UNIVERSITY OF CYPRUS

DEPARTMENT OF EDUCATION

GEOMETRICAL FIGURE APPREHENSION: COGNITIVE PROCESSES AND STRUCTURE

DOCTORAL DISSERTATION

PARASKEVI MICHAEL

2013

DEPARTMENT OF EDUCATION

GEOMETRICAL FIGURE APPREHENSION: COGNITIVE PROCESSES AND STRUCTURE

Paraskevi Michael

Submitted to the faculty of the Department of Education
in partial fulfillment of the requirements
for the degree
Doctor of Philosophy
in the Department of Education,
University of Cyprus
May, 2013

To all the” lighthouses”

that illuminated my journey

and helped me reach

my final destination.

UNIVERSITY OF CYPRUS

DEPARTMENT OF EDUCATION

Dissertation Acceptance

This is to certify that the dissertation prepared

By Paraskevi Michael

Entitled Geometrical Figure Apprehension: Cognitive Processes and Structure

Complies with the University regulations and meets the standards of the University for originality and quality for the degree of Doctor of Philosophy.

The dissertation was successfully presented and defended to the examining committee on Thursday, 11th of April, 2013.

Examining Committee:

Athanasios Gagatsis (Supervisor), ………………………………

Professor, Department of Education, University of Cyprus

Constantinos Christou (Chair), ………………………………

Professor, Department of Education, University of Cyprus

Raymond Duval (Member), ………………………………

Emeritus Professor, Université du Littoral Côte-d'Opale, Boulogne & Centre, IUFM Nord Pas-de Calais, Lille

Demetra Pitta – Pantazi (Member), ………………………………

Associate Professor, Department of Education, University of Cyprus

Alain Kuzniak (Member), ………………………………

Professor, Laboratoire de Didactique André Revuz, Université Paris 7 Diderot

ABSTRACT

The way of looking at any figure constructed with specific tools is a crucial cognitive factor in solving problems and in reasoning and proving in geometry (Duval, 2012). There are many factors that can inhibit or favor discriminating the proper way of looking at geometrical figures, which can be studied experimentally (Duval, 2006). In addition, the universal problem regarding the transition from one educational level to another has occupied the research community in the recent years (Mullins & Irvin, 2000), according to the difficulties the students seem to face during this transition. Therefore, there is obviously a need to identify the cognitive processes and the type of apprehension that the students, from different age groups and educational levels, mobilize during the resolution of geometrical tasks and examine whether this selection leads to a proper way of "looking" at the geometrical figure.

The research was organized mainly in reference to two theoretical frameworks. The first one concerns the distinction between four kinds of figure apprehension proposed by Duval (1988) within the register of geometrical visualization: perceptual apprehension, sequential apprehension, operative apprehension and discursive apprehension. The second theoretical framework focuses on the evolution of the global objectives in the teaching of geometry throughout the curriculum. Houdement and Kuzniak (2003) have proposed the notion of Geometrical Paradigms in analyzing this evolution in the institutional organization of teaching.The synthesis of the aforementioned ideas of the literature formed the basis for conducting this research study.

The general aim of this research study was to investigate the structure and the cognitive processes of the geometrical figure apprehension of the lower and the upper secondary school students. In particular, the students' perceptual, operative, sequential and discursive apprehension were explored. The examination was based on four main axes:

1. The cognitive structure of the geometrical figure apprehension in the lower and the upper secondary school.
2. The relationships between the four types of geometrical figures apprehension.
3. The comparison between the lower and the upper secondary school students' geometrical figure apprehension.
4. The lower and the upper secondary school students' mistakes and ideas about the geometrical figure apprehension.

This examination was based on a combination of quantitative and qualitative data. The quantitative data were collected using a test comprising of 16 tasks, which was administered to 881 students, aged 15 to 17, of lower (Grade 9) and upper (Grade 10, Grade 11) urban and rural secondary schools, in Cyprus. In particular, the participants were 312 students from Grade 9, 304 students from Grade 10, 125 students from Grade 11a and 140 students from Grade 11b. The qualitative data were collected from 9 students with task – based interviews, based on the solution of four tasks.

The data analysis provided important information about the students' geometrical figure apprehension. In particular, a structural model was constructed and verified, which determined the importance of the perceptual, the operative, the sequential and the discursive apprehension for the apprehension of a geometrical figure. The structure of the geometrical figure apprehension was found invariant for the students of the two educational level and of each grade.

The various dimensions of the geometrical figure apprehension were further examined, for tracing the relationships between the four types of geometrical figures apprehension. Interrelations were traced between the different types of apprehension. Strong relations were found between the operative and the discursive apprehension, revealing the importance of visualization in geometrical reasoning. Significant relations also emerged between the discursive and the sequential apprehension, highlighting the importance of the knowledge of mathematical properties in a construction and a reasoning process. Furthermore, the role of the perceptual apprehension occurred very important for the mobilization of the discursive apprehension and the operative apprehension.

The students' geometrical figure apprehension mainly evolves from one grade to a next one and from one educational level to the following one. The sequential and the discursive apprehension tasks seem to create the most of the difficulties for the students, whereas the students are more able to solve the operative and the perceptual apprehension tasks. In addition the intervention of the perceptual apprehension in the solution of the tasks, in a way that it overrides the proper type of apprehension, leads the students to wrong answers or to correct answers that were reached through wrong procedures.

The students' mistakes were mostly related to the perceptual apprehension. Based on the relations between the students' mistakes in the tasks about the recognition of proof and their responses in the tasks about the production of proofs, the students' type of geometrical paradigm in which their geometrical work takes place was determined. In fact, the geometrical work of the students in the lower secondary school and the first grade of the upper secondary school seems to be mostly situated within a paradigm of a mixed type of geometry (GI/GII), possessing mostly characteristics of the Natural Geometry, whereas the geometrical work of the rest of the students in the upper secondary school appears to be mostly related to a mixed type of geometry (GII/GI) which mainly comprises of the characteristics of the Natural Axiomatic Geometry.

ΠΕΡΙΛΗΨΗ

Ο τρόπος που βλέπουμε ένα οποιοδήποτε σχήμα, κατασκευασμένο με συγκεκριμένα όργανα, αποτελεί ένα κρίσιμο γνωστικό παράγοντα για την επίλυση προβλημάτων, για το γεωμετρικό συλλογισμό και τη γεωμετρική απόδειξη (Duval, 2012). Υπάρχουν πολλοί παράγοντες που μπορούν να παρεμποδίσουν ή να ευνοήσουν τη διάκριση του κατάλληλου τρόπου με τον οποίο πρέπει να βλέπουμε τα γεωμετρικά σχήματα, οι οποίοι μπορούν να μελετηθούν εμπειρικά (Duval, 2006). Επιπλέον, το παγκόσμιο πρόβλημα σχετικά με τη μετάβαση από ένα εκπαιδευτικό επίπεδο σε ένα άλλο έχει απασχολήσει την ερευνητική κοινότητα τα τελευταία χρόνια (Mullins & Irvin, 2000), ως προς τις δυσκολίες που οι μαθητές φαίνεται να αντιμετωπίζουν κατά τη μετάβαση αυτή. Ως εκ τούτου, είναι εμφανές ότι υπάρχει ανάγκη να προσδιοριστούν οι γνωστικές διαδικασίες και ο τύπος σύλληψης γεωμετρικού σχήματος που ενεργοποιείται κατά την επίλυση γεωμετρικών προβλημάτων από μαθητές διαφόρων ηλιακών ομάδων και εκπαιδευτικών βαθμίδων και να εξεταστεί κατά πόσο αυτή η επιλογή οδηγεί τους μαθητές στο να δουν ένα γεωμετρικό σχήμα με τον κατάλληλο τρόπο.

Η έρευνα αυτή οργανώθηκε με αναφορά σε δύο κυρίως θεωρητικά πλαίσια. Η πρώτη αφορά στη διάκριση μεταξύ τεσσάρων τύπων σύλληψης γεωμετρικών σχημάτων, που προτείνεται από τον Duval (1988), στα πλαίσια της οπτικοποίησης στη γεωμετρία: η αντιληπτική σύλληψη, η ακολουθιακή σύλληψη, η λειτουργική σύλληψη και η λεκτική σύλληψη. Το δεύτερο θεωρητικό πλαίσιο εστιάζεται στην εξέλιξη των γενικών στόχων του Αναλυτικού Προγράμματος για τη διδασκαλία της γεωμετρίας. Οι Houdement και Kuzniak (2003) έχουν προτείνει την ιδέα των γεωμετρικών παραδειγμάτων, στα πλαίσια της ανάλυσης του τρόπου εξέλιξης της θεσμικής οργάνωσης της διδασκαλίας. Η σύνθεση των παραπάνω ιδεών, όπως προκύπτουν μέσα από τη βιβλιογραφία, αποτέλεσε τη βάση για τη διεξαγωγή της παρούσας ερευνητικής μελέτης.

Ο γενικός στόχος αυτής της ερευνητικής μελέτης ήταν να διερευνήσει τη δομή και τις γνωστικές διαδικασίες της σύλληψης γεωμετρικού σχήματος, σε μαθητές Γυμνασίου και Λυκείου. Ειδικότερα, διερευνήθηκαν η αντιληπτική, η λειτουργική, η ακολουθιακή και η λεκτική σύλληψη γεωμετρικών σχημάτων. Η μελέτη βασίστηκε σε τέσσερις βασικούς άξονες διερεύνησης:

1. Η γνωστική δομή της σύλληψης γεωμετρικού σχήματος των μαθητών Γυμνασίου και Λυκείου.

2. Οι σχέσεις μεταξύ των τεσσάρων τύπων σύλληψης γεωμετρικών σχημάτων.

3. Η σύγκριση της σύλληψης γεωμετρικών σχημάτων μεταξύ των μαθητών Γυμνασίου και Λυκείου.

4. Τα λάθη και οι ιδέες των μαθητών Γυμνασίου και Λυκείου σχετικά με τη σύλληψη γεωμετρικού σχήματος.

Η μελέτη αυτή βασίστηκε στο συνδυασμό ποσοτικών και ποιοτικών δεδομένων. Τα ποσοτικά δεδομένα συλλέχθηκαν με τη χρήση ενός δοκιμίου που περιλάμβανε 16 γεωμετρικά έργα. Το δοκίμιο χορηγήθηκε σε 881 μαθητές, ηλικίας 15 έως 17, από Γυμνάσια και Λύκεια αστικών και αγροτικών περιοχών της Κύπρου. Συγκεκριμένα, οι συμμετέχοντες ήταν 312 μαθητές Γ΄ Γυμνασίου, 304 μαθητές Α΄ Λυκείου, 125 μαθητές Β΄ Λυκείου με μαθηματικά κοινού κορμού και 140 μαθητές από Β΄ Λυκείου με μαθηματικά κατεύθυνσης. Τα ποιοτικά δεδομένα συλλέχθηκαν από 9 μαθητές, από τους οποίους πάρθηκαν συνεντεύξεις με βάση την επίλυση τεσσάρων έργων.

Από την ανάλυση των δεδομένων προέκυψαν σημαντικά στοιχεία για τη σύλληψη γεωμετρικού σχήματος των μαθητών. Συγκεκριμένα, κατασκευάστηκε και επαληθεύτηκε ένα δομικό μοντέλο, με το οποίο προσδιορίστηκε η σημασία της αντιληπτικής, της λειτουργικής, της ακολουθιακής και της λεκτικής σύλληψης γεωμετρικού σχήματος. Η δομή του γεωμετρικού σχήματος σύλληψης διατηρείται αμετάβλητη για τους μαθητές των δύο εκπαιδευτικών βαθμίδων και της κάθε τάξης.

