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GROUP ANALYSIS OF SYSTEMS OF EVOLUTION EQUATIONS

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Abstract

Nowadays, one of the most important tools for the solution of differential equations is the application of Lie symmetry methods. Solutions of nonlinear partial differential equations (PDEs) can be constructed directly from the symmetries or via similarity reductions. However, finding Lie symmetries of PDEs and generally for systems of PDEs, is not an easy task, especially when arbitrary elements appear in the equations. Hence, in order to avoid numerous calculations, some useful restrictions on the functional form of the coefficient functions of the Lie generator, are needed.

The target of the present thesis is to find some useful a-priori restrictions on the form of the generator, to reduce the number of calculations required in group classification. We deal with evolution equations. To achieve this goal, in chapter 2 some basic, necessary definitions are given, that enable us to develop our theory. We describe the notion of Lie groups of transformations, the infinitesimal transformations. We explain what is meant by the terms invariance of a PDE, similarity reductions, nonclassical symmetries and equivalence transformations.

In the next chapter, we exhibit known results for two types of generalized nonlinear scalar PDEs. Specifically, for the nonlinear heat equation without presenting any calculations, we mention out the equivalence transformations, Lie symmetries and invariant solutions. Also, for the generalized Burgers equation, equivalence transformations and Lie symmetries are given. These two equations motivate us to extend these results, for systems of diffusion equations, later in the thesis.

Chapter 4 is the chapter in which the wanted, aforementioned restrictions on the form of the Lie generator are derived. We recall some results from the papers of Tu [93] and Bluman [12]. Motivated by this work for scalar evolution PDEs, we extend similar results to systems of evolution equations. That is, we firstly present restrictions on the form of the coefficient function τ of the generator

$$\Gamma = \tau\left(x,t,u,v\right)\frac{\partial}{\partial t} + \xi\left(x,t,u,v\right)\frac{\partial}{\partial x} + \eta\left(x,t,u,v\right)\frac{\partial}{\partial u} + \mu\left(x,t,u,v\right)\frac{\partial}{\partial v}.$$

We examine when this coefficient function is a function only of t. We have also found counterexamples in which τ depends not only on t. These are interesting examples, that need to be considered in some future work. Furthermore, restrictions on the form of the coefficient functions ξ, η and μ, are given, in the case where $\tau = \tau(t)$ is valid.

Chapters 5 and 6 contain applications of chapter 4 on two special classes of systems of evolution equations. Group classification of systems of diffusion equations is the purpose of chapter 5, while in chapter 6 we examine Burgers-type systems. For both systems, Lie symmetries, as a result of the previous restrictions, and equivalence transformations, that help us to simplify the form of the PDEs, are given. We have studied similarity reductions for two special cases of systems of diffusion equations, whilst we have found some examples of nonclassical reductions and a linearizable case of Burgers systems.

We finally present, in chapter 7, symmetry analysis of a two-dimensional Burgers system. Lie invariance algebra and its subalgebras, followed by the complete point symmetry group, Lie reductions of codimension one and two and also Lie symmetries of the reduced systems of PDEs, complete this thesis.

The last chapter of the thesis, is a description of what we are planning to do in the next few years. Problems that might admit generalizations are listed to be carried out. These are problems appeared in chapters 4, 5, 6 and 7 of the thesis and need further study!

Περίληψη

Στις μέρες μας, ένα από τα πιο σημαντικά εργαλεία για την επίλυση των Διαφορικών Εξισώσεων είναι η εφαρμογή των μεθόδων των συμμετριών του Lie. Λύσεις μιας μη-γραμμικής Μερικής Διαφορικής Εξίσωσης (ΜΔΕ) μπορούν να κατασκευαστούν απευθείας από τις συμμετρίες ή μέσω των μετασχηματισμών υποβιβασμού τάξεως. Ωστόσο, η εύρεση των συμμετριών Lie ΜΔΕ και γενικότερα συστημάτων ΜΔΕ, δεν είναι εύκολη διαδικασία, ειδικότερα στην περίπτωση που στις εξισώσεις εμφανίζονται συναρτήσεις των εξαρτημένων ή/και ανεξάρτητων μεταβλητών. Ως εκ τούτου, προς αποφυγή πολυάριθμων υπολογισμών, απαιτούνται κάποιοι χρήσιμοι περιορισμοί στην συναρτησιακή μορφή των συντελεστών του γεννήτορα.

Ο στόχος της παρούσας διατριβής είναι η εύρεση χρήσιμων, εκ των προτέρων περιορισμών, στην μορφή του γεννήτορα, για να μειωθεί ο αριθμός των υπολογισμών που απαιτούνται στην ταξινόμηση των συμμετριών. Θα ασχοληθούμε με εξισώσεις εξέλιξης. Προς επίτευξη του σκοπού αυτού, στο κεφάλαιο 2, δίνονται κάποιοι βασικοί, χρήσιμοι ορισμοί, που θα μας βοηθήσουν να αναπτύξουμε την θεωρία μας. Γίνεται περιγραφή της έννοιας των ομάδων μετασχηματισμών Lie, των απειροστών μετασχηματισμών. Ακολούθως, επεξηγούμε τι σημαίνει αναλλοίωτη Διαφορική Εξίσωση, μετασχηματισμοί υποβιβασμού τάξεως μιας Διαφορικής Εξίσωσης, μη-κλασσικές συμμετρίες και μετασχηματισμοί ισοδυναμίας.

Στο επόμενο κεφάλαιο, παρουσιάζουμε γνωστά αποτελέσματα για δύο κατηγορίες γενικευμένων μη-γραμμικών βαθμωτών ΜΔΕ. Συγκεκριμένα, για την μη-γραμμική εξίσωση της θερμότητας, χωρίς να παρουσιάσουμε τους υπολογισμούς, δίνουμε τους μετασχηματισμούς ισοδυναμίας, τις συμμετρίες Lie και τις αναλλοίωτες λύσεις. Επίσης, για την γενικευμένη εξίσωση του Burgers, αναφέρουμε τους μετασχηματισμούς ισοδυναμίας και τις συμμετρίες Lie.

Το κεφάλαιο 4, είναι το κεφάλαιο στο οποίο παρουσιάζονται οι ζητούμενοι, προαναφερθέντες περιορισμοί στην μορφή του γεννήτορα. Ανακαλούμε κάποια αποτελέσματα από τις δημοσιεύσεις των Tu [93] και Bluman [12]. Υποκινούμενοι από αυτή την δουλειά, για βαθμωτές ΜΔΕ εξέλιξης, επεκτείνουμε παρόμοια αποτελέσματα και για συστήματα εξισώσεων εξέλιξης. Αναλυτικότερα, πρώτα

παρουσιάζουμε τους περιορισμούς στην μορφή του συντελεστή τ του γεννήτορα

$$\Gamma = \tau\left(x,t,u,v\right)\frac{\partial}{\partial t} + \xi\left(x,t,u,v\right)\frac{\partial}{\partial x} + \eta\left(x,t,u,v\right)\frac{\partial}{\partial u} + \mu\left(x,t,u,v\right)\frac{\partial}{\partial v}.$$

Εξετάζουμε πότε αυτός ο συντελεστής είναι συνάρτηση του t, μόνον. Έχουμε επίσης βρει αντιπαραδείγματα στα οποία το τ, δεν εξαρτάται μόνο από το t. Αυτά είναι ενδιαφέροντα παραδείγματα, τα οποία πρέπει να εξετασθούν σε κάποια μελλοντική εργασία. Επιπλέον, δίνονται και περιορισμοί στην μορφή των συντελεστών ξ, η και μ, στην περίπτωση που ισχύει $\tau = \tau(t)$.

Τα κεφάλαια 5 και 6, αποτελούν εφαρμογές των αποτελεσμάτων του κεφαλαίου 4, σε δύο ειδικές περιπτώσεις συστημάτων εξισώσεων εξέλιξης. Ο σκοπός του κεφαλαίου 5 είναι η ταξινόμηση των συμμετριών συστημάτων εξισώσεων διάχυσης, ενώ στο κεφάλαιο 6 εξετάζουμε συστήματα τύπου Burgers. Και για τα δύο συστήματα, παρουσιάζουμε τις συμμετρίες Lie, ως αποτέλεσμα των προηγούμενων περιορισμών, και τους μετασχηματισμούς ισοδυναμίας, που μας βοηθούν να απλοποιήσουμε την μορφή των ΜΔΕ. Έχουμε μελετήσει τους μετασχηματισμούς υποβιβασμού τάξεως για δύο συγκεκριμένες περιπτώσεις συστημάτων εξισώσεων διάχυσης, ενώ έχουμε βρει και κάποια παραδείγματα μη-κλασσικών συμμετριών, καθώς επίσης και μία γραμμικοποιήσιμη περίπτωση συστήματος τύπου Burgers.

Τέλος, στο κεφάλαιο 7, παρουσιάζουμε ανάλυση συμμετριών ενός δι-διάστατου συστήματος Burgers. Η διατριβή ολοκληρώνεται με την αναλλοίωτη άλγεβρα Lie και τις υποάλγεβρες αυτής, ακολουθούμενη από την πλήρη σημειακή ομάδα συμμετριών, τους υποβιβασμούς Lie συνδιαστάσεων 1 και 2, καθώς επίσης και τις συμμετρίες Lie των υποβιβασμένων συστημάτων ΜΔΕ.

Το τελευταίο κεφάλαιο της διατριβής, αποτελεί περιγραφή της έρευνας που πρόκειται να γίνει στο εγγύς μέλλον. Απαριθμούνται προβλήματα, τα οποία πιθανόν να επιδέχονται γενικεύσεων, που θα εξεταστούν. Αυτά είναι προβλήματα που εμφανίστηκαν στα κεφάλαια 4, 5, 6 και 7 της διατριβής και για τα οποία απαιτείται επιπλέον μελέτη!

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Contents

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Chapter 1

Introduction

Marious Sophus Lie (1842–1899), was a Norwegian mathematician, who first established the group analysis of differential equations. Lie's theories are powerful tools for understanding the physical laws of Nature.

Ovsiannikov in the late 1950's and 1960's and Bluman in the late 1960's and 1970's continued Lie's work, to develop symmetry methods for differential equations. Nowadays, there are several comprehensive accounts of the basic theory as well as more recent applications and generalizations, based on the publication of the texts of Ovsiannikov [72], Bluman and Kumei [13], Bluman and Anco [14], Bluman, Anco and Cheviakov [15], Olver [68], Ibragimov [37] and Fushchich [30].

Transformation methods are one of the most powerful tools currently available in the area of nonlinear PDEs. While there is no existing general theory for solving such equations, many special cases have yielded to appropriate changes of variables. Point transformations are the ones which are mostly used. These are transformations in the space of the dependent and the independent variables of a PDE. Probably the most useful point transformations of PDEs are those which form a continuous Lie group of transformations and which leave the equation invariant. Symmetries of this PDE are then revealed, perhaps enabling new solutions to be found directly or via similarity reductions. The classical method of finding Lie symmetries is to first find infinitesimal transformations, with the benefit of linearization, and then to extend these to groups of finite transformations.

The investigation of nonlinear diffusion equations by means of symmetry methods began in 1959 with Ovsiannikov's work [71] in which the author performed the group classification of the class of equations of the form

$$u_t = (f(u)u_x)_x \,.$$

In Chapter 3 we recall the known results of the above second-order nonlinear diffusion equation. We present the equivalence transformations, Lie symmetries and invariant solutions. We also give the equivalence transformations and Lie symmetries for the generalized Burgers equation

$$u_t = u_{xx} + k(u)u_x.$$

In the 1980's Tu [93] and Bluman [12], presented some a-priori restrictions on the form of the coefficient functions of the Lie symmetry generator for scalar PDEs.

Motivated by these results, in Chapter 4, we give some a-priori restrictions on the form of the coefficient functions of the Lie symmetry generator for systems of evolution equations. These restrictions make the problem of group classification of systems of evolution equations, especially when arbitrary elements exist, easier.

In Chapters 5 and 6, we apply these restrictions to give the complete group classification of the system of diffusion equations

$$u_t = [f(u,v)u_x]_x\,, \quad v_t = [g(u,v)v_x]_x$$

and the systems of Burgers equations

$$u_t = \lambda_1 u_{xx} + f(u,v)u_x + \epsilon_1 v v_x, \quad v_t = \lambda_2 v_{xx} + k(u,v)v_x + \epsilon_2 u u_x, \quad \epsilon_1\epsilon_2 \neq 0$$

and

$$u_t = \lambda_1 u_{xx} + f(u,v)u_x, \quad v_t = \lambda_2 v_{xx} + k(u,v)v_x,$$

respectively.

In Chapter 7 we consider the two-dimensional Burgers system

$$u_t + uu_x + vu_y - \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}) = 0, \quad v_t + uv_x + vv_y - \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}) = 0$$

where Re is the Reynolds number. We present Lie symmetries, Lie reductions, Lie invariance algebra and complete point symmetry group.

Finally, in Chapter 8 we list some open problems that need to be considered, in the future.

The calculations involved in this thesis have been facilitated by the computer algebraic package "MAPLE" [102].

Chapter 2

Basic Definitions

2.1 Lie Groups of Transformations

2.1.1 Groups

Definition 2.1. A *group* is a pair $(G,*)$ that consists of a non-empty set G and a binary operation, $*$: $G \times G \longrightarrow G$, satisfying the following axioms:

(i) Closure:

$$\forall\ g, h \in G \Longrightarrow g * h \in G.$$

(ii) Associativity:

$$\forall\ g, h, k \in G \Longrightarrow g * (h * k) = (g * h) * k.$$

(iii) Identity Element: There is a (unique) element, $e \in G$, called the identity element, such that

$$\forall\ g \in G \Longrightarrow e * g = g * e = g.$$

(iv) Inverse element: For each element $g \in G$ there is a (unique) inverse, $g^{-1} \in G$, such that

$$g * g^{-1} = g^{-1} * g = e.$$

Definition 2.2. A group G is called *abelian* if $\forall\ g, h \in G \Longrightarrow g * h = h * g$.

Definition 2.3. A *subgroup* of G is a non-empty subset of G, which forms a group itself under the same operation.

2.1.2 Examples of Groups

Example 2.1. (Q,+) i.e. the additive group of rational numbers. Here $e = 0$ and $q^{-1} = -q$.

Example 2.2. (R^+,$\cdot$) i.e. the multiplicative group of all positive real numbers. Here $e = 1$ and $g^{-1} = \frac{1}{g}$.

2.1.3 Groups of Transformations

Definition 2.4. The term *space transformation* denotes a function, $T : R^4 \longrightarrow R^4$, defined via

$$x' = \phi\left(x, t, u, v\right), \ t' = \chi\left(x, t, u, v\right), \ u' = \psi\left(x, t, u, v\right), \ v' = \omega\left(x, t, u, v\right),$$

where ϕ, χ, ψ and ω are known functions. Geometrically, T transforms a point (x, t, u, v) to another point (x', t', u', v'), in the same coordinate system.

If the equations defining the transformation T, can be solved with respect to x, t, u, v, then the resulting transformation is called the inverse transformation, T^{-1}, which is defined via

$$x = \Phi\left(x', t', u', v'\right), \ t = X\left(x', t', u', v'\right), \ u = \Psi\left(x', t', u', v'\right), \ v = \Omega\left(x', t', u', v'\right).$$

The composition of these two transformations, gives the identical transformation, i.e.

$$x' = x, \ t' = t, \ u' = u, \ v' = v.$$

We now consider transformations in which the functions ϕ, χ, ψ and ω depend also on a real parameter ϵ. We assume that the parameter ϵ, varies continuously in an open interval, such that $\mid \epsilon \mid < \epsilon_0$. Then, the transformations compose a group of transformations T_ϵ, given via

$$x' = \phi\left(x, t, u, v, \epsilon\right), \ t' = \chi\left(x, t, u, v, \epsilon\right), \ u' = \psi\left(x, t, u, v, \epsilon\right), \ v' = \omega\left(x, t, u, v, \epsilon\right),$$

where ϕ, χ, ψ and ω are analytic functions.

Definition 2.5. A set of transformations of the above form, forms a *one-parameter Lie group of transformations* if it satisfies the following axioms:

α) $T_0 = I$ ($T_{\epsilon_0} = I$) (existence of identity element)

β) $T_\epsilon^{-1} = T_{\epsilon^{-1}}$ (existence of inverse element)

γ) $T_\gamma\left(T_\delta T_\epsilon\right) = \left(T_\gamma T_\delta\right) T_\epsilon$ (associativity)

δ) $T_\delta T_\epsilon = T_{\phi(\delta,\ \epsilon)}$ (closure)

Each value of the parameter ϵ corresponds to a particular member of the group of transformations. Transformations T_ϵ belong to the one-parameter group of transformations.

2.1.4 Examples of One-parameter Lie Groups of Transformations

Example 2.3. Group of Translations:

$$x' = x,\ t' = t + \epsilon,\ u' = u,\ v' = v.$$

Here $T_0 = I$. Hence $\epsilon_0 = 0$. Also $T_\epsilon^{-1} = T_{-\epsilon}$, so $\epsilon^{-1} = -\epsilon$. Finally, from the closure, we have $T_\delta T_\epsilon = T_{\delta+\epsilon}$. Therefore, $\phi(\delta, \epsilon) = \delta + \epsilon$. This transformation represents a translation in the direction of t, at a distance ϵ.

Example 2.4. Group of Rotations:

$$x' = x\cos\epsilon - t\sin\epsilon,\ t' = x\sin\epsilon + t\cos\epsilon,\ u' = u + \epsilon,\ v' = v,$$

where $T_0 = I$, which means $\epsilon_0 = 0$. Furthermore, $\epsilon^{-1} = -\epsilon$, since $T_\epsilon^{-1} = T_{-\epsilon}$. Again, $T_\delta T_\epsilon = T_{\delta+\epsilon}$, so $\phi(\delta, \epsilon) = \delta + \epsilon$. Such a transformation describes a rotation in the xt-plane at an angle ϵ, and a translation in the u-direction, at a distance ϵ.

Example 2.5. Group of scalings:

$$x' = \epsilon x,\ t' = \epsilon^2 t,\ u' = u,\ v' = \epsilon v.$$

In this case $\epsilon_0 = 1$, due to the fact that $T_1 = I$. Furthermore, $T_\epsilon^{-1} = T_{\frac{1}{\epsilon}}$, which means $\epsilon^{-1} = \frac{1}{\epsilon}$. Here, $\phi(\delta, \epsilon) = \delta\epsilon$ because $T_\delta T_\epsilon = T_{\delta\epsilon}$.

2.2 Infinitesimal Transformations

We consider a one-parameter Lie group of transformations T_ϵ, with identity $\epsilon_0 = 0$. Using Taylor's expansion about $\epsilon_0 = 0$, we obtain

$$\begin{aligned} x' &= x + \epsilon\xi(x,t,u,v) + O\left(\epsilon^2\right) \\ t' &= t + \epsilon\tau(x,t,u,v) + O\left(\epsilon^2\right) \\ u' &= u + \epsilon\eta(x,t,u,v) + O\left(\epsilon^2\right) \\ v' &= v + \epsilon\mu(x,t,u,v) + O\left(\epsilon^2\right), \end{aligned} \tag{2.1}$$

where,

$$\xi = \left.\frac{\partial\phi}{\partial\epsilon}\right|_{\epsilon=0},\ \tau = \left.\frac{\partial\chi}{\partial\epsilon}\right|_{\epsilon=0},\ \eta = \left.\frac{\partial\psi}{\partial\epsilon}\right|_{\epsilon=0},\ \mu = \left.\frac{\partial\omega}{\partial\epsilon}\right|_{\epsilon=0}.$$

In these equations we ignore terms of order two and higher. This first order transformation, is called *infinitesimal transformation*. The functions $\xi,\ \tau,\ \eta,\ \mu$ are called the *infinitesimal functions* of the transformation.

The form of the corresponding Lie group of transformations, in finite form, can be found when the infinitesimal functions are known. Their form is the solution of the system of Ordinary Differential Equations (ODEs)

$$\begin{aligned} \frac{dx'}{d\epsilon} &= \xi\left(x',t',u',v'\right) \\ \frac{dt'}{d\epsilon} &= \tau\left(x',t',u',v'\right) \\ \frac{du'}{d\epsilon} &= \eta\left(x',t',u',v'\right) \\ \frac{dv'}{d\epsilon} &= \mu\left(x',t',u',v'\right) \end{aligned} \tag{2.2}$$

subject to the initial conditions,

$$x'=x, \quad t'=t, \quad u'=u, \quad v'=v, \quad \text{for} \;\; \epsilon=0.$$

The above result is known as the *First Fundamental Theorem of Lie.*

2.2.1 Infinitesimal Generators

Definition 2.6. The *infinitesimal generator* of the one-parameter Lie group of transformations (2.1) is the linear differential operator

$$\Gamma = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial v}.$$

For any differentiable function, $F(x,t,u,v)$, we have

$$\Gamma F = \xi\frac{\partial F}{\partial x} + \tau\frac{\partial F}{\partial t} + \eta\frac{\partial F}{\partial u} + \mu\frac{\partial F}{\partial v}.$$

Example 2.6. We consider the group of rotations in the xt-plane

$$x' = x\cos\epsilon - t\sin\epsilon, \;\; t' = x\sin\epsilon + t\cos\epsilon.$$

The *infinitesimal functions* for the transformation are

$$\left.\frac{dx'}{d\epsilon}\right|_{\epsilon=0} = -x\sin\epsilon - t\cos\epsilon|_{\epsilon=0} = -t, \quad \left.\frac{dt'}{d\epsilon}\right|_{\epsilon=0} = x\cos\epsilon - t\sin\epsilon|_{\epsilon=0} = x$$

and the *infinitesimal generator* has the form

$$\Gamma = -t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}.$$

Hence the system (2.2) has the following form

$$\frac{dx'}{d\epsilon} = -t', \quad \frac{dt'}{d\epsilon} = x'$$

subject to the initial conditions

$$x'=x, \; t'=t \;\; \text{when} \; \epsilon=0.$$

2.2.2 Invariant Functions

Definition 2.7. An infinitely differentiable function $F(x,t,u,v)$ is called an *invariant function* of the Lie group of transformations (2.1) if $F(x',t',u',v') = F(x,t,u,v)$ identically in x,t,u,v and ϵ in a neighborhood of $\epsilon = 0$.

Remark 2.1. Given an invariant function $F(x,t,u,v)$, any function $\Phi(F(x,t,u,v))$ is also invariant.

Theorem 2.1. *A function $F(x,t,u,v)$ is an absolute invariant of the Lie group of transformations* (2.1) *with the generator Γ if and only if it is a solution of the homogeneous PDE*

$$\Gamma F(x,t,u,v) = 0,$$

where,

$$\Gamma = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial v}.$$

Example 2.7. If we consider the group of rotations, a function $F(x,t)$ is invariant if and only if

$$\Gamma F(x,t) \equiv -t\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial t} = 0.$$

Using the method of characteristics, one can solve the aforementioned first-order linear PDE, that is,

$$\frac{dx}{-t} = \frac{dt}{x} = \frac{dF}{0}.$$

The solution has the form

$$F = \Psi\left(x^2 + t^2\right).$$

Hence any function of the form $\Psi\left(x^2 + t^2\right)$ remains invariant under the group of rotations.

2.3 Invariance of a PDE

We would like to examine when a PDE remains invariant under the action of the infinitesimal transformation. It is necessary to know how the derivatives are transformed. For second order

PDEs, we define the following extension transformations:

$$
\begin{aligned}
u'_{x'} &= u_x + \epsilon\eta^x\left(x,t,u,u_x,u_t\right) + O\left(\epsilon^2\right)\\
u'_{t'} &= u_t + \epsilon\eta^t\left(x,t,u,u_x,u_t\right) + O\left(\epsilon^2\right)\\
u'_{x'x'} &= u_{xx} + \epsilon\eta^{xx}\left(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}\right) + O\left(\epsilon^2\right)\\
u'_{x't'} &= u_{xt} + \epsilon\eta^{xt}\left(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}\right) + O\left(\epsilon^2\right)\\
u'_{t't'} &= u_{tt} + \epsilon\eta^{tt}\left(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}\right) + O\left(\epsilon^2\right),
\end{aligned}
\tag{2.3}
$$

where the extended infinitesimal functions have the form:

$$
\begin{aligned}
\eta^x &= D_x\left(\eta\right) - u_x D_x\left(\xi\right) - u_t D_x\left(\tau\right)\\
\eta^t &= D_t\left(\eta\right) - u_x D_t\left(\xi\right) - u_t D_t\left(\tau\right)\\
\eta^{xx} &= D_x\left(\eta^x\right) - u_{xx} D_x\left(\xi\right) - u_{xt} D_x\left(\tau\right)\\
\eta^{xt} &= D_t\left(\eta^x\right) - u_{xx} D_t\left(\xi\right) - u_{xt} D_t\left(\tau\right)\\
&= D_x\left(\eta^t\right) - u_{xt} D_x\left(\xi\right) - u_{tt} D_x\left(\tau\right)\\
\eta^{tt} &= D_t\left(\eta^t\right) - u_{xt} D_t\left(\xi\right) - u_{tt} D_t\left(\tau\right).
\end{aligned}
\tag{2.4}
$$

Here, D_x and D_t denote the total derivative operators, with respect to x and t, respectively. The first and second prolongations of the extended infinitesimal generator are defined as:

$$
\Gamma^{(1)} = \Gamma + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t}
\tag{2.5}
$$

$$
\Gamma^{(2)} = \Gamma^{(1)} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xt} \frac{\partial}{\partial u_{xt}} + \eta^{tt} \frac{\partial}{\partial u_{tt}}
\tag{2.6}
$$

respectively.

A transformation is called a Lie symmetry of a second order PDE,

$$E\left(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}\right) = 0,$$

if the PDE has the same form in the new variables x', t', u'. That is,

$$E\left(x',t',u',u'_{x'},u'_{t'},u'_{x'x'},u'_{x't'},u'_{t't'}\right) = 0.$$

The PDE

$$E\left(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}\right) = 0,$$

admits a Lie symmetry of the form

$$\Gamma = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u},$$

if and only if,

$$\Gamma^{(2)} E\big|_{E=0} = 0.$$

This is a multi-variable polynomial in the variables $u_x, u_t, u_{xx}, u_{xt}, u_{tt}$. Equating the coefficients of $u_x, u_t, u_{xx}, u_{xt}, u_{tt}$, gives an over-determined system of PDEs for the infinitesimal functions $\xi\,(x,t,u)$, $\tau\,(x,t,u)$ and $\eta\,(x,t,u)$. The solution of this system provides us with the required Lie symmetries of the PDE.

2.3.1 Similarity Reductions

Lie symmetries lead to the construction of transformations which reduce the number of independent variables of a system of PDEs by at least one. Especially, in the case of a PDE with two independent variables the reduction gives an ordinary differential equation. In the case of an ODE the order of the equation is reduced by one. Such transformations are called similarity transformations or similarity reductions. In the case of a system of PDEs with two independent and two dependent variables they can be constructed from the solution of the invariant surface conditions

$$\begin{aligned} &\tau(x,t,u,v)u_t + \xi(x,t,u,v)u_x = \eta(x,t,u,v), \\ &\tau(x,t,u,v)v_t + \xi(x,t,u,v)v_x = \mu(x,t,u,v). \end{aligned} \tag{2.7}$$

This solution is obtained by solving the characteristic system,

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta} = \frac{dv}{\mu}.$$

Now, if $\frac{\xi(x,t,u,v)}{\tau(x,t,u,v)}$ is independent of u and v, then the solution of (2.7) has the form

$$\begin{aligned} \omega(x,t) &= \text{constant}, \\ u(x,t) &= F(x,t,\omega,\phi(\omega)), \\ v(x,t) &= G(x,t,\omega,\psi(\omega)), \end{aligned} \tag{2.8}$$

where F and G are known functions. Equation (2.8) is the *invariant solution* and the function $\omega(t,x)$ is called the *similarity variable* that constitutes the independent variable of the ODE that we obtain from the transformation. The functions $\phi(\omega)$ and $\psi(\omega)$ are the unknown dependent variables of the ODEs.

2.4 Nonclassical Reductions

The method of nonclassical reductions is a generalization of the classical method of Lie reductions for *obtaining invariant solutions of PDEs.* In this method we, furthermore, require the invariance of the PDE

$$E(x,t,u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots)=0, \tag{2.9}$$

under the invariant surface condition, produced by the infinitesimal generator

$$\Gamma=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.10}$$

As a result an over-determined nonlinear system of PDEs for the determination of the coefficients $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$ is obtained.

Although the term "nonclassical symmetry" is used, it is not a symmetry of a given PDE (2.9) unless the infinitesimal coefficients yielding an infinitesimal generator (2.10) yield a point symmetry of (2.9). Otherwise a mapping resulting from such an infinitesimal generator maps *no* solution of (2.9) into a different solution of it. In other words the nonclassical method is not a "symmetry" method but an extension of Lie's symmetry method ("classical method") for the purpose of finding specific solutions of PDEs.

From the nature of the constraint invariant surface condition equation (2.7), without loss of generality, in using the nonclassical method, two simplifying cases need only be considered when solving the determining equations for finding the form of the infinitesimal coefficients, namely $\tau\neq 0$ and $\tau=0$. In the case $\tau(x,t,u)\neq 0$ we can assume, without loss of generality, that $\tau=1$. Also, when $\tau=0$, without loss of generality, we can take $\xi=1$. In this latter case the invariant conditions result in a single nonlinear PDE in $\eta(x,t,u)$. Here we only consider the case where $\tau=1$. For recent applications of this method see [52] and references therein.

2.5 Equivalence Transformations

Equivalence transformations are nondegenerate point transformations, that preserve the differential structure of the class under study and change only its arbitrary elements. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group. There exist two methods for the calculation of equivalence transformations, the direct method which was first used by Lie [55] and the Lie infinitesimal method which was introduced by Ovsiannikov [72]. Although, the direct method involves considerable computational difficulties, it has the advantage of finding the most general equivalence group and also unfolds all form-preserving [46] (also known as admissible [75]) transformations admitted by this class of equations. For recent applications of the direct method one can refer, for example, to references [94–98].

There are different kinds of equivalence groups. The *usual equivalence group*, which has been used for solving group classification problems since late the 50's, consists of the non-degenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables do not involve arbitrary elements of the class under consideration [72]. The notion of the *generalized equivalence group*, where transformations of variables of given DEs explicitly depend on arbitrary elements, was introduced by Meleshko [60, 61] in the mid nineties. The *extended equivalence group* is an equivalence group whose transformations include nonlocalities with respect to arbitrary elements (e.g., if new arbitrary elements are expressed via integrals of old ones) [38]. The *generalized extended equivalence group* possesses the properties of both generalized and extended equivalence groups [94, 95, 97, 98].

Chapter 3

Group analysis of generalized nonlinear equations

3.1 Introduction

The investigation of nonlinear heat (or diffusion if u represents mass concentration) equations by means of symmetry methods began in 1959 with Ovsiannikov's work [71] in which the author performed the group classification of the class of equations of the form

$$u_t = (f(u)u_x)_x \,. \tag{3.1}$$

Equation (3.1) describes the stationary motion of a boundary layer of fluid over a flat plate and a vortex of incompressible fluid in a porus medium with polytropic relation between gas density and pressure.

Another equation that is of considerable interest in mathematical physics is the nonlinear diffusion-convection equation

$$u_t = (f(u)u_x)_x + k(u)u_x.$$

Lie symmetries of the above equation have been considered in [23, 70, 76]. In the case where $f(u) = 1$ it coincides with the generalized Burgers equation

$$u_t = u_{xx} + k(u)u_x. \tag{3.2}$$

Lie symmetries of (3.2) have been derived in [45].

In this Chapter we present the known results for equations (3.1) and (3.2). We give the equivalence transformations and the group classification of Lie symmetries. We also present examples of invariant solutions.

3.2 Nonlinear heat equation

3.2.1 Equivalence transformations

We find that equation (3.1), admits the equivalence transformations

$$t' = c_1 t + c_2, \quad x' = c_3 x + c_4, \quad u' = c_5 u + c_6, \quad f' = \frac{c_3^2}{c_1} f,$$

where $c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6$ are arbitrary constants and $c_1 c_3 c_5 \neq 0$.

3.2.2 Lie Symmetries

A second-order PDE admits Lie point symmetries if and only if

$$\Gamma^{(2)} E\Big|_{E=0} = 0,$$

where $\Gamma^{(2)}$ is the second prolongation of the generator

$$\Gamma = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u},$$

which is given by the relation

$$\begin{aligned}\Gamma^{(2)} &= \Gamma + \left[D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau)\right]\frac{\partial}{\partial u_t} + \left[D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau)\right]\frac{\partial}{\partial u_x} \\ &+ \left[D_x(\eta^x) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau)\right]\frac{\partial}{\partial u_{xx}}.\end{aligned}$$

Here D_t and D_x represent the total derivative operators with respect to t and x respectively and η^x is the coefficient function of $\frac{\partial}{\partial u_x}$.

