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Error Analysis of the Bergman Kernel Method with Singular Basis Functions

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Διατριβή η οποία υποβλήθηκε προς απόκτηση
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Dedicated to my wife *Yiota* and my daughter *Antria*

Thanks for all!

Περίληψη

Στην παρούσα διατριβή υποθέτουμε ότι το G είναι ένα φραγμένο χωρίο *Jordan* με κατά τμήμα αναλυτικό σύνορο μέσα στο μιγαδικό επίπεδο και έστω ότι $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ συμβολίζει το συμπλήρωμα του $\overline{G}$. Παρουσιάζουμε θεωρητικές εκτιμήσεις και αριθμητικά αποτελέσματα σχετικά με την εφαρμογή της μεθόδου του *Bergman* (BKM) χρησιμοποιώντας μία βάση που περιέχει αλγεβρικές συναρτήσεις. Η μέθοδος αυτή είναι γνωστή ως BKM/AB για την προσέγγιση της σύμμορφης απεικόνισης f_0 από το G στον κανονικοποιημένο δίσκο. Εδώ σημειώνουμε ότι παρουσιάζουμε δύο ειδών σφάλματα για την προσέγγιση της f_0 απο τα βέλτιστα $L^2(G)$ πολυώνυμα: το $L^2(G)$-σφάλμα και το $L^\infty(G)$-σφάλμα. Με αυτό τον τρόπο παρέχουμε πλήρη θεωρητική αιτιολόγηση της μεθόδου BKM/AB.

Στο πρώτο κεφάλαιο περιγράφουμε τη σύγκλιση της BKM για τις διάφορες περιπτώσεις του συνόρου Γ του G. Πιο συγκεκριμένα, όσον αφορά τη σύγκλιση της μεθόδου, στην περίπτωση που η f_0 έχει αναλυτική επέκταση σε όλο το Γ, τότε αυτό είναι συνέπεια του θεωρήματος του *Walsh* περί μέγιστης σύγκλισης [27, §4.7, §5.3]. Στην περίπτωση όπου το σύνορο Γ είναι κατά τμήμα αναλυτικό και η f_0 έχει ιδιομορφίες στο Γ (γωνίες), τότε τα πιο ακριβή αποτελέσματα για το σφάλμα στην προσέγγιση της f_0 οφείλονται στον *D. Gaier* [7].

Οι *Levin*, *Παπαμιχαήλ* και *Σιδερίδης* ήταν οι πρωτοι που πρότειναν στο [12] ότι το σφάλμα όσον αφορά την προσέγγιση της f_0 στο $\overline{G}$ από πολυώνυμα χαμηλού βαθμού εξαρτάται τόσο από τις γωνιές του σύνορου Γ όσο και από τους πόλους της f_0 στο Ω. Ως εκ τούτου, προκειμένου να βελτιωθεί η αριθμητική απόδοση της BKM για την προσέγγιση της f_0, εισάγουν μια μέθοδο που βασίζεται στην ορθοκανονικοποίηση ενός συστήματος που αποτελείται από μονώνυμα και ιδιόμορφες συναρτήσεις που ανακλούν μαζί τις γωνιές και τους πόλους της f_0 στο Γ και στο Ω. Η επέκταση αυτή είναι γνωστή ως BKM/AB. Στο

δεύτερο κεφάλαιο ορίζουμε τον συμβολισμό που θα χρησιμοποιηθεί για την κατασκευή της μεθόδου BKM/AB και συζητάμε την κατάλληλη επιλογή των συναρτήσεων που αποτελούν την βάση. Το τρίτο κεφάλαιο είναι αφιερωμένο στη θεωρητική αιτιολόγηση των διαφόρων BKM και BKM/AB σφαλμάτων, στην περίπτωση όπου η f_0 έχει αναλυτική επέκταση σε όλο το Γ, ως εκ τούτου μόνο συναρτήσεις που ανακλούν τους πόλους χρησιμοποιούνται στην BKM/AB. Σε αυτή την περίπτωση παράγονται άνω και κάτω φράγματα για τα BKM και BKM/AB σφάλματα.

Η θεωρητική πιστοποίηση της μεθόδου BKM/AB με συναρτήσεις που ανακλούν μόνο τις ιδιομορφίες της f_0 στο σύνορο Γ δόθηκε στο [16]. Ο σκοπός του κεφαλαίου 4 είναι να παράξει θεωρητικά αποτελέσματα για την προσέγγιση της f_0 χρησιμοποιώντας συναρτήσεις που ανακλούν ταυτόχρονα, τις ιδιομορφίες της f_0 στο σύνορο Γ και στο Ω. Πιο συγκεκριμένα, έχουμε εξάγει άνω φράγματα για τα BKM και BKM/AB σφάλματα. Στο κεφάλαιο 5 μελετάμε τα BKM και BKM/AB σφάλματα εσωτερικά του G για τις δυο διαφορετικές περιπτώσεις των κεφαλαίων 3 και 4, όπου και εξάγουμε άνω φράγματα. Τέλος, στο κεφάλαιο 6 παρουσιάζουμε αριθμητικά αποτελέσματα που πιστοποιούν τα θεωρητικά αποτελέσματα των κεφαλαίων 3, 4 και 5.

Abstract

In this thesis we assume that G is a bounded Jordan domain in the complex plane with piecewise analytic boundary and let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ denote the complement of $\overline{G}$. We present theoretical estimates and numerical evidence for certain phenomena, regarding the application of the Bergman kernel method (BKM) with algebraic and pole singular basis functions, denoted as BKM/AB for approximating the conformal mapping f_0 of G onto the normalized disk. More precisely, we obtain two sided-estimates for the $L^2(G)$ and $L^\infty(G)$-error, in the best $L^2(G)$-polynomial approximation to f_0. In this way, we complete the task of providing full theoretical justification of this method.

In the first chapter we describe the convergence of the BKM for several cases of the boundary Γ of G. More specifically, regarding the convergence of the method, in the case when f_0 has analytic continuation across Γ, then this is a consequence of Walsh's theory of maximal convergence [27, §4.7, §5.3]. In the case when Γ is piecewise analytic without cusps and f_0 has singularities on Γ, then the most precise results for the error in approximating f_0 are due to D. Gaier [7].

Levin, Papamichael and Siderides were the first to suggest in [12] that the error in approximating f_0 on $\overline{G}$ by polynomials of low degree will depend on both the boundary and the pole singularities of f_0 in Ω. Hence, in order to improve the numerical performance of the BKM for approximating f_0, they introduced a method which is based on orthonormalizing a system of functions consisting of monomials and singular terms that reflect both corner and pole singularities of f_0 on Γ and in Ω. This extension is known as BKM/AB. In Chapter 2 we set up the notation and recall the BKM/AB. Chapter 3 is devoted to the theoretical justification of the various BKM and BKM/AB errors, in

cases when f_0 has an analytic continuation across Γ, hence only basis functions reflecting poles are used in the BKM/AB. In this case we derive upper and lower estimates for the BKM/AB errors. Also we derive upper and lower estimates for the BKM errors.

The theoretical justification of the BKM/AB with basis function that reflect the corner singularities of f_0 was given in [16], by means of sharp estimates for the associated BKM/AB errors. The purpose of Chapter 4 is to derive theoretical results that justify the use of basis functions that reflect both corner and pole basis functions. More specifically, we derive upper estimates for the BKM/AB errors. In addition, we derive more informative estimates for the BKM errors. In Chapter 5 we study the BKM and BKM/AB error interior the domain G for the two different cases of Chapter 3 and 4, and we derive upper estimates. Finally, in Chapter 6, we present numerical computations that illustrate the theory of Chapter 3, 4 and 5.

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Preface / Acknowledgements

This thesis is based on studies of the years 2007 - 2011 at the department of Mathematics and Statistics of the University of Cyprus. The main part of this thesis consists of results that appear in two research papers [14] and [13] that I co-author with my supervisor Professor Nikos Stylianopoulos. With this thesis we complete the task that was put forward by Yu. E. Khokhlov, reviewer of the introductory paper [12] of the BKM/AB in the Mathematical Reviews, who concluded that: "*A proof of the convergence of the numerical method given and an investigation of its convergence rate are lacking, so the results obtained are of a heuristic nature*".

I would like to thank all people who have helped and inspired me during my doctoral study. First and foremost, I want to thank my supervisor, Nikos Stylianopoulos, whose encouragement, supervision and support from the preliminary to the concluding level enabled me to develop an understanding of the subject. Without his guidance and persistent help this dissertation would not have been possible. I could not have found someone better for getting me through the graduate experience and letting me have the freedom to explore new objects of Mathematics.

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Chapter 1

Preliminary results

1.1 Introduction

The purpose of this chapter is to present the preliminary results needed in the analysis of the method considered in this thesis. The presentation is as follows:

(i) In Section 1.2 we introduce the conformal mapping of a bounded Jordan domain G onto the normalized disk.

(ii) A constructive characterization of the above conformal map in terms of a Hilbert space of analytic functions is given in Section 1.3.

(iii) The Bergman kernel method (BKM) for approximating the conformal mapping is described in Section 1.4.

(iv) Well-known results for the BKM errors for approximating the conformal mapping are given in Section 1.5.

1.2 Conformal mapping

Let G be a bounded, simply-connected domain in the complex plane $\mathbb{C}$ whose boundary $\Gamma := \partial G$ is a Jordan curve and let $\Omega := \overline{\mathbb{C}} \backslash \overline{G}$ denote the complement of $\overline{G}$ with respect to the extended complex plane. Fix $z_0 \in G$ and let f_0 denote the *Interior Conformal Map* of G onto the disk $D(0, r_0) := \{z : |z| < r_0\}$, normalized by the conditions $f_0(z_0) = 0$ and $f_0'(z_0) = 1$. The quantity $r_0 := r_0(G, z_0)$ is called the conformal radius of G with respect to z_0.

The unique existence of the interior conformal map f_0 is a consequence of the Riemann's mapping theorem. In order to be more specific, let G be a simply connected domain in the extended complex plane, whose boundary contains more than one point and let z_0 be an arbitrary point of G. Then there exists a unique function $w = f_{z_0}(z)$ which maps G conformally onto the disk $|w| < 1$ and satisfies the conditions

$$f_{z_0}(z_0) = 0, \quad f_{z_0}'(z_0) > 0. \tag{1.2.1}$$

Clearly, $f_0(z) = f_{z_0}(z)/f_{z_0}'(z_0)$ and $r_0 = 1/f_{z_0}'(z_0)$.

Also, let Φ denote the *Exterior Conformal Map* of Ω onto $\Delta := \{w : |w| > 1\}$, normalized so that near infinity,

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \ldots, \ \ \gamma > 0. \tag{1.2.2}$$

($\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$.) Note that $\gamma = 1/\text{cap}(\Gamma)$, where $\text{cap}(\Gamma)$ denote the (logarithmic) capacity of Γ.

1.3 The Hilbert space $L_a^2(G)$

For the inner product

$$\langle f, g \rangle := \int_G f(z)\, \overline{g(z)}\, dA(z), \tag{1.3.1}$$

where dA denotes the differential of the area measure on $\mathbb{C}$, we consider the space $L_a^2(G)$ of all square integrable functions which are analytic in G, i.e.,

$$L_a^2(G) := \big\{ f : f \text{ analytic in } G,\ \langle f, f \rangle < \infty \big\}. \tag{1.3.2}$$

This space with inner product defined as in (1.3.1) is a Hilbert space with corresponding norm

$$\|f\|_{L^2(G)} := \langle f, f \rangle^{\frac{1}{2}}. \tag{1.3.3}$$

In this section we state a number of well-known results, concerning the space $L_a^2(G)$, which are needed for the analysis in subsequent chapters. The detailed proof of most of these results can be found in [2], [3], [4], [15] and [24].

Lemma 1.3.1. *Suppose $f \in L_a^2(G)$, $z_0 \in G$, and $d_{z_0} := \mathrm{dist}(z_0, \Gamma)$ denotes the distance of z_0 from Γ. Then,*

$$|f(z_0)| \leq \frac{\|f\|_{L^2(G)}}{\sqrt{\pi} d_{z_0}}. \tag{1.3.4}$$

Proof. We have $\|f\|^2_{L^2(G)} \geq \int_D |f(z)|^2 dA(z)$, where D is the disk with radius $R = d_{z_0}$ and center z_0. By considering the Taylor series expansion of $f(z)$ in D we have

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots .$$

Set $z = z_0 + re^{i\theta}$, then

$$\begin{aligned} \int_D |f(z)|^2 dA(z) &= \int_0^R \int_0^{2\pi} \left| \sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \right|^2 r d\theta dr \\ &= 2\pi \sum_{k=0}^{\infty} \frac{|a_k|^2}{2(k+1)} R^{2(k+1)}. \end{aligned} \tag{1.3.5}$$

Relation (1.3.5) implies that

$$\int_D |f(z)|^2 dA(z) \geq \pi |a_0|^2 R^2 = \pi |f(z_0)|^2 d_{z_0}^2,$$

and inequality (1.3.4) is established. □

We note that for each compact subset B of G, inequality (1.3.4) implies that

$$|f(z)| \leq \frac{\|f\|_{L^2(G)}}{\sqrt{\pi} d} \quad (z \in B),$$

where $d = \mathrm{dist}(B, \Gamma)$.

Theorem 1.3.1. *With $\langle f, g \rangle$ defined as in (1.3.1), $L_a^2(G)$ is a Hilbert space.*

Proof. Let $\{f_n(z)\}_{n=1}^{\infty}$ be a Cauchy sequence of functions in $L_a^2(G)$, that is, given an $\epsilon > 0$, we can find an $N(\epsilon) \in \mathbb{N}$ such that

$$\|f_n - f_m\|^2_{L^2(G)} < \epsilon, \quad m, n \geq N(\epsilon).$$

For each compact subset B of G, Lemma 1.3.1 implies that

$$|f_n(z) - f_m(z)|^2 < \frac{\epsilon}{\pi d}, \quad z \in B,$$

where $d = \text{dist}(B, \Gamma)$. This means that for each compact subset B of G, the sequence $\{f_n(z)\}_{n=1}^{\infty}$ convergence uniformly to an analytic function F. The inequality $\|f_n - f_m\|^2_{L^2(G)} < \epsilon$ further implies $\int_B |f_n - f_m|^2 dA(z) < \epsilon$. Now by letting $m \to \infty$, we obtain that $\int_B |f_n(z) - F(z)|^2 dA(z) < \epsilon$; hence $f_n(z) - F(z) \in L^2_a(G)$. Since $f_n(z) \in L^2_a(G)$ then $F(z) \in L^2_a(G)$ and that $\|f_n - F\|_{L^2(G)} \to 0$, $(n \to \infty)$. In other words, each Cauchy sequence in $L^2_a(G)$ converges. □

Definition 1.3.1. *If H is a Hilbert space, we say that a subset $S \subset H$ is an ON system (orthonormal system) in H if*

$$\langle u, v \rangle = \begin{cases} 1, & if \quad u = v, \\ 0, & if \quad u \neq v, \end{cases}$$

whenever $u, v \in S$.

Definition 1.3.2. *Suppose S is a subset of a Hilbert space H. Then, S is complete if whenever $y \in H$ with $\langle y, x \rangle = 0$ for all $x \in S$, implies that $y = 0$.*

Suppose $\{u_1, u_2, \ldots, u_n\} \subset H$ is a linearly independent set with n elements. The standard way to construct an ON system $\{v_1, v_2, \ldots, v_n\}$ is the *Cram-Schmidt* process (GS); see e.g. [4, p. 6]. Now we return to our special Hilbert space $L^2_a(G)$ and choose $u_j = z^{j-1}$, $j = 1, 2, \ldots$. If G is bounded, the u_j clearly belong to $L^2_a(G)$, and therefore we can use the GS process to construct uniquely determined polynomials

$$P_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots. \tag{1.3.6}$$

These polynomials are called the *Bergman polynomials* of G and are orthonormal with respect to the inner product (1.3.1), i.e.,

$$\int_G P_m(z)\overline{P_n(z)} dA(z) = \delta_{m,n}. \tag{1.3.7}$$

Definition 1.3.3. *A domain $G \subset \mathbb{C}$ has the PA property if the polynomials are dense in $L^2_\alpha(G)$.*

Here PA stands for polynomial approximation.

Definition 1.3.4. *A simply-connected, bounded domain G is said to be a Caratheodory domain if the boundary ∂G of G is also the boundary of the complement of $\overline{G} = G \cup \partial G$.*

All domains considered in this thesis are Jordan domains and hence, by the Jordan curve theorem (see [15, p. 70]), they are Caratheodory domains. This implies by the Theorem 1 in [4, p. 17], (see e.g. [15, p. 117]) that the Bergman polynomials have the PA property in $L^2_a(G)$ and hence by Theorem 2 in [4, p. 25] the Bergman polynomials $\{P_n\}_{n=0}^{\infty}$ form a *complete orthonormal system* (CON) in the Hilbert space $L^2_a(G)$.

For the practical determination of Bergman polynomials $\{P_n\}_{n=0}^{\infty}$, the Cram-Schmidt method requires the evaluation of the double integrals $\langle z^{i-1}, z^{j-1}\rangle$, where $i, j = 1, 2, \ldots$. These double integrals can be converted into one-dimensional integrals as outline below:

- If the boundary of G is starlike with $r = r(\phi)$ $(0 \leq \phi \leq 2\pi)$ representing the boundary Γ in polar coordinates, the integral becomes

$$\begin{aligned} \langle z^{k-1}, z^{l-1}\rangle &= \int_G z^{k-1}\,\overline{z}^{l-1}\,dA(z) \\ &= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{r(\phi)} \rho^{k+l-1} e^{i\phi(k-l)} dp d\phi \\ &= \frac{1}{k+l}\int_{\phi=0}^{2\pi} [r(\phi)]^{k+l} e^{i\phi(k-l)} d\phi. \end{aligned} \tag{1.3.8}$$

- Assume that G has a piecewise smooth, positively oriented boundary Γ, and suppose that the functions f and g are analytic in G and continuous in $\overline{G}$. An application of Green's formula leads to

$$\int_G f(z)\overline{g(z)}'dA(z) = \frac{1}{2i}\int_\Gamma f(z)\overline{g(z)}dz.$$

  Applying this to the inner products $\langle z^{k-1}, z^{l-1}\rangle$, we transform the area integral into a line integral, i.e.,

$$\begin{aligned} \langle z^{k-1}, z^{l-1}\rangle &= \int_G z^{k-1}\,\overline{z}^{\,l-1}\,dA(z) \\ &= \frac{1}{2il}\int_\Gamma z^{k-1}\overline{z}^{\,l} dz. \end{aligned} \tag{1.3.9}$$

Suppose H is a Hilbert space with corresponding norm $\|\cdot\|$ and $\{v_j\}$ is an ON system in H. For each $x \in H$, we form the *Fourier coefficients* $\gamma_j = \langle x, v_j \rangle$. The following theorem gives the minimum property of the Fourier coefficients.

Theorem 1.3.2. *Let $c_j \in \mathbb{C}$ and $x \in H$. Then,*

(i) The quantity $\|x - \sum_{j=1}^{n} c_j v_j\|^2$, is minimum if and only if $c_j = \gamma_j, \quad j = 1, 2, \ldots, n$.

(ii) The minimum in part (i) equals $\|x\|^2 - \sum_{j=1}^{n} |\gamma_j|^2$.

(iii) For each $x \in H$, Bessel's inequality holds: $\sum_{j=1}^{n} |\gamma_j|^2 \leq \|x\|^2$.

Proof. The three assertions are a consequence of the following computations:

$$\begin{aligned} \|x - \sum_{j=1}^{n} c_j v_j\|^2 &= \langle x - \sum_{j=1}^{n} c_j v_j, x - \sum_{j=1}^{n} c_j v_j \rangle \\ &= \|x\|^2 - \sum_{j=1}^{n} c_j \overline{\gamma}_j - \sum_{j=1}^{n} \overline{c}_j \gamma_j + \sum_{j=1}^{n} |c_j|^2 \\ &= \|x\|^2 - \sum_{j=1}^{n} |\gamma_j|^2 + \sum_{j=1}^{n} (\gamma_j - c_j)(\overline{\gamma}_j - \overline{c}_j) \\ &= \|x\|^2 + \sum_{j=1}^{n} |\gamma_j - c_j|^2 - \sum_{j=1}^{n} |\gamma_j|^2. \end{aligned} \tag{1.3.10}$$

□

For the next theorem we shall require that $\{v_j\}$ is a complete orthonormal system (CON).

Theorem 1.3.3. *The following statements are equivalent.*

(i) $\{v_j\}$ is a CON system.

(ii) For each $x \in H$, the relation $\|x - \sum_{j=1}^{n} \gamma_j v_j\| \to 0, \quad n \to \infty$.

(iii) For each $x \in H$, Perseval's identity holds: $\sum_{j=1}^{n} |\gamma_j|^2 = \|x\|^2$.

Proof. The equivalence of the above statements follows from the Theorem 1.3.2. □

Now suppose $H = L_a^2(G)$, where, to begin with, G is a bounded, simply-connected domain in complex place $\mathbb{C}$ whose boundary Γ is a Jordan curve and let $\{\phi_n\}$ is an ON system of functions in $L_a^2(G)$. For each $f \in L_a^2(G)$, the Fourier coefficients are

$$\gamma_j = \langle f, \phi_j \rangle = \int_G f(z)\overline{\phi_j(z)}dA(z), \quad j = 1, 2, \dots,$$

and the Fourier series of f becomes

$$f \sim \sum_{j=1}^{\infty} \gamma_j \phi_j.$$

If $\{\phi_j\}$ forms a CON system, Theorem 1.3.3 implies that

$$\|f - \sum_{j=1}^{n} \gamma_j \phi_j\|_{L^2(G)} \to 0, \quad n \to \infty. \tag{1.3.11}$$

Theorem 1.3.4. *If $\{\phi_j\}$ form a CON system in $L_a^2(G)$ and $\sum_{j=1}^{\infty} \gamma_j \phi_j$ is the corresponding Fourier series of the function $f \in L_a^2(G)$, then the Fourier series $\sum_{j=1}^{\infty} \gamma_j \phi_j$ converges to f uniformly on each compact subset B of G.*

Proof. If $d := \text{dist}(B, \Gamma)$ denotes the distance from B to the boundary Γ, then Lemma 1.3.1 implies that

$$|f(z) - \sum_{j=1}^{n} \gamma_j \phi_j(z)| \leq \frac{\|f - \sum_{j=1}^{n} \gamma_j \phi_j\|_{L^2(G)}}{\sqrt{\pi} d}, \quad z \in B,$$

and because of (1.3.11) our assertions follows. □

If H is a Hilbert space and M is a bounded linear functional on H, then by the Riesz representation theorem, there exist a uniquely determined element $u \in H$ such that

$$M(x) = \langle x, u \rangle, \quad \forall\ x \in H.$$

In cases where $H = L_a^2(G)$, Lemma 1.3.1 gives that $|f(z_0)| \leq \|f\|_{L^2(G)}/(\sqrt{\pi} d_{z_0})$, where $d_{z_0} = \text{dist}(z_0, \Gamma)$, and hence

$$M(f) := f(z_0), \quad f \in L_a^2(G),$$

is a bounded linear functional in $L_a^2(G)$, for each fixed $z_0 \in G$. Thus, there exist a uniquely determined $u_{z_0} \in L_a^2(G)$ such that

$$f(z_0) = \langle f, u_{z_0} \rangle, \quad f \in L_a^2(G).$$

We use the notation $K(z, z_0)$ instead of u_{z_0}, where $K(z, z_0)$ is called the *Bergman kernel function* of G with respect to z_0 and is the unique function of $L_a^2(G)$ satisfying the reproducing property

$$\langle g, K(\cdot, z_0) \rangle = g(z_0), \text{ for all } g \in L_a^2(G). \tag{1.3.12}$$

Now, it is easy to find the Fourier series expansion of $K(\cdot, z_0)$ with respect to some CON system ϕ_j, because by (1.3.12), the Fourier coefficient are

$$\gamma_j = \langle K(\cdot, z_0), \phi_j \rangle = \overline{\phi_j(z_0)}, \quad j = 1, 2, \ldots. \tag{1.3.13}$$

Theorem 1.3.4 and (1.3.13) gives the following result.

Theorem 1.3.5. *Let $\{\phi_j\}$ be an arbitrary CON system, the Bergman kernel function $K(\cdot, z_0)$ has the Fourier series expansion*

$$K(z, z_0) = \sum_{j=1}^{\infty} \overline{\phi_j(z_0)} \phi_j(z), \quad z, z_0 \in G. \tag{1.3.14}$$

For each fixed $z_0 \in G$, the series convergence uniformly on each compact subset B of G.

In view of the Theorem 1.3.5 and the fact that the Bergman polynomials form a complete orthonormal system in $L_a^2(G)$ we have

$$K(z, z_0) = \sum_{j=1}^{\infty} \overline{P_j(z_0)} P_j(z), \tag{1.3.15}$$

locally uniformly with respect to $z \in G$.

The result of the following theorem is needed for establishing a connection between the Bergman kernel function $K(\cdot, z_0)$ and the interior conformal map f_0 of Section 1.2.

Theorem 1.3.6. *Suppose that $G \subset \mathbb{C}$ is a simply-connected domain and f_0 is the interior conformal map which maps G one-one conformably onto the disk $D(0, r_0) := \{z : |z| <$*

$r_0\}$, normalized by the conditions $f_0(z_0) = 0$ and $f_0'(z_0) = 1$, where z_0 is some fixed point in G. Then the kernel $K(\cdot, z_0)$ is related to the mapping function f_0 by means of

$$f_0'(z) = \frac{K(z, z_0)}{K(z_0, z_0)} \quad \textit{and} \quad f_0(z) = \frac{1}{K(z_0, z_0)} \int_{z_0}^{z} K(\zeta, z_0) d\zeta. \tag{1.3.16}$$

Proof. The details of the proof are as follows.

Assume f_{z_0} is the conformal map of Section 1.2 which satisfies the contitions (1.2.1) and let $G_p := \{z : |f_{z_0}(z)| < p,\ 0 < p < 1\}$. Since $\overline{f_{z_0}(z)} f_{z_0}(z) = p^2$ for $z \in \partial G_p$, Green's formula gives for any $g \in L_a^2(G)$

$$\begin{aligned} \int_{G_p} g(z) \overline{f_{z_0}'(z)} dA(z) &= \frac{1}{2i} \int_{\partial G_p} g(z) \overline{f_{z_0}(z)} dz \\ &= \frac{p^2}{2i} \int_{\partial G_p} \frac{g(z)}{f_{z_0}(z)} dz. \end{aligned} \tag{1.3.17}$$

Using the residue theorem the last integral equals with $2\pi i g(z_0)/f_{z_0}'(z_0)$ and hence if we let $p \to 1$,

$$\langle g, f_{z_0}' \rangle = \frac{\pi g(z_0)}{f_{z_0}'(z_0)},$$

i.e.

$$\langle g, \frac{1}{\pi} f_{z_0}'(z_0) f_{z_0}' \rangle = g(z_0),$$

The uniqueness of the reproducing kernel $K(z, z_0)$ and the fact that $f_0(z) = f_{z_0}(z)/f_{z_0}'(z_0)$ implies the first relation in (1.3.16). The second relation is an easy consequence of the first relation. □

Theorem 1.3.6 yields the two relation,

$$K(z, z_0) = \frac{1}{\pi r_0^2} f_0'(z) \quad \text{and} \quad r_0 = \frac{1}{\sqrt{\pi K(z_0, z_0)}}. \tag{1.3.18}$$

where r_0 is the conformal radius of Section 1.2.