Οι διάφορες διαστάσεις της σύλληψης γεωμετρικού σχήματος εξετάστηκαν περαιτέρω, ώστε να εντοπιστούν οι σχέσεις μεταξύ των τεσσάρων τύπων σύλληψης. Αλληλεπιδράσεις εντοπίστηκαν μεταξύ των διαφόρων τύπων σύλληψης. Ισχυρές σχέσεις βρέθηκαν μεταξύ της λειτουργικής και της λεκτικής σύλληψης, καταδεικνύοντας τη σημασία της οπτικοποίησης στο γεωμετρικό συλλογισμό. Σημαντικές, επίσης, ήταν οι σχέσεις που προέκυψαν μεταξύ της λεκτικής και της ακολουθιακής σύλληψης, τονίζοντας τη σημασία της γνώσης των μαθηματικών ιδιοτήτων για τις γεωμετρικές κατασκευές και τις διαδικασίες συλλογισμού για τις γεωμετρικές αποδείξεις. Επιπλέον, ο ρόλος της αντιληπτικής σύλληψης προέκυψε καθοριστικός για την ενεργοποίηση της λειτουργικής και της λεκτικής σύλληψης γεωμετρικού σχήματος.

Η σύλληψη γεωμετρικού σχήματος των μαθητών κυρίως αναπτύσσεται από τη μια εκπαιδευτική βαθμίδα στην επόμενη. Οι κυριότερες δυσκολίες των μαθητών προκύπτουν στα έργα λεκτικής και ακολουθιακής σύλληψης, ενώ εμφανίζονται ικανότεροι στην επίλυση έργων αντιληπτικής και λειτουργικής σύλληψης. Επιπλέον, η παρέμβαση της αντιληπτικής σύλληψης στη λύση των έργων, με τρόπο που να υπερισχύει έναντι του απαιτούμενου τύπου σύλληψης, οδηγεί τους μαθητές σε λανθασμένες απαντήσεις ή σε ορθές απαντήσεις που προκύπτουν μέσα από λανθασμένες διαδικασίες.

Τα λάθη των μαθητών σχετίζονται, ως επί το πλείστον, με την αντιληπτική σύλληψη. Με βάση τις σχέσεις μεταξύ των λαθών των μαθητών στα έργα αναγνώρισης της απόδειξης και τις απαντήσεις τους στα έργα παραγωγής απόδειξης, καθορίστηκε ο τύπος γεωμετρικού παραδείγματος εντός του οποίου οι μαθητές εργάζονται στη Γεωμετρία. Συγκεκριμένα, η γεωμετρική δραστηριότητα των μαθητών του Γυμνασίου και της Α΄ τάξης του Λυκείου φαίνεται να εντοπίζεται κυρίως σε ένα παράδειγμα ενός μικτού τύπου Γεωμετρίας (Γεωμετρία 1/2), το οποίο περιέχει περισσότερα χαρακτηριστικά από την Εμπειρική Γεωμετρία (Γεωμετρία 1), ενώ η γεωμετρική δραστηριότητα των υπολοίπων μαθητών του Λυκείου φαίνεται ως επί το πλείστον να συνδέεται με ένα μικτό τύπο γεωμετρίας (Γεωμετρία 2/1), ο οποίος διακατέχεται περισσότερο από χαρακτηριστικά της Εμπειρικής Αξιωματικής Γεωμετρίας.

ACKNOWLEDGMENTS

The accomplishment of the particular Thesis is something related to a lot of effort, time and devotion, but also with the fulfillment of personal inquiries regarding the research domain. During this four year period of doctoral studies the contribution of many important people was crucial. Therefore the least I could do is to express my thanks to them through these written words.

I would first of all like to thank my supervisor and advisor Athanasios Gagatsis for all the help, the guidance and the support. His personal interest and his advices were decisive during my postgraduate studies. Professor Gagatsis’ academic and research experience, his scientific articulation but also his pleasant personality made our cooperation a very special experience, during which I had a lot of opportunities for learning, for gaining many experiences and for improving myself as a students and a researcher.

I would like to thank the rest members of the committee for taking time to read my work and provide guidance. Many thanks to Assistant Professor Demetra Pitta-Pantazi, to Professor Constantinos Christou and to Professor Alain Kuzniak for their helpful remarks and advises for the formulation of the final text of the Thesis.

I thank Lecturer Iasonas Lamprianou and Professor Leonidas Kyriakides as well, for their help and guidance regarding the statistical analyses that were performed for the needs of this Thesis.

Furthermore, I thank three very good friends, Lecturer Iliada Elia, Dr. Eleni Deliyianni and Dr. Annita Monoyiou for helping me and for always being willing to discuss, answer my questions and exchange ideas concerning my work.

Finally, I express my thanks to my beloved parents Maria, Chrysanthos, Giota and Michalis, for being always next to me, believing in me and supporting me in every way. I could not, of course, forget the contribution of all my dear friends (Anna, Stella, Georgia, Maria, Galatia, Elena, Ioli, Chara, Xenia and Fotini) who encouraged me and gave me strength with their love. I am grateful to share this accomplishment with my family and friends.

I left on purpose a very important person to express my thanks last, because of his crucial contribution for the accomplishment of this Thesis. I refer to Emeritus Professor Raymond Duval, whose help was precious for this effort. Professor Duval spent much time reading my work and discussing with me and always provided an encouraging word with his feedback. I greatly appreciate his willingness to work with me for accomplishing this Thesis. It was a great honor for me to have the chance to work with Professor Duval, as this cooperation was one of the stronger experiences in my life. During our meetings I gained a lot of important knowledge and I had the chance to meet a great man! I will never forget the excitement I felt when professor Gagatsis announced me that Professor Duval will be a member of the advising committee for this Thesis and the motivation he was providing me every time we met and discussed. MANY THANKS.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION TO THE RESEARCH

Introduction

Geometry is the mathematics of space (Bishop, 1983) and the study of geometry helps students represent and make sense of both the world in which we live and the world of mathematics. Geometry occupies a particular place within mathematics; it appears as a model of physical space, and it follows that the objects it deals with (e.g. lines, planes, points) are supposed to be directly taken from sensory experience, unlike in the other areas of mathematics (Parzysz, 1991). The NCTM (2000) characterizes geometry and the perception of space as key factors for learning mathematics, as they provide ways to reflect on and interpret our natural environment. Among the Curriculum and Evaluation Standards for geometry for grades 9-12, students are expected to:

1) Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships:

– analyze properties and determine attributes of two and three dimensional objects;
– explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them;
– establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others;

2) Specify locations and describe spatial relationships using coordinate geometry and other representational systems:

– use Cartesian coordinates and other coordinate systems, such as navigational, polar, or spherical systems, to analyze geometric situations;
– investigate conjectures and solve problems involving two- and three-dimensional objects represented with Cartesian coordinates.

3) Apply transformations and use symmetry to analyze mathematical situations:

- understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices;
- use various representations to help understand the effects of simple transformations and their compositions.

4) Use visualization, spatial reasoning, and geometric modeling to solve problems:

- draw and construct representations of two- and three-dimensional geometric objects using a variety of tools;
- visualize three-dimensional objects and spaces from different perspectives and analyze their cross sections;
- use vertex-edge graphs to model and solve problems;
- use geometric models to gain insights into, and answer questions in, other areas of mathematics;
- use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture.

However, geometry is typically regarded as a difficult branch of mathematics for many students. During the past thirty years, several mathematics educators have investigated students' geometrical reasoning based on different theoretical frames. For example, van Hiele (1986) developed a model referring to levels of geometrical thinking. Fischbein (1993) introduced the theory of figural concepts and Duval (1988) reported the cognitive analysis of geometrical thinking, discriminating four apprehensions for a geometrical figure. Duval (1999) stressed also the fact that in geometry, a very important factor that influences understanding is that there will be no confusion between the mathematical objects and their representation (Duval, 1999). Next, Houdement and Kuzniak (2003) proposed three different types of Elementary Geometry.

Over the last years in Cyprus small research attempts have been made in the study of geometry (i.e. Panayidou, Tsianni, & Gagatsis 2004; Vourgias, Peskias & Chrysostomou-Vourgia, 2003), based mainly on the analysis of Duval (1995, 1998). These researchers attempted to study the role of the geometrical figure in problem solving in geometry. More recently a theoretical model about the different types of apprehension of the geometrical figure, in primary and secondary school, was confirmed by Deliyianni, Elia, Gagatsis, Monoyiou and Panaoura (2010). However there is a need for a systematic attempt for the investigation of specific factors that relate to the geometrical figure apprehension, such as its structure and the specific cognitive processes that are involved.

It is documented that instructional frameworks drawn from research-based knowledge of students' thinking have a vital role in bridging the gap between learning and teaching and, therefore, in the effectiveness of mathematical teaching programs (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). Consequently, the gathering of empirical data allows the comparison between groups of students in a critical transition point such the transition from lower to upper secondary education. These data can contribute as a valuable source of information regarding aspects of teaching in the two educational levels, as well as the difficulties faced by students of different age groups. Keeping in mind the problems that occur during the transition from one educational level to another universally (Mullins & Irvin, 2000), this study also examines the invariance of the structural relations between the various types of geometrical figure apprehension in both lower and upper secondary education. Taking also into consideration the different geometrical paradigms proposed by Houdement and Kuzniak (2003), this examination turns to the identification of the geometrical paradigm in which students of each level work and the examination of their transition from one type of paradigm to another. The identification of students' geometrical paradigm will be conducted through the examination of their performance in the tasks of a test, their general performance in each different type of apprehension as well as through the analysis of their consistency in the resolution of the tasks. The identification of the students' geometrical paradigm work may function as an epistemological tool for teachers, in order to choose the proper teaching practices and face their students' difficulties (Kuzniak & Rauscher, 2011).

The Cognitive and Educational Problem

The way of looking at any figure constructed with specific tools is a crucial cognitive factor in solving problems and in reasoning and proving in geometry. Figures are the blind spot in mathematics education as much for theories as for teaching. There is an equivocal use of the verb "to see" regarding figures in geometry, because there are two levels of cognitive functioning or recognition, which are not distinguished. Therefore, a strong no congruence often occurs between what is seen and what is named or stated in the utterance of the problem (Duval, 2012). Consequently, a wide gap is created in this case, giving rise to recurrent and very often insuperable difficulties for most students at any level of the curriculum. These difficulties are well known to teachers, but knowledge of the factors that cause them is still fuzzy.

Teaching should be organized to make students aware of the ways of how to see and think in a relevant way in elementary geometry. For this educational goal, research should focus on the cognitive conditions that allow students to learn to seek and find out any geometry problem by themselves any geometry problem (Duval, 2012). Based on the above a number of questions are raised about the source of students' difficulties and how these difficulties can be overcome:

– Is the way of looking at figures the same in geometry as the one for any iconic or diagrammatic representation outside geometry, or is it quite different?

– If the way of looking at figures in geometry is different from the way we look at iconic or diagrammatic representations outside geometry, how can this way be analyzed?

– Should this way be analyzed mainly from the viewpoint of a conceptual understanding of what is represented in the constructed figure for deducing new properties by "reading" the figure?

– Related to the above supposition, what is the cognitive and heuristic interest of visualization through figures in learning geometry?

– On the other hand, can the introduction of a figure in the context of concrete problems help students overcome the gap between a very often deceptive perception of figures and their required mathematical comprehension? Or would it reinforce the students' difficulties in all their later learning?

Therefore, there is obviously a need to identify the cognitive processes and the type of apprehension that students mobilize during the resolution of geometrical tasks and examine whether this selection leads to a proper way of "looking" at the geometrical figure. There are many factors that can inhibit or favor discrimination of these visual operations, which can be studied experimentally (Duval, 2006). Thus this study constitutes an attempt to provide a more coherent idea about the factors that comprise or influence the apprehension of the geometrical figure.

During the transition from lower secondary school to upper secondary school negative effects on the emotional and cognitive domain are identified (National Research Council, 2004; Mullins & Irvin, 2000). In Cyprus, despite the shift in the level of upper secondary school, the Commission for Educational Reform (2005) notices a gap between Grade 10 and Grade 11 in the high school curriculum. Thus, the examination of the geometrical figure apprehension will be in respect of students' transition from lower to upper secondary school.

Aim of the Research and Theoretical Choices

The aim of this research is to analyze the way students in lower and upper secondary education can look at a geometrical figure. For this investigation a device of geometrical tasks is organized by referring to two theoretical frameworks.