In this case we have that

$$E = u_t - f(u)u_{xx} - \frac{df(u)}{du}u_x^2 = 0$$

and equation (3.1) admits Lie point symmetries if and only if

$$\Gamma^{(2)}[u_t - f(u)u_{xx} - \frac{df(u)}{du}u_x^2] = 0 \tag{3.3}$$

for $u_t = f(u)u_{xx} + \frac{df(u)}{du}u_x^2$.

Eliminating u_t and also substituting

$$u_{tx} = f(u)u_{xxx} + 3\frac{df(u)}{du}u_x u_{xx} + \frac{d^2 f(u)}{du^2}u_x^3,$$

equation (3.3) becomes a multi-variable polynomial in $u_x,\ u_{xx}$ and u_{xxx}. The coefficients of different powers of these variables must be zero. These give the determining equations for the coefficients ξ, τ and η.

The coefficients of u_{xxx} and $u_x u_{xxx}$ give respectively

$$\tau_x = \tau_u = 0, \tag{3.4}$$

which implies that $\tau(x,t,u) = \tau(t)$. The coefficient of $u_x u_{xx}$ gives

$$\xi_u = 0, \tag{3.5}$$

which means that $\xi(x,t,u) = \xi(x,t)$. The coefficients of u_{xx}, u_x^2, u_x and the term which is independent of the derivatives, give respectively, the following equations:

$$\eta f_u + (\tau_t - 2\xi_x) f = 0, \tag{3.6}$$

$$\eta_{uu} f + (\tau_t - 2\xi_x + \eta_u) f_u + \eta f_{uu} = 0, \tag{3.7}$$

$$(2\eta_{xu} - \xi_{xx}) f + 2\eta_x f_u + \xi_t = 0, \tag{3.8}$$

$$(\tau_t - 2\xi_x + \eta_{xx}) f + \eta f_u - \eta_t = 0. \tag{3.9}$$

When we solve these equations (3.6)–(3.9), we observe that for the case where f is arbitrary, the symmetry Lie algebra is three-dimensional and is spanned by

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = 2t\partial_t + x\partial_x.$$

Note 3.1. Throughout the thesis, we use both notations for partial derivatives, $\frac{\partial}{\partial x}$ or ∂_x, to present the form of Lie symmetries.

An additional fourth Lie symmetry exists in the cases where $f(u) = e^u$, which is

$$X_4 = x\partial_x + 2\partial_u$$

and $f(u) = u^n$, which is

$$X_4 = \frac{n}{2} x\partial_x + u\partial_u.$$

Finally, for the specific value of the parameter $n = -\frac{4}{3}$, that is, $f(u) = u^{-\frac{4}{3}}$, equation (3.1) admits a fifth symmetry

$$X_5 = -x^2\partial_x + 3xu\partial_u.$$

3.2.3 Invariant Solutions

The primary use of Lie symmetries is to obtain a reduction of variables. Similarity variables appear as constants of integration in the solution of the characteristic equations

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}u}{\eta}.$$

Reductions could be obtained from any symmetry which is an arbitrary linear combination, i.e.

$$a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5.$$

In the case for which f is arbitrary, the optimal system and the corresponding similarity reductions that transform (3.1) into an ODE are given by the operators

$$\begin{aligned}
X_1 &: \quad u = \phi(\omega), \quad \omega = x,\\
X_2 &: \quad u = \phi(\omega), \quad \omega = t,\\
X_3 &: \quad u = \phi(\omega), \quad \omega = \frac{x^2}{t},\\
X_1 + X_2 &: \quad u = \phi(\omega), \quad \omega = x - t.
\end{aligned}$$

In the case where $f(u) = e^u$, the optimal system includes the following additional operators

$$\begin{aligned}
X_1 + X_4 &: \quad u = 2t + \phi(\omega), \quad \omega = xe^{-t},\\
X_4 &: \quad u = \ln(t) + \phi(\omega), \quad \omega = x,\\
CX_3 + X_4 &: \quad u = \frac{1}{C}\ln(x) + \phi(\omega), \quad \omega = tx^{\frac{1}{C}-2},\\
X_2 - \frac{1}{2}(X_3 - X_4) &: \quad u = x + \phi(\omega), \quad \omega = te^x.
\end{aligned}$$

When $f(u) = u^n, \quad n \neq 0, -\frac{4}{3}$, the optimal system also widens and includes the additional operators:

$$\begin{aligned}
X_1 + X_4 &: \quad u = e^t\phi(\omega), \quad \omega = xe^{-\frac{nt}{2}},\\
CX_3 + X_4 &: \quad u = \begin{cases} x^{\frac{1}{C+\frac{n}{2}}}\phi(\omega), & \omega = tx^{-\frac{2C}{C+\frac{n}{2}}}, \quad \text{if } C \neq -\frac{n}{2},\\ t^{-\frac{1}{n}}\phi(\omega), & \omega = x, \quad \text{if } C = -\frac{n}{2}, \end{cases},\\
X_2 - \frac{n}{2}X_3 + X_4 &: \quad u = e^x\phi(\omega), \quad \omega = te^{nx}.
\end{aligned}$$

Lastly, when $f(u) = u^{-\frac{4}{3}}$, for which a fifth symmetry exists, we obtain the following additional reductions:

$$
\begin{aligned}
X_1 + X_4 \quad &: \quad u = e^t \phi(\omega), \quad \omega = xe^{\frac{2t}{3}}, \\
CX_3 + X_4 \quad &: \quad u = \begin{cases} x^{\frac{1}{C-\frac{2}{3}}} \phi(\omega), & \omega = tx^{-\frac{2C}{C-\frac{2}{3}}}, \quad \text{if } C \neq \frac{2}{3}, \\ t^{\frac{3}{4}} \phi(\omega), & \omega = x, \qquad\quad \text{if } C = \frac{2}{3}, \end{cases}, \\
X_5 \quad &: \quad u = x^{-3} \phi(\omega), \quad \omega = t, \\
X_2 + X_5 \quad &: \quad u = x^{-3} \phi(\omega), \quad \omega = t - \frac{1}{x}, \\
X_3 + X_5 \quad &: \quad u = x^{-3} \phi(\omega), \quad \omega = \frac{tx^2}{(x+1)^2}.
\end{aligned}
$$

The results of Section 3.2 can be found in [71].

3.3 Generalized Burgers equation

3.3.1 Equivalence transformations

We find that equation (3.2), admits the equivalence transformations

$$t' = c_4^2 t + c_1, \quad x' = c_4 x + c_6 t + c_2, \quad u' = c_5 u + c_3, \quad k' = \frac{1}{c_4} k - \frac{c_6}{c_4^2},$$

where $c_1, \; c_2, \; c_3, \; c_4, \; c_5, \; c_6$ are arbitrary constants and $c_4 c_5 \neq 0$.

3.3.2 Lie Symmetries

In this case we have that

$$E = u_t - u_{xx} - k(u)u_x = 0$$

and equation (3.2) admits Lie point symmetries if and only if

$$\Gamma^{(2)}[u_t - u_{xx} - k(u)u_x] = 0 \tag{3.10}$$

for $u_t = u_{xx} + k(u)u_x$.

Eliminating u_t and also substituting

$$u_{tx} = u_{xxx} + \frac{dk(u)}{du} u_x^2 + k(u) u_{xx},$$

equation (3.10) becomes a multi-variable polynomial in $u_x, \; u_{xx}$ and u_{xxx}. The coefficients of different powers of these variables must be zero. These give the determining equations for the coefficients ξ, τ and η.

The coefficients of u_{xxx} and $u_x u_{xxx}$ give respectively

$$\tau_x = \tau_u = 0, \tag{3.11}$$

which implies that $\tau(x,t,u) = \tau(t)$. The coefficient of $u_x u_{xx}$ gives

$$\xi_u = 0, \tag{3.12}$$

which means that $\xi(x,t,u) = \xi(x,t)$. The coefficients of $u_{xx},\ u_x^2,\ u_x$ and the term independent of the derivatives give, respectively, the following equations:

$$\tau_t - 2\xi_x = 0, \tag{3.13}$$

$$\eta_{uu} = 0, \tag{3.14}$$

$$2\eta_{xu} + \eta k_u + (\tau_t - \xi_x)k + \xi_t - \xi_{xx} = 0, \tag{3.15}$$

$$\eta_x k + \eta_{xx} - \eta_t = 0. \tag{3.16}$$

The solution of the above system gives the Lie symmetries admitted by equation (3.2). For an arbitrary function k, equation (3.2), admits the Lie symmetries

$$X_1 = \partial_t, \quad X_2 = \partial_x.$$

If $k(u) = e^u$, there exists a third Lie symmetry, given by

$$X_3 = 2t\partial_t + x\partial_x - \partial_u.$$

A third Lie symmetry, also exists in the cases where $k(u) = u^n$, which is

$$X_3 = 2nt\partial_t + nx\partial_x - u\partial_u,$$

and when $k(u) = \ln u$

$$X_3 = t\partial_x - u\partial_u.$$

Finally, five Lie symmetries are admitted for $k(u) = u$, which is the case when equation (3.2) coincides with Burgers equation. Here, we have

$$X_3 = t^2\partial_t + tx\partial_x - (tu + x)\partial_u, \quad X_4 = 2t\partial_t + x\partial_x - u\partial_u, \quad X_5 = t\partial_x - \partial_u.$$

Chapter 4

On the simplification of the form of Lie transformation groups

4.1 Introduction

There have been many attempts to compute Lie symmetries of differential equations using different systems of computer algebras, such as MATHEMATICA, MAPLE, MACSYMA, REDUCE, AXIOM, MuPAD etc, as well as different symbolic manipulation packages [17, 21, 31, 85, 100] (see also detailed review in [32, 33]). These programs, although powerful, are not guaranteed to complete their task. They have a number of essential disadvantages, for example, restrictions on nonlinearities. This is particularly true, when we have the problem of finding the Lie symmetries for a class of PDEs instead of a single PDE. In this case, we have unspecified functions, for example of independent or/and dependent variables, appearing in the PDE which are known as arbitrary elements. The problem of finding the Lie symmetries of such a class of PDEs is known as group classification and it is more complicated since in the determining system, in addition to the coefficient functions, we have the appearance of the arbitrary elements. In order to simplify the group classification, it is important to have a-priori knowledge of the form of the coefficient functions.

The results of the present chapter, appear in [49].

4.2 Known results for the general class of scalar evolution equations

The idea of the present chapter was previously adopted by various authors in the literature for scalar PDEs. For example, Tu [93] proved that the Lie symmetry generator

$$\Gamma = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{4.1}$$

admitted by the general class of evolution equations

$$u_t = H(x,t,u,u_1,u_2,\ldots,u_n), \quad H_{u_n} \neq 0,\ n \geq 2$$

has the simplified form

$$\Gamma = \tau(t)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}.$$

Bluman [12] presented some general results for the nature of the infinitesimal Lie generator (4.1) for linear PDEs. He derived certain criteria to determine whether or not the coefficient functions $\tau(x,t,u)$ and $\xi(x,t,u)$ depend only on the independent variables t and x and also criteria that examine whether the coefficient function $\eta(x,t,u)$ is linear in u. Certain restrictions on the form of (4.1) for wave-type equations are presented in [47]. Corresponding results for the nature of general point transformations for evolution equations can be found, for example, in references [40, 46, 57, 77].

4.3 System of evolution equations with two dependent and two independent variables

We consider the system of evolution equations

$$\begin{aligned} u_t &= H(x,t,u,v,u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_m), \quad n \geq 0, \\ v_t &= G(x,t,u,v,u_1,u_2,\ldots,u_r,v_1,v_2,\ldots,v_p), \quad p \geq 0, \end{aligned} \tag{4.2}$$

where

$$u_i = \frac{\partial^i u}{\partial x^i},\ v_i = \frac{\partial^i v}{\partial x^i},\ i = 1, 2, \ldots.$$

In the subsequent analysis, $u_0 = u$ and $v_0 = v$. We also assume that,

$$\frac{\partial H}{\partial u_n} \neq 0,\ \frac{\partial H}{\partial v_m} \neq 0,\ \frac{\partial G}{\partial u_r} \neq 0,\ \frac{\partial G}{\partial v_p} \neq 0.$$

The condition $n \geq 0$ implies that H is always a function of u and/or its derivatives with respect to x. A similar conclusion can be drawn for the condition $p \geq 0$. Systems of the form (4.2) have considerable interest in mathematical physics and in other disciplines of mathematical applications. Such examples are given in the analysis that follows.

Although in the present work we consider systems of PDEs, the results are also useful for searching for nonlocal (potential) symmetries for scalar PDEs. If a PDE can be written in a conserved form, then by introducing a potential variable (new dependent variable), we can write the PDE as a system of two PDEs. This system might lead to nonlocal symmetries for the original PDE. For more details see Ref. [10,11,13]. Also, complex scalar PDEs can be written as a system of two real equations by separating real and imaginary parts. Such complex equations are the Schrödinger type equations which are of considerable interest in mathematical physics. See, for example, in [53, 64, 75, 78, 86].

The results that are derived in the present chapter will also be useful for calculations of the equivalence transformations of systems of evolution equations using the Lie infinitesimal method which was introduced by Ovsiannikov [72]. Such transformations are used, for example, in determining the differential invariants of differential equations. Examples of constructing differential invariants for a system of PDEs of the form (4.2) can be found, for example, in [30,58].

4.4 Restrictions on the form of the coefficient function τ

The Lie symmetry generator admitted by the system of evolution equations (4.2) has the form

$$\Gamma = \tau\left(x,t,u,v\right)\frac{\partial}{\partial t} + \xi\left(x,t,u,v\right)\frac{\partial}{\partial x} + \eta\left(x,t,u,v\right)\frac{\partial}{\partial u} + \mu\left(x,t,u,v\right)\frac{\partial}{\partial v} \tag{4.3}$$

which corresponds to the one-parameter Lie group of transformations

$$\begin{aligned} t' &= t + \epsilon\tau\left(x,t,u,v\right) + O(\epsilon^2) \\ x' &= x + \epsilon\xi\left(x,t,u,v\right) + O(\epsilon^2) \\ u' &= u + \epsilon\eta\left(x,t,u,v\right) + O(\epsilon^2) \\ v' &= v + \epsilon\mu\left(x,t,u,v\right) + O(\epsilon^2). \end{aligned}$$

These transformations leave the system of evolution equations (4.2) invariant. In this section we present those forms of the class (4.2) that admit Lie symmetries when the coefficient function τ depends only on t.

The corresponding extended generator has the form

$$\Gamma^{\text{ext}} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial v} + \eta^t\frac{\partial}{\partial u_t} + \mu^t\frac{\partial}{\partial v_t} + \sum_{i=1}^{\max(n,r)}\eta^{x^i}\frac{\partial}{\partial u_i} + \sum_{i=1}^{\max(m,p)}\mu^{x^i}\frac{\partial}{\partial v_i}, \tag{4.4}$$

where the coefficients of the extended generator are defined as follows

$$\begin{aligned}
\eta^{t} &= D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau),\\
\eta^{x} &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
\eta^{xx} &= D_x(\eta^{x}) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau),\\
&\vdots\\
\eta^{x^{n+1}} &= D_x\left(\eta^{x^{n}}\right) - u_{x^{n+1}} D_x(\xi) - u_{x^{n}t} D_x(\tau),
\end{aligned}$$

$$\begin{aligned}
\mu^{t} &= D_t(\mu) - v_x D_t(\xi) - v_t D_t(\tau),\\
\mu^{x} &= D_x(\mu) - v_x D_x(\xi) - v_t D_x(\tau),\\
\mu^{xx} &= D_x(\mu^{x}) - v_{xx} D_x(\xi) - v_{xt} D_x(\tau),\\
&\vdots\\
\mu^{x^{n+1}} &= D_x\left(\mu^{x^{n}}\right) - v_{x^{n+1}} D_x(\xi) - v_{x^{n}t} D_x(\tau),
\end{aligned}$$

where D_t and D_x are the total derivatives with respect to t and x, respectively, i.e

$$\begin{aligned}
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xx}\frac{\partial}{\partial v_x} + v_{xt}\frac{\partial}{\partial v_t}\\
&+ u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{xxt}\frac{\partial}{\partial u_{xt}} + u_{xtt}\frac{\partial}{\partial u_{tt}} + v_{xxx}\frac{\partial}{\partial v_{xx}} + v_{xxt}\frac{\partial}{\partial v_{xt}} + v_{xtt}\frac{\partial}{\partial v_{tt}} + \ldots \qquad (4.5)
\end{aligned}$$

and

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_x} + v_{tt}\frac{\partial}{\partial v_t}\\
&+ u_{xxt}\frac{\partial}{\partial u_{xx}} + u_{xtt}\frac{\partial}{\partial u_{xt}} + u_{ttt}\frac{\partial}{\partial u_{tt}} + v_{xxt}\frac{\partial}{\partial v_{xx}} + v_{xtt}\frac{\partial}{\partial v_{xt}} + v_{ttt}\frac{\partial}{\partial v_{tt}} + \ldots . \qquad (4.6)
\end{aligned}$$

Now using mathematical induction, it can be shown that

$$\begin{aligned}
\eta^{x^{k}} &= D_x^{k}\eta - \sum_{j=1}^{k}\binom{k}{j-1}\left(D_x^{k+1-j}\xi\right)u_j - \sum_{i=0}^{k-1}\binom{k}{i}\left(D_x^{k-i}\tau\right)u_{x^{i}t},\\
\mu^{x^{k}} &= D_x^{k}\mu - \sum_{j=1}^{k}\binom{k}{j-1}\left(D_x^{k+1-j}\xi\right)v_j - \sum_{i=0}^{k-1}\binom{k}{i}\left(D_x^{k-i}\tau\right)v_{x^{i}t}. \qquad (4.7)
\end{aligned}$$

We note that in the form of $\eta^{x^{n}}$, $D_x^{n}\eta$ gives

$$\eta_u u_n + \eta_v v_n + (\text{lower order terms}),$$

the first sum gives

$$-n\left(D_x\xi\right)u_n - u_x\left(\xi_u u_n + \xi_v v_n\right) + \ldots$$

and the second sum gives

$$-\left(\tau_u u_n+\tau_v v_n\right)H-n\left(D_x\tau\right)\left(\frac{\partial H}{\partial u_n}u_{2n-1}+\frac{\partial H}{\partial v_m}v_{n+m-1}\right)+\ldots.$$

Similar expressions are obtained from the form of μ^{x^m} in the first determining equation and from η^{x^r},μ^{x^p} in the second determining equation.

Here we require that

$$\Gamma^{\text{ext}}(H-u_t)=0,\quad \Gamma^{\text{ext}}(G-v_t)=0$$

modulo the system (4.2), where Γ^{ext} is defined by (4.4). Hence, we have

$$\tau\frac{\partial H}{\partial t}+\xi\frac{\partial H}{\partial x}+\eta\frac{\partial H}{\partial u}+\mu\frac{\partial H}{\partial v}-\eta^t+\sum_{i=1}^{n}\eta^{x^i}\frac{\partial H}{\partial u_i}+\sum_{i=1}^{m}\mu^{x^i}\frac{\partial H}{\partial v_i} = 0, \tag{4.8}$$

$$\tau\frac{\partial G}{\partial t}+\xi\frac{\partial G}{\partial x}+\eta\frac{\partial G}{\partial u}+\mu\frac{\partial G}{\partial v}-\mu^t+\sum_{i=1}^{r}\eta^{x^i}\frac{\partial G}{\partial u_i}+\sum_{i=1}^{p}\mu^{x^i}\frac{\partial G}{\partial v_i} = 0, \tag{4.9}$$

where

$$u_t=H,\ u_{x^it}=D_x^iH,\ v_t=G,\ v_{x^it}=D_x^iG. \tag{4.10}$$

Using (4.10), equations (4.8) and (4.9) become two multi-variable polynomials in the variables x,t,u,v and the derivatives of u and v with respect to x. The results in the present section are determined by collecting the coefficients of the appropriate variables in (4.8) and (4.9) and setting them equal to zero. The derived results are presented in the following theorems.

We consider the general case where all indices $n,\ m,\ r,\ p\geq 2$.

Theorem 4.1. *If the indices $n,\ m,\ r,\ p\geq 2$ and they are not all equal, then system (4.2) admits Lie symmetries of the form (4.3), where*

$$\tau=\tau(t).$$

If $n=m=r=p\geq 2$ and in addition

$$\left[\left(\frac{\partial H}{\partial u_n}\right)^2+\frac{\partial H}{\partial v_n}\frac{\partial G}{\partial u_n}\right]^2+\left[\frac{\partial H}{\partial u_n}+\frac{\partial G}{\partial v_n}\right]^2\neq 0 \tag{4.11}$$

then

$$\tau=\tau(t).$$

Note 4.1. If restriction (4.11) holds then its symmetric

$$\left[\left(\frac{\partial G}{\partial v_n}\right)^2+\frac{\partial H}{\partial v_n}\frac{\partial G}{\partial u_n}\right]^2+\left[\frac{\partial H}{\partial u_n}+\frac{\partial G}{\partial v_n}\right]^2\neq 0$$

also holds. In the case restriction (4.11) is not satisfied, then we can write

$$\frac{\partial H}{\partial u_n}=-\frac{\partial G}{\partial v_n},\qquad \begin{vmatrix} \frac{\partial H}{\partial u_n} & \frac{\partial G}{\partial u_n} \\ \frac{\partial H}{\partial v_n} & \frac{\partial G}{\partial v_n} \end{vmatrix}=0. \tag{4.12}$$

In other words, if the system (4.2) admits Lie symmetries with the coefficient function τ not depending only on t, then H and G satisfy the conditions (4.12).

Proof: Writing only the highest order terms, the first determining equation has the form

$$-\binom{n}{n-1}(D_x\tau)\,u_{x^{n-1}t}\frac{\partial H}{\partial u_n}+\ldots-\binom{m}{m-1}(D_x\tau)\,v_{x^{m-1}t}\frac{\partial H}{\partial v_m}+\ldots$$

which can be simplified, using (4.10), to

$$\begin{aligned}&-n\,(D_x\tau)\left(\frac{\partial H}{\partial u_n}u_{2n-1}+\frac{\partial H}{\partial v_m}v_{n+m-1}+\ldots\right)\frac{\partial H}{\partial u_n}+\ldots\\&-m\,(D_x\tau)\left(\frac{\partial G}{\partial u_r}u_{r+m-1}+\frac{\partial G}{\partial v_p}v_{p+m-1}+\ldots\right)\frac{\partial H}{\partial v_m}+\ldots\end{aligned} \tag{4.13}$$

Similarly, the second determining equation gives

$$\begin{aligned}&-r\,(D_x\tau)\left(\frac{\partial H}{\partial u_n}u_{n+r-1}+\frac{\partial H}{\partial v_m}v_{m+r-1}+\ldots\right)\frac{\partial G}{\partial u_r}+\ldots\\&-p\,(D_x\tau)\left(\frac{\partial G}{\partial u_r}u_{r+p-1}+\frac{\partial G}{\partial v_p}v_{2p-1}+\ldots\right)\frac{\partial G}{\partial v_p}+\ldots\end{aligned} \tag{4.14}$$

In order to complete the proof, we need to consider three cases:

(i) $n>p$,

(ii) $n<p$ and

(iii) $n=p$,

shown in the following table:

Table 4.1: Proof of theorem 4.1

Relation between n and p	Relation between n, m, r	Equation, term	Coefficient	Restriction	Result
$n > p$		(4.13), v_{n+m-1} or (4.14), u_{n+r-1}	$n\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\frac{\partial H}{\partial u_n}=0$ $r\left(D_x\tau\right)\frac{\partial H}{\partial u_n}\frac{\partial G}{\partial u_r}=0$	$\frac{\partial H}{\partial v_m}\frac{\partial H}{\partial u_n}\neq 0$ $\frac{\partial H}{\partial u_n}\frac{\partial G}{\partial u_r}\neq 0$	$D_x\tau=0\Longrightarrow\tau=\tau(t)$
$n < p$		(4.13), v_{m+p-1} or (4.14), u_{r+p-1}	$m\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial v_p}=0$ $p\left(D_x\tau\right)\frac{\partial G}{\partial u_r}\frac{\partial G}{\partial v_p}=0$	$\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial v_p}\neq 0$ $\frac{\partial G}{\partial u_r}\frac{\partial G}{\partial v_p}\neq 0$	$D_x\tau=0\Longrightarrow\tau=\tau(t)$
$n = p$		(4.13), v_{n+m-1} or (4.14), u_{n+r-1}	$\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\left(n\frac{\partial H}{\partial u_n}+m\frac{\partial G}{\partial v_n}\right)=0$ $\left(D_x\tau\right)\frac{\partial G}{\partial u_r}\left(r\frac{\partial H}{\partial u_n}+n\frac{\partial G}{\partial v_n}\right)=0$	$n\frac{\partial H}{\partial u_n}+m\frac{\partial G}{\partial v_n}\neq 0$ $r\frac{\partial H}{\partial u_n}+n\frac{\partial G}{\partial v_n}\neq 0$	$D_x\tau=0\Longrightarrow\tau=\tau(t)$
$n\frac{\partial H}{\partial u_n}+m\frac{\partial G}{\partial v_n}=0, r\frac{\partial H}{\partial u_n}+n\frac{\partial G}{\partial v_n}=0\Longrightarrow n^2=mr$					
$n = p$	$2n < r+m$	(4.13), u_{r+m-1} or (4.14), v_{m+r-1}	$m\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial u_r}=0$ $r\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial u_r}=0$	$\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial u_r}\neq 0$ $\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial u_r}\neq 0$	$D_x\tau=0\Longrightarrow\tau=\tau(t)$
$n = p$	$2n > r+m$	(4.13), u_{2n-1} or (4.14), v_{2n-1}	$n\left(D_x\tau\right)\left(\frac{\partial H}{\partial u_n}\right)^2=0$ $n\left(D_x\tau\right)\left(\frac{\partial G}{\partial v_n}\right)^2=0$	$\frac{\partial H}{\partial u_n}\neq 0$ $\frac{\partial G}{\partial v_n}\neq 0$	$D_x\tau=0\Longrightarrow\tau=\tau(t)$
$n = p$	$2n = r+m$ $\Longrightarrow$ $n=m=r=p$	(4.13), u_{2n-1} or (4.14), v_{2n-1}	$n\left(D_x\tau\right)\left[\left(\frac{\partial H}{\partial u_n}\right)^2+\frac{\partial H}{\partial v_n}\frac{\partial G}{\partial u_n}\right]=0$ $n\left(D_x\tau\right)\left[\frac{\partial H}{\partial v_n}\frac{\partial G}{\partial u_n}+\left(\frac{\partial G}{\partial v_n}\right)^2\right]=0$	$\left(\frac{\partial H}{\partial u_n}\right)^2+\frac{\partial H}{\partial v_n}\frac{\partial G}{\partial u_n}\neq 0$ (note 4.1)$\Longrightarrow$ $\frac{\partial H}{\partial v_n}\frac{\partial G}{\partial u_n}+\left(\frac{\partial G}{\partial v_n}\right)^2\neq 0$	$D_x\tau=0\Longrightarrow\tau=\tau(t)$

We give a trivial counterexample to show that Theorem 4.1 is not generally true if restriction (4.11) is not valid.

Example 4.1. We show that system

$$u_t = u_n - v_n, \quad v_t = u_n - v_n, \qquad n \geq 2$$

admits the Lie symmetry $x\frac{\partial}{\partial t}$. This shows that τ depends also on x. The form of the symmetry implies that

$$\tau = x,\ \xi = 0,\ \eta = 0,\ \mu = 0,$$

so

$$D_t(\tau) = 0,\ \eta^t = 0,\ \mu^t = 0,$$

$$\frac{\partial H}{\partial x} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial u} = \frac{\partial H}{\partial v} = \frac{\partial G}{\partial x} = \frac{\partial G}{\partial t} = \frac{\partial G}{\partial u} = \frac{\partial G}{\partial v} = 0$$

and

$$\frac{\partial H}{\partial u_n} = 1 \neq 0,\ \frac{\partial H}{\partial v_n} = -1 \neq 0,\ \frac{\partial G}{\partial u_n} = 1 \neq 0,\ \frac{\partial G}{\partial v_n} = -1 \neq 0.$$

Also,

$$\eta^{x^n} = -n\left(u_{2n-1} - v_{2n-1}\right),\ \mu^{x^n} = -n\left(u_{2n-1} - v_{2n-1}\right).$$

Therefore the determining equations (4.8) and (4.9) are satisfied.

Now we consider certain special cases where some of the indices, but not both, of n and p, are less than two.

Theorem 4.2. *If at least one of the four indices is less than two and at least one of n and p is greater than or equal to two, then system (4.2) admits Lie symmetries of the form (4.3), where*

$$\tau = \tau(t).$$

Proof: We split the proof into the following cases:

(i) $n,\ p,\ m \geq 2,\ r \leq 1$ (or its symmetric case $n,\ p,\ r \geq 2,\ m \leq 1$);

(ii) $n,\ p \geq 2,\ r,\ m \leq 1$;

(iii) $n \geq 2,\ p \leq 1$ (or its symmetric case $n \leq 1,\ p \geq 2$);

In these cases the form of the leading terms in each determining equation is more complicated, since we cannot ignore the terms that contain the derivatives $u_n,\ v_n,\ u_m,\ v_m$, e.t.c..

(i) $n,\ p,\ m \geq 2,\ r \leq 1$: We consider separately $r=0$ and $r=1$.

If $r=0$, the determining equations (4.8) and (4.9) have the form (writing only the leading terms)

$$\begin{aligned}
& [-\left(\tau_u u_n + \tau_v v_n\right) H - n\left(D_x\tau\right)\left(\frac{\partial H}{\partial u_n}u_{2n-1} + \frac{\partial H}{\partial v_m}v_{n+m-1} + \dots\right) \\
+ \quad & \eta_u u_n + \eta_v v_n - n\left(D_x\xi\right)u_n - u_x\left(\xi_u u_n + \xi_v v_n\right) + \dots]\frac{\partial H}{\partial u_n} + \dots \\
+ \quad & [-\left(\tau_u u_m + \tau_v v_m\right) G - m\left(D_x\tau\right)\frac{\partial G}{\partial v_p}v_{p+m-1} + \dots \\
+ \quad & \mu_u u_m + \mu_v v_m - m\left(D_x\xi\right)v_m - v_x\left(\xi_u u_m + \xi_v v_m\right) + \dots]\frac{\partial H}{\partial v_m} + \dots
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
& [-\left(\tau_u u_p + \tau_v v_p\right) G - p\left(D_x\tau\right)\frac{\partial G}{\partial v_p}v_{2p-1} + \dots \\
+ \quad & \mu_u u_p + \mu_v v_p - p\left(D_x\xi\right)v_p - v_x\left(\xi_u u_p + \xi_v v_p\right) + \dots]\frac{\partial G}{\partial v_p} + \dots
\end{aligned} \tag{4.16}$$

respectively.

- $n > p$ which means that $n+m-1 > n, m, p, r$. The coefficient of v_{n+m-1} in (4.15) gives

$$n\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\frac{\partial H}{\partial u_n} = 0$$

and since $\frac{\partial H}{\partial v_m}\frac{\partial H}{\partial u_n} \neq 0$, we have $D_x\tau = 0$. That is, $\tau = \tau(t)$.

- $n < p$ which implies that $p+m-1 > n, m, p, r$. The coefficient of v_{p+m-1} in (4.15) gives

$$m\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial v_p} = 0$$

and since $\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial v_p} \neq 0$, we have $D_x\tau = 0$. That is, $\tau = \tau(t)$.

- $n = p$ which indicates that $n+m-1 > n, m, p, r$. The coefficient of v_{n+m-1} in (4.15) gives

$$\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\left(n\frac{\partial H}{\partial u_n} + m\frac{\partial G}{\partial v_n}\right) = 0.$$

If $n\frac{\partial H}{\partial u_n} + m\frac{\partial G}{\partial v_n} \neq 0$ and since $\frac{\partial H}{\partial v_m} \neq 0$ the result follows. If $n\frac{\partial H}{\partial u_n} + m\frac{\partial G}{\partial v_n} = 0$ then

$$H = -\frac{m}{n}u_n\frac{\partial G}{\partial v_n} + \dots$$

which means that H is linear in u_n. We have the following two subcases:

(a) $m \geq 2n$: We have $m > n, p, r, 2n-1$ and the coefficient of u_m in (4.15) gives

$$\mu_u - \xi_u v_x - G\tau_u = 0.$$

The left hand side of this expression is the coefficient of H in the second determining equation (H appears in the expression for μ^t) and hence H disappears from this equation. Noting also that $2n-1>n,p,r$, we can take the coefficient of v_{2n-1} in (4.16) to give $n\left(D_x\tau\right)\left(\frac{\partial G}{\partial v_n}\right)^2=0$. Hence, $\tau=\tau(t)$.

