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SOME ASPECTS OF WEIGHTED KOPPELMAN INTEGRAL REPRESENTATION FORMULAS AND THEIR APPLICATIONS

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Christiana Tryfonos

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The present doctoral dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy of the University of Cyprus. It is a product of original work of my own, unless otherwise mentioned through references, notes, or any other statements.

Christiana Tryfonos

Abstract

The dissertation derives a Koppelman integral representation formula on smooth compact toric varieties representing $(0,q)$ smooth forms taking values in specific line bundles by reducing our construction to the fact that the singular sets of the kernels involved are along the 'exceptional set' of the specific varieties. As an application, we study the vanishing of the Dolbeault cohomology groups of $(0,q)$ forms over smooth compact toric varieties with values in various lines bundles. Even if these results are already known, the novelty here lies on the fact that our method gives an explicit solution to the $\bar{\partial}$-equation on the varieties in question.

We further study the boundary behaviour of a weighted Koppelman integral representation formula on $\mathbb{C}^n$ with a specific choice of weight. Through the use of this specific formula, we manage to recover the extension result in [27] and [30] that if a function f has the 'one dimensional extension property' for every complex line l meeting a domain $D \subset \mathbb{C}^n$ where $1/\bar{t}$ can be extended from $\partial D \cap l$ to $D \cap l$, then f can be extended to a holomorphic function in D which is also continuous on the boundary ∂D. On one hand, the results are close in spirit to those to be found in [27] and [30], but on the other hand are surprising because the kernels involved are not harmonic as the B-M kernel is. Thus, the independence of the results from the choice of the contributing kernels indicates somewhat the topological nature of the results.

Περίληψη

Στην παρούσα διατριβή κατασκευάζουμε έναν ολοκληρωτικό τύπο Koppelman σε ομαλές συμπαγείς τορικές πολλαπλότητες που αναπαριστά $(0,q)$ ομαλές διαφορικές μορφές, οι οποίες λαμβάνουν τιμές σε συγκεκριμένες μονοδιάσταστες διανυσματικές δέσμες. Οι εμπλεκόμενοι πυρήνες της αναπαράστασης κατασκευάζονται έτσι ώστε τα ιδιάζοντα τους σημεία να βρίσκονται στο 'exceptional set' των εν λόγω πολλαπλοτήτων. Ως εφαρμογή, μελετάμε περιπτώσεις στις οποίες οι κλάσεις συνομολογίας Dolbeault των $(0,q)$ μορφών σε ομαλές συμπαγείς πολλαπλότητες που παίρνουν τιμές σε διάφορες διανυσματικές δέσμες είναι τετριμμένες. Αν και τα αποτελέσματα αυτά είναι ήδη γνωστά, η καινοτομία εδώ έγγειται στο γεγονός ότι η μέθοδος μας δίνει μια αναλυτική λύση της $\bar{\partial}$-εξίσωσης στις εν λόγω πολλαπλότητες.

Περαιτέρω μελετούμε τη συνοριακή συμπεριφορά ενός weighted Koppelman τύπου στο $\mathbb{C}^n$ όπου έχουμε χρησιμοποιήσει μια συγκεκριμένη επιλογή βάρους. Μάλιστα μέσω της χρήσης του τύπου αυτού, αναπαράγουμε το αποτέλεσμα επέκτασης που βρίσκεται στα άρθρα [27] και [30], ότι αν μια συνάρτηση f έχει την ' μονοδιάστατη ιδιότητα επέκτασης ' ('one dimension extension property') για κάθε ευθεία που συναντά ένα χωρίο $D \subset \mathbb{C}^n$ και επιπλέον, η $1/\bar{t}$ μπορεί να επεκταθεί από το $\partial D \cap l$ στο $D \cap l$, τότε η f μπορεί να επεκταθεί σε ολόμορφη συνάρτηση στο D η οποία είναι συνεχής στο ∂D. Αφενός, τα αποτελέσματα μας είναι κοντά στο πνεύμα των αντίστοιχων αποτελεσμάτων που βρίσκονται στα άρθρα [27] και [30], αφ' ετέρου είναι εκπληκτικά επειδή οι εμπλεκόμενοι πυρήνες στην περίπτωση μας δεν είναι αρμονικοί όπως τον πυρήνα B-M. Έτσι η ανεξαρτησία των αποτελεσμάτων από την επιλογή των πυρήνων υποδηλώνει την τοπολογική φύση των αποτελεσμάτων.

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To my family

Contents

Introduction

Weighted integral representation formulas for representing holomorphic functions are introduced by M. Andersson and B. Berndtsson in [3], where they also construct solutions of the $\bar{\partial}$-equation. More recently, M. Andersson in [1] provided a general method to get weighted integral representation formulas for $(0, q)$ forms for domains $D \subset\subset \mathbb{C}^n$. The same author in [2] introduce new formulas for matrices of holomorphic functions (as a continuation of [1]) and application to interpolation and division problems.

Koppelman integral representation formulas for $(0, q)$ forms with values in a line bundle on projective space was constructed by B. Berndtsson [5], based on the ideas of G. M. Henkin and P. L. Polyakov [19]. Two years later, T. Hatziafratis in [17] obtained integral representation formulas for the solution of $\bar{\partial}$-equation on domains in algebraic submanifolds which are complete intersections of the complex projective space. More recently, E. Götmark [14], [15] was able to deduce a Weighted Koppelman formula for (p, q) forms on an n-dimensional complex manifold X in the case when the section defining the diagonal of $X \times X$ is global. This method offers a Kopppelman formula for (p, q) forms taking values in line bundles over $\mathbb{P}^n$, but it is also applied on $(0, q)$ forms taking values in $L^k \otimes L^l$ bundle over the cartesian product of projective spaces $\mathbb{P}^n \times \mathbb{P}^m$. The solution to $\bar{\partial}$-problem can also be derived.

The construction of a Koppelman formula on a complex manifold X is achieved by finding suitable kernels K and P that satisfy the current equation $\bar{\partial} K = [\Delta] - P$, $[\Delta]$ being the integration current over the diagonal $\Delta = \{(\zeta, z) : \zeta = z\}$ of $X \times X$.

Other kernels leading to integral representations (not global) for holomorphic functions on an open set associated with a toric variety are obtained by A. Shchuplev, A. Tsikh and A. Yger in [38] and A. Shchuplev in [37]. The integral representation of such functions became also an object for study by A. A. Kytmanov [26], who constructed kernels for holomorphic functions on d-circular domains in $\mathbb{C}^d$ (connected with the two dimensional toric variety) by using an analog of the Fubini-Study form. Later on, A. A. Kytmanov and A. Y. Semusheva [28] derived kernels on toric varieties by generalizing the Bochner-Martinelli form in $\mathbb{C}^n$.

The aims of the thesis

Our contribution is to present a formula of the 'Koppelman type' on some toric projective varieties by reducing our construction to the fact that the singular set of kernels involved is along the 'exceptional sets' of the varieties in question. As a particular case, we are able to recover the Koppelman integral representation formula on $\mathbb{P}^n \times \mathbb{P}^m$ in a Gotmark's work [15], but with different representatives. In general, our method illustrates through examples, such as the Hirzebruch surface, that it provides a general method for representing $(0, q)$ forms which take values in different line bundles over smooth compact toric varieties.

One important consequence of our results is that they allow us to study the vanishing of Dolbeault cohomology groups of $(0, q)$ forms over smooth compact toric varieties with values in various line bundles by constructing explicit solutions to the $\bar{\partial}$-problem.

In the present thesis, we also prove results about the boundary behaviour of a weighted Koppelman formula on $\mathbb{C}^n$ with a specific choice of weight by extending the corresponding results related to the Bochner-Martinelly type integral ([25],[29]). A proof of a Hartog's phenomenon that can be found in [27] and [30] is achieved through this specific weighted Koppelman formula, although the kernels involved fail to be harmonic. More precisely, we reprove by different method the result of A. A. Kytmanov, that a function $f \in \mathcal{C}(\partial D)$ having a 'one dimensional extension property' for every complex line l meeting a domain $D \subset \mathbb{C}^n$ and the function $1/\bar{t}$ can be extended from $\partial D \cap l$ to $D \cap l$, can be extended to a holomorphic function in D which is also continuous on the boundary ∂D.

We begin in Chapter 1 by introducing the notion of toric varieties, and present some results related to them, following [6], [11] and [12]. A number of examples illustrate the construction involved. Moreover, appropriate line bundles and sheaves on the above varieties are constructed. We also present the Weighted Koppelman formula on $\mathbb{C}^n$ and on $\mathbb{P}^n$. Finally the chapter concludes with the notion of the multi-logarithmic residue current and the proof of a generalized Poincaré-Lelong formula.

The second chapter is devoted to the study of the boundary properties of a weighted Koppelman type integral formula with a specific choice of weight. The results obtained are similar to those in [25],[29] for the Bochner-Martinelli type integrals, the most important one being the analogous Jump Theorem for weighted Koppelman type integrals.

The third chapter, containing the main results of the thesis, deals with the construction of a toric Koppelman formula for $(0, q)$ forms on a compact projective toric variety X taking values in a line bundle $V_{\mathcal{L}}$, where $\mathcal{L} = \mathcal{O}_X(D)$ is the sheaf induced by the ample divisor D. This construction exhibits quite complicated combinatorial properties inherited from the nature of the toric varieties. A crucial tool here is the analogue of a Poincaré-Lelong formula for a set of holomorphic functions $\{f_1, \ldots, f_{N-n-1}\}$ of holomorphic functions. Our strategy is to embed the n-dimensional toric variety X into $\mathbb{P}^{N-1}$ and derive a Weighted Koppelman formula whose kernels have singularities exactly at the exceptional set of the toric variety.

Applications of the above to finding explicit generators of cohomology groups and explicit solutions to the $\bar{\partial}$-problem on a toric variety are given in the fourth chapter. By using the dual nature of the Koppelman formula, we also study the cohomology groups for (n, q)-forms on X taking values in the dual bundle and in its k-fold tensor product while an isomorphism related to the cotangent sheaf of $(0, n)$ forms allows to further extend our results.

Chapter 1

Preliminaries

This chapter introduces some notions and results related to multidimensional complex analysis and toric geometry. Geometric, combinatorial and arithmetic aspects of toric varieties which will be used throughout this work, are presented in this introductory chapter. The material used draws heavily on [6], [11] and [12].

In this chapter we present the Weighted Koppelman formula on $\mathbb{C}^n$ (with its proof) and the Koppelman formula for differential forms with values in a line bundle over $\mathbb{P}^n$. The generalization to the toric setting is the main purpose of this thesis. Our approach is inspired by results in [1], [14], [15].

We conclude this chapter by defining the logarithmic residue current connected with a tuple of holomorphic functions $f = (f_1, \ldots, f_p)$ in $\mathbb{C}^n$ and give a generalization of a Poincaré-Lelong formula. This material has been borrowed from [7] and [39].

1.1 Toric Varieties

This section introduces some general notation and explores various aspects related to toric varieties by realising them through multiple examples. Toric varieties constitute a rich class of algebraic varieties which admits special algebraic and geometric properties defined by combinatorial information. There are several definitions for toric varieties but the predominant one states that an n-dimensional toric variety is an irreducible variety X containing a torus $T \simeq (\mathbb{C}^*)^n := (\mathbb{C} \setminus \{0\})^n$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on X. (The algebraic action is an action $T \times X \to X$ given by a morphism.)

Every n-dimensional toric variety $X = X_\Sigma$ is directly related to a set of cones, namely the fan Σ in $\mathbb{R}^n$ of X.

1.1.1 The construction of X

Let M and N be free abelian groups of finite rank such that $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the dual space of N. Picking a $\mathbb{Z}$-basis of N gives an isomorphism $N \simeq \mathbb{Z}^n$ while its dual basis, that is the basis of M, yields that $M \simeq \mathbb{Z}^n$. According to these isomorphisms, a pairing between the two groups becomes the standard dot product on $\mathbb{R}^n$ denoted by

$$\begin{aligned} \langle\, , \rangle \quad &: \quad M \times N \to \mathbb{Z} \\ \langle m, n \rangle \quad &= \quad a_1 b_1 + \cdots a_n b_n, \end{aligned}$$

for $m = (a_1, \ldots, a_n) \in M$ and $n = (b_1, \ldots, b_n) \in N$. The notations $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ are scalar extensions of M and N, respectively.

A subset $\sigma \subset N_{\mathbb{R}}$ is called a *convex polyhedral cone* if there exist $v_1, \ldots, v_s \in N_{\mathbb{R}}$ such that

$$\sigma := \mathrm{Cone}(v_1, \ldots, v_s) = \left\{ \sum_{i=1}^{s} a_i v_i \big| a_i \geq 0 \right\} \subseteq N_{\mathbb{R}}. \tag{1.1}$$

Its dimension is defined as the dimension of the interior of a minimal subspace of $\mathbb{R}^n$ containing σ. A cone is *strongly convex* if and only if $\sigma \cap (-\sigma) = \{0\}$, while a cone is called *smooth* if its minimal set of generators forms part of a $\mathbb{Z}$-basis of N.

A *dual cone* $\check{\sigma}$ of the cone σ is defined by

$$\check{\sigma} = \{ m \in M_{\mathbb{R}} : \langle m, u \rangle \geq 0 \,\text{for all}\, u \in \sigma \}.$$

Hence, every dual cone is associated with a semigroup $S_\sigma = \check{\sigma} \cap M$. This semigroup is finitely generated according to the Gordan's Lemma [6, Proposition 1.2.17]. Then S_σ can be written in the form $S_\sigma = \mathbb{Z}_{\geq 0} m_1 + \cdots + \mathbb{Z}_{\geq 0} m_k$ for $m_1, \ldots, m_k \in S_\sigma$ and $k \in \mathbb{Z}^+$. The vector $m = (c_1, \ldots, c_n) \in S_\sigma$ determines a character which is a group homomorphism $\chi^m : T \to \mathbb{C}^*$ defined by

$$\chi^m (t_1, \ldots, t_n) = t_1^{c_1} \cdots t_n^{c_n}.$$

Hence, χ^m is turned into a Laurent monomial due to the isomorphism $T \simeq (\mathbb{C}^*)^n$. Thus, the elements of the algebra $\mathbb{C}[S_\sigma]$ generated by the characters $\{\chi^{m_i}\}_i$, can be interpreted as the $\mathbb{C}$-valued polynomial functions on S_σ.

An affine toric variety corresponding to $\check{\sigma}$ is the chart U_σ, which is identified with the spectrum of the algebra $\mathbb{C}[S_\sigma]$ (the spectrum being the set of maximal ideals equipped with Zariski topology) due to the close connection between affine varieties and ideals. The chart is written then as

$$U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma].$$

A toric variety is described fully by a set of cones with specific characteristics. Let τ be a *face* of a cone σ, which is a subset of σ such that some a_i in (1.1) are equal to zero.

Definition 1.1.1 The *fan* $\Sigma \subseteq N_\mathbb{R}$ of a toric variety X is a finite collection of strongly convex polyhedral cones such that:
(i) Every face of a cone $\sigma \in \Sigma$, is also in Σ.
(ii) For all σ_1, $\sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each cone σ_1, σ_2 in Σ.

The dimension of the fan is the maximal dimension of its cones. The fan is called *smooth* if every cone in Σ is smooth. In this case, the toric variety X is also smooth. The support of a fan Σ is denoted by

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subseteq N_\mathbb{R}.$$

We say that the fan Σ is *complete* if its support $|\Sigma|$ is all of $N_\mathbb{R}$ and the toric variety X inherits the property of completeness from its fan. *Complete* is equivalent to being *compact* in the classical topology.

Now, the combinatorial data of the fan tell us how to glue together the collection of the affine varieties U_σ for every $\sigma \in \Sigma$ and obtain the toric variety X. More precisely, let $\sigma_1, \sigma_2 \in \Sigma$, $\tau = \sigma_1 \cap \sigma_2$ and $u \in \check{\sigma_1} \cap (-\check{\sigma_2})$. If $m_1, \ldots, m_k$ is a system of generators of $\check{\sigma_1} \cap M$, then without loss of generality, we may assume $m_k = u$. Since $\pm u \in \check{\tau}$ and $\check{\sigma} \subset \check{\tau}$, $\check{\tau} \cap \mathbb{Z}^n$ is generated by $m_1, \ldots, m_k = u, m_{k+1} = -u$. The additive relation $m_k + m_{k+1} = 0$ turns into the multiplicative relation $\chi^{m_k}\chi^{m_{k+1}} = 1$ in $\mathbb{C}[\check{\tau} \cap M]$. Thus, the projection

$$(\chi^{m_1}, \ldots, \chi^{m_k}, \chi^{m_{k+1}}) \mapsto (\chi^{m_1}, \ldots, \chi^{m_k})$$

identifies U_τ with the open subset $U_{\sigma_1} \setminus \{\chi^u = 0\}$. Similarly, one can obtain that $U_\tau \cong U_{\sigma_2} \setminus \{\chi^{-u} = 0\}$. The composition of the above isomorphisms yields that

$$U_{\sigma_1} \setminus \{\chi^u = 0\} \cong U_\tau \cong U_{\sigma_2} \setminus \{\chi^{-u} = 0\}.$$

According to this rule, the affine varieties can be glued along affine varieties associated with their common faces, so that the toric variety X related to the fan arises.

Some examples of well-known toric varieties are the following.

Example 1.1.1 The one-dimensional cones $\sigma_0 = [0, \infty) \subset \mathbb{R}$ and $\sigma_1 = (-\infty, 0] \subset \mathbb{R}$ with $\tau = \sigma_0 \cap \sigma_1 = \{0\}$ are describing the fan of the complex projective space $\mathbb{P}^1$. Since $\check{\sigma_0} = \sigma_0$ and $\check{\sigma_1} = \sigma_1$, the corresponding charts are

$$\begin{aligned} U_{\sigma_0} &= \operatorname{Spec}\left(\mathbb{C}[x]\right) \\ U_{\sigma_1} &= \operatorname{Spec}\left(\mathbb{C}[x^{-1}]\right). \end{aligned}$$

These two charts are glued along their common face $\tau = \{0\}$ since $U_\tau \cong U_{\sigma_0} \setminus \{x = 0\}$ and $U_\tau \cong U_{\sigma_1} \setminus \{x^{-1} = 0\}$ through the isomorphisms $(x, x^{-1}) \mapsto x$ and $(x, x^{-1}) \mapsto x^{-1}$, respectively. Indeed, this is the fan of the complex projective space $\mathbb{P}^1$. Actually, if we look at the homogeneous coordinates (ζ_0, ζ_1) of $\mathbb{P}^1$ and maps $x \mapsto \zeta_1/\zeta_0$, then the standard open cover $\{U_i\}_{i=0}^1$ of $\mathbb{P}^1$ with $U_0 = \{(\zeta_0, \zeta_1) \big| \zeta_0 \neq 0\}$ and $U_1 = \{(\zeta_0, \zeta_1) \big| \zeta_1 \neq 0\}$ identifies with $\{U_{\sigma_i}\}_{i=0}^1$.

Example 1.1.2 Let us consider the two-dimensional fan Σ with cones $\sigma_0 = \operatorname{Cone}(e_1, e_2)$, $\sigma_1 = \operatorname{Cone}(e_2, -e_1 - e_2)$ and $\sigma_2 = \operatorname{Cone}(e_1, -e_1 - e_2)$. Then $\check{\sigma_0} = \operatorname{Cone}(e_1, e_2)$, $\check{\sigma_1} = \operatorname{Cone}(-e_1 + e_2, -e_1)$, $\check{\sigma_2} = \operatorname{Cone}(-e_2, e_1 - e_2)$ and the corresponding charts are given by the spectra of the following rings:

$$\begin{aligned} U_{\sigma_0} &= \operatorname{Spec}(\mathbb{C}[x, y]) \\ U_{\sigma_1} &= \operatorname{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]) \\ U_{\sigma_2} &= \operatorname{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]). \end{aligned}$$

These charts are glued together along their common faces and $\mathbb{P}^2$ is obtained. In particular, let $\tau = \{e_2\}$ be the common face of σ_0 and σ_1 and $e_1 \in \check{\sigma_0} \cap (-\check{\sigma_1})$. Then $U_\tau \cong U_{\sigma_0} \setminus \{x = 0\}$ according the projection $(x, y, x^{-1}) \mapsto (x, y)$ and $U_\tau \cong U_{\sigma_1} \setminus \{x^{-1} = 0\}$ through the mapping $(x^{-1}, x^{-1}y, x) \mapsto (x^{-1}, x^{-1}y)$. Then U_{σ_0} is glued with U_{σ_1}. Similarly, the gluing of the remaining charts is obtained. As in the previous example, the change of coordinates according to the rules $x \mapsto \frac{\zeta_1}{\zeta_0}$ and $y \mapsto \frac{\zeta_2}{\zeta_0}$ identify the standard affine open sets U_i of $\mathbb{P}^2$ with $U_{\sigma_i} \subset X_\Sigma$ for every $i = 0, 1, 2$. Thus, $X_\Sigma \simeq \mathbb{P}^2$.

Example 1.1.3 Generalizing the two previous examples, we describe the construction of the fan of $\mathbb{P}^n$. We consider the standard basis $\{e_i\}_{i=1}^n$ of $N = \mathbb{Z}^n$ and let, also, $e_0 =$

$-e_1 - e_2 - \cdots - e_n$. Then, the n-dimensional cone σ_i for $i = 0, \ldots, n$ is defined by

$$\sigma_i = \text{Cone}(e_0, \ldots, [e_i], \ldots, e_n),$$

where the notation $[e_i]$ denotes that the vector e_i is omitted. As in the previous examples it is easy to verify that the collection $\{U_{\sigma_i}\}_{i=0}^n$ is the usual open cover $\{U_i\}_{i=0}^n$ of $\mathbb{P}^n$.

There are several examples of toric varieties, but one that we will deal with it later on, is the Hirzebruch surface H_r. The index r varies for $r \geq 0$, in such a way that a collection of surfaces arises.

Example 1.1.4 The 2-dimensional Hirzebruch surface H_r is presented by a fan consisting of the four cones $\sigma_1 = \text{Cone}(e_1, e_2)$, $\sigma_2 = \text{Cone}(e_1, -e_2)$, $\sigma_3 = \text{Cone}(-e_1 + re_2, -e_2)$ and $\sigma_4 = \text{Cone}(-e_1 + re_2, e_2)$. Observe that $\check{\sigma_1} = \sigma_1$, $\check{\sigma_2} = \sigma_2$ $\check{\sigma_3} = \text{Cone}(-re_1 - e_2, -e_1)$, $\check{\sigma_4} = \text{Cone}(re_1 + e_2, -e_1)$ such that the corresponding charts are

$$\begin{aligned} U_{\sigma_1} &= \text{Spec}(\mathbb{C}[x, y]) \\ U_{\sigma_2} &= \text{Spec}(\mathbb{C}[x, y^{-1}]) \\ U_{\sigma_3} &= \text{Spec}(\mathbb{C}[x^{-1}, x^{-r}y^{-1}]) \\ U_{\sigma_4} &= \text{Spec}(\mathbb{C}[x^{-1}, x^r y]). \end{aligned}$$

The arising toric variety, after the gluing of the charts U_{σ_i}, is the surface H_r.

Example 1.1.5 A particular case of a Hirzebruch surface for $r = 1$ is denoted by $\mathcal{H}$. Its fan is spanned by the vectors e_1, e_2, $-e_1$, $-e_1 - e_2$. The formation of the four cones is the following: $\sigma_1 = \text{Cone}(e_1, e_2)$, $\sigma_2 = \text{Cone}(-e_1, e_2)$, $\sigma_3 = \text{Cone}(-e_1, -e_1 - e_2)$ and $\sigma_4 = \text{Cone}(-e_1 - e_2, e_1)$. Then $\check{\sigma_1} = \sigma_1$, $\check{\sigma_2} = \sigma_2$ $\check{\sigma_3} = \text{Cone}(-e_1 + e_2, -e_2)$, $\check{\sigma_4} = \text{Cone}(e_1 - e_2, -e_2)$ and

$$\begin{aligned} U_{\sigma_1} &= \text{Spec}(\mathbb{C}[x, y]) \\ U_{\sigma_2} &= \text{Spec}(\mathbb{C}[x^{-1}, y]) \\ U_{\sigma_3} &= \text{Spec}(\mathbb{C}[x^{-1}y, y^{-1}]) \\ U_{\sigma_4} &= \text{Spec}(\mathbb{C}[y^{-1}, xy^{-1}]), \end{aligned}$$

are the charts building the toric variety $\mathcal{H}$, in a manner (gluing) similar to the the previous examples.

All previous examples concern smooth, compact toric varieties since their corresponding fans are smooth and compete. The next example presents a particular case of a weighted projective space which is compact but not smooth.

Example 1.1.6 Another interesting example of a toric variety is the weighted projective space $\mathbb{P}(1,1,2)$ induced by the fan constructed by the vectors e_1, e_2 and $-e_1 - 2e_2$. The generated cones are $\sigma_0 = \text{Cone}(e_1, e_2)$, $\sigma_1 = \text{Cone}(e_2, -e_1 - 2e_2)$ and $\sigma_2 = \text{Cone}(-e_1 - 2e_2, e_1)$ while the corresponding dual cones and their realted charts are $\check{\sigma_0} = \sigma_0 = \text{Cone}(e_1, e_2)$, $\check{\sigma_1} = \text{Cone}(-e_1, -2e_1 + 2e_2)$, $\check{\sigma_2} = \text{Cone}(-e_2, 2e_1 - e_2)$ and

$$\begin{aligned} U_{\sigma_0} &= \text{Spec}(\mathbb{C}[x, y]) \\ U_{\sigma_1} &= \text{Spec}(\mathbb{C}[x^{-1}, x^{-2}y]) \\ U_{\sigma_2} &= \text{Spec}(\mathbb{C}[y^{-1}, x^2y^{-1}]). \end{aligned}$$

After the gluing of the affine varieties U_{σ_i} for $i = 0, 1, 2$, we obtain the desired toric variety $\mathbb{P}(1,1,2)$. Actually, this particular toric variety is not smooth due to the contribution of the non-smooth cone σ_2 in the fan of $\mathbb{P}(1,1,2)$.

It is important to mention that a new toric variety is generated by taking the product of two toric varieties. More precisely, let X_1, X_2 be two toric varieties with their corresponding fans being $\Sigma_1 \subset N_1$, $\Sigma_2 \subset N_2$, respectively. The product of two fans $\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \big| \sigma_i \in \Sigma_i\}$ is also a fan in $N_1 \times N_2$ and the toric variety $X_{\Sigma_1 \times \Sigma_2}$, which corresponds to this fan is isomorphic to the product $X_1 \times X_2$ ([6, Proposition 3.1.14]). An example of such a toric variety is the product of projective spaces, described below.

Example 1.1.7 Let Σ be the fan consisting of the cones $\sigma_{00} = \text{Cone}(e_1, e_2)$, $\sigma_{10} = \text{Cone}(-e_1, e_2)$, $\sigma_{11} = \text{Cone}(-e_1, -e_2)$ and $\sigma_{01} = \text{Cone}(e_1, -e_2)$. Then $\check{\sigma_{ij}} = \sigma_{ij}$, $\forall\, i, j = 0, 1$ and

$$\begin{aligned} U_{\sigma_{00}} &\simeq \text{Spec}(\mathbb{C}[x, y]) \\ U_{\sigma_{10}} &\simeq \text{Spec}(\mathbb{C}[x^{-1}, y]) \\ U_{\sigma_{11}} &\simeq \text{Spec}(\mathbb{C}[x^{-1}, y^{-1}]) \\ U_{\sigma_{01}} &\simeq \text{Spec}(\mathbb{C}[x, y^{-1}]). \end{aligned}$$

Gluing the local charts according to the rule described in example 1.1.1, leads to the identification of spaces $X_\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Similarly, one can realize $\mathbb{P}^n \times \mathbb{P}^m$ as a toric variety, which is the space of main importance in this thesis.

Example 1.1.8 Consider the n-cartesian product of $\mathbb{P}^1$, $(\mathbb{P}^1)^n$ with contributing vectors for the n-dimensional fan the vectors $e_1, \ldots, e_n$ and $-e_1, \ldots, -e_n$. The constructed n-dimensional cones are 2^n and each one is produced by n linearly independent vectors from the above collection of elements.

1.1.2 Quotient construction of a Toric Variety

This section generalizes the construction of projective space as the quotient space of the affine space minus the origin in other words

$$\mathbb{P}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) \Big/ \mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1}$ by scalar multiplication. A toric variety X can be represented as a quotient

$$X_\Sigma = \Big(\mathbb{C}^d \setminus Z(\Sigma)\Big) \Big/ G,$$

where the set $Z(\Sigma)$ is the exceptional set and G is a reductive group. Both objects are defined and described briefly below.

Let $v_1, \ldots, v_d \in \mathbb{Z}^n$ $(d \geq n)$ be the generators of the cones of Σ and $\zeta_1, \ldots, \zeta_d$ be the corresponding homogeneous coordinates. If $\zeta_{\hat{\sigma}} := \prod_{v_j \notin \sigma} \zeta_j$, then one defines the *exceptional set*

$$Z(\Sigma) = \Big\{\zeta \in \mathbb{C}^d : \zeta_{\hat{\sigma}} = 0 \text{ for all } n\text{-cones } \sigma \text{ in } \Sigma\Big\}. \tag{1.2}$$

On the other hand, the *reductive group* G is described by the following:

$$G := \left\{\nu = (\nu_1, \cdots, \nu_d) \in (\mathbb{C}^*)^d : \prod_{i=1}^{d} \nu_i^{\langle e_j, v_i \rangle} = 1 \text{ for } 1 \leq j \leq n\right\}.$$

Example 1.1.9 In the realization of $\mathbb{P}^1 \times \mathbb{P}^1$ as a toric variety with generators $v_1 = e_1$, $v_2 = -e_1$, $v_3 = e_2$ and $v_4 = -e_2$, observe that the exceptional set $Z(\Sigma)$ is generated by the relations $\zeta_1\zeta_3 = 0$, $\zeta_1\zeta_4 = 0$, $\zeta_2\zeta_3 = 0$ and $\zeta_2\zeta_4 = 0$. That is,

$$Z(\Sigma) = \{0\} \times \mathbb{C}^2 \cup \mathbb{C}^2 \times \{0\}.$$

Now, $(\nu_1, \nu_2, \nu_3, \nu_4) \in G$ if and only if $\nu_1\nu_2^{-1} = \nu_3\nu_4^{-1} = 1$. Hence, the reductive group is

$$G = \big\{(\mu, \mu, \lambda, \lambda) \big| \mu, \lambda \in \mathbb{C}^*\big\}$$

and the quotient realization of the above space is given by

$$\mathbb{P}^1 \times \mathbb{P}^1 = \left(\mathbb{C}^4 \setminus (\{0\} \times \mathbb{C}^2 \cup \mathbb{C}^2 \times \{0\})\right) \Big/ G.$$

Example 1.1.10 The exceptional set $Z(\Sigma)$ of the Hirzebruch surface H_r is described by the relations $\zeta_1\zeta_4 = 0$, $\zeta_1\zeta_2 = 0$, $\zeta_2\zeta_3 = 0$ and $\zeta_3\zeta_4 = 0$. Thus,

$$Z(\Sigma) = \{\zeta_1 = \zeta_3 = 0\} \cup \{\zeta_2 = \zeta_4 = 0\}.$$

On the other hand, to determine the group G, take a vector $(\nu_1, \nu_2, \nu_3, \nu_4) \in G$. By calculating the group G acting on $\mathbb{C}^4 \setminus Z(\Sigma)$, we observe that $\nu_1^{-1}\nu_3 = \nu_1^r \nu_2 \nu_4^{-1} = 1$, or equivalently, $\nu_1 = \nu_3$ and $\nu_4 = \nu_1^r \nu_2$. It turns out that

$$G = \left\{(\lambda, \mu, \lambda, \lambda^r \mu) \big| \mu, \lambda \in \mathbb{C}^*\right\}.$$

Thus, H_r is realized as the quotient space $H_r = \left(\mathbb{C}^4 \setminus Z(\Sigma)\right) / G$.

Example 1.1.11 In the particular case of the Hirzebruch surface $\mathcal{H}$ (see example 1.1.5), the exceptional set is

$$Z(\Sigma) = \{\zeta_1 = \zeta_3 = 0\} \cup \{\zeta_2 = \zeta_4 = 0\}$$

with corresponding reductive group being

$$G = \left\{(\lambda\mu, \lambda, \mu, \lambda) \big| \mu, \lambda \in \mathbb{C}^*\right\}.$$

Example 1.1.12 Similarly, in example 1.1.6, we deduce that the exceptional set of $\mathbb{P}(1,1,2)$ is

$$Z(\Sigma) = \{\zeta_0 = \zeta_1 = \zeta_2 = 0\}$$

while its reductive group is described by

$$G = \left\{(\lambda, \lambda, \lambda^2) \big| \lambda \in \mathbb{C}^*\right\}.$$

Thus,

$$\mathbb{P}(1,1,2) = \mathbb{C}^3 \setminus \{0\} / G.$$

1.1.3 Divisors and Sheaves on Toric Varieties

Global information on toric varieties can be collected by using the notions of divisors and sheaves. A *prime divisor* D of an irreducible variety X is an irreducible subvariety of codimension 1. To each such divisor corresponds a ring of the type

$$\mathcal{O}_{X,D} = \{\phi \in \mathbb{C}(X) : \phi \,\text{is defined on } U \subset X \text{ open and} \, U \cap D \neq \emptyset\}$$

where $\mathbb{C}(X)$ is the field of rational functions on X.

The free abelian group generated by the prime divisors is denoted by $\text{Div}\,(X)$ and an element D of $\text{Div}\,(X)$, the so-called Weil divisor, is the sum $D = \sum_i a_i D_i$, where a finite number of coefficients $a_i \in \mathbb{Z}$ are different from zero, while the D_i represent distinct prime divisors of X. If the coefficients a_i are all non negative, then the divisor D is called *effective* and is symbolized by $D \geq 0$. Moreover, two Weil divisors D_1, D_2 on X are said to be linearly equivalent, if there exists a non-zero rational function f such that $\text{div}(f) = D_1 - D_2$. We denote linear equivalence by $D_1 \sim D_2$.

If $f \in \mathbb{C}(X)^*$, then f defines the *principal divisor* of f which equals $\text{div}\,(f) = \sum_D \nu_D(f) D$, where the summation is over all the prime divisors $D \subset X$ and $\nu_D(f)$ is called the order of vanishing of f along D. If $\nu_D(f) = n \geq 0$ then the order of vanishing of f along D is n and when $\nu_D(f) = n < 0$, f has a pole of order $|n|$ along D. The group of principal divisors is denoted by $\text{Div}_0\,(X)$.

A *Cartier* divisor D on a toric variety X is a *locally principal divisor*. That is, for an open cover $\{U_i\}_{i \in I}$ of X, the restriction $D|_{U_i}$ is principal in $\{U_i\}$ for every $i \in I$. Moreover, if $D|_{U_i} = \text{div}\,(f_i)|_{U_i}$ for $i \in I$, then $\{(U_i, f_i)\}_{i \in I}$ are the *local data* for D. The Cartier divisors form the group $\text{CDiv}(X)$. Hence, the Class group and the Picard group are defined as the quotients

$$\text{Cl}\,(X) = \text{Div}\,(X)/\text{Div}_0\,(X) \quad \text{and} \quad \text{Pic}\,(X) = \text{CDiv}\,(X)/\text{Div}_0\,(X)$$

respectively. It is important to mention that the two groups coincide, $\text{Cl}\,(X) = \text{Pic}\,(X)$ if X is smooth [6, Theorem 4.0.22].

In particular, let $\{v_i\}_{i=1}^d$ be the set of generators of the cones of an n-dimensional fan Σ. To each vector v_i there corresponds a variable ζ_i and a divisor

$$D_i = \text{div}(\zeta_i),$$

for $i = 1, \ldots, d$. The variables ζ_i are not necessary independent. The divisor of the character χ^m on X for $m \in M_{\mathbb{R}}$, which is the divisor of a rational function with respect to ζ_i according to the rule $\chi^m = \prod_{i=1}^{d} \zeta_i^{\langle m, v_i \rangle}$, can be expressed by

$$\operatorname{div}(\chi^m) = \sum_{i=1}^{d} \langle m, v_i \rangle D_i. \tag{1.3}$$

Example 1.1.13 Let us continue the example 1.1.3, related to the fan of $\mathbb{P}^n$, where the generators are $v_0 = -e_1 - \cdots - e_n$, $v_1 = e_1, \ldots, v_n = e_n$. If $D_i = \operatorname{div}(\zeta_i)$, where ζ_i is the variable assigned to each vector v_i, for $i = 0, \ldots, n$, then, according to (1.3),

$$0 \sim \operatorname{div}(\chi^{e_j}) = -D_0 + D_j$$

for every $j = 1, \ldots, n$. It turns out that the divisors are equivalent ($D_0 \sim \cdots \sim D_n$) and that $\operatorname{Cl}(\mathbb{P}^n)$ is generated by the class of one of those divisors. Without loss of generality, we can write $\operatorname{Cl}(\mathbb{P}^n) = [D_0]$ which leads to the isomorphism $\operatorname{Cl}(\mathbb{P}^n) \simeq \mathbb{Z}$.

Example 1.1.14 Consider the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ whose fan's generators are the vectors $v_1 = e_1$, $v_2 = -e_1$, $v_3 = e_2$ and $v_4 = -e_2$. Then,

$$\begin{aligned} 0 &\sim \operatorname{div}(\chi^{e_1}) = D_1 - D_2 \\ 0 &\sim \operatorname{div}(\chi^{e_2}) = D_3 - D_4. \end{aligned}$$

Thus $\operatorname{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z}^2$ which is generated by $[D_1] = [D_2]$ and $[D_3] = [D_4]$

Example 1.1.15 The fan of the Hirzebruch surface H_r is generated by $v_1 = -e_1 + re_2$, $v_2 = e_2$, $v_3 = e_1$ and $v_4 = -e_2$. Then, $\operatorname{Cl}(H_r) \simeq \mathbb{Z}^2$ since

$$\begin{aligned} 0 &\sim \operatorname{div}(\chi^{e_1}) = -D_1 + D_3 \\ 0 &\sim \operatorname{div}(\chi^{e_2}) = rD_1 + D_2 - D_4. \end{aligned}$$

Example 1.1.16 Similarly to the previous example, the particular case of the Hirzebruch surface in example 1.1.5 has $\operatorname{Cl}(\mathcal{H}) \simeq \mathbb{Z}^2$ and the classes of its divisors fulfill the relations $[D_2] = [D_4]$ and $[D_1] = [D_3] + [D_4]$.

Example 1.1.17 Since the contributing vectors of the weighted projective space $\mathbb{P}(1, 1, 2)$

are $v_0 = -e_1 - 2e_2$, $v_1 = e_1$ and $v_2 = e_2$, then

$$\begin{aligned} 0 &\sim \operatorname{div}(\chi^{e_1}) = -D_0 + D_1 \\ 0 &\sim \operatorname{div}(\chi^{e_2}) = -2D_0 + D_2. \end{aligned}$$

This means that the equivalence classes satisfy the following relationships $[D_0] = [D_1]$ and $[D_2] = [2D_0]$.

Cartier divisors induce some important sheaves on X constituting the substructure for the rest of the present work. Let $\{U_i\}$ be an open cover of U such that $U = \bigcup_i U_i$.

Definition 1.1.2 A *sheaf* $\mathcal{F} := \mathcal{F}(U)_{U \subset X}$ of $\mathcal{O}_X(U)$ modules is a collection of rational sections on U such that:

- If $V \subset U$ and $f \in \mathcal{F}(U)$ then the restriction $f|_V$ belongs to $\mathcal{F}(V)$.
- If $f_i \in \mathcal{F}(U_i)$, for each i, satisfies the compatibility condition

  $$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \text{ for all } i, j,$$

  then there exists an element $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$ for all i.

By using the gluing property of sheaves, we can find global sections on X from local sections. These functions are called sections of $\mathcal{F}$ over U and the module of section $\mathcal{F}(U)$ can also be expressed as $\Gamma(U, \mathcal{F})$.

In the case of a compact toric variety X, the global holomorphic sections are the constant ones. Thus, if $\mathcal{O}_X$ is the sheaf of holomorphic functions defined by

$$\begin{aligned} U \to \mathcal{O}_X(U) &:= \{f : U \to \mathbb{C} \big| f \text{ is holomorphic on } U\} \\ &= \{f \in \mathbb{C}(X)^* \big| \operatorname{div}(f)|_U \geq 0\} \cup \{0\}, \end{aligned} \tag{1.4}$$

where $U \subseteq X$ open, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.

To every Weil divisor D, one associates a sheaf $\mathcal{O}_X(D)$ which is defined by

$$U \to \mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X)^* \big| (\operatorname{div}(f) + D)|_U \geq 0\} \cup \{0\}.$$

In particular, this is the sheaf of rational functions such that the multiplication of each one by the generating function of the divisor D is a holomorphic function. This notion gives rise to global analysis on compact projective toric varieties, where the global sections are quotients of 'homogeneous' functions.

If $D_1 \sim D_2$ then $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$, ([6, Proposition 4.0.29]).

Now, by taking a Weil divisor D on X, $D = \sum_i a_i D_i$ and by recalling the divisor of the rational function χ^m in (1.3), the relation

$$\operatorname{div}(\chi^m) + D \geq 0$$

is reformulated as

$$\langle m, v_i \rangle + a_i \geq 0, \tag{1.5}$$

for every $i = 1, \ldots, d$. Hence, the polyhedron P_D can be defined.

Definition 1.1.3 The *polyhedron* P_D of a Weil divisor D is defined as

$$P_D := \{ m \in M_{\mathbb{R}} : \langle m, v_i \rangle \geq -a_i, \text{for all } i = 1, \cdots, d \}.$$

This definition provides a direct way to determine the global sections of $\mathcal{O}_X(D)$.

Proposition 1.1.1 *[6, Proposition 4.3.3] If D is a torus-invariant Weil divisor on X, then $\mathcal{O}_X(D)$ is determined by the polyhedron P_D according to the relation*

$$\Gamma(X, \mathcal{O}_X(D)) = \bigoplus_{\operatorname{div}(\chi^m)+D\geq 0} \mathbb{C}\cdot\chi^m = \bigoplus_{m\in P_D\cap M} \mathbb{C}\cdot\chi^m.$$

Example 1.1.18 Since the Class group of $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}$ (see example 1.1.13) and since equivalent divisors induce isomorphic sheaves, it is sufficient to study the sheaf $\mathcal{O}_{\mathbb{P}^n}(kD_0)$ for $k \geq 0$ in order to obtain global sections. The polyhedron P_{kD_0} is k times the convex hull of the vectors $e_1, e_2, \ldots, e_n$, where e_i is the standard basis of $\mathbb{R}^n$. So, by considering characters as Laurent monomials $\chi^m(t) = t_1^{c_1} \cdots t_n^{c_n}$ for $m = (c_1, \ldots, c_n) \in P_{kD_0}$, we get

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kD_0)) \simeq \{ f \in \mathbb{C}[t_1, \ldots, t_n] \big| \deg(f) \leq k \}.$$

Here $\mathbb{C}[t_1, \ldots, t_n]$ is the polynomial ring in variables $t_1, \ldots, t_n$. In the next section, a homogenization of such polynomials yields

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kD_0)) \simeq \{ f \in \mathbb{C}[\zeta_0, \ldots, \zeta_n] \big| f \text{ is homogeneous with} \deg(f) = k \}.$$

Example 1.1.19 By considering $D = kD_2 + lD_4$ with $k, l \geq 0$ on $\mathbb{P}^1 \times \mathbb{P}^1$ (see example 1.1.14), the arising polyhedron P_D is the rectangle with vertices $(0,0),(k,0),(0,l),(k,l)$. Hence, Proposition 1.1.1 implies that for $\chi^m(t) = t_1^{c_1} t_2^{c_2}$ with $m = (c_1, c_2) \in P_D$

$$\begin{aligned} \Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(D)) &\simeq \{f \in \mathbb{C}[t_1, t_2] \big| \deg(f) \leq (k,l)\} \\ &\simeq \{f \in \mathbb{C}[\zeta_1, \zeta_2, \zeta_3, \zeta_4] \big| f \text{ is homogeneous with} \deg(f) = (k,l)\}. \end{aligned}$$

The degree of f is due to the total coordinate ring which is described in the next section.

1.1.4 The Total Coordinate Ring

Recall that in a toric variety X we introduced the variables ζ_i, corresponding to the generators v_i of the fan of X. Then, one considers the ring

$$S = \mathbb{C}[\zeta_1, \ldots, \zeta_d]$$

to be the total coordinate ring of the toric variety X. This ring has a close connection to the algebra and geometry of X. In particular, the grading of S by $\mathrm{Cl}(X)$, gives us the degree of each variable ζ_i and the meaning of a 'homogeneous' polynomial in a toric variety.

More specifically, one considers the short exact sequence

$$0 \to M \xrightarrow{\alpha} \bigoplus_i \mathbb{Z}D_i \xrightarrow{\beta} \mathrm{Cl}(X) \to 0,$$

where the map α sends $m \in M$ to $\mathrm{div}(\chi^m) = \sum_i \langle m, v_i \rangle D_i$, while β sends a Weil divisor to its divisor class in $\mathrm{Cl}(X)$. If $\zeta^a = \prod_{i=1}^d \zeta_i^{a_i}$, then its degree is defined as

$$\deg(\zeta^a) = \deg\left(\prod_{i=1}^d \zeta_i^{a_i}\right) = \big[\sum_i a_i D_i\big].$$

If S_λ is the corresponding graded piece of S for $\lambda = D \in \mathrm{Cl}(X)$, then $S_\lambda \simeq \Gamma(X, \mathcal{O}_X(\lambda))$ ([6, Proposition 5.3.7]). We say that $f \in S_\lambda$ is *homogeneous* of degree λ or f is called *D-homogeneous.*

The global sections $\Gamma(X, \mathcal{O}_X(D))$ of the sheaf $\mathcal{O}_X(D)$ have a connection to the coordinate ring described as follows. If $m \in P_D \cap M$, where $D = \sum_{i=1}^d a_i D_i$, then, following [6], the D-homogenization of χ^m is defined to be the monomial

$$\zeta^{\langle m, D \rangle} = \prod_{i=1}^d \zeta_i^{\langle m, v_i \rangle + a_i} \tag{1.6}$$

where $\zeta^{\langle m,D\rangle}$ belongs to S, due to the P_D-inequalities (1.5).

Example 1.1.20 The total coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is the ring $\mathbb{C}[\zeta_1, \zeta_2, \zeta_3, \zeta_4]$, where the variables ζ_i are not independent. Since $v_1 = e_1$, $v_2 = -e_1$, $v_3 = e_2$ and $v_4 = -e_2$, there is an exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\alpha} \bigoplus_{\rho=1}^{4} \mathbb{Z}D_\rho \xrightarrow{\beta} \mathbb{Z}^2 \longrightarrow 0,$$

where

$$\alpha(a_1, a_2) = a_1 D_1 - a_1 D_2 + a_2 D_3 - a_2 D_4$$

and

$$\beta(a_1 D_1 + a_2 D_2 + a_3 D_3 + a_4 D_4) = (a_1 + a_2, a_3 + a_4).$$

Thus, $\deg(\zeta_1) = \deg(\zeta_2) = (1,0)$, $\deg(\zeta_3) = \deg(\zeta_4) = (0,1)$ and $\deg(\zeta_1^{a_1}\zeta_2^{a_2}\zeta_3^{a_3}\zeta_4^{a_4}) = (a_1 + a_2, a_3 + a_4)$. Thus, from now on, by the term a 'homogeneous polynomial' on $\mathbb{P}^1 \times \mathbb{P}^1$ we will refer to a bihomogeneous polynomial, whose 'degree' is determined by the degree $[\sum_i a_i D_i]$. Namely, let $\lambda = kD_2 + lD_4 \in \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(kD_2 + lD_4) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l)$ be the associated sheaf to $S_{(k,l)}$. Then

$$\Gamma\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l)\right) \simeq S_{(k,l)},$$

which means that the global sections of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l)$ are homogeneous polynomials of degree k in ζ_1, ζ_2 and of degree l in ζ_3, ζ_4. The homogenization of a Laurent polynomial $\chi^m = t_1^a t_2^b$ for $m = (a,b) \in P_{kD_2+lD_4}$ is

$$\zeta^{\langle m,D\rangle} = \zeta_1^a \zeta_2^{k-a} \zeta_3^b \zeta_4^{l-b} = \zeta_2^k \zeta_4^l \left(\frac{\zeta_1}{\zeta_2}\right)^a \left(\frac{\zeta_3}{\zeta_4}\right)^b,$$

with respect to ζ_2, ζ_4. Then, if we recall the example 1.1.19 and take $f \in \Gamma\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l)\right)$, the homogenization of such polynomial is

$$F = \zeta_2^k \zeta_4^l f\left(\frac{\zeta_1}{\zeta_2}, \frac{\zeta_2}{\zeta_4}\right)$$

and the isomorphism in (1.6) is deduced.

1.1.5 Local Coordinates on a Toric Variety

On an n-dimensional toric variety X, let $\{v_1, \ldots, v_d\}$ be the generators of the fan Σ of X. Let $\check{\sigma}$ be a dual cone such that $\check{\sigma} = \mathrm{Cone}(u_1, \ldots, u_n)$. For every $j = 1, \ldots, n$, the monomials

$$\chi^{u_j} = \prod_{i=1}^{d=n+s} \zeta_i^{\langle u_j, v_i \rangle}$$

expressed in homogeneous coordinates $\zeta_1, \cdots, \zeta_d$ of X, are regular in the chart U_σ. They define a system of affine coordinates $(\zeta_1^\sigma, \ldots, \zeta_n^\sigma)$, where $\zeta_j^\sigma = \chi^{u_j}$. Every chart U_σ can be expressed as the quotient

$$U_\sigma = \left\{(\zeta_1, \ldots, \zeta_{n+s}) \in \mathbb{C}^{n+s} \setminus Z(\Sigma),\, \zeta_{n+1} \cdots \zeta_{n+s} \neq 0\right\} \Big/ G.$$

The rational functions ζ_j^σ satisfy $[\zeta_1^\sigma, \ldots, \zeta_n^\sigma, 1, \ldots, 1] = [\zeta_1, \ldots, \zeta_{n+s}]$.
Moreover, since

$$\zeta_j^\sigma = \chi^{u_j} = \prod_{i=1}^{d=n+s} \zeta_i^{\langle u_j, v_i \rangle} = \zeta_j \prod_{i=n+1}^{d} \zeta_i^{\langle u_j, v_i \rangle}$$

and $\prod_{i=n+1}^{d} \zeta_i^{\langle u_j, v_i \rangle} \neq 0$, it is implied that the divisor $D_j = \mathrm{div}(\zeta_j)$ in U_σ is

$$D_j|_{U_\sigma} = \mathrm{div}(\zeta_j^\sigma), \quad \forall\, j = 1, \ldots, d.$$

If $D = \sum_{j=1}^{d=n+s} a_j D_j$, then $D|_{U_\sigma} = (a_1 D_1 + \cdots a_n D_n)|_{U_\sigma}$ because $U_\sigma = X \setminus (D_{n+1} \cup \cdots D_{n+s})$. As a consequence,

$$D|_{U_\sigma} = \mathrm{div}\Big(\prod_{j=1}^{n} (\zeta_j^\sigma)^{a_j}\Big).$$

Example 1.1.21 Let $U_{\sigma_{ij}}$ for $i, j = 0, 1$ be the charts of $\mathbb{P}^1 \times \mathbb{P}^1$. We will concentrate on the chart $U_{\sigma_{00}}$, which corresponds to the cone $\sigma_{00} = \mathrm{Cone}(v_1, v_3) = \mathrm{Cone}(e_1, e_2)$. Similar observations will also hold in the remaining charts. The chart $U_{\sigma_{00}}$ is defined by

$$U_{\sigma_{00}} = \left\{(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mathbb{C}^4 \setminus \left\{\{0\} \times \mathbb{C}^2 \cup \mathbb{C}^2 \times \{0\}\right\},\, \zeta_2 \zeta_4 \neq 0\right\} \Big/ G$$

and $\zeta_1^{\sigma_{00}} = \frac{\zeta_1}{\zeta_2}$, $\zeta_2^{\sigma_{00}} = \frac{\zeta_3}{\zeta_4}$ are the rational functions defining the local coordinates. Moreover,

$$\begin{aligned} D_1|_{U_{\sigma_{00}}} &= \mathrm{div}\left(\frac{\zeta_1}{\zeta_2}\right) \\ D_3|_{U_{\sigma_{00}}} &= \mathrm{div}\left(\frac{\zeta_3}{\zeta_4}\right). \end{aligned}$$

In general, if $D = kD_2 + lD_4$, then

$$\begin{aligned}
D|_{U_{\sigma_{00}}} &= \operatorname{div}(1) \\
D|_{U_{\sigma_{10}}} &= (kD_2)|_{U_{\sigma_{10}}} = \operatorname{div}\left(\frac{\zeta_2}{\zeta_1}\right)^k \\
D|_{U_{\sigma_{11}}} &= (kD_2 + lD_4)|_{U_{\sigma_{11}}} = \operatorname{div}\left(\left(\frac{\zeta_2}{\zeta_1}\right)^k \left(\frac{\zeta_4}{\zeta_3}\right)^l\right) \\
D|_{U_{\sigma_{01}}} &= (lD_4)|_{U_{\sigma_{01}}} = \operatorname{div}\left(\frac{\zeta_4}{\zeta_3}\right)^l .
\end{aligned}$$

1.1.6 Line bundles and transition functions

The notion of vector bundles is a topological construction on a variety X that arises by attaching a vector space to every point of X. A vector bundle contains important global information, a property that makes it a useful and necessary tool in the case of compact toric varieties (where the global holomorphic functions are the constant ones).

Definition 1.1.4 A *vector bundle* V of rank r over a variety X is a manifold which satisfies the following.
There is a morphism $\pi : V \to X$ and an open cover $\{U_i\}$ of X such that:
(i) There is an isomorphism

$$\rho_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^r, \quad \forall\, i \tag{1.7}$$

such that ρ_i followed by a projection pr onto U_i is equal to $\pi|_{\pi^{-1}(U_i)}$, in other words $\mathrm{pr} \circ \rho_i = \pi|_{\pi^{-1}(U_i)}$. The functions ρ_i are called *trivializations.*
(ii) For every i, j, there exist $g_{ij} \in \mathrm{GL}_r(\Gamma(U_i \cap U_j, \mathcal{O}_X))$ such that the composition

$$\rho_i \circ \rho_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^r \longrightarrow (U_i \cap U_j) \times \mathbb{C}^r$$

is given by

$$(x, v) \longmapsto (x, g_{ij} v).$$

The g_{ij} are called the *transition functions* and by construction they satisfy the compatibility conditions:

$$g_{ik} = g_{ij} \circ g_{jk} \quad \text{on } U_i \cap U_j \cap U_k \quad \text{and} \quad g_{ij} = g_{ji}^{-1} \quad \text{on } U_i \cap U_j. \tag{1.8}$$

A vector bundle of rank 1 is called *line bundle.* Bundles are directly related to sections.

Definition 1.1.5 A *(global) section* of a vector bundle V is a mapping

$$s : X \to V$$

such that $\pi \circ s = id_X$.

Since there are no non-trivial holomorphic functions on a complete toric variety, from now on, we are going to work with bundles and sections instead of functions (see also Section 1.1.3).

Vector bundles and sheaves are related as described in the next proposition. More precisely, the notion of transition functions associates naturally the sheaf of a Cartier divisor with a sheaf of sections of a line bundle.

Proposition 1.1.2 *[6, Proposition 6.0.16] If X is a variety with open cover $\{U_i\}$ and for every i, j, we have $g_{ij} \in \mathrm{GL}_r(\Gamma(U_i \cap U_j, \mathcal{O}_X))$ satisfying the compatibility conditions (1.8), then:*
(i)There is a vector bundle of rank r over X, whose transition functions are the g_{ij}.
(ii) A global section $s : X \to V$ is uniquely determined by a collection of r-tuples $s_i \in \mathcal{O}_X^r$ such that on $U_i \cap U_j$, $s_i = g_{ij} s_j$ for all i, j.

For example, let V be a line bundle ($r = 1$) on a smooth toric variety X, $\{U_{\sigma_i}\}_i$ be its covering induced by the cones of its fan and $s : X \to V$ be a global section. If $\rho_{\sigma_i} : \pi^{-1}(U_{\sigma_i}) \simeq U_{\sigma_i} \times \mathbb{C}$ are the corresponding trivializations given in (1.7) and $\tilde{\pi}$ is the projection mapping such that $\tilde{\pi} : U_{\sigma_i} \times \mathbb{C} \to \mathbb{C}$, one can construct functions

$$s_{\sigma_i} : U_{\sigma_i} \to \mathbb{C} \tag{1.9}$$

that are defined on the open set U_{σ_i} of X (and are zero homogeneous with respect to the homogeneity of X) such that

$$s_{\sigma_i} = \tilde{\pi} \circ \rho_i \circ s\big|_{U_{\sigma_i}} \tag{1.10}$$

or, equivalently, $s_{\sigma_i} = \tilde{\rho}_{\sigma_i} \circ s|_{U_{\sigma_i}}$, where $\tilde{\rho}_{\sigma_i} = \tilde{\pi} \circ \rho_{\sigma_i}$. Moreover, observe that on $U_{\sigma_i} \cap U_{\sigma_j}$

$$s_{\sigma_i} = \tilde{\rho}_{\sigma_i} \circ s = \tilde{\rho}_{\sigma_i} \circ \tilde{\rho}_{\sigma_j}^{-1} \circ \tilde{\rho}_{\sigma_j} \circ s = \tilde{\rho}_{\sigma_i} \circ \tilde{\rho}_{\sigma_j}^{-1} \circ s_{\sigma_j} = g_{ij} \circ s_{\sigma_j} = g_{ij} s_{\sigma_j} \tag{1.11}$$

indicating that the functions s_{σ_i}, s_{σ_j} are compatible for every $i, j \in I$ and then s is realized through the use of the family $\{s_{\sigma_i}\}_i$. Since the transition functions are holomorphic, one can observe that the action of the $\bar{\partial}$ on (1.11) yields

$$\bar{\partial}s_{\sigma_i} = g_{ij}\bar{\partial}s_{\sigma_j}$$

on $U_{\sigma_i} \cap U_{\sigma_j}$ and thus the family $\{\bar{\partial}s_{\sigma_i}\}_i$ determines similarly the form $\bar{\partial}s$.

Also, the following theorem is of importance.

Theorem 1.1.1 *[6, Theorem 6.0.18] On a toric variety, the sheaf $\mathcal{L} = \mathcal{O}_X(D)$ of a Cartier divisor is the sheaf of sections of a line bundle $V_{\mathcal{L}} \to X$.*

In particular, if D is a Cartier divisor, then it is locally principal and thus, for an open cover $\{U_i\}$ of X, we get $D|_{U_i} = \mathrm{div}(f_i)|_{U_i}$ for $f_i \in \mathbb{C}(X)^*$. By using the local data of D, $\{U_i, f_i\}$, we can construct a line bundle $V_{\mathcal{L}}$ with transition functions $g_{ij} = f_i/f_j$, since they satisfy the compatibility conditions. Moreover, the functions $(g_{ij})^k$ also satisfy these conditions and they constitute the transition functions of a new line bundle $(V_{\mathcal{L}})^k$, that is the k-fold tensor products of $V_{\mathcal{L}}$ with itself.

Example 1.1.22 The sheaf of sections $\mathcal{O}_{\mathbb{P}^n}(kD_0)$ (see Example 1.1.18) which is also denoted by $\mathcal{O}_{\mathbb{P}^n}(k)$ indicating its independence from any choice of divisor from the collection $\{D_i\}_{i=0}^n$, induces the line bundle denoted by L^k. This bundle is the k-fold tensor product of the hyperplane bundle L of $\mathbb{P}^n$ with itself. Namely, if $\{U_i\}_{i=0}^n$ ($U_i = \{(\zeta_0, \ldots, \zeta_n)|\zeta_i \neq 0\}$) is the standard open covering of $\mathbb{P}^n$, then the local data of D_0 ($D_0 \sim D_1 \sim \cdots \sim D_n$) lead to the transition functions

$$g_{i,j} = \left(\frac{\zeta_j}{\zeta_i}\right)^k$$

of the line bundle L^k, $\pi : L^k \to \mathbb{P}^1$. The sheaf of sections of L^k is the sheaf $\mathcal{O}_{\mathbb{P}^n}(kD_0)$.

Example 1.1.23 Recall the sheaf $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(kD_2+lD_4)$. From the local data of the divisor $D = kD_2 + lD_4$, we observe that the transition functions with respect to the covering $\{U_{\sigma_{ij}}\}_{i,j=0}^1$ (see example 1.1.7) are given by

$$g_{00,10} = \left(\frac{\zeta_1}{\zeta_2}\right)^k,\ g_{00,01} = \left(\frac{\zeta_3}{\zeta_4}\right)^l,\ g_{00,11} = \left(\frac{\zeta_1}{\zeta_2}\right)^k\left(\frac{\zeta_3}{\zeta_4}\right)^l,$$

$$g_{10,01} = \left(\frac{\zeta_2}{\zeta_1}\right)^k\left(\frac{\zeta_3}{\zeta_4}\right)^l,\ g_{10,11} = \left(\frac{\zeta_3}{\zeta_4}\right)^l,\ g_{11,01} = \left(\frac{\zeta_2}{\zeta_1}\right)^k.$$

These transition functions satisfy the conditions (1.8) and hence give a line bundle $\pi : V_{\mathcal{L}} \to \mathbb{P}^1 \times \mathbb{P}^1$ where $V_{\mathcal{L}} = L^k \otimes L^l$ such that the sheaf of sections of this line bundle is the sheaf

$\mathcal{L} = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(kD_2 + lD_4)$.

1.1.7 Ample Divisors

The notion of ample divisors is pivotal for the rest of this work due to the projective embedding that they induce in the case of compact toric varieties.

Let D be a Cartier divisor on a complete toric variety X.

Definition 1.1.6 The divisor D is a *very ample divisor* if
(i) D has no basepoints i.e. for every $p \in X$, there is a section $s \in \Gamma(X, \mathcal{O}(D))$ with $s(p) \neq 0$.
(ii) The mapping

$$\begin{aligned} \phi_D : X &\rightarrow \mathbb{P}^{N-1} \\ p &\mapsto (\chi^{m_1}(p), \ldots, \chi^{m_N}(p)) \end{aligned} \tag{1.12}$$

where $m_i \in P_D \cap \mathbb{Z}^n$, for every $i = 1, \ldots, N$ and $N = |P_D \cap \mathbb{Z}^n|$, is a closed embedding (ϕ_D is an injective, continuous and closed map meaning that $\phi_D(W) \subseteq \mathbb{P}^{N-1}$ is closed for all closed subsets $W \subseteq X$).

The divisor D is *ample* when kD is very ample for some integer $k > 0$. In the particular case of a smooth complete toric variety, the divisor D is ample if and only if it is very ample.

The vectors m_i for every $i = 1, \ldots, N$ are called the *integral points* of the polytope P_D and if X is a compact toric variety, then X is called *projective.*

According to [6] (see Proposition 5.4.7.), $\phi_D : X \rightarrow \mathbb{P}^{N-1}$ is the Zariski closure X (subvariety of $\mathbb{P}^{N-1}$) of the image of $\Phi_D : T \rightarrow \mathbb{P}^{N-1}$ ($\phi_D\big|_T = \Phi_D$), where T is the torus and ϕ_D maps the point $t = (t_1, \ldots, t_n) \in T$ into $(\chi^{m_1}(t), \ldots, \chi^{m_N}(t))$. One can write

$$\phi_D(X) = \overline{\Phi_D(T)} = X. \tag{1.13}$$

Moreover the image $(\chi^{m_1}(p), \ldots, \chi^{m_N}(p))$ of a point p in X through the map ϕ_D can be expressed equivalently as

$$\left(\prod_{i=1}^{d} \zeta_i^{\langle m_1, v_i \rangle}, \ldots, \prod_{i=1}^{d} \zeta_i^{\langle m_N, v_i \rangle} \right)$$

with respect to the variables $(\zeta_1, \ldots, \zeta_d)$, while by applying the D-homogenization argument, the image is given by

$$\left(\prod_{i=1}^{d}\zeta_i^{\langle m_1,v_i\rangle+a_i},\ldots,\prod_{i=1}^{d}\zeta_i^{\langle m_N,v_i\rangle+a_i}\right). \tag{1.14}$$

Hence, these polynomials give an explicit construction of the quotient of the toric variety X by mapping $\mathbb{C}^d \setminus Z(\Sigma)$ to the projective space via these polynomials.

When $D=\sum_{i=1}^{d} a_i D_i$ is a Cartier divisor, there exists $m_\sigma \in M$ such that

$$\langle m_\sigma, v_i\rangle = -a_i \tag{1.15}$$

for every $i=1,\ldots,d$ and for each n-dimensional cone $\sigma \in \Sigma$.

There are simple conditions to determine whether a divisor is an ample divisor.

Proposition 1.1.3 *[6, Theorem 6.1.7] The following are equivalent:*
(i) D has no basepoints.
(ii) $m_\sigma \in P_D$ for every n-dimensional cone $\sigma \in \Sigma$.
(iii) $\{m_\sigma | \sigma \in \Sigma\}$ is the set of vertices of P_D.

Proposition 1.1.4 *[6, Lemma 6.1.13] The divisor D is ample if and only if $m_\sigma \in P_D$ and $m_{\sigma_1} \neq m_{\sigma_2}$ for every two n-dimensional cones $\sigma_1, \sigma_2 \in \Sigma$ with $\sigma_1 \neq \sigma_2$.*

Now, there is an easy way-formula to compute ample divisors on a smooth complete toric variety. Let $\tau = \sigma \cap \sigma'$ for σ, σ' two disjoint n- dimensional cones of Σ and a vector v_j be a generator of the cone σ' but not of σ. Then a necessary and sufficient condition to decide whether a divisor $D=\sum_{i=1}^{d} a_i D_i$ is ample, is the following.

Proposition 1.1.5 *The divisor D is ample if and only if*

$$\langle m_\sigma, v_j\rangle > -a_j \tag{1.16}$$

for every n-dimensional cone σ, where v_j is satisfying the assumptions of the previous paragraph.

Example 1.1.24 Recall the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ with generators $v_1 = e_1$, $v_2 = -e_1$, $v_3 = e_2$ and $v_4 = -e_2$. The Class and Picard groups are equal and

$$\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \{kD_2 + lD_4 | k, l \in \mathbb{Z}\}.$$

The relation (1.15) yields that

$$m_{\sigma_1} = (0,0),\ m_{\sigma_2} = (k,0), m_{\sigma_3} = (k,l), m_{\sigma_4} = (0,l).$$

Hence, by considering the divisor $D = kD_2 + lD_4$, it turns out that D is very ample if and only if $k, l > 0$, according to (1.16).

In general, if $D = \sum_{i=1}^{4} a_i D_i$, then

$$D \sim (a_1 + a_2)D_2 + (a_3 + a_4)D_4,$$

which is ample if and only if $a_1 + a_2 > 0$ and $a_3 + a_4 > 0$.

Example 1.1.25 Let us consider the Hirzebruch surface H_r. By example 1.1.15,

$$\mathrm{Pic}(H_r) = \mathrm{Cl}(H_r) \simeq \{kD_3 + lD_4 | k, l \in \mathbb{Z}\}.$$

The vectors m_{σ_i} are

$$m_{\sigma_1} = (-k,0),\ m_{\sigma_2} = (-k,l), m_{\sigma_3} = (rl,l), m_{\sigma_4} = (0,0).$$

Let $D = kD_3 + lD_4$. Then, (1.16) yields that D is ample if and only if $k, l > 0$.

Now, if $D = \sum_{i=1}^{4} a_i D_i$, then

$$D \sim (a_1 - ra_2 + a_3)D_3 + (a_2 + a_4)D_4.$$

Thus, D is ample if and only if $a_1 - ra_2 + a_3 > 0$ and $a_2 + a_4 > 0$.

Example 1.1.26 In the particular case of the Hirzebruch surface $\mathcal{H}$ of example 1.1.16, the Picard group is given by

$$\mathrm{Pic}(\mathcal{H}) = \mathrm{Cl}(\mathcal{H}) \simeq \{kD_3 + lD_4 | k, l \in \mathbb{Z}\},$$

as in the previous example. The vectors m_{σ_i} are

$$m_{\sigma_1} = (0,0),\ m_{\sigma_2} = (k,0), m_{\sigma_3} = (k,l-k), m_{\sigma_4} = (0,l).$$

Let $D = kD_3 + lD_4$. Then, (1.16) yields that D is ample if and only if $k, l > 0$ and $l > k$, as opposed to the previous example.

Example 1.1.27 By recalling the weighted projective space $\mathbb{P}(1,1,2)$ (see example 1.1.12) and taking the divisor $D = 2D_0$, then $m_{\sigma_0} = (0,0)$, $m_{\sigma_1} = (2,0)$ and $m_{\sigma_2} = (0,1)$. Since m_{σ_i} satisfies the wall inequality (1.16) for every $i = 0,1,2$, the divisor D is ample.

1.2 Weighted Koppelman formula on $\mathbb{C}^n$

Let $\mathcal{E}_{p,q}(\Omega)$ be the space of smooth (p,q) forms in the open set $\Omega \subset \mathbb{C}^n$ while $\mathcal{D}_{p,q}(\Omega) \subset \mathcal{E}_{p,q}(\Omega)$ denotes the subspace of smooth (p,q) forms which are also compactly supported in Ω. In other words, $\mathcal{D}_{p,q}(\Omega)$ is the space of (p,q) test forms on Ω. This space is endowed with the topology of uniform convergence: the sequence $\{\phi_k\} \in \mathcal{D}_{p,q}(\Omega)$ tends to zero if and only if $\operatorname{supp} \phi_k \subset K \subset\subset \Omega$, for a fixed K and both ϕ_k and all its derivatives tend uniformly to zero. A current ψ of bidegree (p,q) on Ω, written as $\psi \in \mathcal{L}_{p,q}(\Omega)$, is a linear continuous form $\psi : \mathcal{D}_{n-p,n-q}(\Omega) \to \mathbb{C}$. If ϕ is a form on $\mathcal{D}_{n-p,n-q}(\Omega)$, the value of ψ at ϕ is denoted by $< \psi, \phi >$. According to distribution theory, every (p,q) form ψ with coefficients in $\mathcal{L}^1_{loc}(\Omega)$ defines a current of bidegree (p,q)

$$< \psi, \phi > := \int_{\Omega} \psi \wedge \phi, \quad \phi \in \mathcal{D}_{n-p,n-q}(\Omega).$$

Thus, the dual pairing $< \psi, \phi >$ is often replaced by the integral notation.