The first one concerns the distinction between four kinds of figure apprehension proposed by Duval (1995, 2005) within the register of geometrical visualization: perceptual apprehension, sequential apprehension, operative apprehension and discursive apprehension. The first type of apprehension corresponds to the spontaneous and common shape recognition. The second type of apprehension corresponds to the construction of figures with the use of particular tools. However, only the third and fourth types of apprehension are relevant in looking at figures in the way that it is required from a mathematical point of view. In fact, operative apprehension is a heuristic reconfiguration of the recognized shapes into other shapes. Discursive apprehension is the verbal recognition of what the recognized shapes actually represent in a mathematical way. It depends on the given elements and on the knowledge of definitions and theorems. But, whatever the expected degree of formulation in teaching is, it always involves a dimensional deconstruction of the shapes recognized.

The second theoretical framework focuses on the evolution of the global objectives in the teaching of geometry throughout the curriculum. Houdement and Kuzniak (2003) have proposed the notion of Geometrical Paradigms in analyzing this evolution in the institutional organization of teaching. Thus, they have distinguished three geometric paradigms: GI for an empirical and pragmatic introduction of geometry in the Primary school, GII for a Natural Axiomatic Geometry, and GIII for a Formal Axiomatic Geometry. But what is taught in the lower secondary school is considered to be a "Mixed Geometry" (GI and GII) (Kuzniak, 2011).

Since this investigation is conducted with lower and upper secondary school students, the aim and scope of the research is to elicit significant observations in answering the following question: "What kinds of figure apprehension are students able to mobilize when they move from lower secondary school (where a Mixed Geometry is taught) to upper secondary school (where a Natural Axiomatic Geometry is taught). In a Mixed Geometry, figures can be used in an empirical way and neither operative apprehension nor a true discursive apprehension is required, whereas in GII the educational standards are quite different.

Based on the above the main aim of the research is summarized in the following figure (Figure 1).

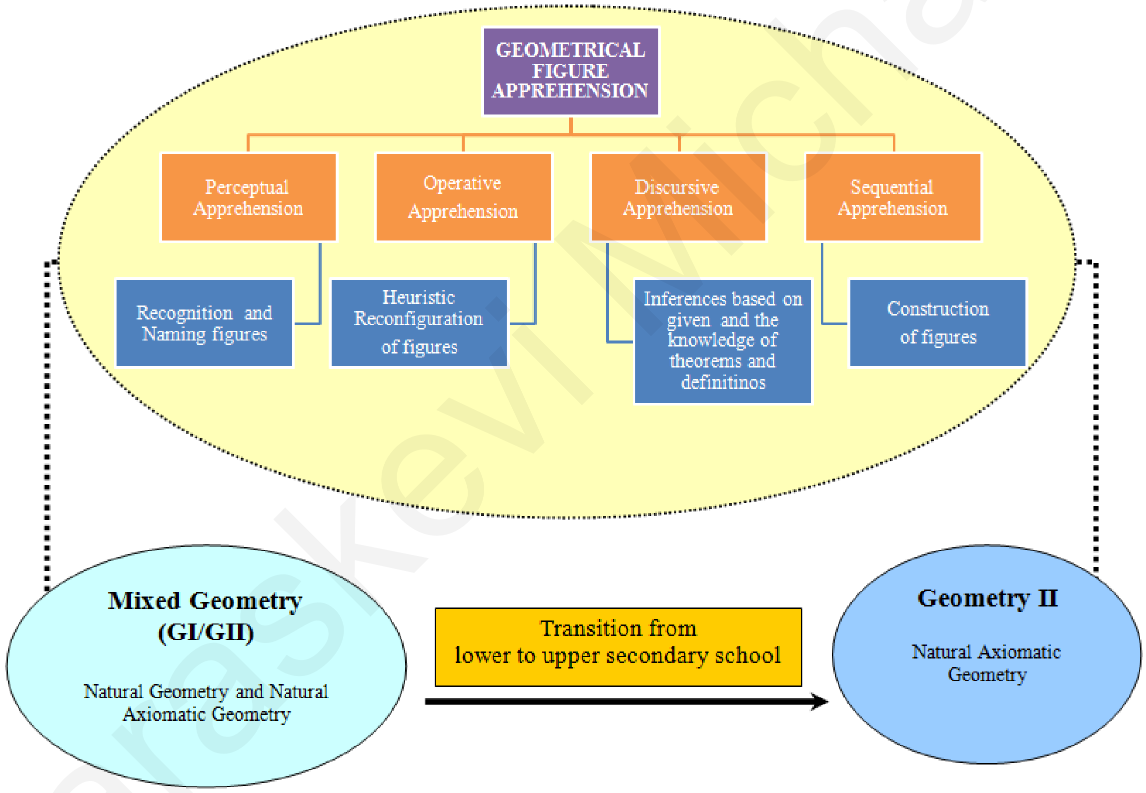


Figure 1. The main aim of the research

Research Questions of the Research

In order to be able to address the main aim of the research students' answers in the tasks will be analyzed according to the following the research questions, which correspond to four main axes of investigation of the study:

1) The cognitive structure of the geometrical figure apprehension in the lower and the upper secondary school:

– What is the cognitive structure of the geometrical figure apprehension in the lower and the upper secondary school?

– What are the similarities and differences between the lower and the upper secondary school students regarding the structure of their geometrical figure apprehension?

2) The relationships between the four types of geometrical figures apprehension:

– What are the relationships between the perceptual, the operative, the discursive and the sequential apprehension for the solution of geometric tasks?

– How do the lower and upper secondary school students behave during the solution of geometrical tasks involving each type of geometrical figure apprehension?

– Is there a development from the perceptual apprehension either to the operative apprehension or to the discursive apprehension, from one grade to the next one, and mainly during the transition from lower to upper secondary school?

– Is the operative apprehension closely connected to the discursive apprehension or they are independent of each other? In other words, are the abilities to solve problems by visualization closely connected to the development of deductive reasoning?

3) The comparison between the lower and the upper secondary school students' geometrical figure apprehension:

– How able are the lower and upper secondary school students to solve geometrical problems corresponding to the perceptual, the operative, the discursive and the sequential apprehension?

– What are the differences between the four groups of students (grade 9, grade 10, grade 11a and grade 11b) concerning their performance in the geometrical figure apprehension tasks?

– What are the predominant kinds of figure apprehension for each kind of geometrical task and at each level of teaching?

– Do students mainly mobilize the same kind of figure apprehension for the solution of geometric tasks at each level of teaching?

4) The lower and the upper secondary school students' mistakes and ideas about the geometrical figure apprehension:

– What are the lower and upper secondary school students' mistakes in the geometrical tasks corresponding to each type of apprehension and what is the reason that causes them?

– Do students get a right answer mainly when the perceptive apprehension is not mathematically deceptive and do they fail when an operative or discursive apprehension is needed?

– What differences exist between the four groups of students (grade 9, grade 10, grade 11a and grade 11b) in regard to their answers in the tasks of the interview?

– Are most students in the lower secondary school still working in the paradigm GI or in the Mixed Geometry (GI/GII) according to the educational standards?

– Are most students in the upper secondary school working in paradigm GII, the Mixed Geometry (GI/GII) paradigm, or only in paradigm GI?

Significance and Originality of the Research

Despite the fact that many changes seem to appear between lower and upper secondary school students' knowledge, abilities and performance (Commission for Educational Reform, 2005), research has paid limited attention to this subject. Specifically, a systematic approach in order to investigate the kinds of figure apprehension that students mobilize in the performance of geometric tasks during their transition from lower to upper secondary school is not found. Also, the structural relation between the four types of geometrical figure apprehension in these educational levels has not been empirically verified yet. Thus, the significance and originality of this study lies on a theoretical, a methodological and a practical perspective.

Concerning the theoretical perspective, the first innovation of this research concerns the verification of a proposed model regarding the structure of the geometrical figure apprehension, in relation to students' transition from lower and upper secondary school. Although the construction of mathematical knowledge and geometrical knowledge in particular, is considered a major issue (Gray, Pinto, Pitta, & Tall, 1999), there is absence of models that describe the construction of students' geometrical knowledge and abilities, such as those acquired through the process of teaching. The proposed model supports empirically Duval's (1995) cognitive analysis about the different apprehensions of a geometrical figure. Also, the model provides information on whether students' transition to a higher grade or a next educational level is related to changes in the structure of the proposed model.

The epistemological situation peculiar to mathematics, the cognitive paradox it creates and the specific problems of understanding that its learning raises create the need to widen the field of questions and observations about thinking processes. Mathematics education requires new cognitive models, more complex than the ones generally accepted (Duval, 2008). Furthermore, this research allows tracing students' cognitive procedures during the solution of geometric tasks and identifying the factors that influence the relevant way of looking at a geometrical figure. In other words it examines whether the type of apprehension students choose is relevant in achieving a correct solution in the geometrical tasks, thus being able to identify factors that function as a source of success or factors that cause difficulties. In addition, the focus on the specific age group of the students provides data for comparing students' geometrical abilities and for pointing out the changes in the students' cognitive processes related to their geometrical thinking during their transition to a higher grade or educational level.

The significance concerning the practical perspective lies in the fact that the results of the research may provide useful information to curriculum designers and teachers in both lower and upper secondary education. The proposed framework about students' geometrical figure apprehension provides useful knowledge for classroom instruction, as it can be considered a step towards attaining curriculum continuity and a smoother transition of students from lower to upper secondary school in mathematical learning. In this research, besides students' abilities regarding the construction, perception and visual processing of the geometrical figure, proof tasks are used in order to examine the differentiation of the theoretical deductive reasoning from others kinds of reasoning. In addition, the elaborate model offers a framework of students' thinking while resolving a wide range of geometrical tasks in a systematic manner within and between the two educational levels. Therefore, the proposed framework may be used as a tool in mathematics instruction and for designing geometry tasks in both lower and upper secondary school. The results appear to be useful from an assessment perspective, as well, as they can provide teachers with specific information about students' thinking in geometry based on prior knowledge and enable them to enhance this thinking by giving appropriate support through the tasks focused on the competences and cognitive processes for the geometrical figure apprehension.

Beyond an assessment of students' performance in geometry over four different Grades, an analysis of the students' personal Geometrical Paradigm is conducted. But mainly the importance of the specific cognitive functioning of visualization in geometry is shown, which is not at all taken in account in the institutional Geometrical Paradigm GI and GII. Most students cannot find out by themselves a way of looking at figures, which runs against the spontaneous way which is relevant outside geometry. In fact, based on the results of this research, the type of Geometry (Kuzniak, 2011) in which students work in each grade is identified and the shift in the type of Geometry they are engaged in while the transition from lower to upper secondary school and from one grade to a next one is shown. The identification of the geometrical paradigm students work in appears to be essential, because according to Kuzniak and Rauscher (2011) "when people share the same paradigm, they can communicate very easily and in an unambiguous way. On the contrary, when they stay in different paradigms, mistakes are frequent and can lead, in some cases, to a total lack of comprehension".

From the methodological perspective, this research provides a valid and reliable measurement tool for lower and upper secondary school students' geometrical figure apprehension and the identification of students' geometrical abilities. The construction of the research instrument and its validation can function as a tool for teachers in examining their students' geometrical thinking and, thus, identify their learning needs. Furthermore, in this research Duval's (2012) way of analyzing mathematical activities is adopted. This method of analysis comprises two different points of view: the mathematical and the cognitive point of view. By this cognitive analysis of the tasks the different cognitive procedures related to students' solutions are revealed and phenomena in which there is an inconsistency between what is expected from the design of the tasks and the way students react are highlighted. The cognitive analysis of the tasks can function as a methodological tool for teacher in order to identify and, thus, try to limit the inconsistency between the way students are working and the way their teachers expect them to do so. It is not an assessment perspective that is used, but the focus is on the cognitive perspective and the main interest is in gaining an insight into students' understanding and misunderstandings though the analysis of their answers. The lack of awareness of students' cognitive capacities and processes can create misunderstandings between students and teachers in the learning of geometry (Kuzniak & Rauscher, 2011).

To sum up, this study contributes to the field of mathematics education research and especially in the field of Geometry, by the formulation of a theoretical background to geometrical apprehension, by providing methodological tools for investigations and by proposing some practical applications. A coherent picture of students' geometrical figure apprehension is provided, which is necessary for teaching approaches focusing at the development of the geometrical understanding. In addition this examination determines whether the students are able to see the geometrical figure in a relevant way and whether this way develops or not at a different educational level. Therefore it is feasible to discuss about the possible factors that influence this relevancy in the way students look at geometrical figures and be able to suggest possible factors that can inhibit or strengthen students' ability to see in a relevant way in geometry. This knowledge contributes in providing suggestions about ways to support students' geometrical understanding in such a critical transition phase of their schooling.

