on $\mathbf{C}^{n}$,

$$
\begin{gather*}
\dot{x}_{1}=f_{1}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \\
\vdots  \tag{22}\\
\dot{x}_{n}=f_{n}\left(x_{1}, x_{2}, \ldots, x_{n}\right)
\end{gather*}
$$

is called a weight-homogeneous vector field of weight $k$ (with respect to a weight vector $v$ ), if $\varpi\left(f_{i}\right)=v_{i}+k=\varpi\left(x_{i}\right)+k$ for $i=1,2, \ldots, n$. A weight-homogeneous vector field of weight 1 is called weight-homogeneous vector field. Furthermore, when all the weights are equal to 1 , this is simply called homogeneous vector field.

A vector field $\dot{x}_{i}=f(x)$ on $\mathbf{C}^{n}$, is called homogeneous if and only if $\varpi\left(f_{i}\right)=2$.

Example 3 We consider the periodic 5-particle Kac-van Moerbeke lattice that is given by the quadratic vector field

$$
\begin{equation*}
\dot{x}_{i}=x_{i}\left(x_{i-1}-x_{i+1}\right), \quad i=1, \ldots, 5, \tag{23}
\end{equation*}
$$

with $x_{i}=x_{i+5}$. This system has the constants of motion,

$$
\begin{align*}
& F_{1}=x_{1}+x_{2}+x_{3}+x_{4}+x_{5}, \\
& F_{2}=x_{1} x_{3}+x_{2} x_{4}+x_{3} x_{5}+x_{4} x_{1}+x_{5} x_{2},  \tag{24}\\
& F_{3}=x_{1} x_{2} x_{3} x_{4} x_{5} .
\end{align*}
$$

If one takes $v=(1,1,1,1,1)$, (23) becomes a homogeneous vector field and the weights of the constants of motion in (24) are $\varpi\left(F_{1}\right)=1, \varpi\left(F_{2}\right)=2$ and $\varpi\left(F_{3}\right)=5$.

Now it is time to see the definition which can be found in [24] of an algebraically completely integrable system. Our objective is complete classification of the algebraically completely integrable Lotka-Volterra systems in three and four dimensions.

## Definition 4

A vector field,

$$
\begin{gather*}
\dot{x}_{1}=f_{1}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \\
\vdots  \tag{25}\\
\dot{x}_{n}=f_{n}\left(x_{1}, x_{2}, \ldots, x_{n}\right),
\end{gather*}
$$

is called an algebraically completely integrable system (a.c.i.) if its solution can be expressed as Laurent series

$$
x_{i}(t)=\frac{1}{t^{v_{i}}} \sum_{k=0}^{\infty} x_{i}^{(k)} t^{k}, \quad i=1,2, \ldots, n,
$$

where $n-1$ of the coefficients $x_{i}^{(k)}$ are free parameters.

### 3.2 Two important propositions

The following proposition is important for two reasons. The first one is that it gives us an induction formula to find the Laurent solution of a weight-homogeneous vector field and the second one is that through it the Kowalevski exponents can be defined, which constitututes the most important tool for this thesis.

## Proposition 1

Suppose that $V$ is a weight-homogeneous vector field on $\mathbf{C}^{n}$ given by

$$
\dot{x}_{i}=f_{i}\left(x_{1}, \ldots, x_{n}\right), \quad i=1,2, \ldots, n,
$$

and suppose that

$$
\begin{equation*}
x_{i}(t)=\frac{1}{t^{v_{i}}} \sum_{k=0}^{\infty} x_{i}^{(k)} t^{k}, \quad i=1,2, \ldots, n, \tag{26}
\end{equation*}
$$

is a weight-homogeneous Laurent solution for this vector field. Then the leading coefficients, $x_{i}^{(0)}$, satisfy the non linear algebraic equations

$$
\begin{gather*}
v_{1} x_{1}^{(0)}+f_{1}\left(x_{1}^{(0)}, \ldots, x_{n}^{(0)}\right)=0, \\
\vdots  \tag{27}\\
v_{n} x_{n}^{(0)}+f_{n}\left(x_{1}^{(0)}, \ldots, x_{n}^{(0)}\right)=0,
\end{gather*}
$$

while the subsequent terms $x_{i}^{(k)}$ satisfy

$$
\begin{equation*}
\left(k \operatorname{Id}_{n}-\mathcal{K}\left(x^{(0)}\right)\right) x^{(k)}=R^{(k)} \tag{28}
\end{equation*}
$$

where $x^{(k)}=\left(\begin{array}{c}x_{1}^{(k)} \\ \vdots \\ x_{n}^{(k)}\end{array}\right)$ and $R^{(k)}=\left(\begin{array}{c}R_{1}^{(k)} \\ \vdots \\ R_{n}^{(k)}\end{array}\right) . R^{(k)}$ is a polynomial, which depends on the variables $x_{1}^{(l)}, \ldots, x_{n}^{(l)}$ with $0 \leq l<k$ only. Also the elements of the $n \times n$ matrix $\mathcal{K}$ are given by

$$
\begin{equation*}
\mathcal{K}_{i, j}:=\frac{\partial f_{i}}{\partial x_{j}}+v_{i} \delta_{i j} \tag{29}
\end{equation*}
$$

where $\delta$ is the Kronecker delta.

Remark 1 The pole order $v_{i}$ of $x_{i}$ in (26) is the ith component of the weight vector that makes the vector field a weight homogeneous one. The number, $v_{i}$, is not necessarily the pole order of $x_{i}$ because some of the $x_{i}^{(0)}$ that can be calculated solving (27) may be equal to zero.

## Definition 5

The set of equations (27) is called the indicial equation of $V$ and its solution set is called the indicial locus and it is denoted by $I$. The $n \times n$ matrix $\mathcal{K}$, defined by (29), is called the Kowalevski matrix and its eigenvalues are called Kowalevski exponents.

Example 4 In the case of the Example 1 the indicial equation is

$$
\begin{align*}
& x_{1}^{(0)}\left(1+x_{5}^{(0)}-x_{2}^{0}\right)=0 \\
& x_{2}^{(0)}\left(1+x_{1}^{(0)}-x_{3}^{0}\right)=0 \\
& x_{3}^{(0)}\left(1+x_{2}^{(0)}-x_{4}^{0}\right)=0  \tag{30}\\
& x_{4}^{(0)}\left(1+x_{3}^{(0)}-x_{5}^{0}\right)=0 \\
& x_{5}^{(0)}\left(1+x_{4}^{(0)}-x_{1}^{0}\right)=0
\end{align*}
$$

The non trivial elements of the indicial locus are $m_{1}, m_{1}^{\prime}, m_{2}, m_{2}^{\prime}, m_{3}, m_{3}^{\prime}, m_{4}, m_{4}^{\prime}, m_{5}$ and $m_{5}^{\prime}$. The labels, 1,2,3,4 and 5, indicate the positions of zeroes. The two of them are

$$
m_{1}=(0,0,-1,1,0) \quad \text { and } \quad m_{1}^{\prime}=(0,-2,1,-1,2)
$$

to which correspond the following Kowalevski matrices:

$$
\mathcal{K}\left(m_{1}\right)=\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0  \tag{31}\\
0 & 2 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 2
\end{array}\right), \mathcal{K}\left(m_{1}^{\prime}\right)=\left(\begin{array}{ccccc}
5 & 0 & 0 & 0 & 0 \\
-2 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 1 \\
-2 & 0 & 0 & 2 & 0
\end{array}\right)
$$

The Kowalevski exponents of $\mathcal{K}\left(m_{1}\right)$ and $\mathcal{K}\left(m_{1}^{\prime}\right)$ are $-1,1,1,2,2$ and $-2,-1,1,2,5$ respectively. It has been proved that this system is indeed integrable and we observe that the positive Kowalevski exponents of $\mathcal{K}\left(m_{1}^{\prime}\right), 1,2$ and 5, are the degrees of the invariant polynomials.

A necessary condition for algebraic integrability is that $n-1$ eigenvalues of $\mathcal{K}$ should be integers. It turns out that the last eigenvalue is always -1 . The eigenvector that corresponds to -1 is also known. This is given from the following Proposition, that can be found in $[1,2,4]$.

## Proposition 2

For any $m$ which belongs to the indicial locus $\mathcal{I}$, except for the trivial element, the Kowalevski matrix $\mathcal{K}(m)$ of a weight homogeneous vector field always has -1 as an eigenvalue. The corresponding eigenspace contains $\left(v_{1} m_{1}, \ldots, v_{n} m_{n}\right)^{T}$ as an eigenvector.

## 4 Properties of Kowalevski exponents

In this section we see some relations about the Kowalevski exponents and the weights of the first integrals of a system and therefore these results also give us some necessary conditions in order to have algebraic completely integrable systems. The following results can be found in $[21,40,46]$ their summary in the article [23].

## Theorem 3

If the weight-homogeneous system $\dot{x}=f(x)$ has $k$ independent algebraic first integrals $I_{1}, \ldots, I_{k}$ of weighted degrees $d_{1}, \ldots, d_{k}$ and Kowalevski exponents $\rho_{2}, \ldots, \rho_{n}$, then there exists a $k \times(n-1)$ matrix $\mathcal{N}$ with integer entries, such that

$$
\begin{equation*}
\sum_{j=2}^{n} \mathcal{N}_{i j} \cdot \rho_{j}=d_{i}, \quad i=1, \ldots, k \tag{32}
\end{equation*}
$$

From this theorem we have the two following corollaries:

## Corollary 1

If the Kowalevski exponents are $\mathbf{Z}$-independent, then there is no rational first integrals.

## Corollary 2

If the Kowalevski exponents are $\mathbf{N}$-independent, then there is no polynomial first integrals.

The next theorem which can be found in [4] gives us a necessary condition for a system to be integrable, that can be examined very easily without any extra calculations except for the computation of the Kowalevski exponents.

## Theorem 4

A necessary condition for a system of the form (22) to be algebraically integrable is that all the Kowalevski exponents be rational numbers, i.e., if at least one Kowalevski exponent is irrational or complex, then (22) cannot have a complete set of algebraic integrals.

We illustrate the Theorem 4 with an example.

Example 5 We determine the values of $\varepsilon \in \mathbf{N} \cup\{0\}$ so that the Hamiltonian vector field derived by the Hamiltonian

$$
\begin{equation*}
H_{\varepsilon}=\frac{1}{2}\left(p_{1}^{2}+p_{2}^{2}\right)+\frac{1}{4}\left(q_{1}^{4}+q_{2}^{4}\right)+\frac{\varepsilon}{2} q_{1}^{2} q_{2}^{2} \tag{33}
\end{equation*}
$$

is integrable.
Hamilton's equations are

$$
\begin{align*}
\dot{q}_{1} & =p_{1} \\
\dot{q}_{2} & =p_{2}  \tag{34}\\
\dot{p}_{1} & =-q_{1}^{3}-\varepsilon q_{1} q_{2}^{2} \\
\dot{p}_{2} & =-q_{2}^{3}-\varepsilon q_{1}^{2} q_{2} .
\end{align*}
$$

Assuming that

$$
q_{1}=c_{1} \tau^{g_{1}}, q_{2}=c_{2} \tau^{g_{2}}, p_{1}=c_{3} \tau^{g_{3}}, p_{2}=c_{4} \tau^{g_{4}}
$$

and substituting into (34) give

$$
\begin{align*}
g_{1} c_{1} \tau^{g_{1}-1} & =c_{3} \tau^{g_{3}} \Rightarrow g_{1}-1=g_{3}, g_{1} c_{1}=c_{3},  \tag{35}\\
g_{2} c_{2} \tau^{g_{2}-1} & =c_{4} \tau^{g_{4}} \Rightarrow g_{2}-1=g_{4}, g_{2} c_{2}=c_{4},  \tag{36}\\
g_{3} c_{3} \tau^{g_{3}-1} & =-c_{1}^{3} \tau^{3 g_{1}}-\varepsilon c_{1} c_{2}^{2} \tau^{g_{1}+2 g_{2}},  \tag{37}\\
g_{4} c_{4} \tau^{g_{4}-1} & =-c_{2}^{3} \tau^{3 g_{2}}-\varepsilon c_{1}^{2} c_{2} \tau^{2 g_{1}+g_{2}} . \tag{38}
\end{align*}
$$

From (35) and (37) we have

$$
\begin{equation*}
\left(g_{1}-1\right) g_{1} c_{1} \tau^{g_{1}-2}=-c_{1}^{3} \tau^{3 g_{1}}-\varepsilon c_{1} c_{2}^{2} \tau^{g_{1}+2 g_{2}} \tag{39}
\end{equation*}
$$

and from (36) and (38)

$$
\begin{equation*}
\left(g_{2}-1\right) g_{2} c_{2} \tau^{g_{2}-2}=-c_{2}^{3} \tau^{3 g_{2}}-\varepsilon c_{1}^{2} c_{2} \tau^{2 g_{1}+g_{2}} . \tag{40}
\end{equation*}
$$

From (39) we have that

$$
\begin{gather*}
g_{1}-2=3 g_{1} \Rightarrow \frac{g_{1}=-1}{\text { or }} \Rightarrow g_{3}=-2, c_{3}=-c_{1}  \tag{41}\\
g_{1}-2=g_{1}+2 g_{2} \Rightarrow g_{2}=-1 \Rightarrow g_{4}=-2, c_{4}=-c_{2}
\end{gather*}
$$

and

$$
\begin{equation*}
2 c_{1}=-c_{1}^{3}-\varepsilon c_{1} c_{2}^{2} \Rightarrow 2=-c_{1}^{2}-\varepsilon c_{2}^{2} \quad\left(c_{1} \neq 0\right) . \tag{42}
\end{equation*}
$$

From (40) we get

$$
\begin{equation*}
2 c_{2}=-c_{2}^{3}-\varepsilon c_{1}^{2} c_{2} \Rightarrow 2=-c_{2}^{2}-\varepsilon c_{1}^{2} \quad\left(c_{2} \neq 0\right) \tag{43}
\end{equation*}
$$

and then

$$
\begin{gather*}
\left(c_{2}^{2}-c_{1}^{2}\right)-\varepsilon\left(c_{2}^{2}-c_{1}^{2}\right)=0 \Rightarrow\left(c_{2}^{2}-c_{1}^{2}\right)(1-\varepsilon)=0  \tag{44}\\
\Longrightarrow \quad \begin{array}{l}
\text { (i) } \quad c_{1}^{2}=c_{2}^{2} \Rightarrow c_{1}^{2}=c_{2}^{2}=-\frac{2}{1+\varepsilon} \\
\text { (ii) } \quad \varepsilon=1 \Rightarrow c_{1}^{2}+c_{2}^{2}=-2
\end{array} \tag{45}
\end{gather*}
$$

Therefore

$$
\begin{array}{ll}
q_{1}=\frac{c_{1}}{\tau}+\ldots+a \tau^{r-1}, & p_{1}=\frac{2 c_{1}}{\tau^{3}}+\ldots+a(r-1)(r-2) \tau^{r-3}  \tag{46}\\
q_{2}=\frac{c_{2}}{\tau}+\ldots+b \tau^{r-1}, & p_{2}=\frac{2 c_{2}}{\tau^{3}}+\ldots+b(r-1)(r-2) \tau^{r-3} .
\end{array}
$$

Substituting them into (34) and equating the coefficients of $\tau^{r-3}$ lead to

$$
\begin{align*}
& a(r-1)(r-2)=-3 c_{1}^{2} a-\varepsilon a c_{2}^{2}-2 \varepsilon c_{1} c_{2} b  \tag{47}\\
& b(r-1)(r-2)=-3 c_{2}^{2} b-\varepsilon b c_{1}^{2}-2 \varepsilon c_{1} c_{2} a \\
& \quad \Rightarrow \mathcal{M}(r)\binom{a}{b}=\text { other terms of } \tau^{r-3}, \tag{48}
\end{align*}
$$

where

$$
\mathcal{M}(r)=\left(\begin{array}{cc}
(r-1)(r-2)+3 c_{1}^{2}+\varepsilon c_{2}^{2} & 2 \varepsilon c_{1} c_{2}  \tag{49}\\
2 \varepsilon c_{1} c_{2} & (r-1)(r-2)+3 c_{2}^{2}+\varepsilon c_{1}^{2}
\end{array}\right) .
$$

In the case of (45i) we have that

$$
\mathcal{M}(r)=\left(\begin{array}{cc}
(r-1)(r-2)+c_{1}^{2}(3+\varepsilon) & 2 \varepsilon c_{1} c_{2}  \tag{50}\\
2 \varepsilon c_{1} c_{2} & (r-1)(r-2)+c_{1}^{2}(3+\varepsilon)
\end{array}\right) .
$$

Non trivial solutions of (48) are desired and so

$$
\begin{gathered}
\operatorname{det} \mathcal{M}(r)=0 \\
\Rightarrow\left[(r-1)(r-2)+c_{1}^{2}(3+\varepsilon)\right]^{2}-4 \varepsilon^{2} c_{1}^{2} c_{2}^{2}=0 \\
\Rightarrow\left[(r-1)(r-2)+c_{1}^{2}(3+\varepsilon)\right]^{2}-4 \varepsilon^{2} c_{1}^{4}=0 .
\end{gathered}
$$

Let $\gamma=(r-1)(r-2)$,

$$
\begin{gathered}
\Rightarrow\left(\gamma+c_{1}^{2}(3+\varepsilon)-2 \varepsilon c_{1}^{2}\right)\left(\gamma+c_{1}^{2}(3+\varepsilon)+2 \varepsilon c_{1}^{2}\right)=0 \\
\Rightarrow\left(\gamma+c_{1}^{2}(3-\varepsilon)\right)\left(\gamma+c_{1}^{2}(1+\varepsilon)\right)=0 \\
\Rightarrow\left(\gamma+c_{1}^{2}(3-\varepsilon)\right)(\gamma-6)=0, \quad \text { since } c_{1}^{2}=-\frac{2}{1+\varepsilon} \\
\left(r^{2}-3 r+2-6\right)\left[r^{2}-3 r+2+c_{1}^{2}(3-\varepsilon)\right]=0 \\
\quad(r+1)(r-4)\left[r^{2}-3 r+2+c_{1}^{2}(3-\varepsilon)\right]=0
\end{gathered}
$$

In order to have rational solutions the discriminant of the third factor has to be a square of an integer number and so

$$
\begin{gather*}
9-4\left(2-\frac{2(3-\varepsilon)}{1+\varepsilon}\right)=1+\frac{8(3-\varepsilon)}{1+\varepsilon}=k^{2} \\
\Rightarrow \ldots \Rightarrow \varepsilon=\frac{32}{k^{2}+7}-1 . \tag{51}
\end{gather*}
$$

Since $\varepsilon \in \mathbf{N} \cup\{0\}$

$$
\begin{aligned}
& k^{2}+7 \in\{1,2,4,8,16,32\} \\
& k^{2} \in\{-6,-5,-3,1,9,25\} \\
& \quad k \in\{1,3,5\} \\
& \Rightarrow \varepsilon \in\{0,1,3\} \text { from }(51) .
\end{aligned}
$$

So $\varepsilon \in\{0,1,3\}$ is a necessary condition for the integrability of the Hamiltonian system derived by the Hamiltonian (33). It is very easy to see that for $\varepsilon=0$ this system is completely integrable because the dynamics

$$
V=\frac{1}{4} q_{1}^{4}+\frac{1}{4} q_{2}^{4}
$$

is separable. The cases in which $\varepsilon=1,3$ are also known as completely integrable systems.

