DEPARTMENT OF MATHEMATICS AND STATISTICS

hp-Finite Element Methods For Fourth-Order Singularly Perturbed Problems

DOCTOR OF PHILOSOPHY DISSERTATION

Philippos Constantinou

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Philippos Constantinou

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Doctoral Candidate: Philippos Constantinou

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The present Doctoral Dissertation was submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy at the ***Department of Mathematics and Statistics*** *and was approved on the 19th December, 2018 by the members of the* ***Examination Committee.***

Examination Committee:

Committee Chair: Georgios Georgiou, Professor,
Department of Mathematics and Statistics,
University of Cyprus

Committee Member: Andreas Karageorghis, Professor,
Department of Mathematics and Statistics,
University of Cyprus

Committee Member: Jens Markus Melenk, Professor,
Institut für Analysis und Scientific Computing,
Technische Universität Wien, Austria

Committee Member: Torsten Olaf Linß, Professor,
Fakultät fúr Mathematik und Informatik,
Fern Universität in Hagen, Germany

Research Supervisor: Christos Xenophontos, Professor,
Department of Mathematics and Statistics,
University of Cyprus

DECLARATION OF DOCTORAL CANDIDATE

The present doctoral dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy of the University of Cyprus. It is a product of original work of my own, unless otherwise mentioned through references, notes, or any other statements.

Philippos Constantinou

To the memory of my mother

Abstract

We study fourth-order singularly perturbed problems in one- and two-dimensions, and the approximation of their solution by the *hp*-Finite Element Method (FEM). The solution to such problems features boundary layers and our goal is twofold: first we want to obtain regularity results for the solution, and second we want to construct robust *hp*-FEM approximations on the so-called *Spectral Boundary Layer Mesh* for its approximation. We are mainly concerned with C^1 conforming FEMs but we also consider a C^0 mixed formulation approximation.

From the point of view of regularity, we provide estimates that are explicit in the differentiation order and the singular perturbation parameter. Both classical differentiability as well as differentiability through asymptotic expansions are derived. Through the latter, we obtain a decomposition of the solution into a smooth part, boundary layers along the boundary, and a (negligible) remainder. Explicit regularity estimates are obtained for each part. The above are achieved for one-dimensional problems with variable (smooth) coefficients as well as two-dimensional problems with constant coefficients posed on smooth domains.

Using the aforementioned results, we construct *hp* approximations that converge independently of the singular perturbation parameter, when the error is measured in the energy norm. In one-dimension the rate is exponential, while in two-dimensions, the rate, in general is spectral (unless certain assumptions are made). Moreover, in two-dimensions, we are faced with the problem of not being able to construct C^1 approximations on curved elements (or even affine, distorted elements). One way to deal with this issue by using a *mixed* formulation, hence C^0 elements suffice. Another, is the use of the *Discontinuous Galerkin FEM*, but only the former is investigated. In all cases studied, numerical results are provided which illustrated the theory.

Περίληψη

Η παρούσα διατριβή αφορά προβλήματα 4ης τάξης, στη μια και στις δύο διαστάσεις, τα οποία είναι διαταραγμένα με ιδιόμορφο/ιδιάζοντα τρόπο. Η λύση τέτοιων προβλημάτων περιέχει συνοριακά στρώματα. Έχουμε δύο στόχους: πρώτα θέλουμε να αποδείξουμε εκτιμήσεις ομαλότητας για τη λύση, οι οποίες είναι ρητές ως προς τη παράμετρο διαταραχής και τη τάξη παραγώγισης. Στη συνέχεια, θέλουμε να προσεγγίσουμε τη λύση τους με την εκδοχή hp της Μεθόδου Πεπερασμένων Στοιχείων (ΜΠΣ) χρησιμοποιώντας το λεγόμενο *Φασματικό Πλέγμα Συνοριακών Στρωμάτων.* Πρώτιστα, μας ενδιαφέρουν C^1 προσεγγίσεις, αν και στο Κεφ.6 θεωρούμε μια C^0 προσέγγιση βασισμένη σε μια *μεικτή* μεταβολική διατύπωση.

Απο τη πλευρά της ομαλότητας, αποδεικνύουμε εκτιμήσεις οι οποίες είναι ρητές ως προς τη παράμετρο διαταραχής και τη τάξη παραγώγισης. Το επιτυγχάνουμε στη περίπτωση της κλασσικής ομαλότητας, όπως επίσης και στην περίπτωση της ομαλότητας μέσω ασυμπτωτικών αναπτυγμάτων. Τα τελευταία μας επιτρέπουν να γράψουμε/αναλύσουμε τη λύση ως ένα άθροισμα όρων που αποτελούνται από το ομαλό μέρος, τα συνοριακά στρώματα (κατά μήκος του συνόρου) και το υπόλοιπο (το οποίο είναι αμελητέο). Ρητές εκτιμήσεις ομαλότητας αποδεικνύονται για το κάθε μέρος στην ανάλυση. Τα πιο πάνω έχουν επιτευχθεί για προβλήματα με μη σταθερούς συντελεστές στη 1-διάσταση και για προβλήματα με σταθερούς συντελεστές στις 2-διαστάσεις, όπου το χωρίο είναι ομαλό.

Χρησιμοποιώντας τις εκτιμήσεις ομαλότητας, κατασκευάζουμε μια κατάλληλη προσέγγιση για τη λύση, με την εκδοχή hp της ΜΠΣ. Αποδεικνύουμε ότι η προσέγγιση συγκλίνει με εκθετικό ρυθμό ανεξάρτητως της παραμέτρου διαταραχής στη νόρμα ενέργειας, στη 1-διάσταση. Στις 2-διαστάσεις, η προσέγγιση δεν μπορεί να έχει C^1 συνέχεια όταν τα στοιχεία του πλέγματος έχουν καμπύλες πλευρές (ή ακόμη και απλώς διαταραγμένες ευθείες). Ξεπερνούμε αυτή την δυσκολία με δύο τρόπους: πρώτα θεωρούμε ότι το χωρίο είναι τετράγωνο, αλλά υποθέτουμε ότι η λύση συμπεριφέρεται έως αν το χωρίο να μην

περιείχε γωνιές (και ως εκ τούτου, και ιδιομορφίες). Ο δεύτερος τρόπος είναι μέσω μιας *μεικτής* διατύπωσης, η οποία επιτρέπει τη χρήση C^0 προσέγγισης. Και για τις δύο κατηγορίες, κατασκευάσαμε συναρτήσεις βάσης με ιεραρχικό τρόπο και τις υλοποιήσαμε στον υπολογιστή (μέσω της MATLAB). Μια τρίτη επιλογή, που δεν θα μελετήσουμε στη παρούσα διατριβή είναι η χρήση της Ασυνεχής Μεθόδου Galerkin, στην οποία δεν χρειάζεται συνέχεια (από στοιχείο σε στοιχείο) των συναρτήσεων βάσης. Και για τις δύο επιλογές που μελετήσαμε, παραθέτουμε αποτελέσματα υπολογισμών που συμφωνούν με τη θεωρία.

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Besides my advisor, I would like to thank the members of my committee: Prof. Georgios Georgiou, Prof. Andreas Karageorghis, Prof. Jens Markus Melenk and Prof. Torsten Olaf Linß for their careful reading of this thesis and their valuable comments.

Moreover, I would like to thank my family and friends whose support have important contribution to finish this work. Special personal acknowledgment goes to Tatiana Galitskaya who never complained about my long working hours and has supported all my endeavors.

It is with deep sadness and pride that I offer this thesis in appreciation of my mother, Constantia Constantinou for her love and fortitude. My mother died in March 2017 after long cancer battle. I dedicate this thesis with gratitude to her memory.

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Chapter 1

Introduction

This thesis lies in the intersection of Numerical Analysis, Finite Element Methods (FEMs) and Singularly Perturbed Problems (SPPs). Our main concern here is to address SPPs that are given by *fourth*-order ordinary or partial differential equations, posed in bounded regions in one and two dimensions.

1.1 Perturbation theory

First, let us present some background information about perturbation theory. This field of applied mathematics, as the entire area of numerical analysis, provides approximations for the solution to certain problems. Usually, we want to solve a differential equation that is given by a mathematical model in order to describe fairly a physical phenomenon. In most cases it is hard to determine the analytical solution and the use of perturbation theory is common in order to obtain an approximation.

As is well known, perturbation theory examines differential equations or systems that share a specific feature: a very small parameter that multiplies one or more terms of the differential equation or system. Perturbation theory allows us to transform each problem to a simpler one. To be more specific, this associated problem does not include the terms with the perturbation parameter. Researchers usually refer to it as a *simplified*, *unperturbed* or *reduced* problem.

The aforementioned class of differential equations can be divided in two categories. The separation is done in accordance with the behaviour of the *simplified* problem.

Namely, in case that the *reduced* problem describes the phenomenon under investigation sufficiently well, it is characterized as a *regularly perturbed* problem. However, it is not always that a perturbed problem behaves this way. Therefore, the scientific community distinguishes those cases and calls the corresponding problems as *singularly perturbed.* We present the following examples to better explain this separation.

Example 1.1.1. This example was given by O' Malley in [55] and represents a regular perturbation. Let $\varepsilon > 0$ be the perturbation parameter and consider the equation

$$u^2 + \varepsilon u - 1 = 0. \tag{1.1.2}$$

The analytical solution to (1.1.2) is given by:

$$u = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 + 4}}{2}. \tag{1.1.3}$$

Now consider the simplified problem, i.e. set $\varepsilon = 0$ in (1.1.2) and note that it has two solutions: $u = \pm 1$. If we set again $\varepsilon = 0$ in (1.1.3) we obtain $u = \pm 1$ once more. In this case the reduced problem is in complete accordance with the initial problem therefore the problem is regularly perturbed.

Example 1.1.4. We next illustrate a simple example of a SPP. For $\varepsilon > 0$, consider the boundary value problem given as

$$\left.\begin{aligned} &\varepsilon v''(x) - v(x) = 1,\ x \in (0, 1), \\ &v(0) = 0,\ v(1) = -1. \end{aligned}\right\} \tag{1.1.5}$$

If we examine the reduced problem, that is

$$\left.\begin{aligned} -v(x) &= 1, \\ v(0) = 0, v(1) &= -1, \end{aligned}\right\}$$

it is clear that the solution $v(x) = -1$, for $x \in [0, 1]$, violates the boundary condition at $x = 0$. The analytical solution to (1.1.5) is

$$v(x, \varepsilon) = -1 + \frac{1}{1 - e^{2/\sqrt{\varepsilon}}}\left(e^{(2-x)/\sqrt{\varepsilon}} - e^{x/\sqrt{\varepsilon}}\right).$$

Obviously, as the perturbation parameter ε tends to zero the solution equals -1. However, the boundary condition at the left endpoint causes *boundary-layer effects*. The reader may see those effects in Figure 1.1. To address this difficulty we must not neglect the derivative term. A natural way to treat this case is by adjusting the variable x in such a way to ensure that $\dfrac{d^2v}{dx^2} = O(\varepsilon^{-1})$. Hence we need to set $x = O(\sqrt{\varepsilon})$. Thus we define $\tilde{x} = x/\sqrt{\varepsilon}$ and the boundary value problem is written in the form

$$\left.\begin{aligned} &\frac{d^2v(\tilde{x})}{d\tilde{x}^2} - v(\tilde{x}) = 1,\ \tilde{x} \in (0, \infty), \\ &v(0) = 0,\ \lim_{\tilde{x}\to\infty} v(\tilde{x}) = -1. \end{aligned}\right\} \tag{1.1.6}$$

The solution to the boundary value problem above is $v(\tilde{x}) = -1 + \mathrm{e}^{-\tilde{x}}$ or $v(x, \varepsilon) = -1 + \mathrm{e}^{-x/\varepsilon}$, which agrees with the exact solution as $\varepsilon \to 0$.

As it is shown by the above example, the peculiarity of SPPs arises from neglecting the term that is multiplied by the petrurbation parameter. In many cases, the perturbation parameter affects the highest order derivative and its omission leads to boundary layer effects. We note that in case $v(1) \neq -1$ then another boundary layer should be expected near $x = 1$.

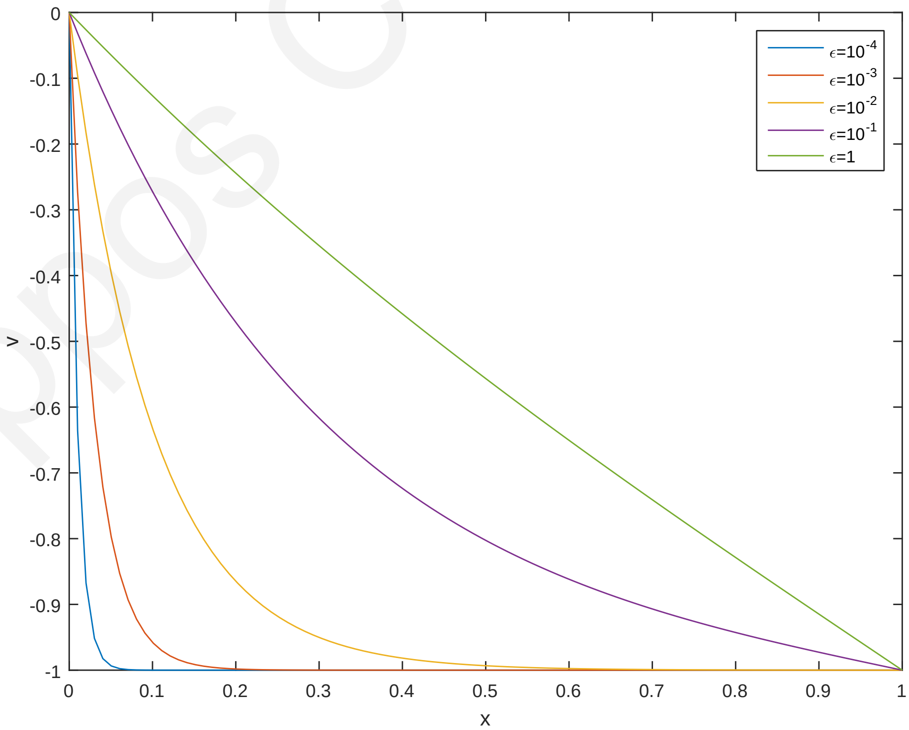


Figure 1.1: Exact solution variation with ε.

Poincaré and Stieltjes [58, 71] worked independently and in 1886 set the origins of perturbation theory through the examination of asymptotic series, which are in general divergent. Prandtl in 1905 [59] examined the motion of a fluid with small viscosity along a body. The first mathematician who studied extensively SPPs is A.N. Tikhonov in the late 1940s and early 1950s. A list of some important publications may be found in [74], [75]. Subsequently, M.I. Vishik and L.A. Lyusternik published a radical paper in 1957 [77], about a well known method (the "Vishik-Lyusternik" method) which treats linear partial differential equations with singular perturbations.

There are many applications of SPPs. For instance, this type of problems can be utilized to describe reaction and convection-diffusion problems, Navier-Stokes equations with small viscosity coefficients, semi-conductor device modeling, plate and shell models for small thickness [45, 47, 65, 67, 79, 80, 81]. Fourth-order diffusion equations appear in many applications such as thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells and phase-field models of multiphase systems [18].

1.2 Finite Element Method

For the approximation of the solution, we apply the Finite Element Method. As we illustrate in the next chapters the *hp* version of the Finite Element Method (FEM) yields optimal results. Let us now discuss briefly the development of the FEM through the last century.

B.G. Galerkin (1915) and others developed the mathematical background for the FEM based on the work of Lord Rayleigh (1870) and W. Ritz (1909), which concerns variational methods and the weighted-residual approach. In 1941, R. Courant had an innovative idea. He introduced the special linear functions defined over triangular regions and utilized the Ritz method for the solution of torsion problems. This idea helped scientists to surpass the main issue in the Ritz method, namely the functions used must fulfill the boundary conditions of the problems. The work of Courant is considered as the origin of the era of FEM since it is very close to the FEM proposed in 1960 by Clough. Back then it was the first time that the phrase "finite element" appeared [20].

Since then, the FEM has been developed and expanded enormously and it is considered as one of the most succesfull approximation methods. Many researchers have worked in this field and have provided some essential and important results. Among the mathematicians who made significant contributions, we mention Babuška, Brezzi, Aziz, Osborn, Kellogg, Arnold, Brenner, Scott [6, 9, 15, 17, 35, 64, 73] and among engineers Argyris, Oden, Szabo, Hughes, Demkowicz, Zienkiewicz [1, 3, 26, 36, 84].

Regarding SPPs, many people have worked on them in the last few decades; among them are the names Vasileva, Butuzov, Bakhvalov, Shishkin [13, 52, 76], O'Malley, Stynes, O'Riordan, Roos, Tobiska, Linß, Kopteva, Schwab, Suri, Melenk, Xenophontos [39, 40, 42, 43, 45, 47, 55, 61, 65, 66, 67, 80, 81]. The bulk of the work is on appropriate discretizations of SPPs so that the method converges independently of the singular perturbation parameter at the optimal rate. While there are several ways to achieve this, the preferred choice among the aforementioned researchers is the use of *layer adapted meshes*, such as the Bakhvalov mesh, the Shishkin mesh or the exponential mesh [13, 68, 69], which may be used in conjunction with finite differences or the h-version of the FEM. If one uses the hp-version of the FEM, then the convergence could be exponential, independently of the singular perturbation parameter, provided the so-called *Spectral Boundary Layer mesh* is used [22, 82].

1.3 The plan of the thesis

Here we present in brief the plan of this thesis, which is divided into seven (7) Chapters. Throughout Chapter 2 we examine a fourth order Singularly Perturbed Boundary Value Problem (SPBVP) in one-dimension with variable coefficients, and we present an extensive analysis of its regularity which provides the main result of the Chapter 2 presented in Theorem 2.4.42.

In Chapter 3, we apply the hp version of FEM to construct approximations of the solution to the problem examined in Chapter 2. The construction is accomplished through the use of the *Spectral Boundary Layer Mesh* (Definition 3.1.11) which yields robust approximation results. We illustrate a set of polynomial functions which make up our C^1-basis in one dimension (see Section 3.1.1), and present a one-dimensional interpolation operator and its approximation properties (Proposition 3.2.1). This work is an extension of [57]. We mention here that the error of the hp-approximation is measured in the energy norm. Our numerical results illustrate robust, exponential convergence.

In Chapter 4, we present the analysis for the regularity of the analogous SPBVP of fourth order that is defined in a smooth bounded region in two-dimensions. Basically, throughout Chapter 4, we illustrate an extension of Morrey's analysis [53] which focuses on fourth order differential equations in 2D. Theorem 4.3.32 gives an important estimate that is independent of ε, in the case $|\alpha| \geq \varepsilon^{-1}$ ($|\alpha|$ is the order of the partial derivatives). Remark 4.4.52 summarizes the regularity of the components of the solution, namely the smooth part, boundary layers and the remainder. Unfortunately, the remainder in our decomposition is not exponentially small, and this has repercussions for the approximation.

In Chapter 5, we consider the problem posed on the reference element $S = (-1, 1)^2$, and construct hierarchical, C^1 basis functions. Then, we define an interpolation operator and study its approximation properties. We also provide *lifting* results that allow us to obtain a global C^1 continuous, piecewise polynomial approximation. Next, assuming certain regularity of the solution to the problem posed on a square (e.g. no corner singularities and exponentially small remainder), we analyze the *hp*-FEM approximation on the *Spectral Boundary Layer mesh*, and we show robust, exponential convergence. The chapter ends with numerical examples.

In Chapter 6, we use a different formulation to approximate the solution to the problem presented in Chapter 4. Namely, we utilize the widely known *Mixed Finite Element* formulation and we show that this method produces robust exponential convergence without requiring a C^1 approximation (hence basis functions), assuming analytic regularity. Theorem 6.2.12 is the main result that gives the convergence in the case of a smooth domain. Several numerical examples are also presented.

Finally, in Chapter 7 we give conclusions and discuss future plans.

1.4 Notation

The notation in this thesis is more or less standard, with just a few exceptions. In this section, we describe the notation used throughout.

The set of integer numbers is denoted by $\mathbb{Z}$, the set of real numbers is denoted by $\mathbb{R}$, the set of complex numbers by $\mathbb{C}$ and for a natural number n we denote the set that consists of all $n-$tuples of real numbers by $\mathbb{R}^n$, that is the well known n-dimensional real space. Also we denote by $\mathbb{T}_l$ the one-dimensional torus of length l, namely $\mathbb{R}/l\mathbb{Z}$ endowed with the usual topology. For the most part, I and J denote arbitrary intervals and Ω and G denote two-dimensional bounded domains unless otherwise specified. We denote the boundary of Ω by $\partial\Omega$ and its closure by $\overline{\Omega}$. The notation $\Omega \subset\subset G$ means that $\overline{\Omega}$ is compact and $\overline{\Omega} \subset G$.

We denote by $\mathbb{P}_q(I)$ the space of polynomials on I, of degree less than or equal to q. We extend this notation to two dimensions and we set $\mathbb{Q}_q(S) = \mathbb{P}_q([-1,1]) \otimes \mathbb{P}_q([-1,1])$, where S is the reference square $S = (-1,1)^2$. The set of functions on Ω with continuous partial derivatives of order $\leq n$ is denoted by $C^n(\Omega)$. By $C^n(\overline{\Omega})$ we denote the set of functions which can, along with their derivatives of order $\leq n$, be continuously extended to $\overline{\Omega}$. With $0 < \mu \leq 1$, the space $C^n_\mu(\Omega)$ is comprised of the functions $u \in C^n(\Omega)$ such that all n-th derivatives of u are Hölder continuous with exponent μ, on each compact subset of Ω. We denote by $C^n_c(\Omega)$ and $C^n_{\mu c}(\Omega)$ the set of functions in $C^n(\Omega)$ and $C^n_\mu(\Omega)$ respectively, which have compact support in Ω. We utilize the Lebesgue spaces $L^p(\Omega)$, of functions f with $\left(\int_\Omega |f|^p\right)^{1/p} < \infty$. By $H^k(\Omega)$ we will denote the Sobolev space of order k on a domain $\Omega \subset \mathbb{R}^2$ with $H^0(\Omega) = L^2(\Omega)$.[a] The usual norms and seminorms are denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. Furthermore we set the spaces

$$H^1_0(\Omega) = \left\{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\right\}$$

and

$$H^2_0(\Omega) = \left\{u \in H^2(\Omega) : u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}.$$

Here $\frac{\partial}{\partial n}$ is the normal derivative directed outwards. By $\langle\cdot,\cdot\rangle_\Omega$ we denote the usual L^2 inner product on Ω.

In this thesis the maximum value of a function f on its domain I will be denoted as $\overline{f} := \sup_{x\in I}|f(x)|$ and the minimum as $\underline{f} := \inf_{x\in I}|f(x)|$. Also positive constants will be denoted by $C, C_1, \tilde{C}, ...$, and may take different values.

[a] In some rare cases we use the notation $H^k_p(\Omega)$ that is referred to the general Sobolev spaces given in Definition 1.5.4

1.5 Preliminaries

For the following definitions let $n, N \in \mathbb{N}$, $k \in \mathbb{Z}$ and assume that all functions below have the form $\bullet : \mathbb{R}^n \to \mathbb{R}^N$.

Definition 1.5.1. A function f is said to be *homogeneous of degree* k if

$$f(\alpha v) = \alpha^k f(v),$$

for all nonzero $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^n$.

Definition 1.5.2. A function f is said to be *positively homogeneous of degree* k if

$$f(\alpha v) = \alpha^k f(v),$$

for all $\alpha \in \mathbb{R}^+$ and $v \in \mathbb{R}^N$.

Definition 1.5.3. A function f is called *essentially homogeneous of degree* k if

- f is positively homogeneous if $k < 0$,
- in the complementary case, f can be written in the form

$$f(v) = f_0(v)\log(|v|) + f_1(v),$$

  where f_0 is a homogeneous polynomial of degree k and f_1 is positively homogeneous of degree k.

We will utilize the following notation: Let α denote a "multi-index", i.e. a vector $(\alpha_1, \cdots, \alpha_n)$ in which each α_i is a non-negative integer. Let $G \subset \mathbb{R}^n$ and $u \in C^{|\alpha|}(G)$, and define

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = (\alpha_1!)\cdots(\alpha_n!), \quad C_\alpha = \frac{|\alpha|!}{\alpha!},$$
$$D^\alpha u = \frac{\partial^{|\alpha|} u}{(\partial x_1)^{\alpha_1}\cdots(\partial x_1)^{\alpha_n}}.$$

Definition 1.5.4. (Space H_p^m). Let $G \subset \mathbb{R}^N$, $N \in \mathbb{N}$. Let $p \in [1, \infty]$ and $m \in \mathbb{N}$. We say a function $f = (f_1, \ldots, f_N) : G \to \mathbb{R}$ belongs to $H_p^m(G)$ if and only if $f \in L^p(G)$ and if for all multiindices $\alpha = (\alpha_1, \dots, \alpha_N)$, $0 \le |\alpha| \le m$, there exist functions $g_\alpha \in L^p(G)$, such that

$$\int_G h(x) g_\alpha(x) dx = (-1)^{|\alpha|} \int_G D^\alpha h(x) f(x) dx, \quad h \in C_c^\infty(G).$$

The following result can be found in [53, Theorem 3.1.1].

Theorem 1.5.5 *[53, Chapter 3] Let $m \in \mathbb{N}$ and $p \in [1, \infty)$. Then, for any multiindex α, $0 \le |\alpha| \le m$, the space $H_p^m(G)$ with norm defined by*

$$\|f\|_p^m = \left(\int_G \left(\sum_{i=1}^N \sum_{0<|\alpha|<m} C_\alpha |D^\alpha f_i|^2 \right)^{p/2} dx \right)^{1/p}, \quad f = (f_1, \dots, f_N), \tag{1.5.6}$$

is a Banach space, where C_α denotes the multinomial coefficient $\frac{|\alpha|!}{\alpha_1! \dots \alpha_N!}$. If $p = 2$, the space is a Hilbert space with the scalar product

$$(u, v)_2^m = \int_G \sum_{i=1}^N \sum_{0 \le |\alpha| \le m} C_\alpha D^\alpha u_i D^\alpha v_i dx.$$

Chapter 2

One dimensional fourth-order SPP with variable coeffiecients

2.1 Introduction

In this chapter we examine a fourth-order singularly perturbed boundary value problem (SPBvp) in one-dimension, presented in (2.2.1) and which models the simplified Orr-Sommerfeld equation from hydrodynamics. In fluid dynamics, researchers have noticed that the linear modes of disturbance to a viscous parallel flow can be described with the aid of an eigenvalue equation, the Orr-Sommerfeld equation (see [56] and [70]). The equation is named after William McFadden Orr and Arnold Sommerfeld, who derived it at the beginning of the 20th century. This specific equation determines certain conditions which the solution to the Navier-Stokes for a parallel, laminar flow must satisfy in order to be stable. The Orr-Sommerfeld equation is

$$Re^{-1}u^{(4)}(x) - \mathrm{i}\lambda\Big(\big(h(x)-\beta\big)u''(x) - \Big(\lambda^2\big(h(x)-\beta\big)+h''(x)\Big)u(x)\Big) = 0, \qquad (2.1.1)$$

where u is the unknown, potential/stream function, h is a given function, Re is the Reynolds number of the base flow, β is a parameter, λ is the wavenumber and i is the imaginary unit. This equation defines an eigenvalue problem, if the parameter β is unknown and is singularly perturbed when the Reynolds number is large enough. The boundary value problem given in (2.2.1) is a simplified version of (2.1.1).

Here we study the behaviour of the solution to the SPBVP given in (2.2.1) below.

Our main task is the construction of a satisfactory approximation, therefore we must first focus on the properties of the solution. Namely, in this chapter we present proper regularity results which allow us to proceed with an $hp-$approximation. We restrict our attention to problems with analytic data, and we note that here we generalize the work that has been presented in [57]. To be more specific, in the aforementioned paper one can find the regularity analysis about the problem under investigation with constant coefficients and we now generalize the results for variable coefficients.

2.2 The model problem and the decomposition of its solution

We consider the following two-point boundary value problem: For a parameter $\varepsilon \in (0,1]$, find $u \in C^4([0,1])$ such that

$$\left.\begin{aligned} \varepsilon^2 u^{(4)}(x) - \big(\alpha(x)u'(x)\big)' + \beta(x)u(x) = f(x), \text{ for } x \in I = (0,1) \\ u(0) = u'(0) = u(1) = u'(1) = 0 \end{aligned}\right\} \tag{2.2.1}$$

where $\alpha(x) \geq c_1 \in \mathbb{R}^+,\ \beta(x) \geq 0\ \forall\, x \in \overline{I}$ and f are given (analytic) functions. Specifically, we assume that there exist positive constants $C_\alpha, C_\beta, C_f, \gamma_\alpha, \gamma_\beta, \gamma_f$ independent of ε, such that for all $n \in \mathbb{N}_0$,

$$\|\alpha^{(n)}\|_{L^\infty(I)} \leq C_\alpha \gamma_\alpha^n n!, \quad \|\beta^{(n)}\|_{L^\infty(I)} \leq C_\beta \gamma_\beta^n n!, \quad \|f^{(n)}\|_{L^\infty(I)} \leq C_f \gamma_f^n n!. \tag{2.2.2}$$

We next present the *variational formulation*, namely we seek a function $u \in H_0^2(I)$ that satisfies the equation

$$\mathcal{B}(u,v) = \mathcal{F}(v), \quad \text{for all } v \in H_0^2(I), \tag{2.2.3}$$

where

$$\mathcal{B}(u,v) := \int_I \Big(\varepsilon^2 u''(\xi)v''(\xi) + \alpha(\xi)u'(\xi)v'(\xi) + \beta(\xi)u(\xi)v(\xi)\Big)d\xi,$$

$$\mathcal{F}(v) := \int_I f(\xi)v(\xi)d\xi.$$

We also define the natural energy norm

$$\|u\|_{\mathcal{E},I} = \Big(\mathcal{B}(u,u)\Big)^{\frac{1}{2}}. \tag{2.2.4}$$

Our goal is to construct $hp-$approximations, hence we must control the dependence of the higher order derivatives of the solution on the peturbation parameter ε. For a constant $\gamma > 0$, one can show in an inductive way, (using equations (2.8)-(2.12) in [72]) the following inequality:

$$\|u^{(n)}\|_{L^\infty(I)} \leq C\gamma^n \max\{n^n, \varepsilon^{1-n}\},\ \forall n \in \mathbb{N}_0. \tag{2.2.5}$$

We observe that, the solution of (2.2.1) is analytic if the data α, β, f are analytic. However, if ε is small, then (2.2.5) does not reveal the entire picture. To address this, we use the method of matched asymptotic expansions, as was done in [38, 41, 78], in order to decompose the solution into a smooth component, boundary layers and a remainder. In this way, we obtain estimates on the derivatives of each part of the solution, separately.

Anticipating that boundary layers will be present near the endpoints of I, we define the stretched variables $\tilde{x} = \frac{x}{\varepsilon}$, $\hat{x} = \frac{1-x}{\varepsilon}$ and make the formal ansatz

$$u \sim \sum_{j=0}^{\infty} \varepsilon^j \Big[U_j(x) + \tilde{U}_j(\tilde{x}) + \hat{U}_j(\hat{x})\Big], \tag{2.2.6}$$

where the functions U_j, $\tilde{U}_j$, $\hat{U}_j$ will be determined shortly. We insert (2.2.6) in the differential equation (2.2.1) and separate the slow (i.e. x) and fast (i.e. $\tilde{x}$, $\hat{x}$) variables. Equating like powers of ε on both sides of the resulting equation, we get:

$$\left.\begin{aligned} -\Big(\alpha(x)U_0'(x)\Big)' + \beta(x)U_0(x) &= f(x), \\ -\Big(\alpha(x)U_1'(x)\Big)' + \beta(x)U_1(x) &= 0, \\ -\Big(\alpha(x)U_j'(x)\Big)' + \beta(x)U_j(x) &= -U_{j-2}^{(4)}(x), \qquad j = 2,3,\cdots \end{aligned}\right\} \tag{2.2.7}$$

$$\left.\begin{aligned} \tilde{U}_0^{(4)}(\tilde{x}) - \Big(\tilde{\alpha}_0(\tilde{x})\tilde{U}_0'(\tilde{x})\Big)' &= 0, \\ \tilde{U}_1^{(4)}(\tilde{x}) - \Big(\tilde{\alpha}_0(\tilde{x})\tilde{U}_1'(\tilde{x})\Big)' &= 0, \\ \tilde{U}_j^{(4)}(\tilde{x}) - \big(\tilde{\alpha}_0(\tilde{x})\tilde{U}_j'(\tilde{x})\big)' &= \tilde{F}_j(\tilde{x}), \quad j = 2,3,\cdots \end{aligned}\right\} \tag{2.2.8}$$

and similarly for $\hat{U}$. In (2.2.8), we used the notation $\tilde{\alpha}_k(\tilde{x}) = \frac{\alpha^{(k)}(0)}{k!}\tilde{x}^k$ and $\tilde{\beta}_k(\tilde{x}) = \frac{\beta^{(k)}(0)}{k!}\tilde{x}^k$ and the right hand side functions are given by

$$\tilde{F}_j(\tilde{x}) := \sum_{k=1}^{A_j} \tilde{\alpha}_k(\tilde{x})\tilde{U}''_{j-k}(\tilde{x}) + \sum_{k=1}^{A_j} \tilde{\alpha}'_k(\tilde{x})\tilde{U}'_{j-k}(\tilde{x}) - \sum_{k=0}^{B_j} \tilde{\beta}_k(\tilde{x})\tilde{U}_{j-2-k}(\tilde{x}),$$

where

$$A_j = \begin{cases} \frac{j}{2}, & \text{if } j \text{ is even,} \\ \frac{j-1}{2}, & \text{if } j \text{ is odd,} \end{cases} \qquad B_j = \begin{cases} \frac{j-2}{2}, & \text{if } j \text{ is even,} \\ \frac{j-3}{2}, & \text{if } j \text{ is odd.} \end{cases}$$

An analogous system as (2.2.8) is obtained for the functions $\hat{U}_j(\hat{x})$, with $\hat{\alpha}_k(\hat{x}) = \frac{\alpha^{(k)}(1)}{k!}\hat{x}^k$ and $\hat{\beta}_k(\hat{x}) = \frac{\beta^{(k)}(1)}{k!}\hat{x}^k$.

In order to satisfy (2.2.1) we add the following boundary conditions to the above systems of equations:

$$U_0(0) = U_0(1) = 0, \tag{2.2.9}$$

$$\tilde{U}_0(0) = 0,\ \tilde{U}_j(0) = -U_j(0),\ j \geq 1, \tag{2.2.10}$$

$$\tilde{U}'_0(0) = 0,\ \tilde{U}'_{j+1}(0) = -U'_j(0),\ j \geq 0, \tag{2.2.11}$$

$$\lim_{\tilde{x}\to\infty} \tilde{U}_j(\tilde{x}) = 0,\ j \geq 0, \tag{2.2.12}$$

$$\hat{U}_0(0) = 0,\ \hat{U}_j(0) = -U_j(1),\ j \geq 1, \tag{2.2.13}$$

$$\lim_{\hat{x}\to\infty} \hat{U}_j(\hat{x}) = 0,\ j \geq 0, \tag{2.2.14}$$

$$\hat{U}'_0(0) = 0,\ \hat{U}'_{j+1}(0) = -U'_j(1),\ j \geq 0. \tag{2.2.15}$$

For $j \in \mathbb{N}_0$, Table 2.1 displays all the boundary value problems satisfied by $U_j, \tilde{U}_j, \hat{U}_j$.

Table 2.1: B.V.P. for functions $U_j, \tilde{U}_j, \hat{U}_j, \forall j \in \mathbb{N}_0$.

Outer solution	Boundary layer at $x = 0$	Boundary layer at $x = 1$
$-(\alpha U'_0)' + \beta U_0 = f$, $U_0(0) = -\tilde{U}_0(0) = 0$, $U_0(1) = -\hat{U}_0(0) = 0$,	$\tilde{U}_0^{(4)} - (\tilde{\alpha}_0\tilde{U}'_0)' = 0$, $\lim_{\tilde{x}\to\infty} \tilde{U}_0(\tilde{x}) = 0$, $\tilde{U}'_0(0) = 0$,	$\hat{U}_0^{(4)} - (\hat{\alpha}_0\hat{U}'_0)' = 0$, $\lim_{\hat{x}\to\infty} \hat{U}_0(\hat{x}) = 0$, $\hat{U}'_0(0) = 0$,
$-(\alpha U'_1)' + \beta U_1 = 0$, $U_1(0) = -\tilde{U}_1(0)$, $U_1(1) = -\hat{U}_1(0)$,	$\tilde{U}_1^{(4)} - (\tilde{\alpha}_0\tilde{U}'_1)' = 0$, $\lim_{\tilde{x}\to\infty} \tilde{U}_1(\tilde{x}) = 0$, $\tilde{U}'_1(0) = -U'_0(0)$,	$\hat{U}_1^{(4)} - (\hat{\alpha}_0\hat{U}'_1)' = 0$, $\lim_{\hat{x}\to\infty} \hat{U}_1(\hat{x}) = 0$, $\hat{U}'_1(0) = -U'_0(1)$,
$-(\alpha U'_j)' + \beta U_j = -U_{j-2}^{(4)}$, $U_j(0) = -\tilde{U}_j(0)$, $U_j(1) = -\hat{U}_j(0)$,	$\tilde{U}_j^{(4)} - (\tilde{\alpha}_0\tilde{U}'_j)' = \tilde{F}_j$, $\lim_{\tilde{x}\to\infty} \tilde{U}_j(\tilde{x}) = 0$, $\tilde{U}'_j(0) = -U'_{j-1}(0)$,	$\hat{U}_j^{(4)} - (\hat{\alpha}_0\hat{U}'_j)' = \hat{F}_j$, $\lim_{\hat{x}\to\infty} \hat{U}_j(\hat{x}) = 0$, $\hat{U}'_j(0) = -U'_{j-1}(1)$.

Next, we define for each $M \in \mathbb{N}_0$, the outer (smooth) expansion u_M^s as

$$u_M^s(x) := \sum_{j=0}^{M} \varepsilon^j U_j(x), \tag{2.2.16}$$

the boundary layer expansion at the left endpoint $\tilde{u}_M^{BL}$, as

$$\tilde{u}_M^{BL}(x) := \sum_{j=0}^{M+1} \varepsilon^j \tilde{U}_j(x/\varepsilon) = \sum_{j=0}^{M+1} \varepsilon^j \tilde{U}_j(\tilde{x}), \tag{2.2.17}$$

and the boundary layer expansion at the right endpoint $\hat{u}_M^{BL}$, as

$$\hat{u}_M^{BL}(x) := \sum_{j=0}^{M+1} \varepsilon^j \hat{U}_j\Big(\frac{1-x}{\varepsilon}\Big) = \sum_{j=0}^{M+1} \varepsilon^j \hat{U}_j(\hat{x}). \tag{2.2.18}$$

Finally, we define the remainder r_M as

$$r_M := u - (u_M^s + \tilde{u}_M^{BL} + \hat{u}_M^{BL}). \tag{2.2.19}$$

Therefore, we decompose the solution u of the problem (2.2.1) into a smooth part u_M, two boundary layer parts $\tilde{u}_M^{BL}, \hat{u}_M^{BL}$ and a smooth remainder r_M, viz.

$$u = u_M^s + \tilde{u}_M^{BL} + \hat{u}_M^{BL} + r_M. \tag{2.2.20}$$

2.3 Preliminaries

In this section we provide some useful propositions and lemmata which are needed to obtain the desired approximation results. The proofs are given in Section 2.5 (Appendix A).

Proposition 2.3.1 *The boundary value problem: find $u \in C^2([0,1])$ satisfying*

$$\begin{aligned} -\big(\alpha(x)u'(x)\big)' + \beta(x)u(x) = g(x), \\ u(0) = g^-,\ u(1) = g^+, \end{aligned} \tag{2.3.2}$$

(for some constants g^-, g^+ and sufficciently smooth α, β, g, where $\alpha(x) \geq c_1 \in \mathbb{R}^+$, $\beta(x) \geq$

$0\ \forall x \in \overline{I}$ *) is equivalent to the problem: find* $v \in C^2([0,1])$ *such that*

$$
\begin{aligned}
-\big(\alpha(x)v'(x)\big)' + \beta(x)v(x) = h(x),\\
v(0) = v(1) = 0,
\end{aligned}
\tag{2.3.3}
$$

where $h(x) := g(x) - \beta(x)\big((g^+ - g^-)x + g^-\big) + \alpha'(x)(g^+ - g^-)$.

Remark 2.3.4. The variational formulation of (2.3.3) reads: Find $u \in H_0^1(I)$, such that

$$
B(u,v) = F(v), \quad \text{for all } v \in H_0^1(I), \tag{2.3.5}
$$

where

$$
B(u,v) = \int_0^1 \alpha(x)u'(x)v'(x)dx + \int_0^1 \beta(x)u(x)v(x)dx, \tag{2.3.6}
$$

and

$$
F(v) = \int_0^1 h(x)v(x)dx. \tag{2.3.7}
$$

Associated with the above problem is the energy norm:

$$
\|u\|_{E,I} := \Big(B(u,u)\Big)^{1/2}. \tag{2.3.8}
$$

Lemma 2.3.9 *Let* $u \in C^2([0,1])$ *be the solution of the boundary value problem* (2.3.3) *and assume that the right hand side* h *is analytic. Then,*

$$
\|u\|_{0,I} \le \|u'\|_{0,I}. \tag{2.3.10}
$$

$$
\|u\|_{0,I} \le \frac{1}{\underline{\beta}}\|h\|_{0,I}, \tag{2.3.11}
$$

$$
\|u'\|_{0,I} \le \frac{1}{\underline{\alpha}}\|h\|_{0,I}, \tag{2.3.12}
$$

where α, β *are the coefficient functions in* (2.3.3).

Lemma 2.3.13 *Consider the problem* (2.3.2) *for some analytic right hand side function* g*. For some constants* C_g, c, $\tilde{c}$ *the solution to the problem satisfies the following inequalities:*

$$
\|u'\|_{0,I} + \|u\|_{0,I} \le \tilde{c}\{C_g + c(|g^+| + |g^-|)\}, \tag{2.3.14}
$$

$$
\|u'\|_{1,I} \le C_g\Big(\frac{1+\tilde{c}c}{\underline{\alpha}} + \tilde{c}\Big) + \Big(\frac{c}{\underline{\alpha}} + 1\Big)\tilde{c}c(|g^+| + |g^-|), \tag{2.3.15}
$$

$$\|u\|_{L^\infty(I)} \leq (2\pi)^{-1}\tilde{c}\{C_g + c(|g^+| + |g^-|)\}, \tag{2.3.16}$$

$$\|u'\|_{L^\infty(I)} \leq (2\pi)^{-1}\Big[C_g\Big(\frac{1+\tilde{c}c}{\underline{\alpha}} + \tilde{c}\Big) + \Big(\frac{c}{\underline{\alpha}} + 1\Big)\tilde{c}c(|g^+| + |g^-|)\Big], \tag{2.3.17}$$

where $\tilde{c} = \max\{\frac{1}{\underline{\beta}}, \frac{1}{\underline{\alpha}}\}$, $c = \overline{\alpha'} + \overline{\beta}$ *and* $C_g > \|g\|_{L^\infty(I)}$.

Proposition 2.3.18 *Let* $\lambda \in \mathbb{C}^+$. *Assume that* $u : (0,\infty) \to \mathbb{C}$ *satisfies*

$$u^{(4)} - \lambda u^{(2)} = f, \ \text{on } (0,\infty), \quad u'(0) = 0, \ \lim_{x\to\infty} u(x) = 0, \tag{2.3.19}$$

where f *is an entire function satisfying, for some* $C_f > 0$,

$$|f(z)| \leq C_f \mathrm{e}^{-Re(\sqrt{\lambda}z)}, \ \text{for all } z \in \mathbb{C}.$$

Then, there exists a $\kappa \in \mathbb{C}^+$ *with* $\kappa^2 = \lambda$ *and for* $x \in (0,\infty)$, u *is uniquely determined (due to the boundary conditions), as*

$$\begin{aligned} u(x) =& \frac{1}{\kappa\lambda}\mathrm{e}^{-\kappa x}\int_0^\infty f(\xi)d\xi - \frac{\mathrm{e}^{-\kappa x}}{2\kappa\lambda}\int_0^\infty \mathrm{e}^{-\kappa\xi}f(\xi)d\xi - \frac{\mathrm{e}^{-\kappa x}}{2\kappa\lambda}\int_0^x \mathrm{e}^{\kappa\xi}f(\xi)d\xi - \\ &- \frac{1}{\lambda}\int_x^\infty \xi f(\xi)d\xi + \frac{x}{\lambda}\int_x^\infty f(\xi)d\xi - \frac{\mathrm{e}^{\kappa x}}{2\kappa\lambda}\int_x^\infty \mathrm{e}^{-\kappa\xi}f(\xi)d\xi. \end{aligned}$$

Lemma 2.3.20 *[51, Lemma 4.8] For* $C_1, d > 0$ *and* $\rho \geq 0$ *the following estimates are valid with* $\tilde{\rho} = \rho/\varepsilon$, $\gamma = 2\max\{1, C_1^2\}$:

$$(C_1 l + \tilde{\rho})^{2l} \leq 2^l (C_1 l)^{2l} + 2^l \tilde{\rho}^{2l} \leq \gamma^l (l^{2l} + \tilde{\rho}^{2l}), \tag{2.3.21}$$

$$\sup_{\rho>0} \rho^n \mathrm{e}^{-\frac{d\rho}{4}} \leq \left(\frac{4n}{\mathrm{e}d}\right)^n. \tag{2.3.22}$$

2.4 Regularity results

We now present the regularity results for the solution u. As mentioned before, we assume that the problem has analytic data and as a result we obtain that the high order derivatives of u are also analytic.

Lemma 2.4.1 *Let* α, β *be analytic functions on* $\overline{I} = [0,1]$ *and* $\alpha(x) \geq c_1 \in \mathbb{R}^+$, $\beta(x) > 0$, $\forall x \in \overline{I}$. *Then there exists* $\gamma_0 > 0$ *such that for all* $\gamma \geq \gamma_0$ *and all* $C_g > \|g\|_{L^\infty(I)}$, *the* *following holds: If g satisfies*

$$\|g^{(n)}\|_{L^\infty(I)} \le C_g \gamma^n n!, \quad \forall n \in \mathbb{N}_0,$$

then the solution u of the boundary value problem

$$\left.\begin{aligned} -\big(\alpha(x)u'(x)\big)' + \beta(x)u(x) = g(x), \qquad x \in I = (0,1),\\ u(0) = g^- \in \mathbb{R},\ u(1) = g^+ \in \mathbb{R}, \end{aligned}\right\} \tag{2.4.2}$$

satisfies

$$\|u^{(n)}\|_{L^\infty(I)} \le \begin{cases} \tilde{C}\{C_g + C(|g^-| + |g^+|)\}, \quad n = 0, 1,\\ \bar{C}\gamma^{n-1}\{C_g + C(|g^-| + |g^+|)\}(n-1)^{n-1}, \quad n \ge 2, \end{cases} \tag{2.4.3}$$

with $C = c$, $\tilde{C} = \frac{1}{2\pi}\max\big\{\frac{1+\tilde{c}c}{\underline{\alpha}} + \tilde{c}, \tilde{c}\big(\frac{c}{\underline{\alpha}} + 1\big)\big\}$, $\bar{C} = \max\big\{1, \frac{1}{\underline{\alpha}}\big\}$ constants independent of n, where $c, \tilde{c}$ are given in Lemma 2.3.13.

Proof. We are going to show that inequality (2.4.3) holds by induction. We choose $\gamma_0 > 0$ to satisfy the following:

$$\gamma_0 \ge \max\{1, \mathrm{e}\gamma_\alpha, \mathrm{e}\gamma_\beta\},$$

and for all $n \in \mathbb{N}_0$, for all $\gamma \ge \gamma_0$,

$$\frac{\bar{C}C_\beta d}{\gamma^2(1 - \mathrm{e}\gamma_\beta/\gamma)} + \frac{\bar{C}C_\alpha d\mathrm{e}\gamma_\alpha}{\gamma(1 - \mathrm{e}\gamma_\alpha/\gamma)} + \frac{\tilde{C}C_\beta d(\mathrm{e}\gamma_\beta)^{n-1}}{\gamma^{n+1}} + \frac{\tilde{C}C_\beta d(\mathrm{e}\gamma_\beta)^n}{\gamma^{n+1}} + \frac{\tilde{C}C_\alpha d(\mathrm{e}\gamma_\alpha)^{n+1}}{\gamma^{n+1}} + \frac{1}{\gamma} \le 1,$$

where $d = \frac{\mathrm{e}^{1/12}\sqrt{2\pi}}{\mathrm{e}}$. The constants $\gamma_\alpha, \gamma_\beta$ are given in (2.2.2) and along with γ, they control the domain of the analyticity of the functions α, β and g, respectively.

Lemma 2.3.13 ensures that (2.4.3) holds for $n = 0, 1$. Taking the n-th derivative of the differential equation in (2.4.2) we get

$$(\alpha u')^{(n+1)} = (\beta u)^{(n)} - g^{(n)}.$$

Using the Leibniz formula for the n-th derivative of a product, we obtain

$$\alpha u^{(n+2)} = \sum_{k=0}^{n} \binom{n}{k} \beta^{(k)} u^{(n-k)} - \sum_{k=0}^{n} \binom{n+1}{k+1} \alpha^{(k+1)} u^{(n+1-k)} - g^{(n)}.$$

Therefore

$$\begin{aligned}\|\alpha u^{(n+2)}\|_{L^\infty(I)} \leq & \sum_{k=0}^{n} \binom{n}{k} \|\beta^{(k)}\|_{L^\infty(I)} \|u^{(n-k)}\|_{L^\infty(I)} + \\ & + \sum_{k=0}^{n} \binom{n+1}{k+1} \|\alpha^{(k+1)}\|_{L^\infty(I)} \|u^{(n+1-k)}\|_{L^\infty(I)} + \|g^{(n)}\|_{L^\infty(I)}.\end{aligned}$$

Since the functions α, β and g are analytic we have

$$\begin{aligned}&\|\alpha u^{(n+2)}\|_{L^\infty(I)} \leq \\ &\leq \sum_{k=0}^{n-2} \binom{n}{k} C_\beta \gamma_\beta^k k! \|u^{(n-k)}\|_{L^\infty(I)} + \binom{n}{n-1} C_\beta \gamma_\beta^{n-1} (n-1)! \|u'\|_{L^\infty(I)} + \\ &\quad + \binom{n}{n} C_\beta \gamma_\beta^n n! \|u\|_{L^\infty(I)} + \sum_{k=0}^{n-1} \binom{n+1}{k+1} C_\alpha \gamma_\alpha^{k+1} (k+1)! \|u^{(n+1-k)}\|_{L^\infty(I)} + \\ &\quad + \binom{n+1}{n+1} C_\alpha \gamma_\alpha^{n+1} (n+1)! \|u'\|_{L^\infty(I)} + C_g \gamma^n n!\end{aligned} \tag{2.4.4}$$

We now assume that (2.4.3) holds for $n+1$ and we are going to show it for $n+2$:

$$\begin{aligned}&\|\alpha u^{(n+2)}\|_{L^\infty(I)} \leq \\ &\leq \{C_g + C(|g^+| + |g^-|)\} \times \\ &\quad \times \Bigg[\sum_{k=0}^{n-2} \binom{n}{k} C_\beta \gamma_\beta^k k! \bar{C} \gamma^{n-1-k} (n-k-1)^{n-k-1} + \tilde{C} \binom{n}{n-1} C_\beta \gamma_\beta^{n-1} (n-1)! + \\ &\quad + \tilde{C} \binom{n}{n} C_\beta \gamma_\beta^n n! + \sum_{k=0}^{n-1} \binom{n+1}{k+1} C_\alpha \gamma_\alpha^{k+1} (k+1)! \bar{C} \gamma^{n-k} (n-k)^{n-k} + \\ &\quad + \tilde{C} \binom{n+1}{n+1} C_\alpha \gamma_\alpha^{n+1} (n+1)! \Bigg] + C_g \gamma^n n!\end{aligned} \tag{2.4.5}$$

By using Stirling's approximation for the factorial:

$$\left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n} \leq \mathrm{e}^{\frac{1}{12n+1}} \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n} \leq n! \leq \mathrm{e}^{\frac{1}{12n}} \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n} \leq \mathrm{e} \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}, \tag{2.4.6}$$

we can prove that

$$\binom{n}{k} \leq \left(\frac{\mathrm{e}n}{k}\right)^k, \quad k \leq n. \tag{2.4.7}$$

Moreover, by utilizing the right hand side inequality of Stirling's approximation we get

$$\frac{1}{n^n} \leq \frac{\mathrm{e}^{1/12n}\sqrt{2\pi n}}{\mathrm{e}^n n!} \leq \frac{d}{n!}, \tag{2.4.8}$$

since

$$\frac{\mathrm{e}^{1/12n}\sqrt{2\pi n}}{\mathrm{e}^n} \leq \frac{\mathrm{e}^{1/12}\sqrt{2\pi}}{\mathrm{e}} = d, \tag{2.4.9}$$

for all $n = 2, 3, \ldots$ Inserting (2.4.7) and (2.4.8) into (2.4.5) we get

$$\begin{aligned}
\|\alpha u^{(n+2)}\|_{L^\infty(I)} \leq{}& \{C_g + C(|g^+| + |g^-|)\} \times \Bigg[\sum_{k=0}^{n-2}\left(\frac{\mathrm{e}n}{k}\right)^k C_\beta \gamma_\beta^k k! \bar{C}\gamma^{n-1-k}(n-k-1)^{n-k-1} + \\
&+ \left(\frac{\mathrm{e}n}{n-1}\right)^{n-1}\tilde{C}C_\beta\gamma_\beta^{n-1}(n-1)! + \left(\frac{\mathrm{e}n}{n}\right)^n \tilde{C}C_\beta\gamma_\beta^n n! + \left(\frac{\mathrm{e}(n+1)}{n+1}\right)^{n+1}\tilde{C}C_\alpha\gamma_\alpha^{n+1}(n+1)! + \\
&+ \sum_{k=0}^{n-1}\left(\frac{\mathrm{e}(n+1)}{k+1}\right)^{k+1} C_\alpha\gamma_\alpha^{k+1}(k+1)!\bar{C}\gamma^{n-k}(n-k)^{n-k}\Bigg] + C_g\gamma^n n! \\
\leq{}& \{C_g + C(|g^+| + |g^-|)\} \times \Bigg[\bar{C}C_\beta d\gamma^{n-1}n^{n-1}\sum_{k=0}^{n-2}\left(\frac{\mathrm{e}\gamma_\beta}{\gamma}\right)^k + \bar{C}C_\alpha d\gamma^n e\gamma_\alpha(n+1)^{n+1}\sum_{k=0}^{n-1}\left(\frac{\mathrm{e}\gamma_\alpha}{\gamma}\right)^k + \\
&+ \tilde{C}C_\beta d(\mathrm{e}\gamma_\beta)^{n-1}n^{n-1} + \tilde{C}C_\beta d(\mathrm{e}\gamma_\beta)^n n^n + \tilde{C}C_\alpha d(\mathrm{e}\gamma_\alpha)^{n+1}(n+1)^{n+1}\Bigg] + C_g\gamma^n(n+1)^{n+1/2}.
\end{aligned}$$

Since the constant γ satisfies $\gamma \geq \max\{1, \mathrm{e}\gamma_\alpha, \mathrm{e}\gamma_\beta\}$, the sums above can be estimated by convergent geometric series. We then have

$$\begin{aligned}
\|\alpha u^{(n+2)}\|_{L^\infty(I)} \leq{}& \big[C_g + C(|g^-| + |g^+|)\big]\gamma^{n+1}(n+1)^{n+1} \times \\
\times&\left[\frac{\bar{C}C_\beta d}{\gamma^2(1-\mathrm{e}\gamma_\beta/\gamma)} + \frac{\bar{C}C_\alpha d\mathrm{e}\gamma_\alpha}{\gamma(1-\mathrm{e}\gamma_\alpha/\gamma)} + \frac{\tilde{C}C_\beta d(\mathrm{e}\gamma_\beta)^{n-1}}{\gamma^{n+1}} + \frac{\tilde{C}C_\beta d(\mathrm{e}\gamma_\beta)^n}{\gamma^{n+1}} + \frac{\tilde{C}C_\alpha d(\mathrm{e}\gamma_\alpha)^{n+1}}{\gamma^{n+1}} + \frac{\gamma^n}{\gamma^{n+1}}\right].
\end{aligned}$$

The specific choice of γ made before, gives

$$\underline{\alpha}\|u^{(n+2)}\|_{L^\infty(I)} \leq \gamma^{n+1}\big[C_g + C(|g^-| + |g^+|)\big](n+1)^{n+1},$$

and therefore

$$\|u^{(n+2)}\|_{L^\infty(I)} \leq \bar{C}\gamma^{n+1}\big[C_g + C(|g^-| + |g^+|)\big](n+1)^{n+1}.$$

This completes the proof. □

Proposition 2.4.10 *Let $j \in \mathbb{N}$, $\kappa, \lambda \in \mathbb{C}^+$, with $\lambda = \kappa^2$ and let F be an entire function* *satisfying, for some* $C_F > 0$, $q \geq \frac{4j}{|\kappa|}$,

$$|F(z)| \leq C_F \mathrm{e}^{-Re(\kappa z)}(q+|z|)^{2j-1}, \quad \textit{for all } z \in \mathbb{C}.$$

Furthermore, let $g_1 \in \mathbb{C}$*, and let* $w : (0,\infty) \to \mathbb{C}$ *be the solution of*

$$w^{(4)} - \lambda w^{(2)} = F \ \textit{on} \ (0,\infty), \quad w'(0) = g_1, \ \lim_{x\to\infty} w(x) = 0. \tag{2.4.11}$$

Then, w can be extended to an entire function (denoted again by w) which satisfies

$$|w(z)| \leq C\left[\frac{C_F}{2j}(q+|z|)^{2j} + \frac{|g_1|}{|\kappa|}\right]\mathrm{e}^{-Re(\kappa z)}, \quad \textit{for all } z \in \mathbb{C}.$$

Proof. The proof follows very closely the proof of Lemma 7.3.6 in [45] and Lemma 4 in [57]. By Proposition 2.3.18, for $z \in (0,\infty)$, w is given by

$$\begin{aligned} w(z) =& \frac{\mathrm{e}^{-\kappa z}}{\lambda^2}\int_0^\infty F(y/\kappa)dy - \frac{\mathrm{e}^{-\kappa z}}{2\lambda^2}\int_0^\infty \mathrm{e}^{-y}F(y/\kappa)dy \\ &- \frac{\mathrm{e}^{-\sqrt{\lambda}z}}{2\lambda^2}\int_0^{\kappa z}\mathrm{e}^{y}F(y/\kappa)dy - \frac{1}{\lambda^2}\int_{\kappa z}^\infty yF(y/\kappa)dy \\ &+ \frac{z}{\kappa\lambda}\int_{\kappa z}^\infty F(y/\kappa)dy - \frac{\mathrm{e}^{\kappa z}}{2\lambda^2}\int_{\kappa z}^\infty \mathrm{e}^{-y}F(y/\kappa)dy - \frac{g_1}{\kappa}\mathrm{e}^{-\kappa z}. \end{aligned}$$

We remove the restriction to $(0,\infty)$ with analytic continuation and we procced by giving bounds on all the above terms. In order to bound the third term, we use as path of integration the straight line connecting 0 and κz to get

$$\begin{aligned} \left|\frac{\mathrm{e}^{-\sqrt{\lambda}z}}{2\lambda^2}\int_0^{\kappa z}\mathrm{e}^{y}F(y/\kappa)dy\right| \leq& \frac{1}{2|\lambda^2|}\mathrm{e}^{-Re(\kappa z)}\int_0^1 C_F(q+t|z|)^{2j-1}|\kappa z|\mathrm{e}^{-Re(\kappa tz)}\mathrm{e}^{Re(\kappa tz)}dt \\ \leq& C_F\frac{|\kappa|}{2|\lambda^2|}\frac{\mathrm{e}^{-Re(\kappa z)}}{2j}\Big((q+|z|)^{2j} - q^{2j}\Big). \end{aligned}$$