Moreover, if Z an $(n-p)$-dimensional complex submanifold of a complex manifold X, then a (p,p)-current of integration over Z is defined by

$$< [Z], \phi > := \int_Z \phi, \quad \phi \in \mathcal{D}_{n-p,n-p}(X).$$

In order to integrate the form ϕ on Z one can use the partition of unity and local coordinates. If $\{U_i\}$ is an open cover of Z and the restriction of ϕ on Z, $\phi\big|_Z$, is supported on a chart $\varphi_i : U_i \to V_i$ where V_i is an open set of $\mathbb{C}^n$ then

$$< [Z], \phi > := \int_Z \phi := \int_{V_i} (\varphi_i^{-1})^* \phi.$$

Moreover, with every closed analytic subset Z of a complex manifold X of dimension $(n-p)$ there is also an associated current of integration defined by integration an $(n-p,n-p)$ test form on X over the regular points of Z.

Currents inherit properties of differential forms like the 'commutative law' of the wedge product such that if $\psi \in \mathcal{L}_{p,q}(\Omega)$ and $\omega \in \mathcal{E}_{r,s}(\Omega)$, a $(p+r, q+s)$- current arises, following the rule

$$< \psi \wedge \omega, \phi > = < \psi, \omega \wedge \phi > = (-1)^{(p+q)(r+s)} < \omega \wedge \psi, \phi >, \quad \phi \in \mathcal{D}_{n-p-r,n-q-s}(\Omega). \quad (1.17)$$

The exterior differentiation of the current $\psi \in \mathcal{L}_{p,q}(\Omega)$ is defined by

$$< \bar{\partial}\psi, \phi > = (-1)^{p+q+1} < \psi, \bar{\partial}\phi >, \quad \phi \in \mathcal{D}_{n-p,n-q-1}(\Omega), \quad (1.18)$$

as a consequence of the classical Stokes formula. According to this rule, $\partial\psi$ and $d\psi$ provide also currents of bidegree $(p+1,q)$ and $(p+1,q+1)$, respectively.

Consider a pair of multivariables $z = (z_1, \ldots, z_n) \in \Omega$ and $\zeta = (\zeta_1, \ldots, \zeta_n) \in \Omega$. For notational convenience, following [1], we are also making use of the settings $\mathcal{L}^m(\Omega) = \bigoplus_{k=0}^n \mathcal{E}_{k,k+m}(\Omega)$ for the space of smooth forms while $\mathcal{L}^m_{curr}(\Omega) = \bigoplus_{k=0}^n \mathcal{L}_{k,k+m}(\Omega)$ for the corresponding space of currents. Let $\eta(\zeta, z) = z - \zeta$. It is a $(0,0)$ form vanishing over the diagonal $\Delta = \{\zeta = z, (\zeta, z) \in \Omega \times \Omega\}$. Define $E^* = \{d\eta_1, \ldots, d\eta_n\} \subset T^*_{1,0}(\Omega \times \Omega)$ to be the dual bundle of $E = \mathrm{span}\{e_1, \ldots, e_n\}$, observe that the basis element of E are the dual elements for the base of E^*.

Following [1], [2], the $(1,0)$- vector field $\delta_{z-\zeta}$ is defined to be

$$\delta_{z-\zeta} = 2\pi i \sum_{i=1}^n (z_i - \zeta_i) \frac{\partial}{\partial \zeta_i}.$$

This contraction acts on smooth differential forms by interior multiplication ($\neg$) according to the rule

$$\frac{\partial}{\partial \zeta_i} \neg d\zeta_j = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker delta. It is easy to observe that $\delta_{\zeta-z}$ anticommutes with the $\bar{\partial}$- operator (i.e. $\delta_{\zeta-z}\bar{\partial} = -\bar{\partial}\delta_{\zeta-z}$). Furthermore, one considers the operator ∇ determined by

$$\nabla := \nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial}.$$

Since $\delta_{\zeta-z}$ lowers degree in z_i by one, while $\bar{\partial}$ increases the conjugate degree by one. Thus ∇ maps $\mathcal{L}^m$ to $\mathcal{L}^{m+1}$.

Definition 1.2.1 A *weight* g is defined to be any smooth form $g \in \mathcal{L}^0(\Omega \times \Omega)$ satisfying both conditions $g_{0,0}(z,z) = 1$ and $\nabla g = 0$.

Introducing weights to the problem of finding a form $u \in \bigwedge(E^* \oplus T^*_{0,1})$ such that $\nabla u = 1 - [\Delta]$, is transforming the problem to finding a form $K \in \bigwedge(E^* \oplus T^*_{0,1})$ satisfying the equation

$$\nabla K = g - [\Delta] \tag{1.19}$$

in the current sense. Here $[\Delta]$ denotes the current of integration over the diagonal Δ of $\Omega \times \Omega$. More precisely, K will belong to $\mathcal{L}^{-1}_{curr}(\Omega \times \Omega)$ since in the right hand side of (1.19), $[\Delta]$ belongs to $\mathcal{L}^{0}_{curr}(\Omega \times \Omega)$ and $g \in \mathcal{L}^0(\Omega \times \Omega)$.

In the case of representing (p, q) forms, a suitable choice of K results from the wedge product of a weight g with a contraction of the Bochner-Martinelli form, namely

$$u := \frac{b}{\nabla b} = \sum_{k=1}^{n} \frac{b \wedge (\bar{\partial} b)^{k-1}}{(\delta_{z-\zeta} b)^k}, \tag{1.20}$$

where z and ζ are both considered to be variables. The $(1, 0)$ form $b(z, \zeta)$ is given by

$$b(z, \zeta) = \frac{1}{2\pi i} \frac{\partial |z - \zeta|^2}{|z - \zeta|^2}, \tag{1.21}$$

where ∂ acts on both ζ and z variables.

In particular, if z is considered to be a fixed point in Ω rather than a variable (this happens in the case of smooth functions), then ∂ indicates the derivative ∂_ζ and hence b can be written in the simpler form

$$b = \frac{1}{2\pi i} \sum_{j=1}^{n} \frac{(\bar{z}_j - \bar{\zeta}_j)}{|z - \zeta|^2} d\zeta_j \tag{1.22}$$

since $\delta_{\zeta - z} b = 1$. Thus, in this particular case, the form u equals

$$u = \sum_{k=1}^{n} b \wedge (\bar{\partial} b)^{k-1}. \tag{1.23}$$

Put $K = (u \wedge g)_{n,n-1}$ and $P = g_{n,n}$ then the equation (1.19) becomes

$$\bar{\partial} K = [\Delta] - P \tag{1.24}$$

and the desired Koppelman formula in $\mathbb{C}^n$ is deduced. More specifically, one has the following Theorem [1].

Theorem 1.2.1 *Assume that $D \subset\subset \Omega$ and $\phi \in \mathcal{E}_{p,q}(\overline{D})$. Then $\phi(z)$ can be represented as* *the sum of integrals*

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar{\partial}\phi + \bar{\partial}_z \int_D K \wedge \phi + \int_D P \wedge \phi, \tag{1.25}$$

by means of the form K and the smooth form P and the integrals are taken over the ζ variable.

Proof. At first, observe that if ϕ has a compact support in D, then the boundary integral vanishes. Since the representation (1.25) has a dual meaning, consider a test form $\psi \in \mathcal{D}_{n-p,n-q}(\Omega)$ that is acting on the right hand side of the renewed (1.25). Then, by making use Stokes theorem and having in mind the equality of currents (1.24), it turns out that

$$\begin{aligned}
& \int_z \left(\int_\zeta K \wedge \bar{\partial}\phi + \bar{\partial}_z \int_\zeta K \wedge \phi + \int_\zeta P \wedge \phi \right) \wedge \psi \\
= & \int_{z,\zeta} K \wedge d\phi \wedge \psi + (-1)^{p+q} \int_{z,\zeta} K \wedge \phi \wedge d\psi + \int_{z,\zeta} P \wedge \phi \wedge \psi \\
= & \int_{z,\zeta} K \wedge d(\phi \wedge \psi) + \int_{z,\zeta} P \wedge \phi \wedge \psi \\
= & \int_{z,\zeta} dK \wedge \phi \wedge \psi + \int_{z,\zeta} P \wedge \phi \wedge \psi \\
= & \int_{z,\zeta} \bar{\partial}K \wedge \phi \wedge \psi + \int_{z,\zeta} P \wedge \phi \wedge \psi \\
= & \int_{z,\zeta} ([\Delta] - P) \wedge \phi \wedge \psi + \int_{z,\zeta} P \wedge \phi \wedge \psi \\
= & \int_{z,\zeta} [\Delta] \wedge \phi \wedge \psi = \int_z \phi \wedge \psi.
\end{aligned}$$

Otherwise, in the case when ϕ does not have a compact support, ϕ can be decomposed into a compactly supported form ϕ_1 in D and a form ϕ_2 that equals to zero in a neighborhood of z such that $\phi = \phi_1 + \phi_2$. According to the previous paragraph, it remains to show that (1.25) holds for ϕ_2. Similarly, one takes a test form ψ with the additional property of having support in the vanishing neighborhood of ϕ_2. Then

$$\begin{aligned}
& \int_z \left(\int_{\partial D} K \wedge \phi_2 \right) \wedge \psi = \int_{z,\zeta} d_\zeta \left(K \wedge \phi_2\right) \wedge \psi \\
= & \int_{z,\zeta} d\left(K \wedge \phi_2\right) \wedge \psi - \int_{z,\zeta} d_z \left(K \wedge \phi_2\right) \wedge \psi \\
= & \int_{z,\zeta} dK \wedge \phi_2 \wedge \psi - \int_{z,\zeta} K \wedge d\phi_2 \wedge \psi - \int_{z,\zeta} d_z \left(K \wedge \phi_2\right) \wedge \psi \\
= & \int_{z,\zeta} ([\Delta] - P) \wedge \phi_2 \wedge \psi - \int_{z,\zeta} K \wedge \bar{\partial}\phi_2 \wedge \psi - \int_{z,\zeta} \bar{\partial}_z \left(K \wedge \phi_2\right) \wedge \psi \\
= & \int_z \phi_2 \wedge \psi - \int_{z,\zeta} P \wedge \phi_2 \wedge \psi - \int_{z,\zeta} K \wedge \bar{\partial}\phi_2 \wedge \psi - \int_{z,\zeta} \bar{\partial}_z \left(K \wedge \phi_2\right) \wedge \psi,
\end{aligned}$$

or, equivalently

$$\int_z \phi_2 \wedge \psi \quad = \quad \int_z \left(\int_{\partial D} K \wedge \phi_2 + \int_D K \wedge \bar{\partial}\phi_2 + \bar{\partial}_z \int_D K \wedge \phi_2 + \int_D P \wedge \phi_2 \right) \wedge \psi.$$

Hence, a Koppelman representation formula for ϕ is obtained by combining the corresponding formulas for ϕ_1 and ϕ_2. $\square$

For the spaces $\mathcal{E}(\overline{D})$ of smooth functions on $\overline{D}$ and $\mathcal{O}(\overline{D})$ of holomorphic functions on $\overline{D}$, one has the following result.

Corollary 1.2.1 If $\phi \in \mathcal{E}(\overline{D})$, the Weighted Koppelman formula is reformulated as follows:

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar{\partial}\phi + \int_D P \wedge \phi, \quad \phi \in \mathcal{E}(\overline{D}). \tag{1.26}$$

Moreover, if $\phi \in \mathcal{O}(\overline{D})$ then

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D P \wedge \phi, \quad \phi \in \mathcal{O}(\overline{D}). \tag{1.27}$$

Proof. In particular, when a function $\phi \in \mathcal{E}(\overline{D})$ for $D \subset\subset \Omega$, the third term of the Koppelman formula does not exist. Then (1.26) is trivial. Moreover, if $\phi \in \mathcal{O}(\overline{D})$ then the second term of (1.26) also vanishes and (1.27) is deduced. $\square$

Remark 1.2.1 The Koppelman representation formula (1.27) is also valid for functions which are continuous on $\overline{D}$ and holomorphic on D, whenever D can be approximated by an increasing (with respect to $\subseteq$) sequence of domains (this is the case when the ∂D is reasonable). More precisely, if ρ is a defining function of D, we can assume that ρ is a $\mathcal{C}^1$ function in a neighborhood U of ∂D such that $D \cap U = \{\rho < 0\}$, $\partial D \cap U = \{\rho = 0\}$ and $d\rho \neq 0$ in U. For a sufficiently small $\epsilon > 0$, if we set $D_\epsilon := \{\rho < -\epsilon\}$, then the Koppelman formula (1.27) is valid on each D_ϵ and a limiting procedure as $\epsilon \to 0$ yields the desired result. Thus, if $\phi \in \mathcal{A}(D) = \mathcal{O}(D) \cap \mathcal{C}(\overline{D})$, then

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D P \wedge \phi. \tag{1.28}$$

1.3 Weighted Koppelman formula on $\mathbb{P}^n$

A Weighted Koppelman formula which provides integral representations for sections on a line bundle, is a generalization of the corresponding formula on $\mathbb{C}^n$. Since $\mathbb{P}^n$ is compact and the global holomorphic functions are the constant ones, the handling of the global object leads to the the use of the line bundle L^k (see Example 1.1.22) with the corresponding sheaf of sections being $\mathcal{O}_{\mathbb{P}^n}(kD_0)$ (the sheaf of meromorphic functions that are isomorphic with k-homogeneous polynomials on $\mathbb{C}^{n+1}$ according to the example 1.1.18).

Let $\mathcal{E}_{0,q}(\Omega, L^k)$ be the space of $(0,q)$ forms on $\Omega \subset \mathbb{P}^n$ taking values in L^k, in other words the space of sections belonging to the vector bundle $\wedge^{0,q}T^*(\Omega) \otimes L^k$. A form on $\mathbb{C}^{n+1}$ is called *projective* if it arises from the pullback of a differential form on $\mathbb{P}^n$ through the canonical projection of $\mathbb{C}^{n+1} \setminus \{0\}$ in $\mathbb{P}^n$. Hence, $\phi \in \mathcal{E}_{0,q}(\Omega, L^k)$ if its pullback to $\mathbb{C}^{n+1}$ is a k-homogeneous projective form of bidegree $(0,q)$. All the forms that appear in the Weighted Koppelman formula on $\mathbb{P}^n$ are projective forms on $\mathbb{C}^{n+1}$, in order to be well-defined.

To decide whether a $(p,0)$-form f on $\mathbb{C}^{n+1}$ is projective or not, E. Götmark formulated in [14] a necessary and sufficient condition which says that f is projective if and only if

$$\delta_\zeta f = 2\pi i \sum_{i=0}^{n} \zeta_i \frac{\partial}{\partial \zeta_i} \neg f = 0.$$

Similarly, a (p,q) form f on $\mathbb{C}^{n+1}$ is projective if and only if it is both δ_ζ and $\delta_{\bar\zeta}$ closed.

Instead of the Bochner-Martinelli form, which plays a fundamental role in the construction of the kernels on $\mathbb{C}^n$, here (following the ideas of Götmark [14]) we consider the $(1,0)$ form

$$v = \bar z \cdot d\zeta - \frac{(\bar z \cdot \zeta)(\bar\zeta \cdot d\zeta)}{|\zeta|^2} \tag{1.29}$$

that takes values in $L^1_{[\zeta]} \otimes L^1_{[z]}$ and

$$u = \frac{v}{\nabla_z v} = \sum_{k=1}^{n} \frac{v \wedge (\bar\partial u)^{k-1}}{(\delta_z v)^k} \tag{1.30}$$

which is a contraction of v. The vector field $\delta_z = 2\pi i \sum_{i=0}^{n} z_i \frac{\partial}{\partial \zeta_i}$ was introduced instead of the corresponding field $\delta_{\zeta - z}$ in $\mathbb{C}^n$, while $\nabla_z = \delta_z - \bar\partial$. An extension of the notion of weights to $\mathbb{P}^n$ also exists. According to [14], a smooth form $g \in \mathcal{L}^0$ is a *projective weight* in $\mathbb{P}^n$ if $\nabla_z g = 0$, $g_{0,0}([z],[z]) = 1$ and g_k takes values in $L^{k-m}_{[\zeta]} \otimes L^{m-k}_{[z]}$ for some fixed m. A specific weight

$$\alpha = \frac{\bar\zeta \cdot z}{|\zeta|^2} - \frac{1}{2\pi i}\bar\partial\left(\frac{\bar\zeta \cdot d\zeta}{|\zeta|^2}\right) \tag{1.31}$$

is used here such that $\alpha_{0,0}$ takes values in the line bundle $L^{-1}_{[\zeta]} \otimes L^1_{[z]}$, while $\alpha_{1,1}$ in $L^0_{[\zeta]} \otimes L^0_{[z]}$. Hence, the kernels K and P are defined by

$$K = (u \wedge \alpha^{n+k})_{n,n-1} \quad \text{and} \quad P = (\alpha^{n+k})_{n,n}$$

respectively and they satisfy the current equation $\bar{\partial} K = [\Delta] - P$. The power of the weight α was chosen such that the integrands in the Weighted Koppelman formula on $\mathbb{P}^n$ take values in the trivial line bundle (being of homogeneity zero in ζ variable) for a form $\phi \in \mathcal{E}_{0,q}(\overline{D}, L^k)$ where $D \subset\subset \Omega$. Hence, the Koppelman formula on $\mathbb{P}^n$ follows:

Theorem 1.3.1 *If $D \subset\subset \Omega$ for some domain $\Omega \subset \mathbb{P}^n$ and $\phi \in \mathcal{E}_{0,q}(\overline{D}, L^k)$, then*

$$\phi(z) = \int_{\partial D} K \wedge \phi + \int_D K \wedge \bar{\partial}\phi + \bar{\partial}_z \int_D K \wedge \phi + \int_D P \wedge \phi, \tag{1.32}$$

where the integrals are taken over the ζ variable.

The form α can be replaced by any other projective weight, g, such that

$$K = (u \wedge g)_{n,n-1} \quad \text{and} \quad P = g_{n,n}. \tag{1.33}$$

Hence, ϕ will take values in different line bundles according to the choice of the projective weight, such that the integrals take values in the trivial line bundle.

1.4 Multi-Logarithmic Residue current

Let $f = (f_1, \ldots, f_p)$ be a p-tuple of holomorphic functions in a domain $G \subset \mathbb{C}^n$ $(p \leq n)$ defining a complete intersection which means that $Z_f = f^{-1}(0) \cap G$ has dimension $n - p$. The *residual current* $\bar{\partial}\left(\frac{1}{f}\right) = \bigwedge_{j=1}^{p} \bar{\partial}\left(\frac{1}{f_j}\right)$ introduced in [7] is defined by

$$< \bar{\partial}\left(\frac{1}{f}\right), \phi > := \lim_{\delta \to 0} \int\limits_{\substack{|f_j(z)| = \epsilon_j(\delta) \\ 1 \leq j \leq p}} \frac{\phi}{f_1 \ldots f_p}, \tag{1.34}$$

where $\phi \in \mathcal{D}_{n,n-p}(G)$ and $\delta \mapsto (\epsilon_1(\delta), \ldots, \epsilon_p(\delta))$ is an admissible path, that is,

$$\lim_{\delta \to 0} \frac{\epsilon_j(\delta)}{\epsilon_{j+1}^k(\delta)} = 0 \quad \text{for any } j \in \{1, \ldots, p-1\} \text{ and any } k \in \mathbb{N}.$$

The limit of (1.34) is independent of the admissible path. For notational convenience let $T^\delta(f) = \{z : |f_j(z)| = \epsilon_j(\delta), j = 1, \ldots, p\}$.

This current is associated with the *multi-logarithmic residue current* which is denoted by

$$< \bar{\partial}\frac{1}{f} \wedge df, \phi > := \lim_{\delta \to 0} \int_{T^\delta(f)} \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_p}{f_p} \wedge \phi, \tag{1.35}$$

for a $\phi \in \mathcal{D}_{n-p,n-p}(G)$.

Following [39], we first define the *geometric multiplicity* $\mu_a(g)$ of a holomorphic mapping $g : \overline{U}_a \to \mathbb{C}^n$ defined in a neighborhood $U_a \subset \mathbb{C}^n$, where a is an isolated zero of g. If the closure $\overline{U}_a$ does not contain any other zero of g except of $z = a$, then there exists a positive number ε such that for almost all ζ belonging to the polydisc $\{\zeta : |\zeta_j| < \varepsilon\}$, the mapping $w = g(z) - \zeta$ has only simple (isolated) zeros in U_a (in other words the Jacobian $J_w = \partial w/\partial z$ is nonzero at these points). Moreover, the number of these zeros is finite and independent of the choice of the point ζ and the neighborhood U_a ([39]). We refer to, the number of such simple zeros as the geometric multiplicity $\mu_a(g)$ of g at a.

Now, in view of the preceding paragraph, let us also define the *multiplicity* $\mu_S(f)$ *of* f *along the irreducible components* S, where S are the irreducible components of Z_f. For each regular point a in Z_f, that is a point for which there is a neighborhood U such that $Z_f \cap U$ is a complex submanifold of U, let L_a be a complex plane transversal to the tangent plane of Z_f at a and $l_{p+1}, \ldots, l_n$ be the corresponding linear functions. The geometric multiplicity $\mu_a(g)$ of the system $g = (f_1, \ldots, f_p, l_{p+1}, \ldots, l_n)$ which remains constant for any transversal plane L_a and for any regular point a lying on one irreducible component S of Z_f due to Rouche's Theorem, defines the multiplicity $\mu_S(f)$. Then,

$$\int_{Z_f} \phi = \sum_S \mu_S(f) \int_S \phi, \quad \phi \in \mathcal{D}_{n-p,n-p}(G).$$

The following theorem constitutes a fundamental tool in our construction. It can be viewed as a generalization of the 'classical' Poincaré-Lelong formula which is the following result for $p = 1$.

Theorem 1.4.1 [7] Let $f = (f_1, \ldots, f_p)$ be a p-tuple of holomorphic functions in a domain $G \subset \mathbb{C}^n$ defining a complete intersection $Z_f = f^{-1}(0) \cap G$. Then,

$$\left(\frac{1}{2\pi i}\right)^p < \bar{\partial}\frac{1}{f} \wedge df, \phi >= \int_{Z_f} \phi,$$

where $\phi \in \mathcal{D}_{n-p,n-p}(G)$.

Proof. Following [39], we consider a holomorphic function ψ on G such that ψ vanishes on the singular points of Z_f located in the support of ϕ. Then $(f, \psi) : G \to \mathbb{C}^{p+1}$ is a complete intersection in a neighborhood of the $\operatorname{supp}\phi$ and according to [39] the identity

$$< \bar{\partial}\frac{1}{f} \wedge df, \phi >= \lim_{r\to 0}\lim_{\delta\to 0} \int_{T^{\delta,r}} \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_p}{f_p} \wedge \phi \tag{1.36}$$

holds, where $T^{\delta,r} = T^{\delta}(f\big|_{D^r})$ is the tube constructed from the restriction of f to the domain

$D^r = \{z \in G : |\psi(z)| > r\}$ (no any singular point of $Z_f \cap \operatorname{supp}\phi$ is contained in the tube). A sequence $\{r_k\}$ that tends to zero can be constructed such that r_k^2 are noncritical values for the restriction of $|\psi|^2$ on the regular part of Z_f. For each k, let $K^k = Z_f \cap \overline{D}^{r_k} \cap \operatorname{supp}\phi$ and $\{U_\nu^k\}$ be a finite open cover for each K^k in G. A parametrization of these open sets allows us to write $U_\nu^k \cap D^{r_k} = U' \times U''$ where U' is a neighborhood of $\mathbb{C}^p$ with coordinates $z' = (z_1, \ldots, z_p)$ while U'' is a neighborhood of zero in $\mathbb{C}^{n-p}$ with coordinates $z'' = (z_{p+1}, \ldots, z_n)$ and $Z_f \cap U_\nu^k = 0' \times U''$.

By using a partition of unity argument subordinate to $\{U_\nu^k\}$, a family of functions $\eta_\nu^k : K^k \to [0,1]$ satisfying $\sum_\nu \eta_\nu^k = 1$ and $\operatorname{supp}(\eta_\nu) \subset U_\nu^k$ is constructed, such that the theorem is reduced to proving the equation

$$\lim_{\delta\to 0} \int_{T^{\delta,r_k}} \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_p}{f_p} \wedge \eta_\nu^k \phi = (2\pi i)^p \mu_S(f) \int_{Z_f \cap U_\nu^k} \eta_\nu^k \phi$$

or, equivalently,

$$\lim_{\delta\to 0} \int_{T^{\delta,r_k} \cap (U' \times U'')} \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_p}{f_p} \wedge \eta_\nu^k \phi = (2\pi i)^p) \mu_S(f) \int_{0' \times U''} \eta_\nu^k \phi. \tag{1.37}$$

In the above integrals, $\mu_S(f)$ is the multiplicity of the mapping f along the irreducible component S intersecting U_ν^k. The integral on the left hand side of (1.37) is modified to an iterated integral according to the following argument. The tube $T^{\delta,r_k} \cap (U' \times U'')$ can be split into the analytic sets $A(\zeta) = \{f = \zeta\} \cap \{U' \times U''\}$ where ζ runs over the distinguished boundary $\Gamma^\delta = \{\zeta \in \mathbb{C}^p : |\zeta_j| = \epsilon_j(\delta), j = 1, \ldots, p\}$. Hence, one can find a proper analytic subset V such that $A(\zeta)$ is smooth in a neighborhood $\mathcal{V}$ of the origin in $\mathbb{C}^p$ outside V (see [39]). Since the path is admissible, Γ^δ does not intersect V and $T^{\delta,r_k} \cap (U' \times U'')$ is decomposed into the nonsingular fibres $A(\zeta)$. Moreover, $f = (f_1, \ldots, f_p)$ does not have zeros on $\partial U' \times U''$ since $0' \times U'' = \{f = 0\} \cap U_\nu^k$ and $A(\zeta)$ for ζ close to zero does not intersect $\partial U' \times U''$. Let $\pi : A(\zeta) \to U''$ denotes the projection mapping of the analytic set $A(\zeta) \subset U' \times U''$ into U'' and set σ equal to the discriminant set of this covering over U'' which is the image of the analytic set $\left\{\dfrac{\partial f}{\partial z'} = 0\right\} \cap A(\zeta)$. The number of sheets of the covering π identifies with the multiplicity of f along $0' \times U''$ $(= \mu_S(f)$ along a component S of the analytic set Z_f intersecting U_ν^k). Then,

$$\lim_{\delta\to 0} \int_{T^{\delta,r_k} \cap (U' \times U'')} \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_p}{f_p} \wedge \eta_\nu^k \phi = \lim_{\delta\to 0} \int_{\Gamma^\delta} \frac{d\zeta_1}{\zeta_1} \wedge \ldots \wedge \frac{d\zeta_p}{\zeta_p} \int_{A(\zeta)} \eta_\nu^k \phi \tag{1.38}$$

where the inner integral on the right hand side will be denoted by $h(\zeta)$ and satisfies

$$h(\zeta) := \int_{A(\zeta)} \eta_\nu^k \phi = \int_{U'' \setminus \sigma} (\eta_\nu^k \phi)(\pi^{-1}(z'')) = \int_{U''} (\eta_\nu^k \phi)(\pi^{-1}(z'')).$$

Note that $h(\zeta)$ is continuous for $\zeta \in \mathcal{V}$. Letting $\zeta \to 0$, since all the sheets of the covering π tend to $0' \times U''$, we get

$$h(0) = \mu_S(f) \int_{0' \times U''} \eta_\nu^k \phi$$

and hence, the right hand side of the (1.38) yields $(2\pi i)^p \mu_S(f) \int_{0' \times U''} \eta_\nu^k \phi$. It turns out that equation (1.37) holds. Thus,

$$\lim_{\delta \to 0} \int_{T^{\delta}(f)} \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_p}{f_p} \wedge \phi = (2\pi i)^p \sum \mu_S(f) \int_S \phi = (2\pi i)^p \int_{Z_f} \phi$$

and the result follows. $\square$

Chapter 2

Boundary properties of functions representable by Weighted Koppelman formula and related Hartogs phenomenon

In the present chapter we establish results concerning the boundary behaviour of weighted Koppelman type integral with a specific choice of weight. Our model are the related results concerning Bohner-Martinelli (B-M) type integrals to be found in [26] and [29]. The differential forms involved in the Koppelman kernel have term contained in B-M integration kernel, but in the case under study the kernels involved are not harmonic. Thus, to some extend the results obtained are on the one hand close in spirit to those found in [30] and [31], but on the other hand they are also surprising because the kernels involved do not have the aforementioned property of B-M kernel.

2.1 Jump Theorems for a Weighted Koppelman formula

A jump theorem for a Weighted Koppelman formula holds for continuous functions satisfying a Hölder condition. This theorem can be generalized on functions that are continuous and integrable in the boundary of a domain D. We also derive a result concerning $\bar{\partial}$-normal derivative of the Koppelman integrals.

2.1.1 A Jump Theorem for continuous functions satisfying a Hölder condition

In this section we assume that D is a domain in $\mathbb{C}^n$ which has a boundary ∂D of class $\mathcal{C}^1$, that is, we can write

$$D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$$

where ρ is a real-valued function of class $\mathcal{C}^1$ in some neighborhood of D and $d\rho \neq 0$ on ∂D.

For a specific choice of a weight in the Weighted Koppelman formula on $\mathbb{C}^n$ (Section 1.2), all the results in this chapter are derived. However, we point out that other weights with similar properties may contribute in deducing similar results. It is straightforward to see that the form

$$h(\zeta, z) \quad := \quad 1 - \nabla\left(\frac{\partial|\zeta - z|^2}{2\pi i(1 + |\zeta - z|^2)}\right).$$

is a weight in $\mathbb{C}^n$ (see Definition 1.2.1) since $h_{0,0}(z,z) = 1$ and $\nabla h = 0$. Moreover, h can be written in the following simpler form

$$h(\zeta, z) \quad = \quad \frac{1}{1 + |\zeta - z|^2} + \bar\partial \frac{\partial|\zeta - z|^2}{2\pi i(1 + |\zeta - z|^2)} \tag{2.1}$$

where $h_{0,0}(\zeta, z) = \frac{1}{1+|\zeta-z|^2} = O(1)$ while $h_{1,1}(\zeta, z) = \bar\partial\frac{\partial|\zeta-z|^2}{2\pi i(1+|\zeta-z|^2)} = O(|\zeta - z|^{-2})$. The weight g is defined to be $g = h^n$. Its component of bidegree $(n-k, n-k)$, namely

$$g_{n-k,n-k} = (h^n)_{n-k,n-k} = \binom{n}{k}(h_{0,0})^k(h_{1,1})^{n-k},$$

satisfies the growth estimate $g_{n-k,n-k} = O(|\zeta - z|^{-2(n-k)})$, for every $k = 1, \dots, n$.

By using this weight g in the construction of the Koppelman kernel, it takes the following form:

$$\begin{aligned} K &= (u \wedge g)_{n,n-1} = \sum_{k=1}^{n} u_{k,k-1} \wedge g_{n-k,n-k} \\ &= \sum_{k=1}^{n} \frac{\binom{n}{k}}{(2\pi i)^n} \frac{\partial|\zeta - z|^2}{|\zeta - z|^2} \wedge \left(\bar\partial \frac{\partial|\zeta - z|^2}{|\zeta - z|^2}\right)^{k-1} \wedge \left(\frac{1}{1 + |\zeta - z|^2}\right)^k \left(\bar\partial \frac{\partial|\zeta - z|^2}{1 + |\zeta - z|^2}\right)^{n-k}. \end{aligned}$$

Quotient rule

$$\bar\partial \frac{\partial|\zeta - z|^2}{|\zeta - z|^2} = \frac{\bar\partial\partial|\zeta - z|^2}{|\zeta - z|^2} - \frac{\partial|\zeta - z|^2}{|\zeta - z|^2} \wedge \frac{\bar\partial|\zeta - z|^2}{|\zeta - z|^2},$$

and multiplication by the form $\partial|\zeta-z|^2/|\zeta-z|^2$, imply that

$$\frac{\partial|\zeta-z|^2}{|\zeta-z|^2}\wedge\bar\partial\,\frac{\partial|\zeta-z|^2}{|\zeta-z|^2}=\frac{\partial|\zeta-z|^2}{|\zeta-z|^2}\wedge\frac{\bar\partial\partial|\zeta-z|^2}{|\zeta-z|^2}.$$

Similarly, one obtains

$$\frac{\partial|\zeta-z|^2}{|\zeta-z|^2}\wedge\bar\partial\,\frac{\partial|\zeta-z|^2}{1+|\zeta-z|^2}=\frac{\partial|\zeta-z|^2}{|\zeta-z|^2}\wedge\frac{\bar\partial\partial|\zeta-z|^2}{1+|\zeta-z|^2}.$$

Hence,

$$\begin{aligned} K &= \sum_{k=1}^{n}\frac{\binom{n}{k}}{(2\pi i)^n|\zeta-z|^{2k}(1+|\zeta-z|^2)^n}\,\partial|\zeta-z|^2\wedge(\bar\partial\partial|\zeta-z|^2)^{n-1} \\ &= \sum_{k=1}^{n}\frac{\binom{n}{k}(n-1)!}{(2\pi i)^n|\zeta-z|^{2k}(1+|\zeta-z|^2)^n}\sum_{j=1}^{n}(-1)^{j-1}(\bar\zeta_j-\bar z_j)d\bar\zeta[j]\wedge d\zeta, \end{aligned} \tag{2.2}$$

where $d\bar\zeta[j]$ denotes the wedge product $d\bar\zeta_1\wedge\ldots\wedge d\bar\zeta_n$ from which the differential $d\bar\zeta_j$ is omitted while $d\zeta=d\zeta_1\wedge\ldots\wedge d\zeta_n$. Observe that

$$K=\left(1-\frac{1}{1+|\zeta-z|^2|}\right)^n\sum_{k=1}^{n}\frac{\binom{n}{k}}{|\zeta-z|^{2k}}\,u_{n,n-1},$$

where $u_{n,n-1}$ is the term of the form u in (1.20) with bidegree $(n,n-1)$ (in other words, $u_{n,n-1}$ is the Bochner-Martinelli kernel). Since

$$\frac{1}{(1+|\zeta-z|^2)^n}\leq\frac{1}{(1+|\zeta-z|^2)^{n-k}}\leq\frac{1}{|\zeta-z|^{2n-2k}} \tag{2.3}$$

for every $k=1,\ldots,n$, it turns out that

$$K(\zeta,z)=O(|\zeta-z|^{1-2n}). \tag{2.4}$$

On the other hand, using the identities $\partial|\zeta-z|^2\wedge\partial|\zeta-z|^2=0$, $\bar\partial|\zeta-z|^2\wedge\bar\partial|\zeta-z|^2=0$ and $n\partial|\zeta-z|^2\wedge\bar\partial|\zeta-z|^2\wedge(\bar\partial\partial|\zeta-z|^2)^{n-1}=|\zeta-z|^2(\bar\partial\partial|\zeta-z|^2)^n$, the kernel P takes the explicit form

$$
\begin{aligned}
P &= g_{n,n} = (h_{1,1})^n = \left(\bar{\partial}\frac{\partial|\zeta - z|^2}{2\pi i(1+|\zeta - z|^2)}\right)^n \\
&= \left(\frac{\bar{\partial}\partial|\zeta - z|^2}{2\pi i(1+|\zeta - z|^2)} - \frac{\partial|\zeta - z|^2 \wedge \bar{\partial}|\zeta - z|^2}{2\pi i(1+|\zeta - z|^2)^2}\right)^n \\
&= \left(\frac{1}{2\pi i}\right)^n \left[\frac{(\bar{\partial}\partial|\zeta - z|^2)^n}{(1+|\zeta - z|^2)^n} - \frac{n\partial|\zeta - z|^2 \wedge \bar{\partial}|\zeta - z|^2 \wedge (\bar{\partial}\partial|\zeta - z|^2)^{n-1}}{(1+|\zeta - z|^2)^{n+1}}\right] \\
&= \left(\frac{1}{2\pi i}\right)^n \frac{(\bar{\partial}\partial|\zeta - z|^2)^n}{(1+|\zeta - z|^2)^{n+1}} \\
&= \left(\frac{1}{2\pi i}\right)^n \frac{(-1)^{\frac{n(n-1)}{2}} n!}{(1+|\zeta - z|^2)^{n+1}}\, d\bar{\zeta} \wedge d\zeta. \qquad (2.5)
\end{aligned}
$$

One observes that the boundary behavior of Weighted Koppelman type integrals will depend mainly on the part K of the kernels involved.

Recall that a function ϕ satisfies the Hölder condition with the exponent $\alpha > 0$ in ∂D if

$$|\phi(\zeta) - \phi(z)| \leq C|\zeta - z|^\alpha, \quad \text{for } \zeta, z \in \partial D, \qquad (2.6)$$

where $C > 0$. For notational convenience, let $\mathcal{C}^\alpha(\partial D)$ be the class of continuous functions that satisfy Hölder condition with exponent α given in (2.6).

Let $\phi \in \mathcal{C}^\alpha(\partial D)$. The next lemma guarantees the extension of ϕ to a neighborhood $V(\partial D)$ of ∂D, as a function satisfying the Hölder condition with the same exponent α. This key lemma is known but its proof is given for the completeness of the thesis.

Lemma 2.1.1 *If $\phi \in \mathcal{C}^\alpha(\partial D)$ then ϕ can be extended as a function satisfying a Hölder condition with the same exponent α in a neighborhood $V(\partial D)$ of ∂D such that $\overline{D} \subset V(\partial D)$.*

In order to prove this extension lemma, we quote an auxiliary result.

Lemma 2.1.2 *Let u be a real function in ∂D such that $u \in \mathcal{C}^\alpha(\partial D)$. Then, u can be extended to $V(\partial D) \supset \overline{D}$, such that Hölder condition holds in $V(\partial D)$ with the same exponent α.*

Proof. Let us define

$$U(\zeta) := \sup_{z \in \partial D} [u(z) - C|z - \zeta|^\alpha], \qquad (2.7)$$

for $\zeta \in V(\partial D) \supset \overline{D}$. We will show that U is the required extension of u. Observe that, $U(\zeta) = u(\zeta)$ for $\zeta \in \partial D$ because $u(z) - C|z - \zeta|^\alpha \leq u(\zeta)$ by the Hölder condition and this upper bound $u(\zeta)$ is attained for $\zeta = z$.

Now, consider ζ and ζ' be any two points in $V(\partial D)$. Hence, $U(\zeta)$ and $U(\zeta')$ are both finite.

Without loss of generality, we can assume that $U(\zeta) \geq U(\zeta')$. Then, U satisfies a Hölder condition in $V(\partial D)$ since

$$\begin{aligned}
0 \leq U(\zeta) - U(\zeta') &= \sup_{z\in\partial D}[u(z) - C|z-\zeta|^\alpha] - \sup_{z\in\partial D}[u(z) - C|z-\zeta'|^\alpha] \\
&\leq \sup_{z\in\partial D}\left[(u(z) - C|z-\zeta|^\alpha) - (u(z) - C|z-\zeta'|^\alpha)\right] \\
&= \sup_{z\in\partial D}\left[C|z-\zeta'|^\alpha - C|z-\zeta|^\alpha\right] \\
&\leq \sup_{z\in\partial D}\left[C\left(|z-\zeta| + |\zeta-\zeta'|\right)^\alpha - C|z-\zeta|^\alpha\right] \\
&\leq C|\zeta-\zeta'|^\alpha, \qquad (2.8)
\end{aligned}$$

where we used the triangle inequality $|z-\zeta'| = |z-\zeta+\zeta-\zeta'| \leq |z-\zeta| + |\zeta-\zeta'|$ and the increasing monotonicity of the function $g(t) = t^\alpha$ for $t > 0$ while $g(t)$ satisfies the inequality $g(t_1) + g(t_2) \geq g(t_1+t_2)$ as a concave function. It turns out that U satisfies the Hölder condition on $V(\partial D)$. □

Now, we can turn back to the proof of Lemma 2.1.1.

Proof Lemma 2.1.1. We begin with the fact that $\phi(z)$ as a complex function can be written in the form $\phi(z) = \mathrm{Re}\phi(z) + i\mathrm{Im}\phi(z)$. The functions $\mathrm{Re}\phi(z)$ and $\mathrm{Im}\phi(z)$ satisfy a Hölder condition with exponent α in $\partial D \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, since

$$|\mathrm{Re}\phi(\zeta) - \mathrm{Re}\phi(z)| \leq |\phi(\zeta) - \phi(z)| \leq C|\zeta - z|^\alpha \quad \text{for} \quad \zeta, z \in \partial D$$

and similarly

$$|\mathrm{Im}\phi(\zeta) - \mathrm{Im}\phi(z)| \leq C|\zeta - z|^\alpha \quad \text{for} \quad \zeta, z \in \partial D.$$

By lemma 2.1.2, $\mathrm{Re}\phi(z)$ and $\mathrm{Im}\phi(z)$ as real functions in ∂D can be extended to $V(\partial D) \supset \overline{D}$ as functions satisfying a Hölder condition on $V(\partial D)$ with the same exponent α and we denote these extended functions by $U(z)$ and $V(z)$, respectively. Hence,

$$\tilde{\phi}(z) = U(z) + iV(z)$$

is the extension of $\phi(z)$ on $V(\partial D) \supset \overline{D}$ such that

$$\begin{aligned}
|\tilde{\phi}(\zeta) - \tilde{\phi}(z)| &= |U(\zeta) - U(z) + i\left(V(\zeta) - V(z)\right)| \\
&\leq |U(\zeta) - U(z)| + |V(\zeta) - V(z)| \\
&\leq 2C|\zeta - z|^\alpha,
\end{aligned}$$

and the result follows. In the sequel by abusing the notation, we will denote $\tilde{\phi}$ by ϕ too. $\square$

Remark 2.1.1 The extension of Hölder condition of an $\phi \in \mathcal{C}^{\alpha}(\partial D)$ also holds outside D. Modifying slightly the previous construction, one can observe that the relation (2.8) also holds for $\zeta, \zeta' \in \mathbb{C}^n \setminus D$. To be more precise, let

$$U(\zeta) := \sup_{z \in \partial D} [u(z) - C|z - \zeta|^{\alpha}], \quad \text{for} \quad \zeta \in \mathbb{C}^n \setminus D$$

be the extension of the real function $u(\zeta)$ in $\mathbb{C}^n \setminus D$ (see Lemma 2.1.2). Then, in view of (2.8), the inequality

$$|U(\zeta) - U(\zeta')| \leq C|\zeta - \zeta'|^{\alpha} \tag{2.9}$$

also holds for $\zeta, \zeta' \in \mathbb{C}^n \setminus D$, if $U(\zeta)$ and $U(\zeta')$ are both finite. The inequality (2.9) is preserved even if one of the values of $U(\zeta)$ and $U(\zeta')$ is finite and the other one is infinite (without loss of generality assume that $\infty = U(\zeta) > U(\zeta')$), since (2.8) also holds under this modification. Then, the case of $U(\zeta)$ and $U(\zeta')$ being both infinite is impossible since by choosing $\zeta' \in \partial D$, $U(\zeta')$ is finite and then $U(\zeta)$ for $\zeta \in \mathbb{C}^n \setminus D$ must be also finite in view of (2.9). It turns out that $U(\zeta)$ is finite for every $\zeta \in \mathbb{C}^n \setminus D$ and analogously to Lemma 2.1.1, ϕ can be extended as a function satisfying a Hölder condition with the same exponent α in $\mathbb{C}^n \setminus D$.

Lemma 2.1.1 allows us to consider the function $\Phi(z)$ defined by

$$\Phi(z) = \int_{\partial D} \phi(\zeta) K(\zeta, z) + \int_{D} \phi(\zeta) P(\zeta, z), \quad z \notin \partial D \tag{2.10}$$

for an $\phi \in \mathcal{C}^{\alpha}(\partial D)$ where K and P are given explicitly in (2.2) and (2.5), respectively.

Let Φ^{+} be the sum of the integrals in (2.10) when $z \in D$, while Φ^{-} denotes the same sum when $z \notin \overline{D}$. Observe that the kernel P which is defined in (2.5) is an (n, n) smooth form with coefficients smooth functions in $z \in \overline{D}$ and thus, it has no singularities. It implies that the integral $\int_D \phi(\zeta) P(\zeta, z)$ is well-defined (since $\phi(\zeta)$ has been extended). However, the kernel K described by (2.2), has singularities when $\{\zeta = z\}$ due to the denominator of u. Thus, we consider the principal value of the boundary integral in the Koppelman type representation (2.10):

$$\text{P. V.} \int_{\partial D} \phi(\zeta) K(\zeta, z) = \lim_{\epsilon \to 0^{+}} \int_{\partial D \setminus B(z, \epsilon)} \phi(\zeta) K(\zeta, z), \quad z \in \partial D. \tag{2.11}$$

Later on, we will prove the existence of this limit.

Following ([26]), let us define $\tau(z)$ to be the solid angle of the tangent cone to the surface ∂D at z. More precisely, define

$$\tau(z) = \lim_{\epsilon \to 0^+} \frac{\operatorname{Vol} S^+(z,\epsilon)}{\operatorname{Vol} S(z,\epsilon)},$$

where $S^+(z,\epsilon)$ is the part of the sphere $S(z,\epsilon) = \{\zeta : |\zeta - z| = \epsilon\}$ lying in D, that is $S^+(z,\epsilon) = D \cap S(z,\epsilon)$. Then, the following lemma holds.

Lemma 2.1.3

$$\text{P.V.} \int_{\partial D} K(\zeta,z) = \tau(z) - \int_D P(\zeta,z),$$

for $z \in \partial D$.

Proof. By the definition of the principal value at $z \in \partial D$, one has

$$\begin{aligned} \text{P.V.} \int_{\partial D} K(\zeta,z) &= \lim_{\epsilon \to 0^+} \int_{\partial D \setminus B(z,\epsilon)} K(\zeta,z) \\ &= \lim_{\epsilon \to 0^+} \left(\int_{\partial (D \setminus B(z,\epsilon))} K(\zeta,z) + \int_{S^+(z,\epsilon)} K(\zeta,z) \right). \end{aligned} \tag{2.12}$$

Notice that the change of the sign of the second integral in (2.12) is due to the opposite orientation of $S(z,\epsilon)$ in order to distinguish it from the respective orientation of ∂D. By applying Stokes' theorem, the first term of the expression (2.12) can be rewritten to the form

$$\int_{\partial (D \setminus B(z,\epsilon))} K = \int_{D \setminus B(z,\epsilon)} \bar{\partial}_\zeta K = \int_{D \setminus B(z,\epsilon)} dK = -\int_{D \setminus B(z,\epsilon)} P,$$

since the kernels K and P satisfy the current equation $dK = [z] - P$ according to (1.24), where $[z]$ is the Dirac measure at z considered as the (n,n) current point evaluation at z. Now, as $\epsilon \to 0^+$, we get that

$$\lim_{\epsilon \to 0^+} \int_{\partial (D \setminus B(z,\epsilon))} K = -\int_D P. \tag{2.13}$$

For the second integral of (2.12), the behavior of the restriction of the kernel K through the sphere $S(z,\epsilon)$ have to be explored. According to [27], the form $d\bar{\zeta}[j] \wedge d\zeta$ is expressed with respect to the Lebesgue surface measure $d\sigma$ after passing to the real coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ due to $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The restrictions of the forms $dx[j] \wedge dy$ and $dx \wedge dy[j]$ to the boundary ∂D equal to $(-1)^j \gamma_j d\sigma$ and $(-1)^{n+j-1} \gamma_{j+n} d\sigma$ where γ_j is the j-th dirrection cosine of the normal vector to ∂D. In particular,

$$\gamma_j = \frac{\partial \rho}{\partial x_j} \frac{1}{2|\text{grad}\rho|} \quad \text{and} \quad \gamma_{n+j} = \frac{\partial \rho}{\partial y_j} \frac{1}{2|\text{grad}\rho|},$$

where $|\text{grad}\rho| = \frac{1}{2}\sqrt{\sum_{k=1}^{n}\left[\left(\frac{\partial\rho}{\partial x_k}\right)^2 + \left(\frac{\partial\rho}{\partial y_k}\right)^2\right]}$. Since, $\frac{\partial\rho}{\partial\bar{\zeta}_j} = \frac{1}{2}\left(\frac{\partial\rho}{\partial x_j} + i\frac{\partial\rho}{\partial y_j}\right)$, after expressing $d\bar{\zeta}[j]\wedge d\zeta$ in terms of $dx[j]\wedge dy$ and $dx\wedge dy[j]$, it follows that

$$d\bar{\zeta}[j]\wedge d\zeta\big|_{\partial D} = 2^{n-1}i^n(-1)^{j-1}\frac{\partial\rho}{\partial\bar{\zeta}_j}\frac{d\sigma}{|\text{grad}\rho|}. \tag{2.14}$$

In particular, if $d\sigma$ is the area element on the sphere then

$$\sum_{j=1}^{n}(-1)^{j-1}(\bar{\zeta}_j - \bar{z}_j)d\bar{\zeta}[j]\wedge d\zeta\big|_{S(z,\epsilon)} = 2^{n-1}i^n\epsilon d\sigma. \tag{2.15}$$

Thus, a substitution of the above form (2.15) into the explicit form of the Koppelman kernel (2.2) yields that

$$K\big|_{S(z,\epsilon)} = \sum_{k=1}^{n}\frac{\binom{n}{k}(n-1)!2^{n-1}i^n\epsilon}{(2\pi i)^n\epsilon^{2k}(1+\epsilon^2)^n}d\sigma$$

which leads to

$$\begin{aligned}
\int_{S^+(z,\epsilon)}K(\zeta,z) &= \sum_{k=1}^{n}\frac{\binom{n}{k}(n-1)!2^{n-1}i^n\epsilon}{(2\pi i)^n\epsilon^{2k}(1+\epsilon^2)^n}\int_{S^+(z,\epsilon)}1d\sigma\\
&= \sum_{k=1}^{n}\frac{\binom{n}{k}\epsilon^{2n-1-(2k-1)}}{(1+\epsilon^2)^n}\frac{(n-1)!}{2\pi^n\epsilon^{2n-1}}\text{Vol}(S^+(z,\epsilon))\\
&= \sum_{k=1}^{n}\binom{n}{k}\frac{\epsilon^{2n-2k}}{(1+\epsilon^2)^n}\frac{\text{Vol}(S^+(z,\epsilon))}{\text{Vol}(S(z,\epsilon))}.
\end{aligned}$$

When $\epsilon\to 0^+$, one observes that the only non-trivial term corresponds to the case $k=n$. Hence,

$$\lim_{\epsilon\to 0^+}\int_{S^+(z,\epsilon)}K(\zeta,z) = \lim_{\epsilon\to 0^+}\frac{\text{Vol}(S^+(z,\epsilon))}{\text{Vol}(S(z,\epsilon))} = \tau(z). \tag{2.16}$$

The result follows from substitution of (2.13) and (2.16) into (2.12). $\square$

Let $F(z)$ be the function defined by the integral

$$F(z) = \int_{\partial D}(\phi(\zeta)-\phi(z))\,K(\zeta,z), \quad z\in\mathbb{C}^n\setminus\partial D. \tag{2.17}$$

Observe that, $F(z)$ has no singularity for $z\in\mathbb{C}^n\setminus\partial D$. On the other hand for $\zeta, z\in\partial D$ one has

$$|\phi(\zeta)-\phi(z)|\,|K(\zeta,z)| \leq C|\zeta - z|^{\alpha+1-2n}d\sigma, \tag{2.18}$$

since $K(\zeta, z) = O(|\zeta - z|^{1-2n})$.

The above statement guarantees the absolute convergence of $F(z)$ for $z \in \partial D$ thus making $F(z)$ to be well-defined over the whole $\mathbb{C}^n$.

Lemma 2.1.4 *If ϕ satisfies a Hölder condition with exponent α, $0 < \alpha < 1$ in $V(\partial D)$, then $F(z)$ satisfies the same Hölder condition in $V(\partial D)$.*

In order to prove this result, let us denote by

$$\Delta_j(\bar{\zeta}, \bar{z}^1, \bar{z}^2) := \frac{(\bar{\zeta}_j - \bar{z}_j^1)}{|\zeta - z^1|^{2k}(1 + |\zeta - z^1|^2)^n} - \frac{(\bar{\zeta}_j - \bar{z}_j^2)}{|\zeta - z^2|^{2k}(1 + |\zeta - z^2|^2)^n} \tag{2.19}$$

and prove the following auxiliary lemma.

Lemma 2.1.5 *Let z^1, z^2 be two points in $V(\partial D)$ such that $|z^1 - z^2| = \delta$ for a small enough $\delta > 0$ and let the ball $B(z^1, 2\delta) \subset V(\partial D)$. If $\zeta = (\zeta_1, \ldots, \zeta_n) \in \partial D \cap \big(B(z^1, 2\delta)\big)^c$ then*

$$\left|\Delta_j(\bar{\zeta}, \bar{z}^1, \bar{z}^2)\right| = \delta O(|\zeta - z^1|^{-2n}),$$

for every $j \in \{1, \ldots, n\}$.

Proof of Lemma 2.1.5. Observe that for every $j \in \{1, \ldots, n\}$, one has

$$\left|\Delta_j(\bar{\zeta}, \bar{z}^1, \bar{z}^2)\right| \leq \frac{|\bar{z}_j^2 - \bar{z}_j^1|}{|\zeta - z^1|^{2k}(1 + |\zeta - z^1|^2)^n} + |\bar{\zeta}_j - \bar{z}_j^2| \left|A_{2k}^n(\zeta, z^1, z^2)\right|, \tag{2.20}$$

where

$$A_{2k}^n(\zeta, z^1, z^2) := \frac{1}{|\zeta - z^1|^{2k}(1 + |\zeta - z^1|^2)^n} - \frac{1}{|\zeta - z^2|^{2k}(1 + |\zeta - z^2|^2)^n}. \tag{2.21}$$

In order to estimate $A_{2k}^n(\zeta, z^1, z^2)$, one uses the binomial expansion of $(1 + |\zeta - z^j|^2)^n$, $j = 1, 2$ to get

$$\begin{aligned}
\left|A_{2k}^n(\zeta, z^1, z^2)\right| &= \left|\frac{1}{|\zeta - z^1|^{2k}(1 + |\zeta - z^1|^2)^n} - \frac{1}{|\zeta - z^2|^{2k}(1 + |\zeta - z^2|^2)^n}\right| \\
&= \frac{\left||\zeta - z^2|^{2k} \sum_{l=0}^n \binom{n}{l} |\zeta - z^2|^{2l} - |\zeta - z^1|^{2k} \sum_{l=0}^n \binom{n}{l} |\zeta - z^1|^{2l}\right|}{|\zeta - z^1|^{2k}|\zeta - z^2|^{2k}(1 + |\zeta - z^1|^2)^n(1 + |\zeta - z^2|^2)^n}.
\end{aligned}$$

Hence, the above expression can be equivalently written as

$$\left|A_{2k}^n(\zeta, z^1, z^2)\right| = \frac{\left|\sum_{l=0}^n \binom{n}{l} \left(\frac{|\zeta - z^2|^{2l}}{|\zeta - z^1|^{2k}} - \frac{|\zeta - z^1|^{2l}}{|\zeta - z^2|^{2k}}\right)\right|}{(1 + |\zeta - z^1|^2)^n(1 + |\zeta - z^2|^2)^n}. \tag{2.22}$$

A geometric sequence argument yields that, for each $0 \leq l \leq n$,

$$\begin{aligned}\left|\frac{|\zeta - z^2|^{2l}}{|\zeta - z^1|^{2k}} - \frac{|\zeta - z^1|^{2l}}{|\zeta - z^2|^{2k}}\right| &= \frac{|\zeta - z^1|^{2l}}{|\zeta - z^2|^{2k}} \frac{\big||\zeta - z^2| - |\zeta - z^1|\big|}{|\zeta - z^1|} \sum_{s=0}^{2l+2k-1} \left|\frac{\zeta - z^2}{\zeta - z^1}\right|^s \\ &\leq \frac{\delta\,|\zeta - z^1|^{2l}}{|\zeta - z^2|^{2k}|\zeta - z^1|} \sum_{s=0}^{2l+2k-1} \left|\frac{\zeta - z^2}{\zeta - z^1}\right|^s.\end{aligned}$$

By combining the above relations, we get

$$\begin{aligned}\left|A^n_{2k}(\zeta, z^1, z^2)\right| &= \left|\frac{1}{|\zeta - z^1|^{2k}(1 + |\zeta - z^1|^2)^n} - \frac{1}{|\zeta - z^2|^{2k}(1 + |\zeta - z^2|^2)^n}\right| \\ &\leq \frac{\delta \sum_{l=0}^{n} \binom{n}{l} |\zeta - z^1|^{2l} \sum_{s=0}^{2l+2k-1} \left|\frac{\zeta - z^2}{\zeta - z^1}\right|^s}{(1 + |\zeta - z^1|^2)^n (1 + |\zeta - z^2|^2)^n |\zeta - z^2|^{2k} |\zeta - z^1|},\end{aligned}$$

The use of (2.3) yields that the first term of (2.20) is less or equal than $\delta/|\zeta - z^1|^{2n}$, while the second term of (2.20) is

$$\begin{aligned}|\bar{\zeta}_j - \bar{z}^2_j|\left|A^n_{2k}(\zeta, z^1, z^2)\right| &\leq \frac{\delta \sum_{l=0}^{n} \binom{n}{l} \frac{|\zeta - z^1|^{2l}}{(1 + |\zeta - z^1|^2)^l} \frac{1}{(1 + |\zeta - z^1|^2)^{n-l}} \sum_{s=0}^{2l+2k-1} \left|\frac{\zeta - z^2}{\zeta - z^1}\right|^s}{|\zeta - z^1||\zeta - z^2|^{2n-1}} \\ &\leq \frac{\delta \sum_{l=0}^{n} \binom{n}{l} \sum_{s=0}^{2l+2k-1} \left|\frac{\zeta - z^2}{\zeta - z^1}\right|^s}{|\zeta - z^1||\zeta - z^2|^{2n-1}}. \qquad (2.23)\end{aligned}$$

Now, since ζ can be chosen close enough to $\partial D \cap \big(B(z^1, 2\delta)\big)^c$ such that the following inequality

$$\frac{2}{3}|\zeta - z^2| \leq |\zeta - z^1| \leq 2|\zeta - z^2|,$$

holds, one can write that

$$\sum_{l=0}^{n} \binom{n}{l} \sum_{s=0}^{2l+2k-1} \left|\frac{\zeta - z^2}{\zeta - z^1}\right|^s \leq \sum_{l=0}^{n} \binom{n}{l} \sum_{s=0}^{2l+2k-1} \left(\frac{3}{2}\right)^s = \text{Constant}.$$

and hence,

$$|\bar{\zeta}_j - \bar{z}^2_j|\left|A^n_{2k}(\zeta, z^1, z^2)\right| \leq \frac{\delta \cdot 2^{2n-1} \cdot \text{Constant}}{|\zeta - z^1|^{2n}} = \delta O(|\zeta - z^1|^{2n})$$

Then, the result arises. $\square$

Proof of Lemma 2.1.4. As in the proof of the preceding lemma, consider two points in $V(\partial D)$, namely z^1, z^2 such that $|z^1 - z^2| = \delta$, for a small enough δ. Let the ball $B(z^1, 2\delta)$ in $V(\partial D)$. By taking the projections of z^j onto $\partial D \cap B(z^1, 2\delta)$, a local compactness argument allows to change the domain of integration from $\partial D \cap B(z^1, 2\delta)$ to the $(2n-1)$-dimensional sphere of radius δ. Then, by using polar coordinates in this sphere, one can observe that

$$\int\limits_{\partial D\cap B(z^1,2\delta)} |\zeta - z^j|^{1-2n} d\sigma \leq C_1. \tag{2.24}$$

Hence, by using (2.18), for $j = 1, 2$

$$\begin{aligned}\left|\int\limits_{\partial D\cap B(z^1,2\delta)} \left(\phi(\zeta) - \phi(z^j)\right) K(\zeta, z^j)\right| &\leq C \int\limits_{\partial D\cap B(z^1,2\delta)} |\zeta - z^j|^{\alpha+1-2n} d\sigma \\ &\leq C(2\delta)^{\alpha} \int\limits_{\partial D\cap B(z^1,2\delta)} |\zeta - z^j|^{1-2n} d\sigma \leq C_2\delta^{\alpha}. \end{aligned}\tag{2.25}$$

The difference between the integrals $F(z^1)$ from $F(z^2)$ along $\partial D \cap (B(z^1, 2\delta))^c$ instead of ∂D, yields that

$$\begin{aligned}&\int\limits_{\partial D\cap(B(z^1,2\delta))^c} \left(\phi(\zeta) - \phi(z^2)\right) K(\zeta, z^2) - \int\limits_{\partial D\cap(B(z^1,2\delta))^c} \left(\phi(\zeta) - \phi(z^1)\right) K(\zeta, z^1) \\ &= \int\limits_{\partial D\cap(B(z^1,2\delta))^c} \left(\phi(\zeta) - \phi(z^2)\right) \left(K(\zeta, z^2) - K(\zeta, z^1)\right) \\ &\quad + (\phi(z^1) - \phi(z^2)) \int\limits_{\partial D\cap(B(z^1,2\delta))^c} K(\zeta, z^1). \end{aligned}\tag{2.26}$$

By using Lemma 2.1.5, since ϕ satisfies the Hölder with exponent α, one can observe that

$$\left|\int\limits_{\partial D\cap(B(z^1,2\delta))^c} \left(\phi(\zeta) - \phi(z^2)\right) \left(K(\zeta, z^2) - K(\zeta, z^1)\right)\right| \leq C_3\delta \int\limits_{\partial D\cap(B(z^1,2\delta))^c} |\zeta - z^1|^{\alpha-2n} d\sigma. \tag{2.27}$$

For the second term of (2.26), it is enough to show that integral $\int_{\partial D\cap(B(z^1,2\delta))^c} K(\zeta, z^1)$ is bounded since then,

$$\left|\phi(z^1) - \phi(z^2)\right| \left|\int\limits_{\partial D\cap(B(z^1,2\delta))^c} K(\zeta, z^1)\right| \leq C_4\delta^{\alpha}, \tag{2.28}$$

where we used the fact that ϕ satisfies a Hölder condition in ∂D. In particular, Lemma 2.1.3 implies that for $z \in \partial D$, where $|\tau(z)| \leq 1$, one has

$$\text{P.V.} \int_{\partial D} K(\zeta, z) = \tau(z) - \int_{D} P(\zeta, z),$$

and

$$\left|\int_D P\right| = \left|\int_D g_{n,n}\right| \;=\; \left|\int_D \left(\frac{1}{2\pi i}\right)^n \frac{(\bar\partial\partial|\zeta-z|^2)^n}{(1+|\zeta-z|^2)^{n+1}}\right|$$
$$= \left|\int_D \left(\frac{1}{\pi}\right)^n \frac{n!dV}{(1+|\zeta-z|^2)^{n+1}}\right|$$
$$\leq \left(\frac{1}{\pi}\right)^n n!\mathrm{Vol}(D) \leq C_5,$$

since $(\bar\partial\partial|\zeta-z|^2)^n = (\bar\partial\partial|\zeta|^2)^n = n!(2i)^n dV$.

A combination of (2.25), (2.27) and (2.28) leads to

$$\begin{aligned}
|F(z^2)-F(z^1)| &\leq C_6\delta^\alpha + C_3\delta \int\limits_{\partial D\cap(B(z^1,2\delta))^c} |\zeta-z^1|^{\alpha-2n}d\sigma \\
&= C_6\delta^\alpha + C_3\delta \int\limits_{\partial D\cap(B(z^1,2\delta))^c} |\zeta-z^1|^{\alpha-1}|\zeta-z^1|^{1-2n}d\sigma \\
&\leq C_6\delta^\alpha + C_7\delta\cdot\delta^{\alpha-1} \int\limits_{\partial D\cap(B(z^1,2\delta))^c} |\zeta-z^1|^{1-2n}d\sigma \leq C_8\delta^\alpha,
\end{aligned}$$

since $\alpha < 1$ and a similar argument to (2.24) provides that the last integral is bounded. □

Hence, the principal value of the boundary integral (2.11) makes sense. More precisely, this integral can be written in the form

$$\begin{aligned}
\mathrm{P.V.}\int_{\partial D} K(\zeta,z)\phi(\zeta) &= \mathrm{P.V.}\int_{\partial D} (\phi(\zeta)-\phi(z))\,K(\zeta,z) + \phi(z)\mathrm{P.V.}\int_{\partial D} K(\zeta,z) \\
&= \int_{\partial D} (\phi(\zeta)-\phi(z))\,K(\zeta,z) + \phi(z)\mathrm{P.V.}\int_{\partial D} K(\zeta,z), \qquad (2.29)
\end{aligned}$$

where the right hand side of (2.29) exists according to Lemma 2.1.3 and Lemma 2.1.4.

For reasons of simplicity, we also introduce the functions

$$G^+(z) = \Phi^+(z) - \int_D (\phi(\zeta)-\phi(z))\,P(\zeta,z), \quad z\in D \qquad (2.30)$$

while

$$G^-(z) = \Phi^-(z) - \int_D (\phi(\zeta)-\phi(z))\,P(\zeta,z), \quad z\in V(\partial D)\setminus\overline{D}. \qquad (2.31)$$

The next theorem ensures that both G^+ and G^- can be extended continuously to ∂D as functions satisfying a Hölder condition such that, for $z \in \partial D$, one has

$$G^+(z) - G^-(z) = \Phi^+(z) - \Phi^-(z).$$

Theorem 2.1.1 *Let D be a bounded domain with piecewise-smooth boundary ∂D and consider a function $\phi \in \mathcal{C}^\alpha(\partial D)$ for $0 < \alpha < 1$. Then, G^+ extends as a function of class $\mathcal{C}^\alpha(\overline{D})$ while G^- extends as a function of class $\mathcal{C}^\alpha(\mathbb{C}^n \setminus D)$. Furthermore, the following equalities concerning the boundary values of G^+ and G^- hold:*

$$\begin{aligned} G^+(z) &= \left(1 + \int_D P(\zeta, z) - \tau(z)\right)\phi(z) + \text{P.V.}\int_{\partial D} \phi(\zeta)K(\zeta, z), \\ G^-(z) &= \left(\int_D P(\zeta, z) - \tau(z)\right)\phi(z) + \text{P.V.}\int_{\partial D} \phi(\zeta)K(\zeta, z), \end{aligned}$$

for $z \in \partial D$. Moreover,

$$\phi(z) \quad = \quad G^+(z) - G^-(z), \quad \textit{for} \quad z \in \partial D. \tag{2.32}$$

Proof. For a function $\phi \in \mathcal{C}^\alpha(\partial D)$, Lemma 2.1.1 implies that ϕ also satisfies the same Hölder condition in $V(\partial D)$. Observe that

$$\begin{aligned} G^+(z) &= \Phi^+(z) - \int_D (\phi(\zeta) - \phi(z))\, P(\zeta, z) \\ &= \int_{\partial D} \phi(\zeta)K(\zeta, z) + \phi(z)\int_D P(\zeta, z) \\ &= \int_{\partial D} (\phi(\zeta) - \phi(z))\, K(\zeta, z) + \phi(z)\left(\int_{\partial D} K(\zeta, z) + \int_D P(\zeta, z)\right), \end{aligned} \tag{2.33}$$

or equivalently, using the definition of F given in (2.17) and Koppelman representation formula

$$G^+(z) = F(z) + \phi(z), \quad z \in D. \tag{2.34}$$

Thus, Lemmas 2.1.1 and 2.1.4 imply that G^+ extends continuously to $\overline{D}$ as a function of class $\mathcal{C}^\alpha(\overline{D})$. Similarly , G^- extends continuously to $\mathbb{C}^n \setminus D$ as a function of class $\mathcal{C}^\alpha(\mathbb{C}^n \setminus D)$ since

$$G^-(z) \quad = \quad \int_{\partial D} (\phi(\zeta) - \phi(z))\, K(\zeta, z) = F(z), \quad z \in V(\partial D) \setminus \overline{D}, \tag{2.35}$$

due to the vanishing of the second term in the right hand side of (2.33) outside of D. Here, we also used Remark 2.1.1.

In order to express $G^{+}(z)$ and $G^{-}(z)$ with respect to $\phi(z)$ for $z \in \partial D$, one can observe that

$$\int_{\partial D\setminus B(z,\epsilon)} \phi(\zeta)K(\zeta,z) \;=\; \int_{\partial D\setminus B(z,\epsilon)} (\phi(\zeta)-\phi(z))\,K(\zeta,z) + \phi(z)\int_{\partial D\setminus B(z,\epsilon)} K(\zeta,z)$$

for ϵ sufficiently small. As $\epsilon \to 0^{+}$, Lemmas 2.1.3 and 2.1.4 imply that

$$\begin{aligned} F(z) &= \int_{\partial D} (\phi(\zeta)-\phi(z))\,K(\zeta,z) \\ &= \text{P.V.}\int_{\partial D} (\phi(\zeta)-\phi(z))\,K(\zeta,z) \\ &= \text{P.V.}\int_{\partial D} \phi(\zeta)K(\zeta,z) - \phi(z)\,\text{P.V.}\int_{\partial D} K(\zeta,z). \end{aligned} \tag{2.36}$$

Hence,

$$G^{+}(z) \;=\; F(z)+\phi(z) = \left(1-\text{P.V.}\int_{\partial D} K(\zeta,z)\right)\phi(z) + \text{P.V.}\int_{\partial D}\phi(\zeta)K(\zeta,z),$$

in view of (2.34). According to Lemma 2.1.3,

$$G^{+}(z) \;=\; \left(1+\int_{D} P(\zeta,z)-\tau(z)\right)\phi(z) + \text{P.V.}\int_{\partial D}\phi(\zeta)K(\zeta,z) \tag{2.37}$$

follows.

Analogously to the case of $G^{+}(z)$ and using (2.17), we deduce that

$$\begin{aligned} G^{-}(z) = F(z) &= \left(-\text{P.V.}\int_{\partial D} K(\zeta,z)\right)\phi(z) + \text{P.V.}\int_{\partial D}\phi(\zeta)K(\zeta,z) \\ &= \left(\int_{D} P(\zeta,z)-\tau(z)\right)\phi(z) + \text{P.V.}\int_{\partial D}\phi(\zeta)K(\zeta,z). \end{aligned} \tag{2.38}$$

By subtracting (2.38) from (2.37), we get the desired Jump Theorem, that is,

$$G^{+}(z)-G^{-}(z) \;=\; \phi(z), \quad \text{for} \quad z\in\partial D.$$

□

Corollary 2.1.1 *If the assumptions of Theorem 2.1.1 are valid, then*

$$\phi(z) = \Phi^{+}(z) - \Phi^{-}(z), \quad z \in \partial D. \tag{2.39}$$

Proof. This is an immediate consequence of (2.32) by substituting the definitions of $G^{+}(z)$

and $G^-(z)$ and then simplifying since both $G^+(z)$ and $G^-(z)$ extend to ∂D. $\square$

2.1.2 Jump Theorem on weighted Koppelman type integrals of continuous functions

Let us consider a bounded domain D with $\mathcal{C}^1$ boundary ∂D and f a continuous function on ∂D. We follow a similar construction as in ([29]) for the jump theorem for continuous functions and Bochner-Martinelli kernel.