Limitations of the Research

There are some certain limitations concerning the design and conduct of the research, mainly due to factors related to sampling and administration of the tests.

The first limitation is related to the fact that no randomness of the sampling procedure was achieved. The reason is that the test was administered in classrooms where teachers were willing to participate in this procedure. Thereafter, it was not possible for the administration of the test to be performed by the researcher in all cases. Despite the fact that no further instructions should be given to students during the performance of the tasks, in the case when the researcher was absent this could not be assured. In addition, there is no certainty that the proper amount of time was provided to the students during the performance of test. These can be factors affecting the reliability of the test, because it cannot be certain that the same conditions held in every classroom during the administration of the test. However, the teachers that administered the test were provided with all the necessary instructions.

As it was difficult to convince teachers to provide their teaching time for the administration of the tests, the content of the test had to be limited. Thus, another limitation concerns the number of tasks that were included in the test, so that the time students would need to complete the test would not exceed two teaching periods. In order to avoid any possible effect of students' tiredness during performance of the tasks, the test was divided into two parts, which were administered to students in two different sessions. Therefore, another limitation is that there were cases in which some students were absent in one of the two sessions of the test, and, thus, there was an effect on the total number of respondents.

A further limitation regards the developmental examination of students' geometrical figure apprehension. For this research only one measurement was conducted and, thus, no longitudinal measurements were performed to examine students' development in their geometrical figure apprehension and their geometrical reasoning. Thus, the description of the results is limited to students' abilities in their current grade.

Thesis Structure and Summary

This first chapter constitutes the introduction to the research, in which the problem is set, the aim and research questions are formulated and the significance, originality and limitations of the study are described.

The second chapter deals with the literature review which relates to the main issues examined in this research. The theoretical framework presented in this chapter constitutes the basis for the design of the research and comprises of the presentation of theoretical frameworks that relate to the cognitive and instructional issues of geometrical thinking. Furthermore, the relevant literature concerning students' transition from one educational level to the next one is also examined, laying emphasis on the transition from lower to upper secondary school.

The third chapter presents the methodology of the research. First information is provided about the participants in the research. Next the instrument used for the research is described and also the way students' answers were codified and scored is explained. In this chapter information about the analysis of the data can also be found, regarding statistical and qualitative analysis.

The next chapter deals with the presentation of the results from the analysis of the qualitative and quantitative data of the research in relation to the research questions that are set. In particular, the results of the descriptive statistics of the tasks are described, the results pertaining to the construct validity of the test and the verification of the proposed model concerning the geometrical figure apprehension are presented, and the implicative relations and the hierarchical similarity connections of the variables are explained.

In the fifth chapter the results are discussed with reference to the theoretical framework that was used for this research and the results from other relevant studies.

The last chapter consists of the conclusions of this research study. Directions for further future research are suggested and implications for the teaching of geometry are also provided.

Operational Definitions

Geometrical figure

There are quite different definitions of geometrical figure. For Duval a geometrical figure constitutes any non – discursive and non – iconic representation, which can be constructed with geometrical tools (ruler, compass, software), in any way the properties that could be represented. The properties represented on the figure depend on the discursive given. For Mesquita (1996) a figure constitutes the external and iconical representation of a concept or a situation in geometry. For Fischbein (1993) geometrical figures are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure also possesses spatial properties like shape, location and magnitude (Fischbein & Nachlieli, 1998). In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations.

Geometrical figure apprehension

Duval (1995, 2005) distinguishes four apprehensions for a geometrical figure: perceptual, sequential, discursive and operative. Specifically, *perceptual apprehension* is the recognition of 2D or 3D shapes at first glance and is associated with names (words). *Sequential apprehension* is required whenever one must construct a figure or describe its construction by using tools. The construction is not the same according to the primitives of the used tools. *Discursive apprehension* is related to the fact that the organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and mathematical properties. Through *operative apprehension* we can get an insight into a problem solution when looking at a figure, as it depends on various ways of modifying a given figure. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three.

Visualization

Duval (1998) refers to visualization as one of three independent cognitive processes in geometry. Visualization is the heuristic and dimensional deconstructive way of looking at a figure. It involves a quick recognition of all the possible figural units (0D, 1D, 2D) and their various configurations.

Geometrical Paradigm

According to Kuzniak (2008), the study of geometry is based on an approach asserting that geometry has undergone significant changes of perspectives equivalent to paradigmatic shifts. Thus three geometrical paradigms are deemed to organize the interplay between intuition, deduction, and reasoning in relation to space. These paradigms reflect the breaks observed between the various academic cycles in the teaching and learning of geometry. A single viewpoint on geometry would miss the complexity of the geometric work, due to different meanings that depend both on the development of mathematics and school institutions. In practice, the field of geometry can be mapped out according to three paradigms, two of which – Geometry I and II – play an important role in today's secondary education (Kuzniak & Vivier, 2009). Geometry I confirms its validation in the material and tangible world; Geometry II is built on a model that approaches reality without being fused with it (Kuzniak & Rauscher, 2011). Furthermore, the transition towards Geometry II based on Geometry I allows the supposition that a Mixed Geometry (GI/GII) is possible (Kuzniak & Rauscher, 2011).

Geometrical Work Space

The Geometrical Work Space (GWS) is the place organized to ensure geometrical work (Kuzniak, 2009). For its definition two planes are introduced. First, there is the components plane that networks three characteristic components (Kuzniak, 2006): the real and local space, the artifacts and a theoretical system of references. To ensure that these components are well used, we need to focus on some cognitive processes involved in geometrical activity (Kuzniak & Vivier, 2009). In adapting Duval (1995), three processes are identified: a visualization process, a construction process and a discursive process.

CHAPTER II

LITERATURE REVIEW

Introduction

In geometry it is necessary to combine the use of at least two representation systems: one for verbal expression of properties or for numerical expression of magnitude and the other for visualization. Thus, geometry is a kind of knowledge area that requires the cognitive joining of two representation registers: on the one hand, the visualization of shapes in order to represent the space and, on the other hand, the language for stating some properties and for deducing from them many others (Duval, 2005).

Geometry is also an important part of the mathematics curriculum. However, students do not display strong conceptual knowledge of this subject (Mistretta, 2000). Geometry may be exciting for mathematicians and for anyone who likes mathematics, but what about the other students who must learn mathematics in their curriculum? Teaching geometry is more complex and often less successful than teaching numerical operations or elementary algebra (Duval, 1998). Carroll (1998) found that junior high and senior high school students often lacked experience in reasoning about geometric ideas. He also argued that middle school students were capable of developing good reasoning about geometric situations when they had substantial exposure to geometry throughout elementary school. Therefore, why teach geometry to all pupils? This question begs another one: How should geometry be taught? In order to put forward some ideas on this basic issue we must take into account the underlying cognitive complexity of geometrical activity (Duval, 1998).

Over the years the issue of the development of the geometrical knowledge has caught the attention of many researchers from the field of mathematics education and the field of cognitive psychology as well. There are a lot of researches and proposed theoretical models about geometry learning, most of them taking into account geometrical figures and the way of using them, but the figure is not the main target of their inquiry. There are authors that are interesting not only in the geometrical figure, but mainly in figures in relation to proving, space etc. (i.e. Bishop, 1983; Parzysz, 1991) and different models that take into account the specific problem of figures and visualization (i.e. Mesquita, 1996; Kurina, 2003).

On the contrary, there are few approaches which can be regarded as focusing explicitly and more specifically on figures and visualization (e.g. van Hiele, 1986; Duval, 1988; Fischbein, 1993; Houdement & Kuzniak, 2003). Thus in this chapter the main points of the van Hiele model of thinking in geometry and Fischbein's theory of figural concepts is presented. The main emphasis is laid on Duval's cognitive model of geometrical figure apprehension and the notions of Geometrical Paradigms and Geometric Work Space suggested by Houdement and Kuzniak (2003), as they constitute the main issues of investigation in this research. Duval focuses on geometrical figures and discriminates different ways of looking at figures. The framework of Houdement and Kuzniak (2003) specifies the nature of geometrical objects, the use of different techniques and the validation mode accepted in each of the three paradigms.

Actually, the first sub-chapter consists of the presentation of important theoretical frameworks, focusing more specifically on the issue of geometrical figures. These frameworks are distinguished into synthetic and global, and those presenting several cognitive variables. The presentation of the approaches in each category is done according to their historical order. Within the category of synthetic and global approaches the different levels of the van Hiele's model of thinking in geometry are firstly described, followed by a summary of Fischbein's theory of figural concepts. The remaining part of this sub-chapter is devoted to Duval's cognitive analysis of geometrical figure apprehension and the notions of Geometrical Paradigms and Geometric Work Space. Actually, the characteristics and the way each of the four different types of geometrical figure apprehension functions is explained.

Subsequently, there is a presentation of some research efforts recently developed in Cyprus, concerning the examination of the geometrical figure apprehension. These researches were based on Duval's approach and tried to examine the dimensions comprising the geometrical figure and compare the reaction of students of different age groups to tasks corresponding to different types of apprehension of geometrical figures.

This chapter ends with a literature review of the transition of students between different educational levels, referring to the different impacts on the students' cognitive and affective domains. The main emphasis is laid on students' transition from lower to upper secondary education, as it is one of the dimensions that are examined in this study.

There is a contrast between physical representations of objects with their corresponding mental images. For example there are differences between a drawing on a paper or a screen and the mental images regarding this drawing. A more important contrast is noticed in geometry, concerning the difference between a drawing of a physical object (a house outlined by a square and a triangle for roof) and a drawing of a geometrical figure (square, triangle) (Duval, 1995):

– A classical definition of the term *representation* is that a representation is something that stands for something else.

– A *figure* can be considered as a representation, but the opposite cannot stand, because not every representation can be considered s a figure. A figure emerges through the presence of traces of spots governed by Gestalt laws and perceptual cues.

– An *image* can be defined as the presentation of something through a physical of physic reproduction relation. The retinal image results from stimuli and the perceived figure results from their organization.

– The term *picture* is used for indicating any figure or image.

Parzysz (1991) referred to figures as instruments of both monstration and demonstration. He mentions some purposes which can be fulfilled by them:

– they *illustrate* definitions (e.g., for parallelogram, pyramid) or theorems (e.g., Pythagorean). This is due to the nature of geometry, whose objects are obviously linked with material realizations (drawings or models which can be drawn).

– they *sum up* a complex set of information: the figure, drawn in order to solve a geometrical problem, allows a simultaneous glance at most of the data presented in the wording.

– they *help in conjecture:* the figure also makes it possible to suggest potential relations between its elements, which will have to be demonstrated afterwards (e.g. in the drawing, this triangle seems to be isosceles: is it true?)

– they *help with proof:* the role of drawings in proofs is essentially of a negative nature, i.e. it provides counter-examples to conjectures (this triangle, which was thought to be isosceles, is certainly not so, for in another instance it is obviously not the case).

In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws (Mesquita, 1996). As a representation, it becomes more economically perceptible compared to the corresponding verbal one because in a figure various relations of an object with other objects are depicted (Lemonidis, 1997). However, the simultaneous mobilization of multiple relationships makes the distinction between what is given and what is required difficult. At the same time, the visual reinforcement of intuition can be so strong that it may narrow the concept image (Mesquita, 1998).

In a geometrical figure internal constraints of organization can be found, which does not exist in a drawing or a physical object, despite the fact that the appearance of the two representations can be common. These internal constraints can be discovered by a construction task of a geometrical figure. Furthermore, a geometrical figure must provide help for the solution of problems and for finding the key idea for a proof. For these reasons the way of looking at a drawing, or even a physical object, is different and thus distinguished from the way we look at a geometrical figure. This difference is maintained whether the representation is on paper or is mental (Duval, 1995).

So, we see that the geometrical figures are creations of notion of the human mind which was inspired by the real objects to create them, but they don't have the same characteristics with them (Lemonidis, 1997). Many researches (Parzyzs, 1988; Duval 1988; Fischbein 1993; Laborde, 1994) point out and separate these different functions of the geometrical figure. Fischbein (1993) talking about this subject introduces the term "figural concepts" since these entities are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. But a geometrical figure also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). Therefore, the double status of external representation in geometry often causes difficulties to students when dealing with geometrical problems due to the interactions between concepts and images in geometrical reasoning (Mesquita, 1998).

