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THE NONLINEAR PARABOLIC THIN OBSTACLE PROBLEM

DOCTOR OF PHILOSOPHY DISSERTATION

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DEPARTMENT OF MATHEMATICS AND STATISTICS

THE NONLINEAR PARABOLIC THIN OBSTACLE PROBLEM

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The present doctoral dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy of the University of Cyprus. It is a product of original work of my own, unless otherwise mentioned through references, notes, or any other statements.

Georgiana Chatzigeorgiou

Περίληψη

Η παρούσα διατριβή χωρίζεται σε δύο μέρη. Αρχικά, θεωρούμε ένα πρόβλημα συνοριακών τιμών με έναν πλήρως μη-γραμμικό παραβολικό τελεστή στο εσωτερικό και με συνοριακές συνθήκες σε μορφή κατά κατεύθυνσης παραγώγου:

$$\begin{cases} F(D^2u) - u_t = f, & \text{in } \ Q_1^+ \\ \beta \cdot Du = g, & \text{on } \ Q_1^* \\ u = u_0, & \text{on } \ \partial_p Q_1^+ \setminus Q_1^*. \end{cases} \tag{0.0.1}$$

Στη συνέχεια, μελετάμε ένα συνοριακό πρόβλημα εμποδίου με έναν πλήρως μη-γραμμικό παραβολικό τελεστή στο εσωτερικό:

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } \ Q_1^+ \\ \max\{u_n, \varphi - u\} = 0, & \text{on } \ Q_1^* \\ u = u_0, & \text{on } \ \partial_p Q_1^+ \setminus Q_1^* \end{cases} \tag{0.0.2}$$

Και στις δύο περιπτώσεις, θεωρούμε την εξίσωση καθώς και τις συνοριακές συνθήκες να ισχύουν μόνο με την έννοια του ιξώδους. Ο τελεστής F είναι ομοιόμορφα ελλειπτικός και με Q_1^+ συμβολίζουμε τον μοναδιαίο παραβολικό ημι-κύλινδρο στον $\mathbb{R}^{n+1}$ με το επίπεδο μέρος του συνόρου του να συμβολίζεται με Q_1^*. Επιπλέον, στο (0.0.1), με β συμβολίζουμε μια δεδομένη διανυσματική συνάρτηση η οποία να ικανοποιεί τη συνθήκη κατά κατεύθυνσης παραγώγου και στο (0.0.2) συμβολίζουμε με φ το λεπτό εμπόδιο.

Ο στόχος μας, και στις δύο περιπτώσεις, είναι να πάρουμε Hölder εκτιμήσεις οι οποίες να ισχύουν μέχρι και το σύνορο, για τη λύση και τις παραγώγους της. Αποτελέσματα αυτής της μορφής αποτελούν την πρωταρχική θεωρητική βάση η οποία μπορεί να χρησιμοποιηθεί για την αυστηρή μαθηματική ανάλυση προβλημάτων που συνδέονται άμεσα με εφαρμογές στις θετικές επιστήμες.

Abstract

The purpose of the present thesis is twofold. Firstly, we consider boundary value problems of oblique derivative type with fully nonlinear parabolic equations inside:

$$\begin{cases} F(D^2u) - u_t = f, & \text{in } \ Q_1^+ \\ \beta \cdot Du = g, & \text{on } \ Q_1^* \\ u = u_0, & \text{on } \ \partial_p Q_1^+ \setminus Q_1^*. \end{cases} \tag{0.0.3}$$

Secondly, we study a boundary obstacle problem with a fully nonlinear parabolic operator in the interior:

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } \ Q_1^+ \\ \max\{u_n, \varphi - u\} = 0, & \text{on } \ Q_1^* \\ u = u_0, & \text{on } \ \partial_p Q_1^+ \setminus Q_1^* \end{cases} \tag{0.0.4}$$

In both cases the equation as well as the boundary conditions are understood in the viscosity sense, F is a uniformly elliptic operator and Q_1^+ is the unit parabolic half-cylinder in $\mathbb{R}^{n+1}$ with the flat part of its boundary denoted by Q_1^*. Moreover in (0.0.3) β is a given vector function satisfying the obliqueness condition while in (0.0.4) φ is the so-called obstacle.

Our main aim in both situations is to derive up to the boundary Hölder estimates for the solution and its derivatives. These results produce the primary theory suitable for a rigorous mathematical analysis for problems with immediate connections to applications in other sciences.

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Dedication

In memory of my grandmother, Maria.

Contents

Chapter 1

Introduction

The subject of this dissertation is included in the general area of linear and fully nonlinear Partial Differential Equations (PDEs) and the theory of Free Boundaries. PDEs are perhaps the most important link between Mathematics and other sciences. Models that appear in Physics, Biology, Finance etc., are described by means of PDEs and the mathematical reasoning is essential for understanding and solving the corresponding problems. The main scope of this project is to develop the mathematical methodology which will be suitable for a rigorous mathematical analysis of questions included in the areas of Boundary Value Problems (BVPs) and of Free Boundary Problems (FBPs). A BVP consists of a PDE, which should be satisfied in the interior of a given domain, together with a set of conditions that should be satisfied on the boundary of this domain (the boundary conditions), which are related again with the unknown solution and its derivatives in a suitable manner. A FBP is again described by a PDE but it is also characterized by the appearance of boundaries whose position and geometry are apriori unknown. That is, the given domain is now splitted by an unknown interface (the free boundary), which is characterized by given conditions (the free boundary conditions) that the unknown solution should satisfy, so that a PDE is satisfied on the one part of the domain while on the other part we may ask to be satisfied a different PDE or differential inequalities, etc., depending on the problem. We also mention that in the study of problems which deal with PDEs it is usually convenient to consider solutions that satisfy the equation in a suitable weaker sense (these solutions may not even be differentiable). The notion of weak solution turns out to be advantageous for many reasons, for example: it is easier to show existence of such solutions, they appear naturally is applications, etc. From the mathematical point of view we want to examine the properties of these solutions and to find out the assumptions on the given data that make these solutions to be smooth.

The purpose of the present thesis is twofold. Firstly, we consider BVPs of oblique derivative type governed by fully nonlinear parabolic operators (Chapter 3). Here both the equation as well as the boundary condition are understood in the viscosity sense. In the second part we study the viscosity solution of a boundary obstacle problem (that is, a FBP) with a fully nonlinear parabolic equation in the interior (Chapter 4). In both cases we derive up to the boundary Hölder estimates for the solution and its derivatives. In the following we give a brief description of the structure of this thesis together with a historical background concerning the problems under study.

In Chapter 2 we introduce the notation as well as the basic definitions and terminology we use throughout the text. Moreover, we prove some main properties of viscosity solutions that will be used in our theory.

In Chapter 3, our main objective is to study the regularity of viscosity solutions of fully nonlinear parabolic equations with oblique boundary conditions of the form

$$\begin{cases} F(D^2u) - u_t = f, & \text{in } Q_1^+ \\ \beta \cdot Du = g, & \text{on } Q_1^* \\ u = u_0, & \text{on } \partial_p Q_1^+ \setminus Q_1^* \end{cases} \tag{1.0.1}$$

where F is a uniformly elliptic convex operator in S_n, f, g and u_0 are given data and $\beta : Q_1^* \to \mathbb{R}^n$ is a given vector function with $\beta_n \geq \delta_0 > 0$. By Q_1^+ we denote the half parabolic cylinder with flat part Q_1^* (detailed definitions are contained in Chapter 2).

A viscosity solution is apriori assumed to be merely continuous and satisfies the equation (or the boundary condition) only in a weak sense which, roughly speaking, says that smooth functions that touches the solution from below/above at a point must be super/sub-solutions of the equation (or the boundary condition) in the classical sense. The notion of viscosity solution was first introduced and studied for first order nonlinear equations, in particular of Hamilton-Jacobi type. Then the notion extended to the case of second order fully nonlinear elliptic/parabolic equations. A first detailed and self-contained survey of the basic theory of viscosity solutions for second order equations is [16]. Here the precise definition of the notion as well as the structural conditions that the equation should satisfy to be compatible with this definition are explained. Also one can find the proofs of existence and comparison results together with an extensive bibliography. The first regularity result regarding viscosity solutions of elliptic equations was given by L. Caffarelli in his seminal work [9] (more details are given in [10]) where the author constructed first two basic tools, an Aleksandrov-Bakelman-Pucci (ABP) estimate and a Harnack inequality. Then he was able to apply an approximation-type technique.

There is a vast literature that concerns oblique derivative boundary value problems for elliptic operators. For the linear elliptic case we refer the reader to the book of G. Lieberman [32] and references therein. In the case of fully nonlinear elliptic operators, existence and uniqueness of viscosity solutions are obtained in [24] (where boundary conditions are in fact more general). Regularity of viscosity solutions was obtained in [33] for the Neumann case and in [27] for the oblique derivative case.

The corresponding theory for linear parabolic equations with oblique derivative boundary data is also well understood. For existence, uniqueness and regularity results we refer to [28], [29], [36], [35], [47] and [20]. For the case when the operator is fully nonlinear parabolic, comparison and existence results for viscosity solutions can be found in [25]. Interior and boundary estimates for fully nonlinear parabolic equations with Dirichlet conditions have been studied by L. Wang in a series of papers (see [44], [45], [46]) where the methods of [9] were successfully adopted in the parabolic framework since the author were able to obtain parabolic analogs of ABP and Harnack estimates. Moreover apriori Hölder estimates for classical solutions appeared in [43], [30]. Our main goal is to investigate the regularity of viscosity solutions.

We prove, under suitable assumptions, Hölder regularity (in the parabolic sense) for u and its first and second derivatives (note that a viscosity solution is only assumed to be continuous so we have to prove also the existence of its derivatives). We start the development of our theory with the Hölder regularity of u (subsection 3.2.2) which is derived through a boundary Harnack-type inequality we construct. Note that the special case of Neumann conditions is considered in subsection 3.2.1 where we prove that the Hölder regularity can be also obtained through a reflection principle. Next we continue with the first and second order regularity (in the parabolic sense) in sections 3.3 and 3.4 respectively. The main idea is to use an approximation method as used (for the elliptic case) in [27] (which is first introduced in [9]). That is, we try to approximate inductively the general problem (1.0.1) by "simpler" ones for which the regularity is known. The "simpler" problems will be special cases of (1.0.1) where the equation as well as the boundary condition are homogeneous and the vector β is constant. To attack the regularity for this type of problems we first examine the regularity for the parabolic Neumann problem (that is, when $\beta = (0, \dots, 0, 1)$) which is obtained by adapting the ideas of [33] in the parabolic framework (subsections 3.3.1, 3.3.2 and 3.4.1). Then, we observe that after a suitable change of variables (introduced in section 2.5) a constant oblique derivative problem can be transformed into a Neumann problem. Moreover, to put through the approximation technique a suitable form of the ABP estimate is needed. This powerful tool is derived in section 3.1.

In Chapter 4 we study the regularity of the viscosity solution of the following thin obstacle problem in a half-cylinder,

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in} \quad Q_1^+ \\ u_y \leq 0, & \text{on} \quad Q_1^* \\ u \geq \varphi, & \text{on} \quad Q_1^* \\ u_y = 0, & \text{on} \quad Q_1^* \cap \{u > \varphi\} \\ u = u_0, & \text{on} \quad \partial_p Q_1^+ \setminus Q_1^* \end{cases} \tag{1.0.2}$$

where, F is a uniformly elliptic convex operator on S_n with ellipticity constants λ and Λ and $\varphi : \overline{Q}_1^* \to \mathbb{R}$, $u_0 : \partial_p Q_1^+ \setminus Q_1^* \to \mathbb{R}$ are given functions. Function φ is the so-called obstacle and $u_0 \geq \varphi$ on $\partial_p Q_1^*$ for compatibility reasons. Our aim is to derive first order regularity for u up to the flat boundary Q_1^*.

The classical obstacle problem as well as the thin obstacle problem are originated in the context of elasticity since model the shape of an elastic membrane which is pushed by an obstacle (which may be very thin) from one side affecting its shape and formation. The same model appears in control theory when trying to evaluate the optimal stopping time for a stochastic process with payoff function. Important cases of obstacle type problems occur when the operators involved are fractional powers of the Laplacian as well as nonlinear operators since they appear, among others, in the analysis of anomalous diffusion, in quasi-geostrophic flows, in biology modeling flows through semi-permeable membranes for certain osmotic phenomena and when pricing American options regulated by assets evolving in relation to jump processes.

Thin (or boundary) obstacle problem (or Signorini's problem) was extensively studied in the elliptic case. For Laplace equation and more general elliptic PDEs in divergence form the problem can be also understood in the variational form, that is as a problem of minimizing a suitable functional over a suitable convex class of functions which should stay above the obstacle on a part of the boundary (or on a sub-manifold of co-dimension at least 1) of the domain of definition. The $C^{1,\alpha}$-regularity of the weak solution for the harmonic case was proved first in 1979 by L. Caffarelli in [8] who treats also the divergence case for regular enough coefficients. Results for more general divergence-type elliptic operators can be found in [42]. For optimal regularity and regularity of the free boundary in the case of linear elliptic equations we refer to [2] and [5] where the harmonic case is studied and to [22], [19], [26] for the case of variable coefficients. Similar results exist also for the case of fractional Laplacians. Regularity of the solution for the classical (thick) obstacle problem was studied in [40], then via the extension problem introduced in [13] the thin obstacle problem was treated in [12]. Finally, for fully nonlinear elliptic operators, regularity of the viscosity solution was proved in [34] (see also [18]) while for optimal and free boundary regularity the only existing work is [38].

The corresponding regularity theory for thin obstacle problems of parabolic type is much less developed. The $C^{1,\alpha}$-regularity of the weak solution was obtained in 1982 by I. Athanasopoulos in [6] who studied the case of heat equation and the case of smooth enough linear parabolic equation. The case of more general linear parabolic operators was examined in [41] and [1]. Optimal and free boundary regularity for the caloric case have been obtained very recently in [4] (see also [17]). Finally for the case of parabolic operators of fractional type we refer the reader to [3] and [11].

Our purpose is to combine the techniques of [8], [6] and [34] adapting them in our fully nonlinear parabolic framework. To achieve this we need up to the boundary Hölder estimates for viscosity solutions of nonlinear parabolic equations with Neumann boundary conditions (as [33] is used in [34]), that is the results of Chapter 3.

We start Chapter 4 discussing the natural assumptions we make on the given data. Our first main accomplishment is to obtain the semi-concavity properties of the solution (section 4.2). We prove Lipschitz continuity in space variables, a lower bound for u_t and for the second tangential derivatives of u (semi-convexity) and an upper bound for the second normal derivative of u (semi-concavity). All these bounds are universal and hold up to the flat boundary Q_1^*. The boundedness of the first and second normal derivatives ensures the existence of u_{y^+} on Q_1^*. Our first intention is to prove that $u_{y^+} \leq 0$ on Q_1^* (which apriori holds only in the viscosity sense). To achieve this we use the penalized problem defined and studied in section 4.1. Finally in section 4.3 we prove the main theorem. To do so we obtain first an estimate in measure (Lemma 76) for u_{y^+} on Q_1^* and subsequently we see how such a property can be carried inside Q_1^+ (Lemma 77). An iterative application of the above two properties gives the regularity of u_{y^+} on Q_1^* around free boundary points (Lemma 78) and then our problem can be treated as a non-homogeneous Neumann problem.

Regarding problem (1.0.2) the demanding questions of the optimal regularity of the solution as well as the regularity of the free boundary are still open. In the linear elliptic theory, optimal regularity obtained first in [2] and the free boundary regularity in [5] where in both cases the methods rely on monotonicity formulas-techniques. In the linear parabolic case one realizes immediately that the presence of time creates even more non-trivial difficulties. The corresponding results in the parabolic case developed very recently (see for instance [4]) where the authors notice first that it is necessary to pay extra attention on the regularity of the time derivative and then to consider a suitable monotonicity formula. We may observe that since monotonicity formulas follow from the structure of the operators, one cannot expect to have such a formulas in the fully nonlinear case, so a different treatment is needed. Even in the elliptic fully nonlinear case the only known result is [38] where the authors show the dichotomy around a free boundary point x_0: Either *1.* $\sup_{B_r(x_0)}(u-\varphi) \geq cr^{2-\epsilon_0}$ or *2.* $\sup_{B_r(x_0)}(u-\varphi) \leq C_\epsilon r^{2-\epsilon}$. In addition around the points where *1.* holds they show that the free boundary is Lipschitz graph. We conclude that optimal and free boundary regularity for problem (1.0.2) are questions which seem to be particularly challenging since one has to deal not only with the lack of monotonicity formulas due to the nonlinear character of the problem but also with the regularity of the time derivative.

Chapter 2

Preliminaries

In this chapter we give the basic definitions and properties needed in the sequel.

2.1 Parabolic Topology and Function Spaces

First, we introduce the notation we follow in the present text.

2.1.1 Notation

We use $X = (x, y)$ to denote a point of $\mathbb{R}^n$, where $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. Moreover, if $t \in \mathbb{R}$ then $P = (X, t)$ denotes a point in $\mathbb{R}^{n+1}$, where X are the space variables and t is the time variable.

The Euclidean ball in $\mathbb{R}^n$ will be denoted by

$$B_r(X_0) := \{X \in \mathbb{R}^n : |X - X_0| < r\}$$

and the elementary cylinder in $\mathbb{R}^{n+1}$ by

$$Q_r(X_0, t_0) := B_r(X_0) \times (t_0 - r^2, t_0].$$

Also, let us define the following half and thin-cylinders, for $r > 0, X_0 \in \mathbb{R}^n_+, t_0 \in \mathbb{R}$

$$Q_r^+(X_0, t_0) := Q_r(X_0, t_0) \cap \{y > y_0\},$$

$$Q_r^-(X_0, t_0) := Q_r(X_0, t_0) \cap \{y < y_0\},$$

$$Q_r^*(X_0, t_0) := Q_r(X_0, t_0) \cap \{y = y_0\}.$$

Note that most of the times when the center is $(0,0)$ we omit it, i.e. we write Q_r instead of $Q_r(0,0)$, B_r instead of $B_r(0)$ etc.

Ω°, $\overline{\Omega}$ and $\partial\Omega$ denote the interior, the closure and the boundary, respectively, of the domain $\Omega \subset \mathbb{R}^{n+1}$ in the sense of the Euclidean topology of $\mathbb{R}^{n+1}$. Also, for a set $\Omega \in \mathbb{R}^{n+1}$ we define its **parabolic interior** to be,

$$int_p(\Omega) := \{(X,t) \in \mathbb{R}^{n+1} : \textit{ there exists } r > 0 \textit{ so that } Q_r^\circ(X,t) \subset \Omega\}$$

and its **parabolic boundary**,

$$\partial_p(\Omega) := \overline{\Omega} \setminus int_p(\Omega).$$

Domains of the form $U \times (t_1, t_2) \in \mathbb{R}^{n+1}$ are called **cylindrical domains** and elementary cylinders consist a special case.

Let us also define the **parabolic distance** for $P_1 = (X,t),\ P_2 = (Y,s) \in \mathbb{R}^{n+1}$,

$$p(P_1, P_2) := \max\{|X - Y|, |t - s|^{1/2}\},$$

where $|\cdot|$ is the Euclidean norm. Note that $Q_r(P_0)$ is the set $\{P \in \mathbb{R}^{n+1} : p(P, P_0) < r, t \leq t_0\}$.

2.1.2 Parabolic Hölder Continuity

Aiming to define the spaces H^α, $H^{\alpha+1}$ and $H^{\alpha+2}$, for $0 < \alpha \leq 1$, we introduce first the following semi-norms. For a function f defined in a domain $\Omega \subset \mathbb{R}^{n+1}$ we define,

$$[f]_{\alpha;\Omega} := \sup_{P_1, P_2 \in \overline{\Omega}, P_1 \neq P_2} \frac{|f(P_1) - f(P_2)|}{p(P_1, P_2)^\alpha}.$$

$$\langle f \rangle_{\alpha+1;\Omega} := \sup_{\substack{(X,t_1),(X,t_2) \in \overline{\Omega} \\ t_1 \neq t_2}} \frac{|f(X,t_1) - f(X,t_2)|}{|t_1 - t_2|^{\frac{\alpha+1}{2}}}.$$

Then we say that,

- $f \in H^\alpha(\overline{\Omega})$ if

$$||f||_{H^\alpha(\overline{\Omega})} := \sup_{\overline{\Omega}} |f| + [f]_{\alpha;\Omega} < +\infty.$$

So f is α-Hölder continuous in X-variables and $\frac{\alpha}{2}$- Hölder continuous in t-variable.

- $f \in H^{\alpha+1}(\overline{\Omega})$ if

$$||f||_{H^{\alpha+1}(\overline{\Omega})} := \sup_{\overline{\Omega}} |f| + \sum_{i=1}^{n} \sup_{\overline{\Omega}} |D_i f| + \sum_{i=1}^{n} [D_i f]_{\alpha;\Omega} + \langle f \rangle_{\alpha+1;\Omega} < +\infty$$

  where $D_i f$ denote the first partial derivatives of f with respect to the space variables assuming that there exist. So, f is $(\alpha+1)$ - Hölder continuous in X-variables while in t-variable f is merely $\frac{\alpha+1}{2}$ - Hölder continuous (with $\frac{1}{2} < \frac{\alpha+1}{2} < 1$). Note that $||f||_{H^{\alpha+1}}$ do not provide anything about the existence of the first time derivative f_t.

- $f \in H^{\alpha+2}(\overline{\Omega})$ if

$$\begin{aligned} ||f||_{H^{\alpha+2}(\overline{\Omega})} := & \sup_{\overline{\Omega}} |f| + \sum_{i=1}^{n} \sup_{\overline{\Omega}} |D_i f| + \sup_{\overline{\Omega}} |f_t| + \sum_{i,j=1}^{n} \sup_{\overline{\Omega}} |D^2_{ij} f| \\ & + [f_t]_{\alpha;\Omega} + \sum_{i,j=1}^{n} [D^2_{ij} f]_{\alpha;\Omega} + \sum_{i=1}^{n} \langle D_i f \rangle_{\alpha+1;\Omega} < +\infty \end{aligned}$$

  where $D_{ij} f$ denote the second partial derivatives of f with respect to the space variables assuming that there exist.

Due to the nonlinear character of the problem we study, most of the times we prove $H^{\alpha+1}$ and $H^{\alpha+2}$-regularity results in the punctual sense at a point. Next, we explain what does this mean. Assume that the function u is defined in $\overline{\Omega}$, where $\Omega := \Omega_0 \times (t_1, t_2)$ is a cylindrical domain of $\mathbb{R}^{n+1}$, we say that u **is punctually** $H^{\alpha+1}$ **at a point** $P_1 \in \overline{\Omega}$ if there exists a polynomial $R_{1;P_1}$ of first order in X, that is $R_{1;P_1}(X) = A_{P_1} + B_{P_1} \cdot (X - X_1)$, where $A_{P_1} \in \mathbb{R}$ and $B_{P_1} \in \mathbb{R}^n$ and some cylinder $\overline{Q}_{r_1}(P_1) \subset \Omega$, so that for any $0 < r < r_1$,

$$|u(X,t) - R_{1;P_1}(X)| \leq K\ r^{1+\alpha}, \quad \text{for every} \ \ (X,t) \in \overline{Q}_r(P_1)$$

for some constant $K > 0$. In the same spirit, we say that u **is punctually** $H^{\alpha+2}$ **at a point** $P_1 \in \overline{\Omega}$ if there exists a polynomial $R_{2;P_1}$ of second order in X and of first order in t, that is $R_{2;P_1}(X,t) = A_{P_1} + B_{P_1} \cdot (X - X_1) + C_{P_1}(t - t_1) + \frac{1}{2}(X - X_1)^{\tau} D_{P_1}(X - X_1)$, where $A_{P_1}, C_{P_1} \in \mathbb{R}$, $B_{P_1} \in \mathbb{R}^n$ and $D_{P_1} \in \mathbb{R}_{n\times n}$ and some cylinder $\overline{Q}_{r_1}(P_1) \subset \Omega$ so that for any $0 < r < r_1$,

$$|u(X,t) - R_{2;P_1}(X,t)| \leq K\ r^{2+\alpha}, \quad \text{for every} \ \ (X,t) \in \overline{Q}_r(P_1)$$

for some constant $K > 0$. Note that when we study points on a flat part of the boundary, the cylinders in the above definitions are replaced by half-cylinders.

The following lemmas state that if $u \in H^{\alpha+1}(\Omega)$ (or $H^{\alpha+2}(\Omega)$) then it is puctually $H^{\alpha+1}$ (or $H^{\alpha+2}$) at every point of Ω.

Lemma 1. *Let $u \in H^{\alpha+1}(\overline{\Omega})$, where $\Omega := \Omega_0 \times (t_1, t_2)$ is a cylindrical domain of $\mathbb{R}^{n+1}$. Then, for any $P_1 \in \overline{\Omega}$,*

$$|u(X,t) - u(X_1,t_1) - \nabla_X u(X_1,t_1) \cdot (X - X_1)| \leq C||u||_{H^{\alpha+1}(\overline{\Omega})}\, p(P_1, P)^{1+\alpha}$$

for every $P \in \overline{\Omega}$.

Proof. First observe that by triangle inequality,

$$\begin{aligned} &|u(X,t) - u(X_1,t_1) - \nabla_X u(X_1,t_1) \cdot (X - X_1)| \\ &\leq |u(X,t) - u(X_1,t) - \nabla_X u(X_1,t) \cdot (X - X_1)| + |u(X_1,t) - u(X_1,t_1)| \\ &\quad + |\nabla_X u(X_1,t) - \nabla_X u(X_1,t_1)||X - X_1|. \end{aligned} \tag{2.1.1}$$

Now, set $v^t(X) := u(X,t) - u(X_1,t) - \nabla_X u(X_1,t) \cdot (X - X_1)$, then

$$v^t(X_1) = 0 \quad \text{and} \quad \nabla_X v^t(X) = \nabla_X u(X,t) - \nabla_X u(X_1,t).$$

The second one yields, $|\nabla_X v^t(X)| \leq ||u||_{H^{\alpha+1}(\overline{\Omega})}|X - X_1|^\alpha$. By mean value theorem we have,

$$\begin{aligned} |v^t(X)| = |v^t(X) - v^t(X_1)| = |\nabla_X v^t(\Xi)||X - X_1| &\leq ||u||_{H^{\alpha+1}(\overline{\Omega})}|\Xi - X_1|^\alpha |X - X_1| \\ &\leq ||u||_{H^{\alpha+1}(\overline{\Omega})}|X - X_1|^{\alpha+1} \end{aligned}$$

where Ξ is lying on the line segment connecting X with X_1.

Hence, returning to (2.1.1),

$$\begin{aligned} &|u(X,t) - u(X_1,t_1) - \nabla_X u(X_1,t_1) \cdot (X - X_1)| \\ &\leq |v^t(X)| + ||u||_{H^{\alpha+1}(\overline{\Omega})}|t - t_1|^{\frac{\alpha+1}{2}} + ||u||_{H^{\alpha+1}(\overline{\Omega})}|t - t_1|^{\frac{\alpha}{2}}|X - X_1| \\ &\leq C||u||_{H^{\alpha+1}(\overline{\Omega})}\, p(P_1, P)^{1+\alpha}. \end{aligned}$$

□

Lemma 2. *Let $u \in H^{\alpha+2}(\overline{\Omega})$, where $\Omega := \Omega_0 \times (t_1, t_2)$ is a cylindrical domain of $\mathbb{R}^{n+1}$. Then, for any $P_1 \in \overline{\Omega}$,*

$$|u(X,t) - u(X_1,t_1) - \nabla_X u(X_1,t_1) \cdot (X - X_1) - u_t(X_1,t_1)(t - t_1) \\ - \frac{1}{2}(X - X_1)^\tau D_X^2 u(X_1,t_1)(X - X_1)| \leq C||u||_{H^{\alpha+2}(\overline{\Omega})}\ p(P_1, P)^{2+\alpha} \tag{2.1.2}$$

for every $P \in \overline{\Omega}$.

Proof. First observe that by triangle inequality and denoting by (I) the left-hand side of (2.1.2) we have

$$(I) \leq |u(X,t) - u(X_1,t) - \nabla_X u(X_1,t) \cdot (X - X_1) - \frac{1}{2}(X - X_1)^\tau D_X^2 u(X_1,t)(X - X_1)| \\ + |u(X_1,t) - u(X_1,t_1) - u_t(X_1,t_1)(t - t_1)| + |\nabla_X u(X_1,t) - \nabla_X u(X_1,t_1)||X - X_1| \\ + \frac{1}{2}|X - X_1|^2 |D_X^2 u(X_1,t) - D_X^2 u(X_1,t_1)| =: (II) + (III) + (IV) + (V).$$

For (II), set

$$v^t(X) := u(X,t) - u(X_1,t) - \nabla_X u(X_1,t) \cdot (X - X_1) - \frac{1}{2}(X - X_1)^\tau D_X^2 u(X_1,t)(X - X_1)$$

then

$$v^t(X_1) = 0 \quad \text{and} \quad v^t_{x_i}(X,t) = u_{x_i}(X,t) - u_{x_i}(X_1,t) - \nabla_X u_{x_i}(X_1,t) \cdot (X - X_1)$$

for any $i = 1, \ldots, n$. Lemma 1 applied to $u_{x_i} \in H^{\alpha+1}$ gives

$$|v^t_{x_i}| \leq C||u||_{H^{\alpha+2}(\overline{\Omega})}|X - X_1|^{1+\alpha}.$$

By mean value theorem we have,

$$|v^t(X)| = |v^t(X) - v^t(X_1)| = |\nabla_X v(\Xi,t)||X - X_1| \\ \leq C||u||_{H^{\alpha+2}(\overline{\Omega})}|\Xi - X_1|^{1+\alpha}|X - X_1| \leq C||u||_{H^{\alpha+2}(\overline{\Omega})}|X - X_1|^{\alpha+2}$$

where Ξ is lying on the line segment connecting X with X_1. That is

$$(II) \leq C||u||_{H^{\alpha+2}(\overline{\Omega})}\ p(P_1, P)^{2+\alpha}.$$

For (III), set $\psi(t) := u(X_1,t) - u(X_1,t_1) - u_t(X_1,t_1)(t-t_1)$, then

$$\psi(t_1) = 0 \quad \text{and} \quad \psi_t(t) = u_t(X_1,t) - u_t(X_1,t_1)$$

thus, $|\psi_t(t)| \leq ||u||_{H^{\alpha+2}(\overline{\Omega})}|t-t_1|^{\alpha/2}$. By mean value theorem we have,

$$\begin{aligned}|\psi(t)| = |\psi(t) - \psi(t_1)| = |\psi_t(\Xi)||t-t_1| &\leq ||u||_{H^{\alpha+2}(\overline{\Omega})}|\Xi - t_1|^{\alpha/2}|t-t_1| \\ &\leq ||u||_{H^{\alpha+2}(\overline{\Omega})}|t-t_1|^{(\alpha+2)/2}\end{aligned}$$

where Ξ is lying between t and t_1. That is $(III) \leq ||u||_{H^{\alpha+2}(\overline{\Omega})}\ p(P_1,P)^{2+\alpha}$.

For (IV) and (V) we have,

$$(IV) \leq ||u||_{H^{\alpha+2}(\overline{\Omega})}|t-t_1|^{(\alpha+1)/2}|X-X_1| \leq ||u||_{H^{\alpha+2}(\overline{\Omega})}\ p(P_1,P)^{2+\alpha}$$

and

$$(V) \leq \frac{1}{2}||u||_{H^{\alpha+2}(\overline{\Omega})}|t-t_1|^{\alpha/2}|X-X_1|^2 \leq C||u||_{H^{\alpha+2}(\overline{\Omega})}\ p(P_1,P)^{2+\alpha}.$$

Combining the above the proof is complete. □

Before we close this subsection, let us note that throughout the text, we will consider integration with respect to the Lebesgue measure in $\mathbb{R}^{n+1}$, we will denote it by $dXdt$. Moreover, we consider the L^p-spaces with respect to this measure. Note also that we use the notation C^k for $k = 0,1,2,\dots$ to denote the standard spaces of functions that are k-times continuously differentiable.

2.2 Viscosity Solutions for Parabolic Equations

In this text we consider **fully nonlinear uniformly parabolic equations** of the form,

$$F(D^2u(X,t),X,t) - u_t(X,t) = f(X,t) \quad \text{for} \quad (X,t) \in \Omega \tag{2.2.1}$$

where, Ω is a bounded domain of $\mathbb{R}^{n+1}$, u is the unknown function defined in Ω and $D^2u(X,t)$ is the $n \times n$ matrix of its second derivatives with respect to the X-variables, $u_t(X,t)$ is the first derivative with respect to the t-variable, f is a given function defined in Ω and the nonlinear operator F is **uniformly elliptic** which means that there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda ||N||_\infty \leq F(M+N,X,t) - F(M,X,t) \leq \Lambda ||N||_\infty \tag{2.2.2}$$

for every $M, N \in S_n$ with $N \geq 0$ and $(X,t) \in \Omega$, where we denote by S_n the space of symmetric $n \times n$ matrices with real entries.

Recall that for any real $n \times n$ matrix M,

$$||M||_\infty = \sup_{|e|=1} |Me|$$

which in case that M is symmetric, using spectral theorem, we can obtain easily that

$$||M||_\infty = \max\{|\lambda_i|, \text{ where } \lambda_i \text{ are the eigenvalues of M}\}.$$

If M is non-negative definite matrix we have that $||M||_\infty$ is its maximum eigenvalue.

The condition (2.2.2) is called **the ellipticity condition** and the constants λ, Λ are called **the ellipticity constants**. From (2.2.2) (replacing M by $M-N$) we may derive that

$$\lambda ||N||_\infty \leq F(M,X,t) - F(M-N,X,t) \leq \Lambda ||N||_\infty \tag{2.2.3}$$

for every $M, N \in S_n$ with $N \geq 0$ and $(X,t) \in \Omega$. Moreover, the ellipticity condition gives that $F(M,X,t)$ is *monotone increasing in M*. Indeed, let $M, N \in S_n$ with $M \leq N$ (which means that $N - M \geq 0$) then by (2.2.2) we have

$$F(M+N-M,X,t) - F(M,X,t) \geq \lambda ||N-M||_\infty \geq 0.$$

In particular, roughly speaking, (2.2.2) and (2.2.3) say that if we add a positive (or negative)-definite matrix then the value of F increases (or decreases) proportionally.

Again by spectral theorem we can derive that any $N \in S_n$ can be written in the form $N = N^+ - N^-$, with $N^+, N^- \geq 0$ and $N^+ N^- = O$. In particular, we get the following equivalence,

Lemma 3. *F is uniformly elliptic if and only if for constants* $0 < \lambda \leq \Lambda$,

$$F(M+N,X,t) \leq F(M,X,t) + \Lambda ||N^+||_\infty - \lambda ||N^-||_\infty$$

for any $M, N \in S_n$ *and any* $(X,t) \in \Omega$.

Proof. For the one direction, let $M, N \in S_n$ then,

$$\begin{aligned} F(M+N,X,t) = F(M-N^- + N^+, X, t) &\leq F(M-N^-, X,t) + \Lambda ||N^+||_\infty \\ &\leq F(M,X,t) - \lambda ||N^-||_\infty + \Lambda ||N^+||_\infty \end{aligned}$$

where we use condition (2.2.2) for $N^+ \geq 0$ and then condition (2.2.3) for $N^- \geq 0$.

Now, for the other direction, let $M, N \in S_n$ with $N \geq 0$. We have that $N^+ = N$, $N^- = O$, thus

$$F(M+N,X,t) \leq F(M,X,t) + \Lambda ||N||_\infty.$$

On the other hand, $(-N)^+ = O$, $(-N)^- = N$, thus $F(M-N,X,t) \leq F(M,X,t) - \lambda ||N||_\infty$ which, if we substitute M with $M+N$, gives $F(M,X,t) \leq F(M+N,X,t) - \lambda ||N||_\infty$, hence

$$F(M+N,X,t) - F(M,X,t) \geq \lambda ||N||_\infty.$$

□

By the above lemma we observe that $F(M,X,t)$ is *Lipschitz continuous in M*. Indeed, since $||N^+||_\infty \leq ||N||_\infty$ and $-\lambda||N^-||_\infty \leq 0$, we have $F(M+N,X,t) \leq F(M,X,t) + \Lambda||N||_\infty$ for any $M, N \in S_n$. Then, let $M_1, M_2 \in S_n$, we apply the above first with $M = M_1$, $N = M_2 - M_1$ and second with $M = M_2$, $N = M_1 - M_2$ to obtain

$$|F(M_1,X,t) - F(M_2,X,t)| \leq \Lambda ||M_1 - M_2||_\infty.$$

We also assume that F and f are continuous in variables (X,t), unless otherwise stated. Note that, F could also depend on derivatives of u of lower order, however in this text we consider the case where the operator F depends only on the second order derivatives of u with respect to X.

Remark 4. *In the rest of this text, when we call a constant* ***universal*** *we mean that it depends only on the dimension and on the ellipticity constants (unless otherwise stated). Note also that although a universal constant may change from the one equation to another we always denote it by C.*

Before we define the notion of viscosity solutions let us give some examples of nonlinear parabolic equations.

- The parabolic Bellman equation

$$\sup_{\alpha\in\mathcal{A}}\{L_\alpha u(X,t)-f_\alpha(X,t)\}-u_t(X,t)=0$$

where $\mathcal{A}$ is any set of indices and for each $\alpha\in\mathcal{A}$, f_α is a real function in Ω and $L_\alpha u=a^{ij}_\alpha(X,t)D^{ij}_X u+b^i_\alpha(X,t)D^i_X u$ is a linear uniformly elliptic operator with bounded measurable coefficients.

- The mean curvature equation

$$\frac{<D^2_X u(X,t)\nabla_X u(X,t),\nabla_X u(X,t)>}{|\nabla_X u(X,t)|^2}-u_t(X,t)=0.$$

Other examples are Pucci operators which will be defined in the next subsection.

We proceed with the definition of viscosity solution of the nonlinear equation (2.2.1). First, let us note that when we say that the function $u:\Omega\subset\mathbb{R}^{n+1}\to\mathbb{R}$ has local maximum (or minimum) at a point (X_0,t_0) we mean that there exists a cylinder $Q_r(X_0,t_0)\subset\Omega$ so that $u(X,t)\leq$ (or $\geq$) $u(X_0,t_0)$ for $(X,t)\in Q_r(X_0,t_0)$.

Motivation. Let u and ϕ be C^1 with respect to t-variable and C^2 with repsect to X-variable and assume that u satisfies (2.2.1) in Ω and that $u-\phi$ has a local maximum at $(X_0,t_0)\in\Omega$. Then $u_t(X_0,t_0)-\phi_t(X_0,t_0)=u_{t^-}(X_0,t_0)-\phi_{t^-}(X_0,t_0)\geq 0$ and $D^2\phi(X_0,t_0)-D^2u(X_0,t_0)\geq 0$. Using the monotoniciy of F we obtain

$$F(D^2\phi(X_0,t_0),x_0,t_0)-\phi_t(X_0,t_0)\geq F(D^2u(X_0,t_0),X_0,t_0)-u_t(X_0,t_0)=f(X_0,t_0).$$

Note. In the following, functions that are defined in a bounded domain $\Omega\subset\mathbb{R}^{n+1}$ and be C^1 with respect to t-variable and C^2 with repsect to X-variable, will be called *test functions.*

Definition 5. *Let $u\in C(\Omega)$, where $\Omega\subset\mathbb{R}^{n+1}$ is a bounded domain.*

- *We say that u is a **viscosity subsolution** of (2.2.1) in Ω if, whenever a test function ϕ is such that the function $u-\phi$ has a local maximun at some point $(X_0,t_0)\in\Omega$ we have that*

$$F(D^2\phi(X_0,t_0),X_0,t_0)-\phi_t(X_0,t_0)\geq f(X_0,t_0). \qquad (2.2.4)$$

- *We say that u is a* ***viscosity supersolution*** *of (2.2.1) in Ω if, whenever a test function ϕ is such that the function $u-\phi$ has a local minimum at some point $(X_0,t_0)\in\Omega$ we have that*

$$F(D^2\phi(X_0,t_0),X_0,t_0)-\phi_t(X_0,t_0)\leq f(X_0,t_0). \qquad (2.2.5)$$

We say that u is a ***viscosity solution*** *of (2.2.1) in Ω if it is both a viscosity subsolution and supersolution.*

Remark 6. *As can be readily seen, if u is a viscosity supersolution of (2.2.1) then $v=-u$ is a viscosity subsolution of $G(D^2v(X,t),X,t)-v_t(X,t)=-f(X,t)$, where $G(M,X,t)=-F(-M,X,t)$ which is also uniformly elliptic with the same constants of ellipticity.*

Definition 7. *Let u and v be two continuous functions defined in a bounded domain $\Omega\subset\mathbb{R}^{n+1}$. We say that v* ***touches*** *u* ***by above*** *at a point $(X_0,t_0)\in\Omega$ if $u(X_0,t_0)=v(X_0,t_0)$ and there exists a cylinder $Q_r(X_0,t_0)\subset\Omega$ such that $u\leq v$ in $Q_r(X_0,t_0)$. Similarly, we define* ***touching by below****.*

Observing that $\phi(X,t)+u(X_0,t_0)-\phi(X_0,t_0)$ touches u by above whenever $u-\phi$ has a local maximum at $(X_0,t_0)\in\Omega$ we can easily deduce that it is enough to consider test functions touching u by above in the definition of viscosity subsolutions (similarly for supersolutions). Also regarding the following we conlcude that it is enough to consider polynomials touching u by above in the definition of viscosity subsolutions.

Lemma 8. *Let $u\in C(\Omega)$, where $\Omega\subset\mathbb{R}^{n+1}$ is a bounded domain. If, whenever a second order parabolic paraboloid $R_2(X,t)=A+B\cdot(X-X_0)+C(t-t_0)+\frac{1}{2}(X-X_0)^\tau D(X-X_0)$ ($A,C\in\mathbb{R},B\in\mathbb{R}^n$ and $D\in\mathbb{R}_{n\times n}$) touches u by above at point $(X_0,t_0)\in\Omega$ we have that*

$$F(D,X_0,t_0)-C\geq f(X_0,t_0),$$

then u is a viscosity subsolution of (2.2.1) in Ω.

Proof. Let $(X_0,t_0)\in\Omega$ and ϕ be a test function that touches u by above at (X_0,t_0). Since ϕ is C^2 in X and C^1 in t, using Taylor's Theorem, we can show that

$$\begin{aligned}\phi(X,t)=&\phi(X_0,t_0)+\phi_t(X_0,t_0)(t-t_0)+\nabla_X\phi(X_0,t_0)\cdot(X-X_0)\\&+\frac{1}{2}(X-X_0)^\tau D_X^2\phi(X_0,t_0)(X-X_0)+\mathbf{o}\left(|X-X_0|^2+|t-t_0|\right)\end{aligned} \qquad (2.2.6)$$

as $(X,t)\to(X_0,t_0)$. To do so we should be careful since different order derivatives with respect to X and t are involved. That is, we expand first in X-variables to get

$$\phi(X,t)=\phi(X_0,t)+\nabla_X\phi(X_0,t)\cdot(X-X_0)+\frac{1}{2}(X-X_0)^\tau D_X^2\phi(X_0,t)(X-X_0)+\mathbf{o}\left(|X-X_0|^2\right)$$

as $X\to X_0$, for any t-level. Next, we expand the C^1-function $\phi(X_0,t)$ and the continuous functions $\nabla_X\phi(X_0,t)$ and $D_X^2\phi(X_0,t)$ in t-variable to obtain,

$$\begin{aligned}\phi(X,t)=&\phi(X_0,t_0)+\phi_t(X_0,t_0)(t-t_0)+\mathbf{o}\left(|t-t_0|\right)+\left(\nabla_X\phi(X_0,t_0)+\mathbf{o}(1)\right)\cdot(X-X_0)\\&+\frac{1}{2}(X-X_0)^\tau\left(D_X^2\phi(X_0,t_0)+\mathbf{o}(1)\right)(X-X_0)+\mathbf{o}\left(|X-X_0|^2\right)\end{aligned}$$

as $X\to X_0$ and $t\to t_0$. To conclude, it remains to observe that

$$\frac{\mathbf{o}(t-t_0)}{|X-X_0|^2+|t-t_0|}\to 0\quad\text{and}\quad\frac{\mathbf{o}\left(|X-X_0|^2\right)}{|X-X_0|^2+|t-t_0|}\to 0$$

as $(X,t)\to(X_0,t_0)$, which is true since $|X-X_0|^2+|t-t_0|$ is greater than or equals to both $|X-X_0|^2$ and $|t-t_0|$. Also,

$$\frac{\mathbf{o}(1)|X-X_0|}{|X-X_0|^2+|t-t_0|}\le\frac{\mathbf{o}(1)|X-X_0|}{|t-t_0|}=\frac{\left(a|t-t_0|+\mathbf{o}(|t-t_0|)\right)|X-X_0|}{|t-t_0|}\to 0$$

and

$$\frac{\mathbf{o}(1)|X-X_0|^2}{|X-X_0|^2+|t-t_0|}\le\frac{\mathbf{o}(1)|X-X_0|^2}{|X-X_0|^2}=\mathbf{o}(1)\to 0$$

as $(X,t)\to(X_0,t_0)$. This shows (2.2.6).

Now, (2.2.6) gives that for any $\epsilon>0$ the paraboloid

$$\begin{aligned}R_2(X,t)=&\phi(X_0,t_0)+\nabla_X\phi(X-X_0)\cdot(X-X_0)+\phi_t(X_0,t_0)(t-t_0)\\&+\frac{1}{2}(X-X_0)^\tau D_X^2\phi(X_0,t_0)(X-X_0)+\epsilon\left(|X-X_0|^2-t+t_0\right)\end{aligned}$$

touches u by above at (X_0,t_0). Therefore from the hypothesis and since

$$D_X^2R_2=D_X^2\phi(X_0,t_0)+2\epsilon I_n,\quad(R_2)_t(X_0,t_0)=\phi_t(X_0,t_0)-\epsilon$$

we have that

$$F(D_X^2\phi(X_0,t_0)+2\epsilon I_n,X_0,t_0)-\phi_t(X_0,t_0)+\epsilon\ge 0.$$

Hence letting $\epsilon \to 0$,

$$F(D_X^2\phi(X_0,t_0), X_0, t_0) - \phi_t(X_0,t_0) \geq 0$$

since F is continuous in M. The proof is complete. □

Remark 9. *Let u be C^1 with respect to t-variable and C^2 with repsect to X-variable then u is a viscosity subsolution of (2.2.1) if and only if it is a classical subsolution.*

Next proposition deals with the stability of viscosity solutions.

Proposition 10. *Assume that $\{F_k\}_{k\in\mathbb{N}}$ are uniformly elliptic operators in Q_1 with the same ellipticity constants $0 < \lambda \leq \Lambda$ and $\{u_k\}_{k\in\mathbb{N}} \subset C(Q_1)$ are such that for every $k \in \mathbb{N}$, u_k is a viscosity subsolution of $F_k(D^2v, X, t) - v_t(X,t) = f(X,t)$ in Q_1. Suppose also that F_k converges to F uniformly in $S_n \times Q_1$ and u_k converges to u uniformly in any $\overline{Q}_\rho(X_0,t_0) \subset Q_1$. Then u is a viscosity subsolution of $F(D^2u, X, t) - u_t(x,t) = f(X,t)$ in Q_1.*

Proof. First we have to check that F is a uniformly elliptic operator. This can be easily obtained using the pointwise convergence and the ellipticity conditions of F_k, since F_k have the same ellipticity constants. Also, we immediately have that $u \in C(Q_1)$ as a locally uniform limit of continuous functions.

Now, let ϕ be a test function that touches u by above at some point $(X_0,t_0) \in Q_1$, we have to show that

$$F(D^2\phi(X_0,t_0)) - \phi_t(X_0,t_0) \geq f(X_0,t_0).$$

Claim. For every $r > 0$ sufficiently small, $\epsilon > 0$ and $k_0 \in \mathbb{N}$ there are $k \geq k_0$, a constant A_k and some $(X_k,t_k) \in Q_r(X_0,t_0) \subset Q_1$ so that the test function

$$\psi_k(X,t) := \phi(X,t) + \frac{\epsilon}{2}(|X-X_0|^2 - t + t_0) + A_k$$

touches u_k by above at (X_k,t_k).

Proof of Claim. We have that for $\rho > 0$ sufficiently small,

$$u(X,t) - \phi(X,t) \leq 0 \quad \text{for } (X,t) \in \overline{Q}_\rho(X_0,t_0) \subset Q_1$$

and

$$u(X_0,t_0)-\phi(X_0,t_0)=0.$$

Then, for $\epsilon>0$ and any $0<r<\rho$,

$$u(X,t)-\phi(X,t)-\frac{\epsilon}{2}(|X-X_0|^2-t+t_0)<0 \quad \text{for } (X,t)\in Q_r(X_0,t_0)\setminus\{(X_0,t_0)\}.$$

We denote by $\tilde{\phi}(X,t):=\phi(X,t)+\frac{\epsilon}{2}(|X-X_0|^2-t+t_0)$ and we consider,

$$c:=\max_{(X,t)\in\partial_p Q_r(X_0,t_0)}\Big(u(X,t)-\tilde{\phi}(X,t)\Big)<0.$$

Then $u-\tilde{\phi}\leq c$, on $\partial_p Q_r(X_0,t_0)$.

Now by the uniform convergence of u_k in $\overline{Q}_r(X_0,t_0)$ we have that there exists some $K\in\mathbb{N}$ such that for any $k\geq\max\{k_0,K\}$,

$$|u_k(X,t)-u(X,t)|<-\frac{c}{4}, \quad \text{for } (X,t)\in\overline{Q}_r(X_0,t_0).$$

Therefore, if $k\geq\max\{k_0,K\}$ and $(X,t)\in\partial_p Q_r(X_0,t_0)$,

$$\begin{aligned}u_k(X,t)-\tilde{\phi}(X,t)&<u(X,t)-\frac{c}{4}-\tilde{\phi}(X,t)\leq c-\frac{c}{4}+u(X_0,t_0)-\tilde{\phi}(X_0,t_0)\\&<\frac{3c}{4}-\frac{c}{4}+u_k(X_0,t_0)-\tilde{\phi}(X_0,t_0)=u_k(X_0,t_0)-\tilde{\phi}(X_0,t_0)+\frac{c}{2}.\end{aligned}$$

Fix $k\geq\max\{k_0,K\}$ and let

$$A_k:=\max_{(X,t)\in\overline{Q}_r(X_0,t_0)}(u_k(X,t)-\tilde{\phi}(X,t)).$$

Since, $u_k(X,t)-\tilde{\phi}(X,t)<u_k(X_0,t_0)-\tilde{\phi}(X_0,t_0)$ for any $(X,t)\in\partial_p Q_r(X_0,t_0)$ then A_k is achieved at some point in $Q_r(X_0,t_0)$, therefore there exists some $(X_k,t_k)\in Q_r(X_0,t_0)$ satisfying

$$u_k(X,t)-\tilde{\phi}(X,t)-A_k\leq 0, \quad \text{for } (X,t)\in\overline{Q}_{r'}(X_k,t_k)\subset Q_r(X_0,t_0)$$

with

$$u_k(X_k,t_k)-\tilde{\phi}(X_k,t_k)-A_k=0$$

and choosing $\psi_k(X,t)=\tilde{\phi}(X,t)+A_k$ the claim follows.

Apply claim for every $r = \frac{1}{m}$ (for sufficiently large $m \in \mathbb{N}$) to get a sequence $(X_{k_m}, t_{k_m}) \to (X_0, t_0)$ and since ψ_{k_m} touches $u_{k_m}(X,t)$ by above at (X_{k_m}, t_{k_m}) we have

$$F_{k_m}(D^2\phi(X_{k_m}, t_{k_m}) + \epsilon I, X_{k_m}, t_{k_m}) - \phi_t(X_{k_m}, t_{k_m}) + \frac{\epsilon}{2} \geq f(X_{k_m}, t_{k_m}).$$

Note that we apply the claim repeatedly and at every step we can take $k_{m+1} \geq k_m$ so $\{u_{k_m}\}_m$ and $\{F_{k_m}\}_m$ form subsequences of $\{u_k\}_k$ and $\{F_k\}_k$ respectively. So, taking $m \to \infty$ we obtain,

$$F(D^2\phi(X_0, t_0) + \epsilon I, X_0, t_0) - \phi_t(X_0, t_0) + \frac{\epsilon}{2} \geq f(X_0, t_0)$$

using the continuity of F_k, f and of the derivatives of ϕ and the uniform convergence of $\{F_k\}$. Finally we take $\epsilon \to 0^+$ and the proof is complete.