### 4.1 Kowalevski exponents and Hamiltonian systems

As we saw in the previous example, the Kowalevski exponents can be applied to Hamiltonian systems. This was expected since they are derived by a vector field, usually a weight homogeneous one. However, in the case of Hamiltonian vector fields there are stronger
results that connect the Kowalevski exponents and complete integrability. In this subsection we present these results for the Kowalevski exponents of a Hamiltonian system in $(q, p)$ coordinates with a natural potential.

## Proposition 3

Let $\dot{x}=f(x)$ be a Hamiltonian system and $H$ its Hamiltonian. If $\rho$ is a Kowalevski exponent, then so is $h-1-\rho$, where $h$ is the weighted degree of the Hamiltonian $H$.

This result was firstly pointed out by Yoshida and given in its final form by Lochak [32].
As shown in [47] and [48] the Kowalevski exponents of the Hamiltonian system with diagonal kinetic energy and homogeneous potential,

$$
H=\frac{1}{2}\left(p_{1}^{2}+\ldots+p_{n}^{2}\right)+V\left(q_{1}, \ldots, q_{n}\right),
$$

where $V(x)$ is homogeneous of degree $k$, but $k \neq 0, \pm 2$, always come by pairs such that

$$
\rho_{i}+\rho_{i+n}=\frac{k+2}{k-2} .
$$

We can now define the difference between two exponents of each pair $\Delta \rho_{i}=\rho_{i}-\rho_{i+n}$. This leads us directly to the following theorem.

## Theorem 5

If the $n$ numbers $\Delta \rho_{i}$ are $\mathbf{Q}$-independent, then the Hamiltonian system has no additional first integral beside the Hamiltonian itself.

### 4.2 Special cases of Hamiltonian systems

We now investigate the integrability of the Hamiltonian system with exponential dynamics. We follow the notation of [18]. The Hamiltonian is given by

$$
\begin{equation*}
H=\frac{1}{2}\langle p, A p\rangle+\sum_{i=1}^{N} c_{i} \mathbf{e}^{\left\langle\nu_{i}, q\right\rangle}, \tag{52}
\end{equation*}
$$

where $p, q, \nu_{i} \in \mathbf{R}^{n},\langle\cdot, \cdot\rangle$ is the scalar product in $\mathbf{R}^{n}$ and $A$ is a symmetric $n \times n$ matrix. The set of vector $\left\{\nu_{1}, \nu_{2}, \ldots \nu_{N}\right\}$ is called the spectrum of the system. Hamilton's equations are

$$
\begin{equation*}
\dot{q}=\frac{\partial H}{\partial p}, \quad \dot{p}=-\frac{\partial H}{\partial q} \tag{53}
\end{equation*}
$$

and, if we change the variables to

$$
\begin{equation*}
a_{i}=-c_{i} \mathbf{e}^{\left\langle\nu_{i}, q\right\rangle}, \quad b_{i}=\left\langle\nu_{i}, p\right\rangle, \tag{54}
\end{equation*}
$$

then the Hamiltonian equations (53) are transformed into

$$
\begin{equation*}
\dot{a}_{k}=a_{k} b_{k}, \quad \dot{b}_{k}=\sum_{i=1}^{N} M_{k i} a_{i}, \quad 1 \leq k \leq N, \tag{55}
\end{equation*}
$$

where $M_{k i}=\left\langle\nu_{k}, \nu_{i}\right\rangle$.
We can see that the vector field (55) is a weight-homogeneous vector field respect to the weight vector $v=\left(v_{1}, v_{2}, \ldots, v_{2 N}\right)$, where

$$
v_{i}=2 \text { and } v_{i+N}=1, \quad \text { for } \quad 1 \leq i \leq N .
$$

Using Proposition 1 we firstly need to find the elements of the indicial locus, that is the set of the solutions of the equations:

$$
\begin{equation*}
2 a_{k}^{(0)}+a_{k}^{(0)} b_{k}^{(0)}=0, \quad b_{k}^{(0)}+\sum_{i=1}^{N} M_{i k} a_{i}^{(0)}=0 \quad \text { for } \quad 1 \leq k \leq N . \tag{56}
\end{equation*}
$$

We find all the elements of the indicial locus by setting all the variables $a_{k}^{(0)}$ equal to zero except $a_{i}^{(0)}$. So we have that

$$
\begin{equation*}
a_{k}^{(0)}=0, \quad a_{i}^{(0)}=\frac{2}{M_{i i}}, \quad b_{k}^{(0)}=-\frac{M_{k i}}{M_{i i}}, \quad b_{i}^{(0)}=-2 \quad \text { for } k \neq i . \tag{57}
\end{equation*}
$$

Similarly we can set some of the variables of $a^{(0)}=\left(a_{1}^{(0)}, a_{2}^{(0)}, \ldots a_{N}^{(0)}\right)^{T}$ equal to zero. If we set the first $m$ variables equal to zero, we have the following results:

$$
\begin{gather*}
a_{j}^{(0)}=0, \quad\left(\begin{array}{c}
a_{m+1}^{(0)} \\
\vdots \\
a_{N}^{(0)}
\end{array}\right)=\left(M^{(m)}\right)^{-1}\left(\begin{array}{c}
2 \\
\vdots \\
2
\end{array}\right),  \tag{58}\\
b_{j}^{(0)}=-\sum_{k=m+1}^{N} M_{j k} a_{k}^{(0)}, \quad b_{m+1}^{(0)}=\ldots=b_{N}^{(0)}=-2, \\
\text { for } 1 \leq j \leq m,
\end{gather*}
$$

where $M^{(m)}$ is the diagonal submatrix of $M$, with the first $m$ rows and the last $m$ columns deleted. For $m=N-1$ the solution of (58) is also the solution of (57).

All these calculations is useful in finding the Kowalevski exponents in order to have a clue whether or not the Hamiltonian system derived from the Hamiltonian (52) is integrable. So the formulas (58) for finding the indicial locus allow us to write the Kowalevski matrix in a block form:

$$
\begin{align*}
& \mathcal{K}=\left(\begin{array}{cc}
U & C \\
M & E
\end{array}\right), \text { where } \\
& U=\operatorname{diag}\left(2+b_{1}^{(0)}, \ldots, 2+b_{N}^{(0)}\right),  \tag{59}\\
& C=\operatorname{diag}\left(a_{1}^{(0)}, \ldots, a_{N}^{(0)}\right), \\
& E=\operatorname{diag}(1, \ldots, 1) .
\end{align*}
$$

The most common case of this system is when $A=I d_{n}$, that is the identity $n \times n$ matrix. These systems become more interesting when the spectrum is the set of the simple roots for a simple Lie Algebra $\mathcal{G}$, from which it follows that $N=\operatorname{rank} \mathcal{G}$.

### 4.2.1 Kozlov-Treshchev Birkhoff system

We consider the Hamiltonian system derived by the Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i=1}^{n} p_{i}^{2}+\sum_{i=1}^{n-1} \mathbf{e}^{q_{i}-q_{i+1}}+\mathbf{e}^{-q_{1}-q_{2}}+\mathbf{e}^{q_{n}}+\mathbf{e}^{2 q_{n}}, \quad n \geq 4, \tag{60}
\end{equation*}
$$

in its simplest form, for $n=4$. Due to (52) we have that

$$
\begin{array}{ll}
A=\mathrm{Id}_{4} & c_{i}=1, \text { for } i=1,2, \ldots, 6 \\
\nu_{1}=(1,-1,0,0) & \nu_{2}=(0,1,-1,0) \\
\nu_{3}=(0,0,1,-1) & \nu_{4}=(-1,-1,0,0) \\
\nu_{5}=(0,0,0,1) & \nu_{6}=(0,0,0,2) .
\end{array}
$$

Solving the corresponding equations (27) we can find all the elements of the indicial locus of this system. In the Table 1 we can see that for the non trivial element of the indicial locus the number -1 is always a Kowalevski exponent and this can be considered as an application of the Theorem 2. In addition the results (Theorems 3, 4 and 5, Proposition 3 and Corollaries 1 and 2) we have seen in the Section 4 are also true for this system.

So Table 1 is a strong evidence that this system is integrable. In fact it has been recently proved that it is completely integrable. The proof can be found in [16].

Table 1: Kowalevski exponents of Kozlov-Treshchev Birkhoff system

| Vector $a^{(0)}$ | Kowalevski <br> Exponents | Vector $a^{(0)}$ | Kowalevski <br> Exponents |
| :--- | :--- | :--- | :--- |
| $(0,0,0,0,0,0)$ | $1,1,1,1,1,1,2,2,2,2,2,2$ | $\left(0,0,0,0,0, \frac{1}{2}\right)$ | $-1,1,1,1,1,1,2,2,2,2,2,3$ |
| $(0,0,0,0,2,0)$ | $-2,-1,1,1,1,1,1,2,2,2,2,4$ | $(0,0,0,1,0,0)$ | $-1,1,1,1,1,1,2,2,2,2,2,3$ |
| $(0,0,1,0,0,0)$ | $-1,1,1,1,1,1,2,2,2,3,3,4$ | $(0,1,0,0,0,0)$ | $-1,1,1,1,1,1,2,2,2,3,3,3$ |
| $(1,0,0,0,0,0)$ | $-1,1,1,1,1,1,2,2,2,2,2,3$ | $\left(0,0,0,1,0, \frac{1}{2}\right)$ | $-1,-1,1,1,1,1,1,2,2,2,3,3$ |
| $(0,0,3,0,0,2)$ | $-3,-1,1,1,1,1,1,2,2,2,4,5$ | $\left(0,1,0,0,0, \frac{1}{2}\right)$ | $-1,-1,1,1,1,1,1,2,2,3,3,4$ |
| $\left(1,0,0,0,0, \frac{1}{2}\right)$ | $-1,-1,1,1,1,1,1,2,2,2,3,3$ | $(0,0,0,1,2,0)$ | $-2,-1,-1,1,1,1,1,2,2,2,3,4$ |
| $\left(0,2,0,2,0, \frac{1}{2}\right)$ | $-2,-1,-1,1,1,1,1,2,2,3,4,5$ | $\left(1,0,0,1,0, \frac{1}{2}\right)$ | $-1,-1,-1,1,1,1,1,2,2,2,3,4$ |
| $\left(0,5,8,0,0, \frac{9}{2}\right)$ | $-5,-3,-1,1,1,1,1,2,4,6,7,7$ | $\left(2,2,0,0,0, \frac{1}{2}\right)$ | $-2,-1,-1,1,1,1,1,2,2,3,4,5$ |
| $(0,0,3,1,0,2)$ | $-3,-1,-1,1,1,1,1,2,2,2,4,6$ | $(1,0,3,0,0,2)$ | $-3,-1,-1,1,1,1,1,2,2,2,4,6$ |
| $(0,0,4,1,6,0)$ | $-3,-2,-1,1,1,1,1,2,2,2,4,7$ | $(1,0,0,1,2,0)$ | $-1,-1,-1,1,1,1,2,2,2,4,4,6$ |
| $(0,6,10,0,12,0)$ | $-5,-3,-2,-1,1,1,1,2,4,6,8,8$ | $(1,0,4,0,6,0)$ | $-3,-2,-1,-1,1,1,1,2,2,2,4,7$ |


| Vector $a^{(0)}$ | Kowalevski <br> Exponents | Vector $a^{(0)}$ | Kowalevski <br> Exponents |
| :--- | :--- | :--- | :--- |
| $(0,0,4,0,6,0)$ | $-3,-2,-1,1,1,1,1,2,2,2,4,6$ | $(0,1,0,0,2,0)$ | $-1,-1,1,1,1,1,1,1,2,2,3,3,5$ |
| $(1,0,0,0,2,0)$ | $-2,-1,-1,1,1,1,1,2,2,2,3,4$ | $(0,0,1,1,0,0)$ | $-1,-1,1,1,1,1,2,2,2,3,4,4$ |
| $(1,0,0,1,0,0)$ | $-1,-1,1,1,1,1,2,2,2,2,2,4$ | $(0,2,2,0,0,0)$ | $-2,-1,1,1,1,1,2,3,4,4,4,6$ |
| $(1,0,1,0,0,0)$ | $-1,-1,1,1,1,1,2,2,2,3,4,4$ | $(2,2,0,0,0,0)$ | $-2,-1,1,1,1,1,2,2,2,3,4,4$ |
| $(2,2,0,0,2,0)$ | $-2,-2,-1,-1,1,1,1,2,2,3,4,6$ | $(1,0,1,1,0,0)$ | $-1,-1,-1,1,1,1,2,2,2,3,4,5$ |
| $(3,4,3,0,0,0)$ | $-3,-2,-1,1,1,1,2,3,4,5,6,8$ | $(1,0,3,1,0,2)$ | $-3,-1,-1,-1,1,1,1,2,2,2,4,7$ |
| $\left(3,4,0,3,0, \frac{1}{2}\right)$ | $-3,-2,-1,-1,1,1,1,2,2,3,4,7$ | $(7,12,15,0,0,8)$ | $-7,-5,-3,-1,1,1,1,2,4,6,8,14$ |
| $(1,0,4,1,6,0)$ | $-3,-2,-1,-1,-1,1,1,2,2,2,4,8$ | $(8,14,18,0,20,0)$ | $-7,-5,-3,-2,-1,1,1,2,4,6,8,16$ |

The following Toda lattices of types of the Lie Algebras $A_{3}, B_{3}, C_{3}, D_{4}$ and $G_{2}$. The root systems of these Lie Algebras can be in the books [27] and [43]. The following systems are found in [36], [37], [38] and [42]. Some further investigation on Hamiltonian systems can be seen in [13] and [29]. In [45] there are some properties of Poisson Brackets.

### 4.2.2 Toda lattice of type $A_{3}$

We consider the Hamiltonian system derived by the Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i=1}^{4} p_{i}^{2}+\sum_{i=1}^{3} \mathbf{e}^{q_{i}-q_{i+1}} . \tag{61}
\end{equation*}
$$

Due to (52) we have that

$$
\begin{array}{ll}
A=\mathrm{Id}_{4} & c_{i}=1, \text { for } i=1,2,3 \\
\nu_{1}=(1,-1,0,0) & \nu_{2}=(0,1,-1,0) \\
\nu_{3}=(0,0,1,-1) . &
\end{array}
$$

The indicial locus of this system and the corresponding Kowalevski exponents of this system are shown in the Table 2

| Vector $a^{(0)}$ | Kowalevski <br> Exponents | Vector $a^{(0)}$ | Kowalevski <br> Exponents |
| :--- | :--- | :--- | :--- |
| $(0,0,0)$ | $1,1,1,2,2,2$ | $(0,0,1)$ | $-1,1,1,2,2,3$ |
| $(0,1,0)$ | $-1,1,1,2,3,3$ | $(1,0,0)$ | $-1,1,1,2,2,3$ |
| $(0,2,2)$ | $-2,-1,1,2,3,4$ | $(1,0,1)$ | $-1,-1,1,2,2,4$ |
| $(2,2,0)$ | $-2,-1,1,2,3,4$ | $(3,4,3)$ | $-3,-2,-1,2,3,4$ |

Table 2: Kowalevski exponents of Toda lattice of type $A_{3}$

From Table 2, as in the previous subsection, there is strong evidence that this system is integrable.

### 4.2.3 Toda lattice of type $B_{3}$

We consider the Hamiltonian system derived by the Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i=1}^{3} p_{i}^{2}+\sum_{i=1}^{2} \mathbf{e}^{q_{i}-q_{i+1}}+\mathbf{e}^{q_{3}} . \tag{62}
\end{equation*}
$$

Due to (52), we have that

$$
\begin{array}{ll}
A=\mathrm{Id}_{3} & c_{i}=1, \text { for } i=1,2,3 \\
\nu_{1}=(1,-1,0) & \nu_{2}=(0,1,-1) \\
\nu_{3}=(0,0,1) . &
\end{array}
$$

The indicial locus of this system and the corresponding Kowalevski exponents of this system are shown in the table below.

| Vector $a^{(0)}$ | Kowalevski <br> Exponents | Vector $a^{(0)}$ | Kowalevski <br> Exponents |
| :--- | :--- | :--- | :--- |
| $(0,0,0)$ | $1,1,1,2,2,2$ | $(0,0,2)$ | $-1,1,1,2,2,4$ |
| $(0,1,0)$ | $-1,1,1,2,3,3$ | $(1,0,0)$ | $-1,1,1,2,2,3$ |
| $(0,4,6)$ | $-3,-1,1,2,4,6$ | $(1,0,2)$ | $-1,-1,1,2,2,5$ |
| $(2,2,0)$ | $-2,-1,1,2,3,4$ | $(6,10,12)$ | $-5,-3,-1,2,4,6$ |

Table 3: Kowalevski exponents of Toda lattice of type $B_{3}$

From this table, as in the previous cases, there is strong evidence that this system is integrable.

### 4.2.4 Toda lattice of type $C_{3}$

We consider the Hamiltonian system derived by the Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i=1}^{3} p_{i}^{2}+\sum_{i=1}^{2} \mathbf{e}^{q_{i}-q_{i+1}}+\mathbf{e}^{2 q_{3}} \tag{63}
\end{equation*}
$$

Again due to (52) we have that

$$
\begin{array}{ll}
A=\mathrm{Id}_{3} & c_{i}=1, \text { for } i=1,2,3 \\
\nu_{1}=(1,-1,0) & \nu_{2}=(0,1,-1) \\
\nu_{3}=(0,0,2) . &
\end{array}
$$

The indicial locus of this system and the corresponding Kowalevski exponents of this system are shown in the Table 4.

| Vector $a^{(0)}$ | Kowalevski <br> Exponents | Vector $a^{(0)}$ | Kowalevski <br> Exponents |
| :--- | :--- | :--- | :--- |
| $(0,0,0)$ | $1,1,1,2,2,2$ | $\left(0,0, \frac{1}{2}\right)$ | $-1,1,1,2,2,3$ |
| $(0,1,0)$ | $-1,1,1,2,3,4$ | $(1,0,0)$ | $-1,1,1,2,2,3$ |
| $(0,3,2)$ | $-3,-1,1,2,4,5$ | $\left(1,0, \frac{1}{2}\right)$ | $-1,-1,1,2,2,4$ |
| $(2,2,0)$ | $-2,-1,1,2,3,6$ | $\left(5,8, \frac{9}{2}\right)$ | $-5,-3,-1,2,4,6$ |

Table 4: Kowalevski exponents of Toda lattice of type $C_{3}$

Again there is a strong evidence that this system is integrable.

