University of Cyprus

Department of Mathematics & Statistics

Cessation of circular and annular Couette flows of a Newtonian liquid with dynamic wall slip

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Title of dissertation

Cessation of circular and annular Couette flow of a Newtonian liquid with dynamic wall slip

(Παύση της κυκλικής και δακτυλιοειδούς ροής Couette ενός Νευτώνειου υγρού με δυναμική ολίσθηση στα τοιχώματα)

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Abstract: Analytical solutions are derived for the cessation Newtonian Couette flows with wall slip obeying a dynamic slip model. The circular Couette flow problem will be the first one that will be studied and afterwards the problem of annular Couette flow will be solved. In circular Couette flow, there are two rotating vertical coaxial cylinders of infinite length and the inner cylinder is rotating. In annular Couette flow, there are two horizontal coaxial cylinders of infinite length and the outer cylinder is sliding. The steady-state solution with no-slip at the walls along with the application of Navier and dynamic slip at the walls will be the way the solution of these two problems will be derived with dynamic slip being the most important part. This slip equation allows for a relaxation time in the development of wall slip by means of a time-dependent term which forces the eigenvalue parameter to appear in the boundary conditions. The resulting spatial problem corresponds to a Sturm–Liouville problem different from that obtained using the static Navier slip condition. The orthogonality condition of the associated eigenfunctions is derived and the solutions are provided for the circular and annular Couette flow.

Keywords: Newtonian fluid ▪ Couette flow ▪ Cessation flow ▪ Navier slip ▪ Dynamic slip

Περίληψη: Αναλυτικές λύσεις προκύπτουν για την παύση των Νευτώνειων Couette ροών με συνθήκη ολίσθηση να εφαρμόζεται στα τοιχώματα που ακολουθεί το μοντέλο δυναμικής ολίσθησης. Το πρόβλημα της κυκλικής ροής Couette θα είναι το πρώτο που θα μελετηθεί και στη συνέχεια θα λυθεί το πρόβλημα της δακτυλιοειδούς ροής Couette. Στην κυκλική ροή Couette, υπάρχουν δύο περιστρεφόμενοι κάθετοι ομοαξονικοί κύλινδροι άπειρου μήκους και ο εσωτερικός κύλινδρος περιστρέφεται. Στη δακτυλιοειδή ροή Couette, υπάρχουν δύο οριζόντιοι ομοαξονικοί κύλινδροι άπειρου μήκους και ο εξωτερικός κύλινδρος ολισθαίνει. Η λύση μόνιμης ροής χωρίς ολίσθηση στα τοιχώματα μαζί με την εφαρμογή Navier και δυναμικής ολίσθησης στα τοιχώματα θα είναι ο τρόπος με τον οποίο θα προκύψει η λύση των δύο προβλημάτων με τη δυναμική ολίσθηση να είναι το πιο σημαντικό κομμάτι. Αυτή η εξίσωση ολίσθησης επιτρέπει ένα χρόνο χαλάρωσης στην ανάπτυξη της ολίσθησης του τοίχου μέσω ενός χρονικά εξαρτώμενου όρου που αναγκάζει την παράμετρο ιδιοτιμής να εμφανίζεται στις συνοριακές συνθήκες. Το χωρικό πρόβλημα που προκύπτει αντιστοιχεί σε ένα πρόβλημα Sturm–Liouville διαφορετικό από αυτό που προκύπτει χρησιμοποιώντας τη στατική συνθήκη ολίσθησης Navier. Η συνθήκη ορθογωνιότητας των σχετικών ιδιοσυναρτήσεων υπολογίζεται και παρέχονται οι λύσεις για την κυκλική και δακτυλιοειδή ροή Couette.

Λέξεις κλειδιά: Νευτώνεια ροή ▪ Ροή Couette ▪ Παύση ροής ▪ Navier ολίσθηση ▪ Δυναμική ολίσθηση

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Chapter 1: Introduction

The goal of this dissertation is to study the circular and annular Newtonian Couette flows with wall slip laws applied into the two cylinders of our problems. Slip at the wall can be occurred not only with non-Newtonian but also with Newtonian fluids. Several slip laws have been used in the literature, but we are going to see three of them here, the no-slip, the Navier slip and the dynamic slip law.

1.1 Wall slip laws

In rheology, the term "wall slip" refers to the phenomenon where a material flowing through a pipe or channel does not exhibit the same flow behavior near the wall as it does in the bulk of the material. Instead, the material near the wall may slip or slide along the surface, resulting in a different flow profile and a reduction in the effective viscosity of the material. (Talmon and Meshkati, 2022)

Wall slip is an important issue in many industrial processes, as it can lead to inaccurate measurements and inconsistent product quality. To account for wall slip, researchers have developed a number of "wall slip laws" that describe how the flow behavior of a material changes near the wall.

Several slip laws have been used in the literature, but we are going to see three of them here, the no-slip, the Navier slip and the dynamic slip.

No-slip boundary condition

The no-slip condition is a fundamental principle in fluid mechanics that describes the behavior of fluid flow at solid surfaces. According to this law, the velocity of a fluid at a solid boundary is zero, or the fluid particles "stick" to the surface of the solid. This condition applies to both liquids and gases.

At the microscopic level, the no-slip condition arises from the interaction between fluid particles and the molecules of the solid surface. When a fluid particle comes into contact with a solid surface, it experiences a force that causes it to slow down and eventually come to a stop. The force arises from a combination of molecular interactions such as van der Waals forces, electrostatic forces, and chemical interactions between the fluid and the solid surface.

The no-slip condition has important consequences for fluid flow at solid boundaries. For example, it means that the velocity of a fluid near a wall is zero, which in turn affects the flow profile of the fluid. It also means that the transport of momentum and heat across a solid boundary is limited, which can have important implications for heat transfer and fluid mixing.

In practical applications, the no-slip condition is often used to model fluid flow in pipes, channels, and other confined geometries. By assuming that the velocity of the fluid is zero at the walls, engineers can simplify the mathematical description of the flow and make predictions about pressure drops, flow rates, and other fluid properties.

If the velocity of a solid boundary is U_s, then the fluid particles adjacent to this boundary have the same velocity (Yilbas, 2018):

$$u_F = U_{S.} \quad \textbf{(1.1)}$$

Hence, if the wall is fixed (not moving), then $u_F = 0$.

Navier-slip boundary condition

The Navier slip law is a mathematical model used to describe the behavior of fluids at solid surfaces. It assumes that the fluid in contact with the surface does not adhere to it completely, but instead has a finite slip velocity along the surface. This slip velocity is characterized by a slip length, which is the distance over which the velocity profile of the fluid changes from its value at the surface to its value in the bulk.

The Navier slip law is named after Claude-Louis Navier, a French physicist and engineer who developed the theory of fluid mechanics in the 19th century. The law is based on the Navier-Stokes equations, which are a set of partial differential equations that describe the motion of fluids.

The Navier slip law is important in the study of fluid flow in microchannels and in the design of microfluidic devices, where the effects of surface interactions become more pronounced due to the small size of the channels. It is also used in the analysis of flow in porous media and in the modeling of boundary layers in fluid flow problems.

Here fluid particles are allowed to slip at the wall. Let us denote by u_w the relative velocity of the fluid particles to that of the wall,

$$u_w = |u_F - U_s|. \quad \textbf{(1.2)}$$

Navier's slip law (Navier, 1827) states that the slip velocity is proportional to the wall shear stress, τ_w , i.e.,

$$\tau_w = \beta u_w, \quad \textbf{(1.3)}$$

where β is the slip coefficient. The no-slip condition is recovered when $\beta \to \infty$; clearly, wall slip becomes stronger as β is reduced.

The coefficient in the Navier slip boundary condition, also known as the slip length, is a material property that depends on the surface characteristics of the solid boundary and the properties of the fluid flowing over it. It represents the distance from the boundary at which the fluid velocity becomes equal to the velocity of the solid boundary, and is defined as:

$$\beta = \delta - \delta_0,$$

where δ is the hydrodynamic boundary layer thickness, and δ_0 is the slip length for a perfectly smooth surface.

The value of β depends on various factors such as the surface roughness of the solid boundary, the viscosity of the fluid, the temperature of the fluid, and the velocity of the fluid. Generally, the slip length increases with increasing surface roughness and decreasing fluid viscosity. In addition, the slip length may also depend on the direction of flow and the type of fluid-solid interaction.

The value of β is typically determined experimentally or through molecular dynamics simulations and can vary widely depending on the specific system being studied.

Dynamic wall slip boundary condition

The dynamic slip law at the walls is a concept in fluid mechanics that describes the behavior of fluids flowing over a solid surface. It refers to the relationship between the velocity of the fluid at the wall and the shear stress that develops there.

In general, the behavior of fluids near a solid wall is influenced by a phenomenon known as the "no-slip" boundary condition, which states that the fluid velocity at the wall is zero. However, in some cases, this assumption does not hold true, and the fluid may slip over the wall to some extent.

The dynamic slip law is important in a wide range of applications, such as microfluidics, nanofluidics, and surface science, where the behavior of fluids near surfaces is of great interest. Understanding the dynamic slip law can help engineers and scientists design more efficient fluid systems and develop better models of fluid behavior.

When a fluid exhibits dynamic wall slip, the slip velocity at the wall depends on the history of the fluid motion. This dependence on past motion is often referred to as a “memory effect” and can be modeled using a memory parameter.

One possible approach to modeling dynamic wall slip with a memory parameter is to use a generalized Navier slip boundary condition, which accounts for the slip velocity at the wall as a function of both the current fluid velocity and its history. The specific form of the slip boundary condition will depend on the underlying physical mechanisms that lead to wall slip.

For example, in the case of a viscoelastic fluid, the slip velocity at the wall may depend on the deformation history of the fluid. In this case, a memory parameter can be introduced to describe the time-dependent behavior of the slip velocity.

Overall, the modeling of dynamic wall slip with a memory parameter can be a complex problem, requiring a detailed understanding of the underlying physics and appropriate mathematical models.

When slip is dynamic, the slip velocity does not adjust instantaneously to the wall shear stress. Eq. (1.3) is generalized by introducing a memory parameter or relaxation time λ (Hatzikiriakos and Dealy, 1991):

$$u_w + \lambda \frac{du_w}{dt} = \frac{\tau_w}{\beta} \ . \qquad \textbf{(1.4)}$$

In steady flow, Eq. (1.4) is equivalent to Navier’s slip law, which is given by Eq. (1.3).

1.2 Newtonian flows with dynamic wall slip

Throughout the literature, several problems have been solved and many analytical solutions have been derived for Newtonian flows with wall slips. In this chapter we are going to review two flows that have been solved in the past for Newtonian fluids with dynamic wall slip.

The first flow is from the paper Kaoullas et al. (2015) and gives us analytical solutions for the problem "Start-up and cessation Newtonian Poiseuille and Couette flows with dynamic slip". In this paper the authors derive analytical solutions for the start-up and cessation Newtonian Poiseuille and Couette flows with wall slip obeying a dynamic slip model. More specifically the authors study the start-up and cessation flows of axisymmetric Poiseuille flow, plane Poiseuille flow, plane Couette flow and circular Couette flow. (The last one is also part of this dissertation, so it is described in detail in subchapter 2.5). The authors conclude with the observation that "under a dynamic slip condition, the slip velocity rather than depending on the instantaneous value of the wall shear stress, also depends on its past states. This effect delays the evolution of the slip velocity, and also the flow development".

The second flow is from the paper of Abou-Dina et al. (2020) and gives us analytical solutions for the problem "Newtonian plane Couette flow with dynamic wall slip". The authors consider the flow of a Newtonian fluid contained between infinite, horizontal parallel plates, placed at a distance H apart. The fluid is assumed to be at rest and suddenly the upper plate starts moving horizontally at a speed V while the lower one is kept fixed.

When dynamic wall slip is considered, the authors derive analytical solutions with two methods, the standard separation of variables (Fourier) method and the one-sided Fourier method. The method that will be used in our problems is the method of separation of variables. The authors conclude their paper stating that "the fact that reaching a steady state in the presence of dynamic wall slip may take very long times is very important and can be used in rheometry. The analytical solution presented in this paper may be useful in calculating the slip relaxation coefficients from transient experiments in both Newtonian and generalized-Newtonian (e.g. power-law) fluids. More systematic experimental data on both Newtonian and non-Newtonian fluids will be most useful in understanding better the implications of dynamic slip in practice".

1.3 Objective and outline of the thesis

The objective is to derive analytical solutions for the circular and annular Couette cessation flows of a Newtonian fluid exhibiting dynamic wall slip. The latter is considered for the first time.

In Chapter 2, we are studying the analytical solutions of a Newtonian fluid in circular Couette flow, where the inner cylinder is rotating with an angular velocity Ω. In the beginning we apply the no-slip law and after that we repeat the process with the Navier slip law at the walls. Then we derive analytical solutions for the cessation of circular Couette flows with no-slip and Navier slip laws applied in our problems respectively. We continue with solving the problem, cessation of circular Couette flow but this time with dynamic slip at the walls. In the last subchapter we present a table with analytical solutions for the velocity of our fluid in the case that the outer cylinder is rotating instead of the inner.

In Chapter 3, we are studying the analytical solutions of a Newtonian fluid in annular Couette flow, where the outer cylinder is sliding with a velocity V when the inner cylinder is fixed. We proceed applying the same wall slips and derive analytical solution for this problem and in the last subchapter we present another table where the inner cylinder is sliding, and the outer cylinder is fixed.

In Chapter 4, we summarize our conclusions of this dissertation and we state any recommendations for future work.

Chapter 2: Circular Couette flow

In fluid dynamics, the Taylor–Couette flow is a type of flow where a fluid is contained between two concentric cylinders, with one of the cylinders rotating while the other is stationary. Taylor demonstrated that by increasing the angular velocity of the inner cylinder beyond a specific limit, the flow of the fluid between the cylinders becomes unsteady, resulting in the emergence of a new state that consists of symmetrical toroidal vortices, referred to as Taylor vortex flow. As a result, as the cylinder's angular speed is raised, the system experiences a series of disturbances that result in states with more complex patterns in space and time. The subsequent state is referred to as a wavy vortex flow.

When the Reynolds number is low, meaning low angular velocities, the flow is steady and only azimuthal. This flow known as circular Couette flow was named after Maurice Marie Alfred Couette, who employed the apparatus to determine viscosity. The research paper by Sir Geoffrey Ingram Taylor (Taylor,1923), which examined the stability of Couette flow, was a significant milestone in the progress of hydrodynamic stability theory. Taylor demonstrated that the no-slip condition is the correct boundary condition for viscous flows at a solid boundary, which was previously in dispute by the scientific community.

Circular Couette flow has wide applications in various fields, including, Magnetic fields, Heat transfer, Rheology and Chemical Engineering.

In Magnetic fields, the stability of the circular Couette flow is being examined in a system consisting of two cylinders that rotate around the same axis, with a ferrofluid filling the gap between them. A uniform magnetic field is applied in the same direction as the cylinder axis. Various models are being used to analyze the stability of this flow, with consideration given to the polydispersity of the ferrofluid to differing extents.(A. Leschhorn et al., 2009)

In Heat transfer, the stability of heated, incompressible Taylor-Couette flow has been investigated through numerical simulations. The study focuses on the impact of the centrifugal and gravitational potentials. The flow occurs between two cylinders that are concentric and differentially heated, with the inner cylinder allowed to rotate.(R. Kedia et al., 1998)

In Rheology, the impact of non-Newtonian rheology on mixing efficiency is not yet fully understood. To shed light on this topic, researchers conducted a study using particle image velocimetry and flow visualization to analyze the effect of shear-thinning rheology on a Taylor-Couette reactor.(Cagney and Balabani, 2019)

In Chemical Engineering, Taylor-Couette flows, which occur between two concentric cylinders, have many potential applications, especially in small-scale two-phase devices for solvent extraction. To explore this further, an experimental device was created with two cylinders, one rotating and one fixed, and the option to add pressure-driven axial flow. Taylor-Couette flow progresses to turbulence via a series of hydrodynamic instabilities, which can significantly impact mixing and the axial dispersion coefficient. These flow bifurcations can also lead to flawed modeling of the interaction between flow and mass transfer, making them a crucial factor to consider. (Nemri et al., 2013)

In this chapter we study the steady, axisymmetric, torsional flow of an incompressible Newtonian liquid between two rotating vertical coaxial cylinders of infinite length with radii R and κR where $0 < \kappa < 1$ and the inner cylinder has angular velocity Ω, so we assume $u_r = u_z = 0$, $du_\theta/d\theta = 0$, $dp/d\theta = 0$ and $g = -ge_z$. In Fig.1 we can see the geometry of circular Couette flow which will interest us in this chapter.

As a result of our assumptions, the θ-momentum equation gives:

$$\frac{\partial u_\theta}{\partial t} = v\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{1}{r^2}u_\theta\right) \qquad \textbf{(2.1)}$$

where, $v = \eta/\rho$ is the kinematic viscosity.

The steady-state solution is found by setting $\partial u_\theta/\partial t = 0$ and integrating twice.

The general form of the angular velocity u_θ is given by (Papanastasiou et al.,1999)

$$u_\theta(r) = c_1 r + \frac{c_2}{r}, \qquad \textbf{(2.2)}$$

and the wall shear stress is given by

$$\tau_{r\theta} = \tau_{\theta r} = -\frac{2\eta c_2}{r^2} = \eta r \frac{d}{dr}\left(\frac{u_\theta}{r}\right), \qquad \textbf{(2.3)}$$

$$\tau_w = \beta\, u_w\,. \qquad \textbf{(2.4)}$$

In each of the following paragraphs we are going to find these constants and derive the analytical solution of the velocity.

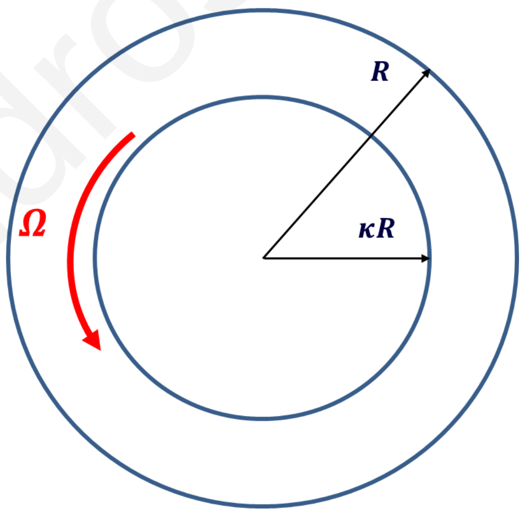

Fig. 1. Geometry of circular Couette flow

2.1 The steady-state circular Couette flow with no slip at the walls

The geometry of the steady-state circular Couette flow with no slip at the walls can be seen in *Fig. 2*. For this problem the inner cylinder is rotating with angular velocity Ω and there is no slip at the walls. As a result, the boundary conditions are:

$$r = \kappa R \quad u_\theta = \Omega \kappa R, \tag{2.5}$$

$$r = R \quad u_\theta = 0. \tag{2.6}$$

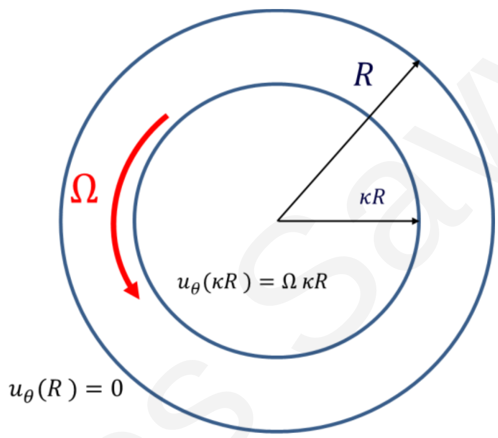

Fig. 2. Geometry of the steady state circular Couette flow with the no-slip laws applied at the walls

Appling the boundary conditions in equation **(2.2)** we get:

$$u_\theta(\kappa R) = \Omega \kappa R = c_1 \kappa R + \frac{c_2}{\kappa R} \ , \tag{2.7}$$

$$u_\theta(R) = 0 = c_1 R + \frac{c_2}{R} \ . \tag{2.8}$$

Solving the above system, we get:

$$c_2 = \frac{\Omega \kappa^2 R^2}{1-\kappa^2} \ , \quad c_1 = -\frac{\Omega \kappa^2}{1-\kappa^2} \ .$$

Substituting the above in equation **(2.2)** we get that the velocity is given by:

$$u_\theta(r) = \frac{\Omega \kappa^2 R}{1-\kappa^2} \left(\frac{R}{r} - \frac{r}{R} \right) . \tag{2.9}$$

The velocity in both walls will be zero because of the no slip condition:

$$u_{w_1} = u_{w_2} = 0 \ . \tag{2.10}$$

Substituting $c_2 = \Omega \kappa^2 R^2 / (1-\kappa^2)$ in equation **(2.3)** we get that the wall shear stress is given by:

$$\tau_{r\theta} = -\frac{2\eta\Omega\kappa^2 R^2}{(1-\kappa^2)r^2} \ . \tag{2.11}$$

The wall shear stress in each of the walls will be

$$\tau_{w_1} = \frac{2\eta\Omega}{(1-\kappa^2)} \quad , \quad \tau_{w_2} = \kappa^2 \tau_{w_1} \ . \tag{2.12}$$

We close this section by finding the dimensionless equations for velocity and wall shear stresses which are going to help us in the sequel.