1.4 Bergman kernel method

The Bergman kernel method (BKM) is an orthonormalization method for computing approximations to the conformal map $f_0(z)$. It is based on the fact that the kernel

$K(z, z_0)$ is given explicitly in terms of the Bergman polynomials $\{P_n(z)\}_{n=0}^{\infty}$. Thus, the partial sums of the Fourier series expansion of $K(z, z_0)$ are given by

$$K_n(z, z_0) := \sum_{j=0}^{n} \langle K(\cdot, z_0), P_j \rangle P_j(z) = \sum_{j=0}^{n} \overline{P_j(z_0)} P_j(z), \quad n \in \mathbb{N}. \tag{1.4.1}$$

The polynomials $\{K(z, z_0)\}_{n=0}^{\infty}$ are the so called *kernel polynomials* of G, with respect to z_0. In accordance with (1.3.16), the n-th BKM approximation to f_0 is given by

$$\pi_n(z) := \frac{1}{K_{n-1}(z_0, z_0)} \int_{z_0}^{z} K_{n-1}(\zeta, z_0) d\zeta, \quad n \in \mathbb{N}. \tag{1.4.2}$$

This defines the sequence $\{\pi_n\}_{n=1}^{\infty}$ of the *Bieberbach polynomials* of G with respect to z_0. Also from (1.3.18)

$$r_n = \frac{1}{\sqrt{\pi K_n(z_0, z_0)}}, \quad n \in \mathbb{N}, \tag{1.4.3}$$

is the BKM approximation to the conformal radius of G at z_0.

The polynomials π_n solve the following minimal problem: Let $\mathbb{P}_n$ denote the class of complex polynomials of degree at most n and let

$$\mathbb{P}_n^* := \{p : p \in \mathbb{P}_n, \text{ with } p(z_0) = 0 \text{ and } p'(z_0) = 1\}. \tag{1.4.4}$$

Then, for each $n \in \mathbb{N}$, *the polynomial* π_n *minimizes uniquely the two norms* $\|f_0' - p'\|_{L^2(G)}$ and $\|p'\|_{L^2(G)}$ over all $p \in \mathbb{P}_n^*$; see e.g. [4, p. 34], [3, Kap. III, §1].

Next we observe the followings:

(i) Because of the Theorem 1.3.4 the kernel polynomials $K_n(z, z_0)$ converge to $K(z, z_0)$ uniformly on each compact subset B of G.

(ii) The kernel polynomials $\{K_n(z, z_0)\}$ provide the best $L^2(G)$- approximation to $K(z, z_0)$ out of the space $\mathbb{P}_n$ of complex polynomials of degree at most n, i.e.,

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \leq \|K(\cdot, z_0) - p\|_{L^2(G)}, \tag{1.4.5}$$

for any $p \in \mathbb{P}_n$.

(iii) Finally we observe that the approximations $\{K_n(z, z_0)\}$ can be expressed in the form

$$K_n(z, z_0) = \sum_{j=1}^{n} k_{nj} z^j.$$

Therefore, by the reproducing property $\langle z^i, K_n(z, z_0)\rangle = z_0^i$ the coefficient $k_{n1}, k_{n2}, \ldots, k_{nn}$ satisfying the normal equations

$$\sum_{j=1}^{n} \langle z^j, z^i \rangle k_{ni} = \overline{z_0^i}, \quad i = 1, 2, \ldots, n.$$

1.5 Well-known results for the BKM errors.

Next we consider only the special case where the boundary Γ of G is a Jordan curve and f_0 is analytic on $\overline{G}$.

Let L_R $(R \geq 1)$, denote the level curve in the exterior of G, i.e.,

$$L_R := \{z : |\Phi(z)| = R\}, \tag{1.5.1}$$

so that $L_1 \equiv \Gamma$. With this notation we have the following lemma which describes the growth of polynomials in $\mathbb{C}$. (See Gaier [4, p. 27], Markushevich [15, p. 112])

Lemma 1.5.1. *(Bernstein's Lemma) If $P(z)$ is a polynomial with $\deg(P) = n$ and $|P(z)| \leq 1$ for $z \in \overline{G}$, then $|P(z)| \leq R^n$ for $z \in L_R$.*

Proof. The function

$$F(z) = P(z)/\Phi^n(z)$$

is analytic in $\overline{\mathbb{C}}\backslash G$ and continuous in $\overline{\mathbb{C}}\backslash G$. Hence the maximum value of F is on G and

$$|F(z)| \leq 1, \;\; z \in \overline{\mathbb{C}}\backslash\overline{G}.$$

For $z \in L_R$ we have the required result. □

Regarding the convergence of the method, we note that in cases when f_0 has an analytic continuation across Γ, then this is a consequence of Walsh's theory of maximal convergence [27, §4.7, §5.3]; see also [4, Ch. I]. In order to be more specific,

Theorem 1.5.1. *Let Γ be a Jordan curve. Then the relation*

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} = O\left(\frac{1}{R^n}\right), \tag{1.5.2}$$

holds for any $1 < R < |\Phi(z_1)|$, but for no $R > |\Phi(z_1)|$, where z_1 denotes the nearest singularity of f_0 in Ω. (We use $\|\cdot\|_{L^\infty(\overline{G})}$ to denote the sup-norm on $\overline{G}$.)

For the proof of the above theorem we use the Bernstein's Lemma to show first that there are no polynomials π_n such that the (1.5.2) holds for $R > |\Phi(z_1)|$. The positive part of the statement (1.5.2), that is for $R < |\Phi(z_1)|$, is proved by using the method of interpolation to show first that

$$\|f_0 - \pi_n\|_{L^2(G)} = O\left(\frac{1}{R_1^n}\right) \tag{1.5.3}$$

where $1 < R_1 < |\Phi(z_1)|$. Therefore, using Lemmas 1.3.1 and 1.5.1, implies

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} \leq N\left(\frac{\sigma}{R_1}\right)^n \tag{1.5.4}$$

where $1 \leq \sigma < R_1 < |\Phi(z_1)|$ and N is a positive constant independent on n. The prove of the statement (1.5.2) for $R < p$ follows from (1.5.4).

In cases when Γ is piecewise analytic and f_0 has singularities on Γ, then Levin, Papamichael and Siderides were the first to observe in [12] that the error (1.5.2) depends on the boundary singularities of the mapping function f_0 on Γ, and also on the singularities of the extension of f_0 across the segments of Γ into Ω. Accordingly, in order to improve the numerical performance of the BKM, they extended the method by orthonormalizing a system of basis functions consisting from monomials, as in the BKM, and also from functions that reflect the dominant singularities of f_0 on Γ and in Ω. This extension is known as BKM/AB (AB stands for *augmented basis*). The BKM/AB was used subsequently in [19] and [20].

The most precise results regarding the convergence of the BKM are due to D. Gaier [7]. In particular, under the assumption that Γ is piecewise analytic without cusps, Gaier derived the estimate

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} = O(\log n)\frac{1}{n^s}, \tag{1.5.5}$$

where $s := \lambda/(2-\lambda)$ and $\lambda\pi$ $(0 < \lambda < 2)$ denotes the smallest exterior angle where two analytic arcs of Γ meet. Regarding sharpness of the estimate (1.3.16), it was shown in [6, Thm. 4] that there are cases where the exponent s can not be replaced by a smaller number. However, the factor $\log n$ can be replaced by $\sqrt{\log n}$, see [1] and [16, Rem. 3.1]. A lower estimate of the form

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} \geq c\,\frac{1}{n^s}, \tag{1.5.6}$$

provided that $1/(2-\lambda)$ is not a positive integer, where c is a constant that does not depend on n, was established in [16, Thm. 3.2] by Maymeskul, Saff and Stylianopoulos. The theoretical justification of the BKM/AB with basis function that reflect the corner singularities of f_0 was given in [16], by means of sharp estimates for the associated BKM/AB errors.

The purpose of the present thesis is to derive theoretical results that justify the use of basis functions that reflect (a) pole singularities of f_0 and (b) both corner and pole singularities of f_0. More specifically, we derive upper and lower estimates for the BKM/AB errors in the case (a), and upper estimates for the BKM/AB errors in the case (b). In doing so, we complete the task that was put forward by Yu. E. Khokhlov, reviewer of the introductory paper [12] of the BKM/AB in the Mathematical Reviews, who concluded that: "*A proof of the convergence of the numerical method given and an investigation of its convergence rate are lacking, so the results obtained are of a heuristic nature*".

This thesis is organized as follows: In Chapter 2 we set up the notation and recall the BKM/AB. Chapter 3 is devoted to the study of the various BKM and BKM/AB errors, in cases when f_0 has an analytic continuation across Γ, hence only basis functions reflecting poles are used in the BKM/AB. In Chapter 4, we consider the case when both corner and pole basis functions are included in BKM/AB. In Chapter 5 we study the BKM and BKM/AB error interior the domain G for the two different cases of Chapter 3 and 4. Finally, in Section 6, we present numerical computations that illustrate the theory of Chapter 3, 4 and 5.

Chapter 2

The Bergman kernel method with singular basis functions

2.1 Corner singularities

Throughout this chapter we assume that the boundary Γ of G consist of N analytic arcs that meet at corner points τ_k, $k = 1, 2, \ldots, N$, where they form interior angles $\alpha_k \pi$, $0 < \alpha_k < 2$. Then, we have the following asymptotic expansions for f_0, valid near τ_k:

(i) If α_k is irrational, then

$$f_0(z) = f_0(\tau_k) + \sum_{p,q} B_{p,q}(z - \tau_k)^{p+q/\alpha_k}, \tag{2.1.1}$$

where p and q run over all integers $p \geq 0$, $q \geq 1$ and $B_{0,1} \neq 0$.

(ii) If $\alpha_k = a/b$, with a and b relative prime numbers, then

$$f_0(z) = f_0(\tau_k) + \sum_{p,q,m} B_{p,q,m}(z - \tau_k)^{p+q/\alpha_k}(\log(z - \tau_k))^m, \tag{2.1.2}$$

where p, q and m run over all integers $p \geq 0$, $1 \leq q \leq a$, $1 \leq m \leq p/b$ and $B_{0,1,0} \neq 0$.

(iii) If τ_k is formed by two straight-line segments, then

$$f_0(z) = f_0(\tau_k) + \sum_{l=1}^{\infty} B_l(z - \tau_k)^{l/\alpha_k}, \tag{2.1.3}$$

where $B_1 \neq 0$. Furthermore, (2.1.1) holds in the case when τ_k is formed by two circular arcs, or a straight-line and a circular arc.

In the above, (i) and (ii) are due to Lehman [10], while (iii) emerges easily from the reflection principle; see also [7, §2.1] and [18, pp. 6–7].

It follows from (iii) that if G is a half-disk or a rectangle, then f_0 has a Taylor series expansion valid around each corner, and thus an analytic continuation across Γ into Ω. In this case, the only singularities of f_0 are simple poles in Ω. This shows that the study of the BKM/AB, even with only pole basis function is important in the applications.

For simplicity in the exposition, we shall assume throughout this paper that *no logarithmic terms occur* in the asymptotic expansion of f_0 near the corner τ_k, $k = 1, 2, \ldots, N$. This, for example, will be the case in the expansions (2.1.1) and (2.1.3) above. Nevertheless, our method of study can be adjusted to cover logarithmic singularities as well.

Let M denote the number of corners of Γ for which α_k is not of the special form $1/m$, $m \in \mathbb{N}$. When we present results for corner singularities we shall assume that $M \geq 1$. We index such corners by τ_k, $k = 1, \ldots, M$. That is, if $N > M$, then the mapping function f_0 has an analytic continuation in some neighborhood of the corner τ_N.

For $k = 1, \ldots, M$, we denote by $\{\gamma_j^{(k)}\}_{j=1}^{\infty}$ the increasing arrangement of the possible powers $p + q/\alpha_k$ of $(z - \tau_k)$ that appear in the asymptotic expansion of $f_0(z)$ near τ_k. In particular, if τ_k is formed by two straight-line segments, then $\gamma_j^{(k)} = j/\alpha_k$, $j = 1, 2, \ldots$. Also, if α_k is irrational, or the corner τ_k is formed by two circular arcs, then

$$\begin{aligned}
\gamma_1^{(k)} &= 1/\alpha_k; \\
\gamma_2^{(k)} &= 1/\alpha_k + \min(1/\alpha_k, 1); \\
\gamma_3^{(k)} &= \begin{cases} 1/\alpha_k + 2, & 0 < \alpha_k < 1/2, \\ 2/\alpha_k, & 1/2 < \alpha_k < 1, \\ 1/\alpha_k + 1, & 1 < \alpha_k < 2; \end{cases} \\
&\vdots
\end{aligned}$$

Remark 2.1.1. Under the assumption regarding the no-appearance of logarithmic terms, the asymptotic expansion near τ_k, $k = 1, 2, \ldots, M$, can be written in the form

$$f_0(z) = \sum_{j=0}^{\infty} a_j^{(k)} (z - \tau_k)^{\gamma_j^{(k)}}, \tag{2.1.4}$$

where, $\gamma_0^{(k)} := 0$ and $a_1^{(k)} \neq 0$. Note that, we always have $\gamma_1^{(k)} > 1/2$, and since τ_k is not a special corner, $\gamma_1^{(k)} \notin \mathbb{N}$. Therefore $(z - \tau_k)^{\gamma_1^{(k)}}$ has an algebraic singularity at τ_k. However, when α_k is rational, it is possible that $\gamma_j^{(k)} \in \mathbb{N}$, for indices $j \geq 2$, so that $(z - \tau_k)^{\gamma_j^{(k)}}$ is analytic at τ_k.

2.2 Pole singularities

Since $f_0(z_0) = 0$, $z_0 \in G$, it follows from the reflection principle for analytic arcs that the extension of f_0 across any segment constituting Γ would have a pole or a pole-type singularity at the reflected images of z_0. For example, if Γ consists explicitly from straight-line segments and/or circular arcs, then f_0 has a simple pole (due to the univalency of f_0) at every mirror image of z_0 (with respect to the straight-lines) and at every geometric inverse of z_0 (with respect to the circular arcs), that lies in Ω. More generally, f_0 may have at points $z_j \in \Omega$ a pole, or a poly-type, singularity of the form

$$(z - z_j)^{-k_j/m_j}, \quad k_j, m_j \in \mathbb{N}. \tag{2.2.1}$$

According to [21, § 5.1], the following three special cases occur frequently in the applications:

(i) $k_j = m_j = 1$. In this case, f_0 has a simple pole at z_j.

(ii) $k_j = 2$, $m_j = 1$. In this case, f_0 has a double pole at z_j.

(iii) $k_j = 1$, $m_j = 2$. In this case, f_0 has a rational pole singularity at z_j.

In order to describe the BKM/AB, we assume that the nearest singularities of f_0 in Ω are poles or rational poles, of the form (2.2.1) at points z_j, $j = 1, 2, \ldots \kappa$, where $|\Phi(z_1)| \leq |\Phi(z_2)| \leq \ldots \leq |\Phi(z_\kappa)|$ and that the other singularities of f_0 in Ω occur at points $z_{\kappa+1}, z_{\kappa+2}, \ldots$, where $|\Phi(z_k)| < |\Phi(z_{\kappa+1})| \leq |\Phi(z_{\kappa+2})| \leq \ldots$.

2.3 BKM/AB

Using the above notation, the BKM/AB with n monomials, κ poles and p_k corner singularities at each (non-special) corner τ_k, $k = 1, 2 \ldots, M$, can be summarized as follows:

(i) Start with the augmented system $\{\eta_j\}$ consisting of:

1. the nearest poles or rational poles, i.e., for $j = 1, 2 \ldots \kappa$,
$$\eta_j(z) = \left[\left(\frac{1}{z - z_j}\right)^{k_j/m_j}\right]'; \tag{2.3.1}$$
2. the dominant $r_M := \sum_{k=1}^{M} p_k$ algebraic singular functions, i.e., for each non-special corner τ_k, $k = 1, 2, \ldots, M$,
$$\eta_{\kappa+j}(z) = [(z - \tau_k)^{\gamma_j^{(k)}}]', \quad j = 1, 2, \ldots p_k; \tag{2.3.2}$$
3. the monomials
$$\eta_{\kappa+r_M+j}(z) = (z^j)', \quad j = 1, 2, \ldots, n. \tag{2.3.3}$$

(As it was noted in Remark 2.1.1, it might be possible that $\gamma_j^{(k)} \in \mathbb{N}$. If this happens, we avoid redundancy in the basis by omitting such $\gamma_j^{(k)}$.)

(ii) Orthonormalize $\{\eta_j\}$, by means of the Gram-Schmidt process to produce the orthonormal set $\{\widetilde{P}_j\}$, where
$$\widetilde{P}_j(z) = \sum_{i=1}^{j} b_{j,i}\, \eta_i(z), \quad b_{j,j} > 0. \tag{2.3.4}$$

(iii) Approximate $K(z, z_0)$ by its finite Fourier expansion with respect to $\{\widetilde{P}_j\}$:
$$\widetilde{K}_n(z, z_0) := \sum_{j=1}^{\kappa+r_M+n} \overline{\widetilde{P}_j(z_0)}\widetilde{P}_j(z) = \sum_{j=1}^{\kappa+r_M+n} d_{n,j}\eta_j(z). \tag{2.3.5}$$

(iv) Approximate $f_0(z)$ by
$$\widetilde{\pi}_{n+1}(z) := \frac{1}{\widetilde{K}_n(z_0, z_0)} \int_{z_0}^{z} \widetilde{K}_n(\zeta, z_0) d\zeta = \sum_{j=1}^{\kappa+r_M+n} c_{n,j}\mu_j(z), \tag{2.3.6}$$

where
$$\mu_j(z) := \int_{z_0}^{z} \eta_j(\zeta) d\zeta. \tag{2.3.7}$$

We call the functions $\{\widetilde{P}_j\}$ the *augmented Bergman polynomials* of G, with respect to $\{\eta_j\}$, and the functions $\{\widetilde{\pi}_n\}$ the *augmented Bieberbach polynomials* over the system $\{\mu_j\}$. Clearly, $\widetilde{\pi}_n(z_0) = 0$ and $\widetilde{\pi}'_n(z_0) = 1$, $n \in \mathbb{N}$. Also

$$\widetilde{r}_n = \frac{1}{\sqrt{\pi \widetilde{K}_n(z_0, z_0)}}, \tag{2.3.8}$$

is the BKM/AB approximation to the conformal radius of G at z_0. Note that $\{\widetilde{P}_j\}_{j=1}^{\infty}$ forms a complete orthonormal system in $L^2_a(G)$. Consequently,

$$K(z, z_0) = \sum_{j=1}^{\infty} \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z), \tag{2.3.9}$$

locally uniformly with respect to $z \in G$, cf. (1.3.15).

We conclude this section by presenting a result which shows that the two errors $\|f'_0 - \widetilde{\pi}'_{n+1}\|_{L^2(G)}$ and $\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}$ are of the same order. This fact will be used below in Sections 3 and 4.

In what follows we denote by c, c_1, $c_2, \ldots,$ constants that are independent of n. For quantities $A > 0$, $B > 0$, we use the notation $A \preceq B$ (inequality with respect to the order) if $A \leq cB$. The expression $A \asymp B$ means that $A \preceq B$ and $B \preceq A$ simultaneously.

Lemma 2.3.1. *It holds that,*

$$\|f'_0 - \widetilde{\pi}'_{n+1}\|_{L^2(G)} \asymp \|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}. \tag{2.3.10}$$

Proof. We set $m := \kappa + r_M + n$ and note that (1.3.14), (2.3.5)–(2.3.9), imply:

$$\begin{aligned} f'_0(z) - \widetilde{\pi}'_{n+1}(z) &= \pi r_0^2 \sum_{j=1}^{\infty} \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z) - \{\sum_{j=1}^{m} |\widetilde{P}_j(z_0)|^2\}^{-1} \sum_{j=1}^{m} \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z) \\ &= \sum_{j=1}^{m} \left[\pi r_0^2 - \{\sum_{j=1}^{m} |\widetilde{P}_j(z_0)|^2\}^{-1}\right] \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z) \\ &+ \pi r_0^2 \sum_{j=m+1}^{\infty} \overline{\widetilde{P}_j(z_0)} \widetilde{P}_j(z). \end{aligned}$$

Therefore, by using the orthonormality of $\widetilde{P}_j$ we see that,

$$\|f'_0 - \widetilde{\pi}'_{n+1}\|^2_{L^2(G)} = \sum_{j=1}^{m} \left[\pi r_0^2 - 1/\widetilde{K}_n(z_0, z_0)\right]^2 |\widetilde{P}_j(z_0)|^2 + (\pi r_0^2)^2 \sum_{j=m+1}^{\infty} |\widetilde{P}_j(z_0)|^2.$$

Now, using once more (1.3.18), we obtain, after some trivial calculation, that

$$\|f_0' - \widetilde{\pi}_{n+1}'\|_{L^2(G)}^2 = \Big[K(z_0, z_0) - \widetilde{K}_n(z_0, z_0)\Big] \Big[K(z_0, z_0)\, \widetilde{K}_n(z_0, z_0)\Big]^{-1}.$$

This and (1.3.12), with $g(\cdot) = K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)$, leads to

$$\|f_0' - \widetilde{\pi}_{n+1}'\|_{L^2(G)}^2 = \frac{\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}^2}{K(z_0, z_0)\, \widetilde{K}_n(z_0, z_0)} \tag{2.3.11}$$

and the result (2.3.10) follows from the set of the obvious inequalities,

$$|\widetilde{P}_1(z_0)| = \widetilde{K}_1(z_0, z_0) \leq \widetilde{K}_n(z_0, z_0) \leq K(z_0, z_0) = (\pi r_0^2)^{-1},$$

with constants depending on r_0 and $|\widetilde{P}_1(z_0)|$ only. □

Remark 2.3.1. It is clear from the proof that the result of Lemma 2.3.1 holds true for *any* complete orthonormal system. We note that for the system $\{\widetilde{P}_j\}_{j=1}^{\infty}$ to be complete it suffices that Γ is a bounded Jordan curve. In particular, (2.3.10) holds with π_{n+1} and K_n in the place of $\widetilde{\pi}_{n+1}$ and $\widetilde{K}_n$, i.e.

$$\|f_0' - \pi_{n+1}'\|_{L^2(G)} \asymp \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)}. \tag{2.3.12}$$

Chapter 3

BKM/AB with pole singularities

In this chapter we study the BKM and BKM/AB errors $\|f_0-\pi_n\|_{L^\infty(\overline{G})}$ and $\|f_0-\widetilde{\pi}_n\|_{L^\infty(\overline{G})}$, under the assumption that f_0 *has an analytic continuation across* Γ in Ω and its only singularities are poles, or rational poles, of the type (2.2.1). More precisely, we refine the classical result (1.5.2) for the BKM error, and at the same time we obtain a lower estimate for it. Furthermore, we establish upper and lower estimates for the BKM/AB error. The lower estimates and the refinement are obtained by exploiting the assumption regarding the singularities of f_0 and by using certain important results of E.B. Saff on polynomial interpolation of meromorphic functions [22]. Since the results of [22] were established for domains with smooth boundaries, we show in the next lemma that they hold true for domains with corners.

In order to do so, we use the *Faber polynomials* $\{F_n\}_{n=0}^{\infty}$ of $\overline{G}$. We recall that $F_n(z)$ is defined as the polynomial part of the Laurent series expansion of Φ^n at infinity, i.e.,

$$F_n(z)-\Phi^n(z)=O\left(\frac{1}{z}\right),\ \text{as } z\to\infty. \tag{3.0.1}$$

This, in view of (5.1.1), gives $F_n\in\mathbb{P}_n$ and

$$F_n(z)=\gamma^n z^n+\cdots\,. \tag{3.0.2}$$

We recall that L_R ($R\geq 1$), denote the level curve of index R of Φ, i.e.,

$$L_R:=\{z:|\Phi(z)|=R\}, \tag{3.0.3}$$

so that $L_1 \equiv \Gamma$. Note that L_R, for $R > 1$, is an analytic Jordan curve. We use G_R to denote its interior, i.e., $G_R := \text{int}(L_R)$. The following result gives the exact rate of convergence of the minimum uniform error in approximating meromorphic functions by polynomials.

Lemma 3.0.2. *Assume that the boundary Γ of G is piecewise Dini-smooth and consider a function f which is analytic on $\overline{G_\varrho}$, for some $\varrho > 1$, apart from a finite number of poles on L_ϱ. Let m denote the highest order of the poles of f on L_ϱ. Then,*

$$\inf_{p \in \mathbb{P}_n} \|f - p\|_{L^\infty(\overline{G})} \asymp \frac{n^{m-1}}{\varrho^n}. \tag{3.0.4}$$

A curve Γ is piecewise Dini-smooth if it consists of a finite number of Dini-smooth arcs. An arc $z = z(s)$, where $s \in [a, b]$ stands for the arclength, is called *Dini-smooth* if $z'(s)$ is continuous on $[a, b]$, and if $z'(s)$ has a modulus of continuity ω which satisfies $\int_0^\alpha [\omega(t)/t]\, dt < \infty$, for some $\alpha > 0$. We note, in particular, that a piecewise Dini-smooth curve may have corners or cusps and that a piecewise analytic Jordan curve is also piecewise Dini-smooth.

Proof. We recall the following two facts regarding Faber polynomials:

(i) For any r, R, with $1 < r < R$, it holds

$$F_n(z) = \Phi^n(z)\left\{1 + O\left(\frac{r^n}{R^n}\right)\right\}, \quad z \in L_R, \tag{3.0.5}$$

see e.g. [26, p. 43].

(ii) Under the assumption on Γ, the Faber polynomials are uniformly bounded on $\overline{G}$ (see [8]), i.e.,

$$\|F_n\|_{L^\infty(\overline{G})} \le c(\Gamma), \quad n \in \mathbb{N}, \tag{3.0.6}$$

where $c(\Gamma)$ is a positive constant that depends on Γ only.

Observe that (3.0.5) implies that the sequence $\{F_n(z)\}_{n=1}^\infty$ has no limit point of zeros exterior to $\overline{G}$. Also, from (3.0.5) and (3.0.6) we have for $z \in \overline{G}$ and $t \in L_\varrho$ that,

$$\frac{|F_n(z)|}{|F_n(t)|} \le \frac{c_1(\Gamma)}{\varrho^n}, \quad n \in \mathbb{N}. \tag{3.0.7}$$

Now, following the proof of Theorem 2 of [22] and using the sequence of the Faber polynomials $\{F_n\}$ in the place of $\{\omega_n\}$, we conclude that there exist polynomials $\{p_n\}_{n=1}^{\infty}$, such that

$$\|f - p_n\|_{L^\infty(\overline{G})} \leq c_2(\Gamma)\frac{n^{m-1}}{\varrho^n}, \quad n \in \mathbb{N}, \tag{3.0.8}$$

see also [23, p. 399]. This yields the upper bound in (2.3.2). The lower bound follows at once from Theorem 10 of [22], by observing that Ω is simply-connected and hence its Green function with pole at infinity has no critical points. □

3.1 The method of Andrievskii and Simonenko

In the study of the BKM approximation of the Bieberbach polynomials π_n to the conformal map f_0 the method of Andrievskii and Simonenko; see e.g [6, p. 292] plays a crucial role because enables the transition from the $L^2(G)$- error $\|f_0' - \pi_n'\|_{L^2(G)}$ to the uniform error $\|f_0 - \pi_n\|_{L^\infty(\overline{G})}$.

The following result is the so-called Andrievskii's lemma for polynomials and rational polynomials and is used to the method of Andrievskii and Simonenko. Its proof, for bounded Jordan domains such that the inverse conformal map $g : \mathbb{D} \to G$ satisfies a Lipschitz condition on $\overline{\mathbb{D}}$, can be found in [5]. This condition is certainly satisfied by the type of domains considered below.

Lemma 3.1.1. *Assume that Γ is piecewise analytic without cusps. Then:*

(i) For any $P_n \in \mathbb{P}_n$, with $P_n(z_0) = 0$, it holds

$$\|P_n\|_{L^\infty(\overline{G})} \preceq \sqrt{\log n}\,\|P_n'\|_{L^2(G)}, \quad n \geq 2. \tag{3.1.1}$$

(ii) For any $P_n \in \mathbb{P}_n$, with $P_n(z_0) = 0$, and q a fixed polynomial with no zeros on $\overline{G}$, it holds that

$$\|P_n/q\|_{L^\infty(\overline{G})} \preceq \sqrt{\log n}\,\|(P_n/q)'\|_{L^2(G)}, \quad n \geq 2. \tag{3.1.2}$$

In what follows, we describe the method of Andrievskii and Simonenko:
First, in order to obtain a uniform estimate of the form $\|f_0 - p_n\|_{L^\infty(\overline{G})}$, it suffices to have

an estimate of $\|f_0' - p_n'\|_{L^2(G)}$ for some arbitrary polynomials $p_n \in \mathbb{P}_n^*$. For assume that we have

$$\|f_0' - p_n'\|_{L^2(G)} \preceq \frac{n^r}{R^n},$$

for some positive constants r, R and $n = 2, 3, 4, \ldots$. Then by the minimum property of the Bieberbach polynomials

$$\|f_0' - \pi_n'\|_{L^2(G)} \preceq \frac{n^r}{R^n}, \tag{3.1.3}$$

and it follows that

$$\|f_0 - \pi_n\|_{L^\infty(G)} \preceq \sqrt{\log n}\frac{n^r}{R^n}. \tag{3.1.4}$$

To see this we take (3.1.3) for n with $2^k \leq n \leq 2^{k+1}$ and after some computations:

$$\|\pi_{2^{k+1}}' - \pi_n'\|_{L^2(G)} \preceq \frac{n^r}{R^n}.$$

Then apply Andrievskii's Lemma [5] (see also Lemma 3.1.1(i) of this thesis) to get

$$\|\pi_{2^{k+1}} - \pi_n\|_{L^\infty(G)} \preceq \sqrt{\log 2^{k+1}}\frac{n^r}{R^n},$$

in particular,

$$\|\pi_{2^{k+1}} - \pi_{2^k}\|_{L^\infty(G)} \preceq \sqrt{k+1}\frac{2^{(k+1)r}}{R^{2^k}},$$

for $k = 1, 2, \ldots$.