Fix a point $z^0 \in \partial D$. We construct a right circular double cone V_{z^0} with vertex at the point z^0 and axis the normal to ∂D at this fixed point while the angle between the axis and the generator of the cone, namely β, is less than $\pi/2$. We also take two points $z^+ \in V_{z^0} \cap D$ and $z^- \in V_{z^0} \cap (\mathbb{C}^n \setminus \overline{D})$ such that

$$a|z^+ - z^0| \leq |z^- - z^0| \leq b|z^+ - z^0|, \tag{2.40}$$

for some positive finite constants a and b ($a \leq b$) that are independent on the points $z^\pm$. The next lemma shows that both the Koppelman kernel $K(\zeta, z)$ from (2.2) and the projection kernel $P(\zeta, z)$ from (2.5) with the specific choice of the weight considered in the previous paragraph are not affected by unitary transformations. This fact allows the translation of z^0 to 0 and the tangent plane to ∂D at z^0 to the plane $T = \{w \in \mathbb{C}^n : \operatorname{Im} w_n = 0\}$, while the contributing kernels of the Koppelman representation formula remain unchanged under these transformations, as in the case of B-M kernel ([29]).

Lemma 2.1.6 *The kernels $K(\zeta, z)$ and $P(\zeta, z)$ are invariant with respect to unitary transformations.*

Proof. The forms that are involved in $K(\zeta, z)$ are those in [29] with some extra distances in front that are invariant under unitary transformations. More precisely, under the unitary transformation given by $\zeta = A\zeta'$ (where A is a unitary complex matrix satisfying $A^*A = AA^* = I$ and A^* is the conjugate transpose of A), the distance $|\zeta - z|$ remains the same since A preserves the inner product, $d\zeta = \det A \, d\zeta' = e^{i\theta} \, d\zeta'$ and

$$\sum_{j=1}^{n} (-1)^{j-1} (\bar{\zeta}_j - \bar{z}_j) d\bar{\zeta}[j] = \det A^* \sum_{p=1}^{n} (-1)^{p-1} (\bar{\zeta}'_p - \bar{z}'_p) d\bar{\zeta}'[p]$$

such that

$$\sum_{j=1}^{n} (-1)^{j-1} (\bar{\zeta}_j - \bar{z}_j) d\bar{\zeta}[j] \wedge d\zeta = \sum_{p=1}^{n} (-1)^{p-1} (\bar{\zeta}'_p - \bar{z}'_p) d\bar{\zeta}'[p] \wedge d\zeta'.$$

Moreover, the relation

$$d\bar{\zeta} \wedge d\zeta = \det A \cdot \det A^{*} d\bar{\zeta}' \wedge d\zeta' = d\bar{\zeta}' \wedge d\zeta'$$

shows that the kernel $P(\zeta, z)$ is also invariant under the unitary transformation. $\square$

For reasons of simplicity, we introduce the function

$$Kf(z) = \int_{\partial D} f(\zeta) K(\zeta, z), \quad z \notin \partial D. \tag{2.41}$$

The notations $K^{+}f(z)$ and $K^{-}f(z)$ denote the value of the above function, defined by integral for $z \in D$ and $z \notin D$, correspondingly. In a similar way, we also consider the function

$$Pf(z) = \int_{D} f(\zeta) P(\zeta, z), \quad z \notin \partial D, \tag{2.42}$$

and denote its value by $P^{+}f(z)$, $P^{-}f(z)$ for the interior and exterior points of D, respectively.

Theorem 2.1.2 *If $f \in \mathcal{C}(\partial D)$, then for every point $z^0 \in \partial D$*

$$\lim_{z^{\pm} \to z^0} \left[\left(Kf(z^{+}) + Pf(z^{+}) \right) - \left(Kf(z^{-}) + Pf(z^{-}) \right) \right] = f(z^0).$$

Proof. Following [29], making use the unitary transformation and translation, we can assume that the point $z^0 = 0$ and the tangent plane to ∂D at z^0 is the hyperplane $T = \{w \in \mathbb{C}^n : \operatorname{Im} w_n = 0\}$, while the kernels $K(\zeta, z)$ and $P(\zeta, z)$ are invariant under these transformations, according to Lemma 2.1.6. Hence, by applying the Implicit function Theorem, the parametrization in a neighborhood of 0 on the surface ∂D is, then, given by $\zeta_1 = w_1, \ldots, \zeta_{n-1} = w_{n-1}, \zeta_n = u_n + i\phi(w)$, for $w = (w_1, \ldots, w_{n-1}, u_n) \in T$, where $\phi(w)$ is a function of class C^1 in a neighborhood W of 0 in the hyperplane T such that $\phi(w) = o(|w|)$ and $|\zeta(w)| \leq C_1 |w|$, for $C_1 > 0$.

If $\tilde{z}^{\pm}$ are the corresponding projections of $z^{\pm}$ onto the $\operatorname{Im} w_n$, then

$$|z^{\pm} - \tilde{z}^{\pm}| \leq |\tilde{z}^{\pm}| \tan\beta, \quad |z^{\pm}| \leq \frac{|\tilde{z}^{\pm}|}{\cos\beta}, \quad a|\tilde{z}^{+}| \cos\beta \leq |\tilde{z}^{-}| \leq \frac{b|\tilde{z}^{+}|}{\cos\beta}, \tag{2.43}$$

in view of (2.40).

According to [29], a $(2n-1)$-dimensional ball $B' = B(z_0, \epsilon) \cap T \subset W$ for $z^0 = 0$ is fixed such that

$$|w - \tilde{z}^{\pm}| \leq C_2 |\zeta(w) - z^{\pm}| \tag{2.44}$$

holds for $w \in B'$, where C_2 is a constant independent of the point $z^0 = 0$. By making use that $\tilde{z}^{\pm} = (0, \ldots, 0, iy_n^{\pm})$ where $y_n^{\pm}$ are the imaginary parts of $z_n^{\pm}$, then

$$|\zeta(w)| \leq C_1 |w| \leq C_1 |w - \tilde{z}^{\pm}| \tag{2.45}$$

is also deduced for $w \in B'$.

Now, we derive an inverse inequality which is required in our case. For $w \in B'$, one can obtain the inequality

$$\begin{aligned} |\zeta(w) - z^{\pm}| &\leq |\zeta(w) - w| + |w - \tilde{z}^{\pm}| + |\tilde{z}^{\pm} - z^{\pm}| \\ &\leq |\phi(w)| + |w - \tilde{z}^{\pm}| + |\tilde{z}^{\pm}| \tan\beta \\ &\leq |\phi(w)| + |w - \tilde{z}^{\pm}| + \tan\beta |w - \tilde{z}^{\pm}| \\ &\leq C_3 |w - \tilde{z}^{\pm}| \end{aligned} \tag{2.46}$$

since $|\tilde{z}^{\pm}| \leq |w - \tilde{z}^{\pm}|$, where C_3 is also a constant independent of the point $z^0 = 0$.

The subtraction of $(Kf(z^-) + Pf(z^-))$ from $(Kf(z^+) + Pf(z^+))$ yields that

$$\begin{aligned} &\big(Kf(z^+) + Pf(z^+)\big) - \big(Kf(z^-) + Pf(z^-)\big) \\ &= \int_{\partial D} \big(f(\zeta) - f(z^0)\big)\big(K(\zeta, z^+) - K(\zeta, z^-)\big) + f(z^0) \int_{\partial D} \big(K(\zeta, z^+) - K(\zeta, z^-)\big) \\ &+ \int_D f(\zeta)\big(P(\zeta, z^+) - P(\zeta, z^-)\big). \end{aligned} \tag{2.47}$$

Letting $z^{\pm}$ goes to z^0, the vanishing of the last integral is trivial. The second integral in (2.47) equals to 1 as $z^{\pm}$ tends to z^0, since

$$\int_{\partial D} \big(K(\zeta, z^+) - K(\zeta, z^-)\big) \;=\; 1 + \int_D \big(P(\zeta, z^-) - P(\zeta, z^+)\big)$$

and hence,

$$\lim_{z^{\pm} \to z^0} \int_{\partial D} \big(K(\zeta, z^+) - K(\zeta, z^-)\big) \;=\; 1 + \int_D \lim_{z^{\pm} \to z^0} \big(P(\zeta, z^-) - P(\zeta, z^+)\big) = 1,$$

due to the fact that the location of z^0 ($z^0 \notin D$) allows the change of the order between the limit and integral.

Then, the theorem follows if the limit of first integral in (2.47) vanishes as $z^{\pm} \to z^0$. We split this integral in two integrals. The first one is taken over $\Gamma = B(z^0, \epsilon) \cap \partial D$ while the second one is taken over its complement of Γ in ∂D. It is straight forward to see that

$$\lim_{z^{\pm} \to z^0} \int_{\partial D \setminus \Gamma} \left(f(\zeta) - f(z^0)\right) \left(K(\zeta, z^+) - K(\zeta, z^-)\right) = 0, \tag{2.48}$$

since the limit can pass through the integral ($z^0 \notin \partial D \setminus \Gamma$).

On the other hand,we will see that the integral over Γ behaves in manner similar to the case of Bochner-Martinelli kernel. In order to study the boundary behavior of the integrals involved, taking into account the kernel K (2.2), let

$$\Delta_j(\bar{\zeta}, \bar{z}^+, \bar{z}^-) := \frac{(\bar{\zeta}_j - \bar{z}_j^+)}{|\zeta - z^+|^{2k}(1 + |\zeta - z^+|^2)^n} - \frac{(\bar{\zeta}_j - \bar{z}_j^-)}{|\zeta - z^-|^{2k}(1 + |\zeta - z^-|^2)^n}.$$

Then,

$$\left|\Delta_j(\bar{\zeta}, \bar{z}^+, \bar{z}^-)\right| \;\; \leq \;\; \left|A_j(\zeta, z^+, z^-)\right| + \left|B_j(\bar{z}^+, \bar{z}^-, z^+, z^-)\right|, \tag{2.49}$$

where

$$A_j(\zeta, z^+, z^-) := \frac{\bar{\zeta}_j}{|\zeta - z^+|^{2k}(1 + |\zeta - z^+|^2)^n} - \frac{\bar{\zeta}_j}{|\zeta - z^-|^{2k}(1 + |\zeta - z^-|^2)^n} \tag{2.50}$$

and

$$B_j(\bar{z}^+, \bar{z}^-, z^+, z^-) := \frac{\bar{z}_j^+}{|\zeta - z^+|^{2k}(1 + |\zeta - z^+|^2)^n} - \frac{\bar{z}_j^-}{|\zeta - z^-|^{2k}(1 + |\zeta - z^-|^2)^n}. \tag{2.51}$$

Computations similar to (2.23) give the estimate

$$\left|A_j(\zeta, z^+, z^-)\right| \;\; \leq \;\; \frac{|\zeta_j|(|z^+| + |z^-|)}{|\zeta - z^-|^{2n}} \sum_{l=0}^{n} \binom{n}{l} |\zeta - z^-|^{2l+2k} \sum_{s=0}^{2l+2k-1} \frac{|\zeta - z^-|^{s-(2l+2k)}}{|\zeta - z^+|^{s+1}},$$

where we used the inequality (2.3). The relations (2.44), (2.45) and (2.46) simplify even more the last inequality. Actually

$$\begin{aligned} \left|A_j(\zeta, z^+, z^-)\right| \;\; \leq \;\; & \frac{C_1 C_2^{2n}(|z^+| + |z^-|)}{|w - \tilde{z}^-|^{2n}} \\ & \cdot \sum_{l=0}^{n} \binom{n}{l} C_3^{2l+2k} |w - \tilde{z}^-|^{2l+2k} \sum_{s=0}^{2l+2k-1} \frac{C_2^{2l+2k-s} |w - \tilde{z}^-|^{s-(2l+2k)}}{C_2^{-s-1} |w - \tilde{z}^+|^{s}}. \end{aligned} \tag{2.52}$$

Two more inequalities are provided in order to reduce the modulus $|w-\tilde{z}^+|$ and $|w-\tilde{z}^-|$ to a multiple of a same quantity. Assume that $a_1 := a\cos\beta < 1$. Hence,

$$|w-\tilde{z}^+| \;=\; \sqrt{\sum_{i=1}^{n-1}|w_i|^2+u_n^2+(y_n^+)^2} \geq \sqrt{\sum_{i=1}^{n-1}|w_i|^2+u_n^2+(a_1y_n^+)^2} = |w-a_1\tilde{z}^+|$$

and

$$|w-\tilde{z}^-| \;=\; \sqrt{\sum_{i=1}^{n-1}|w_i|^2+u_n^2+(y_n^-)^2} \geq \sqrt{\sum_{i=1}^{n-1}|w_i|^2+u_n^2+(a_1y_n^+)^2} = |w-a_1\tilde{z}^+|,$$

where we used the inequality (2.43). Thus,

$$|w-\tilde{z}^\pm| \geq |w-a_1\tilde{z}^+|. \tag{2.53}$$

On the other hand,

$$|w-\tilde{z}^+| \;\leq\; \frac{1}{a_1}\sqrt{\sum_{i=1}^{n-1}|w_i|^2+u_n^2+(a_1y_n^+)^2} = \frac{1}{a_1}|w-a_1\tilde{z}^+|$$

while, since $|\tilde{z}^-| \leq b_1|\tilde{z}^+|$ for $b_1 = b/\cos\beta$ and $b_1 > a_1$ ($b \geq a$ and $0 < \cos\beta < 1$),

$$|w-\tilde{z}^-| \;\leq\; \sqrt{\sum_{i=1}^{n-1}|w_i|^2+u_n^2+(b_1y_n^+)^2} \leq \frac{b_1}{a_1}\sqrt{\sum_{i=1}^{n-1}|w_i|^2+u_n^2+(a_1y_n^+)^2} = \frac{b_1}{a_1}|w-a_1\tilde{z}^+|.$$

By letting $C_4 := \max\{1/a_1, b_1/a_1\}$, we get

$$|w-\tilde{z}^\pm| \leq C_4|w-a_1\tilde{z}^+|. \tag{2.54}$$

Combining the inequalities (2.53) and (2.54), one deduces from (2.52) that

$$\begin{aligned}\left|A_j(\zeta,z^+,z^-)\right| \;&\leq\; \frac{C_1C_2^{2n}(|z^+|+|z^-|)}{|w-a_1\tilde{z}^+|^{2n}} \\ &\quad \cdot\sum_{l=0}^{n}\binom{n}{l}C_3^{2l+2k}C_4^{2l+2k}|w-a_1\tilde{z}^+|^{2l+2k}\sum_{s=0}^{2l+2k-1}\frac{C_2^{2l+2k+1}|w-a_1\tilde{z}^+|^{s-(2l+2k)}}{|w-a_1\tilde{z}^+|^s} \\ &\leq\; \frac{d_1|\tilde{z}^+|}{|w-a_1\tilde{z}^+|^{2n}}, \end{aligned} \tag{2.55}$$

where d_1 depends on a, b, C_1, C_2, C_3, C_4 and β.

The second term of (2.49) can be bounded by the same quantity. Actually

$$\left|B_j(\bar{z}^+, \bar{z}^-, z^+, z^-)\right| \quad \leq \quad \frac{|z_j^+|}{|\zeta - z^+|^{2n}} + \frac{|z_j^+|}{|\zeta - z^-|^{2n}} \leq \frac{d_2|\tilde{z}^+|}{|w - a_1\tilde{z}^+|^{2n}}. \qquad (2.56)$$

Hence,

$$\left|\Delta_j(\bar{\zeta}, \bar{z}^+, \bar{z}^-)\right| \leq \frac{d_3|\tilde{z}^+|}{|w - a_1\tilde{z}^+|^{2n}}.$$

Now, if dS is the surface area element on the surface T, then there is a constant d_4 independent of z^0 such that the Lebesgue surface measure on Γ, $d\sigma$, satisfies the inequality $d\sigma \leq d_4 dS$. Moreover, due to the assumption of the continuity of f in ∂D , for each $\delta > 0$, we can choose a ball B' for a radius ϵ independent of z^0 such that

$$|f(\zeta(w)) - f(0)| < \delta, \quad \text{for} \quad w \in B'.$$

Consequently,

$$\begin{aligned} \left|\int_\Gamma (f(\zeta) - f(0))\left(K(\zeta, z^+) - K(\zeta, z^-)\right)\right| \quad &\leq \quad d_5\delta \int_{B'} \frac{a_1|\tilde{z}^+|dS}{(|w|^2 + a_1^2|\tilde{z}^+|^2)^n} \\ &\leq \quad d_5\delta \int_T \frac{a_1|\tilde{z}^+|dS}{(|w|^2 + a_1^2|\tilde{z}^+|^2)^n}, \end{aligned}$$

where this last integral is a constant independent of $\tilde{z}^+$ in view of [29]. Thus,

$$\lim_{z^\pm \to z^0} \int_\Gamma \left(f(\zeta) - f(z^0)\right)\left(K(\zeta, z^+) - K(\zeta, z^-)\right) = 0$$

and the result follows. $\square$

Corollary 2.1.2 *If $f \in \mathcal{C}(\partial D)$ and $(K^+ f + P^+ f)$ is continuous in $\overline{D}$, then $(K^- f + P^- f)$ extends continuously to $\mathbb{C}^n \setminus D$. Conversely, if $(K^- f + P^- f)$ is continuous in $\mathbb{C}^n \setminus D$ then $(K^+ f + P^+ f)$ extends continuously to $\overline{D}$.*

This corollary is a non-trivial result in several complex variables in comparison to the one-dimensional case.

2.1.3 Jump Theorem for the $\bar{\partial}$-Normal Derivative of the weighted Koppelman type integrals

Similarly to the case of the Bochner-Martinelli type integral studied in [29], we define the derivatives

$$\bar{\partial}_n(Kf) = \sum_{k=1}^{n} \frac{\partial(Kf)}{\partial \bar{z}_k}\rho_k \quad \text{and} \quad \bar{\partial}_n(Pf) = \sum_{k=1}^{n} \frac{\partial(Pf)}{\partial \bar{z}_k}\rho_k, \tag{2.57}$$

where ρ is the defining function of a domain $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $\partial D \in \mathcal{C}^2$. Then, according to [29] (more details can be found in Section 2 of [42]) there exists a neighborhood V of ∂D where $\rho \in \mathcal{C}^2(V)$ such that:

1. $|\text{grad}\rho| = \frac{1}{2}$ in V,

2. If $z^{\pm} \in V$ ($z^{+} \in D$ and $z^{-} \in \mathbb{C}^n \setminus D$) are the points on the normal to ∂D at z satisfying $|z^{+} - z| = |z^{-} - z|$, then $\frac{\partial \rho}{\partial z_k}(z^{\pm}) = \frac{\partial \rho}{\partial z_k}(z)$ and $\frac{\partial \rho}{\partial \bar{z}_k}(z^{\pm}) = \frac{\partial \rho}{\partial \bar{z}_k}(z)$ and

$$\rho_k = 2\frac{\partial \rho}{\partial z_k} \qquad \text{and} \qquad \bar{\rho}_k = 2\frac{\partial \rho}{\partial \bar{z}_k},$$

for $k = 1, 2, \ldots, n$.

Hence, the $\bar{\partial}$-normal derivatives of Kf form (2.41) and Pf from (2.42) are defined as follows:

$$\bar{\partial}_n(Kf) = \sum_{k=1}^{n} \frac{\partial(Kf)}{\partial \bar{z}_k}\rho_k = 2\sum_{k=1}^{n} \frac{\partial(Kf)}{\partial \bar{z}_k}\frac{\partial \rho}{\partial z_k} \tag{2.58}$$

and, similarly,

$$\bar{\partial}_n(Pf) = \sum_{k=1}^{n} \frac{\partial(Pf)}{\partial \bar{z}_k}\rho_k = 2\sum_{k=1}^{n} \frac{\partial(Pf)}{\partial \bar{z}_k}\frac{\partial \rho}{\partial z_k}. \tag{2.59}$$

such that

$$\bar{\partial}_n(Kf + Pf) = \sum_{k=1}^{n} \frac{\partial(Kf + Pf)}{\partial \bar{z}_k}\rho_k = 2\sum_{k=1}^{n} \frac{\partial(Kf + Pf)}{\partial \bar{z}_k}\frac{\partial \rho}{\partial z_k}. \tag{2.60}$$

Taking into account the previous section, according to (2.2) and (2.14), one has

$$\begin{aligned} K(\zeta, z)|_{\partial D} &= \frac{(n-1)!}{2\pi^n}\sum_{j=1}^{n} \frac{\binom{n}{j}\sum_{m=1}^{n}(\bar{\zeta}_m - \bar{z}_m)\frac{\partial \rho}{\partial \bar{\zeta}_m}\frac{d\sigma}{|\text{grad}\rho|}}{|\zeta - z|^{2j}(1 + |\zeta - z|^2)^n} \\ &= \frac{(n-1)!}{\pi^n}\sum_{j=1}^{n} \frac{\binom{n}{j}\sum_{m=1}^{n}\frac{\partial \rho}{\partial \bar{\zeta}_m}(\bar{\zeta}_m - \bar{z}_m)d\sigma}{|\zeta - z|^{2j}(1 + |\zeta - z|^2)^n}, \end{aligned}$$

where we used the fact that $|\mathrm{grad}\rho| = 1/2$ in the neighborhood V of ∂D.

In full analogy with (2.21), let us define

$$A_{2j}^{n}(\zeta, z^{+}, z^{-}) := \frac{1}{|\zeta - z^{+}|^{2j}(1 + |\zeta - z^{+}|^{2})^{n}} - \frac{1}{|\zeta - z^{-}|^{2j}(1 + |\zeta - z^{-}|^{2})^{n}}, \tag{2.61}$$

and

$$C(\zeta, z^{+}) := \frac{j}{|\zeta - z^{+}|^{2j+2}(1 + |\zeta - z^{+}|^{2})^{n+1}} + \frac{n + j}{|\zeta - z^{+}|^{2j}(1 + |\zeta - z^{+}|^{2})^{n+1}}. \tag{2.62}$$

Let $C(\zeta, z^{-})$ be the corresponding term where z^{+} is replaced by z^{-} in the preceding expression. Hence, it is easy to observe that

$$C(\zeta, z^{+}) - C(\zeta, z^{-}) \quad = \quad jA_{2j+2}^{n+1}(\zeta, z^{+}, z^{-}) + (n + j)A_{2j}^{n+1}(\zeta, z^{+}, z^{-}). \tag{2.63}$$

By applying the $\bar{\partial}$-normal derivative into the difference of $Kf(z^{-})$ from $Kf(z^{+})$ according to the rule (2.58) and by using the above notations, we get

$$\begin{aligned}
&\bar{\partial}_n(Kf(z^{+})) - \bar{\partial}_n(Kf(z^{-})) \\
&= 2\sum_{k=1}^{n}\left(\int_{\partial D} f(\zeta)\left(\frac{\partial K(\zeta, z^{+})}{\partial \bar{z}_k} - \frac{\partial K(\zeta, z^{-})}{\partial \bar{z}_k}\right)\frac{\partial \rho(z)}{\partial z_k}\right) \\
&= J_1(z^{+}, z^{-}) + J_2(z^{+}, z^{-}),
\end{aligned} \tag{2.64}$$

where

$$J_1(z^{+}, z^{-}) \quad = \quad -\frac{2(n-1)!}{\pi^{n}}\int_{\partial D} f(\zeta)\sum_{k=1}^{n}\frac{\partial \rho(z)}{\partial z_k}\frac{\partial \rho(\zeta)}{\partial \bar{\zeta}_k}\sum_{j=1}^{n}\binom{n}{j}A_{2j}^{n}(\zeta, z^{\pm})d\sigma \tag{2.65}$$

and

$$\begin{aligned}
J_2(z^{+}, z^{-}) \quad = \quad & \frac{2(n-1)!}{\pi^{n}}\int_{\partial D} f(\zeta) \\
& \left[\sum_{k=1}^{n}\frac{\partial \rho}{\partial z_k}(\zeta_k - z_k^{+})\sum_{m=1}^{n}\frac{\partial \rho}{\partial \bar{\zeta}_m}(\bar{\zeta}_m - \bar{z}_m^{+})\sum_{j=1}^{n}\binom{n}{j}C(\zeta, z^{+})\right. \\
& \left.-\sum_{k=1}^{n}\frac{\partial \rho}{\partial z_k}(\zeta_k - z_k^{-})\sum_{m=1}^{n}\frac{\partial \rho}{\partial \bar{\zeta}_m}(\bar{\zeta}_m - \bar{z}_m^{-})\sum_{j=1}^{n}\binom{n}{j}C(\zeta, z^{-})\right] d\sigma.
\end{aligned} \tag{2.66}$$

Lemma 2.1.7 *The integral $J_2(z^{+}, z^{-})$ defined by (2.66) is invariant with respect to unitary transformations.*

Proof. According to Lemma 2.1.6, the distance $|\zeta - z|$ does not change under unitary transformation. No changes happen in the function ρ and in the area element $d\sigma$, too. Following [29], $\sum_{k=1}^{n} \partial\rho/\partial z_k(\zeta_k - z_k)$ and $\sum_{m=1}^{n} \partial\rho/\partial\bar{\zeta}_m(\bar{\zeta}_m - \bar{z}_m)$ are also invariant under unitary transformation since $z'_k = \sum_{j=1}^{n} a_{jk} z_j$ and $\sum_{k=1}^{n} a_{kj} b_{sk} = \delta_{js}$ where $A = \|a_{jk}\|_{j,k=1}^{n}$ is the unitary matrix and $B = \|b_{jk}\|_{j,k=1}^{n}$ is its inverse matrix. □

The next theorem indicates that the jump theorem of the $\bar{\partial}$-normal derivative of the sum of the Koppelman integral $\bar{\partial}_n Kf(z)$ (see 2.58) and Projection integral $\bar{\partial}_n Pf(z)$ (see 2.59) is zero. Following [29], one formulates the following theorem.

Theorem 2.1.3 *Let $f \in \mathcal{C}(\partial D)$ for a domain D with $\partial D \in \mathcal{C}^2$. Then,*

$$\lim_{z^{\pm} \to z} \left[\bar{\partial}_n \left(Kf(z^+) + Pf(z^+) \right) - \bar{\partial}_n \left(Kf(z^-) + Pf(z^-) \right) \right] = 0.$$

The limit is independent of $z \in \partial D$. Thus, if $\bar{\partial}_n \left(Kf(z^+) + Pf(z^+) \right)$ extends continuously to $\overline{D}$, then $\bar{\partial}_n \left(Kf(z^-) + Pf(z^-) \right)$ extends continuously to $\mathbb{C}^n \setminus D$. Conversely, if $\bar{\partial}_n \left(Kf(z^-) + Pf(z^-) \right)$ extends continuously to $\mathbb{C}^n \setminus D$ then $\bar{\partial}_n \left(Kf(z^+) + Pf(z^+) \right)$ extends continuously to $\overline{D}$.

Proof. Firstly, observe that

$$\begin{aligned} &\lim_{z^{\pm} \to z} \left[\bar{\partial}_n \left(Kf(z^+) + Pf(z^+) \right) - \bar{\partial}_n \left(Kf(z^-) + Pf(z^-) \right) \right] \\ &= \lim_{z^{\pm} \to z} \left(\bar{\partial}_n Kf(z^+) - \bar{\partial}_n Kf(z^-) \right) + \lim_{z^{\pm} \to z} \left(\bar{\partial}_n Pf(z^+) - \bar{\partial}_n Pf(z^-) \right). \end{aligned}$$

The second limit vanishes directly since the limit can pass through the integrals. The first limit requires a different approach since the integration is along ∂D.

Without loss of generality, we can assume that $f(z) = 0$ at the point $z \in \partial D$. As in the previous theorem, the point z can be translated to 0 and the tangent plane to ∂D at z is taken to the plane $T = \{w \in \mathbb{C}^n : \mathrm{Im} w_n = 0\}$. Moreover, the equations $\zeta_1 = w_1, \ldots, \zeta_{n-1} = w_{n-1}, \zeta_n = u_n + i\phi(w)$, parameterize ∂D in a neighborhood of 0. The point $w = (w_1, \ldots, w_{n-1}, u_n) \in T$ and $\phi(w)$ is a function of class $\mathcal{C}^2$ in a neighborhood W of the origin satisfying the following inequalities, according to [29] (more details can be found in Section 22 of [40]):

1. $|\phi(w)| \leq C|w|^2$, for $w \in W$,
2. $|\partial\phi/\partial u_j| \leq C_1|w|$, for $j = 1, \ldots, n$,
3. $|\partial\phi/\partial v_j| \leq C_1|w|$, for $j = 1, \ldots, n-1$,

where $u_j = \mathrm{Re} w_j$ and $v_j = \mathrm{Im} w_j$. Let, also, $z^{\pm} = (0, \ldots, 0, \pm i y_n)$. Since $\partial\phi/\partial w_j = -(\partial\rho/\partial w_j)/(\partial\rho/\partial y_n)$ and $|\partial\rho/\partial y_n| \geq C_2 > 0$, for $w \in W$, it is obtained that

$$\left| \frac{\partial\rho}{\partial\zeta_k}(\zeta(w)) \right| \leq C_3|w| \quad \text{and} \quad \left| \frac{\partial\rho}{\partial\bar{\zeta}_k}(\zeta(w)) \right| \leq C_3|w|, \tag{2.67}$$

for $w \in W$ and $k = 1, \ldots, n-1$. Moreover, a constant C_4 exists such that

$$|\zeta(w)| \leq C_4 |w|. \tag{2.68}$$

All the preceding constants are independent of the point z under consideration.

For a fixed $\epsilon > 0$, we consider a ball $B' = B(0, \epsilon) \cap T \subset W$ such that

$$\begin{aligned} |f(\zeta(w))| &< \epsilon, \quad w \in B' && (2.69)\\ \{z \in \mathbb{C}^n : (z_1, \ldots, z_{n-1}, \mathrm{Re} z_n) &\in B', |\mathrm{Im} z_n| < a\} \subset W, \quad \text{for} \quad a > 0, && (2.70)\\ C(2|y_n| + C|w|^2) &\leq d < 1, \quad \text{for} \quad |y_n| < a \quad \text{and} \quad w \in B'. && (2.71) \end{aligned}$$

In view of [29], the equality

$$\frac{1}{|\zeta(w) - z^{\pm}|^2} = \frac{1}{|w - z^{\pm}|^2 \left[1 - (\pm 2\phi y_n - \phi^2)|w - z^{\pm}|^{-2}\right]}$$

that holds leads to following major inequality:

$$\frac{|(\pm 2\phi y_n - \phi^2)|}{|w - z^{\pm}|^2} \leq \frac{C|w|^2(2|y_n| + C|w|^2)}{|w|^2 + y_n^2} \leq C(2|y_n| + C|w|^2) \leq d < 1, \tag{2.72}$$

for $|y_n| \leq a$ and $w \in B'$. Hence, by the convergence of the geometric sequence with first term equals to 1 and ratio be the expression $(\pm 2\phi y_n - \phi^2)/|w - z^{\pm}|^2$, one can get

$$\frac{1}{1 - (\pm 2\phi y_n - \phi^2)|w - z^{\pm}|^{-2}} = 1 + \frac{(\pm 2\phi y_n - \phi^2)}{|w - z^{\pm}|^2} h(w, z),$$

where $h(z, w)$ is uniformly bounded for $|y_n| \leq a$ and $w \in B'$, according to (2.72). Then, it follows that

$$\frac{1}{|\zeta(w) - z^{\pm}|^{2s}} = \frac{1 + \frac{(\pm 2\phi y_n - \phi^2)}{|w - z^{\pm}|^2} h_s(w, z)}{|w - z^{\pm}|^{2s}}, \tag{2.73}$$

for a positive integer s and $h_s(w, z)$ are also uniformly bounded functions for $|y_n| \leq a$ and $w \in B'$.

Weaker inequalities are directly generated by using (2.72) and (2.73):

$$\begin{aligned} |\zeta(w) - z^{\pm}|^2 &= |w - z^{\pm}|^2 \left[1 - \frac{(\pm 2\phi y_n - \phi^2)}{|w - z^{\pm}|^2}\right] \\ &\leq |w - z^{\pm}|^2 \left[1 + \frac{|(\pm 2\phi y_n - \phi^2)|}{|w - z^{\pm}|^2}\right] \\ &< 2|w - z^{\pm}|^2 = 2|w - z^{+}|^2 && (2.74) \end{aligned}$$

and

$$\frac{1}{|\zeta(w) - z^{\pm}|^{2s}} \leq \frac{C_4}{|w - z^{\pm}|^{2s}} = \frac{C_4}{|w - z^{+}|^{2s}}. \tag{2.75}$$

Let $\Gamma = \{\zeta \in \partial D : \zeta = \zeta(w), w \in B'\}$. In order to explore the first integral J_1 of $\bar{\partial}_n(Kf(z^+)) - \bar{\partial}_n(Kf(z^-))$ that is given explicitly in (2.65) over Γ, we first estimate the $A^n_{2j}(\zeta, z^+, z^-)$ involved term. One can observe that

$$\left|A^n_{2j}(\zeta, z^+, z^-)\right| = \frac{\left|\sum_{l=0}^{n} \binom{n}{l} |\zeta - z^+|^{2l} |\zeta - z^-|^{2l} B_{j+l}(\zeta, z^+, z^-)\right|}{(1 + |\zeta - z^+|^2)^n (1 + |\zeta - z^-|^2)^n}, \tag{2.76}$$

where $B_{j+l}(\zeta, z^+, z^-) = \frac{1}{|\zeta - z^+|^{2j+2l}} - \frac{1}{|\zeta - z^-|^{2j+2l}}$ in view of (2.22). Then, the relation (2.73) yields that

$$\left|B_s(\zeta, z^+, z^-)\right| = \left|\frac{1}{|\zeta - z^+|^{2s}} - \frac{1}{|\zeta - z^-|^{2s}}\right| = \frac{C_5(2|y_n| + C|w|^2)}{(|w|^2 + y_n^2)^s}, \tag{2.77}$$

in view of (2.72) since the equality $|w - z^+| = |w - z^-| = |w|^2 + y_n^2$ holds. Therefore, (2.76) can be rewritten as

$$\begin{aligned} \left|A^n_{2j}(\zeta, z^+, z^-)\right| &\leq \sum_{l=0}^{n} \binom{n}{l} \frac{|\zeta - z^+|^{2l}}{(1 + |\zeta - z^+|^2)^{n-j}} \frac{|\zeta - z^-|^{2l}}{(1 + |\zeta - z^-|^2)^{l}} \left|B_{j+l}(\zeta, z^+, z^-)\right| \\ &\leq \frac{\sum_{l=0}^{n} \binom{n}{l} |\zeta - z^+|^{2l} \left|B_{j+l}(\zeta, z^+, z^-)\right|}{|\zeta - z^+|^{2n-2j}} \\ &\leq \frac{C_4 C_5}{|w - z^{\pm}|^{2n-2j}} \sum_{l=0}^{n} \binom{n}{l} \frac{2^l |w - z^{\pm}|^{2l} (2|y_n| + C|w|^2)}{|w - z^+|^{2j+2l}} \\ &\leq \frac{C_6(2|y_n| + C|w|^2)}{(|w|^2 + y_n^2)^n}. \end{aligned} \tag{2.78}$$

Now, if $J_{1,\Gamma}(z^+, z^-)$ denotes the restriction of the integral J_1 described in (2.65) over the surface Γ, then

$$\begin{aligned} \left|J_{1,\Gamma}(z^+, z^-)\right| &= \frac{2(n-1)!}{\pi^n} \left| \int_{\partial D} f(\zeta) \sum_{k=1}^{n} \frac{\partial \rho(z)}{\partial z_k} \frac{\partial \rho(\zeta)}{\partial \bar{\zeta}_k} \sum_{j=1}^{n} \binom{n}{j} A^n_{2j}(\zeta, z^{\pm}) d\sigma \right| \\ &\leq \epsilon C_7 \int_{B'} \frac{2|y_n| + C|w|^2}{(|w|^2 + y_n^2)^n} dS, \end{aligned} \tag{2.79}$$

where dS is the surface element of the plane T satisfying $d\sigma \leq C_8 dS$, for some constant C_8.

Hence, according to [29],

$$\int_{B'} \frac{|y_n| dS}{(|w|^2 + y_n^2)^n} \leq \int_{T} \frac{|y_n| dS}{(|w|^2 + y_n^2)^n} = \text{const}, \tag{2.80}$$

and

$$\int_{B'} \frac{|w|^2 dS}{(|w|^2+y_n^2)^n} \quad \leq \quad \int_{B'} \frac{dS}{(|w|^2+y_n^2)^{n-1}} = \sigma_{2n-1}\int_0^R \frac{d|w|}{(|w|^2+y_n^2)^{n-1}} \leq R\sigma_{2n-1}, \quad (2.81)$$

where R is the radius of the ball B' and σ_{2n-1} is the area of the unit sphere in $\mathbb{R}^{2n-1}$. Here, polar coordinates in B' have been introduced such that $dS = |w|^{2n-2}d|w| \wedge d\omega$ ($d\omega$ is the surface area element in the unit sphere in $\mathbb{R}^{2n-1}$). Consequently, there exists a constant C_9 independent of the z and y_n such that

$$\left|J_{1,\Gamma}(z^+,z^-)\right| < C_9\epsilon.$$

For the integral $J_1(z^+,z^-)$ in the rest of the ∂D, that is $J_{1,\partial D\setminus\Gamma}(z^+,z^-)$, the result is trivial since as $z^\pm \to 0$, the integral $J_{1,\partial D\setminus\Gamma}(z^+,z^-)$ tends directly to zero.

According to Lemma 2.1.7, the integral $J_2(z^+,z^-)$ is invariant with respect to the unitary transformation. Thus, the integral $J_2(z^+,z^-)$ due its explicit form given in (2.66) requires the computation of the following expression in order to be investigated ([29]):

$$\begin{aligned}
&\sum_{k=1}^n \frac{\partial\rho}{\partial z_k}(0)(\zeta_k - z_k^\pm)\sum_{m=1}^n \frac{\partial\rho}{\partial\bar\zeta_m}(\bar\zeta_m - \bar z_m^\pm)\\
&= -\frac{i}{2}\sum_{m=1}^{n-1}\frac{\partial\rho}{\partial\bar\zeta_m}\,\bar\zeta_m(\zeta_n - z_n^\pm) - \frac{i}{2}\frac{\partial\rho}{\partial\bar\zeta_n}\,(u_n^2+\phi^2+y_n^2 \mp 2\phi y_n),
\end{aligned}$$

since $\partial\rho/\partial z_k = 1/2(\partial\rho/\partial x_k - i\partial\rho/\partial y_k) = -i/2\partial\rho/\partial y_k$. Hence, the integral $J_2(z^+,z^-)$ over Γ is reduced to integral

$$\begin{aligned}
J_2(z^+,z^-)|_\Gamma = \frac{2(n-1)!}{\pi^n}\int_\Gamma f(\zeta)&\\
\Bigg[\left(-\frac{i}{2}\sum_{m=1}^{n-1}\frac{\partial\rho}{\partial\bar\zeta_m}\,\bar\zeta_m(\zeta_n - z_n^+) - \frac{i}{2}\frac{\partial\rho}{\partial\bar\zeta_n}\,(u_n^2+\phi^2+y_n^2-2\phi y_n)\right)&\sum_{j=1}^n\binom{n}{j}C(\zeta,z^+)\\
-\left(-\frac{i}{2}\sum_{m=1}^{n-1}\frac{\partial\rho}{\partial\bar\zeta_m}\,\bar\zeta_m(\zeta_n - z_n^-) - \frac{i}{2}\frac{\partial\rho}{\partial\bar\zeta_n}\,(u_n^2+\phi^2+y_n^2+2\phi y_n)\right)&\sum_{j=1}^n\binom{n}{j}C(\zeta,z^-)\Bigg]\,d\sigma.
\end{aligned}$$

Since the equality (2.63) holds, then we can split the above integral into the following three integrals

$$\begin{aligned}
I_{2,1} \quad = \quad & -\frac{i(n-1)!}{\pi^n}\int_{B'} f(\zeta)\left(\sum_{m=1}^{n-1}\frac{\partial\rho}{\partial\bar\zeta_m}\,\bar\zeta_m\zeta_n + \frac{\partial\rho}{\partial\bar\zeta_n}\,(u_n^2+\phi^2+y_n^2)\right)\\
&\left[\sum_{j=1}^n\binom{n}{j} jA_{2+2j}^{n+1}(\zeta,z^+,z^-) + (n+j)A_{2j}^{n+1}(\zeta,z^+,z^-)\right]d\sigma, \qquad (2.82)
\end{aligned}$$

$$I_{2,+} \quad = \quad \frac{i(n-1)!}{\pi^n}\int_{B'} f(\zeta)\left(\sum_{m=1}^{n-1}\frac{\partial\rho}{\partial\bar{\zeta}_m}\,\bar{\zeta}_m z_n^+ + \frac{\partial\rho}{\partial\bar{\zeta}_n}\,(2\phi y_n)\right)\sum_{j=1}^{n}\binom{n}{j}C(\zeta,z^+)d\sigma \tag{2.83}$$

while

$$I_{2,-} \quad = \quad -\frac{i(n-1)!}{\pi^n}\int_{B'} f(\zeta)\left(\sum_{m=1}^{n-1}\frac{\partial\rho}{\partial\bar{\zeta}_m}\,\bar{\zeta}_m z_n^- - \frac{\partial\rho}{\partial\bar{\zeta}_n}\,(2\phi y_n)\right)\sum_{j=1}^{n}\binom{n}{j}C(\zeta,z^-)d\sigma. \tag{2.84}$$

For the last two terms, according to the definitions of $C(\zeta, z^{\pm})$ (see (2.62)) observe that

$$\left|C(\zeta,z^{\pm})\right| \quad \leq \quad \frac{C_{10}}{|\zeta - z^{\pm}|^{2n+2}} = \frac{C_{11}}{(|w|^2+y_n^2)^{n+1}}.$$

Thus,

$$|I_{2,\pm}| \quad \leq \quad M_1\epsilon\int_{B'}\frac{\left(M_2|w|^2|y_n| + M_3|w|^3|y_n|\right)dS}{(|w|^2+y_n^2)^{n+1}} \leq M_4\epsilon\int_T\frac{|y_n|dS}{(|w|^2+y_n^2)^n} = M_5\epsilon.$$

By similar computations with (2.76) and making use (2.77) for $s = j+l+1$, we obtain

$$\left|A^{n+1}_{2j+2}(\zeta,z^+,z^-)\right| \quad = \quad \frac{\left|\sum_{l=0}^{n+1}\binom{n+1}{l}|\zeta-z^+|^{2l}|\zeta-z^-|^{2l}B_{j+l+1}\right|}{(1+|\zeta-z^+|^2)^{n+1}(1+|\zeta-z^-|^2)^{n+1}}.$$

where $|B_{j+l+1}(\zeta,z^+,z^-)| \leq \frac{C_5(2|y_n|+C|w|^2)}{(|w|^2+y_n^2)^{j+l+1}}$. Since, for every $0 \leq l \leq n+1$,

$$\frac{|\zeta-z^+|^{2l}|\zeta-z^-|^{2l}}{(1+|\zeta-z^+|^2)^{n+1}(1+|\zeta-z^-|^2)^{n+1}} \leq \frac{|\zeta-z^+|^{2l}}{|\zeta-z^+|^{2(n+1-j)}} \leq \frac{M_6}{(|w|^2+y_n^2)^{n+1-j-l}} \tag{2.85}$$

holds in view of (2.74) and (2.75), then

$$\left|A^{n+1}_{2j+2}(\zeta,z^+,z^-)\right| \quad \leq \quad \frac{M_7|y_n|}{(|w|^2+y_n^2)^{n+2}} + \frac{M_8}{(|w|^2+y_n^2)^{n+1}}$$

arises. Similarly, one can observe

$$\left|A^{n+1}_{2j}(\zeta,z^+,z^-)\right| \quad \leq \quad \frac{M_9|y_n|}{(|w|^2+y_n^2)^{n+2}} + \frac{M_{10}}{(|w|^2+y_n^2)^{n+1}}$$

Hence, returning back to (2.82), we get

$$
\begin{aligned}
|I_{2,1}| &\leq M_{11}\epsilon \int_{B'} \left[M_{12}|w|^3 + C_3|w|(|w|^2 + C^2|w|^4 + y_n^2)\right] \\
&\qquad \left[\frac{M_{13}|y_n|}{(|w|^2+y_n^2)^{n+2}} + \frac{M_{14}}{(|w|^2+y_n^2)^{n+1}}\right] dS \\
&\leq M_{15}\epsilon \int_{B'} \frac{|y_n| dS}{(|w|^2+y_n^2)^n} + M_{16}\epsilon \int_{B'} \frac{dS}{(|w|^2+y_n^2)^{n-1}} \leq M_{17}\epsilon,
\end{aligned}
$$

where we used the results in (2.80) and (2.81). Thus,

$$\lim_{z^{\pm}\to z} I_2|_{\Gamma} = 0.$$

The integral $I_2|_{\partial D\setminus\Gamma}$ tends directly to zero as $z^{\pm}$ goes to z since the limit can pass through the integral. $\square$

2.2 An application of weighted Koppelman formula for a Hartogs phenomenon in $\mathbb{C}^n$

Let us consider a bounded domain D in $\mathbb{C}^n$ containing the origin with connected smooth boundary ∂D (of class $\mathcal{C}^2$) and a set of one-dimensional complex lines l of the form

$$l = \{\zeta : \zeta_j = z_j + b_j t, j = 1, \ldots, n, t \in \mathbb{C}\},$$

where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{CP}^{n-1}$.

A function $f \in \mathcal{C}(\partial D)$ has a one-dimensional holomorphic continuation property along the complex line l ($l \cap \partial D \neq \emptyset$) ([30],[31]) if there exists a function f_l satisfying the following properties:

1. $f_l \in \mathcal{C}(\overline{D} \cap l)$
2. $f_l = f$ on the set $\partial D \cap l$
3. f_l is a holomorphic function in the interior points of $\overline{D} \cap l$.

The main results in monograph [27] prove that for some domains $D \subseteq \mathbb{C}^n$ functions having the above one-dimensional holomorphic continuation property extend holomorphically into D. We prove the same results using weighted Koppelman formulas instead of B-M kernel used previously. The novelty here is the fact that the Koppelman kernel K is not harmonic as opposed to the B-M kernel, but the same results can be deduced.

The contributing kernels K and P in this paragraph are those introduced in (2.2) and (2.5). Since the Koppelman kernel $K = \sum_{k=1}^{n} u_{k,k-1} \wedge g_{n-k,n-k}$ involves the well-known B-M kernel defined by

$$u_{n,n-1} = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} d\bar{\zeta}[j] \wedge d\zeta \tag{2.86}$$

for $k = n$, the corresponding term in K kernel will be treated similarly as B-M kernel. It is known that $u_{n,n-1}$ is harmonic in $\mathbb{C}^n$, but we give the proof for completeness of the thesis.

Lemma 2.2.1 *The contributing form $u_{n,n-1}$ in (2.86) of the kernel K is harmonic in $\mathbb{C}^n$,*

$$\Delta u_{n,n-1} = \sum_{k=1}^{n} \frac{\partial^2 u_{n,n-1}}{\partial z_k \partial \bar{z}_k} = 0.$$

Proof. Rewriting $u_{n,n-1}$ equivalently as

$$u_{n,n-1} = \frac{(-1)^n (n-2)!}{(2\pi i)^n} \partial \left(\frac{1}{|\zeta - z|^{2n-2}} \right) \wedge \sum_{j=1}^{n} d\bar{\zeta}[j] \wedge d\zeta[j],$$

the harmonicity of $u_{n,n-1}$ arises form the coefficients of this form that are harmonic functions of z in $\mathbb{C}^n$. In particular,

$$\frac{\partial^2}{\partial z_k \partial \bar{z}_k} \left(\frac{1}{|\zeta - z|^{2n-2}} \right) = -\frac{(n-1)}{|\zeta - z|^{2n}} + \frac{n(n-1)|\zeta_k - z_k|^2}{|\zeta - z|^{2n+2}},$$

for every $k = 1, \dots, n$. Then, $\Delta \left(\frac{1}{|\zeta - z|^{2n-2}} \right) = \sum_{k=1}^{n} \frac{\partial^2}{\partial z_k \partial \bar{z}_k} \left(\frac{1}{|\zeta - z|^{2n-2}} \right) = 0$. It turns out that $u_{n,n-1}$ is harmonic. $\square$

At this point, let us introduce the notation

$$\frac{\partial^\alpha u_{n,n-1}}{\partial \bar{z}^\alpha} = \frac{\partial^{\|\alpha\|} u_{n,n-1}}{\partial \bar{z}_1^{\alpha_1} \cdots \partial \bar{z}_n^{\alpha_n}},$$

for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\|\alpha\| = \alpha_1 + \cdots + \alpha_n$.

The next lemma is formulated in [29] with its proof.

Lemma 2.2.2 [29] *The restriction of $u_{n,n-1}$ and $(\zeta_j - z_j)\partial^\alpha u_{n,n-1}/\partial \bar{z}^\alpha$ for $j = 1, \dots, n$, into curve $\partial D \cap l \subset \mathbb{C}$, for a bounded domain D of $\mathbb{C}^n$, is given by the following relations:*

$$u_{n,n-1}(\zeta, z)|_{\partial D \cap l} = \lambda(b) \wedge \frac{dt}{t}$$

and

$$(\zeta_j - z_j)\frac{\partial^\alpha u_{n,n-1}}{\partial \bar{z}^\alpha}\big|_{\partial D\cap l} = \mu(b)\wedge\frac{dt}{\bar{t}^{|\alpha\|}},$$

where $\lambda(b)$ and $\mu(b)$ are differential forms with respect to direction $b \in \mathbb{CP}^{n-1}$.

Proof. Since $db = db_1 \wedge \cdots \wedge db_n = 0$, then

$$\begin{aligned} d\zeta\big|_{\partial D\cap l} &= d\zeta_1\wedge\ldots\wedge d\zeta_n\big|_{\partial D\cap l} \\ &= (b_1 dt + t db_1)\wedge\ldots\wedge(b_n dt + t db_n) \\ &= t^{n-1}\sum_{j=1}^{n}(-1)^{j-1} b_j dt\wedge db[j], \end{aligned} \tag{2.87}$$

where $db[j]$ denotes the wedge product db where db_j is omitted. Similarly, one can obtain

$$\sum_{j=1}^{n}(-1)^{j-1}(\bar{\zeta}_j - \bar{z}_j)d\bar{\zeta}[j]\big|_{\partial D\cap l} = \bar{t}^n\sum_{j=1}^{n}(-1)^{j-1}\bar{b}_j d\bar{b}[j]. \tag{2.88}$$

Hence, the restriction of $u_{n,n-1}$ along $\partial D\cap l$ yields

$$\begin{aligned} u_{n,n-1}\big|_{\partial D\cap l} &= \frac{(n-1)!}{(2\pi i)^n |b|^{2n}|t|^{2n}}\left(\bar{t}^n\sum_{j=1}^{n}(-1)^{j-1}\bar{b}_j d\bar{b}[j]\right)\wedge\left(t^{n-1}\sum_{j=1}^{n}(-1)^{j-1} b_j dt\wedge db[j]\right) \\ &= \lambda(b)\wedge\frac{dt}{t}, \end{aligned} \tag{2.89}$$

where $\lambda(b)$ is an $(n-1, n-1)$ form with respect to b.

For the second part of the lemma, by repeating the steps of the proof that can be found in [29], observe that $u_{n,n-1}$ can be equivalently written in the form

$$u_{n,n-1} = \sum_{j=1}^{n}(-1)^{j-1}\frac{\partial h}{\partial\zeta_j}d\bar{\zeta}[j]\wedge d\zeta,$$

where $h(\zeta,z) = -\frac{(n-2)!}{(2\pi i)^n}\frac{1}{|\zeta-z|^{2n-2}}$. The derivatives

$$\frac{\partial^\alpha h}{\partial\bar{z}^\alpha} = \frac{(-1)^{\|\alpha\|+1}}{(2\pi i)^n}\frac{(n+\|\alpha\|-2)!(\zeta-z)^\alpha}{|\zeta-z|^{2n+2\|\alpha\|-2}},$$

where $(\zeta - z)^{\alpha} = (\zeta_1 - z_1)^{\alpha_1} \cdots (\zeta_n - z_n)^{\alpha_n}$, yield that

$$\begin{aligned}\frac{\partial^{\alpha} u_{n,n-1}}{\partial \bar{z}^{\alpha}} &= \frac{(n + \|\alpha\| - 2)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^j \frac{\alpha_j (\zeta - z)^{\alpha - e_j}}{|\zeta - z|^{2n+2\|\alpha\|-2}} d\bar{\zeta}[j] \wedge d\zeta \\ &\quad + \frac{(n + \|\alpha\| - 1)!}{(n-1)!} \frac{(\zeta - z)^{\alpha}}{|\zeta - z|^{2\|\alpha\|}} u_{n,n-1}(\zeta, z), \qquad (2.90)\end{aligned}$$

where $(\zeta - z)^{\alpha - e_j} = (\zeta_1 - z_1)^{\alpha_1} \cdots (\zeta_j - z_j)^{a_j - e_j} \cdots (\zeta_n - z_n)^{\alpha_n}$. By multiplying $\frac{\partial^{\alpha} u_{n,n-1}}{\partial \bar{z}^{\alpha}}$ with $(\zeta_j - z_j)$ for every $j = 1, \ldots, n$, then the change of coordinates through the substitutions $\zeta_j - z_j = b_j t$ on $\overline{D} \cap l$ yields that

$$(\zeta_j - z_j) \frac{\partial^{\alpha} u_{n,n-1}}{\partial \bar{z}^{\alpha}} |_{\partial D \cap l} = \mu(b) \wedge \frac{dt}{\bar{t}^{\|\alpha\|}}, \qquad (2.91)$$

where $\mu(b)$ is also an $(n-1, n-1)$ form with respect to b. □

Lemma 2.2.3 *Let assume that the function $\frac{1}{t}$ can be extended holomorphically from $\partial D \cap l$ into $D \cap l$. Then $\frac{1}{1+|b|^2|t|^2}$ can be also extended holomorphically from $\partial D \cap l$ into $D \cap l$.*

Proof. Let $\phi(t)$ be the holomorphic extension of $\frac{1}{t}$ from $\partial D \cap l$ to $D \cap l$. Then for $t \in \partial D \cap l$ one has that

$$\frac{1}{1 + |b|^2 |t|^2} = \frac{1}{\bar{t}\left(\frac{1}{\bar{t}} + |b|^2 t\right)} = \frac{\phi(t)}{\phi(t) + |b|^2 t} = \omega(t) > 0, \qquad (2.92)$$

and that $\omega(t)$ is meromorphic on D. By continuity, $\omega(t)$ is also positive in an open 'annulus' U_ϵ. Since $\partial D \cap l$ is compact, we can take a 'smaller' annulus such that $\overline{V}_\epsilon \subset U_\epsilon$. The function $\omega(t)$ is holomorphic in U_ϵ and $\omega(V_\epsilon) \subset (0, +\infty)$. But, by the open mapping theorem, this implies that $\omega(t) = c$ on $\overline{V}_\epsilon$ for some constant $c < 1$, since otherwise the image of V_ϵ would be an open subset of $\mathbb{C}$ sitting inside $\mathbb{R}$ which is impossible. Hence, $\omega(t) = c$ on $\partial D \cap l$.

Therefore,

$$\frac{\phi(t)}{\phi(t) + |b|^2 t} = c > 0 \, (c \neq 1),$$

or, equivalently

$$\phi(t) = \frac{|b|^2 c}{1 - c} t,$$

for every direction $b \in \mathbb{CP}^{n-1}$. Thus, in view of (2.92), $\omega(t)$ has holomorphic extension to $D \cap l$. □

Thus the domains for which $\frac{1}{t}$ has a holomorphic extension are those for which their intersections with complex lines gives discs with varying smoothly radii since $|t|^2 = \frac{1-c}{|b|^2 c}$ on

$\partial D \cap l$.

Lemma 2.2.4 *If $f \in \mathcal{C}(\partial D)$ has a one-dimensional holomorphic continuation property along the complex line l ($l \cap \partial D$) and F is a biholomorphism from $\overline{D}$ to $\overline{G}$, where G is a domain, then $f^* = f \circ F^{-1} : \overline{G} \to \mathbb{C}$ has a one-dimensional continuation property along $F(l)$.*

Proof. First of all, observe that $f^* \in \mathcal{C}(\partial G)$. Let $\epsilon > 0$ and $z_0 \in \partial D$ such that $F(z_0) = w_0 \in \partial G$. Since $f \in \mathcal{C}(\partial D)$, it is continuous at $z_0 = F^{-1}(w_0)$ and there exists $\delta_1 > 0$ such that for $z \in \partial D$,

$$\|z - F^{-1}(w_0)\| < \delta_1 \quad \text{then} \quad \left|f(z) - f\left(F^{-1}(w_0)\right)\right| < \epsilon.$$

But, since F^{-1} is also continuous at w_0, then there exists $\delta_2 > 0$ such that for $w \in \partial G$,

$$\|w - w_0\| < \delta_2 \quad \text{then} \quad \left\|F^{-1}(w) - F^{-1}(w_0)\right\| < \delta_1.$$

Combining these two inequalities, then for $w \in \partial G$

$$\|w - w_0\| < \delta_2 \quad \text{then} \quad \left|f\left(F^{-1}(w)\right) - f\left(F^{-1}(w_0)\right)\right| < \epsilon.$$

Now, let $z_0 \in D \cap l$. Since $F(D \cap l) = F(D) \cap F(l) = G \cap F(l)$, then $w_0 = F(z_0) \in G \cap F(l)$. According to the one-dimensional holomorphic continuation property along l, there exists $f_l \in \mathcal{C}(\overline{D} \cap l)$ which is holomorphic in the interior points of $\overline{D} \cap l$ such that $f_l = f$ on $\partial D \cap l$. Hence, the $1 \times n$-dimensional Jacobian matrix of the partial derivatives of f_l with respect to the i-coordinate z_i of z

$$J_{f_l}(z_0) = \left[\frac{\partial f_l}{\partial z_i}\right]_{z_0} \tag{2.93}$$

exists in a neighborhood of z_0. Moreover, the holomorphicity of $F^{-1} = (F_1^{-1}, \ldots, F_n^{-1})$ on $G \cap F(l)$ in a neighborhood of w_0 implies that the $n \times n$-dimensional Jacobian matrix of F^{-1},

$$J_{F^{-1}}(w_0) = \left[\frac{\partial F_i^{-1}}{\partial w_j}\right]_{w_0} = \left[\frac{\partial F}{\partial z_j}\right]_{z_0}^{-1} = [J_F(z_0)]^{-1} \tag{2.94}$$

is well-defined, since F is a biholomorphism.

Let $f^*_{F(l)} = f_l \circ F^{-1}$. Then $f^*_{F(l)} \in \mathcal{C}(\overline{G} \cap F(l))$ and

$$f^*_{F(l)}\big|_{\partial G \cap F(l)} = f_l \circ F^{-1}\big|_{\partial G \cap F(l)} = f \circ F^{-1}\big|_{\partial G \cap F(l)} = f^*\big|_{\partial G \cap F(l)}$$

since $F^{-1}(\partial G \cap F(l)) = F^{-1}(F(\partial D \cap l)) = \partial D \cap l$ and $f_l = f$ on $\partial D \cap l$. The holomorphicity of $f^*_{F(l)}$ in the interior points of $\overline{G} \cap F(l)$ is deduced according to the fact that the $1 \times n$-dimensional Jacobian matrix of the partial derivatives of $f^*_{F(l)}$ at w_0 is the product of the Jacobian matrices in (2.93) and (2.94), namely

$$J_{f^*_{F(l)}}(w_0) = J_{f_l}(z_0) \cdot J_{F^{-1}}(w_0). \tag{2.95}$$

Thus, $f^*_{F(l)}$ is the required extension of f^*. $\square$

Lemma 2.2.5 *If $f \in \mathcal{C}(\partial D)$ has a one-dimensional holomorphic continuation property for every complex line l meeting D and the function $1/\bar{t}$ can be extended from $\partial D \cap l$ to $D \cap l$, then $K^- f + P^- f = 0$ in $\mathbb{C}^n \setminus \overline{D}$.*

Proof. Assume that z lies in the unbounded component of $\mathbb{C}^n \setminus \overline{D}$, following [25]. By making an orthogonal linear transformation and a translation in $\mathbb{C}^n$, we can assume that $z = 0$ and the complex hyperplane $\{z : z_1 = 0\}$ does not intersect D. The kernels in the Koppelman representation formula stay invariant as shown in Lemma 2.1.6 and the line l is assumed to be $l = \{\zeta : \zeta_j = b_j t, j = 1, \ldots, n, t \in \mathbb{C}\}$ after these transformations. A change of variables defined by

$$\zeta_1 = 1/v_1 \quad \text{and} \quad \zeta_j = v_j/v_1 \quad \text{for} \quad j = 2, \ldots, n \tag{2.96}$$

yields that

$$d\zeta = d\zeta_1 \wedge \cdots d\zeta_n = \begin{vmatrix} -1/v_1^2 & 0 & \ldots & 0 \\ -v_2/v_1^2 & 1/v_1 & \ldots & 0 \\ & & \vdots & \\ -v_n/v_1^2 & 0 & \ldots & 1/v_1 \end{vmatrix} = -\frac{dv_1 \wedge \cdots dv_n}{v_1^{n+1}} = -dv/v_1^{n+1},$$

which is well-defined since $\mathbb{C}^n \backslash \{z : z_1 = 0\}$ is mapped biholomorphically onto $\mathbb{C}^n \backslash \{v : v_1 = 0\}$. Let F being this biholomorphism defined by (2.96) and G be the image of D under F such that the hyperplane $\mathbb{C}^n \setminus \{v : v_1 = 0\}$ does not intersect G and $F(\partial D) = \partial G$.

Consider the $(n+1) \times n$ matrix

$$\begin{pmatrix} \zeta_1 & \zeta_2 & \ldots & \zeta_n \\ \zeta'_{1v_1} & \zeta'_{2v_1} & \ldots & \zeta'_{nv_1} \\ & & \vdots & \\ \zeta'_{1v_n} & \zeta'_{2v_n} & \ldots & \zeta'_{nv_n} \end{pmatrix},$$

where $\zeta'_{l\,v_m}$ denotes the derivative of ζ_l with respect to v_m. If Δ_j denotes the determinant of the above matrix where $(j+1)$-row is omitted, then

$$\Delta_j = \begin{vmatrix} 1/v_1 & v_2/v_1 & \dots & v_n/v_1 \\ -1/v_1^2 & -v_2/v_1^2 & \dots & -v_n/v_1^2 \\ 0 & 1/v_1 & \dots & 0 \\ & & \vdots & \\ 0 & 0 & \dots & 1/v_1 \end{vmatrix}_j = \begin{cases} 1/v_1^n, & j=1 \\ 0, & j\neq 1 \end{cases},$$

and thus

$$\sum_{j=1}^{n}(-1)^{j-1}\bar{\zeta}_j d\bar{\zeta}[j] = \sum_{j=1}^{n}\overline{\Delta}_j d\bar{v}[j] = d\bar{v}[1]/\bar{v}_1^n.$$

Since $K(\zeta,z)$ is transformed into

$$K(\zeta,0)=\sum_{k=1}^{n}\frac{\binom{n}{k}(n-1)!}{(2\pi i)^n|\zeta|^{2k}(1+|\zeta|^2)^n}\sum_{j=1}^{n}(-1)^{j-1}\bar{\zeta}_j d\bar{\zeta}[j]\wedge d\zeta,$$

(under the initial orthogonal linear transformation and translation in $\mathbb{C}^n$) we obtain that, under the change of coordinate system given in (2.96), the Koppelman kernel becomes

$$\begin{aligned} K(\zeta,0) &= \frac{(-1)^n(n-1)!}{(2\pi i)^n}\sum_{k=1}^{n}\frac{\binom{n}{k}|v_1|^{2k}}{(1+|v_1|^2+\dots|v_n|^2)^n}\frac{dv_1}{v_1}\wedge\frac{d\bar{v}[1]\wedge dv[1]}{(1+|v_2|^2+\dots|v_n|^2)^k} \\ &= \frac{(-1)^n(n-1)!}{(2\pi i)^n}\sum_{k=1}^{n}\frac{\binom{n}{k}|v_1|^{2k}}{(1+|v_1|^2+\dots|v_n|^2)^n}\frac{dv_1}{v_1}\wedge\lambda_k(v_2,\dots,v_n), \end{aligned}$$

where $\lambda_k(v_2,\dots,v_n)=\frac{d\bar{v}[1]\wedge dv[1]}{(1+|v_2|^2+\dots|v_n|^2)^k}$.

The lines $l=\{\zeta:\zeta_j=b_jt,\ j=1,\dots,n,\ t\in\mathbb{C}\}$ are mapped into lines[1] $l_{v_1,\dots,v_n}=\{v:v_1=1/b_1t, v_j=b_j/b_1,\ j=2,\dots,n\}$ through the biholomorphism F in (2.96) and hence, in view of Lemma 2.2.4, the one dimensional holomorphic extension property of f along l leads to the one dimensional holomorphic extension property of f^* along the lines $l_{v_1,\dots,v_n}$.

Moreover, since $1/\bar{t}$ can be extended from $\partial D\cap l$ to $D\cap l$, then Lemma 2.2.3 shows that $\frac{1}{1+|b|^2|t|^2}$ can be also extended from $\partial D\cap l$ to $D\cap l$ and in particular $|t|^2=c(b)$, where $c(b)$ is a function which depends on $|b|$. Hence, the biholomorphism F defined in (2.96) extends the holomorphic continuation of $\frac{1}{1+|b|^2|t|^2}$ from $\partial D\cap l$ to $D\cap l$, into the holomorphic extension of $\frac{|v_1|^2}{1+|v_1|^2+|v_2|^2+\dots+|v_n|^2}$ from $\partial G\cap l_{v_1,\dots,v_n}$ to $G\cap l_{v_1,\dots,v_n}$, as a consequence of Lemma 2.2.4. Furthermore, $|v_1|^2=\frac{1}{|b_1|^2|t|^2}=\frac{1}{|b_1|^2c(b)}$ along $\partial G\cap l_{v_1,\dots,v_n}$. Thus, $\frac{|v_1|^{2k}}{(1+|v_1|^2+\dots+|v_n|^2)^n v_1}$ can be also extended from $\partial G\cap l_{v_1,\dots,v_n}$ to $G\cap l_{v_1,\dots,v_n}$ since the hyperplane $\{v:v_1=0\}$ does not

[1] In this case, the line is a curve with no self-intersection.

intersect G.

Therefore, if G' is the projection of ∂G under the mapping $v \mapsto (v_2, \ldots, v_n)$, we reduce the integration ∂D into iterated integration with respect to the direction:

$$\int_{\partial D} f(\zeta)K(\zeta, 0) = \frac{(-1)^n(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} \binom{n}{k} \int_{G'} \lambda_k(v_2, \ldots, v_n)$$
$$\int_{\partial G \cap l_{v_1,\ldots,v_n}} f^*(v) \frac{|v_1|^{2k}}{(1+|v_1|^2+\cdots+|v_n|^2)^n v_1}\, dv_1 = 0.$$

Note that, $\partial G \cap l_{v_1,\ldots,v_n}$ is a smooth curve, since it is the image of $\partial D \cap l$ under F (biholomorphism). Hence, $K^- f = 0$ in $\mathbb{C}^n \setminus \overline{D}$.

Under the assumptions of this Lemma, $P^- f$ also vanishes in $\mathbb{C}^n \setminus \overline{D}$. Actually, the projection kernel $P(\zeta, z)$ is transformed to $P(\zeta, 0) = \frac{(-1)^{\frac{n(n-1)}{2}} n!}{(2\pi i)^n (1+|\zeta|^2)^{n+1}}\, d\bar{\zeta} \wedge d\zeta$ (see (2.5)), for z lying in the unbounded component of $\mathbb{C}^n \setminus \overline{D}$, and

$$\begin{aligned} P(v_1, \ldots, v_n) &= \frac{(-1)^{\frac{n(n-1)}{2}} n!}{(2\pi i)^n (1+|v_1|^2+\cdots+|v_n|^2)^{n+1}}\, d\bar{v} \wedge dv \\ &= \frac{(-1)^{\frac{n(n+1)}{2}} (n-1)!}{(2\pi i)^n}\, \bar{\partial} \left(\frac{dv_1}{v_1(1+|v_1|^2+\cdots+|v_n|^2)^n} \right) d\bar{v}[1] \wedge dv[1], \end{aligned}$$

after the change of variables given in (2.96). Hence, if G'' is the projection of G under the mapping $v \mapsto (v_2, \ldots, v_n)$ and $C_n = \frac{(-1)^{\frac{n(n+1)}{2}} (n-1)!}{(2\pi i)^n}$, then

$$\begin{aligned} &\int_D f(\zeta) P(\zeta, 0) \\ = {}& C_n \int_{G''} d\bar{v}[1] \wedge dv[1] \int_{G \cap l_{v_1,\ldots,v_n}} f^*(v) \bar{\partial} \left(\frac{dv_1}{v_1(1+|v_1|^2+\cdots+|v_n|^2)^n} \right) \\ = {}& C_n \int_{G''} d\bar{v}[1] \wedge dv[1] \int_{\partial G \cap l_{v_1,\ldots,v_n}} f^*(v) \frac{dv_1}{v_1(1+|v_1|^2+\cdots+|v_n|^2)^n} \end{aligned}$$

by Stoke's Theorem. But the last inner integral vanishes, by the same argument which led to the vanishing of the boundary integral. Thus, $P^- f = 0$ in $\mathbb{C}^n \setminus \overline{D}$. □

Theorem 2.2.1 *If $f \in \mathcal{C}(\partial D)$ has a one-dimensional holomorphic continuation property for every complex line l meeting D and the function $1/\bar{t}$ can be extended from $\partial D \cap l$ to $D \cap l$, then there exists a function $F \in \mathcal{O}(D) \cap \mathcal{C}(\overline{D})$ and $F = f$ on ∂D.*

Proof. We will show that the extension of f is given by the weighted Koppelman type integral formula in the case of holomorphic functions (see Remark 1.2.1), where K and P are the kernels which were introduced in this chapter. Namely, let F be defined by

$$F(z) \;=\; \int_{\partial D} f(\zeta)K(\zeta,z) + \int_D f(\zeta)P(\zeta,z) \tag{2.97}$$

where $K = (u \wedge g)_{n,n-1} = \sum_{k=1}^n u_{k,k-1} \wedge g_{n-k,n-k}$, $P = g_{n,n}$, and g is the weight used in the Jump theorem, that is, $g = h^n$ for $h = 1 - \nabla_{\zeta - z}\left(\frac{\partial|\zeta-z|^2}{2\pi i(1+|\zeta-z|^2)}\right)$. Then the kernels K and P are those that are described in (2.2) and (2.5), respectively.

