COMPUTER VISUALIZATION OF THE APPROXIMATIONS OF FUNCTIONS BY SEVERAL TERMS OF THEIR FOURIER SERIES

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ABSTRACT

Fourier trigonometric series are a constant component of the basic course of calculus at Universities. If we teach the topic about the Fourier series in the classical way it is often necessary to give a lot of time to a tedious mechanical performing of some operations. For a given function and a given interval, a student must count the coefficients of the Fourier series by definite integrals, to investigate the convergence of this series, to draw graphs and so on. Then a teacher with his students cannot sufficiently deal with such an important problem as the approximation of functions in a given interval by the Fourier polynomial, it means by the sum of several initial terms of the Fourier series. The reasons for it are the lack of time and the fact that these problems require more complex calculating and graphics tools for their solution. Fourier series occur very frequently in the problems of mathematical physics. The expansions of the functions can help to solve them in various physical problems. Our aim is to show on concrete simple examples that the graphical and numerical investigations Fourier series expansions by *Mathematica* commands and programs may be effectively used.

KEYWORDS
Fourier expansion, Fourier trigonometric series, *Mathematica*

INTRODUCTION

Many nature and technical phenomena have a periodic character. Fourier series are used extensively in engineering, especially for processing images and other signals. Finding the coefficients of a Fourier series is the same as doing a spectral analysis of a function. Fourier trigonometric series are expansions of periodic functions $f(x)$ on an interval $[-l, l]$, $l > 0$, in terms of an infinite sum of sines and cosines. This form is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\frac{\pi}{l}x + b_n \sin n\frac{\pi}{l}x).$$

The Fourier coefficients a_n, b_n are given by the relations

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos n\frac{\pi}{l}x\,dx, \quad n = 0,1,2,...$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin n\frac{\pi}{l}x\,dx, \quad n = 1,2,...$$

The series expansion converges to the function (equal to the original function at points of continuity or the average of the two limits at points of discontinuity)

$$\overline{f}(x_0) = \frac{1}{2}\left[\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x)\right], \text{ for } -l < x_0 < l,$$

$$= \frac{1}{2}\left[\lim_{x \to -l^+} f(x) + \lim_{x \to l^-} f(x)\right], \text{ for } x_0 = -l, l,$$

if the function *f* satisfies some sufficient conditions for convergence of Fourier series. For instance these conditions are such that *f* and *f'* are sectionally continuous in [-*l, l*] (*f* is periodic outside [-*l, l*] with a period 2*l*). In the Fourier series corresponding to an odd (respectively even) function, only sine (respectively cosine) terms can be present (Widder, 1989). A great thing about using Fourier series on periodic functions is that the first few terms are often a very good approximation to the whole function, not just the region around a special point.

USING MATHEMATICA FOR FOURIER EXPANSION

The *Mathematica* system contains a large number of built - in functions. Most versions of *Mathematica* also include a collection of standard add - on packages that define many additional functions in areas such as algebra, calculus, discrete and numerical mathematics, number theory and statistics. One of these packages provides function giving numerical approximation to Fourier transforms and also provides functions for Fourier series, Fourier coefficients and their numerical counterparts. The numerical approximations use *Mathematica*'s numerical integration and summation directly without first trying for an exact solution. The Fourier trigonometric decomposition, which expresses a periodic function as a superposition of trigonometric functions, may be computed using the function *FourierTrigSeries* (Wolfram at all). In certain circumstances we will not want exact or symbolic coefficients, but numerical approximation to the coefficients. In these cases we can use the numerical version of the Fourier series functions. If we want to demonstrate how the precision of the approximation of the function by its Fourier expansion increases in respect of the order of the used Fourier polynomial (in the given interval) we have two possibilities of visualization of it. The first way is more static. We can show successively (but divided to several different screens) the graph of the given function and the graphs of the sum of several initial terms (for example three, six and so on) of its expansion. The second possibility is to use animated graphs which better exploit the strong visual elements of Fourier analyses. The terms of the Fourier expansion are added up one by one for example by click of the mouse. Students see immediately that the more terms there are, the better the approximation is. The graphic possibilities enable to use only the first way in this contribution.