The following, which gives an estimate for the sixth term, has an almost identical proof to that of Lemma 7.3.6 in [45]:

$$\begin{aligned} \left|\frac{\mathrm{e}^{\sqrt{\lambda}z}}{2\lambda^2}\int_{\kappa z}^\infty \mathrm{e}^{-y}F(y/\kappa)dy\right| &= \frac{1}{2|\lambda|^2}\left|\int_0^\infty \mathrm{e}^{-y}F(z+y/\kappa)dy\right| \\ &\leq \frac{1}{2|\lambda|^2}\int_0^\infty \mathrm{e}^{-y}C_F\mathrm{e}^{-Re(\kappa z+y)}\left(q+|z|+\frac{y}{|\kappa|}\right)^{2j-1}dy \\ &\leq \frac{1}{2|\lambda|^2}C_F\mathrm{e}^{-Re(\kappa z)}|\kappa|^{-(2j-1)}\int_0^\infty \mathrm{e}^{-2y}\big(|\kappa|q+|\kappa z|+y\big)^{2j-1}dy \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2|\lambda|^2} C_F \mathrm{e}^{-Re(\kappa z)} |\kappa|^{-(2j-1)} \int_0^\infty \mathrm{e}^{-y} \frac{1}{2} \big(|\kappa| q + |\kappa z| + y/2\big)^{2j-1} dy \\
&\le \frac{1}{2|\lambda|^2} C_F \mathrm{e}^{-Re(\kappa z)} |\kappa|^{-(2j-1)} 2^{-2j} \int_0^\infty \mathrm{e}^{-y} \big(2|\kappa| q + 2|\kappa z| + y\big)^{2j-1} dy \\
&= \frac{C_F}{2} \mathrm{e}^{-Re(\kappa z)} |\kappa|^{-(2j-1)} 2^{-2j} \mathrm{e}^{2|\kappa|(q+|z|)} \Gamma(2j, 2|\kappa|(q+|z|)),
\end{aligned}
$$

where $\Gamma(\cdot\,,\cdot)$ denotes the incomplete Gamma function. We observe that $2|\kappa| q \ge 2j-1$. Thus we may employ the estimate

$$
\Gamma(a,\xi) \le \frac{\mathrm{e}^{-\xi}\xi^a}{|\xi| - a_0}, \quad a_0 = \max\{a-1, 0\},\ Re(\xi) \ge 0,\ |\xi| > a_0,
$$

(which can be found in [54, Chapter 4, Section 10]) to get

$$
\begin{aligned}
\left| \frac{\mathrm{e}^{\sqrt{\lambda} z}}{2|\lambda|^2} \int_{\kappa z}^\infty \mathrm{e}^{-y} F(y/\kappa) dy \right| &\le \frac{1}{2|\lambda|^2} C_F \mathrm{e}^{-Re(\kappa z)} |\kappa| \frac{(q+|z|)^{2j}}{2|\kappa|(q+|z|) - (2j-1)} \\
&\le \frac{1}{2|\lambda|^2} C_F \mathrm{e}^{-Re(\kappa z)} |\kappa| \frac{(q+|z|)^{2j}}{6j+1}.
\end{aligned}
$$

We note that the integral $\int_0^\infty \mathrm{e}^{-y} F(y/\kappa) d\xi$ is treated just like the sixth term. Hence, for the second term, we obtain

$$
\left| \frac{\mathrm{e}^{-\sqrt{\lambda} z}}{2\lambda^2} \int_0^\infty \mathrm{e}^{-y} F(y/\kappa) dy \right| \le \frac{1}{2|\lambda|^2} C_F \mathrm{e}^{-Re(\kappa z)} |\kappa| \frac{(q+|z|)^{2j}}{6j+1}.
$$

The first and the last term of the solution satisfy the desired estimate as they are multiplied by $\mathrm{e}^{-\kappa z}$. To complete the proof we need to bound the term $\frac{z}{\kappa\lambda} \int_{\kappa z}^\infty F(y/\kappa) dy - \frac{1}{\lambda^2} \int_{\kappa z}^\infty y F(y/\kappa) dy$. We have

$$
\begin{aligned}
&\frac{1}{|\lambda|^2} \left| \int_{\kappa z}^\infty (\kappa z - y) F(y/\kappa) dy \right| = \\
&= \frac{1}{|\lambda|^2} \left| \int_{\kappa z}^\infty (y - \kappa z) F(y/\kappa) dy \right| \\
&= \frac{1}{|\lambda|^2} \left| \int_0^\infty t F\Big(\frac{t}{\kappa} + z\Big) dt \right| \\
&\le \frac{C_F \mathrm{e}^{-Re(\kappa z)}}{|\lambda|^2} \int_0^\infty t \left(q + |z| + \frac{t}{|\kappa|} \right)^{2j-1} \mathrm{e}^{-t} dt \\
&= \frac{C_F \mathrm{e}^{-Re(\kappa z)}}{|\lambda|^2} \sum_{n=0}^{2j-1} \frac{1}{|\kappa|^n} \binom{2j-1}{n} (q+|z|)^{2j-1-n} \int_0^\infty t^{n+1} \mathrm{e}^{-t} dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{C_F \mathrm{e}^{-Re(\kappa z)}}{|\lambda|^2} \sum_{n=0}^{2j-1} \frac{1}{|\kappa|^n} \binom{2j-1}{n} (q+|z|)^{2j-1-n} \Gamma(n+2) \\
&\leq \frac{C_F \mathrm{e}^{-Re(\kappa z)}}{|\lambda|^2} (q+|z|)^{2j-1} \sum_{n=0}^{2j-1} \frac{1}{|\kappa|^n} \frac{(n+1)(2j-1)!}{(2j-1-n)!} (q+|z|)^{-n} \\
&\leq \frac{C_F \mathrm{e}^{-Re(\kappa z)}}{|\lambda|^2} (q+|z|)^{2j-1} \sum_{n=0}^{2j-1} \frac{(n+1)}{|\kappa|^n} (2j-1)^n (q+|z|)^{-n} \\
&\leq \frac{C_F \mathrm{e}^{-Re(\kappa z)}}{|\lambda|^2} (q+|z|)^{2j-1} \sum_{n=0}^{2j-1} (n+1)(q/2)^n (q+|z|)^{-n} \\
&\leq \frac{C_F \mathrm{e}^{-Re(\kappa z)}}{|\lambda|^2} (q+|z|)^{2j-1} \sum_{n=0}^{\infty} 2^{-n}(n+1) \\
&\leq \frac{4C_F \mathrm{e}^{-Re(\kappa z)}}{|\lambda|^2} (q+|z|)^{2j-1} \\
&\leq \frac{C_F \mathrm{e}^{-Re(\kappa z)} (q+|z|)^{2j}}{j|\kappa\lambda|}.
\end{aligned}
$$

Combining all the above inequalities we get

$$
|w(z)| \leq C \left[\frac{C_F}{j} (q+|z|)^{2j} + \frac{|g_1|}{|\kappa|} \right] \mathrm{e}^{-Re(\kappa z)}, \quad \text{for all } z \in \mathbb{C}, \tag{2.4.12}
$$

where C is a constant independent of v and j. □

Lemma 2.4.13 *Let $j \in \mathbb{N}$ and let the function V be an entire function which satisfies*

$$
|V(z)| \leq \frac{C\gamma_V^j}{(j-1)!} (q_j + |z|)^{2(j-1)} \mathrm{e}^{-Re(\sqrt{\alpha(0)} z)}, \ \forall z \in \mathbb{C},
$$

for some constants $C, \gamma_V, \alpha(0) > 0$, and with $q_j = \frac{4j}{\sqrt{\alpha(0)}}$. Then, for all $n \in \mathbb{N}$, we have

$$
|V^{(n)}(z)| \leq C \frac{n! \mathrm{e}^{n+1}}{(n+1)^n} \frac{\gamma_V^j}{(j-1)!} \mathrm{e}^{-Re(\sqrt{\alpha(0)} z)} \left(q_j + \frac{n+1}{\sqrt{\alpha(0)}} + |z| \right)^{2(j-1)}, \quad z \in \mathbb{C}. \tag{2.4.14}
$$

Proof. We apply Cauchy's Integral Theorem for derivatives, letting the integration path be a circle of radius $\frac{n+1}{\sqrt{\alpha(0)}}$ about z and we get

$$
\begin{aligned}
|V^{(n)}(z)| &\leq \frac{n!}{2\pi} \oint_D \frac{|V(\zeta)|}{|\zeta - z|^{n+1}} d\zeta \\
&\leq C \frac{\left(\sqrt{\alpha(0)}\right)^n n! e^{n+1}}{(n+1)^n} \frac{\gamma_V^j}{(j-1)!} \mathrm{e}^{-Re(\sqrt{\alpha(0)} z)} \left(q_j + \frac{n+1}{\sqrt{\alpha(0)}} + |z| \right)^{2(j-1)},
\end{aligned} \tag{2.4.15}
$$

for all $k = 0, \dots, j$. □

Theorem 2.4.16 *Consider the triple* $(U_j, \tilde{U}_j, \hat{U}_j)$ *where* $\{U_j\}_{j\in\mathbb{N}_0}$ *are defined by* (2.2.7), (2.2.10), (2.2.13), $\{\tilde{U}_j\}_{j\in\mathbb{N}_0}$ *by* (2.2.8), (2.2.11), (2.2.12) *and* $\{\hat{U}_j\}_{j\in\mathbb{N}_0}$ *by* (2.2.8), (2.2.14), (2.2.15). *Then, there exist constants* $a, \gamma, C, K > 1$*, independent of* ε*, such that, for all* $n \in \mathbb{N}_0$,

$$\|U_j^{(n)}\|_{L^\infty(I)} \le C\gamma^j \frac{a^{2j} j^{2j}}{j!} K^n n!, \qquad \forall j \in \mathbb{N}_0, \tag{2.4.17}$$

$$|\tilde{U}_j(z)| \le C\gamma^j \frac{1}{(j-1)!}(aj + |z|)^{2(j-1)} \mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}, \qquad \forall z \in \mathbb{C},\ j \in \mathbb{N}, \tag{2.4.18}$$

$$|\hat{U}_j(z)| \le C\gamma^j \frac{1}{(j-1)!}(aj + |z|)^{2(j-1)} \mathrm{e}^{-\sqrt{\alpha(1)}Re(z)}, \qquad \forall z \in \mathbb{C},\ j \in \mathbb{N}. \tag{2.4.19}$$

We note that the constant K *is directly dependent on* γ_f *given in* (2.2.2), *(i.e. on the analyticity of* f*).*

Proof. Let γ_0 be the constant of Lemma 2.4.1 and let γ_f given in (2.2.2). We choose $a, \gamma > 1$, $\tilde{\gamma}_f \ge \max\{\gamma_0, \gamma_f\}$ and $K = \max\{1, \mathrm{e}\tilde{\gamma}_f\}$ to satisfy,

$$a \ge \frac{3}{\min\left\{\sqrt{\alpha(0)}, \sqrt{\alpha(1)}\right\}}, \tag{2.4.20}$$

$$\gamma \ge \max\{\gamma_\alpha, \gamma_{\alpha'}, \gamma_\beta\}, \tag{2.4.21}$$

$$\frac{1}{\gamma}\left(1 + \frac{K}{\min\left\{\sqrt{\alpha(0)}, \sqrt{\alpha(1)}\right\}}\right) \le 1, \tag{2.4.22}$$

$$\frac{2}{a^2}\left(\frac{12K^4}{a^2\gamma^2} + 1\right) \le 1. \tag{2.4.23}$$

We observe that the sequences U_j, $\tilde{U}_j$ and $\hat{U}_j$ are intertwined and cannot be analyzed separately therefore one should treat them simultaneously. We are going to prove (2.4.17)–(2.4.19) using induction. By Lemma 2.4.1 we have

$$\|U_0^{(n)}\|_{L^\infty(I)} \le \begin{cases} \tilde{C}C_f, n = 0, 1 \\ \bar{C}C_f\tilde{\gamma}_f^{n-1}(n-1)^{n-1}, n \ge 2. \end{cases} \tag{2.4.24}$$

Here $\tilde{C}$, $\bar{C}$ are the constants of Lemma 2.4.1. Therefore, for $j = 0$, (2.4.17) holds by (2.4.24) and Stirling's approximation. The function $\tilde{U}_0$ given by (2.2.8), (2.2.11) and

(2.2.12) is calculated as $\tilde{U}_0 = 0$. In the same manner $\hat{U}_0 = 0$. Next, we calculate $\tilde{U}_1$, $\hat{U}_1$, by solving the corresponding boundary value problems:

$$\tilde{U}_1(\tilde{x}) = \frac{U_0'(0)}{\sqrt{\alpha(0)}} \mathrm{e}^{-\sqrt{\alpha(0)}\tilde{x}}, \tag{2.4.25}$$

and

$$\hat{U}_1(\hat{x}) = \frac{U_0'(1)}{\sqrt{\alpha(1)}} \mathrm{e}^{-\sqrt{\alpha(1)}\hat{x}}. \tag{2.4.26}$$

This shows that (2.4.18), (2.4.19) hold for $j = 1$, once we extend $\tilde{U}_1$, $\hat{U}_1$ to $\mathbb{C}$. Moreover, one gets

$$|\tilde{U}_1(0)| + |\hat{U}_1(0)| \leq \frac{|U_0'(0)|}{\sqrt{\alpha(0)}} + \frac{|U_0'(1)|}{\sqrt{\alpha(1)}} \leq CK\Big(\frac{1}{\sqrt{\alpha(0)}} + \frac{1}{\sqrt{\alpha(1)}}\Big). \tag{2.4.27}$$

In the above inequality, we used (2.4.17) for $n = 1$, $j = 0$. We consider U_1 that is given by (2.2.7), (2.2.10) and (2.2.13). From Lemma 2.4.1, we get

$$\|U_1^{(n)}\|_{L^\infty(I)} \leq \begin{cases} C\big(|\tilde{U}_1(0)| + |\hat{U}_1(0)|\big), n = 0, 1 \\ C\tilde{\gamma}_f{}^{n-1}\big(|\tilde{U}_1(0)| + |\hat{U}_1(0)|\big)(n-1)^{n-1}, n \geq 2. \end{cases} \tag{2.4.28}$$

From (2.4.27), (2.4.28) and Stirling's approximation we obtain that (2.4.17) holds for $j = 1$.

We now consider the triple $(U_j, \tilde{U}_j, \hat{U}_j)$ and we assume that the desired results hold for j. We will show that they hold also for $j + 1$. Since $\tilde{U}_{j+1}$ satisfies (2.2.8), (2.2.11) and (2.2.12), we may use Proposition 2.4.10, once we show that

$$|\tilde{F}_{j+1}(z)| \leq C\gamma^j e^{-\sqrt{\alpha(0)}Re(z)} \frac{1}{(j-1)!}(t_{j+1} + |z|)^{2j-1},$$

for some number t_{j+1}, depending on $j + 1$. To this end, we recall that

$$\tilde{F}_j(z) = \sum_{k=1}^{A_j} \alpha_k(z)\tilde{U}_{j-k}''(z) + \sum_{k=1}^{A_j} \alpha_k'(z)\tilde{U}_{j-k}'(z) - \sum_{k=0}^{B_j} \beta_k(z)\tilde{U}_{j-2-k}(z).$$

By the induction hypothesis we have that, for all $k = 0, \ldots, j-1$,

$$|\tilde{U}_{j-k}(\tilde{z})| \leq C\gamma^{j-k}\frac{1}{(j-k-1)!}\Big(a(j-k) + |z|\Big)^{2(j-k-1)} \mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}. \tag{2.4.29}$$

From Lemma 2.4.13 we obtain

$$\begin{aligned}
&\left|\tilde{F}_{j+1}(z)\right| \leq \\
&\quad\leq \sum_{k=1}^{A_{j+1}} C_1\gamma^{j+1-k}C_\alpha\gamma_\alpha^k|z|^k\frac{1}{(j-k)!}\Big(a(j+1-k)+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{2(j-k)}\mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}+\\
&\quad+\sum_{k=1}^{A_{j+1}} C_2\gamma^{j+1-k}C_{\alpha'}\gamma_{\alpha'}^k|z|^k\frac{1}{(j-k)!}\Big(a(j+1-k)+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{2(j-k)}\mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}+\\
&\quad+\sum_{k=0}^{B_{j+1}} C_3\gamma^{j-1-k}C_\beta\gamma_\beta^k|z|^k\frac{1}{(j-2-k)!}\Big(a(j-1-k)+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{2(j-2-k)}\mathrm{e}^{-\sqrt{\alpha(0)}Re(z)},
\end{aligned}$$

since the functions α, β are analytic. Next we present some useful inequalities, namely, for $j \geq 2$, we have,

$$\begin{aligned}
\frac{1}{(j-k)!} \leq&\frac{(j-1)^{k-1}}{(j-1)!} \leq \frac{1}{(j-1)!}\Big(aj+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{k-1}, \quad k \leq j-1,\\
\frac{1}{(j-2-k)!} \leq&\frac{(j-1)^{k+1}}{(j-1)!} \leq \frac{1}{(j-1)!}\Big(aj+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{k+1}, \quad k \leq j-2,
\end{aligned}$$

since $a > 1$ and hence, for $k \leq j-1$,

$$|z|^k\frac{1}{(j-k)!}\Big(a(j+1-k)+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{2(j-k)} \leq \frac{1}{(j-1)!}\Big(a(j+1-k)+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{2j-1},$$

and for $k \leq j-2$,

$$\begin{aligned}
|z|^k\frac{1}{(j-2-k)!}&\Big(a(j+1-k)+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{2(j-2-k)}\\
&\leq\frac{C}{(j-1)!}\Big(a(j+1-k)+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{2j-1}.
\end{aligned}$$

This gives

$$\begin{aligned}
|\tilde{F}_{j+1}(z)| \leq& C\gamma^j\frac{1}{(j-1)!}\Big(aj+\frac{3}{\sqrt{\alpha(0)}}+|z|\Big)^{2j-1}\mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}\times\\
&\times\left[\sum_{k=1}^{A_{j+1}} C_1C_\alpha\gamma_\alpha\Big(\frac{\gamma_\alpha}{\gamma}\Big)^{k-1}+\sum_{k=1}^{A_{j+1}} C_2C_{\alpha'}\gamma_{\alpha'}\Big(\frac{\gamma_{\alpha'}}{\gamma}\Big)^{k-1}++\sum_{k=0}^{B_{j+1}}\frac{C_3C_\beta}{\gamma_\beta}\Big(\frac{\gamma_\beta}{\gamma}\Big)^k\right].
\end{aligned}$$

By (2.4.21) we ensure that the sums above are bounded by convergent geometric series, thus

$$|\tilde{F}_{j+1}(z)| \leq C\gamma^j \frac{1}{(j-1)!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |z|\Big)^{2j-1} \mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}, \tag{2.4.30}$$

and by Proposition 2.4.10, we infer that

$$|\tilde{U}_{j+1}(z)| \leq C\left[\frac{\gamma^j}{j}\frac{1}{(j-1)!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |z|\Big)^{2j} + \Big|\frac{U_j'(0)}{\sqrt{\alpha(0)}}\Big|\right]\mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}.$$

We bound now the term $|U_j'(0)|$ using the induction hypothesis corresponding to (2.4.17). We get

$$|\tilde{U}_{j+1}(z)| \leq C\left[\frac{\gamma^j}{j!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |z|\Big)^{2j} + \frac{K\gamma^j a^{2j} j^{2j}}{j!\sqrt{\alpha(0)}}\right]\mathrm{e}^{-\sqrt{\alpha(0)}Re(z)},$$

and thus we have

$$|\tilde{U}_{j+1}(z)| \leq C\gamma^{j+1}\frac{1}{j!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |z|\Big)^{2j}\left[\frac{1}{\gamma} + \frac{K}{\sqrt{\alpha(0)}\gamma}\right]\mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}.$$

We conclude that

$$|\tilde{U}_{j+1}(z)| \leq C\frac{\gamma^{j+1}}{j!}\Big(a(j+1) + |z|\Big)^{2j}\mathrm{e}^{-\sqrt{\alpha(0)}Re(z)}, \tag{2.4.31}$$

by (2.4.20) and (2.4.22). Thus the assertion (2.4.18) holds for $j+1$. In the same manner, one can prove that (2.4.19) holds also. We next proceed with function U_{j+1}, which satisfies the boundary value problem defined by (2.2.7), (2.2.10), (2.2.13). The induction hypothesis for $j-1$, gives

$$\|U_{j-1}^{(4)}\|_{L^\infty(I)} \leq C\gamma^{j-1}\frac{a^{2(j-1)}(j-1)^{2(j-1)}}{(j-1)!}K^4 4!. \tag{2.4.32}$$

By considering the above, we appeal to Lemma 2.4.1 which implies the estimate

$$\|U_{j+1}^{(n)}\|_{L^\infty(I)} \leq \begin{cases} C\big[\|U_{j-1}^{(4)}\|_{L^\infty(I)} + |\tilde{U}_{j+1}(0)| + |\hat{U}_{j+1}(0)|\big], \; n=0,1, \\ C\tilde{\gamma}_f^{n-1}\big[\|U_{j-1}^{(4)}\|_{L^\infty(I)} + |\tilde{U}_{j+1}(0)| + |\hat{U}_{j+1}(0)|\big](n-1)^{n-1}, \; n \geq 2. \end{cases} \tag{2.4.33}$$

We note that we have already proved (2.4.18) and (2.4.19) for $j+1$. Therefore

$$|\tilde{U}_{j+1}(0)| \leq C\gamma^{j+1}\frac{1}{j!}a^{2j}(j+1)^{2j}, \tag{2.4.34}$$

and the same result holds for $\hat{U}_{j+1}(0)$. We combine (2.4.32)–(2.4.34) and we have that

$$\begin{aligned}\|U^{(4)}_{j-1}\|_{L^\infty(I)} + |\tilde{U}_{j+1}(0)| + |\hat{U}_{j+1}(0)| \leq& \\ \leq& C\left[4!K^4\gamma^{j-1}\frac{1}{(j-1)!}a^{2(j-1)}(j-1)^{2(j-1)} + 2\gamma^{j+1}\frac{1}{j!}a^{2j}(j+1)^{2j}\right] \\ \leq& C\gamma^{j+1}\frac{a^{2(j+1)}(j+1)^{2(j+1)}}{(j+1)!}\left[\frac{4!K^4}{a^4\gamma^2} + \frac{2}{a^2}\right] \\ \leq& C\gamma^{j+1}\frac{a^{2(j+1)}(j+1)^{2(j+1)}}{(j+1)!}.\end{aligned}$$

In the last step we used the assumption (2.4.23). The above result and (2.4.33) yield (2.4.17) for $j+1$. We mention that we treat the term $(n-1)^{n-1}$ in (2.4.33) with the aid of Stirling's approximation and we also note that we obtain the desired results only in the case that (2.4.20)–(2.4.23) hold. □

Corollary 2.4.35 *Let the functions* $\tilde{U}_j$, $\hat{U}_j$ *be given by* (2.2.8), (2.2.11), (2.2.12) *and* (2.2.8), (2.2.14), (2.2.15), *respectively. Then, there exist constants* $C, \tilde{K}, \hat{K}, \gamma > 0$ *independent of* j *and* n *such that, for all* $\tilde{x}$, $\hat{x} > 0$, *and* $\forall\, n \in \mathbb{N}$,

$$|\tilde{U}^{(n)}_j(\tilde{x})| \leq C\tilde{K}^n(a^2 \mathrm{e}\gamma)^j j^{j-1}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|}, \tag{2.4.36}$$

$$|\hat{U}^{(n)}_j(\hat{x})| \leq C\hat{K}^n(a^2 \mathrm{e}\gamma)^j j^{j-1}\mathrm{e}^{-\frac{\sqrt{\alpha(1)}}{2}|\hat{x}|}. \tag{2.4.37}$$

Proof. In Theorem 2.4.16 we have shown that $\tilde{U}_j$, $\hat{U}_j$ are entire and moreover they satisfy (2.4.18) and (2.4.19). Using Lemma 2.4.13 we get for $z \in \mathbb{C}$,

$$|\tilde{U}^{(n)}_j(z)| \leq C\frac{\left(\sqrt{\alpha(0)}\right)^n n^n \mathrm{e}^{n+1}}{(n+1)^n}\gamma^j \mathrm{e}^{-Re(\sqrt{\alpha(0)}z)}\frac{1}{(j-1)!}\left(aj + \frac{n+1}{\sqrt{\alpha(0)}} + |z|\right)^{2(j-1)}. \tag{2.4.38}$$

From Lemma 2.3.20 we have for $\tilde{x} > 0$,

$$\begin{aligned}\left(aj + \frac{n+1}{\sqrt{\alpha(0)}} + \tilde{x}\right)^{2(j-1)}\mathrm{e}^{-\sqrt{\alpha(0)}|\tilde{x}|} \leq& \\ \leq& C\left(aj + \frac{n+1}{\sqrt{\alpha(0)}}\right)^{2(j-1)}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|}\end{aligned}$$

$$
\begin{aligned}
&\leq C(aj)^{2(j-1)}\left(1+\frac{(n+1)}{\sqrt{\alpha(0)}aj}\right)^{2(j-1)} \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|}\\
&\leq C(aj)^{2(j-1)}\left(1+\frac{2(n+1)}{2a\sqrt{\alpha(0)}(j-1)}\right)^{2(j-1)} \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|}\\
&\leq C(aj)^{2(j-1)}\mathrm{e}^{\frac{2(n+1)}{a\sqrt{\alpha(0)}}}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|},
\end{aligned}
\tag{2.4.39}
$$

and we get

$$
\begin{aligned}
&\frac{1}{(j-1)!}\left(aj+\frac{n+1}{\sqrt{\alpha(0)}}+\tilde{x}\right)^{2(j-1)}\mathrm{e}^{-\sqrt{\alpha(0)}|\tilde{x}|}\leq\\
&\qquad\leq Ca^{2(j-1)}\frac{j^{2(j-1)}}{(j-1)!}\mathrm{e}^{\frac{2(n+1)}{a\sqrt{\alpha(0)}}}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|}\\
&\qquad\leq C(a^2\mathrm{e})^{j-1}j^{j-1}\mathrm{e}^{\frac{2(n+1)}{a\sqrt{\alpha(0)}}}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|}.
\end{aligned}
\tag{2.4.40}
$$

In the last step the Stirling's approximation has been used. Therefore

$$
\begin{aligned}
|\tilde{U}_j^{(n)}(\tilde{x})| &\leq C\frac{\sqrt{\alpha(0)}n^n\mathrm{e}^{n+1}}{(n+1)^n}\gamma^j\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|}(a^2\mathrm{e})^{j-1}\mathrm{e}^{\frac{2(n+1)}{a\sqrt{\alpha(0)}}}j^{j-1}\\
&\leq C\frac{\mathrm{e}\sqrt{\alpha(0)}}{a^2}\mathrm{e}^{\frac{2(n+1)}{a\sqrt{\alpha(0)}}}\left(\frac{n\mathrm{e}}{n+1}\right)^n(a^2\mathrm{e}\gamma)^j j^{j-1}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|},
\end{aligned}
\tag{2.4.41}
$$

and this completes the proof. □

We next present the main result regarding the regularity of the solution. Theorem 2.4.42 tells us that the solution has an analytic character, the boundary layers do not affect the solution in areas away from the boundary and the remainder is exponentially small.

Theorem 2.4.42 *Assume that* (2.2.2) *holds. Let* $M\in\mathbb{N}_0$*, there are constants* C*,*K*,*K_1*,* $q>0$ *independent of* $\varepsilon\in(0,1]$ *such that the solution* u *of* (2.2.1) *can be written as*

$$
u=u_M^s+\tilde{u}_M^{BL}+\hat{u}_M^{BL}+r_M,
$$

with

$$
\|(u_M^s)^{(n)}\|_{0,I}\leq CK^n n!,\quad \forall\, n\in\mathbb{N}_0, \tag{2.4.43}
$$

$$
\left|\left(\tilde{u}_M^{BL}\right)^{(n)}(x)\right|+\left|\left(\hat{u}_M^{BL}\right)^{(n)}(x)\right|\leq CK_1^n\varepsilon^{1-n}\mathrm{e}^{-\min\left\{\frac{\sqrt{\alpha(0)}}{2},\frac{\sqrt{\alpha(1)}}{2}\right\}dist(x,\partial I)/\varepsilon},\quad \forall\, n\in\mathbb{N}_0, \tag{2.4.44}
$$

$$\|r_M\|_{L^\infty(\partial I)} + \|r'_M\|_{L^\infty(\partial I)} + \|r_M\|_{\varepsilon,I} \leq C\mathrm{e}^{-q/\varepsilon}, \tag{2.4.45}$$

provided $a^2\varepsilon\mathrm{e}\gamma(M+1) < 1$, where a and γ is given in Theorem 2.4.16.

Proof. We mention that we follow [57, Theorem 6]. First we consider (2.4.43). The combination of (2.2.16) and (2.4.17) and Stirling's approximation gives

$$\begin{aligned}
\|(u^s_M)^{(n)}\|_{0,I} \leq \sum_{j=0}^{M} \varepsilon^j \|U_j^{(n)}\|_{0,I} \leq C \sum_{j=0}^{M} \varepsilon^j \gamma^j \frac{a^{2j} j^{2j}}{j!} K^n n! \\
\leq & CK^n n! \sum_{j=0}^{M} (\varepsilon\gamma)^j \frac{a^{2j} j^{2j}}{j!} \leq CK^n n! \sum_{j=0}^{M} (a^2 \varepsilon \mathrm{e}\gamma j)^j \\
\leq & CK^n n! \sum_{j=0}^{\infty} (a^2 \varepsilon \mathrm{e}\gamma M)^j \\
\leq & CK^n n!.
\end{aligned}$$

The assumption $a^2\varepsilon\mathrm{e}\gamma(M+1) < 1$ allows us to obtain the above result since the sum is a converging geometric series. We proceed with (2.4.44). Recall (2.4.36) which, under the assumption mentioned before, gives us

$$\begin{aligned}
\left|\left(\tilde{u}_M^{BL}\right)^{(n)}(\tilde{x})\right| \leq \sum_{j=0}^{M+1} \varepsilon^j |\tilde{U}_j^{(n)}(\tilde{x})| \leq \sum_{j=1}^{M+1} \varepsilon^j \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|} \tilde{K}^n (a^2\mathrm{e}\gamma)^j j^{j-1} \\
\leq & \varepsilon\gamma \tilde{K}^n \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|} \sum_{j=1}^{M+1} \left(a^2 \varepsilon \mathrm{e}\gamma(M+1)\right)^{j-1} \\
\leq & C\varepsilon \tilde{K}^n \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|} \sum_{j=0}^{\infty} \left(a^2 \varepsilon \mathrm{e}\gamma(M+1)\right)^{j} \\
\leq & C\varepsilon \tilde{K}^n \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}|\tilde{x}|}.
\end{aligned}$$

In the same manner one can infer a similar result for $\hat{u}_M^{BL}$. In order to show (2.4.45) we note that

$$\begin{aligned}
r_M(0) = & u(0) - \left(\sum_{j=0}^{M} \varepsilon^j U_j(0) + \sum_{j=0}^{M+1} \varepsilon^j \tilde{U}_j(0) + \sum_{j=0}^{M+1} \varepsilon^j \hat{U}_j(1/\varepsilon) \right) \\
= & -\varepsilon^{M+1} \tilde{U}_M(0) - \sum_{j=0}^{M+1} \varepsilon^j \hat{U}_j(1/\varepsilon),
\end{aligned}$$

since $u(0) = 0,\ U_j(0) + \tilde{U}_j(0) = 0$, for $j \geq 1$. Hence, using (2.4.36), (2.4.37), we have

$$\begin{aligned}
|r_M(0)| \leq& \varepsilon^{M+1}|\tilde{U}_{M+1}(0)| + \sum_{j=1}^{M+1} \varepsilon^j |\hat{U}_j(1/\varepsilon)| \\
\leq& C(a^2\varepsilon \mathrm{e}\gamma)^{M+1}(M+1)^M + \sum_{j=1}^{M+1} \varepsilon^j C \mathrm{e}^{-\frac{\sqrt{\alpha(1)}}{2\varepsilon}} \gamma^j j^{j-1} \\
\leq& C(a^2\varepsilon \mathrm{e}\gamma)^{M+1}(M+1)^M + C\varepsilon\gamma \mathrm{e}^{-\frac{\sqrt{\alpha(1)}}{2\varepsilon}} \sum_{j=1}^{M+1} \big(a^2\varepsilon \mathrm{e}\gamma(M+1)\big)^{j-1} \\
\leq& C(a^2\varepsilon \mathrm{e}\gamma)^{M+1}(M+1)^M + C\varepsilon\gamma \mathrm{e}^{-\frac{\sqrt{\alpha(1)}}{2\varepsilon}} \sum_{j=0}^{\infty} \big(a^2\varepsilon \mathrm{e}\gamma(M+1)\big)^{j} \\
\leq& C\varepsilon\gamma\big(a^2\varepsilon\gamma(M+1)\big)^M + C\varepsilon \mathrm{e}^{-q\frac{\sqrt{\alpha(1)}}{2\varepsilon}},
\end{aligned}$$

for some positive $q \in \mathbb{R}$, independend of ε and clearly bounded away from 0. Again the assumption $a^2\varepsilon\gamma M < 1$ was used. Furthermore,

$$\begin{aligned}
r'_M(0) =& u'(0) - \left(\sum_{j=0}^{M} \varepsilon^j U'_j(0) + \sum_{j=1}^{M+1} \varepsilon^{j-1} \tilde{U}'_j(0) - \sum_{j=1}^{M+1} \varepsilon^{j-1} \hat{U}'_j(1/\varepsilon) \right) \\
=& -\sum_{j=1}^{M+1} \varepsilon^{j-1} \hat{U}'_j(1/\varepsilon),
\end{aligned}$$

since $u'(0) = \tilde{U}'_0(0) = \hat{U}'_0(0) = 0,\ \tilde{U}'_j(0) + U'_{j-1}(0) = 0$. Thus

$$|r'_M(0)| \leq \sum_{j=1}^{M+1} \varepsilon^{j-1} |\hat{U}'_j(1/\varepsilon)|. \tag{2.4.46}$$

By using (2.4.37) once more, we get

$$|r'_M(0)| \leq C\hat{K} \mathrm{e}^{-q\sqrt{\alpha(1)}/\varepsilon}. \tag{2.4.47}$$

In the same way we obtain analogous results for $|r_M(1)|$, $|r'_M(1)|$. We now apply the operator $L := \varepsilon^2 \frac{d^4}{dx^4} - \frac{d}{dx}(\alpha(x)\frac{d^2}{dx^2}) + \beta(x)$ to the function $u - u_M^s$ and obtain,

$$\begin{aligned}
L(u - u_M^s) =& f - \sum_{j=0}^{M} \varepsilon^j L(U_j) \\
=& f - \sum_{j=0}^{M} \varepsilon^j \left(\varepsilon^2 U_j^{(4)} - \big(\alpha(x)U'_j\big)' + \beta(x)U_j \right) \\
=& -\varepsilon^{M+1} U_{M-1}^{(4)} - \varepsilon^{M+2} U_M^{(4)},
\end{aligned} \tag{2.4.48}$$

since $\{U_j\}_{j\in\mathbb{N}_0}$ satisfy (2.2.7). By (2.4.17), we get

$$
\begin{aligned}
&\|L(u-u_M^s)\|_{L^\infty(I)} \leq \\
\leq & C4!K^4\Big(\varepsilon^{M+1}\gamma^{M-1}\frac{a^{2(M-1)}(M-1)^{2(M-1)}}{(M-1)!}+\varepsilon^{M+2}\gamma^M\frac{a^{2M}M^{2M}}{M!}\Big)\\
\leq & CK^4\varepsilon^M\gamma^M\frac{a^{2M}M^{2M}}{M!}\left[\frac{1}{\gamma a^2}+\varepsilon\right]\\
\leq & C\varepsilon^M\gamma^M\frac{a^{2M}M^M}{M!}.
\end{aligned}
$$

Using Stirling's approximation we obtain

$$
\|L(u-u_M^s)\|_{L^\infty(I)} \leq C(a^2\mathrm{e}\varepsilon\gamma M)^M.
$$

We next obtain an estimate for $L\tilde{u}_M^{BL}$:

$$
\begin{aligned}
L\big(\tilde{u}_M^{BL}\big) = & \sum_{j=0}^{M+1}\varepsilon^j\Bigg[\varepsilon^{-2}\Big(\tilde{U}_j^{(4)}(\tilde{x})-\sum_{k=0}^{j}\varepsilon^k\alpha_k(\tilde{x})\tilde{U}''_{j-k}(\tilde{x})-\sum_{k=0}^{j}\varepsilon^k\alpha'_k(\tilde{x})\tilde{U}'_{j-k}(\tilde{x})\Big)\\
&+\sum_{k=0}^{j-2}\varepsilon^k\beta_k(\tilde{x})\tilde{U}_{j-2-k}(\tilde{x})\Bigg]\\
= & \sum_{\substack{1\leq k\leq j\leq M+1,\\ j-2+k>M-1}} -\varepsilon^{j+k-2}\big(\alpha_k(\tilde{x})\tilde{U}''_{j-k}(\tilde{x})+\alpha'_k(\tilde{x})\tilde{U}'_{j-k}(\tilde{x})\big)\\
&+\sum_{\substack{0\leq k\leq j-2\leq M+1,\\ j+k>M-1}}\varepsilon^{k+j}\beta_k(\tilde{x})\tilde{U}_{j-2-k}(\tilde{x}).
\end{aligned}
$$

We utilize the analyticity of αandβ along with the estimate (2.4.18), and with the aid of Lemma 2.4.13 we obtain

$$
\begin{aligned}
\Big\|L\big(\tilde{u}_M^{BL}\big)\Big\|_{L^\infty(I)} \leq & \varepsilon^M\sum_{\substack{1\leq k\leq j\leq M+1,\\ j-2+k>M-1}}\Big\{|\alpha_k(\tilde{x})\tilde{U}''_{j-k}(\tilde{x})|+|\alpha'_k(\tilde{x})\tilde{U}'_{j-k}(\tilde{x})|\Big\}\\
&+\varepsilon^M\sum_{\substack{0\leq k\leq j-2\leq M+1,\\ j+k>M-1}}|\beta_k(\tilde{x})\tilde{U}_{j-2-k}(\tilde{x})|\\
\leq & \varepsilon^M\sum_{\substack{1\leq k\leq j\leq M+1,\\ j-2+k>M-1}}C\Big\{C_\alpha\gamma_\alpha^k|\tilde{x}|^k\gamma^{j-k}\frac{1}{(j-k-1)!}\Big(a(j-k)+\frac{3}{\sqrt{\alpha(0)}}+|\tilde{x}|\Big)^{2(j-k-1)}+\\
&+C_{\alpha'}\gamma_{\alpha'}^k|\tilde{x}|^k\gamma^{j-k}\frac{1}{(j-k-1)!}\Big(a(j-k)+\frac{2}{\sqrt{\alpha(0)}}+|\tilde{x}|\Big)^{2(j-k-1)}\Big\}\mathrm{e}^{-\sqrt{\alpha(0)\tilde{x}}}+\\
&+\varepsilon^M\sum_{\substack{0\leq k\leq j-2\leq M+1,\\ j+k>M-1}}C_\beta\gamma_\beta^k|\tilde{x}|^k\gamma^{j-2-k}\frac{1}{(j-k-3)!}\Big(a(j-2-k)+|\tilde{x}|\Big)^{2(j-k-3)}\mathrm{e}^{-\sqrt{\alpha(0)\tilde{x}}}.
\end{aligned}
$$

Note that, for $j \geq 2$, there holds,

$$
\begin{aligned}
\frac{1}{(j-k-1)!} &\leq \frac{(j-1)^k}{(j-1)!} \leq \frac{1}{(j-1)!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |\tilde{x}|\Big)^k, \quad k \leq j-1, \\
\frac{1}{(j-k-3)!} &\leq \frac{(j-3)^k}{(j-3)!} \leq \frac{1}{(j-3)!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |\tilde{x}|\Big)^k, \quad k \leq j-2,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&|\tilde{x}|^k \frac{1}{(j-k-1)!}\Big(a(j+1-k) + \frac{3}{\sqrt{\alpha(0)}} + |\tilde{x}|\Big)^{2(j-k-1)} \leq \frac{1}{(j-1)!}\Big(a_j + \frac{3}{\sqrt{\alpha(0)}} + |\tilde{x}|\Big)^{2(j-1)}, \\
&|\tilde{x}|^k \frac{1}{(j-k-3)!}(a(j-1-k) + \frac{3}{\sqrt{\alpha(0)}} + |\tilde{x}|)^{2(j-3-k)} \leq \frac{1}{(j-1)!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |\tilde{x}|\Big)^{2j-3}.
\end{aligned}
$$

From (2.4.40) we have

$$
\begin{aligned}
&\frac{1}{(j-1)!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |\tilde{x}|\Big)^k \mathrm{e}^{-\sqrt{\alpha(0)}\tilde{x}} \leq C(a^2\mathrm{e})^{j-1} j^{j-1} \mathrm{e}^{\frac{6}{a\sqrt{\alpha(0)}}} \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}\tilde{x}}, \\
&\frac{1}{(j-3)!}\Big(aj + \frac{3}{\sqrt{\alpha(0)}} + |\tilde{x}|\Big)^k \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}\tilde{x}} \leq C(a^2\mathrm{e})^{j-3}(j-2)^{j-3} \mathrm{e}^{\frac{6}{a\sqrt{\alpha(0)}}} \mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}\tilde{x}},
\end{aligned}
$$

and this gives

$$
\begin{aligned}
\Big\| L\big(\tilde{u}_M^{BL}\big)\Big\|_{L^\infty(I)} \leq{}& \varepsilon^M \sum_{\substack{1\leq k\leq j\leq M+1,\\ j-2+k>M-1}} C\Big[C_\alpha \gamma_\alpha^k \gamma^{j-k}(a^2\mathrm{e})^{j-1} j^{j-1}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}\tilde{x}} + \\
&+ C_{\alpha'}\gamma_{\alpha'}^k \gamma^{j-k}(a^2\mathrm{e})^{j-1} j^{j-1}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}\tilde{x}}\Big] + \\
&+ \varepsilon^{M-1} \sum_{\substack{0\leq k\leq j-2\leq M+1,\\ j+k>M-1}} C_\beta \gamma_\beta^k \gamma^{j-2-k}(a^2\mathrm{e})^{j-3}(j-2)^{j-3}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}\tilde{x}} \\
\leq{}& \varepsilon^M C\Bigg[\sum_{\substack{1\leq k\leq j\leq M+1,\\ j-2+k>M-1}} \gamma^j (a^2\mathrm{e})^{j-1} j^{j-1}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}\tilde{x}}\bigg(C_\alpha\Big(\frac{\gamma_\alpha}{\gamma}\Big)^k + C_{\alpha'}\Big(\frac{\gamma_{\alpha'}}{\gamma}\Big)^k\bigg) + \\
&+ \sum_{\substack{0\leq k\leq j-2\leq M+1,\\ j+k>M-1}} C_\beta \gamma^{j-2}(a^2\mathrm{e})^{j-3}(j-2)^{j-3}\mathrm{e}^{-\frac{\sqrt{\alpha(0)}}{2}\tilde{x}}\Big(\frac{\gamma_\beta}{\gamma}\Big)^k\Bigg].
\end{aligned}
$$

We recall that in Theorem 2.4.16 the constant γ is chosen in such a way to satisfy (2.4.21) and therefore the sums are bounded by convergent geometric series. Thus

$$
\begin{aligned}
\Big\| L\big(\tilde{u}_M^{BL}\big)\Big\|_{L^\infty(I)} &\leq C\varepsilon^M (a^2\mathrm{e})^M \gamma^{M+1}(M+1)^M \mathrm{e}^{-\sqrt{\alpha(0)\tilde{x}}} \\
&\leq C\gamma\big(a^2\mathrm{e}\varepsilon\gamma(M+1)\big)^M.
\end{aligned} \tag{2.4.49}
$$

The term $\|L\big(\hat{u}_M^{BL}\big)\|_{L^\infty(I)}$ satisfies an analogous result. Thus we have

$$\begin{aligned}&\|L(r_M)\|_{L^\infty(I)} \le\\ \le&\|L(u-u_M^s)\|_{L^\infty(I)} + \|L(\tilde{u}_M^{BL})\|_{L^\infty(I)} + \|L(\hat{u}_M^{BL})\|_{L^\infty(I)}\\ \le& C\big(a^2\mathrm{e}\varepsilon\gamma(M+1)\big)^M.\end{aligned}$$

We choose $M+1$ to be the integer part of $\frac{q}{\varepsilon}$, where $q=\frac{1}{\gamma a^2\mathrm{e}^2}$. Then, we get

$$a^2\gamma\mathrm{e}\varepsilon(M+1)\le \mathrm{e}^{-1},$$

and $M\ge \frac{q}{\varepsilon}-2$, therefore

$$\|L(r_M)\|_{L^\infty(I)} \le C\mathrm{e}^{-M} \le C\mathrm{e}^{-q/\varepsilon}.$$

We have shown that r_M has exponentially small values at the endpoints of $[0,1]$ and Lr_M is uniformly bounded by an exponentially small quantity on the interval $(0,1)$. By stability we have the desired result. □

2.5 Appendix A

Here we give the proofs for Section 2.3.

Proof. (Proposition 2.3.1)

We set the linear function $u_0(x) := (g^+ - g^-)x + g^-$ and the function $v = u - u_0$, where u satisfies problem (2.3.2). Therefore, we get

$$-\alpha(x)v''(x) - \alpha'(x)v'(x) + \beta(x)v(x) = g(x) + \alpha'(x)(g^+ - g^-) - \beta(x)u_0.$$

Thus, v satisfies the boundary value problem

$$\begin{gathered}-\big(\alpha(x)v'(x)\big)' + \beta(x)v(x) = h(x),\\ v(0)=v(1)=0,\end{gathered}$$

for $h(x) := g(x) - \beta(x)\big((g^+ - g^-)x + g^-\big) + \alpha'(x)(g^+ - g^-)$. □

Proof. (Lemma 2.3.9)

We first mention that (2.3.10) is the well known Poincaré inequality. By the Fundamental Theorem of Calculus we have

$$\int_0^x u'(t)dt = u(x) - u(0) = u(x),$$

since $u(0) = 0$. Therefore we obtain

$$\begin{aligned}\int_0^1 |u(x)|^2 dx &= \int_0^1 \Big| \int_0^x u'(t)dt \Big|^2 dx \\ &\leq \int_0^1 \Big(\int_0^1 |u'(t)|dt \Big)^2 dx = \Big(\int_0^1 |u'(x)|dx \Big)^2 .\end{aligned}$$

Using now the Cauchy-Schwarz inequality we infer

$$\begin{aligned}\Big(\int_0^1 |u'(x)|dx \Big)^2 &\leq \Big[\Big(\int_0^1 |u'(x)|^2 dx \Big)^{1/2} \Big(\int_0^1 1^2 dx \Big)^{1/2} \Big]^2 \\ &= \int_0^1 |u'(x)|^2 dx.\end{aligned}$$

In order to show the remaining inequalities we work as follows. Using Cauchy's inequality on (2.3.5) we obtain

$$\|u\|_E^2 \leq \|h\|_{0,I} \|u\|_{0,I}. \tag{2.5.1}$$

Also

$$\sqrt{\underline{\beta}} \| u \|_{0,I} \leq \Big(\int_0^1 \beta(x) u^2(x) dx \Big)^{1/2} \leq \|u\|_E,$$

holds and since $\alpha, \beta > 0$ we get

$$\|u\|_{0,I}^2 \leq \frac{1}{\underline{\beta}} \|u\|_E^2. \tag{2.5.2}$$

Combining inequalities (2.5.1), (2.5.2) we infer that

$$\|u\|_{0,I} \leq \frac{1}{\underline{\beta}} \|h\|_{0,I}.$$

To show inequality (2.3.12) we first observe that

$$\sqrt{\underline{\alpha}} \|u'\|_{0,I} \leq \Big(\int_0^1 \alpha(x) \big(u'(x)\big)^2 dx \Big)^{1/2} \leq \|u\|_E,$$

and therefore we obtain

$$\|u'\|_{0,I}^2 \leq \frac{1}{\underline{\alpha}}\|u\|_E^2. \tag{2.5.3}$$

Combining inequalities (2.3.10), (2.5.1) and (2.5.3) we get the desired result. □

Proof. (Lemma 2.3.13)

By Proposition 2.3.1 we know that problem (2.3.2) is equivalent to (2.3.3). It is transparent that

$$\begin{aligned}\|h\|_{0,I} \leq& \|g\|_{0,I} + \overline{\beta}(|g^+|+|g^-|) + \overline{\alpha'}(|g^+|+|g^-|)\\ \leq& \|g\|_{L^\infty(I)} + \big(|g^+|+|g^-|\big)\big(\overline{\beta}+\overline{\alpha'}\big).\end{aligned} \tag{2.5.4}$$

Using (2.3.11) and (2.5.4) we obtain

$$\|u\|_{0,I} \leq \frac{1}{\underline{\beta}}\Big(C_g + c\big(|g^+|+|g^-|\big)\Big), \tag{2.5.5}$$

where $C_g > \|g\|_{L^\infty(I)}$ and $c = \overline{\alpha'} + \overline{\beta}$. Simillarly, from (2.3.12) and (2.5.4) we get

$$\|u'\|_{0,I} \leq \frac{1}{\underline{\alpha}}\Big(C_g + c\big(|g^+|+|g^-|\big)\Big). \tag{2.5.6}$$

Therefore we have, for $\tilde{c} = \max\{\frac{1}{\underline{\alpha}}, \frac{1}{\underline{\beta}}\}$

$$\|u\|_{0,I} + \|u'\|_{0,I} \leq \tilde{c}\{C_g + c(|g^+|+|g^-|)\},$$

and (2.3.14) is verified. We continue to show (2.3.15). We use the differential equation of (2.3.2), to obtain

$$\|\alpha u''\|_{0,I} \leq \|g\|_{0,I} + \|\alpha'\|_{L^\infty(I)}\|u'\|_{0,I} + \|\beta\|_{L^\infty(I)}\|u\|_{0,I}.$$

By (2.3.14) we get

$$\begin{aligned}\|\alpha u''\|_{0,I} \leq& \|g\|_{0,I} + \big(\overline{\alpha'}+\overline{\beta}\big)\tilde{c}\{C_g + c(|g^+|+|g^-|)\}\\ \leq& C_g\big[1+\tilde{c}(\overline{\alpha'}+\overline{\beta})\big] + \big(\overline{\alpha'}+\overline{\beta}\big)\tilde{c}\{c(|g^+|+|g^-|)\}\\ \leq& C_g\big(1+\tilde{c}c\big) + \tilde{c}c^2(|g^+|+|g^-|).\end{aligned}$$

Hence we have that

$$\|u''\|_{0,I} \leq \frac{1}{\underline{\alpha}}[C_g\big(1+\tilde{c}c\big) + \tilde{c}c^2(|g^+|+|g^-|)]. \tag{2.5.7}$$

It holds that

$$\|u'\|_{1,I} \leq \|u''\|_{0,I} + \|u'\|_{0,I}.$$

From (2.5.7) and (2.3.15) we conclude that

$$\begin{aligned}\|u'\|_{1,I} \leq & \frac{1}{\underline{\alpha}}[C_g(1+\tilde{c}c) + \tilde{c}c^2(|g^+| + |g^-|)] + \tilde{c}\{C_g + c(|g^+| + |g^-|)\} \\ \leq & \frac{1}{\underline{\alpha}}C_g(1+\tilde{c}c) + \tilde{c}C_g + \Big(\frac{\tilde{c}c^2}{\underline{\alpha}} + \tilde{c}c\Big)(|g^+| + |g^-|) \\ \leq & C_g\Big(\frac{1+\tilde{c}c}{\underline{\alpha}} + \tilde{c}\Big) + \Big(\frac{c}{\underline{\alpha}} + 1\Big)\tilde{c}c(|g^+| + |g^-|),\end{aligned} \tag{2.5.8}$$

and (2.3.15) follows. We proceed with the remaining inequalities. Using Sobolev's Embedding Theorem and (2.3.15) we obtain

$$\begin{aligned}\sup_{x\in I}|u(x)| = & \|u\|_{L^\infty(I)} \\ \leq & (2\pi)^{-1}\|u\|_{H^1(I)} \\ \leq & (2\pi)^{-1}\big(\|u'\|_{0,I} + \|u\|_{0,I}\big) \\ \leq & (2\pi)^{-1}\tilde{c}\{C_g + c(|g^+| + |g^-|)\}.\end{aligned}$$

This is sufficient for (2.3.16). To show (2.3.17), we appeal again to Sobolev's Embedding Theorem and use (2.3.15) to get

$$\begin{aligned}\|u'\|_{L^\infty(I)} \leq & (2\pi)^{-1}\|u'\|_{H^1(I)} \\ \leq & (2\pi)^{-1}\Big[C_g\Big(\frac{1+\tilde{c}c}{\underline{\alpha}} + \tilde{c}\Big) + \Big(\frac{c}{\underline{\alpha}} + 1\Big)\tilde{c}c(|g^+| + |g^-|)\Big].\end{aligned}$$

□

Proof. (Proposition 2.3.18)

With the aid of the Green's function of the boundary value problem we can determine the solution. For $x \in (0, \infty)$, we seek a solution in the form

$$u(x) = \int_{-\infty}^{\infty} G(x,\xi)f(\xi)d\xi,$$

where G is Green's function. The solutions of the characteristic equation $m^4 - \lambda m^2 = 0$ are given by $m_{1,2} = 0$ and $m_{3,4} = \pm\kappa$, hence the general solution of the differential equation is

$$v(x) = A + Bx + Ce^{\kappa x} + De^{-\kappa x},$$

for some constants A, B, C and D. Therefore the associated Green's function and its derivatives are given as

$$G(x,\xi) = \begin{cases} A_1(\xi) + B_1(\xi)x + C_1(\xi)e^{\kappa x} + D_1(\xi)e^{-\kappa x}, & x < \xi, \\ A_2(\xi) + B_2(\xi)x + C_2(\xi)e^{\kappa x} + D_2(\xi)e^{-\kappa x}, & x > \xi. \end{cases}$$

$$G_x(x,\xi) = \begin{cases} B_1(\xi) + \kappa C_1(\xi)e^{\kappa x} - \kappa D_1(\xi)e^{-\kappa x}, & x < \xi, \\ B_2(\xi) + \kappa C_2(\xi)e^{\kappa x} - \kappa D_2(\xi)e^{-\kappa x}, & x > \xi. \end{cases}$$

$$G_{xx}(x,\xi) = \begin{cases} \lambda C_1(\xi)e^{\kappa x} + \lambda D_1(\xi)e^{-\kappa x}, & x < \xi, \\ \lambda C_2(\xi)e^{\kappa x} + \lambda D_2(\xi)e^{-\kappa x}, & x > \xi. \end{cases}$$

$$G_{xxx}(x,\xi) = \begin{cases} \lambda\kappa C_1(\xi)e^{\kappa x} - \lambda\kappa D_1(\xi)e^{-\kappa x}, & x < \xi, \\ \lambda\kappa C_2(\xi)e^{\kappa x} - \lambda\kappa D_2(\xi)e^{-\kappa x}, & x > \xi. \end{cases}$$

- In order to ensure that G satisfies the given boundary values we should have $|A_2(\xi)| + |B_2(\xi)| + |C_2(\xi)| = 0$ and $B_1(\xi) + \kappa C_1(\xi) - \kappa D_1(\xi) = 0$.

- Continuity of G at $x = \xi$:

$$A_1(\xi) + B_1(\xi)\xi + C_1(\xi)e^{\kappa\xi} + D_1(\xi)e^{-\kappa\xi} - D_2(\xi)e^{-\kappa\xi} = 0.$$

- Continuity of G_x at $x = \xi$:

$$B_1(\xi) + \kappa C_1(\xi)e^{\kappa\xi} - \kappa D_1(\xi)e^{-\kappa\xi} + \kappa D_2(\xi)e^{-\kappa\xi} = 0.$$

- Continuity of G_{xx} at $x = \xi$:

$$\lambda C_1(\xi)e^{\kappa\xi} + \lambda D_1(\xi)e^{-\kappa\xi} - \lambda D_2(\xi)e^{-\kappa\xi} = 0.$$

- Jump condition of G_{xxx} at $x = \xi$:

$$-\lambda\kappa C_1(\xi)e^{\kappa\xi} + \lambda\kappa D_1(\xi)e^{-\kappa\xi} - \lambda\kappa D_2(\xi)e^{-\kappa\xi} = 1.$$

Solving the system we get

$$A_1(\xi) = -\frac{\xi}{\lambda},$$
$$B_1(\xi) = \frac{1}{\lambda},$$
$$C_1(\xi) = -\frac{\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda},$$
$$D_1(\xi) = \frac{2-\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda},$$
$$D_2(\xi) = \frac{2-\mathrm{e}^{\kappa\xi}-\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda},$$

hence, the appropriate Green's function is given by

$$G(x,\xi) = \begin{cases} -\frac{\xi}{\lambda}+\frac{x}{\lambda}-\frac{\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda}\mathrm{e}^{\kappa x}+\frac{2-\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda}\mathrm{e}^{-\kappa x}, & x<\xi, \\ \frac{2-\mathrm{e}^{\kappa\xi}-\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda}\mathrm{e}^{-\kappa x}, & x>\xi. \end{cases}$$

Finally, the solution of (2.3.19) is

$$\begin{aligned} u(x) =& \int_0^x \frac{2-\mathrm{e}^{\kappa\xi}-\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda}\mathrm{e}^{-\kappa x} f(\xi)d\xi \\ &+ \int_x^\infty \left(-\frac{\xi}{\lambda}+\frac{x}{\lambda}-\frac{\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda}\mathrm{e}^{\kappa x}+\frac{2-\mathrm{e}^{-\kappa\xi}}{2\kappa\lambda}\mathrm{e}^{-\kappa x}\right) f(\xi)d\xi \\ =& \frac{2}{2\kappa\lambda}\mathrm{e}^{-\kappa x}\int_0^x f(\xi)d\xi - \frac{\mathrm{e}^{-\kappa x}}{2\kappa\lambda}\int_0^x \mathrm{e}^{\kappa\xi} f(\xi)d\xi - \frac{1}{2\kappa\lambda}\mathrm{e}^{-\kappa x}\int_0^x \mathrm{e}^{-\kappa\xi} f(\xi)d\xi \\ &- \frac{1}{\lambda}\int_x^\infty \xi f(\xi)d\xi + \frac{x}{\lambda}\int_x^\infty f(\xi)d\xi - \frac{\mathrm{e}^{\kappa x}}{2\kappa\lambda}\int_x^\infty \mathrm{e}^{-\kappa\xi} f(\xi)d\xi \\ &+ \frac{2\mathrm{e}^{-\kappa x}}{2\kappa\lambda}\int_x^\infty f(\xi)d\xi - \frac{\mathrm{e}^{-\kappa x}}{2\kappa\lambda}\int_x^\infty \mathrm{e}^{-\kappa\xi} f(\xi)d\xi \\ =& \frac{1}{\kappa\lambda}\mathrm{e}^{-\kappa x}\int_0^\infty f(\xi)d\xi - \frac{\mathrm{e}^{-\kappa x}}{2\kappa\lambda}\int_0^\infty \mathrm{e}^{-\kappa\xi} f(\xi)d\xi - \frac{\mathrm{e}^{-\sqrt{\lambda}x}}{2\kappa\lambda}\int_0^x \mathrm{e}^{\kappa\xi} f(\xi)d\xi \\ &- \frac{1}{\lambda}\int_x^\infty \xi f(\xi)d\xi + \frac{x}{\lambda}\int_x^\infty f(\xi)d\xi - \frac{\mathrm{e}^{\kappa x}}{2\kappa\lambda}\int_x^\infty \mathrm{e}^{-\kappa\xi} f(\xi)d\xi. \end{aligned}$$

□

Chapter 3

1-D *hp*-approximation results

In the present chapter the analysis of the *hp*-FEM approximation is illustrated with respect to the fourth order SPBVP in one dimension studied in Chapter 2. In particular, we study the performance of the *hp* version on the *Spectral Boundary Layer Mesh* (see Definition 3.1.11 ahead) and we show that the method converges at an exponential rate in the natural energy norm, independently of ε. In [57], the case of constant coefficients was considered. We extend the results of [57] to the case of variable coefficients.

3.1 The construction of the Finite Element Space

Recall the variational formulation as given in Chapter 2: Find $u \in H_0^2(I)$ such that

$$\mathcal{B}(u, v) = \mathcal{F}(v), \quad \forall\, v \in H_0^2(I). \tag{3.1.1}$$

The purpose of this section is the construction of finite-dimensional subspaces $S_N \subset H_0^2(I)$ and subsequently the establishment of the corresponding discrete problem of (3.1.1) that reads: Find $u_N \in S_N \subset H_0^2(I)$ such that

$$\mathcal{B}(u_N, v) = \mathcal{F}(v), \quad \forall\, v \in S_N. \tag{3.1.2}$$

We mention that

$$\|u - u_N\|_{\mathcal{E},I} \leq \inf_{v \in S_N} \|u - v\|_{\mathcal{E},I}, \quad \forall v \in S_N, \tag{3.1.3}$$

holds. To define the desired subspaces S_N for the problem above, we utilize a family of functions that have already been proposed in the literature. We refer to [57] for further details.