Firstly, we split the first integral in (2.97) into the sum of integrals

$$\int_{\partial D} f(\zeta)K(\zeta,z) \;=\; \sum_{k=1}^n \int_{\partial D} f(\zeta)u_{k,k-1} \wedge g_{n-k,n-k} = \sum_{k=1}^n A_{n-k,n-k}, \tag{2.98}$$

where $A_{n-k,n-k} = \int_{\partial D} f(\zeta)u_{k,k-1} \wedge g_{n-k,n-k}$. The holomorphicity of the boundary integral is derived by the holomorphicity of each one of the above terms as will be shown below. Following [31], in order to show that $A_{0,0}$ is holomorphic, the functions $A^j_{0,0}$, for $j = 1,\ldots,n$ are introduced:

$$A^j_{0,0} \;=\; \int_{\partial D} f(\zeta)(\zeta_j - z_j)u_{n,n-1} \wedge g_{0,0}. \tag{2.99}$$

Applying the product rule for the Laplace operator $\Delta = \sum_{s=1}^n \frac{\partial^2}{\partial z_s \partial \bar{z}_s}$,

$$\Delta(fh) = h\Delta f + f\Delta h + \sum_{s=1}^n \frac{\partial f}{\partial \bar{z}_s}\frac{\partial h}{\partial z_s} + \sum_{s=1}^n \frac{\partial f}{\partial z_s}\frac{\partial h}{\partial \bar{z}_s},$$

on $f(\zeta)(\zeta_j - z_j)u_{n,n-1}g_{0,0}$, one has that

$$\Delta[f(\zeta)(\zeta_j - z_j)u_{n,n-1}g_{0,0}] = f(\zeta)(\zeta_j - z_j)\Delta(u_{n,n-1}g_{0,0}) - f(\zeta)\frac{\partial(u_{n,n-1}g_{0,0})}{\partial \bar{z}_j}.$$

However, letting the Laplace operator act again on the form $u_{n,n-1}g_{0,0}$, yields

$$\Delta(u_{n,n-1}g_{0,0}) \;=\; u_{n,n-1}\Delta g_{0,0} + \sum_{s=1}^n \frac{\partial u_{n,n-1}}{\partial \bar{z}_s}\frac{\partial g_{0,0}}{\partial z_s} + \sum_{s=1}^n \frac{\partial u_{n,n-1}}{\partial z_s}\frac{\partial g_{0,0}}{\partial \bar{z}_s},$$

due to the harmonicity of $u_{n,n-1}$ (Lemma 2.2.1). A combination of the above relations provides that

$$\begin{aligned}\frac{\partial A_{0,0}}{\partial \bar{z}_j} \;=\;& -\Delta A^j_{0,0} + \int_{\partial D} f(\zeta)(\zeta_j - z_j)u_{n,n-1}\Delta g_{0,0} + \int_{\partial D} f(\zeta)(\zeta_j - z_j)\sum_{s=1}^n \frac{\partial u_{n,n-1}}{\partial \bar{z}_s}\frac{\partial g_{0,0}}{\partial z_s}\\ &+ \int_{\partial D} f(\zeta)(\zeta_j - z_j)\sum_{s=1}^n \frac{\partial u_{n,n-1}}{\partial z_s}\frac{\partial g_{0,0}}{\partial \bar{z}_s}.\end{aligned} \tag{2.100}$$

Each of the above contributing terms will be explored, separately due to the complexity of each term.

For the first term of (2.100), one can observe that for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$

$$\frac{\partial^\alpha (u_{n,n-1} g_{0,0})}{\partial \bar{z}^\alpha} = \sum_{0 \le \gamma \le \alpha} c_\gamma \frac{\partial^\gamma u_{n,n-1}}{\partial \bar{z}^\gamma} \frac{\partial^{\alpha-\gamma} g_{0,0}}{\partial \bar{z}^{\alpha-\gamma}},$$

where $\gamma \le \alpha$ means that $\gamma_1 \le \alpha_1, \ldots, \gamma_n \le \alpha_n$, while c_γ is some constant. Since $g_{0,0} = h^n_{0,0} = 1/(1 + |\zeta - z|^2)^n$,

$$\frac{\partial^\beta g_{0,0}}{\partial \bar{z}^\beta} = \frac{(n + \|\beta\| - 1)!}{(n-1)!} \frac{(\zeta - z)^\beta}{(1 + |\zeta - z|^2)^{n+\|\beta\|}},$$

for a multi-index $\beta = (\beta_1, \ldots, \beta_n)$ and by changing the variables in terms of b and t, it takes the form

$$\frac{\partial^\beta g_{0,0}}{\partial \bar{z}^\beta}\Big|_{\partial D \cap l} = \frac{(n + \|\beta\| - 1)! b^\beta t^{\|\beta\|}}{(n-1)!(1 + |b|^2|t|^2)^{n+\|\beta\|}}. \tag{2.101}$$

Now, by applying Lemma 2.2.2, deduce that

$$(\zeta_j - z_j)\frac{\partial^\alpha (u_{n,n-1} g_{0,0})}{\partial \bar{z}^\alpha}\Big|_{\partial D \cap l} = \sum_{0 \le \gamma \le \alpha} \nu_{\alpha,\gamma}(b) \frac{t^{\|\alpha-\gamma\|}}{\bar{t}^{\|\gamma\|}(1 + |b|^2|t|^2)^{n+\|\alpha-\gamma\|}} \wedge dt,$$

where $\nu_{\alpha,\gamma}(b) = (n + \|\alpha - \gamma\| - 1)! c_\gamma \mu(b) b^{\alpha-\gamma}/(n-1)!$. Thus,

$$\begin{aligned} &\frac{\partial^\alpha}{\partial \bar{z}^\alpha} \int_{\partial D} f(\zeta)(\zeta_j - z_j) u_{n,n-1} g_{0,0} \\ &= \sum_{0 \le \gamma \le \alpha} \int_{\mathbb{CP}^{n-1}} \nu_{\alpha,\gamma}(b) \int_{\partial D \cap l} \frac{t^{\|\alpha-\gamma\|}}{\bar{t}^{\|\gamma\|}(1 + |b|^2|t|^2)^{n+\|\alpha-\gamma\|}} f(z + bt) dt. \end{aligned} \tag{2.102}$$

In view of Lemma 2.2.3, since we assumed that the function $1/\bar{t}$ can be extended holomorphically from $\partial D \cap l$ to $D \cap l$, the vanishing of (2.102) follows.

In order to show that the second term of (2.100) vanishes, we have to compute $\Delta g_{0,0}$. Direct differentiation yields $\frac{\partial g_{0,0}}{\partial \bar{z}_j} = \frac{n(\zeta_j - z_j)}{(1+|\zeta - z|^2)^{n+1}}$. Consequently,

$$\frac{\partial^2 g_{0,0}}{\partial z_j \partial \bar{z}_j} = \frac{-n}{(1 + |\zeta - z|^2)^{n+1}} + \frac{n(n+1)|\zeta_j - z_j|^2}{(1 + |\zeta - z|^2)^{n+2}}.$$

Hence

$$\Delta g_{0,0} = \sum_{j=1}^n \frac{\partial^2 g_{0,0}}{\partial z_j \partial \bar{z}_j} = \frac{-n^2 + n|\zeta - z|^2}{(1 + |\zeta - z|^2)^{n+2}}$$

Direct computations imply that,

$$\Delta g_{0,0}\big|_{\partial D\cap l} \;=\; \frac{-n^2+n|b|^2|t|^2}{(1+|b|^2|t|^2)^{n+2}} = \frac{n}{(1+|b|^2|t|^2)^{n+1}} - \frac{n^2+n}{(1+|b|^2|t|^2)^{n+2}}.$$

Thus, if $\lambda_1(b) = b_j\lambda(b)$

$$\begin{aligned} &\int_{\partial D} f(\zeta)(\zeta_j - z_j)u_{2,1}\Delta g_{0,0} \\ =\;& \int_{\mathbb{CP}^{n-1}} \lambda_1(b)\int_{\partial D\cap l}\left(\frac{n}{(1+|b|^2|t|^2)^{n+1}} - \frac{n^2+n}{(1+|b|^2|t|^2)^{n+2}}\right) f(z+bt)dt = 0, \end{aligned}$$

where we used the Lemma 2.2.3.

In order to explore the third term in (2.100), one can observe that

$$(\zeta_j - z_j)\frac{\partial u_{n,n-1}}{\partial \bar z_s}\big|_{\partial D\cap l} = \mu(b)\wedge\frac{dt}{\bar t}$$

according to Lemma 2.2.2, while a direct differentiation with respect to z_s provides $\frac{\partial g_{0,0}}{\partial z_s} = \frac{n(\bar\zeta_s-\bar z_s)}{(1+|\zeta-z|^2)^{n+1}}$. Thus

$$\frac{\partial g_{0,0}}{\partial z_s}\big|_{\partial D\cap l} \;=\; \frac{n\bar b_s\bar t}{(1+|b|^2|t|^2)^{n+1}}.$$

It turns out that

$$(\zeta_j - z_j)\frac{\partial u_{n,n-1}}{\partial \bar z_s}\frac{\partial g_{0,0}}{\partial z_s}\big|_{\partial D\cap l} \;=\; \frac{\xi_1(b)}{(1+|b|^2|t|^2)^{n+1}}\wedge dt, \tag{2.103}$$

for each $s = 1,\ldots,n$, where $\xi_1(b) = n\bar b_s\mu(b)$. A combination of Fubini's Theorem and Lemma 2.2.3 lead to the vanishing of the the third term of (2.100).

The fourth term of (2.100) is treated like the third one and a similar result arises. Observe that

$$\frac{\partial u_{n,n-1}}{\partial z_s} \;=\; \sum_{j=1}^{n}(-1)^{j-1}\frac{\partial}{\partial\zeta_j}\left(\frac{\partial h}{\partial z_s}\right)d\bar\zeta[j]\wedge d\zeta = \frac{n(\bar\zeta_s - \bar z_s)}{|\zeta-z|^2}\,u_{n,n-1}$$

and $\frac{\partial g_{0,0}}{\partial \bar z_s} = \frac{n(\zeta_s - z_s)}{(1+|\zeta-z|^2)^{n+1}}$, while the change of coordinates in terms of b and t gives

$$(\zeta_j - z_j)\frac{\partial u_{n,n-1}}{\partial z_s}\frac{\partial g_{0,0}}{\partial \bar z_s}\big|_{\partial D\cap l} \;=\; \frac{\xi_2(b)}{(1+|b|^2|t|^2)^{n+1}}\wedge dt,$$

where $\xi_2(b) = n^2 b_j|b_s|^2\lambda(b)/|b|^2$. Hence, the fourth integral in (2.100) also vanishes.

On the other hand, direct differentiation with respect to $\bar z_j$ implies the holomorphicity of the remaining terms $A_{n-k,n-k}$ for $k=1,\dots,n-1$ in the representation of $F(z)$ in (2.98). According to the explicit form of K (see (2.2))

$$u_{k,k-1}\wedge g_{n-k,n-k}=\frac{(n-1)!}{(2\pi i)^n}\frac{\binom{n}{k}\sum_{j=1}^{n}(-1)^{j-1}(\bar\zeta_j-\bar z_j)d\bar\zeta[j]\wedge d\zeta}{|\zeta-z|^{2k}(1+|\zeta-z|^2)^n},\tag{2.104}$$

one can obtain that

$$\begin{aligned}\frac{\partial\left(u_{k,k-1}\wedge g_{n-k,n-k}\right)}{\partial\bar z_s}&=\frac{(n-1)!}{(2\pi i)^n}\frac{\binom{n}{k}\sum_{j=1}^{n}(-1)^j\delta_j^s d\bar\zeta[j]\wedge d\zeta}{|\zeta-z|^{2k}(1+|\zeta-z|^2)^n}\\&+\frac{(n-1)!}{(2\pi i)^n}\frac{\binom{n}{k}(\zeta_s-z_s)[k+(k+n)|\zeta-z|^2]\sum_{j=1}^{n}(-1)^{j-1}(\bar\zeta_j-\bar z_j)d\bar\zeta[j]\wedge d\zeta}{(|\zeta-z|^2)^{k+1}(1+|\zeta-z|^2)^{n+1}},\end{aligned}$$

while a change of coordinates yields

$$\begin{aligned}\left.\frac{\partial\left(u_{k,k-1}\wedge g_{n-k,n-k}\right)}{\partial\bar z_s}\right|_{\partial D\cap l}&=\frac{(n-1)!\binom{n}{k}}{(2\pi i)^n}\frac{\sum_{j=1}^{n}(-1)^j\delta_j^s d\bar b[j]\wedge\sum_{j=1}^{n}(-1)^{j-1}b_j db[j]}{|b|^{2k}}\\&\quad\wedge\frac{dt}{|t|^{2k-2n+2}(1+|b|^2|t|^2)^n}\\&+\frac{(n-1)!\binom{n}{k}}{(2\pi i)^n}\frac{b_s\sum_{j=1}^{n}(-1)^{j-1}\bar b_j d\bar b[j]\wedge\sum_{j=1}^{n}(-1)^{j-1}b_j db[j]}{|b|^{2k+2}}\\&\quad\wedge\frac{[(k+n)|b|^2|t|^2+k]dt}{|t|^{2k-2n+2}(1+|b|^2|t|^2)^{n+1}}.\end{aligned}$$

Hence,

$$\left.\frac{\partial\left(u_{k,k-1}\wedge g_{n-k,n-k}\right)}{\partial\bar z_s}\right|_{\partial D\cap l}=\xi_3(b)\frac{|t|^{2n-2k-2}dt}{(1+|b|^2|t|^2)^n}+\xi_4(b)\frac{|t|^{2n-2k}dt}{(1+|b|^2|t|^2)^{n+1}},$$

where $\xi_3(b)$ and $\xi_4(b)$ are functions of b depending on degree k of the forms involved. The equality

$$\frac{|t|^{2(n-p)}}{(1+|b|^2|t|^2)^{n+q}}=\sum_{i=1}^{n+1-p}\frac{c_{i,p,q}(b)}{(1+|b|^2|t|^2)^{q+p-1+i}},\tag{2.105}$$

holds for every $q\geq -p$, where $c_{i,p,q}(b)$ are functions of b depending on i, p and q, leads to the vanishing of the following integrals for every $k=1,\dots,n-1$

$$\begin{aligned}\frac{\partial A_{n-k,n-k}}{\partial\bar z_s}&=\int_{\mathbb{CP}^{n-1}}\xi_3(b)\int_{\partial D\cap l}\frac{|t|^{2n-2k-2}}{(1+|b|^2|t|^2)^n}f(z+bt)dt\\&+\int_{\mathbb{CP}^{n-1}}\xi_4(b)\int_{\partial D\cap l}\frac{|t|^{2n-2k}}{(1+|b|^2|t|^2)^{n+1}}f(z+bt)dt,\end{aligned}$$

due to Lemma 2.2.3.

The holomorphicity of the second term in (2.97) related to the kernel $P(\zeta, z)$ follows from direct differentiation of P and an argument similar to the one that has been used in Lemma 2.2.5. In particular, after differentiating the kernel P in (2.5) with respect to $\bar{z}_s$, one gets

$$\frac{\partial P}{\partial \bar{z}_s} = \frac{(-1)^{\frac{n(n-1)}{2}}(n+1)!(\zeta_s - z_s)}{(2\pi i)^n(1+|\zeta - z|^2)^{n+2}} d\bar{\zeta} \wedge d\zeta,$$

where $d\bar{\zeta} \wedge d\zeta = d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$. Its restriction on $D \cap l$ is

$$\begin{aligned}\left.\frac{\partial P}{\partial \bar{z}_s}\right|_{D\cap l} &= \frac{(-1)^{\frac{(n-1)(n+2)}{2}}(n+1)!}{(2\pi i)^n} \frac{b_s t|t|^{2n-2}}{(1+|b|^2|t|^2)^{n+2}} \\ &\quad \left(\sum_{j=1}^{n}(-1)^{j-1}\bar{b}_j d\bar{b}[j]\right) \wedge \left(\sum_{j=1}^{n}(-1)^{j-1}b_j db[j]\right) d\bar{t} \wedge dt \\ &= \sum_{i=1}^{n} l_i(b)\bar{\partial}_t \left(\frac{1}{(1+|b|^2|t|^2)^{i+1}} dt\right),\end{aligned}$$

where

$$l_i(b) = \frac{(-1)^{\frac{n(n+1)}{2}}(n+1)! b_s c_{i,1,2}(b)}{(2\pi i)^n (i+1)|b|^2} \left(\sum_{j=1}^{n}(-1)^{j-1}\bar{b}_j d\bar{b}[j]\right) \wedge \left(\sum_{j=1}^{n}(-1)^{j-1}b_j db[j]\right),$$

by using (2.105). Hence, Stoke's Theorem yields

$$\int_D f(\zeta)\frac{\partial P}{\partial \bar{z}_s} = \sum_{i=1}^{n} \int_{\mathbb{CP}^{n-1}} l_i(b) \int_{\partial D \cap l} \frac{1}{(1+|b|^2|t|^2)^{i+1}} f(z+bt)dt = 0.$$

It turns out that $F = K^+ f + P^+ f \in \mathcal{O}(D)$ and $f = F$ on ∂D in view of Lemma 2.2.5 and Theorem 2.1.2. □

Remark 2.2.1 Lemma 2.2.5 and Theorem 2.2.1 can be formulated under the weaker assumption of the one-dimensional holomorphic continuation property of f along almost complex lines l meeting a germ of a generic manifold that lies in a neighborhood of zero inside D. The idea of the proof follows along the same lines as in [27] with obvious modification.

The ball $B(0, r)$ is a sufficient domain where this Hartogs phenomenon is valid.

Example 2.2.1 ([30]) Let us consider the ball $B(0, r)$ and $S = \partial B(0, r)$. In order to apply the previous Theorem, we have to show the function $1/\bar{t}$ can be extended holomorphically from $S \cap l$ to $B(0, r) \cap l$. Firstly, it is easy to observe that intersection between the ball $B(0, r)$ and the line l ($z \in B(0, r)$, $|b| = 1$) is a circle G defined by

$$G := \{t : |t + \langle z, \bar{b}\rangle|^2 < r^2 - |z|^2 + |\langle z, \bar{b}\rangle|^2\}.$$

This is verified directly, if we substitute the equation of the line l into the equation of the ball $B(0,r)=\{|z|^2=r^2\}$. Then,

$$|z|^2+\left\langle z,\bar{b}\bar{t}\right\rangle+\langle bt,\bar{z}\rangle+|t|^2<r^2,$$

which implies that

$$|t+\left\langle z,\bar{b}\right\rangle|^2 \;=\; |t|^2+\left\langle z,\bar{b}\bar{t}\right\rangle+\langle bt,\bar{z}\rangle+|\left\langle z,\bar{b}\right\rangle|^2<r^2-|z|^2+|\left\langle z,\bar{b}\right\rangle|^2. \quad (2.106)$$

On the boundary of the circle, the above relation becomes equality such that

$$\frac{1}{\bar{t}}=\frac{t+\left\langle z,\bar{b}\right\rangle}{r^2-|z|^2-t\left\langle \bar{z},b\right\rangle}.$$

The denominator of this function does not vanish in G, since otherwise, it would imply that $t=\frac{r^2-|z|^2}{\langle \bar{z},b\rangle}$. But this is impossible, since, if we substitute t into (2.106), then

$$\frac{(r^2-|z|^2)^2}{|\left\langle \bar{z},b\right\rangle|^2}+r^2-|z|^2<0$$

while, on the other hand, $|z|<r$. Thus, $1/\bar{t}$ is holomorphically extended from $S\cap l$ into $B(0,r)\cap l$ and consequently, Theorem 2.2.1 holds in the case of the ball $B(0,r)\subset\mathbb{C}^2$.

Example 2.2.2 Let D be a bounded domain such that at least half of its boundary is a semi-sphere and the rest of its boundary belongs to the class $\mathcal{C}^2$. Let S be the part the sphere. Then, the line meeting semi-sphere cover all the directions of the lines l meeting D. Thus, the intersection of $\partial D\cap l$ contains part of $S\cap l$ and in view of Example 2.2.1, $1/\bar{t}$ can also be extended from $\partial D\cap l$ to $D\cap l$. It turns out that Theorem 2.2.1 is also valid on such domains.

Theorem 2.2.1 can be also extended on some quasi-circular domains. A domain D in $\mathbb{C}^n$ is called *quasi-circular* if $(e^{im_1\theta}z_1,\dots,e^{im_n\theta}z_n)\in D$ for any $(z_1,\dots,z_n)\in D$, where $m_i\in\mathbb{Z}^+$ and $\theta\in\mathbb{R}$. If $m_1=\cdots=m_n$, then D is called a *circular* domain.

Circular and quasi-circular domains are special cases of invariant domains under an action of a compact Lie group. In particular, consider a quasi-circular domain G and let ρ be a holomorphic linear action of the unit circle S^1 (S^1 is a Lie subgroup of $GL_1(\mathbb{C})$) on $\mathbb{C}^n$ given by $\rho(\lambda)(z)=(\lambda^{m_1}z_1,\dots,\lambda^{m_n}z_n)$, for $\lambda\in S^1$. Then, $\mathcal{O}(\mathbb{C}^n)^\rho:=\{f\in\mathcal{O}(\mathbb{C}^n):f\circ\rho(\lambda)=f \text{ for all } \lambda\in \mathrm{S}^1\}=\mathbb{C}$. If $m_1=\cdots=m_n$, the above statement shows that circular domains are also invariant under the action ρ.

Theorem 2.2.2 [24] *Let G_j be compact Lie groups for which there are continuous represen-*

tations $\rho_j : G_j \to GL(\mathbb{C}^n)$. Let, also, Ω_j be G_j-invariant bounded domains in $\mathbb{C}^n$ under the action ρ_j such that $\mathcal{O}(\mathbb{C}^n)^{\rho_j} = \mathbb{C}$. Suppose that $g : \Omega_1 \to \Omega_2$ is a proper holomorphic mapping and the two domains contain 0. Then

(1) g can extend holomorphically to an open neighborhood of $\overline{\Omega}_1$.

(2) If in addition $g^{-1}(0) = \{0\}$, then g is a polynomial mapping.

Corollary 2.2.1 *Let $D \subset \mathbb{C}^n$ be an invariant bounded domain under an action ρ of a compact Lie group. If D is biholomorphic to $B(0,r)$ and $f \in \mathcal{C}(\partial D)$ has a one-dimensional holomorphic continuation property for every complex line l meeting D, then there exists a function $F \in \mathcal{C}(\overline{D}) \cap \mathcal{O}(D)$ and $F = f$ on ∂D.*

Proof. According to Theorem 2.2.2, the biholomorphism can be extended from $\overline{D}$ to $\overline{B(0,r)}$. Hence, a combination of Theorem 2.2.1 and Lemma 2.2.4 leads to the desired result.

Example 2.2.3 Let $\mathbb{B}^2$ be the unit ball in $\mathbb{C}^2$ and $\Omega_k = \{(w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2 - w_1^k| < 1\}$. Then $\phi_k(z_1, z_2) = (z_1, z_2 + z_1^k)$ is a biholomorphism from $\mathbb{B}^2$ to Ω_k which can be hibolomorphically extended to the boundaries (Theorem 2.2.2). Hence, Theorem 2.2.1 also holds on Ω_k.

Example 2.2.4 Let us consider the Cartan domain

$$\boldsymbol{II}_3 := \{Z \in \mathbb{M}(3 \times 3, \mathbb{C}) : Z^t = -Z, \mathbb{I}_3 - ZZ^* > 0^2\},$$

where Z^* denotes the complex-conjugate to the transposed matrix of Z (Z^t). A linear biholomorphism between $\boldsymbol{II}_3$ and the three-dimensional unit ball $\mathbb{B}_3$ exists as follows [23]:

$$\begin{aligned} F : \mathbb{B}_3 &\to \boldsymbol{II}_3, \\ (z_1, z_2, z_3) &\mapsto \begin{pmatrix} 0 & z_1 & z_2 \\ -z_1 & 0 & z_3 \\ -z_2 & -z_3 & 0 \end{pmatrix}. \end{aligned}$$

This biholomorphism is extended holomorphically to the closures of these two circular domains according to Theorem 2.2.2. Hence, Theorem 2.2.1 holds on $\boldsymbol{II}_3$.

Example 2.2.5 Theorem 2.2.1 is also valid on another Cartan type domain, denoted by

$$\boldsymbol{I}_{1,n} := \{Z \in \mathbb{M}(1 \times n, \mathbb{C}) : \mathbb{I}_1 - ZZ^* > 0\},$$

[2] If $A \in \mathbb{M}(m \times m, \mathbb{C})$, $A > 0$ means that A is positive definite i.e. $X^t A \overline{X} = \sum_{j,k=1}^m a_{j,k} X_j \overline{X}_k > 0$ for all $X \in (\mathbb{C}^m)^*$.

since $\boldsymbol{I}_{1,n} \simeq \mathbb{B}_n$.

Chapter 3

Koppelman formula on Toric Varieties

The aim in this chapter is to derive a toric Koppelman integral formula representing smooth $(0, q)$-forms on an n-dimensional smooth compact toric variety X taking values in line bundle $V_{\mathcal{L}}$. The bundle $V_{\mathcal{L}}$ is the induced line bundle of an ample divisor D with $|P_D \cap \mathbb{Z}^n| = N$ and $N \leq 2n + 2$ (see Section 1.1.6). This formula is also extended for forms taking values in the k-fold tensor product of $V_{\mathcal{L}}$, $(V_{\mathcal{L}})^k = \underbrace{V_{\mathcal{L}} \otimes \cdots \otimes V_{\mathcal{L}}}_{k-\text{times}}$.

As a particular case of our result, we are able to recover partly the Koppelman integral formula on $\mathbb{P}^n \times \mathbb{P}^m$ proved by E. Götmark in [15], but with different representatives. However, our construction has the added benefit of yielding koppelman integral representation formulas on some toric projective varieties by taking into account that the singular set of kernels involved has to be along the exceptional set of the varieties in question.

3.1 The projective embedding

Let $D = \sum_{i=1}^{d} a_i D_i$ be an ample divisor on an n-dimensional smooth compact toric variety X. The notion of ample divisors has a fundamental role in the present thesis since they induce a projective embedding (1.12) of X into $\mathbb{P}_{N-1}$

$$\phi_D : X \to \mathbb{P}^{N-1}$$

and a Weighted Koppelman formula on the projective space $\mathbb{P}^{N-1}$ exists.

Let us recall the polyhedron (definition 1.1.3)

$$P_D = \{m \in M_{\mathbb{R}} : \langle m, v_i \rangle \geq -a_i, \text{for all } i = 1, \cdots, d\},$$

where $n+1 \leq d \leq N$ and the integral points $\{m_1, \ldots, m_N\} = P_D \cap \mathbb{Z}^n$. The definition of the polyhedron guarantees that one of the integral points is the zero vector and we set $m_1 = (0, \ldots, 0) \in \mathbb{R}^n$, since a_i will be chosen to be non-negative. Between the integral points, there exists $N-n-1$ equalities such that each one is constructed by four vectors from the above collection of points according to the following way. For each $j = 1, \ldots, N-n-1$, we set m_{j2} to be the vectors whose distance to origin is more than one unit (these vectors are exactly $N-n-1$). Then, for each m_{j2}, one can always find vectors m_{j3} and m_{j4} from the collection of the integral points such that these three vectors form a triangle or a degenerate triangle (if $m_{j3} = m_{j4}$). Thus, the three vectors fulfill the equation

$$m_{j3} + m_{j4} - m_{j2} = 0$$

or, equivalently, if we denote $m_{j1} = m_1 = (0, \ldots, 0)$, then

$$m_{j1} + m_{j2} = m_{j3} + m_{j4} \tag{3.1}$$

for every $j = 1, \ldots, N-n-1$.

As mentioned in Section 1.1.7 (according to [6]), the toric variety X is the Zariski closure of the closed embedding ϕ_D such that $\phi_D(X) = \overline{\Phi_D(T)} = X$. In the following proposition, we will prove that X is a complete intersection in $\mathbb{P}^{N-1}$ arising by the zero set of polynomials induced from equations (3.1).

Proposition 3.1.1 *The toric variety X is realized globally as the zero set of $N-n-1$ 2-homogeneous holomorphic polynomials $f_j : \mathbb{C}^N \to \mathbb{C}$, $j = 1, \ldots, N-n-1$.*

Proof. Recall that $P_D \cap M = \{m_1, \ldots, m_N\}$ where $\chi^{m_i} = \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle}$ for $i = 1, \ldots, N$ (recall that v_k for $k = 1, \ldots, d$ are the generators of the fan of X and η_k the corresponding homogeneous variable for each v_k) span $\Gamma(X, \mathcal{O}_X(D)) = \mathcal{O}_X(D)$ and let

$$\begin{aligned} \phi_D : X &\rightarrow \mathbb{P}^{N-1} \\ p &\mapsto (\chi^{m_1}(p), \ldots, \chi^{m_N}(p)) \end{aligned} \tag{3.2}$$

be the corresponding embedding of the ample divisor D. According to the preceding discussion, one can construct $N-n-1$ equalities described in (3.1) that are translated into the sections

$$f_j \quad = \quad \chi^{m_{j1}}\chi^{m_{j2}} - \chi^{m_{j3}}\chi^{m_{j4}} \tag{3.3}$$

for every $j = 1, \dots, N-n-1$. Then each $f_j : \mathbb{C}^N \to \mathbb{C}$ is interpreted as 2-homogeneous polynomial, by corresponding for each χ^{m_i} the variable z_i, $i = 1, \dots, N$ and X is realized as the intersection of zero sets of these polynomials. One can say that these polynomials are defined locally in a chart U_σ of the toric variety X by changing the characters χ^{m_i} for each $i = 1, \dots, N$ to the homogeneous coordinates of X through the rule $\chi^{m_i} = \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle}$.

Now, through the homogenization argument described in (1.6), one can construct polynomials f_j having a global nature on X by corresponding each z_i to the homogenization of χ^{m_i}, that is $\prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle + a_k}$. Then, X is considered to be a complete intersection in $\mathbb{P}^{N-1}$ given globally by the intersection of the zero sets of those $(N-n-1)$ 2-homogeneous polynomials.

$\square$

The following Lemma is crucial, as it allows to connect the sections on X with the 'restrictions' of those from $\mathbb{P}^{N-1}$.

Lemma 3.1.1 *Let $\phi_D : X \to \mathbb{P}^{N-1}$ be the closed projective embedding defined in* (3.2). *Then*

$$\mathcal{O}_X(D) \simeq \phi_D^* \mathcal{O}_{\mathbb{P}^{N-1}}(1).$$

Proof. The set $\{\prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle + a_k}\}_{i=1}^{N}$ of linearly independent global sections of the line bundle $V_{\mathcal{L}}$ (corresponding to the divisor D) being the homogenization of the set $\{\chi^{m_i}\}_{i=1}^{N}$, constitutes a coordinate system on X embedded in P^{N-1} by ϕ_D from (3.2). Indeed, if $V_i = \left\{p \in X : \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle + a_k}(p) \neq 0\right\}$, then $\{V_i\}$ defines an open covering of X. Moreover, the map $V_i \times \mathbb{C} \to \pi^{-1}(V_i) \subset V_{\mathcal{L}}$ such that $(p, \lambda) \mapsto \lambda \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle + a_k}(p)$ is an isomorphism between open sets of $X \times \mathbb{C}$ and open sets of the line bundle $V_{\mathcal{L}}$. Under the embedding ϕ_D, the transition functions on $V_i \cap V_j$ of the line bundle $V_{\mathcal{L}}$ are the functions

$$g_{ij}^X = \frac{\prod_{k=1}^{d} \eta_k^{\langle m_j, v_k \rangle + a_k}}{\prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle + a_k}} = \frac{\prod_{k=1}^{d} \eta_k^{\langle m_j, v_k \rangle}}{\prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle}} = \frac{\chi^{m_j}}{\chi^{m_i}}.$$

The transition functions g_{ij}^X are the pull back of the transition functions $g_{ij}^{\mathbb{P}^{N-1}} = \frac{z_j}{z_i}$ on $U_i \cap U_j$ of the hyperplane bundle L in $\mathbb{P}^{N-1}$, where z_i are the projective variables of $\mathbb{P}^{N-1}$ for $i = 1 \dots, N$ and $\{U_i\} = \{z \in \mathbb{P}^{N-1} : z_i \neq 0\}$ is the standard open covering of $\mathbb{P}^{N-1}$. Then, $\mathcal{O}_X(D) \simeq \phi_D^* \mathcal{O}_{\mathbb{P}^{N-1}}(1)$. $\square$

Let $\mathcal{E}_{0,q}(X, V_{\mathcal{L}})$ be the space of $(0, q)$ forms on X taking values in the corresponding line bundle $V_{\mathcal{L}}$ of an ample divisor D such that $\mathcal{L} = \mathcal{O}_X(D)$. We say that ϕ is a $(0, q)$ smooth form taking values in $V_{\mathcal{L}}$ by writing $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$, if ϕ is a section of the vector bundle $\bigwedge^{0,q} T^*(X) \otimes V_{\mathcal{L}}$ where $T^*(X)$ is the cotangent bundle of the smooth toric variety X. Recalling the notation from Section 1.1.6, if $\pi' : \bigwedge^{0,q} T^*(X) \otimes V_{\mathcal{L}} \to X$ is the corresponding morphism of the bundle, $\{U_{\sigma_i}\}_i$ is the covering of X and

$$\rho_{\sigma_i} : (\pi')^{-1}(U_{\sigma_i}) \to U_{\sigma_i} \times \mathbb{C} \tag{3.4}$$

are the trivializations of this bundle, one can construct a family of $(0, q)$ forms $\{\phi_{\sigma_i}\}_i$ defined on $U_{\sigma_i} \subset X$. These forms $\{\phi_{\sigma_i}\}_i$ are zero homogeneous (with respect to the homogeneity on X) and satisfy

$$\phi_{\sigma_i}\big|_{U_{\sigma_i}} = \tilde{\rho}_{\sigma_i} \circ \phi, \tag{3.5}$$

where $\tilde{\rho}_{\sigma_i} = \tilde{\pi} \circ \rho_{\sigma_i}$ and $\tilde{\pi} : U_{\sigma_i} \times \mathbb{C} \to \mathbb{C}$ is the projection mapping. Moreover,

$$\phi_{\sigma_i} = g_{ij}\phi_{\sigma_j}, \quad \text{on} \quad U_{\sigma_i} \cap U_{\sigma_j} \tag{3.6}$$

Hence, the family $\{\phi_{\sigma_i}\}_i$ determines $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$ in view of Proposition 1.1.2.

Moreover, in view of Section 1.1.3, the family $\{\bar{\partial}\phi_{\sigma_i}\}_i$ determines $\bar{\partial}\phi$ since $\bar{\partial}\phi_{\sigma_i} = g_{ij}\bar{\partial}\phi_{\sigma_j}$ on $U_{\sigma_i} \cap U_{\sigma_j}$, after the action of the $\bar{\partial}$-operator on (3.6). Observe that $\bar{\partial}\phi \in \mathcal{E}_{0,q+1}(X, V_{\mathcal{L}})$ takes values in the same bundle with ϕ which indicates that the bundle $V_{\mathcal{L}}$ is closed under the $\bar{\partial}$-operator.

Taking into account the notion of projective forms (see Section 1.3), we define, by analogy, the toric projective forms.

Definition 3.1.1 Let d be the number of generators of the smooth toric variety X. We say that a form on $\mathbb{C}^d$ is *toric projective* if it is the pullback of a form on X through the projection

$$\begin{aligned} \pi : \mathbb{C}^d \setminus Z(\Sigma) &\to X \\ (\eta_1, \ldots, \eta_d) &\mapsto [\eta_1, \ldots, \eta_d]. \end{aligned}$$

Hence, one can adapt the corresponding homogeneity of a projective form in our case. We say that a form ϕ belongs to the space $\mathcal{E}_{0,q}(X, V_{\mathcal{L}})$ if its pullback to $\mathbb{C}^d$ through π is a D-homogeneous toric projective form on $\mathbb{C}^d$ (according to the degree of a monomial which is defined in Section 1.1.4).

In particular, let π_{U_σ} be the restriction of π to a chart U_σ of X such that

$$\begin{aligned} \pi_{U_\sigma} : \mathbb{C}^d \setminus Z(\Sigma) &\rightarrow U_\sigma \\ (\eta_1, \dots, \eta_d) &\mapsto (\eta_1^\sigma, \dots, \eta_n^\sigma, 1, \dots, 1), \end{aligned}$$

where $\eta_j^\sigma = \chi^{u_j} = \prod_{i=1}^d \eta_i^{\langle u_j, v_i \rangle}$. The vectors $\{v_i\}_{i=1}^d$ and $\{u_j\}_{j=1}^n$ are the generators of the fan of X and of the dual cone $\check{\sigma}$, respectively (see Section 1.1.5).

If $D = \sum_{i=1}^d a_i D_i$ and $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$, then on the corresponding chart U_σ we have that $\pi^*\phi$ is a differential form in $\mathbb{C}^d \setminus Z(\Sigma)$ locally described by

$$\pi^*\phi = \pi_{U_\sigma}^*\phi = \sum_{1 \le i_1 < \ldots < i_q \le n} g_{i_1 \ldots i_q}(\eta_1, \dots, \eta_d)\, d(\bar{\eta}_{i_1})^\sigma \wedge \cdots \wedge d(\bar{\eta}_{i_q})^\sigma,$$

where $g_{i_1 \ldots i_q}(\eta_1, \dots, \eta_d)$ are D-homogeneous rational functions with respect to the homogeneous coordinates $\eta_1, \dots, \eta_d$ of X. Without loss of generality, we assume that the coefficients $g_{i_1 \ldots i_q}(\eta_1, \dots, \eta_d)$ do not contain any $\bar{\eta}_i$. A homogenization of the above form with respect to D (see (1.6) or [43]) yields

$$\pi_{U_\sigma}^*\phi = \sum_{1 \le i_1 < \ldots < i_q \le n} \prod_{i=1}^d \zeta_i^{a_i} \tilde{g}_{i_1 \ldots i_q}(\eta_1^\sigma, \dots, \eta_n^\sigma)\, d(\bar{\eta}_{i_1})^\sigma \wedge \cdots \wedge d(\bar{\eta}_{i_q})^\sigma. \tag{3.7}$$

Actually, the form (3.7) is the restriction of the global form to the chart U_σ.

The following Lemma is crucial since it verifies that $V_{\mathcal{L}} \cong L$, where $V_{\mathcal{L}}$ is the line bundle of the toric variety X corresponding to the sheaf $\mathcal{L} = \mathcal{O}_X(D)$ while L is the hyperplane bundle of $\mathbb{P}^{N-1}$ corresponding to the sheaf $\mathcal{O}_{\mathbb{P}^{N-1}}(D_0)$.

Lemma 3.1.2 *If $\phi \in \mathcal{E}_{0,q}(X, (V_{\mathcal{L}})^k)$, then through the embedding ϕ_D, ϕ can be extended to a $(0,q)$ form on $\mathbb{P}^{N-1}$ taking values in L^k, denoted by φ and writing $\varphi \in \mathcal{E}_{0,q}(\mathbb{P}^{N-1}, L^k)$ such that $\varphi|_X = \phi$.*

Proof. In view of the preceding paragraph, the pullback of $\phi \in \mathcal{E}_{0,q}(X, (V_{\mathcal{L}})^k)$ is a kD-homogeneous toric projective form on $\mathbb{C}^d \setminus Z(\Sigma)$ such that

$$\pi_{U_\sigma}^*\phi = \sum_{1 \le i_1 < \ldots < i_q \le n} \left(\prod_{i=1}^d \eta_i^{a_i} \right)^k \tilde{g}_{i_1 \ldots i_q}(\eta_1^\sigma, \dots, \eta_n^\sigma)\, d(\bar{\eta}_{i_1})^\sigma \wedge \cdots \wedge d(\bar{\eta}_{i_q})^\sigma$$

on a chart U_σ of X, where $\eta_1^\sigma, \dots, \eta_n^\sigma$ are the local coordinates of the chart U_σ. Under the embedding ϕ_D, the $(0,q)$ form ϕ on X is extended to a form φ on $\mathbb{P}^{N-1}$ of the same bidegree such that if $\tilde{\pi} : \mathbb{C}^N \setminus \{0\} \to \mathbb{P}^{N-1}$ is the projection mapping of $\mathbb{C}^N$ to $\mathbb{P}^{N-1}$ and

$$\begin{aligned} \tilde{\pi}_{U_1} : \mathbb{C}^N \setminus \{0\} &\to \mathbb{P}^{N-1} \\ (\zeta_1, \dots, \zeta_N) &\mapsto (1, \frac{\zeta_2}{\zeta_1}, \dots, \frac{\zeta_N}{\zeta_1}) \end{aligned}$$

is its restriction of $\tilde{\pi}$ on the chart $U_1 = \{\zeta \in \mathbb{P}^{N-1} : \zeta_1 \neq 0\}$, then

$$\tilde{\pi}^*_{U_1}\varphi = \sum_{1 \le j_1 < \ldots < j_q \le N-1} \zeta_1^k \tilde{g}_{i_1 \ldots i_q}\left(\frac{\zeta_2}{\zeta_1}, \dots, \frac{\zeta_N}{\zeta_1}\right) d\left(\frac{\bar{\zeta}_{j_i}}{\bar{\zeta}_1}\right) \wedge \cdots \wedge d\left(\frac{\bar{\zeta}_{j_q}}{\bar{\zeta}_1}\right) \tag{3.8}$$

with respect to the projective variables $\zeta_1, \dots, \zeta_N$ of $\mathbb{P}^{N-1}$. This follows from the identification that exists between the projective variables $\zeta_1, \dots, \zeta_N$ of $\mathbb{P}^{N-1}$ (left hand side of (3.9)) and the homogeneous coordinates $\eta_1, \dots, \eta_d$ of X (right hand side of (3.9)) through the map

$$(\zeta_1, \dots, \zeta_N) \quad \mapsto \quad \left(\prod_{i=1}^d \eta_i^{\langle m_1, v_i \rangle + a_i}, \dots, \prod_{i=1}^d \eta_i^{\langle m_N, v_i \rangle + a_i}\right) = \left(\prod_{i=1}^d \eta_i^{a_i}, \dots, \prod_{i=1}^d \eta_i^{\langle m_N, v_i \rangle + a_i}\right). \tag{3.9}$$

It turns out that $\varphi \in \mathcal{E}_{0,q}(\mathbb{P}^{N-1}, L^k)$, since its pullback to $\mathbb{C}^N$ is a k-homogeneous projective form according to (3.8) (see Section 1.3). $\square$

Example 3.1.1 Let us consider the (very) ample divisor $D = D_2 + D_4$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Then, $P_D \cap M = \{(0, 0, (1, 0), (0, 1), (1, 1)\}$ and the corresponding closed embedding is given by

$$\begin{aligned} \phi_{D_2+D_4} : \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^3 \\ p &\longmapsto (\chi^{(0,0)}, \chi^{(1,0)}, \chi^{(1,1)}, \chi^{(0,1)}) = \left(1, \frac{\eta_1}{\eta_2}, \frac{\eta_1\eta_3}{\eta_2\eta_4}, \frac{\eta_3}{\eta_4}\right) \end{aligned} \tag{3.10}$$

on the chart where $\eta_2 \neq 0$ and $\eta_4 \neq 0$. The embedding is independent of the choice of the chart. One can observe that the image of the embedding can be written in the form $(\eta_2\eta_4, \eta_1\eta_4, \eta_1\eta_3, \eta_2\eta_3)$, indicating the global character of the embedding. This is the well-known Segre embedding. Thus, $\mathbb{P}^1 \times \mathbb{P}^1$ is realized as the zero set of the polynomial

$$f = \chi^{(0,0)}\chi^{(1,1)} - \chi^{(0,1)}\chi^{(1,0)}$$

in $\mathbb{P}^3$. The line bundle corresponds to the divisor $D = D_2 + D_4$ and thus is $V_{\mathcal{L}} = L^1 \otimes L^1$. The preceding Lemma implies that the embedding $\phi_{D_2+D_4}$ in (3.10) guarantees the extension of $\phi \in \mathcal{E}_{0,q}(\mathbb{P}^1 \times \mathbb{P}^1, L^k \otimes L^k)$ to a form $\varphi \in \mathcal{E}_{0,q}(\mathbb{P}^3, L^k)$ such that $\varphi|_{\mathbb{P}^1 \times \mathbb{P}^1} = \phi$.

Now, if we take $D = kD_2 + lD_4$ for $k, l > 0$, then the corresponding embedding is defined by

$$
\begin{aligned}
\phi_{kD_2+lD_4}:\mathbb{P}^1\times\mathbb{P}^1 &\longrightarrow \mathbb{P}^{kl+k+l}\\
p &\longmapsto (\chi^{(0,0)},\chi^{(1,0)},...,\chi^{(k,0)},\chi^{(1,1)},...,\chi^{(k,1)},\dots\chi^{(1,l)},\dots,\chi^{(k,l)})\\
&= \left(1,\frac{\eta_1}{\eta_2},...,\left(\frac{\eta_1}{\eta_2}\right)^k,\dots,\left(\frac{\eta_1}{\eta_2}\right)\left(\frac{\eta_3}{\eta_4}\right)^l,...,\left(\frac{\eta_1}{\eta_2}\right)^k\left(\frac{\eta_3}{\eta_4}\right)^l\right)
\end{aligned}
\tag{3.11}
$$

on the chart where $\eta_2\neq 0$ and $\eta_4\neq 0$. As before, the embedding can be expressed in a global form by multiplying each coordinate of the right hand side of (3.11) by $\eta_2^k\eta_4^l$. Observe that this map is the composition of the Veronese embedding (ν_k,ν_l) with Segre embedding. The zero set of the $(kl+k+l-2)$ polynomials that are derived from the integral points of $P_{kD_2+lD_4}$ leads to the variety $\mathbb{P}^1\times\mathbb{P}^1$. Observe that the restriction of the embedding (3.11) in the case when $k=l$, an $\phi\in\mathcal{E}_{0,q}(\mathbb{P}^1\times\mathbb{P}^1,L^k\otimes L^k)$ is extended to a form $\varphi'\in\mathcal{E}_{0,q}(\mathbb{P}^{k^2+2k},L^1)$ (Lemma 3.1.2). But as k becomes larger, the dimension of the projective space increases rapidly. However, this embedding allows the extension of differential forms on $\mathbb{P}^1\times\mathbb{P}^1$ taking values in $L^k\otimes L^l$ into differential forms on $\mathbb{P}^{kl+k+l}$ taking values in L^1, even if $k\neq l$.

The two extensions $\varphi\in\mathcal{E}_{0,q}(\mathbb{P}^3,L^k)$ and $\varphi'\in\mathcal{E}_{0,q}(\mathbb{P}^{k^2+2k},L^1)$ of $\phi\in\mathcal{E}_{0,q}(\mathbb{P}^1\times\mathbb{P}^1,L^k\otimes L^k)$ arising form the embedding $\phi_{D_2+D_4}$ and $\phi_{kD_2+kD_4}$, repsectively are compatible since by taking $\psi=\phi_{kD_2+kD_4}\circ\phi^{-1}_{D_2+D_4}$, one can extend $\varphi\in\mathcal{E}_{0,q}\left(\mathbb{P}^3,L^k\right)$ to $\varphi'\in\mathcal{E}_{0,q}\left(\mathbb{P}^{k^2+2k},L^1\right)$.

Example 3.1.2 Consider, now, the cartesian product of projective spaces $\mathbb{P}^n\times\mathbb{P}^m$. This toric variety is associated with a fan $\Sigma\subseteq\mathbb{R}^{n+m}$ produced by the vectors $e_1,e_2,\dots,e_n,-e_1-\dots-e_n,e_{n+1},e_{n+2},\dots,e_{n+m},-e_{n+1}-\dots-e_{n+m}$, where $\{e_i\}$ is the standard basis of $\mathbb{R}^{n+m}$. Then for each ray from the above, we denote its generator by v_i for every $i=1,\dots,n+m+2$. While by η_i, we denote the corresponding homogeneous coordinate and by $D_i=\mathrm{div}(\eta_i)$ the corresponding divisor. As mentioned in Section 1.1.3 the divisor of a character χ^l on $\mathbb{P}^n\times\mathbb{P}^m$, for $l\in\mathbb{Z}^{n+m}$ is given by

$$
\mathrm{div}(\chi^l)=\sum_{i=1}^{n+m}\langle l,v_i\rangle\, D_i.
$$

Then,

$$
\begin{aligned}
0 &\sim \mathrm{div}\left(\chi^{e_i}\right)=D_i-D_{n+1},\ \text{for}\ 1\leq i\leq n\quad\text{and}\\
0 &\sim \mathrm{div}\left(\chi^{e_{n+j}}\right)=D_{n+j+1}-D_{n+m+2}\ \text{for}\ 1\leq j\leq m.
\end{aligned}
$$

Thus $\mathrm{Cl}(\mathbb{P}^n\times\mathbb{P}^m)\simeq\mathbb{Z}^2$ which is generated by $[D_1]=\dots=[D_{n+1}]$ and $[D_{n+2}]=\dots=[D_{n+m+2}]$. Without loss of generality, we can choose the Cartier divisor $D=D_{n+1}+D_{n+m+2}$

in order to take global information for sections of $\mathbb{P}^n \times \mathbb{P}^m$ taking values in the corresponding line bundle of the sheaf $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^m}(D)$, that is in line bundle $L^1 \otimes L^1$.

The associated polyhedron P_D is described by the vectors $l \in \mathbb{R}^{n+m}$ satisfying the following: $\langle l, v_i \rangle \geq 0$, for each $\rho \neq n+1, n+m+2$, while $\langle l, v_{n+1} \rangle \geq -1$ and $\langle l, v_{n+m+2} \rangle \geq -1$. These inequalities are equivalent to:

$$\begin{aligned}
&\langle l, e_i \rangle \geq 0, \text{ for each } 1 \leq i \leq n+m \\
&\langle l, -e_1 - \cdots - e_n \rangle \geq -1 \quad \text{and} \\
&\langle l, -e_{n+1} - \cdots - e_{n+m} \rangle \geq -1.
\end{aligned}$$

If $l = (a_1, \ldots, a_{n+m})$, then $a_i \geq 0$ for each $1 \leq i \leq n+m$, $-a_1 - \cdots - a_n \geq -1$ and $-a_{n+1} - \cdots - a_{n+m} \geq -1$, or equivalently, $a_i \geq 0$ for each $1 \leq i \leq n+m$, $a_1 + \cdots + a_n \leq 1$ and $a_{n+1} + \cdots + a_{n+m} \leq 1$.

Let us consider the vectors

$$m_{i,j} := (0, \ldots, \underset{i-\text{position}}{\underset{\uparrow}{1}}, \ldots, 0, 0, \ldots, \underset{j-\text{position}}{\underset{\uparrow}{1}}, \ldots, 0) \in \mathbb{R}^{n+m}, \tag{3.12}$$

for $0 \leq i \leq n$ and $n+1 \leq j \leq n+m$ or $j = 0$. The zero value of i or j means that there is no unit in the first n or the last m positions of $m_{i,j}$, respectively. The argument in the previous paragraph shows that P_D is an $(n+m)$-dimensional cube with $(nm+n+m+1)$ vertices, that are precisely the vectors $m_{i,j}$ for the above possible values of i and j.

Now, the Wall inequality in (1.16) indicates that the divisor $D = D_{n+1} + D_{n+m+2}$ on $\mathbb{P}^n \times \mathbb{P}^m$ is very ample. Thus, the closed embedding defined by

$$\phi_D : \quad \mathbb{P}^n \times \mathbb{P}^m \quad \longrightarrow \mathbb{P}^{nm+n+m}$$

where

$$\begin{aligned}
\phi_D(p) \quad &= (\chi^{m_{0,0}}(p), \chi^{m_{1,0}}(p), \ldots, \chi^{m_{n,0}}(p), \chi^{m_{0,n+1}}(p), \ldots, \chi^{m_{0,n+m}}(p), \\
&\quad , \chi^{m_{1,n+1}}(p), \ldots, \chi^{m_{1,n+m}}(p), \ldots, \chi^{m_{n,n+1}}(p), \ldots, \chi^{m_{n,n+m}}(p))
\end{aligned} \tag{3.13}$$

is induced.

We define polynomials $f_{i,j} : \mathbb{C}^{nm+n+m+1} \to \mathbb{C}$ for $1 \leq i \leq n$ and $n+1 \leq j \leq n+m$, given by

$$
\begin{aligned}
f_{1,n+1} &= \chi^{m_{0,0}} \cdot \chi^{m_{1,n+1}} - \chi^{m_{1,0}} \cdot \chi^{m_{0,n+1}} \\
&\vdots \\
f_{1,n+m} &= \chi^{m_{0,0}} \cdot \chi^{m_{1,n+m}} - \chi^{m_{1,0}} \cdot \chi^{m_{0,n+m}} \\
&\vdots \\
f_{n,n+1} &= \chi^{m_{0,0}} \cdot \chi^{m_{n,n+1}} - \chi^{m_{n,0}} \cdot \chi^{m_{0,n+1}} \\
&\vdots \\
f_{n,n+m} &= \chi^{m_{0,0}} \cdot \chi^{m_{n,n+m}} - \chi^{m_{n,0}} \cdot \chi^{m_{0,n+m}}.
\end{aligned} \tag{3.14}
$$

Then, the associated variety is described by the zero set of the above system of polynomials due to the following trivial equalities between the generating vectors of the polyhedron P_D:

$$
\begin{aligned}
m_{0,0} + m_{1,n+1} &= m_{1,0} + m_{0,n+1} \\
&\vdots \\
m_{0,0} + m_{1,n+m} &= m_{1,0} + m_{0,n+m} \\
&\vdots \\
m_{0,0} + m_{n,n+1} &= m_{n,0} + m_{0,n+1} \\
&\vdots \\
m_{0,0} + m_{n,n+m} &= m_{n,0} + m_{0,n+m}.
\end{aligned}
$$

Introducing the variables z_{ij} to replace the characters $\chi^{m_{ij}}$, it becomes obvious to see that these polynomials are holomorphic with homogeneity two. The cardinality nm of the induced polynomials, that is the number of the integral points that have distance more than 1 from the zero vector, coincides with the codimension of the associated variety and is $nm+n+m-(n+m) = nm$. The embedding ϕ_D in (3.13) gives rise to the extension of the $(0,q)$ forms on $\mathbb{P}^n \times \mathbb{P}^m$ taking values in $L^k \otimes L^k$ into $(0,q)$ forms on $\mathbb{P}^{nm+n+m}$ taking values in L^k, in view of Lemma 3.1.2.

Example 3.1.3 In the particular case of the Hirzebruch surface $\mathcal{H}$ (see example 1.1.26), the very ample divisor $D = D_3 + 2D_4$ induces the embedding

$$
\begin{aligned}
\phi_{D_3+2D_4} : X &\longrightarrow \mathbb{P}^4 \\
p &\longmapsto (\chi^{(0,0)}, \chi^{(1,0)}, \chi^{(1,1)}, \chi^{(0,1)}, \chi^{(0,2)}) = \left(1, \frac{\eta_1}{\eta_3\eta_4}, \frac{\eta_1\eta_2}{\eta_3\eta_4^2}, \frac{\eta_2}{\eta_4}, \frac{\eta_2^2}{\eta_4^2}\right),
\end{aligned} \tag{3.15}
$$

since $P_D \cap M = \{(0,0),(1,0),(1,1),(0,1),(0,2)\}$. This embedding has a local form that holds on the chart of $\mathcal{H}$ where $\eta_3 \neq 0$ and $\eta_4 \neq 0$. However, this local representation can be extended to $\mathcal{H}$ by taking the image of $\phi_{D_3+2D_4}$ to be $(\eta_3\eta_4^2, \eta_1\eta_4, \eta_1\eta_2, \eta_2\eta_3\eta_4, \eta_2^2\eta_3)$. Hence, X is expressed by the zero set of the system of polynomials

$$\begin{aligned} f_1 &= \chi^{(0,0)}\chi^{(1,1)} - \chi^{(0,1)}\chi^{(1,0)} \\ f_2 &= \chi^{(0,0)}\chi^{(0,2)} - \chi^{(0,1)}\chi^{(0,1)}. \end{aligned} \tag{3.16}$$

Following Lemma 3.1.2, an $\phi \in \mathcal{E}_{0,q}\left(\mathcal{H}, (V_{\mathcal{L}})^k\right)$ is extended to an $\varphi \in \mathcal{E}_{0,q}(\mathbb{P}^4, L^k)$, where $V_{\mathcal{L}}$ is the corresponding line bundle of the sheaf $\mathcal{L} = \mathcal{O}_X(D_3 + 2D_4)$.

3.2 A toric weight in X

Let $t = (t_1, \ldots, t_n)$ and $\tau = (\tau_1, \ldots, \tau_n)$ be two vector variables belonging in the algebraic torus $T = (\mathbb{C} \setminus \{0\})^n$ and recall from Section 3.1 that the vectors $m_1, \ldots, m_N$ are the integral points of the polyhedron P_D. A parametrization of $\mathbb{P}^{N-1}$ given by $\chi^{m_i} = t^{m_i} = t_1^{m_i^1} \cdots t_n^{m_i^n}$ for each $i = 1, \ldots, N$ ($N \geq n+1$), where m_i^a for $a = 1, \ldots, n$ denotes the a-coordinate of the vector m_i, modifies the map (3.2) to the following equivalent form:

$$\phi_D(t) = (t^{m_1}, \ldots, t^{m_N}). \tag{3.17}$$

This is the map $\Phi_D = \phi_D\big|_T$ introduced in Section 1.1.7.

Let us define

$$P(t) = \sum_{i=1}^{N} t^{m_i} = \sum_{i=1}^{N} t_1^{m_i^1} \cdots t_n^{m_i^n} \tag{3.18}$$

to be the Laurent polynomial in the torus T generated by the integral points of $P_D \cap \mathbb{Z}^n$ with coefficients equal to one. We introduce the particular $(1,1)$ form, called *toric weight* α^T, defined by

$$\alpha^T = \alpha^T_{0,0} + \alpha^T_{1,1}$$

where

$$\alpha^T_{0,0} = \frac{P(\bar{\tau} \cdot t)}{P(|\tau|^2)} = \frac{\sum_{i=1}^N (\bar{\tau} \cdot t)^{m_i}}{\sum_{i=1}^N |\tau|^{2m_i}} = \frac{\sum_{i=1}^N (\bar{\tau}_1 t_1)^{m_i^1} \cdots (\bar{\tau}_n t_n)^{m_i^n}}{\sum_{i=1}^N |\tau_1|^{2m_i^1} \cdots |\tau_n|^{2m_i^n}} \tag{3.19}$$

and

$$\alpha_{1,1}^T = -\frac{1}{2\pi i}\bar{\partial}\partial \ln P(|\tau|^2).$$

Observe that over the diagonal $t = \tau$ of $T \times T$, one has that $\alpha_{0,0}(t,t) = 1$

Elementary computations, allow to write $\alpha_{1,1}^T$ explicitly. Namely,

$$\begin{aligned}\partial\tau^{m_i} &= \partial\left(\tau_1^{m_i^1}\cdots\tau_n^{m_i^n}\right) = \sum_{a=1}^n m_i^a \tau_a^{m_i^a-1}\left(\tau^{m_i}[a]\right)d\tau_a \\ &= \sum_{a=1}^n m_i^a \tau_1^{m_i^1}\cdots\tau_n^{m_i^n}\frac{d\tau_a}{\tau_a} = \sum_{a=1}^n m_i^a \tau^{m_i}\frac{d\tau_a}{\tau_a},\end{aligned} \tag{3.20}$$

where $\tau^{m_i}[a]$ denotes that $\tau_a^{m_i^a}$ is omitted from the product $\tau^{m_i} = \tau_1^{m_i^1}\cdots\tau_n^{m_i^n}$. Similarly,

$$\partial\bar{\tau}^{m_i} = \sum_{b=1}^n m_i^b \bar{\tau}^{m_i}\frac{d\bar{\tau}_b}{\bar{\tau}_b}. \tag{3.21}$$

Furthermore,

$$\begin{aligned}\partial P(|\tau|^2) &= \partial\left(\sum_{i=1}^N |\tau_1|^{2m_i^1}\cdots|\tau_n|^{2m_i^n}\right) = \sum_{a=1}^n\sum_{i=1}^N m_i^a \tau_a^{m_i^a-1}\bar{\tau}_a^{m_i^a}\left(|\tau|^{2m_i}[a]\right)d\tau_a \\ &= \sum_{a=1}^n\sum_{i=1}^N m_i^a |\tau|^{2m_i}\frac{d\tau_a}{\tau_a}\end{aligned} \tag{3.22}$$

and

$$\bar{\partial} P(|\tau|^2) = \sum_{b=1}^n\sum_{i=1}^N m_i^b |\tau|^{2m_i}\frac{d\bar{\tau}_b}{\bar{\tau}_b}. \tag{3.23}$$

Hence,

$$\begin{aligned}\alpha_{1,1}^T &= -\frac{1}{2\pi i}\,\bar{\partial}\partial\ln P(|\tau|^2) = -\frac{1}{2\pi i}\,\bar{\partial}\frac{\partial\, P(|\tau|^2)}{P(|\tau|^2)} \\ &= -\frac{1}{2\pi i}\,\bar{\partial}\frac{\sum_{a=1}^n\sum_{i=1}^N m_i^a|\tau|^{2m_i}\frac{d\tau_a}{\tau_a}}{\sum_{i=1}^N|\tau|^{2m_i}} \\ &= -\frac{1}{2\pi i}\sum_{b=1}^n\sum_{a=1}^n\left[\frac{\sum\limits_{i=1}^N m_i^a m_i^b|\tau|^{2m_i}}{\sum\limits_{i=1}^N|\tau|^{2m_i}} - \frac{\left(\sum\limits_{i=1}^N m_i^a|\tau|^{2m_i}\right)\left(\sum\limits_{i=1}^N m_i^b|\tau|^{2m_i}\right)}{\left(\sum\limits_{i=1}^N|\tau|^{2m_i}\right)^2}\right]\frac{d\bar{\tau}_b}{\bar{\tau}_b}\wedge\frac{d\tau_a}{\tau_a}.\end{aligned} \tag{3.24}$$

Corollary 3.2.1 *The contributing vectors from the collection of elements* $\{m_1,\ldots,m_N\}$ *in* *the toric weight $\alpha^T_{1,1}$ are:*

1. *along the form component $d\bar{\tau}_b \wedge d\tau_a$ for $a \neq b$ and $1 \leq a, b \leq n$. That is, for every pair of vectors $\{m_i, m_j\}$ such that both the a-coordinate and the b-coordinate of the vectors m_i and m_j are different and no other pair of vectors $\{m_{k_p}, m_{l_p}\}$, $1 \leq p \leq \ell$ for an index ℓ satisfying*
$$m_i + m_j = m_{k_p} + m_{l_p}, \quad \forall\, 1 \leq p \leq \ell$$
and
$$(m_i^a - m_j^a)(m_i^b - m_j^b) = -\sum_{p=1}^{\ell}(m_{k_\ell}^a - m_{l_p}^a)(m_{k_p}^b - m_{l_p}^b),$$
are present.

2. *along the form component $d\bar{\tau}_a \wedge d\tau_a$ for $1 \leq a \leq n$. For every pair of vectors $\{m_i, m_j\}$ such that the components of m_i and m_j in the a-coordinate are different, the corresponding term in the form, is non-trivial.*

Proof. This is a consequence of the definition of $\alpha^T_{1,1}$. Observe that the coefficient of the form $\frac{d\bar{\tau}_b}{\bar{\tau}_b} \wedge \frac{d\tau_a}{\tau_a}$ for fixed a and b can be written equivalently as

$$-\frac{1}{2\pi i\left(\sum_1^n |\tau|^{2m_i}\right)^2}\left[\sum_{\substack{1\leq i,j\leq n\\ i\neq j}} |\tau|^{2(m_i+m_j)}(m_i^a - m_j^a)(m_i^b - m_j^b)\right] \tag{3.25}$$

and the first result follows. If $a = b$, each term of the above summation vanishes if and only if $m_i^a = m_j^a$.

Example 3.2.1 In $\mathbb{P}^n \times \mathbb{P}^m$, the toric weight $\alpha^T_{1,1}$ has only terms containing differentials coming both either from $\mathbb{P}^n$ or from $\mathbb{P}^m$. Moreover, the weight is decomposed in two forms where one depends only on $\mathbb{P}^n$ and the other one only on $\mathbb{P}^m$.

More precisely, let us recall the vectors constructing the polyhedron of $\mathbb{P}^n \times \mathbb{P}^m$ (see (3.12))

$$m_{i,j} = (0, \ldots, 1, \ldots, 0, 0, \ldots, 1, \ldots, 0),$$

for $0 \leq i \leq n$, $n+1 \leq j \leq n+m$ or $j = 0$. Fix a pair (a, b) with $1 \leq a \leq n$ and $n+1 \leq b \leq n+m$. Without loss of generality, let $a = 1$ and $b = n+1$ (the other cases are treated similarly). According to the first part of Corollary 3.2.1, in order to compute the coefficient of the form $d\bar{\tau}_b \wedge d\tau_a$ in $\alpha^T_{1,1}$, it is sufficient to look at the pairs of vectors that have different coordinates in positions $a = 1$ and $b = n+1$. These pairs of vectors are given below:
(a) $m_{0,0}$ with $m_{1,n+1}$ and $m_{0,n+1}$ with $m_{1,0}$,

(b) $m_{1,0}$ with $m_{k,n+1}$ and $m_{k,0}$ with $m_{1,n+1}$ for $2 \leq k \leq n$,

(c) $m_{0,n+1}$ with $m_{1,l}$ and $m_{0,l}$ with $m_{1,n+1}$ for $n+2 \leq l \leq n+m$,

(d) $m_{1,n+1}$ with $m_{k,l}$ and $m_{1,l}$ with $m_{k,n+1}$ for $2 \leq k \leq n$ and $n+2 \leq l \leq n+m$.

However, observe that each of the above cases gives a trivial result. For example, the vectors in case (a) satisfy the equalities

$$m_{0,0} + m_{1,n+1} = m_{0,n+1} + m_{1,0}$$

and

$$(m^1_{1,n+1} - m^1_{0,0})(m^{n+1}_{1,n+1} - m^{n+1}_{0,0}) = -(m^1_{0,n+1} - m^1_{1,0})(m^{n+1}_{0,n+1} - m^{n+1}_{1,0}).$$

Similar equations are valid in the remaining cases and the result follows.

Furthermore, from the initial definition of the toric weight $\alpha^T_{1,1}$, one has

$$\begin{aligned}
P(|\tau|^2) &= \sum_{i,j} |\tau|^{2m_{i,j}} \\
&= |\tau|^{2m_{0,0}} + \sum_{i=1}^{n} |\tau|^{2m_{i,0}} + \sum_{j=n+1}^{n+m} |\tau|^{2m_{0,j}} + \sum_{\substack{1\leq i\leq n \\ n+1\leq j\leq n+m}} |\tau|^{2m_{i,j}} \\
&= \left(1 + \sum_{i=1}^{n} |\tau|^{2m_{i,0}}\right)\left(1 + \sum_{j=n+1}^{n+m} |\tau|^{2m_{0,j}}\right) \\
&= \left(1 + \sum_{i=1}^{n} |\tau_i|^2\right)\left(1 + \sum_{j=n+1}^{n+m} |\tau_j|^2\right).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\alpha^T_{1,1} &= -\frac{1}{2\pi i}\bar{\partial}\partial \ln \mathrm{P}(|\tau|^2) \\
&= -\frac{1}{2\pi i}\bar{\partial}\partial \ln\left(1 + \sum_{i=1}^{n} |\tau_i|^2\right) - \frac{1}{2\pi i}\bar{\partial}\partial \ln\left(1 + \sum_{j=n+1}^{n+m} |\tau_j|^2\right) \\
&= -\frac{1}{2\pi i}\sum_{i=1}^{n} \bar{\partial}\left(\frac{\bar{\tau}_i d\tau_i}{1 + \sum_{i=1}^{n} |\tau_i|^2}\right) - \frac{1}{2\pi i}\sum_{j=n+1}^{n+m} \bar{\partial}\left(\frac{\bar{\tau}_j d\tau_j}{1 + \sum_{j=n+1}^{n+m} |\tau_j|^2}\right). \qquad (3.26)
\end{aligned}$$

However in general toric varieties, such a separation of variables is not possible.

Example 3.2.2 If we look, now, at the particular case of the Hirzebruch surface in example 3.1.3, none of the terms of the form present in the weight $\alpha^T_{1,1}$ vanishes. To be more precise, the contributing vectors in the terms of the form along $d\bar{\tau}_1 \wedge d\tau_2$ and $d\bar{\tau}_2 \wedge d\tau_1$ are the pairs $\{m_2, m_5\}$ and $\{m_3, m_5\}$ in view of Corollary 3.2.1, since $\{m_1, m_3\}$ and $\{m_2, m_4\}$ (the pairs that have different components in every position) satisfy the equalities

$$m_1 + m_3 = m_2 + m_4$$

and

$$(m_1^1 - m_3^1)(m_1^2 - m_3^2) = -(m_2^1 - m_4^1)(m_2^2 - m_4^2).$$

Hence, by substituting the contributing vectors m_i in $\alpha_{1,1}^T$ (see (3.24)) and simplifying each term in the toric weight, one obtains:

$$\begin{aligned}
\alpha_{1,1}^T \quad = \quad & -\frac{1}{2\pi i}\frac{1+2|\tau_2|^2+2|\tau_2|^4+|\tau_2|^6}{(1+|\tau_1|^2+|\tau_2|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^4)^2}d\bar{\tau}_1 \wedge d\tau_1 \\
& -\frac{1}{2\pi i}\frac{-\tau_1\bar{\tau}_2|\tau_2|^2(2+|\tau_2|^2)}{(1+|\tau_1|^2+|\tau_2|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^4)^2}d\bar{\tau}_1 \wedge d\tau_2 \\
& -\frac{1}{2\pi i}\frac{-\bar{\tau}_1\tau_2|\tau_2|^2(2+|\tau_2|^2)}{(1+|\tau_1|^2+|\tau_2|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^4)^2}d\bar{\tau}_2 \wedge d\tau_1 \\
& -\frac{1}{2\pi i}\frac{(1+|\tau_1|^2)(1+|\tau_1|^2+4|\tau_2|^2+|\tau_2|^4)}{(1+|\tau_1|^2+|\tau_2|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^4)^2}d\bar{\tau}_2 \wedge d\tau_2. \qquad (3.27)
\end{aligned}$$

The construction of the particular kernel in the toric representation, that we will present later on, involves the form $(\alpha_{1,1}^T)^n$ which is the wedge n-product of $\alpha_{1,1}^T$. This form is well-defined in the torus $T \subseteq X$.