Now since

$$f_0(z) - \pi_n(z) = [\pi_{2^{k+1}}(z) - \pi_n(z)] + \sum_{j=k+1}^{\infty} [\pi_{2^{j+1}}(z) - \pi_{2^j}(z)], \quad (z \in G),$$

we have

$$\|f_0 - \pi_n\|_{L^\infty(G)} \leq c_1\sqrt{\log n}\frac{n^r}{R^n} + c_2\sum_{j=k+1}^{\infty}\sqrt{j+1}\frac{2^{(j+1)r}}{R^{2^j}}.$$

for some positive constant c_1, c_2. Finally the latter sum is bounded by

$$\sum_{j=k+1}^{\infty}\sqrt{j+1}\frac{2^{(j+1)r}}{R^{2^j}} \preceq \sqrt{k}\frac{2^{kr}}{R^{2^k}} \preceq \sqrt{\log n}\frac{n^r}{R^n},$$

which establishes (3.1.4).

Remark 3.1.1. It is clear from the proof that the method of Adrievskii and Simonenko holds true with the rational polynomials p_n/q in the place of p_n, (q is the fixed polynomial described is Lemma 3.1.1(ii)).

3.2 BKM

The next theorem complements the classical result (1.5.2) of Walsh, in the sense that it provides a lower estimate and, in addition, uses the precise $\varrho = |\Phi(z_1)|$ in the denominator, instead of any R, with $1 < R < \varrho$. This is done by utilizing extra information on the nature of the singularities of f_0 in Ω.

Theorem 3.2.1. *Assume that Γ is piecewise analytic without cusps. Assume further that the conformal map f_0 has an analytic continuation across Γ, such that f_0 is analytic on $\overline{G_\varrho}$, for some $\varrho > 1$, apart from a finite number of poles on L_ϱ. Let m denote the highest order of the poles of f_0 on L_ϱ. Then,*

$$\frac{n^{m-1}}{\varrho^n} \preceq \|f_0 - \pi_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m \sqrt{\log n}}{\varrho^n}, \quad n \geq 2. \tag{3.2.1}$$

Proof. We observe first that the kernel $K(z, z_0)$ shares the same analytic properties with f_0 on $\overline{G_\varrho}$, apart from an unit increase on the order of its poles on L_ϱ. Therefore, using Lemma 3.0.2 with $f \equiv K(\cdot, z_0)$, we conclude that

$$\|K(\cdot, z_0) - p_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m}{\varrho^n}, \quad n \in \mathbb{N}, \tag{3.2.2}$$

for some sequence of polynomials $\{p_n\}_{n=1}^{\infty}$. Since the $L^2(G)$-norm is dominated by the $L^\infty(\overline{G})$-norm, (3.2.2) leads to the estimate

$$\|K(\cdot, z_0) - p_n\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}. \tag{3.2.3}$$

Then, the minimum property of the kernel polynomials implies that

$$\|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}, \tag{3.2.4}$$

which, in conjunction with Remark 2.3.1, yields the estimate

$$\|f_0' - \pi_n'\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}. \tag{3.2.5}$$

Now, we use Andrievskii's Lemma 3.1.1(i) and employ the method of Andrievskii and Simonenko, described in Section 3.1 of this thesis or see e.g. [6, §2.1]. This method enables the transition from an upper bound of the error $\|f_0' - \pi_n'\|_{L^2(G)}$ to a similar bound for the error $\|f_0 - \pi_n\|_{L^\infty(\overline{G})}$, with the extra cost of a $\sqrt{\log n}$ factor, and leads to the upper estimate in (3.2.1). The lower estimate follows immediately from Lemma 3.0.2 taking $f \equiv f_0$ and the fact that

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} \geq \inf_{p \in \mathbb{P}_n} \|f_0 - p\|_{L^\infty(\overline{G})}.$$

□

Remark 3.2.1. There are cases where Γ is piecewise analytic with corners and still f_0 has an analytic (though not one-to-one) continuation in an open set containing $\overline{G}$. This is the case for example, when G is a rectangle.

The following pointwise estimate is useful in the study of the distribution of the zeros of the Bergman polynomials; see e.g., [11], [17], [23] and [9].

Corollary 3.2.1. *With the assumptions of Theorem 3.2.1 it holds,*

$$|P_n(z_0)| \preceq \frac{n^m}{\varrho^n}, \quad n \in \mathbb{N}, \quad (z_0 \in G). \tag{3.2.6}$$

Proof. The result emerges easily from (3.2.4), using the reproducing property of $K(\cdot, z_0)$, the fact that P_{n+1} is orthogonal to any polynomial in $\mathbb{P}_n$, and the Cauchy-Schwarz inequality:

$$\begin{aligned} |P_{n+1}(z_0)| &= |\langle P_{n+1}, K(\cdot, z_0)\rangle| = |\langle P_{n+1}, K(\cdot, z_0) - K_n(\cdot, z_0)\rangle| \\ &\leq \|P_{n+1}\|_{L^2(G)} \, \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \\ &= \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)}. \end{aligned}$$

□

We end this section by providing an estimate for the error in the resulting BKM approximation r_n to the conformal radius r_0.

Corollary 3.2.2. *With the assumptions of Theorem 3.2.1 it holds,*

$$|r_0 - r_n| \preceq \frac{n^{2m}}{\varrho^{2n}}. \tag{3.2.7}$$

Proof. It follow from (1.3.18) and (1.4.3) that

$$\begin{aligned} r_n - r_0 &= \frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{K_n(z_0,z_0)}} - \frac{1}{\sqrt{K(z_0,z_0)}}\right) \\ &= \frac{1}{\sqrt{\pi}}\frac{\sqrt{K(z_0,z_0)} - \sqrt{K_n(z_0,z_0)}}{\sqrt{K_n(z_0,z_0)K(z_0,z_0)}} \\ &= \frac{1}{\sqrt{\pi}}\frac{K(z_0,z_0) - K_n(z_0,z_0)}{\sqrt{K_n(z_0,z_0)K(z_0,z_0)}\left(\sqrt{K(z_0,z_0)} + \sqrt{K_n(z_0,z_0)}\right)}. \end{aligned}$$

Also, from the minimum properties of Fourier expansion for $K(z, z_0)$ and the reproducing property; see e.g [2, p. 172],

$$\|K(\cdot, z_0) - K_n(\cdot, z_0)\|^2_{L^2(G)} = K(z_0, z_0) - K_n(z_0, z_0)$$

and hence, the result (3.2.7) follows by working in the way as in the proof of Lemma 2.3.1. □

Remark 3.2.2. The result above confirms theoretically the experimental observation made [12] by Levin, Papamichael and Siderides that the BKM error in approximation f_0 by polynomials depends on the index of the nearest singularity of f_0 in Ω even for small values of $n \in \mathbb{N}$. More precisely, it follows from (3.2.1) and (3.2.4) that if $m = 1$, then

$$c_1 \frac{1}{|\Phi(z_1)|^n} \leq \|f_0 - \pi_n\|_{L^\infty(\overline{G})} \leq c_2 \frac{n\sqrt{\log n}}{|\Phi(z_1)|^n}. \tag{3.2.8}$$

$$\|K(\cdot, z) - K_n(\cdot, z)\|_{L^2(G)} \leq c_3 \frac{n}{|\Phi(z_1)|^n}. \tag{3.2.9}$$

3.3 BKM/AB with pole singularities

We exploit now the specific assumptions on the singularities of the analytic extension of f_0 studied in Section 2.2. More precisely, the assumption that the nearest singularities of f_0 are κ poles, each one of order k_j at z_j, $j = 1, 2, \ldots \kappa$, where $|\Phi(z_1)| \leq |\Phi(z_2)| \leq \cdots \leq |\Phi(z_\kappa)|$, and that the other singularities of f_0 occur at points $z_{\kappa+1}, z_{\kappa+2}, \ldots$, where $|\Phi(z_\kappa)| < |\Phi(z_{\kappa+1})| \leqslant |\Phi(z_{\kappa+2})| \leqslant \cdots$. Therefore, for the BKM/AB we consider the system $\{\eta_j\}$, defined by the singular functions in (1.3.18), with $m_j = 1$, $j = 1, 2, \ldots, \kappa$, and the n monomials in (1.4.2). Accordingly, we let $\mathbb{P}_n^{A_1}$ denote the following space of augmented polynomials:

$$\mathbb{P}_n^{A_1} := \{p : p(z) = \sum_{j=1}^{\kappa+n} t_j \eta_j(z),\ t_j \in \mathbb{C}\}. \tag{3.3.1}$$

We note that the associated augmented kernel polynomial $\widetilde{K}_n(z, z_0)$ is the best approximation to $K(z, z_0)$ in $L^2(G)$ out of the space $\mathbb{P}_n^{A_1}$, i.e.,

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \leq \|K(\cdot, z_0) - p\|_{L^2(G)}, \tag{3.3.2}$$

for any $p \in \mathbb{P}_n^{A_1}$.

It is clear that, the Remark 3.1.1, holds with $\widetilde{\pi}_n$ and $\widetilde{\pi}_n'$ in the place of rational polynomials π_n/q and $(\pi_n/q)'$, i.e.,

if we assume

$$\|f_0' - \widetilde{\pi}_n'\|_{L^2(G)} \preceq \frac{n^r}{R^n} \tag{3.3.3}$$

for some positive constants r, R and $n = 1, 2, \ldots$, it follows that

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(G)} \preceq \sqrt{\log n}\,\frac{n^r}{R^n}. \tag{3.3.4}$$

The next theorem provides an estimate for the error in the resulting BKM/AB approximation $\widetilde{\pi}_n$ to f_0.

Theorem 3.3.1. *Assume that Γ is piecewise analytic without cusps and set $\varrho := |\Phi(z_{\kappa+1})|$. Then,*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \tag{3.3.5}$$

for any R, with $1 < R < \varrho$, but for no $R > \varrho$.

Proof. Observe that $K(\cdot, z_0)$ has poles of order $k_j + 1$ at each $z_j, j = 1, 2, \ldots, \kappa$, and set $Q(z) := \prod_{j=1}^{k}(z - z_j)^{k_j+1}$. Then, the function $K(z, z_0)Q(z)$ is analytic in the interior G_ϱ of the level curve of L_ϱ, and from Walsh's maximal convergence theorem [27, §4.7] it follows that, for any R, with $1 < R < \varrho$, there exists a sequence of polynomial $\{p_n\}_{n=1}^{\infty}$, such that

$$\|K(\cdot, z_0)Q - p_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}. \tag{3.3.6}$$

Let now $d := \min_{j=1,2,\dots,\kappa}\{|z - z_j| : z \in \Gamma\}$ denote the distance of Γ from the poles $\{z_j\}_{j=1}^{\kappa}$, and set $\xi := \sum_{j=1}^{\kappa} k_j$. Then, $|Q(z)| \geq d^{\kappa+\xi}$, $z \in \Gamma$, and (3.3.6) gives

$$\|K(\cdot, z_0) - \frac{p_n}{Q}\|_{L^\infty(\overline{G})} \leq \frac{c}{d^{\kappa+\xi}}\frac{1}{R^n}.$$

Since the $L^2(G)$-norm is dominated by the $L^\infty(\overline{G})$-norm, we see that there exist a sequence of rational functions $\{Q_n\}_{n=1}^{\infty}$, with $Q_n \in \mathbb{P}_n^{A_1}$, such that,

$$\|K(\cdot, z_0) - Q_n\|_{L^2(G)} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}.$$

Therefore, using the minimum property (3.3.2) of the augmented kernel polynomials, we have

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{3.3.7}$$

and this, in conjunction with the equivalence Lemma 2.3.1, yields the estimate

$$\|f_0' - \widetilde{\pi}_n'\|_{L^2(G)} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}. \tag{3.3.8}$$

Next, we recall that

$$\widetilde{\pi}_n(z) = \sum_{j=1}^{\kappa} c_{n,j}\left[\frac{1}{(z - z_j)^{k_j}} - \frac{1}{(z_0 - z_j)^{k_j}}\right] + \sum_{j=1}^{n} c_{n,\kappa+j}\left[z^j - z_0^j\right], \tag{3.3.9}$$

i.e.,

$$\widetilde{\pi}_n(z) = \frac{P(z)}{q(z)}, \text{ where } q(z) := \prod_{j=1}^{\kappa}(z - z_j)^{k_j}, \tag{3.3.10}$$

and $P(z)$ is a polynomial of degree $n + \xi$.

Then, the transition from the $L^2(G)$-norm to the $L^\infty(\overline{G})$-norm is done as in the proof of Theorem 3.2.1, where now, in view of (3.3.10), Lemma 3.1.1(ii) is applicable (see the method of Andrievskii and Simonenko in Section 3.1). This leads to,

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \frac{\sqrt{\log n}}{R^n}, \quad n \geq 2, \tag{3.3.11}$$

and (3.3.5) follows with a different, but still less than ϱ, R.

Finally, the fact that (3.3.5) holds for no $R > \varrho$ is evident from [27, Thm. 6, Ch. IV], since the contrary assumption would lead to the contradicting conclusion that f_0 has no singularities on L_ϱ; see the next remark. □

Remark 3.3.1. From (3.3.9) it is clear that $\widetilde{\pi}_n(z) = \widetilde{q}_\kappa(z) + p_n(z)$, where $\widetilde{q}_\kappa$ is defined by the nearest κ poles of f_0 in Ω and $p_n \in \mathbb{P}_n$. Hence, (3.3.5) gives

$$\|(f_0 - \widetilde{q}_\kappa) - p_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n},$$

for any $1 < R < \varrho$, and Theorem 6 in [27, Ch. V] implies that the function $f_0 - \widetilde{q}_\kappa$ is analytic in G_ϱ. This shows that the the rational polynomial $\widetilde{q}_\kappa$, constructed by the BKM/AB considered above, *cancels out the specific poles of f_0 that contains.* In particular, this provides the theoretical justification for the heuristic observation made to that effect by Papamichael and Warby in [20, p. 652].

A finer estimate than (3.3.5) can be obtained if the singularities of f_0 on $L_{|\Phi(z_{\kappa+1})|}$ are a finite number of poles.

Theorem 3.3.2. *Assume that Γ is piecewise analytic without cusps and set $\varrho := |\Phi(z_{\kappa+1})|$. Assume, in addition to Theorem 3.3.1, that f_0 has a finite number of poles and no other singularities on L_ϱ and let m denote their highest order. Then,*

$$\frac{n^{m-1}}{\varrho^n} \preceq \|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m\sqrt{\log n}}{\varrho^n}, \quad n \geq 2. \tag{3.3.12}$$

Proof. The upper estimate follows by working in the same way as in the proof of Theorem 3.3.1, but using here the precise result of Lemma 3.0.2, in the place of Walsh's theorem in (3.3.6). In order to be more specific: Observe that $K(\cdot, z_0)$ has poles of order $k_j + 1$ at each $z_j, j = 1, 2, \ldots, \kappa$, and set $Q(z) := \prod_{j=1}^{k}(z - z_j)^{k_j+1}$. Then, the function $K(z, z_0)Q(z)$ is analytic in the interior G_ϱ of the level curve of L_ϱ, and from Lemma 3.0.2 it follows that, there exists a sequence of polynomial $\{p_n\}_{n=1}^\infty$, such that

$$\|K(\cdot, z_0)Q - p_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m}{\varrho^n}, \quad n \in \mathbb{N}. \tag{3.3.13}$$

Let now $d := \min_{j=1,2,\ldots,\kappa}\{|z - z_j| : z \in \Gamma\}$ denote the distance of Γ from the poles $\{z_j\}_{j=1}^\kappa$, and set $\xi := \sum_{j=1}^\kappa k_j$. Then, $|Q(z)| \geq d^{\kappa+\xi}$, $z \in \Gamma$, and (3.3.13) gives

$$\|K(\cdot, z_0) - \frac{p_n}{Q}\|_{L^\infty(\overline{G})} \leq \frac{c}{d^{\kappa+\xi}}\frac{n^m}{\varrho^n}.$$

Since the $L^2(G)$-norm is dominated by the $L^\infty(\overline{G})$-norm, we see that there exist a sequence of rational polynomials $\{Q_n\}_{n=1}^{\infty}$, with $Q_n \in \mathbb{P}_n^{A_1}$, such that,

$$\|K(\cdot, z_0) - Q_n\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}, \quad n \in \mathbb{N}.$$

Therefore, using the minimum property (3.3.2) of the augmented kernel polynomials, we have

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}, \quad n \in \mathbb{N}, \tag{3.3.14}$$

and this, in conjunction with the equivalence Lemma 2.3.1, yields the estimate

$$\|f_0' - \widetilde{\pi}_n'\|_{L^2(G)} \preceq \frac{n^m}{\varrho^n}, \quad n \in \mathbb{N}. \tag{3.3.15}$$

Next, we recall that

$$\widetilde{\pi}_n(z) = \sum_{j=1}^{\kappa} c_{n,j} \left[\frac{1}{(z - z_j)^{k_j}} - \frac{1}{(z_0 - z_j)^{k_j}} \right] + \sum_{j=1}^{n} c_{n,\kappa+j} \left[z^j - z_0^j \right], \tag{3.3.16}$$

i.e.,

$$\widetilde{\pi}_n(z) = \frac{P(z)}{q(z)}, \text{ where } q(z) := \prod_{j=1}^{\kappa} (z - z_j)^{k_j}, \tag{3.3.17}$$

and $P(z)$ is a polynomial of degree $n + \xi$.

Then, the transition from the $L^2(G)$-norm to the $L^\infty(\overline{G})$-norm is done as in the proof of Theorem 3.2.1, where now, in view of (3.3.17), Lemma 3.1.1 and relations (3.3.3) and (3.3.4) are applicable. This leads to the required result,

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m \sqrt{\log n}}{\varrho^n}, \quad n \geq 2. \tag{3.3.18}$$

To obtain the lower estimate, observe that $q\widetilde{\pi}_n$ is a polynomial of degree $n + \xi$ (see (3.3.10)) and that the function qf_0 is analytic on $\overline{G_\varrho}$, apart from a finite number of poles on L_ϱ. Hence, from Lemma 3.0.2 we have, for $n \in \mathbb{N}$,

$$\|qf_0 - q\widetilde{\pi}_n\|_{L^\infty(\overline{G})} \succeq \inf_{p \in \mathbb{P}_{n+\xi}} \|qf_0 - p\|_{L^\infty(\overline{G})} \succeq \frac{n^{m-1}}{\varrho^n}, \tag{3.3.19}$$

which yields the estimate

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \geq \frac{c}{\|q\|_{L^\infty(\overline{G})}} \frac{n^{m-1}}{\varrho^n}.$$

and hence the required result. □

The next result follows by working in the same way as in the proof of Corollaries 3.2.1 and 3.2.2 of Section 3.2. We recall that $\widetilde{P}_n(z)$ denote the augmented Bergman polynomials out of the space $\mathbb{P}_n^{A_1}$ and $\widetilde{r}_n$ denote the BKM/AB approximation in the space $\mathbb{P}_n^{A_1}$ to the conformal radius r_0.

Corollary 3.3.1. *With the assumptions of Theorem 3.3.2 it holds,*

$$|\widetilde{P}_n(z_0)| \preceq \frac{n^m}{\varrho^n}, \quad n \in \mathbb{N}, \quad (z_0 \in G). \tag{3.3.20}$$

and

$$|r_0 - \widetilde{r}_n| \preceq \frac{n^{2m}}{\varrho^{2n}}, \quad n \in \mathbb{N}. \tag{3.3.21}$$

In the more general case, where the nearest κ singularities of f_0 in Ω are rational poles of the type (2.2.1), we have the following result regarding the associated kernel polynomials $\widetilde{K}_n(\cdot, z_0)$.

Theorem 3.3.3. *Assume that Γ is piecewise analytic without cusps and set $\varrho := |\Phi(z_{\kappa+1})|$. Then,*

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{1}{R^n}, \tag{3.3.22}$$

for any R, $1 < R < \varrho$.

Proof. Set $Q(z) := \prod_{j=1}^{k}(z - z_j)^{k_j/m_j+1}$. Then, the function $K(z, z_0)Q(z)$ is analytic in the interior G_ϱ of the level curve of L_ϱ, and from Walsh's maximal convergence theorem [27, §4.7] it follows that, for any R, with $1 < R < \varrho$, there exists a sequence of polynomial $\{p_n\}_{n=1}^{\infty}$, such that

$$\|K(\cdot, z_0)Q - p_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}. \tag{3.3.23}$$

Let now $d := \min_{j=1,2,\ldots,\kappa}\{|z - z_j| : z \in \Gamma\}$ denote the distance of Γ from the poles $\{z_j\}_{j=1}^{\kappa}$, and set $\xi := \sum_{j=1}^{\kappa} k_j$. Then, $|Q(z)| \geq d^{\kappa+\xi}$, $z \in \Gamma$, and (3.3.23) gives

$$\|K(\cdot, z_0) - \frac{p_n}{Q}\|_{L^\infty(\overline{G})} \leq \frac{c}{d^{\kappa+\xi}} \frac{1}{R^n}.$$

Since the $L^2(G)$-norm is dominated by the $L^\infty(\overline{G})$-norm, we see that there exist a sequence of rational polynomials $\{Q_n\}_{n=1}^{\infty}$, with $Q_n \in \mathbb{P}_n^{A_1}$, such that,

$$\|K(\cdot, z_0) - Q_n\|_{L^2(G)} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}.$$

Therefore, using the minimum property (3.3.2) of the augmented kernel polynomials, we have the required result (3.3.22) □

Similar corollary as 3.3.1 of this section holds in this case. That is,

Corollary 3.3.2. *With the assumptions of Theorem 3.3.3 it holds,*

$$|\widetilde{P}_n(z_0)| \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}, \quad (z_0 \in G). \tag{3.3.24}$$

and

$$|r_0 - \widetilde{r}_n| \preceq \frac{1}{R^{2n}}, \quad n \in \mathbb{N}. \tag{3.3.25}$$

Chapter 4

BKM with pole and corner singularities

In this chapter we assume that f_0 *has a singularity on* Γ and study the BKM and BKM/AB errors, corresponding to a variety of different syntheses of the system $\{\eta_j\}$ of basis functions. In stating the results we use the notation and the assumptions set up in Sections 2.1 and 2.2.

4.1 BKM

Our first result is a straightforward consequence of Theorem 3.1 of [23] and Lemma 3.1.1 above. In this section, it is clear from the proof of the method of Andrievskii and Simonenko of Section 3.1 that if we assume

$$\|f_0' - \pi_n'\|_{L^2(G)} \leq c_1 \frac{1}{n^a} + c_2 \frac{1}{R^n} \tag{4.1.1}$$

for some positive constants c_1, c_2, a, R and $n = 1, 2, \dots$, it follows that

$$\|f_0 - \pi_n\|_{L^\infty(G)} \leq c_3 \sqrt{\log n} \frac{1}{n^a} + c_4 \frac{1}{R^n}. \tag{4.1.2}$$

(see e.g [6, p. 292] for the term $\frac{1}{n^a}$.)

Theorem 4.1.1. *Assume that* Γ *is piecewise analytic without cusps and set* $\varrho := |\Phi(z_1)|$ *and* $s := \min\{(2-\alpha_k)/\alpha_k : 1 \leq k \leq M\}$. *Then,*

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} \leq c_1 \sqrt{\log n} \frac{1}{n^s} + c_2 \frac{1}{R^n}, \quad n \geq 2. \tag{4.1.3}$$

for any R, $1 < R < \varrho$.

Proof. The conformal map f_0 can be extended analytically, by means of the reflection principle, beyond Γ to a larger Jordan domain $\widetilde{G}$, such that the boundary $\partial\widetilde{G}$ of $\widetilde{G}$ consists of analytic arcs to be fixed below. For this, we recall our assumptions on the position of the nearest pole z_1, of f_0 in Ω and pick up a point ζ_1 near z_1, but interior to the level curve L_ϱ, with $\varrho := |\Phi(z_1)|$. Thus from the Theorem 3.1 of [23] there exist a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$, of degree at most n, such that

$$\|K(\cdot, z_0) - p_n\|_{L^2(G)} \le c_1\frac{1}{n^s} + c_2\frac{1}{R^n}, \quad n \ge 2. \tag{4.1.4}$$

for any R, $1 < R < |\Phi(\zeta_1)|$. Observe that ζ_1 can be chosen arbitrarily close to z_1. Thus, from the minimum property of the kernel polynomials $K_n(\cdot, z_0)$ we have

$$\|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \le c_3\frac{1}{n^s} + c_4\frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.1.5}$$

for any R, $1 < R < \varrho$, and the transition from the $L^2(G)$-error in (4.1.5) to the $L^\infty(\overline{G})$-error in (4.1.3), goes along the same lines as in the proof of Theorem 3.3.1. (see (4.1.1) and (4.1.2)). □

Remark 4.1.1. Clearly, as $n \to \infty$, (4.1.3) yields the result (1.3.16). However, Theorem 4.1.1 does more: It captures, in a very precise form, the dependance of the BKM error $\|f_0 - \pi_n\|_{L^\infty(\overline{G})}$ for "small" values of n, on both the corner and pole singularities of f_0. This dependance has been testified numerically in [12] and has given rise to the introduction of the BKM/AB.

The following result is a simple consequence of (4.1.5). The proof is similar to the proofs of Corollaries 3.2.1 and 3.2.2.

Corollary 4.1.1. *Under the assumptions of Theorem 4.1.1 the following holds,*

$$|P_n(z_0)| \le c_1\frac{1}{n^s} + c_2\frac{1}{R^n}, \quad n \in \mathbb{N}, \quad (z_0 \in G). \tag{4.1.6}$$

and

$$|r_0 - r_n| \le \left(c_3\frac{1}{n^s} + c_4\frac{1}{R^n}\right)^2, \quad n \in \mathbb{N}. \tag{4.1.7}$$

Remark 4.1.2. For small values of n and for values of R near 1, the dominant term in (4.1.6) is the second one, provided that the two constants c_1 and c_2 are of the same magnitude. This can be observed numerically in Section 6.3, in Table 6.3.10, where for n between 10 and 50, the values of $|P_n(z_0)|$ decay with geometric rate.

Remark 4.1.3. Since $|P_n(z_0)| \leq \|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)}$, it follows from Corollary 4.1.1 that, if for small values of n, $P_n(z_0)$ decays geometrically to zero, then the most “serious" singularity of $K(\cdot, z_0)$, and hence of f_0, is the nearest pole in Ω and not an algebraic singularity on the boundary, as the asymptotic estimate (1.5.5) would suggest. On the other hand, given that f_0 has a singularity on Γ, Theorem 2.1 of [11] implies that any point of Γ is a point of accumulation of the zeros of the sequence $\{P_n\}_{n=1}^{\infty}$. Therefore, an easy way to check whether a pole singularity is more serious than an algebraic singularity, for a range of values of n, is by plotting the zeros of P_n for the same range: *If the zeros stay away from a specific part of the boundary, this indicates that $P_n(z_0)$ decays geometrically and therefore the presence of a pole singularity near that part.* We refer to [23, Examples 2, 3], where (4.2.18) was used as the tool for explaining the misleading nature of such plots.