The geometrical figures are visual representations that are coded with letters or marks indicating given properties. But letters and marks refer to a statement. Also, we always have, be it explicitly or implicitly, a dual representation. In any dual representation, letters and marks anchor the reference points of the statement in the visual representation, but they belong to the statement and not to the visual representation. The visual representation works in a way completely independent of what the given encoded imposes, because for the same statement we have several possible visual variations (Duval, 2008).

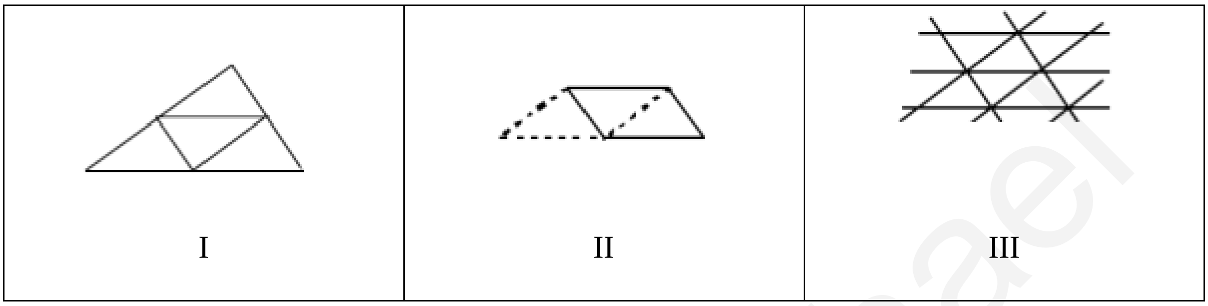


Figure 2. Visual variations of a geometrical figure

In the three figures above (Figure 3), if (I) is given as the starting dual representation then (II) must be recognized as a subfigure of (I) for solving the problem, and (II) (III) can be also be given as starting figures within a dual representation. Analyzing any geometric figure requires that we analyze the visual variations in which it appears as a transitory foreground amongst many others. It is the requirement whose complexity is seriously misunderstood, because without this discrimination, any figure, even the most simple, becomes misleading or opaque. This complexity is due to the possibility of quite different ways to split up the content of a geometric figure into figural units. For distinguishing them we have to take into account the number of dimensions. Thus, we get these two basic ways of splitting up (Duval, 2008). This discrimination goes against the immediate perceptive organization. It is through this double possibility of splitting up that geometric figures constitute a particular kind of semiotic representations with a powerful potentiality for visual treatment:

1) The figural units have a number of dimensions equal to that of the starting figure: 2D/ 2D in the framework of plane geometry. In this case, the figural units correspond to the shapes, which are closed outlines. For example, in Figure 3 above, we can discriminate in (I) two figural units (the two parallelograms (II) or three triangles). This is the spontaneous perceptive interpretation.

2) The figural units have a number of dimensions less than the one of the starting figure: 2D / 1D (or 0D). In this case, the figural units are straight lines free of any closed

outlines. For example, in Figure 3, we can discriminate in (I) six figural units which are the straight lines making up the network (III) underlying (I) and (II) (Duval, 2008).

Models focusing more specifically on the issue of geometrical figures

As geometry aims to encompass the understanding of diverse visual phenomena, it is important to clarify what is meant by visualization and geometrical reasoning, which are necessary in solving mathematical problems involving visual phenomena, and how such reasoning develops (Jones, 1998). What follows is an overview of different theoretical models which have been put forward as useful frameworks in describing and understanding the development of geometrical thinking.

The synthetic and global approaches

The van Hiele model of thinking in geometry

Van Hiele developed his model for the first time in his thesis in 1959. He proposed a model of a sequential and hierarchical progression of geometrical thinking, partly inspired by the Piagetian model of cognitive development. But the progress from one level to the next level is more dependent on educational experiences than on age or maturation. In fact the van Hiele (1986) model suggests five levels:

Level 1 (Visualization): Students recognize figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.

Level 2 (Analysis): Students see figures as collections of properties. They can recognize and name properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties the student knows, but not discern which properties are necessary and which are sufficient to describe the object.

Level 3 (Abstraction): Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and offer informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. The role and significance of formal deduction, however, is not understood.

Level 4 (Deduction): Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class.

Level 5 (Rigor): Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems.

In fact the first three levels are obviously about visualization and the distinction between figures. In this model the crucial point of articulation between figures and reasoning is located at the third level. Clements and Battista (1992) completed this approach of visualization by proposing the existence of Level 0, which they call *pre-recognition*. Students at this level notice only a subset of the visual characteristics of a shape, resulting in an inability to distinguish between figures. For example, they may distinguish between triangles and quadrilaterals, but may not be able to distinguish between a rhombus and a parallelogram.

In this model, a student cannot achieve one level of understanding without having mastered all the previous levels. However a common situation that occurs is that the teacher is thinking at a different van Hiele level than the students. In fact there are cases in which high school geometry teachers think at the fourth or fifth van Hiele level (Mason, 1998). Therefore, teachers need to remember that, although teacher and student may both use the same word, they may interpret it quite differently. Thus, the teacher should evaluate how the student is interpreting a topic in order to communicate effectively.

However, different critics were found for this model. While research is generally supportive of the van Hiele levels as useful in describing students' geometric concept development, Clements (2001) comments that it remains uncertain how well the theory reflects children's mental representations of geometric concepts, as various problems have been identified with the specification of the levels. For example, the labeling of the lowest level as 'visual' is used when visualization is demanded at all levels, and the fact that learners appear to show signs of thinking from more than one level in the same or different tasks, in different contexts. In fact what is striking is that the distinction into three levels of visualization is mainly made according to the absence or presence of reasoning and its role. Also, criticizing this model, Jones (2000) argues that the usefulness of this model in respect to other approaches to plane geometry (such as via vectors or transformations) and to other geometries (such as spherical geometry) is not clear. As a consequence of these various factors, the van Hiele model appears to be of only limited use in determining the geometry curriculum and how this should be sequenced for teaching. Despite these criticisms, what is striking is the fact that visualization is not characterized in itself, but is characterized according to the absence or the first emergence of reasoning.

The theory of figural concepts

Fischbein first developed the idea of concept and figure as a cognitive rule in 1963, but he really applied this idea for characterizing geometrical figures in 1993 in his paper “The theory of figural concepts”, published in the journal Educational Studies in Mathematics. In fact he attempted to interpret geometrical figures as mental entities which possess simultaneously conceptual and figural properties. He observed that while a geometrical figure can be described as having intrinsically conceptual properties (in that it is controlled by a theory), it is not solely a concept, but it is an image too. As he says, “it possesses a property which usual concepts do not possess, namely it includes the mental representation of space property”.

Although there is usually an interaction between images and concepts in mental activity, they seem to be basically incompatible. This is because a concept does not possess spatial properties, as it is ideal and abstract. On the other hand, an image is not reducible to an idea because of its sensorial properties. Nevertheless, a third category of mental representations can be identified, which possess simultaneously both categories of properties: the geometrical figures. So, Fischbein argues that all geometrical figures represent mental constructs which possess, simultaneously, conceptual and figural properties and that geometrical reasoning is characterized by the interaction between the figural and the conceptual aspect.

A geometrical figure is an abstract, ideal entity, a general representation of a category of objects. A point, a line, a plane, a circle, a square, a cube, etc. have no material consistency, no weight, color, density, etc. They are abstract, ideal entities. Points, lines, planes, circles, cubes etc. (in the geometrical, mathematical sense) possess these properties and, consequently, they are concepts. But they are not pure concepts as they possess, in addition to their conceptual properties, also spatial properties (Fischbein & Nachlieli, 1998).

Therefore, he introduces the term “figural concepts” since these entities are simultaneously concepts and spatial representations. This term was used for emphasizing the fact that we deal with a particular type of mental entities which are not reducible, neither to usual images nor to genuine concepts. The properties of figures are completely fixed by definitions in the frame of a certain axiomatic system. Although a figural concept consists of a unitary entity (a concept expressed figurally), it potentially remains under the double and sometimes contradictory influence of the two systems to which it may be related - the conceptual and the figural one. Generality, abstraction, lack of material substance and ideality reflect conceptual characteristics. But a geometrical figure also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998).

Mariotti (1995), in discussing Fischbein’s notion of figural concept, stresses the dialectic relationship between a geometrical figure and a geometrical concept. She argues that geometry is a field in which it is necessary for images and concepts to interact, but from the student’s perspective, there can be tension between the two. In fact, this notion of figural concept works well with people who are very familiar with geometry, but this is not the case with young learners. For most students there is a big gap between the perception of the figure and the concept. Therefore, the first challenge of teaching is how to make students realize this connection.

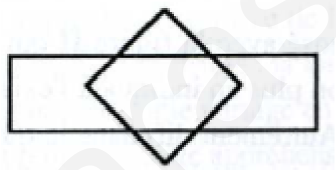
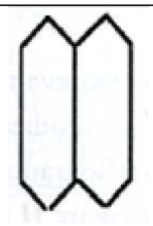
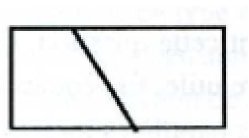
Approaches based on discriminating several cognitive variables

The geometrical figure apprehension

Duval tried to analyze the different possible ways of seeing a figure in geometry firstly in 1988. In order to analyze the heuristic role of geometrical figures, he suggested that a figure must be considered to be a cognitive “apprehension”. The word “apprehension” was chosen to highlight the fact that there are several ways of looking at a figure. So he first distinguished four apprehensions of a geometrical figure: the perceptual apprehension, the operative apprehension, the discursive apprehension and the sequential apprehension. The first one constitutes the spontaneous way of looking at a figure for students and very often it runs against the mathematical way of looking at figures. Only the operative and the discursive ways are the ones required in mathematics, because the operative way is the heuristic use of figures in solving problems, whereas the discursive way corresponds to the way of looking at figures according to the given properties in order to deduce new properties. For the functioning of a drawing as a geometrical figure, perceptual apprehension and at least a second type of apprehension must be evoked. Each of these four types' is related to specific laws of organization and processing of the visual stimulus array (Duval, 1995). Next, each type of apprehension is presented in a more detailed way.

1) The perceptual apprehension

A figure is an organization of elements, which, according to the number of its dimensions, can be points, lines or planes. The points and the lines are respectively characterized by their discrete or continuous character. The areas are characterized by their form that is by their contour: a close line or a sequence of points is sufficient to detach an area of a homogenous field. The perceptual organization of a figure follows the law of closure or continuity: when different lines form a simple and closed contour, they are shown as a figure in the background (Duval, 1988). Thus, the three figures below are recognized at a glance as:

		
1. The superposition of two forms, a square and a rectangle	**2.** An assemblage of two forms that are touched	**3.** The division of a form, a rectangle into two parts

People recognize something (a shape, a representation of an object, etc) in a plane or in depth at first glance. Figural organization laws and pictorial cues determine what the perceived figure shows (Duval, 1994). The perception of the figure results from an unconscious integration and thus the perceived figure can differ from the retinal image.

The retinal image can change, while the perceived properties of the figure (its shape, its size, its brightness, etc) remain the same. This is what is called perceptual constancies. People are also able to name what they recognize (it is a … point, line, triangle, circle…), but to discriminate and recognize in the perceived figure several sub – figures as well. These sub – figures are like possible components that do not necessarily depend on the construction of the figure, because a geometrical figure can involve more figural units or more sub – figures than those used during its construction (Duval, 1995).

Any designed figure in the context of a geometrical activity is the subject of two often opposite attitudes: the one is immediate and automatic – the perceptual apprehension of forms, and the other is controlled and relevant to learning – the discursive interpretation of figural elements. The lag between the discursive interpretation of a figure, required in a geometric situation, and the perceptual apprehension, finds partly its origins in the laws of perceptual organization. These two attitudes are often found in opposition because a figure shows the objects that stand independently from all the expressions and that objects named by the expression of hypotheses are not necessarily those that appear spontaneously. In addition, the distinction between the hypotheses and those that can be deduced has no meaning when they are kept within the perceptual apprehension of the figure. The problem with the geometrical figures is entirely within this lag between the perceptual apprehension and the necessary interpretation controlled by the hypotheses (Duval, 1988).