We divide both parts of our equation with ΩR and we set $u_\theta^* = u_\theta/\Omega R$ and $r^* = r/R$ to get:

$$u_\theta^* = \frac{\kappa^2}{1-\kappa^2}\left(\frac{1}{r^*} - r^*\right) \ . \tag{2.13}$$

Likewise,

$$\tau_{r\theta}^* = -\frac{2\kappa^2}{r^{*2}(1-\kappa^2)} \ , \tag{2.14}$$

where

$$\tau_{r\theta}^* = \frac{\tau_{r\theta}}{\eta\Omega} \ .$$

Additionally,

$$\tau_{w_1}^* = \frac{2}{(1-\kappa^2)} \quad , \quad \tau_{w_2}^* = \kappa^2 \tau_{w_1} \ . \tag{2.15}$$

■

2.2 The steady-state circular Couette flow with Navier slip at the walls

The geometry of the steady-state circular Couette flow with Navier slip at the walls can be seen in *Fig. 3*. For this problem the inner cylinder is rotating with angular velocity Ω and at the walls we have Navier slip, so the velocities at the walls will not be zero. As a result, the boundary conditions are:

$$r = \kappa R \quad u_\theta = \Omega\kappa R - \mathrm{u}_{\mathrm{w}_1}\,, \tag{2.16}$$

$$r = R \quad u_\theta = u_{w_2}\,. \tag{2.17}$$

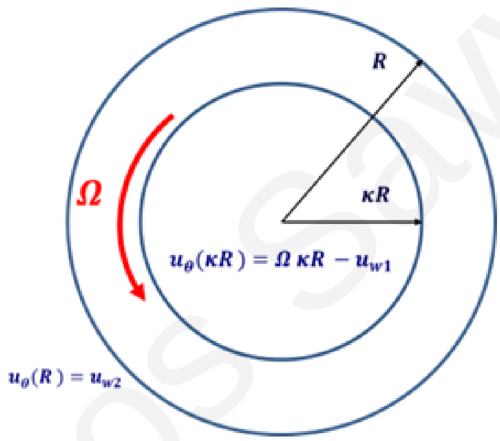

Fig. 3. Geometry of the steady-state circular Couette flow with the Navier slip laws applied at the walls

Appling the boundary conditions in equation **(2.2)** we get:

$$u_\theta(\kappa R) = \Omega\kappa R - u_{w_1} = c_1\kappa R + \frac{c_2}{\kappa R}\,, \tag{2.18}$$

$$u_\theta(R) = u_{w_2} = c_1 R + \frac{c_2}{R}\,. \tag{2.19}$$

By Eq. **(2.4)** we have:

$$u_{w_1} = \frac{2\eta c_2}{\beta\kappa^2 R^2}\,, \tag{2.20}$$

$$u_{w_2} = \frac{2\eta c_2}{\beta R^2}\,. \tag{2.21}$$

From Eqs. **(2.18)** and **(2.20)** we get

$$\Omega\kappa R - \frac{2\eta c_2}{\beta\kappa^2 R^2} = c_1\kappa R + \frac{c_2}{\kappa R}\,, \tag{2.22}$$

and from Eqs. **(2.19)** and **(2.21)** we get

$$c_1 = \frac{2\eta c_2}{\beta R^3} - \frac{c_2}{R^2} \Rightarrow c_1 = \frac{(2\eta - \beta R)c_2}{\beta R^3}\,. \tag{2.23}$$

Equations **(2.22)** and **(2.23)** give:

$$c_1 = \frac{\Omega\kappa^3(2\eta - \beta R)}{2\eta\,(1+\kappa^3) + \beta\kappa R(1-\kappa^2)} \quad , \qquad c_2 = \frac{\Omega\beta\kappa^3 R^3}{2\eta\,(1+\kappa^3) + \beta\kappa R(1-\kappa^2)} \; .$$

Therefore, the velocity is given by:

$$u_\theta(r) = \frac{\Omega\kappa^3 R}{2\eta\,(1+\kappa^3) + \beta\kappa R(1-\kappa^2)}\left((2\eta - \beta R)\frac{r}{R} + \beta R\frac{R}{r}\right) \; . \tag{2.24}$$

Substituting $c_2 = \Omega\beta\kappa^3 R^3/(2\eta\,(1+\kappa^3) + \beta\kappa R(1-\kappa^2))$ in equation **(2.3)** we get that the wall shear stress is given by:

$$\tau_{r\theta} = -\frac{2\eta\Omega\beta\kappa^3 R^3}{2\eta\,(1+\kappa^3) + \beta\kappa R(1-\kappa^2)}\frac{1}{r^2} \, , \tag{2.25}$$

and the wall shear stress in each of the walls is:

$$\tau_{w_1} = \frac{2\eta\Omega\beta\kappa\, R}{2\eta\,(1+\kappa^3) + \beta\kappa R(1-\kappa^2)} \quad , \quad \tau_{w_2} = \kappa^2\tau_{w_1} . \tag{2.26}$$

As a result, from Eq. **(2.4)**, the velocity in each of the walls will be:

$$u_{w_1} = \frac{2\eta\Omega\kappa\, R}{2\eta\,(1+\kappa^3) + \beta\kappa R(1-\kappa^2)} \quad , \quad u_{w_2} = \kappa^2 u_{w_1} . \tag{2.27}$$

We close by finding the dimensionless equations for our velocities and wall shear stresses. For our velocities we are dividing both parts of our equation with ΩR and we set $u_\theta^* = u_\theta/\Omega R$, $r^* = r/R$ and $B = \eta/\beta R$ so we get:

$$u_\theta^* = \frac{\kappa^3}{2B(1+\kappa^3) + \kappa(1-\kappa^2)}\left((2B-1)r^* + \frac{1}{r^*}\right) \; . \tag{2.28}$$

In Fig. 4, we can see the evolution of the velocity profile in circular Couette flow with Navier slip law applied in the walls for various values of *B* (Philippou et al.,2017). We observe that when we raise the value of *B*, the velocity starts decreasing slower each time. Fig. 5 shows the evolution of the velocity at the inner wall, also for various values of *B*. For better understanding of our result, we use a semilog scale. (Georgiou and Xenophontos,2007)

The velocities in the wall are:

$$u_{w_1}^* = \frac{2\kappa B}{2B(1+\kappa^3) + \kappa(1-\kappa^2)} \quad , \quad u_{w_2} = \kappa^2 u_{w_1} \; . \tag{2.29}$$

Likewise,

$$\tau_{r\theta}^* = -\frac{2\kappa^3}{2B(1+\kappa^3) + \kappa(1-\kappa^2)}\frac{1}{r^{*2}} \qquad \text{where} \qquad \tau_{r\theta}^* = \tau_{r\theta}/\eta\Omega \; , \tag{2.30}$$

(2.31)

$$\tau_{w_1}^* = \frac{2\kappa}{2B(1+\kappa^3)+\kappa(1-\kappa^2)} \quad , \quad \tau_{w_2} = \kappa^2 \tau_{w_1}.$$

Remark: When $B = 0$ $(\beta \to \infty)$ Eqs. **(2.28), (2.30)** and **(2.31)** are reduced to Eqs. **(2.13), (2.14)** and **(2.15)** respectively

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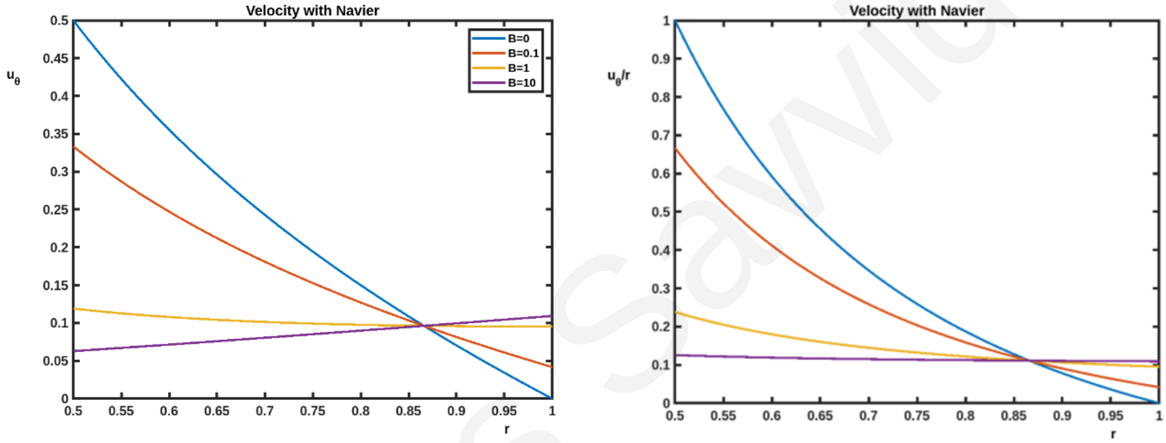
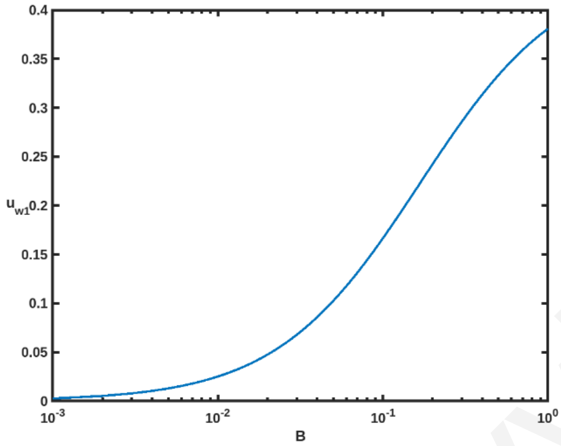
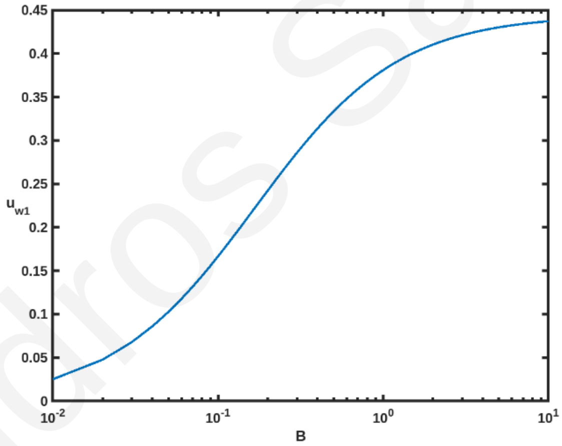


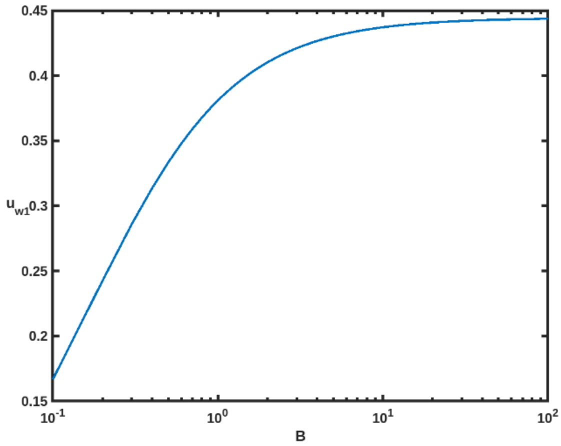
Fig. 4. Evolution of the velocity profile in circular Couette flow with Navier slip law applied in the walls for *B*= 0 , 0.1 , 1 and 10



(a)



(b)



(c)

Fig. 5 Evolution of the velocity at the inner wall when: **(a)** *B*= 0, 1 ; **(b)** *B*= 0, 10 ; **(c)** *B*= 0, 100

2.3 Cessation of circular Couette flow with no slip at the walls

The geometry of the cessation of circular Couette flow with no slip at the walls can be seen in *Fig. 6*. For this problem the inner cylinder is rotating with angular velocity Ω and there is no slip at the walls. When $t = 0$ the angular velocity ceases to exist. As a result, the boundary and initial conditions are:

$$r = \kappa R \quad u_\theta = 0 \quad t \geq 0 \ , \tag{2.32}$$

$$r = R \quad u_\theta = 0 \quad t > 0 \ , \tag{2.33}$$

$$u_\theta(r,0) = \frac{\Omega \kappa^2 R}{1-\kappa^2}\left(\frac{R}{r} - \frac{r}{R}\right) \ . \tag{2.34}$$

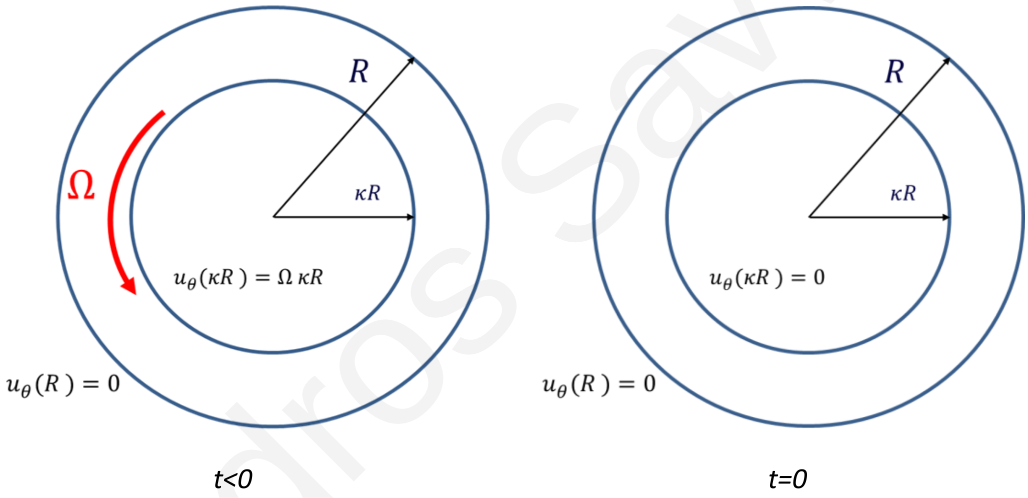


Fig. 6 Geometry of cessation of circular Couette flow with the no slip laws applied at the walls

We solve this initial boundary value problem with the method of separation of variables,

Let

$$u_\theta(r,t) = Y(r)T(t). \tag{2.35}$$

Substituting into Eq. **(2.1)** we get

$$Y(r)T'(t) = \nu\left(Y''(r)T(t) + \frac{1}{r}Y'(r)T(t) - \frac{1}{r^2}Y(r)T(t)\right)$$

Dividing by $\nu Y(r)T(t)$ we get

$$\frac{T'(t)}{\nu T(t)} = \frac{Y''(r)}{Y(r)} + \frac{1}{r}\frac{Y'(r)}{Y(r)} - \frac{1}{r^2}\frac{Y(r)}{Y(r)}$$

Because each side of the equation depends on different variables then each one should be equal with the same constant. Let this constant be $const. = -\frac{a^2}{R^2}$.

As a result, we get two new equations:

$$T'(t) = -\frac{a^2}{R^2} vT(t) \Rightarrow$$

$$T(t) = Ae^{-\frac{a^2}{R^2}vt} \quad . \tag{2.36}$$

Additionally

$$Y''(r) + \frac{1}{r} Y'(r) - \frac{1}{r^2}\left(1 - \frac{a^2}{R^2}r^2\right)Y(r) = 0 \Rightarrow$$

$$Y(r) = c_1 J_1\left(\frac{ar}{R}\right) + c_2 Y_1\left(\frac{ar}{R}\right) \quad . \tag{2.37}$$

Where J_1 and Y_1 are, first order Bessel functions of the first and second kind respectively. We let:

$$Z_1\left(\frac{ar}{R}\right) = J_1\left(\frac{ar}{R}\right) + \beta Y_1\left(\frac{ar}{R}\right) \, , \tag{2.38}$$

with β being a new constant.

From boundary conditions we get: $\beta = -\frac{J_1(a)}{Y_1(a)}$,

$$Z_1(\kappa\alpha) = J_1(\kappa\alpha) + \beta Y_1(\kappa\alpha) = 0 \, , \tag{2.39}$$

$$Z_1(\alpha) = J_1(\alpha) + \beta Y_1(\alpha) = 0 \, . \tag{2.40}$$

With superposition of the solution:

$$u_\theta(r,t) = \sum_{k=1}^{\infty} C_k Z_{1k}\left(\frac{\alpha_k r}{R}\right) e^{-\frac{a_k^2}{R^2}vt} \quad . \tag{2.41}$$

When $t = 0$ we get from equation **(2.41)** and the initial condition, the following:

$$\frac{\Omega\kappa^2 R}{1-\kappa^2}\left(\frac{R}{r} - \frac{r}{R}\right) = \sum_{\kappa=1}^{\infty} C_\kappa Z_{1\kappa}\left(\frac{\alpha_\kappa r}{R}\right) \quad . \tag{2.42}$$

The orthogonality condition states that:

$$\int_{\kappa R}^{R} Z_1^2\left(\frac{\alpha_n r}{R}\right) r \, dr = \left[\frac{r^2}{2R^2}\left\{Z_1'\left(\alpha_\kappa \frac{r}{R}\right)\right\}^2 + \frac{r^2}{2R^2}\left(1 - \frac{R^2}{\alpha_\kappa^2 r^2}\right)\left\{Z_1\left(\alpha_\kappa \frac{r}{R}\right)\right\}^2\right]_\kappa^1 \, .$$

In order to use the orthogonality condition, we multiply **(2.42)** by $rZ_1\left(\frac{\alpha_n r}{R}\right)$ and integrate from κR to R:

$$C_\kappa \int_{\kappa R}^{R} Z_1^2\left(\frac{\alpha_n r}{R}\right) r \, dr = \int_{\kappa R}^{R} \frac{\Omega\kappa^2 R}{1-\kappa^2}\left(\frac{R}{r} - \frac{r}{R}\right) r \, Z_1\left(\frac{\alpha_n r}{R}\right) \, dr \, .$$

Now let $\frac{r}{R} = \xi$ so if $r \in [\kappa R, R] \Rightarrow \xi \in [\kappa, 1]$ and if $\frac{r}{R} = \xi$ then $\frac{1}{R}dr = d\xi$.

So,

$$C_\kappa R^2 \int_\kappa^1 Z_1^2(\alpha_\kappa \xi)\, \xi \, d\xi = \int_\kappa^1 \frac{\Omega \kappa^2 R^3}{1-\kappa^2}\left(\frac{1}{\xi} - \xi\right) \xi \, Z_1(\alpha_\kappa \xi)\, d\xi \Rightarrow$$

$$C_\kappa = \frac{\Omega \kappa^2 R}{1-\kappa^2} \frac{\int_\kappa^1 (1-\xi^2)\, Z_1(\alpha_\kappa \xi)\, d\xi}{\int_\kappa^1 Z_1^2(\alpha_\kappa \xi)\, \xi \, d\xi} \quad . \tag{2.43}$$

We calculate:

$$I_1 := \int_\kappa^1 \xi \; Z_1^2(\alpha_\kappa \xi)\, d\xi = \left[\frac{\xi^2}{2}\{Z_1'(\alpha_\kappa \xi)\}^2 + \frac{\xi^2}{2}\left(1 - \frac{1}{\alpha_\kappa^2 \xi^2}\right)\{Z_1(\alpha_\kappa \xi)\}^2\right]_\kappa^1$$

$$= \left[\frac{1}{2}\left(Z_0(\alpha_\kappa) - \frac{1}{\alpha_\kappa} Z_1(\alpha_\kappa)\right)^2 + \frac{1}{2}\left(1 - \frac{1}{\alpha_\kappa^2}\right) Z_1^2(\alpha_\kappa) - \frac{\kappa^2}{2}\left(Z_0(\kappa\alpha_\kappa) - \frac{1}{\kappa\alpha_\kappa} Z_1(\kappa\alpha_\kappa)\right)^2 \right.$$

$$\left. - \frac{1}{2}\left(1 - \frac{1}{\alpha_\kappa^2}\right) Z_1^2(\kappa\alpha_\kappa)\right] = \frac{1}{2} Z_0^2(\alpha_\kappa) - \frac{\kappa^2}{2} Z_0^2(\kappa\alpha_\kappa)$$

$$I_2 := \int_\kappa^1 Z_1(\alpha_\kappa \xi)\, d\xi = \int_{\kappa\alpha_\kappa}^{\alpha_\kappa} Z_1(u) \frac{du}{\alpha_\kappa} = \frac{1}{\alpha_\kappa}[-Z_0(u)]_{\kappa\alpha_\kappa}^{\alpha_\kappa} = \frac{1}{\alpha_\kappa}[Z_0(\kappa\alpha_\kappa) - Z_0(\alpha_\kappa)]$$

and

$$I_3 := \int_\kappa^1 \xi^2 \, Z_1(\alpha_\kappa \xi)\, d\xi = \int_{\kappa\alpha_\kappa}^{\alpha_\kappa} \frac{u^2}{\alpha_\kappa^2} Z_1(u) \frac{du}{\alpha_\kappa} = \frac{1}{\alpha_\kappa^3}[u^2 Z_2(u)]_{\kappa\alpha_\kappa}^{\alpha_\kappa}$$

$$= \frac{1}{\alpha_\kappa}[Z_2(\alpha_\kappa) - \kappa^2 Z_2(\kappa\alpha_\kappa)]$$

$$= \frac{1}{\alpha_\kappa}\left[\frac{2}{\alpha_\kappa} Z_1(\alpha_\kappa) - Z_0(\alpha_\kappa) - \kappa^2\left(\frac{2}{\kappa\alpha_\kappa} Z_1(\kappa\alpha_\kappa) - Z_0(\kappa\alpha_\kappa)\right)\right]$$

$$= \frac{1}{\alpha_\kappa}\left(-Z_0(\alpha_\kappa) + \kappa^2 Z_0(\kappa\alpha_\kappa)\right)$$

Substituting in **(2.43)** we have:

$$C_\kappa = \frac{\Omega \kappa^2 R}{1-\kappa^2} \; \frac{\frac{1}{\alpha_\kappa}[Z_0(\kappa\alpha_\kappa) - Z_0(\alpha_\kappa) + Z_0(\alpha_\kappa) - \kappa^2 Z_0(\kappa\alpha_\kappa)]}{\frac{1}{2}\,[Z_0^2(\alpha_\kappa) - \kappa^2 Z_0^2(\kappa\alpha_\kappa)]}$$

$$= \frac{2\Omega\kappa^2 R(1-\kappa^2) Z_0(\kappa\alpha_\kappa)}{\alpha_\kappa(1-\kappa^2)[Z_0^2(\alpha_\kappa) - \kappa^2 Z_0^2(\kappa\alpha_\kappa)]} \; .$$

As a result,

$$C_k = \frac{2\Omega\kappa^2 Z_{0k}(\kappa a_k)R}{a_k[Z_{0k}^2(a_k) - \kappa^2 Z_{0k}^2(\kappa a_k)]} \ .$$

The velocity is (see Eq. **(2.41)**):

$$u_\theta(r,t) = \sum_{k=1}^{\infty} \frac{2\Omega\kappa^2 Z_{0k}(\kappa a_k)R}{a_k[Z_{0k}^2(a_k) - \kappa^2 Z_{0k}^2(\kappa a_k)]} Z_{1k}\left(\frac{a_k r}{R}\right) e^{-\frac{a_k^2}{R^2}\nu t} \ . \qquad \textbf{(2.44)}$$