4.2 BKM/AB with corner singularities

We recall that the boundary Γ of G consist of N analytic arcs that meet at corner points τ_k, $k = 1, 2, \ldots, N$, where they form interior angles $\alpha_k \pi$, $0 < \alpha_k < 2$. From our assumptions on Γ, it follows that the conformal map f_0 can be extended analytically, by means of the reflection principle, beyond Γ to a larger Jordan domain $\widetilde{G}$, such that the boundary $\partial\widetilde{G}$ of $\widetilde{G}$ consists of analytic arcs to be fixed below. For this, we recall our assumptions on the position of the nearest poles z_j, $j = 1, \ldots, \kappa$, of f_0 in Ω and pick up a point ζ_1 near z_1, but interior to the level curve L_ϱ, with $\varrho := |\Phi(z_1)|$. Next, we draw the level curve $L_{\widetilde{\varrho}}$, with $\widetilde{\varrho} := |\Phi(\zeta_1)|$ and fix on it points ζ_k, $k = 2, \ldots, N$, "between" τ_k and τ_{k+1}, where we set $\tau_{N+1} = \tau_1$. We connect each non-special corner τ_k, $k = 1, \ldots, M$, with the two adjacent ζ_k's, by using two analytic arcs. Next, we denote by l_k the two arcs emanating from τ_k and call l_N the part (or parts) of the level line $L_{\widetilde{\varrho}}$ that joins together those consecutive points ζ_k that have only one connection with τ_k. See Figure 4.1, for a possible arrangement of corners τ_k, points ζ_k, and arcs l_k and l_N. Finally, we define $\widetilde{G}$ by taking $\partial\widetilde{G} := \{\cup_{k=1}^{M} l_k\} \cup l_N$.

The above construction is such that:

(i) $\partial\widetilde{G}$ is a piecewise analytic Jordan curve that meets Γ at the non-special corner τ_k, $k = 1, \ldots, M$.

(ii) f_0 is continuous on $\widetilde{G} \cup \partial\widetilde{G}$ and analytic in $\widetilde{G}$ and on $\partial\widetilde{G}$, except for the endpoints τ_k.

(iii) The asymptotic expansion (2.1.4) holds for $z \in l_k$, $k = 1, \ldots, M$, in the sense that, for any $p_k \in \mathbb{N}_0$,

$$\begin{aligned} f_0(z) &= \sum_{j=0}^{p_k} a_j^{(k)} (z - \tau_k)^{\gamma_j^{(k)}} + \widetilde{f}_{\gamma_{p_k+1}^{(k)}, \tau_k}(z), \\ \widetilde{f}_{\gamma_{p_k+1}^{(k)}, \tau_k}(z) &= O\left((z - \tau_k)^{\gamma_{p_k+1}^{(k)}} \right). \end{aligned} \tag{4.2.1}$$

We consider now the application of BKM/AB with only corner singularities, where we use $p_k \in \mathbb{N}_0$ singular function for each non-special corner τ_k, $k = 1, 2, \ldots, M$. In order to measure the BKM/AB error we set

$$\nu_k := \min\{j > p_k : \ \gamma_j^{(k)} \notin \mathbb{N}, \ a_j^{(k)} \neq 0\}, \tag{4.2.2}$$

and assume that at least one of ν_k's is finite, otherwise the results become trivial. The associated BKM/AB system $\{\eta_j\}$ is thus defined by $r_M = \sum_{k=1}^{M} p_k$ singular functions of the form (1.3.11) and n monomials (1.4.2). Accordingly, we let $\mathbb{P}_n^{A_2}$ denote the space of augmented polynomials:

$$\mathbb{P}_n^{A_2} := \{p : p(z) = \sum_{j=1}^{r_M+n} t_j \eta_j(z), \ t_j \in \mathbb{C}\}. \tag{4.2.3}$$

Clearly, the associated augmented polynomial $\widetilde{K}_n(z, z_0)$ is the best approximation to $K(z, z_0)$ in $L^2(G)$ out of the space $\mathbb{P}_n^{A_2}$.

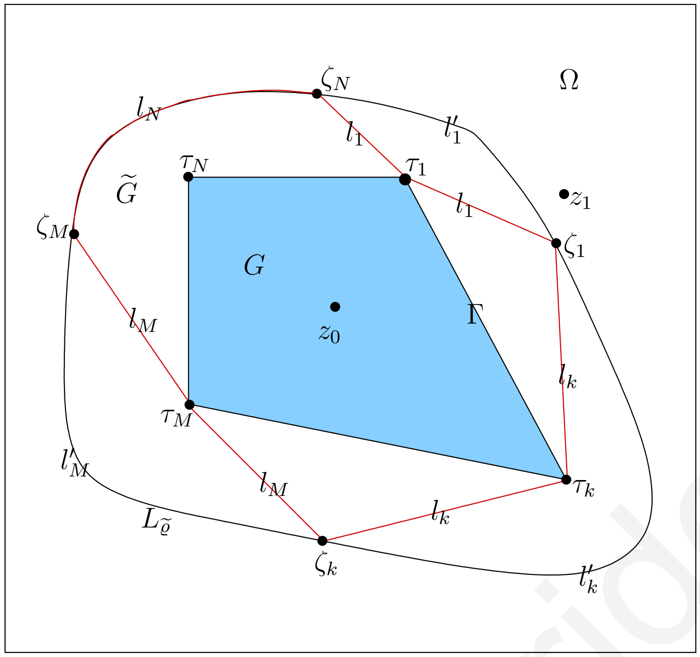

Figure 4.1: The domain $\widetilde{G}$ in the proof of Theorem 4.2.1.

The following lemma is a simple consequence of a result of Andrievskii and Gaier [1, Lemma 6] and it will be used in the proof of the concluding theorem of this section where we establish the BKM/AB error in approximating f_0 by the augmented polynomials $\widetilde{\pi}_n$ derived from $\mathbb{P}_n^{A_2}$. It's proof follows easily from Lemma 6 in [1] by taking into account the remark follows Theorem 1 in [1].

Lemma 4.2.1. *(Andrievskii and Gaier [1, Lemma 6])*
Assume that Γ is piecewise analytic without cusps and let l be any one of the arcs l_k defined at the begging of the section. Assume further that g is a function analytic on $l\backslash\{\tau_k\}$, such that

$$|g(z)| \preceq |z-\tau_k|^{1/\alpha_k}, \text{ for } z \in l,$$

and set

$$F(z) = \int_l \frac{g(z)}{t-z} dt, \quad z \in G.$$

Then, there exist a sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ satisfying

$$\|F - q_n\|_{L^2(G)} \preceq \frac{1}{n^s}, \quad n \in \mathbb{N}, \tag{4.2.4}$$

where $s := (2-\alpha_k)/\alpha_k$.

The next result of Maymeskul, Saff and Stylianopoulos [16, Corollary 2.5] is a version of Andrievsii's lemma (see Lemma 3.1.1) for functions with anti-derivatives in $\mathbb{P}_n^{A_2}$. It will be used below for the transition from the $L^2(G)$-norm to the $L^\infty(\overline{G})$-norm.

Lemma 4.2.2. *(Maymeskul, Saff and Stylianopoulos [16, Corollary 2.5])*
Assume that Γ *is piecewise analytic without cusps and let* $t_k \in \Gamma$, $k = 1, 2, \ldots, m$. *Also, let* $P \in \mathbb{P}_n$ *and assume further that for some constants* $a_{n,k,j}$, $k = 1, 2, \ldots, m$, $j = 1, 2, \ldots, r_k$, *the function*

$$L(z) := P(z) + \sum_{k=1}^{m}\sum_{j=1}^{r_k} a_{n,k,j} f_{\beta_j^{(k)},t_k}(z),$$

where $f_{\beta_j^{(k)},t_k}(z) := (z-t_k)^{\beta_j^{(k)}}$, *with* $\beta_j^{(k)} > 0$ *non-integer, satisfies:* $L(z_0) = 0$ *and* $\|L'\|_{L^2(G)} \le M$. *Then,*

$$\|L\|_{L^\infty(\overline{G})} \le CM\sqrt{\log n}, \tag{4.2.5}$$

where C *is a constant independent of* n *and of* $\{\{a_{n,k,j}\}_{j=1}^{r_k}\}_{k=1}^{m}$.

Let $\widetilde{\pi}_n$ denote the BKM/AB approximation resulting from $\mathbb{P}_n^{A_2}$. Then we have the following:

Theorem 4.2.1. *Assume that* Γ *is piecewise analytic without cusps and set* $\varrho := |\Phi(z_1)|$ *and* $s^\star := \min\{(2-\alpha_k)\gamma_{\nu_k}^{(k)} :\ 1 \le k \le M\}$. *Then,*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \le c_1\sqrt{\log n}\frac{1}{n^{s^\star}} + c_2\frac{1}{R^n}, \quad n \ge 2, \tag{4.2.6}$$

for any R, $1 < R < \varrho$.

Proof. Using Cauchy's integral formula for the derivative of the extension of f_0 we have, for $z \in G$,

$$\begin{aligned} f_0'(z) &= \frac{1}{2\pi i}\int_{\partial\widetilde{G}} \frac{f_0(t)}{(t-z)^2}dt \\ &= \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{f_0(t)}{(t-z)^2}dt + \frac{1}{2\pi i}\int_{l_N}\frac{f_0(t)}{(t-z)^2}dt. \end{aligned} \tag{4.2.7}$$

For each τ_k, $k = 1, \ldots, M$, we consider the first terms up to p_k, of the Lehman expansion (4.2.1) for f_0:

$$F_k(z) := \sum_{j=0}^{p_k} a_j^{(k)} (z - \tau_k)^{\gamma_j^{(k)}}. \tag{4.2.8}$$

Since the function $F_k(z)$, is analytic in $\widetilde{G}$ and continuous on $\partial\widetilde{G}$ we have, as in (4.2.7), for $z \in G$,

$$F_k'(z) = \frac{1}{2\pi i} \sum_{r=1}^{M} \int_{l_r} \frac{F_k(t)}{(t-z)^2} dt + \frac{1}{2\pi i} \int_{l_N} \frac{F_k(t)}{(t-z)^2} dt.$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{M} F_k'(z) &= \frac{1}{2\pi i} \sum_{k=1}^{M} \sum_{r=1}^{M} \int_{l_r} \frac{F_k(t)}{(t-z)^2} dt + \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_N} \frac{F_k(t)}{(t-z)^2} dt \\
&= \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_k} \frac{F_k(t)}{(t-z)^2} dt + \frac{1}{2\pi i} \sum_{k=1}^{M} \sum_{\substack{r=1 \\ r \neq k}}^{M} \int_{l_r} \frac{F_k(t)}{(t-z)^2} dt \\
&\quad + \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_N} \frac{F_k(t)}{(t-z)^2} dt.
\end{aligned}$$

Hence, for $z \in G$,

$$f_0'(z) - \sum_{k=1}^{M} F_k'(z) = g(z) + h(z), \tag{4.2.9}$$

where,

$$g(z) := \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_k} \frac{f_0(t) - F_k(t)}{(t-z)^2} dt, \tag{4.2.10}$$

and

$$\begin{aligned}
h(z) := &\frac{1}{2\pi i} \int_{l_N} \frac{f_0(t)}{(t-z)^2} dt - \frac{1}{2\pi i} \sum_{k=1}^{M} \sum_{\substack{r=1 \\ r \neq k}}^{M} \int_{l_r} \frac{F_k(t)}{(t-z)^2} dt \\
&- \frac{1}{2\pi i} \sum_{k=1}^{M} \int_{l_N} \frac{F_k(t)}{(t-z)^2} dt.
\end{aligned} \tag{4.2.11}$$

Now, we denote by l_r', $r = 1, \ldots, M$, the part of the level line $L_{\widetilde{\varrho}}$ that shares the same endpoints with l_r, so that $L_{\widetilde{\varrho}} = \{\cup_{r=1}^{M} l_r'\} \cup l_N$ and $l_r \cup l_r'$ is the boundary of a Jordan domain in Ω; see Figure 4.1. Since, for $k \neq r$, the function $F_k(z)$, $z \in G$, is analytic in the interior of $l_r \cup l_r'$ and continuous on $l_r \cup l_r'$, we can replace in (4.2.11) the path of integration l_r by l'_r, with suitable orientation, i.e., for $z \in G$,

$$
\begin{aligned}
h(z) = \frac{1}{2\pi i}\int_{l_N}\frac{f_0(t)}{(t-z)^2}dt - \frac{1}{2\pi i}\sum_{k=1}^{M}\sum_{\substack{r=1\\ r\neq k}}^{M}\int_{l'_r}\frac{F_k(t)}{(t-z)^2}dt \\
- \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_N}\frac{F_k(t)}{(t-z)^2}dt.
\end{aligned}
\tag{4.2.12}
$$

Observe that, by construction, f_0 is continuous on l_N and F_k is continuous on $l_N \cup l'_r$, for $k \neq r$. Thus, the function h in (4.2.12) is analytic in $G_{\widetilde{\varrho}}$ and by Walsh's maximal convergence theorem there exist a sequence of polynomials $\{t_n\}_{n=1}^{\infty}$ such that,

$$
\|h - t_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.2.13}
$$

where $1 < R < \widetilde{\varrho}$. Since we can choose ζ_1 arbitrarily close to z_1, (4.2.13) is valid for any $1 < R < \varrho$.

The function g in (4.2.10) consists of sums of integrals of the type,

$$
G(z) = \int_{l_k}\frac{g_k(t)}{(t-z)^2}dt,
$$

where in view of (4.2.1) and (4.2.8) we have , for $t \in l_k$,

$$
|g_k(t)| \preceq |t - \tau_k|^{\gamma^{(k)}_{p_k+1}}.
$$

Hence, by using the result of Lemma 6 in [1] (see also Lemma 4.2.1), in conjunction with the remark following Theorem 1 of the same paper and the triangle inequality, we conclude that there exists a sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ satisfying

$$
\|g - q_n\|_{L^2(G)} \preceq \frac{1}{n^{\widetilde{s}}}, \quad n \in \mathbb{N}, \tag{4.2.14}
$$

where $\widetilde{s} := \min\{(2-\alpha_k)\gamma^{(k)}_{p_k+1} : \ k = 1, 2, \ldots, M\}$. This, combined with (4.2.9), (4.2.13) and the triangle inequality, yields

$$
\|f'_0 - \sum_{k=1}^{M} F'_k - (t_n + q_n)\|_{L^2(G)} \leq c_1\frac{1}{n^{\widetilde{s}}} + c_2\frac{1}{R^n}, \quad n \in \mathbb{N}. \tag{4.2.15}
$$

Note that $\widetilde{s} = s^{\star}$, if $\gamma^{(k)}_{p_k+1} \notin \mathbb{N}$ for the index k for which the minimum is attained in the definition of $\widetilde{s}$. In the opposite case, where for the same index k, it holds $\gamma^{(k)}_{p_k+1} \in \mathbb{N}$, we get $\widetilde{s} = s^\star$ in (4.2.15) by simply subtracting from $g(z)$ and adding to $h(z)$, in the right hand side of (4.2.9), the derivative of the Cauchy integral on l_N, with density function $a^{(k)}_{p_k+1}(z-\tau_k)^{\gamma^{(k)}_{p_k+1}}$. This observation and (1.3.14) imply that there exists a sequence of augmented polynomials $\{\widetilde{p}_n\}$, with $\widetilde{p}_n \in \mathbb{P}_n^{A_2}$, such that,

$$\|K(\cdot,z_0) - \widetilde{p}_n\|_{L^2(G)} \leq c_3\frac{1}{n^{s^\star}} + c_4\frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.2.16}$$

and the rest goes in similar lines as in the proof of Theorem 3.2.1, except here we use the version of Andrievskii's lemma for functions with anti-derivatives in the space $\mathbb{P}_n^{A_2}$, given in [16, Corollary 2.5] (see also Lemma 4.2.2). These yield,

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \leqslant c_5\sqrt{\log n}\frac{1}{n^{s^\star}} + c_6\sqrt{\log n}\frac{1}{R^n}, \quad n \geq 2, \tag{4.2.17}$$

and (4.2.6) follows with a different, but still less than ϱ, R. □

Remark 4.2.1. Note that $n^{s^\star} \leq R^n$, as $n \to \infty$. Therefore from (4.2.6) we recover the result of [16, Thm. 3.1]. However, Theorem 4.2.1 above gives, in addition, the precise dependence of the BKM/AB error on the pole singularities of f_0 for small values of n. We also note the lower estimate

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \geq c\frac{1}{n^{s^\star}}, \quad n \in \mathbb{N},$$

established in [16, Thm. 3.2].

The following result is a simple consequence of the minimum property of the kernel polynomials $K_n(\cdot,z_0)$ out of the space $\mathbb{P}_n^{A_2}$. The proof follows by working in the same way as in the proof of Corollaries 3.2.1 and 3.2.2. We recall that $\widetilde{P}_n(z)$ denote the augmented Bergman polynomials out of the space $\mathbb{P}_n^{A_2}$ and $\widetilde{r}_n$ denote the BKM/AB approximation in the space $\mathbb{P}_n^{A_2}$ to the conformal radius r_0.

Corollary 4.2.1. *Under the assumptions of Theorem 4.2.1 the following holds,*

$$|\widetilde{P}_n(z_0)| \leq c_1\frac{1}{n^{s^\star}} + c_2\frac{1}{R^n}, \quad n \in \mathbb{N}, \quad (z_0 \in G). \tag{4.2.18}$$

and

$$|r_0 - \widetilde{r}_n| \leq \left(c_3\frac{1}{n^{s^\star}} + c_4\frac{1}{R^n}\right)^2, \quad n \in \mathbb{N}. \tag{4.2.19}$$

4.3 BKM/AB with pole and corner singularities

We consider now the application of the BKM/AB with both pole and corner singular basis function of the form studied in Sections 3.3 and 4.2. Regarding poles we recall, in particular, our assumptions in Section 3.3. That is, the nearest singularities of f_0 in Ω are κ poles, each one of order k_j at z_j, $j = 1, 2, \dots \kappa$, where $|\Phi(z_1)| \leq |\Phi(z_2)| \leq \cdots \leq |\Phi(z_\kappa)|$, while the other singularities of f_0 occur at points $z_{\kappa+1}, z_{\kappa+2}, \dots$, where $|\Phi(z_\kappa)| < |\Phi(z_{\kappa+1})| \leqslant |\Phi(z_{\kappa+2})| \leqslant \cdots$. Therefore, for the BKM/AB we consider the system $\{\eta_j\}$, defined by:

(i) the κ pole singular functions (1.3.18), with $m_j = 1$, $j = 1, 2, \dots, \kappa$;

(ii) the $r_M = \sum_{k=1}^{M} p_k$ corner singular functions of the form (1.3.11);

(iii) and the n monomials (1.4.2).

Accordingly, we let $\mathbb{P}_n^{A_3}$ denote the space,

$$\mathbb{P}_n^{A_3} := \{p : p(z) = \sum_{j=1}^{\kappa+r_M+n} t_j \eta_j(z),\ t_j \in \mathbb{C}\},$$

and note that the associated augmented polynomial $\widetilde{K}_n(z, z_0)$ is the best approximation to $K(z, z_0)$ in $L^2(G)$ out of the space $\mathbb{P}_n^{A_3}$.

The following result is a version of Andrievsii's lemma for functions with anti-derivatives in $\mathbb{P}_n^{A_3}$. It will be used below, in the proof of the concluding theorem of this section in the transition from the $L^2(G)$-norm to the $L^\infty(\overline{G})$-norm, where we establish the BKM/AB error in approximating f_0 by the augmented polynomials $\widetilde{\pi}_n$ derived from $\mathbb{P}_n^{A_3}$.

Lemma 4.3.1. *Assume that Γ is piecewise analytic without cusps and let $t_k \in \Gamma$, $k = 1, 2, \dots, m$. Also, let $P \in \mathbb{P}_n$ and q be a fixed polynomial with no zeros on $\overline{G}$. Assume further that for some constants $a_{n,k,j}$, $k = 1, 2, \dots, m$, $j = 1, 2, \dots, r_k$, the function*

$$L(z) := \frac{P(z)}{q(z)} + \sum_{k=1}^{m} \sum_{j=1}^{r_k} a_{n,k,j} f_{\beta_j^{(k)}, t_k}(z),$$

where $f_{\beta_j^{(k)},t_k}(z) = (z - t_k)^{\beta_j^{(k)}}$, *with* $\beta_j^{(k)} > 0$ *non-integer, satisfies:* $L(z_0) = 0$ *and* $\|L'\|_{L^2(G)} \le M$. *Then,*

$$\|L\|_{L^\infty(\overline{G})} \le CM\sqrt{\log n}, \tag{4.3.1}$$

where C *is a constant independent of* n *and of* $\{\{a_{n,k,j}\}_{j=1}^{r_k}\}_{k=1}^{m}$.

Proof. The proof is based on Andrievskii's lemma for singular algebraic functions given in [16, Corollary 2.5] (see also Lemma 4.2.2 above) and relies on the results contained in [16, §2]. The details of the derivation are as follows:

First, we note that our assumption implies that Γ is a quasiconformal curve. Then, it is straightforward to verify that the results of Theorems 2.1 and 2.2 (and hence the result of Corollary 2.2) in [16] hold true for functions of the form $q^2(z)f_{\beta,\tau}(z)$, where $f_{\beta,\tau}(z) := (z-\tau)^\beta$, with $\tau \in \Gamma$ and $\beta > 0$ non-integer. That is,

$$\inf_{p \in \mathbb{P}_n} \|q^2 f_{\beta,\tau} - p\|_{L^2(G)} \asymp \frac{1}{n^{(2-a)(\beta+1)}}, \tag{4.3.2}$$

where $\alpha\pi$ $(0 < \alpha < 2)$ denotes the interior angle of Γ at τ.

With (4.3.2) at hand it is, again, straightforward to verify consequentially that the results of Theorem 2.3, Corollaries 2.3 and 2.4, Lemma 2.3 and Corollary 2.5, of [16], hold true if we replace $f'_{\beta,\tau}$ by $q^2 f'_{\beta,\tau}$. In particular, Corollary 2.5 of [16] applied to the function

$$S(z) := \int_{z_0}^{z} q^2(z)L'(z)dz,$$

where the path of integration $[z_0, z]$ is any rectifiable arc in G, gives that

$$\|S\|_{L^\infty(\overline{G})} \le c_1\sqrt{\log n}\,\|S'\|_{L^2(G)}.$$

(Note that $S(z_0) = 0$ and $S'(z) = q^2(z)L'(z)$.) Therefore, our hypothesis on $\|L'\|_{L^2(G)}$ yields the inequality

$$\|S\|_{L^\infty(\overline{G})} \le c_2\sqrt{\log n}\,M. \tag{4.3.3}$$

On the other hand we have,

$$L(z) = \int_{z_0}^{z} q^{-2}(z)S'(z)dz = q^{-2}(z)S(z) + 2\int_{z_0}^{z} q^{-3}(z)q'(z)S(z)dz,$$

which implies

$$\|L\|_{L^\infty(\overline{G})} \le c_3 \|S\|_{L^\infty(\overline{G})}$$

and (4.3.1) follows from (4.3.3); cf. [5, p. 122]. $\square$

The concluding result of this section provides the theoretical justification for the use of the BKM/AB, with both corner and pole singularities. For the next theorem we recall that the nearest singularities of f_0 in Ω are κ poles, each one of order k_j at z_j, $j = 1, 2, \ldots \kappa$, where $|\Phi(z_1)| \le |\Phi(z_2)| \le \cdots \le |\Phi(z_\kappa)|$, while the other singularities of f_0 occur at points $z_{\kappa+1}, z_{\kappa+2}, \ldots$, where $|\Phi(z_\kappa)| < |\Phi(z_{\kappa+1})| \leqslant |\Phi(z_{\kappa+2})| \leqslant \cdots$.

Let $\widetilde{\pi}_n$ denote the BKM/AB approximation to f_0 resulting from the space $\mathbb{P}_n^{A_3}$. Then we have the following:

Theorem 4.3.1. *Assume that Γ is piecewise analytic without cusps and set $\varrho := |\Phi(z_{\kappa+1})|$ and $s^\star := \min\{(2-\alpha_k)\gamma_{\nu_k}^{(k)} :\ 1 \le k \le M\}$. Then,*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \le c_1 \sqrt{\log n}\frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n}, \quad n \ge 2, \tag{4.3.4}$$

for any R, $1 < R < \varrho$.

Proof. As in the proof of Theorem 3.3.1, we set $Q(z) := \prod_{j=1}^{\kappa}(z-z_j)^{k_j+1}$. The result (4.3.4) will emerge by working as in the proof of Theorem 4.2.1. The basic idea is to consider, in a bigger domain $\widetilde{G}$, the anti-derivatives F and G_k of the functions Qf_0' and QF_k', respectively, in the place of the functions f_0 and F_k. The details of the derivation are as follows:

We note that the function Qf_0' shares the same analytic properties with f_0', apart from the fact that it has the singularities at the points z_j, $j = 1, \ldots, \kappa$, all removed. Therefore, the function

$$F(z) := \int_{z_0}^{z} Q(\zeta) f_0'(\zeta) d\zeta, \tag{4.3.5}$$

can be extended analytically to a larger domain $\widetilde{G}$ than the one considered in Section 4.2. This larger domain $\widetilde{G}$ is obtained by choosing the point ζ_1 close to the nearest pole $z_{\kappa+1}$ of Qf_0' in Ω, but inside the level curve L_ϱ, where now $\varrho := |\Phi(z_{\kappa+1})|$. The remaining part of the construction of $\widetilde{G}$ is exactly the same as in Section 4.2.