By following the technique developed by A. Shchuplev in [37], the form $(\alpha_{1,1}^T)^n$ can be written as a linear combination of forms. The coefficients of this linear combination involve determinants of minor of matrices whose lines (columns) contain vectors $\{m_1, \dots, m_N\}$.

Proposition 3.2.1 *Let A be the matrix*

$$A = \begin{pmatrix} 1 & \dots & 1 \\ m_1^1 & \dots & m_N^1 \\ \vdots & \dots & \vdots \\ m_1^n & \dots & m_N^n \end{pmatrix} \qquad (3.28)$$

and A_j be the determinant of the minor matrix obtained by $(n+1)$ columns of the $(n+1)\times N$ matrix A. Then, the (n,n)-form $(\alpha_{1,1}^T)^n$ equals to

$$\begin{aligned}
(\alpha_{1,1}^T)^n \quad = \quad & \left(-\frac{1}{2\pi i}\right)^n \frac{n!}{|\tau_1|^2\cdots|\tau_n|^2 P(|\tau|^2)^{n+1}} \cdot \\
& \sum_{\substack{|J|=n+1 \\ 1\le j_1\le\cdots\le j_{n+1}\le N}} \det{}^2(A_J)|\tau|^{2m_{j_1}+\cdots+2m_{j_{n+1}}} d\bar{\tau}\wedge d\tau
\end{aligned} \qquad (3.29)$$

where $d\bar{\tau} \wedge d\tau = d\bar{\tau}_1 \wedge d\tau_1 \wedge \cdots \wedge d\bar{\tau}_n \wedge d\tau_n$. The sum is taken over all increasing subsets of the index set $1 \leq j_1 < \cdots < j_{n+1} \leq N$.

Proof. From (3.24), observe at first that $(\alpha_{1,1}^T)^n$ is the product of

$$\left(-\frac{1}{2\pi i}\right)^n \frac{n!}{|\tau_1|^2 \cdots |\tau_n|^2 P(|\tau|^2)^n} d\bar{\tau}_1 \wedge d\tau_1 \wedge \cdots \wedge d\bar{\tau}_n \wedge d\tau_n$$

and the $(n+1) \times (n+1)$ determinant

$$\begin{vmatrix} 1 & \sum_i m_i^1 |\tau|^{2m_i} & \dots & \sum_i m_i^n |\tau|^{2m_i} \\ \frac{\sum_i m_i^1 |\tau|^{2m_i}}{\sum_i |\tau|^{2m_i}} & \sum_i m_i^1 m_i^1 |\tau|^{2m_i} & \dots & \sum_i m_i^1 m_i^n |\tau|^{2m_i} \\ \vdots & \dots & \dots & \vdots \\ \frac{\sum_i m_i^n |\tau|^{2m_i}}{\sum_i |\tau|^{2m_i}} & \sum_i m_i^n m_i^1 |\tau|^{2m_i} & \dots & \sum_i m_i^n m_i^n |\tau|^{2m_i} \end{vmatrix}.$$

By taking $\frac{1}{\sum_i |\tau|^{2m_i}}$ as a common factor of the first column of the determinant, we get that the determinant is equal to

$$\frac{1}{\sum_i |\tau|^{2m_i}} \begin{vmatrix} \sum_i |\tau|^{2m_i} & \sum_i m_i^1 |\tau|^{2m_i} & \dots & \sum_i m_i^n |\tau|^{2m_i} \\ \sum_i m_i^1 |\tau|^{2m_i} & \sum_i m_i^1 m_i^1 |\tau|^{2m_i} & \dots & \sum_i m_i^1 m_i^n |\tau|^{2m_i} \\ \vdots & \dots & \dots & \vdots \\ \sum_i m_i^n |\tau|^{2m_i} & \sum_i m_i^n m_i^1 |\tau|^{2m_i} & \dots & \sum_i m_i^n m_i^n |\tau|^{2m_i} \end{vmatrix}.$$

Hence, the matrix whose determinant is the above one, is written as the product of two matrices

$$\begin{pmatrix} \tau^{m_1} & \dots & \dots & \tau^{m_N} \\ m_1^1 \tau^{m_1} & \dots & \dots & m_N^1 \tau^{m_N} \\ \vdots & \dots & \dots & \vdots \\ m_1^n \tau^{m_1} & \dots & \dots & m_N^n \tau^{m_N} \end{pmatrix} \begin{pmatrix} \bar{\tau}^{m_1} & m_1^1 \bar{\tau}^{m_1} & \dots & m_1^n \bar{\tau}^{m_1} \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ \bar{\tau}^{m_N} & m_N^1 \bar{\tau}^{m_N} & \dots & m_N^n \bar{\tau}^{m_N} \end{pmatrix}$$

where the first matrix has dimension $(n+1) \times N$, while the second one has dimension $N \times (n+1)$.

Cauchy-Binet formula [13, p.9], implies that the determinant of the above product is expressed by the sum

$$\sum_{1\le j_1<\ldots<j_{n+1}\le N}\begin{vmatrix} \tau^{m_{j_1}} & \ldots & \tau^{m_{j_{n+1}}} \\ m^1_{j_1}\tau^{m_{j_1}} & \ldots & m^1_{j_{n+1}}\tau^{m_{j_{n+1}}} \\ \vdots & \ldots & \vdots \\ m^n_{j_1}\tau^{m_{j_1}} & \ldots & m^n_{j_{n+1}}\tau^{m_{j_{n+1}}} \end{vmatrix}\begin{vmatrix} \bar\tau^{m_{j_1}} & m^1_{j_1}\bar\tau^{m_{j_1}} & \ldots & m^n_{j_1}\bar\tau^{m_{j_1}} \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ \bar\tau^{m_{j_{n+1}}} & m^1_{j_{n+1}}\bar\tau^{m_{j_{n+1}}} & \ldots & m^n_{j_{n+1}}\bar\tau^{m_{j_{n+1}}} \end{vmatrix}.$$

Factoring out the term $\tau^{m_{j_i}}$ from the i-column of the first determinant and $\bar\tau^{m_{j_i}}$ from the i-row of the second column for every $i = 1, \ldots, n+1$, the above summation is equal to

$$\sum_{1\le j_1<\ldots<j_{n+1}\le N}|\tau|^{2m_{j_1}+\cdots+2m_{j_{n+1}}}\begin{vmatrix} 1 & \ldots & 1 \\ m^1_{j_1} & \ldots & m^1_{j_{n+1}} \\ \vdots & \ldots & \vdots \\ m^n_{j_1} & \ldots & m^n_{j_{n+1}} \end{vmatrix}\begin{vmatrix} 1 & m^1_{j_1} & \ldots & m^n_{j_1} \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ 1 & m^1_{j_{n+1}} & \ldots & m^n_{j_{n+1}} \end{vmatrix}$$

or, equivalently,

$$\sum_{|J|=n+1}|\tau|^{2m_{j_1}+\cdots+2m_{j_{n+1}}}\det{}^2(A_J).$$

Thus, the result follows. □

The form $(\alpha^T_{1,1})^n$ has a similar form with the Fubini-Study form ([37]) ω of a toric variety defined by

$$\omega = \frac{1}{n!}\left(\frac{i}{2}\right)^n\left(\partial\bar\partial\ln P(|\tau|^2)\right)^n$$

where $P(t) = \sum_i c_i t^{m_i}$ is a Laurent polynomial in the n-dimensional torus T with m_i be only the vertices of the polyhedron P_D and c_i be some non-negative coefficients. Observe that, when the integral points coincide with the vertices of the polyhedron P_D (for example, this happen in the case of $\mathbb{P}^1\times\mathbb{P}^1$ with $D = D_2 + D_4$), then

$$(\alpha^T_{1,1})^n = n!\left(-\frac{1}{\pi}\right)^n\omega, \tag{3.30}$$

with c_i equal to 1.

Corollary 3.2.2 *The collection of vectors* $\{m_{j_1}, \ldots, m_{j_{n+1}}\}$ *from the* N *elements* $\{m_1, \ldots, m_N\}$ *contributes a non-trivial term to* $(\alpha^T_{1,1})^n$ *if and only if the* $(n+1)$ *extended vectors* $\{(1, m^1_{j_1}, \ldots, m^n_{j_1}), \ldots, (1, m^1_{j_{n+1}}, \ldots, m^n_{j_{n+1}})\}$ *in* $\mathbb{R}^{n+1}$ *are linearly independent.*

Proof. This is a direct consequence of Proposition 3.2.1 since every term of the summation vanishes if and only if $\det(A_J) = 0$.

□

Let us denote by $A_J^{p_1,\dots,p_k}$ the $(n+1-k)\times(n+1-k)$ matrix constructed by $(n+1-k)$ columns of the matrix A where $p_1+1,\dots,p_k+1$ rows are omitted while the index $1\le p_1<\ldots<p_k\le n$ is increasing. Hence, similar computations to those in Corollary 3.2.1 lead to the form $(\alpha_{1,1}^T)^{n-k}$.

Proposition 3.2.2 *The $(n-k,n-k)$-form $(\alpha_{1,1}^T)^{n-k}$ equals to*

$$(\alpha_{1,1}^T)^{n-k}=\left(-\frac{1}{2\pi i}\right)^{n-k}\frac{(n-k)!}{P(|\tau|^2)^{n-k+1}}$$
$$\sum_{\substack{|J|=n-k+1\\ 1\le j_1\le\ldots j_{n-k+1}\le N}}\ \sum_{\substack{1\le p_1<\ldots<p_k\le n\\ 1\le q_1<\ldots<q_k\le n}}\frac{\det(A_J^{p_1,\dots,p_k})\det(A_J^{q_1,\dots,q_k})|\tau|^{2m_{j_1}+\cdots+2m_{j_{n-k+1}}}}{\tau[p]\bar\tau[q]}d\bar\tau[q]\wedge d\tau[p] \tag{3.31}$$

where $\tau[p]$ is the product $\tau_1\cdots\tau_n$ where $\tau_{p_1},\dots,\tau_{p_k}$ are omitted while $\bar\tau[q]$ is the product $\bar\tau_1\cdots\bar\tau_n$, where $\bar\tau_{q_1},\dots,\bar\tau_{q_k}$ are omitted and $d\bar\tau[q]\wedge d\tau[p]$ results from deleting the differentials $d\tau_{p_1},\dots,d\tau_{p_k}$ and $d\bar\tau_{q_1},\dots,d\bar\tau_{q_k}$ in $d\bar\tau\wedge d\tau$ and writing the remaining terms of $d\bar\tau$ and $d\tau$ in increasing order of indices, alternately.

Proof. As in the previous proposition,

$$(\alpha_{1,1}^T)^{n-k}=\left(-\frac{1}{2\pi i}\right)^{n-k}\frac{(n-k)!}{P(|\tau|^2)^{n-k}}\times$$

$$\times\sum_{\substack{1\le p_1<\ldots<p_k\le n\\ 1\le q_1<\ldots<q_k\le n}}\left|\begin{matrix}1 & \sum_i m_i^1|\tau|^{2m_i} & \dots & \sum_i m_i^n|\tau|^{2m_i}\\ \frac{\sum_i m_i^1|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}} & \sum_i m_i^1m_i^1|\tau|^{2m_i} & \dots & \sum_i m_i^1m_i^n|\tau|^{2m_i}\\ \vdots & \dots & \dots & \vdots\\ \frac{\sum_i m_i^n|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}} & \sum_i m_i^nm_i^1|\tau|^{2m_i} & \dots & \sum_i m_i^nm_i^n|\tau|^{2m_i}\end{matrix}\right|_{\substack{p_1,\dots,p_k\\ q_1,\dots,q_k}}\frac{d\bar\tau[q]\wedge d\tau[p]}{\tau[p]\bar\tau[q]},$$

where $|\cdot|_{\substack{p_1,\dots,p_k\\ q_1,\dots,q_k}}$ denotes the determinant of a matrix where $p_1+1,\dots,p_k+1$ columns and $q_1+1,\dots,q_k+1$ rows are deleted. Hence,

$$(\alpha_{1,1}^T)^{n-k}=\left(-\frac{1}{2\pi i}\right)^{n-k}\frac{(n-k)!}{P(|\tau|^2)^{n-k+1}}\times$$

$$\times\sum_{\substack{1\le p_1<\ldots<p_k\le n\\ 1\le q_1<\ldots<q_k\le n}}\left|\begin{matrix}\sum_i|\tau|^{2m_i} & \sum_i m_i^1|\tau|^{2m_i} & \dots & \sum_i m_i^n|\tau|^{2m_i}\\ \sum_i m_i^1|\tau|^{2m_i} & \sum_i m_i^1m_i^1|\tau|^{2m_i} & \dots & \sum_i m_i^1m_i^n|\tau|^{2m_i}\\ \vdots & \dots & \dots & \vdots\\ \sum_i m_i^n|\tau|^{2m_i} & \sum_i m_i^nm_i^1|\tau|^{2m_i} & \dots & \sum_i m_i^nm_i^n|\tau|^{2m_i}\end{matrix}\right|_{\substack{p_1,\dots,p_k\\ q_1,\dots,q_k}}\frac{d\bar\tau[q]\wedge d\tau[p]}{\tau[p]\bar\tau[q]}.$$

Cauchy-Binet formula implies that the above determinant can be rewritten as

$$\sum_{|J|=n-k+1} \begin{vmatrix} \tau^{m_{j_1}} & \dots & \tau^{m_{j_{n+1}}} \\ m^1_{j_1}\tau^{m_{j_1}} & \dots & m^1_{j_{n+1}}\tau^{m_{j_{n+1}}} \\ \vdots & \dots & \vdots \\ m^n_{j_1}\tau^{m_{j_1}} & \dots & m^n_{j_{n+1}}\tau^{m_{j_{n+1}}} \end{vmatrix}_{q_1,\dots,q_k} \begin{vmatrix} \bar{\tau}^{m_{j_1}} & \dots & m^n_{j_1}\bar{\tau}^{m_{j_1}} \\ \vdots & \dots & \vdots \\ \vdots & \dots & \vdots \\ \bar{\tau}^{m_{j_{n+1}}} & \dots & m^n_{j_{n+1}}\bar{\tau}^{m_{j_{n+1}}} \end{vmatrix}_{p_1,\dots,p_k}$$

or, equivalently,

$$\sum_{|J|=n-k+1} |\tau|^{2m_{j_1}+\dots+2m_{j_{n-k+1}}} \begin{vmatrix} 1 & \dots & 1 \\ m^1_{j_1} & \dots & m^1_{j_{n+1}} \\ \vdots & \dots & \vdots \\ m^n_{j_1} & \dots & m^n_{j_{n+1}} \end{vmatrix}_{q_1,\dots,q_k} \begin{vmatrix} 1 & m^1_{j_1} & \dots & m^n_{j_1} \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ 1 & m^1_{j_{n+1}} & \dots & m^n_{j_{n+1}} \end{vmatrix}_{p_1,\dots,p_k}$$

$$= \sum_{\substack{|J|=n-k+1 \\ 1\le j_1\le\dots\le j_{n-k+1}\le N}} \det(A_J^{p_1,\dots,p_k})\det(A_J^{q_1,\dots,q_k})|\tau|^{2m_{j_1}+\dots+2m_{j_{n-k+1}}}.$$

Then, (3.31) follows. $\square$

In particular, according to the preceding proposition

$$(\alpha^T_{1,1})^{n-1} = \left(-\frac{1}{2\pi i}\right)^{n-1} \frac{(n-1)!}{P(|\tau|^2)^n}$$
$$\sum_{\substack{|J|=n \\ 1\le j_1\le\dots\le j_n\le N}} \sum_{\substack{1\le p\le n \\ 1\le q\le n}} \frac{\det(A_J^p)\det(A_J^q)|\tau|^{2m_{j_1}+\dots+2m_{j_n}}}{\tau[p]\bar{\tau}[q]} d\bar{\tau}[q]\wedge d\tau[p]. \qquad (3.32)$$

Corollary 3.2.3 *For p and q fixed in $(\alpha^T_{1,1})^{n-1}$, the collection of vectors $\{m_{j_1},\dots,m_{j_n}\}$ from the N elements $\{m_1,\dots,m_N\}$ contributes a non-trivial term in the form containing the terms $d\bar{\tau}[q]\wedge d\tau[p]$ if and only if all the following conditions are valid:*

1. *The n-vectors $\{(1,m_{j_1})_p,\dots,(1,m_{j_n})_p\}$ in $\mathbb{R}^n$ are linearly independent. Recall that $(1,m_{j_i})_p$ denotes the extended vector $(1,m^1_{j_i},\dots,m^n_{j_i})$, where $(p+1)$-th coordinate is omitted.*

2. *The n-vectors $\{(1,m_{j_1})_q,\dots,(1,m_{j_n})_q\}$ in $\mathbb{R}^n$ are linearly independent.*

3. *There doesn't exist any other collection of n-vectors $\{m_{j_{r_1}},\dots,m_{j_{r_n}}\}$, $1\le r\le \ell$ for an index ℓ satisfying*

$$m_{j_1}+\dots+m_{j_n} = m_{j_{r_1}}+\dots+m_{j_{r_n}}, \quad \forall\, 1\le r\le \ell$$

and

$$\det(A_J^p)\det(A_J^q) = -\sum_{r=1}^{\ell} \det(A_{J_r}^p)\det(A_{J_r}^q).$$

Proof. This is an immediate consequence of expression (3.32). □

3.3 Integral representation on a smooth Toric Variety

Let X be an n-dimensional smooth compact toric variety and D be an ample divisor that induces a line bundle $V_{\mathcal{L}}$ where $\mathcal{L} = \mathcal{O}_X(D)$ is the sheaf in which sections and forms take values. Our aim is to derive an integral representation Koppelman formula for these forms in the dual sense, by constructing the kernels for the representation having their singular points along the exceptional set of the toric variety.

A crucial point in our construction is the use of the result $\mathcal{O}_X(D) \simeq \phi_D^* \mathcal{O}_{\mathbb{P}^{N-1}}(1)$ (Lemma 3.1.1). This is the starting point of our construction, since a form on X that takes values in the line bundle $V_{\mathcal{L}}$ ($\mathcal{L} = \mathcal{O}_X(D)$) can be extended to a form on $\mathbb{P}^{N-1}$ which takes values in the hyperplane bundle L corresponding to the sheaf $\mathcal{O}_{\mathbb{P}^{N-1}}(1)$. This procedure reduces all computations to ones on $\mathbb{P}^{N-1}$ (where N is the number of the integral points of the polyhedron P_D). Taking into account the existing Koppelman formula on projective space (Section 1.3), we are able to construct a toric Koppelman formula on some projective toric varieties. The integrals in the toric Koppelman formula that take place on X are well-defined since the zero homogeneity of the integrands on $\mathbb{P}^{N-1}$ with respect to the projective variables $\zeta_1, \ldots, \zeta_N$ of $\mathbb{P}^{N-1}$ is 'translated' to the corresponding zero homogeneity of the integrands on X with respect to the homogeneous coordinates $\eta_1, \ldots, \eta_d$ of X, through the embedding ϕ_D.

For this derivation, we prove an analogue of the generalization of the Poinacaré-Lelong formula (see Theorem 1.4.1) which shrinks the support of the integrands from $\mathbb{P}^{N-1}$ down to X. This is illustrated through the use of a wedge product of vector fields and a specific projective weight that are introduced later in the text.

Let us recall the toric variables $t = (t_1, \ldots, t_n)$ and $\tau = (\tau_1, \ldots, \tau_n)$ which were introduced in Section 3.2. Through the embedding ϕ_D in (3.17), one can extend the algebraic torus T to the projective space $\mathbb{P}^{N-1}$ and write

$$(t^{m_1}, \ldots, t^{m_N}) = (z_1, \ldots, z_N) \in \mathbb{P}^{N-1} \tag{3.33}$$

and

$$(\tau^{m_1}, \ldots, \tau^{m_N}) = (\zeta_1, \ldots, \zeta_N) \in \mathbb{P}^{N-1}. \tag{3.34}$$

The preceding rules state the correspondence between the projective and toric variables.

Let us recall the polynomials (3.3)

$$f_j = \chi^{m_{j1}}\chi^{m_{j2}} - \chi^{m_{j3}}\chi^{m_{j4}}, \qquad j = 1, \ldots, N-n-1.$$

As we have assigned to each $\chi^{m_{ji}} = t^{m_{ji}}$ the variable z_{ji}, then every polynomial $f_j : \mathbb{C}^N \to \mathbb{C}$ takes the form

$$f_j(z) = z_{j1}z_{j2} - z_{j3}z_{j4}, \qquad j = 1, \ldots, N-n-1. \tag{3.35}$$

In view of Section 1.4, we are going to modify the generalization of the Poinacaré-Lelong formula (see Theorem 1.4.1) in the case of algebraic varieties lying in a projective space which are complete intersections. In particular, if $f = (f_1, \ldots, f_{N-n-1})$ is the tuple of 2-homogeneous polynomials in $\mathbb{C}^N$ given explicitly in (3.35), then the toric variety $X = Z_f = f^{-1}(0)$ which is the intersection of the hypersurfaces $\{f_j = 0\}$ for $j = 1, \ldots, N-n-1$ with $\dim X = n$ is a complete intersection in $\mathbb{P}^{N-1}$ due to Proposition 3.1.1.

Taking into account that differentiation is not 'closed' with respect to forms in the projective space, a new operator replacing d is introduced following [18]. This leads to the integration current on X through the logarithmic residue current that resembles in some sense the Poincaré-Lelong formula.

Following [18], we define the operator

$$D := d - 2\partial \log |\zeta|^2$$

on $\mathbb{C}^N \setminus \{0\}$. Observe that, for each $j = 1, \ldots, N-n-1$, Df_j is a projective form on $\mathbb{C}^N \setminus \{0\}$ as opposed to the corresponding form df_j. More precisely,

$$\begin{aligned} \delta_\zeta D f_j &= \delta_\zeta \left(df_j - 2 f_j \partial \log |\zeta|^2 \right) \\ &= 2\pi i \sum_{i=1}^{N} \zeta_i \frac{\partial}{\partial \zeta_i} \neg \left(df_j - 2 f_j \sum_{k=1}^{N} \frac{\bar{\zeta}_k}{|\zeta|^2} d\zeta_k \right) \\ &= 2\pi i \left(\sum_{i=1}^{N} \zeta_i \frac{\partial f_j}{\partial \zeta_i} - 2 f_j \right) = 0, \end{aligned} \tag{3.36}$$

since by differentiating $f_j(\lambda\zeta) = \lambda^2 f_j(\zeta)$ for $\zeta = (\zeta_1, \ldots, \zeta_N)$ with respect to $\lambda \in \mathbb{C}$, one can observe that

$$\sum_{i=1}^{N} \zeta_i \frac{\partial f_j(\lambda\zeta)}{\partial \zeta_i} = 2\lambda f_j(\zeta). \tag{3.37}$$

Hence, the vanishing of (3.36) follows from (3.37) for $\lambda = 1$ and Df_j is a projective form on $\mathbb{C}^N$ by using the necessary and sufficient condition given in Section 1.3.

For notational convenience, let

$$Df = \bigwedge_{j=1}^{N-n-1} Df_j$$

the $(N-n-1, 0)$ projective form on $\mathbb{C}^N$ (recall that this is a a form arises form the pullback of a form in $\mathbb{P}^{N-1}$ through the canonical projection).

In view of Section 1.4, we introduce the residual current

$$\bar{\partial}\left(\frac{1}{f}\right) = \bigwedge_{j=1}^{N-n-1} \bar{\partial}\left(\frac{1}{f_j}\right)$$

(see (1.34)) and the multi-logarithmic residue current

$$\bar{\partial}\frac{1}{f} \wedge df = \bigwedge_{j=1}^{N-n-1} \bar{\partial}\left(\frac{1}{f_j}\right) \wedge \bigwedge_{j=1}^{N-n-1} df_j \tag{3.38}$$

in $\mathbb{C}^N$ (see 1.35). In order for the multi-logarithmic residue current to be well defined in $\mathbb{P}^{N-1}$, the involved forms must be projective. Thus, instead of using multi-logarithmic residue current in (3.38), we introduce the corresponding current in $\mathbb{C}^N$ given by

$$\bar{\partial}\frac{1}{f} \wedge Df = \bigwedge_{j=1}^{N-n-1} \bar{\partial}\left(\frac{1}{f_j}\right) \wedge \bigwedge_{j=1}^{N-n-1} Df_j \tag{3.39}$$

where the forms df_j in (3.38) are replaced by the projective forms Df_j. However, the two currents in (3.38) and (3.39) coincide and an analogue of Theorem 1.4.1 holds. This is the content of the following theorem.

Theorem 3.3.1 *Let $f = (f_1, \ldots, f_{N-n-1})$ be the 2-homogeneous polynomials in $\mathbb{C}^N$ described by (3.35), defining the toric variety $X = f^{-1}(0)$ as a complete intersection in $\mathbb{P}^{N-1}$. Then,*

$$\left(\frac{1}{2\pi i}\right)^{N-n-1} < \bar{\partial}\frac{1}{f} \wedge Df, \varphi >= \int_X \varphi,$$

holds for $\varphi \in \mathcal{D}_{n,n}(\mathbb{P}^{N-1}, L^0)$, where L^0 is the trivial line bundle.

Proof. It is sufficient to observe that the equality

$$
\begin{aligned}
& \bar{\partial}\frac{1}{f} \wedge Df = \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_{N-n-1}} \wedge Df_1 \wedge \ldots \wedge Df_{N-n-1} \\
= \; & \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_{N-n-1}} \wedge (df_1 - 2f_1 \partial \log |\zeta|^2) \wedge \ldots \wedge (df_{N-n-1} - 2f_{N-n-1}\partial \log |\zeta|^2) \\
= \; & \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_{N-n-1}} \wedge df_1 \wedge \ldots \wedge df_{N-n-1} \\
= \; & \bar{\partial}\frac{1}{f} \wedge df
\end{aligned}
$$

holds on $\mathbb{C}^N \setminus \{0\}$ in the current sense in view of the definition of the logarithmic residue current in (1.35), since $f = (f_1, \ldots, f_{N-n-1}) = 0$ on X. Then, the required result is a consequence of Theorem 1.4.1. The current integration of the form $\varphi \in \mathcal{D}_{n,n}(\mathbb{P}^{N-1}, L^0)$ over X makes sense, since in view of Lemma 3.1.2, the restriction of φ on X gives a section $\phi \in \mathcal{D}_{n,n}(X, (V_{\mathcal{L}})^0)$ through the map (3.2) of embedding of X into the projective space $\mathbb{P}^{N-1}$. □

Following [4], we assign to each $f_j(z) = z_{j1}z_{j2} - z_{j3}z_{j4}$, $j = 1 \ldots, N-n-1$, its projective Hefer form defined in $\mathbb{C}^N \setminus \{0\} \times \mathbb{C}^N \setminus \{0\}$ by

$$
\begin{aligned}
H^j(z,\zeta) \; := \; & \frac{1}{4\pi i}\Bigg[(z_{j2} + \alpha\zeta_{j2})\left(d\zeta_{j1} - \frac{\bar{\zeta}\cdot d\zeta}{|\zeta|^2}\zeta_{j1}\right) + (z_{j1} + \alpha\zeta_{j1})\left(d\zeta_{j2} - \frac{\bar{\zeta}\cdot d\zeta}{|\zeta|^2}\zeta_{j2}\right) + \\
& - (z_{j4} + \alpha\zeta_{j4})\left(d\zeta_{j3} - \frac{\bar{\zeta}\cdot d\zeta}{|\zeta|^2}\zeta_{j3}\right) - (z_{j3} + \alpha\zeta_{j3})\left(d\zeta_{j4} - \frac{\bar{\zeta}\cdot d\zeta}{|\zeta|^2}\zeta_{j4}\right)\Bigg].
\end{aligned}
\tag{3.40}
$$

This form has terms of bidegree $(1,0)$ and $(2,1)$ due to the contribution of the weight α given explicitly in (1.31).

The polynomials f_j do affect our representation, and this is reflected through the use of these Hefer forms H^j defined in (3.40) and more specifically, through their terms of bidegree $(1,0)$, as it will be shown later.

For notational simplicity, let us define the form

$$
H := \bigwedge_{j=1}^{N-n-1} H^j \tag{3.41}
$$

which has terms of bidegree $(N-n-1+k, k)$ for every $k = 0, 1, \ldots, N-n-1$ due to the terms of bigeree $(1,0)$ and $(2,1)$ that constitute each H^j.

Proposition 3.3.1 *There is a wedge product of vector fields $\gamma_{N-n-1} \wedge \ldots \wedge \gamma_1$ defined on*

$\mathbb{P}^{N-1}$ *such that*

$$\gamma_{N-n-1} \wedge \ldots \wedge \gamma_1 \neg (df_1 \wedge \ldots \wedge df_{N-n-1}) = (2\pi i)^{N-n-1},$$

where $\neg$ *denotes interior multiplication.*

Proof. Let us consider the vector fields

$$\gamma_j = \frac{2\pi i}{\zeta_{j1}} \frac{\partial}{\partial \zeta_{j2}}, \quad \text{for} \quad j = 1, \ldots, N-n-1, \tag{3.42}$$

defined on the chart $U_1 = \{\zeta_1 = \zeta_{j1} \neq 0\}$ of $\mathbb{P}^{N-1}$. Thus γ_j are also well-defined on the chart U_σ of X where $\prod_{i=1}^{d} \eta_i^{\alpha_i} \neq 0$ due to the correspondence (3.9). Since, each $f_j(\zeta) = \zeta_{j1}\zeta_{j2} - \zeta_{j3}\zeta_{j4}$, the $(1,0)$ form df_j equals to

$$df_j = \zeta_{j1} d\zeta_{j2} + \zeta_{j2} d\zeta_{j1} - \zeta_{j3} d\zeta_{j4} - \zeta_{j4} d\zeta_{j3}.$$

It is easy to observe that $\gamma_j \neg df_j = 2\pi i$ for each j. However, since the variables ζ_{jk} for $j = 1, \ldots, N-n-1$ and $k = 1, \ldots, 4$ are not independent, some of the vectors γ_j may provide a non-trivial result when they also act on a differential form df_l, for $l \neq j$. Arranging the vectors m_{j2} for $j = 1, \ldots, N-n-1$ (see Section 3.1) in increasing order of distance from the origin and letting the wedge product of vector fields $\gamma_{N-n-1} \wedge \ldots \wedge \gamma_1$ act on $df_1 \wedge \ldots \wedge df_{N-n-1}$, one can deduce that the vector γ_{N-n-1} must act on df_{N-n-1} for a non-trivial result, then γ_{N-n-2} acts only on df_{N-n-2} and so on. Then, each vector field γ_j acts on the corresponding form df_j such that

$$\gamma_{N-n-1} \wedge \ldots \wedge \gamma_1 \neg (df_1 \wedge \ldots \wedge df_{N-n-1}) = \bigwedge_{j=1}^{N-n-1} \gamma_j \neg df_j = (2\pi i)^{N-n-1}.$$

$\square$

For notational convenience denote by $\gamma := \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1$. Now, if we apply the vector field γ on the current $[X]$ and use a combination of Theorem 3.3.1 with the preceding proposition, then the following equation in the current sense is deduced for a smooth test form φ on $\mathbb{P}^{N-1}$ taking values in $L^{2(N-n-1)}$ (recall that $\varphi\big|_X = \phi$, where ϕ is an (n,n) test form taking values in the corresponding line bundle of X due to the identification $V_{\mathcal{L}} \cong L$):

$$
\begin{aligned}
&< \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1 \neg [X], \varphi >= \int_{\mathbb{P}^{N-1}} \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1 \neg [X] \wedge \varphi \\
= \;& - \int_{\mathbb{P}^{N-1}} [X] \wedge \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1 \neg \varphi \\
= \;& - \int_X \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1 \neg \varphi \\
= \;& - \left(\frac{1}{2\pi i}\right)^{N-n-1} \lim_{\delta \to 0} \int_{\substack{|f_j| = \epsilon_j(\delta) \\ 1 \le j \le N-n-1}} \frac{Df_1}{f_1} \wedge \ldots \wedge \frac{Df_{N-n-1}}{f_{N-n-1}} \wedge \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1 \neg \varphi \\
= \;& \left(\frac{1}{2\pi i}\right)^{N-n-1} \lim_{\delta \to 0} \int_{\substack{|f_j| = \epsilon_j(\delta) \\ 1 \le j \le N-n-1}} \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1 \neg \left(\frac{Df_1}{f_1} \wedge \ldots \wedge \frac{Df_{N-n-1}}{f_{N-n-1}} \right) \wedge \varphi \\
= \;& (-1)^{(N-n-1)^2} \lim_{\delta \to 0} \int_{\substack{|f_j| = \epsilon_j(\delta) \\ 1 \le j \le N-n-1}} \frac{\varphi}{f_1 \ldots f_{N-n-1}} \\
= \;& (-1)^{(N-n-1)^2} < \bar{\partial} \left(\frac{1}{f}\right), \varphi >,
\end{aligned}
\tag{3.43}
$$

where the sign $(-1)^{(N-n-1)^2}$ arises due to the sign $(-1)^{N-n-1}$ that is induced from the commutation of each γ_j with the $(0, N-n-1)$ form $\bar{\partial}\frac{1}{f} = \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_{N-n-1}}$. Actually, the current equality (3.43) holds on the chart U_σ of X where the γ_j are well-defined but one can observe the equality can be also obtained on every chart of the covering $\{U_{\sigma_i}\}_i$ of X by introducing an equivalent divisor to D (see Section 1.1.3) and similar vector fields γ_j.

In view of (3.43), in order to change the domain of integration from $\mathbb{P}^{N-1}$ to X in the Weighted Koppelman formula on $\mathbb{P}^{N-1}$ (Section 1.3), we introduce a suitable projective weight involving the current $\bar{\partial}\frac{1}{f}$ which is presented below. This construction extends the approach of [18]. We set

$$
g = \left(f_1(z) \frac{1}{f_1} + H^1 \wedge \bar{\partial} \frac{1}{f_1} \right) \wedge \ldots \wedge \left(f_{N-n-1}(z) \frac{1}{f_{N-n-1}} + H^{N-n-1} \wedge \bar{\partial} \frac{1}{f_{N-n-1}} \right). \tag{3.44}
$$

Observe that g is a projective weight since $g_{0,0}([z],[z]) = 1$, and satisfies the equation $\nabla_z g = 0$. To be more precise,

$$
\begin{aligned}
\nabla_z \left(f_j(z) \frac{1}{f_j} + H^j \wedge \bar{\partial} \frac{1}{f_j} \right) &= (\delta_z - \bar{\partial}) \left(f_j(z) \frac{1}{f_j} + H^j \wedge \bar{\partial} \frac{1}{f_j} \right) \\
&= -f_j(z) \bar{\partial} \frac{1}{f_j} + \nabla_z H^j \wedge \bar{\partial} \frac{1}{f_j}.
\end{aligned}
\tag{3.45}
$$

Since,

$$
\begin{aligned}
\nabla_z \left(d\zeta_j - \frac{\bar{\zeta} \cdot d\zeta}{|\zeta|^2} \zeta_j \right) &= 2\pi i \left(z_j - \frac{\bar{\zeta} \cdot z}{|\zeta|^2} \zeta_j \right) + \bar{\partial} \left(\frac{\bar{\zeta} \cdot d\zeta}{|\zeta|^2} \right) \zeta_j \\
&= 2\pi i (z_j - \alpha \zeta_j)
\end{aligned}
$$

and $\nabla_z \alpha = 0$, due to the explicit form (3.40) of H^j, we get

$$\begin{aligned}
\nabla_z H^j &= \frac{1}{2}\left[(z_{j2} + \alpha\zeta_{j2})(z_{j1} - \alpha\zeta_{j1}) + (z_{j1} + \alpha\zeta_{j1})(z_{j2} - \alpha\zeta_{j2})\right. \\
&\quad \left. -(z_{j4} + \alpha\zeta_{j4})(z_{j3} - \alpha\zeta_{j3}) - (z_{j3} + \alpha\zeta_{j3})(z_{j4} - \alpha\zeta_{j4})\right] \\
&= f_j(z) - \alpha^2 f_j(\zeta).
\end{aligned}$$

Substituting the equality in (3.45), one has

$$\nabla_z \left(f_j(z)\frac{1}{f_j} + H^j \wedge \bar{\partial}\frac{1}{f_j} \right) = -f_j(z)\bar{\partial}\frac{1}{f_j} + f_j(z)\bar{\partial}\frac{1}{f_j} - \alpha^2 f_j(\zeta)\bar{\partial}\frac{1}{f_j} = 0 \tag{3.46}$$

and $\nabla_z g = 0$.

The contributing term of the weight g in our representation will be

$$\begin{aligned}
& \left(H^1 \wedge \bar{\partial}\frac{1}{f_1}\right) \wedge \ldots \wedge \left(H^{N-n-1} \wedge \bar{\partial}\frac{1}{f_{N-n-1}}\right) \\
= \;& c_{N,n} H^1 \wedge \ldots H^{N-n-1} \wedge \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_{N-n-1}} \\
= \;& c_{N,n} H \wedge \bar{\partial}\frac{1}{f},
\end{aligned} \tag{3.47}$$

where $c_{N,n} = (-1)^{\frac{(N-n-2)(N-n-1)}{2}}$, leading to integration on X, in view of (3.43).

For $N \leq 2(n+1)$, let $K = g \wedge \alpha^{2+2n-N} \wedge u$ and $P = g \wedge \alpha^{2+2n-N}$ be the 'modified' kernels in the Weighted Koppelman formula on P^{N-1} (see Theorem 1.3.1) where u and α are given explicitly in (1.30) and (1.31), leading to the required toric representation formula. In order to show that these kernels lead to well-defined integrals in the formula (1.32), we have to check that the sum of the homogeneity of the forms involved (integrands) is zero with respect to the ζ-variable in $\mathbb{P}^{N-1}$.

In particular, the weight α in (1.31), has components of bidegree $(0,0)$ and $(1,1)$ with homogeneities -1 in ζ, 1 in z and 0 both in ζ and z, respectively. Hence,

$$\left(\alpha^{2+2n-N}\right)_{k,k} = \binom{2+2n-N}{k} \alpha_{0,0}^{2+2n-N-k} \alpha_{1,1}^{k} \quad \text{for} \quad k \leq 2+2n-N$$

is $-(2+2n-N-k)$-homogeneous in ζ and $(2+2n-N-k)$-homogeneous in z. Furthermore, one deduces that

$$\bar{\partial}\frac{1}{f} = \bigwedge_{j=1}^{N-n-1} \bar{\partial}\frac{1}{f_j}$$

is $-2(N-n-1)$-homogeneous in ζ and 0 in z, since every polynomial $f_j(\zeta)$ is 2-homogeneous in ζ and does not contain any z. The component with bidegree $(1,0)$ of the forms H^j, $(H^j)_{1,0}$, is

1-homogeneous both in ζ and z, for every $j = 1, \dots, N-n-1$, while the term of bidegree $(2,1)$ of H^j, $(H^j)_{2,1}$ is 2-homogeneous in ζ and 0-homogeneous in z. Adding the homogeneities, one gets that

$$\left(H \wedge \bar{\partial}\frac{1}{f} \wedge \alpha^{2+2n-N}\right)_{N-1,N-1} = \sum_{k=0}^{\min\{n,2+2n-N\}} H_{N-1-k,n-k} \wedge \left(\bar{\partial}\frac{1}{f}\right)_{0,N-n-1} \wedge \left(\alpha^{2+2n-N}\right)_{k,k}$$

is -1-homogeneous in ζ and 1-homogeneous in z. Since $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$ is extended to a $(0,q)$ form φ which is 1-homogeneous in ζ ($\varphi \in \mathcal{E}_{0,q}(\mathbb{P}^{N-1}, L^1)$) by Lemma 3.1.2, the homogeneities of the forms involved in the integrand $P \wedge \varphi$ are balanced and the corresponding integral is well-defined.

On the other hand, the form

$$u_{l,l-1} = \frac{v \wedge (\bar{\partial} v)^{l-1}}{(\delta_z v)^l}, \quad 1 \leq l \leq N-1$$

which takes values in $L^l_{[\zeta]} \otimes L^{-l}_{[z]}$ (since v in (1.29) takes values in $L^1_{[\zeta]} \otimes L^{-1}_{[z]}$), gives us that the form

$$\begin{aligned}\left(H \wedge \bar{\partial}\frac{1}{f} \wedge \alpha^{2+2n-N} \wedge u\right)_{N-1,N-2} \\ = \sum_{k,l} H_{N-1-k-l,n-k-l} \wedge \left(\bar{\partial}\frac{1}{f}\right)_{0,N-n-1} \wedge \left(\alpha^{2+2n-N}\right)_{k,k} \wedge u_{l,l-1}\end{aligned}$$

is -1-homogeneous in ζ and 1-homogeneous in z. Thus, the integrals related to the kernel K are also well-defined.

Remark 3.3.1 In the preceding paragraph, the added condition $N \leq 2(n+1)$ is due to the fact that the power of the weight α, leading to well-defined integrands, must be a non-negative integer. We also claim that the number N of the integral points of the polyhedron P_D satisfies the inequality $N \geq 2n$, in the case of an n-dimensional, non-trivial, smooth compact toric variety X ($X \neq \mathbb{P}^n$). The last statement is based on the classification of smooth compact toric varieties ([20],[35]) and the following discussion justifies this claim.

We assume that X is an n-dimensional, non-trivial, smooth compact toric variety, whose fan is generated by d n-dimensional vectors in $\mathbb{R}^n$. Then, $d \geq n+1$. T. Oda in [35] proved that any smooth and complete fan composed by $n+1$ vectors in $\mathbb{R}^n$ determines $\mathbb{P}^n$. This actually follows from the smoothness of the cones constructing the fan of X since the determinant of the generators of each cone must be equal to 1 or -1. Thus, one can further assume that $d \geq n+2$, according to the initial assumption that X is a non-trivial projective toric variety.

We will prove that the number $\sigma(n)$ of the n-dimensional cones (*maximal cones*), composing the complete fan of X, is greater than or equal to $2n$. Moreover, observe that each of these cones induces a distinct vector and the set of these vectors constitutes the set of vertices of P_D (see Section 1.1.7). Hence, since the number N of the integral points is at least the number of the vertices of P_D, we establish the initial statement that

$$N \geq \sigma(n) \geq 2n.$$

We prove that $\sigma(n) \geq 2n$, for $n \geq 2$ through the method of mathematical induction. Firstly, for $n = 2$, the result is trivial since $\sigma(2) \geq d \geq n+2 = 4$. For $n = 3$, following [35],

$$\sigma(3) = 2d - 4 \geq 2(n+2) - 4 = 2n = 6.$$

In general, for the n-dimensional case, we reduce the proof to the case of $d = n+2$. Then, the generalization of the result follows immediately since $d \geq n+2$.

Following [20], every n dimensional smooth compact toric variety with $d = n+2$ generators and $n \geq 2$ is isomorphic to the toric variety whose fan is composed by the following vectors in $\mathbb{R}^n$:

$$\begin{aligned} u_i &= e_i,\ 1 \leq i \leq r \\ u_{r+1} &= -\sum_{i=1}^{r} v_i \\ v_j &= e_{r+j},\ 1 \leq j \leq s-1 \\ v_s &= \sum_{i=1}^{r} a_i e_i - \sum_{j=1}^{s-1} v_j, \end{aligned} \tag{3.48}$$

where $a_1, \ldots, a_r$ are integers with $0 \leq a_1 \leq a_2 \leq \ldots \leq a_r$ and r, s are positive integers such that $r \geq 1$, $2 \leq s \leq n$ and $r + s = n + 1$. If we set $U = \{u_1, \ldots, u_{r+1}\}$ and $V = \{v_1, \ldots, v_s\}$, then the maximal cones of this toric variety are of the forms

$$\text{Cone}\left((U \cup V) \setminus \{u_i, v_j\}\right),\ 1 \leq i \leq r+1, 1 \leq j \leq s.$$

Hence, $\sigma(n) = (r+1)s$. We assume that

$$\sigma(n) = (r+1)s \geq 2n \tag{3.49}$$

and we will prove that the number of the maximal cones of an $(n+1)$-dimensional smooth compact toric variety X with $d = n+3$, satisfies $\sigma(n+1) \geq 2(n+1)$. The generating vectors are of the form (3.48) that correspond in the $(n+1)$-dimensional case, such that $\sigma(n+1) = (r+2)s$ or $\sigma(n+1) = (r+1)(s+1)$. In view of (3.49),

$$\sigma(n+1) = (r+2)s = (r+1)s + s \geq 2n+2 = 2(n+1)$$

or

$$\sigma(n+1) = (r+1)(s+1) = (r+1)s + (r+1) \geq 2n+2 = 2(n+1).$$

Both cases establish the initial statement.

Theorem 3.3.2 *Let X be an n-dimensional smooth compact toric variety and $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$, where $V_{\mathcal{L}}$ is the line bundle corresponds to an ample divisor $D = \sum_{i=1}^{d} a_i D_i$. For $\mathcal{L} = \mathcal{O}_X(D)$ and $|P_D \cap \mathbb{Z}^n| = N$ such that $N \leq 2(n+1)$, we have*

$$C_{N,n}\phi(z) = \int_X \mathcal{K} \wedge \bar{\partial}\phi + \bar{\partial}_z \int_X \mathcal{K} \wedge \phi + \int_X \mathcal{P} \wedge \phi \tag{3.50}$$

on $X = \bigcap_{i=1}^{d} \left\{ z \in \mathbb{C}^N \setminus \{0\} : f_i(z) = 0 \right\}$, where $C_{N,n} = (-1)^{\frac{(N-n-2)(N-n-1)}{2}+1}$, $\mathcal{K} = \gamma \neg \left(H \wedge \alpha^{2+2n-N} \wedge u \right)$ and $\mathcal{P} = \gamma \neg \left(H \wedge \alpha^{2+2n-N} \right)$ are the kernels of the representation formula and the integrals are taken with respect to the ζ variable.

Before we prove Theorem 3.3.2, we point out that the construction, that leads to the formula (3.50), holds on $\bigcap_{i=1}^{d} \{z \in U_1 : f_i(z) = 0\} = U_\sigma = \{h \in X : \prod_{i=1}^{d} h_i^{a_i} \neq 0\}$, where $U_1 = \{z \in \mathbb{P}^{N-1} : z_1 \neq 0\}$. On every chart of the covering $\{U_{\sigma_i}\}_i$ of X, a similar construction holds by introducing an equivalent divisor D and a similar vector field which is well-defined on the corresponding chart. Moreover, on the intersection $U_{\sigma_i} \cap U_{\sigma_j} \neq \emptyset$ of two charts for every i, j, the two integral representations formulas that arise, are equal (as we will prove in the next section). Thus, the integral representation formula (3.50) is independent of the selected chart. Since the forms ϕ, $\mathcal{K}$ and $\mathcal{P}$ are well-defined on every point in X, one can write that the integrands are taken over the whole variety X indicating the global character of the Theorem 3.3.2. The statements of the theorems 3.4.3, 3.6.1 and 3.6.2 should be understood similarly.

Proof. Let $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$. According to Lemma 3.1.2, ϕ can be extended to an $\varphi \in \mathcal{E}_{0,q}(\mathbb{P}^{N-1}, L)$ and thus, φ can be represented by a Weighted Koppelman representation formula on $\mathbb{P}^{N-1}$ (Theorem 1.3.1). Let

$$K = g \wedge \alpha^{2+2n-N} \wedge u \quad \text{and} \quad P = g \wedge \alpha^{2+2n-N}$$

be the corresponding kernels in the integral representation formula on $\mathbb{P}^{N-1}$, where $g =$

$c_{N,n} H \wedge \bar\partial \frac{1}{f}$ and $c_{N,n} = (-1)^{\frac{(N-n-2)(N-n-1)}{2}}$. Observe, that the boundary integral in (1.32) vanishes due to the compact nature of $\mathbb{P}^{N-1}$. Now, the Koppelman formula on $\mathbb{P}^{N-1}$ implies that φ can be represented by the following formula

$$\begin{aligned}
\varphi(z) &= \int_{\mathbb{P}^{N-1}} g \wedge \alpha^{2+2n-N} \wedge u \wedge \bar\partial\varphi + \bar\partial_z \int_{\mathbb{P}^{N-1}} g \wedge \alpha^{2+2n-N} \wedge u \wedge \varphi \\
&\quad + \int_{\mathbb{P}^{N-1}} g \wedge \alpha^{2+2n-N} \wedge \varphi \\
&= c_{N,n} \int_{\mathbb{P}^{N-1}} H \wedge \bar\partial \frac{1}{f} \wedge \alpha^{2+2n-N} \wedge u \wedge \bar\partial\varphi \\
&\quad + c_{N,n} \bar\partial_z \int_{\mathbb{P}^{N-1}} H \wedge \bar\partial \frac{1}{f} \wedge \alpha^{2+2n-N} \wedge u \wedge \varphi \\
&\quad + c_{N,n} \int_{\mathbb{P}^{N-1}} H \wedge \bar\partial \frac{1}{f} \wedge \alpha^{2+2n-N} \wedge \varphi,
\end{aligned}$$

where all the integrals are well-defined on $\mathbb{P}^{N-1}$ in view of the preceding discussion. Then, (3.43) yields

$$\begin{aligned}
\varphi(z) &= (-1)^{(N-n-1)^2} c_{N,n} \int_{\mathbb{P}^{N-1}} H \wedge \gamma\neg[X] \wedge \alpha^{2+2n-N} \wedge u \wedge \bar\partial\varphi \\
&\quad + (-1)^{(N-n-1)^2} c_{N,n} \bar\partial_z \int_{\mathbb{P}^{N-1}} H \wedge \gamma\neg[X] \wedge \alpha^{2+2n-N} \wedge u \wedge \varphi \\
&\quad + (-1)^{(N-n-1)^2} c_{N,n} \int_{\mathbb{P}^{N-1}} H \wedge \gamma\neg[X] \wedge \alpha^{2+2n-N} \wedge \varphi
\end{aligned}$$

on the chart U_σ of X, where $\gamma = \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1$. However, since the exterior product is graded anti-commutative, we get

$$H \wedge \gamma\neg[X] = (-1)^{(N-n-1)^2} \gamma\neg[X] \wedge H.$$

Therefore,

$$\begin{aligned}
\varphi(z) &= c_{N,n} \int_{\mathbb{P}^{N-1}} \gamma\neg[X] \wedge H \wedge \alpha^{2+2n-N} \wedge u \wedge \bar\partial\varphi \\
&\quad + c_{N,n} \bar\partial_z \int_{\mathbb{P}^{N-1}} \gamma\neg[X] \wedge H \wedge \alpha^{2+2n-N} \wedge u \wedge \varphi \\
&\quad + c_{N,n} \int_{\mathbb{P}^{N-1}} \gamma\neg[X] \wedge H \wedge \alpha^{2+2n-N} \wedge \varphi \\
&= -c_{N,n} \int_X \gamma\neg \left(H \wedge \alpha^{2+2n-N} \wedge u\right) \wedge \bar\partial\varphi \\
&\quad - c_{N,n} \bar\partial_z \int_X \gamma\neg \left(H \wedge \alpha^{2+2n-N} \wedge u\right) \wedge \varphi \\
&\quad - c_{N,n} \int_X \gamma\neg \left(H \wedge \alpha^{2+2n-N}\right) \wedge \varphi. \qquad (3.51)
\end{aligned}$$

Let us denote by

$$\mathcal{K} = \gamma \neg \left(H \wedge \alpha^{2+2n-N} \wedge u \right) \quad \text{and} \quad \mathcal{P} = \gamma \neg \left(H \wedge \alpha^{2+2n-N} \right)$$

the Koppelman kernel and Projection kernel on X, respectively. If we set $C_{N,n} = -c_{N,n} = (-1)^{\frac{(N-n-2)(N-n-1)}{2}+1}$, then the desired representation formula for ϕ on U_σ is obtained since $\varphi|_X = \phi$. □

3.4 Explicit computations of Projection and Koppelman kernels in toric variables

The present section is devoted to the explicit computation of the Projection and Koppelman kernels. The goal is to study the relationship between them as well as the geometry of the 'infinity' of the toric variety.

The forms involved in the kernels $\mathcal{K}$ and $\mathcal{P}$ of the toric Koppelman formula (Theorem 3.3.2) are expressed with respect to the projective variables $(z_1, \ldots, z_N, \zeta_1, \ldots, \zeta_N)$ of $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$ (that take values in some line bundles with respect to z and ζ). This fact does not illustrate the contribution of the homogeneous coordinates $h_1, \ldots, h_d$ and $\eta_1, \ldots, \eta_d$ of $X \times X$ in our construction. Thus, we begin, by letting the vector fields γ_j, $j = 1, \ldots, N-n-1$, act on the involved forms expressed in projective variables on $\bigcap_{i=1}^d \{\zeta \in U_1 : f_i(\zeta) = 0\}$ that corresponds to the chart $U_\sigma = \{\prod_{k=1}^d \eta_k^{a_k} \neq 0\}$ of X. We restrict the contributing terms on $T \times T$ following the rules

$$z_i = t^{m_i} = t_1^{m_i^1} \cdots t_n^{m_i^n} \quad \text{and} \quad \zeta_i = \tau^{m_i} = \tau_1^{m_i^1} \cdots \tau_n^{m_i^n}$$

for every $i = 1, \ldots, N$, where $t = (t_1, \ldots, t_n) \in T$ and $\tau = (\tau_1, \ldots, \tau_n) \in T$. Then, in order to pass to the homogeneous coordinates $h = (h_1, \ldots, h_d) \in \mathbb{C}^d$ and $\eta = (\eta_1, \ldots, \eta_d) \in \mathbb{C}^d$ such that $(h, \eta) \in X \times X$, one can follow the rules

$$t^{m_i} = \prod_{k=1}^{d} h_k^{\langle m_i, v_k \rangle} \quad \text{and} \quad \tau^{m_i} = \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle}, \tag{3.52}$$

for $i = 1, \ldots, N$ and express the involved forms in terms of the homogeneous coordinates (h, η) of $X \times X$ locally in a chart U_σ of X. In this setting, the forms and thus the kernels of the representation are zero homogeneous (in other words take values in the trivial line bundle $(V_\mathcal{L})^0$ of X) and the arising formula corresponds to the form ϕ_σ related to ϕ through (3.5).

Since the relation between ϕ and ϕ_σ can be written as

$$\phi(\eta)\big|_{U_\sigma} = \left(\prod_{k=1}^{d} \eta_k^{a_k}\right) \phi_\sigma(\eta), \tag{3.53}$$

the toric Koppelman formula for representing ϕ_σ on U_σ is adjusted to ϕ by multiplying the kernels by $\frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}}$. Moreover, for a divisor $D = \sum_{k=1}^{d} a_k D_k \sim \sum_{k=1}^{d} b_k D_k$, if $U_{\sigma_i} = \{\prod_{k=1}^{d} \eta_k^{a_k} \neq 0\}$ and $U_{\sigma_j} = \{\prod_{k=1}^{d} \eta_k^{b_k} \neq 0\}$ are two charts of X, then observe that on $U_{\sigma_i} \cap U_{\sigma_j}$ one obtains

$$\begin{aligned} \phi(\eta)\big|_{U_{\sigma_i}} &= \left(\prod_{k=1}^{d} \eta_k^{a_k}\right) \phi_{\sigma_i}(\eta) = \left(\frac{\prod_{k=1}^{d} \eta_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{b_k}}\right) \left(\prod_{k=1}^{d} \eta_k^{b_k}\right) \phi_{\sigma_i}(\eta) \\ &= \left(\prod_{k=1}^{d} \eta_k^{b_k}\right) g_{ij} \phi_{\sigma_i}(\eta) = \left(\prod_{k=1}^{d} \eta_k^{b_k}\right) \phi_{\sigma_j}(\eta) = \phi(\eta)\big|_{U_{\sigma_j}}, \end{aligned} \tag{3.54}$$

where $g_{ij} = \left(\frac{\prod_{k=1}^{d} \eta_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{b_k}}\right)$ are the transition functions of the line bundle $V_{\mathcal{L}}$ corresponding to the divisor D. Hence, the representation formulas are compatible on the intersections of the charts and a well defined local representation for ϕ arises on every chart of the covering of X.

The computations are voluminous in particular for the case of the Koppelman kernel. However, we know in advance that the singularities of the involved kernels are located on the exceptional set $Z(\Sigma)$ of the toric variety X. More precisely, according to [6] (see Lemma 5.4.6)

$$V\left(\prod_{i=1}^{d} \eta_i^{\langle m_j, v_i\rangle + a_i} \big| m_j \in P_D\right) = Z(\Sigma) \tag{3.55}$$

holds, which implies that the zero set of the monomials in the left hand side of (3.55) is identified with the exceptional set of the toric variety. This happened since the variables appearing in these monomials are precisely the variables appearing in the monomials $\eta_{\hat{\sigma}} = \prod_{v_j \notin \sigma} \eta_j$ defining the exceptional set $Z(\Sigma)$ of X (see (1.2)). Thus, since the involved forms of the kernels on $\mathbb{P}^{N-1}$ have denominators $|\zeta|^2 = \sum_{i=1}^{N} |\zeta_i|^2$ vanishing on the exceptional set of the projective space, then the singularities of our kernels are located on the exceptional set of X through the correspondence (3.9) between the projective variables of $\mathbb{P}^{N-1}$ and the homogeneous coordinates of X.

3.4.1 The Projection kernel

We will now derive an explicit form of the Projection kernel

$$\mathcal{P} = \gamma \neg (H \wedge \alpha^{2+2n-N})$$

in Theorem 3.3.2 with respect to the the toric variables $(t, \tau) \in T \times T$. Thus, we present the contributing terms in the Projection kernel with respect to the projective variables $(z_1, \ldots, z_N, \zeta_1, \ldots, \zeta_N)$ of $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$ and hence their toric analogues by parameterizing the projective variables into toric variables through the rules (3.33) and (3.34).

We begin with the term of bidegree $(1,0)$ of the Hefer form $H^j(\zeta, z)$ in (3.40). Recall that it corresponds in $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$ to each polynomial

$$f_j = \chi^{m_{j1}}\chi^{m_{j2}} - \chi^{m_{j3}}\chi^{m_{j4}} = z_{j1}z_{j2} - z_{j3}z_{j4}$$

in (3.3) for $j = 1, \ldots, N-n-1$. In view of (3.40),

$$\begin{aligned}
\left(H^j(z,\zeta)\right)_{1,0} \;=\; \frac{1}{4\pi i}\Bigg[& (z_{j2}d\zeta_{j1} + z_{j1}d\zeta_{j2} - z_{j4}d\zeta_{j3} - z_{j3}d\zeta_{j4}) \\
& -(z_{j2}\zeta_{j1} + z_{j1}\zeta_{j2} - z_{j4}\zeta_{j3} - z_{j3}\zeta_{j4})\frac{\bar{\zeta}\cdot d\zeta}{|\zeta|^2} \\
& +\alpha_{0,0}(\zeta_{j2}d\zeta_{j1} + \zeta_{j1}d\zeta_{j2} - \zeta_{j4}d\zeta_{j3} - \zeta_{j3}d\zeta_{j4}) \\
& -\alpha_{0,0}(\zeta_{j2}\zeta_{j1} + \zeta_{j1}\zeta_{j2} - \zeta_{j4}\zeta_{j3} - \zeta_{j3}\zeta_{j4})\frac{\bar{\zeta}\cdot d\zeta}{|\zeta|^2}\Bigg],
\end{aligned} \tag{3.56}$$

where $\alpha_{0,0} = \bar{\zeta}\cdot z/|\zeta|^2$. Since $f_j(\zeta) = \zeta_{j1}\zeta_{j2} - \zeta_{j3}\zeta_{j4} = 0$ on X, the last two terms of the preceding expression vanish and

$$\begin{aligned}
\left(H^j(z,\zeta)\right)_{1,0} \;=\; \frac{1}{4\pi i}\Bigg[& (z_{j2}d\zeta_{j1} + z_{j1}d\zeta_{j2} - z_{j4}d\zeta_{j3} - z_{j3}d\zeta_{j4}) \\
& -(z_{j2}\zeta_{j1} + z_{j1}\zeta_{j2} - z_{j4}\zeta_{j3} - z_{j3}\zeta_{j4})\frac{\bar{\zeta}\cdot d\zeta}{|\zeta|^2}\Bigg].
\end{aligned} \tag{3.57}$$

On the other hand, every vector field $\gamma_k = \frac{2\pi i}{\zeta_{k1}}\frac{\partial}{\partial \zeta_{k2}}$ acts by interior multiplication on forms following the rule

$$\frac{\partial}{\partial \zeta_i} \neg d\zeta_j = \delta_{ij},$$

on $\bigcap_{i=1}^{d}\{\zeta \in U_1 : f_i(\zeta) = 0\}$ ($U_1 = \{\zeta \in \mathbb{P}^{N-1} : \zeta_1 = \zeta_{k1} \neq 0\}$) that corresponds to the chart $U_\sigma = \{\prod_{k=1}^{d}\eta_k^{a_k} \neq 0\}$ of X, according to (3.9). Thus, if we let the vector field $\gamma_k = \frac{2\pi i}{\zeta_{k1}}\frac{\partial}{\partial \zeta_{k2}}$

act on the Hefer form $\left(H^j(z,\zeta)\right)_{1,0}$ in (3.56), then one can observe that

$$
\begin{aligned}
&\gamma_k \neg (H^j)_{1,0}(\zeta, z) \\
&= \frac{2\pi i}{4\pi i \zeta_{k1}} \left[(z_{j2} + \alpha_{0,0}\zeta_{j2}) \left(-\frac{\bar{\zeta}_{k2}}{|\zeta|^2}\zeta_{j1} \right) + (z_{j1} + \alpha_{0,0}\zeta_{j1}) \left(\delta_k^j - \frac{\bar{\zeta}_{k2}}{|\zeta|^2}\zeta_{j2} \right) \right. \\
&\qquad \left. -(z_{j4} + \alpha_{0,0}\zeta_{j4}) \left(-\frac{\bar{\zeta}_{k2}}{|\zeta|^2}\zeta_{j3} \right) - (z_{j3} + \alpha_{0,0}\zeta_{j3}) \left(-\frac{\bar{\zeta}_{k2}}{|\zeta|^2}\zeta_{j4} \right) \right] \\
&= -\frac{\bar{\zeta}_{k2}}{2\zeta_{k1}|\zeta|^2} \left[(z_{j2}\zeta_{j1} + z_{j1}\zeta_{j2} - z_{j4}\zeta_{j3} - z_{j3}\zeta_{j4}) \right. \\
&\qquad \left. +\alpha_{0,0}(\zeta_{j2}\zeta_{j1} + \zeta_{j1}\zeta_{j2} - \zeta_{j4}\zeta_{j3} - \zeta_{j3}\zeta_{j4}) \right] + \frac{(z_{j1} + \alpha_{0,0}\zeta_{j1})}{2\zeta_{k1}}\delta_k^j.
\end{aligned}
$$

Since the polynomials f_j vanish on X, it turns out that

$$
\begin{aligned}
\gamma_k \neg (H^j)_{1,0}(\zeta, z) &= -\frac{\bar{\zeta}_{k2}}{2\zeta_{k1}|\zeta|^2}(z_{j2}\zeta_{j1} + z_{j1}\zeta_{j2} - z_{j4}\zeta_{j3} - z_{j3}\zeta_{j4}) \\
&\quad +\frac{\delta_k^j}{2\zeta_{k1}} \left(\alpha_{0,0}\zeta_{j1} + z_{j1} \right). \qquad (3.58)
\end{aligned}
$$

Similarly, the contribution of the form $(H^j)_{2,1}(\zeta,z)$(see (3.40)) in our construction is non-trivial only when the corresponding vector field γ_j acts on it. This follows from the fact that

$$
\begin{aligned}
(H^j)_{2,1}(\zeta, z) &= \frac{\alpha_{1,1}}{4\pi i} \left(\zeta_{j2}d\zeta_{j1} + \zeta_{j1}d\zeta_{j2} - \zeta_{j4}d\zeta_{j3} - \zeta_{j3}d\zeta_{j4} \right) \\
&\quad -\frac{\alpha_{1,1}}{4\pi i} \left(\zeta_{j2}\zeta_{j1} + \zeta_{j1}\zeta_{j2} - \zeta_{j4}\zeta_{j3} - \zeta_{j3}\zeta_{j4} \right) \frac{\bar{\zeta}\cdot d\zeta}{|\zeta|^2} = 0 \qquad (3.59)
\end{aligned}
$$

due to the vanishing of the polynomials f_j on X. While the action of a vector field on the Hefer form $(H^j)_{2,1}$ yields

$$
\begin{aligned}
\gamma_k \neg (H^j)_{2,1} &= \frac{2\pi i}{4\pi i} \left[\alpha_{1,1}\frac{\zeta_{j2}}{\zeta_{k1}} \left(-\frac{\bar{\zeta}_{k2}}{|\zeta|^2}\zeta_{j1} \right) + \alpha_{1,1}\frac{\zeta_{j1}}{\zeta_{k1}} \left(-\frac{\bar{\zeta}_{k2}}{|\zeta|^2}\zeta_{j2} \right) \right. \\
&\qquad \left. -\alpha_{1,1}\frac{\zeta_{j4}}{\zeta_{k1}} \left(-\frac{\bar{\zeta}_{k2}}{|\zeta|^2}\zeta_{j3} \right) - \alpha_{1,1}\frac{\zeta_{j3}}{\zeta_{k1}} \left(-\frac{\bar{\zeta}_{k2}}{|\zeta|^2}\zeta_{j4} \right) \right] \\
&= -\frac{\alpha_{1,1}\bar{\zeta}_{k2}}{\zeta_{k1}|\zeta|^2} \left(\zeta_{j1}\zeta_{j2} - \zeta_{j3}\zeta_{j4} \right) = 0 \qquad (3.60)
\end{aligned}
$$

for $k \neq j$ and

$$
\begin{aligned}
\gamma_j \neg (H^j)_{2,1} &= \frac{\alpha_{1,1}}{2} - \frac{\alpha_{1,1}\bar{\zeta}_{j2}}{\zeta_{j1}|\zeta|^2} \left(\zeta_{j1}\zeta_{j2} - \zeta_{j3}\zeta_{j4} \right) \\
&= \frac{\alpha_{1,1}}{2}. \qquad (3.61)
\end{aligned}
$$

Moreover, by recalling that the term $\alpha_{1,1}$ of bidegree $(1,1)$ of the weight α (see (1.31)), is equal to

$$\alpha_{1,1} = -\frac{1}{2\pi i}\bar{\partial}(\bar{\zeta}\cdot d\zeta/|\zeta|^2),$$

it is not difficult to deduce that

$$\begin{aligned}\gamma_k \neg \alpha_{1,1} &= \frac{2\pi i}{\zeta_{k1}}\frac{\partial}{\partial \zeta_{k2}} \neg \left[-\frac{1}{2\pi i}\bar{\partial}(\bar{\zeta}\cdot d\zeta/|\zeta|^2)\right] \\ &= \frac{1}{\zeta_{k1}}\bar{\partial}\left(\frac{\bar{\zeta}_{k2}}{|\zeta|^2}\right). \end{aligned} \tag{3.62}$$

The last equality in (3.62) is due to the fact that $\partial/\partial\zeta_{k2}$ anticommutes with $\bar{\partial}$, meaning that $\frac{\partial}{\partial \zeta_{k2}}\,\bar{\partial} = -\bar{\partial}\frac{\partial}{\partial \zeta_{k2}}$.

Now, in order to express the involved forms in the initial toric variables (t,τ) in $T \times T$, recall that the variables $(z_1,\ldots,z_N,\zeta_1,\ldots,\zeta_N)$ of $\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$ can be replaced by $(t^{m_1},\ldots,t^{m_N},\tau^{m_1},\ldots,\tau^{m_N})$ of $T\times T$, taking into account that $t^{m_i} = t_1^{m_i^1}\cdots t_n^{m_i^n}$ and $\tau^{m_i} = \tau_1^{m_i^1}\cdots\tau_n^{m_i^n}$. The existing relations between the projective and the toric variables lead to the following equalities:

$$\begin{aligned} |\zeta|^2 &= \sum_{i=1}^{N}|\zeta_i|^2 = \sum_{i=1}^{N}|\tau|^{2m_i} = P(|\tau|^2), \\ \bar{\zeta}\cdot d\zeta &= d(|\zeta|^2) = d(P(|\tau|^2)) = \sum_{a=1}^{n}\sum_{i=1}^{N} m_i^a |\tau|^{2m_i}\frac{d\tau_a}{\tau_a}, \\ d\zeta_{ji} &= \sum_{a=1}^{n} m_{ji}^a \tau^{m_{ji}}\frac{d\tau_a}{\tau_a}. \end{aligned} \tag{3.63}$$

To distinguish the forms expressed in the projective variables (z,ζ) in $\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$ from the corresponding forms with respect to toric variables, an index T is added to the exponent of the second one.

Lemma 3.4.1 *The toric analogue of the contributing forms in the Projection kernel $\mathcal{P}$ are* *given by*

$$
\begin{aligned}
(H^j)^T_{1,0}(\tau,t) &= \frac{1}{4\pi i}\sum_{a=1}^{n}\Bigg\{-\frac{\left(\tau^{m_{j1}}t^{m_{j2}}+\tau^{m_{j2}}t^{m_{j1}}-\tau^{m_{j3}}t^{m_{j4}}-\tau^{m_{j4}}t^{m_{j3}}\right)\sum_i m_i^a|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}} \\
&\quad +\left(m^a_{j1}\tau^{m_{j1}}t^{m_{j2}}+m^a_{j2}\tau^{m_{j2}}t^{m_{j1}}-m^a_{j3}\tau^{m_{j3}}t^{m_{j4}}-m^a_{j4}\tau^{m_{j4}}t^{m_{j3}}\right)\Bigg\}\frac{d\tau_a}{\tau_a}, \\
\left(\gamma_k\neg(H^j)_{1,0}\right)^T &= -\frac{\bar\tau^{m_{k2}}}{2\sum_i|\tau|^{2m_i}}\left(\tau^{m_{j1}}t^{m_{j2}}+\tau^{m_{j2}}t^{m_{j1}}-\tau^{m_{j3}}t^{m_{j4}}-\tau^{m_{j4}}t^{m_{j3}}\right) \\
&\quad +\frac{\delta^j_k}{2}(\alpha^T_{0,0}+1), \\
\left(\gamma_k\neg(H^j)_{2,1}\right)^T &= \delta^j_k\frac{\alpha^T_{1,1}}{2}, \\
\left(\gamma_k\neg\alpha_{1,1}\right)^T &= \bar\partial\left(\frac{\bar\tau^{m_{k2}}}{P(|\tau|^2)}\right)=\sum_{b=1}^{n}\left(\frac{m^b_{k2}\bar\tau^{m_{k2}}}{\sum_i|\tau|^{2m_i}}-\frac{\bar\tau^{m_{k2}}\left(\sum_i m_i^b|\tau|^{2m_i}\right)}{\left(\sum_i|\tau|^{2m_i}\right)^2}\right)\frac{d\bar\tau_b}{\bar\tau_b}
\end{aligned}
$$

on $T\times T$, where $\alpha^T_{0,0}$ and $\alpha^T_{1,1}$ are the forms given explicitly in (3.19) *and* (3.24).