(b) $m<2n$: The coefficient of u_n in (4.16) gives

$$\left(\mu_u-\xi_u v_x-G\tau_u\right)\frac{\partial G}{\partial v_n}\left(1+\frac{m}{n}\right)=0.$$

As before, H disappears from this equation. Since $2n-1>n,p,r$, coefficient of v_{2n-1} in (4.16) gives $n\left(D_x\tau\right)\left(\frac{\partial G}{\partial v_n}\right)^2=0$. Hence, $\tau=\tau(t)$.

Now, we consider $r=1$. The determining equations have the form (leading terms)

$$\begin{aligned}
&\Big[-\left(\tau_u u_n+\tau_v v_n\right)H-n\left(D_x\tau\right)\left(\frac{\partial H}{\partial u_n}u_{2n-1}+\frac{\partial H}{\partial v_m}v_{n+m-1}+\ldots\right)\\
+\;&\eta_u u_n+\eta_v v_n-n\left(D_x\xi\right)u_n-u_x\left(\xi_u u_n+\xi_v v_n\right)+\ldots\Big]\frac{\partial H}{\partial u_n}+\ldots\\
+\;&\Big[-\left(\tau_u u_m+\tau_v v_m\right)G-m\left(D_x\tau\right)\left(\frac{\partial G}{\partial u_x}u_m+\frac{\partial G}{\partial v_p}v_{p+m-1}+\ldots\right)\\
+\;&\mu_u u_m+\mu_v v_m-m\left(D_x\xi\right)v_m-v_x\left(\xi_u u_m+\xi_v v_m\right)+\ldots\Big]\frac{\partial H}{\partial v_m}+\ldots
\end{aligned}\tag{4.17}$$

and

$$\begin{aligned}
+\;&\Big[-\left(\tau_u u_p+\tau_v v_p\right)G-p\left(D_x\tau\right)\left(\frac{\partial G}{\partial u_x}u_p+\frac{\partial G}{\partial v_p}v_{2p-1}+\ldots\right)\\
+\;&\mu_u u_p+\mu_v v_p-p\left(D_x\xi\right)v_p-v_x\left(\xi_u u_p+\xi_v v_p\right)+\ldots\Big]\frac{\partial G}{\partial v_p}-\left(D_x\tau\right)H\frac{\partial G}{\partial u_x}+\ldots
\end{aligned}\tag{4.18}$$

respectively.

- $n>p$ which means that $n+m-1>n,m,p,r$. The coefficient of v_{n+m-1} in (4.17) gives

$$n\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\frac{\partial H}{\partial u_n}=0$$

and since $\frac{\partial H}{\partial v_m}\frac{\partial H}{\partial u_n}\neq 0$, we have $D_x\tau=0$. That is, $\tau=\tau(t)$.

- $n<p$ which means that $p+m-1>n,m,p,r$. The coefficient of v_{p+m-1} in (4.17) gives

$$m\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial v_p}=0$$

and since $\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial v_p}\neq 0$, we have $D_x\tau=0$. That is, $\tau=\tau(t)$.

- $n=p$ which means that $n+m-1>n,m,p,r$. The coefficient of v_{n+m-1} in (4.15) gives

$$\left(D_x\tau\right)\frac{\partial H}{\partial v_m}\left(n\frac{\partial H}{\partial u_n}+m\frac{\partial G}{\partial v_n}\right)=0.$$

If $n\frac{\partial H}{\partial u_n}+m\frac{\partial G}{\partial v_n}\neq 0$ and since $\frac{\partial H}{\partial v_m}\neq 0$ the result follows. If $n\frac{\partial H}{\partial u_n}+m\frac{\partial G}{\partial v_n}=0$ then

$$H=-\frac{m}{n}u_n\frac{\partial G}{\partial v_n}+\dots$$

which means that H is linear in u_n. We have the following three subcases:

(a) $m<2n-1$: Since $2n-1>n,m,p,r$, the coefficient of u_{2n-1} in (4.17) gives $n\left(D_x\tau\right)\left(\frac{\partial H}{\partial u_n}\right)^2=0$. Hence, $\tau=\tau(t)$.

(b) $m>2n-1$: We have $m>n,p,r,2n-1$ and hence, coefficient of u_m in (4.17) gives

$$\frac{\partial H}{\partial v_m}\left(\mu_u-\xi_u v_x-G\tau_u-m\left(D_x\tau\right)\frac{\partial G}{\partial u_x}\right)=0.$$

Since $\frac{\partial H}{\partial v_m}\neq 0$, we find

$$\mu_u-\xi_u v_x-G\tau_u=m\left(D_x\tau\right)\frac{\partial G}{\partial u_x}.$$

Using this expression and the coefficient of u_n in (4.18), we deduce that

$$\frac{\partial G}{\partial v_n}\frac{\partial G}{\partial u_x}\left(m-n+\frac{m}{n}+\frac{m^2}{n}\right)\left(D_x\tau\right)=0.$$

Since the bracket is nonzero, the result follows.

(c) $m=2n-1$: We note that $m>n,p,r$. The coefficients of u_n in (4.18) and u_m in (4.17) give, respectively,

$$\frac{\partial G}{\partial v_n}\left[\left(\mu_u-\xi_u v_x-G\tau_u\right)\left(1+\frac{m}{n}\right)+\left(\frac{m}{n}-n\right)\left(D_x\tau\right)\frac{\partial G}{\partial u_x}\right]=0, \tag{4.19}$$

$$-n\left(D_x\tau\right)\left(\frac{\partial H}{\partial u_n}\right)^2+\frac{\partial H}{\partial v_m}\left(\mu_u-\xi_u v_x-G\tau_u-m\left(D_x\tau\right)\frac{\partial G}{\partial u_x}\right)=0. \tag{4.20}$$

Combining these two equations we find that

$$\left(D_x\tau\right)\left(n\left(\frac{\partial H}{\partial u_n}\right)^2+\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial u_x}\left(\frac{5n^2-3n}{3n-1}\right)\right)=0. \tag{4.21}$$

Now, differentiating determining equation (4.18) with respect to v_m to find

$$n\left(D_x\tau\right)\left(\frac{\partial G}{\partial v_n}\right)^2+\frac{\partial H}{\partial v_m}\left(D_x\tau\right)\frac{\partial G}{\partial u_x}+\frac{\partial H}{\partial v_m}\left(\mu_u-\xi_u v_x-G\tau_u\right)=0$$

and using (4.19), this equation simplifies to

$$\left(D_x\tau\right)\left(n\left(\frac{\partial G}{\partial v_n}\right)^2+\frac{n^2+n}{3n-1}\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial u_x}\right)=0. \tag{4.22}$$

Using that $\frac{\partial H}{\partial u_n} = -\frac{m}{n}\frac{\partial G}{\partial v_n}$, the bracket in (4.21) and the bracket in (4.22) cannot be simultaneously equal to zero since this would lead to a contradiction. This can be seen by considering the determinant of the coefficients which can be shown to be nonzero. Hence, $D_x\tau = 0$ and the result follows.

A similar proof is derived for the symmetric case $n,\ p,\ r \geq 2,\ m \leq 1$.

(ii) $n,\ p \geq 2,\ r,\ m \leq 1$: In both cases $m = 0$ and $m = 1$, the coefficient of u_{2n-1} (greatest order derivative of u) in (4.15) gives

$$n\,(D_x\tau)\left(\frac{\partial H}{\partial u_n}\right)^2 = 0 \Longrightarrow D_x\tau = 0 \Longrightarrow \tau = \tau(t).$$

Similar results apply for $r = 0$ and $r = 1$.

(iii) $n \geq 2,\ p \leq 1$: We consider separately $m \geq 2$, $m = 1$ and $m = 0$.

• $m \geq 2$: Since $n + m - 1 > n, m$, the coefficient of v_{n+m-1} in (4.13) gives

$$n\,(D_x\tau)\,\frac{\partial H}{\partial v_m}\frac{\partial H}{\partial u_n} = 0 \Longrightarrow D_x\tau = 0 \Longrightarrow \tau = \tau(t).$$

• $m = 1$: If $r < 2n - 1$, then the coefficient of u_{2n-1} in (4.13) gives $n\,(D_x\tau)\left(\frac{\partial H}{\partial u_n}\right)^2 = 0$ and the result follows. If $r \geq 2n - 1$, then coefficient of u_{n+r-1} in (4.14) gives $r\,(D_x\tau)\,\frac{\partial H}{\partial u_n}\frac{\partial G}{\partial u_r} = 0$ which leads to the result.

• $m = 0$: Now the determining equations are symmetric with equations (4.15) and (4.16), where $r = 0$. In the first determining equation the highest derivative of v is v_n. Taking its coefficient, we find

$$\eta_v - \xi_v u_x - H\tau_v = 0.$$

Therefore the coefficient of G in the first determining equation vanishes and the coefficient of u_{2n-1} in the same equation gives

$$n\,(D_x\tau)\left(\frac{\partial H}{\partial u_n}\right)^2 = 0 \Longrightarrow D_x\tau = 0 \Longrightarrow \tau = \tau(t).$$

A similar proof can be constructed for the symmetric case $n \leq 1,\ p \geq 2$. This completes the proof of Theorem 4.2. □

The same result, as in Theorems 4.1 and 4.2, applies when system (4.2) is separable or semiseparable which is stated in the following theorem.

Theorem 4.3. *If system (4.2) is of a separable or of a semiseparable form and at least one of n and p is greater than or equal to two, then it admits Lie symmetries of the form (4.3), where*

$$\tau = \tau(t).$$

Proof: System (4.2) is of the separable form

$$\begin{aligned} u_t &= H\left(x, t, u, u_1, u_2, \ldots, u_n\right) \\ v_t &= G\left(x, t, v, v_1, v_2, \ldots, v_p\right) \end{aligned} \tag{4.23}$$

The first equation does not involve v and its derivatives and the second equation does not involve u and its derivatives. We assume that at least one of n and p is greater than or equal to 2. If $n \geq 2$ we take the coefficient of u_{2n-1} in the first determining equation, which implies that

$$n\left(D_x\tau\right)\left(\frac{\partial H}{\partial u_n}\right)^2 = 0 \Longrightarrow D_x\tau = 0 \Longrightarrow \tau = \tau(t).$$

If $p \geq 2$ we take the coefficient of v_{2p-1} in the second determining equation, which implies that

$$p\left(D_x\tau\right)\left(\frac{\partial G}{\partial v_p}\right)^2 = 0 \Longrightarrow D_x\tau = 0 \Longrightarrow \tau = \tau(t).$$

Now we consider the case where the system (4.2) is of the semi-separable form

$$\begin{aligned} u_t &= H\left(x, t, u, v, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m\right), \\ v_t &= G\left(x, t, v, v_1, v_2, \ldots, v_p\right), \quad p \geq 1. \end{aligned} \tag{4.24}$$

If $p = 1$, then we assume that $n \geq 2$. The result follows from the coefficient of u_{2n-1} in the first determining equation when $m = 0,\ 1$ and from the coefficient of v_{n+m-1} in the first determining equation when $m \geq 2$. If $p \geq 2$ and $m = 0,\ 1$, then the coefficient of v_{2p-1} in the second determining equation gives the desired result.

We complete the proof by considering the case $p,\ m \geq 2$. Writing the highest order terms, the two determining equations have the form

$$\begin{aligned} & [-\left(\tau_u u_n + \tau_v v_n\right) H - n\left(D_x\tau\right)\left(\frac{\partial H}{\partial u_n} u_{2n-1} + \frac{\partial H}{\partial v_m} v_{n+m-1} + \ldots\right) \\ + \quad & \eta_u u_n + \eta_v v_n - n\left(D_x\xi\right) u_n - u_x\left(\xi_u u_n + \xi_v v_n\right) + \ldots] \frac{\partial H}{\partial u_n} + \ldots \\ + \quad & [-\left(\tau_u u_m + \tau_v v_m\right) G - m\left(D_x\tau\right) \frac{\partial G}{\partial v_p} v_{p+m-1} + \ldots \\ + \quad & \mu_u u_m + \mu_v v_m - m\left(D_x\xi\right) v_m - v_x\left(\xi_u u_m + \xi_v v_m\right) + \ldots] \frac{\partial H}{\partial v_m} + \ldots \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} & [-\left(\tau_u u_p + \tau_v v_p\right) G - p\left(D_x\tau\right) \frac{\partial G}{\partial v_p} v_{2p-1} + \ldots \\ + \quad & \mu_u u_p + \mu_v v_p - p\left(D_x\xi\right) v_p - v_x\left(\xi_u u_p + \xi_v v_p\right) + \ldots] \frac{\partial G}{\partial v_p} + \ldots \end{aligned} \tag{4.26}$$

- $n > p$ which means that $n + m - 1 > n, m, p$. The coefficient of v_{n+m-1} in (4.25) gives

$$n\left(D_x\tau\right) \frac{\partial H}{\partial v_m} \frac{\partial H}{\partial u_n} = 0$$

and since $\frac{\partial H}{\partial v_m}\frac{\partial H}{\partial u_n} \neq 0$, we have $D_x\tau = 0$. That is, $\tau = \tau(t)$.

- $n < p$ which means that $p + m - 1 > n, m, p$. The coefficient of v_{p+m-1} in (4.25) gives

$$m\,(D_x\tau)\,\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial v_p} = 0$$

and since $\frac{\partial H}{\partial v_m}\frac{\partial G}{\partial v_p} \neq 0$, we have $D_x\tau = 0$. That is, $\tau = \tau(t)$.

- $n = p$ which means that $n + m - 1 > n, m, p$. The coefficient of v_{n+m-1} in (4.25) gives

$$(D_x\tau)\,\frac{\partial H}{\partial v_m}\left(n\frac{\partial H}{\partial u_n} + m\frac{\partial G}{\partial v_n}\right) = 0.$$

If $n\frac{\partial H}{\partial u_n} + m\frac{\partial G}{\partial v_n} \neq 0$ and since $\frac{\partial H}{\partial v_m} \neq 0$ the result follows. If $n\frac{\partial H}{\partial u_n} + m\frac{\partial G}{\partial v_n} = 0$ then

$$H = -\frac{m}{n}u_n\frac{\partial G}{\partial v_n} + \dots$$

which means that H is linear in u_n. We have the following three subcases:

(a) $m < 2n-1$. Since $2n-1 > n, m, p$, the coefficient of v_{2n-1} in (4.26) gives $n\,(D_x\tau)\left(\frac{\partial G}{\partial v_n}\right)^2 = 0$. Hence, $\tau = \tau(t)$.

(b) $m > 2n - 1$. We have $m > n, p, 2n - 1$ and hence, differentiating (4.26) with respect to v_m to give

$$\frac{\partial H}{\partial v_m}\,(\mu_u - \xi_u v_x - G\tau_u) = 0.$$

Since $\frac{\partial H}{\partial v_m} \neq 0$, we find

$$\mu_u - \xi_u v_x - G\tau_u = 0.$$

Using this expression we deduce that the coefficient of H in (4.26) is zero. The coefficient of v_{2n-1} in (4.26) implies that $D_x\tau = 0$.

(c) $m = 2n - 1$. The coefficient of u_n in (4.26) gives

$$\frac{\partial G}{\partial v_n}\,(\mu_u - \xi_u v_x - G\tau_u)\left(1 + \frac{m}{n}\right) = 0$$

which implies that

$$\mu_u - \xi_u v_x - G\tau_u = 0.$$

This means that H disappears from the determining equation (4.26). The coefficient of v_{2n-1} in (4.26) implies

$$n\left(\frac{\partial G}{\partial v_n}\right)^2 D_x\tau = 0.$$

Hence, $D_x \tau = 0$ and the result follows. □

An example of uncoupled equations is the system of nonlinear diffusion equations $u_t = (f(u)u_x)_x$ and $v_t = (g(v)v_x)_x$. Lie symmetries for these have been derived in 1986 by Knyazeva and Popov and the results can be found at page 171 in the book [36].

4.5 Further restrictions on the form of the symmetry generator

We consider systems of the class (4.2) that admit Lie symmetries, where $\tau = \tau(t)$. Systems where $n = m = r = p \geq 2$ and the condition (4.11) does not hold must be investigated separately. In this section we present further restrictions on the coefficient functions $\xi(x,t,u,v)$, $\eta(x,t,u,v)$ and $\mu(x,t,u,v)$ for certain forms of the general system (4.2).

In the case where $\tau = \tau(t)$, the coefficient functions of the extended generator have the simplified forms

$$\begin{aligned}
\eta^t &= \eta_t + \eta_u u_t + \eta_v v_t - u_x \left(\xi_t + \xi_u u_t + \xi_v v_t\right) - u_t \tau_t \\
\eta^x &= \eta_x + \eta_u u_x + \eta_v v_x - u_x \left(\xi_x + \xi_u u_x + \xi_v v_x\right) \\
\eta^{xx} &= D_x\left(\eta^x\right) - u_{xx} D_x\left(\xi\right) \\
&\vdots \\
\eta^{x^{n+1}} &= D_x\left(\eta^{x^n}\right) - u_{x^{n+1}} D_x\left(\xi\right)
\end{aligned}$$

$$\begin{aligned}
\mu^t &= \mu_t + \mu_u u_t + \mu_v v_t - v_x \left(\xi_t + \xi_u u_t + \xi_v v_t\right) - v_t \tau_t \\
\mu^x &= \mu_x + \mu_u u_x + \mu_v v_x - v_x \left(\xi_x + \xi_u u_x + \xi_v v_x\right) \\
\mu^{xx} &= D_x\left(\mu^x\right) - v_{xx} D_x\left(\xi\right) \\
&\vdots \\
\mu^{x^{n+1}} &= D_x\left(\mu^{x^n}\right) - v_{x^{n+1}} D_x\left(\xi\right).
\end{aligned}$$

Now, using induction we find

$$\begin{aligned}
\eta^{x^k} &= D_x^k \eta - \sum_{j=1}^{k} \binom{k}{j-1} \left(D_x^{k+1-j} \xi\right) u_j \\
\mu^{x^k} &= D_x^k \mu - \sum_{j=1}^{k} \binom{k}{j-1} \left(D_x^{k+1-j} \xi\right) v_j.
\end{aligned}$$

In the subsequent analysis we require the coefficients of v_n and v_{n-1} in η^{x^n} and the coefficients of u_m and u_{m-1} in μ^{x^m}. We use the following lemma.

Lemma 4.1. *If $\tau = \tau(t)$ and $n, m \geq 2$, then η^{x^n} and μ^{x^m} have the following form*

$$\begin{aligned}
\eta^{x^n} &= (\eta_v - u_x\xi_v)\, v_n + n v_{n-1}\left(\eta_{vx} + \eta_{uv}u_x + \eta_{vv}v_x - \xi_{vx}u_x - \xi_{uv}u_x^2 - \xi_{vv}u_xv_x - \xi_v u_{xx}\right) \\
&+ \Phi(x,t,u,v,u_1,\ldots,u_n,v_1,\ldots,v_{n-2}), \qquad (4.27) \\
\mu^{x^m} &= (\mu_u - v_x\xi_u)\, u_m + m u_{m-1}\left(\mu_{ux} + \mu_{uv}v_x + \mu_{uu}u_x - \xi_{ux}v_x - \xi_{uv}v_x^2 - \xi_{uu}u_xv_x - \xi_u v_{xx}\right) \\
&+ \Psi(x,t,u,v,u_1,\ldots,u_{m-2},v_1,\ldots,v_m), \qquad (4.28)
\end{aligned}$$

respectively, where Φ and Ψ are smooth functions in their arguments.

The proof of the lemma is constructed by induction on the indices n and m, respectively.

Theorem 4.4. *We consider system (4.2) with $n \geq 2$ or/and $p \geq 2$. In the following cases*

(i) $n > p, m, r$ (and its symmetric case $p > n, m, r$)

(ii) $n = p < r < m$ (and its symmetric case $n = p < m < r$)

(iii) $n = p < r = m$ with the restriction $(H_{v_mv_m})^2 + (G_{u_mu_m})^2 \neq 0$

(iv) $n = p > m, r$ with the restriction $H_{u_n} \neq G_{v_n}$

(v) $n = p$ and $m, r \leq n-2$ with the restrictions $H_{u_n} = G_{v_n}$, $\left(H_{u_{n-1}u_{n-1}}\right)^2 + \left(G_{v_{n-1}v_{n-1}}\right)^2 \neq 0$

the system (4.2) admits Lie symmetries with generator of the form (4.3), where the coefficient functions have the restricted forms

$$\tau = \tau(t), \quad \xi = \xi(x,t), \quad \eta = \eta(x,t,u), \quad \mu = \mu(x,t,v).$$

Proof: Using the result for $\tau = \tau(t)$, the determining equations have the form (4.8) and (4.9) with

$$\begin{aligned}
\eta^t &= \eta_t + \eta_u H + \eta_v G - u_x\xi_t - u_x\xi_u H - u_x\xi_v G - \tau_t H \\
\mu^t &= \mu_t + \mu_u H + \mu_v G - v_x\xi_t - v_x\xi_u H - v_x\xi_v G - \tau_t G \qquad (4.29)
\end{aligned}$$

and the forms of η^{x^n}, μ^{x^m} are given in Lemma 4.1. We consider each case separately.

(i) $n > p, m, r$: We differentiate (4.8) and (4.9) with respect to v_n and u_n, respectively, to give

$$(\eta_v - u_x\xi_v)\,\frac{\partial H}{\partial u_n} = 0, \quad (\mu_u - v_x\xi_u)\,\frac{\partial H}{\partial u_n} = 0.$$

Hence, $\xi = \xi(x,t)$, $\eta = \eta(x,t,u)$, $\mu = \mu(x,t,v)$. A similar proof can be constructed for its symmetric case $p > n, m, r$.

(ii) $n = p < r < m$: We differentiate (4.9) with respect to v_m to give

$$(\mu_u - v_x\xi_u)\,\frac{\partial H}{\partial v_m} = 0.$$

Hence, $\mu_u - v_x \xi_u = 0$ which means that H disappears from the determining equation (4.9). Therefore the highest order derivative of v that appears in (4.9) is v_r. We differentiate (4.9) with respect to v_r to give

$$(\eta_v - u_x \xi_v) \frac{\partial G}{\partial u_r} = 0.$$

Hence,

$$\xi = \xi(x,t),\ \eta = \eta(x,t,u),\ \mu = \mu(x,t,v).$$

A similar proof may be constructed for its symmetric case $n = p < m < r$.

(iii) $n = p < r = m$ with the restriction $(H_{v_m v_m})^2 + (G_{u_m u_m})^2 \neq 0$: We differentiate (4.9) with respect to v_m to give

$$(\mu_u - v_x \xi_u) \frac{\partial H}{\partial v_m} = (\eta_v - u_x \xi_v) \frac{\partial G}{\partial u_m}.$$

After further differentiation with respect to v_m or u_m and using the restriction

$$(H_{v_m v_m})^2 + (G_{u_m u_m})^2 \neq 0,$$

we conclude that

$$\mu_u - v_x \xi_u = \eta_v - u_x \xi_v = 0.$$

Hence, $\xi = \xi(x,t),\ \eta = \eta(x,t,u),\ \mu = \mu(x,t,v)$.

(iv) $n = p > m, r$ with the restriction $H_{u_n} \neq G_{v_n}$: We differentiate (4.8) and (4.9) with respect to v_n and u_n, respectively, to give

$$(\eta_v - u_x \xi_v) \frac{\partial G}{\partial v_n} = (\eta_v - u_x \xi_v) \frac{\partial H}{\partial u_n}, \quad (\mu_u - v_x \xi_u) \frac{\partial H}{\partial u_n} = (\mu_u - v_x \xi_u) \frac{\partial G}{\partial v_n}.$$

Since $H_{u_n} \neq G_{v_n}$ the result follows.

(v) $n = p$ and $m, r \leq n-2$ with the restrictions

$$H_{u_n} = G_{v_n},\ \left(H_{u_{n-1} u_{n-1}}\right)^2 + \left(G_{v_{n-1} v_{n-1}}\right)^2 \neq 0:$$

From the condition

$$H_{u_n} = G_{v_n} = K(x,t,u,v,u_1,\ldots,u_r,v_1,\ldots,v_m),$$

we conclude that, if $r, m \leq n-2$,

$$\begin{aligned} H &= K(x,t,u,v,u_1,\ldots,u_r,v_1,\ldots,v_m)u_n + F(x,t,u,v,u_1,\ldots,u_{n-1},v_1,\ldots,v_m), \\ G &= K(x,t,u,v,u_1,\ldots,u_r,v_1,\ldots,v_m)v_n + L(x,t,u,v,u_1,\ldots,u_r,v_1,\ldots,v_{n-1}). \end{aligned}$$

With these forms of H and G, v_n disappears from determining equation (4.8) and u_n from (4.9). We differentiate (4.8) and (4.9) with respect to v_{n-1} and u_{n-1}, respectively, to give

$$\begin{aligned}
(\eta_v - u_x\xi_v)\frac{\partial L}{\partial v_{n-1}} &= (\eta_v - u_x\xi_v)\frac{\partial F}{\partial u_{n-1}} \\
&+ n\left(\eta_{vx} + \eta_{uv}u_x + \eta_{vv}v_x - \xi_{vx}u_x - \xi_{uv}u_x^2 - \xi_{vv}u_xv_x - \xi_v u_{xx}\right)K, \\
(\mu_u - v_x\xi_u)\frac{\partial F}{\partial u_{n-1}} &= (\mu_u - v_x\xi_u)\frac{\partial L}{\partial v_{n-1}} \\
&+ n\left(\mu_{ux} + \mu_{uv}v_x + \mu_{uu}u_x - \xi_{ux}v_x - \xi_{uv}v_x^2 - \xi_{uu}u_xv_x - \xi_u v_{xx}\right)K.
\end{aligned}$$

Using the restriction

$$\left(H_{u_{n-1}u_{n-1}}\right)^2 + \left(G_{v_{n-1}v_{n-1}}\right)^2 \neq 0$$

or equivalently

$$\left(F_{u_{n-1}u_{n-1}}\right)^2 + \left(L_{v_{n-1}v_{n-1}}\right)^2 \neq 0,$$

differentiating both of the above equations with respect to u_{n-1} (or v_{n-1}) yields

$$\mu_u - v_x\xi_u = \eta_v - u_x\xi_v = 0.$$

Hence,

$$\xi = \xi(x,t),\ \eta = \eta(x,t,u),\ \mu = \mu(x,t,v).$$

□

Example 4.2. The form of the following system [42]

$$u_t = u_{xxx} + 6uu_x + 2vv_x, \quad v_t = 2(uv)_x,$$

is such that ($n = 3,\ m = r = p = 1$) Theorems 4.2 and 4.4(i) can be applied. Hence, all Lie symmetries admitted by this system are of the restricted form

$$\tau(t)\partial_t + \xi(x,t)\partial_x + \eta(x,t,u)\partial_u + \mu(x,t,v)\partial_v.$$

Theorem 4.5. *System (4.2) with $n = p \geq 2$, $m = n-1$ and $r \leq n-2$ with the restrictions*

$$H_{u_n} = G_{v_n},\ H_{u_{n-1}u_{n-1}} \neq 0,$$

admits Lie symmetries with a generator of the form (4.3), where

$$\tau = \tau(t), \quad \xi = \xi(x,t,v), \quad \mu = \mu(x,t,v).$$

System (4.2) with $n = p \geq 2$, $r = n-1$ and $m \leq n-2$ with the restrictions

$$H_{u_n} = G_{v_n},\ G_{v_{n-1}v_{n-1}} \neq 0,$$

admits Lie symmetries with a generator of the form (4.3), where

$$\tau = \tau(t), \quad \xi = \xi(x,t,u), \quad \eta = \eta(x,t,u).$$

Note 4.2. We point out that the two parts of the above theorem are related by the discrete symmetry $u \mapsto v$ and $v \mapsto u$ applied to both the system of equations and the symmetry generator.

Proof: From the restriction $H_{u_n} = G_{v_n} = K(x, t, u, v, u_1, \ldots, u_r, v_1, \ldots, v_m)$, we have

$$\begin{aligned} H &= K(x,t,u,v,u_1,\ldots,u_r,v_1,\ldots,v_{n-1})u_n + F(x,t,u,v,u_1,\ldots,u_{n-1},v_1,\ldots,v_{n-1}), \\ G &= K(x,t,u,v,u_1,\ldots,u_r,v_1,\ldots,v_{n-1})v_n + L(x,t,u,v,u_1,\ldots,u_r,v_1,\ldots,v_{n-1}), \end{aligned}$$

where $r \leq n-2$. With these forms of H and G, u_n disappears from determining equation (4.9). We differentiate (4.9) with respect to u_{n-1} to give

$$\begin{aligned} (\mu_u - v_x\xi_u)\frac{\partial F}{\partial u_{n-1}} &= (\mu_u - v_x\xi_u)\frac{\partial G}{\partial v_{n-1}} \\ &+ n\left(\mu_{ux} + \mu_{uv}v_x + \mu_{uu}u_x - \xi_{ux}v_x - \xi_{uv}v_x^2 - \xi_{uu}u_x v_x - \xi_u v_{xx}\right)K. \end{aligned}$$

Using the restriction $H_{u_{n-1}u_{n-1}} \neq 0$ or equivalently $F_{u_{n-1}u_{n-1}} \neq 0$ and differentiating the above equation with respect to u_{n-1} implies that $\mu_u - v_x\xi_u = 0$. Hence,

$$\xi = \xi(x,t,v), \;\; \mu = \mu(x,t,v).$$

A similar proof may be constructed for the other part of Theorem 4.5. $\square$

Example 4.3. It can be shown that the system

$$u_t = \frac{v_x}{u_x^3}u_{xx}, \quad v_t = \frac{v_x}{u_x^3}v_{xx}$$

admits, among others, the two Lie symmetries $u\frac{\partial}{\partial x}$ and $v\frac{\partial}{\partial x}$. Here we have

$$n = p = 2, \;\; m = r = 1 = n-1.$$

Also the system

$$u_t = \frac{u_{xxx}}{u_x^3} - \frac{3u_{xx}^2}{u_x^4} - \frac{v_{xx}}{u_x} + \frac{v_x u_{xx}}{u_x^2}, \quad v_t = \frac{v_{xxx}}{u_x^3} - \frac{3u_{xx}v_{xx}}{u_x^4} - \frac{v_x v_{xx}}{u_x^2} + \frac{v_x^2 u_{xx}}{u_x^3}$$

admits at least the two Lie symmetries, $u\frac{\partial}{\partial x}$ and $v\frac{\partial}{\partial x}$. Here we have

$$n = p = 3, \;\; m = r = 2 = n-1.$$

These two results show that in Theorem 4.5 we need to take either m or r less than $n-1$, otherwise restrictions on the form of ξ do not exist in general. In the above systems we note that $H_{u_n} = G_{v_n}$. This shows that the restriction $H_{u_n} \neq G_{v_n}$ is needed in order to prove Theorem 4.4, case (iv).

Theorem 4.6. *We consider the quasi-linear system*

$$\begin{aligned} u_t &= H_1 u_n + H_2(x, t, u, v, u_1, \dots, u_{n-1}, v_1, \dots, v_m), \\ v_t &= G_1(x, t, u, v, u_1, \dots, u_r, v_1, \dots, v_{n-1}) v_n + G_2(x, t, u, v, u_1, \dots, u_r, v_1, \dots, v_{n-1}), \end{aligned}$$

where $n \geq 2,\ m, r \leq n-1$ *and* H_1 *is a nonzero constant. If* $G_1 \neq H_1$*, then*

$$\tau = \tau(t), \quad \xi = \tfrac{1}{n} x \tau_t(t) + \phi(t), \quad \eta = \eta(x, t, u), \quad \mu = \mu(x, t, v).$$

We point out that Theorem 4.6 is also valid for $G_1 = 0$.