□

2.3 Parabolic S-classes

Now we define Pucci's extremal operators which are special cases of nonlinear uniformly elliptic operators but with some specific useful properties.

Definition 11. *Let $0 < \lambda \leq \Lambda$, $M \in S_n$ and denote by $\lambda_i = \lambda_i(M)$, for $i = 1, \ldots, n$, the eigenvalues of M. We define the* ***Pucci's extremal operators*** *by*

$$\mathcal{M}^-(M, \lambda, \Lambda) := \lambda \sum_{\lambda_i > 0} \lambda_i + \Lambda \sum_{\lambda_i < 0} \lambda_i \tag{2.3.1}$$

and

$$\mathcal{M}^+(M, \lambda, \Lambda) := \Lambda \sum_{\lambda_i > 0} \lambda_i + \lambda \sum_{\lambda_i < 0} \lambda_i. \tag{2.3.2}$$

One can easily verify the following properties.

Lemma 12.

(i) *Let $\mathcal{A}_{\lambda,\Lambda}$ be the subset of S_n containing all matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$. Consider also, for $A \in \mathcal{A}_{\lambda,\Lambda}$, the linear functional $L_A(M) = tr(AM)$, where $M \in S_n$. Then*

$$\mathcal{M}^-(M, \lambda, \Lambda) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} L_A(M) \tag{2.3.3}$$

and

$$\mathcal{M}^+(M,\lambda,\Lambda) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} L_A(M). \tag{2.3.4}$$

(ii) *For any $M \in S_n$,*

$$\mathcal{M}^\pm(\alpha M,\lambda,\Lambda) = \alpha\mathcal{M}^\pm(M,\lambda,\Lambda), \quad \textit{if} \quad \alpha \geq 0$$

and

$$\mathcal{M}^+(\alpha M,\lambda,\Lambda) = \alpha\mathcal{M}^-(M,\lambda,\Lambda), \quad \textit{if} \quad \alpha < 0.$$

(iii) *For any $M, N \in S_n$,*

$$\mathcal{M}^+(M,\lambda,\Lambda) + \mathcal{M}^-(N,\lambda,\Lambda) \leq \mathcal{M}^+(M+N,\lambda,\Lambda) \leq \mathcal{M}^+(M,\lambda,\Lambda) + \mathcal{M}^+(N,\lambda,\Lambda).$$
$$\mathcal{M}^-(M,\lambda,\Lambda) + \mathcal{M}^-(N,\lambda,\Lambda) \leq \mathcal{M}^-(M+N,\lambda,\Lambda) \leq \mathcal{M}^-(M,\lambda,\Lambda) + \mathcal{M}^+(N,\lambda,\Lambda).$$

(iv) *For any $M \in S_n$ with $M \geq 0$,*

$$\lambda||M||_\infty \leq \mathcal{M}^-(M,\lambda,\Lambda) \leq \mathcal{M}^+(M,\lambda,\Lambda) \leq n\Lambda||M||_\infty.$$

Remark 13. *Combining the above properties we can easily obtain that operators $\mathcal{M}^-$ and $\mathcal{M}^+$ are uniformly elliptic with ellipticity constants λ and $n\Lambda$.*

Next we define the parabolic S-classes.

Definition 14. *Let $0 < \lambda \leq \Lambda$, $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain and $f \in C(\Omega)$. Assume also that $u \in C(\Omega)$, then*

- *If u is a viscosity subsolution of $\mathcal{M}^+(D^2u(X,t),\lambda,\Lambda) - u_t(X,t) = f(X,t)$ in Ω, we say that $u \in \underline{S}_p(\lambda,\Lambda,f)$ in Ω,.*
- *If u is a viscosity supersolution of $\mathcal{M}^-(D^2u(X,t),\lambda,\Lambda) - u_t(X,t) = f(X,t)$ in Ω, we say that $u \in \overline{S}_p(\lambda,\Lambda,f)$ in Ω.*

In addition we define,

$$S_p(\lambda,\Lambda,f) := \underline{S}_p(\lambda,\Lambda,f) \cap \overline{S}_p(\lambda,\Lambda,f).$$

Next we present some properties of the behavior of parabolic S-classes under certain transformations.

Proposition 15.

(i) *If* $u \in \underline{S}_p(\lambda, \Lambda, f)$ *in* Ω *(or* $u \in \overline{S}_p(\lambda, \Lambda, f)$ *in* Ω*) then*

$$\alpha u \in \underline{S}_p(\lambda, \Lambda, \alpha f) \text{ in } \Omega \quad (\text{or } \alpha u \in \overline{S}_p(\lambda, \Lambda, \alpha f) \text{ in } \Omega) \quad \text{if } \alpha \geq 0$$

and

$$\alpha u \in \overline{S}_p(\lambda, \Lambda, \alpha f) \text{ in } \Omega \quad (\text{or } \alpha u \in \underline{S}_p(\lambda, \Lambda, \alpha f) \text{ in } \Omega) \quad \text{if } \alpha < 0.$$

(ii) *If* $u \in \underline{S}_p(\lambda, \Lambda, f)$ *in* $Q_1(0,0)$ *then for* $r > 0$ *and* $(X_0, t_0) \in \mathbb{R}^{n+1}$,

$$v(Z,s) := u\left(\frac{Z - X_0}{r}, \frac{s - t_0}{r^2}\right) \in \underline{S}_p\left(\lambda, \Lambda, \frac{1}{r^2} f\left(\frac{Z - X_0}{r}, \frac{s - t_0}{r^2}\right)\right)$$

in $Q_r(X_0, t_0)$. *Similar property holds for* $\overline{S}_p$ *as well.*

(iii) *Let* $u \in \underline{S}_p(\lambda, \Lambda, f)$ *in* Ω *and* ϕ *be* C^1 *with respect to* t*-variable and* C^2 *with repsect to* X*-variables in* Ω *and be such that* $\mathcal{M}^+(D^2\phi(X,t), \lambda, \Lambda) - \phi_t(X,t) \leq g(X,t)$ *for any* $(X,t) \in \Omega$ *then* $u - \phi \in \underline{S}_p(\lambda, \Lambda, f - g)$ *in* Ω. *Similar property holds for* $\overline{S}_p$ *as well.*

Proof.

(i) This is obvious by ***(ii)*** of Lemma 12.

(ii) Let ψ be a test function in $Q_r(X_0, t_0)$ that touches v by above at some point $(Z^*, s^*) \in Q_r(X_0, t_0)$. Then

$$u\left(\frac{Z - X_0}{r}, \frac{s - t_0}{r^2}\right) \leq \psi(Z, s)$$

for (Z,s) in a cylinder centered at (Z^*, s^*) within $Q_r(X_0, t_0)$ and $u\left(\frac{Z^* - X_0}{r}, \frac{s^* - t_0}{r^2}\right) = \psi(Z^*, s^*)$. Now, we consider,

$$\tilde{\psi}(X,t) := \psi(rX + X_0, r^2 t + t_0) \quad \text{for} \quad (X,t) \in Q_1(0,0)$$

and $(X^*, t^*) = \left(\frac{Z^* - X_0}{r}, \frac{s^* - t_0}{r^2}\right)$. Then

$$u(X,t) \leq \tilde{\psi}(X,t)$$

for (X,t) in a cylinder around (X^*,t^*) in $Q_1(0,0)$ and $u(X^*,t^*) = \tilde{\psi}(X^*,t^*)$. Hence,

$$\mathcal{M}^+(D^2\tilde{\psi}(X^*,t^*),\lambda,\Lambda) - \tilde{\psi}_t(X^*,t^*) \geq f(X^*,t^*).$$

In addition,

$$\tilde{\psi}_t(X^*,t^*) = r^2\psi_t(Z^*,s^*) \quad \text{and} \quad D^2\tilde{\psi}(X^*,t^*) = r^2 D^2\psi(Z^*,s^*)$$

and by ***(ii)*** of Lemma 12 we have that

$$\mathcal{M}^+(D^2\psi(Z^*,s^*),\lambda,\Lambda) - \psi_t(Z^*,s^*) \geq \frac{1}{r^2} f\left(\frac{Z^*-X_0}{r}, \frac{s^*-t_0}{r^2}\right).$$

(iii) Consider again a test function ψ that touches $u-\phi$ by above at some point $(X_0,t_0) \in \Omega$. Then

$$u(X,t) \leq \psi(X,t) + \phi(X,t)$$

for (X,t) in a cylinder around (X_0,t_0) in Ω and $u(X_0,t_0) = \psi(X_0,t_0) + \phi(X_0,t_0)$. Hence,

$$\mathcal{M}^+(D^2\psi(X_0,t_0) + D^2\phi(X_0,t_0),\lambda,\Lambda) - \psi_t(X_0,t_0) - \phi_t(X_0,t_0) \geq f(X_0,t_0).$$

Therefore, by ***(iii)*** of Lemma 12 we have that

$$\mathcal{M}^+(D^2\psi(X_0,t_0),\lambda,\Lambda) - \psi_t(X_0,t_0) + \mathcal{M}^+(D^2\phi(X_0,t_0),\lambda,\Lambda) - \phi_t(X_0,t_0) \geq f(X_0,t_0)$$

that is,

$$\begin{aligned} &\mathcal{M}^+(D^2\psi(X_0,t_0),\lambda,\Lambda) - \psi_t(X_0,t_0) \\ &\quad \geq f(X_0,t_0) - \mathcal{M}^+(D^2\phi(X_0,t_0),\lambda,\Lambda) + \phi_t(X_0,t_0) \geq f(X_0,t_0) - g(X_0,t_0) \end{aligned}$$

which completes the proof.

□

Finally, we discuss the relation between parabolic S-classes and arbitrary nonlinear operators of parabolic type.

Proposition 16. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain. Assume that ϕ is a test function* *in Ω and $u \in C(\Omega)$ is a viscosity subsolution of the uniformly parabolic equation (2.2.1) in Ω, then*

$$u - \phi \in \underline{S}_p\left(\frac{\lambda}{n}, \Lambda, f(X,t) - F(D^2\phi(X,t), X, t) + \phi_t(X,t)\right) \quad in \quad \Omega.$$

Similarly, if u is a viscosity supersolution then

$$u - \phi \in \overline{S}_p\left(\frac{\lambda}{n}, \Lambda, f(X,t) - F(D^2\phi(X,t), X, t) + \phi_t(X,t)\right) \quad in \quad \Omega.$$

In particular, if u is a viscosity subsolution of (2.2.1) in Ω, then

$$u \in \underline{S}_p\left(\frac{\lambda}{n}, \Lambda, f(X,t) - F(O, X, t)\right) \quad in \quad \Omega$$

(similarly for supersolutions).

Proof. Let ψ be a test function that touches $u - \phi$ by above at some point $(X_0, t_0) \in \Omega$ then the test function $\psi + \phi$ touches u by above at (X_0, t_0). Hence

$$\begin{aligned} f(X_0,t_0) &\leq F(D^2\psi(X_0,t_0) + D^2\phi(X_0,t_0), X_0, t_0) - \psi_t(X_0,t_0) - \phi_t(X_0,t_0) \\ &\leq F(D^2\phi(X_0,t_0), X_0, t_0) + \Lambda||[D^2\psi(X_0,t_0)]^+||_\infty - \lambda||[D^2\psi(X_0,t_0)]^-||_\infty \\ &\quad - \psi_t(X_0,t_0) - \phi_t(X_0,t_0) \end{aligned}$$

due to Lemma 3. But $||[D^2\psi(X_0,t_0)]^+||_\infty$ equals to the maximum positive eigenvalue of $D^2\psi(X_0,t_0)$ (or it is zero) and $||[D^2\psi(X_0,t_0)]^-||_\infty$ equals to the absolute value of the minimum negative (or it is zero) eigenvalue of $D^2\psi(X_0,t_0)$ so

$$||[D^2\psi(X_0,t_0)]^+||_\infty \leq \sum_{\lambda_i > 0} \lambda_i$$

and

$$n||[D^2\psi(X_0,t_0)]^-||_\infty \geq -\sum_{\lambda_i < 0} \lambda_i$$

where $\lambda_i, i = 1, \ldots, n$, are the eigenvalues of $D^2\psi(X_0,t_0)$. Hence

$$f(X_0,t_0) \leq F(D^2\phi(X_0,t_0), X_0, t_0) - \phi_t(X_0,t_0) + \mathcal{M}^+\left(D^2\psi(X_0,t_0), \frac{\lambda}{n}, \Lambda\right) - \psi_t(X_0,t_0).$$

That is,

$$\mathcal{M}^+\left(D^2\psi(X_0,t_0),\frac{\lambda}{n},\Lambda\right)-\psi_t(X_0,t_0)\geq f(X_0,t_0)-F(D^2\phi(X_0,t_0),X_0,t_0)+\phi_t(X_0,t_0)$$

which completes the proof. $\square$

Remark 17. *By Proposition 10 we have that the classes $\underline{S}_p$, $\overline{S}_p$ and S_p are closed under locally uniform limits.*

2.4 Oblique Boundary Data

Here we adopt the notion of oblique boundary conditions in the viscosity framework. In the following we assume that $\beta : Q_r^* \to \mathbb{R}^n$ is a vector-value function defined on the flat boundary so that $\beta_n(x,t) \geq \delta_0 > 0$ for every $(x,t) \in Q_r^*$ and $\|\beta\|_{L^\infty(Q_r^*)} \leq 1$.

Definition 18. *Let $u \in C\left(Q_r^+ \cup Q_r^*\right)$ and $g \in C\left(Q_r^*\right)$.*

(i) *We say that*

$$\beta \cdot Du \geq g \quad \text{on} \quad Q_r^*$$

in the viscosity sense if whenever we take any point $P_0 = (x_0, 0, t_0) \in Q_r^*$ and a test function ϕ that touches u by above at P_0 in some half-cylider $Q_\rho^+(P_0) \subset Q_r^+$, then we must have that

$$\beta(x_0,t_0)\cdot D\phi(P_0) \geq g(x_0,t_0).$$

(ii) *We say that*

$$\beta \cdot Du \leq g \quad \text{on} \quad Q_r^*$$

in the viscosity sense if whenever we take any point $P_0 = (x_0, 0, t_0) \in Q_r^*$ and a test function ϕ that touches u by below at P_0 in some half-cylider $Q_\rho^+(P_0) \subset Q_r^+$, then we must have that

$$\beta(x_0,t_0)\cdot D\phi(P_0) \leq g(x_0,t_0).$$

If the above two hold at the same time we say that $\beta \cdot Du = g$ on Q_r^* in the viscosity sense.

Proposition 19. *(Rescaling). Let $u \in C\left(Q_1^+ \cup Q_1^*\right)$ and $g \in C\left(Q_1^*\right)$ be so that $\beta \cdot Du \geq g$ (resp. $\leq g$) on Q_1^* in the viscosity sense. Let also*

$$v(Z,s) = u\left(\frac{Z - x_0}{r}, \frac{s - t_0}{r^2}\right)$$

for $(Z,s) = (z,w,s) \in Q_r^+(P_0) \cup Q_r^*(P_0)$, where $P_0 = (x_0, 0, t_0)$. Then $\tilde{\beta} \cdot Dv \geq \frac{1}{r}\tilde{g}$ (resp. $\leq \frac{1}{r}\tilde{g}$) in the viscosity sense on $Q_r^*(P_0)$, where $\tilde{g}(z,s) = g\left(\frac{z-x_0}{r}, \frac{s-t_0}{r^2}\right)$ and $\tilde{\beta}(z,s) = \beta\left(\frac{z-x_0}{r}, \frac{s-t_0}{r^2}\right)$.

Proof. Take any point $P_1 = (z_1, 0, s_1) \in Q_r^*(P_0)$ and a test function ϕ that touches v by above at P_1 in some half-cylider around P_1 in $Q_r^+(P_0)$. Set

$$\psi(X,t) := \phi(rX + x_0, r^2 t + t_0)$$

for $(X,t) \in Q_1^+$ and $P_1' := \left(\frac{z_1 - x_0}{r}, 0, \frac{s_1 - t_0}{r^2}\right)$. Note that $(rX + x_0, r^2 t + t_0) \in Q_r^+(P_0)$. Then ψ touches u by above at P_1' in some half-cylider around P_1' in Q_1^+ and hence, by definition, $\beta\left(\frac{z_1-x_0}{r}, \frac{s_1-t_0}{r^2}\right) \cdot D\psi(P_1') \geq g\left(\frac{z_1-x_0}{r}, \frac{s_1-t_0}{r^2}\right)$. Since $D\psi(P_1') = rD\phi(P_1)$ the proof is complete. □

A special case of the oblique-type condition is when $\beta(x,t) = (0, \ldots, 0, 1) \in \mathbb{R}^n$ for every $(x,t) \in Q_r^*$. This type of condition is called *Neumann boundary condition*. Next we give some properties of the viscosity solutions of nonlinear parabolic Neumann problems.

Proposition 20. *(Closedness). Let $\{u_k\}_{k\in\mathbb{N}} \subset C(Q_1^+ \cup Q_1^*)$ are such that for every $k \in \mathbb{N}$, u_k satisfies in the viscosity sense the following*

$$\begin{cases} F\left(D^2 v(X,t)\right) - v_t(X,t) \geq 0, & (X,t) \in Q_1^+ \\ v_y(x,0,t) \geq 0, & (x,t) \in Q_1^* \end{cases} \tag{2.4.1}$$

Assume that u_k converges to u uniformly in any $\overline{Q}_\rho^+(x_0,0,t_0) \subset Q_1^+ \cup Q_1^*$, then u satisfies (2.4.1) in the viscosity sense.

Proof. First note that $F\left(D^2 u\right) - u_t \geq 0$ in Q_1^+ in the viscosity sense by Proposition 10. So, it remains to obtain the Neumann sub-condition. The argument is similar to the one of Proposition 10 but here we must be more careful since the test function ψ_k, which was constructed in the proof of Proposition 10 may touch u_k at a point which lies either on the flat boundary or at in the interior. This is why assuming only the Neumann sub-condition for u_k's is not enough and we need to know also that u_k's are subsolutions of the equation in the interior. The details follow below.

Take any point $P_0 = (x_0, 0, t_0) \in Q_1^*$ and any test function ϕ that touches u by above at P_0 in $\overline{Q}_\rho^+(P_0) \in Q_1^+$. We want to show that, $\phi_y(P_0) \geq 0$.

We have that,

$$u(X,t) - \phi(X,t) \leq 0, \quad \text{for } (X,t) \in \overline{Q}_\rho^+(P_0) \quad \text{and} \quad u(P_0) - \phi(P_0) = 0.$$

Then, for $\epsilon > 0$ and any $0 < r < \rho$,

$$u(X,t) - \phi(X,t) - \frac{\epsilon}{2}(|X - x_0|^2 - t + t_0) < 0 \quad \text{for } (X,t) \in Q_r^+(P_0) \setminus \{P_0\}.$$

We denote by $\tilde{\phi}(X,t) := \phi(X,t) + \frac{\epsilon}{2}(|X - X_0|^2 - t + t_0)$ and by $A_r(P_0) := \partial_p Q_r^+(P_0) \setminus Q_r^*(P_0)$ and we consider,

$$c := \max_{(X,t) \in A_r(P_0)} \Big(u(X,t) - \tilde{\phi}(X,t) \Big) < 0.$$

Then $u - \tilde{\phi} \leq c$ on $A_r(P_0)$.

By uniform convergence of u_k in $\overline{Q}_r^+(P_0)$ we have that for any $k_0 \in \mathbb{N}$, there exists some $K \in \mathbb{N}$ such that for any $k \geq \max\{k_0, K\}$,

$$|u_k(X,t) - u(X,t)| < -\frac{c}{4}, \quad \text{for } (X,t) \in \overline{Q}_r^+(P_0).$$

Therefore, for $k \geq \max\{k_0, K\}$ and $(X,t) \in A_r(P_0)$,

$$\begin{aligned} u_k(X,t) - \tilde{\phi}(X,t) &< u(X,t) - \frac{c}{4} - \tilde{\phi}(X,t) \leq c - \frac{c}{4} + u(P_0) - \tilde{\phi}(P_0) \\ &< \frac{3c}{4} - \frac{c}{4} + u_k(P_0) - \tilde{\phi}(P_0) = u_k(P_0) - \tilde{\phi}(P_0) + \frac{c}{2}. \end{aligned}$$

Fix $k \geq \max\{k_0, K\}$ and let

$$C_k := \max_{(X,t) \in \overline{Q}_r^+(P_0)} (u_k(X,t) - \tilde{\phi}(X,t)).$$

Since, $u_k(X,t) - \tilde{\phi}(X,t) < u_k(P_0) - \tilde{\phi}(P_0)$ for any $(X,t) \in A_r(P_0)$ then C_k cannot be attained on $A_r(P_0)$, then it is achieved at some point $(X_k, t_k) \in Q_r^+(P_0) \cup Q_r^*(P_0)$.

Therefore, if we take a sequence of radii $r_m = \frac{1}{m}$ (for sufficiently large $m \in \mathbb{N}$) there are points $(X_{k_m}, t_{k_m}) \in Q_r^+(P_0) \cup Q_r^*(P_0)$ so that $(X_{k_m}, t_{k_m}) \to P_0$, as $m \to \infty$ and the test function $\psi_{k_m} := \tilde{\phi} + C_{k_m}$ touches by above u_{k_m} at (X_{k_m}, t_{k_m}). Hence, we treat two cases:

1. If $(X_{k_m}, t_{k_m}) \in Q_r^*(P_0)$, then using the Neumann sub-condition for u_{k_m} we have that $\left(\psi_{k_m}\right)_y (X_{k_m}, t_{k_m}) \geq 0$, therefore

$$\phi_y(X_{k_m}, t_{k_m}) + \epsilon y_{k_m} \geq 0. \tag{2.4.2}$$

2. If $(X_{k_m}, t_{k_m}) \in Q_r^+(P_0)$, then using the equation for u_{k_m} we have that

$$F(D^2\phi(X_{k_m}, t_{k_m}) + \epsilon I) - \phi_t(X_{k_m}, t_{k_m}) + \frac{\epsilon}{2} \geq 0. \tag{2.4.3}$$

Now, if ***1.*** is true for an infinite number of m's then taking a suitable subsequence and the limit in (2.4.2) we derive $\phi_y(P_0) \geq 0$, as desired. Otherwise, ***2.*** will be true for an infinite number of m and so taking subsequences and limits in (2.4.3) we derive, $F(D^2\phi(P_0)) - \phi_t(P_0)) \geq 0$.

To finish the proof we assume that $\phi_y(P_0) < 0$ (to get a contadiction). Then having in mind the dichotomy above we conclude that $F(D^2\phi(P_0)) - \phi_t(P_0) \geq 0$ must hold. For small $\gamma > 0$, we consider the following perturbation of ϕ,

$$\phi_\gamma(X,t) = \phi(X,t) + \gamma y - \frac{y^2}{\gamma}.$$

Observe that if $(X,t) \in Q_{\gamma^2}^+(P_0)$, then $y \leq \gamma^2$ gives $y^2 \leq \gamma^2 y$ and $\frac{y^2}{\gamma} \leq \gamma y$. That is $\gamma y - \frac{y^2}{\gamma} \geq 0$. Therefore, we obtain

$$\phi_\gamma(X,t) \geq \phi(X,t) \geq u(X,t), \quad \text{for} \quad (X,t) \in Q_{\gamma^2}^+(P_0) \quad \text{and} \quad \phi_\gamma(P_0) = \phi(P_0) = u(P_0).$$

This shows that, ϕ_γ is also a test function that touches u by above at P_0 and following the same steps as we did for ϕ we conclude that

$$\left(\phi_\gamma\right)_y (P_0) \geq 0 \quad \text{or} \quad F(D^2\phi_\gamma(P_0)) - \left(\phi_\gamma\right)_t (P_0) \geq 0.$$

A direct computation of these quantities will give the contradiction. Indeed,

$$\left(\phi_\gamma\right)_y(P_0) = \phi_y(P_0) + \gamma.$$

We have assumed that $\phi_y(P_0) < 0$ so choosing $0 < \gamma < -\phi_y(P_0)$ we obtain $\left(\phi_\gamma\right)_y(P_0) < 0$. On the other hand,

$$F(D^2\phi_\gamma(P_0)) - \left(\phi_\gamma\right)_t(P_0)) = F\left(D^2\phi(P_0) - \frac{2}{\gamma}\begin{pmatrix} 0\cdots 0 \\ \vdots \ddots \vdots \\ 0 \cdots 1 \end{pmatrix}\right) - \phi_t(P_0)$$

which by Lemma 3 gives,

$$F(D^2\phi_\gamma(P_0)) - \left(\phi_\gamma\right)_t(P_0) \leq F\left(D^2\phi(P_0)\right) - \frac{2\lambda}{\gamma} - \phi_t(P_0).$$

Recall that $F(D^2\phi(P_0)) - \phi_t(P_0) \geq 0$ so choosing $\gamma > 0$ small enough to satisfy $\frac{2\lambda}{\gamma} > F\left(D^2\phi(P_0)\right) - \phi_t(P_0) + 1$, we obtain $F(D^2\phi_\gamma(P_0)) - \left(\phi_\gamma\right)_t(P_0) < 0$ and the proof is complete. □

Proposition 21. *(Reflection Principle). Let $u, f \in C(Q_1^+ \cup Q_1^*)$. Assume that $u \in S_p(\lambda, \Lambda, f)$ in Q_1^+ and $u_y = 0$ on Q_1^* in the viscosity sense. Consider the reflected function,*

$$u^*(x,y,t) = \begin{cases} u(x,y,t), & \text{if} \quad y \geq 0 \\ u(x,-y,t), & \text{if} \quad y < 0 \end{cases}$$

for $(X,t) \in Q_1$, where $X = (x,y)$ ($x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$). Similarly we consider the reflection f^* of f. Then $u^* \in S_p(\lambda, \Lambda, f^*)$ in Q_1.

Proof. First note that since u and f are continuous up to Q_1^* then we can easily verify that $u^*, f^* \in C(Q_1)$. Note also that u^* and f^* are even functions.

Next we observe that $u^* \in S_p(\lambda, \Lambda, f^*)$ in Q_1^+. Similarly we verify that $u^* \in S_p(\lambda, \Lambda, f^*)$ in Q_1^-. Indeed, let ϕ be a test function that touches u^* by above at some point $(X_0, t_0) \in Q_1^-$. Define $\psi(z,w,t) = \phi(z,-w,t)$ where $(Z,t) = (z,w,t) \in Q_1^+$. Then ψ is a test function that touches u by above at $(Z_0,t_0) = (x_0,-y_0,t_0) \in Q_1^+$ and as a consequence $\mathcal{M}^+(D^2\psi(Z_0,t_0),\lambda,\Lambda) - \psi_t(Z_0,t_0) \geq f(Z_0,t_0)$. On the other hand

$f(Z_0,t_0)=f^*(X_0,t_0)$, $\psi_t(Z_0,t_0)=\phi_t(X_0,t_0)$ and

$$D^2\psi(Z_0,t_0)=\begin{pmatrix} D^2_{n-1}\phi(X_0,t_0) & d^\tau \\ d & \phi_{yy}(X_0,t_0)\end{pmatrix}$$

where $d:=(-\phi_{x_1y}(X_0,t_0),\ldots,-\phi_{x_{n-1}y}(X_0,t_0))$. Note that the matrices $D^2\psi(Z_0,t_0)$ and $D^2\phi(X_0,t_0)$ have the same eigenvalues since,

$$\begin{aligned}\det(D^2\psi(Z_0,t_0)-lI_n)&=\det\begin{pmatrix} D^2_{n-1}\phi(X_0,t_0)-lI_{n-1} & d^\tau \\ d & \phi_{yy}(X_0,t_0)-l\end{pmatrix}\\ &=\det\begin{pmatrix} D^2_{n-1}\phi(X_0,t_0)-lI_{n-1} & -d^\tau \\ -d & \phi_{yy}(X_0,t_0)-l\end{pmatrix}\\ &=\det(D^2\phi(X_0,t_0)-lI_n).\end{aligned}$$

Hence $\mathcal{M}^+(D^2\psi(Z_0,t_0),\lambda,\Lambda)=\mathcal{M}^+(D^2\phi(X_0,t_0),\lambda,\Lambda)$. This shows that $u^*\in\underline{S}_p(\lambda,\Lambda,f^*)$ in Q_1^-. In a same fashion we also prove that $u^*\in\overline{S}_p$ in Q_1^-.

We consider the auxiliary functions,

$$v_\gamma(X,t):=u^*(X,t)+\gamma|y|$$

for $\gamma\in\mathbb{R}$. By Proposition 15 ***(iii)*** and since $u^*\in S_p$ in $Q_1^+\cup Q_1^-$, we have that $v_\gamma\in S_p(\lambda,\Lambda,f^*)$ in $Q_1^+\cup Q_1^-$. In order to get that $v_\gamma\in S_p(\lambda,\Lambda,f^*)$ in Q_1 it remains to study what happens across Q_1^*.

Assume first that $\gamma>0$. We will show that v_γ cannot be touched by above by any test function at any point of Q_1^*. Indeed, let ϕ be a test function in Q_1 that touches v_γ by above at some point $P_0=(x_0,0,t_0)\in Q_1^*$. The idea is to use the viscosity Neumann condition to get a contradiction. We have that $\phi(X,t)-\gamma y$ touches u by above at P_0 in some $Q_\rho^+(P_0)\subset Q_1^+$. Then $\phi_y(P_0)-\gamma\geq 0$, i.e.

$$\phi_y(P_0)\geq\gamma>0.$$

But on the other hand, $\phi(X,t)+\gamma y$ touches u^* by above at P_0 in some $Q_\rho^-(P_0)\subset Q_1^-$. A change of variables implies that $u(X,t)\leq\phi(x',-y,t)-\gamma y$, for $(X,t)\in Q_\rho^+(P_0)$ and of course, $u(P_0) = \phi(P_0)$ (since $y_0 = 0$). Then $-\phi_y(P_0) - \gamma \geq 0$, i.e.

$$\phi_y(P_0) \leq -\gamma < 0$$

a contradiction. Therefore such a test function cannot exist.

Consequently, since $v_\gamma \in \underline{S}_p(\lambda, \Lambda, f^*)$ in $Q_1^+ \cup Q_1^-$ and no test function can touch v_γ by above at any point of Q_1^* in a neighborhood in Q_1 we get that $v_\gamma \in \underline{S}_p(\lambda, \Lambda, f^*)$ in Q_1, when $\gamma > 0$. In a similar way we also get that $v_\gamma \in \overline{S}_p(\lambda, \Lambda, f^*)$ in Q_1, when $\gamma < 0$.

Finally, we observe that,

$$|v_\gamma - u^*| = |\gamma||y| \leq |\gamma| \to 0$$

as $\gamma \to 0$, which means that $v_\gamma \to u^*$, as $\gamma \to 0$, in a uniform way in Q_1. So, we can consider for $k \in \mathbb{N}$ the sequences $\{v_{\frac{1}{k}}\}, \{v_{-\frac{1}{k}}\}$. The first one is a subset of $\underline{S}_p(\lambda, \Lambda, f^*)$ in Q_1 and tends to u^* uniformly in Q_1 as $k \to \infty$, hence by closedness $u^* \in \underline{S}_p(\lambda, \Lambda, f^*)$ in Q_1. The second one is a subset of $\overline{S}_p(\lambda, \Lambda, f^*)$ in Q_1 and tends to u^* uniformly in Q_1 as $k \to \infty$, hence by closedness $u^* \in \overline{S}_p(\lambda, \Lambda, f^*)$ in Q_1. The proof is complete. $\square$

Remark 22. *When we study an oblique derivative problem, we call a constant $C > 0$* ***universal*** *if it depends only on $n, \lambda, \Lambda, \delta_0$ and other constants related to function β.*

2.5 Change of variables

Here we consider the case when the function β is constant. In this case we see that using a suitable change of variables, a viscosity problem of the form

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } Q_1^+ \\ \beta \cdot Du = 0, & \text{on } Q_1^* \end{cases} \tag{2.5.1}$$

can be transformed into a nonlinear Neumann parabolic problem

$$\begin{cases} \tilde{F}(D^2v) - v_t = 0, & \text{in } \tilde{Q_1^+} \\ v_y = 0, & \text{on } Q_1^* \end{cases} \tag{2.5.2}$$

where $\tilde{F}$ is also an elliptic operator on S_n and $\tilde{Q_1^+}$ a suitable "half-set".

More precisely, let $\psi = \psi(z, w, t)$ be a smooth function defined on Q_1^+ and consider

$\phi(x,y,t) := \psi\left(x_1 + \frac{\beta_1}{\beta_n}y, \dots, x_{n-1} + \frac{\beta_{n-1}}{\beta_n}y, y, t\right)$. Then

$$\phi_{x_i} = \psi_{z_i}, \quad \text{for every} \quad i = 1, \dots, n-1, \quad \phi_t = \psi_t \quad \text{and}$$

$$\phi_y = \psi_{z_1}\frac{\beta_1}{\beta_n} + \dots \psi_{z_{n-1}}\frac{\beta_{n-1}}{\beta_n} + \psi_w, \text{ that is, } \phi_y = \frac{1}{\beta_n}\beta \cdot D\psi.$$

Note that if

$$A := \begin{pmatrix} 1 \dots 0 & \frac{\beta_1}{\beta_n} \\ \ddots & \\ 0 \dots 1 & \frac{\beta_{n-1}}{\beta_n} \\ 0 \dots 0 & 1 \end{pmatrix}$$

then $\det A = 1$ and

$$A^{-1} = \begin{pmatrix} 1 \dots 0 & -\frac{\beta_1}{\beta_n} \\ \ddots & \\ 0 \dots 1 & -\frac{\beta_{n-1}}{\beta_n} \\ 0 \dots 0 & 1 \end{pmatrix}.$$

We have that $\phi(x,y,t) := \psi\left(A(x,y), t\right)$ and one can easily check that

$$D^2\phi = A^\tau D^2\psi A \quad \text{and} \quad D^2\psi = (A^{-1})^\tau D^2\phi A^{-1}.$$

Define $\tilde{F}(M) := F((A^{-1})^\tau M A^{-1})$. Then $\tilde{F}$ is elliptic and its ellipticity constants are universal multiplicatives of λ and Λ. Indeed, first note that

$$||A||_\infty := \sup_{e \in \mathbb{R}^n,\ |e|=1} |Ae| = \sup_{e \in \mathbb{R}^n,\ |e|=1} \left| e + \left(\frac{\beta_1}{\beta_n}, \dots, \frac{\beta_{n-1}}{\beta_n}, 1\right)\right| \leq 1 + \frac{1}{\delta_0} = \frac{\delta_0 + 1}{\delta_0} =: C_{\delta_0}$$

and the same hold for $||A^{-1}||_\infty, ||A^\tau||_\infty$ and $||(A^{-1})^\tau||_\infty$. Next, let $M, N \in S_n$ with $N \geq 0$, we have

$$\tilde{F}(M+N) - \tilde{F}(M) = F((A^{-1})^\tau M A^{-1} + (A^{-1})^\tau N A^{-1}) - F((A^{-1})^\tau M A^{-1})$$

and we intent to use the ellipticity of F. To do so, we have to ensure that $(A^{-1})^\tau N A^{-1}$

is symmetric and positive definite. We denote by

$$b_{ij} := (A^{-1})_{ij} = \begin{cases} 1, & \text{if } \ i = j \\ 0, & \text{if } \ i \neq j, \ j < n \\ -\frac{\beta_i}{\beta_n}, & \text{if } \ i < n, \ j = n \end{cases} = ((A^{-1})^\tau)_{ji}.$$

Then

$$\begin{aligned} \left((A^{-1})^\tau N A^{-1}\right)_{ij} &= \sum_{l,k=1}^{n} N_{kl} b_{ki} b_{lj} \\ &= \begin{cases} N_{ij}, & \text{if } \ i < n, \ j < n \\ \sum_{l,k=1}^{n} N_{kl} b_{kn} b_{ln}, & \text{if } \ i = j = n \\ \sum_{k=1}^{n} N_{kj} b_{kn} b_{jj} + \sum_{k=1}^{n} N_{kn} b_{kn} b_{nj}, & \text{if } \ i = n, \ j < n \\ \sum_{l=1}^{n} N_{il} b_{ii} b_{ln} + \sum_{l=1}^{n} N_{nl} b_{ni} b_{ln}, & \text{if } \ j = n, \ i < n \end{cases} \\ &= \left((A^{-1})^\tau N A^{-1}\right)_{ji}, \quad \text{since N is symmetric,} \end{aligned}$$

that is, $(A^{-1})^\tau N A^{-1} \in S_n$. Also $\det\left((A^{-1})^\tau N A^{-1}\right) = \det((A^{-1})^\tau)\det(N)\det(A^{-1}) = \det(N) \geq 0$, since N is non-negative definite. Moreover, all the $1\times 1, \dots, (n-1)\times(n-1)$ upper left corners of $(A^{-1})^\tau N A^{-1}$ has positive determinant since N is positive definite. By Sylvester's criterion $(A^{-1})^\tau N A^{-1} \geq 0$. Therefore,

$$\tilde{F}(M+N) - \tilde{F}(M) \leq \Lambda \left\|(A^{-1})^\tau N A^{-1}\right\|_\infty \leq \Lambda C_{\delta_0}^2 \left\|N\right\|_\infty$$

and

$$\tilde{F}(M+N) - \tilde{F}(M) \geq \lambda \left\|(A^{-1})^\tau N A^{-1}\right\|_\infty \frac{\|A^\tau\|_\infty \|A\|_\infty}{\|A^\tau\|_\infty \|A\|_\infty} \geq \frac{\lambda \|N\|_\infty}{\|A^\tau\|_\infty \|A\|_\infty} \geq \frac{\lambda}{C_{\delta_0}^2} \|N\|_\infty .$$

We observe also that the transformation A maps the hyper-plane $\{y = 0\}$ identically into itself and the half-space $\{y > 0\}$ into itself (so does A^{-1}). So, $\tilde{Q_1^+} := \{(x, y, t) = (A^{-1}(z, w), t), \ \text{ for } \ (z, w, t) \in Q_1^+\}$ lies in the half-space $\{y > 0\}$ and Q_1^* is part of its parabolic boundary.

Note that combining all the above one can ensure that if $u(Z, t)$ is a viscosity solution of (2.5.1) then $v(X, t) = u(AX, t)$ is a viscosity solution of (2.5.2). This fact will be useful later to prove regularity for problems of the form (2.5.1) using the regularity of problems of the form (2.5.2).

Closing this section it is interesting to examine how S_p-classes behave under the above transformation. Using the representation (2.3.3) (resp. (2.3.4)) for $\mathcal{M}^+$ (resp. $\mathcal{M}^-$) we show

$$\mathcal{M}^+\left(D^2\phi(X,t),\tilde{\lambda},\tilde{\Lambda}\right)-\phi_t(X,t)\geq\mathcal{M}^+\left(D^2\psi(Z,t),\lambda,\Lambda\right)-\psi_t(Z,t)$$

$$\left(\text{resp.}\quad\mathcal{M}^-\left(D^2\phi(X,t),\tilde{\lambda},\tilde{\Lambda}\right)-\phi_t(X,t)\leq\mathcal{M}^-\left(D^2\psi(Z,t),\lambda,\Lambda\right)-\psi_t(Z,t)\right)$$

where, $(Z,t)=(AX,t)$, ψ and ϕ defined as above and $\tilde{\lambda}:=\frac{\lambda}{C^2_{\delta_0}}$, $\tilde{\Lambda}:=n\Lambda C^2_{\delta_0}$. Indeed, first we observe that

$$\mathcal{M}^+\left(D^2\psi(Z,t),\lambda,\Lambda\right)-\psi_t(Z,t)=\mathcal{M}^+\left((A^{-1})^\tau D^2\phi(X,t)A^{-1},\lambda,\Lambda\right)-\phi_t(X,t).$$

Therefore, it is sufficient to show that for any $M\in S_n$,

$$\mathcal{M}^+\left((A^{-1})^\tau MA^{-1},\lambda,\Lambda\right)\leq\mathcal{M}^+\left(M,\tilde{\lambda},\tilde{\Lambda}\right).$$

That is it is enough to show that

$$\{L_B\left((A^{-1})^\tau MA^{-1}\right),\ B\in\mathcal{A}_{\lambda,\Lambda}\}\subset\{L_{\tilde{B}}\left(M\right),\ \tilde{B}\in\mathcal{A}_{\tilde{\lambda},\tilde{\Lambda}}\}.$$

So, let $B\in\mathcal{A}_{\lambda,\Lambda}$, then

$$L_B\left((A^{-1})^\tau MA^{-1}\right)=tr\left(B(A^{-1})^\tau MA^{-1}\right)=tr\left(A^{-1}B(A^{-1})^\tau M\right)=L_{\tilde{B}}\left(M\right)$$

for $\tilde{B}:=A^{-1}B(A^{-1})^\tau$. It remains to show that $\tilde{B}\in\mathcal{A}_{\tilde{\lambda},\tilde{\Lambda}}$. We have for $\xi\in\mathbb{R}^n$, $\tilde{B}_{ij}\xi_i\xi_j=L_{\tilde{B}}\left(\xi\xi^\tau\right)=L_B\left((A^{-1})^\tau\xi\xi^\tau A^{-1}\right)$ and the $(n\times n)$-matrix $\xi\xi^\tau$ is symmetric, positive definite and its eigenvalues are $\lambda_1=\cdots=\lambda_{n-1}=0$ and $\lambda_n=|\xi|^2$. Hence,

$$\frac{\lambda}{C^2_{\delta_0}}\left|\xi\right|^2\leq L_B\left((A^{-1})^\tau\xi\xi^\tau A^{-1}\right)\leq n\Lambda C^2_{\delta_0}\left|\xi\right|^2$$

using the ellipticity condition for L_B.

Regarding the above we obtain that if $u\in S_p(\lambda,\Lambda)$ in Q_1^+ then $v\in S_p\left(\tilde{\lambda},\tilde{\Lambda}\right)$ in $\tilde{Q_1^+}$.

2.6 Useful known results

Here we present some basic results for viscosity solutions of nonlinear parabolic equations (appeared in [44] and with more details in [23]) as well as a useful iterative argument.

Theorem 23. *(Harnack inequality). Let f be continuous and bounded in Q_ρ and assume that $u \in S_p(\lambda, \Lambda, f)$ in Q_ρ with $u \geq 0$ in Q_ρ. Then*

$$\sup_{K_{\rho R}} u \leq C \left(\inf_{Q_{\rho R^2}} u + \rho^{\frac{n}{n+1}} ||f||_{L^{n+1}(Q_\rho)} \right) \tag{2.6.1}$$

where $C > 0$ is a universal constant and $K_{\rho R} = \rho K_R$ with

$$K_R := B_{\frac{R^2}{2\sqrt{2}}}(0,0) \times \left[-R^2 + \frac{3}{8}R^4, -R^2 + \frac{4}{8}R^4 \right]$$

for $R := \min\left(\frac{1}{3\sqrt{n}}, 3 - 2\sqrt{2}, \frac{1}{\sqrt{10(m+1)}} \right)$ where $m > 0$ is a universal constant.

Theorem 24. *(Aleksandrov-Bakelman-Pucci-type Maximum Principle). Let f be continuous and bounded in Q_ρ and assume that $u \in \overline{S}_p(\lambda, \Lambda, f)$ in Q_ρ with $u \geq 0$ on $\partial_p Q_\rho$. Then*

$$\sup_{Q_\rho} u^- \leq C \rho^{\frac{n}{n+1}} ||f^+||_{L^{n+1}(Q_\rho)} \tag{2.6.2}$$

where $C > 0$ is a universal constant.

Corollary 25. *Let f be continuous and bounded in Q_ρ and assume that $u \in S_p(\lambda, \Lambda, f)$ in Q_ρ then*

$$||u||_{L^\infty(Q_\rho)} \leq ||u||_{L^\infty(\partial_p Q_\rho)} + C \rho^{\frac{n}{n+1}} ||f||_{L^{n+1}(Q_\rho)} \tag{2.6.3}$$

where $C > 0$ is a universal constant.

Corollary 26. *Let $\Omega \in \mathbb{R}^{n+1}$ be a bounded domain and assume that $u \in S_p(\lambda, \Lambda, 0)$ in Ω then*

$$||u||_{L^\infty(\Omega)} \leq ||u||_{L^\infty(\partial_p \Omega)}. \tag{2.6.4}$$

We close with a useful iteration lemma.

Lemma 27. *(An iteration argument). Let ω and σ be two increasing functions defined in the interval $(0, R]$, $R > 0$ and assume that for every $r \le R$ we have*

$$\omega(\tau r) \le \gamma\omega(r) + \sigma(r) \tag{2.6.5}$$

for some $0 < \gamma < 1$ and $0 < \tau < 1$. Then for every $0 < \mu < 1$ and $r \le R$,

$$\omega(r) \le C\left(\left(\frac{r}{R}\right)^{\alpha}\omega(R) + \sigma(r^{\mu}R^{1-\mu})\right) \tag{2.6.6}$$

where $C = C(\gamma)$ and $\alpha = \alpha(\gamma, \tau, \mu)$ are positive constants. In particular, $\alpha = (1-\mu)\frac{\log\gamma}{\log\tau}$.

Proof. For any $r \le R$ we fix some $r \le r_1 \le R$ (which will be chosen suitably later). Then by (2.6.5) we have

$$\omega(\tau r_1) \le \gamma\omega(r_1) + \sigma(r_1).$$

Therefore,

$$\omega(\tau^2 r_1) = \omega(\tau(\tau r_1)) \le \gamma\omega(\tau r_1) + \sigma(r_1) \le \gamma^2\omega(r_1) + (\gamma+1)\sigma(r_1).$$

Applying repeatedly the above procedure we obtain, for every positive integer m, that

$$\begin{aligned}\omega(\tau^m r_1) &\le \gamma^m\omega(r_1) + \sigma(r_1)\sum_{i=0}^{m-1}\gamma^i \\ &\le \gamma^m\omega(R) + \frac{\sigma(r_1)}{1-\gamma},\end{aligned} \tag{2.6.7}$$

where we used the fact that ω is increasing and that $\sum_{i=0}^{\infty}\gamma^i = \frac{1}{1-\gamma}$ since $0 < \gamma < 1$. Now, since $r \le r_1$ and $0 < \tau < 1$ there exists a positive integer m_0 so that

$$\tau^{m_0} r_1 < r \le \tau^{m_0-1} r_1.$$

Hence from (2.6.7) and the monotonicity of ω,we get

$$\omega(r) \le \omega(\tau^{m_0-1}r_1) \le \gamma^{m_0-1}\omega(R) + \frac{\sigma(r_1)}{1-\gamma}. \tag{2.6.8}$$

Also observe that $\tau^{m_0} < \frac{r}{r_1}$, so $-\log\tau^{m_0} \ge -\log\frac{r}{r_1} \ge 0$ and since $0 < \gamma < 1$, $\gamma^{-\log\tau^{m_0}} \le \gamma^{-\log\frac{r}{r_1}} = \left(\frac{r}{r_1}\right)^{-\log\gamma}$, that is, $\gamma^{-m_0\log\tau} \le \left(\frac{r}{r_1}\right)^{-\log\gamma}$. Moreover $0 < \tau <$

1, in particular $\log\tau < 0$, hence we have that $\gamma^{m_0} \leq \left(\frac{r}{r_1}\right)^{\frac{\log\gamma}{\log\tau}}$. That is, $\gamma^{m_0-1} \leq \frac{1}{\gamma}\left(\frac{r}{r_1}\right)^{\frac{\log\gamma}{\log\tau}}$. From (2.6.8) we obtain

$$\omega(r) \leq \frac{1}{\gamma}\left(\frac{r}{r_1}\right)^{\frac{\log\gamma}{\log\tau}}\omega(R) + \frac{\sigma(r_1)}{1-\gamma}. \tag{2.6.9}$$

Now for $0 < \mu < 1$, set

$$r_1 = r^{\mu}R^{1-\mu}$$

and since $0 < \mu < 1$ we can easily deduce that $r \leq r_1 \leq R$. We return to (2.6.9) and get

$$\begin{aligned}\omega(r) &\leq \frac{1}{\gamma}\left(\frac{r}{R}\right)^{(1-\mu)\frac{\log\gamma}{\log\tau}}\omega(R) + \frac{\sigma(r^{\mu}R^{1-\mu})}{1-\gamma}\\ &\leq C\left(\left(\frac{r}{R}\right)^{\alpha}\omega(R) + \sigma(r^{\mu}R^{1-\mu})\right)\end{aligned}$$

where $C = C(\gamma) > 0$ is a constant and $\alpha = (1-\mu)\frac{\log\gamma}{\log\tau}$. Since $r \leq R$ is arbitrary, we have the desired result. □

Chapter 3

Regularity Theory for Fully Nonlinear Parabolic Equations with Oblique Boundary Data

The purpose of this chapter is to study the regularity of viscosity solutions of fully nonlinear parabolic equations with oblique boundary conditions of the form

$$\begin{cases} F(D^2u) - u_t = f, & \text{in } \; Q_1^+ \\ \beta \cdot Du = g, & \text{on } \; Q_1^* \\ u = u_0, & \text{on } \; \partial_p Q_1^+ \setminus Q_1^* \end{cases} \tag{3.0.1}$$

where F is a uniformly elliptic convex operator in S_n, f, g and u_0 are given data and $\beta : Q_1^* \to \mathbb{R}^n$ is a given vector function with $\beta_n \geq \delta_0 > 0$ and $||\beta||_{L^\infty} \leq 1$ (see Chapter 2 for detailed definitions).

3.1 ABP-estimate with Oblique Boundary Data

We prove an ABPT-type maximum principle corresponding to our oblique derivative problem (see [27], [33] for the elliptic case). Recall that the convex envelope of a function $u \in C\left(\overline{Q}_1^+\right)$ is defined as

$$\begin{aligned} \Gamma(u)(X,t) &:= \sup\{v(X,t) : v \leq u \text{ in } Q_1^+, \; v \text{ is convex in } Z \text{ and decreasing in } s\} \\ &= \sup\{\xi \cdot X + h : \xi \cdot Z + h \leq u(Z,s), \text{ for every } \; Z \in B_1^+, s \in (-1,t]\}. \end{aligned}$$

Moreover for smooth enough v we define the function

$$G(v)(X,t) = (Dv(X,t), v(X,t) - X \cdot Dv(X,t)).$$

Note that $\det D_{(X,t)}G(v) = v_t \ \det D^2 v$.

Theorem 28. *(ABP-estimate in the case of oblique boundary data). Let $f \in C\left(\overline{Q}_r^+\right)$, $g \in C\left(\overline{Q}_r^*\right)$ and $u \in \overline{S}_p(\lambda, \Lambda, f) \cap C\left(\overline{Q}_r^+\right)$ with $\beta \cdot Du \leq g$ on Q_r^* in the viscosity sense. Then,*

$$\inf_{\partial_p Q_r^+ \setminus Q_r^*} u - \inf_{Q_r^+} u \leq Cr \left(\int_{\{u=\Gamma_u\}} \left| f^+(X,t) \right|^{n+1} \, \mathrm{d}X \, \mathrm{d}t \right)^{1/n+1} + Cr \sup_{Q_r^*} g^+ \qquad (3.1.1)$$

where Γ_u is the convex envelope of $-u^-$ in Q_r^+ and $C > 0$ is a universal constant.

Proof. For convenience take $r = 1$ and $\inf_{\partial_p Q_1^+ \setminus Q_1^*} u = 0$ (then $u \geq 0$ on $\partial_p Q_1^+ \setminus Q_1^*$). We denote by $M := \sup_{Q_1^+} u^- > 0$ (excluding the trivial case) then there exists $(X_0, t_0) \in Q_1^+ \cup Q_1^*$ (since $u \geq 0$ on $\partial_p Q_1^+ \setminus Q_1^*$) so that $u^-(X_0, t_0) = M$.

Note that if $\sup_{Q_1^*} g^+ \geq \frac{\delta_0 M}{16}$ then (3.1.1) holds. So we consider that $\sup_{Q_1^*} g^+ < \frac{\delta_0 M}{16}$.

Since $\Gamma_u \in H^2(\overline{Q}_1^+)$ then we can show (for more details, see [44] or [23] and references therein)

- Using area formula

$$|G(\Gamma_u)(Q_1^+)| \leq \int_{Q_1^+ \cap \{u=\Gamma_u\}} -(\Gamma_u)_t \det(D^2\Gamma_u) \; dXdt.$$

- $-(\Gamma_u)_t + \lambda\Delta(\Gamma_u) \leq f^+$, in $\{u = \Gamma_u\}$.