Letting $t^* = \nu t/R^2, r^* = r/R \ , u_\theta^* = u_\theta/\kappa\Omega R$ we are getting the dimensionless velocity

$$u_\theta^*(r^*,t^*) = 2\kappa \sum_{k=1}^{\infty} \frac{Z_{0k}(\kappa a_k)}{a_k[Z_{0k}^2(a_k) - \kappa^2 Z_{0k}^2(\kappa a_k)]} Z_{1k}(a_k r^*)\, e^{-a_k^2 t^*} \qquad \textbf{(2.45)}$$

■

2.4 Cessation of circular Couette flow with Navier slip at the walls

The geometry of the cessation of circular Couette flow with Navier slip at the walls can be seen in *Fig. 7*. For this problem the inner cylinder is rotating with angular velocity Ω and there is Navier slip at the walls. When t=0 the angular velocity ceases to exist. As a result, the boundary and initial conditions are:

$$r = \kappa R \qquad u_\theta = u_{w_1} = BRr\frac{d}{dr}\left(\frac{u_\theta}{r}\right)\Big|_{r=\kappa R} \qquad t \geq 0\ , \tag{2.46}$$

$$r = R \qquad u_\theta = u_{w_2} = -BRr\frac{d}{dr}\left(\frac{u_\theta}{r}\right)\Big|_{r=R} \qquad t > 0\ , \tag{2.47}$$

$$u_\theta(r,0) = \frac{\Omega\kappa^3 R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(\frac{R}{r}-(1-2B)\frac{r}{R}\right)\ . \tag{2.48}$$

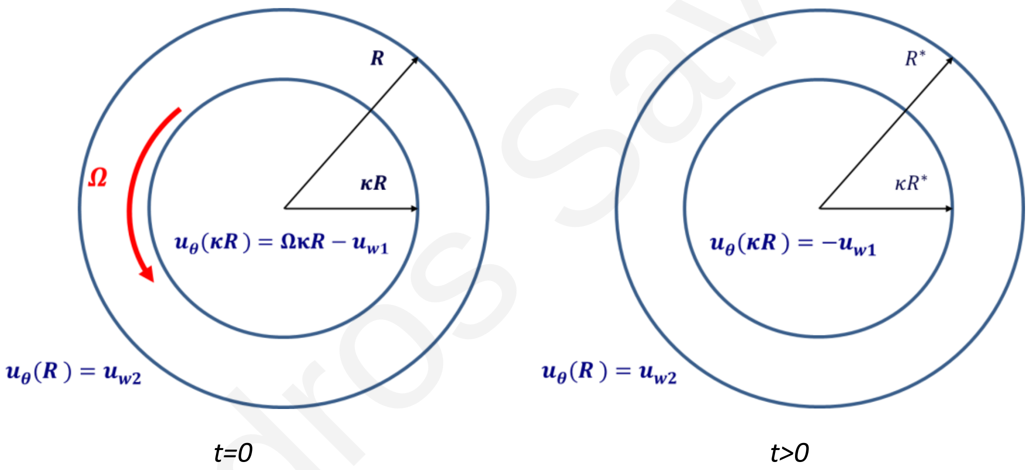


Fig. 7 Geometry of cessation of circular Couette flow with the Navier slip laws applied at the walls

We solve this initial boundary value problem with the method of separation of variables like in the previous paragraph and we get

$$T(t) = Ae^{-\frac{b^2}{R^2}\nu t} \text{ and } Y(r) = B\, Z_1\left(\frac{br}{R}\right)\ ,$$

where Z_1 is given by Eq. **(2.38)**. From the boundary conditions we get:

$$Y(r)T(t) = BRr\frac{d}{dr}\left(\frac{Y(r)T(t)}{r}\right)_{r=\kappa R} = BRr\left[\frac{r\frac{d}{dr}Y(r) - Y(r)}{r^2}\right]_{r=\kappa R}\ ,$$

$$Y(r)T(t) = -BRr\frac{d}{dr}\left(\frac{Y(r)T(t)}{r}\right)_{r=R} = BRr\left[\frac{r\frac{d}{dr}Y(r) - Y(r)}{r^2}\right]_{r=R}\ ,$$

So,

$$Z_1(\kappa b_\kappa) = \left[BR\frac{d}{dr}Z_1\left(b_\kappa\frac{r}{R}\right) - B\frac{R}{r}Z_1\left(b_\kappa\frac{r}{R}\right)\right]_{r=\kappa R},$$

$$\left(\frac{d}{dr}Z_1\left(b_\kappa\frac{r}{R}\right)=\frac{b_\kappa}{R}\left(\frac{1}{2}Z_0\left(b_\kappa\frac{r}{R}\right)-\frac{1}{2}Z_2\left(b_\kappa\frac{r}{R}\right)\right)\right.$$

$$=\frac{b_\kappa}{R}\left(\frac{1}{2}Z_0\left(b_\kappa\frac{r}{R}\right)-\frac{1}{2}\frac{2R}{rb_\kappa}Z_1\left(b_\kappa\frac{r}{R}\right)+\frac{1}{2}Z_0\left(b_\kappa\frac{r}{R}\right)\right)$$

$$\left.=\frac{b_\kappa}{R}Z_0\left(b_\kappa\frac{r}{R}\right)-\frac{1}{r}Z_1\left(b_\kappa\frac{r}{R}\right)\right)\Rightarrow$$

$$Z_1(\kappa b_\kappa)=\frac{BRb_\kappa}{R}Z_0(\kappa b_\kappa)-\frac{B}{\kappa}Z_1(\kappa b_\kappa)-\frac{B}{\kappa}Z_1(\kappa b_\kappa)\Rightarrow$$

$$Bb_\kappa Z_0(\kappa b_\kappa)-\left(1+\tfrac{2B}{\kappa}\right)Z_1(\kappa b_\kappa)=0\ . \qquad \textbf{(2.49)}$$

Likewise,

$$B\ b_\kappa\ Z_0(b_\kappa)+\ (1-2B)\ Z_1(b_\kappa)\ =0\ . \qquad \textbf{(2.50)}$$

From Eq. **(2.50)** we get $Bb_\kappa\big(J_0(b_\kappa)+\gamma_\kappa Y_0(b_\kappa)\big)+(1-2B)\big(J_1(b_\kappa)+\gamma_\kappa Y_1(b_\kappa)\big)=0$

$$\gamma_\kappa=-\ \frac{Bb_\kappa J_0(b_\kappa)+(1-2B)J_1(b_\kappa)}{Bb_\kappa Y_0(b_\kappa)+(1-2B)Y_1(b_\kappa)}\ .$$

Substituting γ_κ into Eq. **(2.49)** we get:

$$Bb_\kappa\left(\big(Bb_\kappa Y_0(b_\kappa)+(1-2B)Y_1(b_\kappa)\big)J_0(\kappa b_\kappa)-\big(Bb_\kappa J_0(b_\kappa)+(1-2B)J_1(b_\kappa)\big)Y_0(\kappa b_\kappa)\right)$$
$$=\left(1+\frac{2B}{\kappa}\right)\Big(\big(Bb_\kappa Y_0(b_\kappa)+(1-2B)Y_1(b_\kappa)\big)J_1(\kappa b_\kappa)$$
$$-\big(Bb_\kappa J_0(b_\kappa)+(1-2B)J_1(b_\kappa)\big)Y_1(\kappa b_\kappa)\Big)\ .$$

With superposition of the solution, we get:

$$u_{\ \theta}(r,t)=\sum_{k=1}^{\infty}D_k Z_{1k}\left(\frac{b_k r}{R}\right)e^{-\frac{b_k^2}{R^2}\nu t}\ . \qquad \textbf{(2.51)}$$

When $t=0$ we get from equation **(2.41)** and the initial condition the following:

$$\sum_{k=1}^{\infty}D_\kappa Z_{1\kappa}\left(\frac{b_\kappa r}{R}\right)=\frac{\Omega\kappa^3 R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(\frac{R}{r}-(1-2B)\frac{r}{R}\right)\ .$$

Using the same orthogonality condition like in the previous paragraph, we multiply by $rZ_1\left(\frac{b_n r}{R}\right)$ and integrate from κR till R:

$$D_\kappa\int_{\kappa R}^{R}\ Z_1^2\left(\frac{b_n r}{R}\right)r\,dr=\int_{\kappa R}^{R}\frac{\Omega\kappa^3 R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(\frac{R}{r}-(1-2B)\frac{r}{R}\right)\ Z_1\left(\frac{b_n r}{R}\right)\ r\ dr\ .$$

Now let $\frac{r}{R}=\xi$ so if $r\in[\kappa R,R]\ \Rightarrow\ \xi\in[\kappa,1]$ and if $\frac{r}{R}=\xi\quad then\ \frac{1}{R}dr=d\xi.$

So

$$D_\kappa R^2 \int_\kappa^1 Z_1^2(b_n\xi)\,\xi\,d\xi = \frac{\Omega\kappa^3 R^3}{2B(1+\kappa^3)+\kappa(1-\kappa^2)} \int_\kappa^1 \left(\frac{1}{\xi}-(1-2B)\xi\right) Z_1(b_n\xi)\,\xi\,d\xi \Rightarrow$$

$$D_\kappa = \frac{\Omega\kappa^3 R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)} \frac{\int_\kappa^1 \left(\frac{1}{\xi}-(1-2B)\xi\right) Z_1(b_n\xi)\,\xi\,d\xi}{\int_\kappa^1 Z_1^2(b_n\xi)\,\xi\,d\xi} \quad . \tag{2.52}$$

We let:

$$I_1 = \int_\kappa^1 Z_1(b_n\xi)\,d\xi = \int_{\kappa b_n}^{b_n} Z_1(u)\frac{du}{b_n} = \frac{1}{b_n}[-Z_0(u)]_{\kappa b_n}^{b_n} = \frac{1}{b_n}[Z_0(\kappa b_n)-Z_0(b_n)] \; ,$$

$$I_2 = \int_\kappa^1 \xi^2 Z_1(b_n\xi)\,d\xi = \int_{\kappa b_n}^{b_n} \frac{u^2}{b_n}Z_1(u)\frac{du}{b_n}$$

$$= \frac{1}{b_n^3}[u^2 Z_2(u)]_{\kappa b_n}^{b_n} = \frac{1}{b_n^3} b_n^2[Z_2(b_n)-\kappa^2 Z_2(\kappa b_n)]$$

$$= \frac{1}{b_n}\left[\frac{2}{b_n}Z_1(b_n) - Z_0(b_n) - \kappa^2\frac{2}{\kappa b_n}Z_1(\kappa b_n) + \kappa^2 Z_0(\kappa b_n)\right]$$

$$= \frac{1}{b_n}\left[\frac{2}{b_n}\frac{(-Bb_n)}{1-2B}Z_0(b_n) - Z_0(b_n) - \frac{2\kappa}{b_n}\frac{Bb_n}{\left(1+\frac{2B}{\kappa}\right)}Z_0(\kappa b_n) + \kappa^2 Z_0(\kappa b_n)\right]$$

$$= \frac{1}{b_n}\left[\frac{-2B-1+2B}{1-2B}Z_0(b_n) - \frac{2\kappa B-\kappa^2-2\kappa B}{1+\frac{2B}{\kappa}}Z_0(\kappa b_n)\right]$$

$$= -\frac{1}{b_n}\frac{\left(1+\frac{2B}{\kappa}\right)Z_0(b_n) - \kappa^2(1-2B)Z_0(\kappa b_n)}{(1-2B)\left(1+\frac{2B}{\kappa}\right)} \; ,$$

$$I_3 = \int_\kappa^1 \xi\,Z_1^2(b_n\xi)\,d\xi = \left[\frac{\xi^2}{2}\{Z_1'(b_\kappa\xi)\}^2 + \frac{\xi^2}{2}\left(1-\frac{1}{b_\kappa^2\xi^2}\right)\{Z_1(b_\kappa\xi)\}^2\right]_\kappa^1$$

$$= \left[\frac{\xi^2}{2}\left\{Z_0(b_\kappa\xi) - \frac{1}{b_\kappa\xi}Z_1(b_\kappa\xi)\right\}^2 + \frac{\xi^2}{2}\frac{(b_\kappa^2\xi^2-1)}{b_\kappa^2\xi^2}\{Z_1(b_\kappa\xi)\}^2\right]_\kappa^1$$

$$= \frac{1}{2}\left\{Z_0(b_\kappa) - \frac{1}{b_\kappa}Z_1(b_\kappa)\right\}^2 + \frac{1}{2}\frac{(b_\kappa^2-1)}{b_\kappa}\{Z_1(b_\kappa)\}^2 - \frac{\kappa^2}{2}\left\{Z_0(\kappa b_\kappa) - \frac{1}{b_\kappa}Z_1(\kappa b_\kappa)\right\}^2 - \frac{(\kappa^2 b_\kappa^2-1)}{2b_\kappa^2}\{Z_1(\kappa b_\kappa)\}^2$$

$$= \frac{1}{2}\left\{Z_0(b_\kappa) + \frac{Bb_\kappa}{b_\kappa(1-2B)} Z_0(b_\kappa)\right\}^2 + \frac{1}{2}\frac{B^2 b_\kappa^2 (b_\kappa^2 - 1)}{(1-2B)^2 b_\kappa^2} Z_0^2(b_\kappa)$$
$$- \frac{\kappa^2}{2}\left\{Z_0(\kappa b_\kappa) - \frac{1}{\kappa b_\kappa}\frac{Bb_\kappa}{\left(1+\frac{2B}{\kappa}\right)} Z_0(\kappa b_\kappa)\right\}^2$$
$$- \frac{(\kappa^2 b_\kappa^2 - 1)}{2b_\kappa^2}\frac{B^2 b_\kappa^2}{\left(1+\frac{2B}{\kappa}\right)^2} Z_0^2(\kappa b_\kappa)$$

$$= \frac{1}{2(1-2B)^2\left(1+\frac{2B}{\kappa}\right)^2}\left[\left(1+\frac{2B}{\kappa}\right)^2 (1-2B+B^2 b_\kappa^2) Z_0^2(b_\kappa)\right.$$
$$\left. - \kappa^2 (1-2B)^2 \left(1+\frac{2B}{\kappa} + B^2 b_\kappa^2\right) Z_0^2(\kappa b_\kappa)\right] .$$

Substituting in equation **(2.52)** we have:

$$D_\kappa = \frac{\Omega \kappa^3 R}{2B(1+\kappa^3) + \kappa(1-\kappa^2)} *$$

$$* \frac{\frac{1}{b_n}[Z_0(\kappa b_n) - Z_0(b_n)] + \frac{(1-2B)}{b_n}\frac{\left[\left(1+\frac{2B}{\kappa}\right)Z_0(b_n) - \kappa^2(1-2B)Z_0(\kappa b_n)\right]}{(1-2B)\left(1+\frac{2B}{\kappa}\right)}}{\frac{1}{2(1-2B)^2\left(1+\frac{2B}{\kappa}\right)^2}\left[\left(1+\frac{2B}{\kappa}\right)^2 (1-2B+B^2 b_\kappa^2) Z_0^2(b_\kappa) - \kappa^2(1-2B)^2\left(1+\frac{2B}{\kappa}+B^2 b_\kappa^2\right)Z_0^2(\kappa b_\kappa)\right]} .$$

As a result,

$$D_k = \frac{2\Omega\kappa^2 R(1-2B)^2 (1+\frac{2B}{\kappa}) Z_{0k}(\kappa b_k)}{b_k\left[\left(1+\frac{2B}{\kappa}\right)^2 (1-2B+B^2 b_\kappa^2) Z_0^2(b_\kappa) - \kappa^2(1-2B)^2\left(1+\frac{2B}{\kappa}+B^2 b_\kappa^2\right)Z_0^2(\kappa b_\kappa)\right]} .$$

The velocity (see Eq. **(2.51)**) is given by:

$$u_\theta(r,t) = \sum_{k=1}^{\infty} D_k Z_{1k}\left(\frac{b_k r}{R}\right) e^{-\frac{b_k^2}{R^2}\nu t} . \qquad \textbf{(2.53)}$$

Now we let $t^* = \nu t/R^2$, $r^* = r/R$, $u_\theta^* = u_\theta/\kappa\Omega R$, $D'_\kappa = D_\kappa/\kappa\Omega R$ we get the dimensionless form:

$$u_\theta^*(r^*, t^*) = \sum_{k=1}^{\infty} D'_\kappa Z_{1k}(b_k r^*)\, e^{-b_k^2 t^*} . \qquad \textbf{(2.54)}$$

<u>Remark:</u> When $B = 0$ $(\beta \to \infty)$ Eq. **(2.54)** is reduced to Eq. **(2.45)**

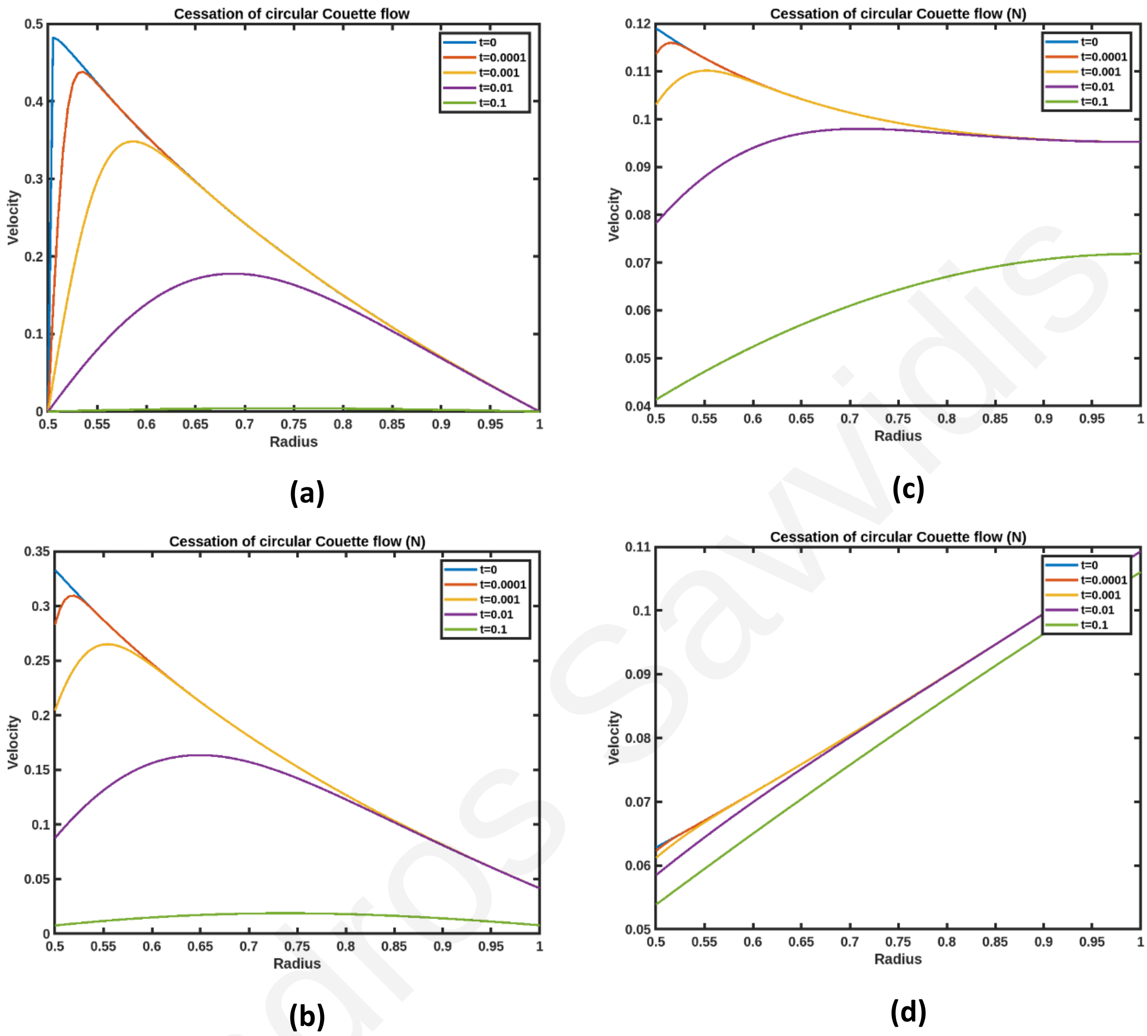


Fig. 8 Evolution of the velocity profile in cessation of circular Couette flow with *κ=0.5* and *t*=0, 0.0001, 0.001, 0.01 and 0.1*:* **(a)** *B=0* (no slip)*;* **(b)** *B=0.1* (weak slip)*;* **(c)** *B=1* (moderate slip)*;* **(d)** *B=10* (strong slip)

In Fig. 8, we can see the evolution of the velocity profile in cessation of circular Couette flow for different values of *B*. As expected the value of the velocity is decreasing but the gradient of the curve of the velocity is increasing. For *B*=10 our curve is linear and that is how we know that the slip is strong.

In Fig. 9, we have the velocity at the walls and as expected the velocity in the inner wall is decreasing in time and the velocity in the outer wall is increasing untill we reach the steady state velocities in the walls.