It follows therefore that (4.3.5) is valid for $z \in \widetilde{G}$, provided the arc of integration $[z_0, z]$ lies on $\widetilde{G} \cup \partial\widetilde{G} \setminus \{\cup_{k=1}^{M} \tau_k\}$ and is rectifiable. (This is always possible because $\partial\widetilde{G}$ is piecewise analytic.) Since the derivative of f_0 near τ_k can be obtained by termwise differentiation of the expansion (4.2.1), (cf. [10, p. 1448]) and since any power in the resulting expansion is bigger than $-\frac{1}{2}$, we see that f_0' is integrable along any rectifiable arc in $\widetilde{G}$ with one endpoint at τ_k. Therefore, integration by parts gives, for $z \in \widetilde{G} \cup \partial\widetilde{G}$,

$$F(z) = Q(z)f_0(z) - \int_{z_0}^{\tau_k} Q'(\zeta)f_0(\zeta)d\zeta - \int_{\tau_k}^{z} Q'(\zeta)f_0(\zeta)d\zeta, \tag{4.3.6}$$

where we made use of the normalization of f_0 at z_0. This shows that F is continuous on $\partial\widetilde{G}$ and analytic and on $\partial\widetilde{G}$, except for the endpoints τ_k. By arguing as in (4.2.7) we have, for $z \in G$,

$$Q(z)f_0'(z) = F'(z) = \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{F(t)}{(t-z)^2}dt + \frac{1}{2\pi i}\int_{l_N}\frac{F(t)}{(t-z)^2}dt. \tag{4.3.7}$$

Similar properties to those of F apply to the anti-derivative

$$G_k(z) := \int_{z_0}^{z} Q(\zeta)F_k'(\zeta)d\zeta, \tag{4.3.8}$$

of QF_k', $k = 1, \ldots, M$. That is, for $z \in \widetilde{G} \cup \partial\widetilde{G}$,

$$\begin{aligned} G_k(z) = Q(z)F_k(z) - Q(z_0)F_k(z_0) \\ - \int_{z_0}^{\tau_k} Q'(\zeta)F_k(\zeta)d\zeta - \int_{\tau_k}^{z} Q'(\zeta)F_k(\zeta)d\zeta, \end{aligned} \tag{4.3.9}$$

and, for $z \in G$,

$$Q(z)F_k'(z) = G_k'(z) = \frac{1}{2\pi i}\sum_{r=1}^{M}\int_{l_r}\frac{G_k(t)}{(t-z)^2}dt + \frac{1}{2\pi i}\int_{l_N}\frac{G_k(t)}{(t-z)^2}dt. \tag{4.3.10}$$

Next, by combining (4.3.6) and (4.3.9) we get,

$$F(z) - d_k - G_k(z) = Q(z)[f_0(z) - F_k(z)] - \int_{\tau_k}^{z} Q'(\zeta)[f_0(z) - F_k(\zeta)]d\zeta, \tag{4.3.11}$$

where

$$d_k := Q(z_0)F_k(z_0) - \int_{z_0}^{\tau_k} Q'(\zeta)[f_0(\zeta) - F_k(\zeta)]d\zeta.$$

This and (4.2.1) lead to

$$|F(z) - d_k - G_k(z)| \preceq |z - \tau_k|^{\gamma^{(k)}_{p_k+1}}, \quad z \in l_k. \tag{4.3.12}$$

By reasoning as in the proof of Theorem 4.2.1 we conclude, by using (4.3.7) and (4.3.10) that, for $z \in G$,

$$Q(z)f_0'(z) - Q(z)\sum_{k=1}^{M} F_k'(z) = g(z) + h(z), \tag{4.3.13}$$

where the singular part

$$g(z) := \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{F(t) - d_k - G_k(t)}{(t-z)^2}dt, \tag{4.3.14}$$

of the splitting (4.3.13) can be approximated, eventually, by a sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ at a polynomial rate, viz.,

$$\|g - q_n\|_{L^2(G)} \preceq \frac{1}{n^{s^\star}}, \quad n \in \mathbb{N}, \tag{4.3.15}$$

with $s^\star := \min\{(2-\alpha_k)\gamma^{(k)}_{\nu_k} : \; 1 \le k \le M\}$ and the analytic part

$$\begin{aligned} h(z) := \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{d_k}{(t-z)^2}dt + \frac{1}{2\pi i}\int_{l_N}\frac{F(t)}{(t-z)^2}dt \\ - \frac{1}{2\pi i}\sum_{k=1}^{M}\sum_{\substack{r=1\\ r\neq k}}^{M}\int_{l_r}\frac{G_k(t)}{(t-z)^2}dt - \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_N}\frac{G_k(t)}{(t-z)^2}dt. \end{aligned} \tag{4.3.16}$$

can be approximated by a sequence of polynomials $\{t_n\}_{n=1}^{\infty}$ at a geometric rate, viz.,

$$\|h - t_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.3.17}$$

where $1 < R < \varrho$. Hence using the triangle inequality we get

$$\|Q(f_0' - \sum_{k=1}^{M} F_k') - (t_n + q_n)\|_{L^2(G)} \le c_1\frac{1}{n^{s^\star}} + c_2\frac{1}{R^n}. \tag{4.3.18}$$

This implies

$$\left\| f_0' - \sum_{k=1}^{M} F_k' - \frac{t_n + q_n}{Q}\right\|_{L^2(G)} \le \frac{c}{d^{\kappa+\xi}}\left[c_1\frac{1}{n^{s^\star}} + c_2\frac{1}{R^n}\right], \tag{4.3.19}$$

where $d := \min_{j=1,2,\ldots,\kappa}\{|z - z_j| : z \in \Gamma\}$ and $\xi := \sum_{j=1}^{\kappa} k_j$. Thus, from (1.3.18) we conclude there exists a sequence of augmented polynomials $\{\widetilde{p}_n\}$, where $\widetilde{p}_n \in \mathbb{P}_n^{A_3}$, such that,

$$\|K(\cdot, z_0) - \widetilde{p}_n\|_{L^2(G)} \le c_3 \frac{1}{n^{s^\star}} + c_4 \frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.3.20}$$

Therefore, using the minimum property of the augmented kernel polynomials, we have

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \le c_3 \frac{1}{n^{s^\star}} + c_4 \frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.3.21}$$

and this, in conjunction with the equivalence Lemma 2.3.1, yields that

$$\|f_0' - \widetilde{\pi}_n'\|_{L^2(G)} \le c_5 \frac{1}{n^{s^\star}} + c_6 \frac{1}{R^n}, \quad n \in \mathbb{N}. \tag{4.3.22}$$

Since,

$$\widetilde{\pi}_n(z) = \frac{P(z)}{q(z)} + \sum_{k=1}^{M} \sum_{j=0}^{p_k} a_{n,k,j} (z - \tau_k)^{\gamma_j^{(k)}}, \tag{4.3.23}$$

where $q(z) := \prod_{j=1}^{\kappa} (z - z_j)^{k_j}$ and $P(z)$ is a polynomial of degree $n + \xi$, the rest goes as the concluding part of the proof of Theorem 4.2.1, except here we use the result of Lemma 4.3.1 in the place of [16, Corollary 2.5]. $\square$

The following result is a simple consequence of the minimum property of the kernel polynomials $\widetilde{K}_n(\cdot, z_0)$ out of the space $\mathbb{P}_n^{A_3}$. We recall that $\widetilde{P}_n(z)$ denote the augmented Bergman polynomials out of the space $\mathbb{P}_n^{A_3}$ and $\widetilde{r}_n$ denote the BKM/AB approximation in the space $\mathbb{P}_n^{A_3}$ to the conformal radius r_0.

Corollary 4.3.1. *With the assumptions of Theorem 4.3.1 it holds,*

$$|\widetilde{P}_n(z_0)| \le c_1 \frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n}, \quad n \in \mathbb{N}, \quad (z_0 \in G). \tag{4.3.24}$$

and

$$|r_0 - \widetilde{r}_n| \le \left(c_3 \frac{1}{n^{s^\star}} + c_4 \frac{1}{R^n} \right)^2, \quad n \in \mathbb{N}. \tag{4.3.25}$$

We end this section providing the theoretical justification for the use of the BKM/AB, with only pole singularities.

Theorem 4.3.2. *Assume Γ is piecewise analytic without cusp and set $\varrho = |\Phi(z_{\kappa+1})|$ and $s := \min\{(2-\alpha_k)\gamma_1^{(k)} : \ 1 \le k \le M\}$. Then,*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \le c_1 \sqrt{\log n}\frac{1}{n^s} + c_2 \frac{1}{R^n}, \quad n \ge 2, \tag{4.3.26}$$

for any R, $1 < R < \varrho$.

Proof. We set $Q(z) := \prod_{j=1}^{\kappa}(z-z_j)^{k_j+1}$ and as in the proof of Theorem 4.3.1, let the function

$$F(z) := \int_{z_0}^{z} Q(\zeta) f_0'(\zeta) d\zeta. \tag{4.3.27}$$

Hence, integration by parts gives, for $z \in \widetilde{G} \cup \partial\widetilde{G}$,

$$F(z) = Q(z)f_0(z) - \int_{z_0}^{\tau_k} Q'(\zeta) f_0(\zeta) d\zeta - \int_{\tau_k}^{z} Q'(\zeta) f_0(\zeta) d\zeta. \tag{4.3.28}$$

Observe, (4.2.1) can be written in the form

$$f_0(z) = f_0(\tau_k) + \widetilde{f}_{\gamma_1^{(k)},\tau_k}(z), \tag{4.3.29}$$

This and (4.3.28) lead to

$$|F(z) - d_k| \preceq |z-\tau_k|^{\gamma_1^{(k)}}, \quad z \in l_k. \tag{4.3.30}$$

where

$$d_k = f_0(\tau_k)Q(z_0) - \int_{z_0}^{\tau_k} Q'(\zeta) \widetilde{f}_{\gamma_1^{(k)},\tau_k}(\zeta) d\zeta.$$

Using Cauchy's integral formula for the F we have for $z \in G$

$$Q(z)f_0'(z) = F'(z) = \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{F(t)}{(t-z)^2}dt + \frac{1}{2\pi i}\int_{l_N}\frac{F(t)}{(t-z)^2}dt. \tag{4.3.31}$$

By reasoning as in the proof of Theorem 4.3.1 we conclude that, for $z \in G$,

$$Q(z)f_0'(z) = g(z) + h(z), \tag{4.3.32}$$

where the singular part

$$g(z) := \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{F(t)-d_k}{(t-z)^2}dt, \tag{4.3.33}$$

of the splitting (4.3.32) can be approximated, eventually, by a sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ at a polynomial rate, viz.,

$$\|g - q_n\|_{L^2(G)} \preceq \frac{1}{n^s}, \quad n \in \mathbb{N}, \tag{4.3.34}$$

with $s := \min\{(2-\alpha_k)\gamma_1^{(k)} : \ 1 \leq k \leq M\}$ and the analytic part

$$h(z) = \frac{1}{2\pi i}\sum_{k=1}^{M}\int_{l_k}\frac{d_k}{(t-z)^2}dt + \frac{1}{2\pi i}\int_{l_N}\frac{F(t)}{(t-z)^2}dt \tag{4.3.35}$$

can be approximated by a sequence of polynomials $\{t_n\}_{n=1}^{\infty}$ at a geometric rate, viz.,

$$\|h - t_n\|_{L^\infty(\overline{G})} \preceq \frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.3.36}$$

where $1 < R < \varrho$. Hence using the triangle inequality we get

$$\|Qf_0' - (t_n + q_n)\|_{L^2(G)} \leq c_1\frac{1}{n^s} + c_2\frac{1}{R^n}. \tag{4.3.37}$$

This implies

$$\left\|f_0' - \frac{t_n + q_n}{Q}\right\|_{L^2(G)} \leq \frac{c}{d^{\kappa+\xi}}\left[c_1\frac{1}{n^s} + c_2\frac{1}{R^n}\right], \tag{4.3.38}$$

where $d := \min_{j=1,2,\dots,\kappa}\{|z - z_j| \ : \ z \in \Gamma\}$ and $\xi := \sum_{j=1}^{\kappa} k_j$. Thus, from (1.3.18) we conclude there exists a sequence of augmented polynomials $\{\widetilde{p}_n\}$, where $\widetilde{p}_n \in \mathbb{P}_n^{A_1}$, such that,

$$\|K(\cdot, z_0) - \widetilde{p}_n\|_{L^2(G)} \leq c_3\frac{1}{n^s} + c_4\frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.3.39}$$

Therefore, using the minimum property of the augmented kernel polynomials, we have

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \leq c_3\frac{1}{n^s} + c_4\frac{1}{R^n}, \quad n \in \mathbb{N}, \tag{4.3.40}$$

and this, in conjunction with the equivalence Lemma 2.3.1, yields that

$$\|f_0' - \widetilde{\pi}_n'\|_{L^2(G)} \leq c_5\frac{1}{n^s} + c_6\frac{1}{R^n}, \quad n \in \mathbb{N}. \tag{4.3.41}$$

Since,

$$\widetilde{\pi}_n(z) = \frac{P(z)}{q(z)}, \tag{4.3.42}$$

where $q(z) := \prod_{j=1}^{\kappa}(z-z_j)^{k_j}$ and $P(z)$ is a polynomial of degree $n+\xi$, the transition from the $L^2(G)$-norm to the $L^\infty(\overline{G})$-norm is done as in the proof of Theorem 3.3.1, where now, in view of (4.3.42), Lemma 3.1.1(ii) is applicable. This leads to,

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \leq c_7 \frac{\sqrt{\log n}}{n^s} + c_8 \frac{\sqrt{\log n}}{R^n}, \quad n \in \mathbb{N} \tag{4.3.43}$$

and Theorem 4.3.2 follows with a different, but still less than ϱ, R. □

The following result is a simple consequence of the minimum property of the kernel polynomials $K_n(\cdot, z_0)$ out of the space $\mathbb{P}_n^{A_1}$ in case where now the boundary Γ is piecewise analytic without cusps. The proof is similar to the proofs of Corollaries 3.2.1 and 3.2.2. We recall that $\widetilde{P}_n(z)$ denote the augmented Bergman polynomials out of the space $\mathbb{P}_n^{A_1}$ and $\widetilde{r}_n$ denote the BKM/AB approximation in the space $\mathbb{P}_n^{A_1}$ to the conformal radius r_0.

Corollary 4.3.2. *With the assumptions of Theorem 4.3.2 it holds,*

$$|\widetilde{P}_n(z_0)| \leq c_1 \frac{1}{n^s} + c_2 \frac{1}{R^n}, \quad n \in \mathbb{N}, \quad (z_0 \in G). \tag{4.3.44}$$

and

$$|r_0 - \widetilde{r}_n| \leq \left(c_3 \frac{1}{n^s} + c_4 \frac{1}{R^n} \right)^2, \quad n \in \mathbb{N}. \tag{4.3.45}$$

Chapter 5

Pointwise error estimates for the BKM and BKM/AB method

Let $\pi_n(z, z_0) = \pi_n(z)$ and $\widetilde{\pi}_n(z, z_0) = \widetilde{\pi}_n(z)$ denote the BKM and BKM/AB approximation to $f_0(z)$ for $z \in G$, given respectively by (1.4.2) and (2.3.6). The purpose of this chapter is to study the pointwise BKM and BKM/AB errors

$$|f_0(z) - \pi_n(z, z_0)| \quad \text{and} \quad |f_0(z) - \widetilde{\pi}_n(z, z_0)|, \;\; z \in G. \tag{5.0.1}$$

This is done by means of the two pointwise errors

$$|K(z, z_0) - K_n(z, z_0)| \quad \text{and} \quad |K(z, z_0) - \widetilde{K}_n(z, z_0)|, \;\; z \in G. \tag{5.0.2}$$

in approximating the kernel $K(z, z_0)$ in terms of the associated kernel polynomials.

For the next two results we assume, as in the introduction, that G is a bounded Jordan domain in $\mathbb{C}$. We state both results for the augmented case $\widetilde{\pi}_n(z, z_0)$ and $\widetilde{K}_n(z, z_0)$, although it is clear that they hold also for the polynomial case $\pi_n(z, z_0)$ and $K_n(z, z_0)$.

Lemma 5.0.2. *For any $z \in G$ the following holds,*

$$\begin{aligned} &|K(z, z_0) - \widetilde{K}_n(z, z_0)| \leq \\ &\|K(\cdot, z) - \widetilde{K}_n(\cdot, z)\|_{L^2(G)} \|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}. \end{aligned} \tag{5.0.3}$$

Proof. Since G is a bounded Jordan domain the augmented polynomials (2.3.4) form a complete orthonormal system in $L_a^2(G)$. Therefore, we have from (1.3.15) and (2.3.5) that for $z \in G$,

$$\begin{aligned} K(z,z_0) - \widetilde{K}_n(z,z_0) &= \sum_{k=1}^{\infty} \widetilde{P}_k(z)\overline{\widetilde{P}_k(z_0)} - \sum_{k=1}^{n} \widetilde{P}_k(z)\overline{\widetilde{P}_k(z_0)} \\ &= \sum_{k=n+1}^{\infty} \widetilde{P}_k(z)\overline{\widetilde{P}_k(z_0)}, \end{aligned} \tag{5.0.4}$$

and the result (5.0.3) follows at once from the Cauchy-Schwarz inequality

$$|K(z,z_0) - \widetilde{K}_n(z,z_0)| \leq \left(\sum_{k=n+1}^{\infty} |\widetilde{P}_k(z)|^2 \right)^{1/2} \left(\sum_{k=n+1}^{\infty} |\widetilde{P}_k(z_0)|^2 \right)^{1/2}, \tag{5.0.5}$$

and the fact that

$$\sum_{k=n+1}^{\infty} |\widetilde{P}_n(\zeta)|^2 = \|K(\cdot,\zeta) - \widetilde{K}_n(\cdot,\zeta)\|_{L^2(G)}^2, \ \zeta \in G. \tag{5.0.6}$$

which follows from the reproducing property (1.3.12). □

Lemma 5.0.3. *Assume that $z \in G$ can be connected by a rectifiable arc $\gamma(z,z_0)$ to a fixed point $z_0 \in G$. Then,*

$$\begin{aligned} &|f_0(z) - \widetilde{\pi}_{n+1}(z,z_0)| \leq \\ &c\big[|\gamma(z,z_0)|\|K(\cdot,z_0) - \widetilde{K}_n(\cdot,z_0)\|_{L^2(G)} \max_{\zeta\in\gamma(z,z_0)} \|K(\cdot,\zeta) - \widetilde{K}_n(\cdot,\zeta)\|_{L^2(G)} \\ &+ \|K(\cdot,z_0) - \widetilde{K}_n(\cdot,z_0)\|_{L^2(G)}^2\big], \end{aligned} \tag{5.0.7}$$

where $|\gamma(z,z_0)|$ denotes the length of $\gamma(z,z_0)$ and c is a positive constant which depends on z_0 but not on n.

Proof. From (1.4.5) and (1.3.14) with $z = z_0$ we have for $\gamma := \gamma(z,z_0)$

$$\begin{aligned} \big[f_0(z) - \widetilde{\pi}_{n+1}(z)\big]\widetilde{K}_n(z_0,z_0) &= f_0(z)\widetilde{K}_n(z_0,z_0) - \int_\gamma \widetilde{K}_n(\zeta,z_0)d\zeta \\ &= f_0(z)\left[\widetilde{K}_n(z_0,z_0) - \frac{1}{\pi r_0^2}\right] + \left[\frac{f_0(z)}{\pi r_0^2} - \int_\gamma \widetilde{K}_n(\zeta,z_0)d\zeta\right] \\ &= \int_\gamma \left[K(\zeta,z_0) - \widetilde{K}_n(\zeta,z_0)\right]d\zeta - f_0(z)\left[K(z_0,z_0) - \widetilde{K}_n(z_0,z_0)\right]. \end{aligned}$$

We note that the application of the reproducing property (1.3.4), gives for any $\zeta \in G$, that

$$K(\zeta,\zeta) - \widetilde{K}_n(\zeta,\zeta) = \|K(\cdot,\zeta) - \widetilde{K}_n(\cdot,\zeta)\|^2_{L^2(G)}. \tag{5.0.8}$$

We also note (see (5.0.6)) the two relations:

$$\sum_{k=n+1}^{\infty} |\widetilde{P}_n(z_0)|^2 = \|K(\cdot,z_0) - \widetilde{K}_n(\cdot,z_0)\|^2_{L^2(G)}, \tag{5.0.9}$$

and

$$\sum_{k=n+1}^{\infty} |\widetilde{P}_n(\zeta)|^2 = \|K(\cdot,\zeta) - \widetilde{K}_n(\cdot,\zeta)\|^2_{L^2(G)}. \tag{5.0.10}$$

Therefore, from the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left|\int_\gamma K(\zeta,z_0) - \widetilde{K}_n(\zeta,z_0)d\zeta\right| = \left|\int_\gamma \sum_{k=n+1}^{\infty} \widetilde{P}_n(\zeta)\overline{\widetilde{P}_n(z_0)}d\zeta\right| \\
&\le \left[\sum_{k=n+1}^{\infty} |\widetilde{P}_n(z_0)|^2\right]^{1/2} \int_\gamma \left[\sum_{k=n+1}^{\infty} |\widetilde{P}_n(\zeta)|^2\right]^{1/2} |d\zeta| \\
&\le \left[\sum_{k=n+1}^{\infty} |\widetilde{P}_n(z_0)|^2\right]^{1/2} |\gamma(z,z_0)| \max_{\zeta\in\gamma(z,z_0)} \left\{\left[\sum_{k=n+1}^{\infty} |\widetilde{P}_n(\zeta)|^2\right]^{1/2}\right\}.
\end{aligned}$$

and the result (5.0.7) follows from (5.0.9) and (5.0.10). □

Remark 5.0.1. It is clear from the proof that the results of Lemmas 5.0.2 and 5.0.3 hold true for any complete orthonormal system. In particular if we use the Bergman polynomials P_n in the place of $\widetilde{P}_n$ then (5.0.3) and (5.0.7) hold for the usually $K_n(z,z_0)$ and $\pi_{n+1}(z,z_0)$, i.e.,

$$\begin{aligned}
&|f_0(z) - \pi_{n+1}(z,z_0)| \le \\
&c\big[|\gamma(z,z_0)|\|K(\cdot,z_0) - K_n(\cdot,z_0)\|_{L^2(G)} \max_{\zeta\in\gamma(z,z_0)} \|K(\cdot,\zeta) - K_n(\cdot,\zeta)\|_{L^2(G)} \\
&+ \|K(\cdot,z_0) - K_n(\cdot,z_0)\|^2_{L^2(G)}\big], \quad z \in G.
\end{aligned} \tag{5.0.11}$$

and

$$\begin{aligned}
&|K(z,z_0) - K_n(z,z_0)| \le \\
&\|K(\cdot,z) - K_n(\cdot,z)\|_{L^2(G)}\|K(\cdot,z_0) - K_n(\cdot,z_0)\|_{L^2(G)}, \quad z \in G.
\end{aligned} \tag{5.0.12}$$

5.1 BKM

Clearly, the relations (1.3.16), (1.3.18) (1.4.1) and (1.4.2) hold when the fixed point $z_0 \in G$ is replaced by another point $z \in G$, and we use the notation f_z for the normalized conformal map $G \to D(0, r_z)$. Note that f_0 and f_z are related by a Mobius transformation. According to (1.3.18), $r_z = r_z(G, z)$ is called the conformal radius of G with respect to z.

We recall that Φ denote the exterior conformal map of Ω onto $\Delta := \{w : |w| > 1\}$, normalized so that near infinity,

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \ldots, \quad \gamma > 0, \tag{5.1.1}$$

and let L_R $(R \geq 1)$ denote the level curve,

$$L_R := \{z : |\Phi(z)| = R\}, \tag{5.1.2}$$

so that $L_1 \equiv \Gamma$. Note that L_R, for $R > 1$, is an analytic Jordan curve. We use G_R to denote its interior, i.e., $G_R := \mathrm{int}(L_R)$. The next theorem provides an estimate for the error in the BKM approximation $\pi_n(z, z_0)$ to $f_0(z)$ and also $K_n(z, z_0)$ to $K(z, z_0)$ for any $z \in G$ under the assumption that f_0, and hence, f_z *has an analytic continuation across* Γ in Ω. We use $\rho(z) > 1$ to denote the index of the nearest poles of f_z on Ω and $m(z)$ to denote the highest order of the poles of f_z. For the next theorem we use $\varrho(z_0)$ to denote $|\Phi(z_1)|$.

Theorem 5.1.1. *Assume that Γ is piecewise analytic without cusps. Assume further that the conformal maps f_0 has an analytic continuation across Γ, such that f_0 and f_z are respectively analytic on $\overline{G_{\varrho(z_0)}}$ and $\overline{G_{\rho(z)}}$, with $\varrho(z_0) > 1$ and $\rho(z) > 1$, apart from a finite number of poles on $L_{\varrho(z_0)}$ and $L_{\rho(z)}$. Let $m(z_0)$ and $m(z)$ denote the highest order of the poles of f_0 and f_z on $L_{\varrho(z_0)}$ and $L_{\rho(z)}$, then, for any $z \in G$,*

$$|f_0(z) - \pi_n(z)| \leq c_1 |\gamma(z, z_0)| \frac{n^{m(z_0)}}{\varrho^n(z_0)} \max_{\zeta \in \gamma(z, z_0)} \frac{n^{m(\zeta)}}{\rho^n(\zeta)} + c_2 \frac{n^{2m(z_0)}}{\varrho^{2n}(z_0)}, \tag{5.1.3}$$

and

$$|K(z, z_0) - K_n(z, z_0)| \leq c_3 \frac{n^{m(z_0)}}{\varrho^n(z_0)} \frac{n^{m(z)}}{\rho^n(z)}, \tag{5.1.4}$$

where $|\gamma(z, z_0)|$ denotes the length of $\gamma(z, z_0)$ and c_1, c_2, c_3 are positive constants which depend on z_0 but not on n.

Proof. From (3.2.4) of Theorem 3.2.1 we have

$$\|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{n^{m(z_0)}}{\varrho^n(z_0)}, \tag{5.1.5}$$

and similar

$$\|K(\cdot, z) - K_n(\cdot, z)\|_{L^2(G)} \preceq \frac{n^{m(z)}}{\rho^n(z)}. \tag{5.1.6}$$

By applying (5.1.5) and (5.1.6) to Remark 5.0.1 we have the required results. (5.1.3) and (5.1.4) □

Remark 5.1.1. If $z = z_0$ then $|\gamma(z, z_0)| = 0$. So the estimate in (5.1.3) takes the form

$$|f_0(z_0) - \pi_n(z_0, z_0)| \leq c_2 \frac{n^{2m(z_0)}}{\varrho^{2n}(z_0)}, \tag{5.1.7}$$

If $z \to \Gamma$ then $\rho(z) \to 1$. So the estimate in (5.1.3) takes the form

$$|f_0(z) - \pi_n(z, z_0)| \leq c_1 |\gamma(z, z_0)| \frac{n^{m(z_0)+m(z)}}{\varrho^n(z_0)}, \tag{5.1.8}$$

5.2 BKM/AB with pole singularities inside the domain

In this section we study the pointwise BKM/AB errors $|K(z, z_0) - \widetilde{K}_n(z, z_0)|$ and $|f_0(z) - \widetilde{\pi}_n(z, z_0)|$ inside the domain G under the assumption that f_0 *has an analytic continuation across* Γ in Ω and its only singularities are poles, or rational poles, of the type (2.2.1). We exploit now the specific assumptions on the singularities of the analytic extension of f_0 studied in Section 2.2. More precisely, the assumption that the nearest singularities of f_0 are κ poles, each one of order k_j at z_j, $j = 1, 2, \ldots \kappa$, where $|\Phi(z_1)| \leq |\Phi(z_2)| \leq \cdots \leq |\Phi(z_\kappa)|$, and that the other singularities of f_0 occur at points $z_{\kappa+1}, z_{\kappa+2}, \ldots$, where $|\Phi(z_\kappa)| < |\Phi(z_{\kappa+1})| \leqslant |\Phi(z_{\kappa+2})| \leqslant \cdots$. We recall the space $\mathbb{P}_n^{A_1}$ of augmented polynomials contains only the singular functions in (2.3.1), with $m_j = 1$, $j = 1, 2, \ldots, \kappa$, and the n monomials in (2.3.3):

$$\mathbb{P}_n^{A_1} := \{p : p(z) = \sum_{j=1}^{\kappa+n} t_j \eta_j(z),\ t_j \in \mathbb{C}\}. \tag{5.2.1}$$

Since f_z is related to f_0 by a Mobius transformation it has exactly the similar behavior in Ω. In particular the nearest singularities of f_z in Ω are κ poles of order k_j, $j = 1, 2, \ldots \kappa$.

The next theorem provides an estimate for the interior error in the resulting BKM/AB approximation $\widetilde{\pi}_n(z, z_0)$ to f_0 and $\widetilde{K}_n(z, z_0)$ to $K(z, z_0)$ inside the domain G. We recall that $\rho(z) > 1$ is the index of the nearest poles of f_z on Ω and $m(z)$ is the highest order of the poles of f_z. For the next theorem we use $\varrho(z_0)$ to denote $|\Phi(z_{\kappa+1})|$.

Theorem 5.2.1. *Assume that Γ is piecewise analytic without cusps. Assume, in addition that f_0 and f_z have a finite number of poles and no other singularities on $L_{\varrho(z_0)}$ and $L_{\rho(z)}$ and let $m(z_0)$, $m(z)$ denote their highest order. Then, for any $z \in G$*

$$|f_0(z) - \widetilde{\pi}_n(z, z_0)| \leq c_1 |\gamma(z, z_0)| \frac{n^{m(z_0)}}{\varrho^n(z_0)} \max_{\zeta \in \gamma(z, z_0)} \frac{n^{m(\zeta)}}{\rho^n(\zeta)} + c_2 \frac{n^{2m(z_0)}}{\varrho^{2n}(z_0)}, \tag{5.2.2}$$

and

$$|K(z, z_0) - \widetilde{K}_n(z, z_0)| \leq c_3 \frac{n^{m(z_0)}}{\varrho^n(z_0)} \frac{n^{m(z)}}{\rho^n(z)}, \tag{5.2.3}$$

where $|\gamma(z, z_0)|$ denotes the length of $\gamma(z, z_0)$ and c_1, c_2, c_3 are positive constants which depend on z_0 but not on n.