3.1.1 The basis functions

We first examine the four cubic *Hermite* polynomials given in the local coordinate system, where $\xi \in [-1, 1]$:

$$h_1(\xi) = \frac{1}{4}(1+\xi)^2(2-\xi) = \frac{1}{4}(-\xi^3 + 3\xi + 2), \tag{3.1.4}$$

$$h_2(\xi) = \frac{1}{4}(1-\xi)^2(2+\xi) = \frac{1}{4}(\xi^3 - 3\xi + 2), \tag{3.1.5}$$

$$h_3(\xi) = \frac{1}{4}(1+\xi)^2(\xi-1) = \frac{1}{4}(\xi^3 + \xi^2 - \xi - 1), \tag{3.1.6}$$

$$h_4(\xi) = \frac{1}{4}(1-\xi)^2(1+\xi) = \frac{1}{4}(\xi^3 - \xi^2 - \xi + 1). \tag{3.1.7}$$

These four *Hermite* polynomials give us the ability to control the values of the interpolant and its first-order derivative at the boundary points and thus they are characterized as **nodal** basis functions. To complete the family of the basis functions we proceed by adding the so called **internal** basis functions.

Suppose $\{L_i(x)\}_{i\in\mathbb{N}_0}$ is the set of *Legendre* polynomials of degree i defined on the interval $I_{ST} = (-1, 1)$. By using the family of *Legendre* polynomials we construct a new family of polynomials over the interval I_{ST}. For $i \geq 5$, the C^1 basis functions can be determined by

$$h_i(x) = \sqrt{\frac{2i-5}{2}} \int_{-1}^{x} \int_{-1}^{t} L_{i-3}(\eta) d\eta dt. \tag{3.1.8}$$

(In fact, this is defined in [64]). For $i \geq 5$, we calculate

$$h_i(x) = \frac{1}{\sqrt{4i-10}} \left[\frac{1}{2i-5} L_{i-5}(x) - \frac{4i-10}{(2i-3)(2i-7)} L_{i-3}(x) + \frac{1}{2i-3} L_{i-1}(x) \right]. \tag{3.1.9}$$

All polynomials given by (3.1.4)–(3.1.8) make up the family $\{h_i\}_{i=1}^{k+1}$ and it is obvious that this family serves as a basis for $\mathbb{P}_k(I_{ST})$, $k \geq 3$. Figure 3.1 presents the first eight aforementioned polynomials and their corresponding first-order derivatives are shown in Figure 3.2. We observe that the following properties hold:

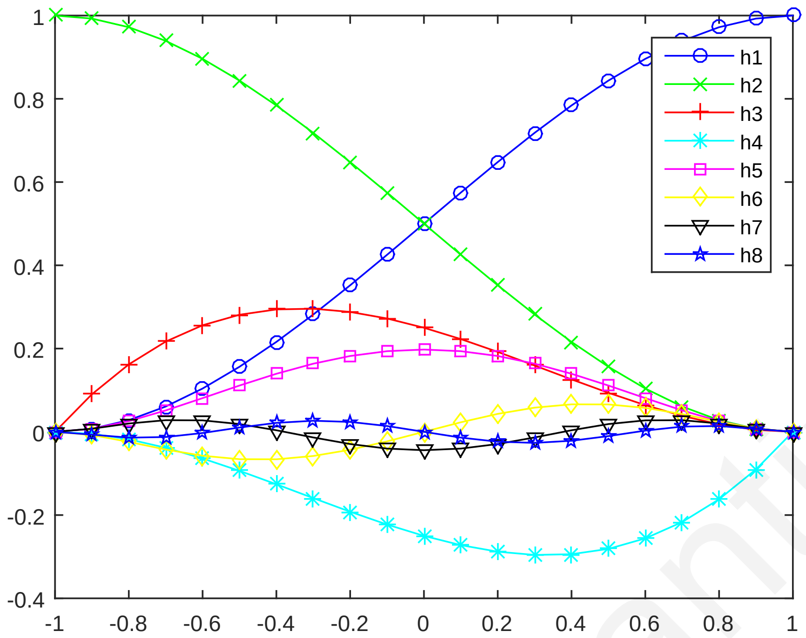


Figure 3.1: The first eight hierarchical C^1 basis functions.

At the boundary points, we have,

$$h_1(-1) = 0,\ h_1(1) = 1,$$
$$h_2(-1) = 1,\ h_2(1) = 0,$$
$$h_i(\pm 1) = 0,\ i \geq 3.$$

The first order derivatives of the basis functions at the boundary points satisfy

$$h_1'(\pm 1) = 0,\ h_2'(\pm 1) = 0,$$
$$h_3'(-1) = 0,\ h_3'(1) = 1,$$
$$h_4'(-1) = 1,\ h_4'(1) = 0$$
$$h_i'(\pm 1) = 0,\ i \geq 5.$$

These properties allow us to determine the values of the interpolant and its first-order derivatives at each vertex node.

The basis $\{h_i\}_{i\in\mathbb{N}}$ has one more essential property: it is *hierarchical*, i.e. the polynomial of degree $p+1$ is obtained as a correction to the polynomial of degree p. Such bases need not be reconstructed when the polynomial degree is increased and it is well suited for hp approximations.

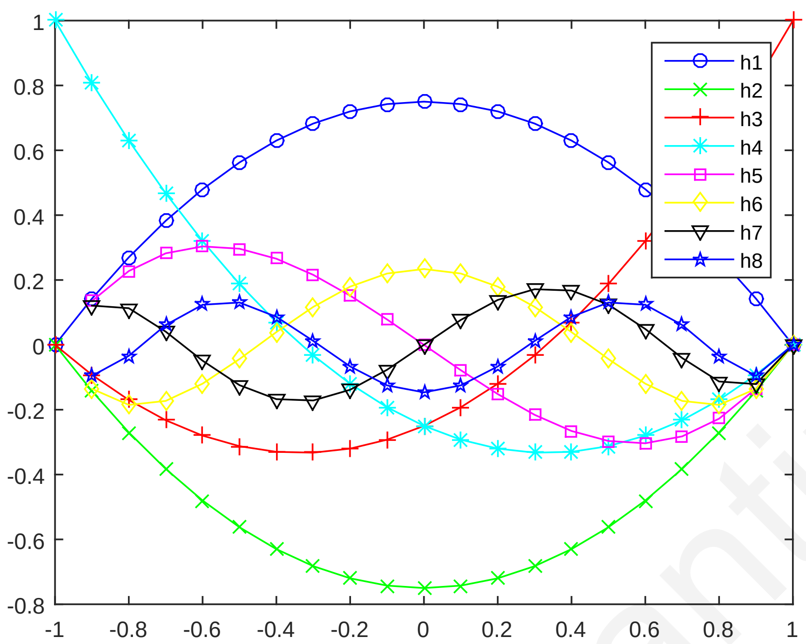


Figure 3.2: The corresponding first order derivatives.

3.1.2 The Finite Element Space

Suppose $\Delta = \{x_0, \ldots, x_m\}$ is an arbitrary partition of the interval I and denote the intervals (x_{i-1}, x_i) by I_i. Also consider the mappings $R_i : I_i \to I_{ST}$ given by

$$R_i(x) = \frac{2x - (x_i + x_{i-1})}{x_i - x_{i-1}}, \tag{3.1.10}$$

and define the space [a]

$$S^p(\Delta) := \{v \in H^2(I) : v \circ R_i^{-1} \in \mathbb{P}_p(I_{ST}),\ i = 1, \ldots, m\}.$$

The desired subspace $S_N \subset H_0^2(I)$ mentioned before, is chosen as $S_N = S_0^p(\Delta) = S^p(\Delta) \cap H_0^2(I)$.

In 1996, C. Schwab and M. Suri [67] introduced a certain type of mesh which became known as the *Spectral Boundary Layer Mesh*. It is comprised of three elements only. Its size can be changed, however we do not add or remove elements. (Hence, we are really using a p-FEM on a moving mesh). Despite that, we call our method hp-FEM in order to be consistent with the bibliography.

[a] Note that we use the same polynomial degree p for all elements. A more general approach is the usage of different polynomials degrees on each subinterval.

Definition 3.1.11. (*Spectral Boundary Layer Mesh*) For $\nu > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \leq 1$, define the spaces $S(\nu, p) = S_0^p(\Delta)$ of piecewise polynomials where the partition is given by

$$\Delta := \begin{cases} \{0, \nu p\varepsilon, 1 - \nu p\varepsilon, 1\}, & \text{if } \nu p\varepsilon < 1/2, \\ \{0, 1\}, & \text{if } \nu p\varepsilon \geq 1/2. \end{cases} \tag{3.1.12}$$

3.2 Approximation in the energy norm

The results presented in this section are an improvement of those found in [57], in the sense that in certain places in [57] various inaccuracies exist. One example is the following proposition (cf. Lemma 7 in [57]).

Proposition 3.2.1 *Let* $u \in H^2([-1,1])$. *Then there exists a linear operator* $\mathcal{I}_p : H^2([-1,1]) \to \mathbb{P}_p([-1,1])$, $p \geq 3$ *such that*

$$\mathcal{I}_p^{(k)} u(\pm 1) = u^{(k)}(\pm 1), \quad k = 0, 1, \tag{3.2.2}$$

and moreover, if $u \in C^\infty([-1,1])$,

$$\|(u - \mathcal{I}_p u)''\|^2_{0,[-1,1]} \leq \frac{(p-\alpha_1)!}{(p-2+\alpha_1)!} \|u^{(\alpha_1+1)}\|^2_{0,[-1,1]}, \tag{3.2.3}$$

$$\|(u - \mathcal{I}_p u)'\|^2_{0,[-1,1]} \leq \frac{1}{(p-1)^2} \frac{(p-\alpha_2)!}{(p-2+\alpha_2)!} \|u^{(\alpha_2+1)}\|^2_{0,[-1,1]}, \tag{3.2.4}$$

$$\|u - \mathcal{I}_p u\|^2_{0,[-1,1]} \leq \frac{1}{(p-1)^4} \frac{(p-\alpha_3)!}{(p-2+\alpha_3)!} \|u^{(\alpha_3+1)}\|^2_{0,[-1,1]} \tag{3.2.5}$$

for any integers $1 \leq \alpha_1, \alpha_2, \alpha_3 \leq p$.

Proof. We make use of Legendre expansions. To be more specific, we form the Legendre expansions of u'' and $\mathcal{I}_p'' u$. Then we obtain that, (see [64, Appendix C])

$$\|u'' - \mathcal{I}_p'' u\|^2_{0,[-1,1]} = \sum_{i=p-1}^{\infty} \frac{2}{2i+1} |a_i|^2.$$

Here the coefficients a_i are given by

$$a_i = \frac{2i+1}{2} \int_{-1}^{1} u''(\xi) L_i(\xi) d\xi. \tag{3.2.6}$$

We next define

$$\mathcal{I}_p'u(\xi) = \int_{-1}^{\xi} \mathcal{I}_p''u(\eta)d\eta + u'(-1),$$

and

$$\mathcal{I}_p u(\xi) = \int_{-1}^{\xi} \mathcal{I}_p'u(\eta)d\eta + u(-1).$$

It is transparent that $\mathcal{I}_p'u(-1) = u'(-1)$ and $\mathcal{I}_p u(-1) = u(-1)$. By the definition of a_i, we note that

$$2a_0 = \int_{-1}^{1} u''(\eta)d\eta = u'(1) - u'(-1), \tag{3.2.7}$$

and using integration by parts,

$$\begin{aligned} 2a_1 =& 3\int_{-1}^{1} \xi u''(\xi)d\xi = 3\Big(u'(1) + u'(-1) - \int_{-1}^{1} u'(\xi)d\xi\Big) \\ =& 3\big(u'(1) + \mathcal{I}_p'u(-1) - u(1) + \mathcal{I}_p u(-1)\big). \end{aligned} \tag{3.2.8}$$

In the same manner, we get

$$2a_0 = \int_{-1}^{1} \mathcal{I}_p''u(\eta)d\eta = \mathcal{I}_p'u(1) - \mathcal{I}_p'u(-1). \tag{3.2.9}$$

The combination of (3.2.7) and (3.2.9) gives us

$$u'(1) - \mathcal{I}_p'u(1) - \big(u'(-1) - \mathcal{I}_p'u(-1)\big) = 0,$$

and thus $u'(1) = \mathcal{I}_p'u(1)$. Using this in (3.2.8) we get

$$2a_1 = 3\big(\mathcal{I}_p'u(1) + \mathcal{I}_p'u(-1) - u(1) + \mathcal{I}_p u(-1)\big). \tag{3.2.10}$$

We proceed to show that $\mathcal{I}_p u(1) = u(1)$.

$$\begin{aligned} \mathcal{I}_p u(1) =& \int_{-1}^{1} \mathcal{I}_p'u(\eta)d\eta + u(-1) \\ =& \int_{-1}^{1}\Big(\int_{-1}^{\eta} \mathcal{I}_p''u(\xi)d\xi + u'(-1)\Big)d\eta + u(-1) \\ =& \sum_{i=0}^{p-2} a_i \int_{-1}^{1}\int_{-1}^{\eta} L_i(\xi)d\xi d\eta + 2u'(-1) + u(-1) \\ =& 2a_0 + \sum_{i=1}^{p-2} a_i \int_{-1}^{1} \frac{1}{2i+1}\big(L_{i+1}(\eta) - L_{i-1}(\eta)\big)d\eta + 2u'(-1) + u(-1) \end{aligned}$$

$$
\begin{aligned}
&= 2a_0 - \frac{2a_1}{3} + 2u'(-1) + u(-1) \\
&= u'(1) + u'(-1) - \frac{2a_1}{3} + u(-1).
\end{aligned}
$$

From (3.2.10) we obtain

$$\mathcal{I}_p u(1) = u'(1) + u'(-1) - \Big(\mathcal{I}_p' u(1) + \mathcal{I}_p' u(-1) - u(1) + \mathcal{I}_p u(-1)\Big) + u(-1) = u(1),$$

and this establishes (3.2.2). With the aid of the identity [64, Lemma 3.10]

$$\int_{-1}^{1} \big|(u'')^{(k)}(\xi)(1-\xi^2)^k\big| d\xi = \sum_{i\geq k} |a_i|^2 \frac{2}{2i+1}\frac{(i+k)!}{(i-k)!},\ k \geq 0,$$

we get the estimate (3.2.3), namely, for $1 \leq k \leq p$,

$$
\begin{aligned}
\|u'' - \mathcal{I}_p'' u\|^2_{0,[-1,1]} &\leq \sum_{i\geq k-1}^{\infty} \frac{2}{2i+1}|a_i|^2 \frac{(i-k+1)!}{(i+k-1)!}\frac{(i+k-1)!}{(i-k+1)!} \\
&\leq \frac{(p-k)!}{(p-2+k)!} \sum_{i\geq k-1}^{\infty} \frac{2}{2i+1}|a_i|^2 \frac{(i+k-1)!}{(i-k+1)!} \\
&\leq \frac{(p-k)!}{(p-2+k)!} \int_{-1}^{1} \big|(u'')^{(k-1)}(\xi)(1-\xi^2)^{k-1}\big| d\xi \\
&\leq \frac{(p-k)!}{(p-2+k)!} \|u^{(k+1)}\|^2_{0,[-1,1]}.
\end{aligned}
$$

In order to get (3.2.4) we observe that

$$\mathcal{I}_p' u(\xi) - u'(\xi) = \int_{-1}^{\xi} \sum_{i=p-1}^{\infty} a_i L_i(t) dt = \sum_{i=p-1}^{\infty} a_i \int_{-1}^{\xi} L_i(t) dt.$$

We note that

$$\psi_i(x) = \int_{-1}^{x} L_i(t) dt = -\frac{1}{i(i+1)}(1-x^2) L_i'(x).$$

The above equality comes up by integrating the Legendre differential equation. By [64, (3.3.9)] we have

$$\int_{-1}^{1} \frac{1}{1-x^2} \psi_i(x)\psi_j(x) dx = \frac{\delta_{ij}}{i(i+1)(2j+1)},$$

where δ_{ij} is the Kronecker delta. Then

$$\|(u - \mathcal{I}_p u)'\|^2_{0,[-1,1]} \leq \int_{-1}^{1} \frac{1}{1-x^2} |u'(x) - \mathcal{I}_p' u(x)|^2 dx$$

$$
\begin{aligned}
&\le \int_{-1}^{1} \frac{1}{1-x^2}\Big(\sum_{i=p-1}^{\infty} a_i\psi_i(x)\Big)^2 dx\\
&\le \sum_{i=p-1}^{\infty} |a_i|^2 \int_{-1}^{1} \frac{\psi_i^2(x)}{1-x^2} dx\\
&\le \sum_{i=p-1}^{\infty} \frac{2}{i(i+1)(2i+1)}|a_i|^2\\
&\le \frac{(p-k)!}{(p-2+k)!}\sum_{i\ge k-1}^{\infty} \frac{2}{i(i+1)(2i+1)}|a_i|^2\frac{(i+k-1)!}{(i-k+1)!}\\
&\le \frac{1}{(p-1)^2}\frac{(p-k)!}{(p-2+k)!}\|u^{(k+1)}\|_{0,[-1,1]}^2.
\end{aligned}
$$

To complete the proof we proceed as follows:

$$
u(x) - \mathcal{I}_p u(x) = \sum_{i=p-1}^{\infty} a_i \int_{-1}^{x}\int_{-1}^{\eta} L_i(t)dtd\eta = \sum_{i=p-1}^{\infty}\sqrt{\frac{2}{2i+1}}a_i h_{i+3}(x),
$$

with h_i given by (3.1.8). It can be shown [85] that

$$
\int_{-1}^{1} h_i(x)h_j(x)dx \begin{cases} \le \frac{C}{i^4}, \text{ if } j = i,\ i+2, \text{ or } i-2 \\ = 0, \text{ otherwise} \end{cases}.
$$

Hence we obtain

$$
\begin{aligned}
\int_{-1}^{1} |u(x) - \mathcal{I}_p u(x)|^2 dx &\le \int_{-1}^{1}\Big(\sum_{i=p-1}^{\infty}\sqrt{\frac{2}{2i+1}}a_i h_{i+3}(x)\Big)^2 dx\\
&\le \sum_{i=p-1}^{\infty}\frac{C}{i^4}\frac{2}{2i+1}|a_i|^2,
\end{aligned}
$$

and this gives us

$$
\begin{aligned}
\|u-\mathcal{I}_p u\|_{0,[-1,1]}^2 \le& C\frac{(p-2-k)!}{(p-2+k)!}\sum_{i\ge k-1}^{\infty}\frac{2}{i^4(2i+1)}|a_i|^2\frac{(i+k-1)!}{(i-k+1)!}\\
\le& \frac{C}{(p-1)^4}\frac{(p-k)!}{(p-2+k)!}\|u^{(k+1)}\|_{0,[-1,1]}^2.
\end{aligned}
$$

□

We next present a Lemma which is necessary for Theorem 3.2.13. We mention that here we borrow the ideas of [83, Lemma 3.1].

Lemma 3.2.11 *Let $p \geq 3$, $\lambda \in (0,1]$. Then, there is a constant $C > 0$ which satisfies*

$$\frac{\big(p-\lambda(p-2)\big)!}{\big(p-2+\lambda(p-2)\big)!} \leq Cp^2(p-2)^{-2\lambda(p-2)}e^{2\lambda(p-2)-1}\left[\frac{(1-\lambda/3)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^{p-2}. \tag{3.2.12}$$

Proof. Recall Stirling's approximation (2.4.6). Using it one may get

$$\begin{aligned}
&\frac{\big(p-\lambda(p-2)\big)!}{\big(p-2+\lambda(p-2)\big)!} \leq \\
&\leq \sqrt{\frac{2\pi\big(p-\lambda(p-2)\big)}{2\pi\big(p-2+\lambda(p-2)\big)}}\frac{\left(\dfrac{p-\lambda(p-2)}{\mathrm{e}}\right)^{p-\lambda(p-2)}}{\left(\dfrac{p-2+\lambda(p-2)}{\mathrm{e}}\right)^{p-2+\lambda(p-2)}}\frac{\mathrm{e}}{\mathrm{e}^{\frac{1}{12(p-2+\lambda(p-2))+1}}} \\
&\leq p^2\mathrm{e}^{2\lambda(p-2)-1}\sqrt{\frac{p\big(1-\lambda(1-2/p)\big)}{(p-2)(1+\lambda)}}\frac{p^{p-2-\lambda(p-2)}}{(p-2)^{p-2+\lambda(p-2)}}\frac{\big(1-\lambda(1-2/p)\big)^{p-\lambda(p-2)}}{(1+\lambda)^{p-2+\lambda(p-2)}} \\
&\leq p^2\mathrm{e}^{2\lambda(p-2)-1}(1-\lambda/3)^2\sqrt{1+\frac{2}{p-2}}\sqrt{\frac{1-\lambda/3}{1+\lambda}}\left(1+\frac{2}{p-2}\right)^{p-2}\times \\
&\qquad \times (p-2)^{-2\lambda(p-2)}\frac{(1-\lambda/3)^{(1-\lambda)(p-2)}}{(1+\lambda)^{(1+\lambda)(p-2)}} \\
&\leq \sqrt{3}\mathrm{e}^2p^2\sqrt{\frac{1-\lambda/3}{1+\lambda}}(p-2)^{-2\lambda(p-2)}\mathrm{e}^{2\lambda(p-2)-1}\left[\frac{(1-\lambda/3)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^{p-2}.
\end{aligned}$$

□

Theorem 3.2.13 *Suppose the space $S(\nu, p)$ is given by Definition 3.1.11 and assume that the functions u and u_N solve (3.1.1) and (3.1.2), respectively. Then the error bound*

$$\|u-u_N\|_{\mathcal{E},I} \leq C\mathrm{e}^{-\sigma\nu p}, \quad 0 < \nu \leq \nu_0 \tag{3.2.14}$$

holds, for some constants $\nu_0, C, \sigma > 0$ depending only on the input data α, β, f.

Proof. [b] Since the value of the number $\nu p\varepsilon$ determines the form of our mesh, one has to study two cases. First, suppose that $\nu p\varepsilon \geq 1/2$ or equivalently $(2\nu\varepsilon)^{-1} \leq p$. We note that in this case the mesh is comprised of one only element, $\Delta = \{0, 1\}$. For $p \geq 3$, by appealing to Proposition 3.2.1, one is able to find a polynomial $\mathcal{I}_p u \in \Pi_p$ that satisfies (3.2.2)–(3.2.5). The combination of those inequalities yields

$$\|u-\mathcal{I}_pu\|^2_{\mathcal{E},I} \leq C\frac{(p-\alpha)!}{(p-2+\alpha)!}\left(\frac{1}{(p-1)^4}+\frac{1}{(p-1)^2}+\varepsilon^2\right)\left\|u^{(\alpha+1)}\right\|^2_{0,I}. \tag{3.2.15}$$

[b] The proof follows ideas that are presented in [83, Lemma 4]

Set $\alpha = \lambda(p-2) \in \mathbb{N}$, for $\lambda \in (0,1)$ to be selected shortly. From Lemma 3.2.11 we have the estimate (3.2.12). Also if $\lambda > 2\nu$ the bound (2.2.5) gives

$$\begin{aligned}\|u^{(\lambda(p-2)+1)}\|_{0,I} \leq & C\gamma^{\lambda(p-2)+1}\max\{(\lambda(p-2)+1)^{\lambda(p-2)+1}, \varepsilon^{-\lambda(p-2)}\} \\ \leq & C\gamma^{\lambda(p-2)+1}(\lambda(p-2)+1)^{\lambda(p-2)+1},\end{aligned}$$

since $p > \frac{1}{2\nu\varepsilon}$. We combine the above inequalities to get

$$\begin{aligned}&\|u - \mathcal{I}_p u\|^2_{\varepsilon,I} \leq \\ &\leq Cp^2(p-2)^{-2\lambda(p-2)}\mathrm{e}^{2\lambda(p-2)-1}\left[\frac{(1-\lambda/3)^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}}\right]^{p-2}\gamma^{2(\lambda(p-2)+1)}(\lambda(p-2)+1)^{2(\lambda(p-2)+1)} \\ &\leq Cp^2\mathrm{e}^{2\lambda(p-2)-1}\left[\frac{(1-\lambda/3)^{1-\lambda}}{(1+\lambda)^{1+\lambda}}\right]^{p-2}\gamma^{2(\lambda(p-2)+1)}(\lambda(p-2)+1)^2(\lambda+1/(p-2))^{2\lambda(p-2)} \\ &\leq Cp^4\mathrm{e}^{2\lambda(p-2)-1}\gamma^{2(\lambda(p-2)+1)}\lambda^{2\lambda(p-2)}\left[\frac{(1-\lambda/3)^{1-\lambda}}{(1+\lambda)^{1+\lambda}}\right]^{p-2}\left[(1+1/\lambda(p-2))^{\lambda(p-2)}\right]^2 \\ &\leq Cp^4\left[\frac{(1-\lambda/3)^{1-\lambda}}{(1+\lambda)^{1+\lambda}}(\mathrm{e}\gamma\lambda)^{2\lambda}\right]^{p-2}.\end{aligned}$$

The choice[c] $\lambda \leq (\mathrm{e}\gamma)^{-1} \in (0,1)$ yields, for $\omega := |\ln w|$ and $w := \frac{(1-\lambda/3)^{1-\lambda}}{(1+\lambda)^{1+\lambda}} < 1$, the estimate

$$\|u - \mathcal{I}_p u\|_{\varepsilon,I} \leq Cp^4\mathrm{e}^{-\omega\nu(p-2)},$$

and the desired result follows from (3.1.3). In the case $\nu p\varepsilon < 1/2$, the four mesh points discretize the domain into three elements, namely $\Delta = \{0, \nu p\varepsilon, 1-\nu p\varepsilon, 1\}$. The solution u is decomposed (see Chapter 2) as

$$u = u_M + \tilde{u}_M + \hat{u}_M + r_M,$$

and each term must be handled separetely.

For $p \geq 3$, from Proposition 3.2.1 there exists a polynomial $\mathcal{I}_p u_M$ such that

$$\|u_M - \mathcal{I}_p u_M\|_{\varepsilon,I} \leq C\Big(\varepsilon^2 + \frac{1}{(p-1)^2} + \frac{1}{(p-1)^4}\Big)\frac{(p-\alpha)!}{(p-2+\alpha)!}\left\|w_M^{(\alpha+1)}\right\|^2_{0,I}.$$

From (2.4.43), the inequality $n! \leq n^n$, $n \in \mathbb{N}$ and by following the same procedure as

[c] As a consequence of this choice, the constant ν in the definition of the mesh must satisfy $\nu \leq \frac{1}{2\mathrm{e}\gamma}$.

in the case presented before, we get, by setting $\alpha = \lambda_1(p-2)$ for some $\lambda_1 \in (0,1)$,

$$\begin{aligned}
&\|u_M - \mathcal{I}_p u_M\|^2_{\mathcal{E},I} \leq \\
\leq & C\left(\varepsilon^2 + \frac{1}{(p-1)^2} + \frac{1}{(p-1)^4}\right)\frac{(p-\lambda_1(p-2))!}{(p-2+\lambda_1(p-2))!}K_1^{2(\lambda_1(p-2)+1)}[(\lambda_1(p-2)+1)!]^2 \\
\leq & Cp^2(p-2)^{-2\lambda_1(p-2)}\mathrm{e}^{2\lambda_1(p-2)-1}\left[\frac{(1-\lambda_1/3)^{(1-\lambda_1)}}{(1+\lambda_1)^{(1+\lambda_1)}}\right]^{p-2} K_1^{2(\lambda_1(p-2)+1)}(\lambda_1(p-2)+1)^{2(\lambda_1(p-2)+1)} \\
\leq & CK_1^2p^2(\lambda_1(p-2)+1)^2\left[\frac{(1-\lambda_1/3)^{1-\lambda_1}}{(1+\lambda_1)^{1+\lambda_1}}(\mathrm{e}K_1)^{2\lambda_1}\right]^{p-2}(\lambda_1+1/(p-2))^{2\lambda_1(p-2)} \\
\leq & Cp^4\left[\frac{(1-\lambda_1/3)^{1-\lambda_1}}{(1+\lambda_1)^{1+\lambda_1}}(\mathrm{e}\lambda_1 K_1)^{2\lambda_1}\right]^{p-2}.
\end{aligned}$$

As a result, for $\lambda_1 \leq \frac{1}{\mathrm{e}K_1}$, $\omega_1 := |\ln \tilde{w}_1|$ and $\tilde{w}_1 := \frac{(1-\lambda_1/3)^{1-\lambda_1}}{(1+\lambda_1)^{1+\lambda_1}} < 1$ we have

$$\|u_M - \mathcal{I}_p u_M\|_{\mathcal{E},I} \leq Cp^4\mathrm{e}^{-\omega_1\nu(p-2)}. \tag{3.2.16}$$

Regarding the boundary layer components one is obliged to analyse them into separate cases.[d] For the interval $I^*_\varepsilon = (0,\nu p\varepsilon)$ we have an approximation given again by Proposition 3.2.1 such that

$$\|\tilde{u}_M - \mathcal{I}_p\tilde{u}_M\|^2_{\mathcal{E},I^*_\varepsilon} \leq C(\nu p\varepsilon)^{2\alpha-2}\left(\varepsilon^2 + \frac{(\nu p\varepsilon)^2}{(p-1)^2} + \frac{(\nu p\varepsilon)^4}{(p-1)^4}\right)\frac{(p-\alpha)!}{(p-2+\alpha)!}\left\|\tilde{u}_M^{(\alpha+1)}\right\|^2_{0,I^*_\varepsilon}.$$

Setting $\alpha = \lambda_2(p-2)$, for some $\lambda_2 \in (0,1)$ and applying (2.4.44), yields

$$\begin{aligned}
&\|\tilde{u}_M - \mathcal{I}_p\tilde{u}_M\|^2_{\mathcal{E},I^*_\varepsilon} \leq \\
\leq & C(\nu p\varepsilon)^{2\lambda_2(p-2)-2}K^{2\lambda_2(p-2)+2}\varepsilon^2\left(1 + \frac{(\nu p)^2}{(p-1)^2} + \frac{(\nu p)^4\varepsilon^2}{(p-1)^4}\right)\times \\
& \times\frac{(p-\lambda_2(p-2))!}{(p-2+\lambda_2(p-2))!}\int_0^{\nu p\varepsilon}\varepsilon^{-2\lambda_2(p-2)}\mathrm{e}^{-2\sqrt{\alpha(0)}dist(x,\partial I)/\varepsilon}dx \\
\leq & \nu p\varepsilon C(\nu p)^{2\lambda_2(p-2)-2}K^{2\lambda_2(p-2)+2}p^2(p-2)^{-2\lambda_2(p-2)}\mathrm{e}^{2\lambda_2(p-2)-1}\left[\frac{(1-\lambda_2/3)^{(1-\lambda_2)}}{(1+\lambda_2)^{(1+\lambda_2)}}\right]^{p-2} \\
\leq & \nu p\varepsilon C\nu^{-2}\nu^{2\lambda_2(p-2)}\left(\frac{p}{p-2}\right)^{2\lambda_2(p-2)}\mathrm{e}^{2\lambda_2(p-2)-1}K^{2\lambda_2(p-2)+2}\left[\frac{(1-\lambda_2/3)^{1-\lambda_2}}{(1+\lambda_2)^{1+\lambda_2}}\right]^{p-2} \\
\leq & C\frac{\varepsilon p}{\nu}(2\lambda_2\nu)^{2\lambda_2(p-2)}\left(1+\frac{2}{2\lambda_2(p-2)}\right)^{2\lambda_2(p-2)}\mathrm{e}^{2\lambda_2(p-2)-1}K^{2\lambda_2(p-2)+2}\left[\frac{(1-\lambda_2/3)^{1-\lambda_2}}{(1+\lambda_2)^{1+\lambda_2}}\right]^{p-2} \\
\leq & C\mathrm{e}^2K^2\frac{\varepsilon p}{\nu}\left[\frac{(1-\lambda_2/3)^{1-\lambda_2}}{(1+\lambda_2)^{1+\lambda_2}}(2\lambda_2\mathrm{e}\nu K)^{2\lambda_2}\right]^{p-2}.
\end{aligned}$$

[d]We omit the analysis of $\hat{u}_M$ since it is similar to the one we illustrate.

For $\omega_2 := |\ln \bar{w}_2|$ and $\bar{w}_2 := \frac{(1-\lambda_2)^{1-\lambda_2}}{(1+\lambda_2)^{1+\lambda_2}}$ we obtain

$$\|\tilde{u}_M - \mathcal{I}_p \tilde{u}_M\|^2_{\mathcal{E},I^*_\varepsilon} \leq \varepsilon C \mathrm{e}^{-\omega_2 \nu (p-2)}, \tag{3.2.17}$$

provided $\nu \leq (2\lambda_2 \mathrm{e} K)^{-1}$.

On the interval $I \setminus I^*_\varepsilon = (\nu p \varepsilon, 1)$ we use $\tilde{u}_M$'s cubic interpolant $I_3 \tilde{u}_M$, and get

$$\|\tilde{u}_M - I_3 \tilde{u}_M\|^2_{\mathcal{E},I\setminus I^*_\varepsilon} \leq \|\tilde{u}_M\|^2_{\mathcal{E},I\setminus I^*_\varepsilon} + \|I_3 \tilde{u}_M\|^2_{\mathcal{E},I\setminus I^*_\varepsilon}. \tag{3.2.18}$$

From the regularity result (2.4.44) we get

$$\|\tilde{u}_M\|^2_{\mathcal{E},I\setminus I^*_\varepsilon} \leq \int_{\nu p \varepsilon}^1 \big(\varepsilon^2 |\tilde{u}''_M|^2 + |\tilde{u}'_M|^2 + |\tilde{u}_M|^2\big) dx \leq \; C \mathrm{e}^{-\nu p},$$

and

$$\|I_3 \tilde{u}_M\|^2_{\mathcal{E},I\setminus I^*_\varepsilon} \leq C(|\tilde{u}_M(\nu p \varepsilon)|^2 + |\tilde{u}'_M(\nu p \varepsilon)|^2 + |\tilde{u}_M(1)|^2 + |\tilde{u}'_M(1)|^2) \leq C \mathrm{e}^{-\nu p},$$

Combining the bounds on the intervals $(0, \nu p \varepsilon)$, $(\nu p \varepsilon, 1)$ by (3.1.3) we obtain

$$\|\tilde{u}_M - \mathcal{I}_p \tilde{u}_M\|^2_{\mathcal{E},I} \leq C \mathrm{e}^{-\nu p},$$

as desired. □

Remark 3.2.19. The constant ν_0 in the previous theorem can be specified if we know the constants of analyticity of the data. However, the choice $\nu_0 = 1$, i.e. the mesh is chosen as $\{0, p\varepsilon, 1-p\varepsilon, 1\}$ if $p\varepsilon \leq 1/2$, gives the best result in all numerical experiments performed by us and in the literature.

To summarize, it has been illustrated that when then *hp*-FEM is applied to fourth-order SPBVPs, the error measured in the energy norm decays exponentially. However, we must mention a weak spot. The natural choice of the energy norm is not *balanced.*

To see this, suppose the energy norm of a layer function $l : [0,1] \to [0,1]$ such that $l'(x) = \exp\big(-dist(x, \partial I)/\varepsilon\big)$, $\forall\, x \in [0,1]$ is measured. We find that the boundary layer term contributes $O(\varepsilon^{1/2})$ to $\varepsilon \|l''\|_{0,I}$. On the other hand, the smooth part contributes $O(1)$ to the corresponding term. Hence, each component has different orders of magnitude and that is the reason researchers call the energy norm *unbalanced.* Moreover, we observe that as the perturbation parameter ε tends to zero, the energy norm of the layer vanishes. Therefore researchers prefer to choose a balanced norm in order to avoid this phenomenon. The balanced norm is defined as

$$|||u|||_I^2 = \varepsilon \|u''\|_{0,I}^2 + \|u\|_{1,I}^2,$$

and in the next section we will numerically test the following conjectures:

$$|||u - u_N|||_I \leq C\mathrm{e}^{-\beta p}, \quad \beta \in \mathbb{R}^+, \tag{3.2.20}$$

$$||u^{(\kappa)} - u_N^{(\kappa)}||_{\infty,I} \leq C\mathrm{e}^{-\sigma p}, \quad \sigma \in \mathbb{R}^+, \quad \kappa = 0, 1, \tag{3.2.21}$$

$$||u^{(\kappa)} - u_N^{(\kappa)}||_{\infty,[0,\nu p\varepsilon]\cup[1-\nu p\varepsilon,1]} \leq C\varepsilon^{1-\kappa}\mathrm{e}^{-\delta p}, \quad \delta \in \mathbb{R}^+, \quad \kappa = 0, 1. \tag{3.2.22}$$

Similar results have been established for second order SPPs [50] and we believe they hold in our case too.

3.3 Numerical Results

Now we present some numerical results to test the conjectures (3.2.20)–(3.2.22). Numerical results illustrating robust exponential convergence in the energy norm may be found in [57].

Example 3.3.1. We first provide a numerical approximation for a simple form of (2.2.1) and (2.2.3), i.e. we choose constant coefficients and we set them as $\alpha(x) = \beta(x) = f(x) = 1$. Using the exact solution for the calculations, in Figure 3.3 we present the error in the balanced norm versus the number of degrees of freedom in a semilog scale. The perturbation parameter takes the values $\varepsilon = 10^{-j}$, $j = 3, \ldots, 8$. As Figure 3.3 illustrates, the method yields robust exponential convergence suggesting that (3.2.20) holds. We also present the numerical results for the error of the approximation and its first-order derivative in the maximum norm (see Figures 3.4 and 3.5). These numerical results show that the error in u improves as $\varepsilon \to 0$, however the error in u' remains unaffected as $\varepsilon \to 0$. This fact suggests that (3.2.21), (3.2.22) also hold.

Example 3.3.2. Next we consider a variable coefficient problem with $\alpha(x) = \mathrm{e}^{-x}$, $\beta(x) = 0$ and $f(x) = \mathrm{e}^{-x^2} + 1$. As was done in Example 3.3.1, in Figures 3.6, 3.7

and 3.8 we show the now estimated error in the balanced and maximum norms versus the number of degrees of freedom, in a semi-log scale. We note that here the error is calculated with the aid of a reference approximation which is obtained with polynomials of degree $2p$, since the exact solution is not availiable. We again reach the same conclusions as in the previous example.

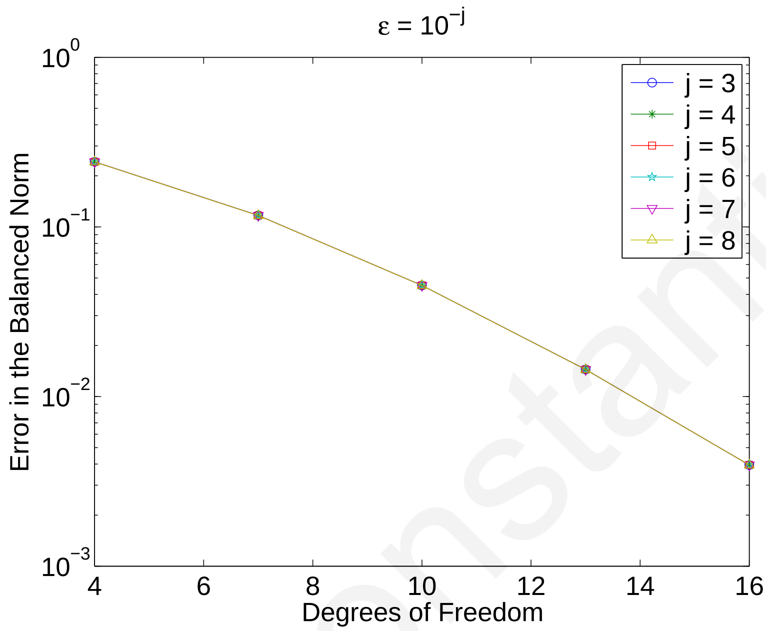


Figure 3.3: Balanced norm convergence for Example 3.3.1

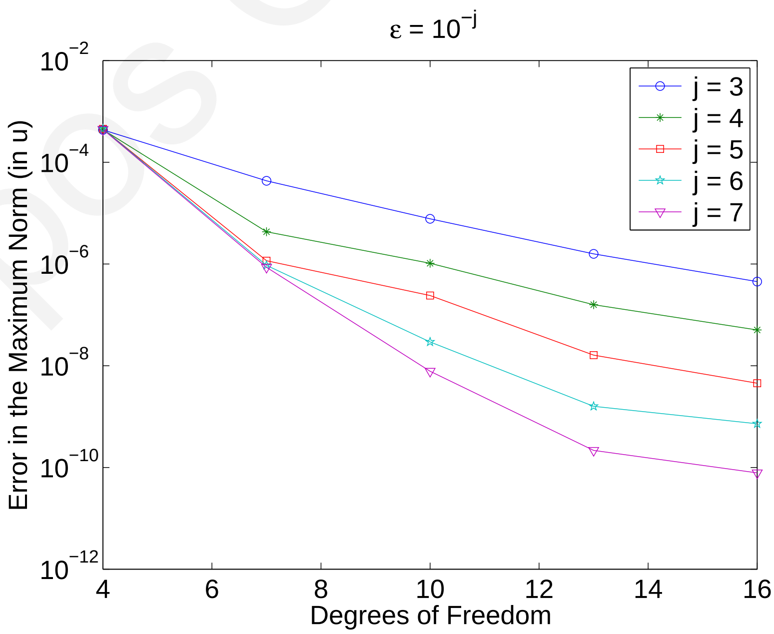


Figure 3.4: Maximum norm convergence in u, within the layer region, for Example 3.3.1

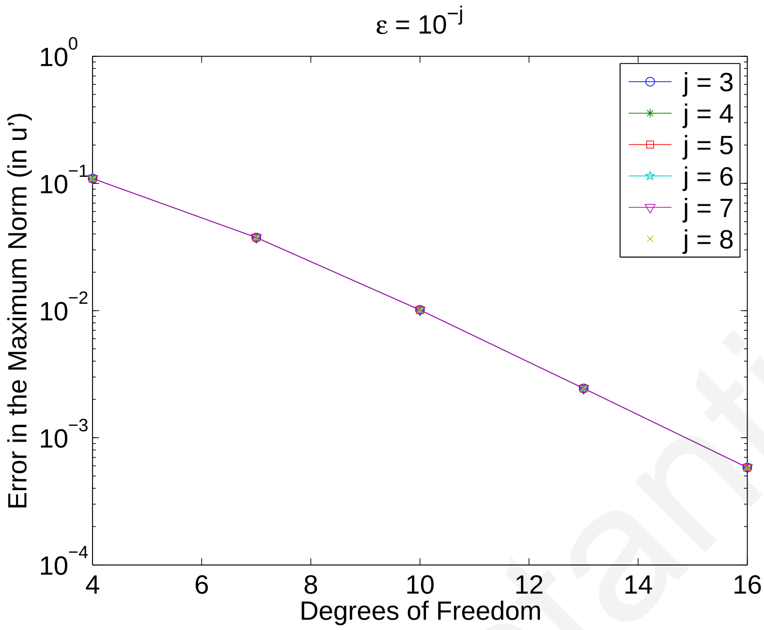


Figure 3.5: Maximum norm convergence in u', within the layer region, for Example 3.3.1

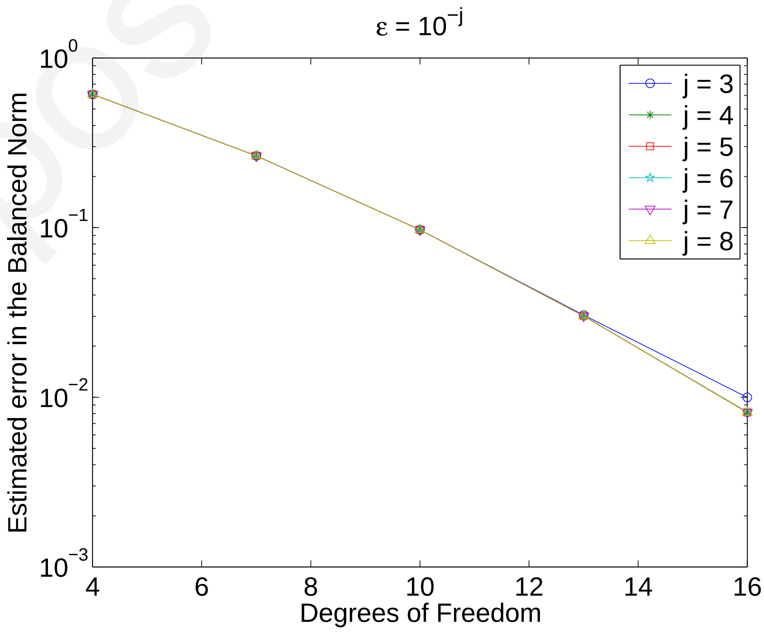


Figure 3.6: Balanced norm convergence for Example 3.3.2

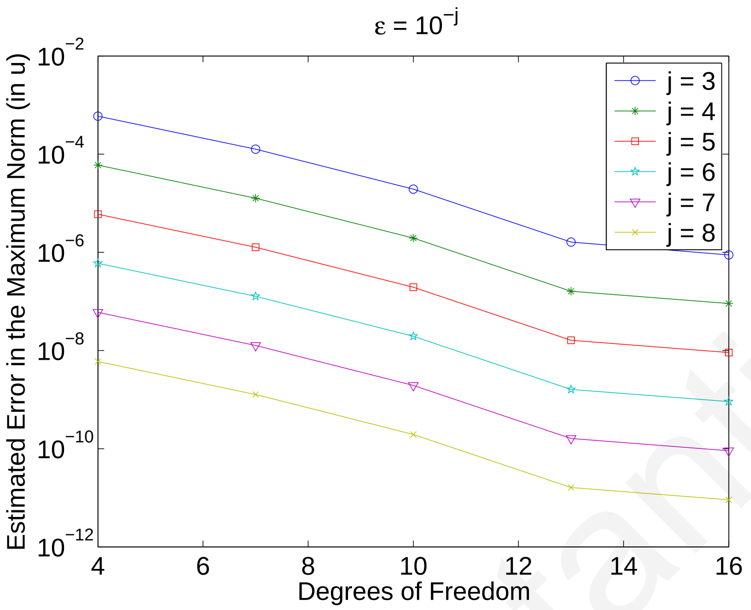


Figure 3.7: Maximum norm convergence in u, within the layer region, for Example 3.3.2

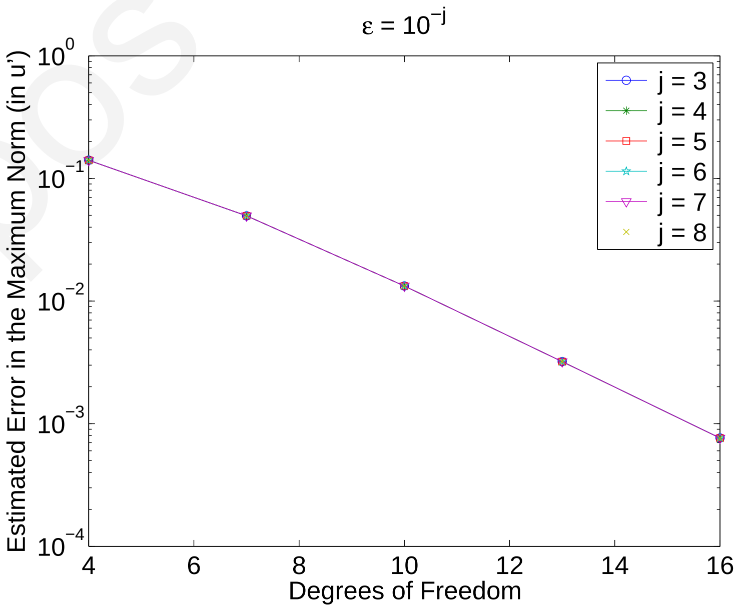


Figure 3.8: Maximum norm convergence in u', within the layer region, for Example 3.3.2

Chapter 4

Regularity results for a fourth-order SPP in two-dimensions

4.1 Introduction

In this chapter we study the regularity of the solution to a Dirichlet boundary value problem for a fourth-order singularly perturbed equation defined on a bounded arbitrary smooth domain in two dimensions. Namely, for $\varepsilon > 0$, we consider the model problem: find $u \in C^4(\overline{\Omega})$ such that

$$\left.\begin{aligned} \Lambda_\varepsilon u = \varepsilon^2 \Delta^2 u - b\Delta u + cu = f, &\quad \text{in } \Omega \subset \mathbb{R}^2, \\ u = \frac{\partial u}{\partial n} = 0, &\quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.1.1}$$

where b and c are positive constants, the right-hand-side function f is assumed to be analytic on a neighborhood of $\overline{\Omega}$, in the sense that

$$\|D^\alpha f\|_{\infty,\Omega} \leq C_f \gamma_f^{|\alpha|} \max\{|\alpha|^{|\alpha|}, \varepsilon^{1-|\alpha|}\}, \quad \text{for all } \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0 \times \mathbb{N}_0, \tag{4.1.2}$$

and Ω is a bounded domain with an analytic boundary, i.e. $\partial\Omega$ is a closed, nonselfintersecting, analytic curve. As usual, Δ is the Laplacian operator, $\Delta^2 = \Delta\Delta$ is the Biharmonic operator, and $\frac{\partial}{\partial n}$ is the normal derivative (directed outwards).

We focus on fourth-order partial differential problems with analytic data defined on smooth domains. In order to generalize the regularity results for the corresponding problems over nonsmooth domains, one must address many issues that arise. For instance, the corners of the domain may complicate the problem and reduce the regularity of the solution. This kind of problems are still open. We recall that for second order problems analogous research has been obtained by several researchers (see for example [10],[19],[35],[42],[45] and [81]) and we expect that similar results also hold for fourth order SPPs on nonsmooth domains. This, however, is not pursued further in this dissertation.

Our goal is to control the high order partial derivatives of the solution u to the problem (4.1.1), explicitely in ε, under the assumption that the input data are analytic too. We adapt results from classical elliptic theory to the data of our problem and we obtain a bound on the higher-order partial derivatives of the solution, in the form:

$$\|D^\alpha u\|_{L^2(\Omega)} \leq CK^{|\alpha|} \max\{|\alpha|^{|\alpha|}, \varepsilon^{1-|\alpha|}\}, \quad \text{for all } \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0 \times \mathbb{N}_0, \tag{4.1.3}$$

with $C, K > 0$ independent of ε. In the case $\varepsilon \geq \frac{1}{|\alpha|}$, the above estimate is sufficient to obtain the desired approximation result.

However, in the complementary case this bound is not helpful, since the error grows as the perturbation parameter tends to zero. To overcome this difficulty we apply the technique from asymptotic expansion theory, that has been used in Chapter 2. The idea is the same as in 1-D, i.e. we decompose the solution into three parts, a smooth part, boundary layers and a remainder. We handle each part separately, namely we present the asymptotic expansions for each part and with the appropriate treatment, we present bounds on their high order partial derivatives. We mention that we use boundary fitted coordinates from [6] and we describe them in Section 4.4.

Variational problem: We recast (4.1.1) into a variational form that reads: Find $u \in H_0^2(\Omega)$ such that

$$B_\varepsilon(u, v) = F(v), \quad \text{for all } v \in H_0^2(\Omega), \tag{4.1.4}$$

where the bilinear form B_ε and the linear functional F are defined as

$$B_\varepsilon(u, v) = \varepsilon^2 \langle \Delta u, \Delta v \rangle_\Omega + \langle b \nabla u, \nabla v \rangle_\Omega + \langle cu, v \rangle_\Omega, \tag{4.1.5}$$

and

$$F(v) = \langle f, v \rangle_\Omega.$$

We define the energy norm as

$$\|u\|^2_{E,\Omega} := B_\varepsilon(u,u), \tag{4.1.6}$$

and it follows that the bilinear form (4.1.5) is strongly coercive with respect to this norm, i.e. for some constant $C > 0$,

$$B_\varepsilon(u,u) \geq \|u\|^2_{E,\Omega}, \quad \forall u \in H^2_0(\Omega). \tag{4.1.7}$$

Moreover

$$\|u\|_{E,\Omega} \leq \|f\|_{L^2(\Omega)}. \tag{4.1.8}$$

4.2 Some auxiliary results

Before we present the main results, we collect some auxiliary lemmata.

Lemma 4.2.1 *[46, Lemma B.4] Let $I \subset \mathbb{R}$ be a closed bounded interval and assume that the functions f, g are analytic on I. Then there are constants C, K_1, $K_2 > 0$ such that*

$$\|D^p(f^n g)\|_{L^\infty(I)} \leq Cp! K_1^n K_2^p \qquad \forall\, n,\, p\, \in\, \mathbb{N}_0.$$

Moreover, the constant $K_1 > \|f\|_{L^\infty(I)}$ may be chosen arbitrarily close to $\|f\|_{L^\infty(I)}$.

Lemma 4.2.2 *[49, Lemma 2.7] For every $q \geq 0$ and for every $M \in \mathbb{N}_0$,*

$$\sum_{j=0}^{M} q^j \leq 2\big(1 + (4q)^M\big).$$

Lemma 4.2.3 *[45, Lemma 7.3.13] For every M, $a \geq 0$, $\alpha \in (0,1)$ there holds*

$$\sup_{r>0} (M + a + r)^M \mathrm{e}^{-\alpha r} \leq M^M \alpha^{-M} \mathrm{e}^{(1-\alpha)M} \mathrm{e}^{\alpha a}.$$

Proposition 4.2.4 *For some positive constants b and c we consider the second order*

Dirichlet boundary value problem: Find $u \in C^2(\overline{\Omega})$ *such that*

$$\begin{aligned} -b\Delta u + cu &= f, \quad \text{in } \Omega \subseteq \mathbb{R}^2, \\ u &= g, \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2.5}$$

Here we suppose that f *is analytic in a neighborhood of* $\overline{\Omega}$ *and* g *is analytic on* $\partial\Omega$*. The domain* Ω *is assumed to be a bounded Lipschitz domain, and* $\partial\Omega$ *is assumed to be a closed, nonselfintersecting, analytic curve. For an analytic function* h*, the problem* (4.2.5) *is equivalent to the BVP: Find* $u \in C^2(\overline{\Omega})$ *such that*

$$\begin{aligned} -b\Delta v + cv &= h, \quad \text{in } \Omega \subseteq \mathbb{R}^2, \\ v &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{4.2.6}$$

where h *is given by* $h = f - cG$*. Here* G *denotes an analytic extension of* g *into* Ω *(as defined in the proof).*

Proof. The boundary data g is analytic; therefore we extend it analytically into Ω as follows: We define the extended function G by

$$\begin{aligned} -\Delta G &= 0 \text{ on } \Omega, \\ G &= g \text{ on } \partial\Omega. \end{aligned}$$

Standard elliptic theory gives that G is analytic on $\overline{\Omega}$. We proceed by setting v as

$$v = u - G. \tag{4.2.7}$$

For $h = f - cG$, the function v satisfies (4.2.6). □

Remark 4.2.8. We consider the problem: Find u such that

$$\begin{aligned} -b\Delta u + cu &= f, \quad \text{in } \Omega \subseteq \mathbb{R}^2, \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{4.2.9}$$

Here b and c are again assumed to be positive constants. The weak solution of (4.2.9) solves the problem: Find $u \in H_0^1(\Omega)$ such that

$$\mathfrak{B}(u, v) = F(v), \text{ for all } v \in H_0^1(\Omega),$$

where the bilinear form is defined as $\mathfrak{B}(u,v) := b\langle \nabla u, \nabla v\rangle + c\langle u, v\rangle$, and the linear form F as $F(v) := \langle f, v\rangle$.

The associated energy norm is given by $\|u\|^2_{\mathfrak{E},\Omega} = \mathfrak{B}(u,u)$. It is easy to see that

$$c\|u\|^2_{0,\Omega} \leq \|u\|^2_{\mathfrak{E},\Omega}, \quad b\|\nabla u\|^2_{0,\Omega} \leq \|u\|^2_{\mathfrak{E},\Omega}. \tag{4.2.10}$$

Moreover by Cauchy's inequality we obtain that $\mathfrak{B}(u,u) \leq \|f\|_{0,\Omega}\|u\|_{0,\Omega}$, and therefore we have

$$\|u\|^2_{\mathfrak{E},\Omega} \leq \|f\|_{0,\Omega}\|u\|_{0,\Omega} \leq \frac{1}{\sqrt{c}}\|f\|_{0,\Omega}\|u\|_{\mathfrak{E},\Omega},$$

which leads to

$$\|u\|_{\mathfrak{E},\Omega} \leq \frac{1}{\sqrt{c}}\|f\|_{0,\Omega}. \tag{4.2.11}$$

From the differential equation of (4.2.9), we have $-b\Delta u = f - cu$ and hence we get

$$b\|\Delta u\|_{0,\Omega} = \|f - cu\|_{0,\Omega} \leq \|f\|_{0,\Omega} + c\|u\|_{0,\Omega}. \tag{4.2.12}$$

Combining (4.2.10), (4.2.11) and (4.2.12) we obtain

$$\|\Delta u\|_{0,\Omega} \leq \frac{2}{b}\|f\|_{0,\Omega}. \tag{4.2.13}$$

Theorem 4.2.14 *[49, Theorem 3.1] Let f, g be analytic functions on $\overline{\Omega}$ and $\partial\Omega$, respectively and let $u \in C^2(\overline{\Omega})$ satisfy of the boundary value problem:*

$$\begin{aligned} -b\Delta u + cu &= f, \quad in\ \Omega \subset \mathbb{R}^2, \\ u &= g, \quad on\ \partial\Omega, \end{aligned} \tag{4.2.15}$$

where $\partial\Omega$ is an analytic curve. Then there are C and $\tilde{\gamma} > 0$ depending only on f and g and the geometry of Ω such that

$$\|D^\alpha u\|_{L^2(\Omega)} \leq C\tilde{\gamma}^{|\alpha|}\max\{|\alpha|, (b/c)^{-1/2}\}^{|\alpha|}(1 + \|u\|_{E,\Omega}), \quad \forall \alpha \in \mathbb{N}^2_0. \tag{4.2.16}$$

Let G be a bounded domain of class C^1 and let u, v be of class C^4 on $\overline{G} = G \cup \partial G$. From Green's theorem we have

$$\int_G (u\Delta v - v\Delta u)dx = \int_{\partial G}\Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big)dS, \tag{4.2.17}$$

where n is the exterior normal. We define (see for instance [2]) the fundamental singularity

$$K^*(r) := \frac{1}{8\pi} r^2(\log r - 1) \tag{4.2.18}$$

of the operator Δ^2 at $x = 0$ which satisfies the differential equation

$$\Delta^2 K^* = \delta \text{ in } \mathbb{R}^2, \tag{4.2.19}$$

in the sense of distributions, where δ is the Dirac measure supported at the origin.

Now, suppose G is bounded and of class C^1, $x_0 \in G$, $u \in C^4(G)$ and there holds $\Delta^2 u = f$, on G. Suppose $\overline{B(x_0, \rho)} \subset G$ and apply (4.2.17) to the domain $G \setminus \overline{B(x_0, \rho)}$, with $v(x) = K^*(x - x_0)$. We obtain

$$\int_{\partial B(x_0,\rho)} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big) dS = \int_{\partial G} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big) dS + \int_{G \setminus B(x_0,\rho)} v(x) f(x) dx. \tag{4.2.20}$$

Letting $\rho \to 0$ in (4.2.20) we get

$$u(x_0) = \int_{\partial G} \Big(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\Big) dS + \int_G K^*(x - x_0) f(x) dx. \tag{4.2.21}$$

In (4.2.21) with $v(x) = K^*(x - x_0)$, we see that if u is of class C^4 on G with

$$\Delta^2 u(x) = f(x), \tag{4.2.22}$$

where G is of a class C^1, then the boundary integrals are biharmonic so that the function U defined by

$$U(x) = \int_G K^*(x - x_0) f(\xi) d\xi, \tag{4.2.23}$$

would differ from u by a biharmonic function and hence would also be a solution of (4.2.22).

Definition 4.2.1 *The equation* (4.2.22) *is known as nonhomogeneous biharmonic equation and the function U in* (4.2.23) *is called the potential of f.*

The above are based on Morrey who studied integrals of the form

$$U(x) = \int_G K(x - \xi) f(\xi) d\xi, \tag{4.2.24}$$

and the results which we present next, are taken from [53].