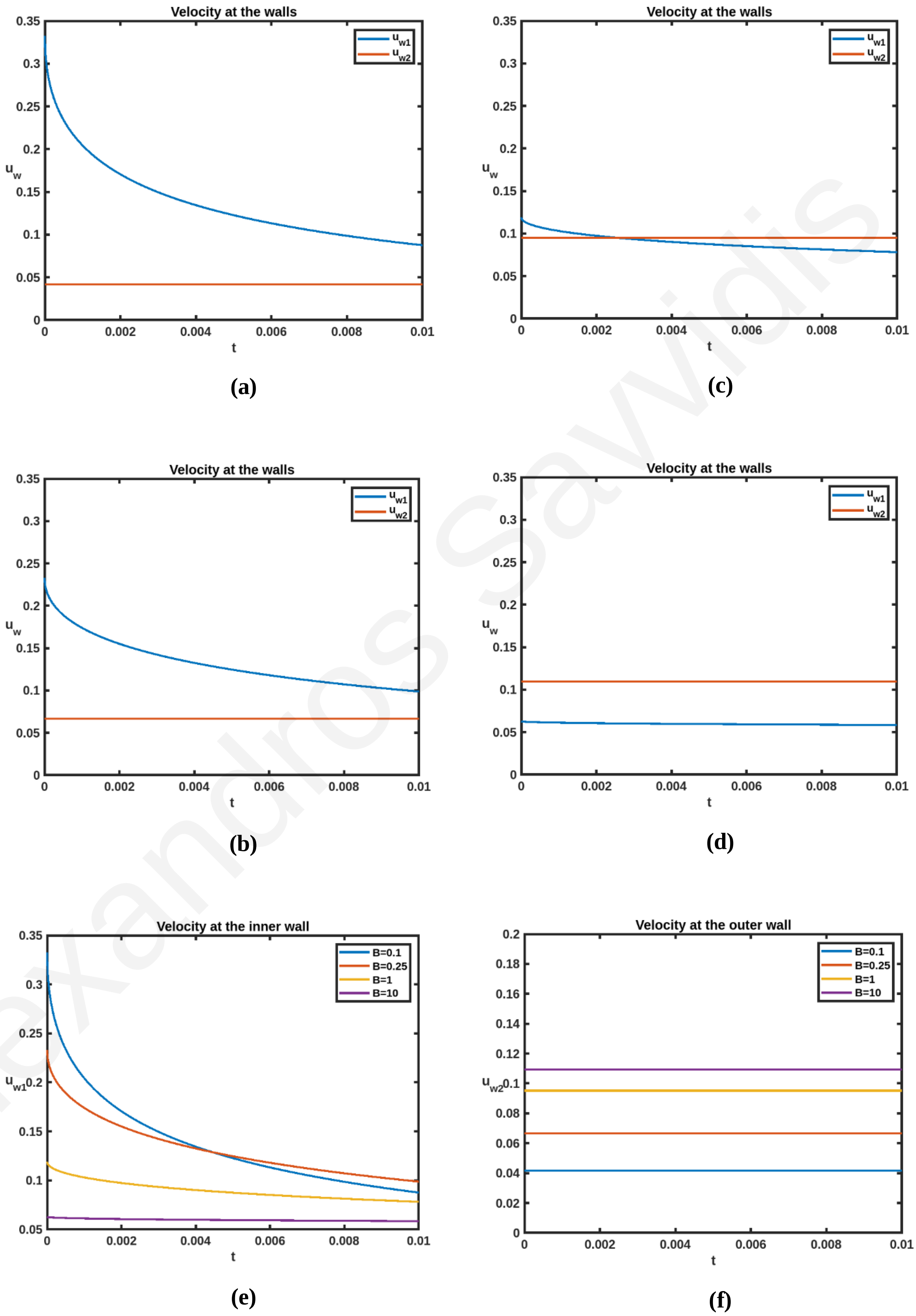


Fig. 9 Evolution of the velocity at the walls when: **(a)** *B=0.1* ; **(b)** *B=0.25* ; **(c)** *B=1* ; **(d)** *B=10* and again for the same values of *B,* all the velocities: **(e)**u_{w_1} ; **(f)** u_{w_2}

2.5 Cessation of circular Couette flow with dynamic slip at the walls

The geometry of the cessation of circular Couette flow with dynamic slip at the walls can be seen in *Fig. 10*. For this problem, the inner cylinder is rotating with angular velocity Ω and there is dynamic slip at the walls. When t=0 the angular velocity ceases to exist. In cessation flow, the velocity of the fluid at both walls will be decreasing , which implies that u_{w_1} will be increasing and u_{w_2} will be decreasing with time (Kaoullas and Georgiou, 2015). As a result, the boundary conditions are:

$$r = \kappa R \quad u_\theta = u_{w_1} = BRr\frac{d}{dr}\left(\frac{u_\theta}{r}\right) - \Lambda\frac{du_\theta}{dt}|_{r=\kappa R} \quad t \geq 0, \tag{2.55}$$

$$r = R \quad u_\theta = u_{w_2} = -BRr\frac{d}{dr}\left(\frac{u_\theta}{r}\right) - \Lambda\frac{du_\theta}{dt}|_{r=R} \quad t > 0. \tag{2.56}$$

In addition,

$$u_{w_1} - \Lambda\frac{du_{w_1}}{dt} = \frac{\tau_{w_1}}{\beta} \quad and \quad u_{w_2} + \Lambda\frac{du_{w_2}}{dt} = \frac{\tau_{w_2}}{\beta}$$

Then the initial condition is

$$u_\theta(r,0) = \frac{\Omega\kappa^3 R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(\frac{R}{r} - (1-2B)\frac{r}{R}\right) \tag{2.57}$$

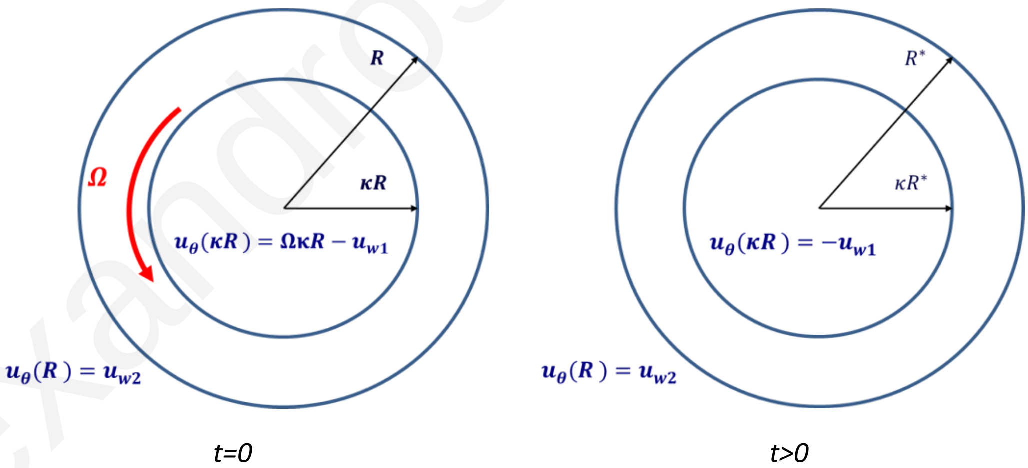


Fig. 10. Geometry of cessation of circular Couette flow with the dynamic slip laws applied at the walls

We solve this initial boundary value problem with the method of separation of variables like in the two previous paragraphs and we get,

$$T(t) = Ae^{-\frac{\lambda^2}{R^2}\nu t} \text{ and } Y(r) = B\, Z_1\left(\frac{\lambda r}{R}\right),$$

where Z_1 is given by Eq. **(2.38)** and from boundary conditions we get:

$$Y(r)T(t) = BRr\frac{d}{dr}\left(\frac{Y(r)T(t)}{r}\right)_{r=\kappa R} + \left(\Lambda\frac{\lambda_\kappa^2}{R^2}vT(t)Y(r)\right)_{r=\kappa R}$$

$$= BRr\left[\frac{r\frac{d}{dr}Y(r) - Y(r)}{r^2}\right]_{r=\kappa R} + \left(\Lambda\frac{\lambda_\kappa^2}{R^2}vT(t)Y(r)\right)_{r=\kappa R},$$

$$Y(r)T(t) = -BRr\frac{d}{dr}\left(\frac{Y(r)T(t)}{r}\right)_{r=R} + \left(\Lambda\frac{\lambda_\kappa^2}{R^2}vT(t)Y(r)\right)_{r=R}$$

$$= BRr\left[\frac{r\frac{d}{dr}Y(r) - Y(r)}{r^2}\right]_{r=R} + \left(\Lambda\frac{\lambda_\kappa^2}{R^2}vT(t)Y(r)\right)_{r=R}.$$

So,

$$Z_1(\kappa b_\kappa) = \left[BR\frac{d}{dr}Z_1\left(\lambda_\kappa\frac{r}{R}\right) - B\frac{R}{r}Z_1\left(\lambda_\kappa\frac{r}{R}\right)\right]_{r=\kappa R} + \left(\Lambda\frac{\lambda_\kappa^2}{R^2}vZ_1\left(\lambda_\kappa\frac{r}{R}\right)\right)_{r=\kappa R}$$

$$\left(\frac{d}{dr}Z_1\left(\lambda_\kappa\frac{r}{R}\right) = \frac{\lambda_\kappa}{R}\left(\frac{1}{2}Z_0\left(\lambda_\kappa\frac{r}{R}\right) - \frac{1}{2}Z_2\left(\lambda_\kappa\frac{r}{R}\right)\right)\right.$$

$$= \frac{\lambda_\kappa}{R}\left(\frac{1}{2}Z_0\left(\lambda_\kappa\frac{r}{R}\right) - \frac{1}{2}\frac{2R}{r\lambda_\kappa}Z_1\left(\lambda_\kappa\frac{r}{R}\right) + \frac{1}{2}Z_0\left(\lambda_\kappa\frac{r}{R}\right)\right)$$

$$\left. = \frac{\lambda_\kappa}{R}Z_0\left(\lambda_\kappa\frac{r}{R}\right) - \frac{1}{r}Z_1\left(\lambda_\kappa\frac{r}{R}\right)\right) \Rightarrow$$

$$Z_1(\kappa\lambda_\kappa) = \frac{BR\lambda_\kappa}{R}Z_0(\kappa\lambda_\kappa) - \frac{B}{\kappa}Z_1(\kappa\lambda_\kappa) - \frac{B}{\kappa}Z_1(\kappa\lambda_\kappa) + \Lambda\frac{\lambda_\kappa^2}{R^2}vZ_1(\kappa\lambda_\kappa) \Rightarrow$$

$$B\lambda_\kappa Z_0(\kappa\lambda_\kappa) - \left(1 + \tfrac{2B}{\kappa} - \Lambda v\tfrac{\lambda_\kappa^2}{R^2}\right)Z_1(\kappa\lambda_\kappa) = 0\ . \quad \textbf{(2.58)}$$

Similarly,

$$B\ \lambda_\kappa\ Z_0(\lambda_\kappa) + \left(1 - 2B - \Lambda v\tfrac{\lambda_\kappa^2}{R^2}\right)\ Z_1(\lambda_\kappa)\ = 0\ . \quad \textbf{(2.59)}$$

From **(2.59)** we get $B\lambda_\kappa\big(J_0(\lambda_\kappa) + \delta_\kappa Y_0(\lambda_\kappa)\big) + \left(1 - 2B - \Lambda v\frac{\lambda_\kappa^2}{R^2}\right)\big(J_1(\lambda_\kappa) + \delta_\kappa Y_1(\lambda_\kappa)\big) = 0$,

$$\delta_\kappa = -\ \frac{B\lambda_\kappa J_0(\lambda_\kappa) + \left(1 - 2B - \Lambda v\frac{\lambda_\kappa^2}{R^2}\right)J_1(\lambda_\kappa)}{B\lambda_\kappa Y_0(\lambda_\kappa) + \left(1 - 2B - \Lambda v\frac{\lambda_\kappa^2}{R^2}\right)Y_1(\lambda_\kappa)}$$

Substituting δ_κ into Eq. **(2.58)** we get:

$$B\lambda_\kappa\left(\left(B\lambda_\kappa Y_0(\lambda_\kappa)+\left(1-2B-\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)Y_1(\lambda_\kappa)\right)J_0(\kappa\lambda_\kappa)-\left(B\lambda_\kappa J_0(\lambda_\kappa)+\left(1-2B-\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)J_1(\lambda_\kappa)\right)Y_0(\kappa\lambda_\kappa)\right)=\left(1+\frac{2B}{\kappa}-\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)\left(\left(B\lambda_\kappa Y_0(\lambda_\kappa)+\left(1-2B-\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)Y_1(\lambda_\kappa)\right)J_1(\kappa\lambda_\kappa)-\left(B\lambda_\kappa J_0(\lambda_\kappa)+\left(1-2B-\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)J_1(\lambda_\kappa)\right)Y_1(\kappa\lambda_\kappa)\right).$$

With superposition of the solutions, we get:

$$u_{\theta}(r,t)=\sum_{k=1}^{\infty}E_k Z_{1k}\left(\frac{\lambda_k r}{R}\right)e^{-\frac{\lambda_k^2}{R^2}vt}\ . \tag{2.60}$$

We now derive the appropriate condition for the eigenfunctions Z_1(Mindlin and Goodman, 1950) :

$$rX_n''(r)+X_n'(r)+\left(\frac{\lambda_n^2}{R^2}r-\frac{1}{r}\right)X_n(r)=0\ . \tag{2.61}$$

From B.C.s we have:

$$X_n(\kappa R)=BRX_n'(\kappa R)-\frac{B}{\kappa}X_n(\kappa R)+\lambda_n^2\frac{\Lambda v}{R^2}X_n(\kappa R)\ , \tag{2.62}$$

$$X_n(R)=-BRX_n'(R)+BX_n(R)+\lambda_n^2\frac{\Lambda v}{R^2}X_n(R)\ \ . \tag{2.63}$$

We now consider the one-dimensional problem in r dimension:

$$rX_m''(r)+X_m'(r)+\left(\frac{\lambda_m^2}{R^2}r-\frac{1}{r}\right)X_m(r)=0\ , \tag{2.64}$$

$$X_m(\kappa R)=BRX_m'(\kappa R)-\frac{B}{\kappa}X_m(\kappa R)+\lambda_m^2\frac{\Lambda v}{R^2}X_m(\kappa R), \tag{2.65}$$

$$X_m(R)=-BRX_m'(R)+BX_m(R)+\lambda_m^2\frac{\Lambda v}{R^2}X_m(R). \tag{2.66}$$

Such that X_m, X_n *and* λ_m, λ_n are distinct $(m\neq n)$

$$\left(rX_n''(r)+X_n'(r)+\left(\frac{\lambda_n^2}{R^2}r-\frac{1}{r}\right)X_n(r)=0\ \Rightarrow\ \left(rX_n'(r)\right)'+\left(\frac{r}{R^2}\lambda_n^2-\frac{1}{r}\right)X_n(r)\right)$$

Multiplying Eq. **(2.61)** by X_m and integrating by parts gives:

$$\int_{\kappa R}^{R}\left(rX_n'(r)\right)'X_m(r)dr+\int_{\kappa R}^{R}\left(\frac{r}{R^2}\lambda_n^2-\frac{1}{r}\right)X_n(r)X_m(r)dr=0 \tag{2.67}$$

Similarly, we multiply **(2.64)** by X_n, integrate by parts and then subtract from **(2.67)** to get:

$$R[X_n'(R)X_m(R)-X_n(R)X_m'(R)]-\kappa R[X_n'(\kappa R)X_m(\kappa R)-X_n(\kappa R)X_m'(\kappa R)]+\frac{(\lambda_n^2-\lambda_m^2)}{R^2}\int_{\kappa R}^{R}rX_n(r)X_m(r)dr=0$$

Using now the b.c.s **(2.62)** , **(2.63)** , **(2.65)** and **(2.66)** we get:

$$(\lambda_n^2-\lambda_m^2)\left[\frac{\Lambda v}{B}\big(X_n(R)X_m(R)+\kappa X_n(\kappa R)X_m(\kappa R)\big)+\int_{\kappa R}^{R} rX_n(r)X_m(r)dr\right]=0\ .$$

Since λ_m *and* λ_n are distinct there holds:

$$\frac{\Lambda v}{B}\big(X_n(R)X_m(R)+\kappa X_n(\kappa R)X_m(\kappa R)\big)+\int_{\kappa R}^{R} rX_n(r)X_m(r)dr=\delta_{m,n}N_n\ , \qquad \textbf{(2.68)}$$

where

$$N_n=\frac{\Lambda v}{B}\Big({X_n}^2(R)+\kappa {X_n}^2(\kappa R)\Big)+\int_{\kappa R}^{R} rX_n^2(r)dr\ , \qquad \textbf{(2.69)}$$

and $\delta_{m,n}$ is the Kronecker delta.

□

In order to find the coefficients E_κ,**(2.60)** must be supplemented by an extra term, thus multiplying it by $rZ_1\left(\frac{\lambda_n r}{R}\right)$ when $t=0$, using the initial condition and integrating from κR to R gives:

$$\sum_{\kappa=1}^{\infty}E_\kappa\int_{\kappa R}^{R} rZ_{1\kappa}\left(\frac{\lambda_\kappa r}{R}\right)Z_1\left(\frac{\lambda_n r}{R}\right)dr$$
$$=\int_{\kappa R}^{R}\frac{\Omega\kappa^3R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(\frac{R}{r}-(1-2B)\frac{r}{R}\right)rZ_1\left(\frac{\lambda_n r}{R}\right)dr\ .$$

Consider,

$$\frac{\Lambda v}{B}\frac{\Omega\kappa^3R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(\frac{R}{r}-(1-2B)\frac{r}{R}\right)Z_1\left(\frac{\lambda_n r}{R}\right)=\frac{\Lambda v}{B}\sum_{\kappa=1}^{\infty}E_\kappa Z_{1\kappa}\left(\frac{\lambda_\kappa r}{R}\right)Z_1\left(\frac{\lambda_n r}{R}\right)\ .$$

When $r=R$,

$$\frac{\Lambda v}{B}\frac{2B\Omega\kappa^3R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}Z_1(\lambda_n)=\frac{\Lambda v}{B}\sum_{\kappa=1}^{\infty}E_\kappa Z_{1\kappa}(\lambda_\kappa)Z_1(\lambda_n)\ .$$

When $r=\kappa R$,

$$\frac{\Lambda v}{B}\frac{\Omega\kappa^3R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(\frac{1}{\kappa}-(1-2B)\kappa\right)Z_1(\kappa\lambda_n)=\frac{\Lambda v}{B}\sum_{\kappa=1}^{\infty}E_\kappa Z_{1\kappa}(\kappa\lambda_\kappa)Z_1(\kappa\lambda_n)\ .$$

So,

$$\sum_{\kappa R}^{R} E_\kappa \left(\int_{\kappa R}^{R} r Z_{1\kappa}\left(\frac{\lambda_\kappa r}{R}\right) Z_1\left(\frac{\lambda_n r}{R}\right) dr + \frac{\Lambda v}{B}\left[Z_{1\kappa}(\lambda_\kappa) Z_1(\lambda_n) + \kappa Z_{1\kappa}(\kappa\lambda_\kappa) Z_1(\kappa\lambda_n)\right] \right)$$

$$= \int_{\kappa R}^{R} \frac{\Omega\kappa^3 R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(\frac{R}{r} - (1-2B)\frac{r}{R}\right) r Z_1\left(\frac{\lambda_n r}{R}\right) dr$$

$$+ \frac{\Lambda v}{B}\frac{\Omega\kappa^3 R}{2B(1+\kappa^3)+\kappa(1-\kappa^2)}\left(2BZ_1(\lambda_n) + (1-\kappa^2+2B\kappa^2) Z_1(\kappa\lambda_n)\right).$$

We are following the same method of solution as in the previous paragraphs and the constants E_κ are given by*:*

$$E_\kappa = \frac{2\Omega\kappa^3 R}{\lambda_\kappa L} Z_1(\kappa\lambda_\kappa) \ , \qquad \textbf{(2.70)}$$

where

$$L = \frac{2\Lambda \nu}{R^2}\kappa\lambda_\kappa\left[Z_1^2(\lambda_\kappa) + \kappa Z_1^2(\kappa\lambda_\kappa)\right] + B\kappa\lambda_\kappa\left[Z_0^2(\lambda_\kappa) + Z_1^2(\lambda_\kappa) - \kappa^2(Z_0^2(\kappa\lambda_\kappa) + Z_1^2(\kappa\lambda_\kappa)\right]$$

$$-2B\kappa\left[Z_0(\lambda_\kappa)Z_1(\lambda_\kappa) - \kappa(Z_0(\kappa\lambda_\kappa)Z_1(\kappa\lambda_\kappa)\right] \ . \qquad \textbf{(2.71)}$$

So, the solution of our problem is:

$$u_\theta(r,t) = \sum_{k=1}^{\infty} E_k Z_1\left(\frac{\lambda_k r}{R}\right) e^{-\frac{\lambda_k^2}{R^2}\nu t} \ . \qquad \textbf{(2.72)}$$

Dividing u_θ by ΩR, and setting $u_\theta^* = u_\theta/\Omega\kappa R$, $r^* = r/R$, $t^* = \nu t/R^2$, $E_k' = E_k/\kappa\Omega R$ we get:

$$u_\theta^*(r^*,t^*) = \sum_{k=1}^{\infty} E_k' Z_1(\lambda_k r^*)\, e^{-\lambda_k^2 t^*} \qquad \textbf{(2.73)}$$

where

$$E_k' = E_k/\Omega\kappa R$$

In Fig. 12, we can see the evolution of the velocity profile in cessation circular Couette flow for weak slip (B=0.1) and moderate slip (B=1), and how the velocity changes for different values of Λ.

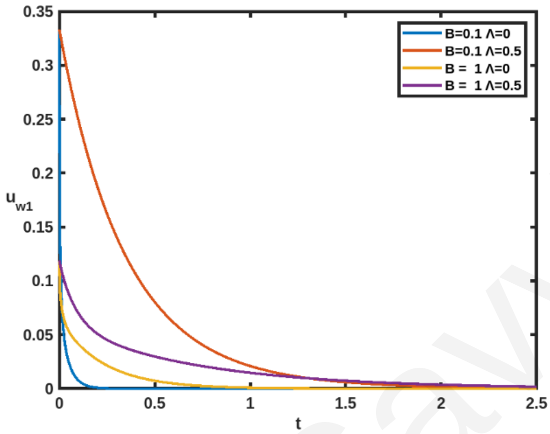
The slip velocities are given by:

$$u_{w_1}^*(t^*) = \sum_{k=1}^{\infty} E_k' Z_1(\kappa\lambda_k)\, e^{-\lambda_k^2 t^*} \ , \qquad \textbf{(2.74)}$$

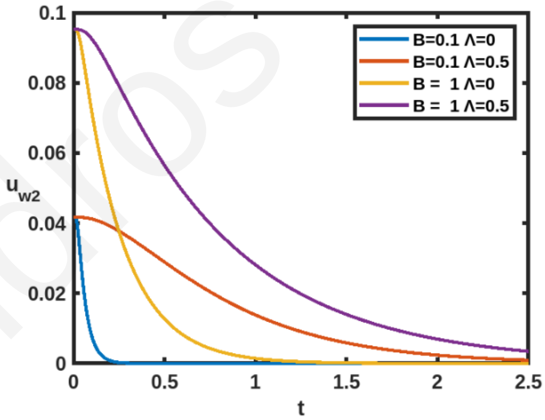
$$u_{w_2}^*(t^*) = \sum_{k=1}^{\infty} E_k' Z_1(\lambda_k)\, e^{-\lambda_k^2 t^*} . \qquad \textbf{(2.75)}$$

Remark: When $\Lambda = 0$, Eq. **(2.73)** is reduced to Eq. **(2.54)**, and when $B = 0$ $(\beta \to \infty)$ Eq. **(2.54)** is reduced to Eq. **(2.45).**

■



(a)



(b)

Fig. 11. Evolution of the slip velocity in cessation of circular Couette flow for different values of Λ and *κ=0.5:* **(a).** u_{w_1}; **(b).** u_{w_2}

In Fig. 11, which depicts the velocity at the walls, we see that the velocity in the inner wall (Fig. 11a) as the value of B is increasing, our curves for different Λ tend to get closer but on the other hand in the outer wall (Fig. 11b), the curves tend to get further from each other.