The autonomous perceptual structure of the geometrical figure can be shown by the following example:

Figure I was proposed to 14 – 15 years old students in France with the expressions below: (Duval, 1988): "*AC and A΄C΄, AB and A΄B΄, BC and B΄C΄ are parallel lines. Show that A is the middle point of B΄C΄*". Just before this question the same problem was given with Figure II.

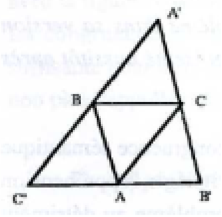


Figure I

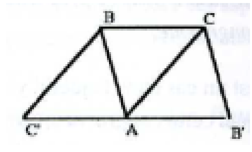


Figure II

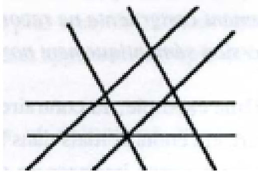


Figure III

Actually, figure I appears as a triangle inscribed in another triangle or like a small triangle posed on a bigger triangle. Figure II is like two overlapping parallelograms and figure III looks like a superposition of lines or a network of parallel lines. The perceptual apprehension of figure II gives the objects which are referred to in the utterance of the task. On the contrary, figure III is semantically the most congruent figure in relation to the verbal part of the task, in which parallel lines are mentioned. The passage from the representation that is semantically congruent has led to a very sharp fall in students' rates of achievement. More than half of the students who had solved the problem in its semantically congruent version did not recognize the problem when presented just after a version semantically non congruent (Duval, 1988).

This example shows that a geometrical figure keeps an autonomous perceptual structuring. The objects that appear can be different from the type of objects that the geometrical situation requires to be seen. A large majority of students remain in the perceptual apprehension. These are those who do not suspect that a figure must be seen through properties or conditions formulated as hypotheses. This is evident during their attitude before solving a problem: they read the verbal part of the problem, they construct the figure and then they focus on the figure without returning to the verbal description. This omission of the verbal part of the problem shows the absence of what is called the discursive interpretation of figures. It is for this reason that the problems that are accessible to students are those in which the verbal part of the problem is semantically congruent with the constructed figure (Duval, 1988).

2) The sequential apprehension

Sequential apprehension is required in the case when a construction of a figure or the description of its construction is needed. The different figural units that emerge by the construction of the figure occur following a specific order. The organization of the elementary figural units does not depend on pictorial cues and laws but on technical constraints and on mathematical properties. These technical constraints depend on the tools that are used for the construction of figures, which can be a ruler and compass or available primitives in geometrical software. On the other hand, a figure created by freehand drawing does not have any instrumental constraints (Duval, 1995).

The importance of the choice of instruments in activities of reproduction is too often unknown. Indeed, the so-called tasks of reproduction can be radically different (from a cognitive and a geometrical point of view as well) depending on the type of instrument that is given to reproduce a figure. There are different examples that show that it is not as much the task of reproduction that is important as the type of instrument chosen for this reproduction. The variety of instruments that can be used in teaching lead to posing four very important questions (Duval & Godin, 2005):

1) Which instruments to use and in what order to use them during the organization of the sequence of activities?

2) Must the construction of a type of figure be associated with the use of a specific instrument or, on the contrary, should it be possible to construct a figure with completely different instruments?

3) To make deferrals in length, should we use tools that permit physical measurements?

4) Since the instruments lead to the visual isolation of either 2D shapes or 1D shapes, how can a choice of instruments pass students from a perceptual 2D vision in a 1D vision which is required for identifying geometric properties?

The instruments that we take to reproduce a given figure control the way we look at a figure. We see immediately that certain instruments (templates, stencils) retain the 2D perceptual priority, while others may only be used if one is able to replace this perception, the visualization of a network of 1D shapes (ruler, compass) (Duval & Godin, 2005).

However, technical constraints can provide feedback, due to the fact that the intended figure cannot be realized while the relationships between mathematical properties and technical constraints are not respected. Therefore, the geometrical figure can function as a model on which actions on representative and observed results are related to operations on the mathematically represented object (Duval, 1995). A sequence of activities of reproduction, organized according to a variation of instruments, can lead students to gradually change their look. But for such activities the figures for reproduction must meet four criteria (Duval & Godin, 2005):

1) The proposed figures should be compositions of forms, not just a "common figures", that is to say, the typical form of a geometrically remarkable polygon (triangle, square, rectangle ...). These compositions of forms can be created by juxtaposition or superposition.

2) These compositions must respect alignments because alignments compliance is important to encourage the activity that is essential for learning to pass from the prevailing recognition of surfaces to the recognition of lines in the analysis of the figures. So students can move gradually from an analysis of figures with a composition of surfaces to an analysis of a composition of lines. Such compositions may also be determined by metric relations.

3) The choice of a figure, that is to say a composition that can be analyzed as a composition of surfaces or as a composition of lines, cannot be separated from the type of the instrument associated with the activity proposed to students (reproduction, restoration, etc).

4) The possibility of control of the equity of two figures by superposition of two figures is important for giving meaning to any activities of reproduction.

Sometimes retours happen through other figures that don't belong to the intended figure. In such cases the sequential apprehension may cause a rupture with perceptual apprehension intentions (Duval, 1995). Thus, we come to wonder about what determines the kinds of figures we should choose. There are two criteria (Duval & Godin, 2005):

1) The figures used should be seen as a composition of forms by juxtaposition and as a composition by superposition.

2) The chosen figures should be made to extend lines or building new lines to make the reproduction.

There is also the chance of reducing the apprehension of the possibilities given by the technical constraints. The scan of possibilities for software can be broad or narrow according to design and intentions (Duval, 1995).

3) The heuristic exploration of a figure: the operative apprehension

The semantic congruence opens or closes the entrance in problems. To elucidate this very essential aspect we must take into consideration not the perceptual apprehension of the figures, but their operative apprehension (Duval, 1988). This type of apprehension is less familiar than the previous ones. However, it is the type of apprehension through which we can get an insight into a solution of a problem when looking at a figure (Duval, 1994).

The operative apprehension of a given figure must be distinguished from its other three apprehensions, as there is a special difficulty for this type of apprehension. The difficulty relates to the fact that for a given figure various figural modifications are possible, as well as many operations to bring them out (Duval, 1995). In operative apprehension the given figure becomes a starting point in order to investigate others configurations that can be obtained by one of these visual operations. In this respect, operative apprehension can develop several strings of figures from a given figure. According to the stated problem, one string shows an insight into the solution. Also, the ability to think of drawing some units more on the given figure is one of the outward signs of operative apprehension (Duval, 1999).

Operative apprehension is an apprehension that focuses on the possible modifications on a starting figure and, therefore, on the perceptual reorganizations that that these modifications create. For each type of modification there are many operations. The heuristic productivity of a figure in a geometrical problem is due to the fact that there is congruence between one of these operations and the possible mathematical treatments of the posed problem. If we can always associate a figure with a described geometrical situation, the figure does not necessarily have a heuristic function in every situation. This is due to two very different reasons (Duval, 1988):

1) The first one concerns the non-congruence between the mathematical treatment and the operative apprehension. Almost all problems involving the homothetic properties present such difficulties. The obstacles that students face during the utilization of the transformations in plane geometry relate also to the non-congruence between the mathematical treatment of the problem and the operative apprehension of a figure.

2) The second reason concerns the case in which there is congruence between the operative apprehension and the mathematical treatment of the problem. If we want to initiate the large majority of students to geometry, this type of problem and the set of factors influencing visibility must particularly regain attention.

Operative apprehension depends on the various ways of modifying a given figure (Duval, 1988, 1995). All figures can be modified in many ways (Table 1), but the most natural way is what is called the mereologic modification. This type of modification is the most often required in the problems given in primary and lower secondary school. With this modification the whole given figure can be divided into parts of various shapes (bands, squares, rectangles and any other form) and these parts can be combined in another whole figure or new subfigures can appear (Duval, 1988, 1995). In this way, the shapes that appeared at the first glance are changed: a parallelogram is changed into a rectangle, or a parallelogram can appear by combining triangles.

The most typical operation is called “reconfiguration” (Duval, 1999). An example of this operation is presented in figure 5. In this example the apprehension of this transformation within the only starting figure can be inhibited by the visual difficulty of double use of one sub-figure.

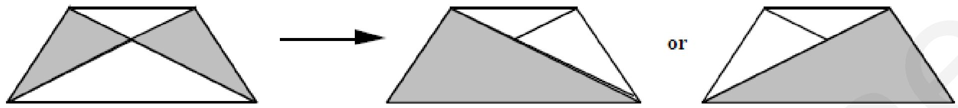


Figure3. Figural processing by reconfiguration (Duval, 1999)

The operations of an intermediate reconfiguration intervene in the first geometrical problems that can be proposed to students, whose solution does not depend on the knowledge and the use of an explicit corpus of definitions and theorems. The operation of intermediate reconfiguration constitutes the heuristic productivity of figures. Thus, we could regroup all the problems in which this operation is congruent with a mathematical treatment possible in a class of problems accessible to students because they do not require, in an explicit way, the implementation of any definition or theorem. However, intermediate reconfiguration is not the only operation of the operative apprehension that is related to mereologic modifications. There is also extension, which is based on an inverse mereologic modification rather than the one implied in the intermediate reconfiguration: for example a triangle can become a part of a parallelogram. The figure is divided and unfolded in the plane (Duval, 1988).

In a given figure the operation of intermediate configuration can be performed in many ways. There are different factors influencing the discernment of the appropriate application of this operation, but particularly four factors are distinguished (Duval, 1988):

1) The fact that the division of the figure in basic parts is given from the start or otherwise must be found.

2) The relevant regrouping of elementary parts form a sub-figure that is convex or not. A non -convex sub-figure is more difficult to be detached from a figure than a convex one, because the perceptual law of a unit of a contour is no longer respected.

3) The relevant regrouping may require the substitution of the basic auxiliary parts to those parts referred to in the utterance of the problem.

4) The fact that the same elementary parts can simultaneously belong to two intermediate regroupings to compare. This is what is called the doubling of the objects (Duval, 1983), which was observed to constitute a real obstacle for certain students. The students were not able to see and understand that the same object can be at the same time within two regroupings posed as different that were sought for comparison.

Table 1

The Complexity of the Operative Apprehension of Figures

Type of configural modification	Operations constituting the heuristic production	Factors influencing the visibility
Mereologic modifications	Intermediate reconfiguration Extension	Convex or non convex character of the basic parts
Optic modifications	Superposability Anamorphosis	Partial recovery Orientation
Place modifications	Rotation Translation	Stability of the surface of the perceptual field for the support of figures

The operative apprehension of a given figure is different both from perceptual and discursive operation:

1) Operative apprehension is different from perceptual apprehension because perception fixes at first glance the vision of some shapes and this evidence makes them steady (Duval, 1999). The operative apprehension of figures is of a different nature as it is not limited to a perceptive manipulation of forms. Certain observations have shown that the students that search for a long time without seeing something implement the same procedures with the ones applied from the students who have immediately seen. It is, therefore, important not to confuse, during the cognitive analysis of geometrical problems the heuristic productivity of a figure and the visibility of the operations related to this productivity. The heuristic productivity depends on the congruence between an operation and the possible mathematical treatment (Duval, 1988).

2) The heuristic character of figures depends on operative apprehension. But not all figures are congruent with the geometric situations that they are supposed to represent. The discursive apprehension is inseparable from a double reference at a semantic network of mathematic objects and an axiomatic network. The analysis of these two forms of apprehension opens the perspectives not only for the classification of geometric problems, but also for a different approach for geometric activities for students (Duval, 1988).