Proof: From Theorems 4.1, 4.2 or 4.3, we have $\tau = \tau(t)$. Also from Theorem 4.4, we have $\eta = \eta(x, t, u)$ and $\mu = \mu(x, t, v)$. The coefficient of u_n in the first determining equation gives

$$H_1[n\xi_x(x, t) - \tau(t)] = 0.$$

Hence, $\xi(t, x) = \frac{1}{n} x \tau_t(t) + \phi(t)$. □

Example 4.4. We consider the coupled Burgers system of the form [16]

$$u_t = \lambda_1 u_{xx} + u u_x + f(u, v) v_x, \quad v_t = \lambda_2 v_{xx} + v v_x + f(u, v) u_x,$$

where $\lambda_1 \neq \lambda_2$. Using Theorem 4.6, this system admits Lie symmetries which are all of the form $\tau(t)\partial_t + (\frac{1}{2} x \tau_t + \phi(t))\partial_x + \eta(x, t, u)\partial_u + \mu(x, t, v)\partial_v$. Lie symmetries of such a form are also admitted by the coupled Drinfeld–Sokolov–Satsuma–Hirota system [2, 22, 84]

$$u_t = -u_{xxx} + 6 u u_x + 6 v_x, \quad v_t = 2 v_{xxx} - 6 u v_x.$$

We note that in the above example for the Burgers system with $\lambda_1 = \lambda_2$ and for the following system [90]

$$u_t + u_{xxx} + 2 u u_x + 2 v u_x = 0, \ v_t + v_{xxx} + 2 v v_x + 2 u v_x = 0$$

Theorem 4.6 does not apply.

4.6 Conclusion

Motivated by the results of Tu [93] and Bluman [12], we have presented a number of results, concerning the form of the infinitesimal generator. A next step could be the generalization of these results in the cases where we have systems with two independent and n dependent variables and also for equations with one dependent and n independent variables.

Chapter 5

Group classification of systems of diffusion equations

5.1 Introduction

Systems of diffusion equations of the form

$$u_t = [f(u,v)u_x + h(u,v)v_x]_x\,, \quad v_t = [k(u,v)u_x + g(u,v)v_x]_x\,, \tag{5.1}$$

are of considerable interest in mathematical biology and in soil science. For instance, such systems describe the movement of water in a homogeneous unsaturated soil, to cases describing the combined transport of water vapour and heat under a combination of gradients of soil temperature and volumetric water content [43, 73]. In such problems $u(x,t)$ and $v(x,t)$ are, respectively, the soil temperature and volumetric water content at depth x and time t.

In the present chapter, we consider the special case of the class (5.1)

$$u_t = [f(u,v)u_x]_x\,, \quad v_t = [g(u,v)v_x]_x\,, \tag{5.2}$$

where we assume that f and g, being the diffusivity coefficients, are non-zero smooth functions in their arguments. Such systems have also been used to model successfully physical situations, such as transport in porous media with variable transmissivity [24] and river pollution [59]. Further applications can be found in plasma physics [80, 81]. We study this system from the Lie point of view. We carry out the Lie group classification. That is, we find the all forms of $f(u,v)$ and $g(u,v)$ such that system (5.2) admits Lie symmetries. The problem was initiated in [41] and completed here.

The results of the present chapter, appear in [48].

5.2 Equivalence Transformations

To derive the equivalence group of the class under consideration we use the direct method [46]. The details of the calculations are omitted for brevity and we only present the results.

Theorem 5.1. *The usual equivalence group $G^{\sim}$ of class (5.2) consists of the transformations*

$$\begin{aligned} &t' = \alpha_1 t + \alpha_2, \quad x' = \beta_1 x + \beta_2, \quad u' = \gamma_1 u + \gamma_2, \quad v' = \delta_1 v + \delta_2, \\ &f' = \alpha_1^{-1}\beta_1^2 f, \quad g' = \alpha_1^{-1}\beta_1^2 g, \end{aligned} \tag{5.3}$$

where $\alpha_1\beta_1\gamma_1\delta_1 \neq 0$.

It turns out that in the case where the arbitrary elements are equal, the usual equivalence group is wider and the results are presented in the following theorem.

Theorem 5.2. *The usual equivalence group $G^{\sim}_{f=g}$ of class (5.2), where $f = g$, consists of the transformations*

$$\begin{aligned} &t' = \alpha_1 t + \alpha_2, \quad x' = \beta_1 x + \beta_2, \quad u' = \gamma_1 u + \gamma_2 v + \gamma_3, \quad v' = \delta_1 u + \delta_2 v + \delta_3, \\ &f' = \alpha_1^{-1}\beta_1^2 f, \end{aligned} \tag{5.4}$$

where $\alpha_1\beta_1(\gamma_1\delta_2 - \gamma_2\delta_1) \neq 0$.

Equivalence transformations are used to simplify the analysis, with the understanding that these equivalence transformations are included in the conclusions. For example, if

$$f(u,v) = (\mu_1 u + \mu_2)^n(\mu_3 v + \mu_4)^m, \text{ with } \mu_1\mu_3 \neq 0,$$

we can use scalings and translations of u and v to take, without loss of generality, $f(u,v) = u^n v^m$.

5.3 Lie Symmetries

The classical method for finding Lie point symmetries is well known, see for example in [13–15, 30, 37, 68, 72]. Here we search for generators

$$\Gamma = \tau(x,t,u,v)\frac{\partial}{\partial t} + \xi(x,t,u,v)\frac{\partial}{\partial x} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \mu(x,t,u,v)\frac{\partial}{\partial v}$$

corresponding to the infinitesimal transformations

$$t' = t + \epsilon\tau(x,t,u,v), \quad x' = x + \epsilon\xi(x,t,u,v), \quad u' = u + \epsilon\eta(x,t,u,v), \quad v' = v + \epsilon\mu(x,t,u,v)$$

to the first order of ϵ. These transformations are such that when the nth extension $\Gamma^{(n)}$ of Γ, where n is the order of the corresponding equation in the system, is applied to the equations of the system the resulting equations are identically zero, modulo the system under consideration. Theorem 4.2 implies that

$$\tau = \tau(t).$$

Here we require that

$$\Gamma^{(2)}\left\{u_t - f u_{xx} - f_u u_x^2 - f_v u_x v_x\right\} = 0, \quad \Gamma^{(2)}\left\{v_t - g v_{xx} - g_u u_x v_x - g_v v_x^2\right\} = 0, \tag{5.5}$$

identically, modulo the system (5.2). The resulting equations, before using system (5.2), are two identities in the variables

$$x, t, u, v, u_x, u_t, v_x, v_t, u_{xx}, v_{xx}.$$

Eliminating u_t and v_t, using the system (5.2), then we have two identities in the variables

$$x, t, u, v, u_x, v_x, u_{xx}, v_{xx}.$$

In order to use Theorem 4.4(iv), we spilt the analysis into two cases:

1: $f \neq g$,

2: $f = g$.

Case 1: $f \neq g$. Using Theorem 4.4(iv), we conclude that

$$\xi = \xi(x,t), \;\; \eta = \eta(x,t,u), \;\; \mu = \mu(x,t,v).$$

These forms simplify the system (5.5). We derive the system of the determining equations that correspond to the first invariant condition in (5.5). The coefficients of u_x^2, $u_x v_x$, u_{xx}, u_x, v_x and the term independent of derivatives give, respectively,

$$\begin{aligned}
&\eta_{uu} f + (\tau_t + \eta_u - 2\xi_x) f_u + \mu f_{uv} + \eta f_{uu} = 0,\\
&(\tau_t + \mu_v - 2\xi_x) f_v + \eta f_{uv} + \mu f_{vv} = 0,\\
&\eta f_u + \mu f_v + (\tau_t - 2\xi_x) f = 0,\\
&2\eta_x f_u + \mu_x f_v + (2\eta_{ux} - \xi_{xx}) f + \xi_t = 0,\\
&\eta_x f_v = 0, \quad \eta_t - \eta_{xx} f = 0.
\end{aligned} \tag{5.6}$$

Similarly, the coefficients of v_x^2, $u_x v_x$, v_{xx}, v_x, u_x and the term independent of derivatives in the second equation in (5.5) give, respectively,

$$
\begin{aligned}
&\mu_{vv}g + (\tau_t + \mu_v - 2\xi_x)g_v + \eta g_{uv} + \mu g_{vv} = 0,\\
&(\tau_t + \eta_u - 2\xi_x)g_u + \mu g_{uv} + \eta g_{uu} = 0,\\
&\eta g_u + \mu g_v + (\tau_t - 2\xi_x)g = 0,\\
&\eta_x g_u + 2\mu_x g_v + (2\mu_{vx} - \xi_{xx})g + \xi_t = 0,\\
&\mu_x g_u = 0, \quad \mu_t - \mu_{xx}g = 0.
\end{aligned} \tag{5.7}
$$

The solution of the above determining system with twelve equations provides us with the desired results. However, since the arbitrary elements f and g are functions of two variables, it makes our task more difficult than the group classifications where the arbitrary elements depend only on one dependent variable.

We differentiate the third equation in (5.6) and subtract the resulting equation from the first equation to give $\eta_{uu}f = 0$. In a similar manner, from the first and third equations in (5.7) we obtain $g\mu_{vv} = 0$. Hence,

$$\eta(x,t,u) = A_1(x,t)u + A_2(x,t), \quad \mu(x,t,v) = A_3(x,t)v + A_4(x,t).$$

Now, from the third equations in (5.6) and in (5.7), respectively, we deduce that the functions $f(u,v)$ and $g(u,v)$ satisfy a first order quasi-linear partial differential equation of the form

$$(\lambda_1 u + \lambda_2)\frac{\partial \phi}{\partial u} + (\lambda_3 v + \lambda_4)\frac{\partial \phi}{\partial v} + \lambda_5 \phi = 0. \tag{5.8}$$

If the coefficients in the above partial differential equation vanish, then we deduce that $f(u,v)$ and $g(u,v)$ are arbitrary functions and $\tau = 2c_1 t + c_2,\ \xi = c_1 x + c_3,\ \eta = \mu = 0$. Hence, for arbitrary f and g, the system (5.2) admits the Lie symmetries

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = 2t\partial_t + x\partial_x, \tag{5.9}$$

where this basis is denoted by $A^{\ker}$. The next step is to find those forms of $f(u,v)$ and $g(u,v)$ which are such that the system (5.2) admits additional Lie symmetries. These forms of $f(u,v)$ and $g(u,v)$ are solutions of the PDE (5.8). Finding all possible forms of $f(u,v)$ and $g(u,v)$ and then substitution into the systems (5.6) and (5.7) enables us to determine the forms of $\tau,\ \xi,\ \eta$ and μ. Hence, the desired Lie symmetries can be obtained. Equivalence transformations (5.3) are used to simplify the PDE (5.8). That is, fixing certain constants in (5.8). Persistence leads to the results listed in Table 5.1. We point out that the solution of PDE (5.8) contains an arbitrary function, such as in cases 1, 2, 3, 4 and 8 in Table 5.1. However, for certain forms of

Table 5.1: Group classification of the system (5.2), where $f \neq g$.

n	$f(u,v)$	$g(u,v)$	Additional Lie symmetries
1.	$u^n F(e^{mv}u)$	$u^n G(e^{mv}u)$	$X_4 = mnt\partial_t - mu\partial_u + \partial_v$
2.	$u^n F(v^m/u)$	$u^n G(v^m/u)$	$X_4 = mnt\partial_t - mu\partial_u - v\partial_v$
3.	$e^u F(e^{nu}v)$	$e^u G(e^{nu}v)$	$X_4 = t\partial_t - \partial_u + nv\partial_v$
4.	$e^u F(v+\varepsilon u)$	$e^u G(v+\varepsilon u)$	$X_4 = t\partial_t - \partial_u + \varepsilon\partial_v$
5.	$u^n v^m$	$au^n v^m$	$X_4 = mx\partial_x + 2v\partial_v,\ X_5 = nx\partial_x + 2u\partial_u$
6.	$u^n e^v$	$au^n e^v$	$X_4 = x\partial_x + 2\partial_v,\ X_5 = nx\partial_x + 2u\partial_u$
7.	e^{u+v}	ae^{u+v}	$X_4 = t\partial_t - \partial_u,\ X_5 = t\partial_t - \partial_v$
8.	$F(v)$	$G(v)$	$X_4 = \partial_u,\ X_5 = u\partial_u$
9.	v^m	av^m	$X_4 = \partial_u,\ X_5 = u\partial_u,\ X_6 = mt\partial_t - v\partial_v$
10.	e^v	ae^v	$X_4 = \partial_u,\ X_5 = u\partial_u,\ X_6 = t\partial_t - \partial_v$

Here $\varepsilon = 0, \pm 1$, $a \neq 0, 1$, a is a constant; F, G are arbitrary functions in their arguments, where $F \neq G$.

this function, system (5.2) admits additional Lie symmetries. Such cases are cases 5, 6, 7, 9 and 10 in Table 5.1.

For completeness, we present the results in the case that the system (5.2) consists of two uncoupled equations. That is, $f = f(u)$ and $g = g(v)$. It is known that the nonlinear diffusion equation $u_t = (f(u)u_x)_x$ admits 3 Lie symmetries if $f(u)$ is an arbitrary function, 4 symmetries if $f(u) = u^n$ or $f(u) = e^u$, 5 symmetries if $f(u) = u^{-\frac{4}{3}}$ and 6 symmetries and an infinite-dimensional symmetry if $f(u) = 1$ [71]. All cases for the uncoupled system can be extracted from the above table, with the exception of three cases, which we list below. For example, in Table 5.1, entry 4, $F(v+\epsilon u) = 1$, $G(v+\epsilon u) = e^{\lambda(v+\epsilon u)}$ and choosing $\epsilon = -\frac{1}{\lambda}$ provides the four Lie symmetries for the uncoupled system $u_t = (e^u u_x)_x$ and $v_t = (e^v v_x)_x$. The remaining cases are

(i) $f(u)$ arbitrary and $g(v) = 1$:
$X_4 = v\partial_v,\ X_\alpha = \alpha(t,x)\partial_v$, where $\alpha_t = \alpha_{xx}$;
(ii) $f(u) = u^{-\frac{4}{3}}$ and $g(v) = v^{-\frac{4}{3}}$:
$X_4 = 4t\partial_t + 3u\partial_u + 3v\partial_v,\ X_5 = x^2\partial_x - 3xu\partial_u - 3xv\partial_v$;
(iii) $f(u) = 1$ and $g(v) = a$:
$X_4 = 2t\partial_x - xu\partial_u - \frac{xv}{a}\partial_v,\ X_5 = u\partial_u,\ X_6 = v\partial_v,$
$X_7 = 4t^2\partial_t + 4xt\partial_x - (x^2+2t)u\partial_u - (x^2+2at)\frac{v}{a}\partial_v,$
$X_\alpha = \alpha(t,x)\partial_u,\ X_\beta = \beta(t,x)\partial_v,$
where $\alpha_t = \alpha_{xx}$ and $\beta_t = a\beta_{xx}$.

The above results for the uncoupled equations were derived in 1986 by Knyazeva and Popov which can be found on page 171 of the book [36].

Case 2: $f = g$. Here, we cannot make use of Theorem 4.4(iv), since $f = g$. Instead, we use the coefficients of $u_x u_{xx}$ and $v_x u_{xx}$ in the first identity of (5.5) to give $\xi_u = \xi_v = 0$. Hence,

$$\xi = \xi(x,t),$$

as in the previous case. These restricted forms of τ and ξ simplify the identities (5.5). In fact, identities (5.5) become two multivariate polynomials in the four variables u_x, v_x, u_{xx} and v_{xx}. The coefficients of these variables will provide us the determining system which is solved to give the forms of the arbitrary elements $f(u,v)$ and $g(u,v)$ and also the coefficient functions $\tau(t)$, $\xi(x,t)$, $\eta(x,t,u,v)$ and $\mu(x,t,u,v)$.

As in the previous case, we list the two systems of determining equations. The coefficients of u_x^2, $u_x v_x$, u_{xx}, u_x, v_x^2, v_x and the term independent of derivatives in the first equation in (5.5) give, respectively,

$$\begin{aligned}
&\eta_{uu} g + (\tau_t + \eta_u - 2\xi_x) g_u + \mu_u g_v + \mu g_{uv} + \eta g_{uu} = 0,\\
&2\eta_{uv} g + \eta_v g_u + (\tau_t + \mu_v - 2\xi_x) g_v + \eta g_{uv} + \mu g_{vv} = 0,\\
&(\tau_t - 2\xi_x) g + \eta g_u + \mu g_v = 0,\\
&2\eta_x g_u + \mu_x g_v + (2\eta_{ux} - \xi_{xx}) g + \xi_t = 0,\\
&\eta_{vv} g = 0, \quad 2\eta_{vx} g + \eta_x g_v = 0, \quad \eta_t - \eta_{xx} g = 0.
\end{aligned} \tag{5.10}$$

Finally, the coefficients of v_x^2, $u_x v_x$, v_{xx}, v_x, u_x^2, u_x and the term independent of derivatives in the second equation in (5.5) give, respectively,

$$\begin{aligned}
&\mu_{vv} g + \eta_v g_u + (\tau_t + \mu_v - 2\xi_x) g_v + \eta g_{uv} + \mu g_{vv} = 0,\\
&2\mu_{uv} g + \mu_u g_v + (\tau_t + \eta_u - 2\xi_x) g_u + \mu g_{uv} + \eta g_{uu} = 0,\\
&(\tau_t - 2\xi_x) g + \eta g_u + \mu g_v = 0,\\
&\eta_x g_u + 2\mu_x g_v + (2\mu_{vx} - \xi_{xx}) g + \xi_t = 0,\\
&\mu_{uu} g = 0, \quad 2\mu_{ux} g + \mu_x g_u = 0, \quad \mu_t - \mu_{xx} g = 0.
\end{aligned} \tag{5.11}$$

We note that the third equations in the determining system (5.10) and in (5.11) are identical. The solution of the above systems that consist of thirteen equations provide us with the desired results. We differentiate the third equation in (5.10) and the resulting equation is subtracted form the first and the second equation to give, respectively, $\eta_{uu} = \eta_{uv} = 0$. Using these conclusions and the fifth equation, we obtain

$$\eta(x,t,u,v) = A_1(x,t)u + A_2(x,t)v + A_3(x,t).$$

Similarly, from the first, second, third and fifth equations in the system (5.11) we obtain

$$\mu(x,t,u,v) = A_4(x,t)u + A_5(x,t)v + A_6(x,t).$$

With the forms of η and μ the third equation in (5.10) implies that $g(u,v)$ satisfies a first order quasi-linear partial differential equation of the form

$$(p_1 u + p_2 v + p_3)\frac{\partial g}{\partial u} + (q_1 u + q_2 v + q_3)\frac{\partial g}{\partial v} + rg = 0, \tag{5.12}$$

unless $\tau_t - 2\xi_x = \eta = \mu = 0$, which implies that $g(u,v)$ is an arbitrary function and hence, the system (5.2) admits $A^{\ker}$ which consists with the three Lie symmetries given by (5.9). If $g(u,v)$ is any solution of PDE (5.12), then the system (5.2) admits the fourth Lie symmetry

$$X_4 = rt\partial_t + (p_1 u + p_2 v + p_3)\partial_u + (q_1 u + q_2 v + q_3)\partial_v. \tag{5.13}$$

Now we search for functions $g(u,v)$ which are such that the system (5.2) admits additional Lie symmetries. In order to achieve this, we need to find all possible solutions of PDE (5.12). However solving this quasi linear PDE is not an easy task. Here we make use of the equivalence transformations of the system (5.2) in the case $f = g$, which are given by equation (5.4). In particular, if we consider equation (5.12) with the variables u, v, g being primed, then the application of equivalence transformation (5.4), with $t' = t,\ x' = x$, transforms equation (5.12) into

$$\begin{aligned}
&[(\gamma_1\delta_2 p_1 + \delta_1\delta_2 p_2 - \gamma_1\gamma_2 q_1 - \gamma_2\delta_1 q_2)\, u + \\
&\left(\gamma_2\delta_2 p_1 + \delta_2^2 p_2 - \gamma_2^2 q_1 - \gamma_2\delta_2 q_2\right) v + \\
&(\gamma_3\delta_2 p_1 + \delta_2\delta_3 p_2 + \delta_2 p_3 - \gamma_2\gamma_3 q_1 - \gamma_2\delta_3 q_2 - \gamma_2 q_3)]\, \frac{\partial g}{\partial u} + \\
&\left[\left(\gamma_1^2 q_1 + \gamma_1\delta_1 q_2 - \gamma_1\delta_1 p_1 - \delta_1^2 p_2\right) u + \right. \\
&(\gamma_1\gamma_2 q_1 + \gamma_1\delta_2 q_2 - \gamma_2\delta_1 p_1 - \delta_1\delta_2 p_2)\, v + \\
&(\gamma_1\gamma_3 q_1 + \gamma_1\delta_3 q_2 + \gamma_1 q_3 - \gamma_3\delta_1 p_1 - \delta_1\delta_3 p_2 - \delta_1 p_3)]\, \frac{\partial g}{\partial v} + \\
&r(\gamma_1\delta_2 - \gamma_2\delta_1)g = 0,
\end{aligned}$$

where $\gamma_1\delta_2 - \gamma_2\delta_1 \neq 0$. Our goal is to simplify, as much as possible, the above equation. We try to fix the constants that appear in the equivalence transformations. This leads to various cases depending on whether certain constants are zero or nonzero. In fact, we find that, instead of solving (5.12), we can equivalently solve, separately, the following PDEs

$$\frac{\partial g}{\partial u}+q_3\frac{\partial g}{\partial v}+rg=0, \tag{5.14}$$

$$\frac{\partial g}{\partial u}+(q_1u+q_2v)\frac{\partial g}{\partial v}+rg=0, \tag{5.15}$$

$$u\frac{\partial g}{\partial u}+(q_1u+q_3)\frac{\partial g}{\partial v}+rg=0, \tag{5.16}$$

$$u\frac{\partial g}{\partial u}+(q_1u+q_2v)\frac{\partial g}{\partial v}+rg=0, \tag{5.17}$$

$$v\frac{\partial g}{\partial u}+(q_1u+q_2v)\frac{\partial g}{\partial v}+rg=0. \tag{5.18}$$

The solutions of the above PDEs provide us all the possible forms of $g(u,v)$ which are substituted into the systems (5.10) and (5.11). Then, the forms of τ, ξ, η and μ are determined and, consequently, the desired Lie symmetries are classified. The results are tabulated in Table 5.2. The constants that appear in the form of $g(u,v)$ depend on the nonzero constants that appear in PDE (5.12) and, consequently, the constants in the symmetry X_4 will be changed. For this reason we also list X_4 in Table 5.2.

The solution of PDE (5.14) is $g(u,v)=e^{mu}\phi(v+\varepsilon u)$. We substitute this form of g into the systems (5.10) and (5.11). The solution of these two systems gives the forms of τ, ξ, η and μ and, consequently, the corresponding Lie symmetries are derived. Without presenting any detailed analysis, we state that the results are tabulated in the entries 8-12 in Table 5.2.

Equation (5.15), depending on the relation between the constants, has a solution of the form $g(u,v)=e^{mu}\phi(v+\varepsilon u^2)$ or of the form $g(u,v)=e^{mu}\phi(v+\varepsilon u+\varepsilon/\delta)$. The first type of solution produces the result that is tabulated in the entry 6 in Table 5.2, while the second solution of g does not give any new cases. Solving PDE (5.16), we find $g(u,v)=u^m\phi(v+\varepsilon u+\delta\ln u)$ and substituting this form of $g(u,v)$ into the systems (5.10) and (5.11) we obtain results which are special cases of entries 7 and 8. Equation (5.17) has either a solution of the form $g(u,v)=u^m\phi[u^n(v+\varepsilon u)]$ or of the form $g(u,v)=u^m\phi(\frac{v}{u}+\delta\ln u)$. The first form of g leads to the results tabulated in the entries 1-5 of Table 5.2, while the second form provides the result that is tabulated in the entry 7. Finally, the solution of (5.18) does not lead to any new cases.

For completeness, we state that in the case where $g(u,v)=1$, that is, system (5.2) becomes two uncoupled linear heat equations, the additional admitted Lie symmetries are

$$X_4=2t\partial_x-x(u\partial_u+v\partial_v),\ X_5=2t^2\partial_t+2tx\partial_x-\tfrac{1}{2}(2t+x^2)(u\partial_u+v\partial_v),\ X_6=u\partial_u,$$

$$X_7=v\partial_v,\ X_8=v\partial_u,\ X_9=u\partial_v,\ X_\alpha=\alpha\partial_u,\ X_\beta=\beta\partial_v,$$

Table 5.2: Group classification of the system (5.2), where $f = g$.

n	$g(u,v)$	Additional Lie symmetries
1.	$(v+\varepsilon u)^n e^{\frac{ru}{v+\varepsilon u}}$	$X_4 = nt\partial_t - u\partial_u - v\partial_v,\ X_5 = rt\partial_t - (v+\varepsilon u)\partial_u + \varepsilon(v+\varepsilon u)\partial_v$
2.	$[(v+\varepsilon u)^2 + \delta^2 u^2]^n e^{p\tan^{-1}\frac{v+\varepsilon u}{\delta u}}$	$X_4 = 2nt\partial_t - u\partial_u - v\partial_v,$ $X_5 = (p\delta + 2\varepsilon n)t\partial_t + v\partial_u - [(\delta^2+\varepsilon^2)u + 2\varepsilon v)\partial_v$
3.	$(v+\varepsilon u-\delta u)^m(v+\varepsilon u+\delta u)^n$	$X_4 = (m+n)t\partial_t - u\partial_u - v\partial_v,$ $X_5 = [(m+n)\varepsilon + (m-n)\delta]t\partial_t + v\partial_u + ([(\delta^2-\varepsilon^2)u - 2\varepsilon v)\partial_v$
4.	$[(v+\varepsilon u)^2 + \delta u]^n$	$X_4 = 2nt\partial_t - 2u\partial_u - (v-\varepsilon u)\partial_v,$ $X_5 = 2\varepsilon nt\partial_t + 2v\partial_u - (\varepsilon^2 u + 3\varepsilon v + \delta)\partial_v$
5.	$(v+\varepsilon u+\delta u^2)^n$	$X_4 = 2nt\partial_t - u\partial_u - (\varepsilon u + 2v)\partial_v,$ $X_5 = 2nt\partial_t + (1-u)\partial_u - [(\varepsilon+2\delta)u + 2v + \varepsilon]\partial_v$
6.	$e^{v+\varepsilon u+\delta u^2}$	$X_4 = \varepsilon t\partial_t - \partial_u + 2\delta u\partial_v,\ X_5 = t\partial_t - \partial_v$
7.	$(\gamma u+\delta v)^m(v+\varepsilon u)^n$	$X_4 = (m+n)t\partial_t - u\partial_u - v\partial_v,$ $X_5 = (\gamma n + \delta\varepsilon m)t\partial_t + \delta v\partial_u - (\gamma\varepsilon u + \gamma v + \delta\varepsilon v)\partial_v$
8.	$e^{\gamma u+\delta v}(v+\varepsilon u)^n$	$X_4 = (\delta\varepsilon - \gamma)t\partial_t + \partial_u - \varepsilon\partial_v,$ $X_5 = n(\gamma - \delta\varepsilon)t\partial_t + \delta(v+\varepsilon u)\partial_u - \gamma(v+\varepsilon u)\partial_v$
9.	$\phi(v+\varepsilon u)$	$X_4 = \partial_u - \varepsilon\partial_v,\ X_5 = u\partial_u - \varepsilon u\partial_v,\ X_6 = v\partial_u - \varepsilon v\partial_v$
10.	$e^{v+\varepsilon u}$	$X_4 = \partial_u - \varepsilon\partial_v,\ X_5 = u\partial_u - \varepsilon u\partial_v,\ X_6 = v\partial_u - \varepsilon v\partial_v,\ X_7 = t\partial_t - \partial_v$
11.	$(v+\varepsilon u)^n$	$X_4 = \partial_u - \varepsilon\partial_v,\ X_5 = u\partial_u - \varepsilon u\partial_v,\ X_6 = v\partial_u - \varepsilon v\partial_v,$ $X_7 = nt\partial_t - (v+\varepsilon u)\partial_v$
12.	$(v+\varepsilon u)^{-2}$	$X_4 = \partial_u - \varepsilon\partial_v,\ X_5 = u\partial_u - \varepsilon u\partial_v,\ X_6 = v\partial_u - \varepsilon v\partial_v,$ $X_7 = 2t\partial_t + (v+\varepsilon u)\partial_v,\ X_8 = x(v+\varepsilon u)(\partial_u - \varepsilon\partial_v)$

Here $\varepsilon = 0, \pm1$ and $r \neq 0,\ \delta \neq 0,\ p,\ \gamma$ are arbitrary constants; ϕ is an arbitrary function.

where $\alpha(x,t)$ and $\beta(x,t)$ are solutions of the linear heat equation, $u_t = u_{xx}$.

We point out that the member of the equivalence transformation (5.4),

$$t' = t, \quad x' = x, \quad u' = u, \quad v' = u \pm v$$

maps $v + \varepsilon u$ with $\varepsilon = \pm1$ in Table 5.2, to the corresponding cases with $\varepsilon = 0$.

Case 12, of Table 5.2 is a member of the system (5.2) which can be linearized by a nonlocal mapping. For this mapping and also for potential symmetries for this case see [39].

5.4 Similarity reductions

Lie symmetries can be employed to derive similarity reductions [13, 68, 72]. These are transformations that reduce the number of independent variables of a system of PDEs by one. In the case of an ordinary differential equation, the order of the equation can be reduced by one. Here we have a system of PDEs in two independent variables and hence, the reduced system consists of two ordinary differential equations. In order to construct a similarity reduction that corresponds to a Lie symmetry generator $\Gamma = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial v}$, we need to find a solution of the characteristic system

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta} = \frac{dv}{\mu}.$$

The complete list of similarity reductions that correspond to Lie symmetries can be achieved by using subalgebras from the so-called optimal system [68]. Alternatively, all possible solutions of the above characteristic system can be found for the linear combination of the basis of Lie symmetries.

In the case $f(u, v) = u^n v^m$ and $g(u, v) = au^n v^m$, we have the system

$$u_t = [u^n v^m u_x]_x, \; v_t = a[u^n v^m v_x]_x \tag{5.19}$$

which admits the Lie symmetries

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = 2t\partial_t + x\partial_x, \quad X_4 = mx\partial_x + 2v\partial_v, \quad X_5 = nx\partial_x + 2u\partial_u.$$

All similarity reductions for this case are tabulated in Table 5.3.

The second example is the member of the system (5.2), where $f(u, v) = g(u, v) = (v + \delta u^2)^n$,

$$u_t = [(v + \delta u^2)^n u_x]_x, \; v_t = [(v + \delta u^2)^n v_x]_x. \tag{5.20}$$

This system admits the five Lie symmetries

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = 2t\partial_t + x\partial_x, \quad X_4 = 2nt\partial_t - u\partial_u - 2v\partial_v,$$

$$X_5 = 2nt\partial_t + (1 - u)\partial_u - 2(\delta u + v)\partial_v.$$

The corresponding results are provided in Table 5.4. The constants that appear in the reductions are arbitrary with the exception of those that appear in the denominator and which are nonzero.

Table 5.3: Similarity reductions for the system $u_t = [u^n v^m u_x]_x,\ v_t = a[u^n v^m v_x]_x$.