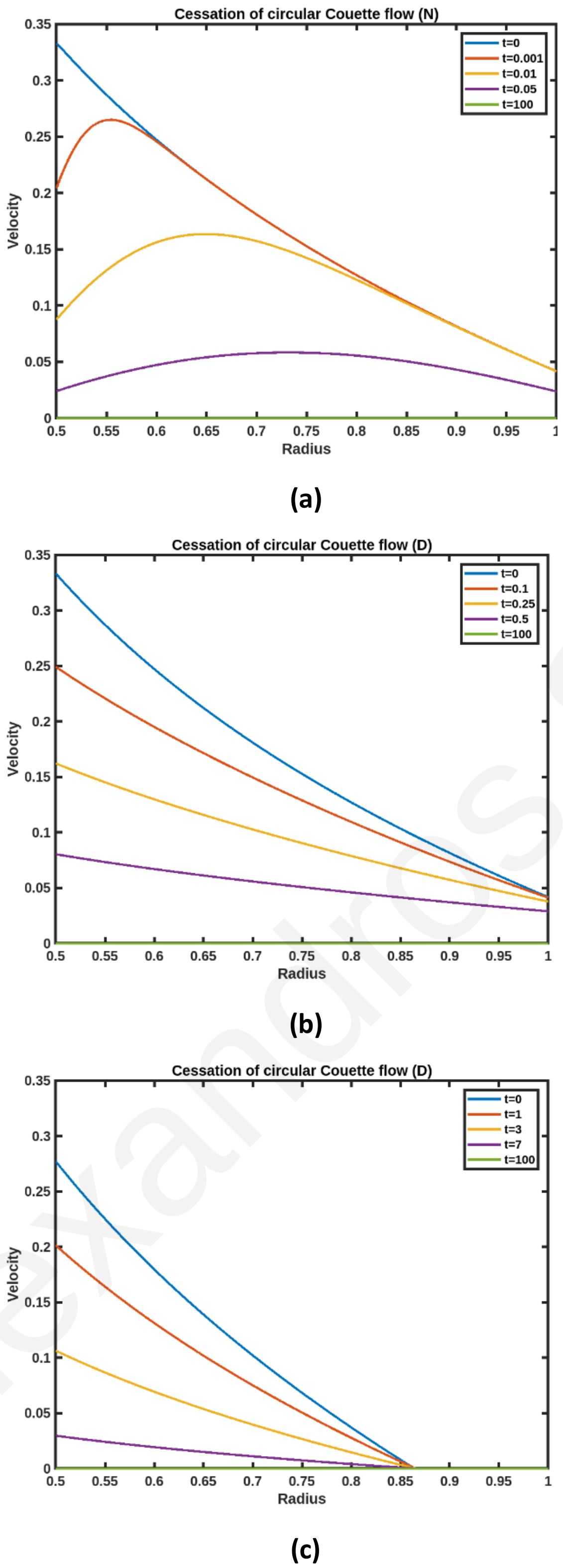


Fig. 12. Evolution of the velocity profile in cessation circular Couette flow with *κ=0.5* and *B=0.1:* **(a)** *Λ=0;* **(b)** *Λ=0.5;* **(c)** *Λ=5;*

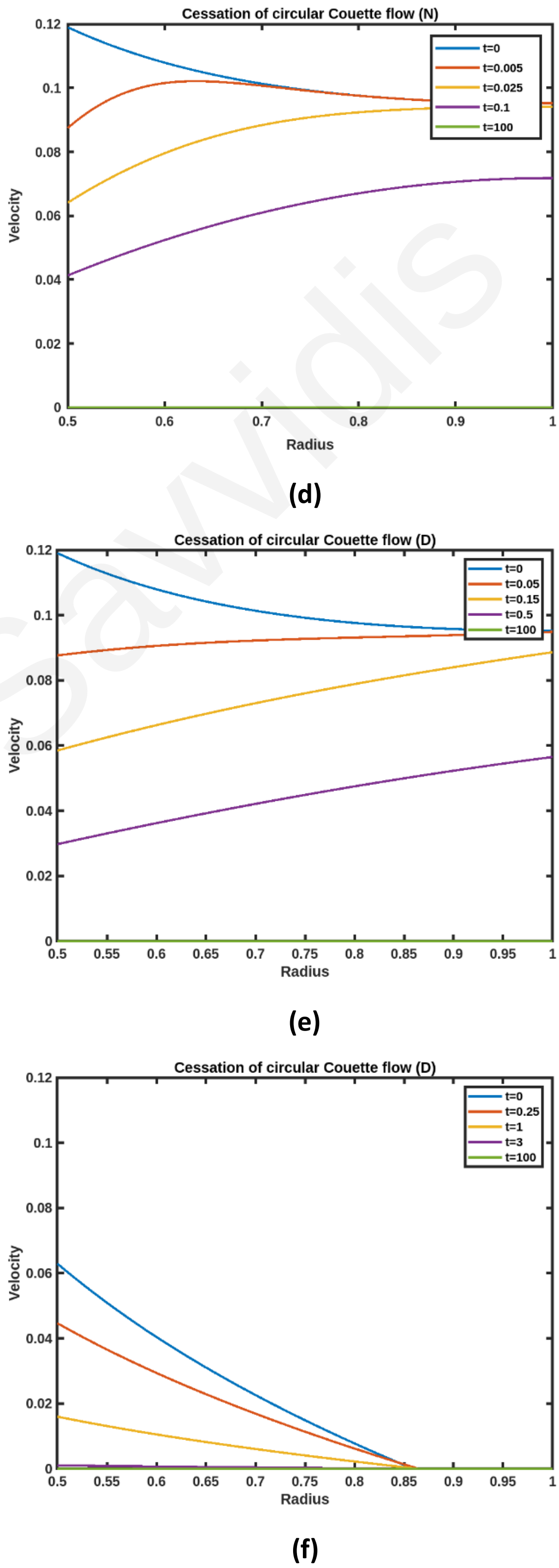


Fig. 13. Evolution of the velocity profile in cessation circular Couette flow with *κ=0.5* and *B=1:* **(a)** *Λ=0;* **(b)** *Λ=0.5;* **(c)** *Λ=5;*

2.6 Appendix: Solutions when the outer cylinder is rotating

In the previous sections we derived analytical solutions for the steady state and cessation problem of circular Couette flow when the inner cylinder is rotating with angular velocity Ω the outer cylinder is fixed and there are three types of slip laws applied at the walls: (a) No-slip law; (b) Navier slip law; (c) Dynamic slip law.

In this paragraph we are going to provide analytical solutions for the same problem but now instead of the inner cylinder, the outer cylinder is rotating with angular velocity Ω and the inner cylinder is fixed.

For the steady state of circular Couette flow with Navier slip at the walls the velocity is given by:

$$u_\theta(r) = \frac{\Omega R}{2B(1+\kappa^3)+\kappa-\kappa^3}\left[(2B+\kappa)\frac{r}{R}-\kappa^3\frac{R}{r}\right]. \tag{2.76}$$

If we let $B=0$ $(\beta \to \infty)$ the result will be the solution for the steady state circular Couette flow with no-slip at the walls:

$$u_\theta(r) = \frac{\Omega R}{1-\kappa^2}\left(\frac{r}{R}-\kappa^2\frac{R}{r}\right). \tag{2.77}$$

For the cessation of circular Couette flow with dynamic slip at the walls the velocity is:

$$u_\theta(r,t) = \sum_{k=1}^{\infty} E_\kappa Z_{1k}\left(\lambda_k\frac{r}{R}\right)e^{-\frac{\lambda_k^2}{R^2}\nu t}, \tag{2.78}$$

and the coefficients in **(2.78)** are given by:

$$E_\kappa = \frac{2\Omega R}{\lambda_\kappa L}Z_1(\lambda_\kappa), \tag{2.79}$$

where

$$\begin{aligned} L = {} & 2\Lambda\nu\frac{\lambda_\kappa}{R^2}\left(Z_1^2(\lambda_\kappa)+\kappa Z_1^2(\kappa\lambda_\kappa)\right)+B\lambda_\kappa[Z_0^2(\lambda_\kappa) \\ & +Z_1^2(\lambda_\kappa)-\kappa^2\left(Z_0^2(\kappa\lambda_\kappa)+Z_1^2(\kappa\lambda_\kappa)\right)] \\ & -2B\left(Z_0(\lambda_\kappa)Z_1(\lambda_\kappa)-\kappa Z_0(\kappa\lambda_\kappa)Z_1(\kappa\lambda_\kappa)\right). \end{aligned} \tag{2.80}$$

If we let $\Lambda=0$ then the result will be the solution for the cessation of circular Couette flow with Navier slip at the walls and the velocity will be given by:

$$u_\theta(r,t) = \sum_{k=1}^{\infty} D_k Z_{1k}\left(\frac{b_k r}{R}\right)e^{-\frac{b_k^2}{R^2}\nu t}. \tag{2.81}$$

where

$$D_\kappa = \frac{2(1-2B)\left(1+\frac{2B}{\kappa}\right)^2 \Omega R\left(-Z_0(b_\kappa)\right)}{b_k\left[\left(1+\frac{2B}{\kappa}\right)^2\left(1-2B+B^2 b_\kappa^2\right)Z_0^2(b_\kappa) - \kappa^2(1-2B)^2\left(1+\frac{2B}{\kappa}+B^2 b_\kappa^2\right)Z_0^2(\kappa b_\kappa)\right]}. \tag{2.82}$$

Now if we let $B = 0 \quad (\beta \to \infty)$ in Eq. **(2.82)** the result will be the solution for the cessation of annular Couette flow with no-slip at the walls:

$$u_\theta(r,t) = \sum_{k=1}^{\infty} C_k Z_{1k}\left(\frac{a_k r}{R}\right) e^{-\frac{a_k^2}{R^2}\nu t}, \tag{2.83}$$

where

$$C_\kappa = \frac{2\Omega R\left(-Z_0(\alpha_\kappa)\right)}{\lambda_\kappa\left[Z_0^2(\lambda_\kappa) - \kappa^2 Z_0^2(\kappa\lambda_\kappa)\right]}. \tag{2.84}$$

■

Chapter 3: Annular Couette flow

Annular Couette flow is a flow pattern that occurs when a fluid is confined between two coaxial cylinders usually with the inner cylinder sliding and the outer cylinder fixed. This flow pattern has various applications in fluid dynamics, such as in Engineering, Biological Systems, Lubrication and Fluid dynamics education.

In Engineering, numerous engineering applications involve rotating components, such as rotating heat pipes, electrical motors, and turbogenerators. The flow of fluid between two concentric cylinders, with one or both cylinders in rotation, is known as Taylor-Couette flow, and has been the subject of extensive research over the years. (Nouri-Borujerdi and. Nakhchi, 2017)

In Biological Systems, flow-induced damage to blood is commonly seen in artificial organs within the bloodstream, specifically hemolysis of red blood cells. The severity of this damage is influenced by shear forces and exposure time. This study focuses on establishing a correlation between these flow-dependent properties and actual hemolysis. To achieve this goal, researchers developed a Couette device. (Paul et al., 2003)

In Lubrication, it is well known that fluid flows in seals and bearings turn from laminar regime into turbulent one when their Reynolds number becomes higher than a critical value. In (Zhang et al., 2003), the primary turbulence models utilized for hydrodynamic lubrication issues were assessed, with an explanation of their development and fundamental principles. To evaluate their efficacy, the models' predictions of flow fields in turbulent Couette flows and shear-induced countercurrent flows were compared to existing measurements. Zhang and Zhang's combined k-ε model demonstrated particularly impressive outcomes, with surpassingly satisfactory results.

In Fluid dynamics education, annular Couette flow is often used as a teaching tool in fluid dynamics courses. The simplicity of the flow allows students to easily understand concepts such as boundary layers, laminar and turbulent flow, and fluid viscosity. It can also be used to demonstrate the principles of flow visualization and measurement techniques.(White, 2006)

In this chapter we are going to study the steady, axisymmetric, rectinal flow of an incompressible Newtonian liquid between two horizontal coaxial cylinders of infinite length with radii R and κR where $0 < \kappa < 1$ and the outer cylinder is sliding with velocity V, so we assume that $u_r = u_\theta = 0$, $du_z/d\theta = 0$ and $dp/dz = 0$.

As a result of our assumptions, the z-momentum equation gives:

$$\frac{\partial u_z}{\partial t} = v\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right), \tag{3.1}$$

where $v = \eta/\rho$ is the kinematic viscosity.

The steady-state solution is found by setting $\partial u_z/\partial t = 0$ and integrating twice.

The general form of the velocity u_z is given by (Papanastasiou et al., 1999)

$$u_z(r) = c_1 \ln r + c_2 , \tag{3.2}$$

and the wall shear stress is given by

$$\tau_{rz} = \tau_{zr} = \frac{\eta c_1}{r} = \eta \frac{du_z}{dr}, \tag{3.3}$$

$$\tau_w = \beta\, u_w . \tag{3.4}$$

In each of the following paragraphs we are going to find these constants and derive the analytical solution of the velocity.

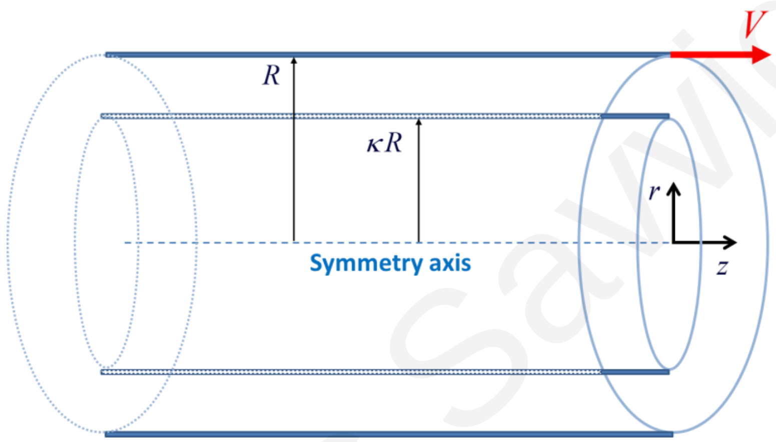


Fig. 1. Geometry of annular Couette flow

3.1 The steady-state annular Couette flow with no slip at the walls

The geometry of the steady-state annular Couette flow with no slip at the walls can be seen in *Fig. 2*. For this problem the outer cylinder is sliding with velocity V and there is no slip at the walls. As a result, the boundary conditions are:

$$r = \kappa R, \quad u_z = 0, \tag{3.5}$$
$$r = R, \quad u_z = V. \tag{3.6}$$

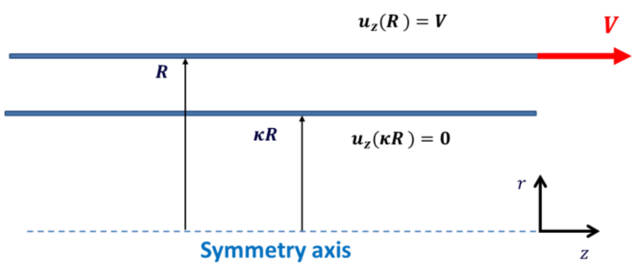


Fig. 2. Geometry of the steady-state annular Couette flow with no-slip laws applied at the walls

Appling the boundary conditions in Eq. **(3.2)** we get:

$$u_z(\kappa R) = 0 = c_1 \ln \kappa R + c_2, \tag{3.7}$$

$$u_z(R) = V = c_1 \ln R + c_2. \tag{3.8}$$

Solving the above system, gives:

$$c_2 = -\frac{V \ln \kappa R}{\ln(1/\kappa)} \quad , \quad c_1 = \frac{V}{\ln(1/\kappa)} \quad .$$

Going back now to our equation **(3.2)** we get that the velocity is given by:

$$u_z(r) = \frac{\ln\left(\frac{r}{\kappa R}\right)}{\ln(1/\kappa)} V \, . \tag{3.9}$$

The velocity along both walls is zero because of the no slip condition:

$$u_{w_1} = u_{w_2} = 0 \, . \tag{3.10}$$

Substituting $c_1 = V/\ln\frac{1}{\kappa}$ in equation **(3.3)** we get that the wall shear stress is:

$$\tau_{rz} = \frac{\eta V}{\ln(1/\kappa)} \frac{1}{r}. \tag{3.11}$$

And the wall shear stress in each of the walls will be

$$\tau_{w_1} = \frac{\eta V}{\kappa R \ln(1/\kappa)} \quad , \quad \tau_{w_2} = \kappa \, \tau_{w_1}. \tag{3.12}$$

Finally, we will find the dimensionless equations for the velocity and wall shear stresses which are going to be utilized in the sequel.

For the velocity we divide both parts of the equation by V and set $u_z^* = u_z/V$ and $r^* = r/R$. This gives:

$$u_z^* = \frac{\ln\left(\frac{r^*}{\kappa}\right)}{\ln(1/\kappa)}. \tag{3.13}$$

Similarly,

$$\tau_{rz}^* = \frac{1}{r^* \ln(1/\kappa)}, \tag{3.14}$$

where

$$\tau_{rz}^* = \frac{\tau_{rz} R}{\eta V}.$$

Additionally,

$$\tau^*_{w_1} = \frac{1}{\kappa \ln(1/\kappa)} \quad , \quad \tau^*_{w_2} = \kappa\, \tau_{w_1}. \qquad \textbf{(3.15)}$$

■

3.2 The steady-state annular Couette flow with Navier slip at the walls

The geometry of the steady-state annular Couette flow with Navier slip at the walls can be seen in *Fig. 3*. For this problem the outer cylinder is sliding with velocity V and at the walls we have Navier slip, so the velocities at the walls will not be zero. As a result, the boundary conditions are:

$$r = \kappa R, \quad u_z = u_{w_1}, \qquad \textbf{(3.16)}$$
$$r = R, \quad u_z = V - u_{w_2}. \qquad \textbf{(3.17)}$$

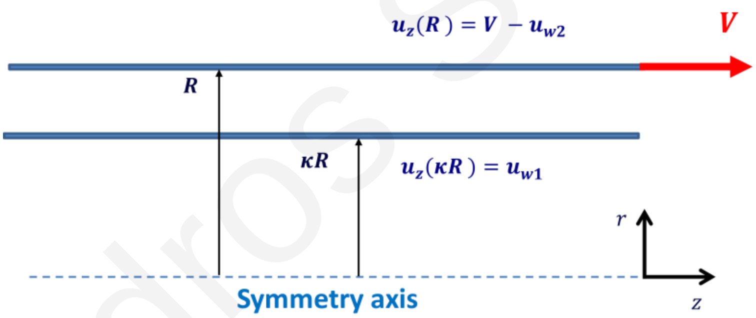


Fig. 3. Geometry of steady-state annular Couette flow with Navier slip laws applied at the walls

Appling the boundary conditions in Eq **(3.2)** we get:

$$u_z(\kappa R) = u_{w_1} = c_1 \ln \kappa R + c_2 \ , \qquad \textbf{(3.18)}$$
$$u_z(R) = V - u_{w_2} = c_1 \ln R + c_2 \, . \qquad \textbf{(3.19)}$$

From Eq. **(3.4)** we get:

$$u_{w_1} = \frac{\eta c_1}{\beta \kappa R}, \qquad \textbf{(3.20)}$$
$$u_{w_2} = \frac{\eta c_1}{\beta R}. \qquad \textbf{(3.21)}$$

Eqs. **(3.18)** and **(3.20)** give

$$\frac{\eta c_1}{\beta \kappa R} = c_1 \ln \kappa R + c_2 \, , \qquad \textbf{(3.22)}$$

and from Eqs. **(3.19)** and **(3.20)** we get

$$V - \frac{\eta c_1}{\beta R} = c_1 \ln R + c_2 \ . \qquad \textbf{(3.23)}$$

Equations **(3.22)** and **(3.23)** give us further:

$$c_1 = \frac{\beta\kappa R}{\eta(\kappa+1) \ + \ \beta\kappa R \ln\frac{1}{\kappa}} V \qquad , \qquad c_2 = \frac{\eta - \beta\kappa R \ln \kappa R}{\eta(\kappa+1) \ + \ \beta\kappa R \ln\frac{1}{\kappa}} V \ .$$

Therefore, the velocity is given by

$$u_z(r) = \frac{\beta\kappa R V}{\eta(\kappa+1) \ + \ \beta\kappa R \ln\frac{1}{\kappa}} \ln\frac{r}{\kappa R} + \frac{\eta V}{\eta(\kappa+1) \ + \ \beta\kappa R \ln\frac{1}{\kappa}} \ . \qquad \textbf{(3.24)}$$

Setting $c_1 = \beta\kappa R V / (\eta(\kappa+1) \ + \ \beta\kappa R \ln\frac{1}{\kappa})$ in Eq. **(3.3)**, we get that the wall shear stress is:

$$\tau_{rz} = \frac{\eta\beta\kappa R}{\eta(\kappa+1) \ + \ \beta\kappa R \ln\frac{1}{\kappa}} V \frac{1}{r} \ . \qquad \textbf{(3.25)}$$

The wall shear stress in each of the walls is

$$\tau_{w_1} = \frac{\eta\beta V}{\eta(\kappa+1) + \ \beta\kappa R \ln\frac{1}{\kappa}} \quad , \quad \tau_{w_2} = \kappa\tau_{w_1} \ . \qquad \textbf{(3.26)}$$

As a result, from Eq. **(3.4)**, the velocity in each of the walls is given by:

$$u_{w_1} = \frac{\eta V}{\eta(\kappa+1) + \ \beta\kappa R \ln\frac{1}{\kappa}} \quad , \quad u_{w_2} = \kappa u_{w_1} \ . \qquad \textbf{(3.27)}$$

Finally, we are going to find the dimensionless equations for the velocities and wall shear stresses. For the velocity we divide both parts of the equation with with V and set $u_z^* = \frac{u_z}{V}$, $r^* = \frac{r}{R}$ and $B = \frac{\eta}{\beta R}$ .This gives:

$$u_z^* = \frac{\kappa \ln\frac{r^*}{\kappa} + B}{B(\kappa+1) + \kappa \ln\frac{1}{\kappa}} \qquad \textbf{(3.28)}$$