Theorem 4.2.25 *[53, Theorem 3.4.2] Let m be a positive integer and let the function K be essentially homogeneous of degree $m-n$. Moreover, suppose K belongs to the space $C^{m+1}(\mathbb{R}^n \setminus \{0\})$, suppose f is defined on $\mathbb{R}^n$ and u is defined by*

$$u(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy, \; y \neq x \in \mathbb{R}^n.$$

1. *Let $0 < \mu \leq 1$. If $f \in C_c^{\mu}(\mathbb{R}^n)$, then $u \in C_{\mu}^m(\mathbb{R}^n)$ and*

$$D^{\alpha}u(x) = \begin{cases} \int_{\mathbb{R}^n} D^{\alpha}K(x-y)f(y)dy, & 0 \leq |\alpha| \leq m-1, \\ C_{\alpha}f(x) + \lim_{\rho\to 0} \int_{\mathbb{R}^n\setminus B(x,\rho)} D^{\alpha}K(x-y)f(y)dy, & |\alpha| = m, \end{cases} \tag{4.2.26}$$

where $C_{\alpha} = -\int_{\Sigma} D^{\beta}K(-\eta)\eta_{\gamma}d\Sigma, \quad \alpha = \beta+\gamma, \; |\beta| = m-1, \; |\gamma| = 1, \; \Sigma = \partial B(0,1).$

2. *If $f \in L_p(\mathbb{R}^n)$ and has compact support on $G \subset\subset \mathbb{R}^n$ then $u \in H_p^m(D)$ for any bounded domain D, the formula (4.2.26) holds almost everywhere and*

$$\|u\|_{p,D}^{m} \leq C(n, N, m, p, D, K)\|f\|_{p,G}^{0}. \tag{4.2.27}$$

4.3 Growth estimates

In this section, we give an important result (Theorem 4.3.32) regarding the derivatives of the solution to the fourth order elliptic Dirichlet SPP, (4.1.1). As already mentioned, we are interested in the fourth order SPP with constant coefficients, however throughout this section we assume that b and c are analytic functions. This is necessary in order to obtain the desired result on the smooth domain. To be more specific, we will first give the results on the disc and half disk. We then need to generalize them on an arbitrary smooth domain. To achieve this task (see Theorem 4.3.32) we will use a certain conformal mapping. The fourth order SPP with constant coefficients is transformed under this mapping into a fourth order SPP with variable coefficients. For this reason we provide the analysis for the problem with the variable coefficients.

Here, we adopt ideas from [49], [53] and we use similar notation as in [53]. We note that Morrey [53] first studied the general second order elliptic Dirichlet problem. Melenk and Schwab [49] considered the Dirichlet boundary value problem for second order SPP in two dimensions. We address the same goal that Melenk and Schwab have achieved in [49] which in our case concerns the fourth order SPP instead. Namely, we control the dependence of the growth of the higher order partial derivatives of the solution on the perturbation parameter and the differentiation order.

The following notation as mentioned before, follows [53]. First, we set

$$[n] := \max\{1, n\}, \qquad \text{for all } n \in \mathbb{Z}.$$

Then, we define for a positive number ρ, the discs B_ρ and half discs $\mathcal{H}_\rho$ in two-dimensions by

$$B_\rho := B_\rho(0) \subset \mathbb{R}^2, \qquad \mathcal{H}_\rho := \{(x, y) \in B_\rho \,|\, y > 0\}.$$

Suppose $\tilde{x} \in \mathbb{R}^2$ and $D \subset \mathbb{R}^2$. We set the distance of $\tilde{x}$ from D as

$$d(\tilde{x}, D) = \inf\{\|\tilde{x} - \tilde{y}\|_2 : \tilde{y} \in D\},$$

where $\|\cdot\|_2$ is the usual Euclidean norm.

For a real number $\rho_0 > 0$ and a smooth function $u \in C^\infty(B_{\rho_0})$, we introduce the following notation:

We consider a two dimensional vector $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and we demand each of its components to be non-negative integers. We also define

$$|\alpha| = \alpha_1 + \alpha_2, \qquad \alpha! = (\alpha_1!)(\alpha_2!), \tag{4.3.1}$$

$$\left|\nabla^n u(x, y)\right|^2 := \sum_{|\alpha|=n} \frac{|\alpha|!}{\alpha!} \left|D^\alpha u(x, y)\right|^2 = \sum_{\alpha_1+\alpha_2=n} \frac{n!}{(\alpha_1!)(\alpha_2!)} \left|\frac{\partial^n u(x, y)}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\right|^2, \quad \forall\, n \in \mathbb{N}_0. \tag{4.3.2}$$

Moreover in order to control the derivatives of the solution to the problem we introduce the following notation:

$$\breve{N}_{\rho_0,n}(u) := \frac{1}{[n]!} \sup_{\rho_0/2 \leq \rho < \rho_0} (\rho_0 - \rho)^{4+n} \|\nabla^{n+4} u\|_{L^2(B_\rho)}, \quad \forall\, n \in \mathbb{N}_0 \cup \{-4, -3, -2, -1\},$$

$$\breve{M}_{\rho_0,n}(u) := \frac{1}{n!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{4+n}\|\nabla^n u\|_{L^2(B_\rho)}, \quad \text{for all } n\in\mathbb{N}_0.$$

As can be seen the only difference between the two quantities above is located in the order of the partial derivatives.

Similarly, we introduce notation about smooth functions defined on the half space and vanishing on the boundary. The first notation (with two indices) is used to control the tangential derivatives of the function. The second one controls both tangential and normal derivatives. We let $u\in C^\infty(\overline{\mathcal{H}}_{\rho_0})$ and $u=\frac{\partial u}{\partial\eta}=0$ along $\partial\mathcal{H}_{\rho_0}$. Then define

$$\breve{N}'_{\rho_0,n}(u) := \begin{cases} \frac{1}{n!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{4+n}\left\|\nabla^4\left(\frac{\partial^n u}{\partial x^n}\right)\right\|_{L^2(\mathcal{H}_\rho)}, & n\ge 0,\\ \sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{4+n}\|\nabla^{4+n}u\|_{L^2(\mathcal{H}_\rho)}, & n=-4,-3,-2,-1,\end{cases}$$

and

$$\breve{N}'_{\rho_0,n,m}(u) := \frac{1}{[n+m]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{4+n+m}\left\|\frac{\partial^{n+m+4}u}{\partial x^n\partial y^{m+4}}\right\|_{L^2(\mathcal{H}_\rho)}, \quad n\ge 0,\ m\ge -4.$$

Finally, for all $n\in\mathbb{N}_0$ and $f\in C^\infty(\overline{\mathcal{H}}_{\rho_0})$, the next quantity controls the tangential and all derivatives, respectively, of the smooth function defined on the half space:

$$\breve{M}'_{\rho_0,n}(f) := \frac{1}{n!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{4+n}\left\|\frac{\partial^n f}{\partial x^n}\right\|_{L^2(\mathcal{H}_\rho)},$$

and

$$\tilde{M}_{\rho_0,n}(f) := \frac{1}{n!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{4+n}\|\nabla^n f\|_{L^2(\mathcal{H}_\rho)}.$$

Here we mention that for every $u\in C^\infty(\overline{\mathcal{H}}_{\rho_0})$,

$$\breve{N}'_{\rho_0,n,i}(u)\le\breve{N}'_{\rho_0,n+i}(u), \quad \text{for all } n\ge -4 \text{ and } i=-4,-3,-2,-1, \tag{4.3.3}$$

and

$$\breve{M}'_{\rho_0,n}(f)\le\tilde{M}'_{\rho_0,n}(f), \quad \text{for all } n\in\mathbb{N}_0,\ f\in C^\infty(\overline{\mathcal{H}}_{\rho_0}). \tag{4.3.4}$$

The proofs of the auxiliary results presented in this section, appear in the Appendix at the end of the chapter.

Lemma 4.3.5 *Let $\rho, \rho_0 \in \mathbb{R}, 0 < \rho < \rho_0$. Suppose $u \in H^3(B_{\rho_0})$, $u\big|_{B_\rho} \in H^4(B_\rho)$, and*

$$\Delta^2 u = f, \quad \text{on } B_{\rho_0}.$$

Then there exists a constant C such that

$$\int_{B_\rho} |\nabla^4 u|^2 dx \leq C\Big[\int_{B_{\rho+\delta}} f^2 dx + \int_{B_{\rho+\delta}} \Big(\delta^{-2}|\nabla^3 u|^2 + \delta^{-4}|\nabla^2 u|^2 + \delta^{-6}|\nabla u|^2 + \delta^{-8}|u|^2\Big) dx\Big], \tag{4.3.6}$$

for all $\delta \in (0, \rho_0 - \rho)$.

Proof. First we define $\eta : \mathbb{R}^2 \to \mathbb{R}$ as

$$\eta(\tilde{x}) := \begin{cases} 1, & \tilde{x} \in B_\rho \subset B_{\rho+\delta}, \\ 1 - 2\delta^{-1}\text{dist}(\tilde{x}, B_\rho), & \text{for } 0 \leq \text{dist}(\tilde{x}, B_\rho) \leq \delta/2, \\ 0, & \text{otherwise,} \end{cases}$$

and by standard mollification we obtain a new function, denoted again by η, such that $\eta \in C^\infty(B_{\rho+\delta})$. We set $v = \eta u$ and we note that

$$\begin{aligned} \Delta^2 v =& \eta f + 6\big(\Delta\eta\big)\big(\Delta u\big) + u\Delta^2\eta + \\ &+ 4\Big[\frac{\partial \eta}{\partial x}\Big(\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2}\Big) + \frac{\partial \eta}{\partial y}\Big(\frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial x^2 \partial y}\Big)\Big] + \\ &+ 4\Big[\frac{\partial u}{\partial x}\Big(\frac{\partial^3 \eta}{\partial x^3} + \frac{\partial^3 \eta}{\partial x \partial y^2}\Big) + \frac{\partial u}{\partial y}\Big(\frac{\partial^3 \eta}{\partial y^3} + \frac{\partial^3 \eta}{\partial x^2 \partial y}\Big)\Big], \end{aligned}$$

and hence

$$|\Delta^2 v| \leq C\Big\{|\eta||f| + \big|\Delta\eta\big|\big|\Delta u\big| + |u|\big|\Delta^2\eta\big| + \big|\nabla\eta\big|\big|\nabla^3 u\big| + \big|\nabla u\big|\big|\nabla^3\eta\big|\Big\}.$$

Using (4.2.27) of Theorem 4.2.25 and the definition of η, we obtain the desired result. □

Lemma 4.3.7 *Let $\rho, \rho_0 \in \mathbb{R}, 0 < \rho < \rho_0$. Suppose $u \in H^3(\mathcal{H}_{\rho_0})$, $u\big|_{H_\rho} \in H^4(\mathcal{H}_\rho)$, and*

$$\begin{aligned} \Delta^2 u &= f, && \text{on } \mathcal{H}_{\rho_0}, \\ u = \frac{\partial u}{\partial \eta} &= 0, && \text{on } \partial\mathcal{H}_{\rho_0}. \end{aligned}$$

Then there is a constant C such that

$$\int_{\mathcal{H}_\rho} |\nabla^4 u|^2 dx \le C\Big[\int_{\mathcal{H}_{\rho+\delta}} f^2 dx + \int_{\mathcal{H}_{\rho+\delta}} \Big(\delta^{-2}|\nabla^3 u|^2 + \delta^{-4}|\nabla^2 u|^2 + \delta^{-6}|\nabla u|^2 + \delta^{-8}|u|^2\Big) dx\Big], \tag{4.3.8}$$

for all $\delta \in (0, \rho_0 - \rho)$.

Proof. We extend the solution u and the right hand side function f to the disk B_{ρ_0} using the reflection given by the following formula, (which satisfies the boundary conditions, see e.g. [62, Theorem 3.1])

$$v(x,y) = -v(x,-y) - 2y\frac{\partial v}{\partial y}(x,-y) - y^2 \Delta v(x,-y), \quad \text{ for } (x,y) \in B_{\rho_0} \setminus \mathcal{H}_{\rho_0}.$$

The desired result follows directly from the previous lemma. □

Lemma 4.3.9 *Let $0 < \rho_0 \le 1$. Suppose $u, f \in C^\infty(B_{\rho_0})$ and u satisfies*

$$\Delta^2 u = f, \tag{4.3.10}$$

on B_{ρ_0}. We suppose also that $\breve{M}_{\rho_0,n}(f) < \infty$ and $\breve{N}_{\rho_0,n}(u) < \infty$, for each n. Then, there is a constant C such that

$$\breve{N}_{\rho_0,n}(u) \le C\Big(\breve{M}_{\rho_0,n}(f) + \breve{N}_{\rho_0,n-1}(u) + \breve{N}_{\rho_0,n-2}(u) + \breve{N}_{\rho_0,n-3}(u) + \breve{N}_{\rho_0,n-4}(u)\Big), \quad n \ge 0. \tag{4.3.11}$$

Proof. Let $n \in \mathbb{N}$. We note that the partial derivatives $\frac{\partial^n u}{\partial x^n}$, $\frac{\partial^n u}{\partial y^n}$ satisfy the biharmonic equations $\Delta^2(\frac{\partial^n u}{\partial x^n}) = \frac{\partial^n f}{\partial x^n}$, $\Delta^2(\frac{\partial^n u}{\partial y^n}) = \frac{\partial^n f}{\partial y^n}$, respectively. Hence we apply Lemma 4.3.5 for each $\frac{\partial^n u}{\partial x^n}$, $\frac{\partial^n u}{\partial y^n}$, we sum up the estimates and we obtain

$$\Big(\breve{N}_{\rho_0,n}(u)\Big)^2 \le C(n!)^{-2} \sup_{\rho_0/2 \le \rho < \rho_0} (\rho_0 - \rho)^{8+2n}\Big[\int_{B_{\rho_0+\delta}} |\nabla^n f|^2 dx + \int_{B_{\rho_0+\delta}} \Big(\delta^{-2}|\nabla^{n+3}u|^2 + \\ + \delta^{-4}|\nabla^{n+2}u|^2 + \delta^{-6}|\nabla^{n+1}u|^2 + \delta^{-8}|\nabla^n u|^2\Big) dx\Big].$$

With the aid of the estimates (4.5.10) and by setting $\delta = \frac{\rho_0 - \rho}{n+1}$ for $n \ge 4$, we get

$$\Big(\breve{N}_{\rho_0,n}(u)\Big)^2 \le C\Big[\Big(1+\frac{1}{n}\Big)^{2n+8}\Big(\breve{M}_{\rho_0,n}(f)\Big)^2 + \Big(1+\frac{1}{n}\Big)^{2n+8}\Big(\breve{N}_{\rho_0,n-1}(u)\Big)^2 + \\ + \frac{(n+1)^2}{(n-1)^2}\Big(1+\frac{1}{n}\Big)^{2n+6}\Big(\breve{N}_{\rho_0,n-2}(u)\Big)^2 + \frac{(n+1)^4}{(n-1)^2(n-2)^2}\Big(1+\frac{1}{n}\Big)^{2n+4}\Big(\breve{N}_{\rho_0,n-3}(u)\Big)^2 +$$

$$+\frac{(n+1)^6}{(n-1)^2(n-2)^2(n-3)^2}\Big(1+\frac{1}{n}\Big)^{2n+2}\Big(\breve{N}_{\rho_0,n-4}(u)\Big)^2\Bigg],$$

which completes the proof. Similar results hold in the cases $n = 0, 1, 2, 3$.[a] □

Proposition 4.3.12 *Let $\rho_0 \in (0, 1]$. Assume b, c and f are analytic functions on B_{ρ_0} and satisfy, for some positive constants C_b, C_c, C_f, γ, γ_b, γ_c, the bounds below:*

$$\left.\begin{aligned}\Big\|\nabla^n b\Big\|_{L^\infty(B_{\rho_0})} &\leq C_b\gamma_b^n n!, \qquad \forall n \in \mathbb{N}_0,\\ \Big\|\nabla^n c\Big\|_{L^\infty(B_{\rho_0})} &\leq C_c\gamma_c^n n!, \qquad \forall n \in \mathbb{N}_0,\end{aligned}\right\} \tag{4.3.13}$$

$$\|\nabla^n f\|_{L^2(B_{\rho_0})} \leq C_f\gamma^n\big(n^n\rho_0^{-n} + \max\{n^n, \varepsilon^{1-n}\}\big), \quad \forall n \in \mathbb{N}_0. \tag{4.3.14}$$

Then, for any function u that satisfies,

$$\varepsilon^2\Delta^2 u - b\Delta u + cu = f, \quad \text{on } B_{\rho_0} \subset \mathbb{R}^2, \tag{4.3.15}$$

there exists a constant $K \geq 1$, independent of ε and ρ_0, such that for all $n \geq -4$,

$$\breve{N}_{\rho_0,n}(u) \leq C_u K^{n+4}\frac{\max\{[n]^{n+4}, (\rho_0/\varepsilon)^{n+3}\}}{[n]!}, \tag{4.3.16}$$

where,

$$\begin{aligned}C_u = \|u\|_{0,B_{\rho_0}} &+ \min\{1, \rho_0/\varepsilon\}\varepsilon\|\nabla u\|_{0,B_{\rho_0}} + \min\{1, (\rho_0/\varepsilon)^2\}\varepsilon^2\|\nabla^2 u\|_{0,B_{\rho_0}}\\ &+ \min\{1, (\rho_0/\varepsilon)^3\}\varepsilon^3\|\nabla^3 u\|_{0,B_{\rho_0}} + C_f\min\{1, (\rho_0/\varepsilon)^4\}.\end{aligned} \tag{4.3.17}$$

Proof. We, again, use induction. By the definition of C_u, the statement holds for $n = -4, -3, -2$ and -1, since $K \geq 1$. Suppose that for $n \geq 0$, it is also true for every $0 \leq n' < n$. We apply Lemma 4.3.9 to the equation $\Delta^2 u = \varepsilon^{-2}(f + b\Delta u - cu)$ and we get,

$$\begin{aligned}\breve{N}_{\rho_0,n}(u) \leq &C\Big(\varepsilon^{-2}\breve{M}_{\rho_0,n}(f) + \varepsilon^{-2}\breve{M}_{\rho_0,n}(b\Delta u) + \varepsilon^{-2}\breve{M}_{\rho_0,n}(cu) +\\ &+ \breve{N}_{\rho_0,n-1}(u) + \breve{N}_{\rho_0,n-2}(u) + \breve{N}_{\rho_0,n-3}(u) + \breve{N}_{\rho_0,n-4}(u)\Big).\end{aligned}$$

By Lemma 4.5.1 we obtain

$$\breve{N}_{\rho_0,n}(u) \leq C\Bigg(\varepsilon^{-2}\breve{M}_{\rho_0,n}(f) + \varepsilon^{-2}C_b\sum_{k=0}^{n}\Big(\frac{\gamma_b\rho_0}{2}\Big)^{n-k}\frac{[k-2]!}{[k]!}\Big(\frac{\rho_0}{2}\Big)^2\breve{N}_{\rho_0,k-2}(u) +$$

[a] We take $\delta = (\rho_0 - \rho)/2$

$$+\varepsilon^{-2}C_c\sum_{k=0}^{n}\Big(\frac{\gamma_c\rho_0}{2}\Big)^{n-k}\frac{[k-4]!}{[k]!}\Big(\frac{\rho_0}{2}\Big)^4\breve{N}_{\rho_0,k-4}(u)+$$
$$+\breve{N}_{\rho_0,n-1}(u)+\breve{N}_{\rho_0,n-2}(u)+\breve{N}_{\rho_0,n-3}(u)+\breve{N}_{\rho_0,n-4}(u)\Big)$$

and with the aid of the inequalities (4.5.14) and the induction hypothesis, we get

$$\breve{N}_{\rho_0,n}(u)\leq C\Big(\varepsilon^{-2}M_{d_0,n}(f)+\big(C_u\frac{1}{n!}M_1K^{n+4}\big)\max\big\{[n]^{n+4},\big(\rho_0/\varepsilon\big)^{n+3}\big\}\Big).$$

Here

$$\begin{aligned}M_1 =&K^{-1}+K^{-2}+K^{-3}+K^{-4}+\frac{C_b}{4}\sum_{k=0}^{n}\Big(\frac{\gamma_b\rho_0}{2}\Big)^{n-k}K^{k-n-2}+\frac{C_c}{16}\sum_{k=0}^{n}\Big(\frac{\gamma_c\rho_0}{2}\Big)^{n-k}K^{k-n-4}\\ \leq&K^{-1}+K^{-2}+K^{-3}+K^{-4}+\frac{K^{-2}}{1-\frac{\gamma_b\rho_0}{2K}}\frac{C_b}{4}+\frac{K^{-4}}{1-\frac{\gamma_c\rho_0}{2K}}\frac{C_c}{16}.\end{aligned}\tag{4.3.18}$$

Above we notice that the assumption $\dfrac{\max\{\gamma_b,\gamma_c\}\rho_0}{2K}\leq 1$ has been used and therefore the sums are bounded by convergent geometric series. Now, we can see that

$$\begin{aligned}\varepsilon^{-2}M_{\rho_0,n}(f)=&\varepsilon^{-2}\frac{1}{[n]!}\sup_{\rho_0/2\leq\rho<\rho_0}(\rho_0-\rho)^{4+n}C_f\gamma^n\big(n^n\rho_0^{-n}+\max\{n^n,\varepsilon^{1-n}\}\big)\\ \leq&\frac{1}{4}\varepsilon^{-2}\rho_0^4\frac{1}{n!}C_f\Big(\frac{\gamma}{2}\Big)^n\big(n^n+\max\{(\rho_0 n)^n,(\rho_0/\varepsilon)^{n-1}\}\big).\end{aligned}$$

Using (4.5.16) and (4.5.17) we get

$$\varepsilon^{-2}M_{\rho_0,n}(f)\leq\frac{1}{2}\Big(\frac{\gamma}{2}\Big)^nC_f\min\big\{1,(\rho_0/\varepsilon)^4\big\}\frac{\max\big\{[n]^{n+4},\big(\rho_0/\varepsilon\big)^{n+3}\big\}}{n!}.$$

Thus

$$N_{\rho_0,n}(u)\leq C_uM_2K^{n+4}\frac{\max\big\{[n]^{n+4},\big(\rho_0/\varepsilon\big)^{n+3}\big\}}{n!},$$

where $M_2=C\Big(\frac{1}{2}\Big(\frac{\gamma}{2K}\Big)^nK^{-4}C_f+M_1\Big)$.[b] The appropriate choice of the constant K ensures that M_2 is smaller than one and the proof is complete. □

Lemma 4.3.19 *Let* $0<\rho_0\leq 1$ *and suppose* $u,f\in C^\infty(\overline{\mathcal{H}}_{\rho_0})$, u *satisfies* (4.3.10) *on* $\mathcal{H}_{\rho_0}$ *and* $u=\frac{\partial u}{\partial\eta}=0$ *along* $\partial\mathcal{H}_{\rho_0}\cap\{(x,y)|y=0\}$. *Then there is a constant* C *such that,*

[b] The constant M_1 is defined in (4.3.18)

for $n \geq 0$,

$$\breve{N}'_{\rho_0,n}(u) \leq C\Big(\breve{M}'_{\rho_0,n}(f) + \breve{N}'_{\rho_0,n-1}(u) + \breve{N}'_{\rho_0,n-2}(u) + \breve{N}'_{\rho_0,n-3}(u) + \breve{N}_{\rho_0,n-4}(u)\Big). \tag{4.3.20}$$

Proof. We apply the estimate (4.3.8) for $\mathcal{H}_\rho$, to $\frac{\partial^n u}{\partial x^n}$ for $n \geq 0$:

$$\int_{\mathcal{H}_\rho}\Big|\nabla^4\Big(\frac{\partial^n u}{\partial x^n}\Big)\Big|^2 dx \leq C\Big(\int_{\mathcal{H}_{\rho_0+\delta}}\Big|\frac{\partial^n f}{\partial x^n}\Big|^2 dx + \int_{\mathcal{H}_{\rho_0+\delta}}(\delta^{-2}\Big|\nabla^3\Big(\frac{\partial^n u}{\partial x^n}\Big)\Big|^2 + \delta^{-4}\Big|\nabla^2\Big(\frac{\partial^n u}{\partial x^n}\Big)\Big|^2 + \\ + \delta^{-6}\Big|\nabla\Big(\frac{\partial^n u}{\partial x^n}\Big)\Big|^2 + \delta^{-8}\Big|\frac{\partial^n u}{\partial x^n}\Big|^2 dx)\Big).$$

By combining the above inequality and estimates (4.5.12) we obtain the result in the same way as in Lemma 4.3.9. □

Lemma 4.3.21 *Let* $\rho_0 \in (0,1]$ *and let* u *satisfy*

$$\varepsilon^2\Delta^2 u - b\Delta u + cu = f \text{ in } \mathcal{H}_{\rho_0}, \quad u = \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\mathcal{H}_{\rho_0} \cap \{(x,y)|y=0\}, \tag{4.3.22}$$

where the functions b, c *and* f *are analytic on* $\mathcal{H}_{\rho_0}$ *and satisfy, for some constants* $C_f, C_b, C_c, \gamma, \gamma_b, \gamma_c > 0$,

$$\left.\begin{aligned} \left\|\frac{\partial^{n_1+n_2} b}{\partial x^{n_1}\partial y^{n_2}}\right\|_{L^\infty(\mathcal{H}_{\rho_0})} &\leq C_b\gamma_b^{n_1+n_2} n_1! n_2!, \quad \forall\, n_1,\, n_2 \in \mathbb{N}_0, \\ \left\|\frac{\partial^{n_1+n_2} c}{\partial x^{n_1}\partial y^{n_2}}\right\|_{L^\infty(\mathcal{H}_{\rho_0})} &\leq C_c\gamma_c^{n_1+n_2} n_1! n_2!, \quad \forall\, n_1,\, n_2 \in \mathbb{N}_0, \end{aligned}\right\} \tag{4.3.23}$$

$$\|\nabla^n f\|_{L^2(\mathcal{H}_{\rho_0})} \leq C_f\gamma^n\big(n^n\rho_0^{-n} + \max\{n^n, \varepsilon^{1-n}\}\big), \quad \forall\, n \in \mathbb{N}_0. \tag{4.3.24}$$

Also let C_u *be defined as in* (4.3.17). *Then there exists a constant* $K_1 > 0$ *independent of* ε *and* ρ_0 *such that,*

$$\breve{N}'_{\rho_0,n}(u) \leq C_u K_1^{n+4}\frac{\max\{[n]^{n+4}, (\rho_0/\varepsilon)^{n+3}\}}{[n]!}, \quad n \geq -4. \tag{4.3.25}$$

Proof. We use the same strategy as in Proposition 4.3.12, namely the strong induction method on n. It is not hard to verify that (4.3.25) holds true for $n = -4, -3, -2$ and -1. By appealing to Lemma 4.3.19 for the equation $\Delta^2 u = \varepsilon^{-2}(f + b\Delta u - cu)$ and using Lemma 4.5.5 we get

$$\begin{aligned}\breve{N}'_{\rho_0,n}(u) \leq & C\Big(\varepsilon^{-2}\breve{M}'_{\rho_0,n}(f) + C_b\sum_{k=0}^{n}\Big(\frac{\gamma_b\rho_0}{2}\Big)^{n-k}\frac{[k-2]!}{[k]!}\Big(\frac{\rho_0}{2}\Big)^2\breve{N}'_{\rho_0,k-2}(u) + \\ & + C_c\sum_{k=0}^{n}\Big(\frac{\gamma_c\rho_0}{2}\Big)^{n-k}\frac{[k-4]!}{[k]!}\Big(\frac{\rho_0}{2}\Big)^4\breve{N}'_{\rho_0,k-4}(u) + \breve{N}'_{\rho_0,n-1}(u) + \\ & + \breve{N}'_{\rho_0,n-2}(u) + \breve{N}'_{\rho_0,n-3}(u) + \breve{N}'_{\rho_0,n-4}(u)\Big).\end{aligned}$$

We proceed by considering that the induction hypothesis holds for all $-4 < n' < n$. Therefore, we obtain

$$\breve{N}'_{\rho_0,n}(\Delta u) \leq C\varepsilon^{-2}\breve{M}'_{\rho_0,n}(f) + C_u M_1 K_1^{n+4}\frac{\max\{[n]^{n+4}, \big(\rho_0/\varepsilon\big)^{n+3}\}}{n!}.$$

Here M_1 is defined as in (4.3.18). The proof can be completed by handling the term $\breve{M}'_{\rho_0,n}(f)$ in the same way we have treated the term $\breve{M}_{\rho_0,n}(f)$ in Proposition 4.3.12. □

Proposition 4.3.26 *Suppose that the hypotheses of Lemma 4.3.21 hold. Then there are constants K_1, $K_2 > 0$ independent of ε and ρ_0 such that for all $n \geq 0$, $m \geq -4$*

$$\breve{N}'_{\rho_0,n,m}(u) \leq C_u K_1^{n+4}K_2^{m+4}\frac{\max\{[n+m+2]^{n+m+4}, \big(\rho_0/\varepsilon\big)^{n+m+3}\}}{[n+m]!}, \tag{4.3.27}$$

where C_u is defined as in (4.3.17) .

Proof. Here we again use the strong induction method on m. From Lemma 4.3.21 and (4.3.3) we confirm that (4.3.27) holds for $m = -4, -3, -2, -1$ and for all n. Taking the $\frac{\partial^{n+m}}{\partial x^n \partial y^m}$ -derivative of the differential equation we have

$$\begin{aligned}\frac{\partial^{n+m+4}u}{\partial x^n\partial y^{m+4}} = & \varepsilon^{-2}\Big(\frac{\partial^{n+m}f}{\partial x^n\partial y^m} + \frac{\partial^{n+m+2}(bu)}{\partial x^{n+2}\partial y^m} + \frac{\partial^{n+m+2}(bu)}{\partial x^n\partial y^{m+2}} - \frac{\partial^{n+m}(cu)}{\partial x^n\partial y^m}\Big) - \\ & - 2\frac{\partial^{n+m+4}u}{\partial x^{n+2}\partial y^{m+2}} - \frac{\partial^{n+m+4}u}{\partial x^{n+4}\partial y^m},\end{aligned}$$

and an application of (4.5.8) yields

$$\begin{aligned}\Big|\frac{\partial^{n+m+4}u}{\partial x^n\partial y^{m+4}}\Big| \leq & \varepsilon^{-2}\Bigg[\Big|\frac{\partial^{n+m}f}{\partial x^n\partial y^m}\Big| + C_b\sum_{k=0}^{n+2}\sum_{l=0}^{m}(n+m+2)^{n+m+2-k-l}\gamma_b^{n+m+2-k-l}\Big|\frac{\partial^{k+l}u}{\partial x^k\partial y^l}\Big| + \\ & + C_b\sum_{k=0}^{n}\sum_{l=0}^{m+2}(n+m+2)^{n+m+2-k-l}\gamma_b^{n+m+2-k-l}\Big|\frac{\partial^{k+l}u}{\partial x^k\partial y^l}\Big| +\end{aligned}$$

$$+ C_c \sum_{k=0}^{n} \sum_{l=0}^{m} (n+m)^{n+m-k-l} \gamma_c^{n+m-k-l} \left| \frac{\partial^{k+l} u}{\partial x^k \partial y^l} \right| \Bigg] +$$
$$+ 2 \left| \frac{\partial^{n+m+4} u}{\partial x^{n+2} \partial y^{m+2}} \right| + \left| \frac{\partial^{n+m+4} u}{\partial x^{n+4} \partial y^{m}} \right|.$$

Hence by taking the L^2-norm we obtain

$$\left\| \frac{\partial^{n+m+4} u}{\partial x^{n} \partial y^{m+4}} \right\|_{L^2(\mathcal{H}_{\rho_0})} \leq C \Bigg[\varepsilon^{-2} \Bigg(C_b \sum_{k=0}^{n+2} \sum_{l=0}^{m} (n+m+2)^{n+m+2-k-l} \gamma_b^{n+m+2-k-l} \left\| \frac{\partial^{k+l} u}{\partial x^k \partial y^l} \right\|_{L^2(\mathcal{H}_{\rho_0})} +$$
$$+ C_b \sum_{k=0}^{n} \sum_{l=0}^{m+2} (n+m+2)^{n+m+2-k-l} \gamma_b^{n+m+2-k-l} \left\| \frac{\partial^{k+l} u}{\partial x^k \partial y^l} \right\|_{L^2(\mathcal{H}_{\rho_0})} + \left\| \frac{\partial^{n+m} f}{\partial x^n \partial y^m} \right\|_{L^2(\mathcal{H}_{\rho_0})} +$$
$$+ C_c \sum_{k=0}^{n} \sum_{l=0}^{m} (n+m)^{n+m-k-l} \gamma_c^{n+m-k-l} \left\| \frac{\partial^{k+l} u}{\partial x^k \partial y^l} \right\|_{L^2(\mathcal{H}_{\rho_0})} \Bigg) + \left\| \frac{\partial^{n+m+4} u}{\partial x^{n+4} \partial y^{m}} \right\|_{L^2(\mathcal{H}_{\rho_0})} +$$
$$+ 2 \left\| \frac{\partial^{n+m+4} u}{\partial x^{n+2} \partial y^{m+2}} \right\|_{L^2(\mathcal{H}_{\rho_0})} \Bigg]. \tag{4.3.28}$$

Using the notation $\breve{N}'_{\rho_0,n,m}$ and $\tilde{M}_{\rho_0,n}$ we can rewrite the above inequality as

$$\breve{N}'_{\rho_0,n,m}(u) \leq$$
$$\leq C \Bigg[\varepsilon^{-2} \Bigg(C_b \sum_{k=0}^{n+2} \sum_{l=0}^{m} \frac{[k+l-4]!}{[n+m]!} (n+m+2)^{n+m+2-k-l} \gamma_b^{n+m+2-k-l} \times$$
$$\times \sup_{\rho_0/2 \leq \rho < \rho_0} (\rho_0 - \rho)^{n+m+4-k-l} \breve{N}'_{\rho_0,k,l-4}(u) +$$
$$+ C_b \sum_{k=0}^{n} \sum_{l=0}^{m+2} \frac{[k+l-4]!}{[n+m]!} (n+m+2)^{n+m+2-k-l} \gamma_b^{n+m+2-k-l} \times$$
$$\times \sup_{\rho_0/2 \leq \rho < \rho_0} (\rho_0 - \rho)^{n+m+4-k-l} \breve{N}'_{\rho_0,k,l-4}(u) +$$
$$+ C_c \sum_{k=0}^{n} \sum_{l=0}^{m} \frac{[k+l-4]!}{[n+m]!} (n+m)^{n+m-k-l} \gamma_c^{n+m-k-l} \times$$
$$\times \sup_{\rho_0/2 \leq \rho < \rho_0} (\rho_0 - \rho)^{n+m+4-k-l} \breve{N}'_{\rho_0,k,l-4}(u) + \tilde{M}_{\rho_0,n+m}(f) \Bigg) +$$
$$+ 2 \breve{N}'_{\rho_0,n+2,m-2}(u) + \breve{N}'_{\rho_0,n+4,m-4}(u) \Bigg],$$

and we simplify it as

$$\breve{N}'_{\rho_0,n,m}(u) \leq C \Bigg[\varepsilon^{-2} \Bigg(C_b \frac{\gamma_b^2 \rho_0^4}{16} \sum_{k=0}^{n+2} \sum_{l=0}^{m} \Big(\frac{\gamma_b \rho_0}{2} \Big)^{n+m-k-l} \frac{[k+l-4]!}{[n+m]!} \times$$
$$\times (n+m+2)^{n+m+2-k-l} \breve{N}'_{\rho_0,k,l-4}(u) +$$

$$
\begin{aligned}
&+ C_b \frac{\gamma_b^2 \rho_0^4}{16} \sum_{k=0}^{n} \sum_{l=0}^{m+2} \Big(\frac{\gamma_b \rho_0}{2}\Big)^{n+m-k-l} \frac{[k+l-4]!}{[n+m]!} \times \\
&\qquad\qquad \times (n+m+2)^{n+m+2-k-l} \breve{N}'_{\rho_0,k,l-4}(u) + \\
&+ C_c \frac{\rho_0^4}{16} \sum_{k=0}^{n} \sum_{l=0}^{m} \Big(\frac{\gamma_c \rho_0}{2}\Big)^{n+m-k-l} \frac{[k+l-4]!}{[n+m]!} (n+m)^{n+m-k-l} \breve{N}'_{\rho_0,k,l-4}(u) + \\
&+ \tilde{M}_{\rho_0,n+m}(f) \Bigg) + 2\breve{N}'_{\rho_0,n+2,m-2}(u) + \breve{N}'_{\rho_0,n+4,m-4}(u) \Bigg].
\end{aligned}
$$

We suppose that (4.3.27) holds for all $0 \le l < m$, hence

$$
\begin{aligned}
\breve{N}'_{\rho_0,k,l-4}(u)[k+l-4]! \le & C C_u K_1^{k+4} K_2^{l} \frac{\max\big\{(k+l-2)^{k+l}, (\rho_0/\varepsilon)^{k+l-1}\big\}}{[k+l-4]!} [k+l-4]! \\
\le & C_u K_1^{k+4} K_2^{l} \max\big\{(n+m+2)^{k+l}, (\rho_0/\varepsilon)^{k+l-1}\big\}.
\end{aligned}
$$

We also recall that

$$
\frac{\rho_0^4}{\varepsilon^2} \le \max\Big\{(n+m+2)^2, \Big(\frac{\rho_0}{\varepsilon}\Big)^4\Big\},
$$

and we obtain

$$
\breve{N}'_{\rho_0,n,m}(u) \le C C_u M_3 K_1^{n+4} K_2^{m+4} \frac{\max\big\{[n+m+2]^{n+m+4}, \big(\rho_0/\varepsilon\big)^{n+m+3}\big\}}{[n+m]!} + \varepsilon^{-2}\Big(\tilde{M}_{\rho_0,n+m}(f)\Big). \tag{4.3.29}
$$

Here the constant M_3 equals

$$
\begin{aligned}
&K_1^4 K_2^{-4} + 2K_1^2 K_2^{-2} + \frac{C_b \gamma_b^2}{16} \sum_{k=0}^{n+2} \sum_{l=0}^{m} K_1^{k-n} K_2^{l-m-4} \Big(\frac{\gamma_b \rho_0}{2}\Big)^{n+m-k-l} + \\
&\quad + \frac{C_b \gamma_b^2}{16} \sum_{k=0}^{n} \sum_{l=0}^{m+2} K_1^{k-n} K_2^{l-m-4} \Big(\frac{\gamma_b \rho_0}{2}\Big)^{n+m-k-l} + \frac{C_c}{16} \sum_{k=0}^{n} \sum_{l=0}^{m} K_1^{k-n} K_2^{l-m-4} \Big(\frac{\gamma_c \rho_0}{2}\Big)^{n+m-k-l}.
\end{aligned}
$$

It follows from the assumption $\dfrac{\max\{\gamma_b,\, \gamma_c\}\rho_0}{2K_2} < 1$, that

$$
M_3 \le K_1^4 K_2^{-4} + 2K_1^2 K_2^{-2} + \frac{2C_b \gamma_b^2 + C_c}{16 K_2^4} \cdot \frac{1}{\Big(1 - \dfrac{\max\{\gamma_b, \gamma_c\}\rho_0}{2K_1}\Big)\Big(1 - \dfrac{\max\{\gamma_b, \gamma_c\}\rho_0}{2K_2}\Big)}.
$$

We note that to obtain the above result we have used (4.5.14). Next, using (4.3.24) we have

$$
\tilde{M}_{\rho_0,n+m}(f) \le \frac{C C_f}{[n+m]!} \Big(\frac{\rho_0}{2}\Big)^4 \Big(\frac{\gamma}{2}\Big)^{n+m} \Big((n+m)^{(n+m)} + \max\big\{\big(\rho_0(n+m)\big)^{n+m}, (\rho_0/\varepsilon)^{n+m-1}\big\}\Big), \tag{4.3.30}
$$

and with the aid of (4.5.16) and (4.5.17) we obtain

$$\varepsilon^{-2}\Big(\tilde{M}_{\rho_0,n+m}(f)\Big) \le CC_f\Big(\frac{\gamma}{2}\Big)^{n+m}\min\{1,(\rho_0/\varepsilon)^4\}\frac{\max\{(n+m+2)^{n+m+4},(\rho_0/\varepsilon)^{n+m+3}\}}{[n+m]!}.$$

Thus,

$$\breve{N}'_{\rho_0,n,m}(u) \le C_u M_4 K_1^{n+4}K_2^{m+4}\frac{\max\{[n+m+2]^{n+m+2},\big(\rho_0/\varepsilon\big)^{n+m+1}\}}{[n+m]!}.$$

where,

$$M_4 = M_3 + C_f K_1^{-4}K_2^{-4}\Big(\frac{\gamma}{2K_1}\Big)^n\Big(\frac{\gamma}{2K_2}\Big)^m\min\{1,(\rho_0/\varepsilon)^4\}.$$

By the choice of K_2, the coefficient M_4 can be bounded by one and this gives us the desired result. □

Lemma 4.3.31 *[49, Lemma 3.6] Let $U, V \subset \mathbb{R}^2$ be bounded open sets. We consider the function $h = (h_1, h_2) : \overline{V} \to \mathbb{R}^2$ which is analytic and injective on $\overline{V}$, det $h' \neq 0$ on $\overline{V}$ and $h(V) \subset U$. Let $f : \overline{U} \to \mathbb{C}$ be analytic on $\overline{U}$ and assume that it satisfies for some $\varepsilon, C_f, \gamma > 0$,*

$$\|\nabla^n f\|_{L^2(U)} \le C_f\gamma^n \max(n^n, \varepsilon^{1-n}), \qquad \textit{for all } n \in \mathbb{N}_0.$$

Then there are $C, K > 0$ depending only on C_f, γ and the map h, such that

$$\|\nabla^n(f\circ h)\|_{L^2(V)} \le CK^n \max(n^n, \varepsilon^{1-n}), \qquad \textit{for all } n \in \mathbb{N}_0.$$

Theorem 4.3.32 *Suppose that the function f and the curve $\partial\Omega$ are analytic and suppose moreover that u is the solution of*

$$\begin{aligned}\varepsilon^2\Delta^2 u - b\Delta u + cu = f, \ \textit{in } \Omega,\\ u = \frac{\partial u}{\partial \eta} = 0, \ \textit{on } \partial\Omega,\end{aligned} \tag{4.3.33}$$

where b, c are positive constants. Then there are constants C, $K > 0$ independent of ε such that

$$\Big\|D^\alpha u\Big\|_{L^2(\Omega)} \le CK^{|\alpha|}\max\{|\alpha|^{|\alpha|},\varepsilon^{1-|\alpha|}\}, \qquad \forall\alpha\in\mathbb{N}_0^2. \tag{4.3.34}$$

Proof. Propositions 4.3.12 and 4.3.26 constitute the major part of this proof. In the interior we obtain the desired result by a straightforward application of Proposition

4.3.12, since for $d_0 \in (0,1)$ and $x_0 \in \Omega$, we consider the ball $B_{d_0}(x_0) \subset \Omega$. In order to complete the proof we examine the boundary. Proposition 4.3.26 is very helpful here, however we are obliged to introduce a conformal mapping in order to use it, since the boundary must be flattenned locally. For $x_0 \in \partial\Omega$, there is a conformal mapping $\phi : \Omega \cap B_{2d_0}(x_0) \to \mathcal{H}_{2d_0}$ [44, Theorem 1] that leaves the term with the biharmonic tensor invariant. The remaining terms under the transformation, may aquire non-constant coefficients. Therefore the transformed functions $v = u \circ \phi^{-1}$ solve

$$\begin{aligned} \varepsilon^2\Delta^2 v - b\big|(\phi^{-1})'\big|^2\Delta v + c\big|(\phi^{-1})'\big|^4 v &= \big|(\phi^{-1})'\big|^4\big(f \circ \phi^{-1}\big), \quad \text{in } \mathcal{H}_{2d_0}, \\ v = \frac{\partial v}{\partial n} = 0, \qquad & \text{on } \partial\mathcal{H}_{2d_0} \cap \{(x,y)|y=0\}. \end{aligned}$$

and using Proposition 4.3.26, Lemma 4.3.31 and a compactness argument we obtain (4.3.34). □

4.4 Analysis of the asymptotic expansion

4.4.1 Solution decomposition

Recall problem (4.1.1):

$$\left.\begin{aligned} \Lambda_\varepsilon u = \varepsilon^2\Delta^2 u - b\Delta u + cu = f, \quad & \text{in } \Omega \subset \mathbb{R}^2, \\ u = \frac{\partial u}{\partial n} = 0, \qquad & \text{on } \partial\Omega. \end{aligned}\right\}$$

We will show that the solution to (4.1.1) satisfies, for some $M \in \mathbb{N}_0$, the following decomposition

$$u_\varepsilon = u_M^s + \chi u_M^{BL} + r_M, \tag{4.4.1}$$

where M is the expansion order, u_M^s denotes the *smooth part*, u_M^{BL} the *boundary layer*, r_M the *remainder* and χ is the *cutoff* function given by (4.4.3) ahead. To this end, in order to treat the boundary layer we define *boundary-fitted coordinates* [6] as follows:

Suppose $\mathbf{z}(\theta) = \big(X(\theta), Y(\theta)\big)$, $\theta \in [0,l)$ is an analytic l-periodic parametrization of the boundary $\partial\Omega$. For the remainder of this chapter, let $\rho_0 > 0$ be fixed such that

$$0 < \rho_0 < \frac{1}{\|\kappa\|_{L^\infty(\mathbb{T}_l)}}, \tag{4.4.2}$$

where $\kappa(\theta)$ denotes the curvature of $\partial\Omega$ at $\mathbf{z}(\theta)$. Then one can find a diffeomorphism Ψ of $(0, \rho_0) \times \mathbb{R}/l$ on

$$\Omega_0 = \{\mathbf{z} - \rho\mathbf{n_z} | \, \mathbf{z} \in \partial\Omega, \ 0 < \rho < \rho_0\},$$

where $\mathbf{n_z}$ is the outward unit normal at $\mathbf{z} \in \partial\Omega$. The mapping Ψ is given as

$$(\rho, \theta) \to \mathbf{z} - \rho\mathbf{n_z} = \big(X(\theta) - \rho Y'(\theta), Y(\theta) + \rho X'(\theta)\big).$$

Figure 4.1 helps us understand better the definition of boundary-fitted coordinates.

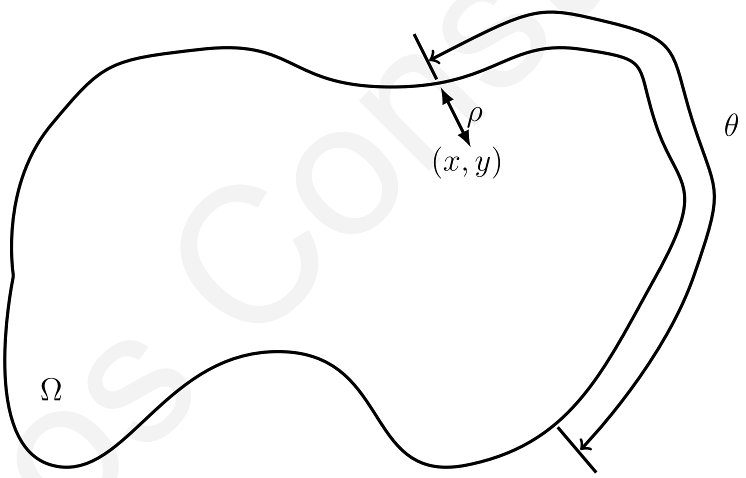


Figure 4.1: Boundary fitted coordinates.

In what follows we will utilize the following function:

$$\sigma(\rho, \theta) := \frac{1}{1 - \kappa(\theta)\rho}.$$

We also introduce a smooth cutoff function $\chi : [0, \infty) \times \mathbb{T}_l \to \{0, 1\}$ as follows. Suppose $0 < \rho_1 < \rho_0$ and define

$$\chi = \begin{cases} 1, & 0 \leq \rho \leq \rho_1 \\ 0, & (\rho_1 + \rho_0)/2 \leq \rho. \end{cases} \tag{4.4.3}$$

Anticipating layers along $\partial\Omega$, we define the *stretched variable* $\tilde{\rho} = \rho/\varepsilon$. We then make the formal ansatz

$$u \sim \sum_{j=0}^{\infty} \varepsilon^j \{u_j(x,y) + \tilde{v}_j(\tilde{\rho},\theta)\}, \tag{4.4.4}$$

with $u_j, \tilde{v}_j$ to be determined, and insert it in the differential equation of (4.1.1). As is usually done, we equate like powers of ε on both sides and obtain the following for the functions u_j:

$$b\Delta u_j - cu_j = f_j, \quad j \in \mathbb{N}_0, \tag{4.4.5}$$

where f_j is defined as

$$f_j := \begin{cases} -f,\ j = 0, \\ 0,\ j = 1, \\ \Delta^2 u_{j-2},\ j \geq 2. \end{cases} \tag{4.4.6}$$

To handle the terms $\tilde{v}_j$, we follow [6] and express the Laplacian tensor in boundary fitted coordinates as:

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \sigma^2(\rho,\theta)\frac{\partial^2}{\partial \theta^2} + \kappa(\theta)\sigma(\rho,\theta)\frac{\partial}{\partial \rho} - \rho\kappa'(\theta)\sigma^3(\rho,\theta)\frac{\partial}{\partial \theta}. \tag{4.4.7}$$

Similarly, using the stretched variable notation $\tilde{\rho} = \rho/\varepsilon$, the tensor Λ_ε defined in (4.1.1) can be rewritten as

$$\Lambda_\varepsilon = \sum_{i=0}^{4}\sum_{j=0}^{4} c_{ij}(\varepsilon,\tilde{\rho},\theta)\frac{\partial^{i+j}}{\partial\tilde{\rho}^i\partial\theta^j}, \tag{4.4.8}$$

where,

$$c_{00} = c,$$

$$c_{10} = \varepsilon^{-1}b\kappa(\theta)\sigma(\tilde{\rho},\theta) - \varepsilon\big[\kappa^3(\theta) + \kappa''(\theta)\big]\sigma^3(\tilde{\rho},\theta) - 3\varepsilon^2\tilde{\rho}\big[\kappa(\theta)\kappa''(\theta) + \big(\kappa'(\theta)\big)^2\big]\sigma^4(\tilde{\rho},\theta) + \\ + \varepsilon^3\tilde{\rho}^2(\kappa'(\theta))^2\kappa(\theta)\sigma^5(\tilde{\rho},\theta),$$

$$c_{01} = -\varepsilon b\tilde{\rho}\kappa'(\theta)\sigma^3(\tilde{\rho},\theta) + 5\varepsilon^2\kappa'(\theta)\kappa(\theta)\sigma^4(\tilde{\rho},\theta) + 9\varepsilon^3\tilde{\rho}\big[\kappa^2(\theta)\kappa'(\theta) + \kappa^{(3)}(\theta)\big]\sigma^5(\tilde{\rho},\theta) + \\ + 10\varepsilon^4\tilde{\rho}^2\kappa'(\theta)\kappa''(\theta)\sigma^6(\tilde{\rho},\theta) + 15\varepsilon^5\tilde{\rho}\big(\kappa'(\theta)\big)^3\sigma^7(\tilde{\rho},\theta),$$

$$c_{20} = -\kappa^2(\theta)\sigma^3(\tilde{\rho},\theta) - \varepsilon^{-2}b, \qquad c_{11} = 2\varepsilon^2\tilde{\rho}\kappa(\theta)\kappa'(\theta)\sigma^4(\tilde{\rho},\theta),$$

$$c_{02} = -b\sigma^2(\tilde{\rho},\theta) + 4\varepsilon^2\kappa^2(\theta)\sigma^4(\tilde{\rho},\theta) + 4\varepsilon^3\tilde{\rho}\kappa''(\theta)\sigma^5(\tilde{\rho},\theta) + 15\varepsilon^4\tilde{\rho}^2\big(\kappa'(\theta)\big)^2\sigma^6(\tilde{\rho},\theta),$$

$$c_{30} = -2\varepsilon^{-1}\kappa(\theta)\sigma^3(\tilde{\rho},\theta),\ c_{21} = 2\varepsilon\tilde{\rho}\kappa'(\theta)\sigma^3(\tilde{\rho},\theta),\ c_{12} = 2\varepsilon\kappa(\theta)\sigma^3(\tilde{\rho},\theta),$$

$$c_{03} = 6\varepsilon^3\tilde{\rho}\kappa'(\theta)\sigma^5(\tilde{\rho},\theta), c_{40} = \varepsilon^{-2},\ c_{22} = 2\sigma^2(\tilde{\rho},\theta),\ c_{04} = \varepsilon^2\sigma^4(\tilde{\rho},\theta),$$

and for all other cases $c_{ij} = 0$. It is helpful to rewrite the tensor Λ_ε in the form

$$\Lambda_\varepsilon = \sum_{j=0}^{\infty} \varepsilon^{j-2} \Lambda_j. \tag{4.4.9}$$

This can been done by expanding the tensor in a power series of ε. Note that the sum given in (4.4.9) converges under the assumption $|\varepsilon \tilde{\rho} \kappa(\theta)| < 1$. Here, the operators Λ_j are given by

$$\Lambda_0 = \frac{\partial^4}{\partial \tilde{\rho}^4} - b\frac{\partial^2}{\partial \tilde{\rho}^2}, \quad \Lambda_1 = -2\kappa(\theta)\frac{\partial^3}{\partial \tilde{\rho}^3} + \kappa(\theta)\frac{\partial}{\partial \tilde{\rho}},$$

$$\Lambda_2 = 2\frac{\partial^2}{\partial \tilde{\rho}^2}\frac{\partial^2}{\partial \theta^2} - 2\kappa^2(\theta)\tilde{\rho}\frac{\partial^3}{\partial \tilde{\rho}^3} + \kappa^2(\theta)\frac{\partial^2}{\partial \tilde{\rho}^2} - \frac{\partial^2}{\partial \theta^2} + \kappa^2(\theta)\tilde{\rho}\frac{\partial}{\partial \tilde{\rho}} + cI,$$

$$\begin{aligned}\Lambda_3 = {} & 2\binom{2}{1}\kappa(\theta)\tilde{\rho}\frac{\partial^4}{\partial \tilde{\rho}^2 \partial \theta^2} - 2\binom{2}{0}\kappa^3(\theta)\tilde{\rho}^2\frac{\partial^3}{\partial \tilde{\rho}^3} + 2\kappa(\theta)\frac{\partial^3}{\partial \tilde{\rho}\partial \theta^2} + 2\kappa'(\theta)\tilde{\rho}\frac{\partial^3}{\partial \tilde{\rho}^2 \partial \theta} - \\ & - \binom{2}{1}\kappa^3(\theta)\tilde{\rho}\frac{\partial^2}{\partial \tilde{\rho}^2} + \big(-\kappa^3(\theta) - \kappa''(\theta)\big)\frac{\partial}{\partial \tilde{\rho}} - b\binom{2}{1}\kappa(\theta)\tilde{\rho}\frac{\partial^2}{\partial \theta^2} + b\binom{2}{0}\kappa^3(\theta)\tilde{\rho}^2\frac{\partial}{\partial \tilde{\rho}} - \\ & - b\kappa'(\theta)\tilde{\rho}\frac{\partial}{\partial \theta},\end{aligned}$$

$$\begin{aligned}\Lambda_4 = {} & 2\binom{3}{1}\kappa^2(\theta)\tilde{\rho}^2\frac{\partial^4}{\partial \tilde{\rho}^2 \partial \theta^2} + \frac{\partial^4}{\partial \theta^4} - 2\binom{3}{0}\kappa^4(\theta)\tilde{\rho}^3\frac{\partial^3}{\partial \tilde{\rho}^3} + 2\binom{3}{2}\kappa^2(\theta)\tilde{\rho}\frac{\partial^3}{\partial \tilde{\rho}\partial \theta^2} + \\ & + 2\binom{3}{2}\kappa'(\theta)\kappa(\theta)\tilde{\rho}^2\frac{\partial^3}{\partial \tilde{\rho}^2 \partial \theta} - \binom{3}{1}\kappa^4(\theta)\tilde{\rho}^2\frac{\partial^2}{\partial \tilde{\rho}^2} + 2\kappa'(\theta)\kappa(\theta)\tilde{\rho}\frac{\partial^2}{\partial \tilde{\rho}\partial \theta} + 4\kappa^2(\theta)\frac{\partial^2}{\partial \theta^2} + \\ & + \left(-\binom{3}{2}\kappa^4(\theta)\tilde{\rho} - \binom{3}{2}\kappa''(\theta)\kappa(\theta)\tilde{\rho} - 3\big(\kappa'(\theta)\big)^2\tilde{\rho} - \kappa''(\theta)\kappa(\theta)\tilde{\rho}\right)\frac{\partial}{\partial \tilde{\rho}} + 5\kappa'(\theta)\kappa(\theta)\frac{\partial}{\partial \theta} - \\ & - b\binom{3}{1}\kappa^2(\theta)\tilde{\rho}^2\frac{\partial^2}{\partial \theta^2} + b\binom{3}{0}\kappa^4(\theta)\tilde{\rho}^3\frac{\partial}{\partial \tilde{\rho}} - b\binom{3}{2}\kappa'(\theta)\kappa(\theta)\tilde{\rho}^2\frac{\partial}{\partial \theta},\end{aligned}$$

$$\begin{aligned}\Lambda_5 = {} & 2\binom{4}{1}\kappa^3(\theta)\tilde{\rho}^3\frac{\partial^4}{\partial \tilde{\rho}^2 \partial \theta^2} + \binom{4}{3}\kappa(\theta)\tilde{\rho}\frac{\partial^4}{\partial \theta^4} - 2\binom{4}{0}\kappa^5(\theta)\tilde{\rho}^4\frac{\partial^3}{\partial \rho^3} + 2\binom{4}{2}\kappa^3(\theta)\tilde{\rho}^2\frac{\partial^3}{\partial \tilde{\rho}\partial \theta^2} + \\ & + 2\binom{4}{2}\kappa'(\theta)\kappa^2(\theta)\tilde{\rho}^3\frac{\partial^3}{\partial \tilde{\rho}^2 \partial \theta} + 6\kappa'(\theta)\tilde{\rho}\frac{\partial^3}{\partial \theta^3} - \binom{4}{1}\kappa^5(\theta)\tilde{\rho}^3\frac{\partial^2}{\partial \tilde{\rho}^2} + 2\binom{4}{3}\kappa'(\theta)\kappa^2(\theta)\tilde{\rho}^2\frac{\partial^2}{\partial \tilde{\rho}\partial \theta} + \\ & + \left(4\binom{4}{3}\kappa^3(\theta)\tilde{\rho} + 4\kappa''(\theta)\tilde{\rho}\right)\frac{\partial^2}{\partial \theta^2} + \Bigg(-\binom{4}{2}\kappa^5(\theta)\tilde{\rho}^2 - \binom{4}{2}\kappa''(\theta)\kappa^2(\theta)\tilde{\rho}^2 - \\ & - 3\binom{4}{3}\big(\kappa'(\theta)\big)^2\kappa(\theta)\tilde{\rho}^2 - \binom{4}{3}\kappa''(\theta)\kappa^2(\theta)\tilde{\rho}^2 - 3\big(\kappa'(\theta)\big)^2\kappa(\theta)\tilde{\rho}^2\big)\Bigg)\frac{\partial}{\partial \tilde{\rho}} + \\ & + \left(5\binom{4}{3}\kappa'(\theta)\kappa^2(\theta)\tilde{\rho} + 9\kappa'(\theta)\kappa^2(\theta)\tilde{\rho} + \kappa^{(3)}\tilde{\rho}\right)\frac{\partial}{\partial \theta} - b\binom{4}{1}\kappa^3(\theta)\tilde{\rho}^3\frac{\partial^2}{\partial \theta^2} + b\binom{4}{0}\kappa^5(\theta)\tilde{\rho}^4\frac{\partial}{\partial \tilde{\rho}} - \\ & - b\binom{4}{2}\kappa'(\theta)\kappa^2(\theta)\tilde{\rho}^3\frac{\partial}{\partial \theta},\end{aligned}$$

$$\Lambda_6 = 2\binom{5}{1}\kappa^4(\theta)\tilde{\rho}^4\frac{\partial^4}{\partial \tilde{\rho}^2 \partial \theta^2} + \binom{5}{3}\kappa^2(\theta)\tilde{\rho}^2\frac{\partial^4}{\partial \theta^4} - 2\binom{5}{0}\kappa^6(\theta)\tilde{\rho}^5\frac{\partial^3}{\partial \rho^3} + 2\binom{5}{2}\kappa^4(\theta)\tilde{\rho}^3\frac{\partial^3}{\partial \tilde{\rho}\partial \theta^2} +$$

$$+2\binom{5}{2}\kappa'(\theta)\kappa^3(\theta)\tilde{\rho}^4\frac{\partial^3}{\partial\tilde{\rho}^2\partial\theta}+6\binom{5}{4}\kappa'(\theta)\kappa(\theta)\tilde{\rho}^2\frac{\partial^3}{\partial\theta^3}-\binom{5}{1}\kappa^6(\theta)\tilde{\rho}^4\frac{\partial^2}{\partial\tilde{\rho}^2}+$$
$$+2\binom{5}{3}\kappa'(\theta)\kappa^3(\theta)\tilde{\rho}^3\frac{\partial^2}{\partial\tilde{\rho}\partial\theta}+\left(4\binom{5}{3}\kappa^4\tilde{\rho}^2+4\binom{5}{4}\kappa''(\theta)\kappa(\theta)\tilde{\rho}^2+15\big((\kappa'(\theta))^2\tilde{\rho}^2\right)\frac{\partial^2}{\partial\theta^2}+$$
$$+\left(-\binom{5}{2}\kappa^6(\theta)\tilde{\rho}^3-\binom{5}{2}\kappa''(\theta)\kappa^3(\theta)\tilde{\rho}^3-3\binom{5}{3}\big(\kappa'(\theta)\big)^2\kappa^2(\theta)\tilde{\rho}^3-\binom{5}{3}\kappa''(\theta)\kappa^3(\theta)\tilde{\rho}^3-\right.$$
$$\left.-3\binom{5}{4}\big(\kappa'(\theta)\big)^2\kappa^2(\theta)\tilde{\rho}^3\right)\frac{\partial}{\partial\tilde{\rho}}+\left(5\binom{5}{3}\kappa'(\theta)\kappa^3(\theta)\tilde{\rho}^2+9\binom{5}{4}\kappa'(\theta)\kappa^3(\theta)\tilde{\rho}^2+\right.$$
$$\left.+\binom{5}{4}\kappa^{(3)}(\theta)\kappa(\theta)\tilde{\rho}^2+10\kappa'(\theta)\kappa''(\theta)\tilde{\rho}^2\right)\frac{\partial}{\partial\theta}-b\binom{5}{1}\kappa^4(\theta)\tilde{\rho}^4\frac{\partial^2}{\partial\theta^2}+b\binom{5}{0}\kappa^6(\theta)\tilde{\rho}^5\frac{\partial}{\partial\tilde{\rho}}-$$
$$-b\binom{5}{2}\kappa'(\theta)\kappa^3(\theta)\tilde{\rho}^4\frac{\partial}{\partial\theta},$$

and for $j = 7, 8, \ldots$

$$\Lambda_j := \sum_{l=0}^{4}\sum_{k=0}^{4} b^j_{lk}(\tilde{\rho},\theta)\frac{\partial^{l+k}}{\partial\tilde{\rho}^l\partial\theta^k}.$$

The functions b^j_{lk} are defined as:

$$b^j_{22}=2\binom{j-1}{1}\kappa^{j-2}(\theta)\tilde{\rho}^{j-2},\quad b^j_{04}=\binom{j-1}{3}\kappa^{j-4}(\theta)\rho^{j-4},$$

$$b^j_{30}=-2\binom{j-1}{0}\kappa^{j}(\theta)\rho^{j-1},\quad b^j_{03}=6\binom{j-1}{4}\kappa'(\theta)\kappa^{j-5}(\theta)\tilde{\rho}^{j-4},$$

$$b^j_{12}=2\binom{j-1}{2}\kappa^{j-2}(\theta)\tilde{\rho}^{j-3},\quad b^j_{21}=2\binom{j-1}{2}\kappa'(\theta)\kappa^{j-3}(\theta)\tilde{\rho}^{j-2},$$

$$b^j_{20}=-\binom{j-1}{1}\kappa^{j}(\theta)\tilde{\rho}^{j-2},\quad b^j_{11}=2\binom{j-1}{3}\kappa'(\theta)\kappa^{j-3}(\theta)\tilde{\rho}^{j-3},$$

$$b^j_{02}=4\binom{j-1}{3}\kappa^{j-2}(\theta)\tilde{\rho}^{j-4}+4\binom{j-1}{4}\kappa''(\theta)\kappa^{j-5}(\theta)\tilde{\rho}^{j-4}+$$
$$+15\binom{j-1}{5}\kappa'(\theta)^2\kappa^{j-6}(\theta)\tilde{\rho}^{j-4}-b\binom{j-1}{1}\kappa^{j-2}(\theta)\tilde{\rho}^{j-2},$$

$$b^j_{10}=-\binom{j-1}{2}\kappa^{j}(\theta)\tilde{\rho}^{j-3}-\binom{j-1}{2}\kappa''(\theta)\kappa^{j-3}(\theta)\tilde{\rho}^{j-3}-$$
$$-3\binom{j-1}{3}\big(\kappa'(\theta)\big)^2\kappa^{j-4}(\theta)\tilde{\rho}^{j-3}-\binom{j-1}{3}\kappa''(\theta)\tilde{\rho}^{j-3}\kappa^{j-3}(\theta)-$$
$$-3\binom{j-1}{4}\big(\kappa'(\theta)\big)^2\kappa^{j-4}(\theta)\tilde{\rho}^{j-3}+b\binom{j-1}{0}\kappa^{j}(\theta)\tilde{\rho}^{j-1},$$

$$b^j_{01}=5\binom{j-1}{3}\kappa'(\theta)\kappa^{j-3}(\theta)\tilde{\rho}^{j-4}+9\binom{j-1}{4}\kappa'(\theta)\kappa^{j-3}(\theta)\tilde{\rho}^{j-4}+$$

$$+\binom{j-1}{4}\kappa^{(3)}(\theta)\kappa^{j-5}(\theta)\tilde{\rho}^{j-4}+10\binom{j-1}{5}\kappa'(\theta)\kappa''(\theta)\kappa^{j-6}(\theta)\tilde{\rho}^{j-4}+$$
$$+15\binom{j-1}{6}\left(\kappa'(\theta)^3\kappa^{j-7}(\theta)\tilde{\rho}^{j-4}-b\binom{j-1}{2}\kappa'(\theta)\tilde{\rho}^{j-2}\kappa^{j-3}(\theta),\right.$$

and $b^j_{kl}=0$ for all other cases.