	ξ	Similarity solution	Reduced equations
1	x	$u = t^k\phi(\xi)$ $v = t^{-\frac{1+nk}{m}}\psi(\xi)$	$\phi'' + n\phi^{-1}(\phi')^2 + m\psi^{-1}\phi'\psi' - k\phi^{1-n}\psi^{-m} = 0$ $ma\psi'' + m^2a\psi^{-1}(\psi')^2 + mna\phi^{-1}\phi'\psi' + (1+nk)\phi^{-n}\psi^{1-m} = 0$
2	t	$u = e^{-\frac{mk}{n}x}\phi(\xi)$ $v = e^{kx}\psi(\xi)$	$n^2\phi' - m^2k^2\phi^{n+1}\psi^m = 0$ $\psi' - ak^2\phi^n\psi^{m+1} = 0$
3	t	$u = x^k\phi(\xi)$ $v = x^{\frac{2-nk}{m}}\psi(\xi)$	$\phi' - (k+k^2)\phi^{n+1}\psi^m = 0$ $m^2\psi' - a(kn-2)(kn-m-2)\phi^n\psi^{m+1} = 0$
4	$x - ct$	$u = e^{-\frac{mk}{n}t}\phi(\xi)$ $v = e^{kt}\psi(\xi)$	$n\phi'' + n^2\phi^{-1}(\phi')^2 + mn\psi^{-1}\phi'\psi' + nc\phi^{-n}\psi^{-m}\phi' + mk\phi^{1-n}\psi^{-m} = 0$ $a\psi'' + am\psi^{-1}(\psi')^2 + an\phi^{-1}\phi'\psi' + c\phi^{-n}\psi^{-m}\psi' - k\phi^{-n}\psi^{1-m} = 0$
5	$\frac{x}{\sqrt{t}}$	$u = t^{-\frac{mk}{n}}\phi(\xi)$ $v = t^k\psi(\xi)$	$2n\phi'' + 2n^2\phi^{-1}(\phi')^2 + 2nm\psi^{-1}\phi'\psi' + n\xi\phi^{-n}\psi^{-m}\phi' + 2mk\phi^{1-n}\psi^{-m} = 0$ $2a\psi'' + 2am\psi^{-1}(\psi')^2 + 2an\phi^{-1}\phi'\psi' + \xi\phi^{-n}\psi^{-m}\psi' - 2k\phi^{-n}\psi^{1-m} = 0$
6	$\frac{x}{t^k}$	$u = t^{k_1}\phi(\xi)$ $v = t^{\frac{2k-1-nk_1}{m}}\psi(\xi)$	$\phi'' + n\phi^{-1}(\phi')^2 + m\psi^{-1}\phi'\psi' + k\xi\phi^{-n}\psi^{-m}\phi' - k_1\phi^{1-n}\psi^{-m} = 0$ $am\psi'' + am^2\psi^{-1}(\psi')^2 + anm\phi^{-1}\phi'\psi' + km\xi\phi^{-n}\psi^{-m}\psi' + (1+nk_1-2k)\phi^{-n}\psi^{1-m} = 0$
7	xe^{-kt}	$u = e^{2k_1t}\phi(\xi)$ $v = e^{2\frac{k-nk_1}{m}t}\psi(\xi)$	$\phi'' + n\phi^{-1}(\phi')^2 + m\psi^{-1}\phi'\psi' + k\xi\phi^{-n}\psi^{-m}\phi' - 2k_1\phi^{1-n}\psi^{-m} = 0$ $am\psi'' + am^2\psi^{-1}(\psi')^2 + amn\phi^{-1}\phi'\psi' + 2(k_1n-k)\phi^{-n}\psi^{1-m} = 0$
8	$e^{-t}x^k$	$u = e^{\frac{2t}{nk}}\phi(\xi)$ $v = \psi(\xi)$	$nk^3\phi'' + n^2k^3\phi^{-1}(\phi')^2 + nmk^3\psi^{-1}\phi'\psi' + nk^2(k-1)\xi^{-1}\phi' + nk\xi^{\frac{2}{k}-1}\phi^{-n}\psi^{-m}\phi' - 2\xi^{\frac{2}{k}-2}\phi^{1-n}\psi^{-m} = 0$ $ak^2\psi'' + amk^2\psi^{-1}(\psi')^2 + ank^2\phi^{-1}\phi'\psi' + ak(k-1)\xi^{-1}\psi' + \xi^{\frac{2}{k}-1}\phi^{-n}\psi^{-m}\psi' = 0$
9	e^xt^k	$u = e^{k_1x}\phi(\xi)$ $v = e^{\frac{1-nkk_1}{mk}x}\psi(\xi)$	$\phi'' + n\phi^{-1}(\phi')^2 + m\psi^{-1}\phi'\psi' + (nk_1 + 2k_1 + 1 + \frac{1}{k})\xi^{-1}\phi' - k\xi^{-\frac{1}{k}-1}\phi^{-n}\psi^{-m}\phi' + k_1m\xi^{-1}\phi\psi^{-1}\psi' + (k_1^2 + \frac{k_1}{k})\xi^{-2}\phi = 0$ $\psi'' + m\psi^{-1}(\psi')^2 + n\phi^{-1}\phi'\psi' + \frac{1}{km}(2 + 2m + km - 2kk_1n - kk_1mn)\xi^{-1}\psi' - \frac{k}{a}\xi^{-\frac{1}{k}-1}\phi^{-n}\psi^{-m}\psi' + \frac{n}{km}(1-kk_1n)\xi^{-1}\psi\phi^{-1}\phi'$ $+\frac{1}{k^2m^2}(1-kk_1n)(1-kk_1n+m)\xi^{-2}\psi = 0$

Table 5.4: Similarity reductions for the system $u_t = [(v+\delta u^2)^n u_x]_x,\ v_t = [(v+\delta u^2)^n v_x]_x$.

	ξ	Similarity solution	Reduced equations
1	x	$u = kt + \phi(\xi)$	$\phi'' + n\psi^{-1}\psi'\phi' - k\psi^{-n} = 0$
		$v = -\delta(kt+\phi(\xi))^2 + \psi(\xi)$	$\psi'' + n\psi^{-1}(\psi')^2 - 2\delta(\phi')^2 = 0$
2	x	$u = \lambda + t^{-\frac{1}{2n}}\phi(\xi)$	$2n\phi'' + 2n^2(\psi+\delta\phi^2)^{-1}(\psi+\delta\phi^2)'\phi' + \phi(\psi+\delta\phi^2)^{-n} = 0$
		$v = -2\delta\lambda(\lambda + t^{-\frac{1}{2n}}\phi(\xi)) + \delta\lambda^2 + t^{-\frac{1}{n}}\psi(\xi)$	$n\psi'' + n^2(\psi+\delta\phi^2)^{-1}(\psi+\delta\phi^2)'\psi' + \psi(\psi+\delta\phi^2)^{-n} = 0$
3	t	$u = kx + \phi(\xi)$	$\phi' = 0$
		$v = -\delta(kx+\phi(\xi))^2 + \psi(\xi)$	$\psi' + 2\delta k^2\psi^n = 0$
4	t	$u = k + x^{\frac{1}{n}}\phi(\xi)$	$n^2\phi' - (n+1)(\psi+\delta\phi^2)^n\phi = 0$
		$v = -2\delta k(k + x^{\frac{1}{n}}\phi(\xi)) + \delta k^2 + x^{\frac{2}{n}}\psi(\xi)$	$n^2\psi' - 2(n+2)(\psi+\delta\phi^2)^n\psi = 0$
5	$x-ct$	$u = kx + \phi(\xi)$	$\phi'' + n\psi^{-1}\psi'(k+\phi') + c\psi^{-n}\phi' = 0$
		$v = -\delta(kx+\phi(\xi))^2 + \psi(\xi)$	$\psi'' + n\psi^{-1}(\psi')^2 + c\psi^{-n}\psi' - 2\delta(k+\phi')^2 = 0$
6	$\frac{x}{\sqrt{t}}$	$u = \ln\|x\| + \phi(\xi)$	$\phi'' + n\psi^{-1}\psi'\phi' + n\xi^{-1}\psi^{-1}\psi' + \frac{1}{2}\xi\psi^{-n}\phi' - \xi^{-2} = 0$
		$v = -\delta(\ln\|x\| + \phi(\xi))^2 + \psi(\xi)$	$\psi'' + n\psi^{-1}(\psi')^2 - 2\delta(\phi')^2 + \frac{1}{2}\xi\psi^{-n}\psi' - 4\delta\xi^{-1}\phi' - 2\delta\xi^{-2} = 0$
7	te^{kx}	$u = \lambda + t^{-\frac{1}{2n}}\phi(\xi)$	$nk^2\phi'' + n^2k^2(\psi+\delta\phi^2)^{-1}(\psi+\delta\phi^2)'\phi' + nk^2\xi^{-1}\phi' - n\xi^{-1}(\psi+\delta\phi^2)^{-n}\phi' + \frac{1}{2}\xi^{-2}(\psi+\delta\phi^2)^{-n}\phi = 0$
		$v = -2\delta\lambda(\lambda + t^{-\frac{1}{2n}}\phi(\xi)) + \delta\lambda^2 + t^{-\frac{1}{n}}\psi(\xi)$	$nk^2\psi'' + n^2k^2(\psi+\delta\phi^2)^{-1}(\psi+\delta\phi^2)'\psi' + nk^2\xi^{-1}\psi' - n\xi^{-1}(\psi+\delta\phi^2)^{-n}\psi' + \xi^{-2}(\psi+\delta\phi^2)^{-n}\psi = 0$
8	xe^{kt}	$u = \lambda + x^{\frac{1}{n}}\phi(\xi)$	$n^2\phi'' + n^3(\psi+\delta\phi^2)^{-1}(\psi+\delta\phi^2)'\phi' + 2\delta n(2n+1)\xi^{-1}(\psi+\delta\phi^2)^{-1}\phi^2\phi' + 2n(n+1)\xi^{-1}(\psi+\delta\phi^2)^{-1}\psi\phi'$
			$-n^2k\xi^{-1}(\psi+\delta\phi^2)^{-n}\phi' + n^2\xi^{-1}(\psi+\delta\phi^2)^{-1}\phi\psi' + (n+1)\xi^{-2}\phi = 0$
		$v = -2\delta\lambda(\lambda + x^{\frac{1}{n}}\phi(\xi)) + \delta\lambda^2 + x^{\frac{2}{n}}\psi(\xi)$	$n^2\psi'' + n^3(\psi+\delta\phi^2)^{-1}(\psi+\delta\phi^2)'\psi' + 2\delta n(n+2)\xi^{-1}(\psi+\delta\phi^2)^{-1}\phi^2\psi' + 4n(n+1)\xi^{-1}(\psi+\delta\phi^2)^{-1}\psi\psi'$
			$-n^2k\xi^{-1}(\psi+\delta\phi^2)^{-n}\psi' + 4\delta n^2\xi^{-1}(\psi+\delta\phi^2)^{-1}\psi\phi\phi' + 2(n+2)\xi^{-2}\psi = 0$
9	$xt^{\frac{1}{2(kn-1)}}$	$u = \lambda + x^k\phi(\xi)$	$\phi'' + n(\psi+\delta\phi^2)^{-1}(\psi+\delta\phi^2)'\phi' + 2k\xi^{-1}[n+1+n\delta(\psi+\delta\phi^2)^{-1}\phi^2]\phi' - \frac{1}{2(kn-1)}\xi^{1-2kn}(\psi+\delta\phi^2)^{-n}\phi'$
			$+kn\xi^{-1}(\psi+\delta\phi^2)^{-1}\phi\psi' + k(2kn+k-1)\xi^{-2}\phi = 0$
		$v = -2\delta\lambda(\lambda + x^k\phi(\xi)) + \delta\lambda^2 + x^{2k}\psi(\xi)$	$\psi'' + n(\psi+\delta\phi^2)^{-1}(\psi+\delta\phi^2)'\psi' + 2k\xi^{-1}[2(n+1) - n\delta(\psi+\delta\phi^2)^{-1}\phi^2]\psi' - \frac{1}{2(kn-1)}\xi^{1-2kn}(\psi+\delta\phi^2)^{-n}\psi'$
			$+4kn\delta\xi^{-1}(\psi+\delta\phi^2)^{-1}\phi\psi\phi' + 2k(2kn+2k-1)\xi^{-2}\psi = 0$

The solutions of the reduced ODEs listed in the Tables 5.3 and 5.4 provide us with special solutions of the original PDEs. Only in some special cases the systems of ODEs listed in Tables 5.3 and 5.4 can be solved analytically. Generally, such reduced systems along with appropriate initial/boundary comditions can be solved numerically. Below we give examples, where we obtain exact solutions for the reduced ODEs.

Example 5.1. We use case 4 of Table 5.3 with $k = 0$. In other words, we look for traveling wave solutions for the system (5.19). The reduced system of the ODEs has the form

$$[\phi^n \psi^m \phi']' + c\phi' = 0, \quad a[\phi^n \psi^m \psi']' + c\psi' = 0.$$

The solution of this system is given explicitly by

$$\phi(\xi) = \nu_3(\psi(\xi) - \nu_2)^a + \nu_1, \qquad \int \frac{\psi^m[\nu_3(\psi - \nu_2)^a + \nu_1]^n}{\nu_2 - c\psi} \mathrm{d}\psi = \xi + \nu_4,$$

where the ν_i are constants of integration. For the special case where $\nu_1 = \nu_2 = 0$, we have

$$\psi(\xi) = (\nu_3'\xi + \nu_4')^{\frac{1}{m+an}},$$

if $a \neq -\frac{m}{n}$ and

$$\psi(\xi) = \nu_3' e^{\nu_4'\xi},$$

if $a = -\frac{m}{n}$.

Example 5.2. We consider case 5 of Table 5.3 which is a special case of case 6. Choosing $m = -n$ and $k = -\frac{1}{2}$ the reduced system of ODEs, after one integration can be written in the form

$$2\phi^n \psi^m \phi' + \xi\phi = \nu_1, \quad 2a\phi^n \psi^m \psi' + \xi\psi = \nu_2.$$

For vanishing constants of integration ν_1 and ν_2, we find

$$\phi(\xi) = \nu_3 \psi^a,$$

where $\psi(\xi) = (\nu_4 - \frac{n(a-1)}{4a\nu_3^n}\xi^2)^{\frac{1}{n(a-1)}}$, if $a \neq 1$ and $\psi(\xi) = \nu_4 e^{-\frac{1}{4\nu_3^n}\xi^2}$, if $a = 1$.

Example 5.3. We look for traveling wave solutions for the system (5.20). We consider the case 5 of Table 5.4 with $k = 0$. Integrating the first reduced ODE and taking the constant of integration equal to zero, we find $\phi'\psi^n = -c\phi$. Substituting into the second reduced ODE and integrating twice, we find

$$\psi(\xi) = \delta c\phi^2 + \nu_1\phi + \nu_2.$$

Now, the first integrating equation gives

$$\int \frac{(\delta c\phi^2 + \nu_1\phi + \nu_2)^n}{\phi}\mathrm{d}\phi = -c\xi + \nu_3.$$

Example 5.4. Case 4 of Table 5.4, gives the exact solution

$$\psi(\xi) = \nu_1\phi(\xi)^{\frac{2(n+2)}{n+1}}, \quad \int \frac{\mathrm{d}\phi}{\phi(\nu_1\phi^{\frac{2(n+2)}{n+1}} + \delta\phi^2)^n} = \frac{n+1}{n^2}\xi + \nu_2.$$

5.5 Conclusion

The complete Lie group classification of class (5.2) has been achieved. The difficulty of the problem lies in the fact that the two arbitrary elements depend on two variables. The solutions of such problems are rare in the literature. The work in the present chapter aims to be an inspiration for the complete group classification of the general class (5.1). Furthermore, the problem of classification of potential symmetries for the class (5.1) needs consideration. Finally, we need to point out the importance of the derivation of the equivalence transformations admitted by the class under consideration. Such transformations simplify the problem of group classification. In the last few years, an algebraic method was introduced to solve group classification problems [7,8]. It is based on the subgroup analysis of the corresponding equivalence group. This method can be used to solve the group classification problems that are considered in this thesis. For recent applications of the algebraic method see [69, 74, 99].

Chapter 6

Lie symmetry analysis of Burgers-type systems

6.1 Introduction

The nonlinear diffusion-convection equations of the form

$$u_t = [d(u)u_x]_x + k(u)u_x, \tag{6.1}$$

where $d(u)$ and $k(u)$ are arbitrary smooth functions, have considerable applications in mathematical physics, chemistry and biology [19, 20, 62, 63, 92]. A number of authors derived Lie symmetries for the class (6.1). However, its complete and strong Lie group classification was presented in [76]. In the case $d(u)$ =constant= λ, we have the generalized Burgers equation

$$u_t = \lambda u_{xx} + k(u)u_x \tag{6.2}$$

whose most famous member is the Burgers equation $u_t = u_{xx} + uu_x$ which has, among others, applications in nonlinear acoustics [20]. Group classification of (6.2) was carried out in [45] (see also in [76]).

A generalization of the class (6.2) is the following system, written in the vector form

$$\mathbf{U}_t = \mathbf{\Lambda}\mathbf{U}_{xx} + \mathbf{K}(\mathbf{U})\mathbf{U}_x,$$

where $\mathbf{U}$ is the vector $[u_1(x,t), u_2(x,t), \ldots, u_n(x,t)]^T$, $\mathbf{\Lambda}$ is an $n \times n$ matrix with constant elements and $\mathbf{K}(\mathbf{U})$ is an $n \times n$ matrix with its elements being functions of the dependent variables $u_1, u_2, \ldots, u_n$. In the case where $n = 2$ and $\mathbf{\Lambda}$ being a diagonal matrix the above vector equation is equivalent to the Burgers-type system

$$u_t = \lambda_1 u_{xx} + f(u,v)u_x + g(u,v)v_x, \quad v_t = \lambda_2 v_{xx} + h(u,v)u_x + k(u,v)v_x, \tag{6.3}$$

where $f(u,v),\ g(u,v),\ h(u,v),\ k(u,v)$ are arbitrary smooth functions in their arguments and λ_1, λ_2 are arbitrary constants.

The Group classification for the special case of (6.3), where $f(u,v) = u$ and $k(u,v) = v$ was considered in [16]. In the present chapter we consider the following two members of the general system (6.3).

$$u_t = \lambda_1 u_{xx} + f(u,v)u_x + \epsilon_1 v v_x, \quad v_t = \lambda_2 v_{xx} + k(u,v)v_x + \epsilon_2 u u_x, \quad \epsilon_1\epsilon_2 \neq 0, \tag{6.4}$$

and

$$u_t = \lambda_1 u_{xx} + f(u,v)u_x, \quad v_t = \lambda_2 v_{xx} + k(u,v)v_x. \tag{6.5}$$

We present the group classification for these two Burgers-type systems.

The idea of group classification was introduced by Ovsiannikov when he considered the nonlinear diffusion equation ($k(u) = 0$ in (6.1)) [71]. Solving group classification problems is important from both the mathematical and physical point of view. Following the physical laws, for example from the Galilean or special relativity principles, models are often constrained with a-priori requirements to symmetry properties.

The Lie algorithm in the group classification of a class of differential equation leads to a complicated over-determined system of PDEs with respect to the coefficient functions of the infinitesimal operator and the arbitrary functions (elements) that appear in the class. The appearance of the arbitrary elements in the over-determined system makes the solution of the problem much more complicated than finding the Lie symmetries of a single system of differential equations. In the present problem, the arbitrary elements f and k depend on two variables which make the classification even more difficult than the usual ones. A similar problem where there exist two arbitrary elements depending in two variables was considered recently in [48]. Another good example, is considered in [89], where certain results of group classification of complex three-dimensional diffusion-type equations are presented. Also, in [65–67], the group classification of reaction-diffusion systems can be found. Examples of group classification of single equations that contain arbitrary elements depend on two variables can be found in [8, 56] that appeared recently in the literature.

In the next section we present the equivalence transformations for the two systems (6.4) and (6.5) which are used to simplify the calculations of Lie symmetries. In section 3, we present the classification of Lie symmetries for the two systems. The results are summarized in three tables. In section 4, we give examples of nonclassical reductions. Finally, we give an example of a Hopf-Cole type transformation that linearizes a specific Burger-type system. The results of the last two sections could be the starting point for further investigations.

The results of the present chapter, appear in [50].

6.2 Equivalence transformations

In this section we derive the equivalence transformations of the classes (6.4) and (6.5) which play an important role in the theory of Lie group classification. These are nondegenerate point transformations, that preserve the differential structure of the class of differential equations under study and change only its arbitrary elements (functions $f(u,v)$ and $k(u,v)$). We can also say that equivalence transformations connect two members of the same class of PDEs. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group. There exist two methods for the calculation of equivalence transformations, the direct method which was first used by Lie [55] and the Lie infinitesimal method which was introduced by Ovsiannikov [72]. Although, the direct method involves considerable computational difficulties, it has the advantage of finding the most general equivalence group and also unfolds all form-preserving [46] (also known as admissible [75]) transformations admitted by this class of equations. For recent applications of the direct method one can refer, for example, to the recent references [94, 95, 97, 98].

The derived equivalence transformations are employed to simplify the forms of the arbitrary elements with the understanding that these equivalence transformations are included in the conclusions.

We calculate the equivalence group of the class under consideration by employing the direct method. The details of the calculations are omitted for brevity and we only present the results.

Theorem 6.1. *The usual equivalence group $G^{\sim}$ of class (6.4) consists of the transformations*

$$\begin{aligned} &t' = \alpha_1^2 t + \alpha_2, \quad x' = \alpha_1 x + \beta_1 t + \beta_2, \quad u' = \frac{u}{\alpha_1}, \quad v' = \frac{v}{\alpha_1}, \\ &f' = \alpha_1^{-1} f - \beta_1 \alpha_1^{-2}, \quad k' = \alpha_1^{-1} k - \beta_1 \alpha_1^{-2}, \end{aligned} \tag{6.6}$$

where $\alpha_1 \neq 0$.

In the above theorem, we assume that $\lambda_1' = \lambda_1,\ \lambda_2' = \lambda_2,\ \epsilon_1' = \epsilon_1,\ \epsilon_2' = \epsilon_2$. Clearly, system (6.4) admits the discrete symmetry

$$t' = t,\ x' = x,\ u' = v,\ v' = u,\ \lambda_1' = \lambda_2,\ \lambda_2' = \lambda_1,\ \epsilon_1' = \epsilon_2,\ \epsilon_2' = \epsilon_1,\ f' = k,\ k' = f.$$

Theorem 6.2. *The usual equivalence group $G^{\sim}$ of class (6.5) consists of the transformations*

$$t' = \alpha_1^2 t + \alpha_2, \quad x' = \alpha_1 x + \beta_1 t + \beta_2, \quad u' = \gamma_1 u + \gamma_2, \quad v' = \delta_1 v + \delta_2,$$
$$f' = \alpha_1^{-1} f - \beta_1 \alpha_1^{-2}, \quad k' = \alpha_1^{-1} k - \beta_1 \alpha_1^{-2}, \tag{6.7}$$

where $\alpha_1 \gamma_1 \delta_1 \neq 0$.

It turns out that in the case where the arbitrary elements are equal, the usual equivalence group is wider.

Theorem 6.3. *The usual equivalence group $G^{\sim}_{f=k}$ of class (6.5), where $f(u,v) = k(u,v)$ and $\lambda_1 = \lambda_2 = 1$, consists of the transformations*

$$t' = \alpha_1^2 t + \alpha_2, \quad x' = \alpha_1 x + \beta_1 t + \beta_2, \quad u' = \gamma_1 u + \gamma_2 v + \gamma_3, \quad v' = \delta_1 u + \delta_2 v + \delta_3,$$
$$f' = \alpha_1^{-1} f - \beta_1 \alpha_1^{-2}, \quad k' = \alpha_1^{-1} k - \beta_1 \alpha_1^{-2}, \tag{6.8}$$

where $\alpha_1(\gamma_1\delta_2 - \gamma_2\delta_1) \neq 0$.

In the above theorems we have used the term usual equivalence group. For the notion of generalized equivalence transformations one can refer to the references [94, 95, 97, 98].

6.3 Lie symmetries

6.3.1 Group classification for the class (6.4)

The Lie method for finding point symmetries is well established in the last decades. Several textbooks exist that describe the method. See for example, in [13–15, 30, 37, 68, 72]. Here we search for generators

$$\Gamma = \tau(x,t,u,v)\frac{\partial}{\partial t} + \xi(x,t,u,v)\frac{\partial}{\partial x} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \mu(x,t,u,v)\frac{\partial}{\partial v}$$

corresponding to the infinitesimal transformations

$$t' = t + \epsilon\tau(x,t,u,v), \quad x' = x + \epsilon\xi(x,t,u,v), \quad u' = u + \epsilon\eta(x,t,u,v), \quad v' = v + \epsilon\mu(x,t,u,v)$$

to the first order of ϵ.

We require that

$$\Gamma^{(2)}\left\{u_t - \lambda_1 u_{xx} - f(u,v)u_x - \epsilon_1 v v_x\right\} = 0, \quad \Gamma^{(2)}\left\{v_t - \lambda_2 v_{xx} - k(u,v)v_x - \epsilon_2 u u_x\right\} = 0, \tag{6.9}$$

identically, modulo the system (6.4). Eliminating u_{xx} and v_{xx} from system (6.4), equations (6.9) become two multi-variable polynomials in u_t, u_x, v_t and v_x. The coefficients of the different combinations of powers of these four variables must be zero, giving the determining system. This system needs to be solved for $\tau,\ \xi,\ \eta$ and μ in terms of $t,\ x,\ u$ and v and also for f and k in terms u and v.

We need to consider two cases:

1. $\lambda_1 \neq \lambda_2$ and

2. $\lambda_1 = \lambda_2$.

Case 1: Based on Theorem 4.6, if $\lambda_1 \neq \lambda_2$ then

$$\tau = \tau(t), \quad \xi = \tfrac{1}{2}x\tau_t(t) + \phi(t), \quad \eta = \eta(x,t,u), \quad \mu = \mu(x,t,v).$$

Using these simplified forms of the coefficient functions, the coefficients of $u_x^2,\ u_x,\ v_x$ and the terms independent of derivatives in the first identity in (6.9) lead to the four equations of the following overdetermined system. The other four equations are the coefficients of $v_x^2,\ v_x,\ u_x$ and the terms independent of derivatives in the second identity in (6.9).

$$\begin{aligned}
&\eta_{uu} = 0,\\
&2\eta\frac{\partial f}{\partial u} + 2\mu\frac{\partial f}{\partial v} + \tau_t f + 4\lambda_1\eta_{xu} + x\tau_{tt} + 2\phi_t = 0,\\
&v\eta_u - v\mu_v - \mu - \tfrac{1}{2}v\tau_t = 0,\\
&\eta_t - \lambda_1\eta_{xx} - f\eta_x - \epsilon_1 v\mu_x = 0,\\
&\mu_{vv} = 0,\\
&2\eta\frac{\partial k}{\partial u} + 2\mu\frac{\partial k}{\partial v} + \tau_t k + 4\lambda_2\mu_{xv} + x\tau_{tt} + 2\phi_t = 0,\\
&u\mu_v - u\eta_u - \eta - \tfrac{1}{2}u\tau_t = 0,\\
&\mu_t - \lambda_2\mu_{xx} - k\mu_x - \epsilon_2 u\eta_x = 0.
\end{aligned}$$

Without presenting any more detailed analysis, we deduce that

$$\tau = 2c_1t + c_2, \quad \xi = c_1x + c_3t + c_4, \quad \eta = -c_1u, \quad \mu = -c_1v$$

and $f(u,v)$ and $k(u,v)$ satisfy a first order PDE of the form

$$c_1uh_u + c_1vh_v - c_1h - c_3 = 0.$$

Investigating all possible solutions of the above PDE, we obtain the following results:

1. f and k arbitrary: $X_1 = \partial_t,\ X_2 = \partial_x$.

2. $f = v\phi\left(\frac{u}{v}\right) + 1$ and $k = 1$: $X_1,\ X_2,\ X_3 = 2t\partial_t + (x-t)\partial_x - u\partial_u - v\partial_v$.

3. $f = v\phi\left(\frac{u}{v}\right),\ k = v\psi\left(\frac{u}{v}\right)$: $X_1,\ X_2,\ Y_3 = 2t\partial_t + x\partial_x - u\partial_u - v\partial_v$.

Case 2: If $\lambda_1 = \lambda_2$, using the mapping

$$t' = \lambda_1 t,\ x' = x,\ u' = u,\ v' = v,\ f' = \frac{f}{\lambda_1},\ k' = \frac{k}{\lambda_1},\ \epsilon_1' = \frac{\epsilon_1}{\lambda_1},\ \epsilon_2' = \frac{\epsilon_2}{\lambda_1},$$

we can take $\lambda_1 = \lambda_2 = 1$. The coefficients of $u_x^2,\ u_x v_x$ and v_x^2 in the identities (6.9) imply that both η and μ are linear in u and v. Hence, we deduce that

$$\tau = \tau(t),\ \xi = \tfrac{1}{2}x\tau_t + \phi(t),$$

$$\eta = A_1(x,t)u + A_2(x,t)v + A_3(x,t),$$

$$\mu = B_1(x,t)u + B_2(x,t)v + B_3(x,t).$$

Here we require that $\eta_v = A_2 \neq 0$ and $\mu_u = B_1 \neq 0$ because otherwise we obtain the three results of Case 1. The coefficients of $u_x,\ v_x$ and the term independent of these derivatives in the identities (6.9) lead to the following overdetermined system

$$(A_{1x}u + A_{2x}v + A_{3x})f + (A_{1xx} - A_{1t})u + (A_{2xx} - A_{2t})v + A_{3xx} - A_{3t} + \epsilon_1 v(B_{1x}u + B_{2x}v + B_{3x}) = 0, \tag{6.10}$$

$$A_2(f-k) + 2A_{2x} + \epsilon_1(\tfrac{1}{2}\tau_t v - A_1 v + B_1 u + 2B_2 v + B_3) = 0, \tag{6.11}$$

$$(A_1u + A_2v + A_3)f_u + (B_1u + B_2v + B_3)f_v + \tfrac{1}{2}\tau_t f + 2A_{1x} + \tfrac{1}{2}\tau_{tt}x + \phi_t + \epsilon_1 B_1 v - \epsilon_2 A_2 u = 0, \tag{6.12}$$

$$(B_{1x}u + B_{2x}v + B_{3x})k + (B_{1xx} - B_{1t})u + (B_{2xx} - B_{2t})v + B_{3xx} - B_{3t} + \epsilon_2 u(A_{1x}u + A_{2x}v + A_{3x}) = 0, \tag{6.13}$$

$$B_1(k-f) + 2B_{1x} + \epsilon_2(\tfrac{1}{2}\tau_t u + 2A_1 u + A_2 v + A_3 - B_2 u) = 0, \tag{6.14}$$

$$(A_1u + A_2v + A_3)k_u + (B_1u + B_2v + B_3)k_v + \tfrac{1}{2}\tau_t k + 2B_{2x} + \tfrac{1}{2}\tau_{tt}x + \phi_t + \epsilon_2 A_2 u - \epsilon_1 B_1 v = 0, \tag{6.15}$$

We note from (6.11) that if $A_2 = 0$, then $B_1 = 0$ (or from (6.14), if $B_1 = 0$, then $A_2 = 0$). Hence both conditions $\eta_v \neq 0$ and $\mu_u \neq 0$ must hold.

From determining equations (6.11) and (6.14), we deduce that f and k are connected by the relation

$$f = k + \mu_1 u + \mu_2 v + \mu_3. \tag{6.16}$$

We substitute the above form of f into equations (6.10) - (6.15). The coefficient of u in (6.11) and the coefficient of v in (6.14) give the homogeneous linear system in A_2 and B_1,

$$\mu_1 A_2 + \epsilon_1 B_1 = 0, \quad \epsilon_2 A_2 - \mu_2 B_1 = 0.$$

Since $A_2 \neq 0$ and $B_1 \neq 0$, the determinant of the coefficients must vanish. We find that

$$\epsilon_1\epsilon_2 + \mu_1\mu_2 = 0.$$

Hence, $\mu_1 \neq 0$ and $\mu_2 \neq 0$. Using the above relation, the system gives

$$B_1 = -\frac{\mu_1}{\epsilon_1}A_2.$$

Now equation (6.11) is linear in v and (6.14) is linear in u which lead to the results

$$A_1 = \frac{-3\epsilon_1\mu_2\tau_t + (4\epsilon_1\mu_1 - 4\mu_2^2)A_2}{6\epsilon_1\mu_2},\ B_2 = \frac{-3\epsilon_1\mu_2\tau_t + (2\epsilon_1\mu_1 - 4\mu_2^2)A_2}{6\epsilon_1\mu_2},$$
$$A_3 = \frac{\mu_3A_2 - 2A_{2x}}{\mu_2},\ B_3 = -\frac{\mu_3A_2 + 2A_{2x}}{\epsilon_1}.$$

We subtract equations (6.10) and (6.13) to find that

$$\epsilon_1 = -\frac{\mu_2^2}{\mu_1}$$

and

$$\mu_3(\mu_2\tau_t + 4\mu_1A_2) = 0.$$

Hence, we need to split the analysis into two cases: $\mu_3 \neq 0$ and $\mu_3 = 0$.

4. If $\mu_3 \neq 0$, then

$$A_2 = -\frac{\mu_2\tau_t}{4\mu_1}.$$

Equation (6.12) or (6.15) gives $\tau = c_1t + c_2$ and differentiation of equation (6.10) with respect to t gives $\phi = c_3t + c_4$. Collecting all the above results and using equation (6.10), we find that if $k(u, v)$ is a solution of the PDE

$$\mu_2(3\mu_1u+\mu_2v+\mu_3)k_u+\mu_1(\mu_1u+3\mu_2v+\mu_3)k_v-2\mu_1\mu_2k+\mu_1\mu_2(\mu_1u+\mu_2v-2\mu_4) = 0, \quad (6.17)$$

$f(u, v)$ is given by the relation (6.16) and

$$\epsilon_1 = -\frac{\mu_2^2}{\mu_1},\quad \epsilon_2 = \frac{\mu_1^2}{\mu_2},$$

then the system (6.4) with $\lambda_1 = \lambda_2 = 1$ admits 3 Lie symmetries, where the third has the form

$$Z_3 = 2t\partial_t + (x + \mu_4t)\partial_x - \tfrac{1}{2\mu_1}(3\mu_1u + \mu_2v + \mu_3)\partial_u - \tfrac{1}{2\mu_2}(\mu_1u + 3\mu_2v + \mu_3)\partial_v.$$

If $\mu_3 = 0$, solving equations (6.10), (6.12) and (6.15), we find three more cases that produce additional symmetries.