Proof. From (3.3.7) of Theorem 3.3.2 we have

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \preceq \frac{n^{m(z_0)}}{\varrho^n(z_0)}. \tag{5.2.4}$$

Now for the estimate

$$\|K(\cdot, z) - \widetilde{K}_n(\cdot, z)\|_{L^2(G)}, \tag{5.2.5}$$

we set $Q(t) := \prod_{j=1}^{k} (t - z_j)^{k_j+1}$, where k_j is defined at the beginning of the section and observe that, the function $K(t, z)Q(t)$ is analytic in the interior of the level curve of $L_{\rho(z)}$. Hence from Lemma 3.0.2 it follows that, there exists a sequence of polynomial $\{p_n\}_{n=1}^{\infty}$, such that

$$\|K(\cdot, z)Q - p_n\|_{L^\infty(\overline{G})} \preceq \frac{n^{m(z)}}{\rho^n(z)}. \tag{5.2.6}$$

(The singular functions of the form $[(\frac{1}{t-z_j})^{k_j/m_j+1}]'$ doesn't cancel out the nearest singularities of $K(\cdot, z)$.) Let now $d := \min_{j=1,2,\ldots,\kappa}\{|t - z_j| : t \in \Gamma\}$ denote the distance of Γ from the poles $\{z_j\}_{j=1}^{\kappa}$, and set $\xi := \sum_{j=1}^{\kappa} k_j$. Then, $|Q(t)| \geq d^{\kappa+\xi}$, $t \in \Gamma$, and (5.2.6) gives

$$\|K(\cdot, z) - \frac{p_n}{Q}\|_{L^\infty(\overline{G})} \leq \frac{c}{d^{\kappa+\xi}} \frac{n^{m(z)}}{\rho^n(z)}.$$

Since the $L^2(G)$-norm is dominated by the $L^\infty(\overline{G})$-norm, we see that there exist a sequence of rational polynomials $\{Q_n\}_{n=1}^\infty$, with $Q_n \in \mathbb{P}_n^{A_1}$, such that,

$$\|K(\cdot, z) - Q_n\|_{L^2(G)} \preceq \frac{n^{m(z)}}{\rho^n(z)}, \quad n \in \mathbb{N}.$$

Therefore, using the minimum property (3.3.2) of the augmented kernel polynomials, we have

$$\|K(\cdot, z) - \widetilde{K}_n(\cdot, z)\|_{L^2(G)} \preceq \frac{n^{m(z)}}{\rho^n(z)}, \quad n \in \mathbb{N}. \tag{5.2.7}$$

By applying (5.2.4) and (5.2.7) to Lemmas 5.0.2, 5.0.3 we have the required results of (5.2.2) and (5.2.3). □

Remark 5.2.1. If $z = z_0$ then $|\gamma(z, z_0)| = 0$ and the estimate in (5.2.2) takes the form

$$|f_0(z_0) - \widetilde{\pi}_n(z_0, z_0)| \leq c_2 \frac{n^{2m(z_0)}}{\varrho^{2n}(z_0)}. \tag{5.2.8}$$

If $z \to \Gamma$ then $\rho \to 1$ and the estimate in (5.2.2) takes the form

$$|f_0(z) - \widetilde{\pi}_n(z, z_0)| \leq c_1 |\gamma(z, z_0)| \frac{n^{m(z_0)+m(z)}}{\varrho^n(z_0)}. \tag{5.2.9}$$

5.3 BKM/AB with corner singularities inside the domain

We consider now the application of the BKM/AB with corner singular basis function. This form of BKM/AB was studied in Section 4.2. We recall the associated space $\mathbb{P}_n^{A_2}$ of augmented polynomials contains only the r_M singular functions of the form (2.3.2), and the n monomials (2.3.3):

$$\mathbb{P}_n^{A_2} := \{p : p(z) = \sum_{j=1}^{r_M+n} t_j \eta_j(z),\ t_j \in \mathbb{C}\}. \tag{5.3.1}$$

In this case we note that the associated augmented kernel polynomials $\widetilde{K}_n(t, z_0)$, $\widetilde{K}_n(t, z)$ belong to the same space $\mathbb{P}_n^{A_2}$. The next theorem provides an estimate for the error in the resulting BKM/AB approximation $\widetilde{\pi}_n(z, z_0)$ to $f_0(z)$ and $\widetilde{K}_n(z, z_0)$ to $K(z, z_0)$ inside the domain G. We recall that $\rho(z)$ is the index of the nearest pole of f_z on Ω and we use $\varrho(z_0)$ to denote $|\Phi(z_1)|$.

Theorem 5.3.1. *Assume that Γ is piecewise analytic without cusps and as in Theorem 4.2.1 set $s^\star := \min\{(2-\alpha_k)\gamma_{\nu_k}^{(k)} : 1 \le k \le M\}$ where $\gamma_j^{(k)}$ is defined in Section 2.1 and ν_k is given by (4.2.2). Then, for any $z \in G$*

$$|f_0(z) - \widetilde{\pi}_n(z)| \le c_1 \frac{1}{n^{2s^\star}} + c_2 \frac{1}{n^{s^\star} R^n(z_0)} + c_3 \frac{1}{n^{s^\star}} \max_{\zeta\in\gamma} \frac{1}{R^n(\zeta)} + c_4 \frac{1}{R^n(z_0)} \max_{\zeta\in\gamma} \frac{1}{R^n(\zeta)} + c_5 \frac{1}{R^{2n}(z_0)}, \tag{5.3.2}$$

and

$$|K(z,z_0) - \widetilde{K}_n(z,z_0)| \le c_6 \frac{1}{n^{2s^\star}} + c_7 \frac{1}{n^{s^\star} R^n(z_0)} + c_8 \frac{1}{n^{s^\star} R^n(z)} + c_9 \frac{1}{R^n(z_0) R(z)}, \tag{5.3.3}$$

where $R(z_0)$, $R(z)$, $1 < R(z_0) < \varrho(z_0)$ and $1 < R(z) < \rho(z)$ and c_i $i = 1, 2, \dots 9$ are positive constants which depend on z_0 but not on n.

Proof. Since f_0 and f_z have exactly the same singular behavior on Γ both $s^\star$ in the errors $\|K(\cdot,z) - \widetilde{K}_n(\cdot,z)\|_{L^2(G)}$, $\|K(\cdot,z_0) - \widetilde{K}_n(\cdot,z_0)\|_{L^2(G)}$ in (4.2.16) of Theorem 4.2.1 are the same. We have

$$\|K(\cdot,z_0) - \widetilde{K}_n(\cdot,z_0)\|_{L^2(G)} \le c_1 \frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n(z_0)}, \tag{5.3.4}$$

and

$$\|K(\cdot,z) - \widetilde{K}_n(\cdot,z)\|_{L^2(G)} \le c_3 \frac{1}{n^{s^\star}} + c_4 \frac{1}{R^n(z)}, \tag{5.3.5}$$

where $R(z_0)$, $R(z)$, $1 < R(z_0) < \varrho(z_0)$ and $1 < R(z) < \rho(z)$. Then applying (5.3.4) and (5.3.5) to Lemmas 5.0.2 and 5.0.3, we have the required results. □

Remark 5.3.1. It is clear that, (5.3.2) and (5.3.3) hold for the usual $K_n(z,z_0)$ and $\pi_n(z,z_0)$ of the space $\mathbb{P}_n$ in the place of $\widetilde{K}_n(z,z_0)$ and $\widetilde{\pi}_n(z,z_0)$, i.e., for any $z \in G$

$$|f_0(z) - \widetilde{\pi}_n(z)| \le c_1 \frac{1}{n^{2s}} + c_2 \frac{1}{n^{s} R^n(z_0)} + c_3 \frac{1}{n^{s}} \max_{\zeta\in\gamma} \frac{1}{R^n(\zeta)} + c_4 \frac{1}{R^n(z_0)} \max_{\zeta\in\gamma} \frac{1}{R^n(\zeta)} + c_5 \frac{1}{R^{2n}(z_0)}, \tag{5.3.6}$$

and

$$|K(z,z_0) - \widetilde{K}_n(z,z_0)| \le c_6 \frac{1}{n^{2s}} + c_7 \frac{1}{n^{s} R^n(z_0)} + c_8 \frac{1}{n^{s} R^n(z)} + c_9 \frac{1}{R^n(z_0) R(z)}, \tag{5.3.7}$$

where $R(z_0)$, $R(z)$, $1 < R(z_0) < \varrho(z_0)$ and $1 < R(z) < \rho(z)$ and as in Theorem 4.1.1 $s := \min\{(2-\alpha_k)/\alpha_k : 1 \le k \le M\}$ and c_i $i = 1, 2, \dots 9$ are positive constants depend on z_0 but not on n.

5.4 BKM/AB with pole and corner singularities inside the domain

We consider now the application of the BKM/AB with both pole and corner singular basis function inside the domain. This form of BKM/AB studied in Sections 3.3 and 4.2. We recall the associated space $\mathbb{P}_n^{A_3}$ of augmented polynomials contains the κ pole singular functions in (2.3.1), with $m_j = 1$, $j = 1, 2, \ldots, \kappa$, the r_M corner singular functions of the form (2.3.2), and the n monomials (2.3.3):

$$\mathbb{P}_n^{A_3} := \{p : p(z) = \sum_{j=1}^{\kappa+r_M+n} t_j \eta_j(z),\ t_j \in \mathbb{C}\}. \tag{5.4.1}$$

The next theorem provide an estimate for the error in the resulting BKM/AB approximation $\widetilde{\pi}_n(z, z_0)$ to $f_0(z)$ and $\widetilde{K}_n(z, z_0)$ to $K(z, z_0)$ inside of the domain G. For the next theorem we recall that $\rho(z)$ is the index of the nearest pole of f_z on Ω and we use $\varrho(z_0)$ to denote $|\Phi(z_{\kappa+1})|$.

Theorem 5.4.1. *Assume that Γ is piecewise analytic without cusps and and as in Theorem 4.3.1 set $s^\star := \min\{(2-\alpha_k)\gamma_{\nu_k}^{(k)} :\ 1 \le k \le M\}$. Then, for any $z \in G$*

$$|f_0(z) - \widetilde{\pi}_n(z)| \le c_1 \frac{1}{n^{2s^\star}} + c_2 \frac{1}{n^{s^\star} R^n(z_0)} + c_3 \frac{1}{n^{s^\star}} \max_{\zeta\in\gamma} \frac{1}{R^n(\zeta)} + c_4 \frac{1}{R^n(z_0)} \max_{\zeta\in\gamma} \frac{1}{R^n(\zeta)} + c_5 \frac{1}{R^{2n}(z_0)}, \tag{5.4.2}$$

and

$$|K(z, z_0) - \widetilde{K}_n(z, z_0)| \le c_6 \frac{1}{n^{2s^\star}} + c_7 \frac{1}{n^{s^\star} R^n(z_0)} + c_8 \frac{1}{n^{s^\star} R^n(z)} + c_9 \frac{1}{R^n(z_0) R(z)}, \tag{5.4.3}$$

where $R(z_0)$, $R(z)$, $1 < R(z_0) < \varrho(z_0)$ and $1 < R(z) < \rho(z)$ and c_i $i = 1, 2, \ldots 9$ are positive constants which depend on z_0 but not on n.

Proof. Since f_0 and f_z have exactly the same singular behavior on Γ both $s^\star$ in the errors $\|K(\cdot, z) - \widetilde{K}_n(\cdot, z)\|_{L^2(G)}$, $\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}$ in (4.3.21) of Theorem 4.3.1 are the same. We have

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)} \le c_1 \frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n(z_0)}, \tag{5.4.4}$$

and

$$\|K(\cdot, z) - \widetilde{K}_n(\cdot, z)\|_{L^2(G)} \leq c_3 \frac{1}{n^{s^\star}} + c_4 \frac{1}{R^n(z)}, \tag{5.4.5}$$

where $R(z_0)$, $R(z)$, $1 < R(z_0) < \varrho(z_0)$ and $1 < R(z) < \rho(z)$, because $\widetilde{K}_n(\cdot, z) \in \mathbb{P}_n^{A_3}$.

By applying the (5.4.4) and (5.4.5) to the Lemmas 5.0.2, 5.0.3 we have the required results. □

Remark 5.4.1. Note that $n^{s^\star} \leq R^n$, as $n \to \infty$. Therefore Theorems (5.3.1) and (5.4.1) becomes

$$|f_0(z) - \widetilde{\pi}_n(z, z_0)| \leq c_1 \frac{1}{n^{2s^\star}}, \tag{5.4.6}$$

and

$$|K(z, z_0) - \widetilde{K}_n(z, z_0)| \leq c_2 \frac{1}{n^{2s^\star}}, \tag{5.4.7}$$

where $c_1,\ c_2$ are positive constants which depend on z_0 but not on n.

Chapter 6

Numerical results

In this section we present numerical examples, that illustrate the convergence results predicted by the theory of Sections 3, 4 regarding the following six errors:

$$E_{n,2}(K,G) := \|K(\cdot,z_0) - K_n(\cdot,z_0)\|_{L^2(G)}, \tag{6.0.1}$$

$$E_{n,\infty}(f_0,G) := \|f_0 - \pi_n\|_{L^\infty(\overline{G})}, \tag{6.0.2}$$

$$\widetilde{E}_{n,2}(K,G) := \|K(\cdot,z_0) - \widetilde{K}_n(\cdot,z_0)\|_{L^2(G)}, \tag{6.0.3}$$

$$\widetilde{E}_{n,\infty}(f_0,G) := \|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})}, \tag{6.0.4}$$

$$E_n(r_0,G) := |r_0 - r_n|, \tag{6.0.5}$$

$$\widetilde{E}_n(r_0,G) := |r_0 - \widetilde{r}_n|, \tag{6.0.6}$$

and the four pointwise errors inside of G of Section 5,

$$E_n(K,G) := |K(z,z_0) - K_n(z,z_0)|, \quad z \in G, \tag{6.0.7}$$

$$E_n(f_0,G) := |f_0(z) - \pi_n(z)|, \quad z \in G, \tag{6.0.8}$$

$$\widetilde{E}_n(K,G) := |K(z,z_0) - \widetilde{K}_n(z,z_0)|, \quad z \in G, \tag{6.0.9}$$

$$\widetilde{E}_n(f_0,G) := |f_0(z) - \widetilde{\pi}_n(z)|, \quad z \in G. \tag{6.0.10}$$

We do this by considering two different geometries: (a) lens-shaped domains; and (b) circular sectors. In both cases the normalized conformal map f_0, and hence the kernel function $K(\cdot, z_0)$, are known explicitly in terms of elementary functions. In addition, we present results illustrating the decay of the two sequences of points $\{P_n(z_0)\}_{n=1}^{\infty}$ and $\widetilde{P}_n(z_0)_{n=1}^{\infty}$ of the Bergman polynomials.

6.1 Computational details

Let $\{\eta_j\}$ denote the set of linearly independent functions defined in (1.3.18)–(1.4.2). For the application of the BKM/AB (or BKM), we compute the associated orthonormal set $\{\widetilde{P}_j\}$ by using the Arnoldi variant of the Gram-Schmidt (GS) process studied in [25], rather than the conventional GS, which is based on the orthonormalization of the monomials $\{z^j\}$, as it is suggested in [12] and [20]. In the Arnoldi GS we construct first the polynomial part of the set $\{\widetilde{P}_j\}$ by orthonormalizing consequently the functions $1, z\widetilde{P}_0, z\widetilde{P}_1, \ldots, z\widetilde{P}_{n-1}$. Then, we orthonormalize the singular basis functions (1.3.18) and (1.3.11). As it is shown in [25], in this way we avoid the instability difficulties associated with the application of the conventional GS method. For a comprehensive report of experiments testifying the instability of the conventional GS in BKM and BKM/AB we refer to [20, §5].

The GS process, requires the computation of inner products of the form

$$\langle \eta_k, \eta_l \rangle = \int_G \eta_k(z)\,\overline{\eta_l(z)}\,dA(z). \tag{6.1.1}$$

For our purposes here, we compute these inner products by using Green's formula in order to transform the area integral into a line integral. For instance, when $\eta_k = z^k$, $\eta_l = z^l$, we have

$$\langle z^k, z^l \rangle = \frac{1}{2(l+1)i} \int_\Gamma z^k\,\overline{z}^{l+1}\,dz. \tag{6.1.2}$$

In all cases considered below this leads to explicit formulas for the inner products (6.1.1).

Regarding the computation of the errors (6.0.1)–(6.0.10) we note the following:

(i) The two errors $\|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)}$ and $\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}$ are computed by using Parseval's identity, i.e.,

$$\|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)}^2 = K(z_0, z_0) - K_n(z_0, z_0), \tag{6.1.3}$$

and

$$\|K(\cdot, z_0) - \widetilde{K}_n(\cdot, z_0)\|_{L^2(G)}^2 = K(z_0, z_0) - \widetilde{K}_n(z_0, z_0). \tag{6.1.4}$$

(ii) Estimates for the two errors $\|f_0 - \pi_n\|_{L^\infty(\overline{G})}$ and $\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})}$ are obtained by using the exact formula for f_0 and then sampling the differences $f_0 - \pi_n$ and $f_0 - \widetilde{\pi}_n$ on 100 uniformly distributed points on each analytic arc forming the boundary Γ.

(iii) Estimates for the two errors $|r_0 - r_n|$ and $|r_0 - \widetilde{r}_n|$ are obtained by using the exact value for r_0 and then sampling the differences $|r_0 - r_n|$ and $|r_0 - \widetilde{r}_n|$.

(iv) In all the numerical examples we present, the conformal map f_0 is known explicitly in terms of elementary functions. Hence, estimates for the four pointwise error estimates $|f_0(z) - \pi_n(z)|$, $|f_0(z) - \widetilde{\pi}_n(z)|$, and $|K(z, z_0) - K_n(z, z_0)|$, $|K(z, z_0) - \widetilde{K}_n(z, z_0)|$, for $z \in G$, are obtained by using the exact formula for f_0 and then computing the differences $|f_0(z) - \pi_n(z)|$, $|f_0(z) - \widetilde{\pi}_n(z)|$, and $|K(z, z_0) - K_n(z, z_0)|$, $|K(z, z_0) - \widetilde{K}_n(z, z_0)|$ for some fixed point $z \in G$.

All results were obtained with Maple 11, using the systems facility for 64-digit floating point arithmetic, on a pentium PC.

6.2 BKM and BKM/AB approximation

6.2.1 Lens-shaped domains

Let $G_{a,b}$ denote the lens-shaped domain, whose boundary Γ consists of two circular arcs Γ_a and Γ_b that join together the points i and $-i$ (Γ_a being to the left of Γ_b) and form angles a and b, respectively, with the linear segment $[-i, i]$. (Thus, with the notation of Section 2.1 we have $\alpha_1 = \alpha_2 = \alpha$, where $\alpha := (a + b)/\pi$.) Let f_0 denote the normalized conformal map from $G_{a,b}$ onto $D(0, r_0)$, with $f_0(0) = 0$ and $f_0'(0) = 1$. By working as in [17, §4], it is easy to check that, if $a + b = k\pi/m$, where $k, m \in \mathbb{N}$, then f_0 is given by

$$f_0(z) = r_0 \frac{\left[\frac{z-i}{z+i}\right]^{\frac{m}{k}} - (-1)^{\frac{m}{k}}}{\left[\frac{z-i}{z+i}\right]^{\frac{m}{k}} - (-1)^{\frac{m}{k}} e^{-2ia\frac{m}{k}}}, \quad z \in \overline{G}_{a,b}, \tag{6.2.1}$$

where $r_0 = (k/m)\sin(ma/k)$. Also,

$$K(z,0) = -\frac{4m^2}{\pi k^2}\frac{[(z-i)(z+i)]^{\frac{m}{k}-1}}{\left[e^{ia\frac{m}{k}}(-i)^{\frac{m}{k}}(z-i)^{\frac{m}{k}} - e^{-ia\frac{m}{k}}(i)^{\frac{m}{k}}(z+i)^{\frac{m}{k}}\right]^2}, \tag{6.2.2}$$

and thus

$$K(0,0) := \frac{m^2}{\pi k^2}\frac{1}{\sin^2(ma/k)}.$$

It is also easy to verify that the formulas (71)–(73) of [17] work as well for the exterior conformal map $\Phi : \overline{\mathbb{C}}\backslash\overline{G}_{a,b} \to \Delta$ consider here. That is, $w = \Phi(z)$ is given by the composition of the following three transformations:

$$\xi(z) := e^{i((m-k)\pi/m+a)}\frac{z-i}{z+i}, \tag{6.2.3}$$

$$t(\xi) := \xi^{m/(2m-k)}, \quad \arg\xi \in (-k\pi/m, (2m-k)\pi/m], \tag{6.2.4}$$

$$w(t) := \frac{1-\lambda_a t}{t-\lambda_a}, \quad \lambda_a := e^{i((m-k)\pi+ma)/(2m-k)}. \tag{6.2.5}$$

We consider separately the following three cases:

(i) $\alpha = 1/2$, with $a = \pi/6$ and $b = \pi/3$;

(ii) $\alpha = 1/2$, with $a = \pi/4$ and $b = \pi/4$;

(iii) $\alpha = 2/13$, with $a = \pi/13$ and $b = \pi/13$.

Cases (i) and (ii): In the first two cases the conformal map f_0 is a rational function, and hence it has an analytic continuation across Γ into Ω. When $a = \pi/6$, (see Figure (6.1)) then the two nearest singularities of f_0 in Ω are the two simple poles at $z_1 = -\sqrt{3}/3$ and $z_2 = \sqrt{3}$, where $|\Phi(z_1)| \approx 1.347$ and $|\Phi(z_2)| \approx 2.532$. Accordingly, in our experiments, we use the singular function $[1/(z-z_1)]'$. This cancels out the nearest singularity at z_1. In the symmetric case, (see Figure (6.2)) where $a = b = \pi/4$, we have

$$f_0(z) = \frac{-2iz}{z^2-1},$$

and the only singularities of f_0 are the two simple poles at $z_1 = -1$ and $z_2 = 1$, where $|\Phi(z_1)| = |\Phi(z_2)| = \sqrt{3}$. In this case, we use the singular function $[z/(z^2 - z_1^2)]'$, which

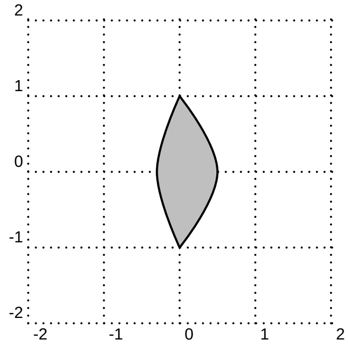

Figure 6.1: Lens-shaped domain, Case (i).

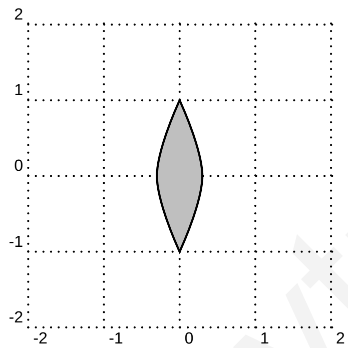

Figure 6.2: Lens-shaped domain, Case (ii).

takes care of both poles at z_1 and z_2. It follows from Remark 3.3.1 that this cancels out all the singularities of f_0.

We recall from Theorems 3.2.1 and 3.3.2 (and their proof) the four estimates,

$$E_{n,2}(K,G) \preceq \frac{n}{|\Phi(z_1)|^n}, \tag{6.2.6}$$

$$\frac{1}{|\Phi(z_1)|^n} \preceq E_{n,\infty}(f_0,G) \preceq \frac{n\sqrt{\log n}}{|\Phi(z_1)|^n}, \tag{6.2.7}$$

and

$$\widetilde{E}_{n,2}(K,G) \preceq \frac{n}{|\Phi(z_2)|^n}, \tag{6.2.8}$$

$$\frac{1}{|\Phi(z_2)|^n} \preceq \widetilde{E}_{n,\infty}(f_0,G) \preceq \frac{n\sqrt{\log n}}{|\Phi(z_2)|^n}. \tag{6.2.9}$$

Below, we present numerical results that illustrate the laws of the above errors and rates. In presenting the numerical results we use the following notation:

- ϱ : This denotes the order of approximation (the base of n) in the errors (6.2.6)–(6.2.9).

- ϱ_n : This denotes the estimate of ϱ, corresponding to n, and is determined as follows: With E_n denoting any of the two errors $E_{n,2}(K,G)$ or $\widetilde{E}_{n,2}(K,G)$, we assume that

$$E_n \approx c\frac{n}{\varrho^n} \tag{6.2.10}$$

and seek to estimate ϱ by means of the formula,

$$\varrho_n = \left(\frac{n}{n-m}\frac{E_{n-m}}{E_n}\right)^{\frac{1}{m}}. \tag{6.2.11}$$

(Here we take $m = 4$, or $m = 5$.) If E_n denotes either of the two errors $E_{n,\infty}(f_0,G)$ or $\widetilde{E}_{n,\infty}(f_0,G)$, then we assume that

$$E_n \approx c\frac{n\sqrt{\log n}}{\varrho^n}, \tag{6.2.12}$$

and seek to estimate ϱ by means of the formula,

$$\varrho_n = \left(\frac{n}{n-m}\frac{\sqrt{\log n}}{\sqrt{\log(n-m)}}\frac{E_{n-m}}{E_n}\right)^{\frac{1}{m}}, \tag{6.2.13}$$

with $m = 4$, or $m = 5$.

- $\varrho_n^\star$: With E_n denoting either of the errors $E_{n,\infty}(f_0,G)$ or $\widetilde{E}_{n,\infty}(f_0,G)$, we also test the law

$$E_n \approx c\frac{1}{\varrho^n}, \tag{6.2.14}$$

thereby estimating ϱ by means of

$$\varrho_n^\star = \left(\frac{E_{n-m}}{E_n}\right)^{\frac{1}{m}}. \tag{6.2.15}$$

The presented results show some evidence of the advantage of the BKM/AB over the BKM. In addition, they indicate a close agreement between the theoretical and the computed order of approximation. In Tables 6.2.1 and 6.2.3, the results associated with the errors $E_{n,2}(K,G)$ and $\widetilde{E}_{n,2}(K,G)$ indicate the convergence of ϱ_n to ϱ. Regarding

	BKM: $\varrho \approx 1.347$		BKM/AB: $\varrho \approx 2.532$	
n	$E_{n,2}(K,G)$	ϱ_n	$\widetilde{E}_{n,2}(K,G)$	ϱ_n
5	4.4e-01	-	2.7e-02	-
10	1.3e-01	1.47	3.6e-04	2.72
15	3.5e-02	1.41	4.1e-06	2.65
20	8.9e-03	1.39	4.6e-08	2.60
25	2.2e-03	1.38	4.9e-10	2.59
30	5.4e-04	1.37	5.2e-12	2.57
35	1.3e-04	1.36	5.4e-14	2.57

Table 6.2.1: BKM approximations to K: Lens-shaped, Case (i).

the errors $E_{n,\infty}(f_0,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$, the results of the Tables 6.2.2 and 6.2.4 show that $\varrho_n^\star$ converges faster to ϱ than ϱ_n. This suggest, at least for the geometry under consideration, a behavior of the type (6.2.14) for the errors $E_{n,\infty}(f_0,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$. As it is predicted by Remark 3.3.1, in Case (ii) the two errors $\widetilde{E}_{n,2}(K,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$ vanish. This was testified in our experiments, in the sense that the computed errors $\widetilde{E}_{n,2}(K,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$ were zero within machine precision, thus they are not quoted in Tables 6.2.3 and 6.2.4.

Case (iii): In this case (see Figure 6.3) the conformal map f_0 has a branch point singularity at each of the two corners $\tau_1 = i$ and $\tau_2 = -i$, and therefore Lehman's expansions (2.1.4) are valid with $\gamma_1^{(1)} = \gamma_1^{(2)} = 13/2$ and $\gamma_2^{(1)} = \gamma_2^{(2)} = 1 + 1/\alpha = 15/2$. This gives $(2-\alpha)/\alpha = 12$ and $(2-\alpha)(1+1/\alpha) = 180/13 = 13.84\cdots$. Furthermore, it follows from (6.2.1) that the nearest singularities of f_0 in Ω, are the two simple poles at $z_1 = \tan(\pi/13)$ and $z_2 = -\tan(\pi/13)$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.119$, and the next singularity occurs at a point z_3, where $|\Phi(z_3)| \approx 2.055$.

Therefore, from Theorem 4.1.1 we have that,

$$E_{n,2}(K,G) \leqslant c_1 \frac{1}{n^{12}} + c_2 \frac{2}{R^n}, \tag{6.2.16}$$

n	BKM: $\varrho \approx 1.347$			BKM/AB: $\varrho \approx 2.532$		
	$E_{n,\infty}(f_0, G)$	$\varrho_n^\star$	ϱ_n	$\widetilde{E}_{n,\infty}(f_0, G)$	$\varrho_n^\star$	ϱ_n
5	2.5e-01	-	-	1.4e-02	-	-
10	6.8e-02	1.299	1.54	1.3e-04	2.541	3.03
15	1.6e-02	1.331	1.47	1.3e-06	2.528	2.79
20	3.8e-03	1.342	1.43	1.2e-08	2.537	2.71
25	8.5e-04	1.346	1.42	1.1e-10	2.532	2.67
30	1.9e-04	1.347	1.41	1.1e-12	2.532	2.64
35	4.3e-05	1.347	1.40	1.1e-14	2.532	2.62

Table 6.2.2: BKM approximations to f_0: Lens-shaped, Case (i).