4) Discursive apprehension

By examining the congruence between the figure and the utterance of the problem on the one hand, and between the figure and the mathematical treatments on the other hand, the problem of the status of the geometrical figures is ignored. Indeed, the only acceptable and relevant properties of a geometrical figure depend, each time, on what is denoted as hypothesis (Duval, 1988). A figure is seen in relation to denomination or a hypothesis that make certain properties explicit. Perceptual apprehension cannot determine the mathematical properties represented in a drawing (Duval, 1995). Some mathematical properties must be given through speech (denomination and hypothesis) and others can be derived from the given properties. Therefore, it cannot be said that a mathematical property "is seen" in a figure. The discursive apprehension of a figure corresponds to the explicitness of other mathematical properties of a figure besides those that are indicated from the stated hypothesis. This explicitness is of a deductive nature. The epistemological function of discursive apprehension is demonstration (Duval, 1994).

The absence of denomination and hypothesis in a drawing makes it an ambiguous representation and, thus, the properties that are seen are not the same for everyone (Duval, 1995). What the perceived figure shows is what is seen without any conscious analysis. Shapes or objects are recognized with the derivation of any meanings to occur through perception. What the perceived figure represents is determined by speech acts (through denomination, definition, primitive commands in a menu). In the case when the properties on a drawing are seen without the determination of speech, the following situation can occur in a classroom: Some students discuss two lines and claim that "these two lines are *almost* parallel" (Duval, 1994). This seems to be a good description of a drawing and, therefore, the claims for and against parallelism appear to be equally valid. In fact the issue was based on students' reasoning, resulting from the first speech determination of some traces in the drawing. Thus, in any geometrical situation the perceptual recognition of properties must remain under the control of statements. What the figure can represent depends on the deductive dependence between statements. For this reason it is possible to have a gap between what the figure *shows* and what it *represents* (Duval, 1995).

This implies a subordination of perceptual apprehension to discursive apprehension and, consequently, the restriction of perceptual apprehension: a geometrical figure does not show, at first, from its form and lines, but from those that are said (Duval, 1988). Discursive apprehension can change without any corresponding changes to perceptual apprehension, since for the same drawing what need to be modified are the first determinations of speech or of the given hypotheses (Duval, 1995). The same figure, from the perceptual point of view, can be a different geometrical figure if we modify the statement of the stated hypothesis (Duval, 1988).

This subordination of perceptual apprehension to discursive apprehension can be regarded as theorizing the figural representation: the geometrical figure becomes a fragment of theoretical discourse. The comprehension of such theorizing of geometrical figures, in which the perceptual apprehension must be subjected to the discursive apprehension, constitutes one of the access thresholds for demonstration. It is well known that students find it useless, and sometimes absurd, to demonstrate a property that is seen from the figure (Duval, 1988).

The discursive apprehension also differentiates radically the tasks of demonstration and the tasks of construction. And this discursive apprehension has a different nature from the description of a construction procedure. In a construction task the figure is in a certain way independent from all the statements. Perceptual apprehension can function as a control register for judging whether the execution of the task is acceptable or not. The same figure can illustrate different geometrical situations, meaning situations in which the hypotheses are not the same (Duval, 1988).

5) The geometrical figure apprehension and the cognitive analysis of mathematical activity

After the description of each type of apprehension, some additional remarks can be made about the relation between them. First of all, the sequential apprehension is explicitly solicited in tasks of construction or in tasks of description for the reproduction of a given figure. However, using figures in geometry creates a fusion of these different apprehensions, so they come to be treated as one (Duval, 1995).

The perceptual, the operative and the discursive apprehension are not always clearly distinguished. However, the resolution of problems very often requires their interactions and the realization of the distinction between these three types of apprehension of figures (Duval, 1988). Secondly, for all these kinds of operations didactical variables can be defined to organize tasks and help students develop in the way they deal with a figure in geometry (Duval, 1995, 2005). With this model we have some cognitive variables about visualization and the role of figures in learning geometry and about the process of development in the way of looking at a figure. Finally, what is crucial in this model is that Duval analyzes figures not in reference to the introduction of reasoning, but mainly by laying emphasis on visualization. A factor that differentiates this model form the ones previously described is that whereas these models focus on reasoning, Duval focuses on visualization.

When we analyze mathematical activities from a cognitive point of view, we analyze the transformation of some semiotic representations into others ones. There are two kinds of transformation (Duval, 2007). Either representation is transformed into another representation of the same register, or it is transformed into a representation of another register, but referring to the same object. These two kinds of transformation are respectively called treatment and conversion. Treatment depends on cognitive processes that are specific to each register. Conversion depends on the cognitive distance between the respective content of representation produced in two different registers.

Geometry requires both figural treatment for heuristic purposes and discursive treatment for valid reasoning from given. So the operative apprehension is a figural reorganization of figural units 2D. There are cognitive factors that inhibit or trigger the figural heuristic recognition. Inhibition results from the prevailing perceptual recognition. So the discursive apprehension starts from the given properties implicitly or explicitly between 1D or 0D figural units. It goes against the perceptual apprehension which makes figural 2D Or 3D recognition prevail over 1D or 0D recognition. Conversion can be made either from apprehension focusing mainly on 2D units, either the perceptual one or the operative, or from apprehension focusing on 0D or 1D figural units.

Geometry also requires conversions between the figural register and language, i.e. to work in two registers. Without such conversion and synergy from the articulation between these two registers, there is no true comprehension, no ability to solve problems. But conversion can occur from three ways (Duval, 2006):

— from perceptual apprehension focusing on the first recognition of 2D or 3D figural units . In this case we have only recognition of typical figures with their name.

— from operative apprehension focusing on all the possible 2D or 3D figural units in a geometrical configuration. So conversion leads to explanation and argumentation, but not to demonstration.

— from discursive apprehension focusing on the possible 0D and 1D figural units and making given properties prevailing for deducing.

For the construction of the research instrument of this study for collecting quantitative data we have confined ourselves to study the kind of geometrical figure apprehension which is prevailing for students. In text books the cognitive variation in the geometrical figure apprehension is completely ignored.

The notion of Geometrical Paradigms

During the conference CERME3 (2003), for the first time, Houdement and Kuzniak presented a paper in which they referred to geometrical paradigms. Houdement and Kuzniak (2003) have focused on figures not in relation to visualization, nor directly to reasoning, but to their importance according to what they call the “different paradigms” of geometry, and on “Geometric Work Space” (GWS). Thus, they proposed that elementary geometry appears to be split into three paradigms, characterizing different forms of geometry: Geometry I (Natural Geometry), Geometry II (Natural Axiomatic Geometry) and Geometry III (Formalist Axiomatic Geometry). These paradigms reflect various stages in the succession of academic cycles. Each stage is characterized by specific practices and challenges in the teaching and learning of the discipline (Kuzniak & Rauscher, 2011). The first two paradigms, play a part in today’s secondary education (Kuzniak & Vivier, 2009).

Specifically, the idea of geometrical paradigms was inspired by the notion of paradigm introduced by Kuhn, in his work on the structure of scientific revolutions. In a global view, a paradigm is what the members of a scientific community share, and, a scientific community consists of people who share a paradigm (Houdement, 2007; Kuzniak, 2012). It consists of all the beliefs, techniques and values shared by a scientific group. The scientific activity of a researcher is guided by the paradigm on which he is working. A paradigm is composed of a theory to guide observation, activity and judgement and to permit new knowledge production (Houdement, 2007). It indicates the correct way for putting and starting the resolution of a problem.

The first paradigm that was proposed was *Natural Geometry (Geometry I)*, which finds its validation in the material and tangible world (real and sensitive world); hence, its name natural geometry. In this geometry, valid assertions are generated using arguments based upon perception and experiment (Kuzniak & Rauscher, 2011). To produce new knowledge in this paradigm, all methods are allowed: evidence, real or virtual experience and, of course, reasoning. Little distinction is made between model and reality and all arguments are allowed to justify an assertion and convince others of its correctness. Assertions are proven by moving back and forth between the model and the real: The most important thing is to develop convincing arguments (Kuzniak, 2012).

The objects of Geometry I are material objects, graphic lines on a paper sheet or virtual lines on a computer screen. Even material, the lines are always consecutive in a first representation of reality. Objects of the sensitive space can be schematised in a micro-space by a network of lines. The straight line is a model and, thus it refuses bumps; the circle is perfect, all its points are at the same distance from the centre. The chosen graphic objects (and their properties) are often in for the first time the most convenient to describe reality, hence the name of Natural Geometry for Geometry I (Houdement, 2007). In this paradigm the ordinary techniques are the drawing techniques with ordinary geometrical tools: ruler, set square, compasses but also folding, cutting, superposing (Houdement, 2007). Proofs could lean on drawings or observations made with common measurement and drawing tools such as rulers, compasses and protractors. Folding or cutting the drawing to obtain visual proofs is also allowed. The development of this geometry was historically motivated by practical problems. The perspective of Geometry I is of a technological nature (Kuzniak, 2012).

Natural Axiomatic Geometry (Geometry II), whose archetype is classic Euclidean Geometry, is built on a model that approaches reality. In Natural Axiomatic Geometry (one model is Euclid's Geometry) the objects are no more material but ideal. Definitions and axioms are necessary to create the objects, but in this paradigm they are as close as possible to the intuition of the sensitive space, therefore the name of Natural Axiomatic Geometry (Houdement, 2007). Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. The system of axioms could be incomplete and partial: The axiomatic process is work in progress with modeling as its perspective. In this geometry, objects such as figures exist only by their definition even if this definition is often based on some characteristics of real and existing objects (Kuzniak, 2012)

Geometry II remains a model of reality. But, once the axioms are fixed, demonstrations inside the system are required to progress and to reach certainty. In this paradigm the text assumes great importance, and all the objects should be defined by texts; drawings are only illustrations, accompaniments of textual propositions. As it is convenient, the expert works with drawings, but they know how to read theses drawing and how all the indications they put on the drawing are validated by the text (Kuzniak, 2012).

To these two approaches, it is necessary to add a third Geometry. *Formal Axiomatic Geometry (Geometry III)* which is scarcely present in compulsory schooling (Houdement, 2007), but is the implicit reference of teacher trainers when they have studied mathematics at university, which is very influenced by this formal and logical approach. In Geometry III, the system of axioms itself, disconnected from reality, is central (Kuzniak, 2008). The system of axioms is complete and unconcerned with any possible applications in the world. It is more concerned with logical problems and tends to complete "intuitive" axioms without any "call" for perceptive evidence such as convexity. Moreover, axioms are organized in families which structure geometrical properties: affine, euclidean, projective, etc. (Kuzniak, 2012).

Each paradigm is global and coherent enough to define and structure geometry as a discipline and to set up respective working spaces suitable to solve a wide range of problems (Kuzniak & Vivier, 2009). These various paradigms - and this is originality of this approach - are not organized into a hierarchy, but their perspectives are different and so the nature and the handling of problems change from one to the next. (Kuzniak, 2008).

The passage from one type of Geometry to another is really complex: it involves a change of theory. This change can be seen as a revolution or as a dialectic and progressive evolution. At least two transitions are not of the same nature. The first (from Geometry I to Geometry II) concerns the nature of the objects and of space. The second (from Geometry II to Geometry III) is more of an epistemological character. During elementary school the first transition is certainly the more crucial one and one could think about the opportunity to teach Geometry II soon to many lower secondary school students (Houdement & Kuzniak, 2003)

Also, the passage from one type of geometry to the other one is not once definitively established at a specific moment of the curriculum, and the transition seems ceaselessly put back on every new notion (Kuzniak, 2009). The new notions are introduced and structured around geometric objects which can be seen also as objects of sensitive space: triangles, circles, polygons (Kuzniak, 2011). Furthermore, the constant emphasis on a transition towards Geometry II based on Geometry I can let us suppose that a mixed Geometry is possible. We will, thus, speak of a mixed Geometry (GI / GII).

The paradigms set the geometric horizon assigned by the user with a problem, which determines the appropriate approach to the problem. It depends on the type of the tasks that influence the choice of paradigm. It is, therefore, important to observe the geometric practices proposed in school mathematics but also in other disciplines (including physics or technology) (Kuzniak, 2006). Also, when people share the same paradigm, they can communicate very easily and in an unambiguous way. By contrast, when they stay in different paradigms, misunderstandings are frequent and can lead, in certain cases, to a total lack of comprehension (Kuzniak, 2012).