### 4.2.5 Toda lattice of type $D_{4}$

We consider the Hamiltonian system derived by the Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i=1}^{4} p_{i}^{2}+\sum_{i=1}^{3} \mathbf{e}^{q_{i}-q_{i+1}}+\mathbf{e}^{q_{3}+q_{4}} \tag{64}
\end{equation*}
$$

Due to (52) we have that

$$
\begin{array}{ll}
A=\mathrm{Id}_{4} & c_{i}=1, \text { for } i=1,2,3,4 \\
\nu_{1}=(1,-1,0,0) & \nu_{2}=(0,1,-1,0) \\
\nu_{3}=(0,0,1,-1) & \nu_{4}=(0,0,1,1) .
\end{array}
$$

The indicial locus of this system and the corresponding Kowalevski exponents of this system are shown in the Table 5.

| Vector $a^{(0)}$ | Kowalevski <br> Exponents | Vector $a^{(0)}$ | Kowalevski <br> Exponents |
| :--- | :--- | :--- | :--- |
| $(0,0,0,0)$ | $1,1,1,1,2,2,2,2$ | $(0,0,0,1)$ | $-1,1,1,1,2,2,2,3$ |
| $(0,0,1,0)$ | $-1,1,1,1,2,2,2,3$ | $(0,1,0,0)$ | $-1,1,1,1,2,3,3,3$ |
| $(1,0,0,0)$ | $-1,1,1,1,2,2,2,3$ | $(0,0,1,1)$ | $-1,-1,1,1,2,2,2,4$ |
| $(0,2,0,2)$ | $-2,-1,1,1,2,3,4,4$ | $(0,2,2,0)$ | $-2,-1,1,1,2,3,4,4$ |
| $(1,0,0,1)$ | $-1,-1,1,1,2,2,2,4$ | $(1,0,1,0)$ | $-1,-1,1,1,2,2,2,4$ |
| $(2,2,0,0)$ | $-2,-1,1,1,2,3,4,4$ | $(0,4,3,3)$ | $-3,-2,-1,1,2,3,4,6$ |
| $(1,0,1,1)$ | $-1,-1,-1,1,2,2,2,5$ | $(3,4,0,3)$ | $-3,-2,-1,1,2,3,4,6$ |
| $(3,4,3,0)$ | $-3,-2,-1,1,2,3,4,6$ | $(6,10,6,6)$ | $-5,-3,-3,-1,2,4,4,6$ |

Table 5: Kowalevski exponents of Toda lattice of type $D_{4}$

As in the previous cases, there is a strong evidence that this system is integrable.

### 4.2.6 Toda lattice of type $G_{2}$

We consider the Hamiltonian system derived by the Hamiltonian.

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i=1}^{2} p_{i}^{2}+\mathbf{e}^{\sqrt{3} q_{1}-3 q_{2}}+\mathbf{e}^{2 q_{2}} \tag{65}
\end{equation*}
$$

Due to (52) we have that

$$
\begin{array}{ll}
A=\operatorname{Id}_{2} & c_{i}=1, \text { for } i=1,2 \\
\nu_{1}=(\sqrt{3},-3) & \nu_{2}=(0,2)
\end{array}
$$

This Hamiltonian system can be derived also from the Hamiltonian

$$
\begin{equation*}
H=\frac{1}{2} \sum_{i=1}^{3} p_{i}^{2}+\mathbf{e}^{q_{1}-q_{2}}+\mathbf{e}^{-2 q_{1}+q_{2}+q_{3}} . \tag{66}
\end{equation*}
$$

Due to (52), we have that

$$
\begin{array}{ll}
A=\mathrm{Id}_{3} & c_{i}=1, \text { for } i=1,2 \\
\nu_{1}=(1,-1,0) & \nu_{2}=(-2,1,1)
\end{array}
$$

The indicial locus of this system and the corresponding Kowalevski exponents of this system are shown in the Table 6.

| Vector $a^{(0)}(65)$ | Vector $a^{(0)}(66)$ | Kowalevski <br> exponents |
| :--- | :--- | :--- |
| $(0,0)$ | $(0,0)$ | $1,1,2,2$ |
| $\left(0, \frac{1}{2}\right)$ | $\left(0, \frac{1}{3}\right)$ | $-1,1,2,5$ |
| $\left(\frac{1}{6}, 0\right)$ | $(1,0)$ | $-1,1,2,3$ |
| $\left(\frac{5}{3}, 3\right)$ | $\left(6, \frac{10}{3}\right)$ | $-5,-1,2,6$ |

Table 6: Kowalevski exponents of Toda lattice of type $G_{2}$

From this table, as in the previous cases, there is strong evidence that this system is integrable.

In fact the complete integrability of all of these systems we have just seen is well known. Adler and Moerbeke found in [3] a necessary and sufficient condition for the algebraic complete integrability of this kind of systems. Let $N$ be the matrix whose rows are the vectors $\nu_{i}$ of each system. If the matrix $N$ has a full rank, then the corresponding system is algebraically completely integrable if and only if

$$
2\left(N N^{T}\right)_{i j}\left(N N^{T}\right)_{i j}^{-1} \quad \forall i \neq j
$$

are non positive integers.

All the above systems, except Kozlov-Treshchev Birkhoff, are algebraically completely integrable. The matrix $W=w_{i j}$ where

$$
w_{i j}=\left(N N^{T}\right)_{i j}\left(N N^{T}\right)_{i j}^{-1}
$$

which corresponds to each systems is displayed below.
Toda lattice of type $A_{3}: \quad W=\left(\begin{array}{ccc}3 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3\end{array}\right)$
Toda lattice of type $B_{3}$ and $C_{3}: W=\left(\begin{array}{ccc}4 & -2 & 0 \\ -2 & 8 & -4 \\ 0 & -4 & 6\end{array}\right)$
$\begin{array}{ll}\text { Toda lattice of type } D_{4}: & W=\left(\begin{array}{cccc}4 & -2 & 0 & 0 \\ -2 & 8 & -2 & -2 \\ 0 & -2 & 4 & 0 \\ 0 & -2 & 0 & 4\end{array}\right) \\ \text { Toda lattice of type } G_{2}: & W=\left(\begin{array}{cc}8 & -6 \\ -6 & 8\end{array}\right) .\end{array}$

## 5 Lotka-Volterra systems

The Lotka-Volterra equations are

$$
\begin{equation*}
\dot{x}_{j}=\sum_{k=1}^{n} a_{j k} x_{j} x_{k}, \text { for } j=1,2, \ldots, n, \tag{67}
\end{equation*}
$$

where the matrix $A=\left(a_{i j}\right)$ is a fixed matrix. In this thesis we restrict our attention to the case where $A$ is a skew-symmetric. In [20] the Hamiltonian formulation is obtained based on Volterra's work using a symplectic realization from $\mathbf{R}^{2 n} \mapsto \mathbf{R}^{n}$. He defined the variables

$$
q_{i}(t)=\int_{0}^{t} u_{i}(s) d s
$$

and

$$
p_{i}(t)=\ln \left(\dot{q}_{i}\right)-\frac{1}{2} \sum_{k=1}^{n} a_{i k} q_{k},
$$

for $i=1,2, \ldots, n$, for a skew-symmetric matrix $A$, as in our case. Now the number of variables is doubled and Volterra's transformation is

$$
\begin{align*}
\mathbf{R}^{2 n} & \mapsto \mathbf{R}^{n}  \tag{68}\\
\left(q_{1}, \ldots, q_{n}, p_{1}, \ldots, p_{i}\right) & \mapsto\left(x_{1}, \ldots, x_{n}\right),
\end{align*}
$$

where

$$
x_{i}=e^{p_{i}+\frac{1}{2} \sum_{k=1}^{n} a_{i k} q_{k}} \text { for } i=1,2, \ldots, n .
$$

The Hamiltonian in these $(q, p)$ coordinates becomes

$$
H=\sum_{i=1}^{n} x_{i}=\sum_{i=1}^{n} \dot{q}_{i}=\sum_{i=1}^{n} \mathrm{e}^{p_{i}+\frac{1}{2} \sum_{k=1}^{n} a_{i k} q_{k}} .
$$

The vector field, (67), for which $A$ is a skew-symmetric can be written as

$$
\begin{align*}
& \dot{q}_{i}=\frac{\partial H}{\partial p_{i}}=\left\{q_{i}, H\right\},  \tag{69}\\
& \dot{p}_{i}=-\frac{\partial H}{\partial q_{i}}=\left\{p_{i}, H\right\},
\end{align*}
$$

$i=1,2, \ldots, n$, and the bracket $\{\cdot, \cdot\}$ is the Poisson canonical one, that is:

$$
\left\{q_{i}, p_{j}\right\}=\delta_{i j}=\left\{\begin{array}{ll}
1, & \text { if } i=j \\
0, & \text { if } i \neq j
\end{array}, \quad i, j=1,2, \ldots, n\right.
$$

All other Poisson Brackets are equal to zero. The corresponding Poisson Bracket in the $x$ coordinates is

$$
\left\{x_{i}, x_{j}\right\}=a_{i j} x_{i} x_{j}, \quad i, j=1,2, \ldots, n .
$$

Equations (67) are obtained by using this Poisson bracket and the Hamiltonian, $H=$ $x_{1}+x_{2}+\ldots+x_{n}$.

The Lotka - Volterra equations were studied by many authors in its various aspects, e.g. complete integrability [10] Poisson and bi-Hamiltonian formulation ([13] and [29]), stability of solutions and Darboux polynomials ([12] and [39]).

### 5.1 The three-dimensional case

Firstly we search for necessary conditions for algebraic integrability on the three-dimensional case of this system. For $n=3$ we have the matrix

$$
A=\left(\begin{array}{ccc}
0 & a & b  \tag{70}\\
-a & 0 & c \\
-b & -c & 0
\end{array}\right)
$$

From (29) the Kowalevski matrix is

$$
\left(\begin{array}{ccc}
a x_{2}^{(0)}+b x_{3}^{(0)}+1 & a x_{1}^{(0)} & b x_{1}^{(0)}  \tag{71}\\
-a x_{2}^{(0)} & -a x_{1}^{(0)}+c x_{3}^{(0)}+1 & c x_{2}^{(0)} \\
-b x_{3}^{(0)} & -c x_{3}^{(0)} & -b x_{1}^{(0)}-c x_{2}^{(0)}+1
\end{array}\right)
$$

where $x^{(0)}=\left(x_{1}^{(0)}, x_{2}^{(0)}, x_{3}^{(0)}\right)$ is an element of the indicial locus, that is a solution of the simultaneous equation (27), which in this problem is written as

$$
\begin{align*}
& x_{1}^{(0)}+a x_{1}^{(0)} x_{2}^{(0)}+b x_{1}^{(0)} x_{3}^{(0)}=0, \\
& x_{2}^{(0)}-a x_{1}^{(0)} x_{2}^{(0)}+c x_{2}^{(0)} x_{3}^{(0)}=0,  \tag{72}\\
& x_{3}^{(0)}-b x_{1}^{(0)} x_{3}^{(0)}-c x_{2}^{(0)} x_{3}^{(0)}=0 .
\end{align*}
$$

In Table 7 we can see the corresponding Kowalevski exponents, the eigenvalues of (71), for each element of the indicial locus.

| Vector $x^{(0)}$ | Kowalevski <br> exponents | Vector $x^{(0)}$ | Kowalevski <br> exponents |
| :---: | :---: | :---: | :---: |
| $(0,0,0)$ | $1,1,1$ | $\left(0, \frac{1}{c},-\frac{1}{c}\right)$ | $-1,1, \frac{a-b+c}{c}$ |
| $\left(\frac{1}{b}, 0,-\frac{1}{b}\right)$ | $-1,1,-\frac{a-b+c}{b}$ | $\left(\frac{1}{a},-\frac{1}{a}, 0\right)$ | $-1,1, \frac{a-b+c}{a}$ |

Table 7: Kowalevski exponents of $3 \times 3$ Lotka-Volterra equations

Kowalevski exponents must be integers in order to have an integrable system. So we have to solve the simultaneous equations

$$
\begin{equation*}
\frac{a-b+c}{a}=k_{1}, \quad \frac{a-b+c}{c}=k_{2}, \quad-\frac{a-b+c}{b}=k_{3}, \tag{73}
\end{equation*}
$$

where $k_{1}, k_{2}, k_{3} \in \mathbf{Z}$. The case $b=a+c$ for which $k_{1}=k_{2}=k_{3}=0$ is investigated below. Solving (73) we find that

$$
\begin{align*}
& k_{3}=\frac{k_{1} k_{2}}{k_{1} k_{2}-k_{1}-k_{2}}, \begin{cases}c=\frac{k_{1}}{k_{2}} a, & b=\frac{k_{1}+k_{2}-k_{1} k_{2}}{k_{2}} a \\
a=\frac{k_{2}}{k_{1}} c, & b=\frac{k_{1}+k_{2}-k_{1} k_{2}}{k_{1}} c\end{cases}  \tag{74}\\
& k_{2}=\frac{k_{1} k_{3}}{k_{1} k_{3}-k_{1}-k_{3}}, \quad \begin{cases}b=-\frac{k_{1}}{k_{3}} a, & c=\frac{k_{1}+k_{3}-k_{1} k_{3}}{k_{3}} a \\
a=-\frac{k_{3}}{k_{1}} b, & c=\frac{k_{1}+k_{3}-k_{1} k_{3}}{k_{1}} b\end{cases}  \tag{75}\\
& k_{1}=\frac{k_{2} k_{3}}{k_{2} k_{3}-k_{2}-k_{3}}, \quad \begin{cases}b=-\frac{k_{2}}{k_{3}} c, & a=\frac{k_{2} k_{3}-k_{2}-k_{3}}{k_{3}} c \\
c=-\frac{k_{3}}{k_{2}} b, & a=\frac{k_{2}+k_{3}-k_{2} k_{3}}{k_{2}} b\end{cases} \tag{76}
\end{align*}
$$

since the Kowalevski exponents cannot be zero. We examine the solution

$$
\begin{equation*}
k_{3}=\frac{k_{1} k_{2}}{k_{1} k_{2}-k_{1}-k_{2}}, \quad b=\frac{k_{1}+k_{2}-k_{1} k_{2}}{k_{2}} a, \quad c=\frac{k_{1}}{k_{2}} a . \tag{77}
\end{equation*}
$$

to check for which integer values for $k_{1}$ and $k_{2}$ the fraction,

$$
\begin{equation*}
k_{3}=\frac{k_{1} k_{2}}{k_{1} k_{2}-k_{1}-k_{2}}, \tag{78}
\end{equation*}
$$

is an integer. Firstly if we take positive values for both $k_{1}$ and $k_{2}$, then

$$
k_{1} k_{2}-k_{1}-k_{2}>0 \Rightarrow k_{1}>2 \text { or } k_{2}>2
$$

Proof
If $k_{1} k_{2}-k_{1}-k_{2}>0, k_{1} \leq 2$ and $k_{2} \leq 2$, then we have

- If $k_{1}=1$, then

$$
k_{1} k_{2}-k_{1}-k_{2}=k_{2}-1-k_{2}=-1<0
$$

This is a contradiction.

- If $k_{1}=2$, then

$$
k_{1} k_{2}-k_{1}-k_{2}=2 k_{2}-2-k_{2}=k_{2}-2 \leq 0 .
$$

This is a contradiction.
The proof for $k_{2}=1$ and $k_{2}=2$ is similar.

Therefore for these values of $k_{1}$ and $k_{2}$

$$
\begin{align*}
\frac{k_{1} k_{2}}{k_{1} k_{2}-k_{1}-k_{2}}=1+\frac{k_{1}+k_{2}}{k_{1} k_{2}-k_{1}-k_{2}} \in \mathbf{Z} & \Longleftrightarrow k_{1} k_{2}-k_{1}-k_{2} \leq k_{1}+k_{2}  \tag{79}\\
& \Longleftrightarrow k_{1} \leq \frac{2 k_{2}}{k_{2}-2} .
\end{align*}
$$

Now the function

$$
f\left(k_{2}\right)=\frac{2 k_{2}}{k_{2}-2}
$$

is decreasing since its derivative is

$$
f^{\prime}\left(k_{2}\right)=-\frac{4}{\left(k_{2}-2\right)^{2}} .
$$

Since $f(3) \leq 6$,

$$
f\left(k_{2}\right) \leq 6 \text { for } k_{2} \geq 3 \Longrightarrow k_{1} \leq 6
$$

Now we observe that the fraction (78) is symmetric with respect to $k_{1}$ and $k_{2}$. So for positive values of $k_{1}$ and $k_{2}$ it is enough to check its values for $1 \leq k_{1} \leq k_{2} \leq 6$.

$$
\text { For } 1 \leq k_{1} \leq k_{2} \leq 6 \begin{cases}k_{1}=1 & \Longrightarrow \forall k_{2} \in \mathbf{N}  \tag{80}\\ k_{1}=2 & \Longrightarrow k_{2} \in\{3,4,6\} \\ k_{1}=3 & \Longrightarrow k_{2} \in\{3,6\} \\ k_{1}=4 & \Longrightarrow k_{2}=4\end{cases}
$$

Secondly we take positive values for $k_{1}$ and negative values for $k_{2}$. Let now, $k_{2}=-x, x>$ 0 . Then

$$
k_{3}=\frac{-k_{1} x}{-k_{1} x-k_{1}+x}=\frac{k_{1} x}{k_{1} x+k_{1}-x} .
$$

For $k_{1}=1$ the Kowalevski exponent, $k_{3}$, is an integer. We have also that

$$
\begin{equation*}
k_{1} x+k_{1}-x=k_{1}+\left(k_{1}-1\right) x>0, \quad \forall k_{1} \geq 1, \quad \forall x>0, \tag{81}
\end{equation*}
$$

and, since $k_{3}$ is an integer, we have that

$$
\begin{equation*}
k_{1} x+k_{1}-x \leq k_{1} x \Longleftrightarrow k_{1}-x \leq 0 \Longleftrightarrow k_{1} \leq x \tag{82}
\end{equation*}
$$

Writing $k_{1} x$ in the form

$$
k_{1} x=1 \cdot\left(k_{1} x+k_{1}-x\right)+x-k_{1},
$$

we can see that

$$
0 \leq x-k_{1}<k_{1} x
$$

by using the results of (81) and (82). That means that $k_{1} x+k_{1}-x$ cannot be a divisor of $k_{1} x$ if $k_{1} \neq x$ and so $k_{3}$ is not an integer. Then in this case $k_{3}$ is an integer only if

$$
k_{1} \in\{1, x\} \Longrightarrow k_{1} \in\left\{1,-k_{2}\right\}
$$

If we take negative values for $k_{1}$ and positive values for $k_{2}$, the result is similar, i.e., $k_{3}$ is an integer only if

$$
k_{2} \in\{1, x\} \Longrightarrow k_{2} \in\left\{1,-k_{1}\right\}
$$

because of the symmetry of the fraction (78).
If we take negative values for both $k_{1}$ and $k_{2}$, then

$$
k_{3}=\frac{x y}{x y+x+y}=1-\frac{x+y}{x y+x+y},
$$

where $k_{1}=-x$ and $k_{2}=-y$ with $x, y>0$. We have that $x y>0$ and $x y+x+y>0$ so that the Kowalevski exponent is an integer if

$$
x y+x+y \leq x+y \Longleftrightarrow x y \leq 0,
$$

that is a contradiction and so in this case $k_{3}$ cannot be an integer.
We examined the case that $k_{1} k_{2}-k_{1}-k_{2} \neq 0$.

The case

$$
k_{1} k_{2}-k_{1}-k_{2}=0 \Rightarrow k_{2}=\frac{k_{1}}{k_{1}-1}=1+\frac{1}{k_{1}-1} .
$$

In order to have integer values of $k_{2}$,

$$
k_{1}-1 \in\{ \pm 1\} \Rightarrow k_{1} \in\{0,2\}
$$

So for non zero Kowalevski exponents we have that $k_{1}=k_{2}=2$. For this case the triple,

$$
-1,1, k_{3},
$$

does not appear in the set of the Kowalevski exponents, since its denominator is zero.
Many of the cases that are obtained below are isomorphic in the following sense.
We firstly see that the Lotka-Volterra equations in $n$ dimensions can be transformed into a simpler form if the elements of the matrix $A=\left(a_{i j}\right)$ in (67) are multiples of a constant parameter. Precisely, if

$$
a_{i j}=C_{i j} a, \quad \text { where } C_{i j} \in \mathbf{R}, \quad i, j=1,2, \ldots, n,
$$

then the Lotka-Volterra system (67) can be written as

$$
\dot{u}_{i}=\sum_{j=1}^{n} C_{i j} u_{i} u_{j}, \quad i=1,2 \ldots, n,
$$

using the transformation

$$
\begin{equation*}
u_{i}=a \cdot x_{i}, \quad i=1,2 \ldots, n . \tag{83}
\end{equation*}
$$

Secondly we consider two systems isomorphic if there is an invertible linear transformation mapping one to other. Special cases of isomorphic systems are those that are derived from a given system by applying a permutation, $\sigma \in S_{n}$, setting

$$
X_{i} \longmapsto x_{\sigma(i)}, \quad i=1,2, \ldots, n .
$$

We illustrate an example for $n=3$.