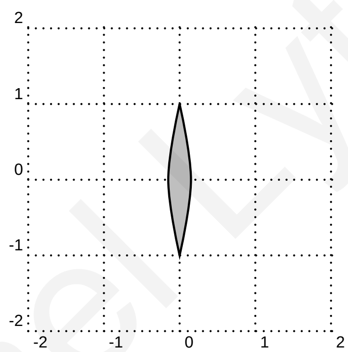

Figure 6.3: Lens-shaped domain, Case (iii).

and

$$E_{n,\infty}(f_0, G) \leqslant c_3 \sqrt{\log n} \frac{1}{n^{12}} + c_4 \frac{1}{R^n}, \tag{6.2.17}$$

where $1 < R < |\Phi(z_1)|$. In order to decide which singular functions to include in the BKM/AB the following estimates, valid for $n = 32$, are relevant; see also Theorem 4.3.1:

$$\frac{1}{n^{(2-\alpha)/\alpha}} \approx 8.7 \times 10^{-19}, \quad \frac{1}{|\Phi(z_1)|^n} \approx 2.7 \times 10^{-2},$$
$$\frac{1}{n^{(2-\alpha)(1+1/\alpha)}} \approx 1.4 \times 10^{-21}, \quad \frac{1}{|\Phi(z_2)|^n} \approx 1.0 \times 10^{-10}.$$

The estimates in the first line indicate that for $n = 32$ (even for bigger values of n), the dominant term in the errors (6.2.16) and (6.2.17) is $c_2 \frac{1}{R^n}$. As it is suggested by the

	BKM: $\varrho \approx 1.732$	
n	$E_{n,2}(K,G)$	ϱ_n
4	2.7e-01	-
8	4.0e-02	1.92
12	5.3e-03	1.83
16	6.7e-04	1.80
20	8.3e-05	1.78
24	1.0e-05	1.78
28	1.2e-06	1.77
32	1.4e-07	1.76
36	1.7e-08	1.75

Table 6.2.3: BKM approximations to K: Lens-shaped, Case (ii).

estimate in the second line, we use in our BKM/AB approximations only the singular function $[z/(z^2 - z_1^2)]'$, which takes care of the two symmetric poles at z_1 and z_2, and we include no basis functions reflecting the corner singularities of f_0 on Γ. Then, from Theorem 4.3.1 we have for the resulting approximations that

$$\widetilde{E}_{n,2}(K,G) \le c_1 \frac{1}{n^{12}} + c_2 \frac{1}{R^n}, \tag{6.2.18}$$

$$\widetilde{E}_{n,\infty}(f_0,G) \leqslant c_3 \sqrt{\log n} \frac{1}{n^{12}} + c_4 \frac{1}{R^n}, \tag{6.2.19}$$

where $1 < R < |\Phi(z_2)|$.

Below, we present numerical results that illustrate the rates in (6.2.16)–(6.2.19). In presenting the numerical results we use the following notation:

- ϱ : This denotes the order of approximation (the base of n) in the errors (6.2.16)–(6.2.19).

- ϱ_n : This denotes the estimate of ϱ, corresponding to n, and is determined as follows: With E_n denoting any of the four errors $E_{n,2}(K,G)$, $\widetilde{E}_{n,2}(K,G)$, $E_{n,\infty}(f_0,G)$ or

n	BKM: $\varrho \approx 1.732$		
	$E_{n,\infty}(f_0, G)$	$\varrho_n^\star$	ϱ_n
4	1.3e-01	-	-
8	1.6e-02	1.685	2.11
12	1.8e-03	1.718	1.95
16	2.0e-04	1.729	1.89
20	2.3e-05	1.732	1.83
24	2.5e-06	1.732	1.83
28	2.8e-07	1.732	1.81
32	3.1e-08	1.732	1.80
36	3.4e-09	1.732	1.79

Table 6.2.4: BKM approximations to f_0: Lens-shaped, Case (ii).

$\widetilde{E}_{n,\infty}(f_0, G)$ we assume that

$$E_n \approx c\frac{1}{\varrho^n}, \tag{6.2.20}$$

and seek to estimate ϱ by means of the formula,

$$\varrho_n = \left(\frac{E_{n-4}}{E_n}\right)^{\frac{1}{4}}. \tag{6.2.21}$$

The results quoted in Tables 6.2.5 and 6.2.6, show the remarkable approximation achieved by the BKM/AB by using as little as 32 monomials. Moreover, they highlight the significance of Theorem 4.3.1, as it is compared to the estimate (1.3.16), in the sense that they confirm fully the theoretical prediction that the two poles at z_1 and z_2 are the most serious singularities of f_0 for small values of n; see also Remark 4.2.1.

n	BKM: $\varrho \approx 1.119$		BKM/AB: $\varrho \approx 2.055$	
	$E_{n,2}(K,G)$	ϱ_n	$\widetilde{E}_{n,2}(K,G)$	ϱ_n
4	2.8819	-	7.3e-02	-
8	2.3812	1.049	5.6e-03	1.898
12	1.3864	1.145	3.9e-04	1.934
16	0.9188	1.108	2.6e-05	1.965
20	0.5961	1.114	1.7e-06	1.974
24	0.3812	1.118	1.1e-07	1.982
28	0.2413	1.121	7.2e-09	1.981
32	0.1538	1.119	4.6e-10	1.992

Table 6.2.5: BKM approximations to K: Lens-shaped, Case (iii).

6.2.2 Circular sector

Let G_α denote the symmetric circular sector of radius 2 and opening angle $\alpha\pi$, $0 < \alpha < 2$, at the origin, i.e.,

$$G_\alpha := \{z : |z| < 2,\ -\alpha\pi/2 < \arg z < \alpha\pi/2\}.$$

Let f_0 denote the normalized conformal map from G_α onto $D(0, r_0)$, with $f_0(1) = 0$ and $f_0'(1) = 1$. For each value of the parameter α the conformal map $f_0(z)$ can be computed by means of the transformations (see [16, p. 532]):

$$f_0(z) = \left[\frac{2\alpha(4^{1/\alpha} - 1)}{4^{1/\alpha} + 1}\right] \frac{t - d}{td - 1}, \tag{6.2.22}$$

where

$$t = \left(\frac{iz^{1/\alpha} + 2^{1/\alpha}}{iz^{1/\alpha} - 2^{1/\alpha}}\right)^2 \quad \text{and} \quad d = \left(\frac{i + 2^{1/\alpha}}{i - 2^{1/\alpha}}\right)^2. \tag{6.2.23}$$

This gives

$$r_0 = \frac{2\alpha(4^{1/\alpha} - 1)}{4^{1/\alpha} + 1} \quad \text{and} \quad K(1,1) = \frac{1}{\pi}\left(\frac{4^{1/\alpha} + 1}{2\alpha(4^{1/\alpha} - 1)}\right)^2. \tag{6.2.24}$$

The normalized exterior map $\Phi : \overline{\mathbb{C}} \backslash \overline{G}_\alpha \to \Delta$ is given, as can be easily verified, by the

n	BKM: $\varrho \approx 1.119$		BKM/AB: $\varrho \approx 2.055$	
	$E_{n,\infty}(f_0, G)$	ϱ_n	$\widetilde{E}_{n,\infty}(f_0, G)$	ϱ_n
4	0.8820	-	1.2e-02	-
8	0.3817	1.232	6.1e-04	2.101
12	0.2044	1.170	3.4e-05	2.053
16	0.1180	1.147	2.0e-06	2.029
20	0.0702	1.139	1.2e-07	2.028
24	0.0424	1.135	7.0e-09	2.028
28	0.0259	1.131	4.1e-10	2.028
32	0.0160	1.128	2.5e-11	2.028

Table 6.2.6: BKM approximations to f_0: Lens-shaped, Case (iii).

composition of the following three transformations:

$$\xi(z) := \frac{i(2^{1-1/\alpha}z^{1/\alpha} - 2i)}{2^{1-1/\alpha}z^{1/\alpha} + 2i}, \quad \arg z \in (-\pi, \pi], \tag{6.2.25}$$

$$t(\xi) := \xi^{2/3}, \quad \arg \xi \in (-\pi/2, 3\pi/2], \tag{6.2.26}$$

$$w(t) := \frac{1 - e^{i\pi/3}t}{t - e^{i\pi/3}}. \tag{6.2.27}$$

We consider separately the following two cases:

(i) $\alpha = 1$ (half-disk);

(ii) $\alpha = 3/2$ (three-quarter disk).

Case (i): When $\alpha = 1$, then the domain G_α is the half-disk

$$G_1 = \{z : |z| < 2, \Re z > 0\}.$$

In this case (see Figure (6.4)) the conformal map f_0 has an analytic continuation across Γ into Ω. The nearest singularities of f_0 in Ω, are the two simple poles at $z_1 = -1$ and $z_2 = 4$, where $|\Phi(z_1)| \approx 1.452$ and $|\Phi(z_2)| \approx 2.212$. Accordingly, in our experiments we

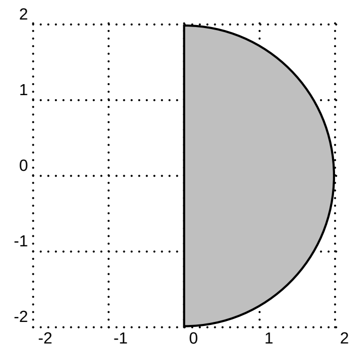

Figure 6.4: Half-disk.

use the singular function $[1/(z - z_1)]'$, which cancels out the nearest pole at z_1. This case is similar to the lens-shaped domain with $\alpha = 1/2$. Hence, the errors $E_{n,2}(K, G)$, $E_{n,\infty}(f_0, G)$, $\widetilde{E}_{n,2}(K, G)$ and $\widetilde{E}_{n,\infty}(f_0, G)$ satisfy respectively (6.2.6), (6.2.7), (6.2.8) and (6.2.9). Our purpose here, is to illustrate that the error bounds in (6.2.6)–(6.2.9) reflect the actual errors. We do so by computing estimates to ϱ_n and $\varrho_n^\star$ of ϱ by using (6.2.10)–(6.2.15).

In Table 6.2.7, the results associated with the errors $E_{n,2}(K, G)$ and $\widetilde{E}_{n,2}(K, G)$ indicate clearly the convergence of ϱ_n to ϱ. Regarding the errors $E_{n,\infty}(f_0, G)$ and $\widetilde{E}_{n,\infty}(f_0, G)$, the results of Table 6.2.8 show that $\varrho_n^\star$ converges faster to ϱ than ϱ_n. This suggest a behavior of the type (6.2.14) for $E_{n,\infty}(f_0, G)$ and $\widetilde{E}_{n,\infty}(f_0, G)$. In both tables the numbers confirm the remarkable advantage of the BKM/AB over the BKM.

Case (ii): In this case f_0 (see Figure (6.5)) has a branch point singularity at the point $\tau_1 = 0$ with

$$f_0(z) = f_0(0) + \sum_{j=1}^{\infty} a_j z^{j/\alpha}, \quad a_1 \neq 0,$$

valid for z close to 0. The nearest singularity of f_0 in Ω is a simple pole at $z_1 = 4$, where $|\Phi(z_1)| \approx 2.04$.

For the application of BKM, Theorem 4.1.1 gives that

$$E_{n,2}(K, G) \leqslant c_1 \frac{1}{n^{1/3}} + c_2 \frac{1}{R^n}, \tag{6.2.28}$$

and

$$E_{n,\infty}(f_0, G) \leqslant c_3 \sqrt{\log n} \frac{1}{n^{1/3}} + c_4 \frac{1}{R^n}, \tag{6.2.29}$$

	BKM: $\varrho \approx 1.452$		BKM/AB: $\varrho \approx 2.212$	
n	$E_{n,2}(K,G)$	ϱ_n	$\widetilde{E}_{n,2}(K,G)$	ϱ_n
5	7.1e-02	-	2.8e-02	-
10	1.6e-02	1.55	7.2e-04	2.39
15	2.8e-03	1.54	1.7e-05	2.29
20	5.1e-04	1.49	3.6e-07	2.29
25	8.7e-05	1.49	7.6e-09	2.26
30	1.5e-05	1.48	1.6e-10	2.24
35	2.4e-06	1.47	3.2e-12	2.24
40	4.0e-07	1.47	6.4e-14	2.24
45	6.6e-08	1.47	1.3e-15	2.23
50	1.1e-08	1.46	2.6e-17	2.23

Table 6.2.7: BKM approximations to K: Half-disk.

where $1 < R < |\Phi(z_1)|$.

Since $1/|\Phi(z_1)|^{50} \approx 3.3 \times 10^{-16}$, and in view of Theorem 4.3.1, we include in our basis only singular functions that reflect the branch point singularity of f_0 at τ_1. More precisely, in order to keep the contribution of both sources of error balanced, we choose to use the first 15 singular function of the form $z^{j/\alpha-1}$, where $j/\alpha \notin \mathbb{N}$. This gives $s^\star = 23/3$ in Theorem 4.3.1, and hence the following estimates for the errors in the resulting BKM/AB approximations,

$$\widetilde{E}_{n,2}(K,G) \le c_1 \frac{1}{n^{23/3}} + c_2 \frac{1}{R^n}, \tag{6.2.30}$$

and

$$\widetilde{E}_{n,\infty}(f_0,G) \leqslant c_3 \sqrt{\log n} \frac{1}{n^{23/3}} + c_4 \frac{1}{R^n}, \tag{6.2.31}$$

where $1 < R < |\Phi(z_1)|$.

Below, we present numerical results that illustrate the rates in (6.2.30)–(6.2.31), where we use the following notation:

n	BKM: $\varrho \approx 1.452$			BKM/AB: $\varrho \approx 2.212$		
	$E_{n,\infty}(f_0, G)$	$\varrho_n^\star$	ϱ_n	$\widetilde{E}_{n,\infty}(f_0, G)$	$\varrho_n^\star$	ϱ_n
5	1.1e-01	-	-	4.0e-02	-	-
10	2.2e-02	1.401	1.64	7.7e-04	2.20	2.62
15	3.4e-03	1.446	1.60	1.5e-05	2.20	2.42
20	5.3e-04	1.450	1.55	2.8e-07	2.22	2.37
25	8.3e-05	1.452	1.53	5.2e-09	2.22	2.34
30	1.3e-05	1.452	1.51	9.8e-11	2.21	2.31
35	2.0e-06	1.452	1.51	1.8e-12	2.22	2.30
40	3.1e-07	1.452	1.50	3.5e-14	2.20	2.27
45	4.8e-08	1.452	1.49	6.6e-16	2.21	2.27
50	7.4e-09	1.452	1.49	1.2e-17	2.22	2.27

Table 6.2.8: BKM approximations to f_0: Half-disk.

- σ : This denotes the exponent of $1/n$ in the errors (6.2.30)–(6.2.31).
- σ_n : This denotes the estimate of σ corresponding to n, and is determined as follows: With E_n denoting any of the two errors $E_{n,2}(K, G)$, $\widetilde{E}_{n,2}(K, G)$, we assume that

$$E_n \approx c\frac{1}{n^\sigma} \tag{6.2.32}$$

and seek to estimate σ by means of the formula

$$\sigma_n = \log\left(\frac{E_{n-5}}{E_n}\right) / \log\left(\frac{n}{n-5}\right). \tag{6.2.33}$$

If E_n denotes either of the two errors $E_{n,\infty}(f_0, G)$ or $\widetilde{E}_{n,\infty}(f_0, G)$, then we assume that

$$E_n \approx c\sqrt{\log n}\frac{1}{n^\sigma}, \tag{6.2.34}$$

and seek to estimate σ by means of the formula

$$\sigma_n = \frac{\log\left(\frac{E_{n-5}}{E_n}\right) - \frac{1}{2}\log\left[\frac{\log(n-5)}{\log n}\right]}{\log\left(\frac{n}{n-5}\right)}. \tag{6.2.35}$$

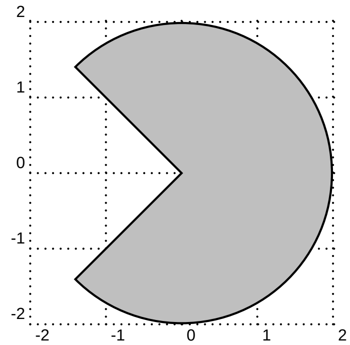

Figure 6.5: 3/4-disk.

In addition, we check a behavior of the form (6.2.20) for the errors $\widetilde{E}_{n,2}(K,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$, by computing ϱ_n as in (6.2.21), with 5 in the place of 4.

Our purposes here is to show that the change of the dominant term in both (6.2.30) and (6.2.31) can actually be detected in the computed errors. This is indeed the case in the results quoted in Table 6.2.9. More precisely, the results associated with the errors $\widetilde{E}_{n,2}(K,G)$ and $\widetilde{E}_{n,\infty}(f_0,G)$ indicate the convergence of ϱ_n to ϱ for values of n up to 50 and the convergence of σ_n to σ for values larger than 50. Furthermore, the results show that the two constants c_1 and c_2 in (6.2.30) and c_3 and c_4 in (6.2.31) are, respectively, of the same magnitude.

6.3 Rates of decrease of the Bergman polynomials.

First, we present results illustrating the rate of decrease of the sequence $\{P_n(1)\}$ for the circular sector considered in Section 6.2.2, with $\alpha = 2/5$ (see Figure (6.6)). In this case, the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{2i\pi/5}$, $z_2 = e^{-2i\pi/5}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.145$.

From the proof of Corollary 3.2.1 and (6.2.28) we have that

$$|P_n(z_0)| \leq \|K(\cdot,z_0) - K_n(\cdot,z_0)\|_{L^2(G)} \leq c_1\frac{1}{n^{(2-\alpha)/\alpha}} + c_2\frac{1}{R^n}, \tag{6.3.1}$$

where $1 < R < |\Phi(z_1)|$, Accordingly, we check to detect the decay in the following two forms:

$$|P_n(1)| \approx c\frac{1}{\varrho^n} \quad \text{and} \quad |P_n(1)| \approx c\frac{1}{n^{\sigma}}, \tag{6.3.2}$$

n	BKM/AB: $\sigma \approx 7.67$ $\varrho \approx 2.04$					
	$\widetilde{E}_{n,2}(K,G)$	σ_n	ϱ_n	$\widetilde{E}_{n,\infty}(f_0,G)$	σ_n	ϱ_n
20	7.2e-05	-	-	8.2e-05	-	-
25	1.6e-05	6.74	1.35	1.5e-05	7.62	1.40
30	2.9e-06	9.31	1.41	2.6e-06	9.70	1.42
35	2.2e-07	16.84	1.67	1.8e-07	17.37	1.71
40	1.0e-08	22.81	1.84	7.7e-09	23.44	1.86
45	4.1e-10	27.41	1.90	2.8e-10	28.17	1.94
50	1.3e-11	32.83	1.99	1.0e-11	31.25	1.95
55	7.5e-12	5.84	1.12	5.3e-12	7.05	1.14
60	2.6e-12	11.84	1.23	2.0e-12	11.20	1.22
65	1.3e-12	8.68	1.15	9.9e-13	8.73	1.15
70	7.4e-13	7.50	1.12	5.9e-13	7.04	1.11
75	4.4e-13	7.58	1.11	3.5e-13	7.57	1.11
80	2.7e-13	7.66	1.10	2.1e-13	7.62	1.10

Table 6.2.9: BKM approximations to f_0 and K: 3/4-disk.

with $\varrho = |\Phi(z_1)|$ and, in view of the remark made in [16, pp. 530–531], $\sigma = (2-\alpha)/\alpha + 1/2$. We do so, by estimating ϱ and σ, respectively, by means of the formulas

$$\varrho_n = \left(\frac{|P_{n-10}(1)|}{|P_n(1)|} \right)^{\frac{1}{10}}, \tag{6.3.3}$$

and

$$\sigma_n = \log\left(\frac{|P_{n-10}(1)|}{|P_n(1)|} \right) / \log\left(\frac{n}{n-10} \right). \tag{6.3.4}$$

The results listed in Table 6.3.10 show clearly the transition from one dominant term to the other in (6.3.1) for values of n around 50.

We end, by presenting results that illustrate the rate of decrease of the augmented sequence $\{\widetilde{P}_n(1)\}$, for the circular sector considered in Section 6.2.2, where now we consider the two cases $\alpha = 3/4$ and $\alpha = 4/5$ (see Figures (6.7) and (6.8)). When $\alpha = 3/4$,

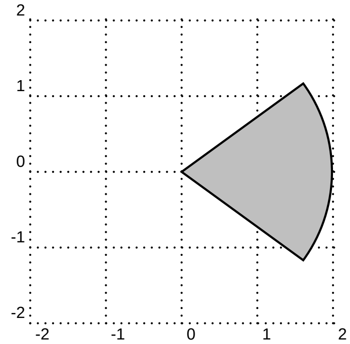

Figure 6.6: Circular sector, $\alpha = 2/5$

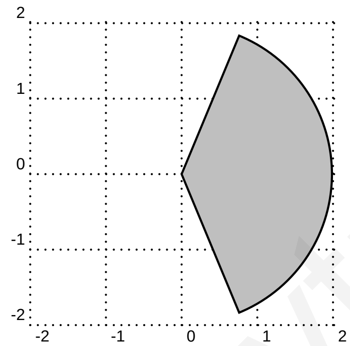

Figure 6.7: Circular sector, $\alpha = 3/4$

then the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{3i\pi/4}$ and $z_2 = e^{-3i\pi/4}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.349$. When $\alpha = 4/5$, then the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{4i\pi/5}$ and $z_2 = e^{-4i\pi/5}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.372$.

In both cases, we construct the sequence $\{\widetilde{P}_n(z)\}$ by augmenting the monomial basis functions with the singular function $z^{1/\alpha-1}$, which reflects the branch point singularity of f_0 at $\tau_1 = 0$, and we seek to detect the decay of the sequence $\{\widetilde{P}_n(1)\}$ in the form

$$|\widetilde{P}_n(1)| \approx c\frac{1}{n^{\sigma}},$$

where, in view of Theorem 4.3.1 and [16, pp. 530–531], $\sigma = 2(2-\alpha)/\alpha + 1/2$. As above, we estimate σ by means of the formula

$$\sigma_n = \log\left(\frac{|\widetilde{P}_{n-10}(1)|}{|\widetilde{P}_n(1)|}\right) / \log\left(\frac{n}{n-10}\right). \tag{6.3.5}$$

The results listed in Tables 6.3.11 and 6.3.12 indicate clearly the convergence of σ_n to

n	$\sigma = 4.5$		$\varrho \approx 1.145$
	$\lvert P_n(1)\rvert$	σ_n	ϱ_n
10	2.6e-02	-	-
20	1.2e-03	4.51	1.37
30	7.6e-06	12.38	1.65
40	1.7e-06	5.14	1.16
50	4.0e-07	6.57	1.16
60	1.8e-07	4.35	1.08
70	9.1e-08	4.49	1.07
80	5.0e-08	4.50	1.06
90	2.9e-08	4.50	1.05
100	1.8e-08	4.50	1.05

Table 6.3.10: Rate of decrease of $|P_n(1)|$: Circular sector, $\alpha = 2/5$.

the predicted value of σ, indicating that the argument in [16, pp. 530–531] applies also to the case of the augmented Bergman polynomials.

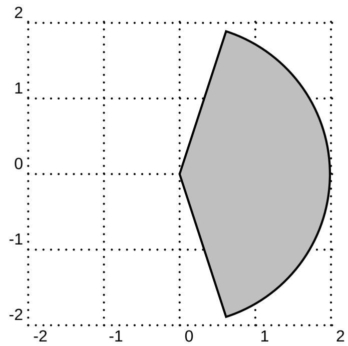

Figure 6.8: Circular sector, $\alpha = 4/5$

	$\sigma \approx 3.833$	
n	$\lvert\widetilde{P}_n(1)\rvert$	σ_n
10	2.8e-03	-
20	7.2e-05	5.30
30	1.1e-05	4.60
40	3.2e-06	4.36
50	1.3e-06	3.86
60	6.6e-07	3.89
70	3.6e-07	3.89
80	2.2e-07	3.88
90	1.4e-07	3.88
100	9.1e-08	3.87

Table 6.3.11: Rate of decrease of $|\widetilde{P}_n(1)|$: Circular sector, $\alpha = 3/4$.

	$\sigma = 3.5$	
n	$\lvert\widetilde{P}_n(1)\rvert$	σ_n
10	2.3e-03	-
20	2.7e-04	3.10
30	2.2e-05	6.17
40	8.7e-06	3.20
50	3.8e-06	3.72
60	2.0e-06	3.66
70	1.1e-06	3.63
80	6.9e-07	3.61
90	4.5e-07	3.59
100	3.1e-07	3.58

Table 6.3.12: Rate of decrease of $|\widetilde{P}_n(1)|$: Circular sector, $\alpha = 4/5$.

6.4 BKM and BKM/AB approximation of the conformal radius r_0

First, we present results illustrating the rate of convergence of the sequence $\{r_n\}$ and $\{\widetilde{r}_n\}$ to the conformal radius r_0 for the lens-shade domain considered in Section 6.2, where we consider the two cases (i) and (ii). We recall that when $a = \pi/6$, then the two nearest singularities of f_0 in Ω are the two simple poles at $z_1 = -\sqrt{3}/3$ and $z_2 = \sqrt{3}$, where $|\Phi(z_1)| \approx 1.347$ and $|\Phi(z_2)| \approx 2.532$. Accordingly, in our experiments, we use the singular function $[1/(z - z_1)]'$. This cancels out the nearest singularity at z_1. In the symmetric case, where $a = b = \pi/4$, the only singularities of f_0 are the two simple poles at $z_1 = -1$ and $z_2 = 1$, where $|\Phi(z_1)| = |\Phi(z_2)| = \sqrt{3}$. In this case, we use the singular function $[z/(z^2 - z_1^2)]'$, which takes care of both poles at z_1 and z_2.

We recall from Corollary 3.2.2 and 3.3.1 the two estimates,

$$E_n(r_0, G) \preceq \frac{n^2}{|\Phi(z_1)|^{2n}}, \tag{6.4.1}$$

and

$$\widetilde{E}_n(r_0, G) \preceq \frac{n^2}{|\Phi(z_2)|^{2n}}. \tag{6.4.2}$$

Below, we present numerical results that illustrate the laws of the above errors and rates. In presenting the numerical results we use the following notation:

- ϱ : This denotes the order of approximation (the base of n) in the errors (6.4.1) and (6.4.2).
- ϱ_n : This denotes the estimate of ϱ, corresponding to n, and is determined as follows: With E_n denoting any of the two errors $E_n(r_0, G)$ or $\widetilde{E}_n(r_0, G)$, we assume that

  $$E_n \approx c\frac{n^2}{\varrho^n}, \tag{6.4.3}$$

  and seek to estimate ϱ by means of the formula,

  $$\varrho_n = \left(\frac{n^2}{(n-m)^2}\frac{E_{n-m}}{E_n}\right)^{\frac{1}{m}}. \tag{6.4.4}$$

  (Here we take $m = 4$, or $m = 5$.)

	BKM: $\varrho \approx 1.814$	
n	$E_n(r_0, G)$	ϱ_n
5	2.7e-02	-
10	2.2e-03	2.19
15	1.6e-04	1.98
20	1.0e-05	1.93
25	6.3e-07	1.90
30	3.8e-08	1.87
35	2.2e-09	1.86

Table 6.4.13: BKM approximations to r_0: Lens-shaped, Case (i).

The presented results show clearly the advantage of the BKM/AB over the BKM. In addition, they indicate a close agreement between the theoretical and the computed order of approximation. In Tables 6.4.13 - 6.4.16, the results associated with the errors $E_n(r_0, G)$ and $\widetilde{E}_n(r_0, G)$ indicate the convergence of ϱ_n to ϱ. As it is predicted by Remark 3.3.1, in Case (ii) the error $\widetilde{E}_n(r_0, G)$ vanish. This was testified in our experiments, in the sense that the computed error $\widetilde{E}_n(r_0, G)$ was zero within machine precision, see Table 6.4.16.