5. If $k(u,v)$ is a solution of the PDE

$$[\mu_2(\mu_1-\nu\mu_2)u+\mu_2^2 v]k_u+[\mu_1^2 u+\mu_2(\mu_1-\nu\mu_2)v]k_v+\nu\mu_2^2 k+\mu_1\mu_2(\mu_1 u+\mu_2 v)=0, \quad (6.18)$$

$f(u,v)$ is given by the relation (6.16) and

$$\epsilon_1=-\frac{\mu_2^2}{\mu_1}, \quad \epsilon_2=\frac{\mu_1^2}{\mu_2},$$

then the system (6.4) with $\lambda_1=\lambda_2=1$ admits a third Lie symmetry

$$W_3=2\nu\mu_2^2 t\partial_t+\nu\mu_2^2 x\partial_x+[\mu_2(\mu_1-\nu\mu_2)u+\mu_2^2 v]\partial_u+[\mu_1^2 u+\mu_2(\mu_1-\nu\mu_2)v]\partial_v.$$

In the special case where $\nu=0$, PDE (6.18) gives the solution

$$k(u,v)=-\mu_1 u+\phi(\mu_1 u-\mu_2 v)$$

and from (6.16),

$$f(u,v)=\mu_2 v+\phi(\mu_1 u-\mu_2 v).$$

The third Lie symmetry takes the form

$$W_3=\mu_2(\mu_1 u+\mu_2 v)\partial_u+\mu_1(\mu_1 u+\mu_2 v)\partial_v.$$

6. If $k=\nu_1 u+\frac{\nu_2\nu_3}{\nu_1}v$ and $f=\nu_3 u+\nu_2 v$, then system (6.4) with $\lambda_1=\lambda_2=1$ and $\epsilon_1=\frac{\nu_2^2(\nu_1-\nu_3)}{\nu_1^2}$, $\epsilon_2=\frac{\nu_1(\nu_1-\nu_3)}{\nu_2}$ admits 4 Lie symmetries,

$$Y_3=2t\partial_t+x\partial_x-u\partial_u-v\partial_v, \quad X_4=\nu_2(\nu_1 u-\nu_2 v)\partial_u-\nu_1(\nu_1 u-\nu_2 v)\partial_v.$$

Finally, we find that $k(u,v)=\nu_1 u+\nu_3$ and $f(u,v)=\nu_2 v+\nu_3$ and the system (6.4) with $\lambda_1=\lambda_2=1$ admits 6 Lie symmetries. Using the equivalence transformations, we can take $\nu_3=0$. Hence, we have the following result:

7. If $k=\nu_1 u$ and $f=\nu_2 v$, the system (6.4) with $\lambda_1=\lambda_2=1$ and $\epsilon_1=\frac{\nu_2^2}{\nu_1}$, $\epsilon_2=\frac{\nu_1^2}{\nu_2}$ admits the Lie symmetries

$$X_3=2t\partial_t+x\partial_x-u\partial_u-v\partial_v,$$
$$X_4=\nu_2(\nu_1 u-\nu_2 v)\partial_u-\nu_1(\nu_1 u-\nu_2 v)\partial_v,$$
$$X_5=4\nu_1\nu_2 t\partial_x-\nu_2[x(\nu_1 u-\nu_2 v)+2]\partial_u+\nu_1[x(\nu_1 u-\nu_2 v)-2]\partial_v,$$
$$X_6=8\nu_1\nu_2 t^2\partial_t+8\nu_1\nu_2 xt\partial_x-\nu_2[(\nu_1(x^2+2t)u-\nu_2(x^2-6t)v+4x]\partial_u$$
$$-\nu_1[(\nu_2(x^2+2t)v-\nu_1(x^2-6t)u+4x]\partial_v$$

The results of the group classification for the class (6.4) are summarized in Table 6.1.

Table 6.1: Group classification of the system (6.4).

n	$f(u,v)$	$k(u,v)$	ϵ_1	ϵ_2	λ_1, λ_2	Lie symmetries
1.	$\forall$	$\forall$	$\forall$	$\forall$	$\forall$	X_1, X_2
2.	$v\phi\left(\frac{u}{v}\right)+1$	1	$\forall$	$\forall$	$\forall$	X_1, X_2, X_3
3.	$v\phi\left(\frac{u}{v}\right)$	$v\psi\left(\frac{u}{v}\right)$	$\forall$	$\forall$	$\forall$	X_1, X_2, Y_3
4.	$k+\mu_1 u+\mu_2 v+\mu_3$	solution of (6.17)	$-\frac{\mu_2^2}{\mu_1}$	$\frac{\mu_1^2}{\mu_2}$	$\lambda_1=\lambda_2=1$	X_1, X_2, Z_3
5.	$k+\mu_1 u+\mu_2 v$	solution of (6.18)	$-\frac{\mu_2^2}{\mu_1}$	$\frac{\mu_1^2}{\mu_2}$	$\lambda_1=\lambda_2=1$	X_1, X_2, W_3
6.	$\nu_3 u+\nu_2 v$	$\nu_1 u+\frac{\nu_2\nu_3}{\nu_1}v$	$\frac{\nu_2^2(\nu_1-\nu_3)}{\nu_1^2}$	$\frac{\nu_1(\nu_1-\nu_3)}{\nu_2}$	$\lambda_1=\lambda_2=1$	X_1, X_2, Y_3, X_4
7.	$\nu_2 v$	$\nu_1 u$	$\frac{\nu_2^2}{\nu_1}$	$\frac{\nu_1^2}{\nu_2}$	$\lambda_1=\lambda_2=1$	$X_1, X_2, X_3, X_4, X_5, X_6$

Here $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2$ and ν_3 are nonzero arbitrary constants.

6.3.2 Group classification for the class (6.5)

Here we require that

$$\Gamma^{(2)}\left\{u_t-\lambda_1 u_{xx}-f(u,v)u_x\right\}=0, \quad \Gamma^{(2)}\left\{v_t-\lambda_2 v_{xx}-k(u,v)v_x\right\}=0, \tag{6.19}$$

identically, modulo the system (6.5).

We consider the cases:

1. $\lambda_1 \neq \lambda_2$ and

2. $\lambda_1 = \lambda_2$.

Case 1: Based on Theorem 4.6, if $\lambda_1 \neq \lambda_2$, then

$$\tau=\tau(t), \quad \xi=\tfrac{1}{2}x\tau_t(t)+\phi(t), \quad \eta=\eta(x,t,u), \quad \mu=\mu(x,t,v).$$

The coefficient of u_x^2 in the first identity in (6.19) gives that $\eta_{uu}=0$ and the coefficient of v_x^2 in the second identity in (6.19) gives $\mu_{vv}=0$. Summarizing, the coefficient functions have the simplified forms

$$\tau=\tau(t), \quad \xi=\tfrac{1}{2}x\tau_t+\phi(t), \quad \eta=A_1(x,t)u+A_2(x,t), \quad \mu=B_1(x,t)v+B_2(x,t)$$

and the identities (6.19) lead to the following determining system

$$(A_{1x}u+A_{2x})f+(\lambda_1 A_{1xx}-A_{1t})u+\lambda_1 A_{2xx}-A_{2t}=0, \tag{6.20}$$

$$(A_1 u+A_2)f_u+(B_1 v+B_2)f_v+\tfrac{1}{2}\tau_t f+2\lambda_1 A_{1x}+\tfrac{1}{2}\tau_{tt}x+\phi_t=0, \tag{6.21}$$

$$(B_{1x}v+B_{2x})k+(\lambda_2 B_{1xx}-B_{1t})v+\lambda_2 B_{2xx}-B_{2t}=0, \tag{6.22}$$

$$(A_1 u+A_2)k_u+(B_1 v+B_2)k_v+\tfrac{1}{2}\tau_t k+2\lambda_2 B_{1x}+\tfrac{1}{2}\tau_{tt}x+\phi_t=0. \tag{6.23}$$

The solution of the system (6.20) - (6.23) provides the forms of the functions $f(u,v)$, $k(u,v)$ and the coefficient functions and therefore the desired Lie symmetries can be obtained. For arbitrary functions $f(u,v)$ and $k(u,v)$, system (6.5) admits the Lie symmetries

$$X_1 = \partial_t,\ X_2 = \partial_x.$$

From equation (6.20) we can determine the form of $f(u,v)$ unless its coefficient is equal to zero. Similarly if the coefficient of $k(u,v)$ in (6.22) is not equal to zero, we can write down its form. These possibilities give the following subcases:

(i) $A_{1x}^2 + A_{2x}^2 \neq 0$ and $B_{1x}^2 + B_{2x}^2 \neq 0$
(ii) $A_{1x}^2 + A_{2x}^2 \neq 0$ and $B_{1x}^2 + B_{2x}^2 = 0$
(iii) $A_{1x}^2 + A_{2x}^2 = 0$ and $B_{1x}^2 + B_{2x}^2 = 0$
We point out that the symmetric case of **(ii)** is omitted.

(i) $A_{1x}^2 + A_{2x}^2 \neq 0$ and $B_{1x}^2 + B_{2x}^2 \neq 0$: From equation (6.20) we deduce that f has the specific form $\frac{p_1 u + p_2}{p_3 u + p_4}$ and, similarly, from (6.22) k has the form $\frac{q_1 v + q_2}{q_3 v + q_4}$. Substitution of these forms into the system (6.20) - (6.23), leads to the conclusion that f is linear in u and k is linear in v. Using the equivalence transformations, we can take $f = u$ or $f = a$ and $k = bv$ or $k = b$, where a and b are arbitrary constants. Clearly, these forms of f and k impose that the system (6.5) consists of two separable differential equations. We state the following results:

(a) $f(u,v) = a$ and $k(u,v) = b$, where a and b are arbitrary constants. System (6.5) which consists of two separable linear equations admits the Lie symmetries

$$\begin{aligned}
&X_1,\ X_2,\ X_3 = 2t\partial_t + x\partial_x - \frac{a}{2\lambda_1}(x+at)u\partial_u - \frac{b}{2\lambda_2}(x+bt)v\partial_v,\\
&X_4 = t\partial_x - \frac{1}{2\lambda_1}(x+at)u\partial_u - \frac{1}{2\lambda_2}(x+bt)v\partial_v,\\
&X_5 = t^2\partial_t + tx\partial_x - \frac{1}{4\lambda_1}\left[(x+at)^2 + 2\lambda_1 t\right]u\partial_u - \frac{1}{4\lambda_2}\left[(x+bt)^2 + 2\lambda_2 t\right]v\partial_v,\\
&X_6 = u\partial_u,\ X_7 = v\partial_v,\ X_{\psi_1} = \psi_1(t,x)\partial_u,\ X_{\psi_2} = \psi_2(t,x)\partial_v,
\end{aligned}$$

where $\psi_1(t,x)$ and $\psi_2(t,x)$ are solutions of the linear PDE

$$\psi_t = \lambda\psi_{xx} + c\psi_x, \tag{6.24}$$

where $(\lambda, c) = (\lambda_1, a)$ and $(\lambda, c) = (\lambda_2, b)$, respectively.

(b) $f(u,v) = a$ and $k(u,v) = bv$. System (6.5) which consists of two separable equations, one being a linear PDE and the other one is the Burgers equation. The admitted Lie symmetries are the following:

$$
\begin{aligned}
&X_1,\ X_2,\ X_3 = 2t\partial_t + x\partial_x - \frac{a}{2\lambda_1}(x+at)u\partial_u - v\partial_v,\\
&X_4 = t\partial_x - \frac{1}{2\lambda_1}(x+at)u\partial_u - \frac{1}{b}\partial_v,\\
&X_5 = t^2\partial_t + tx\partial_x - \frac{1}{4\lambda_1}\left[(x+at)^2 + 2\lambda_1 t\right]u\partial_u - \frac{1}{b}(btv+x)\partial_v,\\
&X_6 = u\partial_u,\ X_{\psi_1} = \psi_1(t,x)\partial_u,
\end{aligned}
$$

where $\psi_1(t,x)$ is a solution of the linear PDE (6.24) with $(\lambda, c) = (\lambda_1, a)$.

(c) $f(u,v) = u$ and $k(u,v) = bv$. Here we have a system of two separable Burgers equations that admits the Lie symmetries

$$
\begin{aligned}
&X_1,\ X_2,\ X_3 = 2t\partial_t + x\partial_x - u\partial_u - v\partial_v,\\
&X_4 = t\partial_x - \partial_u - \frac{1}{b}\partial_v,\ X_5 = t^2\partial_t + tx\partial_x - (tu+x)\partial_u - \frac{1}{b}(btv+x)\partial_v.
\end{aligned}
$$

(ii) $A_{1x}^2 + A_{2x}^2 \neq 0$ and $B_{1x}^2 + B_{2x}^2 = 0$: Equation (6.22) implies that the functions B_1 and B_2 are both constants. From equations (6.20) and (6.21) we find, as in the previous subcase, $f = a$ or $f = u$. In the case $f = a$, in order to satisfy the condition $A_{1x}^2 + A_{2x}^2 \neq 0$, we need to take $k_u = 0$. From equation (6.23) we deduce that $\tau = 2c_1t + c_2$, $\phi = c_3t + c_4$ and $k(v)$ satisfies an ordinary differential equation of the form $(\nu_1 v + \nu_2)k_v + \nu_3 k = \nu_4$. All possible solutions of this latter equation provide the following results which correspond to separable systems with the first equation being linear.

(a) $f(u,v) = a$ and $k(u,v) = k(v)$, where $k(v)$ is an arbitrary function.

$$X_1,\ X_2,\ X_3 = u\partial_u,\ X_\psi = \psi(t,x)\partial_u,$$

where $\psi(t,x)$ is a solution of the linear equation $\psi_t = \lambda_1\psi_{xx} + a\psi_x$.

(b) $f(u,v) = a$ and $k(u,v) = v^n$.

$$X_1,\ X_2,\ X_3 = u\partial_u,\ X_4 = 4\lambda_1 nt\partial_t + 2\lambda_1 nx\partial_x - an(x+at)u\partial_u - 2\lambda_1 v\partial_v,\ X_\psi = \psi(t,x)\partial_u.$$

(c) $f(u,v) = a$ and $k(u,v) = \mathrm{e}^{nv}$.

$$X_1,\ X_2,\ X_3 = u\partial_u,\ X_4 = 4\lambda_1 nt\partial_t + 2\lambda_1 nx\partial_x - an(x+at)u\partial_u - 2\lambda_1\partial_v,\ X_\psi = \psi(t,x)\partial_u.$$

(d) $f(u,v) = a$ and $k(u,v) = \ln v$.

$$X_1,\ X_2,\ X_3 = u\partial_u,\ X_4 = 2\lambda_1 t\partial_x - (x+at)u\partial_u - 2\lambda_1 v\partial_v,\ X_\psi = \psi(t,x)\partial_u.$$

In the case $f(u,v) = u$, in order to satisfy the condition $A_{1x}^2 + A_{2x}^2 \neq 0$, the determining equations (6.20) - (6.23) are satisfied only if $k(u,v) = u$. We find that system (6.5) admits 7 Lie symmetries and the results are tabulated in entry 22 of Table 6.2.

(iii) $A_{1x}^2 + A_{2x}^2 = 0$ and $B_{1x}^2 + B_{2x}^2 = 0$: From equations (6.20) and (6.22) we find that A_1, A_2, B_1 and B_2 are constant functions. Differentiation of equation (6.21) (or (6.23)) with respect to x and t, respectively, gives that $\tau(t)$ and $\phi(t)$ are linear functions. Summarizing, we state that the coefficient functions have the form

$$\tau = 2c_1 t + c_2, \quad \xi = c_1 x + c_3 t + c_4, \quad \eta = c_5 u + c_6, \quad \mu = c_7 v + c_8.$$

Finally, equations (6.21) and (6.23) which become

$$\begin{aligned} &(c_5 u + c_6)\frac{\partial f}{\partial u} + (c_7 v + c_8)\frac{\partial f}{\partial v} + c_1 f + c_3 = 0, \\ &(c_5 u + c_6)\frac{\partial k}{\partial u} + (c_7 v + c_8)\frac{\partial k}{\partial v} + c_1 k + c_3 = 0, \end{aligned} \tag{6.25}$$

need to be satisfied. From equations (6.25) we deduce that the functions $f(u,v)$ and $k(u,v)$ satisfy a first order quasi-linear partial differential equation of the form

$$(\mu_1 u + \mu_2)\frac{\partial \phi}{\partial u} + (\mu_3 v + \mu_4)\frac{\partial \phi}{\partial v} + \mu_5 \phi + \mu_6 = 0. \tag{6.26}$$

If $f(u,v)$ and $k(u,v)$ are any arbitrary solutions of the PDE (6.26), then in addition to the two Lie symmetries X_1 and X_2, system (6.5) admits at least one third Lie symmetry of the form

$$2\mu_5 t\partial_t + (\mu_5 x + \mu_6 t)\partial_x + (\mu_1 u + \mu_2)\partial_u + (\mu_3 v + \mu_4)\partial_v.$$

Now the question is: Which forms of $f(u,v)$ and $k(u,v)$ lead to more than one Lie symmetry of the above form?

The possible forms of $f(u,v)$ and $k(u,v)$ (solutions of (6.26)) are:

(i) $f(u,v) = au + \phi(u + \epsilon v),\ k(u,v) = au + \psi(u + \epsilon v)$,

(ii) $f(u,v) = \mathrm{e}^{nu}\phi(u + \epsilon v),\ k(u,v) = \mathrm{e}^{nu}\psi(u + \epsilon v)$,

(iii) $f(u,v) = au + \phi(v\mathrm{e}^{mu}),\ k(u,v) = au + \psi(v\mathrm{e}^{mu})$,

(iv) $f(u,v) = \mathrm{e}^{nu}\phi(v\mathrm{e}^{mu}),\ k(u,v) = \mathrm{e}^{nu}\psi(v\mathrm{e}^{mu})$,

(v) $f(u,v) = a\ln u + \phi(u\mathrm{e}^{mv}),\ k(u,v) = a\ln u + \psi(u\mathrm{e}^{mv})$,

(vi) $f(u,v) = u^n\phi(u\mathrm{e}^{mv}),\ k(u,v) = u^n\psi(u\mathrm{e}^{mv})$,

(vii) $f(u,v) = a\ln u + \phi(vu^m),\ k(u,v) = a\ln u + \psi(vu^m)$,

(viii) $f(u,v) = u^n\phi(vu^m),\ k(u,v) = u^n\psi(vu^m)$, where ϕ and ψ are arbitrary functions in their arguments. Writing in (iv) $\phi(ve^{mu}) = (ve^{mu})^s\tilde{\phi}(ve^{mu})$ and similarly for ψ, we note that these forms of $f(u,v)$ and $k(u,v)$ are symmetric (interchange u and v) with those of (vi). However since this observation is not so obvious, we keep both forms in (iv) and (vi).

Consecutive substitutions of the above eight forms of $f(u,v)$ and $k(u,v)$ into equations (6.25) lead to the conclusion that for arbitrary ϕ and ψ system (6.5) admits a third Lie symmetry. The results are tabulated in the entries 1 - 8 of Table 6.2. Additional Lie symmetries exist for specific forms of ϕ and ψ. Without presenting the detailed analysis, we state that we obtain the results in the entries 9 - 21 of table 6.2.

Note 6.1. The above solutions for $f(u,v)$ and $k(u,v)$ that satisfy the PDE (6.26) do not include the cases where one of the functions is constant. If $k(u,v) = \nu$, then $f(u,v)$ takes one of the forms (ii), (iv), (vi) and (viii) plus the constant ν. Using the equivalence transformations (6.7) we can take, without loss of generality, $\nu = 0$. The required Lie symmetries can be obtained by setting $\psi = 0$ in the corresponding cases.

Case 2: If $\lambda_1 = \lambda_2$ and using the mapping $t' = \lambda_1 t,\ x' = x,\ u' = u,\ v' = v,\ f' = \frac{f}{\lambda_1},\ k' = \frac{k}{\lambda_1}$, we can take $\lambda_1 = \lambda_2 = 1$. System (6.5) takes the form

$$u_t = u_{xx} + f(u,v)u_x, \quad v_t = v_{xx} + k(u,v)v_x \tag{6.27}$$

The coefficients of $u_x^2,\ u_x v_x$ and v_x^2 in the first identity in (6.19) give $\eta_{uu} = \eta_{uv} = \eta_{vv} = 0$ and from the corresponding coefficients in the second identity we get $\mu_{uu} = \mu_{uv} = \mu_{vv} = 0$. Therefore the coefficient functions η and μ have the form

$$\begin{aligned} \eta(x,t,u,v) &= A_1(x,t)u + A_2(x,t)v + A_3(x,t), \\ \mu(x,t,u,v) &= B_1(x,t)u + B_2(x,t)v + B_3(x,t). \end{aligned}$$

The coefficients of u_x, v_x and the term independent of derivatives u_x and v_x in equations (6.19) give the following six identities:

$$(A_{1x}u + A_{2x}v + A_{3x})f + (A_{1xx} - A_{1t})u + (A_{2xx} - A_{2t})v + A_{3xx} - A_{3t} = 0, \tag{6.28}$$

$$A_2(f - k) + 2A_{2x} = 0, \tag{6.29}$$

$$(A_1u + A_2v + A_3)f_u + (B_1u + B_2v + B_3)f_v + \tfrac{1}{2}\tau_t f + 2A_{1x} + \tfrac{1}{2}\tau_{tt}x + \phi_t = 0, \tag{6.30}$$

$$(B_{1x}u + B_{2x}v + B_{3x})k + (B_{1xx} - B_{1t})u + (B_{2xx} - B_{2t})v + B_{3xx} - B_{3t} = 0, \tag{6.31}$$

$$B_1(k - f) + 2B_{1x} = 0, \tag{6.32}$$

$$(A_1u + A_2v + A_3)k_u + (B_1u + B_2v + B_3)k_v + \tfrac{1}{2}\tau_t k + 2B_{2x} + \tfrac{1}{2}\tau_{tt}x + \phi_t = 0. \tag{6.33}$$

Table 6.2: Group classification of the system (6.5), where λ_1 and λ_2 are arbitrary.

n	$f(u,v)$	$k(u,v)$	Additional Lie symmetries
1.	$au+\phi(u+\epsilon v)$	$au+\psi(u+\epsilon v)$	$X_3=\epsilon at\partial_x-\epsilon\partial_u+\partial_v$
2.	$e^{nu}\phi(u+\epsilon v)$	$e^{nu}\psi(u+\epsilon v)$	$X_3=2\epsilon nt\partial_t+\epsilon nx\partial_x-\epsilon\partial_u+\partial_v$
3.	$au+\phi(ve^{mu})$	$au+\psi(ve^{mu})$	$X_3=at\partial_x-\partial_u+mv\partial_v$
4.	$e^{nu}\phi(ve^{mu})$	$e^{nu}\psi(ve^{mu})$	$X_3=2nt\partial_t+nx\partial_x-\partial_u+mv\partial_v$
5.	$a\ln u+\phi(ue^{mv})$	$a\ln u+\psi(ue^{mv})$	$X_3=mat\partial_x-mu\partial_u+\partial_v$
6.	$u^n\phi(ue^{mv})$	$u^n\psi(ue^{mv})$	$X_3=2mnt\partial_t+mnx\partial_x-mu\partial_u+\partial_v$
7.	$a\ln u+\phi(vu^m)$	$a\ln u+\psi(vu^m)$	$X_3=at\partial_x-u\partial_u+mv\partial_v$
8.	$u^n\phi(vu^m)$	$u^n\psi(vu^m)$	$X_3=2nt\partial_t+nx\partial_x-u\partial_u+mv\partial_v$
9.	$e^{u+\epsilon v}$	$be^{u+\epsilon v}$	$X_3=2t\partial_t+x\partial_x-\partial_u$ $X_4=\epsilon\partial_u-\partial_v$
10.	$(u+\epsilon v)^n$	$b(u+\epsilon v)^n$	$X_3=2nt\partial_t+nx\partial_x-u\partial_u-v\partial_v$ $X_4=\epsilon\partial_u-\partial_v$
11.	v^ne^{mu}	bv^ne^{mu}	$X_3=2nt\partial_t+nx\partial_x-v\partial_v$ $X_4=n\partial_u-mv\partial_v$
12.	v^nu^m	bv^nu^m	$X_3=2nt\partial_t+nx\partial_x-v\partial_v$ $X_4=nu\partial_u-mv\partial_v$
13.	$av+\ln u$	$av+\ln u$	$X_3=at\partial_x-\partial_v$ $X_4=au\partial_u-\partial_v$
14.	$av+e^{nu}$	$av+be^{nu}$	$X_3=2nt\partial_t+nx\partial_x-\partial_u-nv\partial_v$ $X_4=at\partial_x-\partial_v$
15.	$av+u^n$	$av+bu^n$	$X_3=2nt\partial_t+nx\partial_x-u\partial_u-nv\partial_v$ $X_4=at\partial_x-\partial_v$
16.	$a\ln v+\ln u$	$a\ln v+\ln u$	$X_3=t\partial_x-u\partial_u$ $X_4=au\partial_u-v\partial_v$
17.	$\phi(u)$	$\psi(u)$	$X_3=v\partial_v, X_4=\partial_v$
18.	$u+\epsilon v$	$u+\epsilon v$	$X_3=2t\partial_t+x\partial_x-u\partial_u-v\partial_v$ $X_4=\epsilon\partial_u-\partial_v$ $X_5=t\partial_x-\partial_u$
19.	u^n	bu^n	$X_3=v\partial_v, X_4=\partial_v$ $X_5=2nt\partial_t+nx\partial_x-u\partial_u$
20.	e^{nu}	be^{nu}	$X_3=v\partial_v, X_4=\partial_v$ $X_5=2nt\partial_t+nx\partial_x-\partial_u$
21.	$\ln u$	$\ln u$	$X_3=v\partial_v, X_4=\partial_v$ $X_5=t\partial_x-u\partial_u$
22.	u	u	$X_3=v\partial_v, X_4=\partial_v$ $X_5=2t\partial_t+x\partial_x-u\partial_u, X_6=t\partial_x-\partial_u$ $X_7=t^2\partial_t+xt\partial_x-(ut+x)\partial_u$

Here $a,\ b,\ m$ and n are arbitrary constants. In 9, 10 and 17 - 22, $\lambda_1\neq\lambda_2$ since for $\lambda_1=\lambda_2$ the corresponding forms of the system admit additional Lie symmetries (see table 6.3). In 13 - 16 $a\neq 0$ and in 18 $\epsilon\neq 0$.

We point out that if $A_2=B_1=0$, then we obtain the results of the previous case.

From equation (6.29) or equation (6.32), if $A_2^2+B_1^2\neq 0$, then we deduce that

$$f(u,v)=k(u,v)+\lambda.$$

If $\lambda\neq 0$, the determining system (6.28) - (6.33) is satisfied only in the case where k is constant and, consequently, f is also a constant. If $f(u,v)=\mu_1$ and $k(u,v)=\mu_2$, then the linear separable system (6.27) admits nine Lie symmetries and two infinite-dimensional symmetries.

$$X_3 = 4t\partial_t + 2x\partial_x - \mu_1(\mu_1 t + x)u\partial_u - \mu_2(\mu_2 t + x)v\partial_v,$$
$$X_4 = 4t^2\partial_t + 4tx\partial_x - [2t + (\mu_1 t + x)^2]u\partial_u - [2t + (\mu_2 t + x)^2]v\partial_v,$$
$$X_5 = 2t\partial_x - (\mu_1 t + x)u\partial_u - (\mu_2 t + x)v\partial_v,$$
$$X_6 = v\exp\{\tfrac{1}{4}(\mu_2 - \mu_1)[(\mu_2 + \mu_1)t + 2x]\}\partial_u,$$
$$X_7 = u\exp\{-\tfrac{1}{4}(\mu_2 - \mu_1)[(\mu_2 + \mu_1)t + 2x]\}\partial_v,$$
$$X_8 = u\partial_u,\ X_9 = v\partial_v,\ X_{\psi_1} = \psi_1(t,x)\partial_u,\ X_{\psi_2} = \psi_2(t,x)\partial_v,$$

where $\psi_1(t,x)$ and $\psi_2(t,x)$ are solutions of the linear PDE (6.24), where $(\lambda, c) = (1, \mu_1)$ and $(\lambda, c) = (1, \mu_2)$, respectively.

Now we examine the subcase where $\lambda = 0$. If the functions $A_1(x,t)$, $A_2(x,t)$, $A_3(x,t)$, $B_1(x,t)$, $B_2(x,t)$, $B_3(x,t)$ are not all constants then $k(u,v)$ is either constant or linear in u and v. If $f(u,v) = k(u,v) = \mu$, then the corresponding Lie symmetries are obtained from the previous case by setting $\mu_1 = \mu_2 = \mu$. In the case where $k(u,v)$ is a linear function, using the equivalence transformations, we can take $f(u,v) = k(u,v) = v + \epsilon u$. System (6.27) admits nine Lie symmetries

$$X_3 = 2t\partial_t + x\partial_x - u\partial_u - v\partial_v,$$
$$X_4 = t^2\partial_t + xt\partial_x - (\epsilon ut + vt + x)\partial_v,$$
$$X_5 = \epsilon t\partial_x - \partial_u, \quad X_6 = t\partial_x - \partial_v,$$
$$X_7 = v\partial_u - \epsilon v\partial_v, \quad X_8 = u\partial_u - \epsilon u\partial_v,$$
$$X_9 = (\epsilon ut + vt + x)\partial_u - \epsilon(\epsilon ut + vt + x)\partial_v.$$

If the functions $A_1(x,t)$, $A_2(x,t)$, $A_3(x,t)$, $B_1(x,t)$, $B_2(x,t)$, $B_3(x,t)$ are all constants, we find that the coefficient functions have the form

$$\tau = 2c_1 t + c_2, \quad \xi = c_1 x + c_3 t + c_4, \quad \eta = c_5 u + c_6 v + c_7, \quad \mu = c_8 u + c_9 v + c_{10}$$

and we only have to satisfy the determining equation (6.33) which is identical to (6.30),

$$(c_5 u + c_6 v + c_7)\frac{\partial k}{\partial u} + (c_8 u + c_9 v + c_{10})\frac{\partial k}{\partial v} + c_1 k + c_3 = 0. \tag{6.34}$$

We deduce that $k(u,v)$ satisfies a quasi-linear partial differential equation of the form

$$(p_1 u + p_2 v + p_3)\frac{\partial k}{\partial u} + (q_1 u + q_2 v + q_3)\frac{\partial k}{\partial v} + rk + s = 0. \tag{6.35}$$

For an arbitrary form of $k(u,v)$ system (6.27) admits the Lie symmetries $X_1 = \partial_t$ and $X_2 = \partial_x$. If $k(u,v)$ is a solution of the PDE (6.35), then system (6.27) admits at least one more Lie symmetry of the form

$$2rt\partial_t + (rx+st)\partial_x + (p_1u + p_2v + p_3)\partial_u + (q_1u + q_2v + q_3)\partial_v. \tag{6.36}$$

Now our task is to find those forms of $k(u,v)$ that admit more than one Lie symmetry of the form (6.36). In order to achieve this goal, we need to find all possible solutions of the PDE (6.35). However, solving this quasi linear PDE is not an easy task. We make use of the equivalence transformations of the system (6.5) in the case where $f(u,v) = k(u,v)$ and $\lambda_1 = \lambda_2 = 1$ (Theorem 6.3), which is given by equation (6.8) in order to simplify PDE (6.35). We deduce that we can, equivalently, solve the following PDEs:

$$\frac{\partial k}{\partial u} + q_3\frac{\partial k}{\partial v} + rk = 0, \tag{6.37}$$

$$\frac{\partial k}{\partial u} + q_3\frac{\partial k}{\partial v} + s = 0, \tag{6.38}$$

$$\frac{\partial k}{\partial u} + (q_1u + q_2v)\frac{\partial k}{\partial v} + rk = 0, \tag{6.39}$$

$$\frac{\partial k}{\partial u} + (q_1u + q_2v)\frac{\partial k}{\partial v} + s = 0, \tag{6.40}$$

$$u\frac{\partial k}{\partial u} + (q_1u + q_3)\frac{\partial k}{\partial v} + rk = 0, \tag{6.41}$$

$$u\frac{\partial k}{\partial u} + (q_1u + q_3)\frac{\partial k}{\partial v} + s = 0, \tag{6.42}$$

$$u\frac{\partial k}{\partial u} + (q_1u + q_2v)\frac{\partial k}{\partial v} + rk = 0, \tag{6.43}$$

$$u\frac{\partial k}{\partial u} + (q_1u + q_2v)\frac{\partial k}{\partial v} + s = 0, \tag{6.44}$$

$$v\frac{\partial k}{\partial u} + (q_1u + q_2v)\frac{\partial k}{\partial v} + rk = 0, \tag{6.45}$$

$$v\frac{\partial k}{\partial u} + (q_1u + q_2v)\frac{\partial k}{\partial v} + s = 0. \tag{6.46}$$

In the subsequent analysis, where we solve the above equations, certain constants are renamed without stating it.