Proof. The toric Hefer form $(H^j)^T_{1,0}$ on $T\times T$ is obtained by the corresponding projective Hefer form $H^j(\zeta,z)$ on $\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$ given in (3.57), after restricting the projective variables (z,ζ) to the toric variables (t,τ) under the rules $z_i=t^{m_i}$ and $\zeta_i=\tau^{m_i}$ and using the relations (3.63).

The $(0,0)$ smooth form $\left(\gamma_k\neg(H^j)_{1,0}\right)^T$ on $T\times T$ arises from the corresponding projective form $\gamma_k\neg(H^j)_{1,0}(\zeta,z)$ in (3.58) by following the same rules, where we also used that $m_{j1}=(0,\dots,0)$ and that $\alpha_{0,0}=\bar\zeta\cdot z/|\zeta|^2$ can be expressed with respect to the toric variables in the form $\alpha^T_{0,0}=P(\bar\tau\cdot t)/P(|\tau|^2)$.

Similarly, in view of (3.60) and (3.61), the restriction on the toric variables yields the toric form $(\gamma_k\neg(H^j)_{2,1})^T$ since the form

$$
\alpha_{1,1}=-\frac{1}{2\pi i}\bar\partial(\bar\zeta\cdot d\zeta/|\zeta|^2)
$$

on $\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$ is transformed into the form on $T\times T$

$$
\begin{aligned}
\alpha^T_{1,1} &= -\frac{1}{2\pi i}\partial\bar\partial P(|\tau|^2) \\
&= -\frac{1}{2\pi i}\sum_{b=1}^{n}\sum_{a=1}^{n}\left[\frac{\sum_{i=1}^{N}m_i^a m_i^b|\tau|^{2m_i}}{\sum_{i=1}^{N}|\tau|^{2m_i}}-\frac{\left(\sum_{i=1}^{N}m_i^a|\tau|^{2m_i}\right)\left(\sum_{i=1}^{N}m_i^b|\tau|^{2m_i}\right)}{\left(\sum_{i=1}^{N}|\tau|^{2m_i}\right)^2}\right]\frac{d\bar\tau_b}{\bar\tau_b}\wedge\frac{d\tau_a}{\tau_a},
\end{aligned}
$$

by using the derivation laws (3.20) and (3.21) of toric variables, leading to the toric weight $\alpha^T_{1,1}$ (see (3.24)), which was introduced in Section 3.2.

At last, $\gamma_k\neg\alpha_{1,1}$ in (3.62) is transformed to $\left(\gamma_k\neg\alpha_{1,1}\right)^T$, after passing to the toric variables

$(t, \tau) \in T \times T$. $\square$

Proposition 3.4.1 *The vectors from the collection of elements $\{m_1, \ldots, m_N\}$ generating the toric variety and contribute to the coefficient of the form $d\bar{\tau}_b$ in $(\gamma_k \neg \alpha_{1,1})^T$, for a fixed b, are those which have different b-coordinate with the vector m_{k2}.*

Proof. It is a straight forward result since

$$(\gamma_k \neg \alpha_{1,1})^T = \frac{\bar{\tau}^{m_{k2}}}{\left(\sum_i |\tau|^{2m_i}\right)^2} \sum_{b=1}^{n} \left[\sum_{i=1}^{N} \left(m_{k2}^b - m_i^b \right) |\tau|^{2m_i} \right] \frac{d\bar{\tau}_b}{\bar{\tau}_b}, \tag{3.64}$$

according to the Proposition 3.4.1. For a fixed b, the coefficient of the form $\frac{d\bar{\tau}_b}{\bar{\tau}_b}$ is $\sum_{i=1}^{N} \left(m_{k2}^b - m_i^b \right) |\tau|^{2m_i}$ and each term of the above summation vanishes if and only if $m_{k2}^b = m_i^b$. Then, the result is trivial. $\square$

Remark 3.4.1 The contributing forms in the toric Koppelman formula can be written with respect to the homogeneous coordinates $(h_1, \ldots, h_d, \eta_1, \ldots, \eta_d)$ of $X \times X$ on the chart $U_\sigma = \left\{ \prod_{k=1}^{d} \eta_k^{a_k} \neq 0 \right\}$ of X by using the rules (3.52), namely

$$t^{m_i} = \prod_{k=1}^{d} h_k^{\langle m_i, v_k \rangle} \quad \text{and} \quad \tau^{m_i} = \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle}$$

such that the forms arising are zero homogeneous as quotients of homogeneous functions of the same degree with respect to ζ and z. For example, after replacing the toric coordinates with the homogeneous coordinates of X into (3.24), the toric form $\alpha_{1,1}^T$ takes the following form that contains only the homogeneous coordinates $\zeta_1, \ldots, \zeta_d$ since $\alpha_{1,1}^T$ does not contain the variable t:

$$\begin{aligned} \alpha_{1,1}^T &= -\frac{1}{2\pi i} \sum_{b=1}^{n} \sum_{a=1}^{n} \left[\frac{\sum\limits_{i=1}^{N} m_i^a m_i^b \left(\prod\limits_{k=1}^{d} |\eta_k|^{2\langle m_i, v_k \rangle} \right)}{\sum\limits_{i=1}^{N} \left(\prod\limits_{k=1}^{d} |\eta_k|^{2\langle m_i, v_k \rangle} \right)} \right. \\ & \left. \quad - \frac{\left(\sum\limits_{i=1}^{N} m_i^a \left(\prod\limits_{k=1}^{d} |\eta_k|^{2\langle m_i, v_k \rangle} \right) \right) \left(\sum\limits_{i=1}^{N} m_i^b \left(\prod\limits_{k=1}^{d} |\eta_k|^{2\langle m_i, v_k \rangle} \right) \right)}{\left(\sum\limits_{i=1}^{N} \left(\prod\limits_{k=1}^{d} |\eta_k|^{2\langle m_i, v_k \rangle} \right) \right)^2} \right] \\ & \quad \frac{d\left(\prod\limits_{k=1}^{d} \bar{\eta}_k^{\langle e_b, v_k \rangle} \right)}{\prod\limits_{k=1}^{d} \bar{\eta}_k^{\langle e_b, v_k \rangle}} \wedge \frac{d\left(\prod\limits_{k=1}^{d} \eta_k^{\langle e_a, v_k \rangle} \right)}{\prod\limits_{k=1}^{d} \eta_k^{\langle e_a, v_k \rangle}}, \end{aligned}$$

where $e_a = (0, \ldots, \underset{\substack{\uparrow \\ a-\text{position}}}{1}, \ldots, 0)$. Similarly, one can obtain the local writing of the rest of the forms with respect to the homogeneous coordinates of X on U_σ.

Now, we are ready to write down explicitly the Projection kernel of the Koppelman toric representation formula in Theorem 3.3.2 with respect to toric variables.

Theorem 3.4.1 *The explicit form of the Projection kernel $\mathcal{P} = \gamma\neg\left(H \wedge \alpha^{2+2n-N}\right)$ defined by (3.3.2) in toric variables over $T \times T$ is*

$$
\begin{aligned}
\mathcal{P}^T \;=\;& \left(-\frac{1}{2\pi i}\right)^n \frac{n!}{2^{N-n-1}|\tau_1|^2\cdots|\tau_n|^2 P(|\tau|^2)^{n+1}} d\bar{\tau}\wedge d\tau \\
& \Bigg\{(n+1)\alpha^T_{0,0} + \sum_{k=1}^{N-n-1}\left[-\frac{\bar{\tau}^{m_{k2}}\left(\tau^{m_{k1}}t^{m_{k2}} + \tau^{m_{k2}}t^{m_{k1}} - \tau^{m_{k3}}t^{m_{k4}} - \tau^{m_{k4}}t^{m_{k3}}\right)}{\sum_i |\tau|^{2m_i}} + 1\right] \\
& \sum_{|J|=n+1} \det{}^2(A_J)|\tau|^{2m_{j_1}+\cdots+2m_{j_{n+1}}} \\
& + \sum_{k=1}^{N-n-1} \sum_{\substack{1\le p\le n\\ 1\le q\le n}} (-1)^{p-q-1}\left(m^q_{k2}\bar{\tau}^{m_{k2}} - \frac{\bar{\tau}^{m_{k2}}\sum_i m^q_i|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}}\right) \\
& \Bigg[-\frac{\left(\tau^{m_{k1}}t^{m_{k2}} + \tau^{m_{k2}}t^{m_{k1}} - \tau^{m_{k3}}t^{m_{k4}} - \tau^{m_{k4}}t^{m_{k3}}\right)\sum_i m^p_i|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}} \\
& + \left(m^p_{k1}\tau^{m_{k1}}t^{m_{k2}} + m^p_{k2}\tau^{m_{k2}}t^{m_{k1}} - m^p_{k3}\tau^{m_{k3}}t^{m_{k4}} - m^p_{k4}\tau^{m_{k4}}t^{m_{k3}}\right)\Bigg] \\
& \sum_{|J|=n}\det(A_{J,p})\det(A_{J,q})|\tau|^{2m_{j_1}+\cdots+2m_{j_n}}\Bigg\}.
\end{aligned}
\tag{3.65}
$$

Proof. Theorem 3.3.2, implies that the projection kernel $\mathcal{P}$ with respect to the projective variables $(z,\zeta)\in\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$ is given by

$$
\begin{aligned}
\mathcal{P} &= \left[\gamma\neg\left(H\wedge\alpha^{2+2n-N}\right)\right]_{n,n} \\
&= \gamma\neg\left(H\wedge\alpha^{2+2n-N}\right)_{N-1,n} \\
&= \gamma\neg\left(\bigwedge_{j=1}^{N-n-1}(H^j)_{2,1}\wedge\alpha^{2+2n-N}_{1+2n-N}\right) + \gamma\neg\left(\sum_{k=1}^{N-n-1}\bigwedge_{j=1}^{N-n-1}H^j[k]\wedge\alpha^{2+2n-N}_{2+2n-N}\right)
\end{aligned}
\tag{3.66}
$$

where $\gamma = \gamma_{N-n-1}\wedge\ldots\wedge\gamma_1$ and $\bigwedge_{j=1}^{N-n-1} H^j[k]$ is the wedge product of the Hefer forms $(H^j)_{2,1}$ for $1\le j\le N-n-1$ such that the form $(H^k)_{2,1}$ is replaced by the corresponding one $(H^k)_{1,0}$, for some $1\le k\le N-n-1$. These are the contributing terms in $\mathcal{P}$ due to the dimension n of the toric variety and the dimension $N-1$ of the projective space.

Let us denote the first term of (3.66) by $\mathcal{P}_1$ and the second one by $\mathcal{P}_2$. Since the contribution of each $(H^j)_{2,1}$ is non-trivial, when the corresponding vector field γ_j acts on it (see (3.59), (3.60) and (3.61)), we get

$$\begin{aligned}
\mathcal{P}_1 &= \gamma\neg\left(\bigwedge_{j=1}^{N-n-1}(H^j)_{2,1}\wedge\alpha_{1+2n-N}^{2+2n-N}\right)\\
&= \bigwedge_{j=1}^{N-n-1}\gamma_j\neg(H^j)_{2,1}\wedge\alpha_{1+2n-N}^{2+2n-N}\\
&= \left(\frac{\alpha_{1,1}}{2}\right)^{N-n-1}\wedge\binom{2+2n-N}{1+2n-N}(\alpha_{0,0})(\alpha_{1,1})^{1+2n-N}\\
&= \frac{(2+2n-N)}{2^{N-n-1}}\,\alpha_{0,0}(\alpha_{1,1})^n,
\end{aligned}$$

with respect to the projective variables in $\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$. While by passing to the toric variables in $T\times T$ through (3.33) and (3.34), we get

$$\mathcal{P}_1^T = \frac{(2+2n-N)}{2^{N-n-1}}\,\alpha_{0,0}^T(\alpha_{1,1}^T)^n, \tag{3.67}$$

where $\alpha_{0,0}^T=\frac{P(\bar{\tau}\cdot t)}{P(|\tau|^2)}$ and $(\alpha_{1,1}^T)^n$ are given explicitly in $T\times T$ in Proposition 3.2.1.

On the other hand, $\mathcal{P}_2$, which is obtained from the second term of (3.66), describes the geometry of the variety X since the polynomials in (3.3) induce terms in the structure of it by the contribution of the Hefer forms of bidegree $(1,0)$. More precisely, $\mathcal{P}_2$ equals to

$$\begin{aligned}
\mathcal{P}_2 &= \gamma\neg\left(\sum_{k=1}^{N-n-1}\bigwedge_{j=1}^{N-n-1}H^j[k]\wedge\alpha_{2+2n-N}^{2+2n-N}\right)\\
&= \sum_{k=1}^{N-n-1}\gamma_k\neg(H^k)_{1,0}\bigwedge_{\substack{j=1\\ j\neq k}}^{N-n-1}\gamma_j\neg(H^j)_{2,1}\wedge\alpha_{2+2n-N}^{2+2n-N}\\
&\quad-\sum_{k=1}^{N-n-1}(H^k)_{1,0}\bigwedge_{\substack{j=1\\ j\neq k}}^{N-n-1}\gamma_j\neg(H^j)_{2,1}\wedge\gamma_k\neg\alpha_{2+2n-N}^{2+2n-N}\\
&\quad+\sum_{j>k}(H^k)_{1,0}\wedge\gamma_j\neg(\gamma_k\neg(H^j)_{2,1})\wedge\bigwedge_{\substack{p=1\\ p\neq k,j}}^{N-n-1}\gamma_p\neg(H^p)_{2,1}\wedge\alpha_{2+2n-N}^{2+2n-N}\\
&\quad+\sum_{j<k}\gamma_k\neg(\gamma_j\neg(H^j)_{2,1})\wedge(H^k)_{1,0}\wedge\bigwedge_{\substack{p=1\\ p\neq k,j}}^{N-n-1}\gamma_p\neg(H^p)_{2,1}\wedge\alpha_{2+2n-N}^{2+2n-N},
\end{aligned}$$

since the $(N-n-2)$ of the $(N-n-1)$ vector fields γ_j act on the corresponding Hefer forms $(H^j)_{2,1}$. On the other hand, the vector field γ_k provides a non trivial either if it acts on $\alpha_{1,1}$ or on $(H^k)_{1,0}$. Now,

$$\gamma_k\neg\alpha_{2+2n-N}^{2+2n-N}=\gamma_k\neg(\alpha_{1,1})^{2+2n-N}=(2+2n-N)(\alpha_{1,1})^{1+2n-N}\gamma_k\neg\alpha_{1,1},$$

therefore applying the relation (3.61), we get that

$$
\begin{aligned}
\mathcal{P}_2 \;=\; & \sum_{k=1}^{N-n-1} \gamma_k \neg (H^k)_{1,0} \wedge \left(\frac{\alpha_{1,1}}{2}\right)^{N-n-2} \wedge (\alpha_{1,1})^{2+2n-N} \\
& - \sum_{k=1}^{N-n-1} (H^k)_{1,0} \wedge \left(\frac{\alpha_{1,1}}{2}\right)^{N-n-2} \wedge (2+2n-N)(\alpha_{1,1})^{1+2n-N} \wedge \gamma_k \wedge \alpha_{1,1} \\
& + \sum_{j>k} (H^k)_{1,0} \wedge \left(-\gamma_k \neg \frac{\alpha_{1,1}}{2}\right) \wedge \left(\frac{\alpha_{1,1}}{2}\right)^{N-n-3} \wedge (\alpha_{1,1})^{2+2n-N} \\
& + \sum_{j<k} \left(\gamma_k \neg \frac{\alpha_{1,1}}{2}\right) \wedge (H^k)_{1,0} \wedge \left(\frac{\alpha_{1,1}}{2}\right)^{N-n-3} \wedge (\alpha_{1,1})^{2+2n-N},
\end{aligned}
$$

or, equivalently

$$
\begin{aligned}
\mathcal{P}_2 \;=\; & \frac{1}{2^{N-n-2}} \left(\sum_{k=1}^{N-n-1} \gamma_k \neg (H^k)_{1,0}\right) (\alpha_{1,1})^n \\
& - \left(\frac{2+2n-N}{2^{N-n-2}} + \frac{N-n-2}{2^{N-n-2}}\right) \left(\sum_{k=1}^{N-n-1} (H^k)_{1,0} \wedge \gamma_k \neg \alpha_{1,1}\right) (\alpha_{1,1})^{n-1} \\
=\; & \frac{1}{2^{N-n-2}} \left(\sum_{k=1}^{N-n-1} \gamma_k \neg (H^k)_{1,0}\right) (\alpha_{1,1})^n \\
& - \frac{n}{2^{N-n-2}} \left(\sum_{k=1}^{N-n-1} (H^k)_{1,0} \wedge \gamma_k \neg \alpha_{1,1}\right) (\alpha_{1,1})^{n-1}
\end{aligned}
$$

with respect to the projective variables in $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. Hence, we pass to the toric variables (t, τ) in $T \times T$ by applying Proposition 3.4.1 and using the relation (3.32). Thus, the toric analogue $\mathcal{P}_2^T$ of $\mathcal{P}_2$ can be rewritten in the form

$$
\begin{aligned}
\mathcal{P}_2^T \;=\; & \frac{(\alpha_{1,1}^T)^n}{2^{N-n-1}} \sum_{k=1}^{N-n-1} \left[-\frac{\bar{\tau}^{m_{k2}} \left(\tau^{m_{k1}} t^{m_{k2}} + \tau^{m_{k2}} t^{m_{k1}} - \tau^{m_{k3}} t^{m_{k4}} - \tau^{m_{k4}} t^{m_{k3}}\right)}{\sum_i |\tau|^{2m_i}} + \alpha_{0,0}^T + 1 \right] \\
& - \frac{n}{2^{N-n-2}} \sum_{k=1}^{N-n-1} \sum_{\substack{1 \leq p \leq n \\ 1 \leq q \leq n}} \\
& \frac{1}{4\pi i} \left\{ -\frac{\left(\tau^{m_{k1}} t^{m_{k2}} + \tau^{m_{k2}} t^{m_{k1}} - \tau^{m_{k3}} t^{m_{k4}} - \tau^{m_{k4}} t^{m_{k3}}\right) \sum_i m_i^p |\tau|^{2m_i}}{\sum_i |\tau|^{2m_i}} \right. \\
& \left. + \left(m_{k1}^p \tau^{m_{k1}} t^{m_{k2}} + m_{k2}^p \tau^{m_{k2}} t^{m_{k1}} - m_{k3}^p \tau^{m_{k3}} t^{m_{k4}} - m_{k4}^p \tau^{m_{k4}} t^{m_{k3}}\right) \right\} \frac{d\tau_p}{\tau_p} \\
& \wedge \left(\frac{m_{k2}^q \bar{\tau}^{m_{k2}}}{\sum_i |\tau|^{2m_i}} - \frac{\bar{\tau}^{m_{k2}} \left(\sum_i m_i^q |\tau|^{2m_i}\right)}{\left(\sum_i |\tau|^{2m_i}\right)^2} \right) \frac{d\bar{\tau}_q}{\bar{\tau}_q} \\
& \wedge \left(-\frac{1}{2\pi i}\right)^{n-1} \frac{(n-1)!}{P(|\tau|^2)^n} \sum_{|J|=n} \frac{\det(A_{J,p}) \det(A_{J,q}) |\tau|^{2m_{j_1} + \cdots + 2m_{j_n}}}{\tau[p] \overline{\tau[q]}} d\bar{\tau}[q] \wedge d\tau[p],
\end{aligned}
$$

since only the differentials $d\tau_p$ of $(H^k)_{1,0}^T$ and $d\bar{\tau}_q$ of $(\gamma_k \neg \alpha_{11})^T$ provide a non-trivial contribution when they are multiplied by $(\alpha_{1,1}^T)^{n-1}$. Now, in order to put the differentials $d\tau_p$ and

$d\bar{\tau}_q$ into the form $d\bar{\tau}[q] \wedge d\tau[p]$ with correct order, a sign $(-1)^{p-q-1}$ appears. Hence, in view of Proposition 3.2.1, a substitution of the form $(\alpha_{1,1}^T)^n$ into the above expression yields

$$
\begin{aligned}
\mathcal{P}_2^T \quad = \quad & \left(-\frac{1}{2\pi i}\right)^n \frac{n!}{2^{N-n-1}|\tau_1|^2 \cdots |\tau_n|^2 P(|\tau|^2)^{n+1}} d\bar{\tau} \wedge d\tau \\
& \sum_{k=1}^{N-n-1} \left[-\frac{\bar{\tau}^{m_{k2}} \left(\tau^{m_{k1}} t^{m_{k2}} + \tau^{m_{k2}} t^{m_{k1}} - \tau^{m_{k3}} t^{m_{k4}} - \tau^{m_{k4}} t^{m_{k3}}\right)}{\sum_i |\tau|^{2m_i}} + \alpha_{0,0}^T + 1 \right] \\
& \sum_{|J|=n+1} \det{}^2(A_J) |\tau|^{2m_{j_1} + \cdots + 2m_{j_{n+1}}} \\
+ \quad & \left(-\frac{1}{2\pi i}\right)^n \frac{n!}{2^{N-n-1}|\tau_1|^2 \cdots |\tau_n|^2 P(|\tau|^2)^{n+1}} d\bar{\tau} \wedge d\tau \\
& \sum_{k=1}^{N-n-1} \sum_{\substack{1 \le p \le n \\ 1 \le q \le n}} (-1)^{p-q-1} \left(m_{k2}^q \bar{\tau}^{m_{k2}} - \frac{\bar{\tau}^{m_{k2}} \sum_i m_i^q |\tau|^{2m_i}}{\sum_i |\tau|^{2m_i}} \right) \\
& \Bigg\{ -\frac{\left(\tau^{m_{k1}} t^{m_{k2}} + \tau^{m_{k2}} t^{m_{k1}} - \tau^{m_{k3}} t^{m_{k4}} - \tau^{m_{k4}} t^{m_{k3}}\right) \sum_i m_i^p |\tau|^{2m_i}}{\sum_i |\tau|^{2m_i}} \\
& + \left(m_{k1}^p \tau^{m_{k1}} t^{m_{k2}} + m_{k_2}^p \tau^{m_{k2}} t^{m_{k1}} - m_{k3}^p \tau^{m_{k3}} t^{m_{k4}} - m_{k4}^p \tau^{m_{k4}} t^{m_{k3}} \right) \Bigg\} \\
& \sum_{|J|=n} \det(A_{J,p}) \det(A_{J,q}) |\tau|^{2m_{j_1} + \cdots + 2m_{j_n}}.
\end{aligned}
$$

Then, by factoring out the above form, $\mathcal{P}_2^T$ is simplified into the expression

$$
\begin{aligned}
\mathcal{P}_2^T \quad = \quad & \left(-\frac{1}{2\pi i}\right)^n \frac{n!}{2^{N-n-1}|\tau_1|^2 \cdots |\tau_n|^2 P(|\tau|^2)^{n+1}} d\bar{\tau} \wedge d\tau \\
& \sum_{k=1}^{N-n-1} \Bigg\{ \left[-\frac{\bar{\tau}^{m_{k2}} \left(\tau^{m_{k1}} t^{m_{k2}} + \tau^{m_{k2}} t^{m_{k1}} - \tau^{m_{k3}} t^{m_{k4}} - \tau^{m_{k4}} t^{m_{k3}}\right)}{\sum_i |\tau|^{2m_i}} + \alpha_{0,0}^T + 1 \right] \\
& \sum_{|J|=n+1} \det{}^2(A_J) |\tau|^{2m_{j_1} + \cdots + 2m_{j_{n+1}}} \\
& + \sum_{\substack{1 \le p \le n \\ 1 \le q \le n}} (-1)^{p-q-1} \left(m_{k2}^q \bar{\tau}^{m_{k2}} - \frac{\bar{\tau}^{m_{k2}} \sum_i m_i^q |\tau|^{2m_i}}{\sum_i |\tau|^{2m_i}} \right) \\
& \Bigg[-\frac{\left(\tau^{m_{k1}} t^{m_{k2}} + \tau^{m_{k2}} t^{m_{k1}} - \tau^{m_{k3}} t^{m_{k4}} - \tau^{m_{k4}} t^{m_{k3}}\right) \sum_i m_i^p |\tau|^{2m_i}}{\sum_i |\tau|^{2m_i}} \\
& + \left(m_{k1}^p \tau^{m_{k1}} t^{m_{k2}} + m_{k2}^p \tau^{m_{k2}} t^{m_{k1}} - m_{k3}^p \tau^{m_{k3}} t^{m_{k4}} - m_{k4}^p \tau^{m_{k4}} t^{m_{k3}} \right) \Bigg] \\
& \sum_{|J|=n} \det(A_{J,p}) \det(A_{J,q}) |\tau|^{2m_{j_1} + \cdots + 2m_{j_n}} \Bigg\}. \qquad (3.68)
\end{aligned}
$$

If we add the forms $\mathcal{P}_1^T$ and $\mathcal{P}_2^T$ given by (3.67) and 3.68) respectively, then the explicit form (3.65) of the projection kernel $\mathcal{P}^T$ is deduced.

□

3.4.2 The Koppelman kernel

The computation of the Koppelman kernel is rather complicated since the contributing terms are more numerous than those of the Projection kernel and contain the differentials $d\zeta$, $d\bar{\zeta}$ and $d\bar{z}$ with respect to the projective variables $(z,\zeta) \in \mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. This is a consequence of the existence of the form $u(z,\zeta)$ in $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$ (see (1.30)) taking part in the structure of $\mathcal{K} = \gamma \neg (H \wedge \alpha^{2+2n-n} \wedge u)$. Hence its toric analogue $\mathcal{K}^T$ has differentials of $d\tau, d\bar{\tau}$ and $d\bar{t}$. However, the fact that the singularities of this kernel are located on the exceptional set of X as in the case of the Projection kernel is deduced by the initial construction, without making the explicit computations.

Similarly to the case of the Projection kernel in the previous section, we first compute the contributing forms in $\mathcal{K}$ and the action of each vector field γ_j on them with respect to the projective variables $(z,\zeta) \in \mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. Subsequently, their toric analogues are obtained by applying the rules (3.33) and (3.34).

Let us recall the projective form v involving in the contraction of the form $u = v/\nabla_z v$ (see Section 1.3) given by

$$v = \bar{z} \cdot d\zeta - \frac{(\bar{z} \cdot \zeta)(\bar{\zeta} \cdot d\zeta)}{|\zeta|^2}, \tag{3.69}$$

for $(z,\zeta) \in \mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. Recall the action of the vector fields $\gamma_k = \frac{2\pi i}{\zeta_{k1}} \frac{\partial}{\partial \zeta_{k2}}$ for every $k = 1, \ldots, N-n-1$ holds on $\bigcap_{i=1}^{d} \{\zeta \in U_1 : f_i(\zeta) = 0\} \subset \mathbb{P}^{N-1}$ corresponding to the chart $U_\sigma = \left\{ \prod_{k=1}^{d} \eta_k^{a_k} \neq 0 \right\}$ of X. One can easily observe that the interior multiplication of $\gamma_k = \frac{2\pi i}{\zeta_{k1}} \frac{\partial}{\partial \zeta_{k2}}$ on v yields the $(0,0)$ form

$$\gamma_k \neg v = \frac{2\pi i}{\zeta_{k1}} \left[\bar{z}_{k2} - \frac{\bar{z} \cdot \zeta}{|\zeta|^2} \bar{\zeta}_{k2} \right], \tag{3.70}$$

with respect to the projective variables of $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$, while the action of the derivative $\bar{\partial}$ on v in (3.69), since $\bar{\partial}$ acts on both ζ and z variables, provides the following $(1,1)$ form:

$$\bar{\partial} v = d\bar{z} \cdot d\zeta - \frac{(\zeta \cdot d\bar{z})(\bar{\zeta} \cdot d\zeta)}{|\zeta|^2} - (\bar{z} \cdot \zeta) \bar{\partial} \left(\frac{\bar{\zeta} \cdot d\zeta}{|\zeta|^2} \right).$$

Let C be the $(1,1)$ form in $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$ expressed in the projective coordinates, defined by

$$C := d\bar{z} \cdot d\zeta - \frac{(\zeta \cdot d\bar{z})(\bar{\zeta} \cdot d\zeta)}{|\zeta|^2}. \tag{3.71}$$

One can observe that

$$\bar{\partial} v = C + 2\pi i (\bar{z} \cdot \zeta) \alpha_{1,1}. \tag{3.72}$$

in $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$, where $\alpha_{1,1}$ is the term of bidegree $(1,1)$ of the weight α in (1.31). If γ_k acts on C (see(3.71)), then

$$\gamma_k \neg C = -\frac{2\pi i}{\zeta_{k1}} \left[d\bar{z}_{k2} - \frac{\zeta \cdot d\bar{z}}{|\zeta|^2} \bar{\zeta}_{k2} \right] \tag{3.73}$$

leading to

$$\gamma_k \neg \bar{\partial} v = \gamma_k \neg C + 2\pi i (\bar{z} \cdot \zeta) \gamma_k \neg \alpha_{1,1}, \tag{3.74}$$

in view of (3.72). The form $\gamma_k \neg \alpha_{1,1}$ has already computed in (3.62). Furthermore, the form $u_{i,i-1}$, that is the term of u in (1.30) having bidegree $(i, i-1)$, is given by

$$u_{i,i-1} = \frac{v \wedge (\bar{\partial} v)^{i-1}}{(\delta_z v)^i}, \tag{3.75}$$

where $\delta_z v = 2\pi i \left[|z|^2 - \frac{(\bar{z} \cdot \zeta)(\bar{\zeta} \cdot z)}{|\zeta|^2} \right]$. Now, if we let the vector field $\gamma_J = \gamma_{k_j} \wedge \ldots \wedge \gamma_{k_1}$ act on $u_{i,i-1}$ for an increasing sequence of indices $1 \leq k_1 < \ldots < k_j \leq N-n-1$ with $j \leq i$, one can observe that

$$\begin{aligned} u^{[J]}_{i,i-1} \quad &:= \quad \gamma_J \neg u_{i,i-1} = \gamma_J \neg \frac{\left[v \wedge (\bar{\partial} v)^{i-1} \right]}{(\delta_z v)^i} \\ &= \quad \sum_{l=1}^{j} \frac{(-1)^{\frac{l(l-1)}{2} + \frac{(j-l)(j+l-3)}{2}} (i-1)(i-2) \cdots (i-j+1)}{(\delta_z v)^i} \\ & \qquad \gamma_{k_l} \neg v \bigwedge_{r=1 \neq l}^{j} \gamma_{k_r} \neg \bar{\partial} v \wedge \left(\bar{\partial} v \right)^{i-j} \\ &+ \quad \frac{(-1)^{\frac{j(j+1)}{2}} (i-1)(i-2) \cdots (i-j)}{(\delta_z v)^i} v \wedge \bigwedge_{r=1}^{j} \gamma_{k_r} \neg \bar{\partial} v \wedge \left(\bar{\partial} v \right)^{i-j-1} \end{aligned} \tag{3.76}$$

holds on $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$, since the j $(1,0)$-vector fields act on j of the $i-1$ projective forms $\bar{\partial} v$ (this happens if $j \leq i-1$) or one of them act on v and the remaining $j-1$ vector fields on $j-1$ of the $i-1$ forms $\bar{\partial} v$. According to (3.72), the form $\left(\bar{\partial} v \right)^{i-j}$ equals to

$$\left(\bar{\partial} v \right)^{i-j} = \sum_{k=0}^{i-j} \binom{i-j}{k} (2\pi i)^{i-j-k} (\bar{z} \cdot \zeta)^{i-j-k} C^k (\alpha_{1,1})^{i-j-k}.$$

If we set $P(n,k)$ to be the number of k-permutations of elements $\{1,2,\dots,n\}$, then

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1),$$

and one can rewrite $u_{i,i-1}^{[J]}$ in the form

$$\begin{aligned}
u_{i,i-1}^{[J]} \quad = \quad & \sum_{l=1}^{j} \frac{(-1)^{\frac{l(l-1)}{2}+\frac{(j-l)(j+l-3)}{2}} P(i-1,j-1)}{(\delta_z v)^i} \gamma_{k_l} \neg v \bigwedge_{r=1\neq l}^{j} \gamma_{k_r} \neg \bar{\partial} v \\
& \wedge \sum_{k=0}^{i-j} \binom{i-j}{k} (2\pi i)^{i-j-k} (\bar{z}\cdot\zeta)^{i-j-k} \, C^k (\alpha_{1,1})^{i-j-k} \\
+ \quad & \frac{(-1)^{\frac{j(j+1)}{2}} P(i-1,j)}{(\delta_z v)^i} v \wedge \bigwedge_{r=1}^{j} \gamma_{k_r} \neg \bar{\partial} v \\
& \wedge \sum_{k=0}^{i-j-1} \binom{i-j-1}{k} (2\pi i)^{i-j-k} (\bar{z}\cdot\zeta)^{i-j-1-k} \, C^k (\alpha_{1,1})^{i-j-1-k}. \qquad (3.77)
\end{aligned}$$

In this section, for notational convenience, we also introduce the notation

$$\bigwedge_{m=1}^{i} \gamma_{\tau(k_m)} \neg H_{1,0}^{k_m}[J] \tag{3.78}$$

that denotes the action of the wedge product of$(i-|J|)$ $(|J| \leq i)$ vector fields $\gamma_{k_i} \wedge \ldots \wedge \gamma_{k_{j+1}}$ on i Hefer forms of bidegree $(1,0)$ such that the vector fields act on the $(i-|J|)$ Hefer forms while the rest $|J|$ $(1,0)$-Hefer forms are not affected by any vector field. Arbitrary permutations of vector fields on the $(1,0)$-Hefer forms is allowed in view of (3.58). Let, also

$$\alpha_{1,1}^{[l]} = \bigwedge_{i=1}^{l} \gamma_{k_i} \neg \alpha_{1,1} \tag{3.79}$$

for an increasing sequence of indices $1 \leq k_1 < \ldots < k_l \leq N-n-1$, where each $\gamma_{k_i} \neg \alpha_{1,1}$ has been already computed in (3.62).

Lemma 3.4.2 *The toric analogues of the projective forms v, C,$\bar{\partial} v$ and $u_{i,i-1}$ on $T \times T$ are* *given by*

$$\begin{aligned}
v^T &= \sum_{a=1}^{n}\left[\sum_{i=1}^{N} m_i^a \bar{t}^{m_i}\tau^{m_i} - \frac{\left(\sum\limits_{i=1}^{N}\bar{t}^{m_i}\tau^{m_i}\right)\left(\sum\limits_{i=1}^{N} m_i^a|\tau|^{2m_i}\right)}{\sum\limits_{i=1}^{N}|\tau|^{2m_i}}\right]\frac{d\tau_a}{\tau_a}\\
C^T &= \sum_{b=1}^{n}\sum_{a=1}^{n}\left[\sum_{i=1}^{N} m_i^a m_i^b \bar{t}^{m_i}\tau^{m_i} - \frac{\left(\sum\limits_{i=1}^{N} m_i^b\bar{t}^{m_i}\tau^{m_i}\right)\left(\sum\limits_{i=1}^{N} m_i^a|\tau|^{2m_i}\right)}{\sum\limits_{i=1}^{N}|\tau|^{2m_i}}\right]\frac{d\bar{t}_b}{\bar{t}_b}\wedge\frac{d\tau_a}{\tau_a}\\
(\bar{\partial}v)^T &= C^T + 2\pi i\left(\sum_{i=1}^{N}\bar{t}^{m_i}\tau^{m_i}\right)\alpha_{1,1}^T,
\end{aligned}$$

where $\alpha_{1,1}^T$ *is given explicitly in* (3.24). *Moreover, if*

$$(\delta_z v)^T = 2\pi i\left[\sum_{i=1}^{N}|t|^{2m_i} - \frac{\left(\sum\limits_{i=1}^{N}\bar{t}^{m_i}\tau^{m_i}\right)\left(\sum\limits_{i=1}^{N}t^{m_i}\bar{\tau}^{m_i}\right)}{\sum\limits_{i=1}^{N}|\tau|^{2m_i}}\right],$$

then

$$u_{i,i-1}^T = \frac{v^T\wedge\left((\bar{\partial}v)^T\right)^{i-1}}{\left((\delta_z v)^T\right)^i} = \frac{v^T\wedge\sum_{k=0}^{i-1}\binom{i-1}{k}(C^T)^k\left[2\pi i\left(\sum_i \bar{t}^{m_i}\tau^{m_i}\right)\alpha_{1,1}^T\right]^{i-1-k}}{\left((\delta_z v)^T\right)^i}.$$

Proof. By restricting the coordinates from projective $(z,\zeta)\in\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$ to toric variables $(t,\tau)\in T\times T$ according to the rules defined in (3.33) and (3.34), the results arise from from the forms v, C, $\bar{\partial}v$ and $u_{i,i-1}$ given explicitly with respect to the projective variables in (3.69), (3.71),(3.72) and (3.75), respectively. We also use the relations (3.63). □

Proposition 3.4.2 *The contributing vectors for the coefficient of the form* $d\tau_a$ *in* v^T *given in Lemma* 3.4.2*, are the pairs of vectors from the collection of elements* $\{m_1,\ldots,m_N\}$ *that have different components in a-th coordinate.*

Proof. Notice that the $(1,0)$ form v^T is rewritten equivalently on $T\times T$ as

$$v^T = \frac{1}{\sum\limits_{i=1}^{N}|\tau|^{2m_i}}\sum_{a=1}^{n}\left[\sum_{\substack{1\le i,j\le n\\ i\ne j}}\bar{t}^{m_i}\tau^{m_i}|\tau|^{2m_j}\left(m_i^a - m_j^a\right)\right]\frac{d\tau_a}{\tau_a}. \tag{3.80}$$

Then, the result is trivial. □

According to the next proposition, the form C^T has a similar behavior with $\alpha_{1,1}^T$.

Proposition 3.4.3 *The contributing vectors from the collection of elements $\{m_1, \ldots, m_N\}$ in the form C^T are:*

1. *along the form component $d\bar{t}_b \wedge d\tau_a$ with $1 \le a, b \le n$. Every pair of vectors $\{m_i, m_j\}$ such that both a-th coordinate and bth -coordinate of m_i and m_j are different, the corresponding form is non-trivial.*

2. *along the form component $d\bar{t}_a \wedge d\tau_a$ for $1 \le a \le n$. Every pair of vectors $\{m_i, m_j\}$ such that the components of m_i and m_j in a-th coordinate are different, the corresponding form is non-trivial.*

Proof. The result follows from the definition of the form C^T. More precisely, the results are consequences of the fact that the coefficient of the form $\frac{d\bar{t}_b}{\bar{t}_b} \wedge \frac{d\tau_a}{\tau_a}$ for fixed a and b is also expressed by

$$\frac{1}{\sum_{i=1}^{N} |\tau|^{2m_i}} \left[\left(\sum_{i=1}^{N} m_i^a m_i^b \bar{t}^{m_i} \tau^{m_i} \right) \left(\sum_{i=1}^{N} |\tau|^{2m_i} \right) - \left(\sum_{i=1}^{N} m_i^b \bar{t}^{m_i} \tau^{m_i} \right) \left(\sum_{i=1}^{N} m_i^a |\tau|^{2m_i} \right) \right]$$

$$= \frac{1}{\sum_{i=1}^{N} |\tau|^{2m_i}} \left[\sum_{\substack{1 \le i,j \le n \\ i \neq j}} \bar{t}^{m_i} \tau^{m_i} |\tau|^{2m_j} (m_i^a - m_j^a)(m_i^b - m_j^b) \right].$$

□

The form $(C^T)^{n-k}$can be expressed with determinants, along the lines for computation of the form $(\alpha_{1,1}^T)^{n-k}$. For this purpose, we introduce the $(n+1) \times N$ matrix

$$A(\bar{t}, \bar{\tau}) = \begin{pmatrix} \bar{\tau}^{m_1} & \ldots & \bar{\tau}^{m_N} \\ m_1^1 \bar{t}^{m_1} & \ldots & m_N^1 \bar{t}^{m_N} \\ \vdots & \ldots & \vdots \\ m_1^n \bar{t}^{m_1} & \ldots & m_N^n \bar{t}^{m_N} \end{pmatrix}. \tag{3.81}$$

and denote by $A_{J(\bar{t},\bar{\tau})}^{q_1,\ldots,q_k}$ the $(n+1-k) \times (n+1-k)$ matrix constructed by $(n+1-k)$ columns of the matrix $A(\bar{t}, \bar{\tau})$ in (3.81) where $q_1+1, \ldots, q_k+1$ rows are omitted while the index $1 \le q_1 < \ldots < q_k \le n$ is increasing.

Proposition 3.4.4 *The form $(C^T)^{n-k}$ restricted on $T \times T$ equals to*

$$
(C^T)^{n-k} = \frac{(n-k)!}{\sum\limits_{i=1}^{N} |\tau|^{2m_i}}
$$

$$
\sum_{|J|=n-k+1} \sum_{\substack{1 \le p_1 < \ldots < p_k \le n \\ 1 \le q_1 < \ldots < q_k \le n}} \frac{\det(A_J^{p_1,\ldots,p_k}) \det(A_{J(\bar{t},\bar{\tau})}^{q_1,\ldots,q_k}) \tau^{m_{j_1}+\cdots+m_{j_{n-k+1}}}}{\bar{t}[q]\tau[p]} d\bar{t}[q] \wedge d\tau[p] \tag{3.82}
$$

where $d\bar{t}[q] \wedge d\tau[p]$ is resulting from deleting the differentials $d\tau_{p_1}, \ldots, d\tau_{p_k}$ and $d\bar{t}_{q_1}, \ldots, d\bar{t}_{q_k}$ in $d\bar{t} \wedge d\tau$ and writing the remaining terms of $d\bar{t}$ and $d\tau$ in increasing order of indices, alternatively.

Proof. Similarly to the computation of the form $(\alpha_{1,1}^T)^{n-k}$, we get that $(C^T)^{n-k}$ is the product of $(n-k)!$ with

$$
\sum_{\substack{1 \le p_1 < \ldots < p_k \le n \\ 1 \le q_1 < \ldots < q_k \le n}} \left| \begin{matrix} 1 & \sum_i m_i^1 |\tau|^{2m_i} & \ldots & \sum_i m_i^n |\tau|^{2m_i} \\ \frac{\sum_i m_i^1 \bar{t}^{m_i} \tau^{m_i}}{\sum_i |\tau|^{2m_i}} & \sum_i m_i^1 m_i^1 \bar{t}^{m_i} \tau^{m_i} & \ldots & \sum_i m_i^1 m_i^n \bar{t}^{m_i} \tau^{m_i} \\ \vdots & \ldots & \ldots & \vdots \\ \frac{\sum_i m_i^n \bar{t}^{m_i} \tau^{m_i}}{\sum_i |\tau|^{2m_i}} & \sum_i m_i^n m_i^1 \bar{t}^{m_i} \tau^{m_i} & \ldots & \sum_i m_i^n m_i^n \bar{t}^{m_i} \tau^{m_i} \end{matrix} \right|_{\substack{p_1,\ldots,p_k \\ q_1,\ldots,q_k}} \frac{d\bar{t}[q] \wedge d\tau[p]}{\bar{t}[q]\tau[p]}.
$$

Hence,

$$
(C^T)^{n-k} = \frac{(n-k)!}{\sum_i |\tau|^{2m_i}} \times
$$

$$
\sum_{\substack{1 \le p_1 < \ldots < p_k \le n \\ 1 \le q_1 < \ldots < q_k \le n}} \left| \begin{matrix} \sum_i |\tau|^{2m_i} & \sum_i m_i^1 |\tau|^{2m_i} & \ldots & \sum_i m_i^n |\tau|^{2m_i} \\ \sum_i m_i^1 \bar{t}^{m_i} \tau^{m_i} & \sum_i m_i^1 m_i^1 \bar{t}^{m_i} \tau^{m_i} & \ldots & \sum_i m_i^1 m_i^n \bar{t}^{m_i} \tau^{m_i} \\ \vdots & \ldots & \ldots & \vdots \\ \sum_i m_i^n \bar{t}^{m_i} \tau^{m_i} & \sum_i m_i^n m_i^1 \bar{t}^{m_i} \tau^{m_i} & \ldots & \sum_i m_i^n m_i^n \bar{t}^{m_i} \tau^{m_i} \end{matrix} \right|_{\substack{p_1,\ldots,p_k \\ q_1,\ldots,q_k}} \frac{d\bar{t}[q] \wedge d\tau[p]}{\bar{t}[q]\tau[p]}.
$$

The Cauchy-Binet formula yields that the above determinant can be rewritten as

$$
\sum_{|J|=n-k+1} \left| \begin{matrix} \bar{\tau}^{m_{j_1}} & \ldots & \bar{\tau}^{m_{j_{n+1}}} \\ m_{j_1}^1 \bar{t}^{m_{j_1}} & \ldots & m_{j_{n+1}}^1 \bar{t}^{m_{j_{n+1}}} \\ \vdots & \ldots & \vdots \\ m_{j_1}^n \bar{t}^{m_{j_1}} & \ldots & m_{j_{n+1}}^n \bar{t}^{m_{j_{n+1}}} \end{matrix} \right|_{q_1,\ldots,q_k} \left| \begin{matrix} \tau^{m_{j_1}} & \ldots & m_{j_1}^n \tau^{m_{j_1}} \\ \vdots & \ldots & \vdots \\ \vdots & \ldots & \vdots \\ \tau^{m_{j_{n+1}}} & \ldots & m_{j_{n+1}}^n \tau^{m_{j_{n+1}}} \end{matrix} \right|_{p_1,\ldots,p_k}
$$

$$= \sum_{|J|=n-k+1} \tau^{m_{j_1}+\dots m_{j_{n-k+1}}} \begin{vmatrix} \bar{\tau}^{m_{j_1}} & \dots & \bar{\tau}^{m_{j_{n+1}}} \\ m^1_{j_1}\bar{t}^{m_{j_1}} & \dots & m^1_{j_{n+1}}\bar{t}^{m_{j_{n+1}}} \\ \vdots & \dots & \vdots \\ m^n_{j_1}\bar{t}^{m_{j_1}} & \dots & m^n_{j_{n+1}}\bar{t}^{m_{j_{n+1}}} \end{vmatrix}_{q_1,\dots,q_k} \begin{vmatrix} 1 & m^1_{j_1} & \dots & m^n_{j_1} \\ \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots \\ 1 & m^1_{j_{n+1}} & \dots & m^n_{j_{n+1}} \end{vmatrix}_{p_1,\dots,p_k}$$

and the result follows. □

Lemma 3.4.3 *If the vector field $\gamma_k = 2\pi i/\zeta_{k1}\partial/\partial\zeta_{k2}$ acts on the projective forms v, C and $\bar{\partial}v$ by interior multiplication, then the restriction of the variables of the derived forms with respect to toric variables on $T \times T$ yields*

$$\begin{aligned} (\gamma_k \neg v)^T &= 2\pi i \left[\bar{t}^{m_{k2}} - \frac{\bar{\tau}^{m_{k2}}\left(\sum_i \bar{t}^{m_i}\tau^{m_i}\right)}{\sum_i |\tau|^{2m_i}}\right], \\ (\gamma_k \neg C)^T &= -2\pi i \sum_{b=1}^{n} \left[m^b_{k2}\bar{t}^{m_{k2}} - \frac{\bar{\tau}^{m_{k2}}\left(\sum_i m^b_i \bar{t}^{m_i}\tau^{m_i}\right)}{\sum_i |\tau|^{2m_i}}\right] \frac{d\bar{t}_b}{\bar{t}_b}, \\ (\gamma_k \neg \bar{\partial}v)^T &= (\gamma_k \neg C)^T + 2\pi i \left(\sum_i \bar{t}^{m_i}\tau^{m_i}\right)(\gamma_k \neg \alpha_{1,1})^T, \end{aligned}$$

where $(\gamma_k \neg \alpha_{1,1})^T$ is given in Proposition 3.4.1. *Moreover, the toric analogue of* (3.77) *on $T \times T$ is given by*

$$\begin{aligned} \left(u^{[J]}_{i,i-1}\right)^T &= \sum_{l=1}^{j} \frac{(-1)^{\frac{l(l-1)}{2}+\frac{(j-l)(j+l-3)}{2}} P(i-1,j-1)}{((\delta_z v)^T)^i} (\gamma_{k_l} \neg v)^T \left(\bigwedge_{r=1\neq l}^{j} \gamma_{k_r} \neg \bar{\partial}v\right)^T \\ &\quad \wedge \sum_{k=0}^{i-j} \binom{i-j}{k} (2\pi i)^{i-j-k} \left(\sum_{i=1}^{N} \bar{t}^{m_i}\tau^{m_i}\right)^{i-j-k} (C^T)^k (\alpha^T_{1,1})^{i-j-k} \\ &+ \frac{(-1)^{\frac{j(j+1)}{2}} P(i-1,j)}{((\delta_z v)^T)^i} v^T \wedge \left(\bigwedge_{r=1}^{j} \gamma_{k_r} \neg \bar{\partial}v\right)^T \\ &\quad \wedge \sum_{k=0}^{i-j-1} \binom{i-j-1}{k} (2\pi i)^{i-j-k} \left(\sum_{i=1}^{N} \bar{t}^{m_i}\tau^{m_i}\right)^{i-j-1-k} (C^T)^k (\alpha^T_{1,1})^{i-j-1-k}. \end{aligned}$$

Proof. The forms are derived from the corresponding projective forms (3.70), (3.73), (3.74) and (3.77) through the parameterizations $z_j = t^{m_j}$ and $\zeta_j = \tau^{m_j}$ for each $j = 1, \dots, N$ where $(z, \zeta) \in \mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. □

Proposition 3.4.5 *The contributing vectors in $(\gamma_k \neg v)^T$ are the pairs of vectors $\{m_{k2}, m_j\}$ $(j \neq k2)$ for $j = 1, \dots, N$ while in the coefficient of $d\bar{t}_b$ of the form $(\gamma_k \neg C)^T$ are the same pairs of vectors which satisfy the extra condition that the b-th coordinates of these vectors are* *not both zero.*

Proof. It is an immediate consequence of the fact that $(\gamma_k \neg v)^T$ can be written, equivalently, in the form

$$(\gamma_k \neg v)^T = \frac{2\pi i}{\sum_{i=1}^N |\tau|^{2m_i}} \left[\sum_{j=1}^N \tau^{m_j} \left(\bar{t}^{m_{k2}} \bar{\tau}^{m_j} - \bar{\tau}^{m_{k2}} \bar{t}^{m_j} \right) \right], \tag{3.83}$$

while $(\gamma_k \neg C)^T$ can be expressed in the form

$$(\gamma_k \neg C)^T = -\frac{2\pi i}{\sum_{i=1}^N |\tau|^{2m_i}} \sum_{b=1}^n \left[\sum_{i=1}^N \tau^{m_i} \left(m_{k2}^b \bar{t}^{m_{k2}} \bar{\tau}^{m_i} - m_i^b \bar{\tau}^{m_{k2}} \bar{t}^{m_i} \right) \right] \frac{d\bar{t}_b}{\bar{t}_b}.$$

□

The toric analogue of (3.78) and (3.79) on $T \times T$ are expressed by

$$\left(\bigwedge_{m=1}^{i} \gamma_{\tau(k_m)} \neg H_{1,0}^{k_m}[J] \right)^T = \bigwedge_{m=1}^{i} \left(\gamma_{\tau(k_m)} \neg H_{1,0}^{k_m} \right)^T [J] \tag{3.84}$$

and

$$(\alpha_{1,1}^{[l]})^T = \left(\bigwedge_{i=1}^{l} \gamma_{k_i} \neg \alpha_{1,1} \right)^T = \bigwedge_{i=1}^{l} (\gamma_{k_i} \neg \alpha_{1,1})^T, \tag{3.85}$$

respectively, where their involved forms are given in Lemma 3.4.1.

After the preparation lemmas, we are ready, now to write down the toric analogue $\mathcal{K}^T$ of the kernel $\mathcal{K}$ in the toric Koppelman formula , explicitly. In view of Remark 3.3.1, the number N of the integral points of the polyhedron P_D is at least equals to $2n$, where n is the dimension of X. If we let $\gamma = \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1$ act on the form $(H \wedge \alpha^{2+2n-N} \wedge u)$ of bidegree $(N-1, n)$, then

$$
\begin{aligned}
\mathcal{K} &= \left[\gamma\neg\left(H\wedge\alpha^{2+2n-N}\wedge u\right)\right]_{n,n-1} = \gamma\neg\left(H\wedge\alpha^{2+2n-N}\wedge u\right)_{N-1,n-1} \\
&= \gamma\neg\left(\bigwedge_{j=1}^{N-n-1} H_{2,1}^{j}\wedge\alpha_{2n-N}^{2+2n-N}\wedge u_{1,0}\right) \\
&\quad +\gamma\neg\left(\sum_{i=1}^{N-n-1}\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{j=1}^{N-n-1} H^{j}[k_1,\ldots,k_i]\wedge\alpha_{2n-N}^{2+2n-N}\wedge u_{i+1,i}\right) \\
&\quad +\gamma\neg\left(\sum_{i=1}^{N-n-1}\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{j=1}^{N-n-1} H^{j}[k_1,\ldots,k_i]\wedge\alpha_{1+2n-N}^{2+2n-N}\wedge u_{i,i-1}\right) \\
&\quad +\gamma\neg\left(\sum_{i=2}^{N-n-1}\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{j=1}^{N-n-1} H^{j}[k_1,\ldots,k_i]\wedge\alpha_{2+2n-N}^{2+2n-N}\wedge u_{i-1,i-2}\right) \\
&= \mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_3+\mathcal{K}_4,
\end{aligned}
\tag{3.86}
$$

with respect to the projective variables $(z,\zeta)\in\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$. The $\bigwedge_{j=1}^{N-n-1} H^{j}[k_1,\ldots,k_i]$ is the wedge product of $H_{2,1}^{j}$ where $H_{2,1}^{k_1},\ldots,H_{2,1}^{k_i}$ have been replaced by the corresponding $(1,0)$ forms. Some of the above terns may not appear in $\mathcal{K}$ due to the dimensions n and N.

We are going to examine each one of the four $\mathcal{K}_j$, separately. As we will see, the last three terms describing the geometry of the variety due to the contribution of the $(1,0)$ Hefer forms, are given by a similar formula while the first term, $\mathcal{K}_1$, has the simplest form and its toric analogue $\mathcal{K}_1^T$ on $T\times T$ does not contain any differentials related to the t variable.

Lemma 3.4.4 *The toric analogue $\mathcal{K}_1^T$ of the term $\mathcal{K}_1$ on $T\times T$ is*

$$
\mathcal{K}_1^T = \frac{(2+2n-N)(1+2n-N)}{2^{N-n}}(\alpha_{0,0}^T)^2(\alpha_{1,1}^T)^{n-1}\wedge u_{1,0}^T. \tag{3.87}
$$

Proof. Recall that $\gamma=\gamma_{N-n-1}\wedge\ldots\wedge\gamma_1$. Since only the form $H_{2,1}^{j}$ gives a non-trivial result when the corresponding vector field γ_j acts on it (see (3.59),(3.60), (3.61)), we get

$$
\begin{aligned}
\mathcal{K}_1 &= \gamma\neg\left(\bigwedge_{j=1}^{N-n-1} H_{2,1}^{j}\wedge\alpha_{2n-N}^{2+2n-N}\wedge u_{1,0}\right) \\
&= \bigwedge_{j=1}^{N-n-1}\gamma_j\neg H_{2,1}^{j}\wedge\binom{2+2n-N}{2n-N}(\alpha_{0,0})^2(\alpha_{1,1})^{2n-N}\wedge u_{1,0} \\
&= \left(\frac{\alpha_{1,1}}{2}\right)^{N-n-1}\wedge\binom{2+2n-N}{2n-N}(\alpha_{0,0})^2(\alpha_{1,1})^{2n-N}\wedge u_{1,0} \\
&= \frac{(2+2n-N)(1+2n-N)}{2^{N-n}}(\alpha_{0,0})^2(\alpha_{1,1})^{n-1}\wedge u_{1,0}.
\end{aligned}
$$

Hence, a change of coordinates from the projective variables $(z, \zeta) \in \mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$ to the toric variables $(t, \tau) \in T \times T$ following the rules $z_j = t^{m_j}$ and $\zeta_j = \tau^{m_j}$ for $j = 1, \ldots, N$ yields the desired formula (3.87), where $\alpha_{0,0}^T$ is given in (3.19), $u_{1,0}^T$ arises from Lemma 3.4.2 while $(\alpha_{1,1}^T)^{n-1}$ has been computed in (3.32). $\square$

Observe that neither the weight $\alpha_{1,1}^T$ nor the form $u_{1,0}^T$ have any differentials dt's or $d\bar{t}$'s and the Hefer forms induced by the vanishing polynomials f_j do not take part in the $\mathcal{K}_1$ term. In the case under study, the term $\mathcal{K}_1^T$ exists only when $N = 2n$ since $\mathcal{K}_1^T = 0$ when $N = 2n+1$ or $N = 2n+2$, in view of (3.87).

The rest of the contributing terms in the kernel $\mathcal{K}$ expressed in (3.86) are much more complicated due to the existence of Hefer forms of bidegree $(1,0)$ which do not vanish, independently of the action on them by the vector field γ_j.

At this point, we introduce the notation $\text{sign}(J-l)$, that is, a sign depends on the order of the following:

- the action of the $(i - |J|)$ vector fields (from the collection $\{\gamma_{k_1}, \ldots, \gamma_{k_i}\}$ of the i vector fields) on the $(i-|J|)$ $(1,0)$-Hefer forms (from the collection $\{H_{1,0}^{k_1}, \ldots, H_{1,0}^{k_i}\}$)
- the action of the $(|J| - l)$ vector fields (from the rest $|J|$ vector fields that do not act before) on a term the form u of a suitable bidegree
- the action of the rest l vector fields on l forms $\alpha_{1,1}$.

Hence, the last three terms of (3.86) can be simplified according to the following lemmas.

Lemma 3.4.5 *The toric analogue $\mathcal{K}_2^T$ of the second term $\mathcal{K}_2$ in (3.86) on $T \times T$ is given by*

$$\begin{aligned} \mathcal{K}_2^T \;=\; & \sum_{i=1}^{N-n-1} \frac{(2+2n-N)(1+2n-N)}{2^{N-n-i}} (\alpha_{0,0}^T)^2 \sum_{1 \le k_1 \le \ldots \le k_i \le N-n-1} \\ & \sum_{l=0}^{\min\{i, n-i-1\}} \sum_{\substack{l \le |J| \le i \\ 1 \le j_1 < \ldots < j_i \le k_i}} \text{sign}(J-l) P(n-i-1, l) \\ & \left(\bigwedge_{m=1}^{i} \gamma_{\tau(k_m)} \neg H_{1,0}^{k_m}[J] \right)^T \left(\alpha_{1,1}^{[l]} \right)^T \left(u_{i+1,i}^{[J-l]} \right)^T (\alpha_{1,1}^T)^{n-l-i-1}. \end{aligned} \tag{3.88}$$

Proof. Since

$$\begin{aligned}
\mathcal{K}_2 &:= \gamma\neg\left(\sum_{i=1}^{N-n-1}\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{j=1}^{N-n-1}H^j[k_1,\ldots,k_i]\wedge\alpha_{2n-N}^{2+2n-N}\wedge u_{i+1,i}\right)\\
&= \frac{(2+2n-N)(1+2n-N)}{2}(\alpha_{0,0})^2\\
&\quad \gamma\neg\left(\sum_{i=1}^{N-n-1}\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{j=1}^{N-n-1}H^j[k_1,\ldots,k_i]\wedge(\alpha_{1,1})^{2n-N}\wedge u_{i+1,i}\right),
\end{aligned}$$

where $\gamma=\gamma_{N-n-1}\wedge\ldots\wedge\gamma_1$, we observe that the $N-n-i-1$ Hefer forms of bidegree $(2,1)$ are modified to $\left(\frac{\alpha_{1,1}}{2}\right)^{N-n-i-1}$ after the action of the corresponding vector fields on them with no any sign appears, in view of (3.59), (3.60) and (3.61). Then, if l from the remaining i $(1,0)$-vector fields act on l from the $n-i-1$ forms $\alpha_{1,1}$ $(l\le\min\{i,n-i-1\})$, then $P(n-i-1,l)\alpha_{1,1}^{[l]}(a_{1,1})^{n-l-i-1}$ is deduced. For, $l\le|J|\le i$, the $|J|-l$ vector fields lead to the induced form $u_{i+1,1}^{[J-l]}$ by their action on the form $u_{i+1,1}$. The sum of the forms $\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{m=1}^{i}\gamma_{\tau(k_m)}\neg H_{1,0}^{k_m}[J]$ arising from the action of the last $i-|J|$ vector fields, indicates that each $(1,0)$ vector field can act on any $(1,0)$-Hefer form from the collection $\{H_{1,0}^{k_1},\ldots,H_{1,0}^{k_i}\}$ and the rest $|J|$ $(1,0)$ Hefer forms stay invariant. It turns out that $\mathcal{K}_2$ is written in the form (3.88), after passing to the toric variables. $\square$

One can observe that the term $\mathcal{K}_2^T$ is non trivial only when $N=2n$ since $N\ge 2n$ (Remark 3.3.1), as in the case of the term $\mathcal{K}_1^T$.

Lemma 3.4.6 *The toric analogue $\mathcal{K}_3^T$ of the third term $\mathcal{K}_3$ on $T\times T$ is given by*

$$\begin{aligned}
\mathcal{K}_3^T = &\sum_{i=1}^{N-n-1}\frac{2+2n-N}{2^{N-n-i-1}}\alpha_{0,0}^T\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\\
&\sum_{l=0}^{\min\{i,n-i\}}\sum_{\substack{l\le|J|\le i\\ 1\le j_1<\ldots<j_i\le k_i}}\operatorname{sign}(J-l)P(n-i,l)\\
&\left(\bigwedge_{m=1}^{i}\gamma_{\tau(k_m)}\neg H_{1,0}^{k_m}[J]\right)^T\left(\alpha_{1,1}^{[l]}\right)^T\left(u_{i,i-1}^{[J-l]}\right)^T(\alpha_{1,1}^T)^{n-l-i}.
\end{aligned}\tag{3.89}$$

Proof. The form K_3 is equal to

$$
\begin{aligned}
\mathcal{K}_3 &:= \gamma\neg\left(\sum_{i=1}^{N-n-1}\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{j=1}^{N-n-1}H^j[k_1,\ldots,k_i]\wedge\alpha_{1+2n-N}^{2+2n-N}\wedge u_{i,i-1}\right)\\
&= (2+2n-N)\alpha_{0,0}\\
&\quad\ \gamma\neg\left(\sum_{i=1}^{N-n-1}\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{j=1}^{N-n-1}H^j[k_1,\ldots,k_i]\wedge(\alpha_{1,1})^{1+2n-N}\wedge u_{i,i-1}\right),
\end{aligned}
$$

where $\gamma=\gamma_{N-n-1}\wedge\ldots\wedge\gamma_1$. After letting the suitable $N-n-i-1$ vector fields act on the corresponding $N-n-i-1$ $(H^j)_{2,1}$ Hefer forms for $1\le i\le N-n-i$, the form $\left(\frac{\alpha_{1,1}}{2}\right)^{N-n-i-1}$ appears while the rest i vector fields give a non trivial result by acting on any contributing form in $\mathcal{K}_3$. Namely, any l $(l\le i)$ from the i vector fields act on l terms from the $n-i$ $(l\le\min\{i,n-i\})$ forms $\alpha_{1,1}$ such that $P(n-i,l)\alpha_{1,1}^{[l]}(\alpha_{1,1})^{n-l-i}$ is obtained. The $|J|-l$ $(l\le|J|\le i)$ vector fields act on the form $u_{i,i-1}$, resulting in the $u_{i,i-1}^{[J-l]}$ (see 3.77), while the remaining $i-|J|$ vector fields act on $i-|J|$ $(1,0)$ Hefer $(1,0)$ forms (with respect to arbitrary permutations of vector fields on these Hefer forms) and leave the rest $|J|$ Hefer froms of bidegree $(1,0)$ unchanged. Then, the term $\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\bigwedge_{m=1}^{i}\gamma_{\tau(k_m)}\neg H_{1,0}^{k_m}[J]$ arises and the explicit form of $\mathcal{K}_3^T$ is deduced by passing to the toric variables through the rules (3.33) and (3.34). □

As opposed to the preceding terms comprising the kernel $\mathcal{K}^T$, the form $\mathcal{K}_3^T$ contributes in the toric Koppelman kernel $\mathcal{K}^T$ when $N=2n$ or $N=2n+1$.

Lemma 3.4.7 *The term $\mathcal{K}_4^T$ on $T\times T$ is given by*

$$
\begin{aligned}
\mathcal{K}_4^T =& \sum_{i=2}^{N-n-1}\frac{1}{2^{N-n-i-1}}\sum_{1\le k_1\le\ldots\le k_i\le N-n-1}\\
&\sum_{l=0}^{\min\{i,n-i+1\}}\ \sum_{\substack{l\le|J|\le\min\{i,l+i-1\}\\ 1\le j_1<\ldots<j_{\min\{i,l+i-1\}}\le k_i}}\operatorname{sign}(J-l)P(n-i+1,l)\\
&\left(\bigwedge_{m=1}^{i}\gamma_{\tau(k_m)}\neg H_{1,0}^{k_m}[J]\right)^T\left(\alpha_{1,1}^{[l]}\right)^T\left(u_{i-1,i-2}^{[J-l]}\right)^T(\alpha_{1,1}^T)^{n-l-i+1}.
\end{aligned}
\tag{3.90}
$$

Proof. Similar computations with Lemmas 3.4.5 and 3.4.6 lead to the explicit form of $\mathcal{K}_4^T$. □

The contribution of $\mathcal{K}_4^T$ in the construction of $\mathcal{K}^T$ is non trivial when the difference between the dimension of the projective space $\mathbb{P}^{N-1}$ and the corresponding one of the toric variety X

is at least equals to 2 or, equivalently, when N and n satisfy the inequality

$$N - n \geq 3. \tag{3.91}$$

Theorem 3.4.2 *The toric Koppelman kernel $\mathcal{K}^T$ consists of the four terms $\mathcal{K}_1^T$, $\mathcal{K}_2^T$, $\mathcal{K}_3^T$ and $\mathcal{K}_4^T$ depending on the dimensions n and N, which are described by* (3.87), (3.88), (3.89) *and* (3.90)*, respectively.*

Observe that the contribution of the Hefer $(1,0)$ forms to the terms $\mathcal{K}_2^T$, $\mathcal{K}_3^T$ and $\mathcal{K}_4^T$ induce the geometric characteristics of the toric variety coming from the vanishing polynomials f_j along this variety.

Theorem 3.4.3 *If X is an n-dimensional smooth compact projective toric variety and $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$, where $V_{\mathcal{L}}$ is the induced line bundle of an ample divisor $D = \sum_{k=1}^{d} a_k D_k$ (such that $\mathcal{L} = \mathcal{O}_X(D)$) and $|P_D \cap \mathbb{Z}^n| = N$ satisfying $N \leq 2(n+1)$ then for $t \in T$*

$$C_{N,n}\phi(t) = \int_T \mathcal{K}^T \wedge \bar{\partial}\phi + \bar{\partial}_t \int_T \mathcal{K}^T \wedge \phi + \int_T \mathcal{P}^T \wedge \phi, \tag{3.92}$$

where $\mathcal{P}^T$ and $\mathcal{K}^T$ are given explicitly in Theorems 3.4.1 *and* 3.4.2 *with respect to toric variables. Moreover, for $h \in U_\sigma = \{\prod_{k=1}^{d} h_k^{a_k} \neq 0\}$*

$$C_{N,n}\phi(h) = \int_X \mathcal{K}_{\text{hom}} \wedge \bar{\partial}\phi + \bar{\partial}_h \int_X \mathcal{K}_{\text{hom}} \wedge \phi + \int_X \mathcal{P}_{\text{hom}} \wedge \phi, \tag{3.93}$$

where $\mathcal{P}_{\text{hom}}$ and $\mathcal{K}_{\text{hom}}$ are the 'homogenization' of $\mathcal{P}^T$ and $\mathcal{K}^T$ on U_σ, respectively with respect to the homogeneous coordinates $h = (h_1, \ldots, h_d)$ and $\eta = (\eta_1, \ldots, \eta_d)$ on $X \times X$ given by

$$\mathcal{P}_{\text{hom}} = \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}} \mathcal{P}^T \quad \textit{and} \quad \mathcal{K}_{\text{hom}} = \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}} \mathcal{K}^T,$$

after expressing $\mathcal{P}^T$ and $\mathcal{K}^T$ with respect to the homogeneous coordinates through the rules $t^{m_i} = \prod_{k=1}^{d} h_k^{\langle m_i, v_k \rangle}$ and $\tau^{m_i} = \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k \rangle}$. The kernels $\mathcal{P}_{\text{hom}}$ and $\mathcal{K}_{\text{hom}}$ take values in $(V_{\mathcal{L}}^{\vee})_{[\eta]} \otimes (V_{\mathcal{L}})_{[h]}$ and their singularities are located along the exceptional set of X.

Proof. The first part of the Theorem is a direct consequence of Theorem 3.3.2 after passing to the toric variables $(t, \tau) \in T \times T$, while the involved kernels have been computed explicitly in Theorems 3.4.1 and 3.4.2, respectively.