In Fig.4, we can see the evolution of the velocity profile in annular Couette flow with Navier slip law applied to the walls for various values of B (Chatzimina et al.,2007; Chatzimina et al.,2009). Fig.5 shows the evolution of the velocity at the inner wall, for various values of B, in a semilog scale. (Georgiou and Xenophontos,2007)

The velocities in the wall will be

$$u_{w_1} = \frac{B}{B(\kappa+1) + \ \kappa \ln\frac{1}{\kappa}} \quad , \quad u_{w_2} = \kappa u_{w_1}. \qquad \textbf{(3.29)}$$

Similarly,

$$\tau_{rz}^{*} = \frac{\kappa}{B(\kappa+1) + \kappa \ln\dfrac{1}{\kappa}} \frac{1}{r^{*}} \,, \tag{3.30}$$

where

$$\tau_{rz}^{*} = \frac{\tau_{rz} R}{\eta V} \,.$$

Additionally

$$\tau_{w_1} = \frac{1}{B(\kappa+1) + \kappa \ln\dfrac{1}{\kappa}} \,. \tag{3.31}$$

Remark: When $B = 0 \;\; (\beta \to \infty)$ Eqs. **(3.28)** , **(3.30)** and **(3.31)** are reduced to Eqs.**(3.13)**, **(3.14)** and **(3.15)** respectively

■

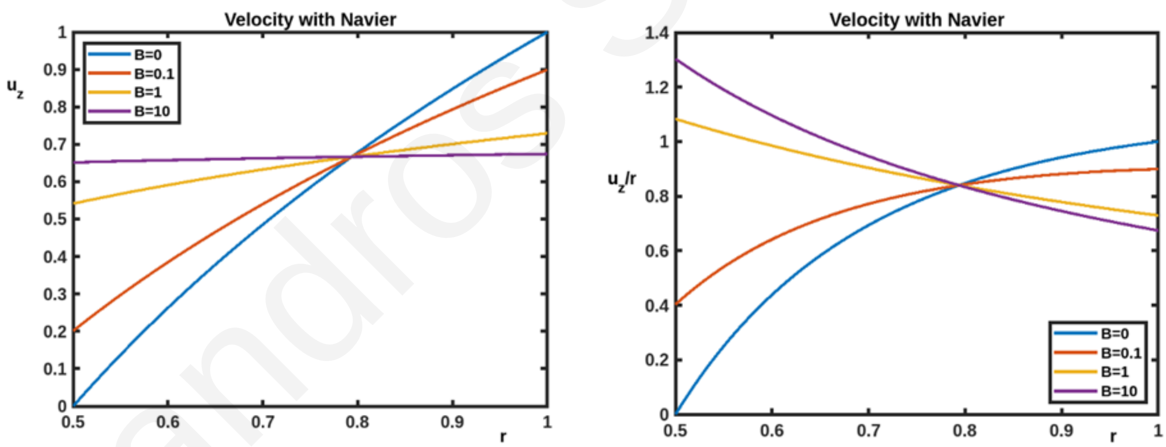
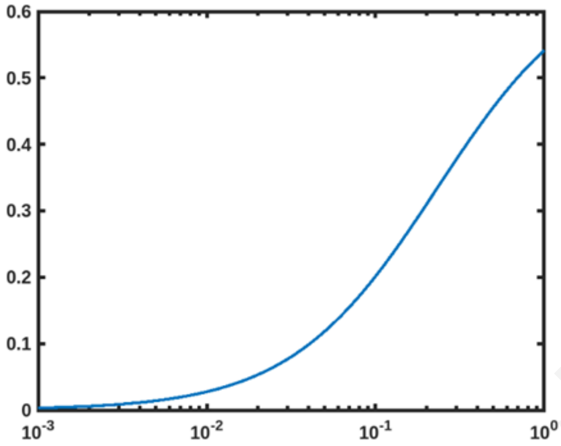
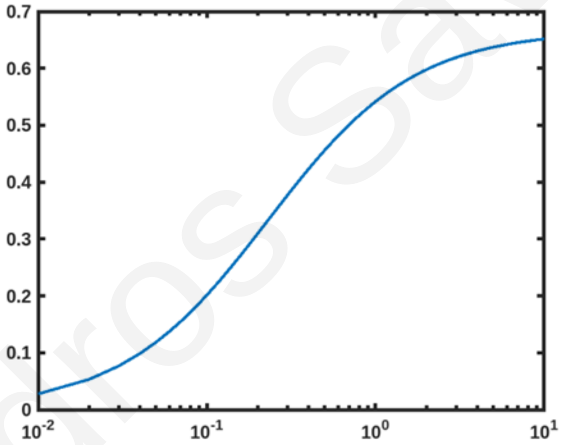


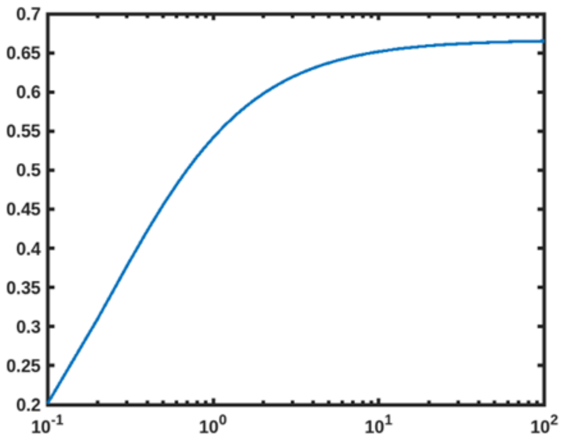
Fig. 4. Evolution of the velocity profile in annular Couette flow with Navier slip law applied in the walls for *B*= 0 , 0.1 , 1 and 10



(a)



(b)



(c)

Fig. 5. Evolution of the velocity at the inner wall when: **(a)** B= 0, 1 ; **(b)** B= 0, 10 ; **(c)** B= 0, 100

3.3 Cessation of annular Couette flow with no slip at the walls

The geometry of the cessation of annular Couette flow with no slip at the walls can be seen in *Fig. 6*. For this problem the outer cylinder is sliding with velocity V and there is no slip at the walls. When *t=0* the velocity ceases to exist. As a result, the boundary and initial conditions are:

$$r = \kappa R\,, \quad u_Z = 0 \quad t \geq 0, \tag{3.32}$$

$$r = R \quad , \quad u_Z = 0 \quad t > 0, \tag{3.33}$$

$$u_z(r,0) = \frac{\ln \dfrac{r}{\kappa R}}{\ln \dfrac{1}{\kappa}} V\,. \tag{3.34}$$

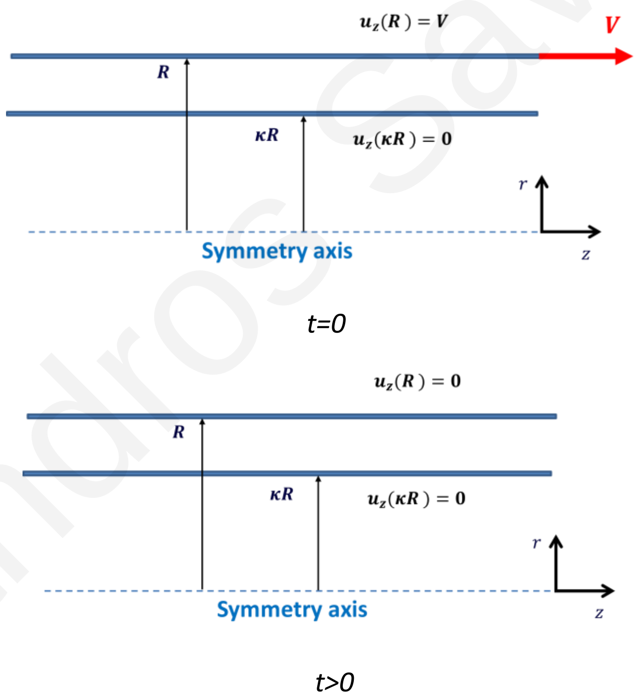


Fig. 6. Geometry of cessation of annular Couette flow with no slip laws applied at the walls

We solve this initial boundary value problem with the method of separation of variables

Let

$$u_z(r,t) = Y(r)T(t). \tag{3.35}$$

Substituting into Eq. **(3.1)** we get

$$Y(r)T'(t) = \nu\left(Y''(r)T(t) + \frac{1}{r}Y'(r)T(t)\right).$$

Dividing by $\nu Y(r)T\,(t)$ we get

$$\frac{T'(t)}{vT(t)} = \frac{Y''(r)}{Y(r)} + \frac{1}{r}\frac{Y'(r)}{Y(r)}.$$

Because each side in the equation depends on different variables, then each one should be equal to the same constant. Let this constant be $const. = -\frac{a^2}{R^2}$

As a result, we get two new equations:

$$T'(t) = -\frac{a^2}{R^2}vT(t) \Rightarrow$$

$$T(t) = Ae^{-\frac{a^2}{R^2}vt}. \qquad \textbf{(3.36)}$$

Additionally

$$Y''(r) + \frac{1}{r}\,Y'(r) + \frac{a^2}{R^2}Y(r) = 0 \Rightarrow$$

$$Y(r) = c_1\,J_0\left(\frac{ar}{R}\right) + c_2\,Y_0\left(\frac{ar}{R}\right), \qquad \textbf{(3.37)}$$

where J_0 and Y_0 are zero order Bessel functions of the first and second kind respectively. The same identities that are applicable for Bessel functions of the first kind are applicable to the second kind too so, we let:

$$Z_0\left(\frac{ar}{R}\right) = J_0\left(\frac{ar}{R}\right) + \beta Y_0\left(\frac{ar}{R}\right), \qquad \textbf{(3.38)}$$

with β being a new constant.

Now from boundary conditions we have $\beta = -\frac{J_1(a)}{Y_1(a)}$,

$$Z_0(\kappa\alpha) = J_0(\kappa\alpha) + \beta Y_0(\kappa\alpha) = 0, \qquad \textbf{(3.39)}$$

$$Z_0(\alpha) = J_0(\alpha) + \beta Y_0(\alpha) = 0. \qquad \textbf{(3.40)}$$

Superposition of the solutions gives:

$$u_z(r,t) = \sum_{k=1}^{\infty} C_k Z_{0k}\left(\frac{\alpha_k r}{R}\right) e^{-\frac{\alpha_k^2}{R^2}vt}. \qquad \textbf{(3.41)}$$

When $t = 0$ we get from Eq. **(3.41)** and the initial condition, the following:

$$\frac{\ln\frac{r}{\kappa R}}{\ln\frac{1}{\kappa}}V = \sum_{\kappa=1}^{\infty} C_\kappa Z_{0\kappa}\left(\frac{\alpha_\kappa r}{R}\right). \qquad \textbf{(3.42)}$$

The orthogonality condition states that:

$$\int_{\kappa R}^{R} Z_0^2\left(\frac{\alpha_n r}{R}\right) r\,dr = \left[\frac{r^2}{2R^2}\left\{Z_0'\left(\alpha_\kappa \frac{r}{R}\right)\right\}^2 + \frac{r^2}{2R^2}\left\{Z_0\left(\alpha_\kappa \frac{r}{R}\right)\right\}^2\right]_\kappa^1.$$

In order to use the orthogonality condition, we multiply Eq. **(3.42)** by $rZ_0\left(\frac{\alpha_n r}{R}\right)$ and integrate from κR to R:

$$C_\kappa \int_{\kappa R}^{R} Z_0^2\left(\frac{\alpha_n r}{R}\right) r\,dr = \int_{\kappa R}^{R} \frac{V}{\ln\frac{1}{\kappa}} \ln\frac{r}{\kappa R}\, r\, Z_0\left(\frac{\alpha_n r}{R}\right)\,dr.$$

Now let $\frac{r}{R} = \xi$ so if $r \in [\kappa R, R] \Rightarrow \xi \in [\kappa, 1]$ and if $\frac{r}{R} = \xi$ *then* $\frac{1}{R}dr = d\xi$.

So

$$C_\kappa R^2 \int_{\kappa}^{1} Z_0^2(\alpha_\kappa \xi)\,\xi\,d\xi = \int_{\kappa}^{1} \frac{V}{\ln\frac{1}{\kappa}} \xi \ln\frac{\xi}{\kappa}\, Z_0(\alpha_\kappa \xi)\,d\xi \Rightarrow$$

$$C_\kappa = \frac{V}{\ln\frac{1}{\kappa}} \frac{\int_\kappa^1 \xi \ln\frac{\xi}{\kappa}\, Z_0(\alpha_\kappa \xi)\,d\xi}{\int_\kappa^1 Z_0^2(\alpha_\kappa \xi)\,\xi\,d\xi}. \qquad \textbf{(3.43)}$$

We calculate from **(3.43)**:

$$I_1 := \int_\kappa^1 \xi\, Z_0^2(\alpha_\kappa \xi)\,d\xi = \left[\frac{\xi^2}{2}\{Z_0'(\alpha_\kappa \xi)\}^2 + \frac{\xi^2}{2}\{Z_0(\alpha_\kappa \xi)\}^2\right]_\kappa^1$$

$$= \left[\frac{1}{2}\left(Z_1(\alpha_\kappa)\right)^2 + \frac{1}{2}\left(1 - \frac{1}{\alpha_\kappa^2}\right) Z_0^2(\alpha_\kappa) - \frac{\kappa^2}{2}\left(Z_1(\kappa\alpha_\kappa)\right)^2 - \frac{1}{2} Z_0^2(\kappa\alpha_\kappa)\right]$$

$$= \frac{1}{2} Z_1^2(\alpha_\kappa) - \frac{\kappa^2}{2} Z_1^2(\kappa\alpha_\kappa)$$

$$I_2 := \int_\kappa^1 \xi \ln\frac{\xi}{\kappa}\, Z_0(\alpha_\kappa \xi)\,d\xi = \frac{1}{\alpha_\kappa^2} \int_{\kappa\alpha_\kappa}^{\alpha_\kappa} \ln\frac{u}{\kappa\alpha_\kappa}\, u Z_0(u)\,du$$

$$= \frac{1}{\alpha_\kappa^2} \int_{\kappa\alpha_\kappa}^{\alpha_\kappa} \ln\frac{u}{\kappa\alpha_\kappa} \left(u Z_1(u)\right)' du = \frac{1}{\alpha_\kappa^2}\left[u \ln\frac{u}{\kappa\alpha_\kappa} Z_1(u)\right]_{\kappa\alpha_\kappa}^{\alpha_\kappa} - \int_{\kappa\alpha_\kappa}^{\alpha_\kappa} \frac{u}{\kappa\alpha_\kappa} \frac{\kappa\alpha_\kappa}{u} Z_1(u)\,du$$

$$= \frac{1}{\alpha_\kappa} \ln\frac{1}{\kappa} Z_1(\alpha_\kappa) - \left[-Z_0(u)\right]_{\kappa\alpha_\kappa}^{\alpha_\kappa} = \frac{1}{\alpha_\kappa} \ln\frac{1}{\kappa} Z_1(\alpha_\kappa)$$

Substituting in Eq. **(3.43)** we have:

$$C_\kappa = \frac{V}{\ln\frac{1}{\kappa}} \frac{\frac{1}{\alpha_\kappa} \ln\frac{1}{\kappa} Z_1(\alpha_\kappa)}{\frac{1}{2} Z_1^2(\alpha_\kappa) - \frac{\kappa^2}{2} Z_1^2(\kappa\alpha_\kappa)}.$$

As a result,

$$C_k = \frac{2VZ_1(\alpha_\kappa)}{\alpha_\kappa[Z_1^2(\alpha_\kappa) - \kappa^2 Z_1^2(\kappa\alpha_\kappa)]}.$$

The velocity in Eq. **(3.41)** is given by

$$u_z(r,t) = \sum_{k=1}^{\infty} \frac{2VZ_1(\alpha_\kappa)}{\alpha_\kappa[Z_1^2(\alpha_\kappa) - \kappa^2 Z_1^2(\kappa\alpha_\kappa)]} Z_{0k}\left(\frac{a_k r}{R}\right) e^{-\frac{a_k^2}{R^2}\nu t}. \quad \textbf{(3.44)}$$

Now we let $t^* = \nu t/R^2$, $r^* = r/R$, $u_z^* = u_z/V$, $C_k^* = C_k/V$, and get the dimensionless form:

$$u_z^*(r^*,t^*) = \sum_{k=1}^{\infty} \frac{2Z_1(\alpha_\kappa)}{\alpha_\kappa[Z_1^2(\alpha_\kappa) - \kappa^2 Z_1^2(\kappa\alpha_\kappa)]} Z_{0k}(a_k r^*)\, e^{-a_k^2 t^*} \quad \textbf{(3.45)}$$

■

3.4 Cessation of annular Couette flow with Navier slip at the walls

The geometry of the cessation of annular Couette flow with Navier slip at the walls can be seen in *Fig. 7*. For this problem the outer cylinder is sliding with velocity V and there is Navier slip at the walls. When t=0 the velocity ceases to exist. As a result, the boundary and initial conditions are:

$$r = \kappa R, \quad u_z = u_{w_1} = BR\frac{du_z}{dr}|_{r=\kappa R} \qquad t \geq 0\ , \quad \textbf{(3.46)}$$

$$r = R\,, \quad u_z = -u_{w_2} = -BR\frac{du_z}{dr}|_{r=R} \quad t > 0, \quad \textbf{(3.47)}$$

$$u_z(r,0) = \frac{\kappa V \ln\frac{r}{\kappa R} + BV}{B(\kappa+1) + \kappa\ln\frac{1}{\kappa}}\ . \quad \textbf{(3.48)}$$

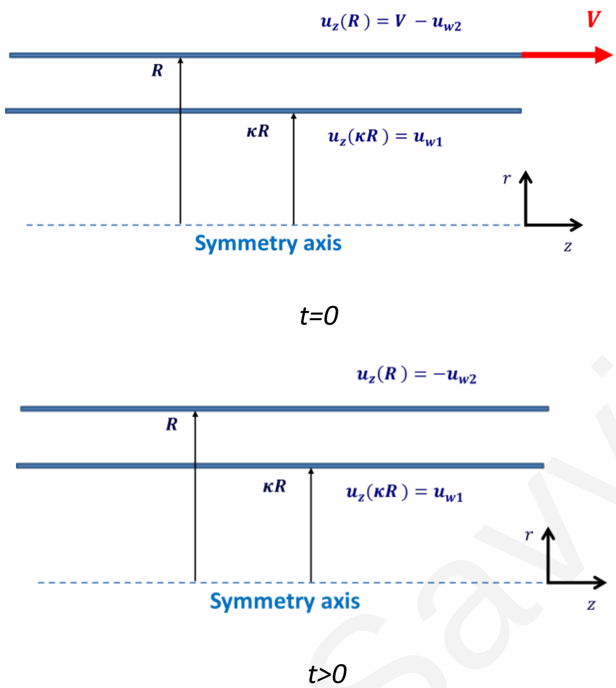


Fig. 7. Geometry of cessation of annular Couette flow with Navier slip laws applied at the walls

We solve this initial boundary value problem with the method of separation of variables like in the previous paragraph and we get

$$T(t) = Ae^{-\frac{b^2}{R^2}\nu t} \text{ and } Y(r) = B\, Z_0\left(\frac{br}{R}\right),$$

where Z_0 is given by Eq. **(3.38).** From the boundary conditions we get:

$$Y(r)T(t) = BR\frac{d}{dr}\big(Y(r)T(t)\big)_{r=\kappa R} = BR\left[T(t)\frac{dY(r)}{dr}\right]_{r=\kappa R},$$

$$Y(r)T(t) = -BRr\frac{d}{dr}\big(Y(r)T(t)\big)_{r=R} = BRr\left[T(t)\frac{dY(r)}{dr}\right]_{r=R}.$$

So,

$$Z_0(\kappa b_\kappa) = \left[BR\frac{dZ_0\left(b_\kappa\frac{r}{R}\right)}{dr}\right]_{r=\kappa R}$$

$$\left(\frac{d}{dr}Z_0\left(b_\kappa\frac{r}{R}\right) = \frac{b_\kappa}{R}\left(\frac{1}{2}Z_{-1}\left(b_\kappa\frac{r}{R}\right) - \frac{1}{2}Z_1\left(b_\kappa\frac{r}{R}\right)\right) = \frac{b_\kappa}{R}\left(-\frac{1}{2}Z_{-1}\left(b_\kappa\frac{r}{R}\right) - \frac{1}{2}Z_1\left(b_\kappa\frac{r}{R}\right)\right)\right.$$
$$\left. = -\frac{b_\kappa}{R}Z_1\left(b_\kappa\frac{r}{R}\right)\right) \Rightarrow$$

$$Z_0(\kappa b_\kappa) = -\frac{BRb_\kappa}{R}Z_1(\kappa b_\kappa) \Rightarrow$$

$$Z_0(\kappa b_\kappa) + Bb_\kappa Z_1(\kappa b_\kappa) = 0\,. \qquad \textbf{(3.49)}$$

Similarly,

$$Z_0(b_\kappa) - \ Bb_\kappa\, Z_1(b_\kappa)\ = 0. \qquad \textbf{(3.50)}$$

From Eq. **(3.50)** we get $J_0(b_\kappa) + \gamma_\kappa Y_0(b_\kappa) - Bb_\kappa\big(J_1(b_\kappa) + \gamma_\kappa Y_1(b_\kappa)\big) = 0$

$$\gamma_\kappa = -\ \frac{J_0(b_\kappa) - Bb_\kappa J_1(b_\kappa)}{Y_0(b_\kappa) - Bb_\kappa Y_1(b_\kappa)}.$$

Substituting γ_κ into Eq. **(3.49)** we get

$$\big(Bb_\kappa Y_1(b_\kappa) - Y_0(b_\kappa)\big)J_0(\kappa b_\kappa) + \big(J_0(b_\kappa) - Bb_\kappa J_1(b_\kappa)\big)Y_0(\kappa b_\kappa)$$
$$= -Bb_\kappa\Big(\big(Bb_\kappa Y_1(b_\kappa) - Y_0(b_\kappa)\big)J_1(\kappa b_\kappa) + \big(J_0(b_\kappa) - Bb_\kappa J_1(b_\kappa)\big)Y_1(\kappa b_\kappa)\Big).$$

Superposition of the solutions, gives:

$$u_z(r,t) = \sum_{k=1}^{\infty} D_\kappa Z_{0k}\left(\frac{b_\kappa r}{R}\right)e^{-\frac{b_k^2}{R^2}\nu t} \qquad \textbf{(3.51)}$$

When $t = 0$ we get from equation **(3.51)** and the initial condition the following:

$$\sum_{k=1}^{\infty} D_\kappa Z_{0\kappa}\left(\frac{b_\kappa r}{R}\right) = \frac{\kappa V \ln\dfrac{r}{\kappa R} + BV}{B(\kappa+1) + \kappa \ln\dfrac{1}{\kappa}}\ .$$

Using the same orthogonality condition like in the previous paragraph, we multiply by $rZ_0\left(\frac{b_n r}{R}\right)$ and integrate from κR to R:

$$D_\kappa \int_{\kappa R}^{R} Z_0^2\left(\frac{b_n r}{R}\right) r\,dr = \int_{\kappa R}^{R} \frac{\kappa V \ln\dfrac{r}{\kappa R} + BV}{B(\kappa+1) + \kappa \ln\dfrac{1}{\kappa}}\ Z_0\left(\frac{b_n r}{R}\right) r\ dr.$$

Now let $\frac{r}{R} = \xi$ so if $r \in [\kappa R, R] \Rightarrow \xi \in [\kappa, 1]$ and if $\frac{r}{R} = \xi$ $then$ $\frac{1}{R}dr = d\xi$.