If a function *f* is given by the unique formula then its Fourier expansion can be obtained by using the function *FourierTrigSeries* for example from the additional package *Calculus`FourierTransform`* of *Mathematica* version 4.1. But a function *f* is often given by various formulas in different parts of its domain. Therefore we had to create a new function *FourierSer* for finding the Fourier expansion of *f* in these cases. The definition of this function has the following form (only for two formulas).

```
FourierSer[f1_, {x_, x1_, x2_}, f2_, {x_, x2_, x3_}, n_] :=
 Module[{a0, a},
  a0 = Integrate[f1, {x, x1, x2}] + Integrate[f2, {x, x2, x3}];
  Do[a[i] = Integrate[f1*Cos[(2*i*Pi*x)/(x3-x1)], {x, x1, x2}] +
            Integrate[f2*Cos[(2*i*Pi*x)/(x3-x1)],{x,x2,x3}];
     b[i] = Integrate[f1*Sin[(2*i*Pi*x)/(x3-x1)],{x,x1,x2}] +
            Integrate[f2*Sin[(2*i*Pi*x)/(x3-x1)],{x,x2,x3}],
            {i,1,n}];
     a0/2 + Sum[a[i]*Cos[(2*i*Pi*x)/(x3 - x1)] +
        b[i]*Sin[(2*Pi*i*x)/(x3 - x1)], {i, 1, n}]]
```

Now let us show three examples of the Fourier trigonometric expansion of typical wave functions. The commentary on every case is done in two parts. The output of the first part is the list of several initial

terms of the Fourier expansion calculated in *Mathematica*. The second part includes the graph of the given function, the graph of the sum of three or six initial terms of its Fourier series.

Example 1. Square wave function

$$f(x) = -2, \quad x \in (-2,0),$$
$$= 2, \quad x \in (0,2).$$

In[1] = FourierSer[-2, {x,-2,0}, 2, {x,0,2}, 12]

```
                Pi x            3 Pi x           5 Pi x              7 Pi x
            8 Sin[-----]    8 Sin[------]    8 Sin[--------]     8 Sin[--------]
                 2               2                2                   2
Out[1]= --------------  +  -------------------  + ------------------- +  --------------------- +
              Pi               3 Pi               5 Pi                7 Pi


                9 Pi x            11 Pi x
            8 Sin[--------]     8 Sin[---------]
                  2                  2
     +  ------------------  +   ---------------------
              9 Pi                  11 Pi
```
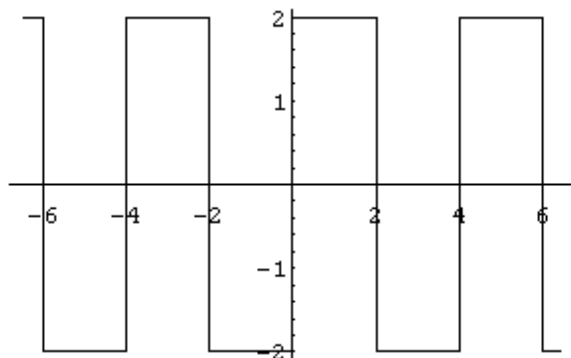

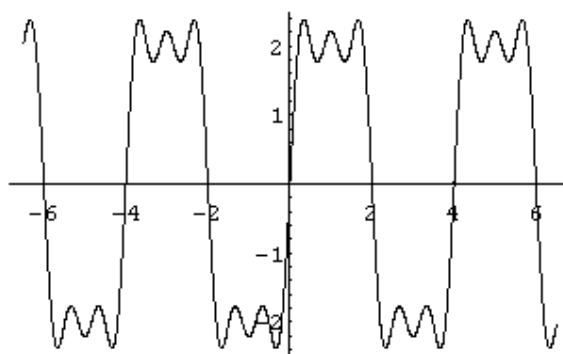Figure 1a. The graph of the given function

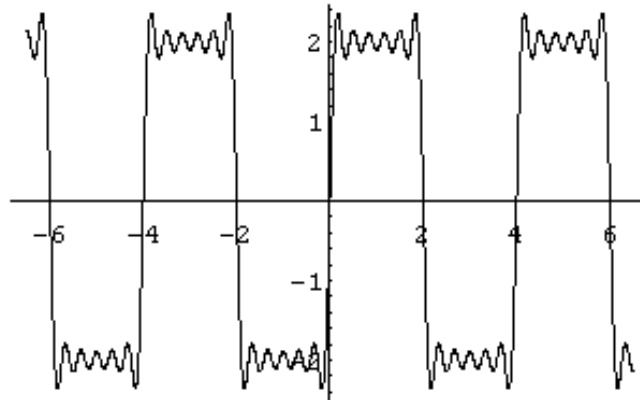

Figure 1b. The graph of the sum of three terms

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Figure 1c. The graph of the sum of six terms