Then we can end up with $|G(\Gamma_u)(Q_1^+)| \leq \int_{Q_1^+ \cap \{u=\Gamma_u\}} (f^+)^{n+1} \; dXdt$.

We consider the set

$$\mathcal{A} := \{(\xi, h) \in \mathbb{R}^n \times \mathbb{R} : |\xi| < \frac{M}{8} < \frac{M}{2} \leq -h \leq \frac{3M}{4}, \xi_n \geq \frac{M}{8}, |\xi'| \leq \frac{\delta_0 M}{16}\}$$

where $\xi' := (\xi_1, \ldots, \xi_{n-1})$. We will show that $\mathcal{A} \subset G(\Gamma_u)(Q_1^+)$. So we take any $(\xi, h) \in \mathcal{A}$ and we consider the polynomial

$$P(X) := \xi \cdot X + h.$$

Then we observe that

1. For every $X \in \overline{B}_2$, $P(X) \le |\xi||X| + h \le 2\frac{M}{8} - \frac{M}{2} = -\frac{M}{4} < 0$.

2. $P(X_0) - u(X_0.t_0) = \xi \cdot X_0 + h + M \ge -|\xi||X_0| + h + M \ge -\frac{M}{8} - \frac{3M}{4} + M = \frac{M}{8} > 0$, that is
$$\max_{\overline{B}_1^+}(P(X) - u(X,t_0)) \ge 0.$$

Define

$$t_1 := \sup\{-1 \le t \le 0 : \text{ for every } s \in [-1,t], \ \max_{\overline{B}_1^+}(P(X) - u(X,s)) < 0\}.$$

Note that -1 belongs in the above set since $u(X,-1) \ge 0$ and $P(X) < 0$. Moreover $t_1 \le t_0 \le 0$ and from the continuity of $P - u$ with respect to s we have that

$$P(X_1) - u(X_1,t_1) = \max_{\overline{B}_1^+}(P(X) - u(X,t_1)) = 0.$$

Then $(X_1,t_1) \in Q_1^+$. Indeed, if $(X_1,t_1) \in \partial_p Q_1^+ \setminus Q_1^*$ then $u(X_1,t_1) \ge 0$ and since $P(X_1) < 0$, $P(X_1) - u(X_1,t_1) < 0$, a contradiction. Also if $(X_1,t_1) \in Q_1^*$, P touches u by below at (X_1,t_1), then $\beta(x_1,t_1) \cdot \xi \le g(x_1,t_1)$ but

$$\begin{aligned}\beta(x_1,t_1) \cdot \xi = \xi_n \beta_n(x_1,t_1) + \xi' \cdot \beta'(x_1,t_1) \ge \delta_0 \xi_n - |\xi'| ||\beta||_{L^\infty} \ge \delta_0 \xi_n - \frac{\delta_0 \xi_n}{2} \\ = \frac{\delta_0 \xi_n}{2} \ge \sup_{Q_1^*} g^+\end{aligned}$$

since $\xi_n > \frac{M}{8} > \frac{2}{\delta_0} \sup_{Q_1^*} g^+$ and we get a contradiction.

Combining the above we have that $P(X) \le -u^-(X,t)$, for every $X \in \overline{B}_1^+$, $-1 < t \le t_1$ and $P(X_1) = -u^-(X_1,t_1)$. Then $P(X) \le \Gamma_u(X,t)$, for every $X \in \overline{B}_1^+$, $-1 < t \le t_1$ and $P(X_1) = \Gamma_u(X_1,t_1)$ with $(X_1,t_1) \in Q_1^+$, then $G(\Gamma_u)(X_1,t_1) = (\xi,h)$. Finally observing that $|\mathcal{A}| = C(\delta_0,n)M^{n+1}$ we finish the proof. $\square$

3.2 Hölder Estimates

3.2.1 H^α-estimates for the Neumann case

In the present subsection, using Harnack inequality, we prove interior Hölder regularity for functions in $S_p(\lambda,\Lambda,f)$ (this result can be found in [44]). Then using reflection property, we prove Hölder regularity up to a flat part of the boundary at which we assume a viscosity Neumann condition.

Theorem 29. *(Interior H^α-regularity). Let f be continuous, bounded in Q_1 and consider a bounded function $u \in S_p(\lambda, \Lambda, f)$ in Q_1. Then for universal constants $C > 0$ and $0 < \alpha < 1$, we have that*

$$u \in H^\alpha(\overline{Q}_{1/2})$$

with an estimate of the form

$$||u||_{H^\alpha(\overline{Q}_{1/2})} \leq C \left(||u||_{L^\infty(Q_1)} + ||f||_{L^{n+1}(Q_1)} \right). \tag{3.2.1}$$

Proof. For $0 < \rho \leq 1$ we consider the quantities $M_\rho := \sup_{Q_\rho} u, m_\rho := \inf_{Q_\rho} u$ and the oscillation of u in Q_ρ, $\omega_\rho := \operatorname{osc}_{Q_\rho} u = \sup_{Q_\rho} u - \inf_{Q_\rho} u$. We apply Harnack inequality (Theorem 23) to the non-negative functions $u - m_\rho, M_\rho - u$ which by ***(iii)*** of Proposition 15, lie in $S_p(\lambda, \Lambda, f)$ in Q_ρ. Hence we obtain

$$\sup_{K_{\rho R}} u - m_\rho \leq C \left(\inf_{Q_{\rho R^2}} u - m_\rho + \rho^{\frac{n+2}{n+1}} ||f||_{L^{n+1}(Q_\rho)} \right)$$

and

$$M_\rho - \inf_{K_{\rho R}} u \leq C \left(M_\rho - \sup_{Q_{\rho R^2}} u + \rho^{\frac{n+2}{n+1}} ||f||_{L^{n+1}(Q_\rho)} \right).$$

Adding the above relations we get

$$\omega_\rho \leq \omega_\rho + \operatorname*{osc}_{K_{\rho R}} u \leq C\omega_\rho - C\omega_{\rho R^2} + 2C\rho^{\frac{n+2}{n+1}} ||f||_{L^{n+1}(Q_\rho)}.$$

Therefore,

$$\omega_{R^2\rho} \leq \frac{C-1}{C}\omega_\rho + 2C\rho^{\frac{n+2}{n+1}} ||f||_{L^{n+1}(Q_1)}.$$

Then, we apply Lemma 27 in $(0, 1]$ with $\tau = R^2 < 1, \gamma = \frac{C-1}{C} < 1, \sigma(\rho) = 2C\rho^{\frac{n+2}{n+1}} ||f||_{L^{n+1}(Q_1)}$ and choosing properly some (universal) $0 < \mu < 1$ so that $0 < \alpha = (1-\mu)\frac{\log \gamma}{\log R^2} < 1$ and $0 < \alpha = (1-\mu)\frac{\log \gamma}{\log R^2} < \mu\frac{n+2}{n+1}$ we finally get

$$\begin{aligned} \omega_\rho &\leq C\rho^\alpha \left(\omega_1 + ||f||_{L^{n+1}(Q_1)} \right) \\ &\leq C\rho^\alpha \left(||u||_{L^\infty(Q_1)} + ||f||_{L^{n+1}(Q_1)} \right) \end{aligned} \tag{3.2.2}$$

for $0 < \rho \leq 1$ and universal constants $0 < \alpha < 1$, $C > 0$.

Now let $(X, t) \in \overline{Q}_{1/2}$. Then one can easily verify that $\overline{Q}_\rho(X, t) \subset Q_1$ for any

$0 < \rho \le \frac{1}{4}$. We observe that, if $(Z,s) \in \overline{Q}_\rho(X,t) \setminus Q_{\rho/2}(X,t)$ then either $|Z-X| > \frac{\rho}{2}$ or $t-s > \frac{\rho^2}{4}$ and so (3.2.2) (properly translated to $Q_\rho(X,t)$) implies,

$$u(X,t) - u(Z,s) \le \omega_{\rho;(X,t)} \le C|X-Z|^\alpha \left(||u||_{L^\infty(Q_1)} + ||f||_{L^{n+1}(Q_1)}\right)$$

or

$$\le C|t-s|^{\alpha/2} \left(||u||_{L^\infty(Q_1)} + ||f||_{L^{n+1}(Q_1)}\right)$$

and as a consequence in any case,

$$|u(X,t) - u(Z,s)| \le C(|X-Z|^\alpha + |t-s|^{\alpha/2}) \left(||u||_{L^\infty(Q_1)} + ||f||_{L^{n+1}(Q_1)}\right) \qquad (3.2.3)$$

for universal constants $0 < \alpha < 1$, $C > 0$. In addition we observe that for any $(Z,s) \in \overline{Q}_{1/4}(X,t) \setminus \{(X,t)\}$, there exists some $0 < \rho \le \frac{1}{4}$ such that $(Z,s) \in \overline{Q}_\rho(X,t) \setminus Q_{\rho/2}(X,t)$ (choose $\rho = \max\left(|Z-X|, \sqrt{|t-s|}\right)$). Hence estimate (3.2.3) holds for every $(Z,s) \in \overline{Q}_{1/4}(X,t)$.

Finally, we obtain estimate (3.2.3) for any $(X,t), (Z,s) \in Q_{1/2}$. We may assume without loss of generality that $s \le t$, then we have two possible cases:

1. If $(Z,s) \in \overline{Q}_{1/4}(X,t)$ the above consideration is in order.
2. If $(Z,s) \notin \overline{Q}_{1/4}(X,t)$, then either $|Z-X| > \frac{1}{4}$ or $t-s > \frac{1}{16}$ therefore

$$|u(X,t) - u(Z,s)| \le 4^\alpha 2 ||u||_{L^\infty(Q_1)} \frac{1}{4^\alpha} \le 4^\alpha 2 ||u||_{L^\infty(Q_1)} |X-Z|^\alpha$$

or

$$\le 4^\alpha 2 ||u||_{L^\infty(Q_1)} |t-s|^{\alpha/2}.$$

In any case we derive estimate (3.2.3) which completes the proof. □

Theorem 30. *(Up to the boundary H^α-regularity). Let f be continuous and bounded in Q_1^+. Assume that $u \in C\left(\overline{Q_1^+}\right)$ is such that*

$$\begin{cases} u \in S_p(\lambda, \Lambda, f), & \textit{in} \quad Q_1^+ \\ u_y = 0, & \textit{on} \quad Q_1^*, \quad \textit{in the viscosity sense.} \end{cases}$$

Then for universal constants $C > 0$ and $0 < \alpha < 1$, we have that

$$u \in H^{\alpha}\left(\overline{Q}^{+}_{1/2}\right)$$

with an estimate

$$||u||_{H^{\alpha}\left(\overline{Q}^{+}_{1/2}\right)} \leq C\left(||u||_{L^{\infty}\left(Q_1^{+}\right)} + ||f||_{L^{n+1}\left(Q_1^{+}\right)}\right). \tag{3.2.4}$$

Proof. Consider the reflected function u^* defined in Proposition 21. Then $u^* \in S_p(\lambda, \Lambda, f^*)$ in Q_1 and by Theorem 29 applied to u^* we obtain

$$||u^*||_{H^{\alpha}(\overline{Q}_{1/2})} \leq C\left(||u^*||_{L^{\infty}(Q_1)} + ||f^*||_{L^{n+1}(Q_1)}\right)$$

for universal constants $C > 0$ and $0 < \alpha < 1$. Now observe that $||u||_{H^{\alpha}\left(\overline{Q}^{+}_{1/2}\right)} \leq ||u^*||_{H^{\alpha}\left(\overline{Q}_{1/2}\right)}$, $||u||_{L^{\infty}\left(Q_1^{+}\right)} = ||u^*||_{L^{\infty}(Q_1)}$ and $||f^*||_{L^{n+1}(Q_1)} = 2^{\frac{1}{n+1}}||f||_{L^{n+1}\left(Q_1^{+}\right)}$ after a change of variables. Combining all the above relations we finish the proof. □

Combining Theorem 30 with Proposition 16 we immediately have the following corollary.

Corollary 31. *Let f be continuous, bounded in Q_1^{+} and assume that the bounded function $u \in C\left(Q_1^{+} \cup Q_1^{*}\right)$ is a viscosity solution of*

$$\begin{cases} F(D^2u) - u_t = f, & \text{in} \quad Q_1^{+} \\ u_y = 0, & \text{on} \quad Q_1^{*}. \end{cases}$$

Then for universal constants $C > 0$ and $0 < \alpha < 1$, we have that

$$u \in H^{\alpha}\left(\overline{Q}^{+}_{1/2}\right)$$

and

$$||u||_{H^{\alpha}\left(\overline{Q}^{+}_{1/2}\right)} \leq C\left(||u||_{L^{\infty}\left(Q_1^{+}\right)} + ||f||_{L^{n+1}\left(Q_1^{+}\right)} + |F(O)|\right). \tag{3.2.5}$$

Using ***(ii)*** of Proposition 15 and then Proposition 21 we can obtain rescaled versions of Theorems 29 and 30.

Corollary 32. *Let f be continuous, bounded in $Q_r(X_0,t_0)$ and consider a bounded function $u \in S_p(\lambda,\Lambda,f)$ in $Q_r(X_0,t_0)$. Then for universal constants $C>0$ and $0<\alpha<1$, we have that*

$$u \in H^\alpha(\overline{Q}_{r/2}(X_0,t_0))$$

and

$$||u||_{H^\alpha\left(\overline{Q}_{r/2}(X_0,t_0)\right)} \leq \frac{C}{r^\alpha}\left(||u||_{L^\infty(Q_r(X_0,t_0))} + r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q_r(X_0,t_0))}\right). \tag{3.2.6}$$

Corollary 33. *Let f be continuous, bounded in $Q_r(X_0,t_0)$ and consider a bounded function $u \in S_p(\lambda,\Lambda,f)$ in $Q_r(X_0,t_0)$. Then for universal constants $C>0$ and $0<\alpha<1$, we have that for any $0<s<r$*

$$u \in H^\alpha(\overline{Q}_s(X_0,t_0))$$

and

$$||u||_{H^\alpha\left(\overline{Q}_s(X_0,t_0)\right)} \leq C\tilde{C}(r,s)\left(||u||_{L^\infty(Q_r(X_0,t_0))} + ||f||_{L^{n+1}(Q_r(X_0,t_0))}\right) \tag{3.2.7}$$

for a constant $\tilde{C}(r,s)>0$.

Proof. If $(X,t) \in Q_s(X_0,t_0)$ then $Q_{r-s}(X,t) \subset Q_r(X_0,t_0)$ and applying Corollary 34 we get the desired estimate for points (Z,s) lying in $\overline{Q}_{\frac{r-s}{2}}(X,t)$. Now for points not in $\overline{Q}_{\frac{r-s}{2}}(X,t)$ we can follow a similar argument as in the proof of Theorem 29. □

By reflection (Proposition 21) we get the following corollaries.

Corollary 34. *Let $f \in C\left(Q_r^+(x_0,t_0)\right)$, bounded and consider a bounded function $u \in C\left(Q_r^+(x_0,t_0) \cup Q_r^*(x_0,t_0)\right)$ be bounded and such that*

$$\begin{cases} u \in S_p(\lambda,\Lambda,f), & \text{in} \quad Q_r^+(x_0,t_0) \\ u_y = 0, & \text{on} \quad Q_r^*(x_0,t_0), \quad \text{in the viscosity sense.} \end{cases}$$

Then for universal constants $C>0$ and $0<\alpha<1$, we have that

$$u \in H^\alpha\left(\overline{Q}^+_{r/2}(x_0,t_0)\right)$$

with an estimate

$$||u||_{H^{\alpha}\left(\overline{Q}^{+}_{r/2}(x_0,t_0)\right)} \leq \frac{C}{r^{\alpha}}\left(||u||_{L^{\infty}\left(Q^{+}_{r}(x_0,t_0)\right)} + r^{\frac{n}{n+1}}||f||_{L^{n+1}\left(Q^{+}_{r}(x_0,t_0)\right)}\right). \qquad (3.2.8)$$

Corollary 35. *Let* $f \in C\left(Q^{+}_{r}(X_0,t_0)\right)$*, bounded and* $u \in C\left(Q^{+}_{r}(x_0,t_0) \cup Q^{*}_{r}(x_0,t_0)\right)$*, bounded satisfying*

$$\begin{cases} u \in S_p(\lambda,\Lambda,f), & \textit{in} \quad Q^{+}_{r}(x_0,t_0) \\ u_y = 0, & \textit{on} \quad Q^{*}_{r}(x_0,t_0), \quad \textit{in the viscosity sense.} \end{cases}$$

Then for universal constants $C > 0$ *and* $0 < \alpha < 1$*, we have that for any* $0 < s < r$*,*

$$u \in H^{\alpha}\left(\overline{Q}^{+}_{s}(x_0,t_0)\right)$$

with an estimate

$$||u||_{H^{\alpha}\left(\overline{Q}^{+}_{s}(x_0,t_0)\right)} \leq C\tilde{C}(r,s)\left(||u||_{L^{\infty}\left(Q^{+}_{r}(x_0,t_0)\right)} + ||f||_{L^{n+1}\left(Q^{+}_{r}(x_0,t_0)\right)}\right) \qquad (3.2.9)$$

for a constant $\tilde{C}(r,s) > 0$*.*

The following boundary Lipschitz-type estimate will be useful in the study of $H^{1+\alpha}$-estimates. It can be proved using a barrier argument (see for instance Lemma 2.1 in [7]).

Proposition 36. *Let* f *be bounded in* Q^{+}_{1} *and* $u \in C\left(Q^{+}_{1} \cup Q^{*}_{1}\right)$ *be bounded and satisfy in the viscosity sense*

$$\begin{cases} u \in S_p(\lambda,\Lambda,f), & \textit{in} \ \ Q^{+}_{1} \\ u = 0, & \textit{on} \ \ Q^{*}_{1}. \end{cases}$$

Then there exists universal constant $C > 0$ *so that*

$$|u(X,t)| \leq C\left(||u||_{L^{\infty}\left(Q^{+}_{1}\right)} + ||f||_{L^{n+1}\left(Q^{+}_{1}\right)}\right)\ y \qquad (3.2.10)$$

for every $(X,t) = (x,y,t) \in \overline{Q}^{+}_{1/2}$*.*

3.2.2 H^α-estimates for the oblique derivative case

In the present subsection we prove Hölder regularity up to the flat boundary for the nonlinear parabolic oblique derivative problem proving first a boundary Harnack-type inequality.

Theorem 37. *(Up to the boundary H^α-regularity). Let f and g be continuous and bounded in Q_1^+ and Q_1^* respectively. Assume that $u \in C\left(\overline{Q_1^+}\right)$ is such that*

$$\begin{cases} u \in S_p(\lambda, \Lambda, f), & \text{in} \quad Q_1^+ \\ \beta \cdot Du = g, & \text{on} \quad Q_1^*, \quad \text{in the viscosity sense.} \end{cases}$$

Then for universal constants $C > 0$ and $0 < \alpha < 1$, we have that

$$u \in H^\alpha\left(\overline{Q}_{1/2}^+\right)$$

with an estimate

$$||u||_{H^\alpha\left(\overline{Q}_{1/2}^+\right)} \leq C\left(||u||_{L^\infty\left(Q_1^+\right)} + ||f||_{L^{n+1}\left(Q_1^+\right)} + ||g||_{L^\infty\left(Q_1^*\right)}\right). \tag{3.2.11}$$

Note that Theorem 30 is a special case of the above but in the previous subsection we gave an alternative proof for the Neumann case through reflection property.

Corollary 38. *(Up to the boundary H^α-regularity-rescaled). Let f and g be continuous and bounded in Q_r^+ and Q_r^* respectively. Assume that $u \in C\left(\overline{Q_r^+}\right)$ is such that*

$$\begin{cases} u \in S_p(\lambda, \Lambda, f), & \text{in} \quad Q_r^+ \\ \beta \cdot Du = g, & \text{on} \quad Q_r^*, \quad \text{in the viscosity sense.} \end{cases}$$

Then for universal constants $C > 0$ and $0 < \alpha < 1$, we have that

$$u \in H^\alpha\left(\overline{Q}_{r/2}^+\right)$$

with an estimate

$$||u||_{H^\alpha\left(\overline{Q}_{r/2}^+\right)} \leq \frac{C}{r^\alpha}\left(||u||_{L^\infty\left(Q_r^+\right)} + r^{\frac{n}{n+1}}||f||_{L^{n+1}\left(Q_r^+\right)} + r||g||_{L^\infty\left(Q_r^*\right)}\right). \tag{3.2.12}$$

Combining the interior Harnack inequality with a barrier argument we get the following boundary Harnack inequality.

Theorem 39. *(Boundary Harnack inequality). Let f and g be continuous and bounded in Q_1^+ and Q_1^* respectively. Assume that $u \in C\left(\overline{Q_1^+}\right)$, $u \geq 0$ is such that*

$$\begin{cases} u \in S_p(\lambda, \Lambda, f), & \text{in} \quad Q_1^+ \\ \beta \cdot Du = g, & \text{on} \quad Q_1^*, \quad \text{in the viscosity sense.} \end{cases}$$

Then for universal constants $C > 0$ and $0 < \rho < 1$, we have for every $0 < r < \frac{1}{2}$,

$$\sup_{K_{\frac{rR}{2}}(A,0)} u \leq C \left(\inf_{H\left(\frac{r}{4}, \rho\right)} u + r^{\frac{n}{n+1}} ||f||_{L^{n+1}(Q_1^+)} + r||g||_{L^\infty(Q_1^*)} \right) \tag{3.2.13}$$

where, $A = (0, \ldots, 0, r) \in \mathbb{R}^n$, $K_R := B_{\frac{R^2}{2\sqrt{2}}}(0,0) \times \left[-R^2 + \frac{3}{8}R^4, -R^2 + \frac{4}{8}R^4\right]$, for $0 < R << 1$ universal defined as in Theorem 23 and

$$H(r, \rho) := \{(X,t) : |x| < \frac{rR^2}{4}, 0 < y < \rho r, -\frac{r^2 R^4}{16} < t \leq 0\}.$$

Proof. For $0 < r < \frac{1}{2}$ note that

$$Q_{r/2}(A, 0) \subset \{(X,t) : |x| < r, \frac{r}{2} < y < \frac{3r}{2}, -r^2 < t \leq 0\}.$$

Then we can apply interior Harnack inequality to u in $Q_{r/2}(A,0)$ (Theorem 23),

$$\sup_{K_{\frac{rR}{2}}(A,0)} u \leq C \left(\inf_{Q_{\frac{rR^2}{2}}(A,0)} u + r^{\frac{n}{n+1}} ||f||_{L^{n+1}(Q_1)} \right).$$

Let

$$H'(r, \rho) := \{(X,t) : |x| < \frac{rR^2}{4}, y = \rho r, -\frac{r^2 R^4}{16} < t \leq 0\}.$$

Note that if we choose $0 < \rho < \frac{\sqrt{3}R^2}{4}$ then $H'(r,\rho) \subset Q_{\frac{rR^2}{2}}(A,0)$. So we want to show that

$$B := \inf_{H'(r,\rho)} u \leq C \left(\inf_{H\left(\frac{r}{4}, \rho\right)} u + r^{\frac{n}{n+1}} ||f||_{L^{n+1}(Q_1^+)} + r||g||_{L^\infty(Q_1^*)} \right). \tag{3.2.14}$$

In other words we want to find a suitable lower bound for u in $H\left(\frac{r}{4}, \rho\right)$. We do this comparing u with a suitable barrier function.

For $\bar{r} := \frac{rR^2}{4}$ we define

$$b(X,t) := B - \frac{B}{4}\left[2 - \frac{y^2}{(\rho r)^2} - \frac{y}{\rho r} + 4\left(\frac{|x|^2 - t}{\bar{r}^2}\right)\right] - \frac{||g||_{L^\infty(Q_1^*)}}{\delta_0}(\rho r - y).$$

We calculate the derivatives of b in $H(r,\rho)$:

$$b_{x_i} = -\frac{2Bx_i}{\bar{r}^2}, \quad b_t = \frac{B}{\bar{r}^2}, \quad b_y = \frac{By}{2(\rho r)^2} + \frac{B}{4\rho r} + \frac{||g||_{L^\infty(Q_1^*)}}{\delta_0}$$

and

$$b_{x_i x_j} = -\frac{2B}{\bar{r}^2}\delta_{ij}, \quad b_{x_i y} = 0, \quad b_{yy} = \frac{B}{2(\rho r)^2}.$$

So, we have in $H(r,\rho)$ that

$$\mathcal{M}^-(D^2 b, \lambda, \Lambda) - b_t = \lambda\frac{B}{2(\rho r)^2} - (n-1)\Lambda\frac{2B}{\bar{r}^2} - \frac{B}{\bar{r}^2} \geq 0$$

if we choose $0 < \rho \leq \sqrt{\frac{\lambda R^4}{32[2(n-1)\Lambda+1]}}$. Hence

$$u - b \in \overline{S}_p(\lambda, \Lambda, f), \quad \text{in} \ \ H(r,\rho).$$

Next, we study b on the parabolic boundary of $H(r,\rho)$. We start with $\overline{H}(r,\rho)\cap\{y=0\}$,

$$\begin{aligned}\beta \cdot Db &= -\frac{2B}{\bar{r}^2}\beta\cdot(x,0) + \frac{B}{4\rho r}\beta_n + \frac{||g||_{L^\infty(Q_1^*)}}{\delta_0}\beta_n \geq -\frac{2B}{\bar{r}^2}1\ \bar{r} + \frac{B}{4\rho r}\delta_0 + ||g||_{L^\infty(Q_1^*)} \\ &\geq \frac{B}{r}\left(-\frac{8}{R^2} + \frac{\delta_0}{4\rho}\right) + ||g||_{L^\infty(Q_1^*)} \geq +||g||_{L^\infty(Q_1^*)}\end{aligned}$$

if we choose $0 < \rho \leq \frac{\delta_0 R^2}{32}$. Hence

$$\beta \cdot D(u-b) \leq 0, \quad \text{on} \ \ \overline{H}(r,\rho)\cap\{y=0\}.$$

We continue with $\partial_p H(r,\rho) \setminus \overline{H}(r,\rho)\cap\{y=0\}$. We have

- On $\{|x| = \bar{r}\}$:

$$\begin{aligned}b(X,t) &= B - \frac{B}{4}\left(1 - \frac{y^2}{(\rho r)^2}\right) - \frac{B}{4}\left(1 - \frac{y}{\rho r}\right) - B + B\frac{t}{\bar{r}^2} - \frac{||g||_{L^\infty(Q_1^*)}}{\delta_0}(\rho r - y) \\ &\leq 0 \leq u(X,t).\end{aligned}$$

- On $\{t = -\bar{r}^2\}$:

$$b(X, -\bar{r}^2) = B - \frac{B}{4}\left(1 - \frac{y^2}{(\rho r)^2}\right) - \frac{B}{4}\left(1 - \frac{y}{\rho r}\right) - B\frac{|x|^2}{\bar{r}^2} - B - \frac{||g||_{L^\infty(Q_1^*)}}{\delta_0}(\rho r - y)$$
$$\leq 0 \leq u(X, -\bar{r}^2).$$

- On $\{y = \rho r\}$:

$$b(x, \rho r, t) = B - B\left(\frac{|x|^2 - t}{\bar{r}^2}\right) \leq B \leq u(x, \rho r, t).$$

Hence

$$u - b \geq 0, \quad \text{on } \partial_p H(r,\rho) \setminus \overline{H}(r,\rho) \cap \{y = 0\}.$$

Therefore from Theorem 28 we have that

$$u - b \geq -r^{\frac{n}{n+1}} ||f||_{L^{n+1}(Q_1^+)}, \quad \text{in } H(r,\rho).$$

Then in $H\left(\frac{r}{4}, \rho\right)$ we have

$$u + Cr||g||_{L^\infty(Q_1^*)} + r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q_1^+)} \geq B - \frac{B}{2}\left[1 + 2\left(\frac{|x|^2 - t}{\bar{r}^2}\right)\right]$$
$$\geq B - \frac{B}{2} - \frac{B}{2}\cdot\frac{1}{8} = \frac{7B}{16}$$

which gives (3.2.14). □

Then Theorem 37 follows in a standard way.

Proof of Theorem 37. For $0 < r \leq \frac{1}{2}$ we consider the quantities

$$M_r := \sup_{Q_r^+} u, \quad m_r := \inf_{Q_r^+} u.$$

Then the functions $v_1 := M_r - u$, $v_2 := u - m_r$ are non-negative in Q_r^+, $v_i \in S_p(\lambda, \Lambda, f)$ in Q_r^+ and $b \cdot Dv_i = g$ on Q_r^*. Then we apply Theorem 39 to v_i

$$M_r - \inf_{K_{\frac{rR}{4}}(A,0)} u \leq C\left(M_r - \sup_{H\left(\frac{r}{8},\rho\right)} u + r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q_r^+)} + r||g||_{L^\infty(Q_r^*)}\right)$$

and

$$\sup_{K_{\frac{rR}{4}}(A,0)} u - m_r \leq C\left(\inf_{H\left(\frac{r}{8},\rho\right)} u - m_r + r^{\frac{n}{n+1}}||f||_{L^{n+1}\left(Q_r^+\right)} + r||g||_{L^\infty(Q_r^*)}\right).$$

Adding the above we get

$$\operatorname*{osc}_{Q_r^+} u \leq \operatorname*{osc}_{Q_r^+} u + \operatorname*{osc}_{K_{\frac{rR}{4}}(A,0)} u \leq C\left(\operatorname*{osc}_{Q_r^+} u - \operatorname*{osc}_{H\left(\frac{r}{8},\rho\right)} u + 2r^{\frac{n}{n+1}}||f||_{L^{n+1}\left(Q_r^+\right)} + 2r||g||_{L^\infty(Q_r^*)}\right)$$

then

$$\operatorname*{osc}_{H\left(\frac{r}{8},\rho\right)} u \leq \gamma \operatorname*{osc}_{Q_r^+} u + 2C\left(r^{\frac{n}{n+1}}||f||_{L^{n+1}\left(Q_r^+\right)} + r||g||_{L^\infty(Q_r^*)}\right)$$

where $\gamma := \frac{C-1}{C} < 1$. Note that $Q^+_{\frac{\rho R^2}{2^5}r} \subset H\left(\frac{r}{8},\rho\right)$ so we can write

$$\operatorname*{osc}_{Q^+_{\frac{\rho R^2}{2^5}r}} u \leq \gamma \operatorname*{osc}_{Q_r^+} u + 2C\left(r^{\frac{n}{n+1}}||f||_{L^{n+1}\left(Q_r^+\right)} + r||g||_{L^\infty(Q_r^*)}\right).$$

Then the result follows in the same way as in the proof of Theorem 29 using Lemma 27. □

3.3 Hölder Estimates for the first derivatives

In this section, we study existence and regularity of the first derivatives of viscosity solutions in the Neumann case (subsection 3.3.2) and then in the general oblique derivative case (subsection 3.3.3). To study the Neumann problem we define suitable difference quotients and apply the Hölder estimates proved in the previous section. To do so we have to explore which problem the difference of two solutions satisfies. This is achieved with the aid of suitable approximate solutions defined in subsection 3.3.1 (the idea had been initially introduced by Jensen for nonlinear elliptic equations). In subsection 3.3.3, first we use the change of variables of section 2.5 and combining with the $H^{1+\alpha}$-estimates for Neumann problems of subsection 3.3.2 we get $H^{1+\alpha}$-estimates for a constant oblique derivative problem. Secondly, we use a standard approximation method (see for example [10], Chapter 8) and approximate a general oblique derivative problem by suitable constant oblique derivative problems.

3.3.1 Approximate sub/super-solutions

Let $u \in C\left(Q_1^+ \cup Q_1^*\right)$, $\epsilon > 0$ and $0 < \rho < \frac{1}{2}$. We define the sub-convolution of u by

$$u^{\epsilon,\rho}(X,t) = \sup_{(Z,s)\in \overline{Q}_\rho^+} \left(u(Z,s) - \frac{1}{\epsilon}|X-Z|^2 - \frac{1}{\epsilon}(t-s)^2 \right)$$

for any $(X,t) \in Q_1^+ \cup Q_1^*$. The super-convolution $u_{\epsilon,\rho}$ is defined accordingly taking infimum and adding (instead of subtracting) the polynomial, that is

$$u_{\epsilon,\rho}(X,t) = \inf_{(Z,s)\in \overline{Q}_\rho^+} \left(u(Z,s) + \frac{1}{\epsilon}|X-Z|^2 + \frac{1}{\epsilon}(t-s)^2 \right).$$

Next we study some basic properties of sub/super-convolutions which will be useful in the sequel.

Lemma 40. ***(i)*** *For $(X_0,t_0) \in Q_1^+ \cup Q_1^*$ there exists a point $(X_0^*, t_0^*) \in \overline{Q}_\rho^+$ so that*

$$u^{\epsilon,\rho}(X_0,t_0) = u(X_0^*,t_0^*) - \frac{1}{\epsilon}|X_0 - X_0^*|^2 - \frac{1}{\epsilon}(t_0 - t_0^*)^2$$

(ii) *$u^{\epsilon,\rho}(X,t) \geq u(X,t)$ for any $(X,t) \in \overline{Q}_\rho^+$.*

(iii) *$0 \leq u^{\epsilon,\rho}(X_0,t_0) - u(X_0,t_0) \leq u(X_0^*,t_0^*) - u(X_0,t_0)$ for any $(X_0,t_0) \in Q_\rho^+$.*

(iv) *$u^{\epsilon,\rho}$ is continuous in $Q_1^+ \cup Q_1^*$.*

(v) *$|X_0 - X_0^*|^2 + (t_0 - t_0^*)^2 \leq \epsilon \operatorname{osc}_{Q_\rho^+} u$, that is if we choose ϵ small enough the point (X_0^*, t_0^*) will be "close" to (X_0,t_0).*

(vi) *$u^{\epsilon,\rho} \to u$ uniformly in $\overline{Q}_\rho^+$, as $\epsilon \to 0^+$.*

(vii) *$(u^{\epsilon,\rho})_y \geq 0$ on Q_1^* in the viscosity sense. (!)*

Analog properties hold for $u_{\epsilon,\rho}$ as well.

Proof. ***(i)-(iii)*** are immediate.

(iv) Take any $(X_1,t_1),(X_2,t_2)\in Q_1^+\cup Q_1^*$, then for any $(Z,s)\in\overline{Q}_\rho^+$ we have

$$\begin{aligned}
u^{\epsilon,\rho}(X_1,t_1) &\geq u(Z,s)-\frac{1}{\epsilon}|X_1-Z|^2-\frac{1}{\epsilon}(t_1-s)^2\\
&\geq u(Z,s)-\frac{1}{\epsilon}|X_2-Z|^2-\frac{1}{\epsilon}|X_1-X_2|^2-\frac{2}{\epsilon}|X_2-Z||X_1-X_2|\\
&\quad -\frac{1}{\epsilon}(t_2-s)^2-\frac{1}{\epsilon}(t_1-t_2)^2-\frac{2}{\epsilon}|t_2-s||t_1-t_2|\\
&\geq u(Z,s)-\frac{1}{\epsilon}|X_2-Z|^2-\frac{1}{\epsilon}(t_2-s)^2-\frac{6}{\epsilon}|X_1-X_2|-\frac{6}{\epsilon}|t_1-t_2|
\end{aligned}$$

using the definition of $u^{\epsilon,\rho}$, the triangle inequality and that $|X_1-X_2|,|X_2-Z|,|t_1-t_2|,|t_1-s|\leq 2$. That is, taking supremum over $\overline{Q}_\rho^+$ we obtain

$$|u^{\epsilon,\rho}(X_1,t_1)-u^{\epsilon,\rho}(X_2,t_2)|\leq\frac{6}{\epsilon}\left(|X_1-X_2|+|t_1-t_2|\right) \tag{3.3.1}$$

for any $(X_1,t_1),(X_2,t_2)\in Q_1^+\cup Q_1^*$. Inequality (3.3.1) implies the continuity of $u^{\epsilon,\rho}$.

(v) Since $u^{\epsilon,\rho}(X_0,t_0)=u(X_0^*,t_0^*)-\frac{1}{\epsilon}|X_0-X_0^*|^2-\frac{1}{\epsilon}(t_0-t_0^*)^2$ we get that

$$|X_0-X_0^*|^2+(t_0-t_0^*)^2=\epsilon\left(u(X_0^*,t_0^*)-u^{\epsilon,\rho}(X_0,t_0)\right)$$

which together with ***(ii)*** implies the result.

(vi) Take any $M>0$. We know that u is uniformly continuous in the compact set $\overline{Q}_\rho^+$, so there exists some $\delta(M)>0$ so that

$$|u(X,t)-u(Z,s)|<M,\quad \text{for any}\quad (X,t),(Z,s)\in\overline{Q}_\rho^+\quad\text{with}\quad |X-Z|,|t-s|<\delta.$$

We choose $0<\epsilon<\frac{\delta^2(M)}{\operatorname{osc}_{\overline{Q}_\rho^+}u}$ (note that if $\operatorname{osc}_{\overline{Q}_\rho^+}u=0$ then u as well as $u^{\epsilon,\rho}$ are both identical zero and the result is obvious). Then taking any $(X_0,t_0)\in\overline{Q}_\rho^+$ we have from ***(v)*** that $|X_0-X_0^*|^2+(t_0-t_0^*)^2\leq\delta^2$, which implies that $|X_0-X_0^*|,|t_0-t_0^*|<\delta$. Therefore $|u(X_0^*,t_0^*)-u(X_0,t_0)|<M$ and by ***(iii)*** we conclude that $0\leq u^{\epsilon,\rho}(X_0,t_0)-u(X_0,t_0)<M$.

(vii) Let ϕ be a test function that touches $u^{\epsilon,\rho}$ by above at some point $(X_0,t_0)=(x_0,0,t_0)\in Q_1^*$. Let $(X_0^*,t_0^*)=(x_0^*,y_0^*,t_0^*)\in\overline{Q}_\rho^+$ be the point in ***(i)***. We have

$$\phi(X,t)\geq u^{\epsilon,\rho}(X,t)\geq u(X_0^*,t_0^*)-\frac{1}{\epsilon}|X-X_0^*|^2-\frac{1}{\epsilon}(t-t_0^*)^2$$

in a half-cylinder around (X_0, t_0). In particular

$$\phi(X_0, t_0) = u^{\epsilon,\rho}(X_0, t_0) = u(X_0^*, t_0^*) - \frac{1}{\epsilon}|X_0 - X_0^*|^2 - \frac{1}{\epsilon}(t_0 - t_0^*)^2.$$

Hence the function $\Phi(y) = \phi(x_0, y, t_0) + \frac{1}{\epsilon}|y - y_0^*|^2 - u(X_0^*, t_0^*) + \frac{1}{\epsilon}|x_0 - x_0^*|^2 + \frac{1}{\epsilon}(t_0 - t_0^*)^2$ is non-negative for small $y > 0$ and zero at $y = 0$. Therefore

$$\Phi'(0) = \lim_{y \to 0^+} \frac{\Phi(y) - \Phi(0)}{y} \geq 0.$$

That is, $\phi_y(X_0, t_0) - \frac{2}{\epsilon}y_0^* \geq 0$. But $y_0^* \geq 0$, thus $\phi_y(X_0, t_0) \geq 0$.

□

Lemma 41. *Assume that u is continuous in $Q_1^+ \cup Q_1^*$ and satisfies the condition $u_y \geq 0$ on Q_1^* in the viscosity sense. Then for any $(X_0, t_0) \in Q_1^+$ the point (X_0^*, t_0^*) of **(i)** in Lemma 40 lies in $\overline{Q}_\rho^+ \setminus Q_\rho^*$.*

Proof. Take any $(X_0, t_0) \in Q_1^+$. We assume that $(X_0^*, t_0^*) \in Q_\rho^*$ in order to arrive to a contradiction. By Lemma 40 we have

$$u^{\epsilon,\rho}(X_0, t_0) = u(X_0^*, t_0^*) - \frac{1}{\epsilon}|X_0 - X_0^*|^2 - \frac{1}{\epsilon}(t_0 - t_0^*)^2$$

and for any $(Z, s) \in \overline{Q}_\rho^+$,

$$u^{\epsilon,\rho}(X_0, t_0) \geq u(Z, s) - \frac{1}{\epsilon}|X_0 - Z|^2 - \frac{1}{\epsilon}(t_0 - s)^2.$$

That is for any $(Z, s) \in \overline{Q}_\rho^+$,

$$u(X_0^*, t_0^*) - \frac{1}{\epsilon}|X_0 - X_0^*|^2 - \frac{1}{\epsilon}(t_0 - t_0^*)^2 \geq u(Z, s) - \frac{1}{\epsilon}|X_0 - Z|^2 - \frac{1}{\epsilon}(t_0 - s)^2.$$

Setting $\phi(Z, s) := u(X_0^*, t_0^*) - \frac{1}{\epsilon}|X_0 - X_0^*|^2 - \frac{1}{\epsilon}(t_0 - t_0^*)^2 + \frac{1}{\epsilon}|X_0 - Z|^2 + \frac{1}{\epsilon}(t_0 - s)^2$ we ensure that

$$\phi \geq u \quad \text{in} \quad \overline{Q}_\rho^+ \quad \text{and} \quad \phi(X_0^*, t_0^*) = u(X_0^*, t_0^*).$$

In addition, since $(X_0^*, t_0^*) \in Q_\rho^*$, the Neumann condition implies that $\phi_y(X_0^*, t_0^*) \geq 0$, But, on the other hand we can compute $\phi_y(X_0^*, t_0^*) = -\frac{2}{\epsilon}(y_0 - y_0^*) = -\frac{2}{\epsilon}y_0 < 0$ and this is a contradiction. □

Lemma 42. *Let $u \in C\left(Q_1^+ \cup Q_1^*\right)$ satisfies in the viscosity sense*

$$\begin{cases} F(D^2u) - u_t \geq 0, & \text{in} \quad Q_1^+ \\ u_y \geq 0, & \text{on} \quad Q_1^*. \end{cases} \tag{3.3.2}$$

Then for any $0 < \rho_1 < \rho < \frac{1}{2}$ there exists some $0 < \epsilon_0 = \epsilon_0(\rho_1, \rho, u)$ such that for any $0 < \epsilon < \epsilon_0$, $u^{\epsilon,\rho}$ is a viscosity subsolution of $F(D^2v) - v_t = 0$ in $Q_{\rho_1}^+$ (hence, $u^{\epsilon,\rho}$ satisfies problem (3.3.2) in $Q_{\rho_1}^+ \cup Q_{\rho_1}^*$ (combining with ***(vii)*** *of Lemma 40)).*

Note that we do not use the Neumann condition of (3.3.2) for u to show that $u^{\epsilon,\rho}$ satisfies the same condition since $u^{\epsilon,\rho}$ satisfies this condition anyway (see ***(vii)***, Lemma 40). But the Neumann condition is needed for u in order to get that $u^{\epsilon,\rho}$ is a subsolution of the equation (regarding Lemma 41). We see this in the following proof.

Proof. Take any point $(X_0, t_0) \in Q_{\rho_1}^+$ and any second order paraboloid $R_2(X,t) = A + B \cdot (X - X_0) + C(t - t_0) + \frac{1}{2}(X - X_0)^\tau D(X - X_0)$ (with $A, C \in \mathbb{R}, B \in \mathbb{R}^n$ and $D \in \mathbb{R}_{n\times n}$) touching $u^{\epsilon,\rho}$ by above at (X_0, t_0). We want to show that $F(D) - C \geq 0$ (see Lemma 8).

Consider the point $(X_0^*, t_0^*) \in \overline{Q}_\rho^+$ for which

$$u^{\epsilon,\rho}(X_0, t_0) = u(X_0^*, t_0^*) - \frac{1}{\epsilon}|X_0 - X_0^*|^2 - \frac{1}{\epsilon}(t_0 - t_0^*)^2$$

and recall that this point cannot lie on Q_ρ^* due to Lemma 41. Consider also the translation

$$\tilde{R}_2(X,t) = R_2(X + X_0 - X_0^*, t + t_0 - t_0^*) + \frac{1}{\epsilon}|X_0 - X_0^*|^2 + \frac{1}{\epsilon}(t_0 - t_0^*)^2.$$

Our aim is to show that for small ϵ this paraboloid touches u at (X_0^*, t_0^*) and then apply the equation for u. Note that $\tilde{R}_2(X_0^*, t_0^*) = R_2(X_0, t_0) + \frac{1}{\epsilon}|X_0 - X_0^*|^2 + \frac{1}{\epsilon}(t_0 - t_0^*)^2 = u^{\epsilon,\rho}(X_0, t_0) + \frac{1}{\epsilon}|X_0 - X_0^*|^2 + \frac{1}{\epsilon}(t_0 - t_0^*)^2 = u(X_0^*, t_0^*)$. Hence it remains to show that $\tilde{R}_2$ stays above u around (X_0^*, t_0^*).

Let $d = \rho - \rho_1 > 0$ and take $\epsilon_0 = \frac{d^4}{16 \operatorname{osc}_{Q_\rho^+} u} > 0$. Then, for $0 < \epsilon \leq \epsilon_0$ we have (by ***(v)*** of Lemma 40) that $|X_0 - X_0^*|^2 + (t_0 - t_0^*)^2 \leq \left(\frac{d}{2}\right)^4$ which ensures that (X_0^*, t_0^*) is an interior point of Q_ρ^+. Indeed, $|X_0 - X_0^*| \leq \frac{d}{2}$, since $\frac{d}{2} < 1$, and so $|X_0^*| < \frac{d}{2} + \rho_1 \leq \rho$. That is, $|X_0^*| < \rho$ and $y_0^* > 0$. Also $|t_0 - t_0^*| \leq \left(\frac{d}{2}\right)^2$, in particular $-\left(\frac{d}{2}\right)^2 + t_0 \leq t_0^*$ and using that $t_0 > -\rho_1^2$ we get that $t_0^* > -\rho^2$. This shows that

$(X_0^*, t_0^*) \in Q_\rho^+$ (and not on its boundary). Therefore, we may choose some $\delta > 0$ so that $Q_\delta(X_0^*, t_0^*) \subset Q_\rho^+$ and $Q_\delta(X_0, t_0) \subset Q_\rho^+$. Now let $(X, t) \in Q_\delta(X_0^*, t_0^*)$, then $(X + X_0 - X_0^*, t + t_0 - t_0^*) \in Q_\delta(X_0, t_0)$ since $|X + X_0 - X_0^* - X_0| = |X - X_0^*| < \delta$ and $t + t_0 - t_0^* - t_0 = t - t_0^*$ with $-\delta^2 < t - t_0^* \leq 0$ and $t_0 - \delta^2 < t + t_0 - t_0^* \leq t_0$. Moreover, we observe that for such a point (X, t),

$$u^{\epsilon,\rho}(X + X_0 - X_0^*, t + t_0 - t_0^*) \geq u(Z, s) - \frac{1}{\epsilon}|Z - X - X_0 + X_0^*|^2 - \frac{1}{\epsilon}(s - t - t_0 + t_0^*)^2$$

for any $(Z, s) \in \overline{Q}_\rho^+$. Hence taking $(Z, s) = (X, t)$,

$$\begin{aligned} R_2(X + X_0 - X_0^*, t + t_0 - t_0^*) &\geq u^{\epsilon,\rho}(X + X_0 - X_0^*, t + t_0 - t_0^*) \\ &\geq u(X, t) - \frac{1}{\epsilon}|X_0 - X_0^*|^2 - \frac{1}{\epsilon}(t_0 - t_0^*)^2. \end{aligned}$$

That is $u(X, t) \leq \tilde{R}_2(X, t)$, for $(X, t) \in Q_\delta(X_0^*, t_0^*)$ as desired. Hence $F(D) - C = F\left(D_X^2 \tilde{R}_2(X_0^*, t_0^*)\right) - \left(\tilde{R}_2\right)_t (X_0^*, t_0^*) \geq 0$ and the proof is complete. □

Now we are able to show the main objective of this subsection. Using the notion of approximate sub/super-solutions we show that the difference of two functions satisfying a Neumann condition is forced to fulfill the same condition. Note that these functions have to be obtained as sub/super-solutions of the equation in the interior.

Proposition 43. *Assume that $u, v \in C\left(Q_1^+ \cup Q_1^*\right)$ satisfy in the viscosity sense*

$$\begin{cases} F(D^2u) - u_t \geq 0, & \text{in } Q_1^+ \\ u_y \geq 0, & \text{on } Q_1^* \end{cases} \quad \text{and} \quad \begin{cases} F(D^2v) - v_t \leq 0, & \text{in } Q_1^+ \\ v_y \leq 0, & \text{on } Q_1^* \end{cases} \tag{3.3.3}$$

Then

$$\begin{cases} u - v \in \underline{S}_p\left(\frac{\lambda}{n}, \Lambda\right), & \text{in } Q_1^+ \\ (u - v)_y \geq 0, & \text{on } Q_1^* \quad \text{(in the viscosity sense).} \end{cases} \tag{3.3.4}$$

Proof. In Theorem 4.6 of [45], L.Wang uses a similar approximate consideration to obtain that $u - v \in \underline{S}_p\left(\frac{\lambda}{n}, \Lambda\right)$ in Q_1^+. Therefore, it remains to examine the Neumann condition.

We define the corresponding approximate sub/super-solutions $u^{\epsilon,\rho}, v_{\epsilon,\rho}$, for which we have that $(u^{\epsilon,\rho} - v_{\epsilon,\rho})_y \geq 0$ on Q_1^* in the viscosity sense. This can be proved using the same idea as in the proof of ***(vii)***, Lemma 40. We are aiming to pass to the limit using Proposition 20. To do se we take any $(X_0, t_0) \in Q_1^*$ and consider $0 < \rho_0 < \rho < 1$ be so that $(X_0, t_0) \in Q_{\rho_0}^* \subset \overline{Q}_{\rho_0}^+ \subset \overline{Q}_{\rho}^+$. Then from Lemma 42 we have that for sufficiently small $\epsilon > 0$, $u^{\epsilon,\rho}, v_{\epsilon,\rho}$ are sub/super-solutions of $F(D^2 w) - w_t = 0$ in $Q_{\rho_0}^+$. So again from Theorem 4.6 of [45], $u^{\epsilon,\rho} - v_{\epsilon,\rho} \in \underline{S}_p\left(\frac{\lambda}{n}, \Lambda\right)$ in $Q_{\rho_0}^+$.

We now apply Proposition 20 to $u^{\epsilon,\rho} - v_{\epsilon,\rho}$ and combining with ***(vi)*** of Lemma 40 we obtain that $(u - v)_y \geq 0$ on $Q_{\rho_0}^*$ and in particular at (X_0, t_0). The proof is complete. □

The above gives a uniqueness result for the nonlinear Neumann problem. A maximum principle which is proved in a previous section is needed.

Proposition 44. *Let* $g \in C\left(\partial_p(Q_1^+) \setminus Q_1^*\right)$ *and* $u, v \in C\left(\overline{Q}_1^+\right)$ *satisfy in the viscosity sense*

$$\begin{cases} F(D^2 w) - w_t = 0, & \textit{in } \; Q_1^+ \\ w_y = 0, & \textit{on } \; Q_1^* \\ w = g, & \textit{on } \; \partial_p(Q_1^+) \setminus Q_1^*. \end{cases} \tag{3.3.5}$$

Then $u = v$ *in* $\overline{Q}_1^+$.

Proof. Let $w := u - v$ then from Proposition 43 w satisfies

$$\begin{cases} w \in S_p\left(\frac{\lambda}{n}, \Lambda\right), & \text{in } \; Q_1^+ \\ w_y = 0, & \text{on } \; Q_1^* \\ w = 0, & \text{on } \; \partial_p(Q_1^+) \setminus Q_1^*. \end{cases}$$

Hence applying Theorem 28 to w and $-w$ we obtain that

$$\operatorname*{osc}_{Q_1^+} w = 0.$$

From the continuity of w we arrive to $w = 0$ in $\overline{Q}_1^+$. □

3.3.2 $H^{1+\alpha}$-estimates for the homogeneous Neumann case

First we give the statement of a theorem concerning interior estimates for the first derivatives proved in Section 4.2. of [45]. Actually, as explained in [45], we have more than typical spatial $H^{1+\alpha}$-estimates and the extra property is related to the t-direction.