Example 6 We prove that the system

$$
\begin{align*}
\dot{x}_{1} & =a x_{1} x_{2}-\frac{a}{3} x_{1} x_{3} \\
\dot{x}_{2} & =-a x_{1} x_{2}+\frac{2 a}{3} x_{2} x_{3}  \tag{84}\\
\dot{x}_{3} & =\frac{a}{3} x_{1} x_{3}-\frac{2 a}{3} x_{2} x_{3}
\end{aligned} \xrightarrow[u_{i}=a \cdot x_{i}]{ } \quad \begin{aligned}
& \dot{x}_{1}=3 x_{1} x_{2}-x_{1} x_{3} \\
& \dot{x}_{2}=-3 x_{1} x_{2}+2 x_{2} x_{3} \\
& \dot{x}_{3}=x_{1} x_{3}-2 x_{2} x_{3}
\end{align*}
$$

is isomorphic to the system

$$
\begin{align*}
& \dot{x}_{1}=a x_{1} x_{2}-2 a x_{1} x_{3} \\
& \dot{x}_{2}=-a x_{1} x_{2}+3 a x_{2} x_{3}  \tag{85}\\
& \dot{x}_{3}=2 a x_{1} x_{3}-3 a x_{2} x_{3}
\end{aligned} \xrightarrow[u_{i}=a \cdot x_{i}]{ } \quad \begin{aligned}
& \dot{x}_{1}=x_{1} x_{2}-2 x_{1} x_{3} \\
& \dot{x}_{2}=-x_{1} x_{2}+3 x_{2} x_{3} \\
& \dot{x}_{3}=2 x_{1} x_{3}-3 x_{2} x_{3}
\end{align*}
$$

Applying $\sigma=\left(\begin{array}{lll}1 & 3 & 2\end{array}\right)$ to the system (84) we have that

$$
\begin{gathered}
\dot{X}_{1}=\dot{x}_{\sigma(1)}=\dot{x}_{3}=x_{1} x_{3}-2 x_{2} x_{3}=X_{2} X_{1}-2 X_{3} X_{1} \\
\dot{X}_{2}=\dot{x}_{\sigma(2)}=\dot{x}_{1}=3 x_{1} x_{2}-x_{1} x_{3}=3 X_{2} X_{3}-X_{2} X_{1}, \\
\dot{X}_{3}=\dot{x}_{\sigma(3)}=\dot{x}_{2}=-3 x_{1} x_{2}+2 x_{2} x_{3}=-3 X_{2} X_{3}+2 X_{3} X_{1}
\end{gathered}
$$

that is the vector field (85).
In Table 8 we can see the different values of $(a, b, c)$ of the solutions (74), (75) and (76) of the simultaneous equations (73) in order to have integer Kowalevski exponents for the Lotka Volterra system (67) in three dimensions derived from the matrix (70). We also can see the elements of the symmetric group $S_{3}$ that makes them isomorphic. Note that $\lambda \in$ Z.

Table 8: Cases with integer Kowalevski exponents

| Vector $(a, b, c)$ | Kowalevski <br> exponents | $\sigma$ |
| :---: | :---: | :---: |
| $\left(a, \frac{a}{\lambda}, \frac{a}{\lambda}\right)$ | $-1,1,1$ |  |
| $(a, a, \lambda a)$ | $-1,1, \lambda$ | $\sigma=\left(\begin{array}{ll}1 & 3\end{array}\right)$ |
| $(a, \lambda a,-a)$ | $-1,1,-\lambda$ | $\sigma=\left(\begin{array}{ll}2 & 3\end{array}\right)$ |
| $\left(a,-\frac{a}{3}, \frac{2 a}{3}\right)$ |  | $\sigma=\left(\begin{array}{lll}1 & 3 & 2\end{array}\right)$ |
| $\left(a,-\frac{2 a}{3}, \frac{a}{3}\right)$ | $-1,1,2$ | $\sigma=\left(\begin{array}{ll}1 & 3\end{array}\right)$ |
| $\left(a,-\frac{3 a}{2}, \frac{a}{2}\right)$ | $-1,1,3$ | $\sigma=\left(\begin{array}{ll}1 & 2\end{array}\right)$ |
| $\left(a,-\frac{a}{2}, \frac{3 a}{2}\right)$ | $-1,1,6$ | $\sigma=\left(\begin{array}{ll}2 & 3\end{array}\right)$ |
| $(a,-2 a, 3 a)$ |  |  |
| $(a,-3 a, 2 a)$ |  | $\sigma=\left(\begin{array}{ll}1 & 2\end{array}\right)$ |


| Vector ( $a, b, c$ ) | Kowalevski exponents | $\sigma$ |
| :---: | :---: | :---: |
| $\begin{gathered} (a, 0, a) \\ (a,-a, 0) \\ (0, b,-b) \end{gathered}$ | -1, 1, 2 | $\begin{array}{r} \sigma=\left(\begin{array}{lll} 1 & 3 & 2 \end{array}\right) \\ \sigma=\left(\begin{array}{lll} 1 & 2 & 3 \end{array}\right) \end{array}$ |
| $\begin{aligned} & \left(a,-\frac{a}{2}, \frac{a}{2}\right) \\ & (a,-a, 2 a) \\ & (a,-2 a, a) \end{aligned}$ | $\begin{aligned} & -1,1,2 \\ & -1,1,4 \end{aligned}$ | $\begin{aligned} \sigma & =\left(\begin{array}{ll} 1 & 3 \end{array}\right) \\ \sigma & =\left(\begin{array}{lll} 1 & 3 & 2 \end{array}\right) \end{aligned}$ |
| ( $a,-a, a)$ | -1, 1,3 |  |

Example 7 The periodic KM system in three dimensions is the system derived from the system

$$
\begin{equation*}
\dot{x}_{i}=\sum_{i=1}^{3} a_{i j} x_{i} x_{j}, \quad i=1,2,3, \tag{86}
\end{equation*}
$$

where $A$ is the $3 \times 3$ skew-symmetric matrix

$$
A=\left(\begin{array}{ccc}
0 & -1 & 1  \tag{87}\\
1 & 0 & -1 \\
-1 & 1 & 0
\end{array}\right)
$$

This system is a special case of the system (67) derived from the matrix (70), where $(a, b, c)=(-1,1,-1)$, that is, one of the cases in Table 8. We can see that the Kowalevski exponents of this system are $-1,1,3$. The system can be written in the Lax-pair form $\dot{L}=[L, B]$, where

$$
L=\left(\begin{array}{ccc}
0 & x_{1} & 1  \tag{88}\\
1 & 0 & x_{2} \\
x_{3} & 1 & 0
\end{array}\right), \quad B=\left(\begin{array}{ccc}
0 & 0 & x_{1} x_{2} \\
x_{2} x_{3} & 0 & 0 \\
0 & x_{3} x_{1} & 0
\end{array}\right)
$$

We have the constants of motion

$$
H_{k}=\operatorname{trace}\left(L^{k}\right), \quad k \in \mathbf{N} .
$$

The functions

$$
\begin{align*}
H_{2} & =x_{1}+x_{2}+x_{3}  \tag{89}\\
H_{3} & =1+x_{1} x_{2} x_{3}
\end{align*}
$$

are independent constants of motion in involution due to the Poisson Bracket derived from the Poisson matrix

$$
\left(\begin{array}{ccc}
0 & -x_{1} x_{2} & x_{1} x_{3}  \tag{90}\\
x_{1} x_{2} & 0 & -x_{2} x_{3} \\
-x_{1} x_{3} & x_{2} x_{3} & 0
\end{array}\right) .
$$

The positive Kowalevski exponents, 1 and 3, are the degrees of the constants of motion of the system. Therefore this system is integrable.

All of these systems are integrable in the sense of Liouville since there exist two constants of motion that are independent and in involution. The function

$$
H_{2}=x_{1}+x_{2}+x_{3}
$$

is the Hamiltonian for these systems because due to the Poisson Bracket that is derived from the Poisson matrix

$$
\left(\begin{array}{ccc}
0 & a x_{1} x_{2} & b x_{1} x_{3}  \tag{91}\\
-a x_{1} x_{2} & 0 & c x_{2} x_{3} \\
-b x_{1} x_{3} & -c x_{2} x_{3} & 0
\end{array}\right) .
$$

the equations (86), where $A=\left(a_{i j}\right)$ is the matrix in (70), can be written in the form

$$
\dot{x}_{i}=\left\{x_{i}, H_{2}\right\}, \quad i=1,2,3 .
$$

Therefore we need one more constant of motion that is independent and in involution with $H_{2}$. The rank of the Poisson matrix is two and there is a Casimir function, $F$, (since the Poisson matrix is degenerate). We also have the integral, $\mathrm{H}_{2}$, so and the system is always integrable. We can see the second constant of motion of each non isomorphic system in the Table 9.

| Vector <br> $(a, b, c)$ | Kowalevski <br> exponents | Casimir <br> function |
| :---: | :---: | :---: |
| $(a, 0, a)$ | $-1,1,2$ | $F=x_{1} x_{3}$ |
| $(a,-a, a)$ | $-1,1,3$ | $F=x_{1} x_{2} x_{3}$ |
| $\left(a,-\frac{a}{2}, \frac{a}{2}\right)$ | $-1,1,2$ | $F=x_{1} x_{2} x_{3}^{2}$ |
| $-1,1,4$ |  |  |
| $(a,-2 a, 3 a)$ | $-1,1,2$ |  |
|  | $-1,1,3$ |  |
|  | $F=x_{1}^{3} x_{2}^{2} x_{3}$ |  |
| $(a, a+c, c)$ | $-1,1,0$ | $F=\frac{x_{1}^{c} x_{3}^{a}}{x_{2}^{a+c}}$ |
| $\left(a, \frac{a}{\lambda}, \frac{a}{\lambda}\right)$ | $-1,1,1$ |  |
|  | $-1,1,-\lambda$ | $F=\frac{x_{3}^{2} x_{1}}{x_{2}}$ |

Table 9: Casimir function for each case

The Casimir function for each case can be written in a general form

$$
F=x_{1}^{c} x_{2}^{-b} x_{3}^{a} .
$$

We would like to classify the algebraic completely integrable Lotka-Volterra equations in three dimensions. In order to use Proposition 1 we have to check for which systems the Laurent solutions are of the form

$$
\begin{equation*}
x_{i}(t)=\frac{1}{t^{v_{i}}} \sum_{k=0}^{\infty} x_{i}^{(k)} t^{k}, \quad i=1,2, \ldots, n, \tag{92}
\end{equation*}
$$

where $v_{i}$ are the components of the weight vector $v$ that makes the vector field

$$
\dot{x}_{i}=f_{i}\left(x_{1}, \ldots, x_{n}\right), \quad i=1,2, \ldots, n,
$$

to be a weight homogeneous one. In this case

$$
f_{i}\left(x_{1}, \ldots, x_{3}\right)=\sum_{k=1}^{3} a_{i k} x_{i} x_{k}, \text { for } i=1,2, \ldots, 3
$$

So we get the weight vector $v=(1,1,1)$.
We have to check whether these systems are a.c.i., which means that the Laurent series of the solution $x_{1}, x_{2}$ and $x_{3}$ must have $n-1=2$ free parameters. If we use the remark in [4], the free parameters appear in a finite number of steps of calculation. The first thing to do is to substitute (92) into equations (67) derived from the matrix (70). After that we equate the coefficients of $t^{k}$. We have already equated the coefficients of $t^{-v_{i}-1}$ by solving the indicial equation to find $x_{i}^{(0)}$. Then we call Step $m(m \in \mathbf{N})$ if we equate the coefficients of $t^{-v_{i}-1+m}$ to find $x_{i}^{(m)}$. According to remark in [4] all the free parameters appear in the first $k_{p}$ Steps, where $k_{p}$ is the largest (positive) Kowalevski exponent of the system.

Table 10: Free parameters for the algebraically completely integrability of each system

| Vector <br> $(a, b, c)$ | Kowalevski <br> exponents | Free <br> parameters |
| :---: | :---: | :---: |
| $(a, 0, a)$ | $-1,1,2$ | $x_{3}^{(1)}, x_{3}^{(2)}$ |
| $(a,-a, a)$ | $-1,1,3$ | $x_{3}^{(1)}, x_{3}^{(3)}$ |
| $\left(a,-\frac{a}{2}, \frac{a}{2}\right)$ | $-1,1,2$ <br> $-1,1,4$ | $x_{3}^{(1)}, x_{3}^{(2)}$ |


| Vector <br> $(a, b, c)$ | Kowalevski <br> exponents | Free <br> parameters |
| :---: | :---: | :---: |
| $(a,-2 a, 3 a)$ | $-1,1,2$ <br> $-1,1,3$ <br> $-1,1,6$ | $x_{3}^{(1)}, x_{3}^{(2)}$ |
| $(a, a+c, c)$ | $-1,1,0$ | $x_{1}^{(1)}, x_{3}^{(0)}$ |
| $\left(a, \frac{a}{\lambda}, \frac{a}{\lambda}\right)$ | $-1,1,1$ <br> $-1,1, \lambda$ <br> $-1,1,-\lambda$ | $x_{1}^{(1)}, x_{2}^{(1)}$ |

We can see that in the three-dimensional case all of the systems displayed in Table 10 are a.c.i. showing the $n-1=2$ free parameters of each case.

Suppose that the Laurent solution of the system is

$$
\begin{align*}
& x_{1}(t)=\frac{1}{t^{\nu_{1}}} \sum_{k=0}^{\infty} x_{1}^{(k)} t^{k}, \text { with } x_{1}^{(0)} \neq 0, \\
& x_{2}(t)=\frac{1}{t^{\nu_{2}}} \sum_{k=0}^{\infty} x_{2}^{(k)} t^{k}, \text { with } x_{2}^{(0)} \neq 0,  \tag{93}\\
& x_{3}(t)=\frac{1}{t^{\nu_{3}}} \sum_{k=0}^{\infty} x_{3}^{(k)} t^{k}, \text { with } x_{3}^{(0)} \neq 0 .
\end{align*}
$$

If $\nu_{1}, \nu_{2}, \nu_{3} \leq 1$, then these systems can be investigated using Proposition 1 since $v=$ $(1,1,1)$. On the other hand, keeping in mind that $H=x_{1}+x_{2}+x_{3}$ is always a constant of motion, we have the following cases:
(i) $\nu_{1}=\nu_{2}=\nu>1$ and $\nu_{3}<\nu$, or
(ii) $\nu_{1}=\nu_{3}=\nu>1$ and $\nu_{2}<\nu$, or
(iii) $\nu_{2}=\nu_{3}=\nu>1$ and $\nu_{1}<\nu$, or
(iv) $\nu_{1}=\nu_{2}=\nu_{3}=\nu>1$.

If we take

$$
A=\left(\begin{array}{ccc}
0 & a & b  \tag{94}\\
-a & 0 & c \\
-b & -c & 0
\end{array}\right)
$$

equations (67) become

$$
\begin{gather*}
\dot{x}_{1}=a x_{1} x_{2}+b x_{1} x_{3},  \tag{95}\\
\dot{x}_{2}=-a x_{1} x_{2}+c x_{2} x_{3},  \tag{96}\\
\dot{x}_{3}=-b x_{1} x_{3}-c x_{2} x_{3} . \tag{97}
\end{gather*}
$$

We examine the cases below
(i) $\nu_{1}=\nu_{2}=\nu>1$ and $\nu_{3}<\nu$

Since $\nu_{1}=\nu_{2}=\nu$, we have that

$$
x_{1}^{(0)}=-x_{2}^{(0)}=\alpha \neq 0
$$

because $H=x_{1}+x_{2}+x_{3}$ is a constant of motion.
We also note that $\nu+\nu_{3}<2 \nu$ and $x_{1}^{(0)} x_{2}^{(0)} \neq 0$. Equating the coefficients of $t^{2 \nu}$ of the LHS and RHS of (95) or (96), we are led to

$$
a x_{1}^{(0)} x_{2}^{(0)}=0 \Longrightarrow a=0 .
$$

As we know that $\nu_{3}+1<\nu+\nu_{3}$, the coefficient of $t^{\nu+\nu_{3}}$ of the RHS of (97) must be equal to zero. So

$$
x_{3}^{(0)}\left(-b x_{1}^{(0)}-c x_{2}^{(0)}\right)=0,
$$

but $x_{3}^{(0)} \neq 0$ and $x_{2}^{(0)}=-x_{1}^{(0)} \neq 0$; therefore

$$
b=c \text {. }
$$

If $b=0 \Rightarrow c=0$, then from (95) and (96) we have that

$$
\dot{x}_{1}=\dot{x}_{2}=0 \Longrightarrow x_{1}, x_{2} \text { are constant functions, }
$$

that is a contradiction because $\nu_{1}=\nu_{2}=\nu>1$.
If $b \neq 0 \Rightarrow c \neq 0$, then the equations (95) and (96) become

$$
\begin{align*}
& \dot{x}_{1}=b x_{1} x_{3},  \tag{98}\\
& \dot{x}_{2}=b x_{2} x_{3} . \tag{99}
\end{align*}
$$

So we have that

$$
(98) \Longrightarrow \nu+1=\nu+\nu_{3} \Rightarrow \nu_{3}=1
$$

since $x_{1}^{(0)} x_{3}^{(0)} \neq 0$ and $x_{2}^{(0)} x_{3}^{(0)} \neq 0$. From equations (98) and (99) we have that

$$
\frac{\dot{x}_{1}}{x_{1}}=\frac{\dot{x}_{2}}{x_{2}}=b x_{3} \Longrightarrow x_{1}=\kappa x_{2}, \quad \kappa \text { is a constant. }
$$

However, we know that

$$
x_{1}^{(0)}=-x_{2}^{(0)} \Longrightarrow \kappa=-1 \Rightarrow x_{1}=-x_{2} .
$$

Equation (97) becomes

$$
\dot{x}_{3}=-b\left(-x_{2}\right) x_{3}-b x_{2} x_{3}=0 \Longrightarrow x_{3}=c, c \text { is a constant. }
$$

This is a contradiction because we proved above that $\nu_{3}=1$ and $x_{3}^{(0)} \neq 0$.
(ii)
$\nu_{1}=\nu_{3}=\nu>1$ and $\nu_{2}<\nu$
It leads to a contradiction, similarly as case (i) does.
(iii)
$\nu_{2}=\nu_{3}=\nu>1$ and $\nu_{1}<\nu$
It leads to a contradiction, similarly as case (i) does.
(iv) $\nu_{1}=\nu_{2}=\nu_{3}=\nu>1$

In this case, for $i=1,2,3$,

$$
x_{i}(t)=\frac{1}{t^{\nu}} \sum_{k=0}^{\infty} x_{i}^{(k)} t^{k},
$$

we have that the degrees of the leading term of the LHS of the equations (95), (96) and (97) are equal to $\nu+1$, but the degrees of the leading term RHS of these equations are equal to $2 \nu$ and so the coefficients of $\frac{1}{t^{\nu+k}}$ of the RHS of these equations must be zero for $k=2,3, \ldots, \nu$.