Example 2. Swath (triangle) wave function

$$f(x) = x + 3, \quad x \in (-3,3).$$

In[2]= FourierTrigSeries[x+3,{x,-3,3}, 6]

```
                 Pi x              2 Pi x                                    4 Pi x
           6 Sin[----]       3 Sin[------]                             3 Sin[--------]
                  3                 3            2 Sin[Pi x]                   3
Out[2]= 3 +  --------------- – ------------------ + ---------------- – --------------- +
                  Pi                 Pi                  Pi                  2 Pi

               5 Pi x
          6 Sin[------]
                 3                  Sin[2 Pi x]
    +   -------------------- – ------------------
              5 Pi                     Pi
```
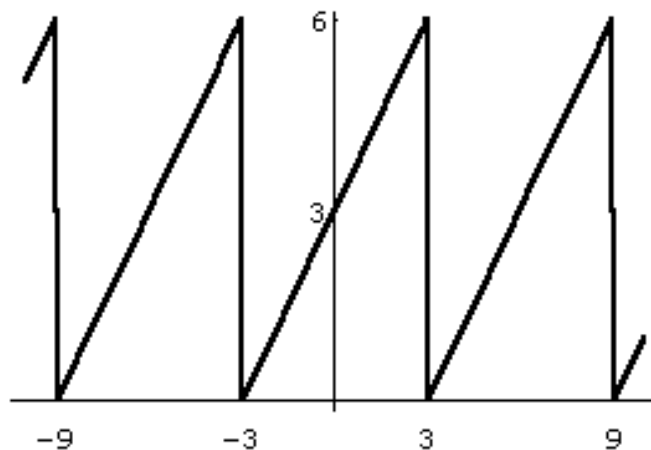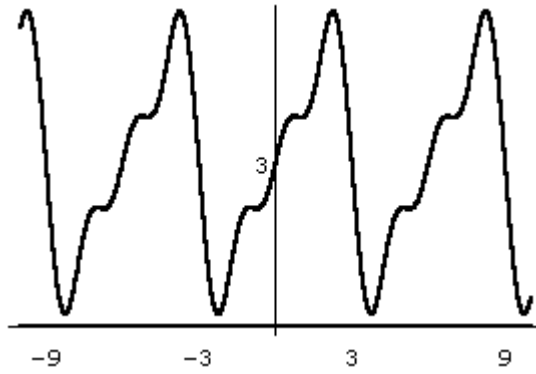

Figure 2a. The graph of the given function
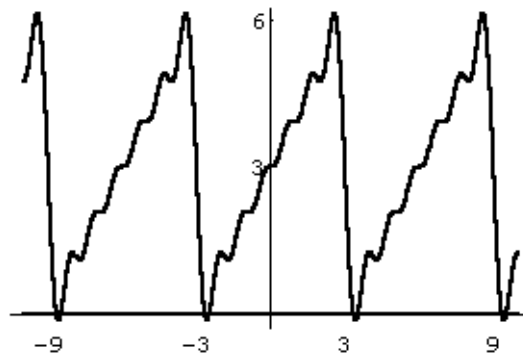
Figure 2b. The graph of the sum of three terms

Figure 2c. The graph of the sum of six terms

Example 3. Rectifier wave function
$f(x) = 0$, $x \in (-\pi, 0)$,
$\quad = \sin x$, $x \in (0, \pi)$.

In[3]= FourierSer[0, {x,-Pi,0}, Sin[x], {x,0,Pi}, 6]

$$\text{Out[2]}= \frac{1}{Pi} - \frac{2\,Cos[2\,x]}{3Pi} - \frac{2\,Cos[4\,x]}{15Pi} - \frac{2\,Cos[6\,x]}{35Pi} -$$

$$-\frac{2\,Cos[8\,x]}{63Pi} - \frac{2\,Cos[10\,x]}{99Pi} - \frac{2\,Cos[12\,x]}{143Pi} + \frac{Sin[x]}{2}$$
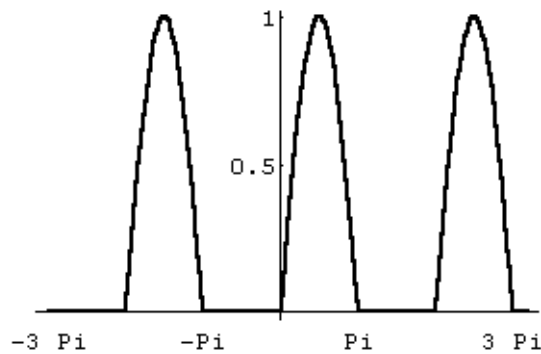