Theorem 45. *(Interior $H^{1+\alpha}$-estimates). Let $u \in C(Q_1)$ be a bounded viscosity solution of $F(D_X^2 u) - u_t = 0$ in Q_1. Then the first derivatives $u_{x_1}, \ldots, u_{x_{n-1}}, u_y$ exist. Moreover there exist universal constants $0 < \alpha < 1, C > 0$ so that $u \in H^{1+\alpha}\left(\overline{Q}_{1/2}\right)$ with an estimate*

$$||u||_{H^{1+\alpha}\left(\overline{Q}_{1/2}\right)} \leq C\left(||u||_{L^\infty(Q_1)} + F(O)\right). \tag{3.3.6}$$

In addition u_t exists and it is H^α in $\overline{Q}_{1/2}$, where the H^α-norm of u_t is bounded by $C\left(||u||_{L^\infty(Q_1)} + F(O)\right)$ as well.

Note that the above ensures the existence of the derivatives at every interior point. So in what follows, it is enough to check the existence of the derivatives at boundary points.

To examine the Neumann problem we need to know the analog theory for the Dirichlet case.

Lemma 46. *Let $u \in C(Q_r^+ \cup Q_r^*)$ be bounded and satisfies in the viscosity sense*

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } Q_r^+ \\ u = 0, & \text{on } Q_r^*. \end{cases} \tag{3.3.7}$$

Then the first derivatives $u_{x_1}, \ldots, u_{x_{n-1}}, u_y$ exist in $\overline{Q}^+_{r/2}$. Moreover there exists universal constant $0 < \alpha < 1$ so that u is punctually $H^{1+\alpha}$ at every point $P_0 \in Q^*_{r/2}$. More precisely for $b_{P_0} = u_y(P_0)$ and any $\tilde{r} \leq \frac{r}{2}$

$$|u(X,t) - b_{P_0} y| \leq C\, \frac{\tilde{r}^{1+\alpha}}{r^{1+\alpha}} \left(||u||_{L^\infty(Q_r^+)} + r^2|F(O)|\right) \tag{3.3.8}$$

for every $(X,t) \in \overline{Q}^+_{\tilde{r}}(P_0)$, where $C > 0$ is a universal constant.

For the proof of Lemma 46 we will need the following.

Lemma 47. *Let f be bounded in Q_1^+ and $u \in C\left(Q_1^+ \cup Q_1^*\right)$ be bounded and satisfies in the viscosity sense*

$$\begin{cases} u \in S_p(\lambda, \Lambda, f), & \text{in } Q_1^+ \\ u = 0, & \text{on } Q_1^*. \end{cases}$$

Then there exist universal constants $0<\alpha<1, C>0$ *so that for any* $0<\rho\leq\frac{1}{2}$

$$\operatorname*{osc}_{Q_\rho^+}\frac{u}{y}\leq C\rho^\alpha\left(\operatorname*{osc}_{Q_{1/2}^+}\frac{u}{y}+||f||_{L^\infty(Q_1^+)}\right). \tag{3.3.9}$$

The idea of the proof of Lemma 47 is based on the proof of Theorem 9.31 in [21] or on its parabolic version appeared in [31] (Lemmata 7.46 and 7.47).

Proof. First we observe that $\frac{u}{y}$ is bounded in $Q_{1/2}^+$ from Proposition 36. Now to obtain estimate (3.3.9) we intend to use the iteration Lemma 27, so we need to prove a relation of the form

$$\operatorname*{osc}_{Q_{\tau\rho}^+}\frac{u}{y}\leq\gamma\operatorname*{osc}_{Q_\rho^+}\frac{u}{y}+C\rho||f||_{L^\infty(Q_1^+)},\quad\text{for every}\quad 0<\rho\leq\frac{1}{2} \tag{3.3.10}$$

where $0<\tau,\gamma<1$ and $C>0$ are universal constants. To do so we use a barrier argument in order to be able to apply Harnack inequality to $\frac{u}{y}$ up to the flat boundary.

First we study the case when $u\geq 0$ in Q_1^+.

Step 1. We get a Harnack-type inequality for $v:=\frac{u}{y}$.

For any $0<\rho\leq\frac{1}{2}$ and $0<\delta\leq 1$ we consider the strip-like set

$$H(\rho,\delta):=\left\{(X,t):|x|<\rho,\frac{\rho\delta}{2}<y<\frac{3\rho\delta}{2},-\rho^2<t\leq 0\right\}.$$

For $A=(0,\dots,0,\rho)$ we consider the cylinder $Q_{\rho/2}(A,0)\subset H(\rho,1)$ and apply Harnack inequality (Theorem 23) inside this interior set. So if $C>0$, $0<R<<1$ and K_r are the ones appeared in Theorem 23 we have,

$$\begin{aligned}\sup_{K_{\frac{\rho R}{2}}(A,0)}v&\leq\frac{2}{\rho}\sup_{K_{\frac{\rho R}{2}}(A,0)}u\leq\frac{2}{\rho}C\left(\inf_{Q_{\frac{\rho R^2}{2}}(A,0)}u+\rho^2\,||f||_{L^\infty(Q_1^+)}\right)\\&\leq\frac{2}{\rho}C\left(\frac{3\rho}{2}\inf_{Q_{\frac{\rho R^2}{2}}(A,0)}v+\rho^2\,||f||_{L^\infty(Q_1^+)}\right).\end{aligned}$$

Hence, defining the following thin set,

$$H'(\rho,\delta):=\left\{(X,t):|x|<\frac{\rho R^2}{4},y=\delta\rho,-\frac{\rho^2R^2}{16}<t\leq 0\right\}$$

which lies inside $Q_{\frac{\rho R^2}{2}}(A,0)$ if we choose $0<\delta<\frac{\sqrt{3}R^2}{4}$. We have

$$\sup_{K_{\frac{\rho R}{2}}(A,0)} v \le C\left(\inf_{H'(\rho,\delta)} v + \rho\,||f||_{L^\infty(Q_1^+)}\right). \tag{3.3.11}$$

<u>Step 2.</u> Through a barrier argument we get an estimate up to the flat boundary,

$$\inf_{H'(\rho,\delta)} v \le C\left(\inf_{\tilde{H}(\frac{\rho}{4},\delta)} v + \rho\,||f||_{L^\infty(Q_1^+)}\right) \tag{3.3.12}$$

where

$$\tilde{H}(\rho,\delta) := \left\{(X,t) : |x| < \frac{\rho R^2}{4}, 0<y<\delta\rho, -\frac{\rho^2 R^2}{16} < t \le 0\right\}.$$

Then (3.3.11) and (3.3.12) will give

$$\sup_{K_{\frac{\rho R}{2}}(A,0)} v \le C\left(\inf_{\tilde{H}(\frac{\rho}{4},\delta)} v + \rho\,||f||_{L^\infty(Q_1^+)}\right). \tag{3.3.13}$$

For convenience we consider the function $\bar{u} := \frac{1}{m}u$, where $m := \inf_{H'(\rho,\delta)} v$ (we assume that $m>0$ excluding the trivial case). Then from ***(i)*** of Proposition 15 we have that $\bar{u} \in S_p(\lambda,\Lambda,\bar{f})$ in Q_1^+, where $\bar{f} := \frac{f}{m}$. Moreover, if we denote by $\bar{v} := \frac{\bar{u}}{y}$ then $\inf_{H'(\rho,\delta)} \bar{v} = 1$ and (3.3.12) for $\bar{v}$ becomes

$$C\left(\inf_{\tilde{H}(\frac{\rho}{4},\delta)} \bar{v} + \rho\,||\bar{f}||_{L^\infty(Q_1^+)}\right) \ge 1. \tag{3.3.14}$$

To show (3.3.14) we define

$$b(X,t) = y\left[1 - \frac{|x|^2}{\tilde{\rho}^2} + \frac{t}{\tilde{\rho}^2} + \left(\frac{1+\rho\,||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right)\left(\frac{y-\delta\rho}{\sqrt{\delta}\rho}\right)\right] \quad \text{for} \quad (X,t) \in \tilde{H}(\rho,\delta)$$

where $\tilde{\rho} := \frac{\rho R^2}{4}$. Our intention is to apply a comparison principle between b and $\bar{u}$. We have to show the following to:

1. $M^-(D^2 b) - b_t \ge \bar{f}$ in $\tilde{H}(\rho,\delta)$ (this will be satisfied in the classical sense due to the smoothness of b). Then from ***(iii)*** of Proposotion 15 we obtain that $\bar{u} - b \in \overline{S}_p(\lambda,\Lambda,0)$ in $\tilde{H}(\rho,\delta)$.

2. $\bar{u} - b \ge 0$ on $\partial_p \tilde{H}(\rho,\delta)$.

First we prove 1: We compute the derivatives of b:

$$b_t = \frac{y}{\tilde{\rho}^2}, \quad b_{x_i} = -\frac{2x_i y}{\tilde{\rho}^2}, \quad \text{for} \quad i = 1, \dots, n-1.$$

$$b_y = 1 - \frac{|x|^2}{\tilde{\rho}^2} + \frac{t}{\tilde{\rho}^2} + \left(\frac{1 + \rho\, ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right)\left(\frac{y - \delta\rho}{\sqrt{\delta}\rho}\right) + \frac{y}{\sqrt{\delta}\rho}\left(\frac{1 + \rho\, ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right).$$

$$b_{x_i x_j} = \begin{cases} 0, & \text{if} \quad i \neq j \\ -\frac{2y}{\tilde{\rho}^2}, & \text{if} \quad i = j \end{cases}, \quad \text{for} \quad i, j = 1, \dots, n-1.$$

$$b_{x_i y} = -\frac{2x_i}{\tilde{\rho}^2}, \quad \text{for} \quad i = 1, \dots, n-1, \quad b_{yy} = \frac{2}{\sqrt{\delta}\rho}\left(\frac{1 + \rho\, ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right).$$

Recall that from Lemma 12

$$\mathcal{M}^-(M, \lambda, \Lambda) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} L_A(M)$$

where $\mathcal{A}_{\lambda,\Lambda}$ be the subset of S_n containing all matrices whose eigenvalues lie in the interval $[\lambda, \Lambda]$ and for $A = (a_{ij}) \in \mathcal{A}_{\lambda,\Lambda}$, L_A is the linear functional $L_A(M) = tr(AM) = \sum_{ij} a_{ij}\mu_{ij}$, where $M = (\mu_{ij}) \in S_n$. So if we show that, for any such linear operator L_A, it holds $L_A(D^2 b) - b_t \geq \bar{f}$ then taking infimum with respect to all suitable A we will have the desired. Take any $A = (a_{ij}) \in \mathcal{A}_{\lambda,\Lambda}$ and observe that from the fact that $\lambda|\xi|^2 \leq \sum_{ij} a_{ij}\xi_{ij} \leq \Lambda|\xi|^2$ for any $\xi \in \mathbb{R}^n$ we obtain $\lambda \leq a_{ii} \leq \Lambda$ and $|a_{in}| \leq \Lambda - \frac{\lambda}{2} =: C_0 > 0$ (taking $\xi_i = 1, \xi_n = 1$ and $\xi_j = 0$ for any other j). So in $\tilde{H}(\rho, \delta)$ we compute

$$\begin{aligned}
L_A(D^2 b) - b_t &= -\frac{2y}{\tilde{\rho}^2}\sum_i^{n-1} a_{ii} - \frac{4}{\tilde{\rho}^2}\sum_i^{n-1} a_{in}x_i + \frac{2a_{nn}}{\sqrt{\delta}\rho}\left(\frac{1 + \rho\, ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right) - \frac{y}{\tilde{\rho}^2} \\
(y < \rho\delta) \;\; &\geq -\frac{2\rho\delta}{\frac{\rho^2 R^4}{16}}(n-1)\Lambda - \frac{4}{\tilde{\rho}^2}\sum_i^{n-1} |a_{in}||x_i| + \frac{2\left(1 + \rho\, ||\bar{f}||_{L^\infty(Q_1^+)}\right)}{\sqrt{\delta}\rho} - \frac{\rho\delta}{\frac{\rho^2 R^4}{16}} \\
(\delta < \sqrt{\delta}, |x| < \tilde{\rho}) \;\; &\geq -\frac{16}{\rho R^4}(1 + 2n\Lambda)\sqrt{\delta} - \frac{4}{\frac{\rho^2 R^4}{16}} C_0 n \frac{\rho R^2}{4} + \frac{2}{\sqrt{\delta}\rho}\left(1 + \rho\, ||\bar{f}||_{L^\infty(Q_1^+)}\right) \\
(\sqrt{\delta}, \rho < 1) \;\; &= -\frac{16}{\rho R^4}(1 + 2n\Lambda)\sqrt{\delta} - \frac{16 C_0 n}{\rho R^2} + \frac{2}{\sqrt{\delta}\rho}\left(1 + \rho\, ||\bar{f}||_{L^\infty(Q_1^+)}\right) =: (G).
\end{aligned}$$

It is enough to choose δ small so that

$$-\frac{16}{R^4}(1+2n\Lambda)\delta - \frac{16C_0 n}{R^2}\sqrt{\delta} + 2 \geq 0. \tag{3.3.15}$$

Indeed if this is the case then $(G) \geq \frac{2}{\sqrt{\delta}}||\bar{f}||_{L^\infty(Q_1^+)} \geq ||\bar{f}||_{L^\infty(Q_1^+)} \geq \bar{f}$ (since $\sqrt{\delta} < 1$). Now, for (3.3.15) we observe that we have a polynomial in $\bar{\delta} := \sqrt{\delta}$. One can realize that this polynomial has two universal roots $\bar{\delta}_1 < 0$, $\bar{\delta}_2 > 0$ and the polynomial is positive in $(\bar{\delta}_1, \bar{\delta}_2)$. So if we choose $0 < \delta < \bar{\delta}_1^2$ we have the desired. This completes the proof of 1.

Now we examine b on $\partial_p \tilde{H}(\rho, \delta)$. We split in the following cases:

- For $y = 0$, $b = 0 = u = \bar{u}$.
- For $y = \delta\rho$, $b(x, \delta\rho, t) = \delta\rho \left(1 - \frac{|x|^2}{\tilde{\rho}^2} + \frac{t}{\tilde{\rho}^2}\right) \leq \delta\rho \leq \bar{u}(x, \delta\rho, t)$, since $\inf_{H'(\rho,\delta)} \frac{\bar{u}}{\delta\rho} = \inf_{H'(\rho,\delta)} \bar{v} = 1$.
- For $t = -\tilde{\rho}^2$, $b(X, -\tilde{\rho}^2) = y \left[-\frac{|x|^2}{\tilde{\rho}^2} + \left(\frac{1+\rho\ ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right)\left(\frac{y-\delta\rho}{\sqrt{\delta}\rho}\right)\right] \leq 0 \leq \bar{u}(X, -\tilde{\rho}^2)$.
- For $|x| = \tilde{\rho}$, $b(X, t) = y \left[\frac{t}{\tilde{\rho}^2} + \left(\frac{1+\rho\ ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right)\left(\frac{y-\delta\rho}{\sqrt{\delta}\rho}\right)\right] \leq 0 \leq \bar{u}(X, t)$.

This yields 2 on $\partial_p \tilde{H}(\rho, \delta)$.

Therefore by maximum principle (Corollary 26) we have that $\bar{u} - b \geq 0$ in $\tilde{H}(\rho, \delta)$ and as a consequence, in $\tilde{H}\left(\frac{\rho}{4}, \delta\right)$ we have an estimate by below for the ratio

$$\begin{aligned}
\frac{\bar{u}(X,t)}{y} &\geq 1 - \frac{|x|^2}{\tilde{\rho}^2} + \frac{t}{\tilde{\rho}^2} + \left(\frac{1+\rho\ ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right)\left(\frac{y-\delta\rho}{\sqrt{\delta}\rho}\right) \\
&\geq 1 - \frac{\left(\frac{\tilde{\rho}}{4}\right)^2}{\tilde{\rho}^2} - \frac{\frac{\tilde{\rho}^2}{16}}{\tilde{\rho}^2} - \left(\frac{1+\rho\ ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}\right)\sqrt{\delta}, \text{ since } |x| < \frac{\tilde{\rho}}{4}, t > -\frac{\tilde{\rho}^2}{16}, y > 0 \\
&\geq \frac{7}{8} - \frac{\sqrt{\delta}}{\lambda} - \frac{\rho\ ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda} \geq \frac{1}{2} - \frac{\rho\ ||\bar{f}||_{L^\infty(Q_1^+)}}{\lambda}
\end{aligned}$$

by choosing $1 \leq \delta \leq \left(\frac{3\lambda}{8}\right)^2$. Hence taking infimum

$$1 \leq C\left(\inf_{\tilde{H}\left(\frac{\rho}{4},\delta\right)} \frac{\bar{u}}{y} + \rho\ ||\bar{f}||_{L^\infty(Q_1^+)}\right).$$

Recalling that $\bar{u} := \frac{1}{m}u$, $\bar{f} := \frac{f}{m}$ for $m := \inf_{H'(\rho,\delta)} v$ we obtain (3.3.12).

Next we treat the general case by removing the assumption on the nonnegativity of u.

Step 3. We denote $M := \sup_{\tilde{H}(2\rho,\delta)} v$ and $m := \inf_{\tilde{H}(2\rho,\delta)} v$. Then the functions $My - u, u - my$ are nonnegative since $v = \frac{u}{y}$ and satisfy the assumptions of lemma. Applying (3.3.13) to these two functions and then adding the two estimates we conclude

$$\operatorname*{osc}_{\tilde{H}\left(\frac{\rho}{4},\delta\right)} v \leq \frac{C-1}{C} \operatorname*{osc}_{\tilde{H}(2\rho,\delta)} v + 2C\rho \, ||f||_{L^\infty(Q_1^+)}$$

which is an analog of (3.3.10) and the proof is completed. □

Remark 48. *Observe that if we consider a function u satisfying the assumptions of Lemma 47 in $Q_r^+ \cup Q_r^*$ (instead of $Q_1^+ \cup Q_1^*$) then using the change of variables $v(Z,s) := u(rZ, r^2 s)$, for $(Z,s) \in Q_1^+ \cup Q_1^*$ (recall the properties of S_p-classes, Proposition 15) we will get that there exist universal constants $0 < \alpha < 1, C > 0$ so that for any $0 < \tilde{r} \leq \frac{r}{2}$*

$$\operatorname*{osc}_{Q_{\tilde{r}}^+} \frac{u}{y} \leq C \left(\frac{\tilde{r}}{r}\right)^\alpha \left(\operatorname*{osc}_{Q_{r/2}^+} \frac{u}{y} + r^2 ||f||_{L^\infty(Q_1^+)} \right). \tag{3.3.16}$$

Moreover, together with a rescaled version of Proposition 36 we also have

$$\operatorname*{osc}_{Q_{\tilde{r}}^+} \frac{u}{y} \leq C \left(\frac{\tilde{r}}{r}\right)^\alpha \left(\frac{1}{r} \operatorname*{osc}_{Q_r^+} u + r^2 ||f||_{L^\infty(Q_1^+)} \right). \tag{3.3.17}$$

Now we are able to prove Lemma 46.

Proof of Lemma 46. First let us examine what Lemma 47 ensures:

- u_y exists on Q_r^*.

  Indeed we show this at $(0,0)$. Let the sequence $\{h_k\}_k$ be so that $h_k \searrow 0$ as $k \to \infty$ and take $m > l$ (large enough) then applying Remark 48 for $(\rho = h_l > h_m)$ we obtain

  $$\frac{u(0,h_m,0)}{h_m} - \frac{u(0,h_l,0)}{h_l} \leq \frac{C}{r^\alpha} K (h_l)^\alpha$$

  where $K := \operatorname{osc}_{\overline{Q_{r/2}^+}} \frac{u}{y} + |F(O)|$. Now let $\epsilon > 0$ and take $\epsilon^* = \left(\frac{\epsilon}{CK}\right)^\alpha$, then there exists some $N \in \mathbb{N}$ so that $h_l < \epsilon^*$ for any $l \geq N$ which implies that $\frac{u(0,h_m,0)}{h_m} - \frac{u(0,h_l,0)}{h_l} < \epsilon$ for any $m > l \geq N$. That is the sequence $\{\frac{u(0,h_k,0)}{h_k}\}$ is a Cauchy sequence and hence it converges to $u_y(0,0)$ (since $u(0,0) = 0$). Similarly using a translated version of Lemma 47 (and Remark 48) we can get the above at any point of Q_r^*.

- $u_y \in H^\alpha\left(Q^*_{r/2}\right)$.

  Indeed, let $h < \frac{\rho}{2}, \rho < \frac{r}{2}$ and $(x_0, t_0), (z_0, s_0) \in Q^*_{\rho/2}$ then again from Remark 48 we have

$$\frac{u(x_0,h,t_0)}{h} - \frac{u(z_0,h,s_0)}{h} \le \frac{C}{r^\alpha} K\rho^\alpha.$$

  Taking $h \to 0$ we obtain $u_y(x_0,0,t_0) - u_y(z_0,0,s_0) \le \frac{C}{r^\alpha} K\rho^\alpha$ which implies

$$\operatorname*{osc}_{Q^*_{\rho/2}} u_y \le \frac{C}{r^\alpha} K\rho^\alpha,$$

  while H^α-estimates can be obtained as in the proof of Theorem 29.

Now let $(X,t) \in Q^+_{\tilde{r}}$ and $h > 0$ small, Remark 48 yields

$$\frac{u(X,t)}{y} - \frac{u(0,h,0)}{h} \le C\left(\frac{\tilde{r}}{r}\right)^\alpha \left(\frac{1}{r}\operatorname*{osc}_{Q^+_r} u + r^2|F(O)|\right).$$

Then letting $h \to 0^+$ we get

$$|u(X,t) - u_y(0,0)y| \le y\; C\left(\frac{\tilde{r}}{r}\right)^\alpha \left(\frac{1}{r}\operatorname*{osc}_{Q^+_r} u + r^2|F(O)|\right)$$

and since $0 < y \le \tilde{r}$ and $r < 1$ we conclude

$$|u(X,t) - u_y(0,0)y| \le\; C\left(\frac{\tilde{r}}{r}\right)^{1+\alpha} \left(||u||_{L^\infty(Q^+_r)} + r^2|F(O)|\right).$$

This proves the estimate around $P_0 = (0,0)$. A suitable translation gives the later for any P_0. □

Theorem 49. *(Boundary $H^{1+\alpha}$-estimates for the Dirichlet problem). Let g be an $H^{1+\alpha}$-function locally on Q^*_1 and $u \in C\left(Q^+_1 \cup Q^*_1\right)$ be bounded and satisfies in the viscosity sense*

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } Q^+_1 \\ u = g, & \text{on } Q^*_1. \end{cases} \tag{3.3.18}$$

Then the first derivatives $u_{x_1}, \dots, u_{x_{n-1}}, u_y$ exist in $\overline{Q}^+_{1/2}$. Moreover there exists universal constant $0 < \alpha_0 < 1$ so that for $\beta = \min\{\alpha, \alpha_0\}$, u is punctually $H^{1+\beta}$ at every point $P_0 \in Q^*_{1/2}$. More precisely, there exists a polynomial $R_{1;P_0}$ of first order *in X, that is $R_{1;P_0}(X) = A_{P_0} + B_{P_0} \cdot (X - X_0)$, where $A_{P_0} = u(P_0) = g(P_0)$ and $B_{P_0} = \big(u_{x_1}(P_0), \dots, u_{x_{n-1}}(P_0), u_y(P_0)\big) = \big(g_{x_1}(P_0), \dots, g_{x_{n-1}}(P_0), u_y(P_0)\big)$ so that*

$$|u(X,t) - R_{1;P_0}(X)| \leq C\left(||u||_{L^\infty(Q_1^+)} + ||g||_{H^{1+\alpha}(\overline{Q}^*_{1/2})} + |F(O)|\right) p(P, P_0)^{1+\beta} \quad (3.3.19)$$

for every $P = (X,t) \in \overline{Q}^+_{1/4}(P_0)$, where $C > 0$ is a universal constant and $p(P, P_0)$ denotes the parabolic distance.

We treat the non-homogeneous case by approximating with suitable homogeneous problems.

Proof. We will show the result around $P_0 = (0,0)$. Note that without the loss of generality we can assume that $u(0,0) = g(0,0) = 0$ and $\nabla_{n-1} g(0,0) = 0$ (since we can consider the transformation $u(X,t) - g(0,0) - \nabla_{n-1} g(0,0) \cdot x$). For convenience let us denote by $K := ||u||_{L^\infty(Q_1^+)} + ||g||_{H^{1+\alpha}(\overline{Q}^*_{1/2})} + |F(O)|$.

We intend to find a number $A \in \mathbb{R}$ so that, for universal $C > 0, 0 < \gamma < 1, \alpha_0 > 0$ and $\beta = \min\{\alpha, \alpha_0\}$, we have

$$\underset{Q^+_{\gamma^k}}{\operatorname{osc}} (u(X,t) - Ay) \leq CK\gamma^{k(1+\beta)}, \quad \text{for any} \quad k \in \mathbb{N}. \quad (3.3.20)$$

Before we continue with the proof lets explain why (3.3.20) suffices. Take any $(X,t) \in Q^+_{1/4}$ then $\rho := \max\{|X|, |t|^{1/2}\} \leq \frac{1}{4}$. Choose $\delta = \frac{\log \rho}{\log \gamma} > 0$ (then $\rho = \gamma^\delta$) and take $k_0 = [\delta] \leq \delta < k_0 + 1$ then $\gamma^{k_0+1} \leq \gamma^\delta = \rho \leq \gamma^{k_0}$. So applying (3.3.20) for k_0 we obtain

$$\begin{aligned} u(X,t) - Ay &= u(X,t) - Ay - (u(0,0) - A \cdot 0) \quad \text{(adding a zero-term)} \\ &\leq \underset{Q^+_{\gamma^{k_0}}}{\operatorname{osc}} (u(X,t) - Ay) \\ &\leq CK\gamma^{k_0(1+\beta)} = \frac{C}{\gamma^{(1+\beta)}} K\gamma^{(k_0+1)(1+\beta)} \leq CK\gamma^{\delta(1+\beta)}. \end{aligned}$$

In a similar way we can get that $u(X,t) - Ay \geq CK\gamma^{\delta(1+\beta)}$. This gives the punctual $H^{1+\alpha}$-estimate at $(0,0)$. Then using translation argument we get the result at every point of $Q^*_{1/2}$.

Now, to prove (3.3.20) we show by induction that there exist universal constants $0 < \gamma << 1, \bar{C} > 0, \alpha_0 > 0$ such that for $\beta := \min\{\alpha, \alpha_0\}$ we can find a number $A_k \in \mathbb{R}$ for any $k \in \mathbb{N}$ so that

$$\underset{Q^+_{\gamma^k}}{\operatorname{osc}} (u(X,t) - A_k y) \leq \bar{C} K\gamma^{k(1+\beta)} \quad (3.3.21)$$

and

$$|A_{k+1} - A_k| \leq CK\gamma^{k\beta}. \tag{3.3.22}$$

First, for $k = 0$, take $A_0 = 0$ and choose any $\bar{C} \geq 2$. Next for the induction we assume that we have found numbers $A_0, \ldots, A_N$ for which (3.3.21) and (3.3.22) hold. Denoting by $r := \gamma^N$ and $B := A_N$ we have

$$\operatorname*{osc}_{Q_r^+} (u(X,t) - By) \leq \bar{C}Kr^{(1+\beta)} \tag{3.3.23}$$

and we want to find a number A_{N+1} so that

$$\operatorname*{osc}_{Q_{\gamma r}^+} (u(X,t) - A_{N+1}y) \leq \bar{C}K\gamma^{(N+1)(1+\beta)}. \tag{3.3.24}$$

Now we consider a suitable problem with homogeneous Dirichlet data on the flat boundary in order to use Theorem 46. Let v be the viscosity solution of

$$\begin{cases} F(D^2v) - v_t = 0, & \text{in } \; Q_r^+ \\ v = 0, & \text{on } \; Q_r^* \\ v = u - By, & \text{on } \; \partial_p Q_r^+ \setminus Q_r^*. \end{cases}$$

Then v satisfies the following:

- Maximum principle (see Theorem 24 or Corollary 25) gives

$$\operatorname*{osc}_{Q_r^+} v \leq \operatorname*{osc}_{Q_r^+} (u(X,t) - By) + Cr^2|F(O)|. \tag{3.3.25}$$

- From Theorem 46 we have that $A := v_y(0,0)$ exists (note that $v_{x_1}(0,0), \ldots,$ $v_{x_{n-1}}(0,0)$ exist as well and equal to 0) and

$$\operatorname*{osc}_{Q_{\tilde{r}}^+} (v(X,t) - Ay) \leq C_0 \left(\frac{\tilde{r}}{r}\right)^{1+\alpha_1} \left(\operatorname*{osc}_{Q_r^+} v + r^2|F(O)|\right) \tag{3.3.26}$$

  for any $\tilde{r} \leq \frac{r}{2}$ and also (see Proposition 36)

$$|A| \leq C\left(\frac{1}{r}\operatorname*{osc}_{Q_r^+} v + r^2|F(O)|\right). \tag{3.3.27}$$

Next, we take $\tilde{r} = \gamma r$ (note that γ is very small) in (3.3.26). Hence

$$\operatorname*{osc}_{Q^+_{\gamma r}} (v(X,t) - Ay) \leq C_0 \gamma^{1+\alpha_1} \operatorname*{osc}_{Q^+_r} v + C_0 r^2 \gamma |F(O)| \tag{3.3.28}$$

since $\gamma^{1+\alpha_1} \leq \gamma$. Now take (universal) $\gamma << 1$ sufficiently small in order to have that $C_0\gamma^{\alpha_1} < 1$. We denote by $1 - \theta := C_0\gamma^{\alpha_1}$, where $0 < \theta < 1$ is a universal constant. Then combining (3.3.28) and (3.3.25) we obtain

$$\operatorname*{osc}_{Q^+_{\gamma r}} (v(X,t) - Ay) \leq (1-\theta)\gamma \operatorname*{osc}_{Q^+_r} (u(X,t) - By) + Cr^2 |F(O)|. \tag{3.3.29}$$

Now to return to u we define $w = u - By - v$. Then

$$\begin{cases} w \in S_p\left(\frac{\lambda}{n}, \Lambda\right), & \text{in } Q^+_r \\ w = g, & \text{on } Q^*_r \\ w = 0, & \text{on } \partial_p Q^+_r \setminus Q^*_r. \end{cases}$$

where the first comes from Theorem 4.6 of [45] and ***(iii)*** of Proposition 15.

Subsequently, applying again maximum principle we obtain

$$\operatorname*{osc}_{Q^+_r} w \leq C||g||_{L^\infty(Q^*_r)}.$$

The regularity we have assumed for g gives the right decay for the oscillation of w. That is, (since $g(0,0) = 0, \nabla_{n-1} g(0,0) = 0$) for $(x,t) \in Q^*_r$

$$\begin{aligned} |g(x,t)| &= |g(x,t) - g(0,0) - \nabla_{n-1} g(0,0) \cdot x| \\ &\leq C||g||_{H^{1+\alpha}(\overline{Q}^*_{1/2})} \left(\max\{|x|, |t|^{1/2}\}\right)^{1+\alpha} \leq C||g||_{H^{1+\alpha}(\overline{Q}^*_{1/2})} r^{1+\alpha}. \end{aligned}$$

Hence we obtain

$$\operatorname*{osc}_{Q^+_r} w \leq Cr^{1+\alpha} ||g||_{H^{1+\alpha}(\overline{Q}^*_{1/2})}. \tag{3.3.30}$$

Adding (3.3.29) and (3.3.30) yields

$$\operatorname*{osc}_{Q^+_{\gamma r}} [u(X,t) - (A+B)y] \leq (1-\theta)\gamma \operatorname*{osc}_{Q^+_r} (u(X,t) - By) + Cr^2|F(O)| + C||g||_{H^{1+\alpha}(\overline{Q}^*_{1/2})} r^{1+\alpha}$$

and using the hypothesis (3.3.23) we get

$$\operatorname*{osc}_{Q^+_{\gamma r}}[u(X,t)-(A+B)y] \leq \bar{C}K(1-\theta)\gamma r^{1+\beta} + CKr^2 + CKr^{1+\alpha}.$$

Recalling that $r=\gamma^N$, the above gives

$$\operatorname*{osc}_{Q^+_{\gamma^{N+1}}}[u(X,t)-(A+B)y] \leq K\left[(1-\theta)\bar{C}\gamma\gamma^{N(1+\beta)} + C\left(\gamma^{2N}+\gamma^{N(1+\alpha)}\right)\right]. \quad (3.3.31)$$

We have to choose the right constants α_0 and $\bar{C}$ in order to obtain (3.3.24). Take α_0 so that $\gamma^{\alpha_0} = 1-\frac{\theta}{2}$ and $\bar{C}$ large enough so that $\frac{\gamma\theta\bar{C}}{4} \geq C$ (note that our choices are independent of N). Then we return to (3.3.31) writing $1-\theta$ as $1-\frac{\theta}{2}-\frac{\theta}{2}$ and recalling that $\beta=\min\{\alpha,\alpha_0\}$,

$$\operatorname*{osc}_{Q^+_{\gamma^{N+1}}}[u(X,t)-(A+B)y] \leq K\left[\left(1-\frac{\theta}{2}\right)\bar{C}\gamma\gamma^{N(1+\beta)} + C\left(\gamma^{2N}+\gamma^{N(1+\alpha)}\right) - \frac{\theta}{2}\bar{C}\gamma\gamma^{N(1+\beta)}\right].$$

Note that

$$C\gamma^{2N} \leq \frac{\gamma\theta\bar{C}}{4}\gamma^{2N} \leq \frac{\gamma\theta\bar{C}}{4}\gamma^{N(1+\beta)}, \quad \text{since} \quad \gamma<1 \ \text{ and } \ \beta+1\leq 2$$

and

$$C\gamma^{N(1+\alpha)} \leq \frac{\gamma\theta\bar{C}}{4}\gamma^{N(1+\alpha)} \leq \frac{\gamma\theta\bar{C}}{4}\gamma^{N(1+\beta)}, \quad \text{since} \quad \gamma<1 \ \text{ and } \ \beta\leq\alpha$$

hence combining all the above we have

$$\operatorname*{osc}_{Q^+_{\gamma^{N+1}}}[u(X,t)-(A+B)y] \leq K\bar{C}\gamma^{\alpha_0+1}\gamma^{N(1+\beta)} \leq K\bar{C}\gamma^{(N+1)(1+\beta)}$$

since $\gamma<1$ and $\beta\leq\alpha_0$. This is (3.3.24) for $A_{N+1}=A+B=A+A_N$. It remains to get (3.3.22) for $k=N$. To do so, we use (3.3.27) together with (3.3.25) and then (3.3.23),

$$|A_{N+1}-A_N| = |A| \leq \frac{C}{r}\bar{C}Kr^{1+\beta} + \bar{C}r^2|F(O)| \leq CK(r^\beta+r) \leq CKr^\beta$$

since $r<1$ and $\beta<1$, that is $|A_{N+1}-A_N| \leq CK\gamma^{N\beta}$ as desired. The inductive proof is completed.

Finally, it remains to get estimate (3.3.20). Observe that

$$\lim_{k\to\infty} |A_{k+1} - A_k| \leq \lim_{k\to\infty} CK\gamma^{k\beta} = 0, \quad \text{since} \quad \gamma < 1.$$

That is, the limit $A_\infty := \lim_{k\to\infty} A_k$ exists and it is the number A of (3.3.20). Indeed, for any $k \in \mathbb{N}$ we have

$$\operatorname*{osc}_{Q^+_{\gamma^k}} (u(X,t) - A_\infty y) \leq \operatorname*{osc}_{Q^+_{\gamma^k}} (u(X,t) - A_k y) + \gamma^k (A_k - A_\infty)$$

where we wrote $u(X,t) - A_\infty y = u(X,t) - A_k y + A_k y - A_\infty y \leq u(X,t) - A_k y + |y|\ |A_k - A_\infty|$ and $|y| \leq \gamma^k$. We have

$$\begin{aligned} \operatorname*{osc}_{Q^+_{\gamma^k}} (u(X,t) - A_\infty y) &\leq \bar{C}K\gamma^{k(1+\beta)} + CK\gamma^k \sum_{j=k}^{\infty} \gamma^{j\beta} \leq \bar{C}K\gamma^{k(1+\beta)} + CK\gamma^k \frac{\gamma^{k\beta}}{1-\gamma^\beta} \\ &= \bar{C}K\gamma^{k(1+\beta)} + \frac{C}{1-\gamma^\beta} K\gamma^{(1+\beta)k} \leq CK\gamma^{k(1+\beta)} \end{aligned}$$

and the proof is complete. □

In order to get (punctual) $H^{1+\alpha}$-regularity for the Neumann problem it is enough (due to Theorem 3.3.19) to show that the restriction of u on Q_1^* is locally $H^{1+\alpha}$. To do so, we need the following lemma.

Lemma 50. *Let $0 < \alpha < 1$, $0 < \beta \leq 1$, $0 < A < B$ and $K > 0$ be constants. Let $u \in L^\infty([A,B])$ with $\|u\|_{L^\infty([A,B])} \leq K$. Let $d = B - A$. Define, for $h \in \mathbb{R}$ with $0 < |h| \leq \frac{d}{2}$,*

$$v_{\beta,h}(l) = \frac{u(l+h) - u(l)}{|h|^\beta}, \quad l \in I_h,$$

where $I_h = [A, B-h]$ if $h > 0$ and $I_h = [A-h, B]$ if $h < 0$. Assume that $v_{\beta,h} \in C^\alpha(I_h)$ and $\|v_{\beta,h}\|_{C^\alpha(I_h)} \leq K$, for any $0 < |h| \leq \frac{d}{2}$. Then we have

1. *if $\alpha + \beta < 1$ then $u \in C^{\alpha+\beta}([A,B])$ and $\|u\|_{C^{\alpha+\beta}([A,B])} \leq CK$*
2. *if $\alpha + \beta > 1$ then $u \in C^{0,1}([A,B])$ and $\|u\|_{C^{0,1}([A,B])} \leq CKd^{\alpha+\beta-1}$*

where the constant C depends only on α and β.

This lemma is proved in [10] (Lemma 5.6) in the interval $[-1,1]$ (that is, with $A = -1, B = 1$). A rescaling argument gives Lemma 50.

Proof. Denote by $D = \frac{A+B}{2}$ and $d' = \frac{d}{2}$ and observe that $l \in [A, B]$ if and only if $k := \frac{l-D}{d'} \in [-1, 1]$. Then define the function

$$\tilde{u}(k) := u(d'k + D), \quad \text{for} \quad k \in [-1, 1].$$

Note first that $\|\tilde{u}\|_{L^\infty([-1,1])} = \|u\|_{L^\infty([A,B])} \leq K$. Moreover, for $0 < |\tilde{h}| \leq 1$ we consider

$$\tilde{v}_{\beta,\tilde{h}}(k) = \frac{\tilde{u}(k+h) - \tilde{u}(k)}{|\tilde{h}|^\beta}, \quad k \in \tilde{I}_{\tilde{h}},$$

where $\tilde{I}_{\tilde{h}} = [-1, 1 - \tilde{h}]$ if $\tilde{h} > 0$ and $\tilde{I}_{\tilde{h}} = [-1 - \tilde{h}, 1]$ if $\tilde{h} < 0$. Hence $0 < d'|\tilde{h}| \leq d'$ and if $k \in \tilde{I}_{\tilde{h}}$ then $d'k + D \in I_{d\tilde{h}}$. Also $\tilde{u}(k+h) = u(d'(k+\tilde{h}) + D) = u(d'k + D + d'\tilde{h})$ then $\tilde{v}_{\beta,\tilde{h}}(k) = (d')^\beta v_{\beta,d\tilde{h}}(d'k + D)$. Taking $k_1, k_2 \in \tilde{I}_{\tilde{h}}$, we have

$$\begin{aligned} |\tilde{v}_{\beta,\tilde{h}}(k_1) - \tilde{v}_{\beta,\tilde{h}}(k_2)| &= (d')^\beta |v_{\beta,d\tilde{h}}(d'k_1 + D) - v_{\beta,d\tilde{h}}(d'k_2 + D)| \\ &\leq (d')^\beta K |d'k_1 - d'k_2|^\alpha = K(d')^{\alpha+\beta} |k_1 - k_2|^\alpha. \end{aligned}$$

That is, $\left\|\tilde{v}_{\beta,\tilde{h}}\right\|_{C^\alpha(\tilde{I}_{\tilde{h}})} \leq K(d')^{\alpha+\beta}$, for any $0 < |\tilde{h}| \leq 1$. Therefore, if $\alpha + \beta < 1$ Lemma 5.6 of [10] gives that for $k_1, k_2 \in [-1, 1]$,

$$|\tilde{u}(k_1) - \tilde{u}(k_2)| \leq C(\alpha, \beta)(d')^{\alpha+\beta} K |k_1 - k_2|^{\alpha+\beta}.$$

So taking any $l_1, l_2 \in [A, B]$ and $k_1 = \frac{l_1 - D}{d'}, k_2 = \frac{l_2 - D}{d'} \in [-1, 1]$, we have that

$$|u(l_1) - u(l_2)| \leq C(\alpha, \beta)(d')^{\alpha+\beta} K \left(\frac{|l_1 - l_2|}{d'}\right)^{\alpha+\beta} \leq C(\alpha, \beta) K |l_1 - l_2|^{\alpha+\beta}.$$

We can argue similarly if $\alpha + \beta > 1$ and the proof is complete. □

Remark 51. *Observe that it can be deduced easily from the proof of Lemma 5.6 in [10] and a rescaling argument as the one above that if $v_{\beta,h}$ is C^α only for negative values of h then we will have the estimates of 1. and 2. in $\left[\frac{A+B}{2}, B\right]$ and not in the whole $[A, B]$. This is useful when we study the t-direction.*

Now we are able to show the main objective of this section, i.e. the up to the flat boundary $H^{1+\alpha}$-regularity for the Neumann problem. Having in mind Theorem 49 we see that it is enough to examine x-directions. To do so we will consider for every direction a suitable difference quotient $v_{\beta,h}$ (similar to the one of Lemma 50) which, due to Theorem 43 will satisfy a Neumann problem and hence H^α-estimates (from the results of the previous chapter). Moreover these estimates will depend on the H^β-estimate of the solution u. Then Lemma 50 will imply larger exponent for the Hölder regularity of u and hence we will be able to consider higher β for the difference quotient. Iterating this procedure we arrive to an H^α-estimate for the difference quotient with $\beta = 1$ which gives the desired. Note also that a similar argument can be applied for the t-direction.

Theorem 52. *(Boundary $H^{1+\alpha}$-estimates for the Neumann problem).*
Let $u \in C\left(Q_1^+ \cup Q_1^*\right)$ be bounded and satisfies in the viscosity sense

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } Q_1^+ \\ u_y = 0, & \text{on } Q_1^*. \end{cases} \tag{3.3.32}$$

Then the first derivatives $u_{x_1}, \ldots, u_{x_{n-1}}, u_y$ exist in $\overline{Q}_{1/2}^+$. Moreover there exists a universal constant $0 < \alpha < 1$ so that u is punctually $H^{1+\alpha}$ at every point $P_0 \in Q_{1/2}^*$. More precisely, there exists a polynomial $R_{1;P_0}$ of first order in X, that is $R_{1;P_0}(X) = A_{P_0} + B_{P_0} \cdot (X - X_0)$, where $A_{P_0} = u(P_0)$ and $B_{P_0} = \big(u_{x_1}(P_0), \ldots, u_{x_{n-1}}(P_0), 0\big)$ so that

$$|u(X,t) - R_{1;P_0}(X)| \leq C\left(||u||_{L^\infty(Q_1^+)} + |F(O)|\right)\, p(P,P_0)^{1+\alpha} \tag{3.3.33}$$

for every $P = (X,t) \in \overline{Q}_{1/4}^+(P_0)$, where $C > 0$ is a universal constant.

In addition, u_t exists and it is H^α in $\overline{Q}_{1/4}^+$ with the corresponding estimate being bounded by above by a term of the form $C\left(||u||_{L^\infty(Q_1^+)} + |F(O)|\right)$.

Proof. For convenience we denote by $K := ||u||_{L^\infty(Q_1^+)} + |F(O)|$.

Lets examine first the x_i-direction, for $i = 1, \ldots, n-1$. For $e_i = (0, \ldots, x_i = 1, \ldots, 0) \in \mathbb{R}^n$, $0 < \beta \leq 1$, $0 < |h| < \frac{1}{8}$ we define

$$v_{\beta,h,i}(X,t) = \frac{u(X + he_i, t) - u(X,t)}{|h|^\beta}, \quad \text{for} \quad (X,t) \in Q_{7/8}^+$$

(note that if $(X,t) \in Q_{7/8}^+$ then $(X + he_i, t) \in Q_1^+$). Define also the following H^α-norm which deals only with x_i-direction

$$||u||_{H_i^\alpha(\Omega)} := ||u||_{L^\infty(\Omega)} + \sup_{\substack{(X,t),(Z,t)\in\Omega \\ x_j = z_j, x_i \neq z_i}} \frac{|u(X,t) - u(Z,t)|}{|x_i - z_i|^\alpha}.$$

Now observe that once we show that the function $w(X,t) = u(X + he_i, t)$ satisfies in the viscosity sense the following

$$\begin{cases} F(D^2 w) - w_t = 0, & \text{in} \quad Q^+_{7/8} \\ w_y = 0, & \text{on} \quad Q^*_{7/8}. \end{cases} \tag{3.3.34}$$

then by Theorem 43 and Proposition 15 we end up with

$$\begin{cases} v_{\beta,h,i} \in S_p\left(\frac{\lambda}{n}, \Lambda\right), & \text{in} \quad Q^+_{7/8} \\ (v_{\beta,h,i})_y = 0, & \text{on} \quad Q^*_{7/8}. \end{cases}$$

For (3.3.34), let ϕ be a test function that touches w by above at some point $(X_0, t_0) \in Q^+_{7/8} \cup Q^*_{7/8}$ and let $\psi(Z,t) = \phi(Z - he_i, t)$. Then ψ touches u by above at $(X_0 + he_i, t_0)$. Note that if $(X_0, t_0) \in Q^+_{7/8}$ then $(X_0 + he_i, t_0) \in Q^+_1$ and $F(D^2\psi(X_0 + he_i, t_0)) - \psi_t(X_0 + he_i, t_0) \geq 0$ while if $(X_0, t_0) \in Q^*_{7/8}$ then $(X_0 + he_i, t_0) \in Q^*_1$ and $\psi_y(X_0 + he_i, t_0) \geq 0$. We finish by observing that the derivatives of ψ at $(X_0 + he_i, t_0)$ are equal to the derivatives of ϕ at (X_0, t_0). We argue similarly for test functions that touch u by below.

Now, take $0 < r < \rho \leq \frac{7}{8}$. Applying Corollary 35 to $v_{\beta,h,i}$ we have

$$||v_{\beta,h,i}||_{H^{\alpha_1}(\overline{Q}^+_r)} \leq C\ C(r,\rho)\ ||v_{\beta,h,i}||_{L^\infty\left(\overline{Q}^+_{\frac{r+\rho}{2}}\right)} \tag{3.3.35}$$

(note that $r < \frac{r+\rho}{2} < \rho \leq \frac{7}{8}$). Next, observe that if $(X,t) \in \overline{Q}^+_{\frac{r+\rho}{2}}$, once we choose

$$0 < |h| < \frac{\rho - r}{2} \tag{3.3.36}$$

we get $(X + he_i, t) \in \overline{Q}^+_\rho$. Also $(X,t) \in \overline{Q}^+_\rho$ therefore

$$|v_{\beta,h,i}(X,t)| = \frac{|u(X + he_i, t) - u(X,t)|}{|h|^\beta} \leq ||u||_{H^\beta_i(\overline{Q}^+_\rho)}.$$

Returning to (3.3.35) we have that

$$||v_{\beta,h,i}||_{H^{\alpha_1}(\overline{Q}^+_r)} \leq C\ C(r,\rho)\ ||u||_{H^\beta_i(\overline{Q}^+_\rho)} \tag{3.3.37}$$

for any $0 < r < \rho \leq \frac{7}{8}$ and h as in (3.3.36).

Moreover observe that Corollary 35 ensures that for some exponent β the norm

$||u||_{H_i^\beta(\overline{Q}_\rho^+)}$ is bounded. More precisely, there exists some universal $0 < \alpha_2 < 1$ so that for any $0 < \rho < 1$,

$$||u||_{H_i^{\alpha_2}(\overline{Q}_\rho^+)} \leq ||u||_{H^{\alpha_2}(\overline{Q}_\rho^+)} \leq C\ C(\rho)K. \tag{3.3.38}$$

Now take $\alpha_0 = \min\{\alpha_1, \alpha_2\}$. We can choose some suitable $0 < \alpha < \alpha_0$ in order to succeed finding a universal integer $m_0 \geq 1$ so that

$$m_0\alpha < 1 \quad \text{and} \quad (m_0+1)\alpha > 1.$$

Indeed, let $m_0 := \left[\frac{1}{\alpha_0}\right] \leq \frac{1}{\alpha_0} < m_0 + 1$, then $m_0\alpha_0 \leq 1$ and $(m_0+1)\alpha_0 > 1$. But since $\frac{1}{\alpha_0} < m_0 + 1$ there exists an $\alpha > 0$ so that $\frac{1}{\alpha_0} < \frac{1}{\alpha} < m_0 + 1$. Then $0 < \alpha < \alpha_0 < 1$ and $m_0\alpha < 1$ and $(m_0+1)\alpha > 1$. Note that for the exponent α chosen above the relations (3.3.37) and (3.3.38) are both true. That is we have

$$||v_{\beta,h,i}||_{H^\alpha(\overline{Q}_r^+)} \leq C\ C(r,\rho)\ ||u||_{H_i^\beta(\overline{Q}_\rho^+)} \tag{3.3.39}$$

for any $0 < r < \rho \leq \frac{7}{8}$ and h as in (3.3.36), as well as

$$||u||_{H_i^\alpha(\overline{Q}_\rho^+)} \leq C\ C(\rho)K. \tag{3.3.40}$$

Recall that our aim is to show that the derivative u_{x_i} exists up to the flat boundary and satisfies some H^α-estimate depending only on CK. This can be obtained if (3.3.39) holds true for $\beta = 1$ with right-hand side term CK in some half-cylinder. To do so we will apply, using Lemma 50, an iteration procedure (which can be started from $\beta = \alpha$, regarding (3.3.40)). The details follow.

We consider the following finite sequence of (universal) radii

$$r_k = \frac{7}{8} - \frac{k}{16m_0}, \quad \text{for} \quad k = 0, 1, \ldots, 2m_0.$$

Note that $r_0 = \frac{7}{8}, r_{2m_0} = \frac{3}{4}$ and $r_{k-1} - r_k = \frac{1}{16m_0}$.

Step 1. (of the iteration): Applying (3.3.39) with $\beta = \alpha, r = r_1, \rho = \frac{7}{8}$ together with (3.3.40) for $\rho = \frac{7}{8}$, we obtain that

$$||v_{\alpha,h,i}||_{H^\alpha(\overline{Q}_{r_1}^+)} \leq CK, \quad \text{for any} \quad 0 < |h| < \frac{1}{16m_0}.$$

Then using the above and Lemma 50 we shall get

$$||u||_{H_i^{2\alpha}(\overline{Q}^+_{r_2})} \leq CK. \tag{3.3.41}$$

That is, we want, for any two $(X,t), (X+Le_i,t) \in \overline{Q}^+_{r_2}$, to have that $|u(X+Le_i,t) - u(X,t)| \leq CK|L|^{2\alpha}$. We split into two cases:

- If $|L| \geq \frac{1}{16m_0}$, then $|u(X+Le_i,t)-u(X,t)| \leq 2K \leq 2K(16m_0)^{2\alpha}|L|^{2\alpha} \leq CK|L|^{2\alpha}$ and we are done.