The coefficients of $\frac{1}{t^{\nu+k}}$ of the RHS of these equations must be zero for $k=2,3, \ldots, \nu$. The coefficients of $\frac{1}{t^{\nu+k}}, k=1,2, \ldots, \nu$, are given by the sums

$$
\begin{equation*}
S_{i, k}=\sum_{\lambda=0}^{\nu-k} x_{i}^{(\lambda)} u_{i, k}^{(\lambda)}, \text { for } i=1,2,3, \tag{100}
\end{equation*}
$$

where

$$
\begin{align*}
& u_{1, k}^{(\lambda)}=a x_{2}^{(\nu-k-\lambda)}+b x_{3}^{(\nu-k-\lambda)}, \\
& u_{2, k}^{(\lambda)}=-a x_{1}^{(\nu-k-\lambda)}+c x_{3}^{(\nu-k-\lambda)},  \tag{101}\\
& u_{3, k}^{(\lambda)}=-b x_{2}^{(\nu-k-\lambda)}-c x_{3}^{(\nu-k-\lambda)} .
\end{align*}
$$

Note that

$$
\begin{equation*}
u_{i, k}^{(\lambda)}=u_{i, j}^{(m)}, \text { if } k+\lambda=j+m . \tag{102}
\end{equation*}
$$

In addition

$$
S_{i, k}=0, \text { for } i=1,2,3 \text { and } k=2,3, \ldots, \nu .
$$

For $k=n$ sum, (100), becomes

$$
S_{i, \nu}=x_{i}^{(0)} u_{i, \nu}^{(0)}=0 \Longrightarrow u_{i, \nu}^{(0)}=0
$$

since $x_{i}^{(0)} \neq 0$.
For $k=\nu-1$ we have that

$$
\begin{align*}
S_{i, \nu-1} & =x_{i}^{(0)} u_{i, \nu-1}^{(0)}+x_{i}^{(1)} u_{i, \nu-1}^{(1)}=0 \\
(102) & \Rightarrow x_{i}^{(0)} u_{i, \nu-1}^{(0)}+x_{i}^{(1)} u_{i, \nu}^{(0)}=x_{i}^{(0)} u_{i, \nu-1}^{(0)}=0  \tag{103}\\
& \Rightarrow u_{i, \nu-1}^{(0)}=0 \text { because } x_{i}^{(0)} \neq 0 .
\end{align*}
$$

Let $m \in\{1,2, \ldots, \nu-1\}$ and assume that $u_{i, k}^{(0)}=0$ for $k>m$.
For $k=m$ we have that

$$
\begin{aligned}
S_{i, m} & =\sum_{\lambda=0}^{\nu-m} x_{i}^{(\lambda)} u_{i, m}^{(\lambda)}=x_{i}^{(0)} u_{i, m}^{(0)}+\sum_{\lambda=1}^{\nu-m} x_{i}^{(\lambda)} u_{i, m}^{(\lambda)} \\
& =x_{i}^{(0)} u_{i, m}^{(0)}+\sum_{\lambda=1}^{\nu-m} x_{i}^{(\lambda)} u_{i, m+\lambda}^{(0)}=x_{i}^{(0)} u_{i, m}^{(0)} .
\end{aligned}
$$

We saw above that $S_{i, m}=0$ if $m>1$ and, since $x_{i}^{(0)} \neq 0$, then $u_{i, m}^{(0)}=0$.
Now we equate the coefficients of $\frac{1}{t^{\nu+1}}$ of the both sides of the equations (95)-(97) and so

$$
S_{i, 1}=x_{i}^{(0)} u_{i, 1}^{(0)}=-\nu x_{i}^{(0)} \Longrightarrow \nu+u_{i, 1}^{(0)}=0 .
$$

Therefore we have that

$$
\begin{align*}
a x_{2}^{(\nu-1)}+b x_{3}^{(\nu-1)} & =-\nu, \\
-a x_{1}^{(\nu-1)}+c x_{3}^{(\nu-1)} & =-\nu,  \tag{104}\\
-b x_{2}^{(\nu-1)}-c x_{3}^{(\nu-1)} & =-\nu .
\end{align*}
$$

These simultaneous equations have solutions only if

$$
b=a+c .
$$

If $a=0 \Rightarrow b=c$ (obviously $b=c \neq 0$ ), then the system is isomorphic to the following

$$
\begin{aligned}
\dot{x}_{1} & =x_{1} x_{3} \\
\dot{x}_{2} & =x_{2} x_{3} \\
\dot{x}_{3} & =-x_{1} x_{3}-x_{2} x_{3}
\end{aligned}
$$

Equating the coefficients of $t^{-2 \nu}(\nu>1)$ in the first and second equations we have that

$$
x_{1}^{(0)} x_{3}^{(0)}=x_{2}^{(0)} x_{3}^{(0)}=0,
$$

that is impossible because $x_{i}^{(0)} \neq 0$, for $i=1,2,3$.

The same happens if $b c=0$. So in the following calculations we assume that $a b c \neq 0$. Firstly we see that there cannot be such a solution with $\nu>2$. Since $b=a+c$ and the function $H=x_{1}+x_{2}+x_{3}$ is a constant of motion, the Lotka-Volterra equations in three dimensions become

$$
\begin{align*}
\dot{x}_{1} & =a k x_{1}-a x_{1}^{2}+c x_{1} x_{3}, \\
\dot{x}_{2} & =-\dot{x}_{1}-\dot{x}_{3},  \tag{105}\\
\dot{x}_{3} & =-c k x_{3}+c x_{3}^{2}-a x_{1} x_{3},
\end{align*}
$$

where $k$ is the constant value of the function $H$. Using Maple we can see that if $k \neq 0$, then the solution is

$$
\begin{gather*}
x_{1}=\frac{k C_{1} \mathbf{e}^{a k t}}{C_{1} \mathbf{e}^{a k t}+a \mathbf{e}^{-c k t}-C_{2}}, x_{3}=\frac{k a \mathbf{e}^{-c k t}}{C_{1} \mathbf{e}^{a k t}+a \mathbf{e}^{-c k t}-C_{2}}, \\
x_{2}=k-x_{1}-x_{3}=-\frac{k C_{2}}{C_{1} \mathbf{e}^{a k t}+a \mathbf{e}^{-c k t}-C_{2}} . \tag{106}
\end{gather*}
$$

Obviously $C_{2} \neq 0$. The pole is $t_{*}$ satisfies

$$
C_{1} \mathbf{e}^{a k t_{*}}+a \mathbf{e}^{-c k t_{*}}-C_{2}=0 \Rightarrow C_{2}=C_{1} \mathbf{e}^{a k t_{*}}+a \mathbf{e}^{-c k t_{*}} \neq 0
$$

Hence using De l' Hôpital Rule we are led to the fact that

$$
\lim _{t \rightarrow t_{*}}\left(t-t_{*}\right) x_{2}(t)=\frac{C_{2}}{a C_{1} \mathbf{e}^{a k t_{*}}-a c \mathbf{e}^{-c k t_{*}}} .
$$

Since the pole order is greater than 1 , we have that

$$
\lim _{t \rightarrow t_{*}}\left(t-t_{*}\right) x_{2}(t)=\infty
$$

Therefore

$$
C_{1}=c \mathbf{e}^{-(a+c) k t}=c \mathbf{e}^{-b k t}
$$

The solution (106) possesses only one arbitrary constant $k$, but we need $n-1=2$. Now if $k=0$ the solutions of (105) are

$$
x_{3}(t)=0, \quad x_{1}(t)=\frac{1}{a t+C_{1}},
$$

or

$$
\begin{equation*}
x_{1}(t)=\frac{C_{1}-c}{a\left(C_{1} t+C_{2}\right)}, \quad x_{3}(t)=-\frac{1}{C_{1} t+C_{2}} . \tag{107}
\end{equation*}
$$

Both solutions lead us to a contradiction since the pole order of $x_{1}$ and $x_{3}$ is greater than 1.

Therefore the case $\nu_{1}=\nu_{2}=\nu_{3}=\nu>1$ cannot give us an algebraic completely integrable system. The solution (107) proves that the case $b=a+c$ is algebraically completely integrable system when $\nu_{1}=\nu_{2}=\nu_{3}=1$.

### 5.2 The four-dimensional case

Now we examine the vector field (67) for $n=4$. In this case we have the matrix

$$
A=\left(\begin{array}{cccc}
0 & a & b & c  \tag{108}\\
-a & 0 & d & e \\
-b & -d & 0 & f \\
-c & -e & -f & 0
\end{array}\right)
$$

There is an important difference between the present and the three-dimensional case. In the latter, the Poisson matrix is not invertible for all values of $a, b$ and $c$ and this explains why can be found two independent integrals for this system. In the four-dimensional case there are some values of $a, b, c, d, e$ and $f$ for which the Poisson matrix,

$$
P=\left(\begin{array}{cccc}
0 & a x_{1} x_{2} & b x_{1} x_{3} & c x_{1} x_{4}  \tag{109}\\
-a x_{1} x_{2} & 0 & d x_{2} x_{3} & e x_{2} x_{4} \\
-b x_{1} x_{3} & -d x_{2} x_{3} & 0 & f x_{3} x_{4} \\
-c x_{1} x_{4} & -e x_{2} x_{4} & -f x_{3} x_{4} & 0
\end{array}\right)
$$

is invertible, particularly if $a f+c d-b e \neq 0$, since

$$
\operatorname{det}(P)=(a f+c d-b e)^{2} x_{1}^{2} x_{2}^{2} x_{3}^{2} x_{4}^{2}
$$

If $P^{-1}$ does not exist, we can find a second Casimir function (independent of $H=x_{1}+$ $x_{2}+x_{3}+x_{4}$ ) that is one of the following six functions

$$
\begin{align*}
F_{1} & =x_{1}^{d+e} x_{2}^{-b-c} x_{3}^{a} x_{4}^{a}, \\
F_{2} & =x_{1}^{-d+f} x_{2}^{b} x_{3}^{-a-c} x_{4}^{b}, \\
F_{3} & =x_{1}^{d} x_{2}^{-b+f} x_{3}^{a-e} x_{4}^{d},  \tag{110}\\
F_{4} & =x_{1}^{-e-f} x_{2}^{c} x_{3}^{c} x_{4}^{-a-b}, \\
F_{5} & =x_{1}^{e} x_{2}^{-c-f} x_{3}^{e} x_{4}^{a-d} \\
\text { and } F_{6} & =x_{1}^{f} x_{2}^{f} x_{3}^{-c-e} x_{4}^{b+d} .
\end{align*}
$$

In order to calculate the Kowalevski exponents we have to use the formula (29). The Kowalevski matrix is

$$
\left(\begin{array}{cccc}
a \chi_{1}+b \chi_{3}+c \chi_{4}+1 & a \chi_{1} & b \chi_{1} & c \chi_{1}  \tag{111}\\
-a \chi_{2} & -a \chi_{1}+d \chi_{3}+e \chi_{4}+1 & d \chi_{2} & e \chi_{2} \\
-b \chi_{3} & -d \chi_{3} & -b \chi_{1}-d \chi_{2}+f \chi_{4}+1 & f \chi_{3} \\
-c \chi_{4} & -e \chi_{4} & -f \chi_{4} & -c \chi_{4}-e \chi_{4}-f \chi_{4}+1
\end{array}\right)
$$

where $x^{(0)}=\left(\chi_{1}, \chi_{2}, \chi_{3}, \chi_{4}\right)$ is an element of the indicial locus, that is a solution of the
simultaneous equation (27), which in this problem it is written as

$$
\begin{align*}
& \chi_{1}+a \chi_{1} \chi_{2}+b \chi_{1} \chi_{3}+c \chi_{1} \chi_{4}=0 \\
& \chi_{2}-a \chi_{1} \chi_{2}+d \chi_{2} \chi_{3}+e \chi_{2} \chi_{4}=0  \tag{112}\\
& \chi_{3}-b \chi_{1} \chi_{3}-d \chi_{2} \chi_{3}+f \chi_{3} \chi_{4}=0 \\
& \chi_{4}-c \chi_{1} \chi_{4}-e \chi_{2} \chi_{4}-f \chi_{3} \chi_{4}=0
\end{align*}
$$

In Table 11 we can see the corresponding Kowalevski exponents, the eigenvalues of (111), to each element of the indicial locus, where

$$
\begin{gather*}
q_{1}=b+f-c, \quad q_{2}=d+f-e,  \tag{113}\\
q_{3}=a+e-c, \quad q_{4}=a+d-b, \\
p=\operatorname{det}(P)=a f+c d-b e .
\end{gather*}
$$

| Vector $x^{(0)}$ | Kowalevski <br> exponents | Vector $x^{(0)}$ | Kowalevski <br> exponents |
| :---: | :---: | :---: | :---: |
| $(0,0,0,0)$ | $1,1,1,1$ | $\left(0,0, \frac{1}{f},-\frac{1}{f}\right)$ | $-1,1, \frac{q_{1}}{f}, \frac{q_{2}}{f}$ |
| $\left(0, \frac{1}{e}, 0,-\frac{1}{e}\right)$ | $-1,1,-\frac{q_{2}}{e}, \frac{q_{3}}{e}$ | $\left(0, \frac{1}{d},-\frac{1}{d}, 0\right)$ | $-1,1, \frac{q_{4}}{d}, \frac{q_{2}}{d}$ |
| $\left(\frac{1}{c}, 0,0,-\frac{1}{c}\right)$ | $-1,1,-\frac{q_{3}}{c},-\frac{q_{1}}{c}$ | $\left(\frac{1}{b}, 0,-\frac{1}{b}, 0\right)$ | $-1,1,-\frac{q_{4}}{b}, \frac{q_{1}}{b}$ |
| $\left(\frac{1}{a},-\frac{1}{a}, 0,0\right)$ | $-1,1, \frac{q_{4}}{a}, \frac{q_{3}}{a}$ | $\left(\frac{q_{2}}{p},-\frac{q_{1}}{p}, \frac{q_{3}}{p},-\frac{q_{4}}{p}\right)$ | $-1,1, \frac{\sqrt{q_{1} q_{2} q_{3} q_{4}}}{p},-\frac{\sqrt{q_{1} q_{2} q_{3} q_{4}}}{p}$ |

Table 11: Kowalevski exponents of $4 \times 4$ Lotka-Volterra equations

We can observe that the sets of Kowalevski exponents, except that one which corresponds to the last element of the indicial locus, are similar to those we had in the three-dimensional case. The method we apply in the four-dimensional case is exactly the same. In particular, we set

$$
\begin{array}{lll}
k_{1}=\frac{q_{1}}{b}, & k_{2}=\frac{q_{1}}{f}, & k_{3}=-\frac{q_{1}}{c}, \\
m_{1}=\frac{q_{2}}{d}, & m_{2}=\frac{q_{2}}{f}, & m_{3}=-\frac{q_{2}}{e},  \tag{114}\\
p_{1}=\frac{q_{3}}{a}, & p_{2}=\frac{q_{3}}{e}, & p_{3}=-\frac{q_{3}}{c}, \\
n_{1}=\frac{q_{4}}{a}, & n_{2}=\frac{q_{4}}{d}, & n_{3}=-\frac{q_{4}}{b} .
\end{array}
$$

If one of the parameters $a, b, c, d, e$ and $f$ is zero, then the corresponding eigenvalues disappear.

Solving these equations simultaneously we find that

$$
\begin{gathered}
a=\frac{p_{2} e}{p_{1}}, \quad b=\frac{\left(p_{1} p_{2}-p_{1}-p_{2}\right) k_{3} e}{p_{1} k_{1}}, \quad c=-\frac{\left(p_{1} p_{2}-p_{1}-p_{2}\right) e}{p_{1}}, \\
d=\frac{\left[\left(\left(-p_{1}-p_{2}+p_{1} p_{2}\right) k_{3}-p_{1} p_{2}+p_{2}\right) k_{1}+\left(-p_{1} p_{2}+p_{1}+p_{2}\right) k_{3}\right] e}{p_{1} k_{1}\left(m_{1}-1\right)}, \\
f=\frac{\left(-p_{1}-p_{2}+p_{1} p_{2}\right)\left(k_{1} k_{3}-k_{3}-k_{1}\right) e}{p_{1} k_{1}},
\end{gathered}
$$

$$
\begin{gathered}
m_{2}=\frac{\left[\left(\left(-p_{1}-p_{2}+p_{1} p_{2}\right) k_{3}-p_{1} p_{2}+p_{2}\right) k_{1}+\left(-p_{1} p_{2}+p_{1}+p_{2}\right) k_{3}\right] m_{1}}{p_{1} k_{1}\left(m_{1}-1\right)} \\
m_{3}=-\frac{\left[\left(\left(-p_{1}-p_{2}+p_{1} p_{2}\right) k_{3}-p_{1} p_{2}+p_{2}\right) k_{1}+\left(-p_{1} p_{2}+p_{1}+p_{2}\right) k_{3}\right] m_{1}}{\left(m_{1}-1\right)\left(-p_{1}-p_{2}+p_{1} p_{2}\right)\left(k_{1} k_{3}-k_{3}-k_{1}\right)} \\
n_{1}=\frac{\left(\left(-p_{1}-p_{2}+p_{1} p_{2}\right) k_{3}-p_{1} p_{2}+p_{2} m_{1}\right) k_{1}+m_{1}\left(-p_{1} p_{2}+p_{1}+p_{2}\right) k_{3}}{k_{1} p_{2}\left(m_{1}-1\right)} \\
n_{2}=\frac{\left(\left(-p_{1}-p_{2}+p_{1} p_{2}\right) k_{3}-p_{1} p_{2}+p_{2} m_{1}\right) k_{1}+m_{1}\left(-p_{1} p_{2}+p_{1}+p_{2}\right) k_{3}}{\left(\left(-p_{1}-p_{2}+p_{1} p_{2}\right) k_{3}-p_{1} p_{2}+p_{2}\right) k_{1}+\left(-p_{1} p_{2}+p_{1}+p_{2}\right) k_{3}} \\
n_{3}=\frac{\left(\left(p_{1}+p_{2}-p_{1} p_{2}\right) k_{3}+p_{1} p_{2}-p_{2} m_{1}\right) k_{1}+m_{1}\left(p_{1} p_{2}-p_{1}-p_{2}\right) k_{3}}{k_{3}\left(-p_{1}-p_{2}+p_{1} p_{2}\right)\left(m_{1}-1\right)} \\
k_{2}=\frac{k_{3} k_{1}}{k_{1} k_{3}-k_{1}-k_{3}}, \quad p_{3}=\frac{p_{1} p_{2}}{p_{1} p_{2}-p_{1}-p_{2}}
\end{gathered}
$$

Starting from the two fractions for $k_{2}$ and $p_{3}$ we can find the values of $k_{1}, k_{3}, p_{1}$ and $p_{3}$ for which $k_{2}$ and $p_{3}$ take integer values using the results we concluded in the threedimensional case. Then we can see that the other fractions take very simple symbolic forms and therefore they can be investigated with elementary techniques of Number Theory. We are led to some values of $a, b, c, d, e$ and $f$, for which the values the above fractions are integers. Furthermore taking theses values we must check for which values of $a, b, c, d, e$ and $f$ the exponent

$$
\frac{\sqrt{(b+f-c)(d+f-e)(a+e-c)(a+d-b)}}{a f+c d-b e}
$$

takes integer values.
For the classification, we consider the following four cases:

1. abcdef $\neq 0$ and $p \neq 0$
2. $a b c d e f \neq 0$ and $\quad p=0$
3. abcdef $=0$ and $\quad p \neq 0$
4. $\quad a b c d e f=0$ and $\quad p=0$.