In order to derive the formula (3.93), by using the relation connecting the toric variables on $T \times T$ with the homogeneous coordinates on $X \times X$, namely $t^{m_i} = \prod_{k=1}^{d} h_k^{\langle m_i, v_k \rangle}$ and

$\tau^{m_i} = \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k\rangle}$, the formula (3.92) can be rewritten as

$$C_{N,n}\phi_\sigma(h) = \int_X \mathcal{K}^T \wedge \bar\partial\phi_\sigma + \bar\partial_h \int_X \mathcal{K}^T \wedge \phi_\sigma + \int_X \mathcal{P}^T \wedge \phi_\sigma \tag{3.94}$$

on the chart U_σ of X, where ϕ_σ is the zero homogeneous (with respect to the homogeneous coordinates of X) $(0,q)$ form as defined in (3.5). In view of Section 3.1, the family $\{\phi_{\sigma_i}\}_i$ determines the form $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$ since on $U_{\sigma_i} \cap U_{\sigma_j}$ for every i, j, the forms ϕ_{σ_i} and ϕ_{σ_j} are compatible (see also (3.54)) while

$$\phi(\eta)\big|_{U_\sigma} = \prod_{k=1}^{d} \eta_k^{a_k}\phi_\sigma(\eta). \tag{3.95}$$

Hence, one can multiply the representation formula (3.94) by $\prod_{k=1}^{d} h_k^{a_k}$ and pass this product into the integrals of the right hand side of (3.94) (since the integrals are taken over the η variable), such that

$$C_{N,n}\prod_{k=1}^{d} h_k^{a_k}\phi_\sigma(h) = \int_X \prod_{k=1}^{d} h_k^{a_k}\mathcal{K}^T \wedge \bar\partial\phi_\sigma + \bar\partial_h \int_X \prod_{k=1}^{d} h_k^{a_k}\mathcal{K}^T \wedge \phi_\sigma + \int_X \prod_{k=1}^{d} h_k^{a_k}\mathcal{P}^T \wedge \phi_\sigma$$

or, equivalently,

$$C_{N,n}\phi(h) = \int_X \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}}\mathcal{K}^T \wedge \bar\partial\phi + \bar\partial_h \int_X \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}}\mathcal{K}^T \wedge \phi + \int_X \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}}\mathcal{P}^T \wedge \phi$$

on the chart U_σ of X. If we set

$$\mathcal{P}_{\text{hom}} = \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}}\mathcal{P}^T \quad \text{and} \quad \mathcal{K}_{\text{hom}} = \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}}\mathcal{K}^T, \tag{3.96}$$

the representation formula (3.93) is obtained.

Recall that the initial kernels $\mathcal{P}$ and $\mathcal{K}$ that we have constructed in Theorem 3.3.2 are -1-homogeneous in ζ variable and 1-homogeneous in z variable in $\mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$ due to the 'endowed' homogeneity of P and K (that can be found in a preceding discussion of Theorem 3.3.2). Hence, we passed to the toric variables from the projective ones using the parameterizations $z^{m_i} = t^{m_i}$ and $\zeta^{m_i} = \tau^{m_i}$, for every $i = 1, \ldots, N$ leading to the kernels $\mathcal{P}^T$ and $\mathcal{K}^T$. The change of coordinates $t^{m_i} = \prod_{k=1}^{d} h_k^{\langle m_i, v_k\rangle}$ and $\tau^{m_i} = \prod_{k=1}^{d} \eta_k^{\langle m_i, v_k\rangle}$ allows to express the kernels $\mathcal{P}^T$ and $\mathcal{K}^T$ with respect to the homogeneous coordinates $\eta = (\eta_1, \ldots, \zeta_d)$ and $h = (h_1, \ldots, h_d)$ on U_σ. This procedure replacing each projective variable of $\mathbb{P}^{N-1}$ by a quotient of homogeneous variables, yields zero homogeneous kernels. Hence, one can factor out $\frac{\prod_{k=1}^{d} \eta_k^{a_k}}{\prod_{k=1}^{d} h_k^{a_k}}$ from $\mathcal{P}^T$ and $\mathcal{K}^T$ (this fraction arises by making common denominator in the expansions of the numerator and denominator of the contributing forms in the kernels and then simplify) such that the remaining factors are the forms $\mathcal{P}_{\text{hom}}$ and $\mathcal{K}_{\text{hom}}$, respectively in view of (3.96). Thus, $\mathcal{P}_{\text{hom}}$ and $\mathcal{K}_{\text{hom}}$ take values in $(V_{\mathcal{L}}^{\vee})_{[\eta]} \otimes (V_{\mathcal{L}})_{[h]}$. The denominator of the kernels in toric form is the polynomial $P(|\tau|^2)$. This polynomial is modified to $\sum_{i=1}^{N} \prod_{k=1}^{d} |\eta_k|^{2(\langle m_i, v_k\rangle + a_k)}$ after changing of coordinates to the homogeneous ones and multiplying by $\prod_{k=1}^{d} |\eta_k|^{2a_k}$ in order to 'clear its denominator'. Actually, $\sum_{i=1}^{N} \prod_{k=1}^{d} |\eta_k|^{2(\langle m_i, v_k\rangle + a_k)}$ is the 'homogenization' of $P(|\tau|^2)$ in view of (1.6). This verifies that the singularities of the kernels are located along the exceptional set $Z(\Sigma)$ of X in view of (3.55), as we initially claimed at the beginning of the present chapter. $\square$

Theorem 3.4.3 is independent from the choice of the chart U_σ (of the covering $\{U_{\sigma_i}\}_i$ of X) since an equivalent divisor can be chosen for each chart and a similar construction yields the desired formula. Recall that, in view of (3.54),

$$\phi\big|_{U_{\sigma_i}} = \phi\big|_{U_{\sigma_j}}$$

on $U_{\sigma_i} \cap U_{\sigma_j}$ of X for every i, j. Thus, if $D = \sum_{k=1}^{d} a_k D_k \sim \sum_{k=1}^{d} b_k D_k$ and $U_{\sigma_i} = \{\prod_{k=1}^{d} h_k^{a_k} \neq 0\}$ and $U_{\sigma_j} = \{\prod_{k=1}^{d} h_k^{b_k} \neq 0\}$ are two corresponding charts, then the representation formula (3.93) for ϕ on U_{σ_i} and the corresponding one on U_{σ_j} are identified on their intersection. Hence,

$$\mathcal{P}_{\text{hom}, U_{\sigma_i}} = \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}} \mathcal{P}_{U_{\sigma_i}}^{T} = \frac{\prod_{k=1}^{d} h_k^{b_k}}{\prod_{k=1}^{d} \eta_k^{b_k}} \mathcal{P}_{U_{\sigma_j}}^{T} = \mathcal{P}_{\text{hom}, U_{\sigma_j}} \tag{3.97}$$

and

$$\mathcal{K}_{\text{hom}, U_{\sigma_i}} = \frac{\prod_{k=1}^{d} h_k^{a_k}}{\prod_{k=1}^{d} \eta_k^{a_k}} \mathcal{K}_{U_{\sigma_i}}^{T} = \frac{\prod_{k=1}^{d} h_k^{b_k}}{\prod_{k=1}^{d} \eta_k^{b_k}} \mathcal{K}_{U_{\sigma_j}}^{T} = \mathcal{K}_{\text{hom}, U_{\sigma_j}} \tag{3.98}$$

on $U_{\sigma_i} \cap U_{\sigma_j}$. The subscript U_{σ_i} and U_{σ_j} denotes the chart where the corresponding kernels are defined. Thus, Theorem 3.4.3 derives a well-defined integral representation formula (3.93) on every chart for the global object $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$.

We remark here that unlike the case of projective spaces or product of projective spaces, bringing the local representation to 'common' denominator does not give a global object.

$\square$

Remark 3.4.2 The required condition $N \leq 2(n+1)$ between the dimensions of the toric variety and of the induced projective space is somehow restrictive. For example, in view of the embedding (3.11), on representing $(0, q)$ forms on $\mathbb{P}^1 \times \mathbb{P}^1$ taking values in $L^k \otimes L^l$ for

$0 \leq q \leq n+m$, this necessary condition implies $kl+k+l \leq 5$. Thus, our Theorem is valid when $k=l=1$ or $k=1, l=2$ or $k=2, l=1$.

Moreover, taking into account the Example 3.1.2 related to the cartesian product $\mathbb{P}^n \times \mathbb{P}^m$, one can observe that when $nm \leq n+m+1$, one can derive a representation formula for $\mathcal{E}_{0,q}(\mathbb{P}^n \times \mathbb{P}^m, L^1 \otimes L^1)$. This is satisfied in the following cases:

- $n=1, m \in \mathbb{Z}^+$ or $n \in \mathbb{Z}^+, m=1$
- $n=2, m=3$ or $n=3, m=2$.

In the particular case of Hirzebruch surface $\mathcal{H}$, according to the Example 3.1.3, since $N=5$ and $n=2$, one can get a toric Koppelman representation formula (in view of Theorem 3.4.2) for $\mathcal{E}_{0,q}(\mathcal{H}, V_{\mathcal{L}})$ where $\mathcal{L} = \mathcal{O}_{\mathcal{H}}(D)$ is the sheaf of the divisor $D = D_3 + 2D_4$ and $0 \leq q \leq 2$.

However, Theorem 3.4.2 cannot be applied on representing smooth forms that take values on line bundles corresponding to ample divisors with bigger D-homogeneity. This happens since the necessary condition $N \leq 2(n+1)$ of the Theorem 3.4.2 is not fulfilled. For example, by considering the divisor $D = D_3 + 3D_4$ then

$$P_D \cap M = \{(0,0), (1,0), (1,1), (1,2), (0,1), (0,2), (0,3)\}$$

(see Examples 1.1.26 and 3.1.3). Thus, the number of the integral points due to the corresponding embedding are $N=7$ while $7 = N > 2(n+1) = 2(2+1)$.

3.5 Examples

In this section we present two examples that illustrate the above construction.

Example 3.5.1 (Koppelman formula for $(0,q)$ forms taking values in $L^1 \otimes L^1$ on $\mathbb{P}^1 \times \mathbb{P}^1$). According to Example 3.1.1 $n=2$, $N=4$ while the contributing vectors are

$$\begin{aligned} m_1 &= m_{11} = (0,0) \\ m_2 &= m_{13} = (1,0) \\ m_3 &= m_{12} = (1,1) \\ m_4 &= m_{14} = (0,1). \end{aligned}$$

Then, the Laurent polynomial $P(t) = 1 + t_1 + t_2 + t_1 t_2 = (1+t_1)(1+t_2)$ is induced and

$$\alpha^T_{0,0} = \frac{P(\bar\tau \cdot t)}{P(|\tau|^2)} = \frac{1 + \bar\tau_1 t_1 + \bar\tau_2 t_2 + \bar\tau_1\bar\tau_2 t_1 t_2}{1 + |\tau_1|^2 + |\tau_2|^2 + |\tau_1|^2|\tau_2|^2} = \frac{(1+\bar\tau_1 t_1)(1+\bar\tau_2 t_2)}{(1+|\tau_1|^2)(1+|\tau_2|^2)}.$$

Since,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

for $|J| = 3$ we obtain that

$$\det A_{(1,2,3)} = 1,\ \det A_{(1,3,4)} = 1,\ \det A_{(1,2,4)} = 1,\ \det A_{(2,3,4)} = 1. \tag{3.99}$$

Similarly, for $|J| = 2$, the following table is obtained.

Table 3.1: Table of det A

$\det A^1_{(1,2)} = 0$	$\det A^2_{(1,2)} = 1$
$\det A^1_{(1,3)} = 1$	$\det A^2_{(1,3)} = 1$
$\det A^1_{(1,4)} = 1$	$\det A^2_{(1,4)} = 0$
$\det A^1_{(2,3)} = 1$	$\det A^2_{(2,3)} = 0$
$\det A^1_{(2,4)} = 1$	$\det A^2_{(2,4)} = -1$
$\det A^1_{(3,4)} = 0$	$\det A^2_{(3,4)} = -1$

By substituting the suitable vectors into the explicit form of the Projection kernel given in Theorem 3.4.1, the kernel $\mathcal{P}^T$ corresponding to smooth $(0,q)$ forms on $\mathbb{P}^1 \times \mathbb{P}^1$ taking values in $L^1 \otimes L^1$ is obtained. After factoring out the contributing terms in $\mathcal{P}^T$, it turns out that

$$\mathcal{P}^T \quad = \quad \left(-\frac{1}{2\pi i}\right)^2 \frac{4(1+\bar\tau_1 t_1)(1+\bar\tau_2 t_2)}{(1+|\tau_1|^2)^3(1+|\tau_2|^2)^3}\, d\bar\tau \wedge d\tau.$$

Since the arising kernel is simple, we are going to find also the kernel $\mathcal{P}_{\text{hom}}$ following Theorem 3.4.3. The rules (3.33) and (3.34) connecting the toric variables with the homogeneous ones on the chart $U_{\sigma_{00}} = \{h_2, h_4 \neq 0\}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ are transformed into

$$\tau_1 = \frac{\eta_1}{\eta_2}, \tau_2 = \frac{\eta_3}{\eta_4} \quad \text{and} \quad t_1 = \frac{h_1}{h_2}, t_2 = \frac{h_3}{h_4}. \tag{3.100}$$

Hence, $\mathcal{P}^T$ is written on the chart $U_{\sigma_{00}}$ into the form

$$\begin{aligned} \mathcal{P}^T \quad = \quad & \left(-\frac{1}{2\pi i}\right)^2 \left(\frac{\eta_2\eta_4}{h_2 h_4}\right) \frac{4(\bar\eta_1 h_1 + \bar\eta_2 h_2)(\bar\eta_3 h_3 + \bar\eta_4 h_4)}{(|\eta_1|^2+|\eta_2|^2)^3(|\eta_3|^2+|\eta_4|^2)^3} \times \\ & (\bar\eta_2 d\bar\eta_1 - \bar\eta_1 d\bar\eta_2) \wedge (\eta_2 d\eta_1 - \eta_1 d\eta_2) \wedge (\bar\eta_4 d\bar\eta_3 - \bar\eta_3 d\bar\eta_4) \wedge (\eta_4 d\eta_3 - \eta_3 d\eta_4). \end{aligned}$$

Following Theorem 3.4.3,

$$\begin{aligned}
\mathcal{P}_{\text{hom}} &= \frac{h_2 h_4}{\eta_2\eta_4}\mathcal{P}^T \\
&= \left(-\frac{1}{2\pi i}\right)^2 \frac{4(\bar{\eta}_1 h_1 + \bar{\eta}_2 h_2)(\bar{\eta}_3 h_3 + \bar{\eta}_4 h_4)}{(|\eta_1|^2+|\eta_2|^2)^3(|\eta_3|^2+|\eta_4|^2)^3} \times \\
&\quad (\bar{\eta}_2 d\bar{\eta}_1 - \bar{\eta}_1 d\bar{\eta}_2)\wedge(\zeta_2 d\eta_1 - \eta_1 d\eta_2)\wedge(\bar{\eta}_4 d\bar{\eta}_3 - \bar{\eta}_3 d\bar{\eta}_4)\wedge(\eta_4 d\eta_3 - \eta_3 d\eta_4)
\end{aligned} \tag{3.101}$$

on the chart $U_{\sigma_{00}}$ of $\mathbb{P}^1 \times \mathbb{P}^1$. Similarly, one can find $\mathcal{P}_{\text{hom}}$ on the remaining three charts of $\mathbb{P}^1 \times \mathbb{P}^1$. Each case yields the same kernel $\mathcal{P}_{\text{hom}}$, thus the Projection kernel (3.101) holds on the whole variety in this specific example.

Comparing our kernel with the corresponding Projection kernel in a Götmark's work [15] representing $(0,q)$ forms taking values in $L^1 \otimes L^1$ on $\mathbb{P}^n \times \mathbb{P}^m$, that is,

$$\begin{aligned}
P &= \left(-\frac{1}{2\pi i}\right)^{n+m}\binom{n+1}{1}\binom{m+1}{1}\left(\frac{\bar{\eta}\cdot z}{|\eta|^2}\right)\left(\frac{\bar{\eta'}\cdot h'}{|\eta'|^2}\right) \\
&\quad \left[\bar{\partial}\left(\frac{\bar{\eta}\cdot d\eta}{|\eta|^2}\right)\right]^n \wedge \left[\bar{\partial}\left(\frac{\bar{\eta'}\cdot d\eta'}{|\eta'|^2}\right)\right]^m,
\end{aligned}$$

where $\eta = (\eta_1, \ldots, \eta_{n+1})$, $\eta' = (\eta_{n+2}, \ldots, \eta_{n+m+2})$, we observe that the two kernels are exactly the same when $n = m = 1$, although the approach to the problem is different.

The Koppelman kernel requires rather long computations. This fact is justified by the existence of contributing forms which have differentials either only in τ variable or both in t and τ variables. Firstly, following Lemmas 3.4.2, the terms that are related with the leading form u are given below:

$$\begin{aligned}
u_{1,0}^T &= \frac{v^T}{(\delta_z v)^T} \\
&= \frac{1}{(\delta_z v)^T}\left[\frac{(1+\bar{t}_2\tau_2)(\bar{t}_1 - \bar{\tau}_1)}{1+|\tau_1|^2}d\tau_1 + \frac{(1+\bar{t}_1\tau_1)(\bar{t}_2 - \bar{\tau}_2)}{1+|\tau_2|^2}d\tau_2\right],
\end{aligned}$$

where

$$(\delta_z v)^T = 2\pi i\left[(1+|t_1|^2)(1+|t_2|^2) - \frac{(1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)(1+\bar{\tau}_1 t_1)(1+\bar{\tau}_2 t_2)}{(1+|\tau_1|^2)(1+|\tau_2|^2)}\right].$$

Moreover, Lemma 3.4.3 yields that

$$(\gamma_1 \neg u_{1,0})^T = \frac{(\gamma_1 \neg v)^T}{(\delta_z v)^T} = \frac{2\pi i}{(\delta_z v)^T}\left[\bar{t}_1\bar{t}_2 - \frac{\bar{\tau}_1\bar{\tau}_2(1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)}{(1+|\tau_1|^2)(1+|\tau_2|^2)}\right].$$

On the other hand, by example 3.2.1 for $n = m = 1$, we get

$$\alpha^T_{1,1} \;\; = \;\; -\frac{1}{2\pi i}\left(\frac{d\bar{\tau}_1 \wedge d\tau_1}{(1+|\tau_1|^2)^2} + \frac{d\bar{\tau}_2 \wedge d\tau_2}{(1+|\tau_2|^2)^2}\right),$$

while the use of Lemma 3.4.2 implies that

$$\begin{aligned} C^T \;\; &= \;\; \frac{1+\bar{t}_2\tau_2}{1+|\tau_1|^2}\, d\bar{t}_1 \wedge d\tau_1 + \frac{\tau_1(\bar{t}_2 - \bar{\tau}_2)}{1+|\tau_2|^2}\, d\bar{t}_1 \wedge d\tau_2 \\ &\quad + \frac{\tau_2(\bar{t}_1 - \bar{\tau}_1)}{1+|\tau_1|^2}\, d\bar{t}_2 \wedge d\tau_1 + \frac{1+\bar{t}_1\tau_1}{1+|\tau_2|^2}\, d\bar{t}_2 \wedge d\tau_2. \end{aligned}$$

Therefore, according to the toric form $(\bar{\partial} v)^T$ (see Lemma 3.4.2), we can write

$$\begin{aligned} (\bar{\partial} v)^T \;\; &= \;\; C^T + 2\pi i \left(\sum_{i=1}^{N} \bar{t}^{m_i}\tau^{m_i}\right)\alpha^T_{1,1} \\ &= \;\; \frac{1+\bar{t}_2\tau_2}{1+|\tau_1|^2}\, d\bar{t}_1 \wedge d\tau_1 + \frac{\tau_1(\bar{t}_2 - \bar{\tau}_2)}{1+|\tau_2|^2}\, d\bar{t}_1 \wedge d\tau_2 \\ &\quad + \frac{\tau_2(\bar{t}_1 - \bar{\tau}_1)}{1+|\tau_1|^2}\, d\bar{t}_2 \wedge d\tau_1 + \frac{1+\bar{t}_1\tau_1}{1+|\tau_2|^2}\, d\bar{t}_2 \wedge d\tau_2 \\ &\quad - \frac{(1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)}{(1+|\tau_1|^2)^2} d\bar{\tau}_1 \wedge d\tau_1 - \frac{(1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)}{(1+|\tau_2|^2)^2} d\bar{\tau}_2 \wedge d\tau_2, \end{aligned}$$

while

$$\begin{aligned} u^T_{2,1} \;\; &= \;\; \frac{v^T \wedge (\bar{\partial} v)^T}{[(\delta_z v)^T]^2} \\ &= \;\; \frac{1}{[(\delta_z v)^T]^2}\Bigg[-\frac{(1+\bar{t}_1\tau_1)^2(1+\bar{t}_2\tau_2)(\bar{t}_2 - \bar{\tau}_2)}{(1+|\tau_1|^2)^2(1+|\tau_2|^2)} d\bar{\tau}_1 \wedge d\tau_1 \wedge d\tau_2 \\ &\quad - \frac{(1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)^2(\bar{t}_1 - \bar{\tau}_1)}{(1+|\tau_1|^2)(1+|\tau_2|^2)^2} d\bar{\tau}_2 \wedge d\tau_2 \wedge d\tau_1 \\ &\quad + \frac{((1+\bar{t}_2\tau_2)(\bar{t}_2 - \bar{\tau}_2)}{(1+|\tau_2|^2)} d\bar{t}_1 \wedge d\tau_1 \wedge d\tau_2 \\ &\quad + \frac{(\bar{t}_1 - \bar{\tau}_1)(1+\bar{t}_1\tau_1)}{(1+|\tau_1|^2)} d\bar{t}_2 \wedge d\tau_1 \wedge d\tau_2\Bigg]. \end{aligned}$$

In view of Lemma 3.4.1, we obtain

$$(\gamma_1 \neg \alpha_{1,1})^T = \frac{\bar{\tau}_2}{(1+|\tau_1|^2)^2(1+|\tau_2|^2)} d\bar{\tau}_1 + \frac{\bar{\tau}_1}{(1+|\tau_1|^2)(1+|\tau_2|^2)^2} d\bar{\tau}_2$$

thus, by using Lemma 3.4.3 one can observe that

$$
\begin{aligned}
(\gamma_1 \neg \bar{\partial} v)^T &= (\gamma_1 \neg C)^T + 2\pi i \left(\sum_i \bar{t}^{m_i} \tau^{m_i} \right) (\gamma_1 \neg \alpha_{1,1})^T \\
&= -2\pi i \left[\bar{t}_2 - \frac{\bar{\tau}_2 |\tau_1|^2 (1+\bar{t}_2\tau_2)}{(1+|\tau_1|^2)(1+|\tau_2|^2)} \right] d\bar{t}_1 - 2\pi i \left[\bar{t}_1 - \frac{\bar{\tau}_1 |\tau_2|^2 (1+\bar{t}_1\tau_1)}{(1+|\tau_1|^2)(1+|\tau_2|^2)} \right] d\bar{t}_2 \\
&\quad + \frac{2\pi i \bar{\tau}_2 (1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)}{(1+|\tau_1|^2)^2(1+|\tau_2|^2)} d\bar{\tau}_1 + \frac{2\pi i \bar{\tau}_1 (1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)}{(1+|\tau_1|^2)(1+|\tau_2|^2)^2} d\bar{\tau}_2
\end{aligned}
$$

and

$$
\begin{aligned}
(\gamma_1 \neg u_{2,1})^T &= \frac{2\pi i}{[(\delta_z v)^T]^2} \left[-\frac{\bar{t}_1 (1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)(\bar{t}_2 - \bar{\tau}_2)}{(1+|\tau_1|^2)^2(1+|\tau_2|^2)} d\bar{\tau}_1 \wedge d\tau_1 \right. \\
&\quad + \frac{\bar{\tau}_1 (1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)^2(\bar{t}_1 - \bar{\tau}_1)}{(1+|\tau_1|^2)^2(1+|\tau_2|^2)^2} d\bar{\tau}_2 \wedge d\tau_1 \\
&\quad + \frac{\bar{\tau}_2 (1+\bar{t}_1\tau_1)^2(1+\bar{t}_2\tau_2)(\bar{t}_2 - \bar{\tau}_2)}{(1+|\tau_1|^2)^2(1+|\tau_2|^2)^2} d\bar{\tau}_1 \wedge d\tau_2 \\
&\quad - \frac{\bar{t}_2 (1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)(\bar{t}_1 - \bar{\tau}_1)}{(1+|\tau_1|^2)(1+|\tau_2|^2)^2} d\bar{\tau}_2 \wedge d\tau_2 \\
&\quad + \frac{\bar{\tau}_1 (1+\bar{t}_2\tau_2)(\bar{t}_2 - \bar{\tau}_2)}{(1+|\tau_1|^2)(1+|\tau_2|^2)} d\bar{t}_1 \wedge d\tau_1 - \frac{\bar{t}_2 (\bar{t}_2 - \bar{\tau}_2)}{(1+|\tau_2|^2)} d\bar{t}_1 \wedge d\tau_2 \\
&\quad \left. - \frac{\bar{t}_1 (\bar{t}_1 - \bar{\tau}_1)}{(1+|\tau_1|^2)} d\bar{t}_2 \wedge d\tau_1 + \frac{\bar{\tau}_2 (1+\bar{t}_1\tau_1)(\bar{t}_1 - \bar{\tau}_1)}{(1+|\tau_1|^2)(1+|\tau_2|^2)} d\bar{t}_2 \wedge d\tau_2 \right].
\end{aligned}
$$

Moreover, according to Lemma 3.4.1

$$
H_{1,0}^T = \frac{1}{4\pi i} \left[\frac{(\tau_2 - t_2)(1+\bar{\tau}_1 t_1)}{1+|\tau_1|^2} d\tau_1 + \frac{(\tau_1 - t_1)(1+\bar{\tau}_2 t_2)}{1+|\tau_2|^2} d\tau_2 \right],
$$

while

$$
(\gamma_1 \neg H_{1,0})^T = \frac{(1+|\tau_1|^2)(1+\bar{\tau}_2 t_2) + (1+|\tau_2|^2)(1+\bar{\tau}_1 t_1)}{2(1+|\tau_1|^2)(1+|\tau_2|^2)}.
$$

Hence, the contributing terms in the Koppelman kernel can be derived briefly, below. The kernel $\mathcal{K}^T$ corresponding to the toric Koppelman representation formula for these forms is given explicitly as follows:

$$
\mathcal{K}^T = \mathcal{K}_1^T + \mathcal{K}_2^T + \mathcal{K}_3^T,
$$

since $n=2$ and $N=4$. The first term is given by

$$\begin{aligned}
\mathcal{K}_1^T &= \frac{(2+2n-N)(1+2n-N)}{2^{N-n}}(\alpha_{0,0}^T)^2(\alpha_{1,1}^T)^{n-1}\wedge u_{1,0}^T \\
&= \frac{1}{2}(\alpha_{0,0}^T)^2(\alpha_{1,1}^T)^1\wedge u_{1,0}^T \\
&= -\frac{(1+\bar{\tau}_1 t_1)^2(1+\bar{\tau}_2 t_2)^2}{4\pi i(\delta_z v)^T(1+|\tau_1|^2)^3(1+|\tau_2|^2)^3} \\
&\quad \left[\frac{(1+\bar{t}_1\tau_1)(\bar{t}_2-\bar{\tau}_2)}{1+|\tau_1|^2}d\bar{\tau}_1\wedge d\tau_1\wedge d\tau_2+\frac{(1+\bar{t}_2\tau_2)(\bar{t}_1-\bar{\tau}_1)}{1+|\tau_2|^2}d\bar{\tau}_2\wedge d\tau_2\wedge d\tau_1\right].
\end{aligned}$$

The second one is given by

$$\begin{aligned}
\mathcal{K}_2^T &= (\alpha_{0,0}^T)^2\sum_{0\le|J|\le 1}\mathrm{sign}(J)P(0,0)(\gamma_1\neg H^1)^T[J](u_{2,1}^{[J]})^T \\
&= (\alpha_{0,0})^2\left[(\gamma_1\neg H_{1,0})^T\wedge u_{2,1}^T-H_{1,0}^T\wedge(\gamma_1\neg u_{2,1})^T\right] \\
&= \frac{(1+\bar{\tau}_1 t_1)^2(1+\bar{\tau}_2 t_2)^3(1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)(\bar{\tau}_2-\bar{t}_2)}{2\left((\delta_z v)^T\right)^2(1+|\tau_1|^2)^5(1+|\tau_2|^2)^4} \\
&\qquad \left[(1+\bar{\tau}_1 t_1)(1+\bar{t}_1\tau_1)+(1+|\tau_1|^2)(1+|t_1|^2)\right]d\bar{\tau}_1\wedge d\tau_1\wedge d\tau_2 \\
&\quad +\frac{(1+\bar{\tau}_1 t_1)^3(1+\bar{\tau}_2 t_2)^2(1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)(\bar{\tau}_1-\bar{t}_1)}{2(\delta_z v)^2(1+|\tau_1|^2)^4(1+|\tau_2|^2)^5} \\
&\qquad \left[(1+\bar{\tau}_2 t_2)(1+\bar{t}_2\tau_2)+(1+|\tau_2|^2)(1+|t_2|^2)\right]d\bar{\tau}_2\wedge d\tau_2\wedge d\tau_1 \\
&\quad +\frac{(1+\bar{\tau}_1 t_1)^3(1+\bar{\tau}_2 t_2)^2(\bar{t}_2-\bar{\tau}_2)}{2(\delta_z v)^2(1+|\tau_1|^2)^3(1+|\tau_2|^2)^4} \\
&\qquad \left[(1+\bar{\tau}_2 t_2)(1+\bar{t}_2\tau_2)+(1+|\tau_2|^2)(1+|t_2|^2)\right]d\bar{t}_1\wedge d\tau_1\wedge d\tau_2 \\
&\quad +\frac{(1+\bar{\tau}_1 t_1)^2(1+\bar{\tau}_2 t_2)^3(\bar{t}_1-\bar{\tau}_1)}{2(\delta_z v)^2)(1+|\tau_1|^2)^4(1+|\tau_2|^2)^3} \\
&\qquad \left[(1+\bar{\tau}_1 t_1)(1+\bar{t}_1\tau_1)+(1+|\tau_1|^2)(1+|t_1|^2)\right]d\bar{t}_2\wedge d\tau_2\wedge d\tau_1.
\end{aligned}$$

The computation deriving the term $\mathcal{K}_3^T$ is the following:

$$\begin{aligned}
\mathcal{K}_3^T &= 2\alpha_{0,0}^T\sum_{l=0}^{1}\sum_{l\le|J|\le 1}\mathrm{sign}(J-l)P(1,l)(\gamma_1\neg H^1)^T[J](\alpha_{1,1}^{[l]})^T(u_{1,0}^{[J-l]})^T(\alpha_{1,1}^T)^{1-l} \\
&= 2\alpha_{0,0}^T\left[(\gamma_1\neg H_{1,0})^T\wedge u_{1,0}^T\wedge\alpha_{1,1}^T-H_{1,0}^T\wedge(\gamma_1\neg u_{1,0})^T\wedge\alpha_{1,1}^T-H_{1,0}^T\wedge(\gamma_1\neg\alpha_{1,1})^T\wedge u_{1,0}^T\right] \\
&= -\frac{(1+\bar{\tau}_1 t_1)(1+\bar{\tau}_2 t_2)^2(\bar{t}_2-\bar{\tau}_2)}{2\pi i(\delta_z v)^T(1+|\tau_1|^2)^4(1+|\tau_2|^2)^3} \\
&\quad \left[(1+|\tau_1|^2)(1+|t_1|^2)+(1+\bar{\tau}_1 t_1)(1+\bar{t}_1\tau_1)\right]d\bar{\tau}_1\wedge d\tau_1\wedge d\tau_2 \\
&\quad -\frac{(1+\bar{\tau}_1 t_1)^2(1+\bar{\tau}_2 t_2)(\bar{t}_1-\bar{\tau}_1)}{2\pi i\delta_z v(1+|\tau_1|^2)^3(1+|\tau_2|^2)^4} \\
&\quad \left[(1+|\tau_2|^2)(1+|t_2|^2)+(1+\bar{\tau}_2 t_2)(1+\bar{t}_2\tau_2)\right]d\bar{\tau}_2\wedge d\tau_2\wedge d\tau_1.
\end{aligned}$$

and $(\delta_z v)^T = 2\pi i\left[(1+|t_1|^2)(1+|t_2|^2)-\frac{(1+\bar{t}_1\tau_1)(1+\bar{t}_2\tau_2)(1+\bar{\tau}_1 t_1)(1+\bar{\tau}_2 t_2)}{(1+|\tau_1|^2)(1+|\tau_2|^2)}\right]$. Note that the fourth term $\mathcal{K}_4^T$ of the Koppelman kernel does not exist in this example since $N-n=2$ which is opposed to inequality (3.91). Similarly, following Theorem 3.4.3 the kernel $\mathcal{K}_{\mathrm{hom}}$ on $U_{\sigma_{00}}$ is given by

$$\mathcal{K}_{\text{hom}} = \frac{h_2 h_4}{\eta_2 \eta_4} \mathcal{K}^T = \frac{h_2 h_4}{\eta_2 \eta_4} \left(\mathcal{K}_1^T + \mathcal{K}_2^T + \mathcal{K}_3^T \right)$$

after expressing the involved terms in the homogeneous coordinates following (3.100). As in the case of $\mathcal{P}_{\text{hom}}$, the same form $\mathcal{K}_{\text{hom}}$ holds on every chart of $\mathbb{P}^1 \times \mathbb{P}^1$ and thus on the whole variety.

One can observe that denominators of the kernels $\mathcal{P}_{\text{hom}}$ and $\mathcal{K}_{\text{hom}}$ are powers of the factors

$$(|\eta_1|^2 + |\eta_2|^2) \quad \text{and} \quad (|\eta_3|^2 + |\eta_4|^2)$$

verifying that the singular points are located along the exceptional set

$$\{\eta_1 = \eta_2 = 0\} \cup \{\eta_3 = \eta_4 = 0\}$$

of $\mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Since, the computations involved in the kernel $\mathcal{K}^T$ are tedious, in the next example, we are going to find explicitly the Projection kernel $\mathcal{P}^T$ while a more general form for the kernel $\mathcal{K}^T$ with explicit expressions of its involved forms are presented.

Example 3.5.2 (Koppelman formula for $(0, q)$ forms taking values in $V_{\mathcal{L}}$ on $\mathcal{H}$). Let us consider the particular case of a Hirzebruch surface, $\mathcal{H}$, that was studied in Example 3.1.3 where $n = 2$, $N = 5$. Since the contributing vectors are

$$\begin{aligned} m_1 &= m_{11} = m_{21} = (0,0) \\ m_2 &= m_{13} = (1,0) \\ m_3 &= m_{12} = (1,1) \\ m_4 &= m_{14} = m_{23} = m_{24} = (0,1) \\ m_5 &= m_{22} = (0,2), \end{aligned}$$

the corresponding Laurent polynomial equals to $P(t) = 1 + t_1 + t_2 + t_1 t_2 + t_2^2$ and

$$\alpha_{0,0}^T = \frac{P(\bar{\tau} \cdot t)}{P(|\tau|^2)} = \frac{1 + \bar{\tau}_1 t_1 + \bar{\tau}_2 t_2 + \bar{\tau}_1 \bar{\tau}_2 t_1 t_2 + \bar{\tau}_2^2 t_2^2}{1 + |\tau_1|^2 + |\tau_1|^2 |\tau_2|^2 + |\tau_2|^2 + |\tau_2|^4}$$

The matrix A is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

For $|J| = 3$ we obtain that

$$\det A_{(1,2,3)} = 1,\ \det A_{(1,3,4)} = 1,\ \det A_{(1,2,4)} = 1,\ \det \mathrm{A}_{(2,3,4)} = 1$$
$$\det A_{(1,2,5)} = 2,\ \det A_{(1,3,5)} = 2,\ \det A_{(1,4,5)} = 0,$$
$$\det A_{(2,3,5)} = 1,\ \det A_{(2,4,5)} = -1,\ \det A_{(3,4,5)} = -1,$$

Similarly, for $|J| = 2$,

Table 3.2: Table of det A

$\det A^1_{(1,2)} = 0$	$\det A^2_{(1,2)} = 1$
$\det A^1_{(1,3)} = 1$	$\det A^2_{(1,3)} = 1$
$\det A^1_{(1,4)} = 1$	$\det A^2_{(1,4)} = 0$
$\det A^1_{(1,5)} = 2$	$\det A^2_{(1,5)} = 0$
$\det A^1_{(2,3)} = 1$	$\det A^2_{(2,3)} = 0$
$\det A^1_{(2,4)} = 1$	$\det A^2_{(2,4)} = -1$
$\det A^1_{(2,5)} = 2$	$\det A^2_{(2,5)} = -1$
$\det A^1_{(3,4)} = 0$	$\det A^2_{(3,4)} = -1$
$\det A^1_{(3,5)} = 1$	$\det A^2_{(3,5)} = -1$
$\det A^1_{(4,5)} = 1$	$\det A^2_{(4,5)} = 0$

Using Theorem 3.4.1, since $n = 2$ and $N = 5$ we get the kernel $\mathcal{P}$ in the case of representing smooth $(0, q)$ forms taking values in $V_{\mathcal{L}}$ on $\mathcal{H}$, where $\mathcal{L}$ is the sheaf induced by the divisor

$D = D_3 + 2D_4$. More precisely,

$$
\begin{aligned}
\mathcal{P}^T \;=\;& \left(-\frac{1}{2\pi i}\right)^2 \frac{1}{2(1+|\tau_1|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^2+|\tau_2|^4)^4} d\bar{\tau}\wedge d\tau \\
& \big\{ \big[-\bar{\tau}_1\bar{\tau}_2(\tau_1-t_1)(\tau_2-t_2) + 3(1+\bar{\tau}_1 t_1+\bar{\tau}_2 t_2+\bar{\tau}_1\bar{\tau}_2 t_1 t_2+\bar{\tau}_2^2 t_2^2) \\
& -\bar{\tau}_2^2(\tau_2-t_2)^2 + 2(1+|\tau_1|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^2+|\tau_2|^4)\big] \\
& \big(1+|\tau_1|^2+5|\tau_2|^2+5|\tau_2|^4+|\tau_1|^2|\tau_2|^2+|\tau_1|^2|\tau_2|^4+|\tau_2|^6\big) \\
& - \big[\bar{\tau}_2(1+|\tau_2|^2+|\tau_2|^4)(\tau_2-t_2)\,\big(t_1\bar{\tau}_1(1+|\tau_2|^2)+(1+|\tau_2|^2+|\tau_2|^4)\big) \\
& + \bar{\tau}_2^2|\tau_1|^2(1+|\tau_2|^2)^2(\tau_2-t_2)^2\big]\,(1+|\tau_1|^2)(1+|\tau_1|^2+4|\tau_2|^2+|\tau_2|^4) \\
& - \big[|\tau_1|^2\bar{\tau}_2(1+|\tau_1|^2-|\tau_2|^4)(\tau_2-t_2)\,\big(t_1\bar{\tau}_1(1+|\tau_2|^2)+(1+|\tau_2|^2+|\tau_2|^4)\big) \\
& - \bar{\tau}_2^2|\tau_1|^2(1+|\tau_1|^2)(1+|\tau_2|^2)(2+|\tau_2|^2)(\tau_2-t_2)^2\big]\,|\tau_2|^2(2+|\tau_2|^2) \\
& - \big[\bar{\tau}_1|\tau_2|^2(1+|\tau_2|^2+|\tau_2|^4)(\tau_1-t_1)\,\big(t_2\bar{\tau}_2(1+|\tau_1|^2+2|\tau_2|^2)+(1+|\tau_1|^2-|\tau_2|^4)\big) \\
& -\bar{\tau}_2^2|\tau_1|^2(1+|\tau_2|^2)(\tau_2-t_2) \\
& \big(t_2\bar{\tau}_2(1+|\tau_1|^2+2|\tau_2|^2)+(2+2|\tau_1|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^2)\big)\big]\,|\tau_2|^2(2+|\tau_2|^2) \\
& - \big[\bar{\tau}_1(1+|\tau_1|^2-|\tau_2|^4)(\tau_1-t_1)\,\big(t_2\bar{\tau}_2(1+|\tau_1|^2+2|\tau_2|^2)+(1+|\tau_1|^2-|\tau_2|^4)\big) \\
& +\bar{\tau}_2(1+|\tau_1|^2)(2+|\tau_2|^2)(\tau_2-t_2) \\
& \big(t_2\bar{\tau}_2(1+|\tau_1|^2+2|\tau_2|^2)+(2+2|\tau_1|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^2)\big)\big] \\
& (1+2|\tau_2|^2+2|\tau_2|^4+|\tau_2|^6)\big\}\,. \qquad (3.102)
\end{aligned}
$$

Hence, the form $\mathcal{P}_{\text{hom}}$ on the chart $U_\sigma = \{\eta_3\eta_4 \neq 0\}$ of $\mathcal{H}$

$$\mathcal{P}_{\text{hom}} = \frac{h_3 h_4^2}{\eta_3\eta_4^2}\mathcal{P}^T,$$

such that $\mathcal{P}^T$ is the form in (3.102) where we replaced

$$\tau_1 = \frac{\eta_1}{\eta_3\eta_4}, \tau_2 = \frac{\eta_2}{\eta_4} \quad \text{and} \quad t_1 = \frac{h_1}{h_3 h_4}, t_2 = \frac{h_2}{h_4}. \qquad (3.103)$$

On the other hand, the contributing kernels in the Koppelman kernel are the forms $\mathcal{K}_3^T$ (Lemma 3.4.6) and $\mathcal{K}_4^T$ (Lemma 3.4.7) since $N = 2n+1$ ($N = 5$, $n = 2$) and the relation $N - n = 3$ satisfies the inequality (3.91), correspondingly.

In view of Lemma 3.4.6,

$$\mathcal{K}_3^T = \sum_{i=1}^{2} \frac{1}{2^{2-i}} \alpha_{0,0}^T \sum_{1\le k_1 \le k_i \le 2} \sum_{l=0}^{\min\{2,2-i\}} \sum_{\substack{l\le |J| \le i \\ 1\le j_1 < j_i \le k_i}} \operatorname{sign}(J-l)$$
$$\left(\bigwedge_{m=1}^{i} \gamma_{\tau(k_m)} \neg H_{1,0}^{k_m}[J]\right)^T \left(\alpha_{1,1}^{[l]}\right)^T \left(u_{i,i-1}^{[J-l]}\right)^T (\alpha_{1,1}^T)^{2-l-i} \qquad (3.104)$$

while Lemma 3.4.7 yields

$$\mathcal{K}_4^T = \sum_{1=k_1<k_2=2} \sum_{l=0}^{1} \sum_{\substack{l\le |J| \le l+1 \\ 1\le j_1 < j_{l+1} \le 2}} \operatorname{sign}(J-l)$$
$$\left(\bigwedge_{m=1}^{2} \gamma_{\tau(k_m)} \neg H_{1,0}^{k_m}[J]\right)^T \left(\alpha_{1,1}^{[l]}\right)^T \left(u_{1,0}^{[J-l]}\right)^T (\alpha_{1,1}^T)^{1-l}. \qquad (3.105)$$

We are going to find explicitly the contributing vectors in $\mathcal{K}_3^T$ and $\mathcal{K}_4^T$. Firstly, according to Lemma 3.4.1

$$(H^1)_{1,0}^T = \frac{1}{4\pi i}\left\{(\tau_2 - t_2)\left[-\frac{\bar\tau_1(\tau_1 - t_1)(1+\tau_2|^2)}{P(|\tau|^2)} + 1\right] d\tau_1 \right.$$
$$\left. + (\tau_1 - t_1)\left[-\frac{\bar\tau_2(\tau_2 - t_2)(1+|\tau_1|^2 + 2|\tau_2|^2)}{P(|\tau|^2)} + 1\right] d\tau_2\right\},$$

$$(H^2)_{1,0}^T = \frac{1}{4\pi i}\left\{\left[-\frac{\bar\tau_1(\tau_2 - t_2)^2(1+\tau_2|^2)}{P(|\tau|^2)}\right] d\tau_1 \right.$$
$$\left. + (\tau_2 - t_2)\left[-\frac{\bar\tau_2(\tau_2 - t_2)(1+|\tau_1|^2 + 2|\tau_2|^2)}{P(|\tau|^2)} + 2\right] d\tau_2\right\}$$

while

$$(\gamma_1 \neg (H^1)_{1,0})^T = -\frac{\bar\tau_1\bar\tau_2(\tau_1 - t_1)(\tau_2 - t_2)}{2P(|\tau|^2)} + \frac{1}{2}(\alpha_{0,0}^T + 1),$$

$$(\gamma_2 \neg (H^1)_{1,0})^T = -\frac{\bar\tau_2^2(\tau_1 - t_1)(\tau_2 - t_2)}{2P(|\tau|^2)},$$

$$(\gamma_1 \neg (H^2)_{1,0})^T = -\frac{\bar\tau_1\bar\tau_2(\tau_2 - t_2)^2}{2P(|\tau|^2)}$$

and

$$(\gamma_2 \neg (H^2)_{1,0})^T = -\frac{\bar\tau_2^2(\tau_2 - t_2)^2}{2P(|\tau|^2)} + \frac{1}{2}(\alpha_{0,0}^T + 1).$$

Recall from Example 3.2.2 that

$$\begin{aligned}
\alpha_{1,1}^T &= -\frac{1}{2\pi i}\frac{1+2|\tau_2|^2+2|\tau_2|^4+|\tau_2|^6}{(P(|\tau|^2))^2}d\bar{\tau}_1\wedge d\tau_1 \\
&\quad -\frac{1}{2\pi i}\frac{-\tau_1\bar{\tau}_2|\tau_2|^2(2+|\tau_2|^2)}{(P(|\tau|^2))^2}d\bar{\tau}_1\wedge d\tau_2 \\
&\quad -\frac{1}{2\pi i}\frac{-\bar{\tau}_1\tau_2|\tau_2|^2(2+|\tau_2|^2)}{(P(|\tau|^2))^2}d\bar{\tau}_2\wedge d\tau_1 \\
&\quad -\frac{1}{2\pi i}\frac{(1+|\tau_1|^2)(1+|\tau_1|^2+4|\tau_2|^2+|\tau_2|^4)}{(P(|\tau|^2))^2}d\bar{\tau}_2\wedge d\tau_2.
\end{aligned}$$

Moreover, due to the action of the vector fields on $\alpha_{1,1}$ (see Lemma 3.4.1), one has

$$(\gamma_1\neg\alpha_{1,1})^T = \frac{\bar{\tau}_1\bar{\tau}_2}{(P(|\tau|^2))^2}\left\{(1+|\tau_2|^2+|\tau_2|^4)\frac{d\bar{\tau}_1}{\bar{\tau}_1}+(1+|\tau_1|^2-|\tau_2|^4)\frac{d\bar{\tau}_2}{\bar{\tau}_2}\right\} \tag{3.106}$$

and

$$(\gamma_2\neg\alpha_{1,1})^T = \frac{\bar{\tau}_2^2}{(P(|\tau|^2))^2}\left\{-|\tau_1|^2(1+|\tau_2|^2)\frac{d\bar{\tau}_1}{\bar{\tau}_1}+(1+|\tau_1|^2)(2+|\tau_2|^2)\frac{d\bar{\tau}_2}{\bar{\tau}_2}\right\} \tag{3.107}$$

such that $(\alpha_{1,1}^{[2]})^T = (\gamma_1\neg\alpha_{1,1})^T\wedge(\gamma_2\neg\alpha_{1,1})^T$. In view of Lemma 3.4.2,

$$\begin{aligned}
v^T &= \left[\bar{t}_1(1+\bar{t}_2\tau_2)-\frac{\bar{\tau}_1(1+|\tau_2|^2)P(\bar{t}\cdot\tau)}{P(|\tau|^2)}\right]d\tau_1 \\
&\quad + \left[\bar{t}_2(1+\bar{t}_1\tau_1+\bar{t}_2\tau_2)-\frac{\bar{\tau}_2(1+|\tau_1|^2+|\tau_2|^2)P(\bar{t}\cdot\tau)}{P(|\tau|^2)}\right]d\tau_2
\end{aligned}$$

and $(\delta_z v)^T = 2\pi i\left[P(|t|^2)-\frac{P(\bar{t}\cdot\tau)P(t\cdot\bar{\tau})}{P(|\tau|^2)}\right]$ such that $u_{1,0}^T = \frac{v^T}{(\delta_z v)^T}$. In addition,

$$\begin{aligned}
C^T &= \left[(1+\bar{t}_2\tau_2)-\frac{|\tau_1|^2(1+\bar{t}_2\tau_2)(1+|\tau_2|^2)}{P(|\tau|^2)}\right]d\bar{t}_1\wedge d\tau_1 \\
&\quad + \left[\bar{t}_1\tau_2-\frac{\bar{\tau}_1\tau_2(1+\bar{t}_1\tau_1+2\bar{t}_2\tau_2)(1+|\tau_2|^2)}{P(|\tau|^2)}\right]d\bar{t}_2\wedge d\tau_1 \\
&\quad + \left[\bar{t}_2\tau_1-\frac{\bar{\tau}_2\tau_1(1+\bar{t}_2\tau_2)(1+|\tau_1|^2+2|\tau_2|^2)}{P(|\tau|^2)}\right]d\bar{t}_1\wedge d\tau_2 \\
&\quad + \left[(1+\bar{t}_1\tau_1+4\bar{t}_2\tau_2)-\frac{|\tau_2|^2(1+\bar{t}_1\tau_1+2\bar{t}_2\tau_2)(1+|\tau_1|^2+2|\tau_2|^2)}{P(|\tau|^2)}\right]d\bar{t}_2\wedge d\tau_2,
\end{aligned}$$

and hence,

$$(\bar{\partial}v)^T = C^T + 2\pi i P(\bar{t}\cdot\tau)\alpha_{1,1}^T$$

while

$$u_{2,1}^T = \frac{v^T\wedge(\bar{\partial}v)^T}{\left((\delta_z v)^T\right)^2}.$$

The form v and C after the action of the vector fields on them through the use of Lemma

3.4.3 are modified to

$$(\gamma_1 \neg v)^T = 2\pi i \left[\bar{t}_1 \bar{t}_2 - \frac{\bar{\tau}_1 \bar{\tau}_2 P(\bar{t}\cdot\tau)}{P(|\tau|^2)} \right] \quad \text{and} \quad (\gamma_2 \neg v)^T = 2\pi i \left[\bar{t}_2^2 - \frac{\bar{\tau}_2^2 P(\bar{t}\cdot\tau)}{P(|\tau|^2)} \right]$$

while

$$(\gamma_1 \neg C)^T \quad = \quad -2\pi i \left[\left(\bar{t}_2 - \frac{\bar{\tau}_2 |\tau_1|^2 (1+\bar{t}_2\tau_2)}{P(|\tau|^2)} \right) d\bar{t}_1 + \left(\bar{t}_1 - \frac{\bar{\tau}_1 |\tau_2|^2 (1+\bar{t}_1\tau_1 + 2\bar{t}_2\tau_2)}{P(|\tau|^2)} \right) d\bar{t}_2 \right]$$

and

$$(\gamma_2 \neg C)^T \quad = \quad -2\pi i \left[-\frac{\tau_1 \bar{\tau}_2^2 (1+\bar{t}_2\tau_2)}{P(|\tau|^2)} d\bar{t}_1 + \left(2\bar{t}_2 - \frac{\bar{\tau}_2 |\tau_2|^2 (1+\bar{t}_1\tau_1 + 2\bar{t}_2\tau_2)}{P(|\tau|^2)} \right) d\bar{t}_2 \right].$$

Hence, one can find the forms

$$(\gamma_i \neg \bar{\partial} v)^T = (\gamma_i \neg C)^T + 2\pi i P(\bar{t}\cdot\tau)(\gamma_i \neg \alpha_{1,1})^T,$$

for $i = 1, 2$. At last, the form $(u_{2,1}^{[1]})^T$ is derived by

$$(\gamma_i \neg u_{2,1})^T = \frac{(\gamma_i \neg v)^T \wedge (\bar{\partial} v)^T}{(\delta_z v)^T} - \frac{v^T \wedge (\gamma_i \neg \bar{\partial} v)^T}{(\delta_z v)^T}$$

for $i = 1$ or $i = 2$ while

$$(u_{2,1}^{[2]})^T = ((\gamma_2 \wedge \gamma_1) \neg u_{2,1}))^T = \sum_{\substack{1 \leq l \leq 2 \\ k \neq l}} \frac{(-1)^{\frac{l(l-1)}{2} + (2-l)(1-l)}}{[(\delta_z v)^T]^2} (\gamma_l \neg v)^T (\gamma_k \neg \bar{\partial} v)^T$$

in view of (3.76).

The substitution of the preceding forms into K_3^T in (3.104) and K_4^T in (3.105) lead to the explicit form of $\mathcal{K}^T$ which is omitted due to the voluminous computations. Hence, if we replace the toric variables with the homogeneous ones through the use of (3.103), we observe that

$$\mathcal{K}_{\text{hom}} = \frac{h_3 h_4^2}{\eta_3 \eta_4^2} \mathcal{K}^T.$$

on the chart $U_\sigma = \{\eta_3 \eta_4 \neq 0\}$ of $\mathcal{H}$.

One again observe here, that the singularities of the kernels are located on the exceptional set

$$\{\eta_1 = \eta_3 = 0\} \cup \{\eta_2 = \eta_4 = 0\}$$

of $\mathcal{H}$ since by passing to the homogeneous coordinates $(\eta_1, \eta_2, \eta_3, \eta_4) \in \mathcal{H}$,

$$P(|\tau|^2) = \sum_{i=1}^{5} \prod_{k=1}^{4} |\eta_k|^{2\langle m_i, v_k\rangle} = 1 + \left|\frac{\eta_1}{\eta_3\eta_4}\right|^2 + \left|\frac{\eta_2}{\eta_4}\right|^2 + \left|\frac{\eta_1\eta_2}{\eta_3\eta_4^2}\right|^2 + \left|\frac{\eta_2}{\eta_4}\right|^4$$

while a 'homogenization' argument (by simplifying the denominators) shows that the denominator of the representation kernels vanishes if and only if

$$\sum_{i=1}^{5} \prod_{k=1}^{4} |\eta_k|^{2(\langle m_i, v_k\rangle + a_k)} \quad = \quad |\eta_3|^2|\eta_4|^4 + |\eta_1|^2|\eta_4|^2 + |\eta_1|^2|\eta_2|^2 + |\eta_2|^2|\eta_3|^2|\eta_4|^2 + |\eta_2|^4|\eta_3|^2 = 0.$$

□

In the next paragraph, we formulate a toric Koppelman formula representing smooth forms taking values in $(V_{\mathcal{L}})^k$ for $k \in \mathbb{Z}^+$, overtaking partly the 'trouble' of large homogeneities.

3.6 The toric Koppelman formula for sections on $(V_{\mathcal{L}})^k$

By extending the toric Koppelman representation formula(Theorem 3.3.2), one can also obtain a formula for $(0,q)$ forms on X taking values in $(V_{\mathcal{L}})^k = \underbrace{V_{\mathcal{L}} \otimes \cdots \otimes V_{\mathcal{L}}}_{k-\text{times}}$. The bundle $(V_{\mathcal{L}})^k$ is a different manifold from the bundle $V_{\mathcal{L}}$ and the generalization of an integral formula for sections taking values in $V_{\mathcal{L}}$ to ones taking values in $(V_{\mathcal{L}})^k$, is non trivial. However, our method in this section not only extends the result of the previous section, but also constructs integral formulas on smooth compact toric varieties on which the previous construction cannot be applied due to the restriction $N \leq 2(n+1)$. This is achieved by using the extension of a section $\phi \in \mathcal{E}_{0,q}\left(X, (V_{\mathcal{L}})^k\right)$ to a section $\varphi \in \mathcal{E}_{0,q}(\mathbb{P}^{N-1}, L^k)$ through the closed embedding ϕ_D (3.2) of X into $\mathbb{P}^{N-1}$ (Lemma 3.1.2), where D is the divisor inducing the line bundle $V_{\mathcal{L}}$, without increasing the number of integral points of the associated polyhedron.

For example, let us recall Example 3.1.1 where the corresponding embedding (3.10) of the divisor $D = D_2 + D_4$ extends the section $\phi \in \mathcal{E}_{0,q}\left(\mathbb{P}^1 \times \mathbb{P}^1, L^k \otimes L^k\right)$ to the section $\varphi \in \mathcal{E}_{0,q}\left(\mathbb{P}^3, L^k\right)$ through the closed embedding $\phi_{D_2+D_4}$, while a section $\phi \in \mathcal{E}_{0,q}\left(\mathbb{P}^1 \times \mathbb{P}^1, L^k \otimes L^k\right)$ can also be extended to the section $\varphi' \in \mathcal{E}_{0,q}\left(\mathbb{P}^{k^2+2k}, L^1\right)$ through the embedding $\phi_{kD_2+lD_4}$ in (3.11) (when $k = l$). Observe that the two extensions coincide when $k = 1$ while for $k > 1$, the two extensions are compatible, since by taking $\psi = \phi_{kD_2+kD_4} \circ \phi^{-1}_{D_2+D_4}$, one can extend $\varphi \in \mathcal{E}_{0,q}\left(\mathbb{P}^3, L^k\right)$ into $\varphi' \in \mathcal{E}_{0,q}\left(\mathbb{P}^{k^2+2k}, L^1\right)$.

The number of integral points belonging to the corresponding polyhedron in the second approach increases rapidly as k becomes larger and the toric Koppelman formula becomes useless for $k > 1$, due to the failure of satisfaction of the necessary condition in Theorem 3.3.2:

$$k^2 + 2k + 1 = N \leq 2(n+1) = 6.$$

On the other hand, the first approach has the benefit that while preserving the number of integral points constant, with the method developed in the previous sections, it leads to integral formulas for sections in $\mathcal{E}_{0,q}\left(\mathbb{P}^1 \times \mathbb{P}^1, L^k \otimes L^k\right)$ for every $k \in \mathbb{N}$ and the involved kernels look like the preceding ones for $k = 1$ (with slight modifications). Some extra terms may be added in their construction, in general. Furthermore, as opposed to the previous case, the 'renewed' condition $N \leq 2n + k + 1$ will appear, generously providing integral representations for large k. Hence, we are able to construct representations for sections taking values in $(V_{\mathcal{L}})^k$ for these values of k .

To avoid confusion between the kernels of this type of integral representation formulas and the preceding one, we denote by $\mathcal{K}^{(k)}$ and $\mathcal{P}^{(k)}$ the Koppelman and Projection kernels correspondingly of the toric Koppelman represenation formula for $(0, q)$ forms on X taking values in $(V_{\mathcal{L}})^k$, with respect to the projective variables $(z, \zeta) \in \mathbb{P}^{N-1} \times \mathbb{P}^{N-1}$. Hence, $\mathcal{K}^{(1)} = \mathcal{K}$ while $\mathcal{P}^{(1)} = \mathcal{P}$.

Theorem 3.6.1 *Let X be an n-dimensional smooth compact toric variety and $\phi \in \mathcal{E}_{0,q}\left(X, (V_{\mathcal{L}})^k\right)$ for $k \geq 1$, where $V_{\mathcal{L}}$ is the induced line bundle of an ample divisor $D = \sum_{k=1}^{d} a_k D_k$ (such that $\mathcal{L} = \mathcal{O}_X(D)$) satisfying $|P_D \cap \mathbb{Z}^n| = N \leq 2n + k + 1$. Then,*

$$C_{N,n}\phi(z) = \int_X \mathcal{K}^{(k)} \wedge \bar{\partial}\phi + \bar{\partial}_z \int_X \mathcal{K}^{(k)} \wedge \phi + \int_X \mathcal{P}^{(k)} \wedge \phi \tag{3.108}$$

on the chart $X = \bigcap_{i=1}^{d} \{z \in \mathbb{C}^N \setminus \{0\} : f_i(z) = 0\}$, where $\mathcal{K}^{(k)} = \gamma\neg\left(H \wedge \alpha^{k+2n+1-N} \wedge u\right)$ and $\mathcal{P}^{(k)} = \gamma\neg\left(H \wedge \alpha^{k+2n+1-N}\right)$ are the kernels of the representation and $\gamma = \gamma_{N-n-1} \wedge \ldots \wedge \gamma_1$.

Proof. According to Lemma 3.1.2, the form $\phi \in \mathcal{E}_{0,q}(X, (V_{\mathcal{L}})^k)$ is extended to a form $\varphi \in \mathcal{E}_{0,q}(\mathbb{P}^{N-1}, L^k)$. The homogeneities of the integrands are balanced in view of the preceding discussion to Theorem 3.3.2 and hence, the proof follows along the lines of the proof of Theorem 3.3.2. $\square$

The number of the contributing terms in the explicit forms of kernels depends on the dimensions n, N and the number k. The forms $\mathcal{P}_1$ and $\mathcal{P}_2$ composing the Projection kernel of the toric Koppelman representation formula for forms taking values in $V_{\mathcal{L}}$ are modified in the case of forms taking values in $(V_{\mathcal{L}})^k$, according to the next remark 3.6.1. Extra terms may be added in the construction of the kernel $\mathcal{P}$. Analogous results hold for the kernel $\mathcal{K}$.

Remark 3.6.1 The forms involved in the kernel $\mathcal{P}_1^{(k)}$ corresponding to the representation of an $\phi \in \mathcal{E}_{0,q}(X, (V_{\mathcal{L}})^k)$ are

$$
\begin{aligned}
\mathcal{P}_1^{(k)} &= \gamma\neg\left(\bigwedge_{j=1}^{N-n-1}(H^j)_{2,1}\wedge(\alpha)_{1+2n-N}^{k+2n+1-N}\right)\\
&= \frac{1}{2^{N-n-1}}\binom{k+2n+1-N}{2n+1-N}(\alpha_{0,0})^k(\alpha_{1,1}^T)^n\\
&= \frac{1}{2n+2-N}\binom{k+2n+1-N}{2n+1-N}(\alpha_{0,0})^{k-1}\mathcal{P}_1, \qquad (3.109)
\end{aligned}
$$

taking into account computations similar to ones involved in derivations of (3.67). Analogously,

$$
\begin{aligned}
\mathcal{P}_2^{(k)} &= \gamma\neg\left(\sum_{k=1}^{N-n-1}\bigwedge_{j=1}^{N-n-1}H^j[k]\wedge(\alpha)_{2+2n-N}^{k+2n+1-N}\right)\\
&= \binom{k+2n+1-N}{2n+2-N}(\alpha_{0,0})^{k-1}\gamma\neg\left(\sum_{k=1}^{N-n-1}\bigwedge_{j=1}^{N-n-1}H^j[k]\wedge(\alpha_{1,1})^{2n+2-N}\right)\\
&= \binom{k+2n+1-N}{2n+2-N}(\alpha_{0,0})^{k-1}\mathcal{P}_2, \qquad (3.110)
\end{aligned}
$$

Combining the above result with the form $\mathcal{P}_2^T$ given explicitly in (3.68), one has that the toric analogue $(\mathcal{P}_2^{(k)})^T$ of $\mathcal{P}_2^{(k)}$ on $T\times T$ is

$$
\begin{aligned}
(\mathcal{P}_2^{(k)})^T = &\left(-\frac{1}{2\pi i}\right)^n\frac{n!\binom{k+2n+1-N}{2n+2-N}(\alpha_{0,0})^{k-1}}{2^{N-n-1}|\tau_1|^2\cdots|\tau_n|^2P(|\tau|^2)^{n+1}}d\bar{\tau}\wedge d\tau\\
&\sum_{k=1}^{N-n-1}\Bigg\{\Bigg[-\frac{\bar{\tau}^{m_{k2}}\left(\tau^{m_{k1}}t^{m_{k2}}+\tau^{m_{k2}}t^{m_{k1}}-\tau^{m_{k3}}t^{m_{k4}}-\tau^{m_{k4}}t^{m_{k3}}\right)}{\sum_i|\tau|^{2m_i}}+\alpha_{0,0}^T+1\Bigg]\\
&\sum_{|J|=n+1}\det{}^2(A_J)|\tau|^{2m_{j_1}+\cdots+2m_{j_{n+1}}}\\
&+\sum_{\substack{1\le p\le n\\1\le q\le n}}(-1)^{p-q-1}\left(m_{k2}^q\bar{\tau}^{m_{k2}}-\frac{\bar{\tau}^{m_{k2}}\sum_i m_i^q|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}}\right)\\
&\Bigg[-\frac{\left(\tau^{m_{k1}}t^{m_{k2}}+\tau^{m_{k2}}t^{m_{k1}}-\tau^{m_{k3}}t^{m_{k4}}-\tau^{m_{k4}}t^{m_{k3}}\right)\sum_i m_i^p|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}}\\
&+\left(m_{k1}^p\tau^{m_{k1}}t^{m_{k2}}+m_{k2}^p\tau^{m_{k2}}t^{m_{k1}}-m_{k3}^p\tau^{m_{k3}}t^{m_{k4}}-m_{k4}^p\tau^{m_{k4}}t^{m_{k3}}\right)\Bigg]\\
&\sum_{|J|=n}\det(A_{J,p})\det(A_{J,q})|\tau|^{2m_{j_1}+\cdots+2m_{j_n}}\Bigg\}.
\end{aligned}
$$

In particular, if $N=n+2$ then no any other terms except of $\mathcal{P}_1^{(k)}$ and $\mathcal{P}_2^{(k)}$ contribute to the construction of $\mathcal{P}^{(k)}$. Then, the expression of the Projection kernel with respect to toric variables is

$$
\begin{aligned}
(\mathcal{P}^{(k)})^T &= (\mathcal{P}_1^{(k)})^T + (\mathcal{P}_2^{(k)})^T \\
&= \left(-\frac{1}{2\pi i}\right)^n \frac{n!\binom{k+2n+1-N}{2n+2-N}(\alpha_{0,0})^{k-1}}{2^{N-n-1}|\tau_1|^2\cdots|\tau_n|^2 P(|\tau|^2)^{n+1}} d\bar{\tau}\wedge d\tau \\
&\quad \Bigg\{\left(\frac{2n+2-N}{k} + N - n - 1\right)\alpha_{0,0} \\
&\quad + \sum_{k=1}^{N-n-1}\left[-\frac{\bar{\tau}^{m_{k2}}\left(\tau^{m_{k1}}t^{m_{k2}} + \tau^{m_{k2}}t^{m_{k1}} - \tau^{m_{k3}}t^{m_{k4}} - \tau^{m_{k4}}t^{m_{k3}}\right)}{\sum_i |\tau|^{2m_i}} + 1\right] \\
&\quad \sum_{|J|=n+1} \det{}^2(A_J)|\tau|^{2m_{j_1}+\cdots+2m_{j_{n+1}}} \\
&\quad + \sum_{k=1}^{N-n-1} \sum_{\substack{1\le p\le n \\ 1\le q\le n}} (-1)^{p-q-1}\left(m_{k2}^q\bar{\tau}^{m_{k2}} - \frac{\bar{\tau}^{m_{k2}}\sum_i m_i^q|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}}\right) \\
&\quad \Bigg[-\frac{\left(\tau^{m_{k1}}t^{m_{k2}} + \tau^{m_{k2}}t^{m_{k1}} - \tau^{m_{k3}}t^{m_{k4}} - \tau^{m_{k4}}t^{m_{k3}}\right)\sum_i m_i^p|\tau|^{2m_i}}{\sum_i|\tau|^{2m_i}} \\
&\quad + \left(m_{k1}^p\tau^{m_{k1}}t^{m_{k2}} + m_{k2}^p\tau^{m_{k2}}t^{m_{k1}} - m_{k3}^p\tau^{m_{k3}}t^{m_{k4}} - m_{k4}^p\tau^{m_{k4}}t^{m_{k3}}\right)\Bigg] \\
&\quad \sum_{|J|=n} \det(A_{J,p})\det(A_{J,q})|\tau|^{2m_{j_1}+\cdots+2m_{j_n}}\Bigg\}.
\end{aligned}
$$

Similar observations lead to the connections between the $\mathcal{K}_1^{(k)}$, $\mathcal{K}_2^{(k)}$, $\mathcal{K}_3^{(k)}$ and $\mathcal{K}_4^{(k)}$ with the corresponding one for $k=1$. Firstly, in view of Lemma 3.4.4 and Lemma 3.4.5,

$$
\mathcal{K}_1^{(k)} = \frac{\binom{k+2n+1-N}{2n-N}}{\binom{2n+2-N}{2n-N}}\alpha_{0,0}^{k-1}\mathcal{K}_1 \quad \text{and} \quad \mathcal{K}_2^{(k)} = \frac{\binom{k+2n+1-N}{2n-N}}{\binom{2n+2-N}{2n-N}}\alpha_{0,0}^{k-1}\mathcal{K}_2 \tag{3.111}
$$

while the forms $\mathcal{K}_3^{(k)}$ and $\mathcal{K}_4^{(k)}$ arising from Lemma 3.4.6 and Lemma 3.4.7 are equal to

$$
\mathcal{K}_3^{(k)} = \frac{\binom{k+2n+1-N}{2n+1-N}}{2n+2-N}\alpha_{0,0}^{k-1}\mathcal{K}_3 \quad \text{and} \quad \mathcal{K}_4^{(k)} = \binom{k+2n+1-N}{2n+2-N}\alpha_{0,0}^{k-1}\mathcal{K}_4, \tag{3.112}
$$

with respect to the projective variables on $\mathbb{P}^{N-1}\times\mathbb{P}^{N-1}$. By transferring the results into toric variables, corresponding relations arise between their toric analogues on $T\times T$.

Moreover, in view of Lemma 3.4.7, if $N=n+2$, then $\mathcal{K}_4^{(k)}$ term does not exist since $i=2$ while $N-n-1=1$. Thus,

$$
\mathcal{K}^{(k)} = \mathcal{K}_1^{(k)} + \mathcal{K}_2^{(k)} + \mathcal{K}_3^{(k)}. \tag{3.113}
$$

Theorem 3.6.2 *Let X be an n-dimensional smooth compact toric variety and $\phi\in\mathcal{E}_{0,q}(X,(V_{\mathcal{L}})^k)$ for $k\ge 1$, where $V_{\mathcal{L}}$ is the induced line bundle of an ample divisor D with*

$\mathcal{L} = \mathcal{O}_X(D)$ *satisfying* $|P_D \cap \mathbb{Z}^n| = N \leq 2n + k + 1$. *Then, for* $h \in U_\sigma = \{\prod_{k=1}^d h_k^{a_k} \neq 0\}$

$$C_{N,n}\phi(h) = \int_X \mathcal{K}^{(k)}_{\mathrm{hom}} \wedge \bar{\partial}\phi + \bar{\partial}_h \int_X \mathcal{K}^{(k)}_{\mathrm{hom}} \wedge \phi + \int_X \mathcal{P}^{(k)}_{\mathrm{hom}} \wedge \phi, \tag{3.114}$$

where $\mathcal{P}^{(k)}_{\mathrm{hom}}$ *and* $\mathcal{K}^{(k)}_{\mathrm{hom}}$ *are the 'homogenization' of* $(\mathcal{P}^{(k)})^T$ *and* $(\mathcal{K}^{(k)})^T$ *on the corresponding chart, respectively with respect to the homogeneous coordinates* $h = (h_1, \ldots, h_d)$ *and* $\eta = (\eta_1, \ldots, \eta_d)$ *on* $X \times X$ *given by*

$$\mathcal{P}^{(k)}_{\mathrm{hom}} = \left(\frac{\prod_{i=1}^d h_i^{a_i}}{\prod_{i=1}^d \eta_i^{a_i}}\right)^k (\mathcal{P}^{(k)})^T \quad \textit{and} \quad (\mathcal{K}^{(k)})_{\mathrm{hom}} = \left(\frac{\prod_{i=1}^d h_i^{a_i}}{\prod_{i=1}^d \eta_i^{a_i}}\right)^k (\mathcal{K}^{(k)})^T,$$

after expressing $(\mathcal{P}^{(k)})^T$ *and* $(\mathcal{K}^{(k)})^T$ *with respect to the homogeneous coordinated through the rules* $t^{m_i} = \prod_{k=1}^d h_k^{\langle m_i, v_k\rangle}$ *and* $\tau^{m_i} = \prod_{k=1}^d \eta_k^{\langle m_i, v_k\rangle}$. *The kernels* $\mathcal{P}^{(k)}_{\mathrm{hom}}$ *and* $\mathcal{K}^{(k)}_{\mathrm{hom}}$ *take values in* $(V_{\mathcal{L}}^\vee)^k_{[\eta]} \otimes (V_{\mathcal{L}})^k_{[h]}$.