- If $|L| < \frac{1}{16m_0}$, we consider the interval $I = \left[-\frac{1}{16m_0}, \frac{1}{16m_0}\right]$ (note that $0, L \in I$). Also, we define

  $$\tilde{u}^{(X,t),i}(l) = u(X+le_i,t), \quad \text{for} \quad l \in I$$

  and

  $$\tilde{v}^{(X,t),i}_{\alpha,h}(l) = \frac{\tilde{u}^{(X,t),i}(l+h) - \tilde{u}^{(X,t),i}(l)}{|h|^\alpha}, \quad \text{for} \quad 0 < |h| < \frac{1}{16m_0}, l \in I_h$$

  where I_h is defined as in Lemma 50. Observe that

  $$\tilde{v}^{(X,t),i}_{\alpha,h}(l) = \frac{u(X+le_i+he_i,t) - u(X+le_i,t)}{|h|^\alpha} = v_{\alpha,h,i}(X+le_i,t).$$

  Now, if $(X,t) \in \overline{Q}^+_{r_2}$ and $l \in I$ then $|X+le_i| \leq r_2 + \frac{1}{16m_0} = r_1$, i.e. $(X+le_i,t) \in \overline{Q}^+_{r_1}$. Hence

  $$||\tilde{v}^{(X,t),i}_{\alpha,h}||_{C^\alpha(I_h)} \leq ||v_{\alpha,h,i}||_{H^\alpha(\overline{Q}^+_{r_1})} \leq CK.$$

  Therefore, Lemma 50 implies

  $$||\tilde{u}^{(X,t),i}||_{C^\alpha(I)} \leq CK$$

  (note that the length of I is a universal number). Then, since $0, L \in I$, we have the desired.

<u>Step 2.</u> (of the iteration): Applying (3.3.39) with $\beta = 2\alpha, r = r_3, \rho = r_2$ together with (3.3.41) we obtain that

$$||v_{2\alpha,h,i}||_{H^\alpha(\overline{Q}^+_{r_3})} \leq CK, \quad \text{for any} \quad 0 < |h| < \frac{1}{16m_0}.$$

Then in the same way as in Step 1 (using Lemma 50) we can derive that

$$||u||_{H_i^{3\alpha}(\overline{Q}^+_{r_4})} \leq CK.$$

We continue this way until:

Step m_0. (of the iteration): Applying (3.3.39) with $\beta = m_0\alpha, r = r_{2m_0-1}, \rho = r_{2m_0-2}$ together with Step $m_0 - 1$ we obtain that

$$||v_{m_0\alpha,h,i}||_{H^\alpha\left(\overline{Q}^+_{r_{2m_0-1}}\right)} \leq CK, \quad \text{for any} \quad 0 < |h| < \frac{1}{16m_0}.$$

Then again as in Step 1 (using Lemma 50) and recalling how the constants α and m_0 have been chosen ($(m_0+1)\alpha > 1$) we can derive that

$$||u||_{H_i^1(\overline{Q}^+_{3/4})} \leq CK. \tag{3.3.42}$$

This last estimate ensures the (almost everywhere) existence of u_{x_i} on $Q^*_{\frac{3}{4}}$ for any $i = 1, \ldots, n-1$. Moreover, applying again (3.3.39) with $\beta = 1, r = \frac{5}{8}, \rho = \frac{3}{4}$ together with (3.3.42) we conclude that

$$||v_{1,h,i}||_{H^\alpha\left(\overline{Q}^+_{5/8}\right)} \leq CK, \quad \text{for any} \quad 0 < |h| < \frac{1}{16m_0}.$$

This gives a suitable H^α-estimate for u_{x_i} on $Q^*_{5/8}$.

Now, observing that u satisfies in the viscosity sense a problem of the form (3.3.18) with $g(x,t) = u(x,0,t)$ and since g is $H^{1+\alpha}$-function on $Q^*_{5/8}$ we can apply Theorem 49 to get the desired result for X-directions.

It remains to examine the t-direction. The proof follows the same lines as above but since some (minor) modifications are needed we present the proof for completeness.

So for $0 < \beta \leq 2$, $-\frac{1}{8} < h < 0$ we define

$$v_{\beta,h}(X,t) = \frac{u(X,t+h) - u(X,t)}{|h|^{\frac{\beta}{2}}}, \quad \text{for} \quad (X,t) \in Q^+_{7/8}$$

(note that if $(X,t) \in Q^+_{7/8}$ then $(X,t+h) \in Q^+_1$). Define also the following H^α-norm which deals only with t-direction

$$||u||_{H_t^\alpha(\Omega)} := ||u||_{L^\infty(\Omega)} + \sup_{(X,t),(X,s)\in\Omega, t\neq s} \frac{|u(X,t) - u(X,s)|}{|t-s|^{\frac{\alpha}{2}}}.$$

Note that we can easily obtain that

$$\begin{cases} v_{\beta,h} \in S_p\left(\frac{\lambda}{n}, \Lambda\right), & \text{in} \quad Q^+_{7/8} \\ (v_{\beta,h})_y = 0, & \text{on} \quad Q^*_{7/8}. \end{cases}$$

Then, taking $0 < r < \rho \leq \frac{7}{8}$ and applying Corollary 35 to $v_{\beta,h}$ we have

$$||v_{\beta,h}||_{H^{\alpha_1}\left(\overline{Q}^+_r\right)} \leq C\ C(r,\rho)\ ||v_{\beta,h,i}||_{L^\infty\left(\overline{Q}^+_{\frac{r+\rho}{2}}\right)} \tag{3.3.43}$$

(note that $r < \frac{r+\rho}{2} < \rho \leq \frac{7}{8}$). Next, observe that if $(X,t) \in \overline{Q}^+_{\frac{r+\rho}{2}}$ and by choosing

$$-\left(\frac{\rho - r}{2}\right)^2 < h < 0 \tag{3.3.44}$$

we get that $(X, t+h) \in \overline{Q}^+_\rho$. Also $(X,t) \in \overline{Q}^+_\rho$ and hence

$$|v_{\beta,h}(X,t)| = \frac{|u(X,t+h) - u(X,t)|}{|h|^{\frac{\beta}{2}}} \leq ||u||_{H^\beta_t\left(\overline{Q}^+_\rho\right)}.$$

Returning back to (3.3.43) we have that

$$||v_{\beta,h}||_{H^{\alpha_1}\left(\overline{Q}^+_r\right)} \leq C\ C(r,\rho)\ ||u||_{H^\beta_t\left(\overline{Q}^+_\rho\right)} \tag{3.3.45}$$

for any $0 < r < \rho \leq \frac{7}{8}$ and h as in (3.3.44).

Moreover we observe that Corollary 35 ensures that there exists a universal $0 < \alpha_2 < 1$ so that for any $0 < \rho < 1$,

$$||u||_{H^{\alpha_2}_t\left(\overline{Q}^+_\rho\right)} \leq ||u||_{H^{\alpha_2}\left(\overline{Q}^+_\rho\right)} \leq C\ C(\rho) K. \tag{3.3.46}$$

Now take $\alpha_0 = \min\{\alpha_1, \alpha_2\}$. We choose a suitable $0 < \alpha < \alpha_0$ in order to obtain a universal integer $m_0 \geq 1$ so that

$$m_0 \frac{\alpha}{2} < 1 \quad \text{and} \quad (m_0 + 1)\frac{\alpha}{2} > 1.$$

Indeed let $m_0 := \left[\frac{2}{\alpha_0}\right] \leq \frac{2}{\alpha_0} < m_0 + 1$, then $m_0 \frac{\alpha_0}{2} \leq 1$ and $(m_0+1)\frac{\alpha_0}{2} > 1$. But since $\frac{2}{\alpha_0} < m_0 + 1$ there exists an $\alpha > 0$ so that $\frac{2}{\alpha_0} < \frac{2}{\alpha} < m_0 + 1$. Then $0 < \alpha < \alpha_0 < 1$ and $m_0 \frac{\alpha}{2} < 1$ and $(m_0+1)\frac{\alpha}{2} > 1$. Note that for the exponent α chosen above the relations

(3.3.45) and (3.3.46) are both true. That is we have

$$||v_{\beta,h}||_{H^\alpha(\overline{Q}_r^+)} \leq C\ C(r,\rho)\ ||u||_{H_t^\beta(\overline{Q}_\rho^+)} \tag{3.3.47}$$

for any $0 < r < \rho \leq \frac{7}{8}$, h as in (3.3.44) and of course

$$||u||_{H_t^\alpha(\overline{Q}_\rho^+)} \leq C\ C(\rho) K. \tag{3.3.48}$$

For the iteration consider the following finite sequence of (universal) radii

$$r_k = \frac{7}{8} - \frac{k}{16m_0}, \quad \text{for} \quad k = 0, 1, \ldots, 2m_0.$$

Note that $r_0 = \frac{7}{8}, r_{2m_0} = \frac{3}{4}$ and $r_{k-1} - r_k = \frac{1}{16m_0}$.

Step 1. (of the iteration): Applying (3.3.47) for $\beta = \alpha, r = r_1, \rho = \frac{7}{8}$ together with (3.3.48) for $\rho = \frac{7}{8}$ we obtain that

$$||v_{\alpha,h}||_{H^\alpha(\overline{Q}_{r_1}^+)} \leq CK \quad \text{for any} \quad -\left(\frac{1}{16m_0}\right)^2 < h < 0.$$

Using the above and Remark 51 we shall get

$$||u||_{H_t^{2\alpha}(\overline{Q}_{r_2}^+)} \leq CK. \tag{3.3.49}$$

That is, we take any two $(X, t_1) \neq (X, t_2) \in \overline{Q}_{r_2}^+$ and since $t_1 \neq t_2$ we can assume without the loss of generality that $t_1 > t_2$ and denote by $t := t_1$ and $t + L := t_2$ (then $L = t_2 - t_1 < 0$) and we aim to get that $|u(X,t) - u(X,t+L)| \leq CK|L|^\alpha$. We split into two cases:

- If $|L| \geq \frac{1}{2}\left(\frac{1}{16m_0}\right)^2$, then $|u(X,t) - u(X,t+l)| \leq 2K \leq 2K2^\alpha(16m_0)^{2\alpha}|L|^\alpha \leq CK|L|^\alpha$ and we are done.
- If $|L| < \frac{1}{2}\left(\frac{1}{16m_0}\right)^2$, we consider the interval $I = \left[-\left(\frac{1}{16m_0}\right)^2, 0\right]$. Define

$$\tilde{u}^{(X,t)}(l) = u(X, t+l), \quad \text{for} \quad l \in I$$

and

$$\tilde{v}^{(X,t)}_{\frac{\alpha}{2},h}(l) = \frac{\tilde{u}^{(X,t)}(l+h) - \tilde{u}^{(X,t)}(l)}{|h|^{\frac{\alpha}{2}}}, \quad \text{for} \quad -\frac{1}{2}\left(\frac{1}{16m_0}\right)^2 < h < 0, l \in I_h$$

where I_h is as in Lemma 50. Then

$$\tilde{v}^{(X,t)}_{\frac{\alpha}{2},h}(l) = \frac{u(X,t+l+h) - u(X,t+l)}{|h|^{\frac{\alpha}{2}}} = v_{\alpha,h}(X,t+l).$$

Now, if $(X,t) \in \overline{Q}^+_{r_2}$, $l \in I$ then $-r_2^2 < t \leq 0$ and $-\left(\frac{1}{16m_0}\right)^2 - r_2^2 < t+l \leq l < 0$. But, $-\left(\frac{1}{16m_0}\right)^2 - r_2^2 = -(r_1-r_2)^2 - r_2^2 = -r_1^2 + 2r_1r_2 - 2r_2^2 \geq -r_1^2$ (using that $r_1 > r_2$), i.e. $(X,t+l) \in \overline{Q}^+_{r_1}$. Then, for $l_1, l_2 \in I_h$

$$\left|\tilde{v}^{(X,t)}_{\frac{\alpha}{2},h}(l_1) - \tilde{v}^{(X,t)}_{\frac{\alpha}{2},h}(l_2)\right| = |v_{\alpha,h}(X,t+l_1) - v_{\alpha,h}(X,t+l_2)| \leq CK|l_1-l_2|^{\frac{\alpha}{2}}.$$

That is,

$$\left\|\tilde{v}^{(X,t)}_{\frac{\alpha}{2},h}\right\|_{C^{\frac{\alpha}{2}}(I_h)} \leq CK$$

and Remark 51 implies

$$||\tilde{u}^{(X,t)}||_{C^{\alpha}(\tilde{I})} \leq CK$$

where $\tilde{I} = \left[-\frac{1}{2}\left(\frac{1}{16m_0}\right)^2, 0\right]$. Since $0, L \in \tilde{I}$, we have the desired.

We repeat the same procedure until:

Step m_0. (of the iteration): Applying (3.3.47) for $\beta = m_0\alpha, r = r_{2m_0-1}, \rho = r_{2m_0-2}$ together with Step $m_0 - 1$ we obtain that

$$||v_{m_0\alpha,h}||_{H^{\alpha}\left(\overline{Q}^+_{r_{2m_0-1}}\right)} \leq CK, \quad \text{for any} \quad -\left(\frac{1}{16m_0}\right)^2 < h < 0.$$

Then again as in Step 1 (using Remark 51) and recalling how the constants α and m_0 have been chosen ($(m_0+1)\frac{\alpha}{2} > 1$) we can derive that

$$||u||_{H^2_t(\overline{Q}^+_{3/4})} \leq CK. \tag{3.3.50}$$

This last estimate ensures the (almost everywhere) existence of u_t in $Q^+_{\frac{3}{4}}$. Moreover, by applying again (3.3.47) for $\beta = 2, r = \frac{1}{2}, \rho = \frac{3}{4}$ together with (3.3.50) gives

$$||v_{1,h}||_{H^\alpha\left(\overline{Q}^+_{5/8}\right)} \leq CK, \quad \text{for any} \quad -\left(\frac{1}{16m_0}\right)^2 < h < 0.$$

The later gives the H^α-estimate for u_t in $Q^+_{1/2}$. □

3.3.3 $H^{1+\alpha}$-estimates for the oblique derivative case

In the following we assume for convenience that $F(O) = 0$. Note that this assumption is not essential in the sense that we can find an operator with the same ellipticity constants satisfying this assumption and subtracting a paraboloid from u the new equation is satisfied. Indeed, from the ellipticity condition we can derive that there exists some $\theta \in \mathbb{R}$ so that $F(\theta I_n) = 0$:

$$F\left(\frac{|F(O)|}{\lambda}I_n\right) - F(O) \geq \lambda\frac{|F(O)|}{\lambda} \geq -F(O) \Rightarrow F\left(\frac{|F(O)|}{\lambda}I_n\right) \geq 0$$

and

$$F(O) - F\left(-\frac{|F(O)|}{\lambda}I_n\right) \geq \lambda\frac{|F(O)|}{\lambda} \geq F(O) \Rightarrow F\left(-\frac{|F(O)|}{\lambda}I_n\right) \leq 0$$

hence from the continuity of F there exists $\theta \in \left[-\frac{|F(O)|}{\lambda}, \frac{|F(O)|}{\lambda}\right]$ so that $F(\theta I_n) = 0$. Now, we consider the operator $G(M) = F(M + \theta I_n)$ then $G(O) = F(\theta I_n) = 0$ and for any $M, N \in S_n, N \geq 0$

$$\lambda \|N\| \leq F(M+N+\theta I_n) - F(M+\theta I_n) \leq \Lambda \|N\| \Rightarrow \lambda \|N\| \leq G(M+N) - G(M) \leq \Lambda \|N\|$$

that is, G is uniformly elliptic with the same ellipticity constants as F. Moreover for any (fixed) $X_0 \in \mathbb{R}^n$ we consider the paraboloid $P(X) = \frac{\theta}{2}|X - X_0|^2$ (then $D^2P = \theta I_n$). Also let u be a solution of $F(D^2u) - u_t = 0$, then $u - P$ is a solution of $G(D^2w) - w_t = 0$. Note also that $u - P$ and $u = u - P + P$ have the same regularity.

First we examine a constant oblique derivative problem using the change of variables of section 2.5.

Theorem 53. *(Boundary $H^{1+\alpha}$-estimates for the constant oblique derivative problem).*

Let $u \in C\left(Q_1^+ \cup Q_1^*\right)$ be bounded and satisfies in the viscosity sense

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } \ Q_1^+ \\ \beta \cdot Du = 0, & \text{on } \ Q_1^* \end{cases} \tag{3.3.51}$$

where β is a constant function. Then the first derivatives $u_{z_1}, \dots, u_{z_{n-1}}, u_w$ exist at $(0,0)$. Moreover there exists a universal constant $0 < \alpha < 1$ so that u is punctually $H^{1+\alpha}$ at $(0,0)$. More precisely, there exists a polynomial R_1 of first order in Z, that is $R_1(Z) = A^0 + B^0 \cdot Z$, where $A^0 = u(0,0)$ and $B^0 = Du(0,0) \in \mathbb{R}^n$ (then, $\beta \cdot B^0 = 0$) so that

$$|u(Z,t) - R_1(Z)| \leq C\ ||u||_{L^\infty(Q_1^+)}\ \left(|Z| + |t|^{1/2}\right)^{1+\alpha} \tag{3.3.52}$$

for every $P = (Z,t) \in \overline{Q}_\rho^+$, where $C > 0$, $0 < \rho < 1$ are universal constants.

In addition, u_t exists and it is H^α in $\overline{Q}_\rho^+$ with the corresponding estimate being bounded by above by a term of the form $C\ ||u||_{L^\infty(Q_1^+)}$.

Proof. Let A be the transformation defined in section 2.5. Define $v(X,t) = u(AX,t)$, for $(X,t) \in Q_r^+$, where $0 < r < \frac{\delta_0}{\delta_0+1} < 1$. Note that $Q_r^+ \subset \tilde{Q_1^+}$. Then

$$\begin{cases} \tilde{F}(D^2v) - v_t = 0, & \text{in } \ Q_r^+ \\ v_y = 0, & \text{on } \ Q_r^*. \end{cases}$$

So applying Theorem 52 to v we have that $v_{x_1}, \dots, v_{x_{n-1}}, v_y$ exist at $(0,0)$ and there exists a polynomial $\tilde{R}_1(X) = \tilde{A}^0 + \tilde{B}^0 \cdot X$, where $\tilde{A}^0 = v(0,0)$ and $\tilde{B}^0 = (v_{x_1}(0,0), \dots, v_{x_{n-1}}(0,0), 0)$ so that

$$|v(X,t) - \tilde{R}_1(X)| \leq C\ ||v||_{L^\infty(Q_r^+)}\ \left(|X| + |t|^{1/2}\right)^{1+\alpha}$$

for every $(X,t) \in \overline{Q}_{r/4}^+$, where $C > 0$, $0 < \alpha < 1$ are universal constants. In addition, v_t exists and it is H^α in $\overline{Q}_{r/4}^+$ with the corresponding estimate being bounded by above by a term of the form $C\ ||v||_{L^\infty(Q_r^+)}$.

Now since $u(Z,t) = v(A^{-1}Z,t)$ then $u_{z_1}, \dots, u_{z_{n-1}}, u_w$ exist at $(0,0)$. Also

$$|v(A^{-1}Z,t) - \tilde{R}_1(A^{-1}Z)| \leq C\ ||u||_{L^\infty(Q_r^+)}\ \left(|A^{-1}Z| + |t|^{1/2}\right)^{1+\alpha}$$

for every $(A^{-1}Z,t) \in \overline{Q}_{r/4}^+$. Note that for $\rho = \frac{\delta_0 r}{4(\delta_0+1)} < 1$ if $(Z,t) \in Q_\rho^+$ then $(A^{-1}Z,t) \in$

$\overline{Q}^+_{r/2}$, that is the above estimate is true for every $(Z,t) \in Q^+_\rho$. Let $R_1(Z) = \tilde{R}_1(A^{-1}Z) = \tilde{A}^0 + \tilde{B}^0 \cdot A^{-1}Z = \tilde{A}^0 + (A^{-1})^\tau \tilde{B}^0 \cdot Z$ and observe that

$$\tilde{A}^0 = v(0,0) = u(0,0) =: A^0$$

and

$$(A^{-1})^\tau \tilde{B}^0 = \left(v_{x_1}(0,0), \dots, v_{x_{n-1}}(0,0), v_y(0,0) - \frac{\beta_1}{\beta_n} v_{x_1}(0,0) - \dots - \frac{\beta_{n-1}}{\beta_n} v_{x_{n-1}}(0,0) \right)$$
$$= \big(u_{z_1}(0,0), \dots, u_{z_{n-1}}(0,0), u_w(0,0)\big) =: B^0$$

and

$$|u(Z,t) - R_1(Z)| \leq C \; ||u||_{L^\infty(Q^+_r)} \; \left(|Z| + |t|^{1/2}\right)^{1+\alpha}$$

for every $(Z,t) \in \overline{Q}^+_\rho$. Furthermore $u_t(Z,t) = v_t(A^{-1}Z,t)$ and $\|u_t\|_{H^\alpha(Q^+_\rho)} \leq C \; \|u\|_{L^\infty(Q^+_1)}$.

□

Theorem 54. *(Boundary $H^{1+\alpha}$-estimates for the general oblique derivative problem). Let g and β be H^γ locally on Q^*_1, $f \in L^q\left(Q^+_1\right)$ with $q > \frac{(n+1)(n+2)}{2}$ and $u \in C\left(Q^+_1 \cup Q^*_1\right)$ be bounded and satisfies in the viscosity sense*

$$\begin{cases} F(D^2u) - u_t = f, & \text{in } \; Q^+_1 \\ \beta \cdot Du = g, & \text{on } \; Q^*_1. \end{cases}$$

Then the first derivatives $u_{x_1}, \dots, u_{x_{n-1}}, u_y$ exist at $(0,0)$. Moreover there exists universal constant $0 < \alpha_0 < 1$ so that for $\alpha = \min\{\alpha_0, \gamma, \frac{2q-(n+1)(n+2)}{q(n+1)}\}$, u is punctually $H^{1+\alpha}$ at $(0,0)$. More precisely, there exists a polynomial $R_{1;0}$ of first order in X, that is $R_{1;0}(X) = A^0 + B^0 \cdot X$, where $A^0 = u(0,0)$ and $B^0 = Du(0,0) \in \mathbb{R}^n$ so that

$$|u(X,t) - R_{1;0}(X)| \leq C \left(||u||_{L^\infty(Q^+_1)} + ||g||_{H^\gamma(\overline{Q}^*_{1/2})} + ||f||_{L^q(\overline{Q}^+_1)} \right) \; \left(|X| + |t|^{1/2}\right)^{1+\alpha} \tag{3.3.53}$$

for every $(X,t) \in \overline{Q}^+_\rho$, where $0 < \rho < 1, C > 0$ are universal constants.

Note that we may assume that $u(0,0) = 0$, considering $u(X,t) - u(0,0)$, then $A^0 = 0$. Also we may assume that $g(0,0) = 0$, considering $u(X,t) - \frac{g(0,0)\; y}{\beta_n(0,0)}$.

Proof. Before we start let us denote for convenience $K := ||u||_{L^\infty(Q^+_1)} + ||g||_{H^\gamma(\overline{Q}^*_{1/2})} + ||f||_{L^q(\overline{Q}^+_1)}$ and $\beta^0 := \beta(0,0) \in \mathbb{R}^n$.

We intend to find some $B^0 \in \mathbb{R}^n$, with $\beta^0 \cdot B^0 = 0$ so that for universal $C > 0, 0 < \eta < 1, 0 < \rho < 1, \alpha_0 > 0$ and $\alpha = \min\{\alpha_0, \gamma, \frac{2q-(n+1)(n+2)}{q(n+1)}\}$ we have

$$\underset{Q^+_{\rho\eta^k}}{\operatorname{osc}} \big(u(X,t) - B^0 \cdot X\big) \leq CK\eta^{k(1+\alpha)}, \quad \text{for any} \quad k \in \mathbb{N} \tag{3.3.54}$$

(see the proof of Theorem 49 where we explain why (3.3.54) suffices). We proceed by induction to show that there exist universal constants $0 < \eta << 1, 0 < \rho << 1, \bar{C} > 0, \alpha_0 > 0$ such that for $\alpha = \min\{\alpha_0, \gamma, \frac{2q-(n+1)(n+2)}{q(n+1)}\}$ we can find a vector $B_k \in \mathbb{R}^n$, with $\beta^0 \cdot B_k = 0$ for any $k \in \mathbb{N}$ so that

$$\underset{Q^+_{\rho\eta^k}}{\operatorname{osc}} (u(X,t) - B_k \cdot X) \leq \bar{C}K\eta^{k(1+\alpha)} \tag{3.3.55}$$

and

$$|B_{k+1} - B_k| \leq CK\eta^{k\alpha}. \tag{3.3.56}$$

Note that the right constants will be deduced from the induction. The details follow.

First, for $k = 0$, take $B_0 = 0$ and choose any $\bar{C} \geq 2$. Next for the induction we assume that we have found vectors $B_0, B_1, \ldots, B_{k_0}$ for which (3.3.55) and (3.3.56) hold. Denoting by $r := \frac{\rho\eta^{k_0}}{2}$ and $B := B_{k_0}$ we have $\beta^0 \cdot B = 0$ and

$$\underset{Q^+_r}{\operatorname{osc}} (u(X,t) - B \cdot X) \leq \frac{4\bar{C}}{\rho^{1+\alpha}} K r^{(1+\alpha)} \tag{3.3.57}$$

and we want to find a vector B_{k_0+1}, with $\beta^0 \cdot B_{k_0+1} = 0$ so that

$$\underset{Q^+_{2\eta r}}{\operatorname{osc}} (u(X,t) - B_{k_0+1} \cdot X) \leq \bar{C}K\eta^{(k_0+1)(1+\alpha)}. \tag{3.3.58}$$

Now we consider a suitable constant oblique derivative problem (as the one of Theorem 53). So let v be the viscosity solution of

$$\begin{cases} F(D^2v) - v_t = 0, & \text{in } Q^+_r \\ \beta^0 \cdot Dv = 0, & \text{on } Q^*_r \\ v = u - B \cdot X, & \text{on } \partial_p Q^+_r \setminus Q^*_r. \end{cases}$$

Then v satisfies the following

- ABP-estimate for the oblique derivative case (see Theorem 28) gives

$$\operatorname*{osc}_{Q_r^+} v \leq \operatorname*{osc}_{Q_r^+} (u(X,t) - B \cdot X) \tag{3.3.59}$$

  since $\beta^0 \cdot Dv = 0$ and $F(O) = 0$.

- From Theorem 53 we have that $\bar{B} = Dv(0,0)$ exists and $\beta^0 \cdot \bar{B} = 0$. Moreover

$$\operatorname*{osc}_{Q_{\tilde{r}}^+} \left(v(X,t) - \bar{B} \cdot X\right) \leq C_0 \left(\frac{\tilde{r}}{r}\right)^{1+\alpha_1} \operatorname*{osc}_{Q_r^+} v \tag{3.3.60}$$

  for any $\tilde{r} \leq \tau\, r$, where $0 < \tau < 1$ universal and also

$$|\bar{B}| \leq \frac{C}{r} \operatorname*{osc}_{Q_r^+} v. \tag{3.3.61}$$

  Next, we take $\tilde{r} = 2\eta r$ (for $0 < \eta < \frac{\tau}{2}$) in (3.3.60). Hence

$$\operatorname*{osc}_{Q_{2\eta r}^+} \left(v(X,t) - \bar{B} \cdot X\right) \leq C_0 \eta^{1+\alpha_1} \operatorname*{osc}_{Q_r^+} v. \tag{3.3.62}$$

Now take (universal) $0 < \eta << 1$ sufficiently small in order to have that $8\ C_0\eta^{\alpha_1} < 1$. We denote by $1 - \theta := 8\ C_0\eta^{\alpha_1}$, where $0 < \theta < 1$ is a universal constant. Then combining (3.3.62) and (3.3.59) and then using (3.3.57) we obtain

$$\operatorname*{osc}_{Q_{2\eta r}^+} \left(v(X,t) - \bar{B} \cdot X\right) \leq \frac{(1-\theta)}{8}\ \eta \operatorname*{osc}_{Q_r^+} (u(X,t) - B \cdot X) \leq \frac{(1-\theta)}{2}\ \eta\ \frac{\bar{C}}{\rho^{1+\alpha}} K r^{1+\alpha}. \tag{3.3.63}$$

Now to return to u we define $w = u - B \cdot X - v$. Then

$$\begin{cases} w \in S_p\left(\frac{\lambda}{n}, \Lambda, f\right), & \text{in } Q_r^+ \\ \beta \cdot Dw = g - \beta \cdot (B + Dv), & \text{on } Q_r^* \\ w = 0, & \text{on } \partial_p Q_r^+ \setminus Q_r^*. \end{cases}$$

Now for $0 < \mu < 1$ (to be chosen universal) we denote by $\bar{r} := r(1-\mu) < r$. We apply again ABP-estimate for the oblique derivative case (Theorem 28)

$$
\begin{aligned}
\operatorname*{osc}_{Q_{\bar{r}}^{+}} w &\leq Cr||f||_{L^{n+1}(Q_r^+)} + Cr||g||_{L^\infty(Q_r^*)} + Cr||\beta \cdot B||_{L^\infty(Q_r^*)} \\
&\quad + Cr||\beta \cdot Dv||_{L^\infty(Q_{\bar{r}}^*)} + \operatorname*{osc}_{\partial_p Q_{\bar{r}}^+ \setminus Q_{\bar{r}}^*} w \\
&=: \mathbf{I} + \mathbf{II} + \mathbf{III} + \mathbf{IV} + \mathbf{V}. \qquad (3.3.64)
\end{aligned}
$$

We want to bound every term **I** - **V** by a term of order $r^{1+\alpha}$. We start with term **I**. Recall that from Hölder inequality we can easily get that if $q_1 < q_2$ and $f \in L^{q_2}\left(Q_r^+\right)$,

$$
||f||_{L^{q_1}\left(Q_r^+\right)} \leq C(n)\ r^{\frac{(n+2)(q_2-q_1)}{q_1 q_2}}\ ||f||_{L^{q_2}\left(Q_r^+\right)}.
$$

Then

$$
||f||_{L^{n+1}(Q_r^+)} \leq C(n)\ r^{\frac{(n+2)(q-n-1)}{q(n+1)}}\ ||f||_{L^q\left(Q_r^+\right)} \leq C(n)\ r^{\frac{n+2}{n+1}-\frac{n+2}{q}}\ ||f||_{L^q\left(Q_r^+\right)}.
$$

Note that $\frac{n+2}{n+1} - \frac{n+2}{q} \geq 1 - \frac{n+2}{q}$ with $0 < 1 - \frac{n+2}{q} < 1$ and $r < 1$. Hence

$$
\mathbf{I} \leq C\ r^{1+\left(1-\frac{n+2}{q}\right)}\ K.
$$

Next, for term **II**, we use the H^γ-regularity of g and the fact that $g(0,0) = 0$, then

$$
\mathbf{II} = Cr||g - g(0,0)||_{L^\infty(Q_r^*)} \leq Crr^\gamma K \leq Cr^{1+\gamma} K.
$$

We continue with term **III**. We use the H^γ-regularity of β and the fact that $\beta^0 \cdot B = 0$,

$$
\mathbf{III} = Cr||\left(\beta - \beta^0\right) \cdot B||_{L^\infty(Q_r^*)} \leq Cr||\beta - \beta^0||_{L^\infty(Q_r^*)}\ |B| \leq Crr^\gamma K \leq Cr^{1+\gamma} K
$$

where we use that $|B| \leq CK$. Indeed, since $|B_0| = 0$, we have $|B| = |B_{k_0} - B_0| \leq \sum_{k=0}^{k_0-1} |B_{k+1} - B_k| \leq CK \sum_{k=0}^{k_0-1} (\eta^\alpha)^k \leq CK\left(\frac{1-(\eta^\alpha)^{k_0}}{1-\eta^\alpha}\right) \leq CK\frac{1}{1-\eta^\alpha} \leq CK$. Next for term **IV**, we use again the H^γ-regularity of β and the fact that $\beta^0 \cdot Dv = 0$ on Q_r^*, we have

$$
\mathbf{IV} \leq Cr||\beta - \beta^0||_{L^\infty(Q_r^*)}\ ||Dv||_{L^\infty(Q_{\bar{r}}^*)} \leq Crr^\gamma \frac{\bar{C}Kr^{1+\alpha}}{r} \leq C_2\rho^\gamma \frac{\bar{C}}{\rho^{1+\alpha}} K r^{1+\alpha}
$$

using again Theorem 53 for v. Finally we examine term **V**. Let $(X_0, t_0) \in \partial_p Q_{\bar{r}}^+ \setminus Q_{\bar{r}}^*$.

- If $|X_0| = \bar{r}$ we choose $\bar{X}_0 \in (\partial B_r)^+$ so that $|X_0 - \bar{X}_0| = \mu r \le \sqrt{2\mu} r$ and $\bar{t}_0 = t_0$.
- If $|X_0| < \bar{r}$ then $t_0 = -(1-\mu)^2 r^2$ and we choose $\bar{t}_0 = -r^2$ then $|t_0 - \bar{t}_0|^{1/2} = r\sqrt{\mu(2-\mu)} \le \sqrt{2\mu} r$ and $\bar{X}_0 = X_0$.

In any case $|X_0 - \bar{X}_0| + |t_0 - \bar{t}_0|^{1/2} \le \sqrt{2\mu} r$ and $(\bar{X}_0, \bar{t}_0) \in \partial_p Q_r^+ \backslash Q_r^*$ that is $w\left(\bar{X}_0, \bar{t}_0\right) = 0$. Then

$$\begin{aligned} |w(X_0,t_0)| &= |w(X_0,t_0) - w\left(\bar{X}_0,\bar{t}_0\right)| \\ &\le |\left(u(X_0,t_0) - B\cdot X_0\right) - \left(u\left(\bar{X}_0,\bar{t}_0\right) - B\cdot \bar{X}_0\right)| + |v(X_0,t_0) - v\left(\bar{X}_0,\bar{t}_0\right)| \end{aligned} \tag{3.3.65}$$

and we bound these terms using H^α-estimates. Indeed, we have that

$$\begin{cases} F(D^2(u - B\cdot X)) - (u - B\cdot X)_t = f, & \text{in } Q_{2r}^+ \\ \beta \cdot D(u - B\cdot X) = g - \beta \cdot B, & \text{on } Q_{2r}^*. \end{cases}$$

Then Corollary 38 gives

$$\begin{aligned} ||u - B\cdot X||_{H^{\alpha_2}(\overline{Q}_r^+)} \le \frac{C}{r^{\alpha_2}} \; & ||u - B\cdot X||_{L^\infty(Q_{2r}^+)} \\ & + \frac{C}{r^{\alpha_2}} \left(r^{\frac{n}{n+1}} ||f||_{L^{n+1}(Q_{2r}^+)} + r||g||_{L^\infty(Q_{2r}^*)} + r||\beta\cdot B||_{L^\infty(Q_{2r}^*)} \right). \end{aligned}$$

Next we apply to v global H^α-estimates. Note that v satisfies a constant oblique derivative problem which can be transformed into a Neumann problem (see section 2.5) then using a reflection principle (see Proposition 21) we can see that we only need global H^α-estimates for the Dirichlet problem. This type of estimates can be found in [45]. Note that the values of v on the parabolic boundary equal to $u - B\cdot X$ which is H^{α_2}. So, for $0 < \alpha_3 << \alpha_2$ universal, we have

$$\begin{aligned} ||v||_{H^{\alpha_3}(\overline{Q}_r^+)} &\le \frac{C}{r^{\alpha_3}} \left(||v||_{L^\infty(Q_r^+)} + r^{\alpha_2} ||u - B\cdot X||_{H^{\alpha_2}(\overline{Q}_r^+)} \right) \\ &\le \frac{C}{r^{\alpha_3}} \; ||u - B\cdot X||_{L^\infty(Q_{2r}^+)} \\ &\quad + \frac{C}{r^{\alpha_3}} \left(r^{\frac{n}{n+1}} ||f||_{L^{n+1}(Q_{2r}^+)} + r||g||_{L^\infty(Q_{2r}^*)} + r||\beta\cdot B||_{L^\infty(Q_{2r}^*)} \right). \end{aligned}$$

Now, we return to (3.3.65).

$$
\begin{aligned}
|w(X_0,t_0)| &\le \left(|X_0-\bar{X}_0|+|t_0-\bar{t}_0|^{1/2}\right)^{\alpha_2}\ \frac{C}{r^{\alpha_2}}\left(||u-B\cdot X||_{L^\infty(Q^+_{2r})}+r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q^+_{2r})}\right)\\
&\quad+\left(|X_0-\bar{X}_0|+|t_0-\bar{t}_0|^{1/2}\right)^{\alpha_2}\frac{C}{r^{\alpha_2}}\left(r||g||_{L^\infty(Q^*_{2r})}+r||\beta\cdot B||_{L^\infty(Q^*_{2r})}\right)\\
&\quad+\left(|X_0-\bar{X}_0|+|t_0-\bar{t}_0|^{1/2}\right)^{\alpha_3}\ \frac{C}{r^{\alpha_3}}\left(||u-B\cdot X||_{L^\infty(Q^+_{2r})}+r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q^+_{2r})}\right)\\
&\quad+\left(|X_0-\bar{X}_0|+|t_0-\bar{t}_0|^{1/2}\right)^{\alpha_3}\frac{C}{r^{\alpha_3}}\left(r||g||_{L^\infty(Q^*_{2r})}+r||\beta\cdot B||_{L^\infty(Q^*_{2r})}\right)\\
&\le C\mu^{\alpha_3/2}\ \ ||u-B\cdot X||_{L^\infty(Q^+_{2r})}\\
&\quad+C\mu^{\alpha_3/2}\left(r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q^+_{2r})}+r||g||_{L^\infty(Q^*_{2r})}+r||\beta\cdot B||_{L^\infty(Q^*_{2r})}\right)\\
&\le \mathbf{VI}+\mathbf{I}'+\mathbf{II}'+\mathbf{III}'.
\end{aligned}
$$

For term $\mathbf{VI}$, we use the hypothesis of the induction, (3.3.57).

$$
\mathbf{VI}\le C_1\mu^{\alpha_3/2}\frac{\bar{C}}{\rho^{1+\alpha}}Kr^{1+\alpha}.
$$

Moreover for term $\mathbf{I}'$, we have

$$
\mathbf{I}'\le C\mu^{\alpha_3/2}r^{\frac{n}{n+1}}C(n)\ r^{\frac{(n+2)(q-n-1)}{q(n+1)}}\ ||f||_{L^q(Q^+_{2r})}\le C\mu^{\alpha_3/2}r^{1+\frac{2q-(n+1)(n+2)}{q(n+1)}}\ K.
$$

We denote by $\alpha(n,q):=\frac{2q-(n+1)(n+2)}{q(n+1)}$ which is positive since $q\ge\frac{(n+1)(n+2)}{2}$. Note also that $\alpha(n,q)<1-\frac{n+2}{q}$. Also, terms $\mathbf{II}'$ and $\mathbf{III}'$ are in fact the same as terms $\mathbf{II}$ and $\mathbf{III}$. That is,

$$
\mathbf{V}\le C_1\mu^{\alpha_3/2}\bar{C}Kr^{1+\alpha}+C\mu^{\alpha_3/2}r^{1+\alpha(n,q)}\ K+C\mu^{\alpha_3/2}r^{1+\gamma}\ K+C_2\rho^\gamma\frac{\bar{C}}{\rho^{1+\alpha}}Kr^{1+\alpha}.
$$

So, returning to (3.3.64), we have

$$
\operatorname*{osc}_{Q^+_{\bar{r}}} w\le CKr^{1+\alpha(n,q)}+CKr^{1+\gamma}+C_1\mu^{\alpha_3/2}\frac{\bar{C}}{\rho^{1+\alpha}}Kr^{1+\alpha}+C_2\rho^\gamma\frac{\bar{C}}{\rho^{1+\alpha}}Kr^{1+\alpha}.
$$

Next combining the above with (3.3.63) and choosing $\mu<1-2\eta$ (then $2\eta<1-\mu$) we get

$$
\begin{aligned}
\operatorname*{osc}_{Q^+_{2\eta r}}\left[u(X,t)-(B+\bar{B})\cdot X\right]&\le\frac{1}{2}(1-\theta)\eta\frac{\bar{C}}{\rho^{1+\alpha}}Kr^{1+\alpha}+CKr^{1+\alpha(n,q)}+CKr^{1+\gamma}\\
&\quad+C_1\mu^{\alpha_3/2}\frac{\bar{C}}{\rho^{1+\alpha}}Kr^{1+\alpha}+C_2\rho^\gamma\frac{\bar{C}}{\rho^{1+\alpha}}Kr^{1+\alpha}. \qquad (3.3.66)
\end{aligned}
$$

Now, recall that our aim is to get relation (3.3.58). We choose the right constants α_0, μ and $\bar{C}$ for this purpose. So, take α_0 so that $\eta^{\alpha_0} = 1-\frac{\theta}{2}$ and $\alpha = \min\{\alpha_0, \gamma, \frac{2q-(n+1)(n+2)}{q(n+1)}\}$. Take $\mu \leq \frac{\eta^{\frac{2(1+\alpha)}{\alpha_3}}}{(4C_1)^{\frac{2}{\alpha_3}}}$ (then $C_1\mu^{\alpha_3/2} \leq \frac{\eta^{1+\alpha}}{4}$), $\rho \leq \frac{\eta^{\frac{1+\alpha}{\gamma}}}{(4C_2)^{\frac{1}{\gamma}}}$ (then $C_2\rho^{\gamma} \leq \frac{\eta^{1+\alpha}}{4}$) and $\bar{C}$ large enough so that $\frac{\eta\theta\bar{C}}{4\rho^{1+\alpha}} \geq 2C$ (note that our choices are all independent of k_0). Then we return to (3.3.66) writing $1-\theta$ as $1-\frac{\theta}{2}-\frac{\theta}{2}$ and recalling that $r = \frac{\rho\eta^{k_0}}{2}$,

$$\begin{aligned}\operatorname*{osc}_{Q^+_{\rho\eta^{k_0+1}}} \left[u(X,t) - (B+\bar{B})\cdot X\right] &\leq \frac{1}{2}\left(1-\frac{\theta}{2}\right)\bar{C}\eta\eta^{k_0(1+\alpha)}K + 2Cr^{1+\alpha}K \\ &\quad - K\frac{\eta\theta}{4}\frac{\bar{C}}{\rho^{1+\alpha}}r^{1+\alpha} + K\bar{C}\frac{\eta^{1+\alpha}}{2}\eta^{k_0(1+\alpha)} \\ &\leq \bar{C}K\eta^{(k_0+1)(1+\alpha)}.\end{aligned}$$

This is relation (3.3.58) for $B_{k_0+1} = B+\bar{B} = B_{k_0}+\bar{B}$ and $\beta^0\cdot B_{k_0+1} = \beta^0\cdot B_{k_0}+\beta^0\cdot\bar{B} = 0$. It remains to get (3.3.56) for $k = k_0$. To do so, we use relation (3.3.61) together with (3.3.59) and then (3.3.57),

$$|B_{k_0+1} - B_{k_0}| = |\bar{B}| \leq \frac{C}{r}\bar{C}Kr^{1+\alpha} \leq CKr^{\alpha} = CK\eta^{k_0\alpha}$$

as we want. So the inductive proof is completed.

Finally, it remains to get estimate (3.3.54). Observe that

$$\lim_{k\to\infty} |B_{k+1} - B_k| \leq \lim_{k\to\infty} CK\eta^{k\alpha} = 0, \quad \text{since} \quad \eta < 1.$$

That is, the limit $B_\infty := \lim_{k\to\infty} B_k$ exists and it is the vector B^0 of (3.3.54). Indeed, $\beta^0\cdot B^0 = 0$ and for any $k\in\mathbb{N}$ we have

$$\operatorname*{osc}_{Q^+_{\rho\eta^k}} (u(X,t) - B_\infty\cdot X) \leq \operatorname*{osc}_{Q^+_{\rho\eta^k}} (u(X,t) - B_k\cdot X) + \rho\eta^k(B_k - B_\infty)$$

here we wrote $u(X,t) - B_\infty\cdot X = u(X,t) - B_k\cdot X + B_k\cdot X - B_\infty\cdot X \leq u(X,t) - B_k\cdot X + |X|\,|B_k - B_\infty|$ and $|X| \leq \rho\eta^k$. We have

$$\begin{aligned}\operatorname*{osc}_{Q^+_{\rho\eta^k}} (u(X,t) - B_\infty\cdot X) &\leq \bar{C}K\eta^{k(1+\alpha)} + CK\eta^k\sum_{j=k}^{\infty}\eta^{j\alpha} \leq \bar{C}K\eta^{k(1+\alpha)} + CK\eta^k\frac{\eta^{k\alpha}}{1-\eta^{\alpha}} \\ &= \bar{C}K\eta^{k(1+\alpha)} + \frac{C}{1-\eta^{\alpha}}K\eta^{(1+\alpha)k} \leq CK\eta^{k(1+\alpha)}\end{aligned}$$

and the proof is completed. □

3.4 Hölder Estimates for the second derivatives

3.4.1 $H^{2+\alpha}$-estimates for the homogeneous Neumann case

Here we prove $H^{2+\alpha}$ boundary estimates. For, we will use first Lemma 47 which applied on the derivative u_y will give the existence and Hölder continuity of the second derivative u_{yy}. Then for the tangential directions, our purpose is to consider the restriction of u on the thin-cylinder Q_1^* and to show that satisfies a suitable parabolic equation there. Hence we will be able to use the interior estimates proved in [45].

First let us formulate here Theorems 4.13 and 1.1 of [45] in the form we are going to use them.

Theorem 55. *(Interior $H^{2+\alpha}$-estimates). Let $u \in C(Q_1)$ be a bounded viscosity solution of $F(D^2u) - u_t = 0$ in Q_1. Assume also that F is convex. Then there exist universal $C > 0, 0 < \alpha < 1$ so that*

$$\|u\|_{H^{2+\alpha}\left(Q_{1/2}\right)} \leq C\left(\|u\|_{L^\infty(Q_1)} + |F(O)|\right). \tag{3.4.1}$$

Also, we have the same result for operators that depend on (X,t), that is of the form $F(M,(X,t))$. For, we define

$$\theta_F(X,t) = \sup_{M\in S_n} \frac{|F(M,(X,t)) - F(M,(0,0))|}{|M|+1}.$$

Theorem 56. *(Interior $H^{2+\alpha}$-estimates for more general operators). Let $u \in C(Q_1)$ be a bounded viscosity solution of $F(D^2u,(X,t)) - u_t = 0$ in Q_1. Assume that any solution v of the equation $F(D^2v + B,(0,0)) - v_t = E$, where B, E are such that $F(B,(0,0)) = E$, satisfies $H^{2+\beta}$-estimates*

$$\|u\|_{H^{2+\beta}\left(Q_{r/2}\right)} \leq \frac{C}{r^{2+\alpha}}\left(\|u\|_{L^\infty(Q_r)} + |F(O,(0,0))|\right). \tag{3.4.2}$$

Assume also that

$$\left(\frac{1}{m_{n+1}(Q_r)}\int_{Q_r}\theta_F^{n+1}\right)^{1/(n+1)} \leq Cr^\alpha. \tag{3.4.3}$$

Then u_t and the second derivatives of u exist in $\overline{Q}_{1/2}$. Moreover there exists universal constant $0<\alpha<\beta$ so that u is punctually $H^{2+\alpha}$ at every point $P_0 \in Q_{1/2}$ and more precisely there exists a polynomial $R_{2;P_0}$ of second order in X and of first order in t, that is $R_{2;P_0}(X,t) = A_{P_0} + B_{P_0}\cdot(X-X_0) + C_{P_0}(t-t_0) + \frac{1}{2}(X-X_0)^\tau D_{P_0}(X-X_0)$, *where* $A_{P_0} = u(P_0), B_{P_0} = \nabla_X u(P_0), C_{P_0} = u_t(P_0)$ *and* $D_{P_0} := D_X^2 u(P_0)$ *so that*

$$|u(X,t) - R_{2;P_0}(X,t)| \leq C \left(||u||_{L^\infty(Q_1^+)} + |F(O,(0,0))| \right) \; p(P,P_0)^{2+\alpha} \qquad (3.4.4)$$

for every $P = (X,t) \in \overline{Q}_{1/2}(P_0)$, *where* $C > 0$ *is a universal constant.*

We continue with an immediate consequence of Lemma 47.

Corollary 57. *Let* f *be bounded in* Q_1^+ *and* $u \in C\left(Q_1^+ \cup Q_1^*\right)$ *be bounded and satisfies*

$$\begin{cases} u \in S_p(\lambda,\Lambda,f), & \text{in } \; Q_1^+ \\ u = 0, & \text{on } \; Q_1^*. \end{cases}$$

Then u_y *exists on* Q_1^* *and for a universal constants* $C > 0, 0 < \alpha < 1$ *we have*

$$|u(X,t) - u_y(x,0,t)y| \leq C \left(\|u\|_{L^\infty\left(\overline{Q}_1^+\right)} + \|f\|_{L^\infty\left(\overline{Q}_1^+\right)} \right) y^{1+\alpha} \qquad (3.4.5)$$

for every $(X,t) \in \overline{Q}_{1/2}^+$. *Moreover,* u_y *is* $H^\alpha\left(\overline{Q}_{1/2}^+\right)$ *with the corresponding norm depending only on universal quantities and* $K := \|u\|_{L^\infty\left(\overline{Q}_1^+\right)} + \|f\|_{L^\infty\left(\overline{Q}_1^+\right)}$.

Proof. Note first that the justification for the existence and H^α-regularity of u_y can be found in the proof of Lemma 46.

Now let $(X,t) \in \overline{Q}_{1/2}^+$. We apply Remark 48 in $\overline{Q}_y^+(x,0,t) \subset Q_1^+$ to obtain for small $h > 0$,

$$\frac{u(X,t)}{y} - \frac{u(x,h,t)}{h} \leq CK \; y^a.$$

So letting $h \to 0$,

$$\frac{u(X,t)}{y} - u_y(x,0,t) \leq CK \; y^a$$

which implies the result. □

Next we apply the above to u_y to obtain the following.

Corollary 58. *Let* $u \in C\left(Q_1^+ \cup Q_1^*\right)$ *be bounded and satisfies in the viscosity sense*

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } \; Q_1^+ \\ u_y = 0, & \text{on } \; Q_1^*. \end{cases}$$

Then u_{yy} exists on Q_1^* and for a universal constants $C > 0, 0 < \alpha < 1$ we have

$$\left| u(X,t) - u(x,0,t) - \frac{1}{2} u_{yy}(x,0,t)\ y^2 \right| \leq C \left\| u \right\|_{L^\infty\left(\overline{Q}_1^+\right)}\ y^{2+\alpha} \qquad (3.4.6)$$

for every $(X,t) \in \overline{Q}_{1/2}^+$. Moreover, u_{yy} is $H^\alpha\left(\overline{Q}_{1/2}^+\right)$ with the corresponding norm depending only on universal quantities and $K := \|u\|_{L^\infty\left(\overline{Q}_1^+\right)}$.

Proof. First we observe that u_y exists in $Q_1^+ \cup Q_1^*$ from Theorem 52 and moreover it satisfies the following

$$\begin{cases} u_y \in S_p\left(\frac{\lambda}{n}, \Lambda\right), & \text{in } \ Q_1^+ \\ u_y = 0, & \text{on } \ Q_1^*. \end{cases}$$

Indeed, let $(X_0, t_0) \in Q_1^+$ and $\overline{Q}_r(X_0, t_0) \subset Q_1^+$. We consider the difference quotient

$$u_h(X,t) = \frac{u(x, y+h, t) - u(X,t)}{h}, \quad \text{for} \quad (X,t) \in \overline{Q}_r(X_0,t_0),\ 0 < h < \frac{r}{2}.$$

Then from Theorem 4.6 in [45] we have that $u_h \in S_p\left(\frac{\lambda}{n}, \Lambda\right)$ in $Q_r(X_0, t_0)$. But u_y is the uniform limit of u_{h_k} as $k \to \infty$ (due to the uniform H^α-estimates which are satisfied by u_h). Then using Proposition 10 we have the result.

Hence we can apply Corollary 57 to u_y. This means that u_{yy} exists and it is $H^\alpha\left(\overline{Q}_{1/2}^+\right)$. Also from (3.4.5) we have

$$-CKy^{1+\alpha} \leq u_y(X,t) - u_{yy}(x,0,t)y \leq CKy^{1+\alpha}$$

for any $(X,t) \in Q_{1/2}^+$. Then we integrate in direction y

$$\begin{aligned} u(X,t) - u(x,0,t) = \int_0^y u_y(x,\rho,t)\ d\rho &\leq \int_0^y \left(u_{yy}(x,0,t)\rho + CK\rho^{1+\alpha} \right)\ d\rho \\ &= u_{yy}(x,0,t)\frac{y^2}{2} + CKy^{2+\alpha} \end{aligned}$$

for any $(X,t) \in Q_{1/2}^+$. Repeating the same argument we have also the other-side inequality and the proof is complete. □

Proposition 59. *Let $u \in C\left(Q_1^+ \cup Q_1^*\right)$ be bounded and satisfies in the viscosity sense*

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } \; Q_1^+ \\ u_y = 0, & \text{on } \; Q_1^*. \end{cases}$$

Consider the restriction of u on Q_1^*, $v(x,t) := u(x,0,t)$. Moreover, denoting by $A(x,t) := u_{yy}(x,0,t)$ (which exists regarding Corollary 58) we consider the operator

$$G(M,x,t) := F\begin{pmatrix} M & 0 \\ 0 & A(x,t) \end{pmatrix} \tag{3.4.7}$$

for $(x,t) \in Q_1^*$ and $M \in S_{n-1}$. Then

$$G\left(D^2v, x, t\right) - v_t = 0, \quad \text{in} \quad Q_1^*$$

in the viscosity sense.