So

$$D_\kappa R^2 \int_{\kappa}^{1} Z_0^2(b_n\xi)\,\xi\,d\xi = \frac{R^2}{B(\kappa+1) + \kappa \ln\dfrac{1}{\kappa}} \int_{\kappa}^{1} (\kappa V \ln\frac{\xi}{\kappa} + BV)\ Z_0(b_n\xi)\,\xi\ d\xi \Rightarrow$$

$$D_\kappa = \frac{1}{B(\kappa+1) + \kappa \ln\dfrac{1}{\kappa}} \frac{\int_\kappa^1 (\kappa V \ln\frac{\xi}{\kappa} + BV)\ Z_0(b_n\xi)\,\xi\ d\xi}{\int_\kappa^1\ Z_0^2(b_n\xi)\,\xi\,d\xi}\ . \qquad \textbf{(3.52)}$$

We calculate from Eq.**(3.52)**:

$$I_1 := \int_{\kappa}^{1} BV\xi\, Z_0(b_n\xi)\; d\xi = BV \int_{\kappa b_n}^{b_n} \frac{u}{b_n}\, Z_0(u)\frac{du}{b_n} = \frac{BV}{b_n^2}[uZ_1(u)]_{\kappa b_n}^{b_n}$$

$$= \frac{BV}{b_n}[Z_1(\kappa b_n) - \kappa Z_1(\kappa b_n)],$$

$$I_2 := \int_{\kappa}^{1} \kappa V\xi \ln\frac{\xi}{\kappa}\; Z_0(b_\kappa\xi)\, d\xi = \kappa V \int_{\kappa b_n}^{b_n} \frac{u}{b_n} \ln\frac{u}{\kappa b_n} Z_0(u)\frac{du}{b_n}$$

$$= \kappa V\left(\frac{1}{b_n}\ln\frac{1}{\kappa}Z_1(b_n) + \frac{1}{b_n^2}Z_0(b_n) - \frac{1}{b_n^2}Z_0(\kappa b_n)\right)$$

$$= \kappa V\left(\frac{1}{b_n}\ln\frac{1}{\kappa}Z_1(b_n) + \frac{B}{b_n}Z_1(b_n) - \frac{B}{b_n}Z_1(\kappa b_n)\right),$$

$$I_3 := \int_{\kappa}^{1} \xi\, Z_0^2(b_n\xi)\; d\xi = \left[\frac{\xi^2}{2}\{Z_0'(b_\kappa\xi)\}^2 + \frac{\xi^2}{2}\{Z_0(b_\kappa\xi)\}^2\right]_\kappa^1$$

$$= \left[\frac{\xi^2}{2}\{Z_1(b_\kappa\xi)\}^2 + \frac{\xi^2}{2}\{Z_0(b_\kappa\xi)\}^2\right]_\kappa^1$$

$$= \frac{1}{2}\{Z_1^2(b_\kappa) - \kappa^2 Z_1^2(\kappa b_\kappa)\} + \frac{1}{2}\{Z_0^2(b_\kappa) - \kappa^2 Z_0^2(\kappa b_\kappa)\}^2$$

$$= \frac{1}{2}Z_1^2(b_\kappa) - \frac{\kappa^2}{2}Z_1^2(\kappa b_\kappa) + \frac{B^2 b_\kappa^2}{2}Z_1^2(b_\kappa) - \frac{\kappa^2 B^2 b_\kappa^2}{2}Z_1^2(\kappa b_\kappa)$$

$$= \frac{1}{2}[(1 + B^2 b_\kappa^2)Z_1^2(b_\kappa) - \kappa^2(1 + B^2 b_\kappa^2)Z_1^2(\kappa b_\kappa)]$$

$$= \frac{1}{2}(1 + B^2 b_\kappa^2)[Z_1^2(b_\kappa) - \kappa^2 Z_1^2(\kappa b_\kappa)].$$

From Eq. **(3.52)** we have:

$$D_\kappa = \frac{1}{B(\kappa + 1) + \kappa \ln\frac{1}{\kappa}} *$$

$$* \frac{\frac{BV}{b_n}[Z_1(\kappa b_n) - \kappa Z_1(\kappa b_n)] + \kappa V\left(\frac{1}{b_n}\ln\frac{1}{\kappa}Z_1(b_n) + \frac{B}{b_n}Z_1(b_n) - \frac{B}{b_n}Z_1(\kappa b_n)\right)}{\frac{1}{2}(1 + B^2 b_\kappa^2)[Z_1^2(b_\kappa) - \kappa^2 Z_1^2(\kappa b_\kappa)]}.$$

As a result,

$$D_k = \frac{2VZ_{1k}(b_k)}{b_k(1+B^2 b_\kappa^2)[Z_1^2(b_\kappa) - \kappa^2 Z_1^2(\kappa b_\kappa)]} \; .$$

The velocity in Eq. **(3.51)** is:

$$u_z(r,t) = \sum_{k=1}^{\infty} \frac{2VZ_{1k}(b_k)}{b_k(1+B^2 b_\kappa^2)[Z_1^2(b_\kappa) - \kappa^2 Z_1^2(\kappa b_\kappa)]} Z_{0k}\left(\frac{b_k r}{R}\right) e^{-\frac{b_k^2}{R^2}\nu t}. \qquad \textbf{(3.53)}$$

Now we let $t^* = \nu t/R^2,\ r^* = r/R,\ u_z^* = u_Z/V, D'_\kappa = D_k/V$ and get the dimensionless form:

$$u_z^*(r^*,t^*) = \sum_{k=1}^{\infty} D'_\kappa Z_{0k}(b_k r^*)\, e^{-b_k^2 t^*} \qquad \textbf{(3.54)}$$

■

In Fig. 8, we can see the evolution of the velocity profile in cessation of circular Couette flow for different values of B. As expected the value of the velocity is decreasing but the gradient of the curve of the velocity is increasing. When we increase the value of B, our curves tend to be closer to the initial curves for $t = 0$.

In Fig. 9, we have the velocity at the walls and as expected the velocity in the outer wall is decreasing by time and the velocity in the inner wall is increasing till we reach our steady state velocities in the walls.

Remark: When $B = 0 \quad (\beta \rightarrow \infty)$ Eq. **(3.54)** is reduced to Eq. **(3.45)**

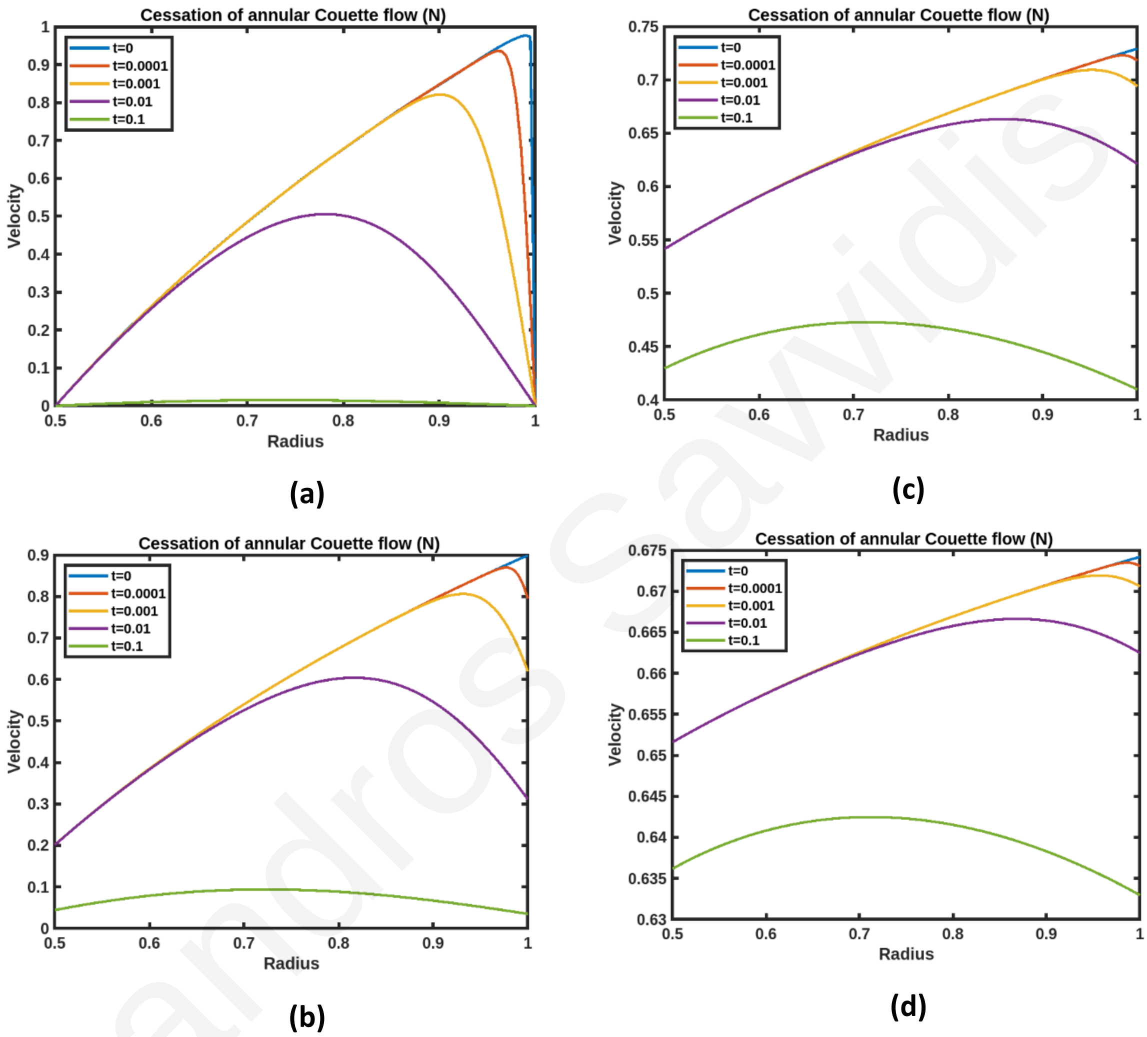


Fig. 8. Evolution of the velocity profile in cessation of annular Couette flow with *κ=0.5* and *t*=0, 0.0001, 0.001, 0.01 and 0.1*:* **(a)** *B=0* (no slip)*;* **(b)** *B=0.1* (weak slip)*;* **(c)** *B=1* (moderate slip)*;* **(d)** *B=10* (strong slip)

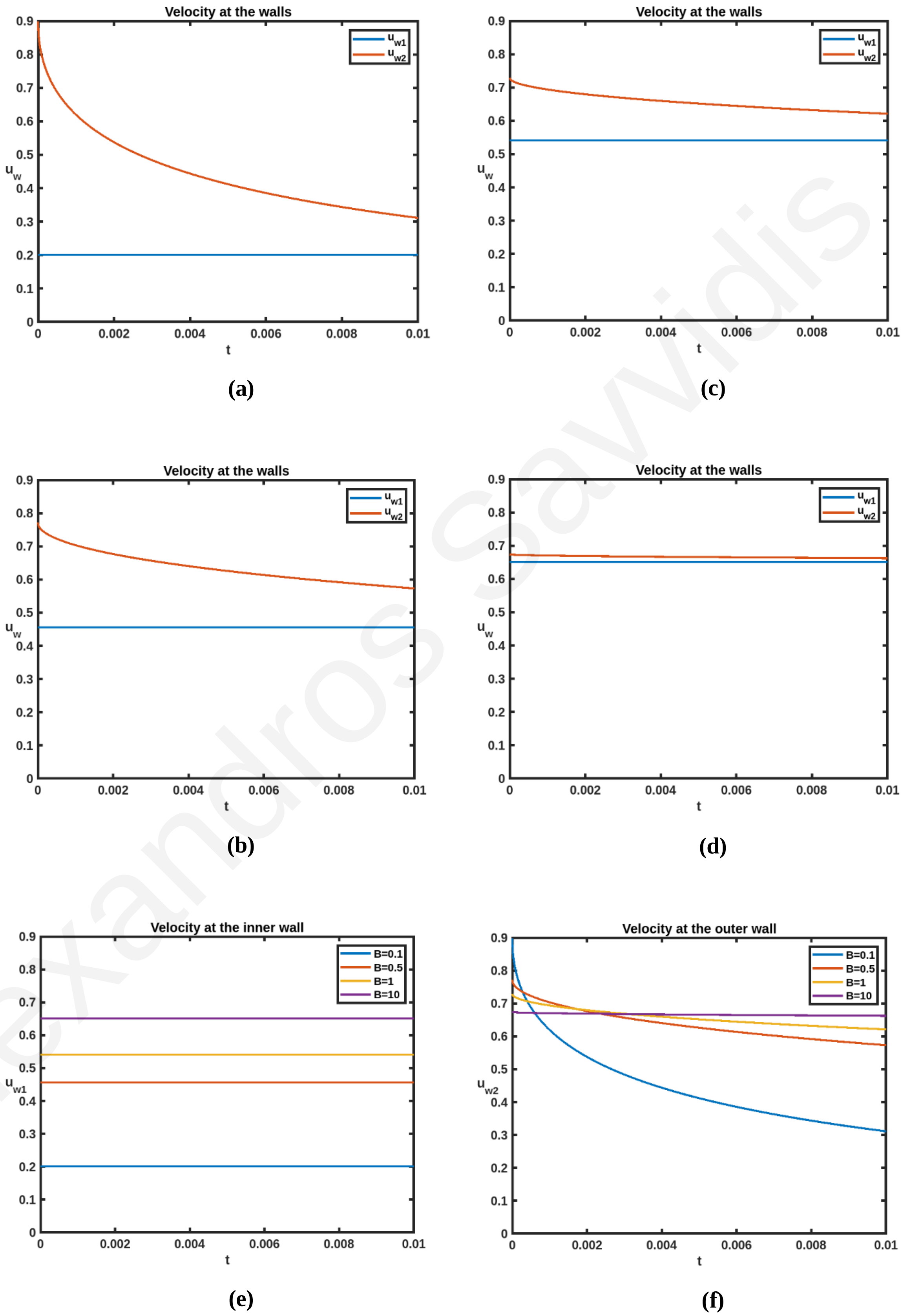


Fig. 9. Evolution of the velocity at the walls when: **(a)** *B=0.1* ; **(b)** *B=0.5* ; **(c)** *B=1* ; **(d)** *B=10* and again for the same values of *B,* all the velocities: **(e)** u_{w_1} ; **(f)** u_{w_2}

3.5 Cessation of annular Couette flow with dynamic slip at the walls

The geometry of the cessation of annular Couette flow with dynamic slip at the walls can be seen in *Fig. 10*. For this problem, the outer cylinder is sliding with velocity V and there is dynamic slip at the walls. When $t = 0$ the velocity ceases to exist. In cessation flow, the velocity of the fluid at both walls will be decreasing , which implies that u_{w_1} will be decreasing and u_{w_2} will be increasing with time. As a result, the boundary and initial conditions are:

$$r = \kappa R, \quad u_z = u_{w_1} = BR\frac{du_z}{dr} + \Lambda\frac{du_z}{dt}|_{r=\kappa R} \qquad t \geq 0\,, \tag{3.55}$$

$$r = R\,, \quad u_z = -u_{w_2} = -BR\frac{du_z}{dr} + \Lambda\frac{du_z}{dt}|_{r=R} \qquad t > 0, \tag{3.56}$$

Since,

$$u_{w_1} - \Lambda\frac{du_{w_1}}{dt} = \frac{\tau_{w_1}}{\beta} \quad and \quad u_{w_2} + \Lambda\frac{du_{w_2}}{dt} = \frac{\tau_{w_2}}{\beta}\,.$$

The initial condition is

$$u_z(r,0) = \frac{\kappa V \ln\dfrac{r}{\kappa R} + BV}{B(\kappa+1) + \kappa\ln\dfrac{1}{\kappa}}\,. \tag{3.57}$$

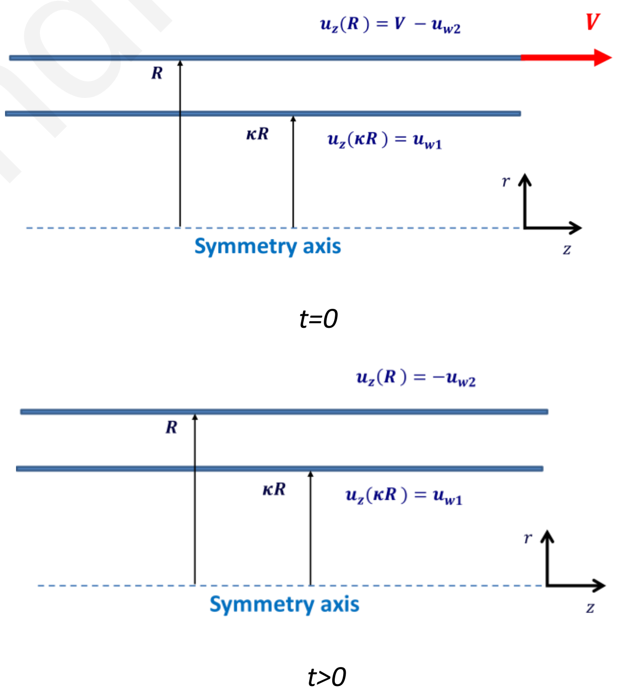


Fig. 10. Geometry of cessation of annular Couette flow with the dynamic slip laws applied at the walls

We solve this initial boundary value problem with the method of separation of variables like in the two previous paragraphs and we get:

$$T(t) = Ae^{-\frac{\lambda^2}{R^2}\nu t} \text{ and } Y(r) = B\, Z_0\left(\frac{\lambda r}{R}\right),$$

where Z_0 is given by Eq. **(3.38)**. From the boundary conditions we get

$$\begin{aligned} Y(r)T(t) &= BR\frac{d}{dr}\big(Y(r)T(t)\big)_{r=\kappa R} + \Lambda\left(-\frac{\lambda_\kappa^2}{R^2}\nu\right)Y(r)T(t) \\ &= \left(BR\left[T(t)\frac{dY(r)}{dr}\right] - \Lambda\left(\frac{\lambda_\kappa^2}{R^2}\nu\right)Y(r)T(t)\right)_{r=\kappa R}, \end{aligned}$$

$$\begin{aligned} Y(r)T(t) &= -BRr\frac{d}{dr}\big(Y(r)T(t)\big)_{r=R} + \Lambda\left(-\frac{\lambda_\kappa^2}{R^2}\nu\right)Y(r)T(t) \\ &= \left(-BR\left[T(t)\frac{dY(r)}{dr}\right] - \Lambda\left(\frac{\lambda_\kappa^2}{R^2}\nu\right)Y(r)T(t)\right)_{r=R}. \end{aligned}$$

So,

$$Z_0(\kappa\lambda_\kappa) = \left[BR\frac{dZ_0\left(\lambda_\kappa \frac{r}{R}\right)}{dr}\right]_{r=\kappa R}$$

$$\begin{aligned} &\left(\frac{d}{dr}Z_0\left(\lambda_\kappa\frac{r}{R}\right) = \frac{\lambda_\kappa}{R}\left(\frac{1}{2}Z_{-1}\left(\lambda_\kappa\frac{r}{R}\right) - \frac{1}{2}Z_1\left(\lambda_\kappa\frac{r}{R}\right)\right) = \frac{b_\kappa}{R}\left(-\frac{1}{2}Z_{-1}\left(\lambda_\kappa\frac{r}{R}\right) - \frac{1}{2}Z_1\left(\lambda_\kappa\frac{r}{R}\right)\right)\right. \\ &\qquad\left. = -\frac{\lambda_\kappa}{R}Z_1\left(\lambda_\kappa\frac{r}{R}\right)\right) \Rightarrow \end{aligned}$$

$$Z_0(\kappa\lambda_\kappa) = -\frac{BR\lambda_\kappa}{R}Z_1(\kappa\lambda_\kappa) - \Lambda\left(\frac{\lambda_\kappa^2}{R^2}\nu\right)Z_0(\kappa\lambda_\kappa) \Rightarrow$$

$$B\lambda_\kappa Z_1(\kappa\lambda_\kappa) + \left(1 + \Lambda\nu\frac{\lambda_\kappa^2}{R^2}\right)Z_0(\kappa\lambda_\kappa) = 0 \;. \qquad \textbf{(3.58)}$$

Similarly,

$$B\lambda_\kappa\, Z_1(\lambda_\kappa) - \left(1 + \Lambda\nu\frac{\lambda_\kappa^2}{R^2}\right)Z_0(\lambda_\kappa) = 0 \;. \qquad \textbf{(3.59)}$$

From **(3.59)** we get $B\lambda_\kappa\big(J_1(\lambda_\kappa) + \delta_\kappa Y_1(\lambda_\kappa)\big) - \left(1 + \Lambda\nu\frac{\lambda_\kappa^2}{R^2}\right)\big(J_0(\lambda_\kappa) + \delta_\kappa Y_0(\lambda_\kappa)\big) = 0$,

$$\delta_\kappa = -\frac{B\lambda_\kappa J_1(\lambda_\kappa) - \left(1 + \Lambda\nu\frac{\lambda_\kappa^2}{R^2}\right)J_0(\lambda_\kappa)}{B\lambda_\kappa Y_1(\lambda_\kappa) - \left(1 + \Lambda\nu\frac{\lambda_\kappa^2}{R^2}\right)Y_0(\lambda_\kappa)}\,.$$