### 5.2.1 First case

The results for abcdef $\neq 0$ and $p \neq 0$ are shown in Table 12, where we can see the values of $a, b, c, d, e$ and $f$ for which the Kowalevski exponents are integers. The values of Kowalevski exponents in each case are given as a proof. We do not give isomorphic cases because the table would be very large. It should be noted that $k \in \mathbf{Z}$.

Table 12: First case of $4 \times 4$ Lotka-Volterra equations

| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(e, 2 k e,-e,-2 e, e,-e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm 4,4 \\ -1,1, \pm k, 2 \\ -1,1, \pm 2 k, 4 \end{gathered}$ |
| $(3 e, 6 k e,-2 e,-3 e, e,-2 e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm 6,6 \\ -1,1, \pm 2 k, 2 \\ -1,1, \pm 3 k, 3 \end{gathered}$ |
| $(e, k e,-e,-e, e,-e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm 3,3 \\ -1,1, \pm k, 3 \end{gathered}$ |
| $(k e,-e, e,-e, e,-2 e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm 2,2 \\ -1,1, \pm k, 4 \end{gathered}$ |
| $(2 k e,-2 e, e,-2 e, e,-3 e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm 2,2 \\ -1,1, \pm k, 3 \\ -1,1, \pm 2 k, 6 \end{gathered}$ |
| $(e, k e, e,-e, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm k, \pm 1 \\ -1,1, k e,-k e \end{gathered}$ |
| $\left(\frac{e}{2},-\frac{e}{2},-\frac{e}{2}, e, e, 2 e\right)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm 2,2 \\ -1,1, \pm 4,4 \end{gathered}$ |
| $(3 e,-2 e, e, e, e,-3 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1, \pm 3,3 \\ -1,1,6,-3 \\ -1,1,1,2 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $\left(-e, k^{2} e, k e, e, e, k e\right)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm k, k \\ -1,1, \pm k, 1 \\ -1,1, \pm k^{2}, k \end{gathered}$ |
| $(e, e,-e, k e, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1,3,3 \\ -1,1, \pm k, 3 \\ -1,1,3 k e,-3 k e \end{gathered}$ |
| $(2 e,-e,-3 e, 3 e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \\ -1,1,3,3 \\ -1,1,-3,6 \\ -1,1,9 e,-9 e \end{gathered}$ |
| $(e,-e,-2 e, 2 e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \\ -1,1,2,2 \\ -1,1,-2,4 \\ -1,1,4,4 \\ -1,1,8 e,-8 e \end{gathered}$ |
| $(2 e,-e,-e, e, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1,2,2 \\ -1,1, \pm 1,4 \\ -1,1,2 e,-2 e \end{gathered}$ |
| ( $x e, x e, e, e k x, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm k x, x \\ -1,1, \pm k, 1 \\ -1,1,-x,-x \\ -1,1, e k x,-e k x \end{gathered}$ |
| $\left(\frac{3 e}{2}, \frac{3 e}{2},-\frac{e}{2}, 3 k e, e, e\right)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm 3 k, 3 \\ -1,1,6,6 \\ -1,1, \pm 2 k, 2 \\ -1,1,6 k e,-6 k e \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(-k x e, k e, e, k e, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm x, 1 \\ -1,1, k, k \\ -1,1, \pm k x,-k \\ -1,1, k e,-k e \end{gathered}$ |
| $(e, e,-2 e,-2 e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,-2,4 \\ -1,1,2,2 \\ -1,1, \pm 4,4 \\ -1,1,1,2 \end{gathered}$ |
| $\left(\frac{2 e}{3},-\frac{e}{3},-\frac{e}{3}, e, e, e\right)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm 1,2 \\ -1,1,3,3 \\ -1,1, \pm 3,6 \\ -1,1,3 e,-3 e \end{gathered}$ |
| $(-e,-e, k e, x e, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm x, \pm k \\ -1,1,-k e x, k e x \end{gathered}$ |
| $(e, e,-2 e, k e, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1,2,2 \\ -1,1, \pm k, 4 \\ -1,1,4 k e,-4 k e \end{gathered}$ |
| $(-e, k e, e, e, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1,-1,-1 \\ -1,1, \pm k, 1 \\ -1,1, k e,-k e \end{gathered}$ |
| $(3 e, 3 e,-2 e, 3 k e, e, e)$ | $\begin{gathered} \pm 1,1,1,1 \\ -1,1, \pm k, 2 \\ -1,1,3,3 \\ -1,1, \pm 3 k, 6 \\ -1,1,6 k e,-6 k e \end{gathered}$ |


| Vector $(a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(2 e, 2 e,-3 e, 2 k e, e, e)$ | $\pm 1,1,1,1$ |
|  | $-1,1, \pm k, 3$ |
|  | $-1,1,2,2$ |
|  | $-1,1, \pm 2 k, 6$ |
|  | $-1,1,6 k e,-6 k e$ |
|  | $\pm 1,1,1,1$ |
|  | $-1,1, \pm 2 k, 4$ |
|  | $-1,1,4,4$ |
|  | $-1,1 \pm k, 2$ |
|  | $-1,1,4 k e,-4 k e$ |

### 5.2.2 Second case

If $a b c d e f \neq 0$ and $p=0$, then we get the results given in Table 13. We recall that isomorphic cases are not shown.

Table 13: Second case of $4 \times 4$ Lotka-Volterra equations

| Vector $(a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(-e,-e, 2 e,-2 e, e,-3 e)$ | $1,1,1,1$ |
|  | $-1,1,2,3$ |
|  | $-2,-1,1,6$ |
| $(e,-e, e, e, e,-2 e)$ | $-1,1,1,3$ |
|  | $1,1,1,1$ |
|  | $-1,1,1,2$ |
|  | $-1,1,1,3$ |
|  | $-1,1,1,4$ |
|  | $-1,1,3,4$ |
|  | $-2,-1,1,3$ |


| Vector $(a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
|  | $1,1,1,1$ |
| $\left(\frac{e}{4},-\frac{e}{4}, \frac{e}{4},-\frac{e}{2}, e,-\frac{e}{2}\right)$ | $-1,0,1,4$ |
|  | $-1,1,1,2$ |
|  | $-1,1,2,4$ |
|  | $-4,-1,1,4$ |
|  | $1,1,1,1$ |
|  | $-1,1,1,2$ |
|  | $-1,1,1,4$ |
|  | $-1,-1,1,6$ |
|  | $-1,1,1,3$ |
|  | $-3,-1,1,4$ |
|  | $-1,1,2,3$ |

We also recall that in this case, $p=a f+c d-b e=0$ and we have the Casimirs (110) that are independent of the function $H=x_{1}+x_{2}+x_{3}+x_{4}$.

### 5.2.3 Third case

The results for $a b c d e f=0$ and $p \neq 0$ are given in Table 14. We recall again that isomorphic cases are not shown.

Table 14: Third case of $4 \times 4$ Lotka-Volterra equations

| Vector $(a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(0,-2 e,-e,-2 e, e, e)$ | $1,1,1,1$ |
|  | $-1,0,0,1$ |
|  | $-1,0,1,1$ |
|  | $-2,-1,0,1$ |
|  | $-2,-1,1,2$ |
| $(0, e,-e, e, e, e)$ | $1,1,1,1$ |
|  | $-1,0,1,3$ |
|  | $-1,0,1,1$ |
|  | $-1,0,0,1$ |
|  | $-1,1,1,2$ |
|  | $-1,1,1,3$ |
|  | $-3,-2,-1,1$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| (0,e,-e,e,e,-2e) | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -2,-1,0,1 \\ -2,-1,1,2 \\ \hline \end{gathered}$ |
| $(e, 0, e, e, 0, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,2,2 \end{gathered}$ |
| ( $0,-e,-e,-e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,-1,0,1 \\ -1,-1,1,2 \\ -1,-1,1,1 \\ -2,-1,-1,1 \end{gathered}$ |
| $(0, e, e,-e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,2,2 \\ -2,-1,0,1 \end{gathered}$ |
| $(0, e,-e, e, e, 2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0, l \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,1,1,2 \\ -1,1,2,2 \\ -4,-2,-1,1 \end{gathered}$ |
| $(0,2 e,-e, 0, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,-1,1,2 \\ -4,-2,-1,1 \end{gathered}$ |
| (0, - ke, e, ke, e, e) | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1, k \\ -1,0,1,-k \\ -1,1,1,2 \\ -1,1, k,-k \\ -2,-1,1,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $\left(0,-\frac{2 e}{3},-e,-\frac{2 e}{3}, e,-\frac{e}{3}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,3 \\ -1,0,1,6 \\ -2,-1,0,1 \\ -2,-1,1,2 \end{gathered}$ |
| ( $\left.0,-\frac{e}{3},-e,-\frac{e}{3}, e,-\frac{2 e}{3}\right)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,3 \\ -1,0,1,6 \\ -2,-1,0,1 \\ -2,-1,1,2 \end{gathered}$ |
| $(0,-2 e,-e,-2 e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,1,2,4 \\ -2,-1,1,2 \\ -4,-1,1,2 \end{gathered}$ |
| $(0,-e,-e,-e, e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,1,1,2 \\ -2,-1,1,2 \\ -4,-1,1,2 \end{gathered}$ |
| $(0,-e, e, e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,3 \\ -1,-1,0,1 \\ -1,-1,1,2 \\ -2,-1,1,3 \\ -3,-1,0,1 \end{gathered}$ |
| $(e, 0,-2 e, e, e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,1,2,4 \\ -2,-1,0,1 \\ -2,-1,1,2 \\ -2,-1,1,4 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| (0, $\left.\frac{e}{2}, e,-\frac{e}{2}, e, \frac{e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,2,2 \\ -2,-1,0,1 \\ -2,-1,1,2 \end{gathered}$ |
| $\left(-\frac{e}{2}, 0, \frac{e}{2}, \frac{e}{2}, e,-\frac{e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,2,2 \\ -1,-1,0,1 \\ -2,-1,0,1 \end{gathered}$ |
| $(0,0, e,-e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,2 \\ -1,-1,0,1 \\ -2,-1,0,1 \end{gathered}$ |
| (0, -e, e, e, e, 2e) | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,1,2,2 \\ -2,-1,0,1 \end{gathered}$ |
| $(e,-e, e, 0,2 e, 2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,1,2,2 \\ -2,-1,0,1 \end{gathered}$ |
| $(e,-e, 0,0, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,2,2 \\ -2,-1,0,1 \\ \hline \end{gathered}$ |
| (0, -e, e, e, e, -2e) | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,2 \\ -2,-1,0,1 \\ -2,-1,1,2 \\ -2,-1,1,4 \\ -4,-1,0,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(0,3 e, e,-3 e, e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,3 \\ -1,1,2,2 \\ -2,-1,0,1 \\ -6,-1,0,1 \end{gathered}$ |
| $(2 e, 0,3 e,-2 e, e,-3 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,3 \\ -1,1,2,2 \\ -2,-1,0,1 \\ -6,-1,0,1 \end{gathered}$ |
| $\left(0, \frac{3 e}{2}, e,-\frac{3 e}{2}, e,-\frac{e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,6 \\ -1,1,2,2 \\ -2,-1,0,1 \\ -3,-1,0,1 \end{gathered}$ |
| $\left(\frac{e}{2}, 0, \frac{3 e}{2},-\frac{e}{2}, e,-\frac{3 e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,6 \\ -1,1,2,2 \\ -2,-1,0,1 \\ -3,-1,0,1 \end{gathered}$ |
| $(0,2 e, e,-2 e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,4 \\ -1,1,2,2 \\ -2,-1,0,1 \\ -4,-1,0,1 \end{gathered}$ |
| $(-e, 0, e,-e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,2 \\ -1,-1,0,1 \\ -1,-1,-1,1 \\ \hline \end{gathered}$ |
| (0,e,e,-e, e, -2e) | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,2 \\ -1,1,2,4 \\ -2,-1,0,1 \\ -2,-2,-1,1 \\ -4,-1,0,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| ( $\left.0, \frac{e}{2}, e,-\frac{e}{2}, e,-\frac{e}{2}\right)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,1,2,4 \\ -1,0,0,1 \\ -1,-1,0,1 \\ -2,-1,0,1 \\ -2,-2,-1,1 \end{gathered}$ |
| $(e, 0,-e, e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,1,2,3 \\ -1,-1,1,2 \\ -1,-1,1,3 \\ -3,-1,0,1 \end{gathered}$ |
| $(e, 0,2 e, e, e, 2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,1,2,2 \end{gathered}$ |
| $\left(\frac{e}{2}, 0,-\frac{e}{2}, \frac{e}{2}, e,-\frac{e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,2,4 \\ -1,-1,1,2 \\ -2,-1,1,2 \\ -4,-1,0,1 \end{gathered}$ |
| $(-e, e, 0,2 e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,1,2,2 \end{gathered}$ |
| $(e, 0,0,0, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,1,2 \\ -1,-1,0,1 \\ \hline \end{gathered}$ |
| (ke, $0, e, k e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1, k \\ -1,1,1,2 \\ -1,1, k, k \\ \hline \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(-3 e, 0,-2 e,-3 e, e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,2,2 \\ -1,0,1,3 \\ -6,-1,0,1 \end{gathered}$ |
| $(-e,-3 e, 0,-2 e, e,-3 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,3 \\ -1,1,2,2 \\ -6,-1,0,1 \end{gathered}$ |
| $\left(-\frac{3 e}{2}, 0,-\frac{e}{2},-\frac{3 e}{2}, e,-\frac{e}{2}\right)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,6 \\ -1,1,2,2 \\ -3,-1,0,1 \end{gathered}$ |
| $\left(-e,-\frac{3 e}{2}, 0,-\frac{e}{2}, e,-\frac{3 e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,6 \\ -1,1,2,2 \\ -3,-1,0,1 \end{gathered}$ |
| $(-2 e, 0,-e,-2 e, e,-e)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,1,2,2 \\ -4,-1,0,1 \end{gathered}$ |
| $\left(-\frac{e}{2}, 0,-\frac{e}{2},-\frac{e}{2}, e,-\frac{e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,4 \\ -1,1,2,4 \\ -2,-1,0,1 \\ -2,-1,1,1 \\ -2,-1,1,2 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $\left(\frac{e}{2}, 0,-\frac{e}{2},-\frac{e}{2}, e, \frac{e}{2}\right)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,-1,1,2 \\ -2,-1,1,2 \\ -4,-2,-1,1 \end{gathered}$ |
| $(2 e, 0,-e,-2 e, e, e)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,1,2 \\ -2,-1,1,4 \\ -4,-2,-1,1 \end{gathered}$ |
| $(-e, 0,-k e, e, e, k e)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1, k \\ -1,0,1,-k \\ -1,1,1,2 \\ -1,1, k, k \\ -2,-1,-1,1 \end{gathered}$ |
| $\left(\frac{e}{2}, 0, \frac{e}{2},-\frac{e}{2}, e,-\frac{e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,1,2,4 \\ -2,-1,1,1 \\ -2,-1,1,2 \end{gathered}$ |
| $(e,-e, 0,2 e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,1,2 \\ -1,1,2,2 \\ -1,1,2,4 \\ -4,-1,0,1 \end{gathered}$ |
| $(-e, 0,0,-e, e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,1,2 \\ -1,1,2,4 \\ -4,-1,0,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(e,-2 e, 0, e, e, 2 e)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,1,2,2 \\ -1,1,2,4 \\ -2,-1,0,1 \end{gathered}$ |
| $(e,-e, 0,-2 e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,0,1 \\ -2,-1,1,2 \end{gathered}$ |
| $(e, 2 e, 0, e, e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,0,1 \\ -2,-1,1,2 \end{gathered}$ |
| $(0,-e, e, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -2,-1,0,1 \\ -2,-1,1,2 \end{gathered}$ |
| $\left(e, \frac{2 e}{3}, 0,-\frac{e}{3}, 0,-\frac{2 e}{3}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,3 \\ -1,0,1,6 \\ -2,-1,1,2 \end{gathered}$ |
| $(e, e, 0,-2 e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,4 \\ -1,1,1,2 \\ -2,-1,0,1 \\ -2,-1,1,2 \\ -4,-1,1,2 \end{gathered}$ |
| (e, $\left.\frac{e}{2}, 0,-\frac{e}{2}, e,-\frac{e}{2}\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -2,-1,1,2 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(2 e, 2 e,-e, 0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,1,2 \\ -2,-1,1,4 \\ -4,-2,-1,1 \end{gathered}$ |
| $(-e,-e, 2 e, 0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,1,2,4 \\ -2,-1,-1,1 \\ -2,-2,-1,1 \end{gathered}$ |
| $(e,-2 e, 0, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,1,2,4 \\ -2,-1,1,1 \end{gathered}$ |
| $(2 e, 2 e, e, 0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,1,2 \end{gathered}$ |
| (2e,e,e, -e,e, 0) | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,1,2 \end{gathered}$ |
| $(-e, e, 0,0, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -2,-1,1,2 \end{gathered}$ |
| $(-2 e,-2 e, e, 0,-3 e, 3 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,3 \\ -2,-1,1,2 \\ -6,-1,0,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(-2 e,-2 e, e, 0, e,-e)$ | $\begin{gathered} \hline 1,1,1,1 \\ 1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,1,2,4 \\ -2,-2,-1,1 \\ -4,-2,-1,1 \end{gathered}$ |
| $(-2 e, e, e, 3 e,-3 e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,3 \\ -2,-1,1,2 \\ -6,-1,0,1 \end{gathered}$ |
| $(-e,-e, 2 e, 0,-3 e, 3 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,6 \\ -2,-1,1,2 \\ -3,-1,0,1 \end{gathered}$ |
| $(0,0, e, k e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1, k \\ -1,1,1,1 \end{gathered}$ |
| $(-e, 2 e, 2 e, 3 e,-3 e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,6 \\ -2,-1,1,2 \\ -3,-1,0,1 \end{gathered}$ |
| $(e, e,-e, 0,2 e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -2,-1,1,2 \\ -4,-1,0,1 \end{gathered}$ |
| $(e, e,-2 e, 0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -2,-1,1,2 \\ -2,-1,1,4 \\ -2,-1,-1,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(-e, e,-e, 0, e, e)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,3 \\ -1,-1,1,2 \\ -2,-1,1,3 \\ -3,-1,-1,1 \end{gathered}$ |
| (2e, -2e, e, 0, e, e) | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,1,2 \\ -2,-1,0,1 \\ -2,-1,1,1 \\ -2,-1,1,2 \end{gathered}$ |
| $(e,-e, e, e, 0, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,3 \\ -1,1,2,3 \\ -1,-1,0,1 \\ -1,-1,1,2 \\ -3,-1,1,1 \end{gathered}$ |
| $(-e, e, e,-e, e, 0)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,3 \\ -1,1,2,3 \\ -1,-1,0,1 \\ -3,-1,0,1 \\ -2,-1,-1,1 \end{gathered}$ |
| $(e,-e, e, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,3 \\ -1,1,1,3 \\ -2,-1,1,1 \\ -3,-1,1,2 \end{gathered}$ |
| $(e,-e, e,-2 e,-2 e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,0,1 \\ -2,-2,-1,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(-2 e,-e, e, e, e, 0)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,0,1 \\ -2,-2,-1,1 \end{gathered}$ |
| $(e, 2 e,-2 e, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -2,-2,-1,1 \end{gathered}$ |
| $\left(0,0,-e,-k^{2} e, e, e\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,1 \\ -k,-1,1,2 \\ -2 k e,-1,1,2 k e \\ -2,-2,-1,1 \end{gathered}$ |
| $(0,0,-e,-e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,3 \\ -1,1,1,3 \\ -2,1,0,1 \\ -3,-1,1,2 \end{gathered}$ |
| $(0,0, e, e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,-1,1,1 \\ -1,-1,1,2 \\ -2,-1,0,1 \end{gathered}$ |
| $(0,0,-e, 2 e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,4 \\ -1,1,1,2 \\ -2,-1,0,1 \\ -4,-1,1,2 \end{gathered}$ |
| $\left(e,-e,-e, 0,0, k^{2} e\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,1 \\ -1,1,2,2 \\ -2 k e,-1,1,2 k e \\ -k,-2,-1,1 \end{gathered}$ |
| $(e, 0,0,-e, e, 2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(0,0,-e,-2 e, e,-e)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -2,-1,0,1 \\ \hline \end{gathered}$ |
| $(0,0, e,-e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,3 \\ -1,1,2,3 \\ -2,-1,0,1 \\ -3,-1,0,1 \end{gathered}$ |
| $(0,0, e,-2 e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,2 \\ -1,1,2,4 \\ -2,-1,0,1 \\ -4,-1,0,1 \end{gathered}$ |
| $(-e, k e, 0, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1, k \\ -1,-1,1,1 \\ -k,-1,0,1 \end{gathered}$ |
| $\left(e,-k^{2} e, 0,-e, e, 0\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,2, k \\ -1,-1,1,1 \\ -2,-1,1,2 \\ -k,-1,1,2 \\ -2 k e,-1,1,2 k e \end{gathered}$ |
| (0,e, $0, e, e, e)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,1,1,1 \\ -1,1,1,2 \end{gathered}$ |
| $(e,-e, 0, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,3 \\ -1,1,2,3 \\ -3,-1,1,1 \end{gathered}$ |