We end this section by presenting results illustrating the rate of convergence of the sequence $\{r_n\}$ and $\{\widetilde{r}_n\}$ to the conformal radius r_0 for the circular sector considered in Section 6.2, where now we consider the two cases $\alpha = 3/4$ and $\alpha = 4/5$. When $\alpha = 3/4$, then the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{3i\pi/4}$ and $z_2 = e^{-3i\pi/4}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.349$ and the next singularity occurs at a point $z_3 = 4$, where $|\Phi(z_3)| \approx 2.866$. When $\alpha = 4/5$, then the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{4i\pi/5}$ and $z_2 = e^{-4i\pi/5}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.372$ and the next singularity occurs at a point $z_3 = 4$, where $|\Phi(z_3)| \approx 2.687$. In both cases, we augment the monomial basis functions with the singular function $z^{1/\alpha - 1}$, which reflects the branch point singularity of f_0 at $\tau_1 = 0$,

	BKM/AB: $\varrho \approx 6.411$	
n	$\widetilde{E}_n(r_0, G)$	ϱ_n
5	9.3e-05	-
10	1.6e-08	7.45
15	2.2e-12	6.96
20	2.7e-16	6.76
25	3.1e-20	6.70
30	3.5e-24	6.65
35	3.7e-28	6.60

Table 6.4.14: BKM/AB approximations to r_0: Lens-shaped, Case (i).

Therefore, from Corollary 4.1.1 we have respectively for $\alpha = 3/4$ and $\alpha = 4/5$ that,

$$E_n(r_0, G) \leqslant \left(c_1 \frac{1}{n^{5/3}} + c_2 \frac{1}{R^n} \right)^2, \tag{6.4.5}$$

and

$$E_n(r_0, G) \leqslant \left(c_3 \frac{1}{n^{3/2}} + c_4 \frac{1}{R^n} \right)^2, \tag{6.4.6}$$

where $1 < R < |\Phi(z_1)|$. In order to decide which singular functions to include in the BKM/AB the following estimates, valid for $n = 60$, are relevant; for $\alpha = 3/4$

$$\frac{1}{n^{(2-\alpha)/\alpha}} \approx 1.1 \times 10^{-3}, \quad \frac{1}{|\Phi(z_1)|^n} \approx 1.6 \times 10^{-8},$$
$$\frac{1}{n^{(2-\alpha)(2/\alpha)}} \approx 1.2 \times 10^{-6}, \quad \frac{1}{|\Phi(z_3)|^n} \approx 3.7 \times 10^{-28}.$$

and for $\alpha = 4/5$

$$\frac{1}{n^{(2-\alpha)/\alpha}} \approx 2.1 \times 10^{-3}, \quad \frac{1}{|\Phi(z_1)|^n} \approx 5.7 \times 10^{-9},$$
$$\frac{1}{n^{(2-\alpha)(2/\alpha)}} \approx 4.6 \times 10^{-6}, \quad \frac{1}{|\Phi(z_2)|^n} \approx 1.8 \times 10^{-26}.$$

The estimates in the first line indicate that for $n = 60$ (even for smaller values of n), the dominant term in the errors (6.4.5) and (6.4.6) is $c_2 \frac{1}{n^{(2-\alpha)/\alpha}}$. As it is suggested

	BKM: $\varrho = 3$	
n	$E_n(r_0, G)$	ϱ_n
4	1.5e-02	-
8	3.1e-04	3.72
12	5.5e-06	3.34
16	8.9e-08	3.24
20	1.3e-09	3.16
24	2.0e-11	3.13
28	2.8e-13	3.11
32	4.0e-15	3.09
36	5.5e-17	3.08

Table 6.4.15: BKM approximations to r_0: Lens-shaped, Case (ii).

by the estimate in the second line, we use in our BKM/AB approximations only the singular function $z^{1/\alpha-1}$, which reflects the branch point singularity of f_0 at $\tau_1 = 0$, and we include no basis functions reflecting the pole singularities of f_0 at z_1 and z_2. Then, from Corollary 4.3.1 we have for the resulting approximations that

$$\widetilde{E}_n(r_0, G) \leq \left(c_1 \frac{1}{n^{10/3}} + c_2 \frac{1}{R^n} \right)^2, \tag{6.4.7}$$

$$\widetilde{E}_n(r_0, G) \leq \left(c_3 \frac{1}{n^3} + c_4 \frac{1}{R^n} \right)^2, \tag{6.4.8}$$

where $1 < R < |\Phi(z_2)|$.

Below, we present numerical results that illustrate the rates in (6.4.5) – (6.4.8). In presenting the numerical results we use the following notation:

- σ : This denotes the exponent of $1/n$ in the errors (6.4.5) – (6.4.8).
- σ_n : This denotes the estimate of σ, corresponding to n, and is determined as follows: With E_n denoting any of the two errors $E_n(r_0, G)$ or $\widetilde{E}_n(r_0, G)$, we assume that

$$E_n \approx c \frac{1}{n^\sigma}, \tag{6.4.9}$$

	BKM/AB:	
n	$E_n(r_0,G)$	ϱ_n
4	1.0e-64	-
8	1.0e-64	-
12	1.0e-64	-
16	1.0e-64	-
20	1.0e-64	-
24	1.0e-64	-
28	1.0e-64	-
32	1.0e-64	-
36	1.0e-64	-

Table 6.4.16: BKM/AB approximations to r_0: Lens-shaped, Case (ii).

As above, we estimate σ by means of the formula

$$\sigma_n = \log\left(\frac{E_{n-10}}{\widetilde{E}_n}\right)/\log\left(\frac{n}{n-10}\right). \tag{6.4.10}$$

The presented results show clearly the advantage of the BKM/AB over the BKM. In addition, they indicate a close agreement between the theoretical and the computed order of approximation. In Tables 6.4.17 - 6.4.20, the results associated with the errors $E_n(r_0,G)$ and $\widetilde{E}_n(r_0,G)$ indicate the convergence of σ_n to σ.

	BKM: $\sigma = 10/3$	
n	$E_n(r_0, G)$	σ_n
10	9.9e-04	-
20	2.3e-05	5.44
30	6.2e-06	3.24
40	2.3e-06	3.36
50	1.1e-06	3.34
60	6.0e-07	3.34
70	3.6e-07	3.34
80	2.3e-07	3.34
90	1.6e-07	3.33
100	1.1e-07	3.33

Table 6.4.17: BKM approximations to r_0: Circular sector, $\alpha = 3/4$.

	BKM/AB: $\sigma = 20/3$	
n	$\widetilde{E}_n(r_0, G)$	σ_n
10	4.4e-01	-
20	1.8e-04	12.78
30	2.6e-08	8.26
40	1.1e-10	7.22
50	2.6e-11	6.68
60	7.5e-12	6.72
70	2.7e-12	6.71
80	1.1e-12	6.70
90	4.9e-13	6.69
100	2.4e-13	6.69

Table 6.4.18: BKM/AB approximations to r_0: Circular sector, $\alpha = 3/4$.

	BKM: $\sigma = 3$	
n	$E_n(r_0, G)$	σ_n
10	1.2e-03	-
20	4.2e-05	5.34
30	1.2e-05	3.14
40	4.9e-06	3.10
50	2.5e-06	3.06
60	1.4e-06	3.04
70	8.9e-07	3.04
80	5.9e-07	3.02
90	4.1e-07	3.02
100	3.0e-07	3.01

Table 6.4.19: BKM approximations to r_0: Circular sector, $\alpha = 4/5$.

	BKM/AB: $\sigma = 6$	
n	$E_n(r_0, G)$	σ_n
10	8.3e-05	-
20	1.0e-07	9.64
30	5.6e-09	7.20
40	9.6e-10	6.12
50	2.4e-10	6.28
60	7.6e-11	6.20
70	3.0e-11	6.16
80	1.3e-11	6.14
90	6.3e-12	6.12
100	3.3e-12	6.10

Table 6.4.20: BKM/AB approximations to r_0: Circular sector, $\alpha = 4/5$.

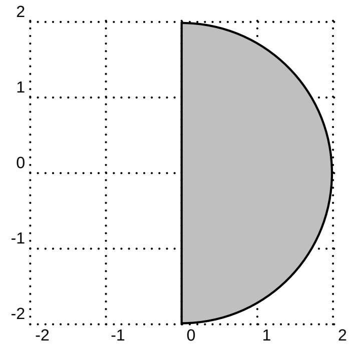

Figure 6.9: Half-disk.

6.5 BKM and BKM/AB approximation inside the domain

6.5.1 Example: Half disk

First, we present results illustrating the rate of convergence of the Bieberbach polynomials $\{\pi_n(z)\}$ and $\{\widetilde{\pi}_n(z)\}$ to the conformal map $f_0(z)$ and the kernel polynomials $K_n(z, z_0)$ and $\widetilde{K}_n(z, z_0)$ to $K(z, z_0)$ interior to the half disk considered in Section 6.2. (see Figure 6.9) As in Section 6.2 we set $z_0 = 1$ and choose $z = 0.25$. We recall that, in this case both the conformal maps f_0 and f_z have an analytic continuation across Γ into Ω and that the nearest singularities of f_0 in Ω are the two simple poles at $z_1 = -1$ and $z_2 = 4$, where $|\Phi(z_1)| \approx 1.452$ and $|\Phi(z_2)| \approx 2.212$. Accordingly, in our experiments we use the singular function $[1/(z - z_1)]'$, which cancels out the nearest pole at z_1. Let now $\gamma = [z, z_0]$. The point on γ for which the $\max\limits_{\zeta\in\gamma}\|K(\cdot,\zeta) - K_n(\cdot,\zeta)\|_{L^2(G)}$ occurs is z. The nearest singularities of f_z are the two simple poles at $z_1^* = -0.25$ and $z_2^* = 16$, where $|\Phi(z_1^*)| \approx 1.101$ and $|\Phi(z_2^*)| \approx 9.922$.

We recall from Theorems 5.1.1 and 5.2.1 the estimates,

$$E_n(f_0, G) \leq c_1 |\gamma(z, z_0)| \frac{n^2}{|\Phi(z_1)\Phi(z_1^*)|^n} + c_2 \frac{n^2}{|\Phi(z_1)|^{2n}}, \quad n \in \mathbb{N}, \tag{6.5.1}$$

$$E_n(K, G) \leq c_3 \frac{n^2}{|\Phi(z_1)\Phi(z_1^*)|^n}, \quad n \in \mathbb{N}, \tag{6.5.2}$$

and

$$\widetilde{E}_n(f_0, G) \leq c_1|\gamma(z, z_0)|\frac{n^2}{|\Phi(z_2)\Phi(z_1^*)|^n} + c_2\frac{n^2}{|\Phi(z_2)|^{2n}}, \quad n \in \mathbb{N}, \tag{6.5.3}$$

$$\widetilde{E}_n(K, G) \leq c_3\frac{n^2}{|\Phi(z_2)\Phi(z_1^*)|^n}, \quad n \in \mathbb{N}. \tag{6.5.4}$$

Below, we present numerical results that illustrate the laws of the above errors and rates. In presenting the numerical results we use the following notation:

- ϱ : This denotes the order of approximation (the base of n) in the errors (6.5.1) - (6.5.4). $\left(\text{Note that } \frac{n^2}{|\Phi(z_1)\Phi(z_1^*)|^n} > \frac{n^2}{|\Phi(z_1)|^{2n}}\right)$
- ϱ_n : This denotes the estimate of ϱ, corresponding to n, and is determined as follows: With E_n denoting any of the errors $E_n(f_0, G)$, $\widetilde{E}_n(f_0, G)$, $E_n(K, G)$, $\widetilde{E}_n(K, G)$ we assume that

$$E_n \approx c\frac{n^2}{\varrho^n} \tag{6.5.5}$$

  and seek to estimate ϱ by means of the formula,

$$\varrho_n = \left(\frac{n^2}{(n-m)^2}\frac{E_{n-m}}{E_n}\right)^{\frac{1}{m}}. \tag{6.5.6}$$

The presented results show clearly the advantage of the BKM/AB over the BKM also in the interior of G. In addition, they indicate a close agreement between the theoretical and the computed order of approximation. In Tables 6.5.21–6.5.24, the results associated with the errors $E_n(f_0, G)$, $\widetilde{E}_n(f_0, G)$ and $E_n(K, G)$ and $\widetilde{E}_n(K, G)$ indicate the convergence of ϱ_n to ϱ.

	BKM: $z = 0.25$, $\varrho \approx 1.60$	
n	$E_n(f_0, G)$	ϱ_n
5	4.7e-02	-
10	5.9e-03	2.00
15	6.0e-04	1.86
20	5.9e-05	1.79
25	5.7e-06	1.75
30	5.4e-07	1.72
35	5.2e-08	1.70
40	5.0e-09	1.69
45	4.8e-10	1.68
50	4.6e-11	1.67

Table 6.5.21: BKM interior approximations to f_0: Half-disk.

	BKM/AB: $z = 0.25$, $\varrho \approx 2.435$	
$2\ n$	$\widetilde{E}_n(f_0, G)$	ϱ_n
5	3.7e-03	-
10	4.6e-05	3.17
15	6.2e-07	2.79
20	7.6e-09	2.70
25	9.3e-11	2.65
30	1.1e-12	2.61
35	1.3e-14	2.58
40	1.6e-16	2.56
45	1.8e-18	2.55
50	2.2e-20	2.53

Table 6.5.22: BKM/AB interior approximations to f_0: Half-disk.

	BKM: $z = 0.25,\ \varrho \approx 1.60$	
n	$E_n(K,G)$	ϱ_n
5	4.3e-02	-
10	7.1e-03	1.22
15	9.3e-04	1.34
20	1.1e-04	1.36
25	1.3e-05	1.40
30	1.5e-06	1.44
35	1.7e-07	1.46
40	1.8e-08	1.48
45	1.9e-09	1.49
50	2.0e-10	1.50

Table 6.5.23: BKM interior approximations to K: Half-disk.

	BKM/AB: $z = 0.25$, $\varrho \approx 2.435$	
n	$\widetilde{E}_n(K, G)$	ϱ_n
5	2.8e-03	-
10	6.7e-05	2.78
15	1.1e-06	2.67
20	1.7e-08	2.59
25	2.4e-10	2.55
30	3.4e-12	2.53
35	4.6e-14	2.51
40	6.0e-16	2.50
45	7.9e-18	2.49
50	1.0e-19	2.49

Table 6.5.24: BKM/AB interior approximations to f_0: Half-disk.

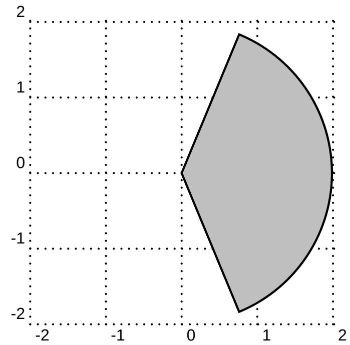

Figure 6.10: Circular sector, $\alpha = 3/4$

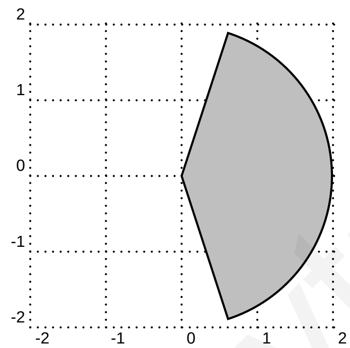

Figure 6.11: Circular sector, $\alpha = 4/5$

6.5.2 Example: Circular sector

In this example we present results illustrating the rate of convergence of the sequence $\{\pi_n(z)\}$ and $\{\widetilde{\pi}_n(z)\}$ to the conformal map $f_0(z)$ and the kernel polynomials $\{K_n(z, z_0)\}$ and $\{\widetilde{K}_n(z, z_0)\}$ to $K(z, z_0)$ interior to the the circular sector considered in Section 6.3, where we consider the two cases $\alpha = 3/4$ and $\alpha = 4/5$. (see Figure 6.10 and 6.11)

As in Section 6.3 we set $z_0 = 1$ and choose $z = 0.5$. We recall that when $\alpha = 3/4$, then the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{3i\pi/4}$ and $z_2 = e^{-3i\pi/4}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.349$ and the next singularity occurs at a point $z_3 = 4$, where $|\Phi(z_3)| \approx 2.866$. Let now $\gamma = [z, z_0]$. The point on γ for which the $\max\limits_{\zeta \in \gamma} \|K(\cdot, \zeta) - K_n(\cdot, \zeta)\|_{L^2(G)}$ occurs is z. The nearest singularities of f_z in Ω are the two simple poles at the symmetric points $z_1^* = \frac{e^{3i\pi/4}}{2}$ and $z_2^* = \frac{e^{-3i\pi/4}}{2}$, with $|\Phi(z_1^*)| = |\Phi(z_2^*)| \approx 1.203$.

When $\alpha = 4/5$, then the nearest singularities of f_0 in Ω are the two simple poles at the symmetric points $z_1 = e^{4i\pi/5}$ and $z_2 = e^{-4i\pi/5}$, where $|\Phi(z_1)| = |\Phi(z_2)| \approx 1.372$ and the next singularity occurs at a point $z_3 = 4$, where $|\Phi(z_3)| \approx 2.687$. For the same z and z_0 as above let $\gamma = [z, z_0]$. The point on γ for which the $\max_{\zeta\in\gamma}\|K(\cdot,\zeta) - K_n(\cdot,\zeta)\|_{L^2(G)}$ occurs is z. In this case the nearest singularities of f_z in Ω are the two simple poles at the symmetric points $z_1^* = \frac{e^{4i\pi/5}}{2}$ and $z_2^* = \frac{e^{-4i\pi/5}}{2}$, with $|\Phi(z_1^*)| = |\Phi(z_2^*)| \approx 1.232$.

Therefore, from Remark 5.3.1 we have respectively for $\alpha = 3/4$ and $\alpha = 4/5$ that,

$$E_n(f_0, G) \leq c_1\frac{1}{n^{10/3}} + c_2\frac{1}{R_1^n n^{5/3}} + c_3\frac{1}{R_2^n n^{5/3}} + c_4\frac{1}{R_1^n R_2^n} + c_5\frac{1}{R_1^{2n}},\ n \in \mathbb{N}, \tag{6.5.7}$$

$$E_n(K, G) \leq c_6\frac{1}{n^{10/3}} + c_7\frac{1}{R_1^n n^{5/3}} + c_8\frac{1}{R_2^n n^{5/3}} + c_9\frac{1}{R_1^n R_2^n},\ n \in \mathbb{N}, \tag{6.5.8}$$

and

$$E_n(f_0, G) \leq c_1\frac{1}{n^3} + c_2\frac{1}{R_1^n n^{3/2}} + c_3\frac{1}{R_2^n n^{3/2}} + c_4\frac{1}{R_1^n R_2^n} + c_5\frac{1}{R_1^{2n}},\ n \in \mathbb{N}, \tag{6.5.9}$$

$$E_n(K, G) \leq c_6\frac{1}{n^3} + c_7\frac{1}{R_1^n n^{3/2}} + c_8\frac{1}{R_2^n n^{3/2}} + c_9\frac{1}{R_1^n R_2^n},\ n \in \mathbb{N}, \tag{6.5.10}$$

where $1 < R_1 < |\Phi(z_1)|$ and $1 < R_2 < |\Phi(z_1^*)|$.

In order to decide which singular functions to include in the BKM/AB the following estimates, valid for $n = 60$, are relevant; for $\alpha = 3/4$

$$\frac{1}{n^{2(2-\alpha)/\alpha}} \approx 1.2 \times 10^{-6}, \quad \frac{1}{n^{(2-\alpha)/\alpha}|\Phi(z_1)|^n} \approx 1.7 \times 10^{-11},$$
$$\frac{1}{n^{(2-\alpha)/\alpha}|\Phi(z_1^*)|^n} \approx 1.6 \times 10^{-8}, \quad \frac{1}{(|\Phi(z_1^*)||\Phi(z_1)|)^n} \approx 2.4 \times 10^{-13},$$
$$\frac{1}{|\Phi(z_1)|^{2n}} \approx 2.5 \times 10^{-16}.$$

and for $\alpha = 4/5$

$$\frac{1}{n^{2(2-\alpha)/\alpha}} \approx 4.6 \times 10^{-6}, \quad \frac{1}{n^{(2-\alpha)/\alpha}|\Phi(z_1)|^n} \approx 1.2 \times 10^{-11},$$
$$\frac{1}{n^{(2-\alpha)/\alpha}|\Phi(z_1^*)|^n} \approx 8.7 \times 10^{-9}, \quad \frac{1}{(|\Phi(z_1^*)||\Phi(z_1)|)^n} \approx 2.3 \times 10^{-14},$$
$$\frac{1}{|\Phi(z_1)|^{2n}} \approx 3.3 \times 10^{-17}.$$

The estimates in the first line indicate that for $n = 60$ (even for smaller values of n), the dominant term in the errors (6.5.7)– (6.5.10) is $c_2\frac{1}{n^{2(2-\alpha)/\alpha}}$. As it is suggested by the estimate in the second line, we use in our BKM/AB approximations only the singular function $z^{1/\alpha-1}$, which reflects the branch point singularity of f_0 at $\tau_1 = 0$, and we not include basis functions reflecting the pole singularities of f_0 at z_1 and z_2. Then, from Theorem 5.3.1 we have respectively for $\alpha = 3/4$ and $\alpha = 4/5$ that,

$$\widetilde{E}_n(f_0, G) \leq c_1\frac{1}{n^{20/3}} + c_2\frac{1}{R_1^n n^{10/3}} + c_3\frac{1}{R_2^n n^{10/3}} + c_4\frac{1}{R_1^n R_2^n} + c_5\frac{1}{R_1^{2n}},\ n \in \mathbb{N}, \tag{6.5.11}$$

$$\widetilde{E}_n(K, G) \leq c_6\frac{1}{n^{20/3}} + c_7\frac{1}{R_1^n n^{10/3}} + c_8\frac{1}{R_2^n n^{10/3}} + c_9\frac{1}{R_1^n R_2^n},\ n \in \mathbb{N}, \tag{6.5.12}$$

and

$$\widetilde{E}_n(f_0, G) \leq c_1\frac{1}{n^6} + c_2\frac{1}{R_1^n n^3} + c_3\frac{1}{R_2^n n^3} + c_4\frac{1}{R_1^n R_2^n} + c_5\frac{1}{R_1^{2n}},\ n \in \mathbb{N}, \tag{6.5.13}$$

$$\widetilde{E}_n(K, G) \leq c_6\frac{1}{n^6} + c_7\frac{1}{R_1^n n^3} + c_8\frac{1}{R_2^n n^3} + c_9\frac{1}{R_1^n R_2^n},\ n \in \mathbb{N}, \tag{6.5.14}$$

where $1 < R_1 < |\Phi(z_2)|$ and $1 < R_2 < |\Phi(z_1^*)|$.

Below, we present numerical results that illustrate the rates in (6.5.7)–(6.5.14). In presenting the numerical results we use the following notation:

- σ : This denotes the exponent of $1/n$ in the errors (6.5.7) – (6.5.14).
- σ_n : This denotes the estimate of σ, corresponding to n, and is determined as follows: With E_n denoting any of the errors $E_n(f_0, G)$, $\widetilde{E}_n(f_0, G)$, $E_n(K, G)$, $\widetilde{E}_n(K, G)$, we assume that

  $$E_n \approx c\frac{1}{n^\sigma} \tag{6.5.15}$$

  As above, we estimate σ by means of the formula

  $$\sigma_n = \log\left(\frac{E_{n-10}}{E_n}\right) / \log\left(\frac{n}{n-10}\right). \tag{6.5.16}$$

The presented results show clearly the advantage of the BKM/AB over the BKM. In addition, they indicate a close agreement between the theoretical and the computed order of approximation. In Tables 6.5.25 - 6.5.32, the results associated with the errors $E_n(f_0, G)$, $\widetilde{E}_n(f_0, G)$ and $E_n(K, G)$, $\widetilde{E}_n(K, G)$ indicate the convergence of σ_n to σ.

	BKM: $z = 0.5,\ \sigma = 10/3$	
n	$E_n(f_0, G)$	σ_n
10	1.5e-03	-
20	3.7e-05	5.31
30	8.0e-06	3.78
40	3.1e-06	3.27
50	1.5e-06	3.38
60	8.0e-07	3.36
70	4.8e-07	3.35
80	3.1e-07	3.35
90	2.1e-07	3.34
100	1.4e-07	3.34

Table 6.5.25: BKM approximations to f_0: Circular sector, $\alpha = 3/4$.

	BKM/AB: $z = 0.5,\ \sigma = 20/3$	
n	$\widetilde{E}_n(f_0, G)$	σ_n
10	1.3e-04	-
20	4.3e-07	8.22
30	2.2e-09	12.99
40	3.2e-10	6.79
50	7.8e-11	6.29
60	2.2e-11	6.88
70	8.0e-12	6.67
80	3.2e-12	6.69
90	1.5e-12	6.70
100	7.3e-13	6.70

Table 6.5.26: BKM/AB approximations to f_0: Circular sector, $\alpha = 3/4$.

	BKM: $z = 0.5,\ \sigma = 10/3$	
n	$E_n(K, G)$	σ_n
10	1.7e-03	-
20	8.5e-05	4.29
30	1.3e-05	4.55
40	5.4e-06	3.14
50	2.6e-06	3.37
60	1.4e-06	3.38
70	8.3e-07	3.36
80	5.3e-07	3.35
90	3.6e-07	3.35
100	2.5e-07	3.34

Table 6.5.27: BKM approximations to K: Circular sector, $\alpha = 3/4$.

	BKM/AB: $z = 0.5$, $\sigma = 20/3$	
n	$\widetilde{E}_n(K, G)$	σ_n
10	5.0e-04	-
20	1.2e-06	8.70
30	6.1e-09	13.03
40	2.9e-10	10.65
50	1.7e-10	2.34
60	4.5e-11	7.31
70	1.6e-11	6.68
80	6.6e-12	6.66
90	3.0e-12	6.69
100	1.5e-12	6.69

Table 6.5.28: BKM/AB approximations to K: Circular sector, $\alpha = 3/4$.

	BKM: $z = 0.5,\ \sigma = 3$	
n	$E_n(f_0, G)$	σ_n
10	1.0e-03	-
20	7.6e-05	3.77
30	1.5e-05	3.99
40	6.0e-06	3.18
50	3.0e-06	3.10
60	1.7e-06	3.09
70	1.1e-06	3.06
80	7.1e-07	3.05
90	5.0e-07	3.04
100	3.6e-07	3.03

Table 6.5.29: BKM approximations to f_0: Circular sector, $\alpha = 4/5$.

	BKM/AB: $z = 0.5$, $\sigma = 6$	
n	$\widetilde{E}_n(f_0, G)$	σ_n
10	1.3e-04	-
20	1.2e-06	6.78
30	1.8e-08	10.31
40	2.2e-09	7.33
50	6.4e-10	5.56
60	2.1e-10	6.18
70	8.0e-11	6.21
80	3.5e-11	6.17
90	1.7e-11	6.14
100	8.9e-12	6.12

Table 6.5.30: BKM/AB approximations to f_0: Circular sector, $\alpha = 4/5$.

	BKM: $z = 0.5$, $\sigma = 3$	
n	$E_n(K,G)$	σ_n
10	8.2e-04	-
20	1.5e-04	2.48
30	2.6e-05	4.29
40	9.8e-06	3.34
50	4.9e-06	3.37
60	2.8e-06	3.09
70	1.7e-06	3.07
80	1.2e-06	3.06
90	8.1e-07	3.05
100	5.9e-07	3.04

Table 6.5.31: BKM approximations to K: Circular sector, $\alpha = 4/5$.

	BKM/AB: $z = 0.5, \;\; \sigma = 6$	
n	$\widetilde{E}_n(K, G)$	σ_n
10	3.5e-04	-
20	2.2e-06	7.31
30	6.5e-08	8.66
40	3.4e-09	10.24
50	1.2e-09	4.80
60	4.0e-10	5.91
70	1.5e-10	6.24
80	6.7e-11	6.19
90	3.2e-11	6.15
100	1.7e-11	6.13

Table 6.5.32: BKM/AB approximations to K: Circular sector, $\alpha = 4/5$.

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