Proof. For convenience we show the result first at $P_0 = (0,0) \in Q_1^*$. So let ϕ be a test function on Q_1^* that touches v from below at $(0,0)$. That is, for (x,t) in some Q_ρ^*,

$$\phi(x,t) \le v(x,t) = u(x,0,t) \quad \text{and} \quad \phi(0,0) = v(0,0) = u(0,0,0).$$

Our aim is to show that

$$F\begin{pmatrix} D^2\phi(0,0) & 0 \\ 0 & A(0,0) \end{pmatrix} - \phi_t(0,0) \le 0.$$

To do so we extend ϕ into Q_1^+ and translate it suitably to turn it into a test function that touches u at some point of Q_r^+ and then use the equation for u.

For small $\epsilon > 0$ we consider,

$$\tilde{\phi}(X,t) = \phi(x,t) + \frac{A(0,0)}{2}y^2 - \epsilon(|X|^2 - t).$$

First, using Corollary 58 we want to obtain that for sufficiently small $r > 0$

$$u(X,t) \ge \tilde{\phi}(X,t) + \frac{\epsilon}{2}(|X|^2 - t), \quad \text{for any} \quad (X,t) \in \overline{Q}_r^+. \tag{3.4.8}$$

Indeed, Corollary 58 implies that for any $(X,t) \in Q^+_{1/2}$,

$$u(X,t) \geq u(x,0,t) + \frac{A(x,t)}{2} y^2 - CKy^{2+\alpha}$$

moreover, A is H^α that is,

$$\frac{|A(0,0) - A(x,t)|}{|x|^\alpha + |t|^{\frac{\alpha}{2}}} \leq CK \;\Rightarrow\; A(0,0) - A(x,t) \leq CK|x|^\alpha + CK|t|^{\frac{\alpha}{2}}.$$

Hence

$$\begin{aligned} u(X,t) &\geq u(x,0,t) + \frac{A(0,0)}{2} y^2 - CK|x|^\alpha y^2 - CK|t|^{\frac{\alpha}{2}} y^2 - CKy^{2+\alpha} \\ &\geq u(x,0,t) + \frac{A(0,0)}{2} y^2 - CK|X|^{2+\alpha} - CK|t|^{\frac{\alpha}{2}} y^2. \end{aligned}$$

Now choose $0 < r < \min\{\rho, \left(\frac{\epsilon}{4CK}\right)^{1/\alpha}\}$, then for $(X,t) \in \overline{Q}^+_r$ we have

$$CK|X|^{2+\alpha} \leq CK|X|^2 r^\alpha \leq CK|X|^2 \frac{\epsilon}{4CK} \leq \frac{\epsilon}{4}(|X|^2 - t)$$

and

$$CK|t|^{\frac{\alpha}{2}} y^2 \leq CK \frac{\epsilon}{4CK} |X|^2 \leq \frac{\epsilon}{4}(|X|^2 - t).$$

So, using also that $\phi \leq u$ on Q^*_ρ we get

$$u(X,t) \geq \phi(x,t) + \frac{A(0,0)}{2} y^2 - \frac{\epsilon}{2}(|X|^2 - t)$$

which by the definition of $\tilde{\phi}$ implies (3.4.8).

Now, from (3.4.8) we have an extension of ϕ that stays below u in Q^+_r. Next, using this we translate suitably $\tilde{\phi}$ in order to achieve $u - \tilde{\phi}$ to have a local minimum. So we consider for $h \in \mathbb{R}$,

$$\tilde{\phi}_h(X,t) = \tilde{\phi}(x, y-h, t).$$

Then

$$\begin{aligned} \tilde{\phi}_h(X,t) &= \phi(x,t) + \frac{A(0,0)}{2}(y-h)^2 - \epsilon(|x|^2 + (y-h)^2 - t) \\ &= \tilde{\phi}(X,t) - A(0,0)yh + \frac{A(0,0)}{2} h^2 + 2\epsilon hy - \epsilon h^2. \end{aligned}$$

Next, we observe that,

$$\tilde{\phi}_h(0,0,0) = \phi(0,0) + \frac{A(0,0)}{2}h^2 - \epsilon h^2$$

and since $\phi(0,0) = u(0,0,0)$,

$$u(0,0,0) - \tilde{\phi}_h(0,0,0) = -\frac{A(0,0)}{2}h^2 + \epsilon h^2$$

and by (3.4.8),

$$u(X,t) - \tilde{\phi}_h(X,t) \geq \frac{\epsilon}{2}(|X|^2 - t) + (A(0,0) - 2\epsilon)hy + u(0,0,0) - \tilde{\phi}_h(0,0,0)$$

for any $(X,t) \in \overline{Q}_r^+$. So we have that

- On $\partial_p Q_r^+ \setminus Q_r^*$,

$$u(X,t) - \tilde{\phi}_h(X,t) \geq \frac{\epsilon}{2}r^2 + (A(0,0) - 2\epsilon)hy + u(0,0,0) - \tilde{\phi}_h(0,0,0). \quad (3.4.9)$$

- On $\overline{Q}_r^*$,

$$(\tilde{\phi}_h)_y = -A(0,0)h + 2\epsilon h. \quad (3.4.10)$$

Subsequently, we split into two cases.

Case 1: If $A(0,0) \leq 0$. We choose $h > 0$ and we have,

- On $\partial_p Q_r^+ \setminus Q_r^*$, using (3.4.9) and that $(A(0,0) - 2\epsilon)h \leq 0, y < r$, so

$$\begin{aligned} u(X,t) - \tilde{\phi}_h(X,t) &\geq \frac{\epsilon}{2}r^2 + (A(0,0) - 2\epsilon)hr + u(0,0,0) - \tilde{\phi}_h(0,0,0) \\ &\geq u(0,0,0) - \tilde{\phi}_h(0,0,0) \end{aligned}$$

  choosing $0 < h \leq \frac{\epsilon r}{2(2\epsilon - A(0,0))}$, (then $\frac{\epsilon}{2}r^2 + (A(0,0) - 2\epsilon)hr \geq 0$).

- On $\overline{Q}_r^*$, by (3.4.10) we know that $(\tilde{\phi}_h)_y > 0$. Also $u_y = 0$, hence $(u - \tilde{\phi}_h)_y < 0$.

The above imply that the minimum of $u - \tilde{\phi}_h$ in $\overline{Q}_r^+$ cannot be attained on $\partial_p \overline{Q}_r^+$, that is it is attained at some point $(X_1, t_1) \in Q_r^+$ and hence it is a local (in the parabolic sense) minimum. Then, we can use the equation at (X_1, t_1), i.e. $F(D^2\tilde{\phi}_h(X_1, t_1)) -$

$(\tilde{\phi}_h)_t(X_1,t_1) \leq 0$. But

$$D^2\tilde{\phi}_h(X_1,t_1) = \begin{pmatrix} D^2\phi(x_1,t_1) - 2\epsilon I_{n-1} & 0 \\ 0 & A(0,0) - 2\epsilon \end{pmatrix}$$

and, $(\tilde{\phi}_h)_t(X_1,t_1) = \phi_t(x_1,t_1) + \epsilon$. So, taking $\epsilon \to 0$ then $r \to 0$ and $(x_1,t_1) \to (0,0)$ and we obtain

$$F\begin{pmatrix} D^2\phi(0,0) & 0 \\ 0 & A(0,0) \end{pmatrix} - \phi_t(0,0) \leq 0.$$

Case 2: If $A(0,0) > 0$. We choose $h = -\bar{h}$, for $\bar{h} > 0$ and $\epsilon < \frac{A(0,0)}{2}$, so

- On $\partial_p Q_r^+ \setminus Q_r^*$, using (3.4.9) and that $(A(0,0) - 2\epsilon)\bar{h} > 0, y < r$ we have,

$$\begin{aligned} u(X,t) - \tilde{\phi}_h(X,t) &\geq \frac{\epsilon}{2}r^2 - (A(0,0) - 2\epsilon)\bar{h}r + u(0,0,0) - \tilde{\phi}_h(0,0,0) \\ &\geq u(0,0,0) - \tilde{\phi}_h(0,0,0) \end{aligned}$$

  choosing $0 < \bar{h} \leq \frac{\epsilon r}{2(A(0,0)-2\epsilon)}$, (then $\frac{\epsilon}{2}r^2 - (A(0,0) - 2\epsilon)\bar{h}r \geq 0$).

- On $\overline{Q}_r^*$, by (3.4.10) we have, $(\tilde{\phi}_h)_y = -h(A(0,0)-2\epsilon) = \bar{h}(A(0,0)-2\epsilon) > 0$. Also $u_y = 0$, hence $(u - \tilde{\phi}_h)_y < 0$.

Then we can argue as in Case 1 and get again that

$$F\begin{pmatrix} D^2\phi(0,0) & 0 \\ 0 & A(0,0) \end{pmatrix} - \phi_t(0,0) \leq 0.$$

Finally note that a similar argument can be applied for test functions that touch v by above. Moreover, to get the result for any point of Q_1^*, we have to translate first Corollary 58 and then to consider $\tilde{\phi}$ and take relation (3.4.8) around the point we study. □

Now we are able to prove the main theorem of this section.

Theorem 60. *(Boundary $H^{2+\alpha}$-estimates for the Neumann problem).*

Let $u \in C\left(Q_1^+ \cup Q_1^*\right)$ be bounded and satisfies in the viscosity sense

$$\begin{cases} F(D^2u) - u_t = 0, & in \;\; Q_1^+ \\ u_y = 0, & on \;\; Q_1^*. \end{cases}$$

Then the second derivatives of u exist in $\overline{Q}^+_{1/2}$. Moreover there exists universal constant $0 < \alpha < 1$ so that u is punctually $H^{2+\alpha}$ at every point $P_0 \in Q^*_{1/2}$. More precisely, there exists a polynomial $R_{2;P_0}$ of second order in X and of first order in t, that is $R_{2;P_0}(X,t) = A_{P_0} + B_{P_0} \cdot (X - X_0) + C_{P_0}(t - t_0) + \frac{1}{2}(X - X_0)^\tau D_{P_0}(X - X_0)$, where $A_{P_0} = u(P_0), B_{P_0} = \left(u_{x_1}(P_0), \ldots, u_{x_{n-1}}(P_0), 0\right), C_{P_0} = u_t(P_0)$ and

$$D_{P_0} := \begin{pmatrix} u_{x_1x_1}(P_0) & \ldots & u_{x_1x_{n-1}}(P_0) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_{x_{n-1}x_1}(P_0) & \ldots & u_{x_{n-1}x_{n-1}}(P_0) & 0 \\ 0 & \ldots & 0 & u_{yy}(P_0) \end{pmatrix}$$

so that

$$|u(X,t) - R_{2;P_0}(X,t)| \leq C\left(||u||_{L^\infty\left(Q_1^+\right)} + |F(O)|\right) \; p(P,P_0)^{2+\alpha} \qquad (3.4.11)$$

for every $P = (X,t) \in \overline{Q}^+_{1/2}(P_0)$, where $C > 0$ is a universal constant.

Proof. Our intention is to combine Corollary 58 and interior $H^{2+\alpha}$-estimates on Q_1^* (through Proposition 59).

So, let $v(x,t) = u(x,0,t)$. Then from Proposition 59 v satisfies in the viscosity sense the equation $G(D^2v(x,t),(x,t)) - v_t(x,t) = 0$ in Q_1^*, where G is defined in (3.4.7). In order to use interior $H^{2+\alpha}$-estimates we have to verify that this equation satisfies the assumptions of Theorem 56:

- First we observe that G has the same ellipticity constants as F. Indeed, for $M, N \in S_{n-1}, N \geq 0$

$$G(M+N,x,t) - G(M,x,t) := F\begin{pmatrix} M+N & 0 \\ 0 & A(x,t) \end{pmatrix} - F\begin{pmatrix} M & 0 \\ 0 & A(x,t) \end{pmatrix}$$

and applying the ellipticity condition of F with the following matrices in S_n

$$\bar{M} := \begin{pmatrix} M & 0 \\ 0 & A(x,t) \end{pmatrix}, \quad \bar{N} := \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

we have what we want.

- Next we have to examine if the quantity θ_G satisfies the assumption (3.4.3). We have

$$\begin{aligned} 0 \leq \theta_G(x,t) &= \sup_{M \in S_{n-1}} \frac{|G(M,(x,t)) - G(M,(0,0))|}{|M|+1} \\ &= \sup_{M \in S_{n-1}} \frac{\left| F\begin{pmatrix} M & 0 \\ 0 & A(x,t) \end{pmatrix} - F\begin{pmatrix} M & 0 \\ 0 & A(0,0) \end{pmatrix} \right|}{|M|+1} \\ (F \text{ Lipschitz}) \quad &\leq \sup_{M \in S_{n-1}} \frac{\left\| \begin{pmatrix} 0 & 0 \\ 0 & A(x,t) - A(0,0) \end{pmatrix} \right\|}{|M|+1} \\ &= |A(x,t) - A(0,0)| \sup_{M \in S_{n-1}} \frac{1}{|M|+1} \leq CK \max\{|x|, |t|^{1/2}\}^{\alpha} \end{aligned}$$

since $\frac{1}{|M|+1} \leq 1$ and the function A is H^α from Corollary 58. Hence this estimate implies immediately assumption (3.4.3).

- Finally, we examine the assumption (3.4.2). So, let w be any viscosity solution of the equation $G(D^2w+B,(0,0))-w_t = E$, where B, E are such that $G(B,(0,0)) = E$. We consider the operator $L(M) := G(M+B,(0,0))-E$, for $M \in S_{n-1}$. Then L is elliptic with the same ellipticity constants as G (hence as F). Indeed, for $M, N \in S_{n-1}, N \geq 0$,

$$\lambda||N|| \leq G(M+N+B,(0,0)) - E + G(M+B,(0,0)) + E \leq \Lambda||N||.$$

Moreover, L is convex since F is, $L(O) = 0$ and w satisfies the homogeneous equation $L(D^2w) - w_t = 0$. Then Theorem 55 implies the assumption (3.4.2).

First we will get the result at $P_0 = (0,0,0)$, for convenience. Regarding the above conversation we can apply Theorem 56 to v and obtain that there exists a polynomial $\tilde{R}_{2;P_0}(x,t) = \tilde{A}_{P_0} + \tilde{B}_{P_0} \cdot x + \tilde{C}_{P_0} t + \frac{1}{2} x^\tau \tilde{D}_{P_0} x$, where $\tilde{A}_{P_0} = v(0,0), \tilde{B}_{P_0} = \nabla_x v(0,0), \tilde{C}_{P_0} =$

$v_t(0,0)$ and $\tilde{D}_{P_0} := D_x^2 v(0,0)$ so that

$$|u(x,0,t) - \tilde{R}_{2;P_0}(x,t)| \leq C\left(||u||_{L^\infty(Q_1^+)} + |F(O)|\right)\ \max\{|x|, |t|^{1/2}\}^{2+\alpha}$$

for every $(x,t) \in \overline{Q}^*_{1/2}$. On the other hand we have estimate (3.4.6) of Corollary 58 which gives for $(X,t) \in \overline{Q}^+_{1/2}$,

$$\begin{aligned}\left|u(X,t) - u(x,0,t) - \frac{1}{2}A(0,0)\ y^2\right| &\leq C\, \|u\|_{L^\infty(\overline{Q}_1^+)}\ y^{2+\alpha} + |A(x,t) - A(0,0)|\,\frac{y^2}{2} \\ &\leq CK\ \max\{|X|, |t|^{1/2}\}^{2+\alpha} + CK\ \max\{|x|, |t|^{1/2}\}^{\alpha}\frac{y^2}{2} \\ &\leq CK\ \max\{|X|, |t|^{1/2}\}^{2+\alpha}\end{aligned}$$

where $K := ||u||_{L^\infty(Q_1^+)} + |F(O)|$ and using the H^α-regularity of $A(x,t) := u_{yy}(x,0,t)$. Then, we take $R_{2;P_0}(X,t) = \tilde{R}_{2;P_0}(x,t) + \frac{A(0,0)}{2}y^2$ and combining the above we have for $(X,t) \in \overline{Q}^+_{1/2}$,

$$\begin{aligned}|u(X,t) - R_{2;P_0}(X,t)| &\leq \left|u(X,t) - u(x,0,t) - \frac{1}{2}A(0,0)\ y^2\right| + |u(x,0,t) - \tilde{R}_{2;P_0}(x,t)| \\ &\leq CK\ \max\{|X|, |t|^{1/2}\}^{2+\alpha}.\end{aligned}$$

For the points of $Q^*_{1/2}$ other than $(0,0,0)$ we use translation. □

3.4.2 $H^{2+\alpha}$-estimates for the oblique derivative case

In the present section we intent to obtain $H^{2+\alpha}$-estimates for the general oblique derivative problem (Theorem 62). We achieve this again using an approximation technique. We "approximate" the general problem by homogeneous problems with a suitable function β in the oblique derivative condition (as in Lemma 65). To get Lemma 65 we need to examine first the case when we have a non-homogeneous oblique condition but with constant β (Lemma 64) which can be done again by approximating the problem with suitable constant oblique derivative problems. Thereafter we first examine a constant oblique derivative problem (Theorem 61) using the change of variables of section 2.5. For convenience we assume that $F(O) = 0$ (see section 3.3.3).

Theorem 61. *(Boundary $H^{2+\alpha}$-estimates for the constant oblique derivative problem).*

Let $u \in C\left(Q_1^+ \cup Q_1^*\right)$ be bounded and satisfies in the viscosity sense

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } Q_1^+ \\ \beta \cdot Du = 0, & \text{on } Q_1^* \end{cases}$$

where β is a constant function. Then the second derivatives of u exist at $(0,0)$. Moreover there exists universal constant $0 < \alpha < 1$ so that u is punctually $H^{2+\alpha}$ at $(0,0)$. More precisely, there exists a polynomial $R_{2;0}$ of second order in Z and of first order in t, that is $R_{2;0}(Z,t) = A^0 + B^0 \cdot Z + C^0 t + \frac{1}{2} Z^\tau D^0 Z$, where $A^0 = u(0,0), B^0 = Du(0,0) \in \mathbb{R}^n, C^0 = u_t(0,0)$ and $D^0 = D^2u(0,0) \in S_n$ so that

$$|u(Z,t) - R_{2;0}(Z,t)| \leq C\ ||u||_{L^\infty\left(Q_1^+\right)}\ \left(|Z| + |t|^{1/2}\right)^{2+\alpha} \tag{3.4.12}$$

for every $(Z,t) \in \overline{Q}_\rho^+$, where $C > 0$ and $0 < \rho < 1$ are universal constants.

Proof. Let A be the transformation defined in section 2.5. Define $v(X,t) = u(AX,t)$, for $(X,t) \in Q_r^+$, where $0 < r < \frac{\delta_0}{\delta_0+1} < 1$. Note that $Q_r^+ \subset \tilde{Q_1^+}$. Then

$$\begin{cases} \tilde{F}(D^2v) - v_t = 0, & \text{in } Q_r^+ \\ v_y = 0, & \text{on } Q_r^* \end{cases}$$

with $\tilde{F}$ convex. So applying Theorem 60 to v we have that the second derivatives of v exist at $(0,0)$ and there exists a polynomial $\tilde{R}_2(X,t) = \tilde{A}^0 + \tilde{B}^0 \cdot X + \tilde{C}^0 t + \frac{1}{2} X^\tau \tilde{D}^0 X$, where $\tilde{A}^0 = v(0,0), \tilde{B}^0 = Dv(0,0) \in \mathbb{R}^n, \tilde{C}^0 = v_t(0,0)$ and $\tilde{D}^0 = D^2v(0,0) \in S_n$ so that

$$|v(X,t) - \tilde{R}_2(X,t)| \leq C\ ||v||_{L^\infty\left(Q_r^+\right)}\ \left(|X| + |t|^{1/2}\right)^{2+\alpha}$$

for every $(X,t) \in \overline{Q}_{r/2}^+$, where $C > 0$, $0 < \alpha < 1$ are universal constants.

Now since $u(Z,t) = v(A^{-1}Z,t)$ then the second derivatives of u exist at $(0,0)$. Also

$$|u(A^{-1}Z,t) - \tilde{R}_2(A^{-1}Z,t)| \leq C\ ||u||_{L^\infty\left(Q_r^+\right)}\ \left(|A^{-1}Z| + |t|^{1/2}\right)^{2+\alpha}$$

for every $(A^{-1}Z,t) \in \overline{Q}_{r/2}^+$. Note that for $\rho = \frac{\delta_0 r}{2(\delta_0+1)} < 1$ if $(Z,t) \in Q_\rho^+$ then $(A^{-1}Z,t) \in \overline{Q}_{r/2}^+$, that is the above estimate is true for every $(Z,t) \in Q_\rho^+$. Let $R_2(Z,t) = \tilde{R}_2(A^{-1}Z,t) = \tilde{A}^0 + \tilde{B}^0 \cdot A^{-1}Z + \tilde{C}^0 t + \frac{1}{2}(A^{-1}Z)^\tau \tilde{D}^0 A^{-1}Z = \tilde{A}^0 + (A^{-1})^\tau \tilde{B}^0 \cdot$

$Z + \tilde{C}^0 t + \frac{1}{2} Z^\tau (A^{-1})^\tau \tilde{D}^0 A^{-1} Z$ and observe that

$$\tilde{A}^0 = v(0,0) = u(0,0) =: A^0$$

and

$$\begin{aligned}(A^{-1})^\tau \tilde{B}^0 &= \left(v_{x_1}(0,0), \ldots, v_{x_{n-1}}(0,0), v_y(0,0) - \frac{\beta_1}{\beta_n} v_{x_1}(0,0) - \cdots - \frac{\beta_{n-1}}{\beta_n} v_{x_{n-1}}(0,0) \right) \\ &= \big(u_{z_1}(0,0), \ldots, u_{z_{n-1}}(0,0), u_w(0,0) \big) =: B^0\end{aligned}$$

and

$$\tilde{C}^0 = v_t(0,0) = u_t(0,0) =: C^0$$

and

$$(A^{-1})^\tau \tilde{D}^0 A^{-1} = D^2 u(0,0) =: D^0$$

and

$$|u(Z,t) - R_2(Z,t)| \leq C \; ||u||_{L^\infty(Q_r^+)} \; \left(|Z| + |t|^{1/2} \right)^{2+\alpha}$$

for every $(Z,t) \in \overline{Q}_\rho^+$. □

Theorem 62. *(Boundary $H^{2+\alpha}$-estimates for the general oblique derivative problem). Let g and β be $H^{1+\gamma}$ locally on Q_1^*, $f \in H^\gamma(Q_1^+)$ and $u \in C(Q_1^+ \cup Q_1^*)$ be bounded and satisfies in the viscosity sense*

$$\begin{cases} F(D^2 u) - u_t = f, & \text{in } \; Q_1^+ \\ \beta \cdot Du = g, & \text{on } \; Q_1^*. \end{cases}$$

Then the second derivatives of u and u_t exist at $(0,0)$. Moreover there exists universal constant $0 < \alpha_0 < 1$ so that for $\alpha = \min\{\alpha_0, \gamma\}$, u is punctually $H^{2+\alpha}$ at $(0,0)$. More precisely, there exists a polynomial $R_{2;0}$ of second order in X and of first order in t, that is $R_{2;0}(X,t) = A^0 + B^0 \cdot X + \Gamma^0 t + \frac{1}{2} X^\tau D^0 X$, where $A^0 = u(0,0), B^0 = Du(0,0) \in \mathbb{R}^n, \Gamma^0 = u_t(0,0)$ and $D^0 = D^2 u(0,0) \in S_n$ so that

$$|u(X,t) - R_{2;0}(X,t)| \leq C \left(||u||_{L^\infty(Q_1^+)} + ||g||_{H^{1+\gamma}(\overline{Q}_{1/2}^*)} + ||f||_{H^\gamma(Q_1^+)} \right) \left(|X| + |t|^{1/2} \right)^{2+\alpha} \tag{3.4.13}$$

for every $(X,t) \in \overline{Q}_\rho^+$, where $0 < \rho < 1, C > 0$ are universal constants.

Note that we may assume that:

1. $u(0,0) = 0$, considering $u(X,t) - u(0,0)$ (then $A^0 = 0$).

2. $g(0,0) = 0$, considering $u(X,t) - \frac{g(0,0)\ y}{\beta_n(0,0)}$.

3. $f(0,0) = 0$, considering $F'(M) := F(M) - f(0,0)$, then $F'(D^2u) - u_t = f - f(0,0)$ in the viscosity sense.

4. $g_{x_i}(0,0) = 0$ for every $i = 1, \dots, n-1$, considering

$$\bar{u}(X,t) := u(X,t) - \frac{y}{\beta_n(0,0)} \sum_{k=1}^{n-1} g_{x_k}(0,0) x_k.$$

Here we have to be more careful since we subtract a second order term. We have to examine what equation u satisfies. Denote by $h(X) := \frac{y}{\beta_n(0,0)} \sum_{k=1}^{n-1} g_{x_k}(0,0)x_k$, then for every $i, j = 1, \dots, n-1$

$$h_{x_i}(X) = \frac{y\ g_{x_i}(0,0)}{\beta_n(0,0)}, \quad h_y(X) = \frac{1}{\beta_n(0,0)} \sum_{k=1}^{n-1} g_{x_k}(0,0) x_k$$

and

$$h_{x_i x_j}(X) = 0, \quad h_{x_i y}(X) = \frac{g_{x_i}(0,0)}{\beta_n(0,0)} = h_{y x_i}(X), \quad h_{yy}(X) = 0.$$

That is

$$M_0 := D^2 h(X) = \begin{pmatrix} 0 & \dots & 0 & \frac{g_{x_1}(0,0)}{\beta_n(0,0)} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \frac{g_{x_{n-1}}(0,0)}{\beta_n(0,0)} \\ \frac{g_{x_1}(0,0)}{\beta_n(0,0)} & \dots & \frac{g_{x_{n-1}}(0,0)}{\beta_n(0,0)} & 0 \end{pmatrix} \in S_n.$$

We have that $F\left(D^2\bar{u} + M_0\right) - \bar{u}_t = f$ in Q_1^+ in the viscosity sense. Moreover, $\beta \cdot D\bar{u} = \bar{g}$, on Q_1^* in the viscosity sense, where $\bar{g}(x,t) := g(x,t) - \frac{\beta_n(x,t)}{\beta_n(0,0)} \sum_{k=1}^{n-1} g_{x_k}(0,0)x_k$ (note that $\bar{g}_{x_i}(0,0) = 0$). Note also that $\bar{F}(M) := F(M + M_0)$ has the same ellipticity constants as F

Before we continue let us make a remark that will be useful in the following proofs.

Remark 63. *Let*

$$D = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Then there exists $\tau_0 \in \mathbb{R}$ *so that* $\bar{F}(\tau_0 D) = 0$. *Moreover,* $|\tau_0| \leq C||g||_{H^{1+\gamma}\left(\overline{Q}^*_{1/2}\right)}$, *where* $C > 0$ *universal.*

Indeed, denote by $l := \frac{|F(M_0)|}{\lambda}$

$$F\left(M_0 + lD\right) - F(M_0) \geq \lambda l \geq -F(M_0) \Rightarrow F\left(M_0 + lD\right) \geq 0$$

and

$$F(M_0) - F\left(M_0 - lD\right) \geq \lambda l \geq F(M_0) \Rightarrow F\left(M_0 + lD\right) \leq 0$$

hence from the continuity of F there exists $\tau_0 \in [-l, l]$ so that $F(M_0 + \tau_0 D) = 0$. Also, $|\tau_0| \leq \frac{|F(M_0) - F(O)|}{\lambda} \leq \frac{\Lambda}{\lambda}||M_0||_\infty \leq C(\lambda, \Lambda, \delta_0)||g||_{H^{1+\gamma}\left(\overline{Q}^*_{1/2}\right)}$.

Note that, in the following we denote $\bar{u}, \bar{g}, \bar{F}$ by u, g, F for convenience.

As we mention in the start, in order to prove Theorem 62 we prove first two special cases.

Lemma 64. *Let* β *be constant* g *be* $H^{1+\gamma}$ *locally on* Q_1^* *and* $u \in C\left(Q_1^+ \cup Q_1^*\right)$ *be bounded and satisfies in the viscosity sense*

$$\begin{cases} F(D^2 u) - u_t = 0, & \text{in } \ Q_1^+ \\ \beta \cdot Du = g, & \text{on } \ Q_1^*. \end{cases}$$

Then the second derivatives of u *and* u_t *exist at* $(0,0)$. *Moreover there exists universal constant* $0 < \alpha_0 < 1$ *so that for* $\alpha = \min\{\alpha_0, \gamma\}$, u *is punctually* $H^{2+\alpha}$ *at* $(0,0)$. *More precisely, there exists a polynomial* $R_{2;0}$ *of second order in* X *and of first order in* t, *that is* $R_{2;0}(X,t) = A^0 + B^0 \cdot X + \Gamma^0 t + \frac{1}{2} X^\tau D^0 X$, *where* $A^0 = u(0,0), B^0 = Du(0,0) \in \mathbb{R}^n, \Gamma^0 = u_t(0,0)$ *and* $D^0 = D^2 u(0,0) \in S_n$ *so that*

$$|u(X,t) - R_{2;0}(X,t)| \leq C\left(||u||_{L^\infty\left(Q_1^+\right)} + ||g||_{H^{1+\gamma}\left(\overline{Q}^*_{1/2}\right)}\right)\ \left(|X| + |t|^{1/2}\right)^{2+\alpha} \quad (3.4.14)$$

for every $(X,t) \in \overline{Q}^+_{1/4}$, *where* $C > 0$ *is a universal constant.*

Proof. Before we start let us denote for convenience $K := ||u||_{L^\infty(Q_1^+)} + ||g||_{H^{1+\gamma}(\overline{Q}^*_{1/2})}$. We intend to find some $R^0(X,t) = B^0 \cdot X + \Gamma^0 t + \frac{1}{2} X^\tau D^0 X$, with $\beta \cdot B^0 = 0$ and $F(D^0) - \Gamma^0 = 0$ so that for universal $C > 0, 0 < \eta < 1, \alpha_0 > 0$ and $\alpha = \min\{\alpha_0, \gamma\}$ we have

$$\operatorname*{osc}_{Q^+_{\eta^k}} \left(u(X,t) - R^0(X,t)\right) \le CK\eta^{k(2+\alpha)}, \quad \text{for any} \quad k \in \mathbb{N}. \tag{3.4.15}$$

Similarly as we explain in the proof of Theorem 49, estimate (3.4.15) is enough.

Now, to get (3.4.15) we show by induction that there exist universal constants $0 < \eta << 1, \bar{C} > 0, \alpha_0 > 0$ such that for $\alpha = \min\{\alpha_0, \gamma\}$ we can find a paraboloid $R_k(X,t) = B_k \cdot X + \Gamma_k t + \frac{1}{2} X^\tau D_k X$, with

$$F(D_k) - \Gamma_k = 0, \quad \beta \cdot B_k = 0 \quad \text{and} \quad \sum_{j=1}^n (D_k)_{ij}\beta_j = 0, \; i = 1, \dots, n-1 \tag{3.4.16}$$

for any $k \in \mathbb{N}$ so that

$$\operatorname*{osc}_{Q^+_{\eta^k}} (u(X,t) - R_k(X,t)) \le \bar{C}K\eta^{k(2+\alpha)} \tag{3.4.17}$$

and

$$||D_{k+1} - D_k|| \le CK\eta^{k\alpha}, \; |\Gamma_{k+1} - \Gamma_k| \le CK\eta^{k\alpha}, \; |B_{k+1} - B_k| \le CK\eta^{k(1+\alpha)}. \tag{3.4.18}$$

Note that the right constants will be deduced from the induction. The details follow.

First, for $k = 0$, take $B_0 = 0$, $\Gamma_0 = 0$ and $(D_0)_{ij} = 0$, for $ij \neq nn$ and $(D_0)_{nn} = \tau_0$ where τ_0 is chosen so that $F(D_0) = 0$ (see Remark 63) and we see that

$$\operatorname*{osc}_{Q^+_1} (u(X,t) - R_0(X,t)) \le 2||u||_{L^\infty(Q_1^+)} + 2||D_0||_\infty \le \bar{C}K$$

choosing $\bar{C}$ large enough.

Next for the induction we assume that we have found paraboloids $R_0, R_1, \dots, R_{k_0}$ for which (3.4.16), (3.4.17) and (3.4.18) hold. Denoting by $r := \eta^{k_0}$ we have

$$\operatorname*{osc}_{Q^+_r} (u(X,t) - R_{k_0}(X,t)) \le \bar{C}Kr^{(2+\alpha)} \tag{3.4.19}$$

and we want to find a paraboloid R_{k_0+1} satisfying (3.4.16), (3.4.18) and

$$\underset{Q^+_{\eta r}}{\operatorname{osc}} \left(u(X,t) - R_{k_0+1}(X,t)\right) \le \bar{C} K \eta^{(k_0+1)(2+\alpha)}. \tag{3.4.20}$$

Now we consider a suitable constant oblique derivative problem (as the one of Theorem 61). So let v be the viscosity solution of

$$\begin{cases} G(D^2 v) - v_t = 0, & \text{in } \ Q^+_r \\ \beta \cdot Dv = 0, & \text{on } \ Q^*_r \\ v = u - R_{k_0}, & \text{on } \ \partial_p Q^+_r \setminus Q^*_r \end{cases}$$

where $G(M) = F(M + D_{k_0}) - \Gamma_{k_0}$ which is an elliptic operator with the same ellipticity constants as F. Also $G(O) = F(D_{k_0}) - \Gamma_{k_0} = 0$. Then v satisfies the following

- ABP-estimate for the oblique derivative case (see Theorem 28) gives

$$\underset{Q^+_r}{\operatorname{osc}}\, v \le \underset{Q^+_r}{\operatorname{osc}} \left(u(X,t) - R_{k_0}(X,t)\right) \tag{3.4.21}$$

  since $\beta \cdot Dv = 0$ and $G(O) = 0$.

- From Theorem 61 we have that $\bar{B} := Dv(0,0)$, $\bar{\Gamma} := v_t(0,0)$, $\bar{D} := D^2 v(0,0)$ exist and for $\bar{R}(X,t) = \bar{B} \cdot X + \bar{\Gamma} t + \frac{1}{2} X^\tau \bar{D} X$ we have

$$\underset{Q^+_{\tilde{r}}}{\operatorname{osc}} \big(v(X,t) - \bar{R}(X,t)\big) \le C_0 \left(\frac{\tilde{r}}{r}\right)^{2+\alpha_1} \underset{Q^+_r}{\operatorname{osc}}\, v \tag{3.4.22}$$

  for any $\tilde{r} \le \rho\, r$, where $0 < \rho < 1$ universal and also

$$|\bar{B}| \le \frac{C}{r} \underset{Q^+_r}{\operatorname{osc}}\, v, \ \ |\bar{\Gamma}| \le \frac{C}{r^2} \underset{Q^+_r}{\operatorname{osc}}\, v, \ \ ||\bar{D}||_\infty \le \frac{C}{r^2} \underset{Q^+_r}{\operatorname{osc}}\, v. \tag{3.4.23}$$

  Note that $\beta \cdot \bar{B} = 0$ from the oblique condition and $F(\bar{D} + D_{k_0}) - \Gamma_{k_0} - \bar{\Gamma} = 0$ using the continuity of F, $D^2 v$ and v_t. Also, $\beta \cdot Dv = 0$ holds in the classical sense on Q^*_1 and we can differentiate this condition with respect to x_i, for any $i = 1, \dots, n-1$ to get $\sum_{j=1}^n \bar{D}_{ij} \beta_j = 0$, for any $i = 1, \dots, n-1$.

Next, we take $\tilde{r} = \eta r$ (for $0 < \eta < \rho$) in (3.4.22). Hence

$$\underset{Q^+_{\eta r}}{\operatorname{osc}} \big(v(X,t) - \bar{R}(X,t)\big) \le C_0 \eta^{2+\alpha_1} \underset{Q^+_r}{\operatorname{osc}}\, v. \tag{3.4.24}$$

Now take (universal) $0 < \eta << 1$ sufficiently small in order to have that $C_0\eta^{\alpha_1} < 1$. We denote by $1-\theta := C_0\eta^{\alpha_1}$, where $0 < \theta < 1$ is a universal constant. Then combining (3.4.24) and (3.4.21) and then using (3.4.19) we obtain

$$\operatorname*{osc}_{Q^+_{\eta r}} \left(v(X,t) - \bar{R}(X,t)\right) \leq (1-\theta)\ \eta^2\ \operatorname*{osc}_{Q^+_r} \left(u(X,t) - R_{k_0}(X,t)\right) \leq (1-\theta)\ \eta^2\ \bar{C}Kr^{2+\alpha}. \tag{3.4.25}$$

Now to return to u we define $w = u - R_{k_0} - v$. Note that

$$D^2(R_{k_0}+v) = D_{k_0} + D^2 v, \quad (R_{k_0}+v)_t = \Gamma_{k_0} + v_t$$

hence, $F(D^2(R_{k_0}+v)) - (R_{k_0}+v)_t = F(D_{k_0}+D^2v) - \Gamma_{k_0} - v_t = 0$. Then from Theorem 4.6 of [45] and ***(iii)*** of Proposition 15 we have that $w \in S_p\left(\frac{\lambda}{n},\Lambda\right)$. Moreover we can easily check that $DR_{k_0} = D_{k_0}X + B_{k_0}$, then on Q^*_r, $\beta\cdot DR_{k_0} = \beta\cdot D_{k_0}X + \beta\cdot B_{k_0} = \sum_{j=1}^n \beta_j \sum_{k=1}^{n-1}(D_{k_0})_{jk}x_k \ + \ 0 = \sum_{k=1}^{n-1}\sum_{j=1}^n \beta_j (D_{k_0})_{jk}x_k = 0$. That is combining the above we have

$$\begin{cases} w \in S_p\left(\frac{\lambda}{n},\Lambda\right), & \text{in } \ Q^+_r \\ \beta\cdot Dw = g, & \text{on } \ Q^*_r \\ w = 0, & \text{on } \ \partial_p Q^+_r \setminus Q^*_r. \end{cases}$$

Next we apply again ABP-estimate for the oblique derivative case (Theorem 28) and then the $H^{1+\gamma}$-estimate for g together with the fact that $g(0,0)=0$ and $Dg(0,0)=0$ to obtain

$$\operatorname*{osc}_{Q^+_r} w \leq Cr||g||_{L^\infty(Q^*_r)} = Cr||g - g(0,0) - Dg(0,0)||_{L^\infty(Q^*_r)} \leq CKrr^{1+\gamma} \leq CKr^{2+\gamma}.$$

Next combining the above with (3.4.24) we get

$$\operatorname*{osc}_{Q^+_{\eta r}} \left[u(X,t) - (R_{k_0}(X,t) + \bar{R}(X,t))\right] \leq (1-\theta)\eta^2\bar{C}Kr^{2+\alpha} + CKr^{2+\gamma}. \tag{3.4.26}$$

Now, recall that our aim is to get relation (3.4.20). We choose the right constants α_0 and $\bar{C}$ for this purpose. So, take α_0 so that $\eta^{\alpha_0} = 1 - \frac{\theta}{2}$ and $\alpha = \min\{\alpha_0,\gamma\}$ and $\bar{C}$ large enough so that $\frac{\eta^2\theta\bar{C}}{2} \geq C$ (note that our choices are all independent of k_0). Then we return to (3.4.26) writing $1-\theta$ as $1-\frac{\theta}{2}-\frac{\theta}{2}$ and recalling that $r=\eta^{k_0}$,

$$\operatorname*{osc}_{Q^+_{\eta^{k_0+1}}}\left[u(X,t)-(R_{k_0}(X,t)+\bar{R}(X,t))\right]\leq K\left[\left(1-\frac{\theta}{2}\right)\bar{C}\eta^2\eta^{k_0(2+\alpha)}+Cr^{2+\alpha}-\frac{\eta^2\theta\bar{C}}{2}r^{2+\alpha}\right]$$
$$\leq\bar{C}K\eta^{(k_0+1)(2+\alpha)}.$$

This is relation (3.4.20) for $R_{k_0+1}=R_{k_0}+\bar{R}$. Note also that $F(D_{k_0}+\bar{D})-(\Gamma_{k_0}+\bar{\Gamma})=0$, $\beta\cdot B_{k_0+1}=\beta\cdot B_{k_0}+\beta\cdot\bar{B}=0$ and for any $i=1,\dots n-1$, $\sum_{j=1}^n(D_{k_0+1})_{ij}\beta_j=\sum_{j=1}^n(D_{k_0})_{ij}\beta_j+\sum_{j=1}^n(\bar{D})_{ij}\beta_j=0$. It remains to get (3.4.18) for $k=k_0$. To do so, we use relation (3.4.23) together with (3.4.21) and then (3.4.19),

$$|B_{k_0+1}-B_{k_0}|=|\bar{B}|\leq\frac{C}{r}\bar{C}Kr^{2+\alpha}\leq CKr^{\alpha}=CK\eta^{k_0(1+\alpha)}$$

$$|D_{k_0+1}-D_{k_0}|=|\bar{D}|\leq\frac{C}{r^2}\bar{C}Kr^{2+\alpha}\leq CKr^{\alpha}=CK\eta^{k_0\alpha}$$

$$|\Gamma_{k_0+1}-\Gamma_{k_0}|=|\bar{\Gamma}|\leq\frac{C}{r^2}\bar{C}Kr^{2+\alpha}\leq CKr^{\alpha}=CK\eta^{k_0\alpha}$$

as we want. So the inductive proof is completed.

Finally, it remains to get estimate (3.4.15). Observe that

$$\lim_{k\to\infty}|B_{k+1}-B_k|\leq\lim_{k\to\infty}CK\eta^{k(1+\alpha)}=0,\ \lim_{k\to\infty}|\Gamma_{k+1}-\Gamma_k|\leq\lim_{k\to\infty}CK\eta^{k\alpha}=0$$

$$\lim_{k\to\infty}|D_{k+1}-D_k|\leq\lim_{k\to\infty}CK\eta^{k\alpha}=0,\quad\text{since}\quad\eta<1.$$

That is the limits $B_\infty:=\lim_{k\to\infty}B_k$, $\Gamma_\infty:=\lim_{k\to\infty}\Gamma_k$ and $D_\infty:=\lim_{k\to\infty}D_k$ exist and $R^0(X,t)=B_\infty\cdot X+\Gamma_\infty t+\frac{1}{2}X^\tau D_\infty X$ satisfies (3.4.15). Indeed, $\beta\cdot B_\infty=0$, $F(D_\infty)-\Gamma_\infty=0$ and for any $k\in\mathbb{N}$ we have

$$\operatorname*{osc}_{Q^+_{\eta^k}}\left(u(X,t)-R^0(X,t)\right)\leq\operatorname*{osc}_{Q^+_{\eta^k}}\left(u(X,t)-R^0(X,t)\right)$$
$$+\eta^k|B_k-B_\infty|+\eta^{2k}|\Gamma_k-\Gamma_\infty|+\frac{1}{2}\eta^{2k}||D_k-D_\infty||$$

here we wrote $u(X,t)-R^0(X,t)=u(X,t)-R_k(X,t)+R_k(X,t)-R^0(X,t)\leq u(X,t)-R_k(X,t)+|B_k-B_\infty|\ |X|+|\Gamma_\infty-\Gamma_k||t|+\frac{1}{2}||D_\infty-D_k|||X|^2$ and $|X|\leq\eta^k$, $|t|\leq\eta^{2k}$.

We have

$$\operatorname*{osc}_{Q^+_{\eta^k}} \left(u(X,t) - R^0(X,t)\right) \le \bar{C}K\eta^{k(2+\alpha)} + CK\eta^k \sum_{j=k}^{\infty} \eta^{j(1+\alpha)} + 2CK\eta^{2k} \sum_{j=k}^{\infty} \eta^{j\alpha}$$
$$\le CK\eta^{k(2+\alpha)}$$

using the sum of geometric series and the proof is completed. □

Lemma 65. *Let β be constant function, $N_0 \in \mathbb{R}^{n\times n}$ with $||N_0||_\infty \le C_1$ and $u \in C\left(Q_1^+ \cup Q_1^*\right)$ be bounded and satisfies in the viscosity sense*

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in } \ Q_1^+ \\ (\beta + N_0 X) \cdot Du = 0, & \text{on } \ Q_1^*. \end{cases}$$

Then the second derivatives of u and u_t exist at $(0,0)$. Moreover there exists universal constant $0 < \alpha < 1$ so that u is punctually $H^{2+\alpha}$ at $(0,0)$. More precisely, there exists a polynomial $R_{2;0}$ of second order in X and of first order in t, that is $R_{2;0}(X,t) = A^0 + B^0 \cdot X + \Gamma^0 t + \frac{1}{2} X^\tau D^0 X$, where $A^0 = u(0,0), B^0 = Du(0,0) \in \mathbb{R}^n, \Gamma^0 = u_t(0,0)$ and $D^0 = D^2u(0,0) \in S_n$ so that

$$|u(X,t) - R_{2;0}(X,t)| \le C||u||_{L^\infty\left(Q_1^+\right)} \left(|X| + |t|^{1/2}\right)^{2+\alpha} \tag{3.4.27}$$

for every $(X,t) \in \overline{Q}^+_{1/4}$, where $C > 0$ depends on universal constants and on C_1.

Before we continue with the proof let us make a useful remark.

Remark 66. *Assume that for any $(X_0,t_0) \in Q^*_{1/2}$ we have that*

$$|u(X,t) - u(X_0,t_0) - Du(X_0,t_0)\cdot(X - X_0)| \le C^* \left(|X - X_0| + |t - t_0|^{1/2}\right)^{1+\alpha}$$

for every $(X,t) \in Q^+_{1/4}(X_0,t_0)$. Then

$$|(Du(X,t) - Du(0,0)) \cdot X| \le C^* \left(|X| + |t|^{1/2}\right)^{1+\alpha}$$

for every $(X,t) \in Q^*_{1/4}$.

Indeed, we take initially $(X_0,t_0) = (0,0)$, then

$$|u(X,t) - u(0,0) - Du(0,0)\cdot X| \le C^* \left(|X| + |t|^{1/2}\right)^{1+\alpha} \tag{3.4.28}$$

for every $(X,t) \in Q^+_{1/4}$. Then we take (X_0, t_0) to be any point (X,t) of $Q^*_{1/4}$, then

$$|u(Z,s) - u(X,t) - Du(X,t)\cdot(Z-X)| \le C^* \left(|Z-X| + |s-t|^{1/2}\right)^{1+\alpha}$$

for every $(Z,s) \in Q^+_{1/4}(X,t)$ and choose $(Z,s) = (0,0) \in Q^+_{1/4}(X,t)$, then

$$|u(0,0) - u(X,t) + Du(X,t)\cdot X| \le C^* \left(|X| + |t|^{1/2}\right)^{1+\alpha}. \tag{3.4.29}$$

Combining now twice (3.4.28) and (3.4.29) we get

$$(Du(X,t) - Du(0,0))\cdot X \le C^* \left(|X| + |t|^{1/2}\right)^{1+\alpha}$$

and

$$((Du(0,0) - Du(X,t))\cdot X \le C^* \left(|X| + |t|^{1/2}\right)^{1+\alpha}$$

which gives the result.

Now, we proceed with the proof of Lemma 65.

Proof. Our intention here is to "convert" our problem into a constant non-homogeneous oblique derivative problem in order to use the result of Lemma 64. To do so we add to u a suitable paraboloid. Note that u satisfies $H^{1+\alpha}$-estimates locally up to the flat boundary and $H^{2+\alpha}$-interior estimates so it is in fact a classical solution.

First we choose $N \in S_n$ so that $N\beta = N_0^\tau Du(0,0)$. Note that such a matrix exists since the above is actually a linear system of n equations and $\frac{n(n+1)}{2}$ variables and the matrix of the system can be shown to have rank equals to n (using that $\beta_n \neq 0$), that is the system has infinitely many solutions. Moreover $||N||_\infty \le C(n,\delta_0)|Du(0,0)|$. Then we define $v(X,t) := u(X,t) + \frac{1}{2}X^\tau N X$. Note that

$$Dv(X,t) = Du(X,t) + NX,\ \ D^2v(X,t) = D^2u(X,t) + N,\ \ v_t(X,t) = u_t(X,t).$$

Then

$$F(D^2v - N) - v_t = 0, \quad \text{in} \quad Q_1^+.$$

Also, for $X \in Q_1^*$,

$$\beta\cdot Dv(X,t) = -N_0X\cdot Du(X,t) + \beta\cdot NX = -(Du(X,t))^\tau N_0 X + (NX)^\tau\beta$$

and since N is symmetric $(NX)^\tau = X^\tau N$, then

$$\begin{aligned}\beta \cdot Dv(X,t) &= -(Du(X,t))^\tau N_0 X + X^\tau N_0^\tau Du(0,0)\\ &= -(Du(X,t))^\tau N_0 X + (Du(0,0))^\tau N_0 X\\ &= (Du(0,0) - Du(x,0,t))^\tau N_0(x,0) =: g(x,t).\end{aligned}$$

We observe also that $v(0,0) = u(0,0) = 0$, $G(M) := F(M-N)$ has the same ellipticity constants as F and

$$||g||_{L^\infty(Q_r^*)} \leq ||N_0||_\infty r ||Du(0,0) - Du(x,0,t)||_{L^\infty(Q_r^*)} \leq C||u||_{L^\infty\left(Q_1^+\right)} r^{1+\alpha}$$

using the $H^{1+\alpha}$-estimates that u satisfies (Theorem 54) and Remark 66.

Therefore we can apply Lemma 64 to v to obtain that there exists $\bar{R}(X,t) = \bar{B} \cdot X + \bar{\Gamma} t + \frac{1}{2} X^\tau \bar{D} X$ so that

$$||v - \bar{R}||_{L^\infty\left(Q_r^+\right)} \leq C\left(||v||_{L^\infty\left(Q_1^+\right)} + ||g||_{H^{1+\alpha}\left(Q_{1/2}^+\right)} + |F(-N)|\right) r^{2+\alpha} \leq C||u||_{L^\infty\left(Q_1^+\right)} r^{2+\alpha}$$

for any $r \leq \frac{1}{4}$. Taking as $R^0(X,t) := \bar{R}(X,t) + \frac{1}{2}X^\tau N X$ the proof is complete. □

Proof of Theorem 62. Before we start let us denote for convenience $K := ||u||_{L^\infty\left(Q_1^+\right)} + ||g||_{H^{1+\gamma}\left(\overline{Q}_{1/2}^*\right)} + ||f||_{H^\gamma\left(Q_1^+\right)}$ and $\beta^0 := \beta(0,0)$, $\beta_{x_i}^0 := \beta_{x_i}(0,0) \in \mathbb{R}^n$. We intend to find some $R^0(X,t) = B^0 \cdot X + \Gamma^0 t + \frac{1}{2} X^\tau D^0 X$, with $\beta^0 \cdot B^0 = 0$ and $F(D^0) - \Gamma^0 = 0$ so that for universal $C > 0, 0 < \eta < 1, 0 < \rho < 1, \alpha_0 > 0$ and $\alpha = \min\{\alpha_0, \gamma\}$ we have

$$\operatorname*{osc}_{Q_{\rho\eta^k}^+} \left(u(X,t) - R^0(X,t)\right) \leq CK\eta^{k(2+\alpha)}, \quad \text{for any} \quad k \in \mathbb{N}. \tag{3.4.30}$$

Similarly as we explain in the proof of Theorem 49, estimate (3.4.30) is enough.