| Vector $(a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(-e, e, 0,-e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,3 \\ -1,1,2,3 \\ -2,-1,0,1 \\ -3,-1,1,1 \end{gathered}$ |
| $(-e, 2 e, 0,-e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,4 \\ -1,1,2,4 \\ -2,-1,0,1 \\ -2,-1,1,1 \end{gathered}$ |
| $\left(0,-k^{2} e,-e, 0, e,-e\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,-1,1,1 \\ -2,-1,1,2 \\ -k,-1,1,2 \\ -2 k e,-1,1,2 k e \\ -k,-2,-1,1 \end{gathered}$ |
| $(0, e,-e, 0, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,3 \\ -1,-1,1,3 \\ -3,-2,-1,1 \end{gathered}$ |
| $(0,-e,-e, 0, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,-1,-1,1 \\ -2,-1,-1,1 \end{gathered}$ |
| (0,e, -e, e, e, 0) | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -2,-2,-1,1 \end{gathered}$ |
| $\left(0,-e, k^{2} e, e, 0, e\right)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,1,2,2 \\ -2,-1,1, k \\ -k,-1,1,2 \\ -2 k e,-1,1,2 k e \\ -1,-1,-1,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(0,-e, e, e, 0,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,3 \\ -2,-1,1,3 \\ -3,-1,-1,1 \end{gathered}$ |
| $(0,-e, 2 e, e, 0,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -2,-1,1,4 \\ -2,-1,-1,1 \end{gathered}$ |
| $(0,-e, 0, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1-1,0,1 \\ -2,-1,1,1 \end{gathered}$ |
| $\left(e, 0,0, e, e, k^{2} e\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,1 \\ -1,1,2,2 \\ -1,1,2, k \\ -2 k e,-1,1,2 k e \end{gathered}$ |
| $(-e, 0,0, e, e,-k e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,1,1,1 \\ -k,-1,0,1 \end{gathered}$ |
| $(-e, 0,0,-e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,1,3 \\ -1,1,2,3 \\ -3,-1,0,1 \end{gathered}$ |
| $(e, 0,0,-e, e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,0,1,4 \\ -1,1,1,2 \\ -4,-1,1,2 \end{gathered}$ |
| $\left(-k^{2} e, 0, e, 0, e,-e\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,1 \\ -1,1,2,2 \\ -2 k e,-1,1,2 k e \\ -k,-2,-1,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(e, 0,-e, 0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,1,3 \\ -2,-1,1,3 \\ -3,-1,0,1 \end{gathered}$ |
| $(2 e, 0,-e, 0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,2 \\ -1,1,1,2 \\ -2,-1,1,4 \\ -4,-1,0,1 \end{gathered}$ |
| $(0,0, e, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,-1,0,1 \end{gathered}$ |
| $(0,0,-e,-e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \\ -2,-1,1,2 \\ -2 e,-1,1,2 e \\ -2,-1,-1,1 \end{gathered}$ |
| $(0,-e, 0,0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,2,2 \\ -1,-1,1,2 \\ -2,-1,1,1 \\ -2 e,-1,1,2 e \end{gathered}$ |
| $(e, 0,0, e, 0, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \\ -1,1,2,2 \\ -2 e,-1,1,2 e \end{gathered}$ |
| $\left(0,0, e, k^{2} e, 0,0\right)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,1 \\ -k e,-1,1, k e \\ -1,-1,-1,1 \end{gathered}$ |
| $(0,2 e,-e, 2 e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \\ -1,0,1,1 \\ -1,0,1,2 \\ -1,1,2,2 \\ -1,1,2,4 \\ -4,-2,-1,1 \end{gathered}$ |


| Vector $(a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
|  | $1,1,1,1$ |
|  | $-1,0,0,1$ |
|  | $\left.-\frac{e}{2},-e,-\frac{e}{2}, e,-\frac{e}{2}\right)$ |
|  | $-1,0,1,4$ |
|  | $-2,-1,0,1$ |
|  | $-2,-1,1,2$ |

### 5.2.4 Fourth case

The results for $a b c d e f=0$ and $p=0$ are given in Table 15. Isomorphic cases are not shown.

Table 15: Fourth case of $4 \times 4$ Lotka-Volterra equations

| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(0, e, e, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \end{gathered}$ |
| $(0,0,0, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,1,1 \end{gathered}$ |
| $(0,0, e, 0, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,0,1 \end{gathered}$ |
| $(e, 0,-e, e, 0, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,2,2 \\ -2,-2,-1,1 \end{gathered}$ |
| $(0,0,0,-e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \\ -2,-1,1,1 \end{gathered}$ |
| $(-e, e, 0,0,-e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,2,2 \\ -2,-1,1,2 \end{gathered}$ |
| $(0,0,0, k e, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,1 \\ -1,1,1, k \end{gathered}$ |
| $(0,0,0, e, k e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1, k \\ -1,-1,1,1 \\ -k,-1,1,1 \end{gathered}$ |
| $(0,0,0,-3 e, e,-2 e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \\ -1,1,1,3 \\ -6,-1,1,1 \end{gathered}$ |


| Vector ( $a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(0,0,0,3 e,-2 e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \\ -1,1,1,6 \\ -3,-1,1,1 \end{gathered}$ |
| $(0,0,0,2 e,-3 e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,3 \\ -1,1,1,6 \\ -2,-1,1,1 \end{gathered}$ |
| $(0,0,0,-2 e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \\ -1,1,1,4 \\ -4,-1,1,1 \end{gathered}$ |
| $(0,0,0, e,-2 e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,4 \\ -2,-1,1,1 \end{gathered}$ |
| $(0,0,0,-e, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,3 \\ -3,-1,1,1 \end{gathered}$ |
| $(e, 0,0,0, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,1,2 \end{gathered}$ |
| $(0,0,-e, 0, e, e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,1,2 \\ -2,-2,-1,1 \end{gathered}$ |
| $(0,0, e, 0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,2,2 \\ -2,-1,0,1 \end{gathered}$ |
| $(0,0,-e, 0, e,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,1,2 \\ -2,-1,0,1 \\ -2,-1,1,2 \end{gathered}$ |
| $(e, 0,0, e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,1,2 \\ -1,1,2,2 \end{gathered}$ |
| $(-e, 0,0,-e, e, 0)$ | $\begin{gathered} 1,1,1,1 \\ -1,1,2,2 \\ -2,-1,1,2 \end{gathered}$ |
| $(e, 0,0,-e, e, 0)$ | $\begin{gathered} \hline 1,1,1,1 \\ -1,0,1,2 \\ -2,-1,1,2 \end{gathered}$ |
| $(e, 0, e, 0,0,-e)$ | $\begin{gathered} 1,1,1,1 \\ -1,0,1,1 \\ -1,1,1,2 \\ -2,-1,0,1 \end{gathered}$ |


| Vector $(a, b, c, d, e, f)$ | Kowalevski exponents |
| :---: | :---: |
| $(e, 0,0,0,0, e)$ | $1,1,1,1$ |
|  | $-1,1,1,1$ |

These are all the non isomorphic cases of the algebraically completely integrability of Lotka-Volterra equation in four dimensions, for which the Kowalevski exponents can be defined.

We have classified the algebraically completely integrable Lotka-Volterra equations in four dimensions, for which the Kowalevski exponents can be defined. However there are some systems for which the Kowalevski exponents cannot be defined. These are those whose Laurent solutions are:

$$
x_{i}(t)=\frac{1}{t^{\nu_{i}}} \sum_{k=0}^{\infty} x_{i}^{(k)} t^{k}, \quad i=1,2, \ldots, n
$$

where at least one of $\nu_{i}$ is greater than 1.
In addition, keeping in mind that $H=x_{1}+x_{2}+x_{3}+x_{4}$ is always a constant of motion, we can see that we have the following cases:
(i) $\nu_{1}=\nu_{2}=\nu>1$ and $\nu_{3}, \nu_{4}<\nu$,
(ii) $\nu_{1}=\nu_{3}=\nu>1$ and $\nu_{2}, \nu_{4}<\nu$,
(iii) $\nu_{1}=\nu_{4}=\nu>1$ and $\nu_{2}, \nu_{3}<\nu$,
(iv) $\nu_{2}=\nu_{3}=\nu>1$ and $\nu_{1}, \nu_{4}<\nu$,
(v) $\nu_{2}=\nu_{4}=\nu>1$ and $\nu_{1}, \nu_{3}<\nu$,
(vi) $\nu_{3}=\nu_{4}=\nu>1$ and $\nu_{1}, \nu_{2}<\nu$,
(vii) $\nu_{1}=\nu_{2}=\nu_{3}=\nu>1$ and $\nu_{4}<\nu$,
(viii) $\nu_{1}=\nu_{2}=\nu_{4}=\nu>1$ and $\nu_{3}<\nu$,
(ix) $\nu_{1}=\nu_{3}=\nu_{4}=\nu>1$ and $\nu_{2}<\nu$,
(x) $\nu_{2}=\nu_{3}=\nu_{4}=\nu>1$ and $\nu_{1}<\nu$,
(xi) $\nu_{1}=\nu_{2}=\nu_{3}=\nu_{4}=\nu>1$.

It is implied that $x_{i}^{(0)} \neq 0$ because otherwise the pole order of $x_{i}$ would not be equal to $\nu_{i}$. It is also implied that the cases $(i)$ up to $(v i)$ have been investigated with the same way, as well as the cases (vii) up to $(x)$.

We take

$$
A=\left(\begin{array}{cccc}
0 & a & b & c  \tag{115}\\
-a & 0 & d & e \\
-b & -d & 0 & f \\
-c & -e & -f & 0
\end{array}\right) .
$$

Equations (67) become

$$
\begin{align*}
& \dot{x}_{1}=a x_{1} x_{2}+b x_{1} x_{3}+c x_{1} x_{4},  \tag{116}\\
& \dot{x}_{2}=-a x_{1} x_{2}+d x_{2} x_{3}+e x_{2} x_{4},  \tag{117}\\
& \dot{x}_{3}=-b x_{1} x_{3}-d x_{2} x_{3}+f x_{3} x_{4},  \tag{118}\\
& \dot{x}_{4}=-c x_{1} x_{4}-e x_{2} x_{4}-f x_{3} x_{4} . \tag{119}
\end{align*}
$$

(i) $\nu_{1}=\nu_{2}=\nu>1$ and $\nu_{3}, \nu_{4}<\nu$

The function $H=x_{1}+x_{2}+x_{3}+x_{4}$ is a constant of motion and so we have that

$$
x_{1}^{(0)}=-x_{2}^{(0)} .
$$

In addition, since $\nu_{1}=\nu_{2}=\nu>1, \nu_{3}, \nu_{4}<\nu$ and $x_{i}^{(0)} \neq 0$ for $i=1,2,3,4$, we can easily see that the coefficients of $t^{2 \nu}$ in the RHSs of (116) and (117) are equal to zero. We have that

$$
a x_{1}^{(0)} x_{2}^{(0)}=0 \Longrightarrow a=0 .
$$

For the same reason the coefficients of $t^{\nu+\nu_{3}}$ and $t^{\nu+\nu_{4}}$ in the RHSs of (118) and (119), respectively, are equal to zero. Therefore

$$
x_{3}^{(0)}\left(-b x_{1}^{(0)}-d x_{2}^{(0)}\right)=0 \Longrightarrow b=d
$$

and

$$
x_{4}^{(0)}\left(-c x_{1}^{(0)}-e x_{2}^{(0)}\right)=0 \Longrightarrow c=e .
$$

Hence, equations (116)-(119) become

$$
\begin{align*}
\dot{x}_{1} & =x_{1}\left(b x_{3}+e x_{4}\right),  \tag{120}\\
\dot{x}_{2} & =x_{2}\left(b x_{3}+e x_{4}\right),  \tag{121}\\
\dot{x}_{3} & =x_{3}\left(-b x_{1}-b x_{2}+f x_{4}\right),  \tag{122}\\
\dot{x}_{4} & =x_{4}\left(-c x_{1}-e x_{2}-f x_{3}\right) . \tag{123}
\end{align*}
$$

From (120) and (121) we obtain that

$$
\frac{\dot{x}_{1}}{x_{1}}=\frac{\dot{x}_{2}}{x_{2}}=b x_{3}+e x_{4} \Longrightarrow x_{1}=\lambda x_{2}, \quad \lambda \text { is a constant. }
$$

But since

$$
x_{1}^{(0)}=-x_{2}^{(0)} \Longrightarrow \lambda=-1 \Rightarrow x_{1}=-x_{2},
$$

then equations (122) and (123) become

$$
\begin{align*}
\dot{x}_{3} & =f x_{3} x_{4}  \tag{124}\\
\dot{x}_{4} & =-f x_{3} x_{4} .
\end{align*}
$$

We have that

$$
\begin{equation*}
\Rightarrow \dot{x}_{3}=-\dot{x}_{4} \Rightarrow x_{3}=-x_{4}+k, \tag{125}
\end{equation*}
$$

where $k$ is a constant. Hence

$$
\nu_{3}=\nu_{4} .
$$

If $f=0$, then $x_{3}$ and $x_{4}$ are constant functions. Equations (121) become

$$
\dot{x}_{1}=\mu x_{1}, \quad \text { where } \mu \text { is a constant },
$$

the solution of which is

$$
x_{1}=A e^{\mu t},
$$

where $A$ is a constant. This is a contradiction because this solution has no pole (we assumed above that $x_{1}$ has pole order $\nu>1$ ). Therefore we have that

$$
f \neq 0 .
$$

This constraint leads us along with equations (124) to the conclusion that

$$
\nu_{3}=\nu_{4}=1
$$

We can equate the non zero coefficients of $\frac{1}{t^{2}}$ of both sides of (124), that is

$$
-x_{3}^{(0)}=f x_{3}^{(0)} x_{4}^{(0)} .
$$

Because of (125) we have that $x_{4}^{(0)}=-x_{3}^{(0)}$, and then

$$
-x_{3}^{(0)}=-f\left[x_{3}^{(0)}\right]^{2} \Longrightarrow x_{3}^{(0)}=\frac{1}{f}=-x_{4}^{(0)} .
$$

When we substitute (125) into the equations (120) and (123), they become

$$
\begin{align*}
& \dot{x}_{1}=b k x_{1}+(-b+e) x_{1} x_{4}  \tag{126}\\
& \dot{x}_{4}=f x_{4}^{2}-f k x_{4} \quad \text { (Bernoulli) } \tag{127}
\end{align*}
$$

respectively. Now from (126), if we equate the coefficients $\frac{1}{t^{\nu+1}}$, we have that

$$
\begin{gathered}
\nu x_{1}^{(0)}=(-b+e) x_{1}^{(0)}\left(-\frac{1}{f}\right) \Rightarrow-\nu=\frac{-b+e}{-f} \\
\nu=\frac{e-b}{f}>1 .
\end{gathered}
$$

We can solve the Bernoulli equation (127) considering two cases, namely $k=0$ or $k \neq 0$.

If $k=0$, then the solution of (127) is

$$
x_{4}(t)=-\frac{1}{f t+C_{1}}, \quad x_{1}(t)=\frac{C_{2}}{\left(f t+C_{1}\right)^{\nu}}
$$

where $C_{1}$ and $C_{2}$ are arbitrary constants. This proves that this case does not lead to an algebraic completely integrable system, because we have only two arbitrary constants (we need $n-1=3$ constants).

If $k \neq 0$, then on solving the Bernoulli equation (127) we find that

$$
x_{4}(t)=\frac{k}{1+C_{1} \mathbf{e}^{k f t}}, \quad x_{1}(t)=\frac{C_{2} \mathrm{e}^{k e t}}{\left(1+C_{1} \mathrm{e}^{k f t}\right)^{\nu}},
$$

where $C_{1}$ and $C_{2}$ are arbitrary constants. We can see that this case does not either lead to an algebraically completely integrable system since its solution has only two arbitrary constants $C_{2}$ and $k$ and we need $n-1=3$.