If $r = 0$ in (6.37), the general solution has the form $k(u,v) = \phi(u + \epsilon v)$. We substitute this form into (6.34) to deduce that $\phi(\xi)$, $\xi = u + \epsilon v$ satisfies an ordinary differential equation of the form

$$(\nu_1\xi + \nu_2)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \nu_3\phi = \nu_4. \tag{6.47}$$

All possible solutions of the above equation lead to the results tabulated in the entries 19, 22 - 24 in Table 6.3. If $r \neq 0$ in (6.37), the general solution has the form $k(u,v) = e^{mu}\phi(u + \epsilon v)$.

From (6.34), we find that $\phi(\xi),\ \xi = u + \epsilon v$ satisfies an ordinary differential equation of the form

$$(\nu_1\xi + \nu_2)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + (\nu_3\xi + \nu_4)\phi = 0.$$

For an arbitrary function ϕ, system (6.27) admits three Lie symmetries. The other possible solutions of the above equations are $\phi(\xi) = e^{m\xi}\xi^n$ and $\phi(\xi) = e^{\xi^2+n\xi}$ which produce an additional fourth Lie symmetry and the results are tabulated in the entries 1 and 12 of Table 6.3, respectively. If $s \neq 0$ in (6.38), the general solution has the form $k(u, v) = mu + \phi(u + \epsilon v)$. In this case, we deduce that $\phi(\xi)$ is a solution of an ordinary differential equation of the form

$$(\nu_1\xi + \nu_2)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \nu_3\phi = \nu_4\xi + \nu_5.$$

The possible solutions of the above ordinary differential equations are $\phi(\xi) = re^{n\xi} + m\xi$, $\phi(\xi) = r\xi^n + m\xi$, $\phi(\xi) = \ln\xi + m\xi$ and $\phi(\xi) = \xi\ln\xi + m\xi$ which all produce a fourth Lie symmetry. The results appear in the entries 3 - 6 of Table 6.3.

If $r = 0$ in (6.39) we find that $k(u, v) = \phi(v + \epsilon u^2)$ or $k(u, v) = \phi((v + \epsilon u)e^{mu})$ depending if the constant q_2 is zero or nonzero. Substitution of the first form into (6.34) implies that ϕ satisfies an ordinary differential equation of the form (6.47). The three possible solutions of this equation produce the results tabulated in the entries 8 - 10 of Table 6.3. The second form provides special cases of the results in the entries 1 and 3 of the same table. If $r \neq 0$, equation (6.39) has the general solution $k(u, v) = e^{ru}\phi(v + \epsilon u^2)$ or $k(u, v) = e^{ru}\phi((v + \epsilon u)e^{mu})$. The first form with ϕ being arbitrary gives three Lie symmetries, while in the case $\phi = e^{v+\epsilon u^2}$ we find four Lie symmetries and the result is a special case of the entry 12 of Table 6.3. The second form with ϕ being arbitrary gives three Lie symmetries, if $\phi = ((v + \epsilon u)e^{mu})^n$, we reproduce the results of entry 1 in Table 6.3 and if $\phi = ((v + \epsilon u)e^{mu})^{-r/m}$, we reproduce the results of entry 22 in Table 6.3.

If $s \neq 0$, equation (6.40) has the general solution $k(u, v) = au + \phi(v + \epsilon u^2)$ or $k(u, v) = au + \phi((v + \epsilon u)e^{mu})$. We substitute the first form into (6.34) to find that, if ϕ is arbitrary we have three Lie symmetries, if $\phi = \sqrt{v + \epsilon u^2}$ we have four Lie symmetries and the results are tabulated in the entry 11 of Table 6.3 and if $\phi = v + \epsilon u^2$ we have five Lie symmetries and the results are tabulated in the entry 20 of Table 6.3. The second form produces three Lie symmetries for arbitrary ϕ, four Lie symmetries for $\phi = \ln((v + \epsilon u)e^{mu})$ which is a recalculation of the results of entry 3 of Table 6.3 and six Lie symmetries for $\phi = -\frac{a}{m}\ln((v + \epsilon u)e^{mu})$ which is recalculation of the results of entry 24 of Table 6.3.

Now we consider equations (6.41) and (6.42). If $r = 0$, the general solution of (6.41) has the form $k(u, v) = \phi(v + \epsilon u + m\ln u)$. For arbitrary ϕ, we obtain three Lie symmetries, for

$\phi = v + \epsilon u + m \ln u$, four Lie symmetries which is a special case of entry 3 in Table 6.3 and in the case where $\phi = u^m e^{v+\epsilon u}$, we obtain a special case of entry 1 in Table 6.3. If $r \neq 0$, the general solution of (6.41) has the form $k(u,v) = u^r \phi(v + \epsilon u + m \ln u)$. For $m \neq 0$, the determining equation (6.34) implies that $\phi = e^{-\frac{r}{m}(v+\epsilon u+m \ln u)}$ or $\phi = e^{v+\epsilon u+m \ln u}$. In both cases we obtain existing results. For $m = 0$, equation (6.34) implies that ϕ is of exponential or power form which do not lead to any new results. If $s \neq 0$, then equation (6.42) has the general solution $k(u,v) = s \ln u + \phi(v + \epsilon u + m \ln u)$. We substitute into (6.34) and we obtain four different cases. If ϕ is an arbitrary function, then system (6.27) admits three Lie symmetries. If $\phi = \ln(v + \epsilon u + m \ln u)$, we find three Lie symmetries for $m \neq 0$ and four for $m = 0$ and the result is a special case of the entry 7 in Table 6.3. If $\phi = v + \epsilon u + m \ln u$, we recalculate the result of a special case of entry 3 in Table 6.3. Finally, if $\phi = -\frac{s}{m}(v+\epsilon u+m \ln u)$, we reproduce a subset of Lie symmetries of entry 25 in Table 6.3.

Equation (6.43) with $r = 0$ has the general solution $k(u,v) = \phi(u^m(v + \epsilon u))$ or $k(u,v) = \phi(\frac{v}{u} + m \ln u)$ depending if $q_2 \neq 1$ or $q_2 = 1$. In the case where $k(u,v) = \phi(u^m(v+\epsilon u))$, equation (6.34) implies that for arbitrary ϕ there exist three Lie symmetries, for $\phi = (u^m(v + \epsilon u))^n$ four symmetries which is a special case of entry 2 in Table 6.3 and for $\phi = \ln(u^m(v + \epsilon u))$ also four symmetries which are special cases of entry 7 in Table 6.3. For the second form, $k(u,v) = \phi(\frac{v}{u} + m \ln u)$, we need to take the subcases $m \neq 0$ and $m = 0$. As before, for arbitrary ϕ we find three Lie symmetries. When $m \neq 0$, additional symmetries exist when $\phi = \frac{v}{u} + m \ln u$ and $\phi = e^{\frac{v}{u}+m \ln u}$ and the results are special cases of the entries 13 and 14 in Table 6.3, respectively. When $m = 0$, we deduce that $\phi(\xi)$, $\xi = \frac{v}{u}$ satisfies an ordinary differential equation of the form

$$(\nu_1 \xi^2 + \nu_2 \xi + \nu_3) \frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \nu_4 \phi = \nu_5. \tag{6.48}$$

If $\nu_1 \neq 0$ and $\nu_4 \neq 0$, the possible solutions of (6.48) are of the forms $\phi(\xi) = e^{r \tan^{-1} \xi}$, $\phi(\xi) = \left(\frac{\xi+\epsilon}{\xi+\delta}\right)^n$ and $\phi(\xi) = e^{\frac{n}{\xi+\epsilon}}$. These forms lead to the special cases tabulated in the entries 2, 13 and 17 of Table 6.3. If $\nu_1 \neq 0$ and $\nu_4 = 0$, we find that $\phi(\xi) = \tan^{-1}(n\xi)$, $\phi(\xi) = \ln\left(\frac{\xi+\epsilon}{\xi+\delta}\right)$ and $\phi(\xi) = \frac{1}{\xi+\epsilon}$. The first two forms lead to results obtained in previous cases. The third form leads to the results tabulated in the entry 21 of Table 6.3. If $\nu_1 = 0$, we find $\phi = \xi^n$ (Table 6.2, entry 12), $\phi = \ln \xi$ (Table 6.2, entry 18), $\phi = e^{n\xi}$ (Table 6.3, entry 13 with $\epsilon = 0$ and $m = 0$) and $\phi = \xi$ (Table 6.3, entry 21 with $\epsilon = 0$).

Equation (6.43) with $r \neq 0$ has the general solution $k(u,v) = u^r \phi(u^m(v + \epsilon u))$ or $k(u,v) = u^r \phi(\frac{v}{u} + m \ln u)$. We substitute the first form $k(u,v) = u^r \phi(\xi)$, $\xi = u^m(v + \epsilon u)$ into (6.34). We find that $\phi = \xi^n$ and $\phi = \xi^{-\frac{r}{m}}$ which lead to results that have already been found in previous cases. The second form for $m \neq 0$ leads to a special case of entry 13 in Table 6.3. If $m = 0$, we find $\phi(\xi) = \xi^{-r}$, $\xi = \frac{v}{u}$ which gives the results of entry 22 in Table 6.3 with $\epsilon = 0$ or $\phi(\xi)$ satisfies an ordinary differential equation of the form

$$(\nu_1\xi^2 + \nu_2\xi + \nu_3)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + (-r\nu_1\xi + \nu_4)\phi = 0. \tag{6.49}$$

If $\nu_1 = 0$, we find that $\phi = \xi^n$ and $\phi = e^{n\xi}$ which give special cases of the results of entry 2 and entry 13 of Table 6.3, respectively. If $\nu_1 \neq 0$, we find the three possible solutions of (6.49): $\phi = (n^2\xi^2 + 1)^{r/2}e^{m\tan^{-1}n\xi}$, $\phi = (\xi + m)^r e^{\frac{n}{\xi+m}}$ and $\phi = (\xi + m)^{r/2+n}(\xi + \epsilon)^{r/2-n}$. Rearranging the constants, we find the corresponding forms of $k(u, v)$ and then we substitute into (6.34) to derive the corresponding Lie symmetries and the results are tabulated in the entries 2, 13 and 17 with $\epsilon = 0$ in the Table 6.3.

Equation (6.44) with $s \neq 0$ has the general solution $k(u, v) = s\ln u + \phi(u^m(v + \epsilon u))$ or $k(u, v) = s\ln u + \phi(\frac{v}{u} + m\ln u)$. We substitute the first form $k(u, v) = s\ln u + \phi(\xi)$, $\xi = u^m(v+\epsilon u)$ into (6.34). We find that $\phi = \xi^n$ and $\phi = \xi^{-\frac{s}{m}}$ which lead to results that have already been found in previous cases. The second case, for $m \neq 0$ reproduces a special case of the results of entry 14 in the Table 6.3. In the subcase where $m = 0$, $\phi(\xi)$, $\xi = \frac{v}{u}$ satisfies an ordinary differential equation of the form

$$(\nu_1\xi^2 + \nu_2\xi + \nu_3)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = r\nu_1\xi + \nu_4. \tag{6.50}$$

If $\nu_1 = 0$, then $\phi = \xi$ or $\phi = \ln\xi$ which lead to special cases of entries 7 and 14 in Table 6.3. If $\nu \neq 0$, ϕ takes one of the following three forms: $\phi = m\tan^{-1}n\xi + \frac{r}{2}\ln(n^2\xi^2 + 1)$, $\phi = \frac{n}{\xi+m} + r\ln(\xi + m)$ and $\phi = (\frac{r}{2} + n)\ln(\xi + m) + (\frac{r}{2} - n)\ln(\xi + \epsilon)$. Rearranging the constants, we substitute the corresponding form of $k(u, v)$ into (6.34). We find the results tabulated in the entries 7, 18 with $\epsilon = 0$ and 14 of Table 6.3.

The solutions of equation (6.45) with $r \neq 0$ and of equation (6.46) with $s \neq 0$ are cumbersome expressions. We neglect these cases and we only consider equation (6.45) with $r = 0$. If $q_2 = 0$, the general solution is of the form $k(u, v) = \phi(u^2 + nv^2)$. For arbitrary ϕ we find three Lie symmetries and for $\phi = (u^2+nv^2)^r$ and $\phi = \ln(u^2+nv^2)$ we find four symmetries and the results coincide with those of entries 17 with $\epsilon = 0$ and 18 in Table 6.3 with $m = \epsilon = 0$, respectively. If $q_2 \neq 0$, then $k(u, v) = \phi(\xi)$, where ξ takes the following three forms: $\xi = (v + \epsilon u)e^{\frac{\epsilon u}{v+\epsilon u}}$, $\xi = (v + \epsilon u)^m(v + \delta u)^n$ and $\xi = [(v + \epsilon u)^2 + \delta^2u^2]e^{r\tan^{-1}(\frac{v+\epsilon u}{\delta u})}$. Substitution into (6.34) implies that in all three cases $\phi = \xi^n$ or $\phi = \ln\xi$. For both forms of ϕ, the first two cases give results obtained earlier. The third case leads to the results tabulated in the entries 17 and 18 of Table 6.3.

Table 6.3: Group classification of the system (6.5), where $\lambda_1 = \lambda_2 = 1$ and $f(u,v) = k(u,v)$.

n	$k(u,v)$	Additional Lie symmetries
1.	$e^{v+\delta u}(v+\epsilon u)^m$	$X_3 = (v+\epsilon u+m)\partial_u - [\delta(v+\epsilon u)+m\epsilon]\partial_v$, $X_4 = 2(\epsilon-\delta)t\partial_t + (\epsilon-\delta)x\partial_x + \partial_u - \epsilon\partial_v$
2.	$(v+\epsilon u)^n(v+\delta u)^m$	$X_3 = [m(v+\epsilon u)+n(v+\delta u)]\partial_u - [m\delta(v+\epsilon u)+n\epsilon(v+\delta u)]\partial_v$, $X_4 = 2(n+m)t\partial_t + (n+m)x\partial_x - u\partial_u - v\partial_v$
3.	$\ln(v+\epsilon u) + m(v+\delta u)$	$X_3 = [m(v+\epsilon u)+1]\partial_u - [m\delta(v+\epsilon u)+\epsilon]\partial_v$, $X_4 = m(\epsilon-\delta)t\partial_x + \partial_u - \epsilon\partial_v$
4.	$e^{n(v+\epsilon u)} + m(v+\delta u)$	$X_3 = 2n(\epsilon-\delta)t\partial_t + n(\epsilon-\delta)x\partial_x + [n(v+\delta u)-1]\partial_u$ $+[\delta - n\epsilon(v+\delta u)]\partial_v$, $X_4 = m(\epsilon-\delta)t\partial_x + \partial_u - \epsilon\partial_v$
5.	$(v+\epsilon u)^n + m(v+\delta u)$	$X_3 = 2n(\epsilon-\delta)t\partial_t + n(\epsilon-\delta)x\partial_x + (n(v+\delta u) - v - \epsilon u)\partial_u$ $+(\delta(v+\epsilon u) - n\epsilon(v+\delta u))\partial_v$, $X_4 = m(\epsilon-\delta)t\partial_x + \partial_u - \epsilon\partial_v$
6.	$(v+\epsilon u)\ln(v+\epsilon u) + m(v+\delta u)$	$X_3 = 2m(\epsilon-\delta)t\partial_t + m(\epsilon-\delta)x\partial_x + [(m\delta - m\epsilon - \epsilon)u - v]\partial_u$ $+[\epsilon^2 u + (m\delta - m\epsilon + \epsilon)v]\partial_v$ $X_4 = m(\epsilon-\delta)t\partial_x + \partial_u - \epsilon\partial_v$
7.	$n\ln(v+\epsilon u) + m\ln(v+\delta u)$	$X_3 = [m(v+\epsilon u)+n(v+\delta u)]\partial_u - [m\delta(v+\epsilon u)+n\epsilon(v+\delta u)]\partial_v$, $X_4 = (n+m)t\partial_x - u\partial_u - v\partial_v$
8.	$(v+\gamma u^2)^n$	$X_3 = \partial_u - 2\gamma u\partial_v, X_4 = 4nt\partial_t + 2nx\partial_x - u\partial_u - 2v\partial_v$
9.	$e^{n(v+\gamma u^2)}$	$X_3 = \partial_u - 2\gamma u\partial_v, X_4 = 2nt\partial_t + nx\partial_x - \partial_v$
10.	$\ln(v+\gamma u^2)$	$X_3 = \partial_u - 2\gamma u\partial_v, X_4 = 2t\partial_x - u\partial_u - 2v\partial_v$
11.	$mu + \sqrt{v+\gamma u^2}$	$X_3 = 2t\partial_t + x\partial_x - u\partial_u - 2v\partial_v, X_4 = mt\partial_x - \partial_u + 2\gamma u\partial_v$
12.	$e^{(v+\delta u)^2 + n(v+\epsilon u)}$	$X_3 = [2(v+\delta u)+n]\partial_u - (2\delta(v+\delta u)+n\epsilon]\partial_v$, $X_4 = 2n(\epsilon-\delta)t\partial_t + n(\epsilon-\delta)x\partial_x - \partial_u + \delta\partial_v$
13.	$(v+\epsilon u)^m e^{\frac{ru}{v+\epsilon u}}$	$X_3 = [ru - m(v+\epsilon u)]\partial_u + [rv + m\epsilon(v+\epsilon u)]\partial_v$, $X_4 = 2mt\partial_t + mx\partial_x - u\partial_u - v\partial_v$
14.	$\frac{ru}{v+\epsilon u} + m\ln(v+\epsilon u)$	$X_3 = [ru - m(v+\epsilon u)]\partial_u + [rv + m\epsilon(v+\epsilon u)]\partial_v$, $X_4 = mt\partial_x - u\partial_u - v\partial_v$
15.	$\frac{ru}{v+\epsilon u} + m(v+\epsilon u)^n$	$X_3 = 2nt\partial_t + nx\partial_x - (n+1)u\partial_u + (n\epsilon u - v)\partial_v$, $X_4 = rt\partial_x - (v+\epsilon u)\partial_u + \epsilon(v+\epsilon u)\partial_v$
16.	$\frac{ru}{v+\epsilon u} + m\frac{\ln(v+\epsilon u)}{v+\epsilon u}$	$X_3 = 2rt\partial_t + rx\partial_x - m\partial_u + [(r(v+\epsilon u) + m\epsilon]\partial_v$, $X_4 = rt\partial_x - (v+\epsilon u)\partial_u + \epsilon(v+\epsilon u)\partial_v$
17.	$[(v+\epsilon u)^2 + \delta^2 u^2]^m e^{r\tan^{-1}(\frac{v+\epsilon u}{\delta u})}$	$X_3 = [r\delta u + 2m(v+\epsilon u)]\partial_u - [2m(\delta^2+\epsilon^2)u + (2m\epsilon - r\delta)v]\partial_v$, $X_4 = 4mt\partial_t + 2mx\partial_x - u\partial_u - v\partial_v$
18.	$m\tan^{-1}\frac{v+\epsilon u}{\delta u} + n\ln[(v+\epsilon u)^2 + \delta^2 u^2]$	$X_3 = [m\delta u + 2n(v+\epsilon u)]\partial_u - [2n(\delta^2+\epsilon^2)u + (2n\epsilon - m\delta)v]\partial_v$, $X_4 = 2nt\partial_x - u\partial_u - v\partial_v$
19.	$\phi(v+\epsilon u)$	$X_3 = \partial_u - \epsilon\partial_v, X_4 = u\partial_u - \epsilon u\partial_v, X_5 = v\partial_u - \epsilon v\partial_v$
20.	$(v+\epsilon u)^2 + m(v+\delta u)$	$X_3 = 4(\epsilon-\delta)t\partial_t + 2(\epsilon-\delta)x\partial_x + ((2\delta-\epsilon)u + v)\partial_u$ $+((\delta - 2\epsilon)v - \delta\epsilon u)\partial_v$, $X_4 = m(\epsilon-\delta)t\partial_x + \partial_u - \epsilon\partial_v$, $X_5 = (2(v+\epsilon u)+m)\partial_u - (2\epsilon(v+\epsilon u)+m\delta)\partial_v$
21.	$\frac{u}{v+\epsilon u}$	$X_3 = u\partial_u + v\partial_v, X_4 = 2t\partial_t + x\partial_x + (v+\epsilon u)\partial_v$, $X_5 = 4\epsilon t\partial_t + (2\epsilon x - t)\partial_x + v\partial_u + \epsilon^2 u\partial_v$
22.	$(v+\epsilon u)^n$	$X_3 = \partial_u - \epsilon\partial_v, X_4 = u\partial_u - \epsilon u\partial_v, X_5 = v\partial_u - \epsilon v\partial_v$, $X_6 = 2nt\partial_t + nx\partial_x - (v+\epsilon u)\partial_v$
23.	$e^{n(v+\epsilon u)}$	$X_3 = \partial_u - \epsilon\partial_v, X_4 = u\partial_u - \epsilon u\partial_v, X_5 = v\partial_u - \epsilon v\partial_v$, $X_6 = 2nt\partial_t + nx\partial_x - \partial_v$
24.	$\ln(v+\epsilon u)$	$X_3 = \partial_u - \epsilon\partial_v, X_4 = u\partial_u - \epsilon u\partial_v, X_5 = v\partial_u - \epsilon v\partial_v$, $X_6 = t\partial_x - (v+\epsilon u)\partial_v$
25.	$v+\epsilon u$	$X_3 = 2t\partial_t + x\partial_x - u\partial_u - v\partial_v$, $X_4 = t^2\partial_t + xt\partial_x - (\epsilon ut + vt + x)\partial_v$, $X_5 = \epsilon t\partial_x - \partial_u, X_6 = t\partial_x - \partial_v, X_7 = v\partial_u - \epsilon v\partial_v$, $X_8 = u\partial_u - \epsilon u\partial_v, X_9 = (\epsilon ut + vt + x)\partial_u - \epsilon(\epsilon ut + vt + x)\partial_v$

Here ϵ, δ, γ, n, m and r are arbitrary constants. In 1 - 7, m, $n \neq 0$ and $\delta \neq \epsilon$. In 8 - 10, n, $\gamma \neq 0$. In 13 - 16 m, $r \neq 0$. In 17, 18 $\delta \neq 0$. In 20 $m \neq 0$ and $\delta \neq \epsilon$.

Note 6.2. We have used the equivalence transformations to simplify the forms of $f(u,v)$ and $k(u,v)$. However in some cases, we have kept the form of these functions in order to verify certain special cases. For example, system (6.5) with $f(u,v) = k(u,v) = v + \epsilon u$ (entry 25 of Table 6.3) can be mapped into the same system with $f(u,v) = k(u,v) = v$ using the member of the equivalence transformations

$$t \mapsto t, \quad x \mapsto x, \quad u \mapsto u, \quad v \mapsto v - \epsilon u.$$

Similarly, we can use the above mapping for entries 1 - 7 ($\delta \mapsto \delta - \epsilon$), 12 - 24 to transform $v + \epsilon u$ into v. In an inverse manner, we can use the inverse transformation to replace v by $v + \epsilon u$ in entries 8 - 11.

6.4 Examples of nonclassical reductions

Bluman and Cole introduced a new method for finding group-invariant solutions of partial differential equations [9] which was called "non-classical reduction". Later, it was also called, by different authors, conditional symmetries, Q-conditional symmetries and reduction operators [27,29,54]. A precise and rigorous definition of nonclassical invariance was first formulated in [26] where they generalized the Lie definition of invariance (see also [101]). The necessary definitions and relevant statements on the theory of nonclassical reductions can be found in [95].

We search for non-classical reductions for the classes (6.4) and (6.5). Non-classical reductions for the special case of (6.3) where $f(u,v) = u$ and $k(u,v) = v$ were obtained in [4,16]. We require invariance of equation (6.4) in conjunction with its invariant surface conditions

$$\tau(x,t,u,v)u_t + \xi(x,t,u,v)u_x = \eta(x,t,u,v),$$

$$\tau(x,t,u,v)v_t + \xi(x,t,u,v)v_x = \mu(x,t,u,v),$$

under the infinitesimal transformations generated by

$$\Gamma = \tau(x,t,u,v)\frac{\partial}{\partial t} + \xi(x,t,u,v)\frac{\partial}{\partial x} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \mu(x,t,u,v)\frac{\partial}{\partial v}.$$

The non-classical method for finding reductions leads to an over-determined nonlinear system of partial differential equations for finding the forms of the coefficient functions $\tau(x,t,u,v)$, $\xi(x,t,u,v)$, $\eta(x,t,u,v)$ and $\mu(x,t,u,v)$ while in the case of classical Lie method the corresponding system consists of linear partial differential equations. Obviously, the derivation of non-classical reductions is not an easy task.

For evolution equations there exist two principally different cases of finding the operator Γ, $\tau \neq 0$ and $\tau = 0$. In the present chapter, we consider the case with $\tau \neq 0$ and without loss of generality we can assume that $\tau = 1$. When $\tau = 0$ (known as "no-go" case), we can take without loss of generality $\xi = 1$.

We apply the second extension of the operator Γ to the system (6.4) and we eliminate the derivatives u_{xx}, v_{xx} using the system and u_t, v_t from the invariant surface conditions. The result is a polynomial in the variables u_x, v_x. The coefficients of various powers of these variables give the following determining system

$$\begin{aligned}
&\xi_{uu} = \xi_{vv} = \xi_{uv} = 0,\\
&\eta_{uu} - 2\xi_{ux} + 2\xi\xi_u + 2\xi_u f + \epsilon_2 u\xi_v = 0,\\
&\mu_{vv} - 2\xi_{vx} + 2\xi\xi_v + 2\xi_v k + \epsilon_1 v\xi_u = 0,\\
&\eta_{vv} + \epsilon_1 v\xi_v = 0,\\
&\mu_{uu} + \epsilon_2 u\xi_u = 0,\\
&2\eta_{uv} - 2\xi_{vx} + 2\epsilon_1 v\xi_u + 2\xi\xi_v + \xi_v f + \xi_v k = 0,\\
&2\mu_{uv} - 2\xi_{ux} + 2\epsilon_2 u\xi_v + 2\xi\xi_u + \xi_u f + \xi_u k = 0,\\
&2\eta_{ux} - \epsilon_2 u\eta_v + \epsilon_1 v\mu_u - 2\eta\xi_u - \xi_{xx} + 2\xi\xi_x + \xi_x f + \eta f_u + \mu f_v + \xi_t = 0,\\
&2\mu_{vx} - \epsilon_1 v\mu_u + \epsilon_2 u\eta_v - 2\mu\xi_v - \xi_{xx} + 2\xi\xi_x + \xi_x k + \eta k_u + \mu k_v + \xi_t = 0,\\
&2\eta_{vx} - \epsilon_1 v\eta_u + \epsilon_1 v\mu_v + \eta_v f - \eta_v k - 2\eta\xi_v + \epsilon_1 v\xi_x + \epsilon_1\mu = 0,\\
&2\mu_{ux} - \epsilon_2 u\mu_v + \epsilon_2 u\eta_u + \mu_u k - \mu_u f - 2\mu\xi_u + \epsilon_2 u\xi_x + \epsilon_2\eta = 0,\\
&\eta_t - \eta_{xx} - \eta_x f - \epsilon_1 v\mu_x + 2\eta\xi_x = 0,\\
&\mu_t - \mu_{xx} - \mu_x k - \epsilon_2 u\eta_x + 2\mu\xi_x = 0.
\end{aligned}$$

The solution of the determining system provides the forms of the functions $f(u, v)$, $k(u, v)$ and also the forms of the coefficient functions $\xi(x, t, u, v)$, $\eta(x, t, u, v)$, $\mu(x, t, u, v)$. We point out that every Lie symmetry generator is also a non-classical generator. Hence, our task is to find reductions that are not equivalent to Lie symmetry reductions. For the system (6.5) the corresponding determining system coincides with the above with $\epsilon_1 = \epsilon_2 = 0$.

Here we present two examples of nonclassical reductions and the complete classification will be considered in a separate work. The system

$$u_t = u_{xx} - \epsilon_1 \frac{v^2}{u} u_x + \epsilon_1 v v_x, \quad v_t = v_{xx} - \epsilon_2 \frac{u^2}{v} v_x + \epsilon_2 u u_x,$$

admits the nonclassical generator

$$X = \partial_t - \frac{3}{x}\partial_x - \frac{3u}{x^2}\partial_u - \frac{3v}{x^2}\partial_v,$$

which produces the similarity mapping

$$u = x\phi(\zeta), \quad v = x\psi(\zeta), \quad \zeta = x^2 + 6t,$$

that transforms the system into the system of ordinary differential equations

$$2\phi\phi'' - \epsilon_1\psi^2\phi' + \epsilon_1\phi\psi\psi' = 0, \quad 2\psi\psi'' - \epsilon_2\phi^2\psi' + \epsilon_2\phi\psi\phi' = 0.$$

This example is analogue to the one presented in [4].

The system

$$u_t = u_{xx} + (v + \epsilon u)u_x, \quad v_t = v_{xx} + (v + \epsilon u)v_x,$$

admits the nonclassical generator

$$X = \partial_t - (v + \epsilon u)\partial_x.$$

As we have seen earlier, the above system with $\epsilon \neq 0$ is equivalent to the system with $\epsilon = 0$. In the case $\epsilon = 0$, we have the implicit similarity reduction

$$u = \phi(\zeta), \quad v = \psi(\zeta), \quad \zeta = tv + x,$$

that reduces the system into the simple linear system

$$\phi'' = 0, \quad \psi'' = 0,$$

which gives the solutions

$$u = \frac{c_3 x + (c_2 c_3 - c_1 c_4)t + c_4}{1 - c_1 t}, \quad v = \frac{c_1 x + c_2}{1 - c_1 t}.$$

6.5 A linearizing Burgers system

It is well known that the Hopf-Cole transformation connects Burgers equation with the linear heat equation. In references [4, 29] the following Hopf-Cole-type mapping appears

$$u' = \frac{u_x}{u}, \quad v' = v_x - \frac{u_x}{u}v,$$

which maps the nonlinear system

$$u'_{t'} = u'_{x'x'} + 2u'u'_{x'}, \quad v'_{t'} = v'_{x'x'} + 2v'u'_{x'},$$

into the linear system

$$u_t = u_{xx}, \quad v_t = v_{xx}. \tag{6.51}$$

Here we present a similar example for the general class (6.5). In particular, the nonlinear system

$$u'_{t'} = u'_{x'x'} + u'u'_{x'}, \quad v'_{t'} = v'_{x'x'} + u'v'_{x'},$$

(which admits 9 Lie symmetries) is connected with the linear system (6.51) under the Hopf-Cole-type mapping

$$u' = \frac{2u_x}{u}, \quad v' = \frac{v}{u}. \tag{6.52}$$

The idea for deriving such linearizing mappings for general classes of Burgers-type systems, as for diffusion-type systems [44, 87, 88], will be considered in a separate work in the near future.

Chapter 7

Symmetry analysis of two-dimensional Burgers system

7.1 Introduction

In the last decades a lot of attention has been paid to study the various forms of Burgers equations [82]. If we ignore the pressure gradient terms from the incompressible Navier-Stokes equations, we obtain the nonlinear system

$$\begin{aligned} &u_t + uu_x + vu_y - \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}) = 0, \\ &v_t + uv_x + vv_y - \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}) = 0 \end{aligned} \tag{7.1}$$

which is known as the two-dimensional Burgers system, where Re is the Reynolds number. We point out that the solution of the system (7.1) will not, necessarily, satisfy the continuity equation. The system (7.1) has been considered, for example, in [34,83] where certain underlying geometric and group theoretical properties were discussed.

It is well known that the Hopf-Cole transformation relates the Burgers equation and the linear heat equation [18, 35]. This transformation can be generalized for multi-dimensional equations [3, 18]. In two dimensions the generalization of Hopf-Cole transformation is

$$u = -\frac{2}{\mathrm{Re}}\frac{\phi_x}{\phi}, \quad v = -\frac{2}{\mathrm{Re}}\frac{\phi_y}{\phi}.$$

which relates the system (7.1) with the additional constraint $u_y = v_x$ and the system

$$\phi_t - \phi_{xx} - \phi_{yy} = 0.$$

Solutions of this linear equation provide solutions of system (7.1) with the use of the Hopf-Cole transformation.