We determine $\tilde{v}_k$ by setting

$$\Lambda_\varepsilon u^{BL}=\sum_{i=-2}^{\infty}\varepsilon^i\sum_{k=0}^{i+2}\Lambda_k\tilde{v}_{i+2-k}=0$$

and require that all coefficients in the power series in ε are zero. We obtain the following system:

$$\left.\begin{aligned}\frac{\partial^4\tilde{v}_j}{\partial\tilde{\rho}^4}-b\frac{\partial^2\tilde{v}_j}{\partial\tilde{\rho}^2}=-\tilde{G}_j,\quad j=0,1,2,\dots,\\ \tilde{G}_0=0,\quad \tilde{G}_j=\sum_{\nu=1}^{7}\tilde{G}^\nu_j,\quad j=1,2,\dots\end{aligned}\right\}\tag{4.4.10}$$

The functions $\{\tilde{G}^\nu_j\}_{\nu=1}^7$ that appear in (4.4.12), for $j=1,2,\dots$ are given by[c]:

$$\tilde{G}^1_j(\tilde{\rho},\theta)=\sum_{\nu=0}^{j-1}\left[-2\binom{\nu}{0}\kappa^{\nu+1}(\theta)\tilde{\rho}^\nu\frac{\partial^3\tilde{v}_{j-1-\nu}}{\partial\tilde{\rho}^3}-\binom{\nu}{1}\kappa^{\nu+1}(\theta)\tilde{\rho}^{\nu-1}\frac{\partial^2\tilde{v}_{j-1-\nu}}{\partial\tilde{\rho}^2}+\right.$$
$$\left.+\left(-\binom{\nu}{2}\kappa^{\nu+1}(\theta)\tilde{\rho}^{\nu-2}+b\binom{\nu}{0}\kappa^{\nu+1}(\theta)\tilde{\rho}^\nu\right)\frac{\partial\tilde{v}_{j-1-\nu}}{\partial\tilde{\rho}}\right],$$

$$\tilde{G}^2_j(\tilde{\rho},\theta)=\sum_{\nu=0}^{j-2}\left[2\binom{\nu+1}{1}\kappa^\nu(\theta)\tilde{\rho}^\nu\frac{\partial^4\tilde{v}_{j-2-\nu}}{\partial\tilde{\rho}^2\partial\theta^2}+2\binom{\nu+1}{2}\kappa^\nu(\theta)\tilde{\rho}^{\nu-1}\frac{\partial^3\tilde{v}_{j-2-\nu}}{\partial\tilde{\rho}\partial\theta^2}+\right.$$
$$\left.+\left(4\binom{\nu+1}{3}\kappa^\nu(\theta)\tilde{\rho}^{\nu-2}-b\binom{\nu+1}{1}\kappa^\nu(\theta)\tilde{\rho}^\nu\right)\frac{\partial^2\tilde{v}_{j-2-\nu}}{\partial\theta^2}\right],$$

$$\tilde{G}^3_j(\tilde{\rho},\theta)=\sum_{\nu=0}^{j-3}\left[2\binom{\nu+2}{2}\kappa'(\theta)\kappa^\nu(\theta)\tilde{\rho}^{\nu+1}\frac{\partial^3\tilde{v}_{j-3-\nu}}{\partial\rho^2\partial\theta}+2\binom{\nu+2}{3}\kappa'(\theta)\kappa^\nu(\theta)\tilde{\rho}^\nu\frac{\partial^2\tilde{v}_{j-3-\nu}}{\partial\rho\partial\theta}+\right.$$
$$+\left(-\binom{\nu+2}{2}-\binom{\nu+2}{3}\right)\kappa''(\theta)\kappa^\nu(\theta)\tilde{\rho}^\nu\frac{\partial\tilde{v}_{j-3-\nu}}{\partial\tilde{\rho}}+$$
$$\left.+\left(5\binom{\nu+2}{3}+9\binom{\nu+2}{4}\right)\kappa'(\theta)\kappa^\nu(\theta)\tilde{\rho}^{\nu-1}\frac{\partial\tilde{v}_{j-3-\nu}}{\partial\tilde{\theta}}\right],$$

$$\tilde{G}^4_j(\tilde{\rho},\theta)=\sum_{\nu=0}^{j-4}\left[\left(-3\binom{\nu+3}{3}-3\binom{\nu+3}{4}\right)\left(\kappa'(\theta)\right)^2\kappa^\nu(\theta)\tilde{\rho}^{\nu+1}\frac{\partial\tilde{v}_{j-4-\nu}}{\partial\tilde{\rho}}+\right.$$

[c] Here the empty sums take the value zero

$$+\binom{\nu+3}{3}\kappa^{\nu}(\theta)\tilde{\rho}^{\nu}\frac{\partial^4\tilde{v}_{j-4-\nu}}{\partial\theta^4}\Bigg],$$

$$\tilde{G}_j^5(\tilde{\rho},\theta)=\sum_{\nu=0}^{j-5}\Bigg[6\binom{\nu+4}{4}\kappa'(\theta)\kappa^{\nu}(\theta)\tilde{\rho}^{\nu+1}\frac{\partial^3\tilde{v}_{j-5-\nu}}{\partial\theta^3}+4\binom{\nu+4}{4}\kappa''(\theta)\kappa^{\nu}(\theta)\tilde{\rho}^{\nu+1}\frac{\partial^2\tilde{v}_{j-5-\nu}}{\partial\theta^2}+$$
$$+\binom{\nu+4}{4}\kappa^{(3)}(\theta)\kappa^{\nu}(\theta)\tilde{\rho}^{\nu+1}\frac{\partial\tilde{v}_{j-5-\nu}}{\partial\tilde{\theta}}\Bigg],$$

$$\tilde{G}_j^6(\tilde{\rho},\theta)=\sum_{\nu=0}^{j-6}\Bigg[15\binom{\nu+5}{5}\big(\kappa'(\theta)\big)^2\kappa^{\nu}(\theta)\tilde{\rho}^{\nu+2}\frac{\partial^2\tilde{v}_{j-6-\nu}}{\partial\theta^2}+$$
$$+10\binom{\nu+5}{5}\kappa'(\theta)\kappa''(\theta)\kappa^{\nu}(\theta)\tilde{\rho}^{\nu+2}\frac{\partial\tilde{v}_{j-6-\nu}}{\partial\theta}\Bigg],$$

and

$$\tilde{G}_j^7(\tilde{\rho},\theta)=\sum_{\nu=0}^{j-7}\Bigg[15\binom{\nu+6}{6}\big(\kappa'(\theta)\big)^3\kappa^{\nu}(\theta)\tilde{\rho}^{\nu+3}\frac{\partial\tilde{v}_{j-7-\nu}}{\partial\theta}\Bigg].$$

To summarize, the functions u_j, $\tilde{v}_j$ that appear in (4.4.4) satisfy

$$b\Delta u_j-cu_j=f_j,\qquad j=0,1,2,\ldots, \tag{4.4.11}$$

$$\frac{\partial^4\tilde{v}_j}{\partial\tilde{\rho}^4}-b\frac{\partial^2\tilde{v}_j}{\partial\tilde{\rho}^2}=-\tilde{G}_j,\quad j=0,1,2,\ldots \tag{4.4.12}$$

In order to satisfy the boundary conditions in (4.1.1), the system given by (4.4.11),(4.4.12) is supplemented with the following:

$$u_0\Big|_{\partial\Omega}=0,\quad u_j=-\left.\tilde{v}_j\right|_{\partial\Omega},\ j=1,2,\ldots \tag{4.4.13}$$

$$\left.\begin{aligned}\frac{\partial}{\partial\tilde{\rho}}\tilde{v}_0(\tilde{\rho},\theta)\Big|_{\partial\Omega}&=0,\\ \frac{\partial}{\partial\tilde{\rho}}\tilde{v}_j(\tilde{\rho},\theta)\Big|_{\partial\Omega}&=-\frac{\partial}{\partial\rho}\tilde{u}_{j-1}(\rho,\theta)\Big|_{\partial\Omega},\quad j=1,2,\ldots\\ \lim_{\tilde{\rho}\to\infty}\tilde{v}_j(\tilde{\rho},\theta)&=0,\quad j=0,1,2,\ldots\end{aligned}\right\} \tag{4.4.14}$$

The function $\tilde{u}_{j-1}$ that appears in (4.4.14) is defined as follows:

$$\tilde{u}_{j-1}(\rho,\theta)=u_{j-1}\left(X(\theta)-\rho Y'(\theta),Y(\theta)+\rho X'(\theta)\right). \tag{4.4.15}$$

Since we assume that $\partial\Omega$ is a smooth (analytic) curve, we have that $X(\theta),Y(\theta)$ are analytic functions, (and therefore $|X^{(k)}(\theta)|,|Y^{(k)}(\theta)|\leq C\gamma_*^k k!\ \forall k=0,1,2,...$ where $\gamma_*>0$ controls the domain of the analyticity of the functions X,Y) as well as $1-$

$\kappa(\theta)\rho$, $(1-\kappa(\theta)\rho)^{-1} \leq C$. Thus for $u_{j-1}(x,y)$ defined in Ω_0, the above change of variables produces

$$\begin{aligned}\frac{\partial u_{j-1}}{\partial x} &= \frac{1}{1-\kappa(\theta)\rho}\left\{\frac{\partial \tilde{u}_{j-1}}{\partial \theta}X'(\theta) - \frac{\partial \tilde{u}_{j-1}}{\partial \rho}\left(Y'(\theta)+\rho X''(\theta)\right)\right\},\\ \frac{\partial u_{j-1}}{\partial y} &= \frac{1}{1-\kappa(\theta)\rho}\left\{\frac{\partial \tilde{u}_{j-1}}{\partial \rho}\left(X'(\theta)-\rho Y''(\theta)\right) + \frac{\partial \tilde{u}_{j-1}}{\partial \theta}Y'(\theta)\right\}.\end{aligned}$$

This shows that the first derivatives with respect to the (physical) x, y variables are bounded by the first derivatives with respect to the ρ, θ variables, and vice versa, if one considers the inverse mapping:

$$\begin{aligned}\frac{\partial \tilde{u}_{j-1}}{\partial \rho} &= \varkappa(\rho,\theta)\left\{-\frac{\partial u_{j-1}}{\partial x}Y'(\theta) + X'(\theta)\frac{\partial u_{j-1}}{\partial y}\right\},\\ \frac{\partial \tilde{u}_{j-1}}{\partial \theta} &= \varkappa(\rho,\theta)\left\{-\frac{\partial u_{j-1}}{\partial x}Y'(\theta) + X'(\theta)\frac{\partial u_{j-1}}{\partial y}\right\},\end{aligned}$$

where

$$\varkappa(\rho,\theta) = \frac{(1-\kappa(\theta)\rho)}{\left(Y'(\theta)+\rho X''(\theta)\right)Y'(\theta) + \left(X'(\theta)-\rho Y''(\theta)\right)X'(\theta)}.$$

Loosely speaking, the regularity of u_j remains 'unaffected' as we transform from the x, y variables to the ρ, θ variables. As a final remark, we note that

$$\left.\frac{\partial \tilde{u}_{j-1}}{\partial \rho}\right|_{\rho=0} = \frac{-Y'(\theta)}{\left(Y'(\theta)\right)^2+\left(X'(\theta)\right)^2}\left.\frac{\partial u_{j-1}}{\partial x}\right|_{\partial\Omega} + \frac{X'(\theta)}{\left(Y'(\theta)\right)^2+\left(X'(\theta)\right)^2}\left.\frac{\partial u_{j-1}}{\partial y}\right|_{\partial\Omega} \quad (4.4.16)$$

hence

$$\left|\frac{\partial \tilde{u}_{j-1}}{\partial \rho}(0,\theta)\right| \leq C\left\|Du_{j-1}\right\|_{L^\infty(\Omega)}. \quad (4.4.17)$$

Lemma 4.4.18 *Let* $\rho_0 > 0$ *given by* (4.4.2)*, let* u *be analytic on* $\overline{\Omega}$ *and let* $\tilde{u}(\rho,\theta) = u(X(\theta)-\rho Y'(\theta), Y(\theta)+\rho X'(\theta))$ (*cf.* (4.4.15))*. Then there exist* $\rho_0', \Theta \in \mathbb{R}^+$*, such that* $\partial\tilde{u}/\partial\rho$ (*cf.* (4.4.16)) *is holomorphic in the complex neighborhood of* $(0,\rho_0')\times[0,\ell]$*, given by* $B_{\rho_0'}(0)\times V_\Theta$*, where*

$$V_\Theta = \{z \in \mathbb{C} : dist(z,[0,\ell]) < \Theta\}. \quad (4.4.19)$$

Proof. Since u is analytic on $\bar{\Omega}$, there is a complex neighborhood of $\partial\Omega$ on which u is holomorphic and bounded. Specifically, due to the fact that $\partial\Omega$ is parametrized by arclength $\theta \in [0,\ell]$, we have that $\tilde{u}(\rho,\theta)$ as well as $\partial\tilde{u}/\partial\rho$, are holomorphic in $B_{\rho_0'}(0) \times V_\Theta$, with V_Θ given by (4.4.19), for some $\rho_0', \Theta \in \mathbb{R}^+$. The result follows. □

Since u_0 satisfies (4.4.11), (4.4.13), with f analytic, standard elliptic regularity theory gives that u_0 is analytic in $\overline{\Omega}$. Lemma 4.4.18 further gives $\rho_0', \theta_1 \in \mathbb{R}^+$ such that $\tilde{u}_0$, as well $\frac{\partial \tilde{u}_0}{\partial \rho}$, are holomporphic in $B_{\rho_0'}(0) \times V_{\theta_1}$.

Next, we easily verify that $\tilde{v}_0 = 0$. Moreover we note that

$$\tilde{G}_1 = \Lambda_1 \tilde{v}_0 = 0,$$

and

$$\tilde{G}_2 = \Lambda_1 \tilde{v}_1 + \Lambda_2 \tilde{v}_0 = (2b+1)\kappa(\theta)\frac{\partial \tilde{u}_0}{\partial \rho}(0,\theta)e^{-\sqrt{b}\tilde{\rho}},$$

thus

$$\tilde{v}_1(\tilde{\rho},\theta) = -\frac{1}{\sqrt{b}}\frac{\partial \tilde{u}_0(0,\theta)}{\partial \rho}e^{-\sqrt{b}\tilde{\rho}}. \tag{4.4.20}$$

This shows that $\tilde{v}_1$ is holomorphic in $\mathbb{C} \times V_{\theta_1}$ as well, and as a consequence, u_1 satisfying (4.4.11), (4.4.13) is analytic in $\overline{\Omega}$. By Lemma 4.4.18 there is $\rho_1', \theta_2 \in \mathbb{R}^+$ such that $\frac{\partial \tilde{u}_1}{\partial \rho}$ is holomorphic in $B_{\rho_1'}(0) \times V_{\theta_2}$. Similarly

$$\tilde{v}_2(\tilde{\rho},\theta) = \left[\kappa(\theta)(2b+1)\left(\frac{5}{4b^2} + \frac{2\tilde{\rho}-1}{b^{3/2}}\right)\frac{\partial \tilde{u}_0(0,\theta)}{\partial \rho} - \frac{1}{\sqrt{b}}\frac{\partial \tilde{u}_1(0,\theta)}{\partial \rho}\right]e^{-\sqrt{b}\tilde{\rho}},$$

and by the same reasoning, we obtain $\theta_3 \in \mathbb{R}^+$ such that $\tilde{v}_2$ is holomorphic in $\mathbb{C} \times V_{\theta_3}$. In general, we get $\theta_1 \geq \theta_2 \geq \theta_3 \geq \ldots$, such that $\tilde{v}_j$ is holomorphic in $\mathbb{C} \times V_{\theta_{j+1}}$.

As we show in the next subsection, the functions $\tilde{v}_j$ can be written in the form

$$\tilde{v}_j(\tilde{\rho},\theta) = \Pi_j(\tilde{\rho},\theta)\mathrm{e}^{-\sqrt{b}\tilde{\rho}}, \tag{4.4.21}$$

where Π_j is a polynomial of degree $2(j-1)$ in $\tilde{\rho}$, with coefficients that depend on θ. Finally, we define for some $M \in \mathbb{N}_0$,

$$u_M^s(x,y) = \sum_{j=0}^{M} \varepsilon^j u_j(x,y), \tag{4.4.22}$$

$$u_M^{BL}(\tilde{\rho},\theta) := \sum_{k=0}^{M+1} \varepsilon^j \tilde{v}_j(\tilde{\rho},\theta), \tag{4.4.23}$$

as the smooth part and boundary layer. The remainder r_M is defined as

$$r_M := u - u_M^s - \chi u_M^{BL}, \tag{4.4.24}$$

where χ is the cut off function given in (4.4.3). By construction the remainder r_M satisfies

$$\left.\begin{aligned} \Lambda_\varepsilon r_M &= g_1 \quad \text{in } \Omega, \\ r_M &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.4.25}$$

where

$$g_1 = -\varepsilon^{M+1}\Delta^2 u_{M-1} - \varepsilon^{M+2}\Delta^2 u_M - \Lambda_\varepsilon(\chi u_M^{BL}).$$

4.4.2 Regularity results

Our goal throughout this subsection is to provide information about the regularity of the functions given by (4.4.22)–(4.4.24). In Theorem 4.4.36 we give the regularity of the functions $\{u_j\}_{j=0}^\infty$, $\{\tilde{v}_j\}_{j=0}^\infty$ that are defined by (4.4.11)–(4.4.14). The desired results about the smooth part u_M^s, the boundary layer u_M^{BL} and the remainder r_M are presented in Theorems 4.4.40, 4.4.46 and 4.4.50, respectively.

We note that using the stretched coordinate $\tilde{\rho}$, the tensor Λ_ε given by (4.1.1) can be written in the form (4.4.8).

To obtain the lemma that follows we apply again, Cauchy's integral theorem for derivatives. We note that the results are a variation of Lemma 7.3.7 in [45].

Recall the definition of V_Θ given by (4.4.19) and set

$$V_\Theta(d) := \{z \in V_\Theta \,|\, \mathrm{dist}(z, \partial V_\Theta) > d\}. \tag{4.4.26}$$

Lemma 4.4.27 *Let $\Theta>0$ and $M \in \mathbb{N}_0$. Assume that U is holomorphic on $\mathbb{C}\times V_\Theta$ and suppose that there exist $C_{U,j} > 0$ (depending on U and $j \in \{0, 1, \dots, M\}$), such that for all $d \in (0, \Theta)$ and for every $(z, \zeta) \in \mathbb{C}\times V_\Theta(d)$, there holds*

$$|U(z, \zeta)| \le C_{U,j} d^{-j}(c_j + |z|)^{2(j-1)} \mathrm{e}^{-Re(\sqrt{b}z)}. \tag{4.4.28}$$

Then for all $d < \Theta$, $(z,\zeta) \in \mathbb{C} \times V_\Theta(d)$, *we have*

$$\left|\frac{\partial^m}{\partial z^m}U(z,\zeta)\right| \leq C_{U,j} m! d^{-j} \mathrm{e}^{\sqrt{b}}(\hat{c}_j + |z|)^{2(j-1)}\mathrm{e}^{-Re(\sqrt{b}z)}, \tag{4.4.29}$$

$$\left|\frac{\partial^n}{\partial \zeta^n}U(z,\zeta)\right| \leq C_{U,j} n! \kappa_1^{-n} d^{-(j+n)}(\bar{c}_j + |z|)^{2(j-1)}\mathrm{e}^{-Re(\sqrt{b}z)}, \tag{4.4.30}$$

$$\left|\frac{\partial^{n+m}}{\partial z^m \partial \zeta^n}U(z,\zeta)\right| \leq C_{U,j} m! n! \kappa_1^{-n} d^{-(j+n)} \mathrm{e}^{\sqrt{b}}(c_j^* + |z|)^{2(j-1)}\mathrm{e}^{-Re(\sqrt{b}z)}, \tag{4.4.31}$$

where c_j, $\hat{c}_j$, $\bar{c}_j$, c_j^* *are constants (depending on* j*) and* $\kappa_1 \in (0,1)$.

Proof. We follow the proof of [45, Lemma 7.3.7]. With the aid of Cauchy's integral theorem for derivatives and by using as path of integration a circle of radius 1 around z, we proceed to show the estimate (4.4.29):

$$\begin{aligned}\left|\frac{\partial^m}{\partial z^m}U(z,\zeta)\right| \leq & \frac{m!}{2\pi}\int_{|t|=1}\left|\frac{U(z+t,\zeta)}{t^{m+1}}\right|dt \\ \leq & \frac{m!}{2\pi}\int_{|t|=1}\frac{d^{-j}C_{U,j}}{|t|^{m+1}}(c_j + |z| + 1)^{2(j-1)}\mathrm{e}^{-Re(\sqrt{b}z)+\sqrt{b}}dt \\ \leq & C_{U,j} d^{-j}\mathrm{e}^{\sqrt{b}} m! (\hat{c}_j + |z|)^{2(j-1)}\mathrm{e}^{-Re(\sqrt{b}z)}.\end{aligned}$$

In order to get the bound for derivatives with respect to the ζ-variable, for $\kappa_1 \in (0,1)$ we choose a different circle, namely, we integrate on $\partial B_{\kappa_1 d}(\zeta)$:

$$\begin{aligned}\left|\frac{\partial^n}{\partial \zeta^n}U(z,\zeta)\right| = & \left|\frac{n!}{2\pi i}\int_{|t|=\kappa_1 d}\frac{U(z,\zeta+t)}{t^{n+1}}dt\right| \\ \leq & \frac{n!}{2\pi}2\pi\kappa_1 d\left(\frac{1}{\kappa_1 d}\right)^{n+1} C_{U,j} d^{-j}(c_j + |z| + \kappa_1 d)^{2(j-1)}\mathrm{e}^{-Re(\sqrt{b}z)} \\ \leq & C_{U,j} n! \left(\frac{1}{\kappa_1 d}\right)^{n} d^{-j}(\bar{c}_j + |z|)^{2(j-1)}\mathrm{e}^{-Re(\sqrt{b}z)} \qquad (4.4.32) \\ \leq & C_{U,j} n! \kappa_1^{-n} d^{-(j+n)}(\bar{c}_j + |z|)^{2(j-1)}\mathrm{e}^{-Re(\sqrt{b}z)}.\end{aligned}$$

Combining the above steps we can get the third estimate. We note that the derivatives with respect to z and to ζ commute. □

Lemma 4.4.33 *Let the curve* $\partial\Omega$ *be analytic. For a positive number* Θ*, there exists* $B > 0$*, such that*

$$|\kappa(\theta)| \leq B < \frac{1}{\rho_0},\ \forall \theta \in \{\phi \in \mathbb{C} :\ |\mathrm{Im}\phi| < \Theta\}, \tag{4.4.34}$$

with ρ_0 satisfying (4.4.2). In addition, for $\nu \in \mathbb{N}_0$, any term written in the following form, can be estimated as follows: There is $C_B > 0$ such that, with $l, m, n \in \{0, 1, 2, 3\}$, there holds

$$\binom{\nu}{i}(1+b)|\kappa'(\theta)|^l|\kappa''(\theta)|^m|\kappa^{(3)}(\theta)|^n|\kappa(\theta)|^\nu \leq C_B 2^\nu B^\nu \quad \forall i = 0, \ldots, \nu. \tag{4.4.35}$$

Proof. We recall that the function κ is continuous. Hence, by restricting the variable Θ, we can find $B > 0$ such that $\|\kappa\|_{L^\infty(\mathbb{T}_L)} \leq B < \frac{1}{\rho_0}$. Moreover the curvature is analytic, thus its derivatives are bounded. We choose a constant C_B in such a way to satisfy

$$(1+b)|\kappa'(\theta)|^l|\kappa''(\theta)|^m|\kappa^{(3)}(\theta)|^n \leq C_B \quad \text{for all } l, m, n \in \{0, 1, 2, 3\}.$$

Now using the above, Lemma 4.2.1 and the estimate for the binomial coefficient:

$$\binom{n}{i} \leq 2^n, \ \forall\, 0 \leq i \leq n,$$

we obtain (4.4.35). □

Theorem 4.4.36 *Let $M \in \mathbb{N}_0$, $j \in \{0, 1, \ldots, M\}$, and $\{u_j\}$, $\{\tilde{v}_j\}$ be given by (4.4.11), (4.4.13) and (4.4.12), (4.4.14), respectively. Then u_j is analytic on a neighborhood of $\overline{\Omega}$, i.e. there exist constants $C_{u,j}$, $C_{v,j}$, $K_j > 0$ depending on j such that, for $n \in \mathbb{N}_0$,*

$$\|\nabla^n u_j\|_{L^\infty(\Omega)} \leq C_{u,j} K_j^n \max\{n, \sqrt{c/b}\}^n. \tag{4.4.37}$$

Moreover, there are $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_M \geq \Theta > 0$, such that for $j \in \{1, 2, \ldots, M\}$,

$$\left|\tilde{v}_j(\tilde{\rho}, \theta)\right| \leq C_{v,j} d^{-j}(q_j + |\tilde{\rho}|)^{2(j-1)} \mathrm{e}^{-\sqrt{b}\tilde{\rho}}, \quad \forall d \in (0, \theta_j), \quad \forall (\tilde{\rho}, \theta) \in \mathbb{C} \times V_{\theta_j}(d), \tag{4.4.38}$$

where $q_j > 1$ are constants (depending on j) and $V_{\theta_j}(d)$ is given by (4.4.26).

Proof. We proceed following the proof of Theorem 4.2.14, namely we are going to use induction. As has been done in 1-D, both sequences must be handled simultaneously.

We first consider u_0. By appealing to Theorem 4.2.14 we may ensure that u_0 satisfies (4.4.37). We have already seen that $\tilde{v}_0 = 0$ and $\tilde{v}_1$ is given by (4.4.20), thus both satisfy

(4.4.38). The function u_1 satisfies

$$\begin{aligned} &b\Delta u_1 - cu_1 = 0, \\ &u_1\big|_{\partial\Omega} = -\tilde{v}_1\big|_{\partial\Omega} = \frac{1}{\sqrt{b}}\frac{\partial \tilde{u}_0(0,\theta)}{\partial\rho}, \end{aligned}$$

with $\tilde{u}_0(0,\theta)$ defined by (4.4.15). We suppose now that (4.4.37), (4.4.38) hold for j and we will prove them for $j+1$. We will prove that the function $\tilde{G}_{j+1}$ appearing in (4.4.12) satisfies the desired condition in order to apply Proposition 2.4.10. First, note that $\tilde{G}_{j+1}$ is defined as a sum involving the functions $\tilde{v}_k$ (and their derivatives), with $k \leq j$. By the induction hypothesis, $\tilde{v}_k$ satisfies (4.4.39) which implies that we have $\theta_k \in \mathbb{R}^+$ such that $\tilde{v}_k$ is holomorphic in $\mathbb{C} \times V_{\theta_k}, k \leq j$. Then, by using Lemma 4.4.27 and the induction hypothesis for (4.4.38), we obtain for every $d \in (0, \theta_j)$,

$$\left|\tilde{G}_{j+1}(\tilde{\rho},\theta)\right| \leq \sum_{\nu=1}^{7}\left|\tilde{G}^{\nu}_{j+1}(\tilde{\rho},\theta)\right| \leq C_{v,j}d^{-(j+1)}(q_j+|\tilde{\rho}|)^{2j-1}\mathrm{e}^{-\sqrt{b}\tilde{\rho}}, \quad \forall(\tilde{\rho},\theta)\in\mathbb{C}\times V_{\theta_j}(d). \tag{4.4.39}$$

The above bound can be obtained by handling each term $\left|\tilde{G}^{\nu}_{j+1}\right|$ separately. We present the calculations for the first term:

$$\begin{aligned} \left|\tilde{G}^{1}_{j+1}(\tilde{\rho},\theta)\right| \leq &\sum_{\nu=0}^{j}\Bigg[2\binom{\nu}{0}|\kappa(\theta)|^{\nu+1}|\tilde{\rho}|^{\nu}\left|\frac{\partial^3\tilde{v}_{j-\nu}}{\partial\tilde{\rho}^3}\right| + \binom{\nu}{1}|\kappa(\theta)|^{\nu+1}|\tilde{\rho}|^{\nu-1}\left|\frac{\partial^2\tilde{v}_{j-\nu}}{\partial\tilde{\rho}^2}\right| + \\ &\qquad + \left(-\binom{\nu}{2}|\kappa(\theta)|^{\nu+1}|\tilde{\rho}|^{\nu-2} + b\binom{\nu}{0}|\kappa(\theta)|^{\nu+1}|\tilde{\rho}|^{\nu}\right)\left|\frac{\partial\tilde{v}_{j-\nu}}{\partial\tilde{\rho}}\right|\Bigg] \\ \leq &C_{v,j}C_B B d^{-j}\mathrm{e}^{\sqrt{b}}\mathrm{e}^{-\sqrt{b}\tilde{\rho}}\times \\ &\times\sum_{\nu=0}^{j}\Big[(2dB)^{\nu}|\tilde{\rho}|^{\nu-2}\Big((12+b)|\tilde{\rho}|^2+4|\tilde{\rho}|+1\Big)(q_{j-\nu}+|\tilde{\rho}|)^{2(j-\nu-1)}\Big] \\ \leq &C_{v,j}C_B B d^{-j}\mathrm{e}^{\sqrt{b}}\mathrm{e}^{-\sqrt{b}\tilde{\rho}}\frac{(2dB)^{j+1}-1}{2dB-1}(q_j+|\tilde{\rho}|)^{2j-1}, \end{aligned}$$

under the assumption $2dB \neq 1$. In the same manner we handle the second term:

$$\begin{aligned} \left|\tilde{G}^{2}_{j+1}(\tilde{\rho},\theta)\right| \leq &\sum_{\nu=0}^{j-1}\Bigg[2\binom{\nu+1}{1}|\kappa(\theta)|^{\nu}|\tilde{\rho}|^{\nu}\left|\frac{\partial^4\tilde{v}_{j-1-\nu}}{\partial\tilde{\rho}^2\partial\theta^2}\right| + 2\binom{\nu+1}{2}|\kappa(\theta)|^{\nu}|\tilde{\rho}|^{\nu-1}\left|\frac{\partial^3\tilde{v}_{j-1-\nu}}{\partial\tilde{\rho}\partial\theta^2}\right| + \\ &+\left(4\binom{\nu+1}{3}|\kappa(\theta)|^{\nu}|\tilde{\rho}|^{\nu-2} + b\binom{\nu+1}{1}|\kappa(\theta)|^{\nu}|\tilde{\rho}|^{\nu}\right)\left|\frac{\partial^2\tilde{v}_{j-1-\nu}}{\partial\theta^2}\right|\Bigg] \\ \leq &C_{v,j}C_B\kappa_1^{-2}d^{-(j+1)}\mathrm{e}^{-\sqrt{b}\tilde{\rho}}\times \end{aligned}$$

$$\times \sum_{\nu=0}^{j-1} (2dB)^{\nu} |\tilde{\rho}|^{\nu-2} \Big((16\mathrm{e}^{\sqrt{b}} + 4b)|\tilde{\rho}|^2 + 8\mathrm{e}^{\sqrt{b}}|\tilde{\rho}| + 16 \Big) (q_j + |\tilde{\rho}|)^{2(j-\nu-2)}$$

$$\leq C_{v,j} C_B \kappa_1^{-2} d^{-(j+1)} \mathrm{e}^{-\sqrt{b}\tilde{\rho}} \frac{(2dB)^j - 1}{2dB - 1} (q_j + |\tilde{\rho}|)^{2j-1}.$$

By working in same way, one may obtain the analogous bounds, for the remaining terms; we omit to present the calculations.

Since $\tilde{v}_j$ solves the BVP given by (4.4.12), (4.4.14) and we have shown that (4.4.39) holds, we may use Proposition 2.4.10 and get

$$\left|\tilde{v}_{j+1}(\tilde{\rho}, \theta)\right| \leq C \left[C_{v,j} d^{-(j+1)} (q_{j+1} + |\tilde{\rho}|)^{2j} + \frac{1}{\sqrt{b}} \left| \frac{-\partial \tilde{u}_j(0,\theta)}{\partial \rho} \right| \right] \mathrm{e}^{-\sqrt{b}\tilde{\rho}}.$$

Note that $\dfrac{\partial \tilde{u}_j}{\partial \rho}$ appears in the estimate above ($\tilde{u}_j$ given by (4.4.15)). By Lemma 4.4.18 $\dfrac{\partial \tilde{u}_j}{\partial \rho}$ is holomorphic in $B_{\rho'_{j-1}}(0) \times V_{\theta_j}$. Then by the induction hypothesis for u_j (and hence $\tilde{u}_j$) we obtain, for all $d \in (0, \theta_{j+1})$, $(\tilde{\rho}, \theta) \in \mathbb{C} \times V_{\theta_{j+1}}(d)$,

$$\left|\tilde{v}_{j+1}(\tilde{\rho}, \theta)\right| \leq C \left[C_{v,j} d^{-(j+1)} (q_{j+1} + |\tilde{\rho}|)^{2j} + \frac{C_{u,j} K}{\sqrt{b}} \max\{1, \sqrt{c/b}\} \right] \mathrm{e}^{-\sqrt{b}\tilde{\rho}}.$$

This implies,

$$\left|\tilde{v}_{j+1}(\tilde{\rho}, \theta)\right| \leq C_{v,j+1} d^{-(j+1)} (q_{j+1} + |\tilde{\rho}|)^{2j} \mathrm{e}^{-\sqrt{b}\tilde{\rho}}, \quad \forall d \in (0, \theta_{j+1}), \quad \forall (\tilde{\rho}, \theta) \in \mathbb{C} \times V_{\theta_{j+1}}(d),$$

thus we ensure that (4.4.38) holds for $j+1$. Next we consider u_{j+1}. This function solves the BVP

$$b\Delta u_{j+1} - c u_{j+1} = \Delta^2 u_{j-1},$$
$$u_{j+1}\big|_{\partial\Omega} = -\tilde{v}_{j+1}\big|_{\partial\Omega}.$$

From the induction hypothesis, we assume that (4.4.37) holds for $j-1$ and we have already shown that (4.4.38) holds for $j+1$. Then, Theorem 4.2.14 ensures that (4.4.37) holds for $j+1$. □

Theorem 4.4.40 *Let $M \in \mathbb{N}_0$ and let u_M^s be defined by* (4.4.22)*. Then there exist $C_M, \bar{K}_M > 0$ depending on M such that*

$$\|\nabla^n u_M^s\|_{L^\infty(\Omega)} \leq C_M \bar{K}_M^n \max\{n, \sqrt{c/b}\}^n, \quad \forall n \in \mathbb{N}_0. \tag{4.4.41}$$

Proof. With the aid of Cauchy-Schwarz inequality and the definition (4.4.22) we have

$$\left|\nabla^n u_M^s\right|^2 \leq M \sum_{j=0}^{M} \varepsilon^{2j} \left|\nabla^n u_j\right|^2.$$

By using (4.4.36) we obtain for $\varepsilon \neq 1$,

$$\begin{aligned}\left|\nabla^n u_M^s\right|^2 &\leq M \sum_{j=0}^{M} \varepsilon^{2j} C_{u,j}^2 K_j^{2n} \max\{n, \sqrt{c/b}\}^{2n} \\ &\leq M C_M \bar{K}_M^{2n} \frac{\varepsilon^{2M+1}-1}{\varepsilon-1} \max\{n, \sqrt{c/b}\}^{2n} \\ &\leq C_M \bar{K}_M^{2n} \max\{n, \sqrt{c/b}\}^{2n}.\end{aligned}$$

□

Lemma 4.4.42 *Let $M \in \mathbb{N}_0$ and $j \in \{0, 1, \ldots, M\}$. Assume that $\tilde{v}_j$ given by* (4.4.12) *and* (4.4.14). *Then there are constants $K_1 > 1$, $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_M \geq \Theta > 0$ and C_j, $q_j > 0$ (depending on j), such that for $m, n \in \mathbb{N}_0$, $\tilde{\rho} > 0$, $\theta \in V_{\theta_j}(d), d \in (0, \theta_j)$, there holds*

$$\left|\frac{\partial^{m+n} \tilde{v}_j(\tilde{\rho}, \theta)}{\partial \tilde{\rho}^m \partial \theta^n}\right| \leq C_j m! n! \mathrm{e}^{\sqrt{b}} K_1^n d^{-(j+n)} \Big(q_{j+1} + |\tilde{\rho}|\Big)^{2(j-1)} \mathrm{e}^{-\sqrt{b}\tilde{\rho}}, \qquad (4.4.43)$$

Proof. The combination of Lemma 4.4.27 and Theorem 4.4.36 gives the result. □

Lemma 4.4.44 *Let $M \in \mathbb{N}_0$ and consider the boundary layer term u_M^{BL} given in* (4.4.23) *and the tensor Λ_ε given by* (4.4.8) *with respect to boundary fitted coordinates. There exist $\Theta > 0$ and $\theta_M > \Theta, C_M, q_M > 0$ depending on M, f and $\partial\Omega$, such that for $d \in (0, \theta_M)$,*

$$\left|\Lambda_\varepsilon u_M^{BL}(\rho, \theta)\right| \leq C_M \varepsilon^{2M} d^{-(2M+2)} (q_M + |\tilde{\rho}|)^{4M} \mathrm{e}^{-\sqrt{b}\tilde{\rho}}, \qquad (4.4.45)$$

for all $(\rho, \theta) \in B_{\rho_0/4}(0) \times V_{\theta_M}$, with ρ_0 satisfying (4.4.2).

Proof. Let $j \in \{0, 1, \ldots, M\}$. With the aid of Lemma 4.4.33 we select the constants C_B, B to ensure that the inequalities $\lambda = \rho_0 B < 1$ and $|\rho\kappa(\theta)| \leq \lambda < 1$, $\forall (\rho, \theta) \in B_{\rho_0/4}(0) \times V_{\theta_j}$ hold. By the definition of Λ_ε and u_M^{BL}, and the property of the functions

$\tilde{v}_j$, i.e. $\sum_{j=0}^{i} L_j \tilde{v}_{i-j} = \sum_{j=0}^{i} L_{i-j}\tilde{v}_j = 0$, we have

$$\Lambda_\varepsilon u_M^{BL}(\rho,\theta) = \sum_{\nu=2M+2}^{\infty} \varepsilon^{\nu-2} \sum_{j=0}^{2M+1} \Lambda_{\nu-j}\tilde{v}_j(\rho/\varepsilon,\theta)$$

$$= -\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\Bigg[-2\binom{\nu-j-1}{0}\kappa^{\nu-j}(\theta)\tilde{\rho}^{\nu-j-1}\frac{\partial^3\tilde{v}_j}{\partial\tilde{\rho}^3} - \binom{\nu-j-1}{1}\kappa^{\nu-j}(\theta)\tilde{\rho}^{\nu-j-2}\frac{\partial^2\tilde{v}_j}{\partial\tilde{\rho}^2} +$$

$$+\left(-\binom{\nu-j-1}{2}\kappa^{\nu-j}(\theta)\tilde{\rho}^{\nu-j-3} + b\binom{\nu-j-1}{0}\kappa^{\nu-j}(\theta)\tilde{\rho}^{\nu-j-1}\right)\frac{\partial\tilde{v}_j}{\partial\tilde{\rho}}\Bigg] -$$

$$-\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\Bigg[2\binom{\nu-j-1}{1}\kappa^{\nu-j-2}(\theta)\tilde{\rho}^{\nu-j-2}\frac{\partial^4\tilde{v}_j}{\partial\tilde{\rho}^2\partial\theta^2} +$$

$$+2\binom{\nu-j-1}{2}\kappa^{\nu-j-2}(\theta)\tilde{\rho}^{\nu-j-3}\frac{\partial^3\tilde{v}_j}{\partial\tilde{\rho}\partial\theta^2} +$$

$$+\left(4\binom{\nu-j-1}{3}\kappa^{\nu-j-2}(\theta)\tilde{\rho}^{\nu-j-4} - b\binom{\nu-j-1}{1}\kappa^{\nu-j-2}(\theta)\tilde{\rho}^{\nu-j-2}\right)\frac{\partial^2\tilde{v}_j}{\partial\theta^2}\Bigg] -$$

$$-\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\Bigg[2\binom{\nu-j-1}{2}\kappa'(\theta)\kappa^{\nu-j-3}(\theta)\tilde{\rho}^{\nu-j-2}\frac{\partial^3\tilde{v}_j}{\partial\rho^2\partial\theta} +$$

$$+2\binom{\nu-j-1}{3}\kappa'(\theta)\kappa^{\nu-j-3}(\theta)\tilde{\rho}^{\nu-j-3}\frac{\partial^2\tilde{v}_j}{\partial\rho\partial\theta} +$$

$$+\left(-\binom{\nu-j-1}{2} - \binom{\nu-j-1}{3}\right)\kappa''(\theta)\kappa^{\nu-j-3}(\theta)\tilde{\rho}^{\nu-j-3}\frac{\partial\tilde{v}_j}{\partial\tilde{\rho}} +$$

$$+\left(5\binom{\nu-j-1}{3} + 9\binom{\nu-j-1}{4}\right)\kappa'(\theta)\kappa^{j-3-\nu}(\theta)\tilde{\rho}^{j-n-4}\frac{\partial\tilde{v}_j}{\partial\tilde{\theta}}\Bigg] -$$

$$-\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\Bigg[\binom{\nu-j-1}{3}\kappa^{\nu-j-4}(\theta)\tilde{\rho}^{\nu-j-4}\frac{\partial^4\tilde{v}_j}{\partial\theta^4} +$$

$$+\left(-3\binom{\nu-j-1}{3} - 3\binom{\nu-j-1}{4}\right)\left(\kappa'(\theta)\right)^2\kappa^{\nu-j-4}(\theta)\tilde{\rho}^{\nu-j-3}\frac{\partial\tilde{v}_j}{\partial\tilde{\rho}}\Bigg] -$$

$$-\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\Bigg[6\binom{\nu-j-1}{4}\kappa'(\theta)\kappa^{\nu-j-5}(\theta)\tilde{\rho}^{\nu-j-4}\frac{\partial^3\tilde{v}_j}{\partial\theta^3} +$$

$$+4\binom{\nu-j-1}{4}\kappa''(\theta)\kappa^{j-5-\nu}(\theta)\tilde{\rho}^{\nu-j-4}\frac{\partial^2\tilde{v}_j}{\partial\theta^2} +$$

$$+\binom{\nu-j-1}{4}\kappa^{(3)}(\theta)\kappa^{\nu-j-5}(\theta)\tilde{\rho}^{\nu-j-4}\frac{\partial\tilde{v}_j}{\partial\tilde{\theta}}\Bigg] -$$

$$-\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\Bigg[15\binom{\nu-j-1}{5}\left(\kappa'(\theta)\right)^2\kappa^{\nu-j-6}(\theta)\tilde{\rho}^{\nu-j-4}\frac{\partial^2\tilde{v}_j}{\partial\theta^2} +$$

$$+10\binom{\nu-j-1}{5}\kappa'(\theta)\kappa''(\theta)\kappa^{\nu-j-6}(\theta)\tilde{\rho}^{\nu-j-4}\frac{\partial\tilde{v}_j}{\partial\theta}\Bigg] -$$

$$-\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\left[15\binom{\nu-j-1}{6}\big(\kappa'(\theta)\big)^3\kappa^{\nu-j-7}(\theta)\tilde{\rho}^{\nu-j-4}\frac{\partial\tilde{v}_j}{\partial\theta}\right].$$

Obviously $\Lambda_\varepsilon u_m^{BL}$ is an 'enormous' term which must be handled. Here we present only the calculations for the first double sum. All the rest can be treated in same manner. We have:

$$\begin{aligned}
&\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\left[2\binom{\nu-j-1}{0}|\kappa(\theta)|^{\nu-j}|\tilde{\rho}|^{\nu-j-1}\left|\frac{\partial^3\tilde{v}_j}{\partial\tilde{\rho}^3}\right|+\right.\\
&\qquad+\binom{\nu-j-1}{1}|\kappa(\theta)|^{\nu-j}|\tilde{\rho}|^{\nu-j-2}\left|\frac{\partial^2\tilde{v}_j}{\partial\tilde{\rho}^2}\right|+\\
&\qquad\left.+\left(\binom{\nu-j-1}{2}|\kappa(\theta)|^{\nu-j}|\tilde{\rho}|^{\nu-j-3}+b\binom{\nu-j-1}{0}|\kappa(\theta)|^{\nu-j}|\tilde{\rho}|^{\nu-j-1}\right)\left|\frac{\partial\tilde{v}_j}{\partial\tilde{\rho}}\right|\right]\\
&=I_1+I_2+I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1=&\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\left[2\binom{\nu-j-1}{0}|\kappa(\theta)|^{\nu-j}|\tilde{\rho}|^{\nu-j-1}\left|\frac{\partial^3\tilde{v}_j}{\partial\tilde{\rho}^3}\right|+\right.\\
&\left.+b\binom{\nu-j-1}{0}|\kappa(\theta)|^{\nu-j}|\tilde{\rho}|^{\nu-j-1}\left|\frac{\partial\tilde{v}_j}{\partial\tilde{\rho}}\right|\right],
\end{aligned}$$

$$I_2=\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\binom{\nu-j-1}{1}|\kappa(\theta)|^{\nu-j}|\tilde{\rho}|^{\nu-j-2}\left|\frac{\partial^2\tilde{v}_j}{\partial\tilde{\rho}^2}\right|,$$

and

$$I_3=\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}\varepsilon^{\nu-2}\binom{\nu-j-1}{2}|\kappa(\theta)|^{\nu-j}|\tilde{\rho}|^{\nu-j-3}\left|\frac{\partial\tilde{v}_j}{\partial\tilde{\rho}}\right|.$$

We begin with I_1 and we appeal again to Lemmata 4.4.27, 4.4.33, 4.4.43 and, for fixed $d\in(0,\theta_M)$, we get

$$I_1\leq\mathrm{e}^{\sqrt{b}}C_B\sum_{j=0}^{2M+1}\sum_{\nu=2M+2}^{\infty}C_{v,j}(2\varepsilon B|\tilde{\rho}|)^{\nu-2}d^{-j}(12+b)2^{1-j}B^{2-j}|\rho|^{1-j}(q_{j+1}+|\tilde{\rho}|)^{2(j-1)}\mathrm{e}^{-\sqrt{b}\tilde{\rho}}.$$

By Lemma 4.4.33 we choose B to satisfy (4.4.34) and since $\rho\leq\frac{\rho_0}{4}$, we have for fixed

$d \in (0, \theta_M)$,

$$I_1 \leq \mathrm{e}^{\sqrt{b}} B^2 (48+4b) C_B \big(2\varepsilon B|\tilde{\rho}|\big)^{2M} \sum_{j=0}^{2M+1} C_{v,j} (2Bd)^{-j} |\tilde{\rho}|^{1-j} (q_{j+1} + |\tilde{\rho}|)^{2(j-1)} \mathrm{e}^{-\sqrt{b}\tilde{\rho}}.$$

Note that, for $0 \leq j \leq 2M+1$,

$$|\tilde{\rho}|^{1-j} \Big(q_j + |\tilde{\rho}|\Big)^{2(j-1)} \leq |\tilde{\rho}|^{-2M} \Big(q_M + |\tilde{\rho}|\Big)^{4M},$$

where q_M is adjusted properly, i.e. it is taken to be the $\max\{q_j \,|\, 0 \leq j \leq 2M+1\}$. Therefore, we get, for fixed $d \in (0, \theta_M)$ and $2dB \neq 1$,

$$\begin{aligned} I_1 \leq & \mathrm{e}^{\sqrt{b}} B^2 (48+4b) C_M C_B \big(2\varepsilon B|\tilde{\rho}|\big)^{2M} \frac{(2Bd)^{-2M-2} - 1}{(2Bd)^{-1} - 1} |\tilde{\rho}|^{-2M} (q_M + |\tilde{\rho}|)^{4M} \mathrm{e}^{-\sqrt{b}\tilde{\rho}} \\ \leq & \mathrm{e}^{\sqrt{b}} (12+b) C_M C_B \varepsilon^{2M} d^{-2M-2} \frac{2B}{1-2Bd} (q_M + |\tilde{\rho}|)^{4M} \mathrm{e}^{-\sqrt{b}\tilde{\rho}}. \end{aligned}$$

We proceed with terms I_2, I_3. Those terms vanish for $\nu - j - 2 < 0$, $\nu - j - 3 < 0$, respectively, hence by appealing to Lemmata 4.4.27, 4.4.33, 4.4.43, and keeping in mind $\rho \leq \frac{\rho_0}{4}$, we bound them as follows:

$$\begin{aligned} I_2 \leq & \sum_{j=0}^{2M} \sum_{\nu=2M+2}^{\infty} \varepsilon^{\nu-2} \binom{\nu-j-1}{1} |\kappa(\theta)|^{\nu-j} |\tilde{\rho}|^{\nu-j-2} \left| \frac{\partial^2 \tilde{v}_j}{\partial \tilde{\rho}^2} \right| + \\ & + \sum_{j=2M+1}^{2M+1} \sum_{\nu=2M+3}^{\infty} \varepsilon^{\nu-2} \binom{\nu-j-1}{1} |\kappa(\theta)|^{\nu-j} |\tilde{\rho}|^{\nu-j-2} \left| \frac{\partial^2 \tilde{v}_j}{\partial \tilde{\rho}^2} \right| \\ \leq & 4B^2 \mathrm{e}^{\sqrt{b}} C_B \big(2\varepsilon B|\tilde{\rho}|\big)^{2M} \sum_{j=0}^{2M} C_{v,j} (2Bd)^{-j} |\tilde{\rho}|^{-j} (q_j + |\tilde{\rho}|)^{2(j-1)} \mathrm{e}^{-\sqrt{b}\tilde{\rho}} + \\ & + 4B^2 \mathrm{e}^{\sqrt{b}} C_M C_B \big(2\varepsilon B|\tilde{\rho}|\big)^{2M+1} (2Bd)^{-(2M+1)} |\tilde{\rho}|^{-(2M+1)} (q_M + |\tilde{\rho}|)^{4M} \mathrm{e}^{-\sqrt{b}\tilde{\rho}}, \end{aligned}$$

and

$$\begin{aligned} I_3 \leq & \sum_{j=0}^{2M-1} \sum_{\nu=2M+2}^{\infty} \varepsilon^{\nu-2} \binom{\nu-j-1}{2} |\kappa(\theta)|^{\nu-j} |\tilde{\rho}|^{\nu-j-3} \left| \frac{\partial \tilde{v}_j}{\partial \tilde{\rho}} \right| + \\ & + \sum_{j=2M}^{2M} \sum_{\nu=2M+3}^{\infty} \varepsilon^{\nu-2} \binom{\nu-j-1}{2} |\kappa(\theta)|^{\nu-j} |\tilde{\rho}|^{\nu-j-3} \left| \frac{\partial \tilde{v}_j}{\partial \tilde{\rho}} \right| + \\ & + \sum_{j=2M+1}^{2M+1} \sum_{\nu=2M+4}^{\infty} \varepsilon^{\nu-2} \binom{\nu-j-1}{2} |\kappa(\theta)|^{\nu-j} |\tilde{\rho}|^{\nu-j-3} \left| \frac{\partial \tilde{v}_j}{\partial \tilde{\rho}} \right| \end{aligned}$$

$$\le 2B^2 e^{\sqrt{b}} C_B \big(2\varepsilon B|\tilde\rho|\big)^{2M} \sum_{j=0}^{2M-1} C_{v,j}(2Bd)^{-j}|\tilde\rho|^{-1-j}(q_j+|\tilde\rho|)^{2(j-1)} e^{-\sqrt{b}\tilde\rho}+$$
$$+\,2B^2 e^{\sqrt{b}} C_M C_B \big(2\varepsilon B|\tilde\rho|\big)^{2M+1}(2Bd)^{-(2M+1)}|\tilde\rho|^{-(2M+1)}(q_M+|\tilde\rho|)^{4M-2} e^{-\sqrt{b}\tilde\rho}+$$
$$+\,2B^2 e^{\sqrt{b}} C_M C_B \big(2\varepsilon B|\tilde\rho|\big)^{2M+2}(2Bd)^{-(2M+2)}|\tilde\rho|^{-(2M+2)}(q_M+|\tilde\rho|)^{4M} e^{-\sqrt{b}\tilde\rho},$$

for fixed $d \in (0, \theta_M)$. We adjust properly the constant q_M and for $0 \le j \le 2M$, we have

$$|\tilde\rho|^{-j}(q_j+|\tilde\rho|)^{2(j-1)} \le |\tilde\rho|^{-2M}(q_M+|\tilde\rho|)^{4M-2}.$$

Also for $0 \le j \le 2M-1$,

$$|\tilde\rho|^{-j-1}(q_j+|\tilde\rho|)^{2(j-1)} \le |\tilde\rho|^{-2M}(q_M+|\tilde\rho|)^{4M-4}.$$

The combination of the previous inequalities gives, for fixed $d \in (0, \theta_M)$,

$$I_2 \le 4B^2 e^{\sqrt{b}} C_B C_M \varepsilon^{2M} d^{-(2M+1)}\Big(\frac{d}{1-2Bd}+\varepsilon\Big)(q_M+|\tilde\rho|)^{4M} e^{-\sqrt{b}\tilde\rho}.$$

Simillary, for fixed $d \in (0, \theta_M)$,

$$I_3 \le 2B^2 e^{\sqrt{b}} C_B C_M \varepsilon^{2M} d^{-(2M+2)}\Big(\frac{2Bd^3}{1-2Bd}+\varepsilon d+\varepsilon^2 d^2\Big)(q_M+|\tilde\rho|)^{4M} e^{-\sqrt{b}\tilde\rho}.$$

□

Theorem 4.4.46 *Let $M \in \mathbb{N}_0$ and u_M^{BL} be defined by* (4.4.23) *and suppose that $\rho_0 > 0$ satisfies* (4.4.2). *In addition, suppose that the right hand function f in* (4.1.1) *and the curve of the boundary of the domain are analytic. Then there exist constant $C_{M+1} > 0$, depending on $M+1$, $K > 1, K_1 > 1, K_2 > 0$ such that the inner expansion u_M^{BL} is analytic on $(0, \rho_0) \times \mathbb{T}_l$ and for every n, $m \in \mathbb{N}_0$, $(\rho, \theta) \in (0, \rho_0) \times \mathbb{T}_l$,*

$$\left|\frac{\partial^{m+n} u_M^{BL}(\rho,\theta)}{\partial\rho^m \partial\theta^n}\right| \le C_{M+1} e^{\frac{m+3}{2}}(m+1)^{1/2} n! K^n \varepsilon^{1-m} e^{-\sqrt{b}\rho/\varepsilon}. \qquad (4.4.47)$$