Remark: The same result holds for the cases (ii) up to (vi) with similar calculations.
(vii)
$\nu_{1}=\nu_{2}=\nu_{3}=\nu>1$ and $\nu_{4}<\nu$
The coefficients of $\frac{1}{t^{2 \nu}}$ in RHS of the equations (116), (117) and (118) are all equal to zero because $\nu>1$. Therefore we have

$$
\begin{align*}
a x_{2}^{(0)}+b x_{3}^{(0)} & =0, \\
-a x_{1}^{(0)}+d x_{3}^{(0)} & =0  \tag{128}\\
-b x_{1}^{(0)}-d x_{2}^{(0)} & =0
\end{align*}
$$

because $x_{i}^{(0)} \neq 0, i=1,2,3$. Also, because of that, as we saw in the last case of the three-dimensional problem, we are led to the relation

$$
b=a+d \text {. }
$$

Equating also the coefficients of $\frac{1}{t^{\nu+\nu_{4}}}$ of the equation (119) we have that

$$
\begin{equation*}
-c x_{1}^{(0)}-e x_{2}^{(0)}-f x_{3}^{(0)}=0 \tag{129}
\end{equation*}
$$

because $\nu>1$ and $x_{4}^{(0)} \neq 0$. From equations (128) and (129) we have that

$$
a f-b e+c d=0 \text {. }
$$

If $a=0$, then from (128) we have that $b=d=0$ because $x_{i}^{(0)} \neq 0$ for $i=1,2,3$. The Lotka-Volterra equations become

$$
\begin{align*}
& \dot{x}_{1}=c x_{1} x_{4} \\
& \dot{x}_{2}=e x_{2} x_{4}  \tag{130}\\
& \dot{x}_{3}=f x_{3} x_{4}, \\
& \dot{x}_{4}=x_{4}\left(-c x_{1}-e x_{2}-f x_{3}\right) .
\end{align*}
$$

If $c e f=0$, then

$$
x_{i}=\text { constant }
$$

for some $i \in\{1,2,3\}$, but this is a contradiction, because we assumed that $\nu>0$. Therefore cef $\neq 0$ and from (130) we have that

$$
\frac{\dot{x}_{1}}{c x_{1}}=\frac{\dot{x}_{2}}{e x_{2}}=\frac{\dot{x}_{3}}{f x_{3}}=x_{4} \Longrightarrow x_{1}^{c}=A x_{2}^{e}=B x_{3}^{f},
$$

where $A$ and $B$ are non zero constants. In addition the pole orders of $x_{1}^{c}, x_{2}^{e}$ and $x_{3}^{f}$ are $c \nu, e \nu$ and $f \nu$ respectively. So

$$
c \nu=e \nu=f \nu \Longrightarrow c=e=f \neq 0 \quad(\nu>1) .
$$

Substituting this relation into (130) we can see that the solution is:

$$
\begin{aligned}
& x_{1}=\frac{C_{2}}{-1+f \mathbf{e}^{C_{3}\left(t+C_{4}\right)}}, \quad x_{2}=\frac{C_{1}}{-1+f \mathbf{e}^{C_{3}\left(t+C_{4}\right)}}, \\
& x_{3}=\frac{-C_{3}+f C_{2}+f C_{1}}{f\left(1-f \mathbf{e}^{C_{3}\left(t+C_{4}\right)}\right)}, \quad x_{4}=\frac{C_{3} e^{C_{3}\left(t+C_{4}\right)}}{f \mathbf{e}^{C_{3}\left(t+C_{4}\right)}-1},
\end{aligned}
$$

where $C_{i}$ are constants for $i=1,2,3,4$. The question here is: "What is the pole order of each $x_{i}$ ?". Before answering this question, we must observe that $C_{1} C_{2} C_{3} \neq 0$ because otherwise $x_{1}$ and $x_{2}$ are constant functions. That cannot be true since $\nu>1$.

The pole $t_{*}$ satisfies the equation

$$
\begin{equation*}
f \mathbf{e}^{C_{3}\left(t_{*}+C_{4}\right)}=1 \tag{131}
\end{equation*}
$$

Using the De l' Hôpital Rule and (131) we can find that

$$
\lim _{t \rightarrow t_{*}}\left(t-t_{*}\right) x_{1}(t)=\frac{C_{2}}{f C_{3} \mathrm{e}^{C_{3}\left(t_{*}+C_{4}\right)}}=\frac{C_{1}}{C_{3}} \in \mathbf{R}
$$

because $C_{3} \neq 0$. This is a contradiction because in this case we assumed that the order of the pole $t_{*}$ is greater than 1.

Therefore if $a=0$, we are not led to an algebraically completely integrable system. What happens if $a \neq 0$ ? Equations (128) make us conclude that $b d \neq 0$ because $x_{i}^{(0)} \neq 0$ for $i=1,2,3$. So

$$
a b d \neq 0 .
$$

Using a similar notation as in the three-dimensional case we are led to important results. Assume firstly that $\nu_{4} \geq 1$. Let

$$
\begin{align*}
& u_{1, k}^{(\lambda)}=a x_{2}^{(\nu-k-\lambda)}+b x_{3}^{(\nu-k-\lambda)}, \\
& u_{2, k}^{(\lambda)}=-a x_{1}^{(\nu-k-\lambda)}+d x_{3}^{(\nu-k-\lambda)}, \\
& u_{3, k}^{(\lambda)}=-b x_{2}^{(\nu-k-\lambda)}-d x_{3}^{(\nu-k-\lambda)},  \tag{132}\\
& u_{4, k}^{(\lambda)}=-c x_{1}^{(\nu-k-\lambda)}-e x_{2}^{(\nu-k-\lambda)}-f x_{2}^{(\nu-k-\lambda)}
\end{align*}
$$

and

$$
\begin{align*}
v_{1, k}^{(\lambda)} & =u_{1, k}^{(\lambda)}+c x_{4}^{\left(\nu_{4}-k-\lambda\right)} \\
v_{2, k}^{(\lambda)} & =u_{2, k}^{(\lambda)}+e x_{4}^{\left(\nu_{4}-k-\lambda\right)}  \tag{133}\\
v_{3, k}^{(\lambda)} & =u_{3, k}^{(\lambda)}+f x_{4}^{\left(\nu_{4}-k-\lambda\right)} .
\end{align*}
$$

Let $S_{i, k}$ be the coefficient of $\frac{1}{t^{\nu+k}}$ in the RHS of $\dot{x}_{i}$ for $i=1,2,3$ in $(116,117,118)$, respectively. $S_{i, k}$ is given by the sum

$$
\begin{equation*}
S_{i, k}=\sum_{\lambda=0}^{\nu-k} x_{i}^{(\lambda)} u_{i, k}^{(\lambda)}, \quad k=\nu_{4}+1, \ldots, \nu, \tag{134}
\end{equation*}
$$

and for $k=1,2, \ldots, \nu_{4}$ the corresponding coefficient is given by

$$
\begin{equation*}
S_{i, k}=\sum_{\lambda=0}^{\nu_{4}-k} x_{i}^{(\lambda)} v_{i, k}^{(\lambda)}+\sum_{\lambda=\nu_{4}-k+1}^{\nu-k} x_{i}^{(\lambda)} u_{i, k}^{(\lambda)} . \tag{135}
\end{equation*}
$$

The corresponding coefficients of $\frac{1}{t^{\nu}+k}$ in the RHS of (119) is given by the sum

$$
\begin{equation*}
S_{4, k}=\sum_{\lambda=0}^{\nu_{4}-k} x_{4}^{(\lambda)} u_{4, k}^{(\lambda)}, \quad \text { for } k=1, \ldots, \nu_{4} . \tag{136}
\end{equation*}
$$

Using calculations similar to those in the corresponding case in three dimensions ( $\nu_{1}=\nu_{2}=\nu_{3}=\nu$ ), since

$$
S_{i, k}=0, \quad \text { for } \quad k=2,3, \ldots, \nu_{i}, \quad i=1,2,3,4,
$$

we are also led to

$$
\begin{align*}
& u_{i, k}^{(0)}=0, \quad \forall k \in\left\{\nu_{4}+1, \ldots, \nu\right\} \\
& v_{i, k}^{(0)}=0, \quad \forall k \in\left\{2,3, \ldots, \nu_{4}\right\} \tag{137}
\end{align*}
$$

These are true $\forall i \in\{1,2,3\}$. For $i=4$,

$$
\begin{equation*}
u_{4, k}^{(0)}=0, \quad \forall k \in\left\{2,3, \ldots, \nu_{4}\right\} . \tag{138}
\end{equation*}
$$

using that

$$
\begin{array}{lll}
u_{i, k}^{(\lambda)}=u_{i, q_{1}}^{\left(q_{2}\right)}, & \text { if } \quad k+\lambda=q_{1}+q_{2},  \tag{139}\\
v_{i, k}^{(\lambda)}=v_{i, q_{1}}^{\left(q_{2}\right)}, & \text { if } & k+\lambda=q_{1}+q_{2} .
\end{array}
$$

We have that

$$
S_{i, 1}=x_{i}^{(0)} v_{i, 1}^{(0)}=-\nu_{i} x_{i}^{(0)} \Longrightarrow v_{i, 1}^{(0)}=-\nu_{i} .
$$

So we have the linear simultaneous equation

$$
\begin{align*}
a x_{2}^{(\nu-1)}+b x_{3}^{(\nu-1)}+c x_{4}^{\left(\nu_{4}-1\right)} & =-\nu, \\
-a x_{1}^{(\nu-1)}+d x_{3}^{(\nu-1)}+e x_{4}^{\left(\nu_{4}-1\right)} & =-\nu,  \tag{140}\\
-b x_{1}^{(\nu-1)}-d x_{2}^{(\nu-1)}+f x_{4}^{\left(\nu_{4}-1\right)} & =-\nu, \\
-c x_{1}^{(\nu-1)}-e x_{2}^{(\nu-1)}-f x_{3}^{(\nu-1)} & =-\nu_{4} .
\end{align*}
$$

We have also that

$$
x_{1}^{(\nu-1)}+x_{2}^{(\nu-1)}+x_{3}^{(\nu-1)}+x_{4}^{\left(\nu_{4}-1\right)}=0
$$

since the LHS of this equation is the coefficient of $\frac{1}{t}$ in $H=x_{1}+x_{2}+x_{3}+x_{4}$, that is constant on the solutions of the Lotka-Volterra equations we work on. The matrix of this linear system is

$$
\left(\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 0  \tag{141}\\
0 & a & b & c & -\nu \\
-a & 0 & d & e & -\nu \\
-b & -d & 0 & f & -\nu \\
-c & -e & -f & 0 & -\nu_{4}
\end{array}\right) .
$$

Applying elementary row transformations, we have the equivalent linear system

$$
\left(\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 0  \tag{142}\\
0 & 1 & \frac{b}{a} & \frac{c}{a} & -\frac{\nu}{a} \\
0 & 0 & 0 & a-c+e & 0 \\
0 & 0 & 0 & 0 & -\frac{c \nu-e \nu-a \nu_{4}}{a} \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Assuming that $x_{3}^{(\nu-1)} \neq 0$ then $c=a+e$. If $x_{3}^{(\nu-1)}=0$ then the system (140) derives a system which has a solution if and only if $c=a+e$. So in both cases

$$
c=a+e \text {. }
$$

Therefore from the fourth row of the system (142) we have that

$$
\nu_{4}=\frac{(c-e) \nu}{a}=\frac{a \nu}{a}=\nu,
$$

that is a contradiction, since in this case $\nu_{4}<\nu$.

Remark: The same result holds for the cases (viii) up to ( $x$ ) with similar calculations.
(xi) $\nu_{1}=\nu_{2}=\nu_{3}=\nu_{4}=\nu>1$

We use the same notation as in the previous case without the need of the $v_{i, k}^{(\lambda)}$. As in the previous case we equate the coefficients of $\frac{1}{t^{\nu+k}}$ for $k=1,2, \ldots, \nu$. For $k=\nu$ we have the linear system

$$
\begin{align*}
a x_{2}^{(0)}+b x_{3}^{(0)}+c x_{4}^{(0)} & =0, \\
-a x_{1}^{(0)}+d x_{3}^{(0)}+e x_{4}^{(0)} & =0, \\
-b x_{1}^{(0)}-d x_{2}^{(0)}+f x_{4}^{(0)} & =0,  \tag{143}\\
-c x_{1}^{(0)}-e x_{2}^{(0)}-f x_{3}^{(0)} & =0 .
\end{align*}
$$

We solve last three equations with respect to $a, b$ and $c$, we find that

$$
\begin{gathered}
a=\frac{d x_{3}^{(0)}+e x_{4}^{(0)}}{x_{1}^{(0)}}, \quad b=\frac{-d x_{2}^{(0)}+f x_{4}^{(0)}}{x_{1}^{(0)}}, \\
c=-\frac{e x_{2}^{(0)}+f x_{3}^{(0)}}{x_{1}^{(0)}} .
\end{gathered}
$$

We can easily check that

$$
a f-b e+c d=0 \text {. }
$$

Having also that $u_{i, k}^{(0)}=0, k=2,3, \ldots, \nu$, we obtain the simultaneous equations

$$
\begin{align*}
a x_{2}^{(\nu-1)}+b x_{3}^{(\nu-1)}+c x_{4}^{(\nu-1)} & =-\nu, \\
-a x_{1}^{(\nu-1)}+d x_{3}^{(\nu-1)}+e x_{4}^{(\nu-1)} & =-\nu, \\
-b x_{1}^{(\nu-1)}-d x_{2}^{(\nu-1)}+f x_{4}^{(\nu-1)} & =-\nu,  \tag{144}\\
-c x_{1}^{(\nu-1)}-e x_{2}^{(\nu-1)}-f x_{3}^{(\nu-1)} & =-\nu .
\end{align*}
$$

We have also that

$$
x_{1}^{(\nu-1)}+x_{2}^{(\nu-1)}+x_{3}^{(\nu-1)}+x_{4}^{(\nu-1)}=0
$$

because this is the coefficient of $\frac{1}{t}$ the function $H=x_{1}+x_{2}+x_{3}+x_{4}$, that is always a constant of motion for this system. The matrix of this linear system is the following:

$$
\left(\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 0  \tag{145}\\
0 & a & b & c & -\nu \\
-a & 0 & d & e & -\nu \\
-b & -d & 0 & f & -\nu \\
-c & -e & -f & 0 & -\nu
\end{array}\right)
$$

that is similar to the matrix (141) with the only difference being that $\nu_{4}=\nu$.

If $a=0$ and $b=0$, then from (143) we have also that $c=0$, that is a contradiction because

$$
\dot{x}_{1}=0 \Longrightarrow x_{1} \text { is a constant. }
$$

Assume that $d=b \neq 0$. Then

$$
a f-b e+c d=0 \Longrightarrow b e=c b \Rightarrow e=c
$$

The matrix (145) is row equivalent to the matrix

$$
\left(\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 0  \tag{146}\\
0 & 0 & 1 & \frac{c}{b} & -\frac{\nu}{b} \\
0 & 0 & 0 & b+f-c & 0 \\
0 & 0 & 0 & \frac{c(b+f-c)}{b} & -\frac{\nu(b+f-c)}{b} \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

This is also the row equivalent of the matrix which derives from which the system (143) if we replace $\nu$ by 0 . We can see very easily that the above linear system has a solution if and only if

$$
f=c-b \text {. }
$$

The solution for $x_{4}$ of this system is:

$$
\begin{equation*}
x_{4}(t)=C_{2}+C_{3} \sin (t \sqrt{m})+C_{4} \cos (t \sqrt{m}), \tag{147}
\end{equation*}
$$

where $m=3 b^{2}+3 d^{2}-2 b d=2\left(d^{2}+b^{2}\right)+(d-b)^{2}$ and $C_{1}, C_{2}, C_{3}, C_{4}$ are constants. This is a contradiction because the function (147) has no poles. So, if we assume that $d \neq b$ and having that $b \neq 0$, then

$$
a f-b e+c d=0 \Rightarrow b e=c d \Longrightarrow e=\frac{c d}{b}
$$

Hence the row equivalent matrix of (145) is

$$
\left(\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 0  \tag{148}\\
0 & 1 & \frac{b}{b-d} & \frac{b+f}{b-d} & \frac{\nu}{b-d} \\
0 & 0 & 1 & \frac{c}{b} & \frac{\nu}{b} \\
0 & 0 & 0 & 0 & \frac{(b-d) \nu}{b} \\
0 & 0 & 0 & 0 & \frac{(b+f-c) \nu}{b}
\end{array}\right) .
$$

This system has no solution since $d \neq b$. Hence we conclude that

$$
a \neq 0 .
$$

Therefore we have that

$$
\begin{equation*}
f=\frac{b e-c d}{a} . \tag{149}
\end{equation*}
$$

The row equivalent matrix of (145) is

$$
\left(\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 0  \tag{150}\\
0 & 1 & \frac{b}{a} & \frac{c}{a} & -\frac{\nu}{a} \\
0 & 0 & a+d-b & a+e-c & 0 \\
0 & 0 & 0 & 0 & -\frac{(a+d-b) \nu}{a} \\
0 & 0 & 0 & 0 & -\frac{(a+e-c) \nu}{a}
\end{array}\right) .
$$

Hence this system has a solution only if

$$
b=a+d, \quad c=a+e
$$

and from the (149) we have that

$$
f=e-d \text {. }
$$

The present ODE system can be solved for any values of $a, d$ and $e$. The simplest form of the solution is

$$
x_{2}=C_{1}+C_{2} \sin (t \sqrt{m})+C_{3} \cos (t \sqrt{m}),
$$

where $C_{1}, C_{2}$ and $C_{3}$ are constants and

$$
\begin{aligned}
& m=-2 e d+3 a^{2}+3 d^{2}+2 d a+3 e^{2}+2 a e \\
& \Longleftrightarrow m=(a+d-e)^{2}+2(a+e)^{2}+2 d^{2} .
\end{aligned}
$$

The contradiction is that this function has no poles.

So the case $\nu_{1}=\nu_{2}=\nu_{3}=\nu_{4}=\nu>1$, cannot provide an algebraically completely integrable system.

As we can see, the Painleve Analysis does not lead to an algebraically completely integrable Lotka-Volterra system in four dimensions, as it happened in the three-dimensional case.

## 6 Conclusion-Open problems

In this thesis we examine the algebraic integrability of Lotka-Volterra systems in three and four dimensions. We restrict attention to systems defined by a skew-symmetric matrix. The basic tool in the classification is the use of Painleve analysis, examination of the eigenvalues of the Kowalevski matrix and other standard Lax pair and Poisson techniques. Some number theoretic techniques are also used. In the four dimensional case the classification involves over 100 cases. All the known integrable cases appear including the open and periodic Kac-van Moerbeke systems and a big number of new cases. The application of Painleve analysis and especially of the ARS algorithm is a useful tool for calculating Laurent solutions and the check for the correct number of free parameters. Another important tool is the use of the Kowalevski exponents. Imposing some integrality conditions on the exponents and using some number theoretic techniques we obtain necessary conditions for the algebraic integrability of the systems. Afterwards we prove that the conditions are not only necessary but also sufficient. We would like to close by stating some open problems as ideas for future research in this area.

1. Generalizations to higher dimensions The first and obvious problem is to try to generalize this results in $\mathbf{R}^{n}$ for $n \geq 5$. We believe that this problem would be quite difficult at the present time. However, some of the systems that appear in the three and four dimensional classification should generalize in an arbitrary dimension.
2. Applications In this thesis we consider a theoretical problem from a mathematical point of view. However the big number of systems in the four dimensional classification should be analyzed from various viewpoints, e.g. Complete Integrability, Lax pair representation, Poisson geometry and also for potential applications in Mathematical Physics.
3. Other systems Another open problem is to do a similar classification for systems for which the defining matrix is not skew-symmetric, e.g. symmetric or a general matrix.
4. Connections with Toda lattices It is well-known that the KM-system is equivalent to the classical Toda lattice. It will be interesting to find a similar connection for the new systems that appear in the four dimensional classification.
5. Negative Exponents As was suggested by Peter Leach it would be interesting to consider the case of negative Kowalevski exponents.
6. Darboux polynomials Is the classification of Darboux polynomials for such systems related to the present classification? We believe that the answer is positive
7. Kowalevski Exponents Looking at the tables in the present thesis it is not absolutely clear what is the relation between the degrees of the invariants and the set of Kowalevski exponents. However, there are many relevant results in the literature.

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$$
\begin{aligned}
& \frac{\mathrm{d} x_{1}}{\mathrm{~d} t}=a_{1} x_{2} x_{3}+a_{2} x_{3} x_{1}+a_{3} x_{1} x_{2} \\
& \frac{\mathrm{~d} x_{2}}{\mathrm{~d} t}=b_{1} x_{2} x_{3}+b_{2} x_{3} x_{1}+b_{3} x_{1} x_{2} \\
& \frac{\mathrm{~d} x_{3}}{\mathrm{~d} t}=c_{1} x_{2} x_{3}+c_{2} x_{3} x_{1}+c_{3} x_{1} x_{2}
\end{aligned}
$$

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