Now, to get (3.4.30) we show by induction that there exist universal constants $0 < \eta << 1, 0 < \rho << 1, \bar{C} > 0, \alpha_0 > 0$ such that for $\alpha = \min\{\alpha_0, \gamma\}$ we can find a paraboloid $R_k(X,t) = B_k \cdot X + \Gamma_k t + \frac{1}{2} X^\tau D_k X$, with

$$\begin{aligned}&F(D_k) - \Gamma_k = 0, \quad \beta^0 \cdot B_k = 0 \quad \text{and}\\ &\sum_{j=1}^{n} \left[(D_k)_{ij}\beta_j^0 + (\beta_j)_{x_i}^0 (B_k)_j\right] = 0, \ i = 1, \dots, n-1\end{aligned} \tag{3.4.31}$$

for any $k \in \mathbb{N}$ so that

$$\operatorname*{osc}_{Q^+_{\rho\eta^k}} (u(X,t) - R_k(X,t)) \leq \bar{C}K\eta^{k(2+\alpha)} \tag{3.4.32}$$

and

$$||D_{k+1} - D_k|| \leq CK\eta^{k\alpha},\ |\Gamma_{k+1} - \Gamma_k| \leq CK\eta^{k\alpha},\ |B_{k+1} - B_k| \leq CK\eta^{k(1+\alpha)}. \tag{3.4.33}$$

First, for $k = 0$, take $B_0 = 0$, $\Gamma_0 = 0$ and $(D_0)_{ij} = 0$, for $ij \neq nn$ and $(D_0)_{nn} = \tau_0$ where τ_0 is chosen so that $F(D_0) = 0$ (see Remark 63) and we see that

$$\operatorname*{osc}_{Q^+_1} (u(X,t) - R_0(X,t)) \leq 2||u||_{L^\infty(Q^+_1)} + 2||D_0||_\infty \leq \bar{C}K$$

choosing $\bar{C}$ large enough.

Next for the induction we assume that we have found paraboloids $R_0, R_1, \ldots, R_{k_0}$ for which (3.4.31), (3.4.32) and (3.4.33) hold. Denoting by $r := \frac{\rho\eta^{k_0}}{2}$ we have

$$\operatorname*{osc}_{Q^+_r} (u(X,t) - R_{k_0}(X,t)) \leq \frac{4\bar{C}}{\rho^{2+\alpha}} K r^{(2+\alpha)} \tag{3.4.34}$$

and we want to find a paraboloid R_{k_0+1} satisfying (3.4.31), (3.4.33) and

$$\operatorname*{osc}_{Q^+_{2\eta r}} (u(X,t) - R_{k_0+1}(X,t)) \leq \bar{C}K\eta^{(k_0+1)(2+\alpha)}. \tag{3.4.35}$$

Now we consider a suitable oblique derivative problem (as the one of Lemma 65). So let v be the viscosity solution of

$$\begin{cases} G(D^2v) - v_t = 0, & \text{in } Q^+_r \\ (\beta^0 + D\beta^0 x) \cdot Dv = 0, & \text{on } Q^*_r \\ v = u - R_{k_0}, & \text{on } \partial_p Q^+_r \setminus Q^*_r \end{cases}$$

where $G(M) = F(M + D_{k_0}) - \Gamma_{k_0}$ which is an elliptic operator with the same ellipticity constants as F. Note that $G(O) = F(D_{k_0}) - \Gamma_{k_0} = 0$. Also by $D\beta^0$ we denote the matrix

$$D\beta^0 := \begin{pmatrix} (\beta_1)_{x_1}(0,0) & \dots & (\beta_1)_{x_{n-1}}(0,0) \\ & \ddots & \\ (\beta_n)_{x_1}(0,0) & \dots & (\beta_n)_{x_{n-1}}(0,0) \end{pmatrix} \in \mathbb{R}^{n\times(n-1)}.$$

Then v satisfies the following

- ABP-estimate for the oblique derivative case (see Theorem 28) gives

$$\operatorname*{osc}_{Q_r^+} v \leq \operatorname*{osc}_{Q_r^+} (u(X,t) - R_{k_0}(X,t)) \tag{3.4.36}$$

  since $(\beta^0 + D\beta^0 x)\cdot Dv = 0$ and $G(O) = 0$.

- From Lemma 65 we have that $\bar{B} := Dv(0,0)$, $\bar{\Gamma} := v_t(0,0)$, $\bar{D} := D^2v(0,0)$ exist and for $\bar{R}(X,t) = \bar{B}\cdot X + \bar{\Gamma}t + \frac{1}{2}X^\tau \bar{D}X$ we have

$$\operatorname*{osc}_{Q_{\tilde{r}}^+} \left(v(X,t) - \bar{R}(X,t)\right) \leq C_0 \left(\frac{\tilde{r}}{r}\right)^{2+\alpha_1} \operatorname*{osc}_{Q_r^+} v \tag{3.4.37}$$

  for any $\tilde{r} \leq \frac{r}{4}$ and also

$$|\bar{B}| \leq \frac{C}{r}\operatorname*{osc}_{Q_r^+} v,\ \ |\bar{\Gamma}| \leq \frac{C}{r^2}\operatorname*{osc}_{Q_r^+} v,\ \ ||\bar{D}||_\infty \leq \frac{C}{r^2}\operatorname*{osc}_{Q_r^+} v. \tag{3.4.38}$$

  Note that $(\beta^0 + D\beta^0\ 0)\cdot\bar{B} = 0$ from the oblique derivative condition, that is $\beta^0\cdot\bar{B} = 0$ and $F(\bar{D} + D_{k_0}) - \Gamma_{k_0} - \bar{\Gamma} = 0$ using the continuity of F, D^2v and v_t. Also, $(\beta^0 + D\beta^0 x)\cdot Dv = 0$ holds in the classical sense on Q_r^* and we can differentiate this condition with respect to x_i, for any $i = 1,\dots,n-1$ to get at $(x,t) = (0,0)$, $\sum_{j=1}^n \left[\bar{D}_{ij}\beta_j^0 + (\beta_j)_{x_i}^0 \bar{B}_j\right] = 0$, for any $i = 1,\dots,n-1$.

Next, we take $\tilde{r} = 2\eta r$ (for $0 < \eta < \frac{1}{8}$) in (3.4.37). Hence

$$\operatorname*{osc}_{Q_{2\eta r}^+} \left(v(X,t) - \bar{R}(X,t)\right) \leq C_0\eta^{2+\alpha_1} \operatorname*{osc}_{Q_r^+} v. \tag{3.4.39}$$

Now take (universal) $0 < \eta << 1$ sufficiently small in order to have that $8\ C_0\eta^{\alpha_1} < 1$. We denote by $1-\theta := 8\ C_0\eta^{\alpha_1}$, where $0 < \theta < 1$ is a universal constant. Then combining (3.4.39) and (3.4.36) and then using (3.4.34) we obtain

$$\operatorname*{osc}_{Q^+_{2\eta r}} \left(v(X,t)-\bar{R}(X,t)\right) \le \frac{(1-\theta)}{8}\,\eta^2 \operatorname*{osc}_{Q^+_r}\left(u(X,t)-R_{k_0}(X,t)\right) \le \frac{(1-\theta)}{2}\,\eta^2\,\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}. \tag{3.4.40}$$

Now to return to u we define $w = u - R_{k_0} - v$. Note that

$$D^2(R_{k_0}+v) = D_{k_0} + D^2 v, \quad (R_{k_0}+v)_t = \Gamma_{k_0} + v_t$$

hence, $F(D^2(R_{k_0}+v))-(R_{k_0}+v)_t = F(D_{k_0}+D^2v)-\Gamma_{k_0}-v_t = 0$. Then $w \in S_p\left(\frac{\lambda}{n},\Lambda,f\right)$ in Q^+_r. Moreover we can easily check that $DR_{k_0} = D_{k_0}X + B_{k_0}$. That is, w satisfies

$$\begin{cases} w \in S_p\left(\frac{\lambda}{n},\Lambda,f\right), & \text{in } \; Q^+_r \\ \beta\cdot Dw = g - \beta\cdot\left(D_{k_0}X + B_{k_0} + Dv\right), & \text{on } \; Q^*_r \\ w = 0, & \text{on } \; \partial_p Q^+_r \setminus Q^*_r. \end{cases}$$

Now for $0<\mu<1$ (to be chosen universal) we denote by $\bar{r} := r(1-\mu) < r$. We apply again ABP-estimate for the oblique derivative case (Theorem 28)

$$\begin{aligned} \operatorname*{osc}_{Q^+_{\bar r}} w &\le Cr||f||_{L^{n+1}(Q^+_r)} + Cr||g||_{L^\infty(Q^*_{\bar r})} + Cr||\beta\cdot(D_{k_0}X+B_{k_0})||_{L^\infty(Q^*_{\bar r})} \\ &\quad + Cr||\beta\cdot Dv||_{L^\infty(Q^*_{\bar r})} + \operatorname*{osc}_{\partial_p Q^+_{\bar r}\setminus Q^*_{\bar r}} w \\ &=: \mathbf{I}+\mathbf{II}+\mathbf{III}+\mathbf{IV}+\mathbf{V}. \end{aligned} \tag{3.4.41}$$

We want to bound every term **I - V** by a term of order $r^{2+\alpha}$. We start with term **I**. We have

$$\mathbf{I} \le Cr||f||_{L^\infty(Q^+_r)}\left(\int_{Q^+_r} 1\; dX\; dt\right)^{\frac{1}{n+1}} \le Cr^2||f||_{L^\infty(Q^+_r)}$$

then using the H^γ regularity of f and the fact that $f(0,0)=0$ we get

$$\mathbf{I} \le Cr^2||f-f(0,0)||_{L^\infty(Q^+_r)} \le CKr^{2+\gamma}.$$

Next, for term **II**, we use the $H^{1+\gamma}$-regularity of g and the fact that $g(0,0)=0$,

$Dg(0,0,)=0$, then

$$\mathbf{II} = Cr||g - g(0,0) - Dg(0,0)\cdot x||_{L^\infty(Q_r^*)} \le Crr^{1+\gamma}K \le CKr^{2+\gamma}.$$

We continue with **III** and we study first the term

$$\mathbf{A} := (\beta^0 + D\beta^0 x)\cdot(D_{k_0}X + B_{k_0}) = \beta^0\cdot D_{k_0}X + D\beta^0 x\cdot D_{k_0}X + D\beta^0 x\cdot B_{k_0}$$

since $\beta^0\cdot B_{k_0}=0$ and

$$\beta^0\cdot D_{k_0}X = \sum_{i=1}^{n}\beta_i^0\sum_{k=1}^{n-1}(D_{k_0})_{ik}x_k$$

and

$$D\beta^0 x\cdot B_{k_0} = \sum_{i=1}^{n}\sum_{k=1}^{n-1}(\beta_i)^0_{x_k}x_k(B_{k_0})_i$$

that is,

$$\beta^0\cdot D_{k_0}X + D\beta^0 x\cdot B_{k_0} = 0.$$

Hence, $\mathbf{A} = D\beta^0 x\cdot D_{k_0}X$. Returning to **III**, we have

$$\begin{aligned}\mathbf{III} &\le Cr||\beta - \beta^0 - D\beta^0 x||_{L^\infty(Q_r^*)}||D_{k_0}X + B_{k_0}||_{L^\infty(Q_r^*)}\\ &\quad + Cr||(\beta^0 + D\beta^0 x)\cdot(D_{k_0}X + B_{k_0})||_{L^\infty(Q_r^*)}\\ &\le Crr^{1+\gamma}(||D_{k_0}||_\infty + |B_{k_0}|) + Crr^2||D\beta^0||_\infty||D_{k_0}||_\infty\end{aligned}$$

using the $H^{1+\gamma}$-regularity of β. Note also that $|B_{k_0}|\le CK$ and $||D_{k_0}||_\infty \le CK$. Indeed, since $|B_0|=0$, we have $|B| = |B_{k_0} - B_0| \le \sum_{k=0}^{k_0-1}|B_{k+1}-B_k| \le CK\sum_{k=0}^{k_0-1}(\eta^{1+\alpha})^k \le CK\left(\frac{1-(\eta^{1+\alpha})^{k_0}}{1-\eta^{1+\alpha}}\right) \le CK\frac{1}{1-\eta^{1+\alpha}} \le CK$ and similarly $||D_{k_0}||_\infty = ||D_{k_0}-D_0||_\infty \le \sum_{k=0}^{k_0-1}||D_{k+1}-D_k||_\infty \le CK\sum_{k=0}^{k_0-1}(\eta^\alpha)^k \le CK\left(\frac{1-(\eta^\alpha)^{k_0}}{1-\eta^\alpha}\right) \le CK$, then $||D_{k_0}||_\infty \le CK + ||D_0||_\infty \le CK$. Then $\mathbf{III}\le CKr^{2+\gamma}$.

Next for term **IV**, we use again the $H^{1+\gamma}$-regularity of β and the fact that $(\beta^0 + D\beta^0 x)\cdot Dv = 0$ on Q_r^*, we have

$$\mathbf{IV} \le Cr||\beta - \beta^0 - D\beta^0 x||_{L^\infty(Q_r^*)}||Dv||_{L^\infty(Q_{\bar r}^*)} \le C\rho^{1+\gamma}\frac{\bar C}{\rho^{2+\alpha}}Kr^{2+\alpha}$$

using Theorem 53 for v. Finally we examine term **V**. Let $(X_0,t_0)\in\partial_p Q_{\bar r}^+\setminus Q_{\bar r}^*$.

- If $|X_0| = \bar r$ we choose $\bar X_0 \in (\partial B_r)^+$ so that $|X_0 - \bar X_0| = \mu r \le \sqrt{2\mu}r$ and $\bar t_0 = t_0$.

- If $|X_0| < \bar{r}$ then $t_0 = -(1-\mu)^2 r^2$ and we choose $\bar{t}_0 = -r^2$ then $|t_0 - \bar{t}_0|^{1/2} = r\sqrt{\mu(2-\mu)} \leq \sqrt{2\mu}r$ and $\bar{X}_0 = X_0$.

In any case $|X_0 - \bar{X}_0| + |t_0 - \bar{t}_0|^{1/2} \leq \sqrt{2\mu}r$ and $(\bar{X}_0, \bar{t}_0) \in \partial_p Q_r^+ \backslash Q_r^*$ that is $w\left(\bar{X}_0, \bar{t}_0\right) = 0$. Then

$$\begin{aligned} |w\left(X_0, t_0\right)| &= |w\left(X_0, t_0\right) - w\left(\bar{X}_0, \bar{t}_0\right)| \\ &\leq |\left(u\left(X_0, t_0\right) - R_{k_0}(X_0, t_0)\right) - \left(u\left(\bar{X}_0, \bar{t}_0\right) - R_{k_0}(\bar{X}_0, \bar{t}_0)\right)| \\ &\quad + |v\left(X_0, t_0\right) - v\left(\bar{X}_0, \bar{t}_0\right)| \end{aligned} \tag{3.4.42}$$

and we bound these terms using H^α-estimates. Indeed, we have that

$$\begin{cases} F(D^2(u - R_{k_0}) + D_{k_0}) - \Gamma_{k_0} - (u - R_{k_0})_t = f, & \text{in } \ Q_{2r}^+ \\ \beta \cdot D(u - R_{k_0}) = g - \beta \cdot (D_{k_0}X + B_{k_0}), & \text{on } \ Q_{2r}^*. \end{cases}$$

Recall that $G(M) = F(M + D_{k_0}) - \Gamma_{k_0}$ has the same ellipticity constants as F and $G(O) = 0$. Then Corollary 38 gives

$$\begin{aligned} ||u - R_{k_0}||_{H^{\alpha_2}\left(\overline{Q}_r^+\right)} \leq \frac{C}{r^{\alpha_2}} &\left(||u - R_{k_0}||_{L^\infty\left(Q_{2r}^+\right)} + r^{\frac{n}{n+1}}||f||_{L^{n+1}\left(Q_{2r}^+\right)}\right) \\ &+ \frac{C}{r^{\alpha_2}}\left(r||g||_{L^\infty\left(Q_{2r}^*\right)} + r||\beta \cdot (D_{k_0}X + B_{k_0})||_{L^\infty\left(Q_{2r}^*\right)}\right). \end{aligned}$$

Next we apply to v global H^α-estimates. Note that the values of v on the parabolic boundary equal to $u - R_{k_0}$ which is H^{α_2}. So, for $0 < \alpha_3 << \alpha_2$ universal, we have

$$\begin{aligned} ||v||_{H^{\alpha_3}\left(\overline{Q}_r^+\right)} &\leq \frac{C}{r^{\alpha_3}} \left(||v||_{L^\infty\left(Q_r^+\right)} + r^{\alpha_2}||u - R_{k_0}||_{H^{\alpha_2}\left(\overline{Q}_r^+\right)}\right) \\ &\leq \frac{C}{r^{\alpha_3}} \ ||u - R_{k_0}||_{L^\infty\left(Q_{2r}^+\right)} \\ &\quad + \frac{C}{r^{\alpha_3}}\left(r^{\frac{n}{n+1}}||f||_{L^{n+1}\left(Q_{2r}^+\right)} + r||g||_{L^\infty\left(Q_{2r}^*\right)} + r||\beta \cdot (D_{k_0}X + B_{k_0})||_{L^\infty\left(Q_{2r}^*\right)}\right). \end{aligned}$$

Now, we return to (3.4.42) and we have that $|w(X_0,t_0)|$ is bounded by

$$
\begin{aligned}
&\left(|X_0-\bar{X}_0|+|t_0-\bar{t}_0|^{1/2}\right)^{\alpha_2}\ \frac{C}{r^{\alpha_2}}\left(||u-R_{k_0}||_{L^\infty(Q_{2r}^+)}+r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q_{2r}^+)}\right)\\
&\quad+\left(|X_0-\bar{X}_0|+|t_0-\bar{t}_0|^{1/2}\right)^{\alpha_2}\frac{C}{r^{\alpha_2}}\left(r||g||_{L^\infty(Q_{2r}^*)}+r||\beta\cdot(D_{k_0}X+B_{k_0})||_{L^\infty(Q_{2r}^*)}\right)\\
&\quad+\left(|X_0-\bar{X}_0|+|t_0-\bar{t}_0|^{1/2}\right)^{\alpha_3}\ \frac{C}{r^{\alpha_3}}\left(||u-R_{k_0}||_{L^\infty(Q_{2r}^+)}+r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q_{2r}^+)}\right)\\
&\quad+\left(|X_0-\bar{X}_0|+|t_0-\bar{t}_0|^{1/2}\right)^{\alpha_3}\ \frac{C}{r^{\alpha_3}}\left(r||g||_{L^\infty(Q_{2r}^*)}+r||\beta\cdot(D_{k_0}X+B_{k_0})||_{L^\infty(Q_{2r}^*)}\right)\\
&\le C\mu^{\alpha_3/2}\ \ ||u-R_{k_0}||_{L^\infty(Q_{2r}^+)}\\
&\quad+C\mu^{\alpha_3/2}\left(r^{\frac{n}{n+1}}||f||_{L^{n+1}(Q_{2r}^+)}+r||g||_{L^\infty(Q_{2r}^*)}+r||\beta\cdot(D_{k_0}X+B_{k_0})||_{L^\infty(Q_{2r}^*)}\right)\\
&\le \mathbf{VI}+\mathbf{I}'+\mathbf{II}'+\mathbf{III}'.
\end{aligned}
$$

For term **VI**, we use the hypothesis of the induction, (3.4.34), then $\mathbf{VI}\le C_1\mu^{\alpha_3/2}\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}$. Moreover for term $\mathbf{I}'$, we have

$$
\begin{aligned}
\mathbf{I}'\le Cr^{\frac{n}{n+1}}||f||_{L^\infty(Q_r^+)}\ \left(\int_{Q_r^+}1\ dX\ dt\right)^{\frac{1}{n+1}} &\le Cr^{\frac{n}{n+1}}||f||_{L^\infty(Q_r^+)}\ C(n)r^{\frac{n+2}{n+1}}\\
&=Cr^2||f||_{L^\infty(Q_r^+)}
\end{aligned}
$$

then using the H^γ regularity of f and the fact that $f(0,0)=0$ we get

$$
\mathbf{I}'\le Cr^2||f-f(0,0)||_{L^\infty(Q_r^+)}\le CKr^{2+\gamma}.
$$

Also, terms $\mathbf{II}'$ and $\mathbf{III}'$ are in fact the same as terms **II** and **III**. That is,

$$
\mathbf{V}\le C_1\mu^{\alpha_3/2}\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}+C\mu^{\alpha_3/2}Kr^{2+\gamma}+C_2\rho^{1+\gamma}\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}.
$$

So, returning to (3.4.41), we have

$$
\operatorname*{osc}_{Q_{\bar{r}}^+} w\le CKr^{2+\gamma}+C_1\mu^{\alpha_3/2}\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}+C_2\rho^{1+\gamma}\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}.
$$

Next combining the above with (3.4.40) and choosing $\mu<1-\eta$ (then $\eta<1-\mu$)

we get

$$\operatorname*{osc}_{Q^+_{\eta r}}\left[u(X,t)-(R_{k_0}+\bar{R})(X,t)\right] \le \frac{1}{2}(1-\theta)\eta^2\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}+CKr^{2+\gamma}$$
$$+C_1\mu^{\alpha_3/2}\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}+C_2\rho^{1+\gamma}\frac{\bar{C}}{\rho^{2+\alpha}}Kr^{2+\alpha}. \tag{3.4.43}$$

Now, recall that our aim is to get relation (3.4.35). We choose the right constants α_0, μ and $\bar{C}$ for this purpose. So, take α_0 so that $\eta^{\alpha_0}=1-\frac{\theta}{2}$ and $\alpha=\min\{\alpha_0,\gamma\}$. Take $\mu\le\frac{\eta^{\frac{2(2+\alpha)}{\alpha_3}}}{(4C_1)^{\frac{2}{\alpha_3}}}$ (then $\mu^{\alpha_3/2}\le\frac{\eta^{2+\alpha}}{4C_1}$), $\rho\le\frac{\eta^{\frac{2+\alpha}{1+\gamma}}}{(4C_2)^{\frac{1}{1+\gamma}}}$ (then $C_2\rho^{1+\gamma}\le\frac{\eta^{2+\alpha}}{4}$) and $\bar{C}$ large enough so that $\frac{\eta\theta\bar{C}}{4\rho^{2+\alpha}}\ge C$ (note that our choices are all independent of k_0). Then we return to (3.4.43) writing $1-\theta$ as $1-\frac{\theta}{2}-\frac{\theta}{2}$ and recalling that $r=\frac{\rho\eta^{k_0}}{2}$,

$$\operatorname*{osc}_{Q^+_{\rho\eta^{k_0+1}}}\left[u(X,t)-(R_{k_0}+\bar{R})(X,t)\right] \le \frac{1}{2}\left(1-\frac{\theta}{2}\right)\bar{C}\eta^2\eta^{k_0(2+\alpha)}K+Cr^{2+\alpha}K$$
$$-K\frac{\eta\theta\bar{C}}{4\rho^{2+\alpha}}r^{2+\alpha}+\bar{C}\frac{\eta^{2+\alpha}}{2}\eta^{k_0(2+\alpha)}$$
$$\le\bar{C}K\eta^{(k_0+1)(2+\alpha)}.$$

This is relation (3.4.35) for $R_{k_0+1}=R_{k_0}+\bar{R}$. Note also that $F(D_{k_0}+\bar{D})-(\Gamma_{k_0}+\bar{\Gamma})=0$, $\beta^0\cdot B_{k_0+1}=\beta^0\cdot B_{k_0}+\beta^0\cdot\bar{B}=0$ and for any $i=1,\dots n-1$,

$$\sum_{j=1}^n\left[(D_{k_0+1})_{ij}\beta^0_j+(\beta_j)^0_{x_i}(B_{k_0+1})_j\right]=\sum_{j=1}^n\left[(D_{k_0})_{ij}\beta_j+(\beta_j)^0_{x_i}(B_{k_0})_j\right]$$
$$+\sum_{j=1}^n\left[(\bar{D})_{ij}\beta_j+(\beta_j)^0_{x_i}(\bar{B})_j\right]=0.$$

It remains to get (3.4.33) for $k=k_0$. To do so, we use relation (3.4.38) together with (3.4.36) and then (3.4.34),

$$|B_{k_0+1}-B_{k_0}|=|\bar{B}|\le\frac{C}{r}\bar{C}Kr^{2+\alpha}\le CKr^{\alpha}=CK\eta^{k_0(1+\alpha)}$$

$$|D_{k_0+1}-D_{k_0}|=|\bar{D}|\le\frac{C}{r^2}\bar{C}Kr^{2+\alpha}\le CKr^{\alpha}=CK\eta^{k_0\alpha}$$

$$|\Gamma_{k_0+1}-\Gamma_{k_0}|=|\bar{\Gamma}|\le\frac{C}{r^2}\bar{C}Kr^{2+\alpha}\le CKr^{\alpha}=CK\eta^{k_0\alpha}$$

as we want. So the inductive proof is completed.

Finally, it remains to get estimate (3.4.30). Observe that

$$\lim_{k\to\infty}|B_{k+1}-B_k|\le\lim_{k\to\infty}CK\eta^{k(1+\alpha)}=0,\ \lim_{k\to\infty}|\Gamma_{k+1}-\Gamma_k|\le\lim_{k\to\infty}CK\eta^{k\alpha}=0$$

$$\lim_{k\to\infty}|D_{k+1}-D_k|\le\lim_{k\to\infty}CK\eta^{k\alpha}=0,\quad \text{since}\quad \eta<1.$$

That is the limits $B_\infty:=\lim_{k\to\infty}B_k$, $\Gamma_\infty:=\lim_{k\to\infty}\Gamma_k$ and $D_\infty:=\lim_{k\to\infty}D_k$ exist and $R^0(X,t)=B_\infty\cdot X+\Gamma_\infty t+\frac{1}{2}X^\tau D_\infty X$ satisfies (3.4.30). Indeed, $\beta^0\cdot B_\infty=0$, $F(D_\infty)-\Gamma_\infty=0$ and for any $k\in\mathbb{N}$ we have

$$\begin{aligned}\operatorname*{osc}_{Q^+_{\rho\eta^k}}\big(u(X,t)-R^0(X,t)\big)&\le\operatorname*{osc}_{Q^+_{\rho\eta^k}}\big(u(X,t)-R^0(X,t)\big)\\&\quad+\rho\eta^k|B_k-B_\infty|+\rho\eta^{2k}|\Gamma_k-\Gamma_\infty|+\frac{1}{2}\rho\eta^{2k}||D_k-D_\infty||\end{aligned}$$

here we wrote $u(X,t)-R^0(X,t)=u(X,t)-R_k(X,t)+R_k(X,t)-R^0(X,t)\le u(X,t)-R_k(X,t)+|B_k-B_\infty|\ |X|+|\Gamma_\infty-\Gamma_k||t|+\frac{1}{2}||D_\infty-D_k|||X|^2$. We have

$$\begin{aligned}\operatorname*{osc}_{Q^+_{\rho\eta^k}}\big(u(X,t)-R^0(X,t)\big)&\le\bar{C}K\eta^{k(2+\alpha)}+CK\eta^k\sum_{j=k}^{\infty}\eta^{j(1+\alpha)}+2CK\eta^{2k}\sum_{j=k}^{\infty}\eta^{j\alpha}\\&\le CK\eta^{k(2+\alpha)}\end{aligned}$$

using the sum of geometric series and the proof is completed. □

Chapter 4

Regularity Theory for the Fully Nonlinear Parabolic Thin Obstacle Problem

In the present chapter we intent to study the regularity of the viscosity solution of the following thin obstacle problem

$$\begin{cases} F(D^2u) - u_t = 0, & \text{in} \quad Q_1^+ \\ u_y \leq 0, & \text{on} \quad Q_1^* \\ u \geq \varphi, & \text{on} \quad Q_1^* \\ u_y = 0, & \text{on} \quad Q_1^* \cap \{u > \varphi\} \\ u = u_0, & \text{on} \quad \partial_p Q_1^+ \setminus Q_1^* \end{cases} \tag{4.0.1}$$

where, F is a uniformly elliptic operator on S_n with ellipticity constants λ and Λ and $\varphi : \overline{Q_1^*} \to \mathbb{R}$, $u_0 : \partial_p Q_1^+ \setminus Q_1^* \to \mathbb{R}$ are given functions with $\varphi \in H^{2+\alpha}(Q_1^*)$ the so-called obstacle and with $u_0 \geq \varphi$ on $\partial_p Q_1^*$ for compatibility reasons. Note that the conditions on Q_1^* can be written also as $\max\{u_y, \varphi - u\} = 0$.

We consider that the solution u of (4.0.1) can be recovered as the minimum viscosity supersolution of

$$\begin{cases} F(D^2v) - v_t \leq 0, & \text{in} \quad Q_1^+ \\ v_y \leq 0, & \text{on} \quad Q_1^* \\ v \geq \varphi, & \text{on} \quad Q_1^* \\ v \geq u_0, & \text{on} \quad \partial_p Q_1^+ \setminus Q_1^* \end{cases} \tag{4.0.2}$$

with u_t locally bounded by above in Q_1^+ (note that under suitable assumptions on F we have that u_t does exist in Q_1^+ once $F(D^2u) - u_t = 0$ in Q_1^+ in the viscosity sense).

The aim of this chapter is to prove that u lies in $H^{1+\alpha}$ up to the flat boundary Q_1^*. To do so we make the following assumptions on F and u_0.

- **Assumptions on** F. First we assume that F is convex on S_n so we have interior $H^{2+\alpha}$-estimates for the viscosity solutions. Moreover considering the following extension of F in $\mathbb{R}^{n\times n}$

$$F(M) = F\left(\frac{M+M^\tau}{2}\right), \quad \text{for} \quad M \in \mathbb{R}^{n\times n}$$

we assume that F is once continuously differentiable in $\mathbb{R}^{n^2}$ and we denote $F_{ij} := \frac{\partial F}{\partial m_{ij}}$. We can easily see that $F_{ij}(M) = F_{ji}(M)$ for any M, indeed let H^{ij} denote the matrix with elements

$$\left(H_h^{ij}\right)_{kl} = \begin{cases} 0, & \text{if} \quad k \neq i \quad \text{or} \quad l \neq j, \\ h, & \text{if} \quad k = i \quad \text{and} \quad l = j \end{cases}$$

where $h \in \mathbb{R}$ and observe that $\left(H_h^{ij}\right)^\tau = H_h^{ji}$ then

$$\begin{aligned} F_{ij}(M) &= \lim_{h\to 0} \frac{F\left(M + H_h^{ij}\right) - F(M)}{h} = \lim_{h\to 0} \frac{F\left(\frac{M+H_h^{ij}+\left(M+H_h^{ij}\right)^\tau}{2}\right) - F(M)}{h} \\ &= \lim_{h\to 0} \frac{F\left(\frac{M+M^\tau}{2} + \frac{\left(H_h^{ji}\right)^\tau + H_h^{ji}}{2}\right) - F(M)}{h} \\ &= \lim_{h\to 0} \frac{F\left(\frac{M+H_h^{ji}+\left(M+H_h^{ji}\right)^\tau}{2}\right) - F(M)}{h} = F_{ji}(M). \end{aligned}$$

We suppose also that $F_{in} = 0$, for any $i = 1, \ldots, n-1$ (then $F_{ni} = 0$ as well).

Finally, we assume for convenience that $F(O) = 0$, an assumption that can be easily removed (see the introduction of subsection 3.3.3).

- **Assumptions on u_0.** Note that we intend to examine the regularity up to flat boundary Q_1^* (and not up to $\partial_p Q_1^+ \setminus Q_1^*$) thus we may assume that $u_0 > \varphi$ on $\partial_p Q_1^*$. Therefore if $v \in C\left(\overline{Q_1^+}\right)$ is the viscosity solution of

$$\begin{cases} F(D^2 v) - v_t = 0, & \text{in} \quad Q_1^+ \\ v_y = 0, & \text{on} \quad Q_1^* \\ v = u_0, & \text{on} \quad \partial_p Q_1^+ \setminus Q_1^* \end{cases} \tag{4.0.3}$$

then due to the continuity of v and φ and the compactness of $\partial_p Q_1^*$ we see that there exists some $0 < \rho < 1$ so that $v > \varphi$ on $Q_1^* \setminus Q_{1-\rho}^*$. Then using the ABP-type estimate of Theorem 28 we get that $u > \varphi$ on $Q_1^* \setminus Q_{1-\rho}^*$ thus $u_y = 0$ on $Q_1^* \setminus Q_{1-\rho}^*$, in the viscosity sense.

We denote by $\Delta^* := \{(x,t) \in Q_1^* : u(x,0,t) = \varphi(x,t)\}$ the *contact set*, by $\Omega^* := \{(x,t) \in Q_1^* : u(x,0,t) > \varphi(x,t)\}$ the *non-contact set* and by $\Gamma = \partial\Delta^* \cap Q_1^*$ the *free boundary*. We assume that $\Delta^* \neq \emptyset$ since otherwise we would have a Neumann boundary value problem for which the regularity is known from the previous chapter. Note that around the points of the $\text{int}(\Delta^*)$ and around the points of Ω^* we can treat our problem as Dirichlet or Neumann problem respectively. That is, around these points the regularity is known. Finally, we denote by $K := ||u||_{L^\infty(Q_1^+)} + ||\varphi||_{H^{2+\alpha}(Q_1^*)}$ and in the following a constant $C > 0$ that depends only on K, n, λ, Λ and ρ will be called *universal*.

4.1 A penalized problem

We start with the study of the following penalized problem

$$\begin{cases} F\left(D^2 u^{(k)}\right) - \left(u^{(k)}\right)_t = 0, & \text{in} \quad Q_1^+ \\ \left(u^{(k)}\right)_y = -k\left(\varphi - u^{(k)}\right)^+ := g^{(k)}, & \text{on} \quad Q_1^* \\ u^{(k)} = u_0, & \text{on} \quad \partial_p Q_1^+ \setminus Q_1^*. \end{cases} \tag{4.1.1}$$

We observe that (4.1.1) is not an obstacle problem. Using ABP-estimate and a barrier argument we obtain independent of k estimates for $u^{(k)}$ and $g^{(k)}$ (Lemmata 67 and 68). Then we can treat (4.1.1) as a non-homogeneous Neumann problem and using suitable Hölder estimates of the previous chapter we obtain the uniform convergence of $u^{(k)}$ to u (Proposition 69) and the existence of $\left(u^{(k)}\right)_y$ in the classical sense (Lemma 71). Note also that for any $k \in \mathbb{N}$, comparing $u^{(k)}$ with the solution v of (4.0.3) we have that $u^{(k)} > \varphi$ on $Q_1^* \setminus Q_{1-\rho}^*$, by Theorem 28.

Lemma 67 (Independent of k estimate for $u^{(k)}$)**.** *For any $k \in \mathbb{N}$,*

$$||u^{(k)}||_{L^\infty(Q_1^+)} \leq \max\{||u||_{L^\infty(\partial_p Q_1^+ \setminus Q_1^*)},\ ||\varphi||_{L^\infty(Q_1^*)}\}. \tag{4.1.2}$$

Proof. First, by Theorem 28 we have

$$\inf_{Q_1^+} u^{(k)} \geq \inf_{\partial_p Q_1^+ \setminus Q_1^*} u^{(k)} = \inf_{\partial_p Q_1^+ \setminus Q_1^*} u$$

since $\left(u^{(k)}\right)_y \leq 0$ in the viscosity sense on Q_1^*. Hence it remains to bound $\sup_{Q_1^+} u^{(k)}$.

Assume that

$$\sup_{Q_1^+} u^{(k)} > \sup_{\partial_p Q_1^+ \setminus Q_1^*} u$$

and let $(X_0, t_0) \in \overline{Q}_1^+$ be such that $u^{(k)}(X_0, t_0) = \sup_{Q_1^+} u^{(k)} =: M$. From weak maximum principle (see [44], Corollary 3.20) we know that

$$||u^{(k)}||_{L^\infty(\overline{Q}_1^+)} \leq ||u^{(k)}||_{L^\infty(\partial_p Q_1^+)}$$

thus we can choose $(X_0, t_0) = (x_0, 0, t_0) \in Q_1^*$. Then a Hopf's-type lemma gives that $u_y^{(k)}(x_0, t_0) < 0$ in the viscosity sense. Therefore $-k\left(\varphi(x_0, t_0) - u^{(k)}(x_0, 0, t_0)\right) < 0$, that is $M = u^{(k)}(x_0, 0, t_0) < \varphi(x_0, t_0) \leq ||\varphi||_{L^\infty(Q_1^*)}$. □

Lemma 68 (Independent of k estimate for $g^{(k)}$)**.** *For any $k \in \mathbb{N}$,*

$$||g^{(k)}||_{L^\infty(Q_1^*)} \leq C\left(K, n, \lambda, \Lambda, \rho\right). \tag{4.1.3}$$

Proof. Note that $g^{(k)} \leq 0$ on Q_1^*, so we need to obtain only a lower bound. Let $(x_0, t_0) \in \overline{Q}_1^*$ be such that $g^{(k)}(x_0, t_0) = \min_{\overline{Q}_1^*} g^{(k)}$ and we may assume that $g^{(k)}(x_0, t_0) < 0$ (excluding the trivial case where $g^{(k)} = 0$ identically). Recall also that $u^{(k)} > \varphi$ on $Q_1^* \setminus Q_{1-\rho}^*$ which implies that $g^{(k)} = 0$ on $Q_1^* \setminus Q_{1-\rho}^*$. That is, $(x_0, t_0) \in Q_{1-\rho}^*$.

We intend to turn the obstacle φ into a suitable test function (whose normal derivative will not depend on k) that touches $u^{(k)}$ by below at (x_0, t_0) and then to use the viscosity condition $\left(u^{(k)}\right)_y = g^{(k)}$ to bound $g^{(k)}(x_0, t_0)$.

We denote by

$$M := \inf_{Q_1^+} u - \sup_{Q_1^*} \varphi$$

and we observe that $M \leq 0$, indeed

$$\inf_{Q_1^+} u \leq \inf_{Q_1^*} u \leq u(x^*, 0, t^*) = \varphi(x^*, t^*) \leq \sup_{Q_1^*} \varphi$$

where (x^*, t^*) is any point of Δ^*. Keep also in mind that by Lemma 67, $M \leq \inf_{Q_1^+} u^{(k)} - \sup_{Q_1^*} \varphi$. We consider b to be the solution of the following Dirichlet boundary value problem

$$\begin{cases} \mathcal{M}^-\left(D^2 b, \frac{\lambda}{n}, \Lambda\right) - b_t = (\Lambda n + 1)\, ||\varphi||_{H^{2+\alpha}(Q_1^*)}, & \text{in} \quad Q_\rho^+ \\ b = M, & \text{on} \quad \partial_p Q_\rho^+ \setminus Q_\rho^* \\ b = 0, & \text{on} \quad Q_{\rho/2}^* \\ b(x,0,t) = \frac{2M}{\rho}\left(\max\left\{|x|, |t|^{\frac{1}{2}}\right\} - \frac{\rho}{2}\right), & \text{on} \quad Q_\rho^* \setminus Q_{\rho/2}^*. \end{cases}$$

Note that $\max\left\{|x|, |t|^{\frac{1}{2}}\right\} = \frac{\rho}{2}$ on $\partial_p Q_{\rho/2}^*$ and $\max\left\{|x|, |t|^{\frac{1}{2}}\right\} = \rho$ on $\partial_p Q_\rho^*$, that is, $b = 0$ on $\partial_p Q_{\rho/2}^*$ and $b = M$ on $\partial_p Q_\rho^*$. Hence the Dirichlet data on $\partial_p Q_\rho^+$ is a continuous function (recall also that the maximum of two continuous functions is continuous). Moreover applying regularity theory for Dirichlet problems in $Q_{\rho/2}^+$, we obtain that $b \in H^{1+\alpha}\left(\overline{Q}_{\rho/4}^+\right)$ with the corresponding estimate depending only on $\rho, n, \lambda, \Lambda, K$. In particular, $|Db(0,0)| \leq C\,(K, n, \lambda, \Lambda, \rho)$.

Next, we consider the function

$$\Phi(X,t) = u^{(k)}(x_0, 0, t_0) - \varphi(x_0, t_0) + \varphi(x,t) + b\left((X,t) - (x_0, 0, t_0)\right)$$

for $(X,t) \in Q_\rho^+(x_0, t_0) \subset Q_1^+$. Since $b(0,0) = 0$, $\Phi(x_0, 0, t_0) = u^{(k)}(x_0, 0, t_0)$. Moreover,

- On $\partial_p Q_\rho^+(x_0,t_0) \setminus Q_\rho^*(x_0,t_0)$:

$$\Phi(X,t) \le u^{(k)}(x_0,0,t_0) - \varphi(x_0,t_0) + \sup_{Q_1^*} \varphi + \inf_{Q_1^+} u^{(k)} - \sup_{Q_1^*} \varphi$$
$$\le \inf_{Q_1^+} u^{(k)} \le u^{(k)}(X,t), \quad \text{since} \quad g^{(k)}(x_0,t_0) < 0.$$

- On $Q_\rho^*(x_0,t_0)$:

$$\Phi(x,0,t) \le u^{(k)}(x,0,t) - \varphi(x,t) + \varphi(x,t) = u^{(k)}(x,0,t)$$

  using that $b \le 0$ on Q_ρ^* and that $g^{(k)}(x_0,t_0) \le g^{(k)}(x,t)$ for any $(x,t) \in \overline{Q}_1^*$ and $g^{(k)}(x_0,t_0) < 0$, that is, $\varphi(x_0,t_0) - u^{(k)}(x_0,0,t_0) \ge \left(\varphi(x,t) - u^{(k)}(x,0,t)\right)^+ \ge \varphi(x,t) - u^{(k)}(x,0,t)$.

That is we have that $\Phi \le u^{(k)}$ on $\partial_p Q_\rho^+(x_0,t_0)$. Note also that if we extend φ in Q_1^+ by $\varphi(X,t) = \varphi(x,t)$ and l_i, $i = 1,\ldots,n$ denote the eigenvalues of $D^2\varphi \in S_n$

$$\mathcal{M}^-\left(D^2\varphi, \frac{\lambda}{n}, \Lambda\right) - \varphi_t = \frac{\lambda}{n}\sum_{l_i>0} l_i + \Lambda \sum_{l_i<0} l_i - \varphi_t \ge -\Lambda \sum_{l_i<0} |l_i| - |\varphi_t|$$
$$\ge -\Lambda n ||D^2\varphi||_\infty - |\varphi_t| \ge -(\Lambda n + 1)||\varphi||_{H^{2+\alpha}(Q_1^*)}.$$

That is, $\mathcal{M}^-\left(D^2 b + D^2\varphi, \frac{\lambda}{n}, \Lambda\right) - b_t - \varphi_t \ge 0$. Thus, $u^{(k)} - \Phi \in \overline{S}_p\left(\frac{\lambda}{n}, \Lambda\right)$ in $Q_\rho^+(x_0,t_0)$. Applying maximum principle we have that $\Phi \le u^{(k)}$ in $Q_\rho^+(x_0,t_0)$. In other words Φ touches $u^{(k)}$ by below at (x_0,t_0). Hence $\Phi_y(x_0,0,t_0) \le g^{(k)}(x_0,t_0)$. But we observe that $\Phi_y(x_0,0,t_0) = b_y(0,0)$ which completes the proof. □

Proposition 69. *$u^{(k)} \to u$ uniformly in $\overline{Q}_1^+$.*

Proof. We split our proof into two steps.

Step 1. We prove equicontinuity of $u^{(k)}$'s. For, it is enough to obtain an independent of k modulus of continuity for $u^{(k)}$ in $\overline{Q}_1^+$. Note that Lemma 67 gives a uniform L^∞-bound for $u^{(k)}$ in $\overline{Q}_1^+$. Also Lemma 68 gives a uniform L^∞-bound for $g^{(k)}$, thus using Theorem 37 we get a uniform H^α-estimate for $u^{(k)}$ in $\overline{Q}_{1-\frac{\rho}{2}}^+$. So it remains to get a uniform modulus of continuity in $\overline{Q}_1^+ \setminus \overline{Q}_{1-\frac{\rho}{2}}^+$.

Note that $\left(u^{(k)}\right)_y$=0 on $Q_1^* \setminus Q_{1-\rho}^*$. Thus if we extend $u^{(k)}$ in $Q_1 \setminus Q_{1-\rho}$ considering its even reflection $\tilde{u}^{(k)}$ with respect to y we have that $\tilde{u}^{(k)} \in S_p(\lambda,\Lambda)$ (see Proposition 21). We observe also that $\tilde{u}^{(k)}|_{\partial_p Q_1} = u_0$ is independent of k and smooth enough and $\tilde{u}^{(k)}|_{\partial_p Q_{1-\rho}}$ satisfy uniform H^α-estimate from the previous discussion. So using global H^α-estimates for Dirichlet problems we get the uniform modulus we desired in $\overline{Q}_1^+ \setminus \overline{Q}_{1-\frac{\rho}{2}}^+$.

Step 2. We explain now why equicontinuity is enough. First recall again that Lemma 67 ensures the existence of a uniform L^∞-bound for $u^{(k)}$ in $\overline{Q}_1^+$. Therefore Arzelá-Ascoli implies that every subsequence of $\{u^{(k)}\}$ has a subsequence that converges uniformly in $\overline{Q}_1^+$. We claim that every uniformly convergent subsequence of $\{u^{(k)}\}$ must converge to u. Thus we should have that $u^{(k)} \to u$ uniformly in $\overline{Q}_1^+$, so to finish the proof it remains to prove this claim.

Let v be the uniform limit of $\{u^{(k_m)}\}$ in $\overline{Q}_1^+$. If we show that v satisfies problem (4.0.1) then $v = u$ by uniqueness. The closedness result of Proposition 20 gives immediately that

$$\begin{cases} F(D^2 v) - v_t = 0, & \text{in} \quad Q_1^+ \\ v_y \leq 0, & \text{on} \quad Q_1^* \end{cases}$$

in the viscosity sense. Additionally, $v = u_0$ on $\partial_p Q_1^+ \setminus Q_1^*$. It remains to check that

1. $v_y = 0$ on $Q_1^* \cap \{v > \varphi\}$, in the viscosity sense.
2. $v \geq \varphi$ on Q_1^*.

For 1. let $(x_0, t_0) \in Q_1^*$ be so that $v(x_0, 0, t_0) > \varphi(x_0, t_0)$. From the continuity of v and φ, there exists some small $\delta > 0$ so that $v(x, 0, t) > \varphi(x, t)$ for any $(x, t) \in \overline{Q}_\delta^*(x_0, t_0)$. Next we use the uniform convergence of $u^{(k_m)}$ to v. Take

$$\varepsilon := \min_{\overline{Q}_\delta^*}(v - \varphi) > 0$$

then there exists $n_0 \in \mathbb{N}$ so that $|u^{(k_m)} - v| < \varepsilon$ in $\overline{Q}_\delta^*(x_0, t_0)$ for any $m \geq n_0$. Hence $u^{(k_m)} - v > -\varepsilon \geq -v + \varphi$, that is $u^{(k_m)} > \varphi$, so $\left(u^{(k_m)}\right)_y = 0$ in $\overline{Q}_\delta^*(x_0, t_0)$ for any $m \geq n_0$. Since $F\left(D^2 u^{(k_m)}\right) - \left(u^{(k_m)}\right)_t = 0$ in $\overline{Q}_\delta^+(x_0, t_0)$ again from the closedness result of Proposition 20 we get that $v_y = 0$ on $\overline{Q}_\delta^*(x_0, t_0)$, in the viscosity sense.

For 2. we assume that there exists some $(x_0, t_0) \in Q_1^*$ such that $v(x_0, 0, t_0) < \varphi(x_0, t_0)$ to get a contradiction. Again using the convergence we have that there exists $n_0 \in \mathbb{N}$ so that $u^{(k_m)}(x_0, 0, t_0) - v(x_0, 0, t_0) < \varphi(x_0, t_0) - v(x_0, 0, t_0)$ for any $m \geq n_0$. Hence $g^{(k_m)}(x_0, 0, t_0) = -k_m\left(\varphi(x_0, t_0) - u^{(k_m)}(x_0, 0, t_0)\right)$, that is, $\varphi(x_0, t_0) -$

$u^{(k_m)}(x_0,0,t_0) = -\frac{1}{k_m} g^{(k_m)}(x_0,0,t_0)$ for any $m \geq n_0$ and $g^{(k)}$ is bounded independently of k by Lemma 68. So taking $m \to \infty$ we get that $\varphi(x_0,t_0) = v(x_0,0,t_0)$ which is a contradiction. □

Proposition 69 gives the following.

Lemma 70. *For any* $0 < \delta < 1$, $Du^{(k)} \to Du$ *uniformly in* $K_\delta := Q_{1-\delta} \cap \{y > \delta\}$.

Proof. Note first that from interior $H^{1+\alpha}$-estimates for viscosity solutions of $F(D^2 v) - v_t = 0$ we know the existence of $Du^{(k)}, Du$ in K_δ and a uniform H^α-estimate for $Du^{(k)}$ (recall that $||u^{(k)}||_{L^\infty(Q_1^+)}$ are uniformly bounded). Therefore using Arzelá-Ascoli we get that every subsequence of $\{Du^{(k)}\}$ has a subsequence that converges uniformly in $\overline{K}_\delta$. Then by standard calculus we know that any uniformly convergent subsequence of $\{Du^{(k)}\}$ should converge to Du (since $u^{(k)} \to u$). This gives the desired. □

Lemma 71. *For any* $0 < \delta < 1$, $u^{(k)} \in H^{1+\alpha}\left(\overline{Q}^+_{1-\delta}\right)$.

Although the $H^{1+\alpha}$-estimates of the above may depend on k, Lemma 71 ensures the existence and regularity of $\left(u^{(k)}\right)_y$ on Q_1^* in the classical sense.

Proof. Using Lemma 68 and Theorem 37 we get a uniform H^α-estimate for $u^{(k)}$ in $\overline{Q}^+_{1-\frac{\delta}{2}}$ which means that $g^{(k)} = -k\left(\varphi - u^{(k)}\right)^+$ is H^α on $\overline{Q}^*_{1-\frac{\delta}{2}}$. Then applying Theorem 54 we get the desired. □

4.2 Semi-concavity of the solution

In this section we obtain some bounds for the first and second derivatives of the solution. The first application of these bounds is about to ensure that u_{y^+} exists on Q_1^*.

Proposition 72. *For any* $0 < \delta < 1$,

(A) $|u_{x_i}|, |u_y| \leq C$, *in* $Q^+_{1-\delta}$, *for any* $i = 1, \ldots, n-1$

(B) $u_{x_i x_i}, u_t \geq -C$, *in* $Q^+_{1-\delta}$, *for any* $i = 1, \ldots, n-1$

(C) $u_{yy} \leq C$, *in* $Q^+_{1-\delta}$

where the constant $C > 0$ *depends only on* $K, n, \lambda, \Lambda, \rho$ *and* δ.

Note that since F is convex, we have that $u_{x_ix_j}$ and u_t exist inside Q_1^+ in the classical sense by interior estimates.

Proof.

For *(A)*, we thicken the obstacle φ. First, we extend φ as a solution inside Q_1^+ and Q_1^- (following the idea of Theorem 1(a) in [2], see also Proposition 2.1 in [18]), that is we consider the viscosity solutions of the Dirichlet problems

$$\begin{cases} F(D^2\tilde{\varphi}) - \tilde{\varphi}_t = 0, & \text{in } Q_1^+ \\ \tilde{\varphi} = \varphi, & \text{on } Q_1^* \\ \tilde{\varphi} = -||u||_{L^\infty(Q_1^+)}, & \text{on } \partial_p Q_1^+ \setminus Q_1^* \end{cases} \quad \text{and} \quad \begin{cases} F(D^2\tilde{\varphi}) - \tilde{\varphi}_t = 0, & \text{in } Q_1^- \\ \tilde{\varphi} = \varphi, & \text{on } Q_1^* \\ \tilde{\varphi} = -||u||_{L^\infty(Q_1^+)}, & \text{on } \partial_p Q_1^- \setminus Q_1^*. \end{cases}$$

For any $0 < \delta < 1$ and since φ is smooth enough we obtain, using Theorem 49, that $\tilde{\varphi}$ is Lipschitz in $Q_{1-\frac{\delta}{2}}$ with a constant that depends only on K, n, λ, Λ and δ. Moreover, we can show that $\tilde{u} \geq \tilde{\varphi}$ in Q_1, where $\tilde{u}$ denotes the even reflection of u in y in Q_1. Indeed, $u - \tilde{\varphi} \in S_p\left(\frac{\lambda}{n}, \Lambda\right)$ in Q_1^+, $u \geq \varphi = \tilde{\varphi}$ on Q_1^*, $u \geq -||u||_{L^\infty(Q_1^+)} = \tilde{\varphi}$ on $\partial_p Q_1^+ \setminus Q_1^*$, thus using maximum principle we get that $u \geq \tilde{\varphi}$ in $\overline{Q}_1^+$ and we repeat the same argument in Q_1^- (note that $F(D^2\tilde{u}) - \tilde{u}_t = 0$ in Q_1^- due to the assumptions on F). Finally, Proposition 21 ensures that $F(D^2\tilde{u}) - \tilde{u}_t \leq 0$ in Q_1 and that $F(D^2\tilde{u}) - \tilde{u}_t = 0$ in $Q_1 \cap \{\tilde{u} > \tilde{\varphi}\}$ in the viscosity sense. That is $\tilde{u}$ satisfies a thick obstacle problem in Q_1 with obstacle $\tilde{\varphi}$ which is Lipschitz in $Q_{1-\frac{\delta}{2}}$. Therefore $\tilde{u} \in H^1(Q_{1-\delta})$ with a constant that depends only on K, n, λ, Λ and δ (see [39], [37]) which gives *(A)*.