Substituting δ_κ into Eq. **(3.58)** we get

$$B\lambda_\kappa\left(\left(\left(1+\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)Y_0(\lambda_\kappa)-B\lambda_\kappa Y_1(\lambda_\kappa)\right)J_1(\kappa\lambda_\kappa)\right.$$
$$\left.+\left(B\lambda_\kappa J_1(\lambda_\kappa)-\left(1+\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)J_0(\lambda_\kappa)\right)Y_1(\kappa\lambda_\kappa)\right)$$
$$=-\left(1+\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)\left[\left(\left(1+\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)Y_0(\lambda_\kappa)-B\lambda_\kappa Y_1(\lambda_\kappa)\right)J_0(\kappa\lambda_\kappa)\right.$$
$$\left.+\left(B\lambda_\kappa J_1(\lambda_\kappa)-\left(1+\Lambda v\frac{\lambda_\kappa^2}{R^2}\right)J_0(\lambda_\kappa)\right)Y_0(\kappa\lambda_\kappa)\right].$$

Superposition of the solutions gives:

$$u_z(r,t)=\sum_{k=1}^{\infty}E_k Z_{0k}\left(\frac{\lambda_k r}{R}\right)e^{-\frac{\lambda_k^2}{R^2}vt}. \qquad \textbf{(3.60)}$$

We next find the appropriate condition for the eigenfunctions Z_0:

$$rX_n''(r)+X_n'(r)+\frac{\lambda_n^2}{R^2}rX_n(r)=0\,. \qquad \textbf{(3.61)}$$

From B.C.s:

$$X_n(\kappa R)=BRX_n'(\kappa R)-\lambda_n^2\frac{\Lambda v}{R^2}X_n(\kappa R), \qquad \textbf{(3.62)}$$

$$X_n(R)=-BRX_n'(R)-\lambda_n^2\frac{\Lambda v}{R^2}X_n(R). \qquad \textbf{(3.63)}$$

And now we consider the one-dimensional problem in r and introduce:

$$rX_m''(r)+X_m'(r)+\frac{\lambda_m^2}{R^2}rX_m(r)=0 \qquad \textbf{(3.64)}$$

$$X_m(\kappa R)=BRX_m'(\kappa R)-\lambda_m^2\frac{\Lambda v}{R^2}X_m(\kappa R) \qquad \textbf{(3.65)}$$

$$X_m(R)=-BRX_m'(R)-\lambda_m^2\frac{\Lambda v}{R^2}X_m(R) \qquad \textbf{(3.66)}$$

Since X_m, X_n and λ_m, λ_n are distinct $(m\neq n)$,

$$\left(rX_n''(r)+X_n'(r)+\frac{\lambda_n^2}{R^2}rX_n(r)=0\;\Rightarrow\;\left(rX_n'(r)\right)'+\frac{r}{R^2}\lambda_n^2X_n(r)\right).$$

Multiplying Eq. **(3.61)** by X_m and integrating by parts gives

$$\int_{\kappa R}^{R}\left(rX_n'(r)\right)'X_m(r)dr+\int_{\kappa R}^{R}\frac{r}{R^2}\lambda_n^2X_n(r)X_m(r)dr=0. \qquad \textbf{(3.67)}$$

Similarly, we multiply Eq. **(3.64)** by X_n, integrate by parts and then subtract it from Eq.**(3.67)** to get:

$$R[X_n'(R)X_m(R) - X_n(R)X_m'(R)] - \kappa R[X_n'(\kappa R)X_m(\kappa R) - X_n(\kappa R)X_m'(\kappa R)] + \frac{(\lambda_n^2 - \lambda_m^2)}{R^2}\int_{\kappa R}^{R} rX_n(r)X_m(r)dr = 0.$$

Using now Eqs. **(3.62)** , **(3.63)** , **(3.65)** and **(3.66)** we get:

$$(\lambda_n^2 - \lambda_m^2)\left[-\frac{\Lambda v}{B}\big(X_n(R)X_m(R) + \kappa X_n(\kappa R)X_m(\kappa R)\big) + \int_{\kappa R}^{R} rX_n(r)X_m(r)dr\right] = 0.$$

Since λ_m *and* λ_n are distinct

$$-\frac{\Lambda v}{B}\big(X_n(R)X_m(R) + \kappa X_n(\kappa R)X_m(\kappa R)\big) + \int_{\kappa R}^{R} rX_n(r)X_m(r)dr = \delta_{m,n}N_n, \qquad \textbf{(3.68)}$$

where

$$N_n = -\frac{\Lambda v}{B}\left({X_n}^2(R) + \kappa {X_n}^2(\kappa R)\right) + \int_{\kappa R}^{R} rX_n^2(r)dr, \qquad \textbf{(3.69)}$$

and $\delta_{m,n}$ is the Kronecker delta.

□

In order to find the coefficients E_κ, Eq.**(3.29)** must be supplemented by an extra term, thus multiplying it by $rZ_0\left(\frac{\lambda_n r}{R}\right)$ when $t = 0$, using the initial condition and integrating from κR till R gives:

$$\sum_{k=1}^{\infty} E_\kappa \int_{\kappa R}^{R} rZ_{0\kappa}\left(\frac{\lambda_\kappa r}{R}\right)Z_0\left(\frac{\lambda_n r}{R}\right)dr = \int_{\kappa R}^{R} \frac{\kappa V \ln\frac{r}{\kappa R} + BV}{B(\kappa+1) + \kappa \ln\frac{1}{\kappa}} rZ_0\left(\frac{\lambda_n r}{R}\right)dr.$$

Also,

$$\frac{\Lambda v}{B}\frac{\kappa V \ln\frac{r}{\kappa R} + BV}{B(\kappa+1) + \kappa \ln\frac{1}{\kappa}} Z_0\left(\frac{\lambda_n r}{R}\right) = \frac{\Lambda v}{B}\sum_{\kappa=1}^{\infty} E_\kappa Z_{0\kappa}\left(\frac{\lambda_\kappa r}{R}\right)Z_0\left(\frac{\lambda_n r}{R}\right).$$

When $r = R$,

$$\frac{\Lambda v}{B}\frac{\kappa V \ln\frac{1}{\kappa} + BV}{B(\kappa+1) + \kappa \ln\frac{1}{\kappa}} Z_0(\lambda_n) = \frac{\Lambda v}{B}\sum_{\kappa=1}^{\infty} E_\kappa Z_{0\kappa}(\lambda_\kappa)Z_0(\lambda_n),$$

And when $r = \kappa R$,

$$\frac{\Lambda v}{B}\frac{BV}{B(\kappa+1)+\kappa\ln\frac{1}{\kappa}}Z_0(\kappa\lambda_n) = \frac{\Lambda v}{B}\sum_{\kappa=1}^{\infty}E_\kappa Z_{1\kappa}(\kappa\lambda_\kappa)Z_1(\kappa\lambda_n).$$

So,

$$\sum_{\kappa R}^{R} E_\kappa\left(\int_{\kappa R}^{R} rZ_{0\kappa}\left(\frac{\lambda_\kappa r}{R}\right)Z_0\left(\frac{\lambda_n r}{R}\right)dr - \frac{\Lambda v}{B}[Z_{0\kappa}(\lambda_\kappa)Z_0(\lambda_n)+\kappa Z_{0\kappa}(\kappa\lambda_\kappa)Z_0(\kappa\lambda_n)]\right)$$
$$= \int_{\kappa R}^{R}\frac{\kappa V\ln\frac{r}{\kappa R}+BV}{B(\kappa+1)+\kappa\ln\frac{1}{\kappa}}rZ_0\left(\frac{\lambda_n r}{R}\right)dr$$
$$-\frac{\Lambda v}{B}\frac{\left(\kappa V\ln\frac{1}{\kappa}+BV\right)Z_0(\lambda_n)+\kappa BVZ_0(\kappa\lambda_n)}{B(\kappa+1)+\kappa\ln\frac{1}{\kappa}}.$$

We are following the same method of solution as the previous paragraphs and the constants E_κ are given by:

$$E_\kappa = \frac{2V}{\lambda_\kappa^2 L}Z_0(\lambda_\kappa), \tag{3.70}$$

where

$$L = B\left(Z_1^2(\lambda_\kappa)+Z_0^2(\lambda_\kappa)\right) - B\kappa^2\left(Z_1^2(\kappa\lambda_\kappa)+Z_0^2(\kappa\lambda_\kappa)\right) - \frac{2\Lambda v}{R^2}\left(Z_0^2(\lambda_\kappa)+\kappa Z_0^2(\kappa\lambda_\kappa)\right). \tag{3.71}$$

So, the solution of our problem is:

$$u_z(r,t) = \sum_{k=1}^{\infty}E_k Z_0\left(\frac{\lambda_k r}{R}\right)e^{-\frac{\lambda_k^2}{R^2}vt}. \tag{3.72}$$

Dividing u_z by V, and setting $u_z^* = u_z/V$, $r^* = r/R$, $t^* = vt/R^2$ we get:

$$u_z^*(r^*,t^*) = \sum_{k=1}^{\infty}\tilde{E}_k Z_0(\lambda_k r^*)\,e^{-\lambda_k^2 t^*}, \tag{3.73}$$

where $\widetilde{E_k} = E_k/V$ and the slip velocities are given by

$$u^*_{w_1}(t^*) = \sum_{k=1}^{\infty} \tilde{E}_k Z_0(\kappa\lambda_k)\, e^{-\lambda_k^2 t^*}, \quad \textbf{(3.74)}$$

$$u^*_{w_2}(t^*) = \sum_{k=1}^{\infty} \tilde{E}_k Z_0(\lambda_k)\, e^{-\lambda_k^2 t^*}. \quad \textbf{(3.75)}$$

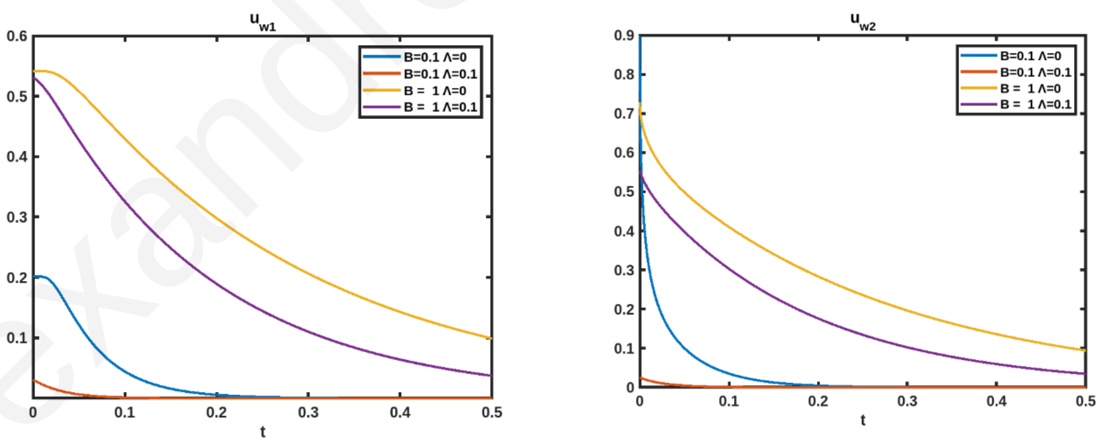
For more information about the method of solution for these types of problems, see (Kaoullas and Georgiou, 2015).

In Fig. 11, which depicts the velocity at the walls, we see that the velocity in the inner wall (Fig. 11b) as the value of B is increasing, our curves for different Λ tend to get closer but on the other hand in the outer wall (Fig. 11a), the curves tend to get further from each other.

In Fig. 12 and Fig. 13, we can see the evolution of the velocity profile in cessation of annular Couette flow for weak slip (B=0.1) and moderate slip (B=1) and how the velocity changes for different values of Λ.

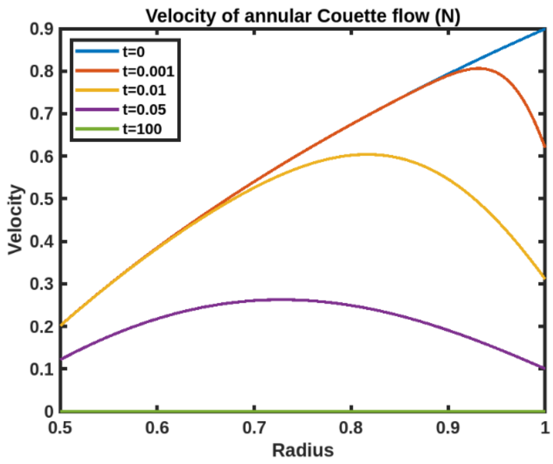
Remark: When $\Lambda = 0$ Eq. **(3.73)** is reduced to Eq. **(3.54)** and when $B = 0 \Leftrightarrow \beta \to \infty$ Eq. **(3.54)** is reduced to Eq. **(3.45)**

■

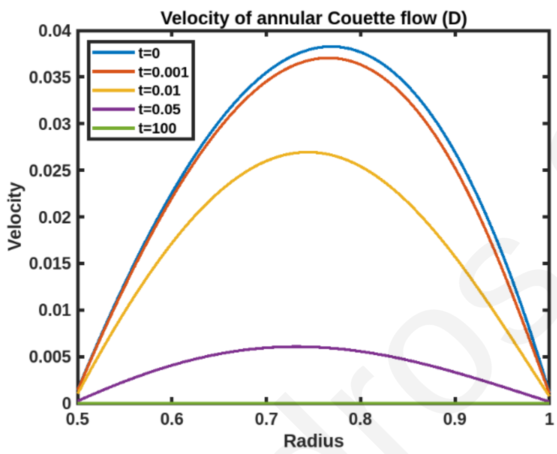


(a) **(b)**

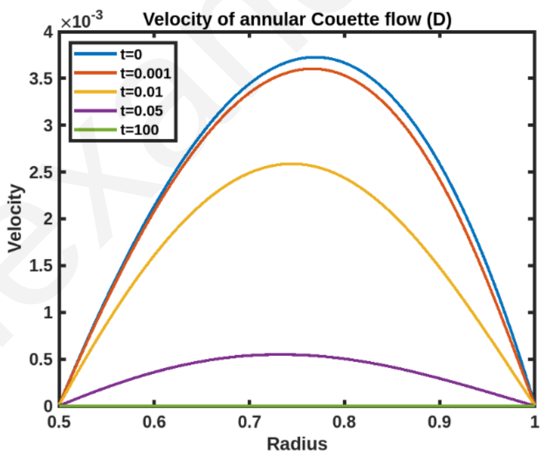
Fig. 11. Evolution of the slip velocity in cessation circular Couette flow for different values of Λ and κ=0.5: **(a)** u_{w_1}; **(b)** u_{w_2}



(a)

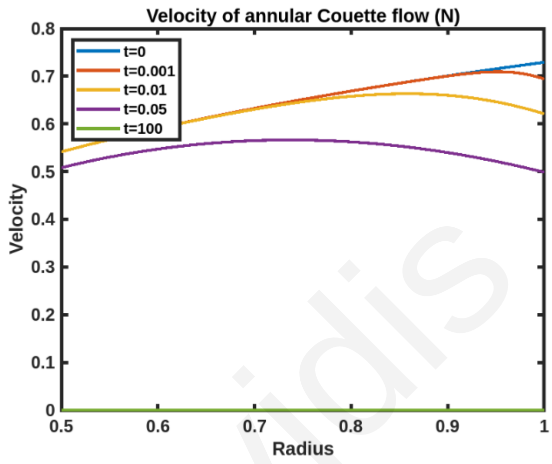


(b)

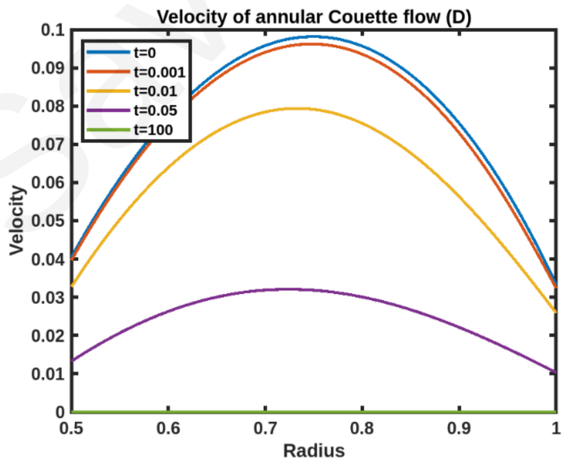


(c)

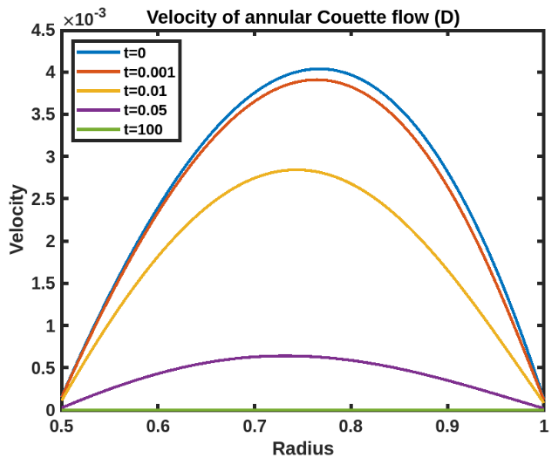
Fig. 12. Evolution of the velocity profile in cessation circular Couette flow with *κ=0.5* and *B=0.1:* **(a)** *Λ=0;* **(b)** *Λ=0.5;* **(c)** *Λ=5;*



(d)



(e)



(f)

Fig. 13. Evolution of the velocity profile in cessation circular Couette flow with *κ=0.5* and *B=1*: **(a)** *Λ=0;* **(b)** *Λ=0.5;* **(c)** *Λ=5;*

3.6 Appendix: Solutions when the inner cylinder is moving

In the previous section we derived analytical solutions for the steady state and cessation problem of annular Couette flow when the outer cylinder is sliding with velocity V, the inner cylinder is fixed, and there are three types of slip laws applied at the walls: (a) No-slip law; (b) Navier slip law; (c) Dynamic slip law.

In this paragraph we are going to provide the analytical solutions for the same problem but now instead of the outer cylinder, now the inner cylinder is sliding with velocity V, and the outer cylinder is fixed.

For the steady state of annular Couette flow with Navier slip at the walls the velocity is given by:

$$u_z = \frac{\kappa V \ln\frac{r}{R} + B\kappa V}{B(\kappa+1) + \kappa \ln \kappa} \ . \tag{3.76}$$

If we let $B = 0 \ \ (\beta \to \infty)$ the result will be the solution for the steady state annular Couette flow with no-slip at the walls:

$$u_z = \frac{V}{ln\kappa} \ln\frac{r}{R} \ . \tag{3.77}$$

For the cessation of annular Couette flow with dynamic slip at the walls the velocity will be:

$$u_z(r,t) = \sum_{k=1}^{\infty} E_k Z_0\left(\frac{\lambda_k r}{R}\right) e^{-\frac{\lambda_k^2}{R^2}\nu t}, \tag{3.78}$$

and the coefficients in **(3.78)** are given by

$$E_\kappa = -\frac{2\kappa V}{\lambda_\kappa^2} Z_0(\kappa\lambda_\kappa), \tag{3.79}$$

where

$$L = B\left(Z_1^2(\lambda_\kappa) + Z_0^2(\lambda_\kappa)\right) - B\kappa^2\left(Z_1^2(\kappa\lambda_\kappa) + Z_0^2(\kappa\lambda_\kappa)\right) + \frac{2\Lambda\nu}{R^2}\left(Z_0^2(\lambda_\kappa) + \kappa Z_0^2(\kappa\lambda_\kappa)\right). \tag{3.80}$$

If we let $\Lambda = 0$ then the result will be the solution for the cessation of annular Couette flow with Navier slip at the walls and the velocity will be given by:

$$u_z(r,t) = \sum_{k=1}^{\infty} D_k Z_{0k}\left(\frac{b_k r}{R}\right) e^{-\frac{b_k^2}{R^2}\nu t}, \tag{3.81}$$

where

$$D_\kappa = -\frac{2\kappa V}{b_\kappa} \frac{Z_1(\kappa b_\kappa)}{(1 + B^2 b_\kappa^2)[Z_1^2(b_\kappa) - \kappa^2 Z_1^2(\kappa b_\kappa)]}. \tag{3.82}$$

Now if we let $B = 0 \quad (\beta \to \infty)$ in Eq. **(3.82),** the result will be the solution for the cessation of annular Couette flow with no-slip at the walls:

$$u_z(r,t) = \sum_{k=1}^{\infty} C_k Z_{0k}\left(\frac{a_k r}{R}\right) e^{-\frac{a_k^2}{R^2}\nu t}, \tag{3.83}$$

where

$$C_\kappa = -\frac{2\kappa V}{\alpha_\kappa}\frac{Z_1(\kappa\alpha_\kappa)}{Z_1^2(\alpha_\kappa) - \kappa^2 Z_1^2(\kappa\alpha_\kappa)}. \tag{3.84}$$

■

Chapter 4: Concluding Remarks and Recommendations

In this thesis analytical solution for the cessation of circular and annular Newtonian Couette flow with dynamic slip along the walls were derived. Initially, the steady-state analytical solutions were presented for no-slip and Navier slip laws at the walls. Then, solutions were derived for the velocity, with the same slip laws applied when the cessation occurred, using the separation of variables method and the the well-known orthogonality condition. However, for the dynamic slip law, the orthogonality condition was found to differ due to the presence of a time-dependent term that causes the eigenvalue parameter to appear in the boundary conditions. The resulting Sturm-Liouville problem was different from that obtained using the static Navier slip condition. The orthogonality condition of the associated eigenfunctions was derived and the solutions were provided for the circular and annular Couette flow.

In the case of dynamic slip both for the circular and annular Couette flow, the slip velocity is not solely influenced by the present value of the wall shear stress, but also by its preceding states. For higher values of Λ, the time that was needed for the velocity to reach the steady-state value, was increasing. Consequently, the development of slip velocity and flow is slowed down due to this phenomenon.

A recommendation for a future research problem would be do derive analytical solutions for the problems of cessation of annular and circular Couette flow but this time with logarithmic wall slip applied to the walls, and compare the results with the problems that were studied in this thesis, with dynamic slip.

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