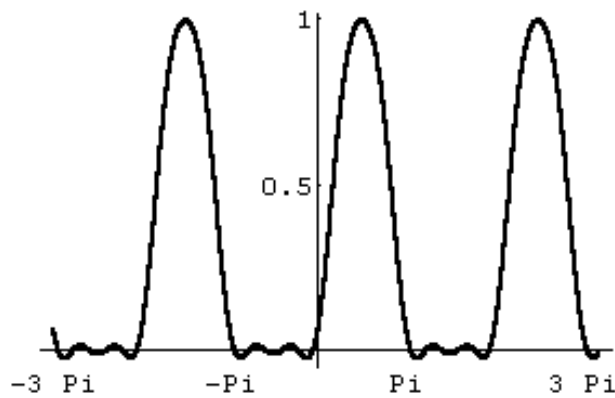Figure 3a. The graph of the given function

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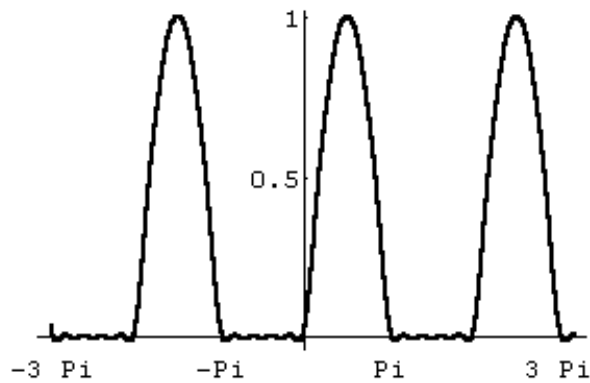Figure 3b. The graph of the sum of three terms

Figure 3c. The graph of the sum of six terms

OTHER CALCULATIONS

The difference between the values of the given function f and the values of its Fourier polynomial F_n on an interval (-l, l) can be also expressed numerically. The mean quadratic deviation can be defined as

$$\delta\,(f,F_n)=\sqrt{\int_{-l}^{l}(f(x)-F_n(x))^2\,dx}\ .$$

Example 4: Evaluate the mean quadratic deviation for the swath (triangle)
$f(x)=x+3,\quad x\in(-3,3)$.
wave function for even n from 2 to 10.

We get these values of the mean quadratic deviation using the function
                    Delta[f_,Fn_,l_]:= Sqrt[Integrate[(f-Fn)^2, {x,-l,l}]]
in the system *Mathematica*:

n	2	4	6	8	10
δ	2.07886	1.55624	1.29622	1.13397	1.02048

It is clearly to see from table that the mean quadratic deviation is a decreasing function of n. It means that the approximation of a function f by its Fourier polynomial is the more precise if it is used the greater value of n. The system *Mathematica* can be also used for other calculations connected with Fourier series. The process of differentiation or integration of Fourier series term by term is such example. One of the interesting cases is the possibility to obtain the Fourier series expansion of a function F on term by term integration of the Fourier series expansion of the function f knowing only this expansion and the function itself can be unknown.

CONCLUDING REMARK

Fourier series analysis is integral to the study physics, mathematics and engineering, but the methods can be difficult to reach and apply. The Application of the system *Mathematica* eliminates time constraints typically associated with the treatment of Fourier series and provides a platform for learning and investigating more interesting and complicated problems. The basic features of this system give teachers an opportunity to explore different methods in depth and help students develop deeper understanding of this numerical technique. The program tools can remove not only tedious numerical calculations but even to use the calculating and graphic possibilities of software to help students better understand the solution of relevant problems. And this is one of the most important tasks of training in mathematics at Universities.

REFERENCES

Wolfram, S. at all: Mathematica 4 Standard Add - on Packages. Champaign, II, Wolfram Media, 1999.

Widder, D., V.: Advanced Calculus. Dover Publications, Inc., New York, 1989.

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