For *(B)*, we denote by $d := \min\{\rho, \delta\}$ and we consider the set $\tilde{Q}^+ := Q^+_{1-\frac{d}{3}} \setminus Q^+_{1-\frac{2d}{3}}$. We observe that $u_y = 0$ on $\tilde{Q}^*$ in the viscosity sense, since $\tilde{Q}^* \subset Q_1^* \setminus Q^*_{1-\rho}$. Thus $H^{2+\alpha}$-estimates of subsection 3.4.1 can be applied in $\tilde{Q}^+$, so we have H^α-estimates for $u_{x_ix_i}$ and u_t on $\partial_p Q^+_{1-\frac{d}{2}} \setminus Q^*_{1-\frac{d}{2}}$. In particular we have uniform bounds for the corresponding difference quotients, that is,

$$\frac{u(x + he_i, y, t) + u(x - he_i, y, t) - 2u(x, y, t)}{h^2} \geq -C \tag{4.2.1}$$

where $\{e_i\}_{1 \leq i \leq n}$ is the orthonormal basis of $\mathbb{R}^n$ and

$$\frac{u(x, y, t - h) - u(x, y, t)}{h} \geq -C \tag{4.2.2}$$

for $(X, t) \in \partial_p Q^+_{1-\frac{d}{2}} \setminus Q^*_{1-\frac{d}{2}}$ and $h > 0$ small enough (depending only on d). Note also that constant $C > 0$ depends only on $K, n, \lambda, \Lambda, \rho$ and δ.

We study (4.2.1) first in order to bound $u_{x_i x_i}$, for $i = 1, \ldots, n-1$. We observe that

$$v(x,y,t) := \frac{u(x+he_i,y,t) + u(x-he_i,y,t)}{2} + Ch^2 \geq u(x,y,t), \quad \text{on} \;\; \partial_p Q^+_{1-\frac{d}{2}} \setminus Q^*_{1-\frac{d}{2}}.$$

Moreover, for $(x,t) \in Q^*_{1-\frac{d}{2}}$,

$$\begin{aligned} v(x,0,t) &= \frac{u(x+he_i,0,t) + u(x-he_i,0,t)}{2} + Ch^2 \\ &\geq \frac{\varphi(x+he_i,t) + \varphi(x-he_i,t)}{2} + Ch^2 \geq \varphi(x,t) \end{aligned}$$

changing C if necessary depending on K. We observe also that the convexity of F ensures that $F(D^2 v) - v_t \leq 0$ in $Q^+_{1-\frac{d}{2}}$ in the viscosity sense. Finally note that $v_y \leq 0$ on $Q^*_{1-\frac{d}{2}}$ in the viscosity sense (can be obtained as Theorem 43). That is v is a viscosity supersolution of (4.0.2) in $Q^+_{1-\frac{d}{2}}$. But u is assumed to be the least supersolution of (4.0.2), thus $v \geq u$ in $Q^+_{1-\frac{d}{2}}$. That is,

$$\frac{u(x+he_i,y,t) + u(x-he_i,y,t) - 2u(x,y,t)}{h^2} \geq -C$$

in $Q^+_{1-\frac{d}{2}}$ and $C > 0$ depends only on $K, n, \lambda, \Lambda, \rho$ and δ, so we take $h \to 0^+$. Next we study (4.2.2) in a similar way in order to bound u_t. We observe that

$$w(x,y,t) := u(x,y,t-h) + Ch \geq u(x,y,t), \quad \text{on} \;\; \partial_p Q^+_{1-\frac{d}{2}} \setminus Q^*_{1-\frac{d}{2}}.$$

Moreover, for $(x,t) \in Q^*_{1-\frac{d}{2}}$,

$$w(x,0,t) = u(x,0,t-h) + Ch \geq \varphi(x,t-h) + Ch \geq \varphi(x,t)$$

changing C if necessary depending on K. Finally, we observe that $F(D^2 w) - w_t = 0$ in $Q^+_{1-\frac{d}{2}}$ and $w_y \leq 0$ on $Q^*_{1-\frac{d}{2}}$ in the viscosity sense. That is w is a viscosity supersolution of (4.0.2). But u is assumed to be the least supersolution of (4.0.2) in $Q^+_{1-\frac{d}{2}}$, thus $w \geq u$ in $Q^+_{1-\frac{d}{2}}$. That is,

$$\frac{u(x,y,t-h) - u(x,y,t)}{h} \geq -C$$

in $Q^+_{1-\frac{d}{2}}$ and $C > 0$ depends only on $K, n, \lambda, \Lambda, \rho$ and δ, so we take $h \to 0^+$.

For *(C)*, we use *(B)* and the equation. To study the equation further we define

$$a_{ij}(X,t) := \int_0^1 F_{ij}\left(hD^2u(X,t)\right)\ dh$$

and we observe that $\frac{d}{dh}\left[F\left(hD^2u(X,t)\right)\right] = \sum_{i,j=1}^n F_{ij}\left(hD^2u(X,t)\right)\ u_{x_ix_j}(X,t)$. That is,

$$\begin{aligned}\sum_{i,j=1}^n a_{ij}(X,t)\ u_{x_ix_j}(X,t) &= \int_0^1 \sum_{i,j=1}^n F_{ij}\left(hD^2u(X,t)\right)\ u_{x_ix_j}(X,t)\ dh \\ &= F\left(D^2u(X,t)\right) - F(O) = F\left(D^2u(X,t)\right).\end{aligned}$$

Thus, $\sum_{i,j=1}^n a_{ij}(X,t)\ u_{x_ix_j}(X,t) - u_t(X,t) = 0$ in Q_1^+. Also, we have that $a_{ij} = a_{ji}$ and that $a_{in} = a_{ni} = 0$, for any $1 \leq i \leq n-1$, from the assumptions on F. Additionally we may observe that using the ellipticity of F we have that for any $M \in S_n$ and $h > 0$

$$\lambda h \leq F\left(M + H_h^{ii}\right) - F(M) \leq \Lambda h$$

so taking $h \to 0^+$ we have that $\lambda \leq F_{ii}(M) \leq \Lambda$. In particular, $\lambda \leq a_{ii}(X,t) \leq \Lambda$, for any $(X,t) \in Q_1^+$, $i = 1,\ldots,n$. So if $A_{n-1}(X,t) := \left(a_{ij}(X,t)\right)_{i,j=1,\ldots,n-1} \in S_{n-1}$ we have

$$\begin{aligned}a_{nn}(X,t)u_{yy}(X,t) &= -\sum_{i,j=1}^{n-1} a_{ij}(X,t)\ u_{x_ix_j}(X,t) + u_t(X,t) \\ &= -\mathrm{tr}\left(A_{n-1}(X,t)\ D_{n-1}^2u(X,t)\right) + u_t(X,t) \\ &= -\mathrm{tr}\left[A_{n-1}(X,t)\left(\ D_{n-1}^2u(X,t) + C\mathrm{I}_{n-1}\right)\right] \\ &\quad + \mathrm{tr}\left(CA_{n-1}(X,t)\right) + u_t(X,t)\end{aligned}$$

where $C > 0$ the constant of *(B)*, so $D_{n-1}^2u(X,t) + C\mathrm{I}_{n-1} \geq 0$ in $Q_{1-\delta}^+$. Also, $\mathrm{tr}\left(A_{n-1}(X,t)\right) = \sum_{i=1}^{n-1} a_{ii}(X,t) \geq \lambda(n-1) \geq 0$, thus

$$\mathrm{tr}\left[A_{n-1}(X,t)\left(\ D_{n-1}^2u(X,t) + C\mathrm{I}_{n-1}\right)\right] = \mathrm{tr}\left[A_{n-1}(X,t)\right]\mathrm{tr}\left[\ D_{n-1}^2u(X,t) + C\mathrm{I}_{n-1}\right] \geq 0$$

and $\mathrm{tr}\left(CA_{n-1}(X,t)\right) = C\sum_{i=1}^{n-1} a_{ii}(X,t) \leq C\Lambda(n-1)$, $a_{nn}(X,t) \geq \lambda$. Hence we have that

$$u_{yy} \leq C\ \frac{\Lambda(n-1)+1}{\lambda}, \quad \text{in} \quad Q_{1-\delta}^+.$$

□

For any $(x,t) \in Q_1^*$ we define

$$\sigma(x,t) := \lim_{y \to 0^+} u_y(x,y,t).$$

Note that Proposition 72 ensures the existence of the above limit for any $(x,t) \in Q_1^*$. Indeed, we consider the function $v(X,t) = u_y(X,t) - Cy$, for $(X,t) \in Q_1^+$. Then using *(A)* and *(C)* of Proposition 72 we obtain that $v > -2C$ and $v_y = u_{yy} - C \leq 0$ in $Q_{1-\delta}^+$, that is, v is monotone decreasing in y and bounded by below, thus $\lim_{y\to 0^+} v(x,y,t)$ exists for $(x,t) \in Q_{1-\delta}^*$, for any $0 < \delta < 1$.

Furthermore we remark that the existence of the above limit ensures (through a simple De L'Hôpital rule) the existence of $\lim_{y\to 0^+} \frac{u(x,y,t)-u(x,0,t)}{y}$, that is u_{y^+} exists on Q_1^* and equals to σ (note also that u_y is continuous in y up to Q_1^*). Thereafter the viscosity condition $u_y \leq 0$ on Q_1^* suggests that we should have

$$\sigma \leq 0, \quad \text{on} \quad Q_1^*. \tag{4.2.3}$$

Now although we know that $u_{y^+} = \sigma$ on Q_1^* in the classical sense, we cannot use the viscosity condition to get (4.2.3) since we do not know if u_{y^+} is continuous in $(x,0,t)$. We prove (4.2.3) in the next lemma using the penalized problems introduced in the previous section to approach u by classical solutions.

Lemma 73. *$\sigma \leq 0$ on Q_1^*.*

Proof. For $k \in \mathbb{N}$ (fixed), we consider the solution $u^{(k)}$ of (4.1.1). We denote by $v := \left(u^{(k)}\right)_y$ which exists in the classical sense and it is continuous in $Q_1^+ \cup Q_1^*$ (due to Lemma 71). Then $v \leq 0$ on Q_1^* and $u^{(k)} > \varphi$, that is, $v = 0$ on $Q_1^* \setminus Q_{1-\delta}^*$ if $0 < \delta < \rho$. Moreover we can use Theorem 52 in $Q_{1-\frac{\delta}{3}}^+ \setminus Q_{1-\frac{2\delta}{3}}^+$ to obtain that

$$v \leq M, \quad \text{on} \quad \partial_p Q_{1-\frac{\delta}{2}}^+ \setminus Q_{1-\frac{\delta}{2}}^*$$

where $M > 0$ is a constant independent of k (recall that by Lemma 67 $u^{(k)}$'s have uniform L^∞-bound).

Next we apply a barrier argument to v. We define the function b to be the viscosity solution of (see also the proof of Lemma 68)

$$\begin{cases} \mathcal{M}^+\left(D^2 b, \frac{\lambda}{n}, \Lambda\right) - b_t = 0, & \text{in} \quad Q^+_{1-\frac{\delta}{2}} \\ b = M, & \text{on} \quad \partial_p Q^+_{1-\frac{\delta}{2}} \setminus Q^*_{1-\frac{\delta}{2}} \\ b = 0, & \text{on} \quad Q^*_{1-\delta} \\ b(x,0,t) = \frac{2M}{\delta}\left(\max\left\{|x|, |t|^{\frac{1}{2}}\right\} - 1 + \delta\right), & \text{on} \quad Q^*_{1-\frac{\delta}{2}} \setminus Q^*_{1-\delta}. \end{cases}$$

As explained in the proof of Lemma 68 the boundary data given in the above problem is a continuous function. We remark that

$$v \leq b, \quad \text{on} \quad \partial_p Q^+_{1-\frac{\delta}{2}} \setminus Q^*_{1-\frac{\delta}{2}}$$

and

$$v \leq 0 \leq b, \quad \text{on} \quad Q^*_{1-\frac{\delta}{2}}.$$

Moreover, we know that $v \in \underline{S}_p\left(\frac{\lambda}{n}, \Lambda\right)$ in $Q^+_{1-\frac{\delta}{2}}$ (this can be obtained if we recover $\left(u^{(k)}\right)_y$ as a uniform limit of difference quotients and then we use Theorem 4.6 of [45] and the closedness of viscosity solutions proved in Proposition 10). Then $v - b \in \underline{S}_p\left(\frac{\lambda}{n}, \Lambda\right)$ in $Q^+_{1-\frac{\delta}{2}}$. So using maximum principle we get that $v \leq b$ in $Q^+_{1-\frac{\delta}{2}}$ and note that function b does not depend on k. On the other hand $\left(u^{(k)}\right)_y \to u_y$ as $k \to \infty$ pointwise in $Q^+_{1-\frac{\delta}{2}}$ by Lemma 70. Hence $u_y \leq b$ in $Q^+_{1-\frac{\delta}{2}}$. Finally, we observe that $b = 0$ on $Q^*_{1-\delta}$, for any $0 < \delta < \rho$ and we take $y \to 0^+$. □

4.3 Regularity of the solution

Recall that at the points of Ω^* (as well as at the points of the interior of Δ^*) the regularity of the solutions is known. Thereafter at these points the viscosity Neumann condition holds in the classical sense, thus $\sigma = 0$ in Ω^*.

In this section we concentrate in studying the regularity of σ around free boundary points in order to treat our problem as a non-homogeneous Neumann boundary value problem around these points. To achieve this we show first Lemma 78 which gives an H^α-estimate for σ in universal neighborhoods of points of Ω^*. Lemma 78 is based on Lemmata 76 and 77 and on semi-concavity of u in y. Lemma 76 says that considering a non-contact point $P_0 \in Q^*_{1/2}$, we can find a universal neighborhood of P_0 which contains a small universal thin-cylinder where σ decays proportionally to its radius. Finally Lemma 77 says that the information we have in this small thin cylinder can be carried to a suitable set inside Q_1^+ and then is carried back in a parabolic neighborhood of P_0 using semi-concavity in y. An iterative application of the above gives Lemma 78.

We start with Lemma 75 which is important in proving Lemma 76. The following remark is used in the proof of Lemma 75.

Remark 74. *For $P_0 := (x_0, t_0) \in \Omega^*$, $K_0 := 2K$ and*

$$\tilde{\varphi}_{P_0}(x,t) := \varphi(x_0,t_0) + D\varphi(x_0,t_0)\cdot(x-x_0) - K_0(t-t_0) + K_0|x-x_0|^2.$$

we have that $\tilde{\varphi}_{P_0} > \varphi$ in $Q_1^* \cap \{t \le t_0\} \setminus \{(x_0,t_0)\}$.

Indeed, let $\Phi = \tilde{\varphi}_{P_0} - \varphi$. Then we observe that

(a) $\Phi(x_0,t_0) = 0$.

(b) $D\Phi(x,t) = D\varphi(x_0,t_0) + 2K_0(x-x_0) - D\varphi(x,t)$, thus $D\Phi(x_0,t_0) = 0$.

(c) $D^2\Phi(x,t) = 2K_0 I_{n-1} - D^2\varphi(x,t) > 0$, that is Φ is convex with respect to x.

(d) $\Phi_t(x,t) = -2K_0 - \varphi_t(x,t) < 0$, that is Φ is monotone decreasing with respect to t.

Then *(c)* (through integration) and *(b)* give that $\Phi(x,t_0) - \Phi(x_0,t_0) > (x-x_0)\cdot D\Phi(x_0,t_0) = 0$ for $x \neq x_0$. Thus by *(a)* we have that $\Phi(x,t_0) > \Phi(x_0,t_0) = 0$, for $x \neq x_0$. On the other hand *(d)* gives that $\Phi(x,t) > \Phi(x,t_0)$ for any $t < t_0$ and any x. Combining the above we get that $\Phi(x,t) > \Phi(x_0,t_0) = 0$, for any $x \neq x_0$ and any $t < t_0$.

Lemma 75. *For $P_0 = (x_0,t_0) \in \Omega^*$, $K_0 := 2K$ and $C_0 > \frac{n}{\lambda}[\Lambda(n-1)+1]$ we define*

$$h_{P_0}(x,y,t) := \varphi(x_0,t_0) + D\varphi(x_0,t_0)\cdot(x-x_0) - K_0(t-t_0) + K_0|x-x_0|^2 - C_0K_0y^2.$$

We consider also any set of the form $\Theta := \tilde{\Theta} \times (t_1,t_0] \subset Q_1$, with $P_0 \in \Theta$, $\tilde{\Theta} \subset \mathbb{R}^n$ a bounded domain containing x_0 and $t_1 < t_0$. Then

$$\sup_{\partial_p\Theta \cap \{y \ge 0\}} (u - h_{P_0}) \ge 0.$$

Proof. Let $w := u - h_{p_0}$ then

1. $w(x_0,0,t_0) = u(x_0,0,t_0) - \varphi(x_0,t_0) > 0$, since $(x_0,t_0) \in \Omega^*$.

2. $w \in \underline{S}_p\left(\frac{\lambda}{n},\Lambda\right)$ in Q_1^+. Indeed, we note that $(h_{P_0})_{ij} = 0$ for $i \neq j$, $(h_{P_0})_{ii} = 2K_0$ for $i < n$, $(h_{P_0})_{nn} = -2C_0K_0$ and $(h_{P_0})_t = -K_0$. Then $\mathcal{M}^+\left(D^2h_{P_0},\frac{\lambda}{n},\Lambda\right) - (h_{P_0})_t = -2C_0K_0\frac{\lambda}{n} + 2K_0\Lambda(n-1) + K_0 < 2K_0\left[-\left[\Lambda(n-1)+1\right] + \Lambda(n-1) + \frac{1}{2}\right] = -K_0 < 0$ in the classical sense. Since $u \in \underline{S}_p\left(\frac{\lambda}{n},\Lambda\right)$ in Q_1^+ we get the desired.

3. $w_y = 0$ on Ω^* in the classical sense. Indeed, we note that $(h_{P_0})_y = -\frac{2K_0n^2\Lambda}{\lambda}\,y$, that is $(h_{P_0})_y = 0$ on Q_1^*. Since $u_y = 0$ on Ω^* in the classical sense we get the desired.

Now we denote by w^* the extension of w in Q_1 considering its even reflection with respect to y and we have that $w^* \in \underline{S}_p\left(\frac{\lambda}{n},\Lambda\right)$ in $Q_1 \setminus \Delta^*$ (see Proposition 21). Then maximum principle (Corollary 26) gives that

$$\sup_{\partial_p(\Theta\setminus\Delta^*)\cap\{y\geq 0\}} w = \sup_{\partial_p(\Theta\setminus\Delta^*)} w^* \geq \sup_{\Theta\setminus\Delta^*} w^* \geq w(x_0,0,t_0) > 0$$

since $(x_0,t_0) \in \Theta\setminus\Delta^*$. Finally we observe that $\partial_p\left(\Theta\setminus\Delta^*\right)\cap\{y\geq 0\} \subset (\partial_p\Theta \cap \{y \geq 0\}) \cup (\Delta^* \cap \{t \leq t_0\})$. On the other hand, $h_{P_0} = \tilde{\varphi}_{P_0} > \varphi$ on $Q_1^* \cap \{t \leq t_0\} \setminus \{(x_0,t_0)\}$ from Remark 74 and $\varphi = u$ on Δ^*, that is $w < 0$ on $\Delta^* \cap \{t \leq t_0\}$ and the proof is complete. □

Lemma 76. *For $\gamma > 0$ we define $\Omega^*_\gamma := \{(x,t) \in Q_1^* : \sigma(x,t) > -\gamma\}$. Let $(x_0,t_0) \in \Omega^* \cap Q^*_{1/2}$, then there exist constants $0 < \bar{C} < \bar{\bar{C}} < 1$ which depend only on K, n, λ, Λ, ρ so that for any $0 < \gamma < \frac{1}{2}$ there exists a thin-cylinder $Q^*_{\bar{C}\gamma}(\bar{x},\bar{t})$ so that*

$$Q^*_{\bar{C}\gamma}(\bar{x},\bar{t}) \subset Q^*_{\bar{\bar{C}}\gamma}(x_0,t_0) \cap \Omega^*_\gamma.$$

Proof. Let $(x_0,t_0) \in \Omega^* \cap Q^*_{1/2}$, we apply Lemma 75 with

$$\Theta := B^*_{C_1\gamma}(x_0) \times (-C_2\gamma, C_2\gamma) \times \left(t_0 - (C_1\gamma)^2, t_0\right]$$

where $0 < C_2 << C_1 << 1$ to be chosen. Then there exists $P_1 = (x_1,y_1,t_1) \in \partial_p\Theta \cap \{y \geq 0\}$ so that

$$u(P_1) - h_{P_0}(P_1) \geq 0. \tag{4.3.1}$$

We split into cases.

<u>*Case 1.*</u> If $|x_1 - x_0| = C_1\gamma$ or $t_1 = t_0 - (C_1\gamma)^2$. Then using (4.3.1) and Remark 74

we have in the first occasion that

$$\begin{aligned}u(P_1) &\geq \varphi(x_0,t_0) + D\varphi(x_0,t_0)\cdot(x_1-x_0) - K_0(t_1-t_0) + \frac{K_0}{2}|x_1-x_0|^2 \\ &\quad + \frac{K_0}{2}|x_1-x_0|^2 - C_0K_0y_1^2 \\ &\geq \varphi(x_1,t_1) + \frac{K_0}{2}(C_1\gamma)^2 - C_0K_0(C_2\gamma)^2\end{aligned}$$

and in the second occasion we have that

$$\begin{aligned}u(P_1) &\geq \varphi(x_0,t_0) + D\varphi(x_0,t_0)\cdot(x_1-x_0) + K_0|x_1-x_0|^2 - \frac{K_0}{2}(t_1-t_0) \\ &\quad - \frac{K_0}{2}(t_1-t_0) - C_0K_0y_1^2 \\ &\geq \varphi(x_1,t_1) + \frac{K_0}{2}(C_1\gamma)^2 - C_0K_0(C_2\gamma)^2.\end{aligned}$$

Thus in any case

$$u(x_1,y_1,t_1) \geq \varphi(x_1,t_1) + C_4\gamma^2 \tag{4.3.2}$$

where $C_4 > 0$ a constant depending only on universal constants and on C_1, C_2 (choosing $0 < C_2 < \sqrt{\frac{C_0}{2}}\ C_1$).

Now take any $(x_2,t_2) \in Q^*_{C_3\gamma}(x_1,t_1)$, that is $|x_1-x_2| < C_3\gamma$ and $t_1-(C_3\gamma)^2 < t_2 \leq t_1$, for C_3 to be chosen. First we intend to transfer the information (4.3.2) from (x_1,y_1,t_1) to (x_2,y_1,t_2) using the tangential semi-convexity of u (Proposition 72). We denote by $\tau = \frac{x_2-x_1}{|x_2-x_1|} \in \mathbb{R}^{n-1}$ and we assume that $(x_2-x_1)\cdot D_{n-1}(u-\varphi)(P_1) \geq 0$ (considering the extension of φ in Q_1^+ where $\varphi^*(x,y,t) = \varphi(x,y)$). We notice that

$$\begin{aligned}&\int_0^{|x_2-x_1|}\int_0^e (u-\varphi)_{\tau\tau}(x_1+\tau h, y_1, t_1)\ dhde \\ &\quad = \int_0^{|x_2-x_1|} \left[(u-\varphi)_\tau(x_1+\tau e, y_1, t_1) - (u-\varphi)_\tau(x_1,y_1,t_1)\right]\ de \\ &\quad = (u-\varphi)(x_2,y_1,t_1) - (u-\varphi)(x_1,y_1,t_1) - |x_2-x_1|(u-\varphi)_\tau(x_1,y_1,t_1)\end{aligned}$$

and

$$\int_{t_2}^{t_1} (u-\varphi)_t(x_2,y_1,h)\ dh = (u-\varphi)(x_2,y_1,t_1) - (u-\varphi)(x_2,y_1,t_2).$$

Combining the above we get

$$\int_0^{|x_2-x_1|}\int_0^e (u-\varphi)_{\tau\tau}(x_1+\tau h, y_1, t_1)\ dhde - \int_{t_2}^{t_1}(u-\varphi)_t(x_2,y_1,h)\ dh$$
$$= (u-\varphi)(x_2,y_1,t_2)-(u-\varphi)(x_1,y_1,t_1)-|x_2-x_1|(u-\varphi)_\tau(x_1,y_1,t_1). \quad (4.3.3)$$

On the other hand using *(B)* of Proposition 72 we have

$$\int_0^{|x_2-x_1|}\int_0^e (u-\varphi)_{\tau\tau}(x_1+\tau h, y_1, t_1)\ dhde \geq -C\int_0^{|x_2-x_1|} e\ de$$
$$\geq -C|x_2-x_1|^2 \geq -C(C_3\gamma)^2$$

and

$$-\int_{t_2}^{t_1}(u-\varphi)_t(x_2,y_1,h)\ dh \geq -C(t_1-t_2) \geq -C(C_3\gamma)^2.$$

Therefore returning to (4.3.3) we have that

$$(u-\varphi)(x_2,y_1,t_2)-(u-\varphi)(x_1,y_1,t_1)-|x_2-x_1|(u-\varphi)_\tau(x_1,y_1,t_1) \geq -C(C_3\gamma)^2.$$

That is,

$$(u-\varphi)(x_2,y_1,t_2) \geq (u-\varphi)(x_1,y_1,t_1)+(x_2-x_1)\cdot D_{n-1}(u-\varphi)(x_1,y_1,t_1) - C(C_3\gamma)^2$$
$$\geq C_4\gamma^2 - C(C_3\gamma)^2 > 0 \quad (4.3.4)$$

chosing $0 < C_3^2 < \frac{C_4}{C}$.

Now (to get a contradiction) we assume that $(x_2,t_2)\notin\Omega^*_\gamma$, that is $\sigma(x_2,t_2)\leq -\gamma < 0$. Then $(x_2,t_2)\in\Delta^*$, that is $u(x_2,0,t_2)=\varphi(x_2,t_2)$. Similarly as before the normal semi-concavity of u (*(C)* of Proposition 72) will allow to transfer this information from $(x_2,0,t_2)$ to (x_2,y_1,t_2). We have

$$Cy_1^2 \geq \int_0^{y_1}\int_0^e u_{yy}(x_2,h,t_2)\ dhde = \int_0^{y_1}\left[u_y(x_2,e,t_2)-\sigma(x_2,t_2)\right]\ dhde$$
$$= u(x_2,y_1,t_2)-u(x_2,0,t_2)-y_1\sigma(x_2,t_2)$$

then

$$u(x_2, y_1, t_2) - \varphi(x_2, t_2) \leq Cy_1^2 + y_1(-\gamma) \leq y_1(CC_2\gamma - \gamma) = y_1\gamma(CC_2 - 1) < 0$$

choosing $0 < C_2 \leq \frac{1}{C}$. This is a contradiction regarding (4.3.4).

Case 2. If $y_1 = C_2\gamma$. Then using (4.3.1) and Remark 74 we have

$$u(x_1, y_1, t_1) \geq \varphi(x_1, t_1) - C_0 K_0 C_2^2 \gamma^2. \tag{4.3.5}$$

We take any $(x_2, t_2) \in Q^*_{C_2\gamma}(x_1, t_1)$, that is $|x_1 - x_2| < C_2\gamma$ and $t_1 - (C_2\gamma)^2 < t_2 \leq t_1$. Assuming that $(x_2 - x_1) \cdot D_{n-1}(u - \varphi)(P_1) \geq 0$ we can repeat the computations of Case 1 slightly modified to obtain

$$(u - \varphi)(x_2, C_2\gamma, t_2) \geq -C^* C_2^2 \gamma^2. \tag{4.3.6}$$

Now (to get a contradiction) we assume that $(x_2, t_2) \notin \Omega^*_\gamma$, that is $\sigma(x_2, t_2) \leq -\gamma < 0$. Then $(x_2, t_2) \in \Delta^*$, that is $u(x_2, 0, t_2) = \varphi(x_2, t_2)$. Similarly as in Case 1 we get that

$$u(x_2, C_2\gamma, t_2) - \varphi(x_2, t_2) \leq CC_2^2\gamma^2 - C_2\gamma^2 \leq \gamma^2(C_6 C_2^2 - C_2) < -C^* C_2^2 \gamma^2$$

where $C_6 := \max\{C, C^*\}$ and choosing $C_2 < (C_6 - C^*)^{-1}$. This is a contradiction regarding (4.3.6).

In any case we have that there exists $0 < C_7 << 1$ depending only on $\rho, n, \lambda, \Lambda, K$ so that if $(x_2, t_2) \in Q^*_{C_7\gamma}(x_1, t_1)$ with $(x_2 - x_1) \cdot D_{n-1}(u - \varphi)(x_1, y_1, t_1) \geq 0$ (which roughly speaking holds at least in the "half" of $Q^*_{C_7\gamma}(x_1, t_1)$) then $(x_2, t_2) \in \Omega^*_\gamma$. Moreover choosing $1 > \bar{\bar{C}} > C_7 + C_1$ it is easy to check that $Q^*_{C_7\gamma}(x_1, t_1) \subset Q^*_{\bar{\bar{C}}\gamma}(x_0, t_0)$. By choosing a thin cylinder $Q^*_{\bar{C}\gamma}(\bar{x}, \bar{t})$ inside $Q^*_{C_7\gamma}(x_1, t_1) \cap \{(x_2 - x_1) \cdot D_{n-1}(u - \varphi)(x_1, y_1, t_1) \geq 0\}$ the proof is complete. □

Now maximum principle and a barrier argument give the following important property.

Lemma 77. *Consider the set* $K_1 := B_1^* \times (0, 1) \times (-1, 0]$ *and assume that* $w \in C(K_1)$

satisfies in the viscosity sense

$$\cdot \begin{cases} \mathcal{M}^-\left(D^2w, \lambda, \Lambda\right) - w_t \le 0, & in \quad K_1 \\ w \ge 0, & in \quad K_1. \end{cases}$$

Suppose that there exists some neighborhood $Q^*_\delta(\bar{x}, \bar{t}) \subset Q^*_1$ *so that*

$$\liminf_{y\to 0^+} w(x,y,t) \ge 1, \quad \textit{for any} \quad (x,t) \in \overline{Q}^*_\delta(\bar{x},\bar{t}).$$

Then, there exists $\varepsilon = \varepsilon(\delta, n, \lambda, \Lambda) > 0$ *so that*

$$w(x,y,t) \ge \varepsilon, \quad \textit{for any} \quad (x,y,t) \in \overline{B}^*_{1/2} \times \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[-\frac{\delta^2}{2}, 0\right].$$

Proof. For any $P' = (x', t') \in \overline{Q}^*_{1-\delta}$ we define the auxiliary function

$$\begin{cases} \mathcal{M}^-\left(D^2 b_{P'}, \frac{\lambda}{n}, \Lambda\right) - (b_{P'})_t = 0, & \text{in} \quad K_1 \\ b_{P'} = 0, & \text{on} \quad \partial_p K_1 \setminus Q^*_\delta(P') \\ b_{P'}(x,0,t) = 1 - \frac{1}{\delta}\max\{|x-x'|, \sqrt{2}|t-t''|^{\frac{1}{2}}\}, & \text{on} \quad Q^*_\delta(P') \end{cases}$$

where $t'' := t' - \frac{\delta^2}{2}$. It can be easily checked that the boundary data given above is a continuous function on $\partial_p K_1$. Moreover applying regularity results for Dirichlet-type boundary value problems (see [45]) we have that $b_{P'}$ is Lipschitz in $\overline{K}_1$ with the corresponding constant depending only on δ and universal quantities (but not on P').

We claim that

$$b_{P'} > 0, \quad \text{in} \quad \overline{B}^*_{1/2} \times \left[\frac{1}{4}, \frac{3}{4}\right] \times \left[-\frac{\delta^2}{2}, 0\right] := K_2. \tag{4.3.7}$$

Indeed, note first that $b_{P'} \ge 0$ on $\partial_p K_1$, thus by maximum principle $b_{P'} \ge 0$ in $\overline{K}_1$. We suppose (to get a contradiction) that there exists some $(x_1, y_1, t_1) \in K_2$ with $b_{P'}(x_1, y_1, t_1) = 0$ which means that $b_{P'}$ attains its minimum over $\overline{K}_1$ at (x_1, y_1, t_1). Then strong maximum principle gives that

$$b_{P'} = 0, \quad \text{on} \quad \overline{K}_1 \cap \{t \le t_1\}.$$

Note that $t_1 \ge -\frac{\delta^2}{2} \ge t' - \frac{\delta^2}{2} > t' - \delta^2$ then there exist $(x,t) \in Q^*_\delta(P')$ such that $t < t_1$, that is $b_{P'}(x,0,t) > 0$ and $t < t_1$ which is a contradiction.

Now let $\varepsilon(P',\delta,n,\lambda,\Lambda) := \min_{K_2} b_{P'} > 0$ and

$$\tilde{\varepsilon}(\delta,n,\lambda,\Lambda) := \inf_{P'\in Q^*_{1-\delta}} \varepsilon(P',\delta,n,\lambda,\Lambda) \geq 0.$$

We want to show that $\tilde{\varepsilon} > 0$. We assume that $\tilde{\varepsilon} = 0$, then there exists $\{P'_j := (x'_j,t'_j)\}_{j\in\mathbb{N}} \subset Q^*_{1-\delta}$ so that $\varepsilon(P'_j,\delta,n,\lambda,\Lambda) \to 0$ as $j\to\infty$. Also for any $j\in\mathbb{N}$ there exists $(X_j,t_j)\in K_2$ so that $\varepsilon(P'_j,\delta,n,\lambda,\Lambda) = b_{P'_j}(X_j,t_j)$. We notice also that $\{P'_j\}$, $\{(X_j,t_j)\}$ are both bounded sequences and therefore there exist convergent subsequences (for which we use the same indices for simplicity). That is

$$P'_j \to P'_\infty \in \overline{Q}^*_{1-\delta}, \quad (X_j,t_j)\to(X_\infty,t_\infty), \quad \text{as } j\to\infty.$$

On the other hand $b_{P'_j}$ are equicontinuous and uniformly bounded in $\overline{K}_1$, thus there exist a uniformly convergent subsequence in $\overline{K}_1$, that is $b_{P'_j}\to b_\infty$ uniformly in $\overline{K}_1$ as $j\to\infty$ (using the same indices for simplicity). To get the contradiction it is enough to show that

$$b_\infty = b_{P'_\infty}, \quad \text{in} \quad \overline{K}_1. \tag{4.3.8}$$

Indeed, if (4.3.8) holds then by uniform convergence we have that $b_{P'_j}(X_j,t_j)\to b_{P'_\infty}(X_\infty,t_\infty)$ as $j\to\infty$ but $b_{P'_j}(X_j,t_j)\to 0$ as $j\to\infty$, thus $b_{P'_\infty}(X_\infty,t_\infty)=0$ which contradicts (4.3.7) since $(X_\infty,t_\infty)\in K_2$. Now to obtain (4.3.8) due to uniqueness it is enough to show that b_∞ solves the same Dirichlet problem as $b_{P'_\infty}$ in $\overline{K}_1$. From closedness of viscosity solutions (Proposition 10) we know that $\mathcal{M}^-\left(D^2 b_\infty,\frac{\lambda}{n},\Lambda\right)-(b_\infty)_t = 0$ in K_1. Also $b_\infty = 0$ on $\partial_p K_1\setminus Q^*_1$. Thus it remains to check the following two

1. $b_\infty(x,0,t) = 1-\frac{1}{\delta}\max\{|x-x'_\infty|,\sqrt{2}|t-t''_\infty|^{\frac{1}{2}}\}$ on $\overline{Q}^*_\delta(P'_\infty)$
2. $b_\infty = 0$ on $Q^*_1\setminus\overline{Q}^*_\delta(P'_\infty)$.

For (x,t) such that $|x-x'_\infty|<\delta$ and $|t-t''_\infty|<\frac{\delta^2}{2}$ then we can choose an integer $m=m(x,t,\delta)>\frac{3}{2\delta}$ so that $(x,t)\in Q^*_{\delta-\frac{1}{m}}(P'_\infty)$. Also, since $x'_j\to x'_\infty$ and $t'_j\to t'_\infty$ as $j\to\infty$, there exists integer $N=N(x,t,\delta)\in\mathbb{N}$ so that for any $j\geq N$

$$|x'_j - x'_\infty| < \frac{1}{m} \quad \text{and} \quad |t''_j - t''_\infty| < \frac{1}{m^2}.$$

Then for any $j\geq N$, $|x-x'_j|\leq|x-x'_\infty|+|x'_j-x'_\infty|<\delta-\frac{1}{m}+\frac{1}{m}=\delta$ and $|t-t''_j|\leq|t-t''_\infty|+$

$|t''_j - t''_\infty| < \frac{1}{2}\left(\delta - \frac{1}{m}\right)^2 + \frac{1}{m^2} = \frac{\delta^2}{2} - \frac{\delta}{m} + \frac{1}{2m^2} + \frac{1}{m^2} < \frac{\delta^2}{2}$, using that $m > \frac{3}{2\delta}$. So we have that $(x,t) \in \overline{Q}^*_\delta(P'_j)$ for any $j \geq N$, that is $b_{P'_j}(x,0,t) = 1 - \frac{1}{\delta}\max\{|x-x'_j|, \sqrt{2}|t-t''_j|^{\frac{1}{2}}\}$ and taking $j \to \infty$ we obtain 1. at (x,t). Note that for (x,t) such that $|x - x'_\infty| = \delta$ or $t'_\infty - \delta^2 = t$ or $t = t'_\infty$ we use the continuity of b_∞. Finally for $(x,t) \in Q^*_1 \setminus \overline{Q}^*_\delta(P'_\infty)$, that is $|x - x'_\infty| > \delta$ or $|t - t''_\infty| > \frac{\delta^2}{2}$ we follow a similar argument as before. We can choose an integer $m = m(x,t,\delta) > \frac{1}{\delta}$ so that $(x,t) \in Q^*_1 \setminus \overline{Q}^*_{\delta+\frac{1}{m}}(P'_\infty)$. Also there exists integer $N = N(x,t,\delta) \in \mathbb{N}$ so that for any $j \geq N$

$$|x'_j - x'_\infty| < \frac{1}{m} \quad \text{and} \quad |t''_j - t''_\infty| < \frac{1}{m^2}.$$

Then for $j \geq N$, if $|x - x'_\infty| > \delta + \frac{1}{m}$ then $|x - x'_j| > \delta + \frac{1}{m} - \frac{1}{m} = \delta$. If $|t - t''_\infty| > \frac{1}{2}\left(\delta + \frac{1}{m}\right)^2$ then $|t - t''_j| > \frac{1}{2}\left(\delta + \frac{1}{m}\right)^2 - \frac{1}{m^2} = \frac{\delta^2}{2} + \frac{\delta}{m} + \frac{1}{2m^2} - \frac{1}{m^2} > \frac{\delta^2}{2}$, using that $m > \frac{3}{2\delta}$. So we have that $(x,t) \in Q^*_1 \setminus \overline{Q}^*_\delta(P'_j)$ for any $j \geq N$, that is $b_{P'_j}(x,0,t) = 0$ and taking $j \to \infty$ we obtain 2. at (x,t). Thus (4.3.8) is true.

We have obtained that $\tilde{\varepsilon} > 0$ and if $\bar{P} = (\bar{x}, \bar{t})$ the given point we have that $b_{\bar{P}} \geq \tilde{\varepsilon}$ in K_2. We use maximum principle to get this information for w as well. So let $v = w - b_{\bar{P}}$ then $v \in \overline{S}_p\left(\frac{\lambda}{n}, \Lambda\right)$ in K_1. Moreover if $(x,t) \in Q^*_\delta(\bar{P})$ then $\liminf_{y\to 0^+} v(x,y,t) \geq 1 - b_{\bar{P}}(x,0,t) \geq 0$ from the definition of $b_{\bar{P}}$ and if $(x,t) \in \partial_p K_1 \setminus Q^*_\delta(\bar{P})$ then $\liminf_{y\to 0^+} v(x,y,t) \geq 0$ since $w \geq 0$. Then maximum principle gives that $w \geq b_{\bar{P}} \geq \tilde{\varepsilon}$ in K_2. □

The next lemma is a consequence of an iterative argument.

Lemma 78. *Let $(x_0,t_0) \in \Omega^* \cap Q^*_{1/2}$, then there exists universal constants $0 < \alpha < 1$, $C > 0$ so that*

$$0 \geq \sigma(x,t) \geq -C\left(|x - x_0| + |t - t_0|^{1/2}\right)^\alpha, \quad \textit{for any } \ (x,t) \in Q^*_{1/2}(x_0,t_0).$$

Proof. Our aim is to show that for any $k \in \mathbb{N}$

$$u_y(X,t) \geq -C\theta^k, \quad \text{for every} \quad (X,t) \in Q^*_{r^k}(x_0,t_0) \times \{y \in (0, r^k)\} \qquad (4.3.9)$$

where $0 < r << \theta < 1$ to be chosen and $C > 0$ universal. We proceed by induction. For $k = 1$ it follows by *(A)* of Proposition 72 choosing the right C. We assume that (4.3.9) holds for some k and we prove it for $k+1$.

We define

$$w = \frac{u_y + C\theta^k}{-\mu r^k + C\theta^k}, \quad \text{in} \quad Q^*_{r^k}(x_0, t_0) \times \{y \in (0, r^k)\}$$

where $0 < \mu < 1$ a small constant to be chosen. Then by the hypothesis of the induction $u_y + C\theta^k \geq 0$ in $Q^*_{r^k}(x_0, t_0) \times \{y \in (0, r^k)\}$ and choosing $r < \theta$ and $\mu < C$ we have that $-\mu r^k + C\theta^k > 0$, that is $w \geq 0$ in $Q^*_{r^k}(x_0, t_0) \times \{y \in (0, r^k)\}$. Moreover, $\mathcal{M}^-\left(D^2 w, \frac{\lambda}{n}, \Lambda\right) - w_t \leq 0$ in $Q^*_{r^k}(x_0, t_0) \times \{y \in (0, r^k)\}$. We observe also that

$$\lim_{y \to 0^+} w(x, y, t) = \frac{\sigma(x, t) + C\theta^k}{-\mu r^k + C\theta^k}, \quad \text{for} \quad (x, t) \in Q^*_{r^k}(x_0, t_0).$$

On the other hand applying Lemma 76 around $(x_0, t_0) \in \Omega^* \cap Q^*_{1/2}$ with $\gamma = \mu r^k < \mu r < \frac{1}{2}$ we get that there exists

$$Q^*_{\bar{C}\mu r^k}(\bar{x}, \bar{t}) \subset Q^*_{\mu r^k}(x_0, t_0) \cap \Omega^*_{\mu r^k}$$

where $0 < \bar{C} < 1$ depends only on K, n, λ, Λ and ρ. Thus $\sigma > -\mu r^k$ on $Q^*_{\bar{C}\mu r^k}(\bar{x}, \bar{t})$. That is,

$$\lim_{y \to 0^+} w(x, y, t) \geq 1, \quad \text{for} \quad (x, t) \in Q^*_{\bar{C}\mu r^k}(\bar{x}, \bar{t}).$$

Therefore, w satisfies the assumptions of Lemma 77 in $Q^*_{r^k}(x_0, t_0) \times (0, r^k)$. So we apply Lemma 77 to the rescaled $W(x, y, t) := w(\mu r^k x + x_0, \mu r^k y, (\mu r^k)^2 t + t_0)$ in K_1 and obtain that

$$w \geq \varepsilon, \quad \text{in} \quad \overline{B}^*_{\frac{\mu r^k}{2}}(x_0) \times \left[\frac{\mu r^k}{4}, \frac{3\mu r^k}{4}\right] \times \left[t_0 - \frac{(\bar{C}\mu r^k)^2}{2}, t_0\right] \tag{4.3.10}$$

where $\varepsilon = \varepsilon(\bar{C}, n, \lambda, \Lambda) > 0$, that is, $u_y \geq -C\theta^k + \varepsilon(C - \mu)\theta^k \geq -C\theta^k + \frac{\varepsilon C\theta^k}{2}$ using that $r < \theta$ and choosing $\mu < \frac{C}{2}$.

Now to fill the gap of $y \in \left(0, \frac{\mu r^k}{4}\right]$ we integrate u_{yy} with respect to y and use *(C)* of Proposition 72. For $(x, t) \in \overline{B}^*_{\frac{\mu r^k}{2}}(x_0) \times \left[t_0 - \frac{(\bar{C}\mu r^k)^2}{2}, t_0\right]$ we have

$$u_y\left(x, \frac{\mu r^k}{2}, t\right) - u_y(x, y, t) = \int_y^{\frac{\mu r^k}{2}} u_{yy}(x, h, t)\, dh \leq C_0 \frac{\mu r^k}{2} - C_0 y$$

where $C_0 > 0$ the constant of Proposition 72. Then

$$u_y(x,y,t) \geq u_y\left(x, \frac{\mu r^k}{2}, t\right) - C_0\frac{\mu r^k}{2} \geq -C\theta^k + \frac{\varepsilon C\theta^k}{2} - C_0\frac{\mu r^k}{2}.$$

Therefore in $\overline{B}^*_{\frac{\mu r^k}{2}}(x_0) \times \left(0, \frac{3\mu r^k}{4}\right] \times \left[t_0 - \frac{(\bar{C}\mu r^k)^2}{2}, t_0\right]$ we have that

$$u_y(x,y,t) \geq -C\theta^k + \frac{\varepsilon C\theta^k}{2} - C_0\mu r^k.$$

We choose $0 < r < \min\left\{\frac{\mu}{2}, \frac{\bar{C}\mu}{\sqrt{2}}\right\} < \frac{1}{2}$ then the above holds in $\overline{B}^*_{r^{k+1}}(x_0) \times (0, r^{k+1}) \times \left[t_0 - (r^{k+1})^2, t_0\right]$. Also using that $r < \theta$ and choosing $\mu < \frac{C\varepsilon}{4C_0}$ and $\theta > 1 - \frac{\varepsilon}{4}$ we have $-C\theta^k + \frac{\varepsilon C\theta^k}{2} - C_0\mu r^k \geq -C\left(\theta^k - \frac{\varepsilon}{4}\theta^k\right) \geq -C\theta^{k+1}$ and the induction is complete.

Taking $y \to 0^+$ in (4.3.10) we have that for any $k \in \mathbb{N}$

$$\sigma(x,t) \geq -C\theta^k, \quad \text{for every} \quad (x,t) \in Q^*_{r^k}(x_0,t_0)$$

where $0 < r << \theta < 1$ and $C > 0$ universal. The above gives the desired regularity for σ. Indeed, for $0 < R < \frac{1}{2}$ we consider the quantity

$$a_{(x_0,t_0)}(R) := \sup_{Q^*_R(x_0,t_0)} |\sigma|.$$

Then we have that $a_{(x_0,t_0)}(r^k) \leq \theta^k$ for any $k \in \mathbb{N}$. We choose $m_0 \in \mathbb{N}$ so that $r^{m_0} < R \leq r^{m_0-1}$, thus $a_{(x_0,t_0)}(R) \leq \theta^{m_0-1}$ and $\theta^{-\log r^{m_0}} \leq \theta^{-\log R} = R^{-\log\theta}$, that is, $\theta^{m_0} \leq R^{\frac{\log\theta}{\log r}}$. Choosing $0 < \alpha := \frac{\log\theta}{\log r} < 1$ we have $a_{(x_0,t_0)}(R) \leq \frac{1}{\theta}\theta^{m_0} \leq CR^\alpha$. Finally, for $(x,t) \in Q^*_{1/2}(x_0,t_0)$ let $R = \max\left\{|x-x_0|, |t-t_0|^{1/2}\right\} < \frac{1}{2}$. Then $(x,t) \in \overline{Q}^*_R(x_0,t_0)$, so $|\sigma(x,t)| \leq CR^\alpha \leq C\left(|x-x_0| + |t-t_0|^{1/2}\right)^\alpha$. □

Now we are ready to obtain the main theorem.

Theorem 79. *Let $P_0 = (x_0,t_0) \in \Gamma^* \cap Q^*_{1/2}$, there exist universal constants $0 < \alpha < 1, C > 0, 0 < r << 1$ and an affine function $R_0(X) = A_0 + B_0 \cdot (X - (x_0,0))$, where $A_0 = u(P_0)$, $B_0 = Du(P_0)$ so that*

$$|u(X,t) - R_0(X)| \leq C\left(|X - (x_0,0)| + |t-t_0|^{1/2}\right)^{1+\alpha}, \quad \textit{for any} \;\; (X,t) \in Q^+_r(P_0).$$

Proof. First we use Lemma 78 to get the regularity of σ around P_0. So Lemma 78 gives the following two:

1. $\sigma(x_0,t_0)=0$. Indeed we know that $\sigma=0$ in Ω^* and since $\Gamma^*=\partial\Omega^*\cap Q_1^*$ there exists $\{(x_k,t_k)\}_{k\in\mathbb{N}}\subset\Omega^*\cap\overline{Q}^*_{1/2}$ so that $(x_k,t_k)\to(x_0,t_0)$ as $k\to\infty$. We have $0\geq\sigma(x_0,t_0)\geq -C\left(|x_0-x_k|+|t_0-t_k|^{1/2}\right)^\alpha$ for any large $k\in\mathbb{N}$. Thus taking $k\to\infty$ we get the desired.

2. $0\geq\sigma(x,t)\geq -C\left(|x-x_0|+|t-t_0|^{1/2}\right)^\alpha$, for any $(x,t)\in\overline{Q}^*_{1/4}(x_0,t_0)$. Indeed, we consider again $\{(x_k,t_k)\}_{k\in\mathbb{N}}\subset\Omega^*\cap\overline{Q}^*_{1/2}$ so that $(x_k,t_k)\to(x_0,t_0)$ as $k\to\infty$. We have $0\geq\sigma(x,t)\geq -C\left(|x-x_k|+|t-t_k|^{1/2}\right)^\alpha$ for any large $k\in\mathbb{N}$ and any $(x,t)\in\overline{Q}^*_{1/4}(x_0,t_0)$ and we let $k\to\infty$.

On the other hand we know that $u_y=\sigma$ on Q_1^* in the classical sense but since u_y is continuous in (X,t) (regarding 2.) this holds also in the viscosity sense. Thus once the Neumann data σ is H^α (regarding 2. again) we can apply Theorem 54 in $\overline{Q}^+_{1/4}(x_0,t_0)$ and get the desired. □

Conclusions

The research presented in this thesis produce the primary theory for problems with immediate connections to applications that involve nonlinear equations for certain types of free boundary problems. More precisely, in the present thesis we succeeded to derive several boundary Hölder estimates for problems that involve fully nonlinear parabolic operators. The text was essentially splitted into two parts.

In the first part (Chapter 3), we concentrated to a fully nonlinear parabolic oblique derivative problem and we derived delicate Hölder estimates of zero, first and second order for the viscosity solutions, up to the flat boundary, under suitable assumptions on the data. These results contribute in the completion of the standard parabolic regularity theory and establish a solid framework which will initiate the study of even more general degenerate/ singular operators and of more general (non-flat) domains of definition. The results of Chapter 3 appear in [15].

In the second part (Chapter 4), we dealt with a fully nonlinear parabolic thin obstacle problem. Under natural assumptions on the given data, we managed to derive first order Hölder estimates, up to the flat boundary, for the corresponding viscosity solution. This result is the initial step towards the understanding of this free boundary problem and opens the way to the study of the even more demanding questions of the optimal regularity of the solution and the regularity of the free boundary. The results of Chapter 4 appear in [14].

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