Proof. From Theorem 4.4.36 we know that $\{\tilde v_j\}_{j\in\mathbb{N}_0}$ are holomorphic on $\mathbb{C}\times V_\Theta$, for some $\Theta > 0$. We apply Cauchy's differentiation formula and for all $n, m \in \mathbb{N}_0$, $\rho \ge 0$, $\theta \in \mathbb{T}_l$

we have that

$$\frac{\partial^{m+n}\tilde{v}_j(\rho/\varepsilon,\theta)}{\partial\rho^m\partial\theta^n} = -\varepsilon^{-m}\frac{m!n!}{4\pi^2}\int_{|r|=r_0}\int_{|s|=\frac{\Theta}{2}}\frac{\tilde{v}_j(\rho/\varepsilon+r,\Theta+s)}{r^{m+1}s^{n+1}}dsdr.$$

By applying (4.4.38) on the functions $\{\tilde{v}_j\}_{j=0}^{\infty}$ and setting $r_0=\frac{m+1}{\sqrt{b}}$ we get

$$\left|\frac{\partial^{m+n}\tilde{v}_j(\rho/\varepsilon,\theta)}{\partial\rho^m\partial\theta^n}\right| \leq C_{v,j}\varepsilon^{-m}\frac{m!n!\mathrm{e}^{m+1}}{(m+1)^m}\Big(\frac{2}{\Theta}\Big)^{n+j}\Big(q_j+\rho/\varepsilon+\frac{m+1}{\sqrt{b}}\Big)^{2(j-1)}\mathrm{e}^{-\sqrt{b}\rho/\varepsilon}.$$

Considering the above calculations and Stirling's formula,

$$m! \leq Cm^m\mathrm{e}^{-m}\big(2\pi(m+1)\big)^{1/2},$$

we obtain,

$$\left|\frac{\partial^{m+n}\tilde{v}_j(\rho/\varepsilon,\theta)}{\partial\rho^m\partial\theta^n}\right| \leq C_{v,j}\varepsilon^{-m}\mathrm{e}\big(2\pi(m+1)\big)^{1/2}n!\Big(\frac{2}{\Theta}\Big)^{n+j}\Big(q_j+\rho/\varepsilon+\frac{m+1}{\sqrt{b}}\Big)^{2(j-1)}\mathrm{e}^{-\sqrt{b}\rho/\varepsilon}.$$

Consider the term

$$\begin{aligned}\Big(q_j+\rho/\varepsilon+\frac{m+1}{\sqrt{b}}\Big)^{2(j-1)}\mathrm{e}^{-\sqrt{b}\rho/\varepsilon} &\leq \Big(q_j+\rho/\varepsilon+\frac{m+1}{\sqrt{b}}\Big)^{2j}\mathrm{e}^{-\sqrt{b}\rho/\varepsilon}\\ &=\big(\sqrt{b}\big)^{-2j}\Big(\sqrt{b}q_j+\frac{\sqrt{b}\rho}{\varepsilon}+m+1\Big)^{2j}\mathrm{e}^{-\frac{\sqrt{b}}{2}\rho/\varepsilon}\mathrm{e}^{-\frac{\sqrt{b}}{2}\rho/\varepsilon}\\ &\leq\big(\sqrt{b}\big)^{-2j}\Big(2j+\sqrt{b}q_j+\frac{\sqrt{b}\rho}{\varepsilon}+m+1\Big)^{2j}\mathrm{e}^{-\frac{1}{2}(\sqrt{b}\rho/\varepsilon+m+1)}\mathrm{e}^{\frac{m+1}{2}}\mathrm{e}^{-\frac{\sqrt{b}}{2}\rho/\varepsilon}.\end{aligned} \tag{4.4.48}$$

By Lemma 4.2.3 we obtain

$$\Big(q_j+\rho/\varepsilon+\frac{m+1}{\sqrt{b}}\Big)^{2(j-1)}\mathrm{e}^{-\sqrt{b}\rho/\varepsilon} \leq \big(\sqrt{b}\big)^{-2j}(2j)^{2j}\mathrm{e}^{j}\mathrm{e}^{\frac{\sqrt{b}q_j}{2}}\mathrm{e}^{\frac{m+1}{2}}\mathrm{e}^{-\frac{\sqrt{b}\rho}{2\varepsilon}}.$$

Hence,

$$\left|\frac{\partial^{m+n}\tilde{v}_j(\rho/\varepsilon,\theta)}{\partial\rho^m\partial\theta^n}\right| \leq C_{v,j}\varepsilon^{-m}\mathrm{e}\big(2\pi(m+1)\big)^{1/2}n!\Big(\frac{2}{\Theta}\Big)^{n+j}\big(\sqrt{b}\big)^{-2j}(2j)^{2j}\mathrm{e}^{j}\mathrm{e}^{\frac{\sqrt{b}q_j}{2}}\mathrm{e}^{\frac{m+1}{2}}\mathrm{e}^{-\frac{\sqrt{b}\rho}{2\varepsilon}}.$$

The term $\left(\frac{2\mathrm{e}}{\Theta}\right)^{j}\left(\frac{2j}{\sqrt{b}}\right)^{2j}\mathrm{e}^{\frac{\sqrt{b}q_j}{2}}$ may be absorbed by the constant $C_{v,j}$ (since it depends on j.) Therefore, for $K = \frac{2}{\Theta}$,

$$\left|\frac{\partial^{m+n}\tilde{v}_j(\rho/\varepsilon,\theta)}{\partial\rho^m\partial\theta^n}\right| \le C_{v,j}\varepsilon^{-m}\mathrm{e}^{\frac{m+3}{2}}\big(2\pi(m+1)\big)^{1/2}n!K^n\mathrm{e}^{-\frac{\sqrt{b}\rho}{2\varepsilon}}. \tag{4.4.49}$$

We now consider the definition of the term u_M^{BL} in (4.4.23) and we use (4.4.49):

$$\begin{aligned}\left|\frac{\partial^{m+n}u_M^{BL}(\rho/\varepsilon,\theta)}{\partial\rho^m\partial\theta^n}\right| &\le \sum_{j=0}^{M+1}\varepsilon^j\left|\frac{\partial^{m+n}\tilde{v}_j(\rho/\varepsilon,\theta)}{\partial\rho^m\partial\theta^n}\right| = \sum_{j=1}^{M+1}\varepsilon^j\left|\frac{\partial^{m+n}\tilde{v}_j(\rho/\varepsilon,\theta)}{\partial\rho^m\partial\theta^n}\right| \\ &\le \sum_{j=1}^{M+1} C_{v,j}\mathrm{e}^{\frac{m+3}{2}}(m+1)^{1/2}n!\varepsilon^{1-m}K^n\varepsilon^{j-1}\mathrm{e}^{-\frac{\sqrt{b}\rho}{2\varepsilon}} \\ &\le \mathrm{e}^{\frac{m+3}{2}}(m+1)^{1/2}n!\varepsilon^{1-m}K^n\mathrm{e}^{-\frac{\sqrt{b}\rho}{2\varepsilon}}\sum_{j=0}^{M}C_{v,j+1}\varepsilon^j \\ &\le C_{M+1}\mathrm{e}^{\frac{m+3}{2}}(m+1)^{1/2}n!\varepsilon^{1-m}K^n\mathrm{e}^{-\frac{\sqrt{b}\rho}{2\varepsilon}}.\end{aligned}$$

□

Theorem 4.4.50 *Let $M \in \mathbb{N}_0$ and r_M be defined by* (4.4.24) *and suppose that the right hand function f in* (4.1.1) *and the curve of the boundary of the domain are analytic. Then there exist $C_{M+1} > 0$, depending on M, and $\beta > 0$ depending on b and the cut-off function χ, such that*

$$\|r_M\|_{E,\Omega} \le C_{M+1}\Big(\big(\varepsilon M^2\big)^{2M} + \mathrm{e}^{-\beta/\varepsilon}\Big). \tag{4.4.51}$$

Proof. We have,

$$\begin{aligned}\Lambda_\varepsilon r_M =& \Lambda_\varepsilon(u-u_M^s) - \Lambda_\varepsilon(\chi u_M^{BL}) = f - \sum_{\nu=0}^{M}\varepsilon^\nu\Lambda_\varepsilon(u_\nu) - \Lambda_\varepsilon(\chi u_M^{BL}) = \\ =& f - \sum_{\nu=0}^{M}\varepsilon^\nu\big(\varepsilon^2\Delta^2u_\nu - b\Delta u_\nu + cu_\nu\big) - \Bigg(\varepsilon^2\chi\Delta^2u_M^{BL} + 6\varepsilon^2\Delta\chi\Delta u_M^{BL} + \varepsilon^2u_M^{BL}\Delta^2\chi - \\ & - bu_M^{BL}\Delta\chi - b\chi\Delta u_M^{BL} + c\chi u_M^{BL} - 2b\left[\frac{\partial\chi}{\partial x}\frac{\partial u_M^{BL}}{\partial x} + \frac{\partial\chi}{\partial y}\frac{\partial u_M^{BL}}{\partial y}\right] + \\ & + 4\varepsilon^2\left[\frac{\partial\chi}{\partial x}\left(\frac{\partial^3u_M^{BL}}{\partial x^3} + \frac{\partial^3u_M^{BL}}{\partial x\partial y^2}\right) + \frac{\partial\chi}{\partial y}\left(\frac{\partial^3u_M^{BL}}{\partial x^2\partial y} + \frac{\partial^3u_M^{BL}}{\partial y^3}\right)\right] + \\ & + 4\varepsilon\left[\frac{\partial u_M^{BL}}{\partial x}\left(\frac{\partial^3\chi}{\partial x^3} + \frac{\partial^3\chi}{\partial x\partial y^2}\right) + \frac{\partial u_M^{BL}}{\partial y}\left(\frac{\partial^3\chi}{\partial x^2\partial y} + \frac{\partial^3\chi}{\partial y^3}\right)\right]\Bigg) \\ =& -\varepsilon^{M+1}\Delta^2u_{M-1} - \varepsilon^{M+2}\Delta^2u_M - \Bigg(6\varepsilon^2\Delta\chi\Delta u_M^{BL} + \varepsilon^2u_M^{BL}\Delta^2\chi - bu_M^{BL}\Delta\chi +\end{aligned}$$

$$+ 4\varepsilon^2\left[\frac{\partial\chi}{\partial x}\left(\frac{\partial^3 u_M^{BL}}{\partial x^3}+\frac{\partial^3 u_M^{BL}}{\partial x\partial y^2}\right)+\frac{\partial\chi}{\partial y}\left(\frac{\partial^3 u_M^{BL}}{\partial x^2\partial y}+\frac{\partial^3 u_M^{BL}}{\partial y^3}\right)\right]+$$
$$+ 4\varepsilon\left[\frac{\partial u_M^{BL}}{\partial x}\left(\frac{\partial^3 \chi}{\partial x^3}+\frac{\partial^3 \chi}{\partial x\partial y^2}\right)+\frac{\partial u_M^{BL}}{\partial y}\left(\frac{\partial^3 \chi}{\partial x^2\partial y}+\frac{\partial^3 \chi}{\partial y^3}\right)\right]-$$
$$- 2b\left[\frac{\partial\chi}{\partial x}\frac{\partial u_M^{BL}}{\partial x}+\frac{\partial\chi}{\partial y}\frac{\partial u_M^{BL}}{\partial y}\right]+\chi\Lambda_\varepsilon u_M^{BL}\Bigg).$$

With the aid of (4.4.37), the first two terms can be bounded as

$$\begin{aligned}\left\|\varepsilon^{M+1}\Delta^2 u_{M-1}+\varepsilon^{M+2}\Delta^2 u_M\right\|_{L^\infty(\Omega)} &\leq \varepsilon^{M+1}\left\|\Delta^2 u_{M-1}\right\|+\varepsilon^{M+2}\left\|\Delta^2 u_M\right\|\\ &\leq \varepsilon^{M+1}\max\{4,\sqrt{c/b}\}^4\left(C_{M-1}K_{M-1}^4+C_M K_M^4\right)\\ &\leq \overline{C}_M\varepsilon^{M+1}\max\{4,\sqrt{c/b}\}^4\end{aligned}$$

To handle the term $\chi\Lambda_\varepsilon u_M^{BL}$ we recall the support properties of χ (given by (4.4.3)). We utilize Lemma 4.4.44 and we have,

$$\left\|\chi\Lambda_\varepsilon u_M^{BL}\right\|_{L^2(\Omega)} \leq C_M\varepsilon^{2M}\left(\int_{\Omega_0}\left[d^{-(2M+2)}(q_M+|\rho/\varepsilon|)^{4M}\mathrm{e}^{-\frac{\sqrt{b}\rho}{\varepsilon}}\right]^2 d\rho d\theta\right)^{1/2}.$$

By following (4.4.48) we may obtain (for $m=1$),

$$\left\|\chi\Lambda_\varepsilon u_M^{BL}\right\|_{L^2(\Omega)} \leq C_M\varepsilon^{2M}\left(\frac{4M}{\sqrt{b}}\right)^{4M}\mathrm{e}^{2M+1/2}\mathrm{e}^{\frac{\sqrt{b}q_M}{2}}\left(\int_{\Omega_0}\left[d^{-(2M+2)}\mathrm{e}^{-\frac{\sqrt{b}\rho}{2\varepsilon}}\right]^2 d\rho d\theta\right)^{1/2}.$$

Letting $d\to\Theta$ (with Θ as in the statement of Theorem 4.4.36) gives

$$\begin{aligned}\left\|\chi\Lambda_\varepsilon u_M^{BL}\right\|_{L^2(\Omega)} &\leq C_M\varepsilon^{2M}\left(\frac{4M}{\sqrt{b}}\right)^{4M}\mathrm{e}^{2M+1/2}\mathrm{e}^{\frac{\sqrt{b}q_M}{2}}\left(\frac{1}{\Theta}\right)^{2M+2}|\Omega_0|^{1/2}\\ &\leq C_M\varepsilon^{2M}M^{4M}\left(\frac{16\mathrm{e}}{b\Theta}\right)^{2M}\\ &\leq C_M\left(\frac{16\varepsilon\mathrm{e}M^2}{b\Theta}\right)^{2M}.\end{aligned}$$

We proceed with the remaining terms. We recall that in a neighborhood of $\rho=0$, the cut-off function $\chi\equiv 1$ and all its derivatives vanish. Since u_M^{BL} and all its derivatives are exponentially small by (4.4.47), we may bound the terms, for $\rho>\rho_0/2$, by using Theorem 4.4.46, as follows:

$$\left\|\Delta\chi\Delta u_M^{BL}\right\|_{L^2(\Omega)} \leq C_{M+1}\varepsilon^{-1}\mathrm{e}^{-\frac{\beta}{\varepsilon}},$$

$$\left\| u_M^{BL} \Delta\chi \right\|_{L^2(\Omega)} \le C_{M+1}\varepsilon \mathrm{e}^{-\frac{\beta}{\varepsilon}},$$

$$\left\| u_M^{BL} \Delta^2\chi \right\|_{L^2(\Omega)} \le C_{M+1}\varepsilon \mathrm{e}^{-\frac{\beta}{\varepsilon}},$$

$$\begin{aligned}\left\| \frac{\partial\chi}{\partial x}\left(\frac{\partial^3 u_M^{BL}}{\partial x^3} + \frac{\partial^3 u_M^{BL}}{\partial x\partial y^2}\right) + \frac{\partial\chi}{\partial y}\left(\frac{\partial^3 u_M^{BL}}{\partial x^2\partial y} + \frac{\partial^3 u_M^{BL}}{\partial y^3}\right)\right\|_{L^2(\Omega)} &\le \left\| \nabla\chi\nabla^3 u_M^{BL}\right\|_{L^2(\Omega)} \\ &\le C_{M+1}\varepsilon^{-2}\mathrm{e}^{-\frac{\beta}{\varepsilon}},\end{aligned}$$

$$\begin{aligned}\left\| \frac{\partial u_M^{BL}}{\partial x}\left(\frac{\partial^3 \chi}{\partial x^3} + \frac{\partial^3 \chi}{\partial x\partial y^2}\right) + \frac{\partial u_M^{BL}}{\partial y}\left(\frac{\partial^3 \chi}{\partial x^2\partial y} + \frac{\partial^3 \chi}{\partial y^3}\right)\right\|_{L^2(\Omega)} &\le \left\| \nabla^3\chi\nabla u_M^{BL}\right\|_{L^2(\Omega)} \\ &\le C_{M+1}\mathrm{e}^{-\frac{\beta}{\varepsilon}},\end{aligned}$$

$$\left\| \frac{\partial\chi}{\partial x}\frac{\partial u_M^{BL}}{\partial x} + \frac{\partial\chi}{\partial y}\frac{\partial u_M^{BL}}{\partial y}\right\|_{L^2(\Omega)} \le \left\| \nabla\chi\nabla u_M^{BL}\right\|_{L^2(\Omega)} \le C_{M+1}\mathrm{e}^{-\frac{\beta}{\varepsilon}},$$

for some $\beta > 0$ depending on b and χ. Combining the above bounds we have, with the aid of stability, the desired result. □

We summarize the main results of this section:

Remark 4.4.52. Let $M \in \mathbb{N}_0$, and let u be the solution of (4.1.1). Then u can be written as:

$$u = u_M^s + \chi u_M^{BL} + r_M,$$

and there are constants $C_{1,M}$, $C_{2,M+1}$, $C_{3,M+1} > 0$ depending on M and ρ_0 independent of $\varepsilon \in (0, 1]$ such that, for all $m, n \in \mathbb{N}_0$, $(\rho, \theta) \in (0, \rho_0) \times \mathbb{T}_l$,

$$\begin{aligned}&\|\nabla^n u_M^s\|_{L^\infty(\Omega)} \le C_{1,M} K_M^n \max\{n, \sqrt{c/b}\}^n, \\ &\left| \frac{\partial^{m+n} u_M^{BL}(\rho,\theta)}{\partial\rho^m\partial\theta^n}\right| \le C_{2,M+1}\mathrm{e}^{\frac{m+3}{2}}(m+1)^{1/2} n! K^n \varepsilon^{1-m}\mathrm{e}^{-\frac{\sqrt{b}\rho}{2\varepsilon}}, \\ &\|r_M\|_{E,\Omega} \le C_{3,M+1}\Big(\big(\varepsilon M^2\big)^{2M} + \mathrm{e}^{-\beta/\varepsilon}\Big),\end{aligned}$$

for some $\beta > 0$.

4.5 Appendix

Here we present some useful lemmas and inequalities. In some cases, we ommit the proof.

Lemma 4.5.1 *Let $0 < \rho_0 \leq 1$ and the integer $n \geq 2$. Suppose that b, c and u are analytic functions defined on the disc B_{ρ_0}, where b and c satisfy the bounds*

$$\begin{aligned} \Big\|\nabla^n b\Big\|_{L^\infty(B_{\rho_0})} &\leq C_b \gamma_b^n n!, \qquad \forall n \in \mathbb{N}_0, \\ \Big\|\nabla^n c\Big\|_{L^\infty(B_{\rho_0})} &\leq C_c \gamma_c^n n!, \qquad \forall n \in \mathbb{N}_0. \end{aligned} \tag{4.5.2}$$

Then we have

$$\breve{M}_{\rho_0,n}(cu) \leq C_c \sum_{k=0}^{n} \Big(\frac{\gamma_c \rho_0}{2}\Big)^{n-k} \frac{[k-4]!}{[k]!} \Big(\frac{\rho_0}{2}\Big)^4 \breve{N}_{\rho_0,k-4}(u), \tag{4.5.3}$$

$$\breve{M}_{\rho_0,n}(b\Delta u) \leq C_b \sum_{k=0}^{n} \Big(\frac{\gamma_b \rho_0}{2}\Big)^{n-k} \frac{[k-2]!}{[k]!} \Big(\frac{\rho_0}{2}\Big)^2 \breve{N}_{\rho_0,k-2}(u). \tag{4.5.4}$$

Proof. With the aid of Leibniz's formula, (the proof can be found in [53, Lemma 5.7.4.])

$$|\nabla^n(cu)| \leq \sum_{k=0}^{n} |\nabla^{n-k} c||\nabla^k u|,$$

we have,

$$\begin{aligned} &\breve{M}_{\rho_0,n}(cu) = \\ &= \frac{1}{[n]!} \sup_{\rho_0/2 \leq \rho < \rho_0} (\rho_0 - \rho)^{n+4} \|\nabla^n(cu)\|_{L^2(B_{\rho_0})} \\ &\leq \frac{1}{[n]!} \sup_{\rho_0/2 \leq \rho < \rho_0} (\rho_0 - \rho)^{n+4} \sum_{k=0}^{n} \binom{n}{k} \|\nabla^{n-k} c\|_{L^\infty(B_{\rho_0})} \|\nabla^k u\|_{L^2(B_{\rho_0})} \\ &\leq \frac{1}{[n]!} \sup_{\rho_0/2 \leq \rho < \rho_0} (\rho_0 - \rho)^{n+4} C_c \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \gamma_c^{n-k} (n-k)! \frac{[k-4]!}{\sup_{\rho_0/2 \leq \rho < \rho_0} (\rho_0 - \rho)^k} \breve{N}_{\rho_0,k-4}(u) \\ &\leq C_c \sum_{k=0}^{n} \Big(\frac{\gamma_c \rho_0}{2}\Big)^{n-k} \frac{[k-4]!}{[k]!} \Big(\frac{\rho_0}{2}\Big)^4 \breve{N}_{\rho_0,k-4}(u). \end{aligned}$$

In a similar manner,

$$\breve{M}_{\rho_0,n}(b\Delta u) =$$

$$
\begin{aligned}
&=\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}\|\nabla^n(b\Delta u)\|_{L^2(B_{\rho_0})}\\
&\le\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}\sum_{k=0}^{n}\binom{n}{k}\|\nabla^{n-k}b\|_{L^\infty(B_{\rho_0})}\|\nabla^k\Delta u\|_{L^2(B_{\rho_0})}\\
&\le\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}C_b\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\gamma_b^{n-k}(n-k)!\frac{[k-2]!}{\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^k}\breve{N}_{\rho_0,k-2}(u)\\
&\le C_b\sum_{k=0}^{n}\left(\frac{\gamma_b\rho_0}{2}\right)^{n-k}\frac{[k-2]!}{[k]!}\left(\frac{\rho_0}{2}\right)^2\breve{N}_{\rho_0,k-2}(u).
\end{aligned}
$$

□

Lemma 4.5.5 *Let $0<\rho_0\le 1$ and the integer $n\ge 2$. Suppose that b, c and u are analytic functions defined on the half disc $\mathcal{H}_{\rho_0}$, where b and c satisfy the bounds (4.3.23). Then we have*

$$\breve{M}'_{\rho_0,n}(cu)\le C_c\sum_{k=0}^{n}\left(\frac{\gamma_c\rho_0}{2}\right)^{n-k}\frac{[k-4]!}{[k]!}\left(\frac{\rho_0}{2}\right)^4\breve{N}'_{\rho_0,k-4}(u), \tag{4.5.6}$$

$$\breve{M}'_{\rho_0,n}(b\Delta u)\le C_b\sum_{k=0}^{n}\left(\frac{\gamma_b\rho_0}{2}\right)^{n-k}\frac{[k-2]!}{[k]!}\left(\frac{\rho_0}{2}\right)^2\breve{N}'_{\rho_0,k-2}(u), \tag{4.5.7}$$

and

$$\left|\frac{\partial^{n_1+n_2}(bu)}{\partial x^{n_1}\partial y^{n_2}}\right|\le C_b\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}(n_1+n_2)^{n_1+n_2-k-l}\gamma_b^{n_1+n_2-k-l}\left|\frac{\partial^{k+l}u}{\partial x^k\partial y^l}\right|. \tag{4.5.8}$$

Proof. Using Leibniz's formula once more, we get

$$
\begin{aligned}
&\breve{M}'_{\rho_0,n}(cu)=\\
&=\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}\left\|\frac{\partial^n(cu)}{\partial x^n}\right\|_{L^2(B_{\rho_0})}\\
&\le\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}\sum_{k=0}^{n}\binom{n}{k}\left\|\frac{\partial^{n-k}c}{\partial x^{n-k}}\right\|_{L^\infty(B_{\rho_0})}\left\|\frac{\partial^k u}{\partial x^k}\right\|_{L^2(B_{\rho_0})}\\
&\le\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}C_c\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\gamma_c^{n-k}(n-k)!\frac{[k-4]!}{\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^k}\breve{N}'_{\rho_0,k-4}(u)\\
&\le C_c\sum_{k=0}^{n}\left(\frac{\gamma_c\rho_0}{2}\right)^{n-k}\frac{[k-4]!}{[k]!}\left(\frac{\rho_0}{2}\right)^4\breve{N}'_{\rho_0,k-4}(u).
\end{aligned}
$$

In a similar manner,

$$\breve{M}'_{\rho_0,n}(b\Delta u)=$$

$$
\begin{aligned}
=&\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}\left\|\frac{\partial^n(b\Delta u)}{\partial x^n}\right\|_{L^2(B_{\rho_0})}\\
\le&\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}\sum_{k=0}^{n}\binom{n}{k}\left\|\frac{\partial^{n-k}b}{\partial x^{n-k}}\right\|_{L^\infty(B_{\rho_0})}\left\|\frac{\partial^k(\Delta u)}{\partial x^k}\right\|_{L^2(B_{\rho_0})}\\
\le&\frac{1}{[n]!}\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^{n+4}C_b\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}\gamma_b^{n-k}(n-k)!\frac{[k-2]!}{\sup_{\rho_0/2\le\rho<\rho_0}(\rho_0-\rho)^k}\breve{N}'_{\rho_0,k-2}(u)\\
\le&C_b\sum_{k=0}^{n}\left(\frac{\gamma_b\rho_0}{2}\right)^{n-k}\frac{[k-2]!}{[k]!}\left(\frac{\rho_0}{2}\right)^2\breve{N}'_{\rho_0,k-2}(u).
\end{aligned}
$$

To obtain the third bound we recall the assumptions on b and we apply again Leibniz's formula:

$$
\begin{aligned}
\left|\frac{\partial^{n_1+n_2}(bu)}{\partial x^{n_1}\partial y^{n_2}}\right|\le&\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\binom{n_1}{k}\binom{n_2}{l}\left|\frac{\partial^{k+l}(bu)}{\partial x^k\partial y^l}\right|\\
\le&C_b\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\binom{n_1}{k}\binom{n_2}{l}(n_1-k)!(n_2-l)!\gamma_b^{n_1+n_2-k-l}\left|\frac{\partial^{k+l}u}{\partial x^k\partial y^l}\right|\\
\le&C_b\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}\frac{n_1!}{k!}\frac{n_2!}{l!}\gamma_b^{n_1+n_2-k-l}\left|\frac{\partial^{k+l}u}{\partial x^k\partial y^l}\right|\\
\le&C_b\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}n_1^{n_1-k}n_2^{n_2-l}\gamma_b^{n_1+n_2-k-l}\left|\frac{\partial^{k+l}u}{\partial x^k\partial y^l}\right|\\
\le&C_b\sum_{k=0}^{n_1}\sum_{l=0}^{n_2}(n_1+n_2)^{n_1+n_2-k-l}\gamma_b^{n_1+n_2-k-l}\left|\frac{\partial^{k+l}u}{\partial x^k\partial y^l}\right|
\end{aligned}
$$

□

Lemma 4.5.9 *Let* $0<\rho_0\le 1$ *and* $u,f\in C^\infty(B_{\rho_0})$. *For each* $\rho_0/2<\rho\le\rho+\delta<\rho_0$, $0<\delta\le\rho$ *the following inequalities hold:*

$$
\begin{aligned}
\int_{B_{\rho+\delta}}|\nabla^n f|^2dx&\le\big(n!\big)^2(\rho_0-\rho-\delta)^{-2(n+4)}\big[\breve{M}_{\rho_0,n}(f)\big]^2, \quad n\ge 1,\\
\int_{B_{\rho+\delta}}|\nabla^{n+1}u|^2dx&\le\big[(n-3)!\big]^2(\rho_0-\rho-\delta)^{-2(n+1)}\big[\breve{N}_{\rho_0,n-3}(u)\big]^2, \quad n\ge 3,\\
\int_{B_{\rho+\delta}}|\nabla^{n+2}u|^2dx&\le\big[(n-2)!\big]^2(\rho_0-\rho-\delta)^{-2(n+2)}\big[\breve{N}_{\rho_0,n-2}(u)\big]^2, \quad n\ge 2,\\
\int_{B_{\rho+\delta}}|\nabla^{n+3}u|^2dx&\le\big[(n-1)!\big]^2(\rho_0-\rho-\delta)^{-2(n+3)}\big[\breve{N}_{\rho_0,n-1}(u)\big]^2, \quad n\ge 1.
\end{aligned}
\tag{4.5.10}
$$

Lemma 4.5.11 *Suppose* $0 < \rho_0 \leq 1$ *and* $u \in C^\infty(\mathcal{H}_{\rho_0})$*. Then,*

$$
\begin{aligned}
|\nabla^3\nabla_x^n u|^2 &\leq |\nabla^4\nabla_x^{n-1}u|^2, \quad n \geq 1; \qquad |\nabla^3\nabla_x^n u|^2 = |\nabla^3 u|^2, \quad n = 0,\\
|\nabla^2\nabla_x^n u|^2 &\leq |\nabla^4\nabla_x^{n-2}u|^2, \quad n \geq 2; \qquad |\nabla^2\nabla_x^n u|^2 \leq |\nabla^{2+n} u|^2, \quad n = 0,1,\\
|\nabla\nabla_x^n u|^2 &\leq |\nabla^4\nabla_x^{n-3}u|^2, \quad n \geq 3; \qquad |\nabla\nabla_x^n u|^2 \leq |\nabla^{n+1} u|^2, \quad n = 0,1,2,
\end{aligned}
\tag{4.5.12}
$$

hold true.

Lemma 4.5.13 *We suppose* $\varepsilon > 0$, $0 < \rho_0 \leq 1$ *and we let* $n \geq 2$ *be an integer. Then we have*

$$
\begin{aligned}
&\frac{n}{n}\frac{\max\{[n-1]^{n+3}, \big(\rho_0/\varepsilon\big)^{n+2}\}}{[n-1]!} \leq \frac{1}{n!}\max\{[n]^{n+4}, \big(\rho_0/\varepsilon\big)^{n+3}\},\\
&\frac{n(n-1)}{n(n-1)}\frac{\max\{[n-2]^{n+2}, \big(\rho_0/\varepsilon\big)^{n+1}\}}{[n-2]!} \leq \frac{1}{n!}\max\{[n]^{n+4}, \big(\rho_0/\varepsilon\big)^{n+3}\},\\
&\frac{n(n-1)(n-2)}{n(n-1)(n-2)}\frac{\max\{[n-3]^{n+1}, \big(\rho_0/\varepsilon\big)^{n}\}}{[n-3]!} \leq \frac{1}{n!}\max\{[n]^{n+4}, \big(\rho_0/\varepsilon\big)^{n+3}\},\\
&\frac{n(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)}\frac{\max\{[n-4]^{n}, \big(\rho_0/\varepsilon\big)^{n-1}\}}{[n-4]!} \leq \frac{1}{n!}\max\{[n]^{n+4}, \big(\rho_0/\varepsilon\big)^{n+3}\},\\
&\varepsilon^{-2}\rho_0^2\frac{1}{[n]!}\frac{[n]!}{[k]!}\max\{[k-2]^{k+2}, \big(\rho_0/\varepsilon\big)^{k+1}\} \leq \frac{1}{n!}\max\{[n]^{n+4}, \big(\rho_0/\varepsilon\big)^{n+3}\},\\
&\varepsilon^{-2}\rho_0^4\frac{1}{[n]!}\frac{[n]!}{[k]!}\max\{[k-4]^{k}, \big(\rho_0/\varepsilon\big)^{k-1}\} \leq \frac{1}{n!}\max\{[n]^{n+4}, \big(\rho_0/\varepsilon\big)^{n+3}\}.
\end{aligned}
\tag{4.5.14}
$$

Lemma 4.5.15 *For the same hypotheses of Lemma 4.5.13, we get*

$$
n^n + \max\{\big(\rho_0 n\big)^n, \big(\rho_0/\varepsilon\big)^{n-1}\} \leq 2\max\{n^n, \big(\rho_0/\varepsilon\big)^{n-1}\}, \tag{4.5.16}
$$

and

$$
(\rho_0/\varepsilon)^4 \leq \min\{1, (\rho_0/\varepsilon)^4\}\max\{n, (\rho_0/\varepsilon)^4\}. \tag{4.5.17}
$$

Chapter 5

2-D *hp*-approximation results

In the present chapter, we construct tools for the *hp*–FEM approximation to solutions of singularly perturbed problems, such as the one studied in Chapter 4. In particular, we construct hierarchical basis functions on the reference square $S = (-1, 1)^2$, and study their approximation properties. With the help of the interpolation operator of Chapter 3, we define an appropriate interpolant by tensor product, and obtain bounds on the interpolation error, for the reference square. Since the mesh contains, more often than not, several elements, we also address the issue of inter-element C^1 continuity. Specifically, we provide appropriate *lifting* results, which are meant to correct the discontinuous approximation. By liftings, we mean polynomial extensions of polynomial traces, which allow us to construct, from local approximations, a globally C^1 continuous piecewise polynomial approximation without degrading the error estimate [12].

When the domain is curved, one has to construct elements with (at least some) curved edges. Unfortunately, it is not possible to construct a C^1 continuous approximation on curved elements. In Chapter 6 we address this issue with an alternative (mixed) formulation of our problem. Here, however, we consider the case of a square domain (hence no curved elements) and we make an Assumption on the regularity of the solution, which is meant to eliminate the (possible) presence of corner singularities and at the same time allow us to focus on the *hp* approximation of boundary layers. The *Spectral Boundary Layer mesh*, we've seen before in one dimension, is extended to two dimensions, and our main result (Theorem 5.3.12) gives robust, exponential convergence in the energy norm.

At the end of the chapter, we present numerical results to corroborate the theory.

We consider again the variational problem under study: Find $u \in H_0^2(\Omega)$ such that

$$B_\varepsilon(u,v) = F(v), \quad \text{for all } v \in H_0^2(\Omega). \tag{5.0.1}$$

The bilinear form B_ε and the linear functional F are defined as

$$B_\varepsilon(u,v) = \varepsilon^2\langle \Delta u, \Delta v\rangle_\Omega + b\langle \nabla u, \nabla v\rangle_\Omega + c\langle u, v\rangle_\Omega \tag{5.0.2}$$

and

$$F(v) = \langle f, v\rangle_\Omega.$$

The energy norm is defined as $\|u\|_{E,\Omega} = \Big(B_\varepsilon(u,u)\Big)^{1/2}$.

As usual, in order to establish an approximation for the weak solution of problem (5.0.1), we replace the Hilbert space $H_0^2(\Omega)$ by a finite dimensional subspace $V_N \subseteq H_0^2(\Omega)$. This subspace is comprised of piecewise differentiable polynomials of a fixed degree associated with a subdivision of the computational domain. We note that the parameter N is related to the discretization of the domain. Consequently, the variational problem is replaced by the following discrete problem: Find $u_N \in V_N$ such that

$$B_\varepsilon(u_N, v_N) = F(v_N), \quad \text{for all } v_N \in V_N. \tag{5.0.3}$$

From C*é*a's lemma, it follows that the finite element approximation u_N to the weak solution $u \in H_0^2(\Omega)$, is the best approximation to u in the energy norm, i.e.

$$\|u - u_N\|_{E,\Omega} \leq \inf_{v_N \in V_N} \|u - v_N\|_{E,\Omega}, \tag{5.0.4}$$

for some constant $C > 0$ independent of f and V_N.

5.1 C^1-Basis functions on the reference element

To describe the approximation u_N it is necessary to construct a basis for the subspace V_N. The basis gives us the ability to define the subspace V_N (i.e. the *Finite Element Space*) and consequently the solution to the discrete problem. Therefore, our task in this subsection is the complete description of that set which makes up the basis functions defined in two dimensions.

We will construct our mesh using quadrilaterals, therefore we begin our analysis by considering the four-node *Bogner-Fox-Schmit* element which is presented in Figure 5.1. This element has sixteen (16) degrees of freedom, four at each vertex node. The degrees of freedom are associated with the values of the solution and its first and second order mixed partial derivatives at the nodes. Therefore we need to choose proper basis functions in order to control the values of the approximation and its derivatives at each vertex node. Those functions must have continuous first order partial derivatives and we must ensure that they possess a hierarhical character. For the corresponding problem in one dimension, a family of functions which satisfies these conditions, has already been introduced in Section 3.1 of Chapter 3. See also [57].

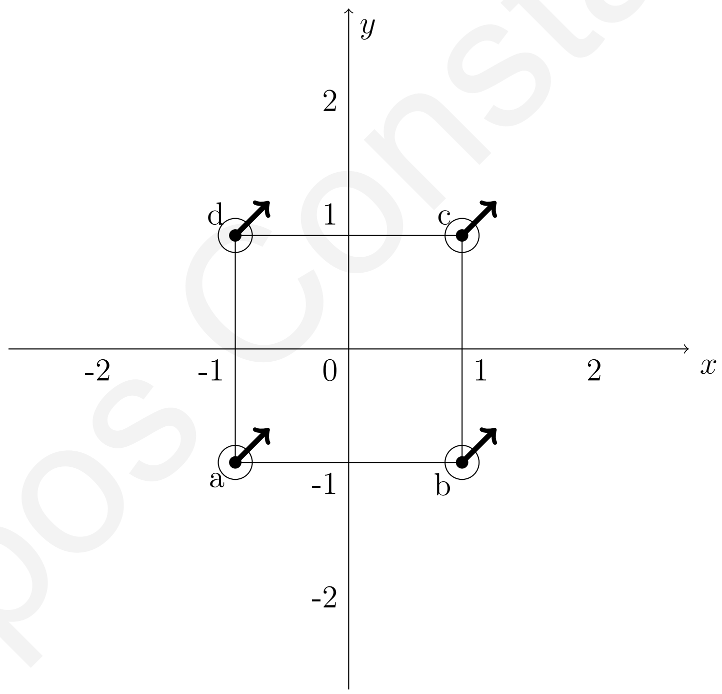


Figure 5.1: Bogner-Fox-Schmit Element

We first examine the four cubic *Hermite* polynomials given by (3.1.4)–(3.1.7). By utilizing these four functions one can describe precisely the *Bogner-Fox-Schmit* element. This occurs by the construction of a two dimensional basis with the aid of tensor products among the *Hermite* polynomials. To improve the perfomance, the extension of the *Bogner-Fox-Schmit* element to a hierarchical high-order element is needed. The extension is achieved through the usage of the polynomials given by (3.1.9) which control the internal values.

We seek to construct hierarchical basis functions that are defined on the reference square $S = \{(\xi, \eta)| -1 \le \xi, \eta \le 1\}$. For $-1 \le \xi, \eta \le 1$ the natural choice of the family $\{h_i(\xi) \times h_j(\eta)\}_{i,j}$ gives us the C^1-basis we seek.

As a result, we define a hierarchical set of basis functions in two dimensions (2D), (by taking tensor products of the one dimensional (1D) basis presented in Chapter 3) and we subdivide them into three categories: *vertex, edge* and *face* (we mention that here we follow the idea given in [29]).

Vertex modes (Nodal basis functions)

First we define the nodal basis functions by using the tensor product among the cubic *Hermite* polynomials. There are sixteen (16) such cases which describe exactly the *Bogner-Fox-Schmit* element. Their formula is given by

$$\phi_i^j(\xi, \eta) = h_{p_1}(\xi) h_{p_2}(\eta), \; i = 1, 2, 3, 4, \; j = a, b, c, d, \tag{5.1.1}$$

where p_1 and p_2 are polynomial degrees and their relation is presented in Table 5.1 below.

Table 5.1: Indices p_1 and p_2 for the nodal basis functions

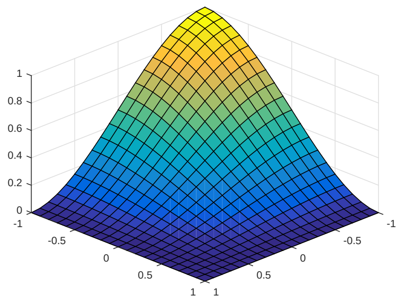
ϕ	ϕ_1^a	ϕ_2^a	ϕ_3^a	ϕ_4^a	ϕ_1^b	ϕ_2^b	ϕ_3^b	ϕ_4^b	ϕ_1^c	ϕ_2^c	ϕ_3^c	ϕ_4^c	ϕ_1^d	ϕ_2^d	ϕ_3^d	ϕ_4^d
p_1	1	1	2	2	3	3	4	4	3	3	4	4	1	1	2	2
p_2	1	2	1	2	1	2	1	2	3	4	3	4	3	4	3	4

As shown in Table 5.1 four nodal basis functions are associated with each vertex and each function matches either the solution, the first order partial derivatives, or the second order mixed derivative.

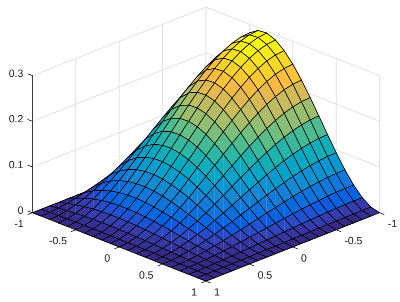
Figure 5.2 displays the corresponding functions which are associated with the vertex at the point a and Figure 5.3 is related with the vertex at d. Their values at the vertex points are essential for our construction.

Edge Modes (Side basis functions)

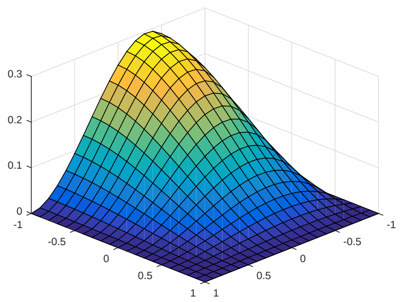
We now proceed with the side basis functions, using polynomials of degree p in each direction. There are $8(p-3)$ such functions and for $i = 1, \ldots, p-3$, we present their



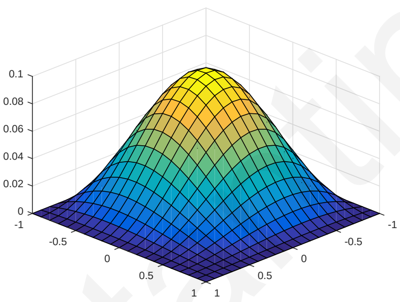
(a) $\phi_1^a(\xi,\eta) = h_1(\xi)h_1(\eta)$



(b) $\phi_2^a(\xi,\eta) = h_1(\xi)h_2(\eta)$



(c) $\phi_3^a(\xi,\eta) = h_2(\xi)h_1(\eta)$



(d) $\phi_4^a(\xi,\eta) = h_2(\xi)h_2(\eta)$

Figure 5.2: Nodal basis functions at vertex a

formulas at each side:

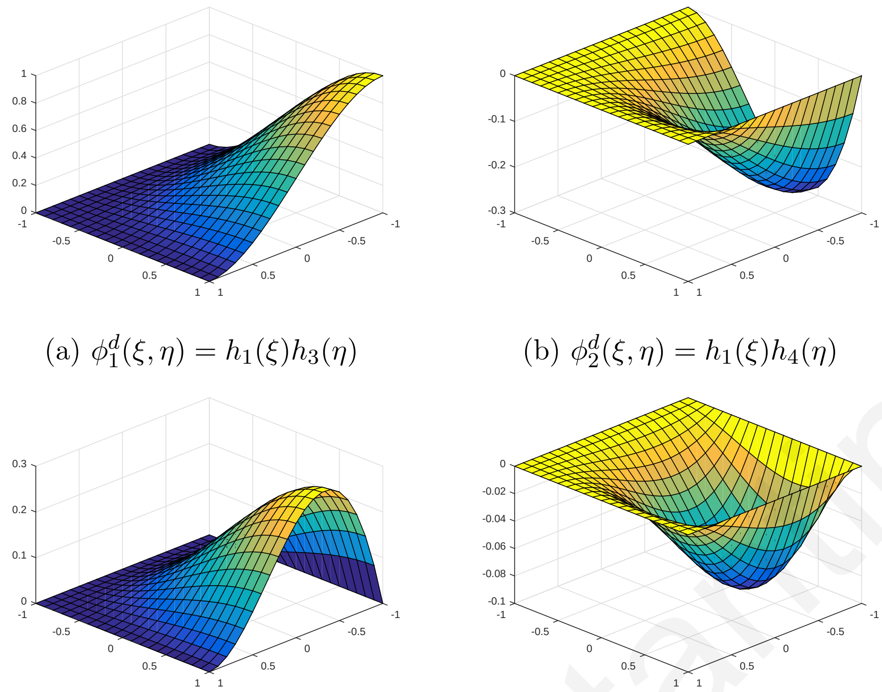
$$\textbf{Side ab:} \begin{cases} \phi_i^{ab_1}(\xi,\eta) = h_{i+4}(\xi)h_1(\eta) \\ \phi_i^{ab_2}(\xi,\eta) = h_{i+4}(\xi)h_2(\eta) \end{cases} \quad \textbf{Side bc:} \begin{cases} \phi_i^{bc_1}(\xi,\eta) = h_3(\xi)h_{i+4}(\eta) \\ \phi_i^{bc_2}(\xi,\eta) = h_4(\xi)h_{i+4}(\eta) \end{cases}$$

$$\textbf{Side cd:} \begin{cases} \phi_i^{cd_1}(\xi,\eta) = h_{i+4}(\xi)h_4(\eta) \\ \phi_i^{cd_2}(\xi,\eta) = h_{i+4}(\xi)h_3(\eta) \end{cases} \quad \textbf{Side da:} \begin{cases} \phi_i^{da_1}(\xi,\eta) = h_2(\xi)h_{i+4}(\eta) \\ \phi_i^{da_2}(\xi,\eta) = h_1(\xi)h_{i+4}(\eta) \end{cases}$$

Figure 5.4 shows some edge shape functions which are related with side ab. As can be observed from the first four figures, the application of such basis function gives us the means to control the values of the interpolant along side ab. Regarding the other four figures below, although it may not be clear, such basis functions let us determine the values of the first partial derivatives. In addition, Figure 5.5 displays the analogous edge shape functions related to side bc.

Face Modes (Internal basis functions)

To complete the construction, we define the internal basis functions in order to control



(a) $\phi_1^d(\xi,\eta) = h_1(\xi)h_3(\eta)$ (b) $\phi_2^d(\xi,\eta) = h_1(\xi)h_4(\eta)$

(c) $\phi_3^d(\xi,\eta) = h_2(\xi)h_3(\eta)$ (d) $\phi_4^d(\xi,\eta) = h_2(\xi)h_4(\eta)$

Figure 5.3: Nodal basis functions at vertex d

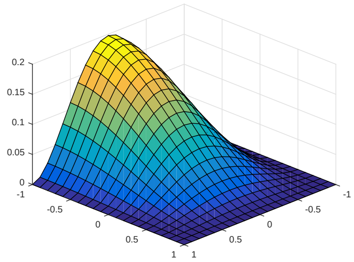
the internal values of the approximation, using polynomials of degree p in each direction. There are $(p-3)^2$ internal basis functions.They are given by the formula:

$$\phi_{i,j}(\xi,\eta) = h_{i+4}(\xi)h_{j+4}(\eta), \quad i,j = 1,\ldots,p-3. \tag{5.1.2}$$

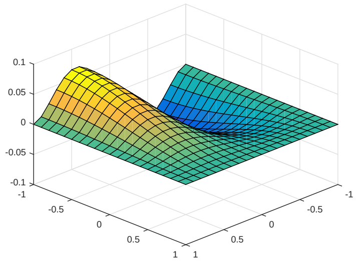
The reader may examine some of the face functions in Figure 5.6. The edge functions along with the face modes complete the description of the hierarchical high order square element. By the definitions of the side and internal functions, it is clear that they appear for polynomials of order strictly greater than 3. Since the basis functions have been fully determined, it is not hard to deduce that the approximation of the solution is given as

$$u_p(\xi,\eta) = \sum_{k=1}^{16} \alpha_k\phi_k(\xi,\eta) + \sum_{l=1}^{8(p-3)} \beta_l\phi_l(\xi,\eta) + \sum_{m=1}^{(p-3)^2} \gamma_m\phi_m(\xi,\eta), \tag{5.1.3}$$

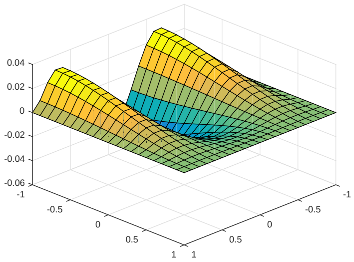
for some constants α_k, β_l and γ_m.



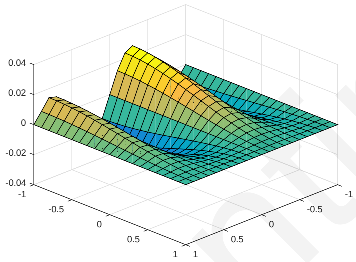
(a) $\phi_1^{ab_1}(\xi,\eta) = h_5(\xi)h_1(\eta)$



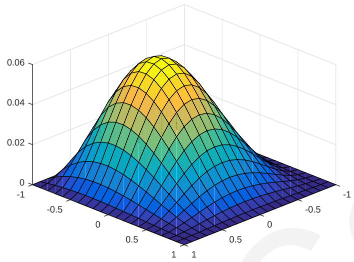
(b) $\phi_2^{ab_1}(\xi,\eta) = h_6(\xi)h_1(\eta)$



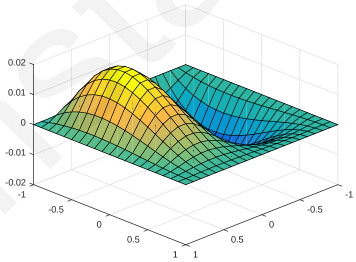
(c) $\phi_3^{ab_1}(\xi,\eta) = h_7(\xi)h_1(\eta)$



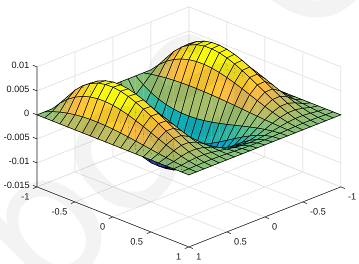
(d) $\phi_4^{ab_1}(\xi,\eta) = h_8(\xi)h_1(\eta)$



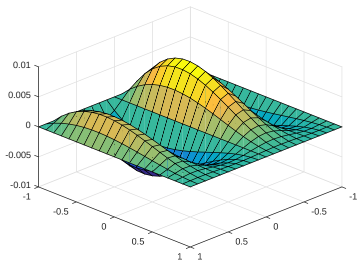
(e) $\phi_1^{ab_2}(\xi,\eta) = h_5(\xi)h_2(\eta)$



(f) $\phi_2^{ab_2}(\xi,\eta) = h_6(\xi)h_2(\eta)$

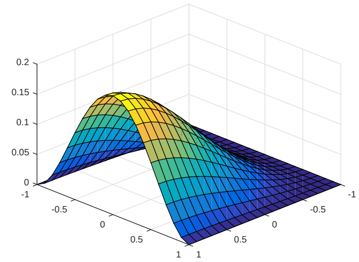


(g) $\phi_3^{ab_2}(\xi,\eta) = h_7(\xi)h_2(\eta)$

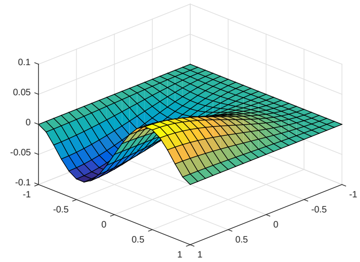


(h) $\phi_4^{ab_2}(\xi,\eta) = h_8(\xi)h_2(\eta)$

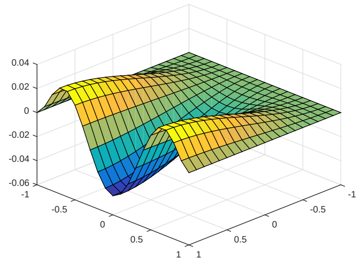
Figure 5.4: Side basis functions at side ab



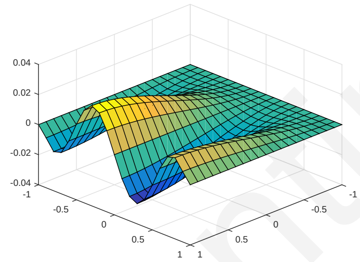
(a) $\phi_1^{bc_1}(\xi,\eta) = h_3(\xi)h_5(\eta)$



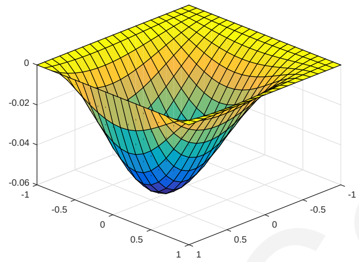
(b) $\phi_2^{bc_1}(\xi,\eta) = h_3(\xi)h_6(\eta)$



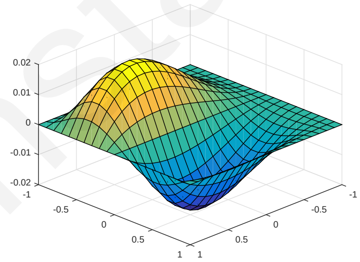
(c) $\phi_3^{bc_1}(\xi,\eta) = h_3(\xi)h_7(\eta)$



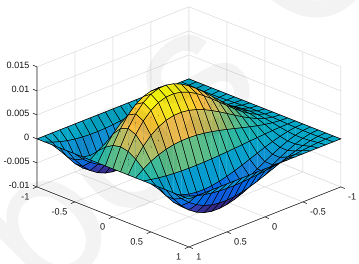
(d) $\phi_4^{bc_1}(\xi,\eta) = h_3(\xi)h_8(\eta)$



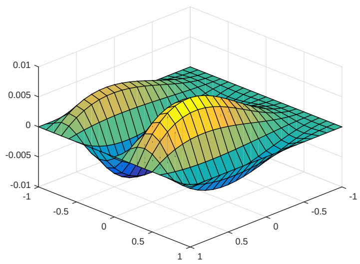
(e) $\phi_1^{bc_2}(\xi,\eta) = h_4(\xi)h_5(\eta)$



(f) $\phi_2^{bc_2}(\xi,\eta) = h_4(\xi)h_6(\eta)$

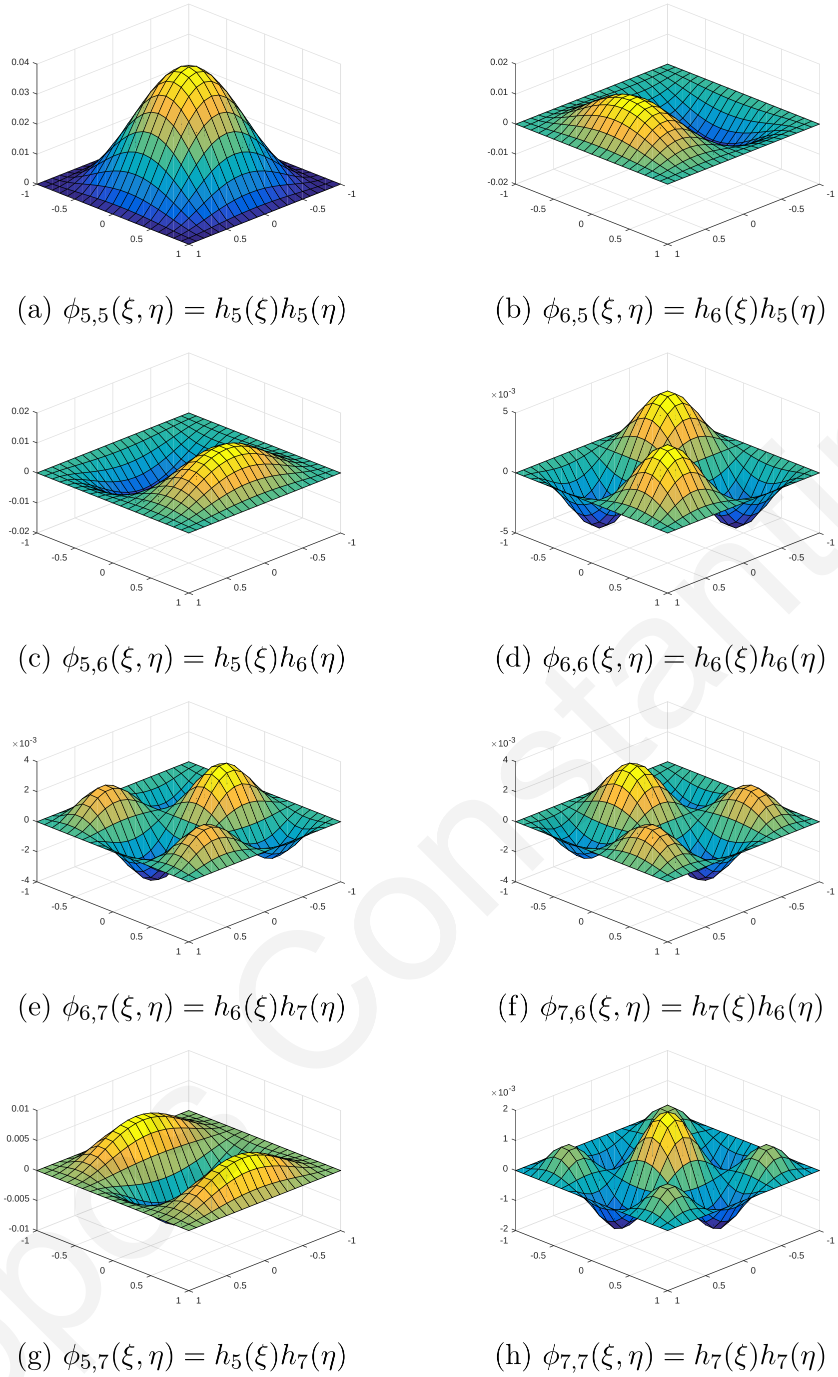


(g) $\phi_3^{bc_2}(\xi,\eta) = h_4(\xi)h_7(\eta)$



(h) $\phi_4^{bc_2}(\xi,\eta) = h_4(\xi)h_8(\eta)$

Figure 5.5: Side basis functions at side bc



(a) $\phi_{5,5}(\xi,\eta) = h_5(\xi)h_5(\eta)$ (b) $\phi_{6,5}(\xi,\eta) = h_6(\xi)h_5(\eta)$

(c) $\phi_{5,6}(\xi,\eta) = h_5(\xi)h_6(\eta)$ (d) $\phi_{6,6}(\xi,\eta) = h_6(\xi)h_6(\eta)$

(e) $\phi_{6,7}(\xi,\eta) = h_6(\xi)h_7(\eta)$ (f) $\phi_{7,6}(\xi,\eta) = h_7(\xi)h_6(\eta)$

(g) $\phi_{5,7}(\xi,\eta) = h_5(\xi)h_7(\eta)$ (h) $\phi_{7,7}(\xi,\eta) = h_7(\xi)h_7(\eta)$

Figure 5.6: Some internal basis functions

5.2 Approximation results on the unit square

The following Lemmas are a simplification of [34, Lemma 3.1], [34, Lemma 3.2] and [34, Lemma 3.6], respectively, for $m = 2$, hence we omit their proofs. In order to be consistent with the publication [34], we will give the following results on the square $(0, 1) \times (0, 1)$ instead of the reference square.

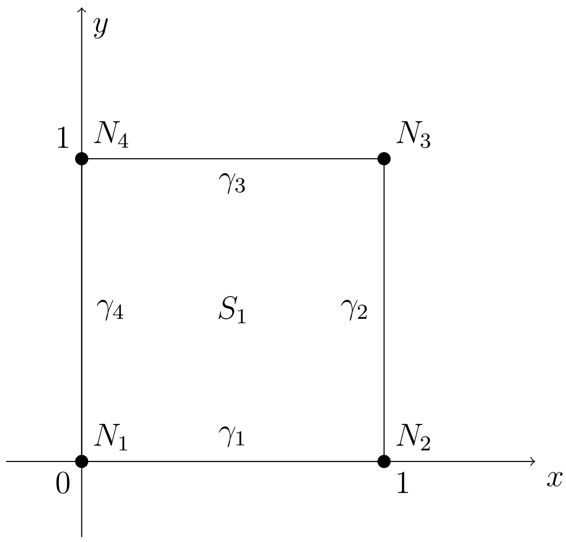


Figure 5.7: Reference square S_1 and its notations

As aforementioned, we denote the square $(0,1) \times (0,1)$ and its sides by S_1 and γ_i respectively. Also, we denote its whole boundary by γ (see Figure 5.7). The usual Sobolev space on γ_i is denoted by $H^k(\gamma_i)$. Let Q be a subset of $\{1,2,3,4\}$ and define $H^k(\gamma) = \Pi_{i\in Q} H^k(\gamma_i)$. Moreover set $\tilde{\gamma} = \{(x,0)\,|\,0 < x < 1\}$.