Proof. This result is a generalization of the representation (3.93) for $\phi \in \mathcal{E}_{0,q}(X, (V_{\mathcal{L}})^k)$. Observe that, here, the equation (3.95) connecting ϕ with the family $\{\phi_{\sigma_i}\}_i$ is reformulated to

$$\phi(\eta)\big|_{U_\sigma} = \left(\prod_{i=1}^d \eta_i^{a_i}\right)^k \phi_\sigma(\eta)$$

and hence the proof is similar to the corresponding one of Theorem 3.4.3. $\square$

Remark 3.6.2 On the one hand, Theorem 3.6.2 constitutes a generalization of Theorem 3.4.3 since by applying Theorem 3.6.2, one can derive integral formulas for sections with bigger homogeneities on varieties satisfying Theorem 3.4.3. In particular, one can take formulas for $(0, q)$ forms taking values in $(L^k)^\lambda \otimes (L^l)\lambda$ on $\mathbb{P}^1 \times \mathbb{P}^1$, whenever $\lambda \geq kl+k+l-4$ for $k, l, \lambda \in \mathbb{Z}^+$. This arises from example 3.1.1 yielding that $N-1 = kl+k+l$. Furthermore, integral formulas can be constructed on smooth $(0, q)$ forms taking values in the bundle $L^k \otimes L^k$ over $\mathbb{P}^n \times \mathbb{P}^m$, whenever $k \geq nm - n - m$ for $k, n, m \in \mathbb{Z}^+$. Here, we used the example 3.1.2 leading that the number of the integral points corresponding to the divisor $D = D_{n+1} + D_{n+m+2}$, is $N = nm + n + m + 1$. The formula 3.114 also holds on the Hirzebruch surface $\mathcal{H}$, for $(0, q)$ smooth forms taking values in $(V_{\mathcal{L}})^k$, where $\mathcal{L} = \mathcal{O}_{\mathcal{H}}(D_3 + 2D_4)$ and $k \in \mathbb{N}$ (see example 3.1.3).

On the other hand, Theorem 3.6.2 also provides integral formulas on smooth compact toric varieties failing to satisfy the necessary condition $N \leq 2(n+1)$ of Theorem 3.4.3 for any line bundle corresponding to an ample divisor. An example of such variety is presented in the following example.

Example 3.6.1 We consider the 3-dimensional smooth compact toric variety X whose fan is generated by the vectors $v_1 = e_1, v_2 = e_2, v_3 = -e_1 - e_2, v_4 = e_3, v_5 = e_1 + 2e_2 - e_3$. The six maximal cones composing its fan are $\sigma_1 = \mathrm{Cone}(e_1, e_2, e_3)$, $\sigma_2 = \mathrm{Cone}(e_1, e_2, e_1 + 2e_2 - e_3)$, $\sigma_3 = \mathrm{Cone}(e_1, -e_1 - e_2, e_3)$, $\sigma_4 = \mathrm{Cone}(e_1, -e_1 - e_2, e_1 + 2e_2 - e_3)$, $\sigma_5 = \mathrm{Cone}(e_2, -e_1 - e_2, e_3)$ and $\sigma_6 = \mathrm{Cone}(e_2, -e_1 - e_2, e_1 + 2e_2 - e_3)$. Since,

$$\mathrm{Pic}(X) = \mathrm{Cl}(X) \simeq \{kD_3 + lD_5 | k, l \in \mathbb{Z}\},$$

we take $D = kD_3 + lD_5$. Then, the condition (1.15) yields that $m_{\sigma_1} = (0,0,0)$, $m_{\sigma_2} = (0,0,l)$, $m_{\sigma_3} = (0,k,0)$, $m_{\sigma_1} = (0,k,2k+l)$, $m_{\sigma_5} = (k,0,0)$ and $m_{\sigma_6} = (k,0,k+l)$ while D is ample if and only of $k, l > 0$. Hence, by choosing $k = l = 1$, the integral points of the corresponding polyhedron are

$$P_D \cap \mathbb{Z}^3 = \{(0,0,0), (0,0,1), (0,1,0), (0,1,3), (1,0,0), (1,0,2), (1,0,1), (0,1,1), (0,1,2)\}.$$

Thus, $N = 9 > 2(n+1) = 8$ and X does not satisfy Theorem 3.4.3, under these conditions. However, for $k \geq 2$, the number N satisfies the inequality $N \leq 2n + k + 1$ and thus Theorem 3.6.2 derives well-defined integral representation formulas for $(0,q)$ forms taking values in $(V_{\mathcal{L}})^k$, where $\mathcal{L} = \mathcal{O}_X(D_3 + D_5)$.

3.7 Examples

This paragraph is a continuation of the Section 3.5, generalizing the examples related to $\mathbb{P}^1 \times \mathbb{P}^1$ and Hirzebruch surface $\mathcal{H}$.

Example 3.7.1 (Koppelman formula for $(0,q)$ forms taking values in $L^k \otimes L^k$ on $\mathbb{P}^1 \times \mathbb{P}^1$). If $\phi \in \mathcal{E}_{0,q}(\mathbb{P}^1 \times \mathbb{P}^1, L^k \otimes L^k)$, then the Projection kernel $(\mathcal{P}^{(k)})^T$ of an $\phi \in \mathcal{E}_{0,q}(\mathbb{P}^1 \times \mathbb{P}^1, L^k \otimes L^k)$ is also obtained in view of Remark 3.6.1. More precisely, by similar computations with Example 3.5.1, the Projection kernel with respect to the toric variables on $T \times T$ is given by

$$\begin{aligned}
(\mathcal{P}^{(k)})^T &= \left(-\frac{1}{2\pi i}\right)^2 \frac{\frac{(k+1)k}{2}(\alpha^T_{0,0})^{k-1}}{(1+|\tau_1|^2)^3(1+|\tau_2|^2)^3}\, d\bar{t} \wedge d\tau \\
&\quad \left[\left(\frac{2+k}{k}\right)(1+\bar{\tau}_1 t_1)(1+\bar{\tau}_2 t_2) + (1+\bar{\tau}_1 t_1)(1+\bar{\tau}_2 t_2)\right] \\
&= \left(-\frac{1}{2\pi i}\right)^2 \frac{(k+1)^2(1+\bar{\tau}_1 t_1)^k(1+\bar{\tau}_2 t_2)^k}{(1+|\tau_1|^2)^{k+2}(1+|\tau_2|^2)^{k+2}}\, d\bar{\tau} \wedge d\tau.
\end{aligned}$$

By using (3.100), one can write $(\mathcal{P}^{(k)})^T$ with respect to the homogeneous coordinates as follows:

$$
\begin{aligned}
(\mathcal{P}^{(k)})^T &= \left(-\frac{1}{2\pi i}\right)^2 \left(\frac{\eta_2\eta_4}{h_2h_4}\right)^k \frac{(k+1)^2(\bar{\eta}_1h_1+\bar{\eta}_2h_2)^k(\bar{\eta}_3h_3+\bar{\eta}_4h_4)^k}{(|\eta_1|^2+|\eta_2|^2)^{k+2}(|\eta_3|^2+|\eta_4|^2)^{k+2}} \times \\
&\quad (\bar{\eta}_2d\bar{\eta}_1-\bar{\eta}_1d\bar{\eta}_2)\wedge(\eta_2d\eta_1-\eta_1d\eta_2)\wedge(\bar{\eta}_4d\bar{\eta}_3-\bar{\eta}_3d\bar{\eta}_4)\wedge(\eta_4d\eta_3-\eta_3d\eta_4),
\end{aligned}
$$

while Theorem 3.6.2 yields

$$
\begin{aligned}
\mathcal{P}^{(k)}_{\mathrm{hom}} &= \left(\frac{h_2h_4}{\eta_2\eta_4}\right)^k (\mathcal{P}^k)^T \\
&= \left(-\frac{1}{2\pi i}\right)^2 \frac{(k+1)^2(\bar{\eta}_1h_1+\bar{\eta}_2h_2)^k(\bar{\eta}_3h_3+\bar{\eta}_4h_4)^k}{(|\eta_1|^2+|\eta_2|^2)^{k+2}(|\eta_3|^2+|\eta_4|^2)^{k+2}} \times \\
&\quad (\bar{\eta}_2d\bar{\eta}_1-\bar{\eta}_1d\bar{\eta}_2)\wedge(\eta_2d\eta_1-\eta_1d\eta_2)\wedge(\bar{\eta}_4d\bar{\eta}_3-\bar{\eta}_3d\bar{\eta}_4)\wedge(\eta_4d\eta_3-\eta_3d\eta_4)
\end{aligned}
$$

on the chart $U_{\sigma_{00}}$ of $\mathbb{P}^1\times\mathbb{P}^1$.

As in Example 1.1.7, we observe that our kernel holds on the whole variety and it is identified with the corresponding one in a Götmark's work [14].

Moreover, according to Remark 3.6.1, the Koppelman kernel of the representation on $T\times T$ is

$$
(\mathcal{K}^{(k)})^T = (\mathcal{K}_1^{(k)})^T + (\mathcal{K}_2^{(k)})^T + (\mathcal{K}_3^{(k)})^T,
$$

where $(\mathcal{K}_1^{(k)})^T$, $(\mathcal{K}_2^{(k)})^T$, $(\mathcal{K}_3^{(k)})^T$ are expressed with respect to $\mathcal{K}_1^T$, $\mathcal{K}_2^T$ and $\mathcal{K}_3^T$ (the explicit forms of the second ones are given in Example 3.5.1), according to the analogous relations of (3.111) and (3.112) in toric variables. More precisely,

$$
(\mathcal{K}_1^{(k)})^T = (\alpha_{0,0}^T)^{k-1}\mathcal{K}_1^T,\ (\mathcal{K}_2^{(k)})^T = (\alpha_{0,0}^T)^{k-1}\mathcal{K}_2^T \quad \text{and} \quad (\mathcal{K}_3^{(k)})^T = \frac{k+1}{2}(\alpha_{0,0}^T)^{k-1}\mathcal{K}_3^T,
$$

where $\alpha_{0,0}^T = \frac{(1+\bar{\tau}_1t_1)(1+\bar{\tau}_2t_2)}{(1+|\tau_1|^2)(1+|\tau_2|^2)}$. The kernel $\mathcal{K}^{(k)}_{\mathrm{hom}}$ on $U_{\sigma_{00}}$ is given by

$$
\mathcal{K}^{(k)}_{\mathrm{hom}} = \left(\frac{h_2h_4}{\eta_2\eta_4}\right)^k (\mathcal{K}^{(k)})^T,
$$

after applying the rules (3.100) in $(\mathcal{K}^{(k)})^T$. In this specific example, the kernel $\mathcal{K}^{(k)}_{\mathrm{hom}}$ constitutes a global form as the kernel $\mathcal{P}^{(k)}_{\mathrm{hom}}$.

Example 3.7.2 (Koppelman formula for $(0,q)$ forms taking values in $(V_{\mathcal{L}})^k$ on $\mathcal{H}$). Let us recall Example 3.5.2. By considering an $\phi\in\mathcal{E}_{0,q}(\mathcal{H},(V_{\mathcal{L}})^k)$ where $\mathcal{L}=\mathcal{O}_{\mathcal{H}}(D_3+2D_4)$, one can observe that the kernel $(\mathcal{P}^{(k)})^T$ constitutes from the forms $(\mathcal{P}_1^{(k)})^T$ and $(\mathcal{P}_2^{(k)})^T$ following the rules (3.109) and (3.110), respectively such that

$$(\mathcal{P}_1^{(k)})^T = (\alpha_{0,0}^T)^{k-1}\mathcal{P}_1^T \quad \text{and} \quad (\mathcal{P}_2^{(k)})^T = (\alpha_{0,0}^T)^{k-1}\mathcal{P}_2^T$$

and an extra term denoted by $(P_3^{(k)})^T$ which arises from the toric analogue of

$$\begin{aligned}\mathcal{P}_3^{(k)} &:= \gamma\neg\left(H_{1,0}^1 \wedge H_{1,0}^2 \wedge \alpha_{2,2}^k\right)\\ &= \binom{k}{k-2}\alpha_{0,0}^{k-2}\gamma\neg\left(H_{1,0}^1 \wedge H_{1,0}^2 \wedge (\alpha_{1,1})^2\right)\end{aligned}$$

where $\gamma = \gamma_2 \wedge \gamma_1$, such that

$$\begin{aligned}(P_3^{(k)})^T &= \binom{k}{k-2}(\alpha_{0,0}^T)^{k-2} \times\\ &\sum_{1=k_1<k_2=2}\sum_{l=0}^{2}\operatorname{sign}l\bigwedge_{m=1}^{2}\left(\gamma_{\tau(k_m)}\neg H_{1,0}^{k_m}[l]\right)^T P(2,l)(\alpha_{1,1}^{[l]})^T(\alpha_{1,1}^T)^{2-l}.\end{aligned} \tag{3.115}$$

The sign$l$ is depending on the order of the action of the l vector fields on the $(1,0)$ Hefer forms and on the order of the action of the $(2-l)$ vector fields on the corresponding number of the forms $\alpha_{1,1}$. The involved forms were given briefly in the Example 3.5.2. Hence, the Projection kernel is the sum

$$\begin{aligned}(\mathcal{P}^{(k)})^T &= (\mathcal{P}_1^{(k)})^T + (\mathcal{P}_2^{(k)})^T + (\mathcal{P}_3^{(k)})^T\\ &= (\alpha_{0,0}^T)^{k-1}\mathcal{P}_1^T + (\alpha_{0,0}^T)^{k-1}\mathcal{P}_2^T + (\mathcal{P}_3^{(k)})^T\\ &= (\alpha_{0,0}^T)^{k-1}\mathcal{P}^T + (\mathcal{P}_3^{(k)})^T,\end{aligned}$$

where $\mathcal{P}^T$ is the Projection kernel for representing $(0,q)$ forms taking values in $V_{\mathcal{L}}$, expressed briefly in (3.102), $(\mathcal{P}_3^{(k)})^T$ is given by (3.115) and $\alpha_{0,0}^T = \frac{1+\bar{\tau}_1 t_1+\bar{\tau}_2 t_2+\bar{\tau}_1\bar{\tau}_2 t_1 t_2+\bar{\tau}_2^2 t_2^2}{1+|\tau_1|^2+|\tau_1|^2|\tau_2|^2+|\tau_2|^2+|\tau_2|^4}$.

On the other hand, the Koppelman kernel of this representation is given by

$$(\mathcal{K}^{(k)})^T \;=\; (\mathcal{K}_3^{(k)})^T + (\mathcal{K}_4^{(k)})^T$$

with no extra terms added such that

$$(\mathcal{K}_3^{(k)})^T = (\alpha_{0,0}^T)^{k-1}\mathcal{K}_3^T \quad \text{and} \quad (\mathcal{K}_4^{(k)})^T = (\alpha_{0,0}^T)^{k-1}\mathcal{K}_4^T,$$

($\mathcal{K}_3^T$ and $\mathcal{K}_4^T$ are expressed in (3.104) and (3.105) respectively) due to the equations given in (3.112). Hence

$$(\mathcal{K}^{(k)})^T = (\alpha_{0,0}^T)^{k-1}\mathcal{K}^T,$$

where $\mathcal{K}^T$ is the Koppelman kernel in the case of representing smooth $(0, q)$ forms taking values in $V_{\mathcal{L}}$.

Chapter 4

Some applications of Koppelman toric integral representation formula

4.1 Cohomology of line bundles on Toric varieties

Using the toric integral representation formula derived in the third chapter, we are going to construct explicit solution of the $\bar{\partial}$-problem on an n-dimensional smooth compact toric variety X. This leads to the vanishing of the Dolbeault cohomology groups $H^{0,q}(X, V_{\mathcal{L}})$ of $(0,q)$-forms, $0 < q \leq n$, that take values in a line bundle $V_{\mathcal{L}}$ corresponding to the sheaf $\mathcal{L} = \mathcal{O}_X(D)$ of an ample divisor D, for some cases.

The Dolbeault cohomology groups $H^{0,q}(X, V_{\mathcal{L}})$ of $(0,q)$ forms taking values in $V_{\mathcal{L}}$ over X is defined by

$$H^{0,q}(X, V_{\mathcal{L}}) := \frac{\mathrm{Ker}\{\bar{\partial} : \mathcal{E}_{0,q}(X, V_{\mathcal{L}}) \to \mathcal{E}_{0,q+1}(X, V_{\mathcal{L}})\}}{\mathrm{Im}\{\bar{\partial} : \mathcal{E}_{0,q-1}(X, V_{\mathcal{L}}) \to \mathcal{E}_{0,q}(X, V_{\mathcal{L}})\}},$$

for the chain complex

$$\cdots \xrightarrow{\bar{\partial}} \mathcal{E}_{0,q-1}(X, V_{\mathcal{L}}) \xrightarrow{\bar{\partial}} \mathcal{E}_{0,q}(X, V_{\mathcal{L}}) \xrightarrow{\bar{\partial}} \mathcal{E}_{0,q+1}(X, V_{\mathcal{L}}) \to \cdots .$$

The Dolbeault operator is the $\bar{\partial}$-operator, acting on smooth forms taking values in $V_{\mathcal{L}}$.

Applying Dolbeault's Theorem ([10]), one can identify the Dolbeault cohomology group $H^{0,q}(X, V_{\mathcal{L}})$ of $(0,q)$-forms with the q-th sheaf cohomology group $H^q(X, \mathcal{O}_X(D))$ of the sheaf $\mathcal{O}_X(D)$, due to an isomorphism that exists between the two groups. Following [6], the q-th cohomology group $H^q(X, \mathcal{F})$ of a sheaf $\mathcal{F}$ on X is realised through the following exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{G}^0 \xrightarrow{d^0} \mathcal{G}^1 \xrightarrow{d^1} \mathcal{G}^2 \xrightarrow{d^2} \cdots$$

that leads to the complex of global sections

$$\Gamma(X, \mathcal{G}^0) \xrightarrow{d^0} \Gamma(X, \mathcal{G}^1) \xrightarrow{d^1} \Gamma(X, \mathcal{G}^2) \xrightarrow{d^2} \cdots$$

such that $d^{q+1} \circ d^q = 0$, for all $q \geq 0$. Hence, $H^q(X, \mathcal{F}) = \mathrm{Ker}(d^q)/\mathrm{Im}(d^{q-1})$, where d^{-1} is the zero map $0 \to \Gamma(X, \mathcal{G}^0)$.

There are several treatments of $H^q(X, \mathcal{O}_X(D))$ and some of them can be found in [6] and [34]. M. Demazure in [9] proved that the group $H^q(X, \mathcal{O}_X(D))$ is trivial for $q > 0$, whenever D is a Cartier divisor with no basepoints (see Definition 1.1.6) on a smooth compact toric variety X. This is known as the *Demazure Vanishing Theorem*. We present in brief this result, following the lines of [34]. Let

$$Z_{D,m} := \{u \in N_{\mathbb{R}} : \langle m, u \rangle \geq \phi_D(u)\},$$

where $\phi_D : |\Sigma| \to \mathbb{R}$ is a linear function such that $\phi_D(v_i) = -a_i$ $(D = \sum_{i=1}^d a_i D_i)$, for each generator v_i, $i = 1, \ldots, d$, of the fan. A natural decomposition of the sheaf cohomology yields that

$$H^q(X, \mathcal{O}_X(D)) = \bigoplus_{m \in M} H^q_{Z_{D,m}}(N_{\mathbb{R}}, \mathbb{C})_{\chi^m}, \tag{4.1}$$

where $H^q_{Z_{D,m}}(N_{\mathbb{R}}, \mathbb{C})_{\chi^m}$ is the cohomology group of $N_{\mathbb{R}}$ with repect to the character χ^m, with support $Z_{D,m}$ and with coefficients in $\mathbb{C}$. The group $H^q_{Z_{D,m}}(N_{\mathbb{R}}, \mathbb{C})$ is defined as the relative cohomology group $H^q(N_{\mathbb{R}} \setminus Z_{D,m}, \mathbb{C})$ through a long exact sequence. Since $N_{\mathbb{R}} \setminus Z_{D,m}$ is either empty (if $m \in P_D$) or a convex set (if $m \in M \setminus P_D$), this set is contractible leading that the group $H^q(N_{\mathbb{R}} \setminus Z_{D,m}, \mathbb{C})$ is trivial for $q > 0$. Thus, $H^q_{Z_{D,m}}(N_{\mathbb{R}}, \mathbb{C})$ is also trivial for all $q > 0$ and $m \in M$. In view of (4.1), $H^q(X, \mathcal{O}_X(D)) = 0$ for $q > 0$. Applying Dolbeault's Theorem, this result can be reformulated as

$$H^{0,q}(X, V_{\mathcal{L}}) = 0, \quad \text{for every} \quad q > 0.$$

Our contribution is an alternative proof of the above result, based on the 'original' definition of the Dolbeault cohomology. Our approach constructs explicit solutions of the $\bar{\partial}$-equation on the varieties under study, thus illustrating the Demazure Vanishing Theorem for some cases from an analytic point of view.

Theorem 4.1.1 *Let X be an n-dimensional smooth compact toric variety and $V_{\mathcal{L}}$ be a line bundle over X corresponding to an ample divisor D with $|P_D \cap \mathbb{Z}^n| = N \leq 2(n+1)$. Then, the cohomology group $H^{0,q}(X, V_{\mathcal{L}})$ of $(0,q)$-forms, $0 < q \leq n$, is trivial.*

Proof. The result follows from the integral representation formula (3.93) of $\phi \in \mathcal{E}_{0,q}(X, V_{\mathcal{L}})$ on every chart U_σ derived in Theorem 3.4.3. It is sufficient to construct a solution of the $\bar{\partial}$-equation $\bar{\partial}_h \omega = \phi$ on every chart U_σ and verify that the solution is compatible on the intersection of the charts. Let ϕ be a $\bar{\partial}$-closed $(0,q)$-form for $0 < q \leq n$ taking values in $V_{\mathcal{L}}$ on X. Since $\bar{\partial}\phi = 0$, the integral $\int_X \mathcal{K}_{\text{hom}} \wedge \bar{\partial}\phi$ vanishes on every chart U_σ of X. The integral $\int_X \mathcal{P}_{\text{hom}} \wedge \phi$ also vanishes on every chart U_σ, because the left hand side of (3.93) is just the form $\phi(h)$ of bidegree $(0,q)$ that contains differentials of h variable, while $\int_X \mathcal{P}_{\text{hom}} \wedge \phi$ includes only differentials of η variable. This is due to the fact that the kernel $\mathcal{P}_{\text{hom}}$ is the homogenization of $\mathcal{P}^T$, where the second one has only differentials of τ variable, in view of Theorem 3.4.1. Hence, in view of (3.93)

$$C_{N,n}\phi(h) = \bar{\partial}_h \int_X \mathcal{K}_{\text{hom}} \wedge \phi \tag{4.2}$$

on U_σ. It implies that the form $\omega_\sigma(h) = C_{N,n} \int_X \mathcal{K}_{\text{hom},U_\sigma} \wedge \phi$ is the solution of the $\bar{\partial}$-equation on U_σ. Moreover, on the intersection $U_{\sigma_i} \cap U_{\sigma_j}$ of two charts, since $\phi|_{U_{\sigma_i}} = \phi|_{U_{\sigma_j}}$ (see (3.54)) and $\mathcal{K}_{\text{hom},U_{\sigma_i}} = \mathcal{K}_{\text{hom},U_{\sigma_j}}$ (see (3.98)), $\omega_{\sigma_i} = \omega_{\sigma_j}$. Therefore, the cohomology group $H^{0,q}(X, V_{\mathcal{L}})$ for $0 < q \leq n$ is trivial. □

A generalization of the above result is also obtained.

Theorem 4.1.2 *The cohomology group $H^{0,q}(X, (V_{\mathcal{L}})^k)$ for $0 < q \leq n$ is trivial, when $N \leq 2n + k + 1$ and k is a positive integer.*

Proof. The proof follows along arguments, similar to those in the proof of Theorem 4.1.1. □

Some illustration examples are presented below.

Example 4.1.1 Let X be the product $\mathbb{P}^1 \times \mathbb{P}^1$ and consider the space of $(0,q)$ forms taking values in $L^k \otimes L^l$. According to the suitable projective embedding (see Example 3.1.1), $N - 1 = kl + k + l$ and since N must satisfy the inequality $N \leq 2(n+1)$ (according to the assumptions of Theorem 4.1.1), then $kl + k + l \leq 5$ ($n = 2$ and $N = 6$). Then, $k = l = 1$ or $k = 1, l = 2$ or $k = 2, l = 1$. Now, by applying Theorem 4.1.1, for the specific choices of k and l, the cohomology groups $H^{0,q}(\mathbb{P}^1 \times \mathbb{P}^1, L^k \otimes L^l)$ for $0 < q \leq 2$ are trivial. Generally, in view of Theorem 4.1.2, $H^{0,q}(\mathbb{P}^1 \times \mathbb{P}^1, (L^k)^\lambda \otimes (L^l)^\lambda)$ is trivial for k, l and λ which are positive integers satisfying the inequality $\lambda \geq kl + k + l - 4$. In view of (4.2), $\phi(z)$ takes the form

$$\phi(h) = \bar{\partial}_h \int_X \mathcal{K}_{\text{hom}} \wedge \phi.$$

Example 4.1.2 According to the Example 3.1.2, the divisor $D = D_{n+1} + D_{n+m+2}$ of $\mathbb{P}^n \times \mathbb{P}^m$ lead to a projective embedding where $N = nm + n + m + 1$. Then, Theorem 4.1.2 provides that $H^{0,q}(\mathbb{P}^n \times \mathbb{P}^m, L^k \otimes L^k) = 0$ for $0 < q \leq n + m$ and k, n, m satisfying the inequality $k \geq nm - n - m$.

Example 4.1.3 In the case of $(\mathbb{P}^1)^3 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the integral vectors $(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1), (1,1,1)$ of the associated polyhedron (for $D = D_2 + D_4 + D_6$) induce an embedding in $\mathbb{P}^7$. Theorem 4.1.2 yields that $H^{0,q}((\mathbb{P}^1)^3, L^k \otimes L^k \otimes L^k) = 0$ for $0 < q \leq 3$ and k being every positive integer.

Example 4.1.4 The divisor $D = D_3 + D_4$ on Hirzebruch surface H_r leads to embedding in $\mathbb{P}^4$ and in $\mathbb{P}^5$ when $r = 1$ and $r = 2$, respectively. Let $V_{\mathcal{L}}^{(r)}$ be the corresponding bundle in each case. Hence, both cases satisfy that $N \leq 2(n+1)$. Applying Theorem 4.1.2, the cohomology groups $H^{0,q}(H_r, (V_{\mathcal{L}}^{(r)})^k) = 0$ for $0 < q \leq 2$, $r = 1, 2$ and k being every positive integer.

Now, recall the Example 3.2.2 of the particular case of a Hirzebruch surface denoted by $\mathcal{H}$.

Example 4.1.5 The application of Theorem 4.1.2 on Example 3.2.2, yields that $H^{0,q}(\mathcal{H}, (V_{\mathcal{L}})^k)$ for $0 < q \leq 2$ and $k \in \mathbb{N}$ is trivial, where $\mathcal{L} = \mathcal{O}_{\mathcal{H}}(D_3 + 2D_4)$.

4.2 Cohomology of the dual bundle on Toric varieties

The dual nature of the Koppelman formula allows also the study of the cohomology group $H^{n,q}(X, V_{\mathcal{L}}^\vee)$ and $H^{n,q}(X, (V_{\mathcal{L}}^\vee)^k)$ for $0 \leq q < n$, where $V_{\mathcal{L}}^\vee$ is the dual bundle of $V_{\mathcal{L}}$ corresponding to the dual sheaf $\mathcal{L}^\vee$ while $(V_{\mathcal{L}}^\vee)^k = \underbrace{V_{\mathcal{L}}^\vee \otimes \cdots \otimes V_{\mathcal{L}}^\vee}_{k-\text{times}}$. The dual sheaf $\mathcal{L}^\vee$ satisfies $\mathcal{L} \otimes \mathcal{L}^\vee = \mathcal{O}_X = \mathbb{C}$ (recall that $\mathcal{O}_X$ is the sheaf of holomorphic functions, that are the constant ones) and since $\mathcal{L} = \mathcal{O}_X(D)$, then $\mathcal{L}^\vee = \mathcal{O}_X(D)^\vee \simeq \mathcal{O}_X(-D)$ (Proposition 8.0.6 in [6]). Thus, $V_{\mathcal{L}}^\vee$ is the line bundle corresponding to the divisor $-D$.

Let us denote by $< .,. >$ the duality pairing in the Koppelman representation formula and ψ be the symmetric form of ϕ, meaning that if $\phi \in \mathcal{E}_{0,n-q}(X, V_{\mathcal{L}})$ for $0 \leq q \leq n$ then $\psi \in \mathcal{E}_{n,q}(X, V_{\mathcal{L}}^\vee)$, in view of the preceding paragraph.

Theorem 4.2.1 *Let X be an n-dimensional smooth compact toric variety and $V_{\mathcal{L}}^{\vee}$ be the dual bundle of $V_{\mathcal{L}}$, where $\mathcal{L} = \mathcal{O}_X(D)$ with $|P_D \cap \mathbb{Z}^n| = N \leq 2(n+1)$. If $\psi \in \mathcal{E}_{n,q}(X, V_{\mathcal{L}}^{\vee})$, $0 \leq q \leq n$, then*

$$C_{N,n}\psi(\eta) = \bar{\partial}_\eta \int_X \mathcal{K}_{\mathrm{hom}} \wedge \psi + \int_X \mathcal{K}_{\mathrm{hom}} \wedge \bar{\partial}_h \psi + \int_X \mathcal{P}_{\mathrm{hom}} \wedge \psi \tag{4.3}$$

on the chart $U_\sigma = \{\prod_{k=1}^{d} \eta_k^{a_k} \neq 0\}$ of X, where $\mathcal{P}_{\mathrm{hom}}$ and $\mathcal{K}_{\mathrm{hom}}$ are the 'homogenization' of $\mathcal{P}^T$ and $\mathcal{K}^T$ on U_σ, respectively with respect to the homogeneous coordinates on X, given in Theorem 3.4.3. The integrals in (4.3) *are taken over the h variable.*

Proof. Let $\phi \in \mathcal{E}_{0,n-q}(X, V_{\mathcal{L}})$ for $0 \leq q \leq n$ and ψ be the symmetric form of ϕ such that $\psi \in \mathcal{E}_{n,q}(X, V_{\mathcal{L}}^{\vee})$. Since $\phi(z)$ on U_σ can be represented by

$$C_{N,n} < \psi, \phi > = < \psi, \mathcal{K}_{\mathrm{hom}} \wedge \bar{\partial}_\eta \phi > + < \psi, \bar{\partial}_h(\mathcal{K}_{\mathrm{hom}} \wedge \phi) > + < \psi, \mathcal{P}_{\mathrm{hom}} \wedge \phi >$$

with respect to the dual pairing notation, according to Theorem 3.4.3 (formula (3.93)) then the properties of currents given in (1.17) and (1.18), derive the following formula:

$$\begin{aligned} C_{N,n} < \psi, \phi > &= (-1)^{n+q} < \mathcal{K}_{\mathrm{hom}} \wedge \psi, \bar{\partial}_\eta \phi > + (-1)^{n+q+1} < \bar{\partial}_h \psi, \mathcal{K}_{\mathrm{hom}} \wedge \phi > \\ &\quad + < \mathcal{P}_{\mathrm{hom}} \wedge \psi, \phi > \\ &= < \bar{\partial}_\eta(\mathcal{K}_{\mathrm{hom}} \wedge \psi), \phi > + < \mathcal{K}_{\mathrm{hom}} \wedge \bar{\partial}_h \psi, \phi > + < \mathcal{P}_{\mathrm{hom}} \wedge \psi, \phi > \end{aligned}$$

on U_σ. Thus, the formula (4.3) is obtained. □

Remark 4.2.1 In view of Theorem 3.6.2, if $\psi \in \mathcal{E}_{n,q}(X, (V_{\mathcal{L}}^{\vee})^k)$ and $N \leq 2n + k + 1$, then

$$C_{N,n}\psi(\tau) = \bar{\partial}_\tau \int_T (\mathcal{K}^{(k)})^T \wedge \psi + \int_T (\mathcal{K}^{(k)})^T \wedge \bar{\partial}_t \psi + \int_T (\mathcal{P}^{(k)})^T \wedge \psi. \tag{4.4}$$

with respect to the toric variables (t, τ) on $T \times T$ while, after passing to the homogeneous coordinates (h, η) on $X \times X$, one has

$$C_{N,n}\psi(\eta) = \bar{\partial}_\eta \int_X \mathcal{K}_{\mathrm{hom}}^{(k)} \wedge \psi + \int_X \mathcal{K}_{\mathrm{hom}}^{(k)} \wedge \bar{\partial}_h \psi + \int_X \mathcal{P}_{\mathrm{hom}}^{(k)} \wedge \psi \tag{4.5}$$

on U_σ.

The Corollary that follows is a realization of the Toric Serre Duality (see Theorem 9.2.10 in [6]) for the varieties under study. In particular, the Toric Serre Duality in our case shows that there exists isomorphism exists between the cohomology groups of $(0, q)$ forms taking values in a line bundle $V_{\mathcal{L}}$ and the $(n, n-q)$ forms of complementary bidegree taking values in the dual bundle $V_{\mathcal{L}}^{\vee}$. Namely,

$$H^{0,q}(X, V_{\mathcal{L}}) \simeq H^{n,n-q}(X, V_{\mathcal{L}}^{\vee}) \quad \text{for} \quad 0 \leq q \leq n, \tag{4.6}$$

where $\mathcal{L} = \mathcal{O}_X(D)$ and $\mathcal{L}^{\vee} = \mathcal{O}_X(-D)$.

Corollary 4.2.1 *Let X be an n-dimensional smooth compact toric variety with D being ample divisor satisfying $|P_D \cap \mathbb{Z}^n| = N$ and $N \leq 2(n+1)$. If $V_{\mathcal{L}}^{\vee}$ is the dual bundle corresponding to the sheaf $L^{\vee} = \mathcal{O}_X(-D)$, then the cohomology group $H^{n,q}(X, V_{\mathcal{L}}^{\vee})$ is trivial, for $0 \leq q < n$. In general, $H^{n,q}(X, (V_{\mathcal{L}}^{\vee})^k)$ is trivial for $0 \leq q < n$ when $N \leq 2n + k + 1$ and k is a positive integer.*

Proof. If we consider ψ to be a $\bar{\partial}$-closed (n, q)-form for $0 \leq q < n$, then the second integral in (4.3) vanishes on every chart U_σ of X. Moreover, $\int_X \mathcal{P}_{\text{hom}} \wedge \psi = 0$ on every chart U_σ since there are not enough $d\bar{h}$'s. Actually, in the third integral of (4.3), $\mathcal{P}_{\text{hom}}$ is an (n, n) form in $d\eta$'s and $d\bar{\eta}$'s since $\mathcal{P}^T$ is an (n, n) form in $d\tau$'s and $d\bar{\tau}$'s and $\psi(\eta)$ is an (n, q) form in $d\eta$'s and $d\bar{\eta}$'s with $q \neq n$. Then,

$$C_{N,n}\psi(\eta) = \bar{\partial}_\eta \int_X \mathcal{K}_{\text{hom}} \wedge \psi \tag{4.7}$$

on U_σ, which means that $\omega_\sigma(\eta) = C_{N,n} \int_X \mathcal{K}_{\text{hom},U_\sigma} \wedge \psi$ is a solution of the $\bar{\partial}$-equation $\bar{\partial}\omega = \psi$ on U_σ. As in Theorem 4.1.1, the solutions ω_{σ_i} and ω_{σ_j} of the $\bar{\partial}$-equation on the two corresponding charts U_{σ_i} and U_{σ_j} respectively, are identified on its intersection $U_{\sigma_i} \cap U_{\sigma_j}$. Hence, the first result follows. The general statement holds similarly by following Remark 4.2.1. $\square$

The Corollary 4.2.1 is illustrated through the following examples.

Example 4.2.1 In view of Example 4.1.1, Corollary 4.2.1 yields that the groups $H^{2,q}(\mathbb{P}^1 \times \mathbb{P}^1, L^{-k} \otimes L^{-l})$ are trivial for $0 \leq q < 2$ and for the following choices of k and l: $k = l = 1$ or $k = 1, l = 2$ or $k = 2, l = 1$. In general, $H^{2,q}(\mathbb{P}^1 \times \mathbb{P}^1, (L^{-k})^\lambda \otimes (L^{-l})^\lambda) = 0$ whenever k, l and λ which are positive integers satisfying the inequality $\lambda \geq kl + k + l - 4$.

Example 4.2.2 Taking into account Example 4.1.2, Corollary 4.2.1 yields that the cohomology groups $H^{n+m,q}(\mathbb{P}^n \times \mathbb{P}^m, L^{-k} \otimes L^{-k})$ are trivial for $0 \leq q < n + m$ k, n, m being positive integers satisfying the inequality $k \geq nm - n - m$.

Example 4.2.3 Following Example 4.2.3, the groups $H^{3,q}((\mathbb{P}^1)^3, L^{-k} \otimes L^{-k} \otimes L^{-k})$ are trivial, where $0 \leq q < 3$ and k positive integer.

Example 4.2.4 Let us recall the Example 4.1.4. If $V_{\mathcal{L}}^{\vee}$ is the induced dual line bundle corresponding to the divisor $-D_3 - D_4$, we observe that the groups $H^{2,q}(H_r, (V_{\mathcal{L}}^{\vee})^k)$ are trivial, whenever $0 \leq q < 2$, $r = 1, 2$ and k is positive integer.

Example 4.2.5 In view of Example 4.1.5, if we apply Corollary 4.2.1 on $\mathcal{H}$, then one can observe that $H^{2,q}(\mathcal{H}, (V_{\mathcal{L}}^{\vee})^k)$ is trivial, whenever $0 \leq q < 2$ and $k \in \mathbb{N}$, where $\mathcal{L}^{\vee} = \mathcal{O}_{\mathcal{H}}(-D_3 - 2D_4)$.

4.3 Further results about Cohomology groups

The present Section is devoted to extend the results of the preceding two sections, through the following Lemma.

Lemma 4.3.1 *Let X be an n-dimensional toric variety, where the set $\{v_i, i = 1, \dots, d\}$ of the generators of the cones of Σ spans $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Then*

$$\bigwedge{}^n T^*_{0,1}(X) \simeq \mathcal{O}_X\left(-\sum_{i=1}^{d} D_i\right)$$

holds, where each D_i is the corresponding divisor of the homogeneous variable z_i for every $i = 1, \dots, d$.

Proof. Fix a basis $e_1, \dots, e_n$ of $M \simeq \mathbb{Z}^n$ and for each subset $\{v_{i_1}, \dots, v_{i_n}\} \subset \{v_i, i = 1, \dots, d\}$ constituting a generator system for an n-dimensional cone σ of the fan Σ of X, we get the $n \times n$ determinant

$$\det(v_I) = \det(\langle e_i, v_{i_k}\rangle).$$

This determinant depends on the ordering of the i_k and it is either equal to 1 or -1 due to the smoothness of each cone. If $h_{i_1}, \dots, h_{i_n}$ are the corresponding homogeneous coordinates of the generating vectors $v_{i_1}, \dots, v_{i_n}$, then the $(n, 0)$ form

$$\Omega = \sum_{|I|=n} \det(v_I) \left(\prod_{i \notin I} h_i\right) dh_{i_1} \wedge \ldots \wedge dh_{i_n} \tag{4.8}$$

is a global form on X taking values in the line bundle $\mathcal{O}_X\left(\sum_{i=1}^{d} D_i\right)$. This means that the bundle $\bigwedge{}^n T^*_{0,1}(X) \otimes \mathcal{O}_X\left(\sum_{i=1}^{d} D_i\right)$ is trivial which turns out that $\bigwedge{}^n T^*_{0,1}(X) \simeq \mathcal{O}_X\left(-\sum_{i=1}^{d} D_i\right)$. □

An alternative proof of Lemma 4.3.1 can be found in [6].

Theorem 4.3.1 *Let X be an n-dimensional smooth compact toric variety, whose polyhedron P_D has N integral points, where $N \le 2n+k+1$. Then, the following holds:*

1. *$H^{n,q}(X, V'_{\mathcal{L}}) = 0$, when $0 < q \le n$ and $\mathcal{L}' = \mathcal{O}_X(kD + \sum_{i=1}^d D_i)$*

2. *$H^{0,q}(X, V''_{\mathcal{L}}) = 0$, when $0 \le q < n$ and $\mathcal{L}'' = \mathcal{O}_X(-kD - \sum_{i=1}^d D_i)$,*

Proof. The first part of the Theorem is a consequence of Lemma 4.3.1 and Theorem 4.1.2. To be more precise, in view of Lemma 4.3.1, we get the isomorphism $\mathcal{E}_{0,q}(X, (V_{\mathcal{L}})^k) \simeq \mathcal{E}_{n,q}(X, V'_{\mathcal{L}})$ where $\mathcal{L} = \mathcal{O}_X(D)$ while $\mathcal{L}' = \mathcal{O}_X(kD + \sum_{i=1}^d D_i)$. This isomorphism is illustrated by considering a form $\psi \in \mathcal{E}_{n,q}(X, V'_{\mathcal{L}})$ which can be written as the wedge product of a form $\phi \in \mathcal{E}_{0,q}(X, (V_{\mathcal{L}})^k)$ with the $(n,0)$-form Ω defined by (4.8). Since $H^{0,q}(X, (V_{\mathcal{L}})^k)$ is trivial for $0 < q \le n$ (Theorem 4.1.2), then $H^{n,q}(X, V'_{\mathcal{L}})$ is also trivial for $0 < q \le n$. In particular,

$$\psi(h) \;=\; \phi(h) \wedge \Omega(h) = C_{N,n}\left(\bar{\partial}_h \int_X \mathcal{K}^{(k)}_{\text{hom}} \wedge \phi\right) \wedge \Omega = C_{N,n}\bar{\partial}_h\left(\left(\int_X \mathcal{K}^{(k)}_{\text{hom}} \wedge \phi\right) \wedge \Omega\right)$$

on U_σ, since Ω does not contain any $\bar{h}$. Then $\omega_\sigma(h) = C_{N,n}\left(\int_X \mathcal{K}^{(k)}_{\text{hom},U_\sigma} \wedge \phi\right) \wedge \Omega(h)$ constitutes a solution of $\bar{\partial}_h \omega_\sigma = \psi(h)$ on U_σ. Since, a solution ω_σ can be found on every chart U_σ, a similar argument with Theorem 4.1.1 and Corollary 4.2.1 yields the first part of the Theorem.

Similarly, in order to prove the second part of theorem, one consider the isomorphism $\mathcal{E}_{0,q}(X, (V_{\mathcal{L}})^k) \simeq \mathcal{E}_{n,q}(X, V'_{\mathcal{L}})$ in an inverse way. According to Corollary 4.2.1, the cohomology groups $H^{n,q}(X, (V^\vee_{\mathcal{L}})^k)$ ($\mathcal{L}^\vee = \mathcal{O}_X(-D)$) are trivial for $0 \le q < n$, when $N \le 2n+k+1$ and hence $H^{0,q}(X, V''_{\mathcal{L}})$ are also trivial for $0 \le q < n$ and $\mathcal{L}'' = \mathcal{O}_X(-kD - \sum_{i=1}^d D_i)$. The explicit solution of the $\bar{\partial}$-equation on U_σ for an $\psi' \in \mathcal{E}_{0,q}(X, V''_{\mathcal{L}})$ arises in the following way. The wedge product of $\psi'(\eta)$ with $\Omega(\eta)$ ($\Omega(\eta)$ is given by (4.8) where h is replaced by η) yields an (n,q) form denoted by $\psi(\eta) = \psi'(\eta) \wedge \Omega(\eta)$, that takes values in $(V^\vee_{\mathcal{L}})^k$. Hence, in view of Corollary 4.2.1, $C_{N,n}\psi(\eta) = \bar{\partial}_\eta \int_X \mathcal{K}^{(k)}_{\text{hom}} \wedge \psi$ on U_σ (by writing the solution with respect to the homogeneous coordinates $\eta_1, \ldots, \eta_d$ of X on U_σ). The solution will arise if we manage to express the kernel $\mathcal{K}^{(k)}_{\text{hom}}$ as $\mathcal{K}^{(k)}_{\text{hom}} = \mathcal{K}' \wedge \Omega$ on U_σ, where $\mathcal{K}'$ is a $(0, n-1)$ form in terms of η and $\Omega(\eta) = \sum_{|I|=n} \det(v_I)\left(\prod_{i \notin I} \eta_i\right) d\eta_{i_1} \wedge \ldots \wedge d\eta_{i_n}$ such that for each $I = \{i_1, \ldots, i_n\}$, the set $\{v_{i_1}, \ldots, v_{i_n}\}$ is a generator system for an n-dimensional cone σ of the fan Σ of X. Then

$$\begin{aligned}\psi' \wedge \Omega = \psi \;&=\; C_{N,n}\bar{\partial}_\eta \int_X \mathcal{K}^{(k)}_{\text{hom}} \wedge \psi = C_{N,n}\bar{\partial}_\eta \int_X \mathcal{K}' \wedge \Omega \wedge \psi \\ &=\; C_{N,n}\bar{\partial}_\eta\left(\int_X (-1)^{n(n+q)}\mathcal{K}' \wedge \psi\right) \wedge \Omega \qquad (4.9)\end{aligned}$$

on U_σ and $\omega_\sigma(\eta) = C_{N,n}\int_X (-1)^{n(n+q)}\mathcal{K}' \wedge \psi$ will be a solution of the equation $\bar{\partial}_\eta \omega = \psi'(\eta)$

on U_σ. In order to find the $(0, n-1)$ form $\mathcal{K}'$, observe that

$$\frac{\sum\limits_{|I|=n} \left(\prod\limits_{i \notin I} \bar{\eta}_i \right) d\eta_{i_{n+1}} \wedge \ldots \wedge d\eta_{i_d}}{\sum\limits_{|I|=n} \prod\limits_{i \notin I} |\eta_i|^2} \wedge \Omega(\eta) = d\eta_1 \wedge \ldots d\eta_d, \tag{4.10}$$

holds on X, since the denominator $\sum_{|I|=n} \prod_{i \notin I} |\eta_i|^2$ vanishes on the exceptional set $Z(\Sigma)$ of X (see the definition of $Z(\Sigma)$ in (1.2)). Then, $\mathcal{K}'$ is the $(0, n-1)$ part of

$$\mathcal{K}^{(k)}_{\text{hom}} \wedge \frac{\sum\limits_{|I|=n} \left(\prod\limits_{i \notin I} \bar{\eta}_i \right) d\eta_{i_{n+1}} \wedge \ldots \wedge d\eta_{i_d}}{\sum\limits_{|I|=n} \prod\limits_{i \notin I} |\eta_i|^2},$$

because $\mathcal{K}' \wedge \Omega = \mathcal{K}^{(k)}_{\text{hom}}$ in view of (4.10). $\square$

Theorem 4.3.1 allows to extend further the examples of the preceding two sections.

Example 4.3.1 In view of first statement in Theorem 4.3.1, the cohomology groups $H^{2,q}(\mathbb{P}^1 \times \mathbb{P}^1, L^{\lambda k+2} \otimes L^{\lambda l+2})$ are trivial for $\lambda \geq kl + k + l - 4$ and $0 < q \leq 2$, since $D_1 \sim D_2$ and $D_3 \sim D_4$. Similarly, the second statement of Theorem 4.3.1 implies that the groups $H^{0,q}(\mathbb{P}^1 \times \mathbb{P}^1, L^{-\lambda k-2} \otimes L^{-\lambda l-2})$ are also trivial for k, l and λ being positive integers satisfying the inequality $\lambda \geq kl + k + l - 4$ and $0 \leq q < 2$.

Example 4.3.2 Applying Theorem 4.3.1, we get that the cohomology groups $H^{n+m,q}(\mathbb{P}^n \times \mathbb{P}^m, L^{k+n+1} \otimes L^{k+m+1}) = 0$ for $0 < q \leq n + m$ and $H^{0,q}(\mathbb{P}^n \times \mathbb{P}^m, L^{-k-n-1} \otimes L^{-k-m-1}) = 0$ for $0 \leq q < n + m$, whenever k, n, m satisfying the inequality $k \geq nm - n - m$. Here, we used that $D_1 \sim \ldots \sim D_{n+1}$ and $D_{n+2} \sim \ldots \sim D_{n+m+2}$.

Example 4.3.3 According to Theorem 4.3.1, $H^{3,q}((\mathbb{P}^1)^3, L^{k+2} \otimes L^{k+2} \otimes L^{k+2}) = 0$ for $0 < q \leq 3$ and $H^{0,q}((\mathbb{P}^1)^3, L^{-k-2} \otimes L^{-k-2} \otimes L^{-k-2}) = 0$ for $0 \leq q < 3$, when k is a positive integer.

Example 4.3.4 For $r = 1, 2$, the groups $H^{2,q}(H_r, V'_{\mathcal{L}})$ are trivial for $0 < q \leq 2$, where $\mathcal{L}' = \mathcal{O}_{H_r}((k + 2 - r)D_3 + (k + 2)D_4)$ and $H^{0,q}(H_r, V''_{\mathcal{L}})$ are also trivial for $0 \leq q < 2$, where $\mathcal{L}'' = \mathcal{O}_{H_r}(-(k + 2 - r)D_3 - (k + 2)D_4)$ and k is every positive integer. The homogeneities arise from the equivalent relations between the divisors: $D_1 \sim D_3$ and $rD_1 + D_2 - D_4 \sim 0$.

Example 4.3.5 If $\mathcal{L}' = \mathcal{O}_{\mathcal{H}}((k+2)D_3 + (2k+3)D_4)$ and $\mathcal{L}'' = \mathcal{O}_{\mathcal{H}}(-(k+2)D_3 - (2k+3)D_4)$ are these specific sheaves on $\mathcal{H}$, then the cohomology groups $H^{2,q}(\mathcal{H}, V'_{\mathcal{L}})$ are trivial for $0 < q \leq 2$ and $H^{0,q}(\mathcal{H}, V''_{\mathcal{L}})$ are also trivial for $0 \leq q < 2$, whenever $k \in \mathbb{N}$. Here, we used that $D_2 \sim D_4$ and $D_1 \sim D_3 + D_4$.

4.4 A realization of the isomorphism between $H^{0,0}(X,(V_{\mathcal{L}})^k)$ and $H^{n,n}(X,(V_{\mathcal{L}}^{\vee})^k)$

Let us consider a holomorphic section ϕ on an n-dimensional smooth compact toric variety X taking values in $V_{\mathcal{L}}$. Then, only the third integral of (3.93) remains to represent ϕ and this statement is formulated in the next theorem.

Theorem 4.4.1 *Let X be an n-dimensional smooth compact toric variety and $V_{\mathcal{L}}$ be the line bundle corresponding to the sheaf $\mathcal{L} = \mathcal{O}_X(D)$, where $D = \sum_{k=1}^{d} a_k D_k$ satisfying $|P_D \cap \mathbb{Z}^n| = N \leq 2(n+1)$. If ϕ is a holomorphic section taking values in $V_{\mathcal{L}}$, then*

$$C_{N,n}\phi(h) = \int_X \mathcal{P}_{\text{hom}} \wedge \phi \tag{4.11}$$

on $U_\sigma = \{\prod_{k=1}^{d} h_k^{a_k} \neq 0\}$, where $\mathcal{P}_{\text{hom}}$ is the 'homogenization' of $\mathcal{P}^T$ on U_σ given in Theorem 3.4.3.

Proof. It is an immediate consequence of Theorem 3.4.3 since $\bar{\partial}\phi = 0$ and the integral $\bar{\partial}_h \int_X \mathcal{K}_{\text{hom}} \wedge \phi$ on U_σ vanishes because $\phi(h)$ does not contain any $d\bar{h}$'s and there is no other term contains $d\bar{h}$'s in the representation to be cancelled out with that term ($\mathcal{P}_{\text{hom}}$ has only differentials of $d\bar{\eta}$'s and $d\eta$'s). Then, the representation formula (4.11) is deduced. $\square$

Theorem 4.4.1 is a realization of the Proposition 1.1.1, that is

$$H^{0,0}(X, V_{\mathcal{L}}) = \Gamma(X, \mathcal{O}_X(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m.$$

In such a way, this confirms the correctness of the construction of the Projection kernel, based on the integral points of the polyhedron P_D.

Remark 4.4.1 If ϕ is a holomorphic section taking values in $(V_{\mathcal{L}})^k$ for $N \leq 2n+k+1$, then

$$C_{N,n}\phi(h) = \int_X \mathcal{P}_{\text{hom}}^{(k)} \wedge \phi \tag{4.12}$$

on U_σ, according to Theorem 3.6.2.

The isomorphism between the cohomology groups $H^{0,0}(X, V_{\mathcal{L}})$ and $H^{n,n}(X, V_{\mathcal{L}}^{\vee})$ verifies that the cohomology group $H^{n,n}(X, V_{\mathcal{L}}^{\vee})$ is also non-trivial. A realization of this isomorphism takes place in the next theorem.

Theorem 4.4.2 *Let ψ be a smooth form of bidegree (n,n) on X taking values in $V_{\mathcal{L}}^{\vee}$. Then*

ψ is $\bar{\partial}$-exact if and only if the duality pair $< \psi, \phi >= 0$ for every holomorphic section ϕ taking values in $V_{\mathcal{L}}$.

Proof. For the necessity, we suppose that ψ is $\bar{\partial}$-exact. This means that there is a form ω of bidegree $(n, n-1)$ such that $\bar{\partial}\omega = \psi$. Applying Stoke's theorem and making use the holomorphicity of ϕ, we get

$$< \psi, \phi >= \int_X \psi\phi = \int_X \bar{\partial}\omega\phi = \int_X \omega \wedge \bar{\partial}\phi = 0,$$

for every holomorphic section ϕ taking values in $V_{\mathcal{L}}$.

For the sufficiency, assume that $< \psi, \phi >= 0$ for every holomorphic section ϕ taking values in $V_{\mathcal{L}}$. Observe that,

$$< \mathcal{P}_{\text{hom}}\psi, \phi >=< \psi, \mathcal{P}_{\text{hom}}\phi >= C_{N,n} < \psi, \phi >= 0$$

on U_σ, where we used Theorem 4.4.1 and the property (1.17) of currents. The current equation holds on every chart U_σ. Using Corollary 4.2.1 and the toric representation formula (4.3), we deduce that ψ is $\bar{\partial}$-exact. □

Theorem 4.4.2 illustrates the isomorphism

$$H^{0,0}(X, V_{\mathcal{L}}) \cong H^{n,n}(X, V_{\mathcal{L}}^{\vee}),$$

which is a particular case of the Toric Serre Duality (4.6) when $q = n$. □

Generally, one can also observe that

$$H^{0,0}(X, (V_{\mathcal{L}})^k) \cong H^{n,n}(X, (V_{\mathcal{L}}^{\vee})^k),$$

when $N \leq 2n + k + 1$, according to a similar argument with the one used in the proof of Theorem 4.4.2.

As we have already seen the cohomology group $H^{n,n}(X, (V_{\mathcal{L}}^{\vee})^k)$ for $N \leq 2n + k + 1$ is non-trivial. However, there is a necessary and sufficient condition leading to the vanishing of the $\bar{\partial}$-equation related to a smooth (n, n) form on X taking values in $(V_{\mathcal{L}}^{\vee})^k$.

Theorem 4.4.3 *Let ψ be a smooth form of bidegree (n, n) on X taking values in $(V_{\mathcal{L}}^{\vee})^k$ where $\mathcal{L} = \mathcal{O}_X(D)$, $k \geq 1$ and $N \leq 2n + k + 1$. Then, the $(n, n-1)$ form*

$$\omega(\eta) = C_{N,n} \int_X \mathcal{K}_{\text{hom}}^{(k)} \wedge \psi, \tag{4.13}$$

where $\mathcal{K}^{(k)}_{\text{hom}}$ *is the 'homogenization' of the Koppelman kernel* $(\mathcal{K}^{(k)})^T$ *on* U_σ *given in Theorem* 3.6.2, *is a solution of the equation* $\bar\partial\omega = \psi$ *on* U_σ *if and only if* ψ *satisfies*

$$\int_X h_1^{a_1}\cdots h_d^{a_d}\psi = 0$$

on U_σ, *for every* $(a_1,\ldots,a_d)\in(\mathbb{Z}_{\geq 0})^d$ *so that* $a_1D_1+\cdots+a_dD_d\sim kD$ *and where* $h_1,\ldots,h_d$ *are the homogeneous coordinates on* X.

Proof. We assume first that $\int_X h_1^{a_1}\cdots h_d^{a_d}\psi = 0$ on U_σ, for every $(a_1,\ldots,a_d)\in(\mathbb{Z}_{\geq 0})^d$ satisfying $a_1D_1+\cdots+a_dD_d\sim kD$. Recall that, according to Theorem 3.6.2, $P^{(k)}_{\text{hom}}$ takes values in $(V_{\mathcal{L}})^k$ with respect to the homogeneous coordinates h on X and it does not contain any $\bar h$. Thus, $\int_X \mathcal{P}^{(k)}_{\text{hom}}\wedge\psi = 0$ on U_σ. Moreover, due to the full bidegree of ψ, we get $\bar\partial_h\psi = 0$ and the second term in the representation (4.5) disappears. Hence, (4.5) yields

$$C_{N,n}\psi(\eta) = \bar\partial_\eta\int_X \mathcal{K}^{(k)}_{\text{hom}}\wedge\psi$$

on U_σ such that $\omega(\eta) = C_{N,n}\int_X \mathcal{K}^{(k)}_{\text{hom}}\wedge\psi$ is a solution of the equation $\bar\partial_\eta\omega = \psi(\eta)$ on U_σ.

On the other hand, if there is a solution ω satisfying the equation $\bar\partial\omega = \psi$ on U_σ, then

$$\int_X h_1^{a_1}\cdots h_d^{a_d}\psi = \int_X h_1^{a_1}\cdots h_d^{a_d}\bar\partial\omega = -\int_X \bar\partial(h_1^{a_1}\cdots h_d^{a_d})\wedge\omega = 0,$$

where we used the property (1.18) of currents as an application of Stoke's theorem.

Example 4.4.1 Let us denote by $\mathcal{O}_{1,1}(\mathbb{P}^n\times\mathbb{P}^m)$ the space of holomorphic sections on $\mathbb{P}^n\times\mathbb{P}^m$ taking values in $L^1\otimes L^1$ when $nm\leq n+m+1$ ($n=1$ and $m=r$ for every $r\in\mathbb{Z}^+$ or $n=r$ and $m=1$ or $n=2$ and $m=3$ or $n=3$ and $m=2$). We consider the divisor $D = D_{n+1}+D_{n+m+2}$ on $\mathbb{P}^n\times\mathbb{P}^m$ (see Example 3.1.2). Hence, the assumptions of Theorem 4.4.1 are satisfied and

$$C_{N,n+m}\phi(h) = \int_X \mathcal{P}_{\text{hom}}\wedge\phi, \tag{4.14}$$

on $U_\sigma = \{h_{n+1}h_{n+m+2\neq 0}\}$, for each $\phi\in\mathcal{O}_{1,1}(\mathbb{P}^n\times\mathbb{P}^m)$, where $C_{N,n+m} = (-1)^{\frac{(nm-1)nm}{2}+1}$. We derive explicitly the Projection kernel $\mathcal{P}^T$ with respect to the toric variables and then $\mathcal{P}_{\text{hom}}$ on the chart U_σ. However, instead of applying Proposition 3.4.1, which describes the final form of the Projection kernel, we are going to use relations (3.67) and (3.68). Actually the weight $\alpha^T_{1,1}$ has been already computed in (3.26) thus avoiding having determinants of big order. First of all,

$$\alpha^T_{1,1} \;=\; -\frac{1}{2\pi i}\sum_{i=1}^{n}\bar\partial\left(\frac{\bar\tau_i d\tau_i}{1+\sum_{i=1}^{n}|\tau_i|^2}\right)-\frac{1}{2\pi i}\sum_{j=n+1}^{n+m}\bar\partial\left(\frac{\bar\tau_j d\tau_j}{1+\sum_{j=n+1}^{n+m}|\tau_j|^2}\right).$$

by (3.26). Then,

$$\begin{aligned}(\alpha^T_{1,1})^{n+m} \;&=\; \left(-\frac{1}{2\pi i}\right)^{n+m}(n+m)!\\ &\bigwedge_{i=1}^{n}\bar\partial\left(\frac{\bar\tau_i d\tau_i}{1+\sum_{i=1}^{n}|\tau_i|^2}\right)\wedge\bigwedge_{j=n+1}^{n+m}\bar\partial\left(\frac{\bar\tau_j d\tau_j}{1+\sum_{j=n+1}^{n+m}|\tau_j|^2}\right)\end{aligned}\tag{4.15}$$

because $d\tau_i \wedge d\tau_i = 0$. Hence, (3.67) yields that

$$\mathcal{P}^T_1 \;=\; \frac{(n+m-nm+1)}{2^{nm}}\,\alpha^T_{0,0}(\alpha^T_{1,1})^{n+m}.$$

Since

$$\alpha^T_{0,0}=\frac{P(\bar\tau\cdot t)}{P(|\tau|^2)}=\frac{(1+\sum_{i=1}^{n}\bar\tau_i t_i)(1+\sum_{j=n+1}^{n+m}\bar\tau_j t_j)}{(1+\sum_{i=1}^{n}|\tau_i|^2)(1+\sum_{j=n+1}^{n+m}|\tau_j|^2)},$$

one has that

$$\begin{aligned}\mathcal{P}^T_1 \;&=\; \left(-\frac{1}{2\pi i}\right)^{n+m}\frac{(n+m-nm+1)(n+m)!}{2^{nm}}\\ &\frac{(1+\sum_{i=1}^{n}\bar\tau_i t_i)(1+\sum_{j=n+1}^{n+m}\bar\tau_j t_j)}{(1+|\tau|^2)(1+|\tau'|^2)}\bigwedge_{i=1}^{n}\bar\partial\left(\frac{\bar\tau_i d\tau_i}{1+|\tau|^2}\right)\wedge\bigwedge_{j=n+1}^{n+m}\bar\partial\left(\frac{\bar\tau_j d\tau_j}{1+|\tau'|^2}\right),\end{aligned}\tag{4.16}$$

where $|\tau|^2=\sum_{i=1}^{n}|\tau_i|^2$ and $|\tau'|^2=\sum_{j=n+1}^{n+m}|\tau_j|^2$.

In order to find the second term of the Projection kernel, $\mathcal{P}^T_2$, we consider Hefer forms $H^{i,j}$ of the polynomials $f_{i,j}$ for $1\le i\le n$ and $n+1\le j\le n+m$ (see (3.14)) and $\gamma_{i,j}$ the corresponding vector fields. Then, since $N=nm+n+m+1$ and the dimension of the toric variety $X=\mathbb{P}^n\times\mathbb{P}^m$ is $n+m$, we get that

$$\begin{aligned}\mathcal{P}^T_2 \;&=\; \frac{1}{2^{nm-1}}\left[\sum_{k=1}^{n}\sum_{l=n+1}^{n+m}(\gamma_{k,l}\neg H^{k,l}_{1,0})^T\right](\alpha^T_{1,1})^{n+m}\\ &-\frac{(n+m)}{2^{nm-1}}\left[\sum_{k=1}^{n}\sum_{l=n+1}^{n+m}(H^{k,l})^T_{1,0}(\gamma_{k,l}\neg\alpha_{1,1})^T\right](\alpha^T_{1,1})^{n+m-1}\end{aligned}\tag{4.17}$$

according to (3.68), where

$$(H^{k,l}_{1,0})^T = \frac{1}{4\pi i}\Bigg\{(\tau_l - t_l)d\tau_k + (\tau_k - t_k)d\tau_l$$
$$-(\tau_k - t_k)(\tau_l - t_l)\left(\sum_{i=1}^{n}\frac{\bar{\tau}_i d\tau_i}{1+|\tau|^2} + \sum_{j=n+1}^{n+m}\frac{\bar{\tau}_j d\tau_j}{1+|\tau'|^2}\right)\Bigg\}$$

while

$$(\gamma_{k,l}\neg H^{k,l}_{1,0})^T = \frac{1}{2}\left[-\frac{\overline{\tau_k\tau_l}(\tau_k - t_k)(\tau_l - t_l)}{(1+|\tau|^2)(1+|\tau'|^2)} + \alpha^T_{0,0} + 1\right].$$

Moreover,

$$(\gamma_{k,l}\neg\alpha_{1,1})^T = \frac{\bar{\tau}_l}{1+|\tau'|^2}\,\bar{\partial}\left(\frac{\bar{\tau}_k}{1+|\tau|^2}\right) + \frac{\bar{\tau}_k}{1+|\tau|^2}\,\bar{\partial}\left(\frac{\bar{\tau}_l}{1+|\tau'|^2}\right).$$

Then,

$$\mathcal{P}^T_2 = \left(-\frac{1}{2\pi i}\right)^{n+m}\frac{(n+m)!}{2^{nm}(1+|\tau|^2)(1+|\tau'|^2)}$$
$$\left\{\left[1+\bar{\tau}\cdot t + (n-1)(1+|\tau|^2)\right]\left[1+\bar{\tau'}\cdot t' + (m-1)(1+|\tau'|^2)\right]\right.$$
$$\left.+nm(1+\bar{\tau}\cdot t)(1+\bar{\tau'}\cdot t')\right\}$$
$$\bigwedge_{i=1}^{n}\bar{\partial}\left(\frac{\bar{\tau}_i d\tau_i}{1+|\tau|^2}\right)\wedge\bigwedge_{j=n+1}^{n+m}\bar{\partial}\left(\frac{\bar{\tau}_j d\tau_j}{1+|\tau'|^2}\right). \tag{4.18}$$

By adding the terms $\mathcal{P}^T_1$ and $\mathcal{P}^T_2$ which are given explicitly in (4.16) and (4.18), respectively, we get a simpler form of the Projection kernel $\mathcal{P}^T$:

$$\mathcal{P}^T = \left(-\frac{1}{2\pi i}\right)^{n+m}\frac{(n+m)!}{2^{nm}(1+|\tau|^2)(1+|\tau'|^2)}$$
$$\left\{(n+m+1)(1+\bar{\tau}\cdot t)(1+\bar{\tau'}\cdot t')\right.$$
$$\left.+\left[1+\bar{\tau}\cdot t + (n-1)(1+|\tau|^2)\right]\left[1+\bar{\tau'}\cdot t' + (m-1)(1+|\tau'|^2)\right]\right\}$$
$$\bigwedge_{i=1}^{n}\bar{\partial}\left(\frac{\bar{\tau}_i d\tau_i}{1+|\tau|^2}\right)\wedge\bigwedge_{j=n+1}^{n+m}\bar{\partial}\left(\frac{\bar{\tau}_j d\tau_j}{1+|\tau'|^2}\right). \tag{4.19}$$

Now, let $\eta = (\eta_1, \ldots, \eta_{n+1})$ and $\eta' = (\eta_{n+2}, \ldots, \eta_{n+m+2})$ being the projective homogeneous coordinates of $\mathbb{P}^n$ and $\mathbb{P}^m$, respectively. A change of coordinates on $\mathbb{P}^n \times \mathbb{P}^m$ from toric to projective homogeneous on U_σ in (4.19), will give us the required formula. Recall that, every coordinate τ_i corresponds to η_i/η_{n+1} for $i = 1, \ldots, n$, while τ_j is equal to η_{j+1}/η_{n+m+2} for

$j = n+1, \ldots, n+m$ on U_σ. Then,

$$\bigwedge_{i=1}^{n} \bar{\partial}\left(\frac{\bar{\tau}_i d\tau_i}{1+|\tau|^2}\right) = \frac{1}{n!}\left[\bar{\partial}\left(\frac{\bar{\eta}\cdot d\eta}{|\eta|^2}\right)\right]^n$$

and

$$\bigwedge_{j=n+1}^{n+m} \bar{\partial}\left(\frac{\bar{\tau}_j d\tau_j}{1+|\tau'|^2}\right) = \frac{1}{m!}\left[\bar{\partial}\left(\frac{\bar{\eta'}\cdot d\eta'}{|\eta'|^2}\right)\right]^m.$$

Hence, the kernel $\mathcal{P}_{\text{hom}}$, being the 'homogenization' of the kernel $\mathcal{P}^T$ representing $\mathcal{O}_{1,1}(\mathbb{P}^n \times \mathbb{P}^m)$ when $nm \leq n+m+1$ is given by [1]

$$\begin{aligned}
\mathcal{P}_{\text{hom}} &= \frac{h_{n+1}h_{n+m+2}}{\eta_{n+1}\eta_{n+m+2}}\mathcal{P}^T \\
&= \left(-\frac{1}{2\pi i}\right)^{n+m}\binom{n+m}{n}\frac{1}{2^{nm}|\eta|^2|\eta'|^2} \\
&\quad \Big[(n+m+1)(\bar{\eta}\cdot h)(\bar{\eta'}\cdot h') \\
&\quad + \left(\bar{\eta}\cdot h + (n-1)|\eta|^2\frac{h_{n+1}}{\eta_{n+1}}\right)\left(\bar{\eta'}\cdot h' + (m-1)|\eta'|^2\frac{h_{n+m+2}}{\eta_{n+m+2}}\right)\Big] \\
&\quad \left[\bar{\partial}\left(\frac{\bar{\eta}\cdot d\eta}{|\eta|^2}\right)\right]^n \wedge \left[\bar{\partial}\left(\frac{\bar{\eta'}\cdot d\eta'}{|\eta'|^2}\right)\right]^m.
\end{aligned}$$

[1]This formula is valid on the chart $U_\sigma = \{\eta_{n+1}\eta_{n+m+2} \neq 0\}$ of $\mathbb{P}^n \times \mathbb{P}^m$ and it is given by a similar form on the remaining charts of this variety.

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