Transformation properties of evolution equations and systems have been widely studied because of the many practical benefits that such knowledge provides and also because of the variety of physical applications for which these equations are model equations. The knowledge of the Lie group of point symmetries is particularly useful in the study of a partial differential equation. While there is no existing general theory for solving nonlinear PDEs, these methods have proved to be very powerful. The Lie group analysis of system (7.1) has been studied by various authors [1, 25, 91]. Although the derivation of Lie symmetries is accurate, the analysis on reductions of the system (7.1) is incomplete. For example, in [25] an optimal system is presented where two Lie symmetries admitted by the system are missing.

We present the complete list of similarity reductions. This goal can be achieved by constructing the optimal system either of one or two-dimensional subalgebras of its Lie symmetry algebra. The optimal system of two-dimensional subalgebras enables us to reduce the system (7.1) directly to system of ordinary differential equations. The optimal system of one-dimensional subalgebras leads to reductions where the reduced systems consist of PDEs in two independent variables. Then, we derive the Lie symmetries for each reduced system and consequently we construct the corresponding optimal system which is used to have the second reduction. Although the second approach involves of more calculations, it has the advantage of unfolding possible missing (hidden) symmetries.

The results of the present chapter, appear in [51].

7.2 Lie invariance algebra and complete point symmetry group

The classical approach for deriving Lie symmetries is well known and established, see for example in references [13, 68, 72]. Firstly, we note that the point transformation

$$x' = \frac{x}{\sqrt{\mathrm{Re}}}, \quad y' = \frac{y}{\sqrt{\mathrm{Re}}}, \quad t' = t, \quad u' = \frac{u}{\sqrt{\mathrm{Re}}}, \quad v' = \frac{v}{\sqrt{\mathrm{Re}}},$$

maps (7.1) into

$$\begin{aligned} &u_t + uu_x + vu_y - (u_{xx} + u_{yy}) = 0, \\ &v_t + uv_x + vv_y - (v_{xx} + v_{yy}) = 0. \end{aligned} \tag{7.2}$$

Therefore, without loss of generality we can take $\mathrm{Re} = 1$.

The Lie symmetry algebra of the system (7.2) can be found in [1, 25, 91]. The maximal Lie invariance algebra of the Burgers system is the so-called reduced (i.e., centerless) special Galilei algebra [28] with space dimension two

$$A^{max} = \langle X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 \rangle,$$

where

$$X_1 = \partial_t, \quad X_2 = 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - v\partial_v,$$

$$X_3 = t^2\partial_t + tx\partial_x + ty\partial_y + (x - tu)\partial_u + (y - tv)\partial_v, \quad X_4 = x\partial_y - y\partial_x + u\partial_v - v\partial_u,$$

$$X_5 = \partial_x, \quad X_6 = \partial_y, \quad X_7 = t\partial_x + \partial_u, \quad X_8 = t\partial_y + \partial_v.$$

The commutation relations of A^{max} and the adjoint actions for the Lie algebra of the system (7.2), are given in Tables 7.1 and 7.2, respectively.

Table 7.1: Commutation relations of A^{max} of the system (7.2)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$2X_1$	X_2	0	0	0	X_5	X_6
X_2	$-2X_1$	0	$2X_3$	0	$-X_5$	$-X_6$	X_7	X_8
X_3	$-X_2$	$-2X_3$	0	0	$-X_7$	$-X_8$	0	0
X_4	0	0	0	0	$-X_6$	X_5	$-X_8$	X_7
X_5	0	X_5	X_7	X_6	0	0	0	0
X_6	0	X_6	X_8	$-X_5$	0	0	0	0
X_7	$-X_5$	$-X_7$	0	X_8	0	0	0	0
X_8	$-X_6$	$-X_8$	0	$-X_7$	0	0	0	0

Table 7.2: Adjoint actions for the Lie algebra of the system (7.2)

Ad	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	X_1	$X_2 - 2\epsilon X_1$	$X_3 - \epsilon X_2 + \epsilon^2 X_1$	X_4	X_5	X_6	$X_7 - \epsilon X_5$	$X_8 - \epsilon X_6$
X_2	$e^{2\epsilon} X_1$	X_2	$e^{-2\epsilon} X_3$	X_4	$e^{\epsilon} X_5$	$e^{\epsilon} X_6$	$e^{-\epsilon} X_7$	$e^{-\epsilon} X_8$
X_3	$X_1 + \epsilon X_2 + \epsilon^2 X_3$	$X_2 + 2\epsilon X_3$	X_3	X_4	$X_5 + \epsilon X_7$	$X_6 + \epsilon X_8$	X_7	X_8
X_4	X_1	X_2	X_3	X_4	$\cos\epsilon X_5 + \sin\epsilon X_6$	$-\sin\epsilon X_5 + \cos\epsilon X_6$	$\cos\epsilon X_7 + \sin\epsilon X_8$	$-\sin\epsilon X_7 + \cos\epsilon X_8$
X_5	X_1	$X_2 - \epsilon X_5$	$X_3 - \epsilon X_7$	$X_4 - \epsilon X_6$	X_5	X_6	X_7	X_8
X_6	X_1	$X_2 - \epsilon X_6$	$X_3 - \epsilon X_8$	$X_4 + \epsilon X_5$	X_5	X_6	X_7	X_8
X_7	$X_1 + \epsilon X_5$	$X_2 + \epsilon X_7$	X_3	$X_4 - \epsilon X_8$	X_5	X_6	X_7	X_8
X_8	$X_1 + \epsilon X_6$	$X_2 + \epsilon X_8$	X_3	$X_4 + \epsilon X_7$	X_5	X_6	X_7	X_8

7.3 Subalgebras of Lie invariance algebra

The classification of subalgebras of Galilei algebras was considered in a number of papers, see [5,6,28] and references therein. We have listed inequivalent subalgebras of A^{max} from the very beginning and compare the obtained list with the list presented in [28].

We classify subalgebras of the algebra A^{max}, up to the equivalence relation generated by the induced adjoint action of the point symmetry group G of the Burgers system on A^{max}. One-dimensional inequivalent subalgebras: In the case when the Lie algebra is solvable, we need to start with the general element of Lie symmetry algebra,

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6 + a_7X_7 + a_8X_8.$$

From Table 7.1 we conclude that the Lie algebra is not solvable, so we cannot make use of the above procedure. In our case we use the Levi decomposition of the specific algebra.

$$A^{max} = \langle X_1, X_2, X_3\rangle \Subset \langle X_4, X_5, X_6, X_7, X_8\rangle,$$

where the subalgebra

$$\langle X_1, X_2, X_3\rangle$$

is called the Levi factor of A^{max} and the subalgebra

$$\langle X_4, X_5, X_6, X_7, X_8\rangle$$

is a radical of A^{max}.

From [68] we find that the optimal system of the Levi factor is

$$\{0\}, \{X_1\}, \{X_2\}, \{X_1 + X_3\}.$$

For each element we add the tail

$$a_4X_4 + a_5X_5 + a_6X_6 + a_7X_7 + a_8X_8$$

and we use the table with the adjoint actions 7.2 to construct the desired optimal system. We obtain the following independent elements: In the case of $\{X_1\}$ we obtain $A^{max\ 1.1}_{\kappa}$, $A^{max\ 1.2}$. The second element leads to $A^{max\ 1.3}_{\kappa}$. The element $\{X_1 + X_3\}$ provides us with the results $A^{max\ 1.4}_{\kappa}$, $A^{max\ 1.5}_{\mu}$. Finally, the element $\{0\}$ gives $A^{max\ 1.6}$, $A^{max\ 1.7}$, $A^{max\ 1.8}$. Below, we list all the derived elements after the application of the adjoint actions.

$$A^{max\ 1.1}_{\kappa} = \langle X_1 + \kappa X_4\rangle_{\kappa\in\{0,1\}}, \quad A^{max\ 1.2} = \langle X_1 + X_8\rangle, \quad A^{max\ 1.3}_{\kappa} = \langle X_2 + 2\kappa X_4\rangle_{\kappa\geqslant 0},$$
$$A^{max\ 1.4}_{\kappa} = \langle X_1 + X_3 + \kappa X_4\rangle_{\kappa\geqslant 0}, \quad A^{max\ 1.5}_{\mu} = \langle X_1 + X_3 + X_4 + \mu(X_7 - X_6)\rangle_{\mu>0},$$
$$A^{max\ 1.6} = \langle X_4\rangle, \quad A^{max\ 1.7} = \langle X_7 - X_6\rangle, \quad A^{max\ 1.8} = \langle X_6\rangle.$$

Two-dimensional inequivalent subalgebras consist of the following: [79]

$$
\begin{aligned}
&A_{\kappa}^{max\ 2.1} = \langle X_1, X_2 + \kappa X_4 \rangle_{\kappa \geqslant 0}, \quad A^{max\ 2.2} = \langle X_1, X_4 \rangle, \quad A^{max\ 2.3} = \langle X_2, X_4 \rangle, \\
&A^{max\ 2.4} = \langle X_1 + X_3, X_4 \rangle, \quad A_{\mu}^{max\ 2.5} = \langle X_1 + X_3 + X_4 + \mu(X_8 + X_5), X_7 - X_6 \rangle_{\mu \geqslant 0}, \\
&A_{\mu}^{max\ 2.6} = \langle X_7 - X_6, X_8 + \mu X_5 \rangle_{\mu > 0}, \\
&A_{\mu\nu}^{max\ 2.7} = \langle X_6, X_1 + \mu X_7 + \nu X_8 \rangle_{\mu,\nu \geqslant 0,\ \mu^2+\nu^2 \in \{0,1\}}, \\
&A^{max\ 2.8} = \langle X_6, X_2 \rangle, \quad A^{max\ 2.9} = \langle X_6, X_5 \rangle, \quad A^{max\ 2.10} = \langle X_6, X_8 \rangle, \\
&A_{\mu}^{max\ 2.11} = \langle X_6, X_7 + \mu X_8 \rangle_{\mu \geqslant 0}, \quad A^{max\ 2.12} = \langle X_6, X_8 + X_5 \rangle.
\end{aligned}
$$

The two-dimensional subalgebras can be used to reduce the initial system (7.2), directly to systems of ODEs. In the case of one-dimensional subalgebras, the system (7.2), is reduced to a system of PDEs in two independent variables. The next step is to determine the Lie symmetries of the reduced systems, which lead to similarity reductions that transform these systems to systems of ODEs. Clearly, the second choice is lengthier. However, in this way it is possible to unfold missing symmetries of the reduced systems of PDEs. The definition and theory of missing symmetries can be found, for example, in [13].

7.4 Lie reductions of codimension one

Ansatzes constructed with one-dimensional subalgebras of A^{max} reduce the system (7.2) to systems of two partial differential equations in two independent variables. This can be achieved, by solving the appropriate invariant surface conditions

$$\tau(x,y,t,u,v)u_t + \xi^x(x,y,t,u,v)u_x + \xi^y(x,y,t,u,v)u_y = \eta(x,y,t,u,v),$$

$$\tau(x,y,t,u,v)v_t + \xi^x(x,y,t,u,v)v_x + \xi^y(x,y,t,u,v)v_y = \mu(x,y,t,u,v),$$

which correspond to the symmetry generator

$$\Gamma = \tau\frac{\partial}{\partial t} + \xi^x\frac{\partial}{\partial x} + \xi^y\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \mu\frac{\partial}{\partial v}.$$

Below, for each of the one-dimensional subalgebras listed in the previous section, we present an ansatz constructed for (u,v) with this subalgebra and the corresponding reduced system. Here

$$\phi = \phi\left(\xi,\eta\right),\ \psi = \psi\left(\xi,\eta\right),$$

are new unknown functions of the invariant independent variables (ξ,η).

1.1. $A^{max\ 1.1}_{\kappa} = \langle X_1 + \kappa X_4 \rangle_{\kappa \in \{0,1\}}$:

$$u = \phi\cos\tau - \psi\sin\tau - \kappa y,$$
$$v = \phi\sin\tau + \psi\cos\tau + \kappa x,$$

where $\quad \xi = x\cos\tau + y\sin\tau, \quad \eta = -x\sin\tau + y\cos\tau, \quad \tau := \kappa t;$

$$\phi\phi_\xi + \psi\phi_\eta - \phi_{\xi\xi} - \phi_{\eta\eta} - 2\kappa\psi - \kappa\xi = 0,$$
$$\phi\psi_\xi + \psi\psi_\eta - \psi_{\xi\xi} - \psi_{\eta\eta} + 2\kappa\phi - \kappa\eta = 0.$$

1.2. $A^{max\ 1.2} = \langle X_1 + X_8 \rangle$: $\quad u = \phi, \quad v = \psi + t, \quad$ where $\quad \xi = x, \quad \eta = y - \dfrac{t^2}{2};$

$$\phi\phi_\xi + \psi\phi_\eta - \phi_{\xi\xi} - \phi_{\eta\eta} = 0,$$
$$\phi\psi_\xi + \psi\psi_\eta - \psi_{\xi\xi} - \psi_{\eta\eta} + 1 = 0.$$

1.3. $A^{max\ 1.3}_{\kappa} = \langle X_2 + 2\kappa X_4 \rangle_{\kappa \geqslant 0}$:

$$u = \frac{1}{\sqrt{|t|}}(\phi\cos\tau - \psi\sin\tau) + \frac{x}{2t} - \kappa\frac{y}{t},$$
$$v = \frac{1}{\sqrt{|t|}}(\phi\sin\tau + \psi\cos\tau) + \frac{y}{2t} + \kappa\frac{x}{t},$$

where $\quad \xi = \dfrac{1}{\sqrt{|t|}}(x\cos\tau + y\sin\tau), \quad \eta = \dfrac{1}{\sqrt{|t|}}(-x\sin\tau + y\cos\tau), \quad \tau := \kappa\ln|t|;$

$$\phi\phi_\xi + \psi\phi_\eta - \phi_{\xi\xi} - \phi_{\eta\eta} - 2\hat\kappa\psi - \left(\kappa^2 + \frac{1}{4}\right)\xi = 0,$$
$$\phi\psi_\xi + \psi\psi_\eta - \psi_{\xi\xi} - \psi_{\eta\eta} + 2\hat\kappa\phi - \left(\kappa^2 + \frac{1}{4}\right)\eta = 0, \quad (\hat\kappa := \kappa\,\mathrm{sgn}\,t).$$

1.4. $A^{max\ 1.4}_{\kappa} = \langle X_1 + X_3 + \kappa X_4 \rangle_{\kappa \geqslant 0}$:

$$u = \frac{1}{\sqrt{t^2+1}}(\phi\cos\tau - \psi\sin\tau) + \frac{tx}{t^2+1} - \frac{\kappa y}{t^2+1},$$
$$v = \frac{1}{\sqrt{t^2+1}}(\phi\sin\tau + \psi\cos\tau) + \frac{ty}{t^2+1} + \frac{\kappa x}{t^2+1},$$

where $\quad \xi = \dfrac{1}{\sqrt{t^2+1}}(x\cos\tau + y\sin\tau), \quad \eta = \dfrac{1}{\sqrt{t^2+1}}(-x\sin\tau + y\cos\tau), \quad \tau := \kappa\tan^{-1}t;$

$$\phi\phi_\xi + \psi\phi_\eta - \phi_{\xi\xi} - \phi_{\eta\eta} - 2\kappa\psi + (1-\kappa^2)\xi = 0,$$
$$\phi\psi_\xi + \psi\psi_\eta - \psi_{\xi\xi} - \psi_{\eta\eta} + 2\kappa\phi + (1-\kappa^2)\eta = 0.$$

1.5. $A^{max\ 1.5}_{\mu} = \langle X_1 + X_3 + X_4 + \mu(X_7 - X_6)\rangle_{\mu>0}$:

$$u = \frac{t\phi + \psi}{t^2+1} + \frac{t(x+\mu)}{t^2+1} - \frac{y}{t^2+1},$$
$$v = \frac{-\phi + t\psi}{t^2+1} + \frac{ty}{t^2+1} + \frac{x-\mu}{t^2+1},$$

where $\xi = \dfrac{tx-y}{t^2+1} - \mu \tan^{-1} t, \quad \eta = \dfrac{x+ty}{t^2+1}$;

$$\phi\phi_\xi + \psi\phi_\eta - \phi_{\xi\xi} - \phi_{\eta\eta} - 2\psi = 0,$$

$$\phi\psi_\xi + \psi\psi_\eta - \psi_{\xi\xi} - \psi_{\eta\eta} + 2\phi + 2\mu = 0.$$

1.6. $A^{max\ 1.6} = \langle X_4\rangle$: $\quad u = \dfrac{x}{r}\phi - \dfrac{y}{r}\psi + \dfrac{x}{r^2}, \quad v = \dfrac{y}{r}\phi + \dfrac{x}{r}\psi + \dfrac{y}{r^2}$,
where $\xi = t, \;\; \eta = r := \sqrt{x^2+y^2}$;

$$\phi_\xi + \phi\phi_\eta - \phi_{\eta\eta} - \frac{\psi^2}{\eta} - \frac{1}{\eta^3} = 0,$$
$$\psi_\xi + \phi\psi_\eta - \psi_{\eta\eta} + \frac{\phi\psi}{\eta} + 2\frac{\psi}{\eta^2} = 0.$$

1.7. $A^{max\ 1.7} = \langle X_7 - X_6\rangle$: $\quad u = \dfrac{\phi - t\psi + tx - y}{t^2+1}, \quad v = \dfrac{t\phi + \psi + x + ty}{t^2+1}$,
where $\xi = \tan^{-1} t, \quad \eta = \dfrac{x+ty}{t^2+1}$;

$$\phi_\xi + \phi\phi_\eta - \phi_{\eta\eta} - 2\psi = 0,$$

$$\psi_\xi + \phi\psi_\eta - \psi_{\eta\eta} + 2\phi = 0.$$

1.8. $A^{max\ 1.8} = \langle X_6\rangle$: $\quad u = \phi, \quad v = \psi, \quad$ where $\quad \xi = t, \quad \eta = x$;

$$\phi_\xi + \phi\phi_\eta - \phi_{\eta\eta} = 0,$$

$$\psi_\xi + \phi\psi_\eta - \psi_{\eta\eta} = 0.$$

The linearizing mapping for this reduced system is given by the Hopf-Cole-type transformation

$$\phi = -2\frac{A_x}{A}, \quad \psi = \frac{B}{A},$$

where the functions A and B satisfy the linear heat equation

$$\alpha_t - \alpha_{xx} = 0.$$

In the next section, we classify the Lie symmetries of the above eight reduced systems.

7.5 Lie symmetries of reduced systems of PDEs

We have selected ansatzes in such a way that the reduced systems are quite simple and can be grouped into two sets depending on their structure, which is convenient for studying their symmetries and finding exact solutions.

We note that the reduced systems 1.1 - 1.5 are of the general form

$$\begin{aligned}
&\phi\phi_\xi + \psi\phi_\eta - \phi_{\xi\xi} - \phi_{\eta\eta} - 2\kappa\psi + \alpha\xi = 0,\\
&\phi\psi_\xi + \psi\psi_\eta - \psi_{\xi\xi} - \psi_{\eta\eta} + 2\kappa\phi + \alpha\eta + \beta = 0,
\end{aligned}$$

where κ, α and β are constants with $\alpha\beta = 0$. We investigate the group classification of the above system. We find that, depending on the values of these parameters, a system of the above form admits the following maximal Lie invariance algebra B:

$$\begin{aligned}
&\alpha \neq 0,\ \beta = 0\colon && B = \langle Y_3\rangle,\\
&\alpha = 0,\ \beta \neq 0\colon && B = \langle Y_1, Y_2\rangle,\\
&\alpha = \beta = 0,\ \kappa \neq 0\colon && B = \langle Y_1, Y_2, Y_3\rangle,\\
&\alpha = \beta = \kappa = 0\colon && B = \langle Y_1, Y_2, Y_3, Y_4\rangle,
\end{aligned}$$

where

$$\begin{aligned}
&Y_1 = \partial_\xi, \quad Y_2 = \partial_\eta, \quad Y_3 = \xi\partial_\eta - \eta\partial_\xi + \phi\partial_\psi - \psi\partial_\phi,\\
&Y_4 = \xi\partial_\xi + \eta\partial_\eta - \phi\partial_\phi - \psi\partial_\psi.
\end{aligned}$$

As a result, the maximal Lie invariance algebras of reduced systems 1.1 - 1.5 are respectively

$$\begin{aligned}
&B^1 = \langle Y_3\rangle \quad \text{if} \quad \kappa = 1 \qquad \text{and} \qquad B^1 = \langle Y_1, Y_2, Y_3, Y_4\rangle \quad \text{if} \quad \kappa = 0,\\
&B^2 = \langle Y_1, Y_2\rangle, \qquad B^3 = \langle Y_3\rangle,\\
&B^4 = \langle Y_3\rangle \quad \text{if} \quad \kappa \neq 1 \qquad \text{and} \qquad B^4 = \langle Y_1, Y_2, Y_3\rangle \quad \text{if} \quad \kappa = 1,\\
&B^5 = \langle Y_1, Y_2\rangle.
\end{aligned}$$

The other three reduced systems are of the form

$$\begin{aligned}
&\phi_\xi + \phi\phi_\eta - \phi_{\eta\eta} + F(\xi, \eta, \phi, \psi) = 0,\\
&\psi_\xi + \phi\psi_\eta - \psi_{\eta\eta} + G(\xi, \eta, \phi, \psi) = 0,
\end{aligned}$$

where the parameter-functions $F = F(\xi, \eta, \phi, \psi)$, $G = G(\xi, \eta, \phi, \psi)$ are at most quadratic in (ϕ, ψ). The maximal Lie invariance algebras of the reduced systems 1.6 - 1.8 have the following structure:

$$B^6 = \langle \partial_\xi,\ 2\xi\partial_\xi + \eta\partial_\eta - \phi\partial_\phi - \psi\partial_\psi,$$
$$\xi^2\partial_\xi + \xi\eta\partial_\eta + (\eta - \xi\phi)\partial_\phi - \xi\psi\partial_\psi\rangle,$$
$$B^7 = \langle \partial_\xi,\ \partial_\eta,\ \cos(2\xi)\partial_\eta - 2\sin(2\xi)\partial_\phi - 2\cos(2\xi)\partial_\psi,$$
$$\sin(2\xi)\partial_\eta + 2\cos(2\xi)\partial_\phi - 2\sin(2\xi)\partial_\psi\rangle,$$
$$B^8 = \langle \partial_\xi,\ 2\xi\partial_\xi + \eta\partial_\eta - \phi\partial_\phi - \psi\partial_\psi,\ \xi^2\partial_\xi + \xi\eta\partial_\eta + (\eta - \xi\phi)\partial_\phi,$$
$$\partial_\eta,\ \xi\partial_\eta + \partial_\phi,\ \partial_\psi,\ \phi\partial_\psi,\ (\eta - \xi\phi)\partial_\psi,\ \psi\partial_\psi\rangle.$$

The question which arises is whether the reduced systems, which correspond to cases $A^{max\ 1.1}_\kappa - A^{max\ 1.8}$, admit hidden (missing) symmetries. The investigation showed that only the linearizable case $A^{max\ 1.8}$, admits hidden symmetries. Since cases $A^{max\ 1.1}_\kappa - A^{max\ 1.7}$ do not admit hidden symmetries, we use two-dimensional subalgebras to reduce the system (7.2), to systems of ODEs. This analysis is carried out in the next section.

7.6 Lie reductions of codimension two

Since the reduced system constructed with the subalgebra $A^{max\ 1.8} = \langle X_6\rangle$ is linearizable, a two-dimensional subalgebra of A^{max} is significant for use in the course of reducing of the system (7.2) only if it does not contain the vector field X_6 or, more generally, a vector field equivalent to X_6. Therefore, only the subalgebras $A^{max\ 2.1}_\kappa$–$A^{max\ 2.6}_\mu$ are significant for Lie reduction among the listed two-dimensional inequivalent subalgebras. Below, for each of these subalgebras, we present an ansatz constructed for (u, v) and the corresponding reduced system. Here $\phi = \phi(\xi)$, $\psi = \psi(\xi)$, are new unknown functions of the invariant independent variable ξ, and $r := \sqrt{x^2 + y^2}$.

2.1. $A^{max\ 2.1}_\kappa = \langle X_1, X_2 + \kappa X_4\rangle_{\kappa\geqslant 0}$:

$$u = \frac{x}{r^2}\phi - \frac{y}{r^2}\psi,\quad v = \frac{y}{r^2}\phi + \frac{x}{r^2}\psi,\quad \text{where}\quad \xi = \tan^{-1}\frac{y}{x} - \kappa\ln r;$$
$$(\psi - \kappa\phi - 2\kappa)\phi' - (\kappa^2+1)\phi'' + 2\psi' - \phi^2 - \psi^2 = 0,$$
$$(\psi - \kappa\phi - 2\kappa)\psi' - (\kappa^2+1)\psi'' - 2\phi' = 0.$$

2.2. $A^{max\ 2.2} = \langle X_1, X_4\rangle$:

$$u = \frac{x\phi - y\psi}{r} + \frac{x}{r^2},\quad v = \frac{y\phi + x\psi}{r} + \frac{y}{r^2},\quad \text{where}\quad \xi = r;$$
$$\phi\phi' - \phi'' - \frac{\psi^2}{\xi} - \frac{1}{\xi^3} = 0,$$
$$\phi\psi' - \psi'' + \frac{\phi\psi}{\xi} + 2\frac{\psi}{\xi^2} = 0.$$

2.3. $A^{max\ 2.3} = \langle X_2, X_4 \rangle$:

$$u = \frac{x\phi - y\psi}{r\sqrt{|t|}} + \frac{x}{r^2} + \frac{x}{2t}, \quad v = \frac{y\phi + x\psi}{r\sqrt{|t|}} + \frac{y}{r^2} + \frac{y}{2t}, \quad \text{where} \quad \xi = \frac{r}{\sqrt{|t|}};$$

$$\phi\phi' - \phi'' - \frac{\psi^2}{\xi} - \frac{1}{\xi^3} - \frac{\xi}{4} = 0,$$

$$\phi\psi' - \psi'' + \frac{\phi\psi}{\xi} + 2\frac{\psi}{\xi^2} = 0.$$

2.4. $A^{max\ 2.4} = \langle X_1 + X_3, X_4 \rangle$:

$$u = \frac{x\phi - y\psi}{r\sqrt{t^2+1}} + \frac{x}{r^2} + \frac{tx}{t^2+1}, \quad v = \frac{y\phi + x\psi}{r\sqrt{t^2+1}} + \frac{y}{r^2} + \frac{ty}{t^2+1}, \quad \text{where} \quad \xi = \frac{r}{\sqrt{t^2+1}};$$

$$\phi\phi' - \phi'' - \frac{\psi^2}{\xi} - \frac{1}{\xi^3} + \xi = 0,$$

$$\phi\psi' - \psi'' + \frac{\phi\psi}{\xi} + 2\frac{\psi}{\xi^2} = 0.$$

2.5. $A_\mu^{max\ 2.5} = \langle X_1 + X_3 + X_4 + \mu(X_8 + X_5), X_7 - X_6 \rangle_{\mu \geqslant 0}$:

$$u = \frac{\phi - t\psi + tx - y + \mu}{t^2+1}, \quad v = \frac{t\phi + \psi + x + ty + \mu t}{t^2+1},$$

$$\text{where} \quad \xi = \frac{x + ty}{t^2+1} - \mu \tan^{-1} t;$$

$$\phi\phi' - \phi'' - 2\psi = 0,$$

$$\phi\psi' - \psi'' + 2\phi + 2\mu = 0.$$

2.6. $A_\mu^{max\ 2.6} = \langle X_7 - X_6), X_8 + \mu X_5 \rangle_{\mu > 0}$:

$$u = \frac{t\phi - \mu\psi + tx - \mu y}{t^2+\mu}, \quad v = \frac{\phi + t\psi + x + ty}{t^2+\mu}, \quad \text{where} \quad \xi = t;$$

$$\phi' = 0, \quad \psi' = 0.$$

The maximal Lie invariance algebras of the above systems of ODEs are the following:

2.1. $\langle \partial_\xi \rangle$; 2.2. $\langle \xi\partial_\xi - \phi\partial_\phi - \psi\partial_\psi \rangle$; 2.3. $\{0\}$; 2.4. $\{0\}$; 2.5. $\langle \partial_\xi \rangle$;

2.6. $\langle \alpha(\xi, \phi, \psi)\partial_\xi + \beta(\phi, \psi)\partial_\phi + \gamma(\phi, \psi)\partial_\psi \rangle$,

where α, β and γ run through the sets of smooth functions of their arguments.

Therefore, all Lie symmetries of the significant reduced systems of ODEs are induced by Lie symmetries of the original system (7.2), and thus they should not be used for the further reductions to systems of algebraic equations.

Chapter 8

Further study

The main purpose of the present thesis was to establish a-priori restrictions on the form of the coefficient functions of the symmetry generator (4.3), in order to avoid difficult and complicated calculations when group classification is needed. This problem becomes more risky, using symbolic manipulation packets, especially when arbitrary elements appear in the system of PDEs under consideration. This goal was achieved with the theorems proved in Chapter 4. The motivation of this work came from the papers of Tu [93] and Bluman [12]. Such restrictions obtained where applied in Chapters 5 and 6, to simplify the procedure of group classification of systems (5.2) and (6.4-6.5), respectively.

In this final chapter of the thesis, we list some open problems that need to be considered in the future.

The first problem which needs further investigation is whether Theorem 4.1 admits a generalization in the case when the number of PDEs of the system under consideration and the number of the dependent variables are equal. That is, if the Jacobian matrix which corresponds to the derivatives of the right-hand sides of the PDEs with respect to the highest order spatial derivatives of the dependent variables, is not a nilpotent matrix of degree two, then $\tau = \tau(t)$.

We would also like to deal with a system in which the above condition is biased. We are planning to examine a system of two PDEs with two independent and two dependent variables, where both the trace and the determinant of the aforementioned Jacobian matrix vanish. An interesting example of such a system is

$$u_t = [(u+cv)^n u_x + c(u+cv)^n v_x]_x\,, \quad v_t = -\left[\frac{1}{c}(u+cv)^n u_x + (u+cv)^n v_x\right]_x,$$

where c is nonzero constant. This is a member of the general class

$$u_t = [f(u,v)u_x + h(u,v)v_x]_x\,, \quad v_t = [k(u,v)u_x + g(u,v)v_x]_x\,,$$

and admits the Lie symmetry

$$X = \phi(u + cv)\partial_t,$$

where ϕ is an arbitrary function of $u + cv$. This is an example of system of PDEs where the coefficient function τ depends not only on t.

We, furthermore, would like to take into account systems (5.1) and (6.3). Such systems are generalizations of systems (5.2) and (6.4)-(6.5), respectively, that were investigated in two separate chapters in the thesis. The Group classification will be carried out in future papers. Moreover, the classification of potential symmetries will be investigated. For example, system

$$u_t = u_{xx} + 2uu_x, \quad v_t = v_{xx} + 2uv_x + 2vu_x, \tag{8.1}$$

which is a member of class (6.3), can be written as a system of four equations,

$$w_x = u, \quad w_t = u^2 + u_x, \quad z_x = v, \quad z_t = 2uv + v_x,$$

by introducing the potential variables w and z. This system admits 9 Lie symmetries and 2 infinite-dimensional symmetries. Eight symmetries project into Lie symmetries of the original system and the remaining symmetry

$$X_9 = v\partial_u - 2vz\partial_v + z\partial_w - z^2\partial_z$$

and the two infinite-dimensional symmetries

$$X_\alpha = (\alpha_x - \alpha u)e^{-w}\partial_u - [(\alpha_x - \alpha u)z + \alpha v]e^{-w}\partial_v + \alpha e^{-w}\partial_w - \alpha z e^{-w}\partial_z,$$

$$X_\beta = (\beta_x - \beta u)e^{-w}\partial_v + \beta e^{-w}\partial_z,$$

where $\alpha(x,t)$ and $\beta(x,t)$ are solutions of the linear heat equation.

Motivated by the Hopf-Cole transformation, we can derive similar mappings for the general system (6.3). An example of such a transformation is

$$u = \frac{\phi_x}{\phi} = (\ln\phi)_x, \quad v = \frac{\phi_x\psi - \phi\psi_x}{\phi^2} = -\left(\frac{\psi}{\phi}\right)_x,$$

which maps the system (8.1) into the linear system

$$\phi_t = \phi_{xx}, \quad \psi_t = \psi_{xx}.$$

Some other interesting open problems that need lengthier investigation contain equivalence transformations of systems of two diffusion equations with two independent and two dependent variables. Our aim is to prove that the point transformation which corresponds to t' depends only on t.

Finally, regarding Chapter 7, we search for exact solutions of the reduced systems of ODEs.

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