Lemma 5.2.1 *Let $S_1 = (0,1)^2$ and suppose $u \in H^3(S_1)$ satisfies the boundary conditions*

$$\begin{aligned} u(0,0) = u(1,0) =& u(0,1) = u(1,1) = 0, \\ \frac{\partial u}{\partial \xi}(0,0) = \frac{\partial u}{\partial \xi}(1,0) =& \frac{\partial u}{\partial \xi}(0,1) = \frac{\partial u}{\partial \xi}(1,1) = 0, \\ \frac{\partial u}{\partial \eta}(0,0) = \frac{\partial u}{\partial \eta}(1,0) =& \frac{\partial u}{\partial \eta}(0,1) = \frac{\partial u}{\partial \eta}(1,1) = 0. \end{aligned}$$

Then $u \in H^2(\gamma)$ and

$$\|u\|_{H^2(\gamma)} \leq C|u|_{H^3(S_1)}. \tag{5.2.2}$$

Lemma 5.2.3 *Let $\psi_1(\xi), \psi_2(\xi)$ be polynomials of degree p on $\tilde{\gamma}$ that satisfy*

$$\psi_1(0) = \psi_1(1) = \psi_2(0) = \psi_2(1) = 0,$$
$$\psi_1'(0) = \psi_1'(1) = \psi_2'(0) = \psi_2'(1) = 0.$$

Then there are polynomials Ψ_1, Ψ_2 of degree p in ξ and degree 3 in η such that

$$\Psi_1(\xi,\eta) = \begin{cases} \psi_1(\xi), & \text{for all } (\xi,\eta) \in \tilde{\gamma}, \\ 0, & \text{for } (\xi,\eta) \text{ belonging to the other sides of } S_1, \end{cases} \tag{5.2.4}$$

$$\frac{\partial \Psi_1}{\partial \xi}(\xi,\eta) = \begin{cases} \psi_1'(\xi), & \textit{for all } (\xi,\eta) \in \tilde{\gamma}, \\ 0, & \textit{for } (\xi,\eta) \textit{ belonging to the other side of } S_1, \end{cases} \tag{5.2.5}$$

$$\frac{\partial \Psi_1}{\partial \eta}(\xi,\eta) = 0, \qquad \textit{for } (\xi,\eta) \textit{belonging to each side of } S_1, \tag{5.2.6}$$

$$\Psi_2(\xi,\eta) = 0, \qquad \textit{for } (\xi,\eta) \textit{ belonging to each side of } S_1, \tag{5.2.7}$$

$$\frac{\partial \Psi_2}{\partial \xi}(\xi,\eta) = 0, \qquad \textit{for } (\xi,\eta) \textit{ belong to all other side of } S_1, \tag{5.2.8}$$

$$\frac{\partial \Psi_2}{\partial \eta}(\xi,\eta) = \begin{cases} \psi_2(\xi), & \textit{for all } (\xi,\eta) \in \tilde{\gamma}, \\ 0, & \textit{for } (\xi,\eta) \textit{ belonging to the other side of } S_1, \end{cases} \tag{5.2.9}$$

and

$$\|\Psi_1\|_{H^2(S_1)} \le C\|\psi_1\|_{H^2(\tilde{\gamma})}, \tag{5.2.10}$$

$$\|\Psi_2\|_{H^2(S_1)} \le C\|\psi_2\|_{H^2(\tilde{\gamma})}. \tag{5.2.11}$$

Proof. Define $\Psi_1(\xi,\eta) = \psi_1(x)(2\eta^3 - 3\eta^2 + 1)$ and $\Psi_2(\xi,\eta) = \psi_2(x)(\eta^3 - 2\eta^2 + \eta)$. It is transparent that the polynomials satisfy the conditions above. □

Next, we recall Proposition 3.2.1, since we seek to take advantage of its results and extend them to two dimensions with the help of a tensor product operator. Some calculations are needed, however it is not hard to deduce the 2D analogous results in the reference square which we present in Proposition 5.2.35 and Lemma 5.2.37. As mentioned before, the approximation results which are obtained by the tensor product of the one dimensional operator are not adequate to deduce the intended result. One must also surpass the difficulty of the interelement discontinuity, if the mesh consists of more than one element. Using a scaling argument (Lemma 5.2.43) and ideas from [34] we construct in Theorem 5.2.46 a C^1 piecewise polynomial that approximates the given function and we show bounds on the error.

We note that one can use the one dimensional operators $\mathcal{I}_p$ on a two dimensional function $u(\xi,\eta)$ with respect to ξ or η. Throughout the rest of this section we denote this by the symbols $\mathcal{I}_p^{(\xi)}$ or $\mathcal{I}_p^{(\eta)}$, respectively.

Lemma 5.2.12 *Let $u \in H^2(S)$ and let $p \geq 3$ be an integer. Then*

$$\left\| \frac{\partial^2 (u - \mathcal{I}_p^{(\xi)} u)}{\partial \xi^2} \right\|_{0,S}^2 \leq \frac{(p-\alpha_1)!}{(p-2+\alpha_1)!} \left\| \frac{\partial^{\alpha_1+1} u}{\partial \xi^{\alpha_1+1}} \right\|_{0,S}^2, \tag{5.2.13}$$

$$\left\| \frac{\partial^2 (u - \mathcal{I}_p^{(\eta)} u)}{\partial \eta^2} \right\|_{0,S}^2 \leq \frac{(p-\beta_1)!}{(p-2+\beta_1)!} \left\| \frac{\partial^{\beta_1+1} u}{\partial \eta^{\beta_1+1}} \right\|_{0,S}^2, \tag{5.2.14}$$

$$\left\| \frac{\partial^2 (u - \mathcal{I}_p^{(\xi)} u)}{\partial \eta^2} \right\|_{0,S}^2 \leq \frac{1}{(p-1)^4} \frac{(p-\alpha_3)!}{(p-2+\alpha_3)!} \left\| \frac{\partial^{\alpha_3+3} u}{\partial \xi^{\alpha_3+1} \partial \eta^2} \right\|_{0,S}^2, \tag{5.2.15}$$

$$\left\| \frac{\partial^2 (u - \mathcal{I}_p^{(\eta)} u)}{\partial \xi^2} \right\|_{0,S}^2 \leq \frac{1}{(p-1)^4} \frac{(p-\beta_3)!}{(p-2+\beta_3)!} \left\| \frac{\partial^{\beta_3+3} u}{\partial \xi^2 \partial \eta^{\beta_3+1}} \right\|_{0,S}^2, \tag{5.2.16}$$

$$\left\| \frac{\partial^2 (u - \mathcal{I}_p^{(\xi)} u)}{\partial \xi \partial \eta} \right\|_{0,S}^2 \leq \frac{1}{(p-1)^2} \frac{(p-\alpha_2)!}{(p-2+\alpha_2)!} \left\| \frac{\partial^{\alpha_2+2} u}{\partial \xi^{\alpha_2+1} \partial \eta} \right\|_{0,S}^2, \tag{5.2.17}$$

$$\left\| \frac{\partial^2 (u - \mathcal{I}_p^{(\eta)} u)}{\partial \xi \partial \eta} \right\|_{0,S}^2 \leq \frac{1}{(p-1)^2} \frac{(p-\beta_2)!}{(p-2+\beta_2)!} \left\| \frac{\partial^{\beta_2+2} u}{\partial \xi \partial \eta^{\beta_2+1}} \right\|_{0,S}^2, \tag{5.2.18}$$

$$\left\| \frac{\partial (u - \mathcal{I}_p^{(\xi)} u)}{\partial \xi} \right\|_{0,S}^2 \leq \frac{1}{(p-1)^2} \frac{(p-\alpha_2)!}{(p-2+\alpha_2)!} \left\| \frac{\partial^{\alpha_2+1} u}{\partial \xi^{\alpha_2+1}} \right\|_{0,S}^2, \tag{5.2.19}$$

$$\left\| \frac{\partial (u - \mathcal{I}_p^{(\eta)} u)}{\partial \eta} \right\|_{0,S}^2 \leq \frac{1}{(p-1)^2} \frac{(p-\beta_2)!}{(p-2+\beta_2)!} \left\| \frac{\partial^{\beta_2+1} u}{\partial \eta^{\beta_2+1}} \right\|_{0,S}^2, \tag{5.2.20}$$

$$\left\| \frac{\partial (u - \mathcal{I}_p^{(\xi)} u)}{\partial \eta} \right\|_{0,S}^2 \leq \frac{1}{(p-1)^4} \frac{(p-\alpha_3)!}{(p-2+\alpha_3)!} \left\| \frac{\partial^{\alpha_3+2} u}{\partial \xi^{\alpha_3+1} \partial \eta} \right\|_{0,S}^2, \tag{5.2.21}$$

$$\left\| \frac{\partial (u - \mathcal{I}_p^{(\eta)} u)}{\partial \xi} \right\|_{0,S}^2 \leq \frac{1}{(p-1)^4} \frac{(p-\beta_3)!}{(p-2+\beta_3)!} \left\| \frac{\partial^{\beta_3+2} u}{\partial \xi \partial \eta^{\beta_3+1}} \right\|_{0,S}^2, \tag{5.2.22}$$

$$\left\| u - \mathcal{I}_p^{(\xi)} u \right\|_{0,S}^2 \leq \frac{1}{(p-1)^4} \frac{(p-\alpha_3)!}{(p-2+\alpha_3)!} \left\| \frac{\partial^{\alpha_3+1} u}{\partial \xi^{\alpha_3+1}} \right\|_{0,S}^2, \tag{5.2.23}$$

$$\left\| u - \mathcal{I}_p^{(\eta)} u \right\|_{0,S}^2 \leq \frac{1}{(p-1)^4} \frac{(p-\beta_3)!}{(p-2+\beta_3)!} \left\| \frac{\partial^{\beta_3+1} u}{\partial \eta^{\beta_3+1}} \right\|_{0,S}^2, \tag{5.2.24}$$

for any integers $1 \leq \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \leq p$.

Proof. One can easily deduce the above estimates with the aid of Proposition 3.2.1.

For instance, to show (5.2.13) we perform the calculations

$$\left|\frac{\partial^2(u-\mathcal{I}_p^{(\xi)}u)}{\partial\xi^2}\right|_{0,S}^2 = \int_{-1}^1\int_{-1}^1\left|\frac{\partial^2(u-\mathcal{I}_p^{(\xi)}u)}{\partial\xi^2}\right|^2 dxdy$$

$$= \int_{-1}^1\left\|\frac{\partial^2}{\partial x^2}u(\cdot,y)-\frac{\partial^2}{\partial x^2}\mathcal{I}_p^{(\xi)}u(\cdot,y)\right\|_{0,[-1,1]}^2 dy \leq \frac{(p-\alpha_1)!}{(p-2+\alpha_1)!}\left\|\frac{\partial^{\alpha_1+1}u}{\partial^{\alpha_1+1}\xi}\right\|_{0,S}^2,$$

where (3.2.3) has been used. One can show all other estimates in a similar way. □

Lemma 5.2.25 *We assume the same conditions as in Lemma 5.2.12. Then,*

$$\left|u-\mathcal{I}_p^{(\xi)}u\right|_{2,S}^2 \leq \frac{(p-\alpha_1)!}{(p-2+\alpha_1)!}\left\|\frac{\partial^{\alpha_1+1}u}{\partial\xi^{\alpha_1+1}}\right\|_{0,S}^2 + \frac{1}{(p-1)^4}\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+3}u}{\partial\xi^{\alpha_3+1}\partial\eta^2}\right\|_{0,S}^2 + \\ + \frac{2}{(p-1)^2}\frac{(p-\alpha_2)!}{(p-2+\alpha_2)!}\left\|\frac{\partial^{\alpha_2+2}u}{\partial\xi^{\alpha_2+1}\partial\eta}\right\|_{0,S}^2, \tag{5.2.26}$$

$$\left|u-\mathcal{I}_p^{(\eta)}u\right|_{2,S}^2 \leq \frac{(p-\beta_1)!}{(p-2+\beta_1)!}\left\|\frac{\partial^{\beta_1+1}u}{\partial\eta^{\beta_1+1}}\right\|_{0,S}^2 + \frac{1}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+3}u}{\partial\xi^2\partial\eta^{\beta_3+1}}\right\|_{0,S}^2 + \\ + \frac{2}{(p-1)^2}\frac{(p-2-\beta_2)!}{(p-2+\beta_2)!}\left\|\frac{\partial^{\beta_2+2}u}{\partial\xi\partial\eta^{\beta_2+1}}\right\|_{0,S}^2, \tag{5.2.27}$$

$$\left|u-\mathcal{I}_p^{(\xi)}u\right|_{1,S}^2 \leq \frac{1}{(p-1)^2}\frac{(p-\alpha_2)!}{(p-2+\alpha_2)!}\left\|\frac{\partial^{\alpha_2+1}u}{\partial\xi^{\alpha_2+1}}\right\|_{0,S}^2 + \frac{1}{(p-1)^4}\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+2}u}{\partial\xi^{\alpha_3+1}\partial\eta}\right\|_{0,S}^2, \tag{5.2.28}$$

$$\left|u-\mathcal{I}_p^{(\eta)}u\right|_{1,S}^2 \leq \frac{1}{(p-1)^2}\frac{(p-\beta_2)!}{(p-2+\beta_2)!}\left\|\frac{\partial^{\beta_2+1}u}{\partial\eta^{\beta_2+1}}\right\|_{0,S}^2 + \frac{1}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+2}u}{\partial\xi\partial\eta^{\beta_3+1}}\right\|_{0,S}^2, \tag{5.2.29}$$

for any integers $1 \leq \alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3 \leq p$.

Proof. We recall that

$$\left|u-\mathcal{I}_p^{(\xi)}u\right|_{2,S}^2 \leq \left|\frac{\partial^2(u-\mathcal{I}_p^{(\xi)}u)}{\partial\xi^2}\right|_{0,S}^2 + \left|\frac{\partial^2(u-\mathcal{I}_p^{(\xi)}u)}{\partial\eta^2}\right|_{0,S}^2 + 2\left|\frac{\partial^2(u-\mathcal{I}_p^{(\xi)}u)}{\partial\xi\partial\eta}\right|_{0,S}^2, \tag{5.2.30}$$

and

$$\left|u-\mathcal{I}_p^{(\xi)}u\right|_{1,S}^2 \leq \left|\frac{\partial(u-\mathcal{I}_p^{(\xi)}u)}{\partial\xi}\right|_{0,S}^2 + \left|\frac{\partial(u-\mathcal{I}_p^{(\xi)}u)}{\partial\eta}\right|_{0,S}^2. \tag{5.2.31}$$

By appealing to Lemma 5.2.12, the desired results follow. □

We want to obtain appropriate bounds in the plane, therefore we need the existence of a two dimensional operator, which is defined for m, $n > 2$, as

$$\mathcal{J}_{m,n} = \mathcal{I}_m^{(\xi)} \otimes \mathcal{I}_n^{(\eta)}. \tag{5.2.32}$$

For simplicity's sake we use the notation $\mathcal{J}_m = \mathcal{J}_{m,m}$. We remark that this operator has been introduced in [63]. For $u \in H^2(S)$, $\mathcal{J}_m$ satisfies, for $k = 0, 1$, the following conditions:

$$\frac{\partial^k}{\partial \xi^k}\big(\mathcal{J}_m u(x_i, y_i)\big) = \frac{\partial^k}{\partial \xi^k}\big(u(x_i, y_i)\big) \tag{5.2.33}$$

and

$$\frac{\partial^k}{\partial \eta^k}\big(\mathcal{J}_m u(x_i, y_i)\big) = \frac{\partial^k}{\partial \eta^k}\big(u(x_i, y_i)\big), \tag{5.2.34}$$

where the vertices of S are denoted by (x_i, y_i).

Proposition 5.2.35 *Let $p > 3$ be integer and let $u \in H^2(S)$. For the integers $1 \le \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \le p$, the projector defined in* (5.2.32) *satisfies* (5.2.33), (5.2.34) *and the bounds below:*

$$\begin{aligned} \left|u - \mathcal{J}_p u\right|^2_{0,S} \le & \frac{1}{(p-1)^4}\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+1}u}{\partial \xi^{\alpha_3+1}}\right\|^2_{0,S} + \frac{1}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+1}u}{\partial \eta^{\beta_3+1}}\right\|^2_{0,S} + \\ & + \frac{1}{(p-1)^8}\frac{(p-\alpha_6)!}{(p-2+\alpha_6)!}\frac{(p-\beta_6)!}{(p-2+\beta_6)!}\left\|\frac{\partial^{\alpha_6+\beta_6+2}u}{\partial \xi^{\alpha_6+1}\partial \eta^{\beta_6+1}}\right\|^2_{0,S}, \end{aligned}$$

$$\begin{aligned} \left|u - \mathcal{J}_p u\right|^2_{1,S} \le & \frac{1}{(p-1)^2}\frac{(p-\alpha_2)!}{(p-2+\alpha_2)!}\left\|\frac{\partial^{\alpha_2+1}u}{\partial \xi^{\alpha_2+1}}\right\|^2_{0,S} + \frac{1}{(p-1)^4}\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+2}u}{\partial \xi^{\alpha_3+1}\partial \eta}\right\|^2_{0,S} + \\ & + \frac{1}{(p-1)^2}\frac{(p-\beta_2)!}{(p-2+\beta_2)!}\left\|\frac{\partial^{\beta_2+1}u}{\partial \eta^{\beta_2+1}}\right\|^2_{0,S} + \frac{1}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+2}u}{\partial \xi \partial \eta^{\beta_3+1}}\right\|^2_{0,S} + \\ & + \frac{1}{(p-1)^6}\frac{(p-\alpha_5)!}{(p-2+\alpha_5)!}\frac{(p-\beta_6)!}{(p-2+\beta_6)!}\left\|\frac{\partial^{\alpha_5+\beta_6+2}u}{\partial \xi^{\alpha_5+1}\partial \eta^{\beta_6+1}}\right\|^2_{0,S} + \\ & + \frac{1}{(p-1)^6}\frac{(p-\alpha_6)!}{(p-2+\alpha_6)!}\frac{(p-\beta_5)!}{(p-2+\beta_5)!}\left\|\frac{\partial^{\alpha_6+\beta_5+2}u}{\partial \xi^{\alpha_6+1}\partial \eta^{\beta_5+1}}\right\|^2_{0,S}, \end{aligned}$$

$$\begin{aligned} \left|u - \mathcal{J}_p u\right|^2_{2,S} \le & \frac{(p-\alpha_1)!}{(p-2+\alpha_1)!}\left\|\frac{\partial^{\alpha_1+1}u}{\partial \xi^{\alpha_1+1}}\right\|^2_{0,S} + \frac{1}{(p-1)^4}\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+3}u}{\partial \xi^{\alpha_3+1}\partial \eta^2}\right\|^2_{0,S} + \\ & + \frac{2}{(p-1)^2}\frac{(p-\alpha_2)!}{(p-2+\alpha_2)!}\left\|\frac{\partial^{\alpha_2+2}u}{\partial \xi^{\alpha_2+1}\partial \eta}\right\|^2_{0,S} + \frac{(p-\beta_1)!}{(p-2+\beta_1)!}\left\|\frac{\partial^{\beta_1+1}u}{\partial \eta^{\beta_1+1}}\right\|^2_{0,S} + \end{aligned}$$

$$+\frac{1}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+3}u}{\partial\xi^2\partial\eta^{\beta_3+1}}\right\|_{0,S}^2+\frac{2}{(p-1)^2}\frac{(p-\beta_2)!}{(p-2+\beta_2)!}\left\|\frac{\partial^{\beta_2+2}u}{\partial\xi\partial\eta^{\beta_2+1}}\right\|_{0,S}^2+$$
$$+\frac{1}{(p-1)^4}\frac{(p-\alpha_4)!}{(p-2+\alpha_4)!}\frac{(p-\beta_6)!}{(p-2+\beta_6)!}\left\|\frac{\partial^{\alpha_4+\beta_6+2}u}{\partial\xi^{\alpha_4+1}\partial^{\beta_6+1}\eta}\right\|_{0,S}^2+$$
$$+\frac{1}{(p-1)^4}\frac{(p-\alpha_6)!}{(p-2+\alpha_6)!}\frac{(p-\beta_4)!}{(p-2+\beta_4)!}\left\|\frac{\partial^{\alpha_6+\beta_4+2}u}{\partial\xi^{\alpha_6+1}\partial\eta^{\beta_4+1}}\right\|_{0,S}^2+$$
$$+\frac{2}{(p-1)^4}\frac{(p-\alpha_5)!}{(p-2+\alpha_5)!}\frac{(p-\beta_5)!}{(p-2+\beta_5)!}\left\|\frac{\partial^{\alpha_5+\beta_5+2}u}{\partial\xi^{\alpha_5+1}\partial\eta^{\beta_5+1}}\right\|_{0,S}^2.$$

Proof. For $i=0,1,2$ the use of triangle inequality leads to

$$\left|u-\mathcal{J}_p u\right|_{i,S}^2\le\left|u-\mathcal{I}_p^{(\xi)}u\right|_{i,S}^2+\left|u-\mathcal{I}_p^{(\eta)}u\right|_{i,S}^2+\left|(I-\mathcal{I}_p^{(\eta)})\otimes(I-\mathcal{I}_p^{(\xi)})u\right|_{i,S}^2.$$

We control the first two seminorms on the right hand side using the appropriate estimates of Lemmas 5.2.12 and 5.2.25. To bound the third seminorm we work as follows. We set $v=\big(I-\mathcal{I}_p^{(\xi)}\big)u$. Therefore

$$E_i:=\left|(I-\mathcal{I}_p^{(\eta)})\otimes(I-\mathcal{I}_p^{(\xi)})u\right|_{i,S}^2=\left|v-\mathcal{I}_p^{(\eta)}v\right|_{i,S}^2,\quad i=0,1,2.$$

With the aid of (5.2.24) and Proposition 3.2.1 we get

$$E_0\le\frac{1}{(p-1)^8}\frac{(p-\alpha_6)!}{(p-2+\alpha_6)!}\frac{(p-\beta_6)!}{(p-2+\beta_6)!}\left\|\frac{\partial^{\alpha_6+\beta_6+2}u}{\partial\xi^{\alpha_6+1}\partial\eta^{\beta_6+1}}\right\|_{0,S}^2.\tag{5.2.36}$$

We note that $\frac{\partial}{\partial y}$ and $\mathcal{I}_p^{(\xi)}$ commute and by using (5.2.29) and Proposition 3.2.1 we obtain

$$E_1\le\frac{1}{(p-1)^6}\frac{(p-\alpha_5)!}{(p-2+\alpha_5)!}\frac{(p-\beta_6)!}{(p-2+\beta_6)!}\left\|\frac{\partial^{\alpha_5+\beta_6+2}u}{\partial\xi^{\alpha_5+1}\partial\eta^{\beta_6+1}}\right\|_{0,S}^2+$$
$$+\frac{1}{(p-1)^6}\frac{(p-\alpha_6)!}{(p-2+\alpha_6)!}\frac{(p-\beta_5)!}{(p-2+\beta_5)!}\left\|\frac{\partial^{\alpha_6+\beta_5+2}u}{\partial\xi^{\alpha_6+1}\partial\eta^{\beta_5+1}}\right\|_{0,S}^2.$$

Similar calculations and using (5.2.27) and Proposition 3.2.1 lead to

$$E_2\le\frac{1}{(p-1)^4}\frac{(p-\alpha_4)!}{(p-2+\alpha_4)!}\frac{(p-\beta_6)!}{(p-2+\beta_6)!}\left\|\frac{\partial^{\alpha_4+\beta_6+2}u}{\partial\xi^{\alpha_4+1}\partial\eta^{\beta_6+1}}\right\|_{0,S}^2+$$
$$+\frac{1}{(p-1)^4}\frac{(p-\alpha_6)!}{(p-2+\alpha_6)!}\frac{(p-\beta_4)!}{(p-2+\beta_4)!}\left\|\frac{\partial^{\alpha_6+\beta_4+2}u}{\partial\xi^{\alpha_6+1}\partial\eta^{\beta_4+1}}\right\|_{0,S}^2+$$
$$+\frac{2}{(p-1)^4}\frac{(p-\alpha_5)!}{(p-2+\alpha_5)!}\frac{(p-\beta_5)!}{(p-2+\beta_5)!}\left\|\frac{\partial^{\alpha_5+\beta_5+2}u}{\partial\xi^{\alpha_5+1}\partial\eta^{\beta_5+1}}\right\|_{0,S}^2.$$

The estimates above for $E_i, i=0,1,2$ and Lemma 5.2.25 give the desired result. □

Lemma 5.2.37 *Suppose $p \geq 3$ is an integer, $u \in H^2(S)$ and suppose $\mathcal{J}_p$ is given by* (5.2.32)*. For $i = 1, 2, 3,$ and for the integers $1 \leq \alpha_i,\ \beta_i \leq p$, the conditions* (5.2.33)*,* (5.2.34) *and the following bounds hold:*

$$\begin{aligned}
\left\|u-\mathcal{J}_p u\right\|_{1,S}^2 \leq{}& \frac{1}{(p-1)^4}\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+1}u}{\partial\xi^{\alpha_3+1}}\right\|_{0,S}^2 + \frac{1}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+1}u}{\partial\eta^{\beta_3+1}}\right\|_{0,S}^2 + \\
&+ \frac{1}{(p-1)^2}\frac{(p-\alpha_2)!}{(p-2+\alpha_2)!}\left\|\frac{\partial^{\alpha_2+1}u}{\partial\xi^{\alpha_2+1}}\right\|_{0,S}^2 + \frac{1}{(p-1)^2}\frac{(p-\beta_2)!}{(p-2+\beta_2)!}\left\|\frac{\partial^{\beta_2+1}u}{\partial\eta^{\beta_2+1}}\right\|_{0,S}^2 + \\
&+ \frac{1}{(p-1)^4}\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+2}u}{\partial\xi^{\alpha_3+1}\partial\eta}\right\|_{0,S}^2 + \frac{1}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+2}u}{\partial\xi\partial\eta^{\beta_3+1}}\right\|_{0,S}^2 \\
&+ \frac{1}{(p-1)(p-2)}\left(\frac{2}{(p-1)^6}+\frac{1}{(p-1)^8}\right)\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+3}u}{\partial\xi^{\alpha_3+1}\partial\eta^2}\right\|_{0,S}^2,
\end{aligned}\tag{5.2.38}$$

$$\begin{aligned}
\left|u-\mathcal{J}_p u\right|_{2,S}^2 \leq{}& \frac{(p-\alpha_1)!}{(p-2+\alpha_1)!}\left\|\frac{\partial^{\alpha_1+1}u}{\partial\xi^{\alpha_1+1}}\right\|_{0,S}^2 + \frac{(p-\beta_1)!}{(p-2+\beta_1)!}\left\|\frac{\partial^{\beta_1+1}u}{\partial\eta^{\beta_1+1}}\right\|_{0,S}^2 + \\
&+ \frac{2}{(p-1)^2}\frac{(p-\alpha_2)!}{(p-2+\alpha_2)!}\left\|\frac{\partial^{\alpha_2+2}u}{\partial\xi^{\alpha_2+1}\partial\eta}\right\|_{0,S}^2 + \frac{2}{(p-1)^2}\frac{(p-\beta_2)!}{(p-2+\beta_2)!}\left\|\frac{\partial^{\beta_2+2}u}{\partial\xi\partial\eta^{\beta_2+1}}\right\|_{0,S}^2 + \\
&+ \left(1+\frac{3}{(p-1)(p-2)}\right)\frac{1}{(p-1)^4}\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+3}u}{\partial\xi^{\alpha_3+1}\partial\eta^2}\right\|_{0,S}^2 + \\
&+ \frac{1}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+3}u}{\partial\xi^2\partial\eta^{\beta_3+1}}\right\|_{0,S}^2,
\end{aligned}\tag{5.2.39}$$

and

$$\begin{aligned}
\left\|u-\mathcal{J}_p u\right\|_{E,S}^2 \leq{}& \varepsilon^2\left(\frac{(p-\alpha_1)!}{(p-2+\alpha_1)!}\left\|\frac{\partial^{\alpha_1+1}u}{\partial\xi^{\alpha_1+1}}\right\|_{0,S}^2 + \frac{(p-\beta_1)!}{(p-2+\beta_1)!}\left\|\frac{\partial^{\beta_1+1}u}{\partial\eta^{\beta_1+1}}\right\|_{0,S}^2\right) + \\
&+ \frac{1}{(p-1)^2}\left(\frac{(p-\alpha_2)!}{(p-2+\alpha_2)!}\left\|\frac{\partial^{\alpha_2+1}u}{\partial\xi^{\alpha_2+1}}\right\|_{0,S}^2 + \frac{(p-\beta_2)!}{(p-2+\beta_2)!}\left\|\frac{\partial^{\beta_2+1}u}{\partial\eta^{\beta_2+1}}\right\|_{0,S}^2\right) + \\
&+ \frac{2\varepsilon^2}{(p-1)^2}\left(\frac{(p-\alpha_2)!}{(p-2+\alpha_2)!}\left\|\frac{\partial^{\alpha_2+2}u}{\partial\xi^{\alpha_2+1}\partial\eta}\right\|_{0,S}^2 + \frac{(p-\beta_2)!}{(p-2+\beta_2)!}\left\|\frac{\partial^{\beta_2+2}u}{\partial\xi\partial\eta^{\beta_2+1}}\right\|_{0,S}^2\right) + \\
&+ \frac{1}{(p-1)^4}\left(\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+1}u}{\partial\xi^{\alpha_3+1}}\right\|_{0,S}^2 + \frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+1}u}{\partial\eta^{\beta_3+1}}\right\|_{0,S}^2\right) + \\
&+ \frac{1}{(p-1)^4}\left(\frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+2}u}{\partial\xi^{\alpha_3+1}\partial\eta}\right\|_{0,S}^2 + \frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+2}u}{\partial\xi\partial\eta^{\beta_3+1}}\right\|_{0,S}^2\right) + \\
&+ \left[\varepsilon^2\left(1+\frac{3}{(p-1)(p-2)}\right)\frac{1}{(p-1)^4} + \frac{1}{(p-1)(p-2)}\left(\frac{2}{(p-1)^6}+\frac{1}{(p-1)^8}\right)\right]\times \\
&\times \frac{(p-\alpha_3)!}{(p-2+\alpha_3)!}\left\|\frac{\partial^{\alpha_3+3}u}{\partial\xi^{\alpha_3+1}\partial\eta^2}\right\|_{0,S}^2 + \frac{\varepsilon^2}{(p-1)^4}\frac{(p-\beta_3)!}{(p-2+\beta_3)!}\left\|\frac{\partial^{\beta_3+3}u}{\partial\xi^2\partial\eta^{\beta_3+1}}\right\|_{0,S}^2.
\end{aligned}\tag{5.2.40}$$

Proof. The result is a direct consequence of the preceding proposition if one sets $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6$, $\beta_4 = \beta_5 = \beta_6 = 1$. □

Corollary 5.2.41 *Assume the conditions and notations of Lemma 5.2.37. For $1 \leq s \leq p$ we have*

$$\begin{aligned}\left\|u - \mathcal{J}_p u\right\|^2_{E,S} \leq & C\frac{(p-s)!}{(p-2+s)!}\left[\left\|\frac{\partial^{s+1}u}{\partial_\xi^{s+1}}\right\|^2_{0,S} + \left\|\frac{\partial^{s+2}u}{\partial_\xi^{s+1}\partial\eta}\right\|^2_{0,S} + \left\|\frac{\partial^{s+3}u}{\partial_\xi^{s+1}\partial\eta^2}\right\|^2_{0,S} + \right.\\ & \left. + \left\|\frac{\partial^{s+1}u}{\partial_\eta^{s+1}}\right\|^2_{0,S} + \left\|\frac{\partial^{s+2}u}{\partial\xi\partial_\eta^{s+1}}\right\|^2_{0,S} + \varepsilon^2\left\|\frac{\partial^{s+3}u}{\partial\xi^2\partial_\eta^{s+1}}\right\|^2_{0,S}\right],\end{aligned} \tag{5.2.42}$$

and (5.2.33), (5.2.34) *also hold.*

Proof. By setting $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = s$, the above result follows in a straightforward way from (5.2.40). □

We next extend the previous result to an arbitrary rectangle.

Lemma 5.2.43 *Let $p \geq 3$. Suppose $\mathcal{D} = (a,b) \times (c,d)$ with $h_1 = b - a$ and $h_2 = d - c$ and let $u \in H^3(\mathcal{D})$. Then there exists a polynomial $\psi = \mathcal{J}_m u$ of order p which satisfies, for $1 \leq s \leq p$, $k = 0, 1$,*

$$\frac{\partial^k}{\partial\xi^k}\big(\psi(x_i, y_i)\big) = \frac{\partial^k}{\partial\xi^k}\big(u(x_i, y_i)\big), \tag{5.2.44}$$

$$\frac{\partial^k}{\partial\eta^k}\big(\psi(x_i, y_i)\big) = \frac{\partial^k}{\partial\eta^k}\big(u(x_i, y_i)\big), \tag{5.2.45}$$

where (x_i, y_i) denotes the vertices of $\mathcal{D}$ and

$$\begin{aligned}\|u - \psi\|^2_{1,\mathcal{D}} \leq & C\frac{(p-s)!}{(p-2+s)!}h^{-2}\left[h_1^{2(s+1)}\left\|\frac{\partial^{s+1}u}{\partial_\xi^{s+1}}\right\|^2_{0,\mathcal{D}} + h_1^{2(s+1)}h_2\left\|\frac{\partial^{s+2}u}{\partial_\xi^{s+1}\partial\eta}\right\|^2_{0,\mathcal{D}} + \right.\\ & + h_1^{2(s+1)}h_2^3\left\|\frac{\partial^{s+3}u}{\partial_\xi^{s+1}\partial\eta^2}\right\|^2_{0,\mathcal{D}} + h_2^{2(s+1)}\left\|\frac{\partial^{s+1}u}{\partial_\eta^{s+1}}\right\|^2_{0,\mathcal{D}} + h_1 h_2^{2(s+1)}\left\|\frac{\partial^{s+2}u}{\partial\xi\partial_\eta^{s+1}}\right\|^2_{0,\mathcal{D}} + \\ & \left. + h_1^3 h_2^{2(s+1)}\left\|\frac{\partial^{s+3}u}{\partial\xi^2\partial_\eta^{s+1}}\right\|^2_{0,\mathcal{D}}\right],\end{aligned}$$

and

$$\varepsilon^2\big|u-\psi\big|^2_{2,\mathcal{D}} \leq C\frac{(p-s)!}{(p-2+s)!}\varepsilon^2 h^{-4}\Bigg[h_1^{2(s+1)}\Bigg\|\frac{\partial^{s+1}u}{\partial_\xi^{s+1}}\Bigg\|^2_{0,\mathcal{D}} + h_1^{2(s+1)}h_2\Bigg\|\frac{\partial^{s+2}u}{\partial_\xi^{s+1}\partial\eta}\Bigg\|^2_{0,\mathcal{D}} + $$
$$+ h_1^{2(s+1)}h_2^3\Bigg\|\frac{\partial^{s+3}u}{\partial_\xi^{s+1}\partial\eta^2}\Bigg\|^2_{0,\mathcal{D}} + h_2^{2(s+1)}\Bigg\|\frac{\partial^{s+1}u}{\partial_\eta^{s+1}}\Bigg\|^2_{0,\mathcal{D}} + h_1 h_2^{2(s+1)}\Bigg\|\frac{\partial^{s+2}u}{\partial\xi\partial_\eta^{s+1}}\Bigg\|^2_{0,\mathcal{D}} + $$
$$+ h_1^3 h_2^{2(s+1)}\Bigg\|\frac{\partial^{s+3}u}{\partial\xi^2\partial_\eta^{s+1}}\Bigg\|^2_{0,\mathcal{D}}\Bigg].$$

Here $h = \min\{h_1, h_2\}$.

Proof. We set $\psi = \mathcal{J}_p u$, where $\mathcal{J}_p$ is defined by (5.2.32). By rescaling the bound (5.2.42) the above estimates follow as an immediate result on the rectangle $\mathcal{D}$. □

We next adjust the C^1-continuity, following [34].

Theorem 5.2.46 *Let* Ω *be the mesh defined in* (5.3.10) *and let* $u \in H^3(\Omega)$. *Then, for* $1 \leq s \leq p$, *there exists a a piecewise* C^1*-continuous polynomial* $\psi \in S(\kappa, p)$ *that satisfies: for* $k = 0, 1$, *the conditions*

$$\frac{\partial^k}{\partial\xi^k}\big(\psi(x_i, y_i)\big) = \frac{\partial^k}{\partial\xi^k}\big(u(x_i, y_i)\big), \tag{5.2.47}$$

$$\frac{\partial^k}{\partial\eta^k}\big(\psi(x_i, y_i)\big) = \frac{\partial^k}{\partial\eta^k}\big(u(x_i, y_i)\big), \tag{5.2.48}$$

where (x_i, y_i) *denotes the vertices of* Ω *and*

$$\|u-\psi\|^2_{1,\Omega} \leq C\frac{(p-s)!}{(p-2+s)!}h^{-2}\Bigg[h_1^{2(s+1)}\Bigg\|\frac{\partial^{s+1}u}{\partial_\xi^{s+1}}\Bigg\|^2_{0,\Omega} + h_1^{2(s+1)}h_2\Bigg\|\frac{\partial^{s+2}u}{\partial_\xi^{s+1}\partial\eta}\Bigg\|^2_{0,\Omega} + $$
$$+ h_1^{2(s+1)}h_2^3\Bigg\|\frac{\partial^{s+3}u}{\partial_\xi^{s+1}\partial\eta^2}\Bigg\|^2_{0,\Omega} + h_2^{2(s+1)}\Bigg\|\frac{\partial^{s+1}u}{\partial_\eta^{s+1}}\Bigg\|^2_{0,\Omega} + h_1 h_2^{2(s+1)}\Bigg\|\frac{\partial^{s+2}u}{\partial\xi\partial_\eta^{s+1}}\Bigg\|^2_{0,\Omega} + $$
$$+ h_1^3 h_2^{2(s+1)}\Bigg\|\frac{\partial^{s+3}u}{\partial\xi^2\partial_\eta^{s+1}}\Bigg\|^2_{0,\Omega}\Bigg], \tag{5.2.49}$$

and (5.2.50)

$$\varepsilon^2\big|u-\psi\big|^2_{2,\Omega} \leq C\frac{(p-s)!}{(p-2+s)!}\varepsilon^2 h^{-4}\Bigg[h_1^{2(s+1)}\Bigg\|\frac{\partial^{s+1}u}{\partial_\xi^{s+1}}\Bigg\|^2_{0,\Omega} + h_1^{2(s+1)}h_2\Bigg\|\frac{\partial^{s+2}u}{\partial_\xi^{s+1}\partial\eta}\Bigg\|^2_{0,\Omega} + $$
$$+ h_1^{2(s+1)}h_2^3\Bigg\|\frac{\partial^{s+3}u}{\partial_\xi^{s+1}\partial\eta^2}\Bigg\|^2_{0,\Omega} + h_2^{2(s+1)}\Bigg\|\frac{\partial^{s+1}u}{\partial_\eta^{s+1}}\Bigg\|^2_{0,\Omega} + h_1 h_2^{2(s+1)}\Bigg\|\frac{\partial^{s+2}u}{\partial\xi\partial_\eta^{s+1}}\Bigg\|^2_{0,\Omega} + $$

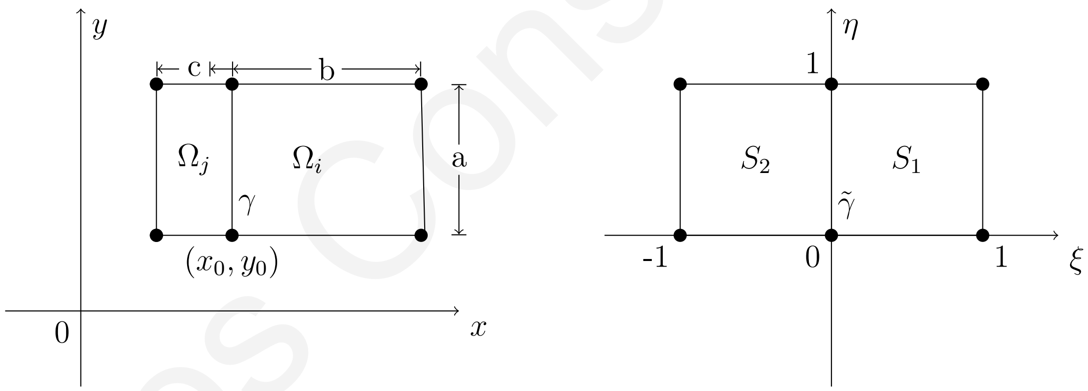
$$+ h_1^3 h_2^{2(s+1)} \left\| \frac{\partial^{s+3} u}{\partial \xi^2 \partial_\eta^{s+1}} \right\|_{0,\Omega}^2 \Bigg]. \tag{5.2.51}$$

Here $h = \min\{h_1, h_2\}$.

Proof. We will only consider the case of two rectangles with a common side, since we will be only considering meshes with rectangular elements. Let Ω_i and Ω_j be the rectangles shown in Figure 5.8(a). Suppose that γ is the common side, of Ω_i and Ω_j and suppose also that R_i and R_j are the mappings of the standard squares S_1 and S_2 (presented in Figure 5.8(b)) onto Ω_i and Ω_j, respectively. We define them as:

$$R_i := \begin{cases} x = c\xi + x_0 \\ y = a\eta + y_0 \end{cases}, \quad R_j := \begin{cases} x = b\xi + x_0 \\ y = a\eta + y_0 \end{cases}. \tag{5.2.52}$$

Let $\zeta_i = \mathcal{J}_p u$, which satisfies (5.2.46) on the rectangle Ω_i. In the same manner



(a) Adjacent elements. (b) Standard squares.

Figure 5.8: Mapping of standard squares onto adjacent elements

there is a polynomial ζ_j that provides the same result on Ω_j. Obviously the piecewise polynomial defined on the union $\Omega_i \cup \Omega_j$ and it is comprised of the polynomials ζ_i, ζ_j and is discontinuous at the intersection γ. We procced to eliminate the jump in the approximation across the edges of the elements.

We set $\psi_1 = (\zeta_i - \zeta_j)|_\gamma$ and hence $\hat{\psi}_1(\eta) = \hat{\psi}_1\big(R_i^{-1}(x,y)\big)$. Then

$$\hat{\psi}_1^{(m)}(\eta) = \frac{\partial^m}{\partial \eta^m}(\hat{\zeta}_i - \hat{\zeta}_j)\big|_{\hat{\gamma}} = \frac{\partial^m}{\partial \eta^m}\hat{\zeta}_i(0,\eta) - \frac{\partial^m}{\partial \eta^m}\hat{\zeta}_j(0,\eta).$$

We recall that, for $0 \le m \le 1$,

$$\frac{\partial^m}{\partial \eta^m}\hat{u}_i(0,\eta) = \frac{\partial^m}{\partial \eta^m}\hat{u}_j(0,\eta), \quad 0 \le \eta \le 1, \tag{5.2.53}$$

since u is C^1-continuous.

From (5.2.33), (5.2.34) and (5.2.53) we have

$$\tilde{\psi}_1^{(m)}(0) = \tilde{\psi}_1^{(m)}(1) = 0, \quad 0 \le m \le 1.$$

With the aid of Lemma 5.2.3 one can find a polynomial $\hat{\Psi}_1(\xi,\eta)$ of degree $\le p$ in η and degree 3 in ξ such that (5.2.4)-(5.2.6) are satisfied. Set

$$\begin{aligned}\hat{\chi}_i =& \hat{\zeta}_i - \hat{\Psi}_1, \quad \text{on } S_1,\\ \hat{\chi}_j =& \hat{\zeta}_j, \qquad\quad \text{on } S_2,\end{aligned}$$

then

$$\begin{aligned}\chi_i =& \hat{\chi}_i\big(R_i^{-1}(x,y)\big) = \zeta_i - \Psi_1 = \zeta_i - \hat{\Psi}_1\big(R_i^{-1}(x,y)\big) \quad \text{on} \quad \Omega_i,\\ \chi_j =& \hat{\chi}_j\big(R_j^{-1}(x,y)\big) \quad \text{on} \quad \Omega_j.\end{aligned}$$

From Lemma 5.2.3 and the definition of ψ_1, we see that

$$(\chi_i - \chi_j)\big|_\gamma = (\hat{\chi}_i - \hat{\chi}_j)\big|_{\hat{\gamma}} = (\hat{\zeta}_i - \hat{\zeta}_j)\big|_{\hat{\gamma}} - \hat{\psi}_1(\eta) = 0.$$

Moreover, we set

$$\psi_2 = \frac{\partial}{\partial x}(\chi_i - \chi_j)\big|_\gamma = \frac{\partial}{\partial x}(\zeta_i - \zeta_j)\big|_\gamma - \frac{\partial}{\partial x}\Psi_1\big|_\gamma. \tag{5.2.54}$$

Note that

$$\hat{\psi}_2(\eta) = \frac{1}{c}\frac{\partial\hat{\zeta}_i(0,\eta)}{\partial\xi} - \frac{1}{b}\frac{\partial\hat{\zeta}_j(0,\eta)}{\partial\xi} - \frac{1}{c}\frac{\partial\hat{\Psi}_1(0,\eta)}{\partial\xi}. \tag{5.2.55}$$

This along with (5.2.33), (5.2.53) and (5.2.55) yields

$$\psi_2(0) = \psi_2(1) = 0.$$

Moreover

$$\hat{\psi}_2'(\eta) = \frac{1}{c}\frac{\partial^2}{\partial\xi\partial\eta}\hat{\zeta}_i(0,\eta) - \frac{1}{b}\frac{\partial^2}{\partial\xi\partial\eta}\hat{\zeta}_j(0,\eta) - \frac{1}{c}\frac{\partial^2}{\partial\xi\partial\eta}\hat{\Psi}_1(0,\eta), \tag{5.2.56}$$

Combining (5.2.33), (5.2.34) gives

$$\frac{\partial^2}{\partial\xi\partial y}\big(\mathcal{J}_m u(x_i, y_i)\big) = \frac{\partial^2}{\partial\xi\partial y}\big(u(x_i, y_i)\big)$$

and with the aid of (5.2.53) and (5.2.55), we get

$$\psi_2^{(1)}(0) = \psi_2^{(1)}(1) = 0.$$

By appealing again to Lemma 5.2.3, we get a polynomial Ψ_2 of degree p in η and degree 3 in ξ such that (5.2.7), (5.2.8) and (5.2.9) hold. We now set

$$\begin{aligned}\hat{\psi}_i =& \hat{\chi}_i - c\hat{\Psi}_2, \qquad \text{on } S_1,\\ \hat{\psi}_j =& \hat{\chi}_j, \qquad\qquad\quad \text{on } S_2,\end{aligned}$$

and

$$\begin{aligned}\psi_i =& \chi_i - c\hat{\Psi}_2\big(R_i^{-1}(x, y)\big), \qquad \text{on } \Omega_i,\\ \psi_j =& \hat{\chi}_j, \qquad\qquad\qquad\qquad\quad \text{on } \Omega_j.\end{aligned}$$

It is clear that

$$(\psi_i - \psi_j)\big|_\gamma = (\hat{\psi}_i - \hat{\psi}_j)\big|_{\hat{\gamma}} = (\hat{\zeta}_i - \hat{\Psi}_1 - c\hat{\Psi}_2 - \hat{\zeta}_j)\big|_{\hat{\gamma}} = 0. \tag{5.2.57}$$

Also from (5.2.6), (5.2.9),

$$\begin{aligned}\frac{\partial}{\partial x}(\psi_i - \psi_j)\Big|_\gamma =& \frac{\partial}{\partial x}(\zeta_i - \Psi_1 - c\Psi_2 - \zeta_j)\big|_\gamma\\ =& \frac{1}{c}\frac{\partial\hat{\zeta}_i(0,\eta)}{\partial\xi} - \frac{1}{b}\frac{\partial\hat{\zeta}_j(0,\eta)}{\partial\xi} - \frac{1}{c}\frac{\partial\hat{\Psi}_1(0,\eta)}{\partial\xi} - \frac{\partial\hat{\Psi}_2(0,\eta)}{\partial\xi} = 0.\end{aligned} \tag{5.2.58}$$

Equations (5.2.57), (5.2.58) ensure the C^1-continuity of ψ_i and ψ_j along γ and we also get

$$\|\hat{u}_j - \hat{\psi}_j\|_{H^2(S_2)} = \|\hat{u}_j - \hat{\zeta}_j\|_{H^2(S_2)} \tag{5.2.59}$$

and

$$\|\hat{u}_i - \hat{\psi}_i\|_{H^2(S_1)} \leq C\Big(\|\hat{u}_i - \hat{\zeta}_i\|_{H^2(S_1)} + \|\hat{\Psi}_1\|_{H^2(S_1)} + \|\hat{\Psi}_2\|_{H^2(S_1)}\Big). \tag{5.2.60}$$

From Lemmas 5.2.1 and 5.2.3 we obtain for $k = 1, 2$,

$$\begin{aligned} \|\hat{\Psi}_k\|_{H^2(S_1)} \leq & C\big(\|\hat{u}_i - \hat{\chi}_i\|_{H^2(\tilde{\gamma})} + \|\hat{u}_j - \hat{\chi}_j\|_{H^2(\tilde{\gamma})}\big) \\ \leq & C\big(\|\hat{u}_i - \hat{\chi}_i\|_{H^3(S_1)} + \|\hat{u}_j - \hat{\chi}_j\|_{H^2(S_1)}\big) \end{aligned} \tag{5.2.61}$$

and therefore ψ_i is the desired approximation on the associated rectangle Ω_i. □

5.3 An *hp*–FEM on the square

We now consider the case when $\Omega = (0,1)^2$ and make certain assumptions on the regularity of the solution in order to construct an *hp*-FEM on the *Spectral Boundary Layer mesh.*

First we recall that from Chapter 4, we have the decomposition

$$u = u_M^s + \chi u_M^{BL} + r_M$$

where $M \in \mathbb{N}_0$ is a *fixed* constant. This fact makes the remainder r_M not neglible, hence it needs to be approximated as well. However, the regularity of r_M, which follows from Chapter 4, is

$$\|r_M\|_{k,\Omega} \leq C_M \varepsilon^{1-k},$$

thus applying the tools of the previous sections is not possible (without inheriting negative powers of ε). One choice is to use standard p–version estimates [27] and obtain, a polynomial $\pi_p r_M$ such that

$$\|r_M - \pi_p r_M\|_{E,\Omega} \leq C p^{2-s+\delta} \;\; \forall s > 0,$$

where $\delta > 0$ is arbitrarily small (see [27] for more details). This error bound would dominate the other two, since the smooth part and the boundary layer part could be approximated at an exponential rate if the *Spectral Boundary Layer mesh* is used.

In order to provide the approximation details for our problem posed on the unit square, we thus make the following assumption.

Assumption 5.3.1. The solution of (5.0.1) satisfies the decomposition

$$u = u^s + \sum_{k=1}^{4} u_k^{BL} + \sum_{k=1}^{4} u_k^{AL}, \tag{5.3.2}$$

where u^s denotes the *smooth part*, $\{u_k^{BL}\}_{k=1}^4$, are the *boundary layers* along each side and $\{u_k^{CL}\}_{k=1}^4$, the *auxiliary* layers near the four corners. In addition, we assume that there exist constants $\gamma, \gamma_i, K_i, C > 0$, $i = 1, \ldots 4$, such that the following bounds on each component of the decomposition (5.3.2) hold:

$$\left\| \frac{\partial^{i+j} u^s}{\partial x^i \partial y^j} \right\|_{0,\Omega} \leq C \gamma^{i+j} (i+j)!, \tag{5.3.3}$$

$$\left| \frac{\partial^{i+j} u_1^{BL}}{\partial x^i \partial y^j}(x,y) \right| \leq C \gamma_1^{i+j} \varepsilon^{1-i} \mathrm{e}^{-x/\varepsilon}, \quad \left| \frac{\partial^{i+j} u_3^{BL}}{\partial x^i \partial y^j}(x,y) \right| \leq C \gamma_3^{i+j} \varepsilon^{1-i} \mathrm{e}^{-(1-x)/\varepsilon}, \tag{5.3.4}$$

$$\left| \frac{\partial^{i+j} u_2^{BL}}{\partial x^i \partial y^j}(x,y) \right| \leq C \gamma_2^{i+j} \varepsilon^{1-j} \mathrm{e}^{-y/\varepsilon}, \quad \left| \frac{\partial^{i+j} u_4^{BL}}{\partial x^i \partial y^j}(x,y) \right| \leq C \gamma_4^{i+j} \varepsilon^{1-j} \mathrm{e}^{-(1-y)/\varepsilon}, \tag{5.3.5}$$

$$\left| \frac{\partial^{i+j} u_1^{AL}}{\partial x^i \partial y^j}(x,y) \right| \leq C K_1^{i+j} \varepsilon^{1-i-j} \mathrm{e}^{-(x+y)/\varepsilon}, \quad \left| \frac{\partial^{i+j} u_3^{AL}}{\partial x^i \partial y^j}(x,y) \right| \leq C K_2^{i+j} \varepsilon^{1-i-j} \mathrm{e}^{-[(1-x)+(1-y)]/\varepsilon} \tag{5.3.6}$$

$$\left| \frac{\partial^{i+j} u_2^{AL}}{\partial x^i \partial y^j}(x,y) \right| \leq C K_3^{i+j} \varepsilon^{1-i-j} \mathrm{e}^{-[(1-x)+y]/\varepsilon}, \quad \left| \frac{\partial^{i+j} u_4^{AL}}{\partial x^i \partial y^j}(x,y) \right| \leq C K_4^{i+j} \varepsilon^{1-i-j} \mathrm{e}^{-[x+(1-y)]/\varepsilon}, \tag{5.3.7}$$

for all $(x, y) \in \overline{\Omega}$. Moreover, there are constants $C, K > 0$, depending only on the data, such that

$$\|\nabla^n u\|_{L^2(\Omega)} \leq C K^n \max\{n^n, \varepsilon^{1-n}\}, \quad \forall n \in \mathbb{N}_0. \tag{5.3.8}$$

We define our mesh Δ which is an appropriate two-dimensional version of the *Spectral Boundary Layer Mesh* (see also [67]).

Definition 5.3.9. Let $\kappa > 0$ be a fixed number and let $p \geq 3$ be the degree of the approximating polynomials. We then set Δ as

$$\Delta = \begin{cases} [0,1] \times [0,1], & 1/2 \leq \kappa p \varepsilon, \\ \cup_{i=1}^{9} \Delta_i, & \kappa p \varepsilon < 1/2. \end{cases} \tag{5.3.10}$$

Namely, in the case $\kappa p \varepsilon < 1/2$, we divide our domain into 9 elements as shown in Figure 5.9. Using this mesh one can capture every part of the solution (smooth part, boundary and auxiliary layers), and it is the minimal mesh that yields robust exponential convergence.

The elements Δ_i are given by:

$$\Delta_1 = [\kappa p\varepsilon, 1-\kappa p\varepsilon]^2,\ \Delta_2 = [0,\kappa p\varepsilon]^2,\ \Delta_3 = [\kappa p\varepsilon, 1-\kappa p\varepsilon]\times[0,\kappa p\varepsilon],$$
$$\Delta_4 = [1-\kappa p\varepsilon, 1]\times[0,\kappa p\varepsilon],\ \Delta_5 = [1-\kappa p\varepsilon, 1]\times[\kappa p\varepsilon, 1-\kappa p\varepsilon],$$
$$\Delta_6 = [1-\kappa p\varepsilon]^2,\ \Delta_7 = [\kappa p\varepsilon, 1-\kappa p\varepsilon]\times[1-\kappa p\varepsilon, 1],$$
$$\Delta_8 = [0,\kappa p\varepsilon]\times[1-\kappa p\varepsilon, 1],\ \Delta_9 = [0,\kappa p\varepsilon]\times[\kappa p\varepsilon, 1-\kappa p\varepsilon].$$

Consider the affine mappings $R_i : S \to \Delta_i$, $i = 1,\ldots,9$ and define the space V_N as

$$V_N \equiv V(\kappa,p) = \{u \in H_0^2(S_1) : u|_{\Delta_i} = \psi_p \circ R_i^{-1},\ \text{for some}\ \psi_p \in \mathbb{Q}_p(S), i = 1,\ldots,9\}. \tag{5.3.11}$$

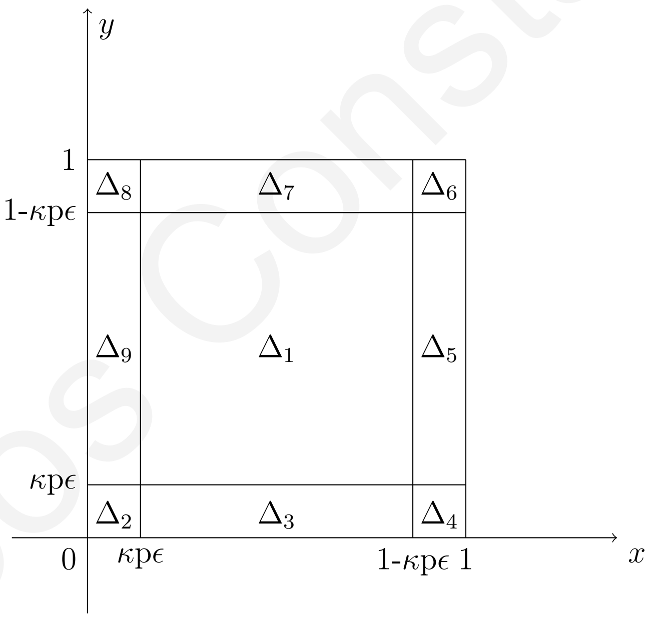


Figure 5.9: The Spectral Boundary Layer Mesh in the case $\kappa p\varepsilon < 0.5$

To conclude, we present the main result, namely we prove that an $hp-$approximation defined on the *Spectral Boundary Layer Mesh* yields robust exponential convergence in the energy norm.

Theorem 5.3.12 *Suppose u solves* (5.0.1) *and let $u_N \in V(\kappa,p)$ be the solution of* (5.0.3) *based on the Spectral Boundary Layer mesh. Then there are positive constants*

C, σ, κ, independent of ε, such that

$$\|u - u_N\|_{E,\Omega} \leq C\mathrm{e}^{-\sigma\kappa p}. \tag{5.3.13}$$

Proof. We study two separate cases: $\kappa p\varepsilon < 1/2$ and $\kappa p\varepsilon \geq 1/2$.

Case 1: We begin by considering the "asymptotic range of p", i.e. the case when $p \geq (2\kappa\varepsilon)^{-1}$. By setting $s = r_1(p-2)$, with $r_1 \in (0,1)$, to be selected shortly, and appealing to (5.3.8) and (5.2.46) [a] we obtain, for $\tilde{K} = \max\{1, K\}$,

$$\left\|u - \mathcal{J}_p u\right\|^2_{E,\Omega} \leq C\frac{(p - r_1(p-2))!}{(p-2+r_1(p-2))!}\tilde{K}^{2(r_1(p-2)+3)}$$
$$\times \max\left\{(r_1(p-2)+3)^{2(r_1(p-2)+3)}, \varepsilon^{1-2(r_1(p-2)-3)}\right\}.$$

We use Lemma 3.2.11 to obtain

$$\frac{\big(p - r_1(p-2)\big)!}{\big(p-2+r_1(p-2)\big)!} \leq Cp^2(p-2)^{-2r_1(p-2)}\mathrm{e}^{2r_1(p-2)-1}\left[\frac{(1-r_1/3)^{(1-r_1)}}{(1+r_1)^{(1+r_1)}}\right]^{p-2},$$

hence

$$\begin{aligned}
&\left\|u - \mathcal{J}_p u\right\|^2_{E,\Omega} \leq \\
\leq &Cp^2\tilde{K}^6\left(\frac{(1-r_1/3)^{1-r_1}}{(1+r_1)^{1+r_1}}\right)^{p-2}(p-2)^{-2r_1(p-2)}\mathrm{e}^{2r_1(p-2)-1}\tilde{K}^{2r_1(p-2)}\times \\
&\times (r_1(p-2)+3)^6(r_1(p-2)+3)^{2r_1(p-2)} \\
\leq &Ce\tilde{K}^6p^8\left(\frac{(1-r_1/3)^{1-r_1}}{(1+r_1)^{1+r_1}}\mathrm{e}^{2r_1}\tilde{K}^{2r_1}\right)^{p-2}(2r_1)^{2r_1(p-2)}\left(\frac{1}{2}+\frac{3}{2r_1(p-2)}\right)^{2r_1(p-2)} \\
\leq &Cp^8\left(\frac{(1-r_1/3)^{1-r_1}}{(1+r_1)^{1+r_1}}(2\mathrm{e}r_1\tilde{K})^{2r_1}\right)^{p-2}.
\end{aligned} \tag{5.3.14}$$

We now choose $r_1 = (2\mathrm{e}\tilde{K})^{-1} \in (0,1)$ and we denote

$$\sigma_1 = |\ln \tau_1|, \; \tau_1 = \frac{(1-r_1/3)^{1-r_1}}{(1+r_1)^{1+r_1}} < 1.$$

The above choices lead to

$$\left\|u - \mathcal{J}_p u\right\|^2_{E,\Omega} \leq Cp^8\mathrm{e}^{-\sigma_1 p}.$$

The factor p^8 may be absorbed in the exponential and adjusting the constants.

[a] $h = 1$ in this case

Case 2: In the pre-asymptotic case $p < (2\kappa\varepsilon)^{-1}$, the mesh is defined as in (5.3.10). In that case we use the decomposition (5.3.2) and we handle each part using (5.3.3)–(5.3.6). We first consider the smooth part. From Lemma 5.2.43 there is an approximation $\mathcal{J}_p u^s$ of u^s that satisfies (5.2.46). We now set $s = r_2(p-2)$, for some $r_2 \in (0,1)$ arbitrary. Using assumption (5.3.3), we get, for $\tilde{K} = \max\{1, \gamma\}$

$$\left\| u^s - \mathcal{J}_p u^s \right\|_{E,\Omega}^2 \leq C \frac{(p - r_2(p-2))!}{(p-2+r_2(p-2))!} \tilde{K}^{2(r_2(p-2)+3)} \big((r_2(p-2)+3)!\big)^2, \qquad (5.3.15)$$

and with the aid of the inequality $j! \leq j^j$, $\forall j \in \mathbb{N}$, we have,

$$\begin{aligned}
&\left\| u^s - \mathcal{J}_p u^s \right\|_{E,\Omega}^2 \leq \\
\leq & C \frac{(p - r_2(p-2))!}{(p-2+r_2(p-2))!} \tilde{K}^{2(r_2(p-2)+3)} \Big((r_2(p-2)+3)^{r_2(p-2)+3} \Big)^2 \\
\leq & C p^2 \tilde{K}^6 \left(\frac{(1-r_2/3)^{1-r_2}}{(1+r_2)^{1+r_2}} \right)^{p-2} (p-2)^{-2r_2(p-2)} \mathrm{e}^{2r_2(p-2)-1} \tilde{K}^{2r_2(p-2)} \times \\
& \times (r_2(p-2)+3)^6 (r_2(p-2)+3)^{2r_2(p-2)} \\
\leq & C \mathrm{e} \tilde{K}^6 p^8 \left(\frac{(1-r_2/3)^{1-r_2}}{(1+r_2)^{1+r_2}} \mathrm{e}^{2r_2} \tilde{K}^{2r_2} \right)^{p-2} (2r_2)^{2r_2(p-2)} \left(\frac{1}{2} + \frac{3}{2r_2(p-2)} \right)^{2r_2(p-2)} \\
\leq & C p^8 \left(\frac{(1-r_2/3)^{1-r_2}}{(1+r_2)^{1+r_2}} (2\mathrm{e} r_2 \tilde{K})^{2r_2} \right)^{p-2} \\
\leq & C p^8 \mathrm{e}^{-\sigma_2(p-2)}.
\end{aligned} \qquad (5.3.16)$$

Here the calculations have been done as in (5.3.14). The symbol σ_2 denotes the number $\sigma_2 = |\ln \tau_2|$, $\tau_2 = \frac{(1-r_2/3)^{1-r_2}}{(1+r_2)^{1+r_2}} < 1$.

We proceed by considering the boundary layers. Here we display the calculations only for the boundary layer u_1^{BL}. We omit all other cases, since one can deduce the same results in a similar way. We partition the domain Ω as $\Omega_1 \cup \Omega_2$, where $\Omega_1 = \Delta_2 \cup \Delta_8 \cup \Delta_9$ and $\Omega_2 = \Omega \setminus \Omega_1$. By Theorem 5.2.46 one finds an approximation ψ^{BL} of u_1^{BL} with the properties mentioned there. By assumption (5.3.4) on the subdomain Ω_1, we have, for $i, j \in \mathbb{N}$,

$$\begin{aligned}
\int_{\Omega_1} \left| \frac{\partial^{i+j} u_1^{BL}}{\partial x^i \partial y^j}(x,y) \right|^2 dxdy \leq & \int_0^1 \int_0^{\kappa p \varepsilon} \left| C \gamma_1^{i+j} \varepsilon^{1-i} \mathrm{e}^{-x/\varepsilon} \right|^2 dxdy \\
\leq & \kappa p \varepsilon C \gamma_1^{2(i+j)} \varepsilon^{2-2i}.
\end{aligned} \qquad (5.3.17)$$

Therefore, for $s, j \in \mathbb{N}$ it holds,

$$\left\| \frac{\partial^{s+j+1} u_1^{BL}}{\partial x^{s+1} \partial y^j} \right\|_{0,\Omega_1}^2 \leq \kappa p \varepsilon C \gamma_1^{2(s+j+1)} \varepsilon^{2-2(s+1)},$$

and

$$\left\| \frac{\partial^{s+j+1} u_1^{BL}}{\partial x^{j} \partial y^{s+1}} \right\|_{0,\Omega_1}^2 \leq \kappa p \varepsilon C \gamma_1^{2(j+s+1)} \varepsilon^{2-2j},$$

For $r_3 \in (0,1)$ arbitrary, we set $s = r_3(p-2)$, in (5.2.46) and with the use of assumption (5.3.4) we obtain, for $\tilde{K} = \max\{1, \gamma_1\}$,

$$\begin{aligned}
&\left\| u_1^{BL} - \psi^{BL} \right\|_{1,\Omega_1}^2 \leq \\
&\leq C(\kappa p \varepsilon)^{-2} \frac{(p - r_3(p-2))!}{(p-2+r_3(p-2))!} \tilde{K}^{2(r_3(p-2)+3)} \times \\
&\times \left[3\kappa p \varepsilon (\kappa p \varepsilon)^{2(r_3(p-2)+1)} \varepsilon^{2-2(r_3(p-2)+1)} + \varepsilon^2 \kappa p \varepsilon + (\kappa p \varepsilon)^2 + (\kappa p \varepsilon)^4 \varepsilon^{-2} \right] \\
&\leq C \frac{(p - r_3(p-2))!}{(p-2+r_3(p-2))!} \tilde{K}^{2(r_3(p-2)+3)} \left[3(\kappa p)^{2(r_3(p-2))+1} + \frac{1}{\kappa p} + 1 + (\kappa p)^2 \right] \\
&\leq C p^3 \left(\frac{p}{p-2} \right)^{2 r_3 (p-2)} \mathrm{e}^{2 r_3 (p-2) - 1} \left[\frac{(1 - r_3/3)^{(1-r_3)}}{(1+r_3)^{(1+r_3)}} \right]^{p-2} \tilde{K}^{2(r_3(p-2)+3)} \times \\
&\times \left[3\kappa^{2(r_3(p-2))+1} + \frac{1}{\kappa p^{2(r_3(p-2))+2}} + \frac{1}{p^{2(r_3(p-2))+1}} + \frac{\kappa^2}{p^{2(r_3(p-2))-1}} \right] \\
&\leq p^3 3^{2 r_3 (p-2)} C \left(\frac{(1 - r_3/3)^{1-r_3}}{(1+r_3)^{1+r_3}} (\kappa \mathrm{e} \tilde{K})^{2 r_3} \right)^{p-2} \\
&\leq p^3 C \left(\frac{(1 - r_3/3)^{1-r_3}}{(1+r_3)^{1+r_3}} (3\kappa \mathrm{e} \tilde{K})^{2 r_3} \right)^{p-2} \\
&\leq C p^3 \varepsilon \mathrm{e}^{-\sigma_3 (p-2)},
\end{aligned} \tag{5.3.18}$$

where

$$\sigma_3 = |\ln \tau_3|, \quad \tau_3 = \frac{(1 - r_3)^{1 - r_3/3}}{(1+r_3)^{1+r_3}} < 1.$$

Furthermore, we have

$$\begin{aligned}
&\varepsilon^2 \left| u_1^{BL} - \psi^{BL} \right|_{2,\Omega_1}^2 \leq \\
&\leq C \varepsilon^2 (\kappa p \varepsilon)^{-4} \frac{(p - r_3(p-2))!}{(p-2+r_3(p-2))!} \tilde{K}^{2(r_3(p-2)+3)} \times \\
&\times \left[3\kappa p \varepsilon (\kappa p \varepsilon)^{2(r_3(p-2)+1)} \varepsilon^{2-2(r_3(p-2)+1)} + \varepsilon^2 \kappa p \varepsilon + (\kappa p \varepsilon)^2 + (\kappa p \varepsilon)^4 \varepsilon^{-2} \right] \\
&\leq C p^3 \varepsilon \mathrm{e}^{-\sigma_3 (p-2)},
\end{aligned} \tag{5.3.19}$$

and therefore, from (5.3.18) and (5.3.19) we conclude that

$$\left\| u_1^{BL} - \psi^{BL} \right\|_{E,\Omega_1}^2 \leq Cp^3 \varepsilon \mathrm{e}^{-\sigma_3 p}. \tag{5.3.20}$$

On the subdomain Ω_2, we have

$$\left\| u_1^{BL} \right\|_{E,\Omega_2}^2 \leq \int_0^1 \int_{\kappa p \varepsilon}^1 \left| C\mathrm{e}^{-x/\varepsilon} \right|^2 dxdy \leq C(1-\kappa p \varepsilon)\mathrm{e}^{-2\kappa p}. \tag{5.3.21}$$

As we have done in Chapter 3 (see (3.2.18)), we approximate u_1^{BL} by its cubic interpolant $I_3 u_1^{BL}$, therefore

$$\left\| u_1^{BL} - I_3 u_1^{BL} \right\|_{E,\Omega_2}^2 \leq \left\| u_1^{BL} \right\|_{E,\Omega_2}^2 + \left\| I_3 u_1^{BL} \right\|_{E,\Omega_2}^2 \leq C\mathrm{e}^{-\nu p}. \tag{5.3.22}$$

From (5.3.20) and (5.3.22) we get

$$\left\| u_1^{BL} - \psi^{BL} \right\|_{E,\Omega}^2 \leq Cp^2 \mathrm{e}^{-\tilde{\sigma}_3 p}. \tag{5.3.23}$$

Next we consider the auxiliary layers. As before we provide the details only for u_1^{AL}, since the rest are similar. We again make a partition of the domain $\Omega = \Omega_3 \cup \Omega_4$ where $\Omega_3 = \Delta_2$ and $\Omega_4 = \Omega \setminus \Delta_2$. On Ω_3 we have the bound

$$\begin{aligned} \int_{\Omega_3} \left| \frac{\partial^{i+j} u_1^{AL}}{\partial x^i \partial y^j}(x,y) \right|^2 dxdy \leq & \int_0^{\kappa p \varepsilon} \int_0^{\kappa p \varepsilon} \left| CK_1^{i+j} \varepsilon^{1-i-j} \mathrm{e}^{-(x+y)/\varepsilon} \right|^2 dxdy \\ \leq & (\kappa p \varepsilon)^2 CK_1^{2(i+j)} \varepsilon^{2-2i-2j}, \end{aligned} \tag{5.3.24}$$

where assumption (5.3.5) was used. Let $r_4 \in (0,1)$ be arbitrary and set $s = r_4(p-2)$. Therefore on Ω_3, we have from Lemma 5.2.43, for $\tilde{K} = \max\{1, K_1\}$,

$$\begin{aligned} & \left\| u_1^{CL} - \psi^{AL} \right\|_{1,\Omega_3}^2 \leq \\ & \leq C(\kappa p \varepsilon)^{-2} \frac{(p - r_4(p-2))!}{(p-2+r_4(p-2))!} \tilde{K}^{2(r_4(p-2)+3)} \times \\ & \times (\kappa p \varepsilon)^2 \Big[(\kappa p \varepsilon)^{2(r_4(p-2)+1)} \varepsilon^{2-2(r_4(p-2)+1)} + (\kappa p \varepsilon)^{2(r_4(p-2)+1)+1} \varepsilon^{2-2(r_4(p-2)+1)-2} + \\ & + (\kappa p \varepsilon)^{2(r_4(p-2)+1)+3} \varepsilon^{2-2(r_4(p-2)+1)-4} + (\kappa p \varepsilon)^{2(r_4(p-2)+1)} \varepsilon^{2-2(r_4(p-2)+1)} + \\ & + (\kappa p \varepsilon)^{2(r_4(p-2)+1)+1} \varepsilon^{2-2(r_4(p-2)+1)-2} + (\kappa p \varepsilon)^{2(r_4(p-2)+1)+3} \varepsilon^{2-2(r_4(p-2)+1)-4} \Big] \\ & \leq 2C \frac{(p - r_4(p-2))!}{(p-2+r_4(p-2))!} \tilde{K}^{2(r_4(p-2)+3)} p^{2(r_4(p-2)+1)+2} \kappa^{2(r_4(p-2)+1)} \left[\frac{\varepsilon^2}{p^2} + \frac{\kappa \varepsilon}{p} + \varepsilon \kappa^3 \right] \end{aligned}$$

$$
\begin{aligned}
&\leq C\varepsilon p^6\Big(\frac{p}{p-2}\Big)^{2r_4(p-2)} \mathrm{e}^{2r_4(p-2)-1}\kappa^{2(r_4(p-2)+1)}\left[\frac{(1-r_4/3)^{(1-r_4)}}{(1+r_4)^{(1+r_4)}}\right]^{p-2}\tilde{K}^{2(r_4(p-2)+3)}\\
&\leq \varepsilon p^6 3^{2r_4(p-2)} C\left(\frac{(1-r_4/3)^{1-r_4}}{(1+r_4)^{1+r_4}}(\kappa \mathrm{e}\tilde{K})^{2r_4}\right)^{p-2}\\
&\leq \varepsilon p^6 C\left(\frac{(1-r_4/3)^{1-r_4}}{(1+r_4)^{1+r_4}}(3\kappa \mathrm{e}\tilde{K})^{2r_4}\right)^{p-2}\\
&\leq \varepsilon C p^6 \mathrm{e}^{-\sigma_4(p-2)},
\end{aligned}
\tag{5.3.25}
$$

where

$$
\sigma_4 = |\ln \tau_4|, \quad \tau_4 = \frac{(1-r_4)^{1-r_4}}{(1+r_4)^{1+r_4}}.
$$

In addition, working as above gives

$$
\varepsilon^2 \Big| u_1^{AL} - \psi^{AL} \Big|^2_{2,\Omega_3} \leq C p^6 \mathrm{e}^{-\sigma_4(p-2)}, \tag{5.3.26}
$$

and thus, using (5.3.25) and (5.3.26) we get

$$
\Big\| u_1^{AL} - \psi^{AL} \Big\|^2_{E,\Omega_3} \leq C p^6 \mathrm{e}^{-\sigma_4 p}, \tag{5.3.27}
$$

On Ω_4, with the aid of (5.3.6), we estimate the integral

$$
\begin{aligned}
\Big\| u_1^{AL} \Big\|^2_{E,\Omega_4} &\leq \int_{\Omega_3} C(\varepsilon \mathrm{e}^{-(x+y)/\varepsilon})^2 dxdy\\
&\leq \int_{\kappa p\varepsilon}^1 \int_{\kappa p\varepsilon}^1 C(\varepsilon \mathrm{e}^{-(x+y)/\varepsilon})^2 dxdy\\
&\leq \varepsilon^2 (\kappa p \varepsilon)^2 C \mathrm{e}^{-4\kappa p}.
\end{aligned}
\tag{5.3.28}
$$

Using the cubic interpolant $I_3 u_1^{AL}$, the following estimate is deduced:

$$
\Big\| u_1^{AL} - \ell u_1^{AL} \Big\|^2_{E,\Omega_4} \leq \Big\| u_1^{AL} \Big\|^2_{E,\Omega_4} + \Big\| I_3 u_1^{AL} \Big\|^2_{E,\Omega_4} \leq C p^2 \mathrm{e}^{-4\kappa p},
$$

and therefore by the above and (5.3.27), we obtain

$$
\Big\| u_1^{AL} - \psi^{AL} \Big\|^2_{E,\Omega} \leq C p^2 \mathrm{e}^{-\tilde{\sigma}_4 p}. \tag{5.3.29}
$$

Using (5.3.16), (5.3.23), (5.3.29) and the analogous bounds for all other boundary and corner layers, we infer

$$\begin{aligned}\left\|u-\psi\right\|_{E,S}^2 &= \left\|\left(u^s+\sum_{k=1}^4 u_k^{BL}+\sum_{k=1}^4 u_k^{AL}\right)-\left(\mathcal{J}_p u^s+\sum_{k=1}^4 \psi_k^{BL}+\sum_{k=1}^4 \psi_k^{AL}\right)\right\|_{E,S}^2\\ &\leq \left\|u^s-\mathcal{J}_p u^s\right\|_{E,S}^2+\left\|\sum_{k=1}^4 (u_k^{BL}-\psi_k^{BL})\right\|_{E,S}^2+\left\|\sum_{k=1}^4 (u_k^{AL}-\sum_{k=1}^4 \psi_k^{AL})\right\|_{E,S}^2\\ &\leq Cp^7 \mathrm{e}^{-\sigma\kappa p}.\end{aligned}$$

From (5.0.4) we get the desired result, after we absorb the powers of p in the exponential. □

5.4 Numerical results

In this section we provide numerical evidence in order to validate the theoretical results presented before. We focus on a function with a boundary layer on the right side of $\Omega = (0,1)^2$, at $x = 1$. We choose $b = c = 1$ and f such that the exact solution to (4.1.1) is given by

$$u(x,y) = x^2y^2(1-x^2)(1-y^2)\mathrm{e}^{-(1-x)/\varepsilon}, \quad \forall\, x,y \in \Omega.$$

In this case the *Spectral Boundary Layer Mesh* is comprised of two elements, as is shown in Figure 5.10. (The parameter κ in the definition of the mesh is set equal to

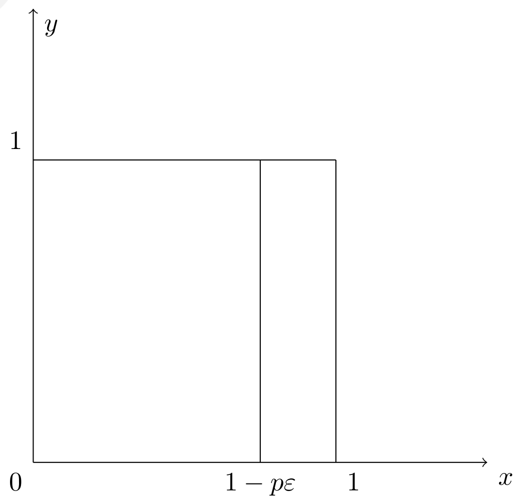


Figure 5.10: The Spectral Boundary Layer mesh for the numerical example

1. The corresponding solutions for the values $\{\varepsilon^j\}_{j=3}^8$ of the perturbation parameter,

are approximated by polynomials of degree $p = 3, \ldots, 11$ (in each variable). In Figure 5.11 we show the percentage relative error in the energy norm versus the square root of the number of degrees of freedom (DOF) in a semilog scale. Figure 5.11 agrees with Theorem 5.3.12.

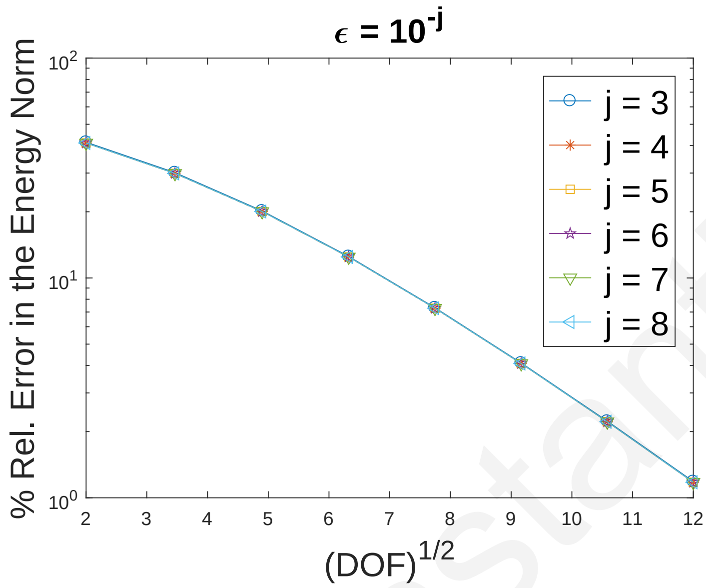


Figure 5.11: Convergence of the *hp*-version on the Spectral Boundary Layer mesh

As a second example, we choose f so that the exact solution is:

$$u(x) = x^2 \sin(\pi x) \sin^2(\pi y)\left(1 - \mathrm{e}^{-\frac{(1-x)}{\varepsilon}}\right), \quad \forall\, x, y \in \Omega = (0,1)^2.$$

We perform the same computations as before, and the results is shown in Figure 5.12. As the figure shows the results illustrate robust exponential, just like the theory predicts. We also see the lack of balance in the energy norm, manifesting itself as better performance as $\varepsilon \to 0$. Unfortunately, round-off error does not allow us to obtain (meaningful) results for smaller ε and/or for more DOF.

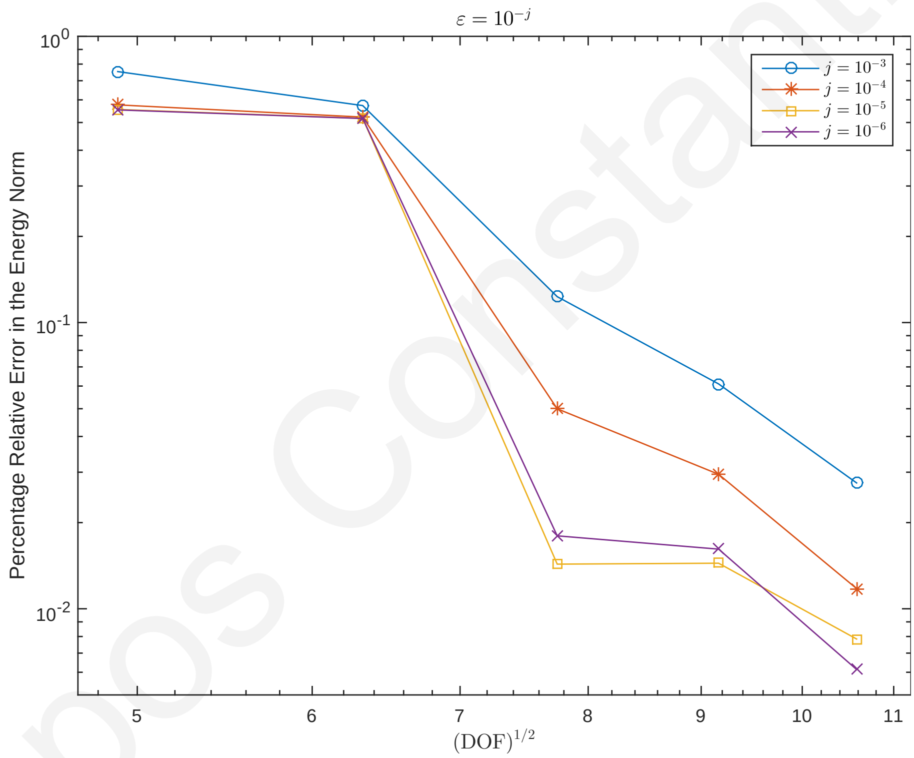


Figure 5.12: Convergence of the *hp*-version on the Spectral Boundary Layer mesh

Chapter 6

2-D Mixed *hp*-approximation results

6.1 Introduction

In this chapter we study the mixed finite element approximation of the fourth order SPP that is under study. Such formulations are preferred in many cases to the standard Finite Element approach for different reasons. A mixed formulation is obtained by the proper treatment of the initial equation. We rewrite the equation as a system, we formulate it into a variational form and then discretize it. Therefore, by utilizing the mixed method we use two spaces to approximate dual variables. Mixed formulations are a powerful tool that can be used to address problems which arise in a variety of scientific fields such as electrical enginnering, structural and fluid mechanics.

The mixed finite element method was introduced by Brezzi [16] and among others, Babuška, Oden, Lee [11, 8], Crouzeix and Raviart [25] have affected the development of the analysis of the mixed methods. The reader can study results about mixed methods in [28, 30] and in the books [14, 15, 17, 33].

We choose to use a mixed method in order to avoid the problem of using affine elements *or* making assumptions on the structure of the solution (like we did in Chapter 5). The mixed formulation allows us to use C^0 basis functions (since we are now discretizing H^1 and not H^2) to approximate the (now) dual variables. The present approach was presented in [31] where the authors studied an h-FEM on a Shishkin-type mesh. In this chapter we extend their results, by considering an hp-FEM approximation on the Spectral Boundary Layer mesh. We mention that the material of this chapter appears in [21], where the following assumption on the regularity of the solution was made.

Assumption 6.1.1. The BVP (4.1.1) has a solution u which can be decomposed as a smooth part u^S, a boundary layer part u^{BL} and a remainder r, viz.

$$u = u^S + \chi u^{BL} + r, \tag{6.1.2}$$

where χ is a smooth cut-off function, satisfying

$$\chi = \begin{cases} 1 & \text{for} \quad 0 < \rho < \rho_0/3 \\ 0 & \text{for} \quad \rho > 2\rho_0/3 \end{cases}.$$

Moreover, there exist constants $C_1, C_2, C_3, K_1, K_2, \omega, \delta > 0$, independent of ε but depending on the data, such that

$$\left\| D^n u^S \right\|_{0,\Omega} \leq C_1 \, |n|! K_1^{|n|} \; \forall \, n \in \mathbb{N}_0^2, \tag{6.1.3}$$

$$\left| \frac{\partial^{m+n} u^{BL}(\rho,\theta)}{\partial \rho^m \partial \theta^n} \right| \leq C_2 n! K_2^{m+n} \varepsilon^{1-m} e^{-\omega\rho/\varepsilon} \; \forall \, m,n \in \mathbb{N}, (\rho,\theta) \in \overline{\Omega}_0, \tag{6.1.4}$$

$$\left\| r \right\|_E \leq C_3 e^{-\delta/\varepsilon}. \tag{6.1.5}$$

Finally, there exist constants $C, K > 0$, depending only on the data, such that

$$\left\| D^n u \right\|_{0,\Omega} \leq C K^{|n|} \max \left\{ |n|^{|n|}, \varepsilon^{1-|n|} \right\} \; \forall \, n \in \mathbb{N}_0^2. \tag{6.1.6}$$

6.2 Mixed formulation and discretization

6.2.1 The mixed formulation of the problem

Let u be the solution of (4.1.1). We select [a]

$$w = \varepsilon \Delta u \in H^2(\Omega) \tag{6.2.1}$$

[a] The fact that $w \in H^2(\Omega)$ is a consequence of the smoothness of f and $\partial\Omega$

and as a consequence we seek a vector $(u, w) \in H_0^1(\Omega) \times H^1(\Omega)$ such that for all $(\phi, \psi) \in H^1(\Omega) \times H_0^1(\Omega)$,

$$\left.\begin{aligned} &\varepsilon\langle \nabla u, \nabla \phi\rangle_\Omega + \langle w, \phi\rangle_\Omega = 0, \\ &b\langle \nabla u, \nabla \psi\rangle_\Omega + c\langle u, \psi\rangle_\Omega - \varepsilon\langle \nabla w, \nabla \psi\rangle_\Omega = \langle f, \psi\rangle_\Omega. \end{aligned}\right\} \tag{6.2.2}$$

Equations in (6.2.2) describe the mixed formulation of (4.1.1). Evidently the corresponding bilinear form is given by

$$\mathcal{B}\big((u, w), (\psi, \phi)\big) = \varepsilon\langle \nabla u, \nabla \phi\rangle_\Omega + \langle w, \phi\rangle_\Omega + b\langle \nabla u, \nabla \psi\rangle_\Omega + c\langle u, \psi\rangle_\Omega - \varepsilon\langle \nabla w, \nabla \psi\rangle_\Omega \tag{6.2.3}$$

and, on that account, the associated energy norm is defined as

$$\big|\big|\big|(u, w)\big|\big|\big|_\Omega^2 := \|w\|_{0,\Omega}^2 + b\|\nabla u\|_{0,\Omega}^2 + c\|u\|_{0,\Omega}^2. \tag{6.2.4}$$

We mention that

$$\big|\big|\big|(u, w)\big|\big|\big|_\Omega^2 = \|u\|_{E,\Omega}^2,$$

since

$$\big|\big|\big|(u, w)\big|\big|\big|_\Omega^2 = \big|\big|\big|(u, \varepsilon\Delta u)\big|\big|\big|_\Omega^2 = \varepsilon^2\|\Delta u\|_{0,\Omega}^2 + b\|\nabla u\|_{0,\Omega}^2 + c\|u\|_{0,\Omega}^2.$$

Lemma 6.2.5 *The bilinear form is coercive with respect to the energy norm i.e.*

$$\big|\big|\big|(u, w)\big|\big|\big|_\Omega^2 \leq \mathcal{B}\big((u, w), (u, w)\big), \quad \forall\, u \in H_0^1(\Omega),\ w \in H^1(\Omega).$$

Proof. This was shown in [31]. □

6.2.2 Discretization by a mixed *hp*-FEM on smooth domains

The discrete version of (6.2.2) reads: find $(u_N, w_N) \in V_1^N \times V_2^N \subset H_0^1(\Omega) \times H^1(\Omega)$ [b] such that $\forall\, (\psi, \phi) \in V_1^N \times V_2^N$,

$$\left.\begin{aligned} &\varepsilon\langle \nabla u_N, \nabla \phi\rangle_\Omega + \langle w_N, \phi\rangle_\Omega = 0, \\ &b\langle \nabla u_N, \nabla \psi\rangle_\Omega + c\langle u_N, \psi\rangle_\Omega - \varepsilon\langle \nabla w_N, \nabla \psi\rangle_\Omega = \langle f, \psi\rangle_\Omega. \end{aligned}\right\} \tag{6.2.6}$$

[b] The finite dimensional subspaces V_1^N, V_2^N are defined in the sequel

We assume Ω is an open domain and $\partial\Omega$ is an analytic curve. Here we make use of the boundary fitted coordinates presented in Section 4.4.

Let the mesh $\Delta = \{\Omega_i\}_{i=1}^N$ be comprised of curvilinear quadrilaterals, subject to the usual conditions (see, e.g. [48]) and associate with each Ω_i a bijective mapping $M_i : S \to \overline{\Omega_i}$. We define the spaces

$$\mathcal{S}^p(\Delta) = \Big\{ u \in H^1(\Omega) : u\Big|_{\Omega_i} \circ M_i \in \mathbb{Q}_p(S),\ i = 1, \ldots, N \Big\},$$
$$\mathcal{S}_0^p(\Delta) = S^p(\Delta) \cap H_0^1(\Omega).$$

and we take $V_1^N = \mathcal{S}_0^p(\Delta), V_2^N = \mathcal{S}^p(\Delta)$, with the mesh Δ chosen following the construction in [48, 50].

We denote by Δ_A a *fixed* (asymptotic) mesh consisting of curvilinear quadrilateral elements Ω_i, $i = 1, \ldots, N_1$. These elements Ω_i are the images of the reference square S under the element mappings $M_{A,i}$, $i = 1, \ldots, N_1 \in \mathbb{N}$ (the subscript A emphasizes that they correspond to the asymptotic mesh). They are assumed to satisfy conditions (M1)–(M3) of [48] in order for the space $\mathcal{S}^p(\Delta)$ to have the necessary approximation properties. Moreover, the element mapings $M_{A,i}$ are assumed to be analytic (with analytic inverse), or equivalently [48]

$$\|(M'_{A,i})^{-1}\|_{\infty,S_{ST}} \leq C, \qquad \|D^\alpha M_{A,i}\|_{\infty,S_{ST}} \leq C\alpha!\gamma^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2, \quad i = 1, \ldots, N_1,$$

for some constants $\gamma, C > 0$. We also assume that the elements do not have a single vertex on the boundary $\partial\Omega$ but only complete, single edges. For convenience, we number the elements along the boundary first, i.e., Ω_i, $i = 1, \ldots, N_2 < N_1$ for some $N_2 \in \mathbb{N}$.

We now give the definition of the appropriate *Spectral Boundary Layer Mesh* $\Delta_{BL} = \Delta_{BL}(\kappa, p)$.

Definition 6.2.7. (Spectral Boundary Layer mesh $\Delta_{BL}(\kappa, p)$). Given parameters $\kappa > 0$, $p \in \mathbb{N}$, $\varepsilon \in (0, 1]$ and the (asymptotic) mesh Δ_A, the Spectral Boundary Layer mesh $\Delta_{BL}(\kappa, p)$ is defined as follows:

- If $\kappa p\varepsilon \geq 1/2$ then we are in the asymptotic range of p and we use the mesh Δ_A.
- If $\kappa p\varepsilon < 1/2$, we need to define so-called needle elements. We do so by splitting the elements Ω_i, $i = 1, \ldots, N_2$ into two elements Ω_i^{need} and Ω_i^{reg}. To this end, split

the reference square S into two elements

$$S^{need} = [0, \kappa p\varepsilon] \times [0, 1], \qquad S^{reg} = [\kappa p\varepsilon, 1] \times [0, 1],$$

and define the elements Ω_i^{need}, Ω_i^{reg} as the images of these two elements under the element map $M_{A,i}$ and the corresponding element maps as the concatenation of the affine maps

$$A^{need} : S \to S^{need}, \qquad (\xi, \eta) \to (\kappa p\varepsilon\xi, \eta),$$
$$A^{reg} : S \to S^{reg}, \qquad (\xi, \eta) \to (\kappa p\varepsilon + (1 - \kappa p\varepsilon)\xi, \eta),$$

with the element map $M_{A,i}$, i.e., $M_i^{need} = M_{A,i} \circ A^{need}$ and $M_i^{reg} = M_{A,i} \circ A^{reg}$. Explicitly:

$$\Omega_i^{need} = M_{A,i}(S^{need}), \qquad \Omega_i^{reg} = M_{A,i}(S^{reg}),$$
$$M_i^{need}(\xi, \eta) = M_{A,i}(\kappa p\varepsilon\xi, \eta), \qquad M_i^{reg}(\xi, \eta) = M_{A,i}(\kappa p\varepsilon + (1 - \kappa p\varepsilon)\xi, \eta).$$

An example of such a mesh construction is illustrated in Figure 6.1 on the unit circle. In total, the mesh $\Delta_{BL}(\kappa, p)$ consists of $N = N_1 + N_2$ elements if $\kappa p\varepsilon < 1/2$. By construction, the resulting mesh

$$\Delta_{BL} = \Delta_{BL}(\kappa, p) = \{\Omega_1^{need}, \ldots, \Omega_{N_1}^{need}, \Omega_1^{reg} \ldots, \Omega_{N_1}^{reg}, \Omega_{N_1+1}, \ldots, \Omega_N\}$$

is a regular admissible mesh in the sense of [48].

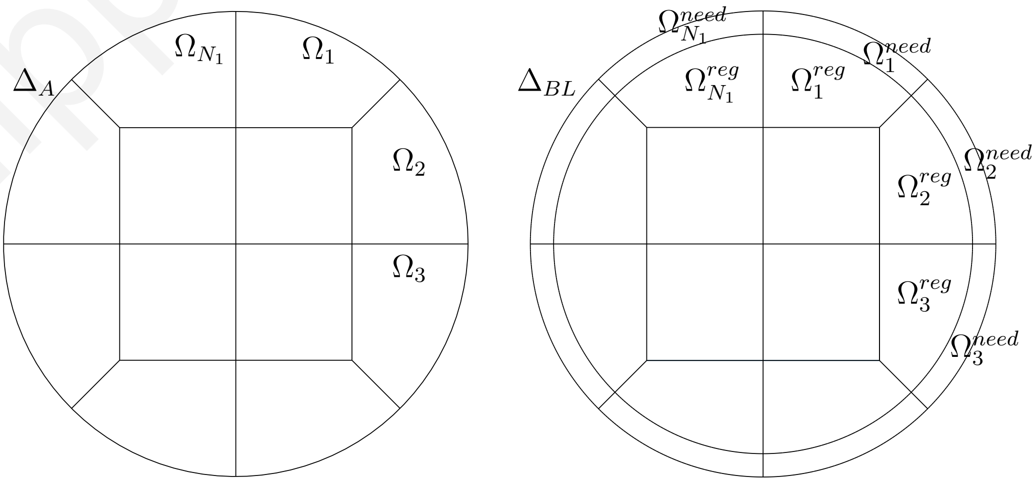


Figure 6.1: Example of an admissible mesh. Left: asymptotic mesh Δ_A. Right: Spectral boundary layer mesh Δ_{BL}.

Error Estimates

Our approximation will be based on the (element-wise) Gauß-Lobatto interpolant from [48, Prop. 3.11] (see also [45]) and its improvement in [50].

Lemma 6.2.8 *Let* (u, z) *be the solution to (6.2.2) and assume that (4.3.14) holds. Then there exist constants* κ_0*,* κ_1*,* C*,* $\beta > 0$ *independent of* $\varepsilon \in (0, 1]$ *and* $p \in \mathbb{N}$*, such that the following is true: For every* p *and every* $\kappa \in (0, \kappa_0]$ *with* $\kappa p \geq \kappa_1$*, there exist* $\pi_p u \in \mathcal{S}_0^p(\Delta_{BL}(\kappa, p)), \pi_p w \in \mathcal{S}^p(\Delta_{BL}(\kappa, p))$ *such that*

$$\max\left\{\|u - \pi_p u\|_{\infty,\Omega}, \|\nabla(u - \pi_p u)\|_{\infty,\Omega}, \|w - \pi_p w\|_{\infty,\Omega}, \varepsilon^{1/2}\|\nabla(w - \pi_p w)\|_{0,\Omega}\right\} \leq C\mathrm{e}^{-\beta p\kappa}$$

Proof. The proof is separated into two cases.

Case 1: $\kappa p\varepsilon \geq 1/2$ (asymptotic case).

In this case we use the asymptotic mesh Δ_A and u satisfies (4.3.34). Inspecting the proof of Corollary 3.5 of [50], we see that we can find $\pi_p u \in \mathcal{S}_0^p(\Delta_A)$ such that

$$\|u - \pi_p u\|_{\infty,\Omega} + \|\nabla(u - \pi_p u)\|_{\infty,\Omega} \leq Cp^2(\ln p + 1)^2\mathrm{e}^{-\beta p\kappa} \tag{6.2.9}$$

(due to the fact that for u the boundary layers are in the derivative, hence we have an extra power of ε in estimate (4.3.34)). For $w = \varepsilon\Delta u$, we have

$$\|D^\alpha w\|_{0,\Omega} \leq C\varepsilon K^{|\alpha|+2}\max\{(|\alpha| + 2)^{|\alpha|+2}, \varepsilon^{1-(|\alpha|+2)}\}\ \ \forall\ |\alpha| \in \mathbb{N}_0^2.$$

and by Corollary 3.5 of [50], there exists a $\pi_p w \in \mathcal{S}^p(\Delta_A)$ such that

$$\|w - \pi_p w\|_{\infty,\Omega} + \varepsilon^{1/2}\|\nabla(w - \pi_p w)\|_{0,\Omega} \leq Cp^2(\ln p + 1)^2\mathrm{e}^{-\beta p\kappa}. \tag{6.2.10}$$

This gives the result in the asymptotic case, once we absorb the powers of p in the exponential term and by adjusting the constants.

Case 2: $\kappa p\varepsilon < 1/2$ (pre-asymptotic case).

In this case we use the *Spectral Boundary Layer* mesh Δ_{BL} and u is decomposed as

$$u = u^S + \chi u^{BL} + r.$$

The approximation of u^S and r is constructed as in Case 1 above (basically taken to be that of [48]) and estimates like (6.2.9) may be obtained. For u^{BL} we use the approximation of Lemma 3.4 in [50], taking advantage of the extra power of ε in the regularity estimates. Ultimately, we get $\pi_p u \in \mathcal{S}_0^p(\Delta_A)$ such that (6.2.9) holds and for $w = \varepsilon\Delta u$, a similar argument gives (6.2.10). $\square$

The previous lemma allows us to measure the error between the solution (u, w) and its interpolant $(\pi_p u, \pi_p w)$. The following lemma allows us to measure the error between the interpolant and the finite element solution (u_N, w_N).

Lemma 6.2.11 *Let* $(u_N, w_N) \in \mathcal{S}_0^p(\Delta_{BL}(\kappa, p)) \times \mathcal{S}^p(\Delta_{BL}(\kappa, p))$ *be the solution to* (6.2.6). *Then there exist polynomials* $\pi_p u \in \mathcal{S}_0^p(\Delta_{BL}(\kappa, p))$, $\pi_p w \in \mathcal{S}^p(\Delta_{BL}(\kappa, p))$ *such that*

$$||| \, (\pi_p u - u_N, \pi_p w - w_N) \, |||^2 \leq C\mathrm{e}^{-\tilde{\beta}p},$$

with $C, \tilde{\beta} > 0$ *a constant independent of* ε *and* p.

Proof. Recall that the bilinear form $B\left((\cdot,\cdot),(\cdot,\cdot)\right)$, given by (6.2.3) is coercive, hence we have, with $\psi = \pi_p u - u_N \in S_0^p(\Delta_{BL}(\kappa, p))$ and $\phi = \pi_p w - w_N \in \mathcal{S}^p(\Delta_{BL}(\kappa, p))$,

$$\begin{aligned} |||(\psi, \phi)|||^2 &\leq B\left((\pi_p u - u, \pi_p w - w), (\psi, \phi)\right) = \varepsilon \left\langle \nabla\left(\pi_p u - u\right), \nabla\phi\right\rangle + \left\langle \pi_p w - w, \phi\right\rangle \\ &\quad + b\left\langle \nabla\left(\pi_p u - u\right), \nabla\psi\right\rangle + c\left\langle \pi_p u - u, \psi\right\rangle - \varepsilon\left\langle \nabla\left(\pi_p w - w\right), \nabla\psi\right\rangle \\ &= : I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Each term is treated using Cauchy-Schwarz and Lemma 6.2.8, except for I_1 which also requires the use of an inverse inequality:

$$\begin{aligned} |I_1| &= \left|\varepsilon\left\langle \nabla\left(\pi_p u - u\right), \nabla\phi\right\rangle\right| \leq \varepsilon \left\|\nabla\left(\pi_p u - u\right)\right\|_{0,\Omega} \left\|\nabla\phi\right\|_{0,\Omega} \\ &\leq C\left\|\nabla\left(\pi_p u - u\right)\right\|_{0,\Omega} \varepsilon(\kappa p\varepsilon)^{-1}p^2 \left\|\phi\right\|_{0,\Omega} \leq Cp\mathrm{e}^{-\beta p}\left\|\phi\right\|_{0,\Omega}, \\ |I_2| &= \left|\left\langle \pi_p w - w, \phi\right\rangle\right| \leq \left\|\pi_p w - w\right\|_{0,\Omega}\left\|\phi\right\|_{0,\Omega} \leq C\mathrm{e}^{-\beta p}\left\|\phi\right\|_{0,\Omega}, \\ |I_3| &= \left|b\left\langle \nabla\left(\pi_p u - u\right), \nabla\psi\right\rangle\right| \leq C\left\|\nabla\left(\pi_p u - u\right)\right\|_{0,\Omega}\left\|\nabla\psi\right\|_{0,\Omega} \leq C\mathrm{e}^{-\beta p}\left\|\nabla\psi\right\|_{0,\Omega}, \\ |I_4| &= \left|c\left\langle \pi_p u - u, \psi\right\rangle\right| \leq C\left\|\pi_p u - u\right\|_{0,\Omega}\left\|\psi\right\|_{0,\Omega} \leq C\mathrm{e}^{-\beta p}\left\|\psi\right\|_{0,\Omega}, \\ |I_5| &= \left|\varepsilon\left\langle \nabla\left(\pi_p w - w\right), \nabla\psi\right\rangle\right| \leq \varepsilon\left\|\nabla\left(\pi_p w - w\right)\right\|_{0,\Omega}\left\|\nabla\psi\right\|_{0,\Omega} \leq C\mathrm{e}^{-\beta p}\left\|\nabla\psi\right\|_{0,\Omega}. \end{aligned}$$

Hence, after absorbing the factor p into the exponential term in the estimate for I_1, we get

$$|||(\psi,\phi)|||_{\Omega}^{2} \leq C\mathrm{e}^{-\tilde{\beta}p}\left(\|\nabla\psi\|_{0,\Omega}+\|\phi\|_{0,\Omega}+\|\psi\|_{0,\Omega}\right) \leq C\mathrm{e}^{-\tilde{\beta}p}|||(\psi,\phi)|||_{\Omega}$$

and the proof is complete. □

Combining Lemmas 6.2.8 and 6.2.11 we establish the main result of this chapter.

Theorem 6.2.12 *Let $(u,w) \in H_0^1(\Omega)\times H^1(\Omega)$, $(u_N,w_N)\in V_1^N \times V_2^N$ be the solutions to (6.2.2) and (6.2.6), respectively. Then there exists a positive constant β, independent of ε but depending on κ, such that*

$$|||\,(u-u_N, w-w_N)\,|||_{\Omega} \leq C\mathrm{e}^{-\beta p}.$$

Proof. The triangle inequality gives

$$|||\,(u-u_N,w-w_N)\,|||_{\Omega} \leq |||\,(u-\pi_p u, w-\pi_p w)\,|||_{\Omega} + |||\,(\pi_p u - u_N, \pi_p w - w_N)\,|||_{\Omega}$$

and we then use Lemmas 6.2.8 and 6.2.11. □

Remark 6.2.1 *We note that the result of the previous theorem gives robust exponential convergence of the hp version of the FEM on the* Spectral Boundary Layer *mesh when the error is measured in the (energy) norm* (6.2.4)*. As was pointed out in [31], this norm is not* balanced *in the sense that, if u^{BL}, u^S are layer and smooth components of the solution, respectively, then*

$$|||(u^{BL},\varepsilon\Delta u^{BL})|||_{\Omega} = O(\varepsilon^{1/2})\ ,\ |||(u^{S},\varepsilon\Delta u^{S})|||_{\Omega} = O(1).$$

This means that as $\varepsilon\to 0$, the energy norm 'does not see the layer'. The convergence of our method in a stronger, balanced norm is beyond the scope of this thesis. Nevertheless, we study it computationally in the next section.

6.3 Numerical results

In this section we present the results of numerical computations for two examples, taken from [21].

Example 6.3.1. We set $b = c = 1$ and $\Omega = (0,1)^2$ and we choose the proper right hand side function f to ensure that the exact solution is given by

$$u(x,y) = X(x)Y(y).$$

Here

$$X(x) = \frac{1}{2}\left(\sin(\pi x) + \frac{\pi\varepsilon}{1-\varepsilon^{-1/\varepsilon}}(\varepsilon^{-x/\varepsilon} + \varepsilon^{(x-1)/\varepsilon} - 1 - \varepsilon^{-1/\varepsilon})\right),$$
$$Y(y) = \left(2y(1-y^2) + \varepsilon\left[ld(1-2y) - 3\frac{q}{l} + \left(\frac{3}{l} - d\right)\varepsilon^{-y//\varepsilon} + \left(\frac{3}{l} + d\right)\varepsilon^{(y-1)//\varepsilon}\right]\right),$$

with $l = 1 - \varepsilon^{-1/\varepsilon}$, $q = 2 - l$ and $d = 1/(q - 2\varepsilon l)$. (cf. [31, 32]). As can be seen, this function has boundary layers along each side of Ω (and no corner singularities), thus the appropriate mesh to approximate the solution is the *Spectral Boundary Layer Mesh* that is comprised of nine elements and it is shown at Figure 5.9. For the computations we take $\kappa = 1$, as this choice gave the smallest errors.

For $\varepsilon = 10^{-j}, j = 3, \dots, 9$, we approximate the solution by using polynomials of degree $p = 1, \dots, 20$ in each variable and we illustrate the percentage relative error in the energy norm versus the polynomial degree in a semi-log scale in Figure 6.2. As it is shown in the figure the exponential convergence of the method is visible, namely we observe straight lines (as p is increased). As $\varepsilon \to 0$, the errors get smaller, which is a manifestation of the lack of balance in the norm, even though we have robustness.

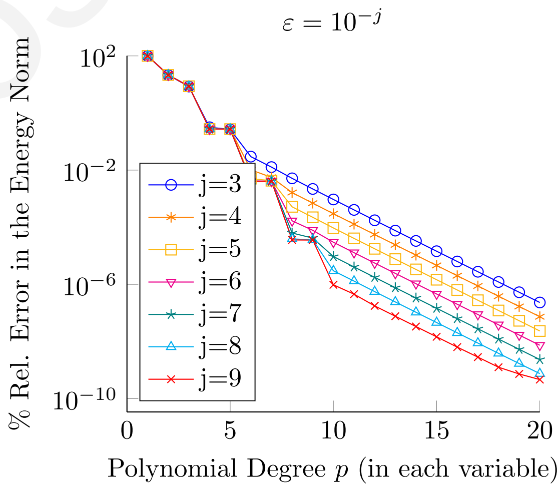


Figure 6.2: Energy norm convergence

Example 6.3.2. In this example we consider a smooth domain Ω with boundary Γ given by a curve $\gamma(\phi)$ using polar coordinates. We consider the so called Cranioid-curve with

$$\gamma(\phi) = \left(\frac{1}{4}\sin(\phi) + \frac{1}{2}\sqrt{1 - 0.9\cos(\phi)^2} + \frac{1}{2}\sqrt{1 - 0.7\cos(\phi)^2}\right) \cdot \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix} \quad (6.3.3)$$

and $\phi \in [0, 2\pi)$. On this domain Ω we choose $b = c = 1$ and as right-hand side $f(x, y) = 10x$. Figure 6.3 shows the (approximated) solutions to this problem for a

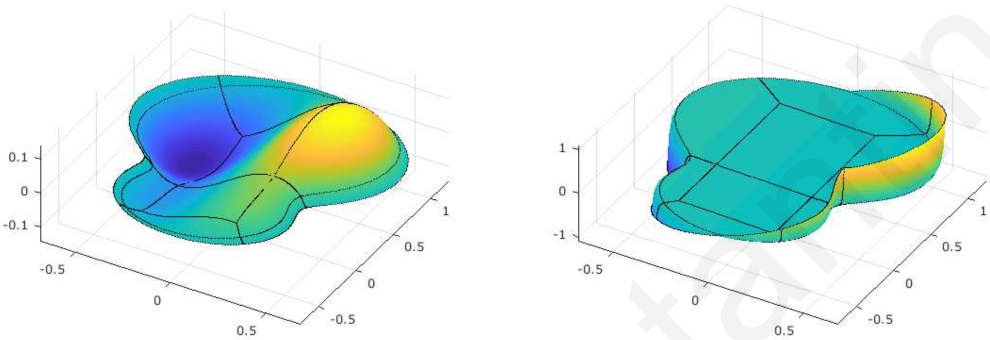


Figure 6.3: Approximate solutions u (left) and w (right), and mesh, for $\varepsilon = 10^{-2}$.

rather large value of $\varepsilon = 10^{-2}$. The solutions show the expected behaviour with a visible layer structure only for w. Note that the mesh consists of eight coarse and six needle, curved quadrilaterals in the boundary layer region. Here the width of the numerical layer region (and therefore of the quadrilaterals) is set to $\kappa p \varepsilon$ with $\kappa = 1$.

As the exact solution is unknown we use a numerically computed reference solution in its place. It is computed on a mesh generated by once refining the shown mesh in Figure 6.3 and with a polynomial degree $p = 18$ that is larger by two than the maximal one used for the simulations.

The results obtained in the energy-norm can be seen in the left picture of Figure 6.4. We observe a robust exponential convergence, visible as a straight decay in the semilog plot. The error curves for different values of ε lie on top of each other (different to our previous example but similar to the example of Ch.5, cf. Figure 5.11). Regarding Remark 6.2.1 we also investigated the error component $\|w - w_h\|_{0,\Omega}$ separately. The right picture of Figure 6.4 shows the error curves for this part, normalised by $|||(u, w)|||_{\Omega}$. Obviously, this part of the error decays proportionally to $\varepsilon^{1/2}$ and exponentially in p. Thus a balancing as indicated in Remark 6.2.1 will also give robust error measures.

Example 6.3.4. Finally, we choose $b = c = f = 1$ and $\Omega = (-1, 1)^2 \setminus (-1, 0)^2$, that is an L-shaped domain. This example is meant to examine what happens when the

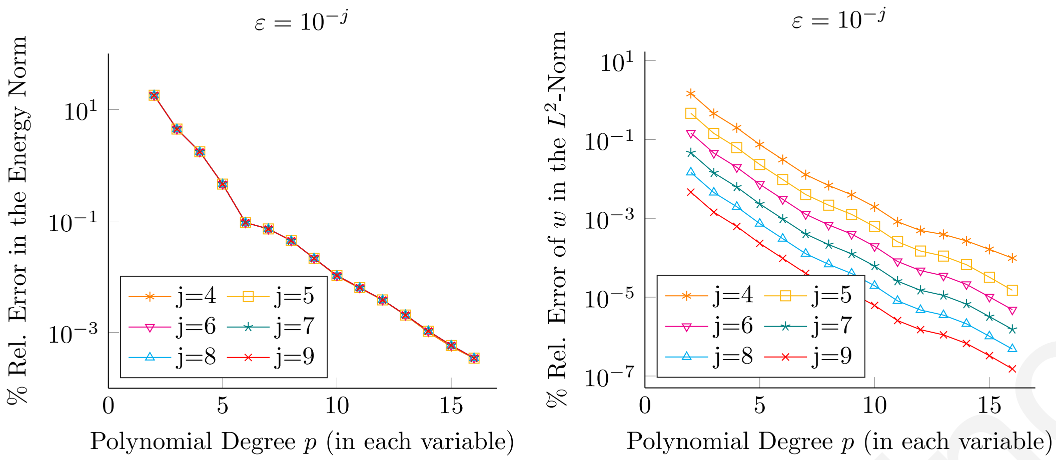


Figure 6.4: Convergence of the solutions on Cranioid-domain

domain is a polygon and the data does not satisfy any compatibility conditions (thus the solution contains corner singularities). In second-order singularly perturbed problems, the corner singularities have support only in the layer region [35]. For fourth-order singularly perturbed problems, this is still an open question but we note the following: the limiting problem is (essentially) a Poisson-like problem and it will feature its own (classical) corner singularities. As a result, the Spectral Boundary Layer mesh will need to include geometric refinement toward the re-entrant corner, in addition to the needle elements along the boundary.

Figure 6.5 shows two meshes: a mesh that includes *only* boundary layer refinement (left) and a mesh with *both* boundary layer and geometric refinement (right). The latter uses three refinements inside the layer region and two outside, with radio 0.15.

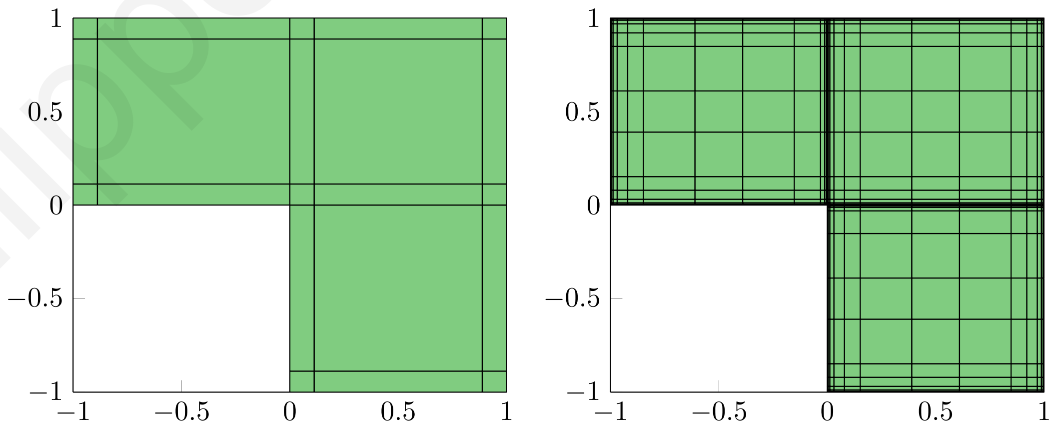


Figure 6.5: The meshes used in Example 6.3.4

Figure 6.6 shows the comparison of the two schemes. In particular, we show the percentage relative error in the energy norm versus the polynomial degree p, in a semilog scale. Since there is no exact solution availiable, we used a reference solution obtained with $p = 21$. Both seem to yield robust exponential convergence once ε is small enough, but the method that uses geometric refinement seems to give better results (at the expense, of course, of using much more degrees of freedom). Based on this experiment, we feel that this issue deserves further study (theoretical and computational) and we intend to do so in the near future.

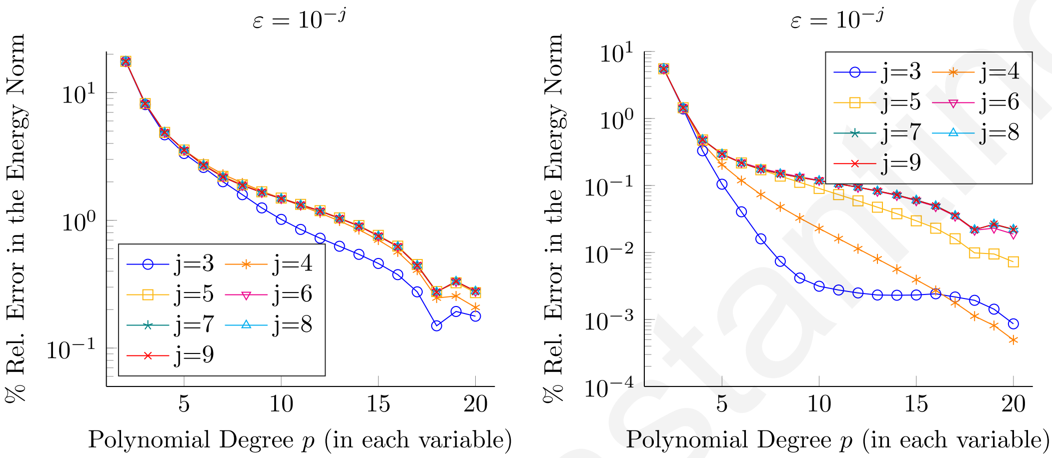


Figure 6.6: Energy norm convergence for Example 6.3.4

Acknowledgment: I would like to thank Dr. S. Franz and Dr. L. Ludwig for performing the computations that appeared in this chapter (and also in [21]).

Chapter 7

Conclusions and future work

In this dissertation we studied fourth order singularly perturbed problems in one- and two-dimensions. For the latter, we assumed that the domain was smooth and that no corner singularities are present. In one-dimension, we established appropriate regularity results, which enabled us to construct a robust, exponentially converging *hp*-FEM approximation measured in the energy norm. In two-dimensions, we studied the regularity of the solution in the case of smooth domains (and data) and obtained estimates which are explicit in the differentiation order and in ε. Then, we provided the numerical analysis for the approximation of the solution of 2-D problems in two cases: over a square under some assumptions and over an arbitrary smooth domain. For the former, we used the standard Galerkin FEM while for the latter, we used a mixed formulation.

One direction for future work is the study of the problem over polygonal domains. Regularity results that show the structure of the solution and establish a decomposition into a smooth part, boundary layers and corner singularities, would be welcomed by the research community. One thing to note is that we expect the corner singularities to extend beyond the layer region, which is different from what happens in second order problems (see [35]). The reason for this is the fact that the limiting problem is a second order PDE (similar to the Laplacian), thus the limiting solution will have singularities independently of ε.

Another direction for future research has to do with the approximation of the solution by appropriate methods. We showed that when the element mappings are affine, a C^1 approximation is possible and with the use of the *Spectral Boundary Layer* mesh, exponential convergence, independently of ε is attained, under certain assumptions on the regularity of the solutions. However, for smooth domains, curved elements must be used and it is not possible to construct a C^1 approximation on elements with non-affine maps. One solution, presented in Chapter 6, uses a *mixed formulation*, hence only requiring C^0 continuity along element boundaries. If one sticks to a non-mixed Galerkin formulation (like the one presented in Chapter 5), then treating curved elements would require a new approach. Two possible approaches are the following:

1. *The Discontinuous Galerkin FEM*, in which inter-element continuity is not required [4, 5, 7, 60].
2. *Isogeometric Analysis*, which combines high regularity basis functions and Galerkin's approach [23, 24, 37].

Both seem very promising, albeit on different sides of the spectrum, and they both deserve to be studied in the near future. We should mention that for the second approach, we are not aware of any work involving singularly perturbed problems.

A final suggestion is the study of *balanced* norm estimates. As mentioned in Section 3.2 we expect that a balanced norm will provide optimal results and therefore we intend to do so in one-dimension and in two-dimensions, at least over smooth domains.

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