The notion of Geometric Work Space

The Geometrical Working Space (GWS) is the place organized to ensure the geometrical work (Kuzniak, 2008, 2009) and, thus, the work of people solving geometry problems (geometricians) (Kuzniak, 2011; Kuzniak, 2012). It establishes the reference to the complex setting in which the problem solver acts (Kuzniak & Rauscher, 2011). The concept of Workspace of Geometry intervenes naturally until there is a reflection of the conceived elementary geometry, as a result of an interaction between an individual and geometric problems. The workspace must realize the articulations between different elements from which the most characteristic are a theoretical framework, the geometric objects and a material space (Kuzniak, 2006).

To define the Geometric Work Space two planes are introduced that are at the same time material and intellectual: the components plane and the cognitive plane (Kuzniak & Rauscher, 2011).In the components plane there is a networking of three characteristic components (Kuzniak, 2006) of the geometrical activity into its purely mathematical dimension (Kuzniak, 2012, 2008). These three interacting components are the following ones:

1. The real and local space as material support with a set of concrete objects;
2. Artifacts such as drawing instruments (rulers, compass, etc.) and software available to the geometrician;
3. A theoretical system of references made of properties organized in a way depending on the geometrical paradigm. This theoretical system defines the logico-deductive system in which the theoretical model(s) and the mathematical object(s) are organized.

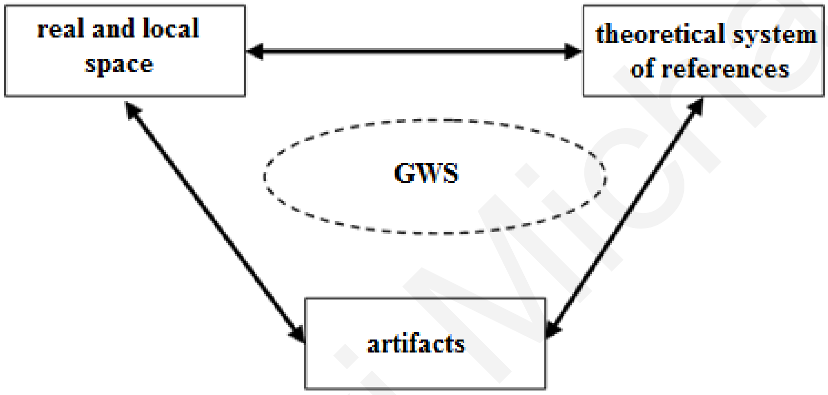


Figure 4. The components of the geometric work space

By themselves, the components are not sufficient to define the global meaning of a GWS, which depends on the function that its designer and its users gave to it (Kuzniak, 2011). The GWS function can evolve in connection with the social and economic context which influences the educational institutions where geometry is taught. Moreover, this function strongly depends on the cognitive ability of a particular user (Kuzniak, 1999). To ensure that the components are well used we need to focus on some cognitive processes involved in the geometrical activity (Kuzniak & Vivier, 2009). At each pole of the GFS a cognitive component is superimposed appropriate to the geometrical activity of the students (Bulf, 2009).

Therefore, a second level was introduced, centered on the cognitive articulation of the GWS components, to understand how groups, and also particular individuals, use and exploit the geometrical knowledge in their practice of the domain (Kuzniak, 2012). The cognitive plane was introduced to clarify the cognitive processes involved in geometry and to describe the cognitive activity of a particular user. Duval's approach was used to clarify the cognitive processes involved in problem solving in geometry. In adapting Duval (1995, 1998), geometry involves three kinds of cognitive processes which fulfill specific epistemological functions:

1) *visualization processes* with regard to space representation for the illustration of a statement, for the heuristic exploration of a complex situation, for a synoptic glance over it, or for a subjective verification;

2) *construction processes* by tools: construction of configurations can work like a *model* in that the actions on the representative and the observed results are related to the mathematical objects which are represented;

3) *reasoning*, in relation to discursive processes for extension of knowledge, for proof, for explanation.

These different processes can be performed separately. So, visualization does not depend on construction: there is access to figures, whatever way they are constructed. And even if construction leads to visualization, construction processes depend only on connections between mathematical properties and the technical constraints of the used tools. Ultimately, if visualization is an intuitive aid that is sometimes necessary for finding a proof, reasoning depends exclusively on the corpus of propositions (definitions, axioms, theorems) which is available. And in some cases visualization can be misleading or impossible. However, these three kinds of cognitive processes are closely connected and their synergy is cognitively necessary for proficiency in geometry (Duval, 1998).

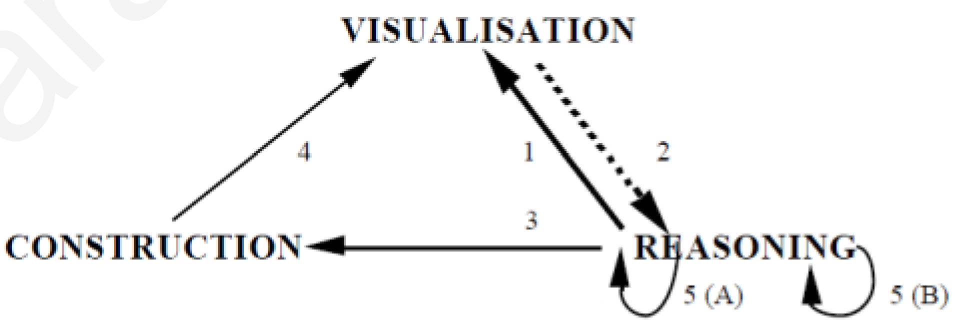


Figure5. The underlying cognitive interactions involved in geometrical activity (Duval, 1998)

In Figure 5 each arrow represents the way a kind of cognitive process can support another kind in any task. Arrow 2 is dotted because visualization does not always help reasoning. Arrow 5(B) emphasizes that reasoning B can develop in an independent way. In many cases we can have a longer circuit (Duval, 1998).

As the GWS is created within the framework of school institutions, we need to introduce different levels in order to describe the diversity existing in school education (Kuzniak, 2011). When the problem is posed to a real individual (the pupil, the student or the teacher), their treatment takes place in the workspace that will be important to be studied (Kuzniak, 2009). This person handles the problem with his/her personal GWS, which generally depends on the knowledge of the person but also on the institution where the person works (Houdement, 2007). It is formed in a progressive way depending on the individual and sometimes cannot be operational. However, not only students are concerned by this notion but also teachers are responsible for shaping it. Indeed, they have to have clear consciousness of the nature of the GWS to avoid some misunderstandings resulting from a vague and implicit management of the interplay between paradigms (Kuzniak, 2011).

Compared to the previous models falling into the synthetic and global approaches, the model of the geometrical paradigm and the GWS somehow interweaves the dimensions of visualization and reasoning. Whatever the paradigm is, visualization is very important for the GWS and its relation with reasoning is evident in the cognitive plane that is introduced, where these two cognitive processes are reciprocally connected.

Research on the geometrical figure apprehension in Cyprus

Even though previous research studies investigated extensively the role of external representations in geometry (e.g. Duval, 1998; Mesquita, 1996; Kurina, 2003), the cognitive processes underlying the four apprehensions of a geometrical figure proposed by Duval (1995) have not been empirically verified yet. Based on the cognitive model proposed by Duval, in Cyprus two research attempts were recently made, aiming at examining the structure of the geometrical figure apprehension.

Actually, Deliyianni, Elia, Gagatsis, Monoyiou and Panaoura (2010) tried to determine the structure of the geometrical figure apprehension. Finally, they have confirmed a three level hierarchy about the role of perceptual, operative and discursive apprehension in geometrical figure apprehension for primary and secondary school students (Figure 6).

The first order – factors of the model revealed the differential effect of perceptual and recognition abilities, of the ways of figure modification and of the measurement concept on the three second-order factors that correspond respectively to perceptual, operative and discursive apprehension of a geometrical figure. Finally, the effect of the three second – order factors on a third-order factor that corresponds to the geometrical figure apprehension was confirmed. In addition, the application of the structural equations modeling allowed the examination of the invariance of this structure across primary and secondary school students.

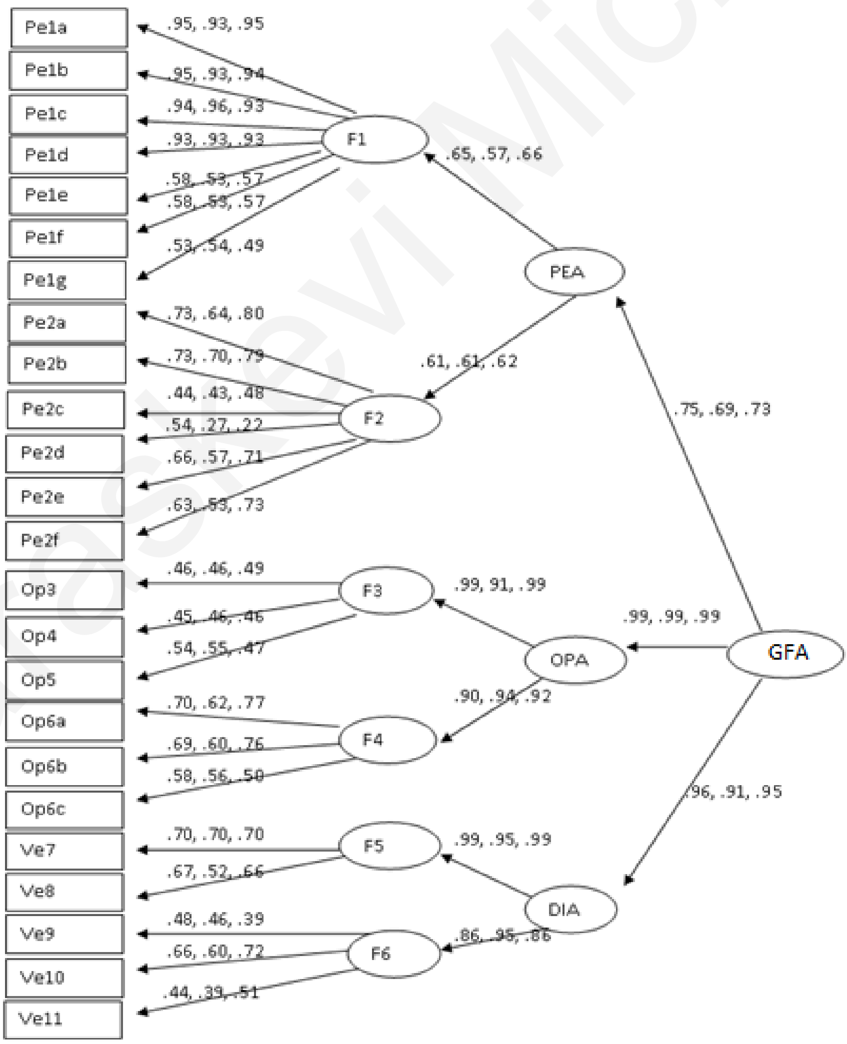


Figure 6. The CFA model of the geometrical figure apprehension for primary and secondary school students

The structure of geometrical figure apprehension appears to remain constant for primary and secondary school students and actually the potency of this structure seems to increase according to age. The model points out the important role of each type of geometrical figure apprehension, taking into account that, even though coordination between them is needed, each one is distinct from the other (Duval, 1999). The elaborated model, thus, offers teachers a framework of students' thinking while solving a wide range of geometrical tasks in a systematic manner within and between the two educational levels.

Furthermore, the results of their study revealed differences in primary and secondary school students 'performance in the geometrical figure apprehension tasks. Particularly, secondary school students' performance was higher in all the dimensions of the geometrical figure understanding relative to the primary school students' performance. Concerning the way students behaved during the geometrical tasks solution process, it was observed that the behavior of primary and secondary school students was similar during the solution process of some of the tasks. This finding revealed that geometrical figure understanding stability existed to a certain extent in these students' behavior. However, in some cases differences were observed in the way the two age groups of students dealt with geometrical figure understanding tasks. To be specific, secondary school students behaved in a consistent way during the solution of the perceptual, operative and discursive tasks. By contrast, primary school students dealt with perceptual tasks in isolation, indicating a compartmentalized way of thinking.

Moving a step forward, Michael, P., Elia, I., Gagatsis, A. and Kalogirou, P. (2010) investigated the role the mereologic, the optic and the place way modifications exert on operative figure apprehension of primary school students and they verified a model which lent support to Duval's (1995) conceptualization of the cognitive processes underlying operative figure apprehension.

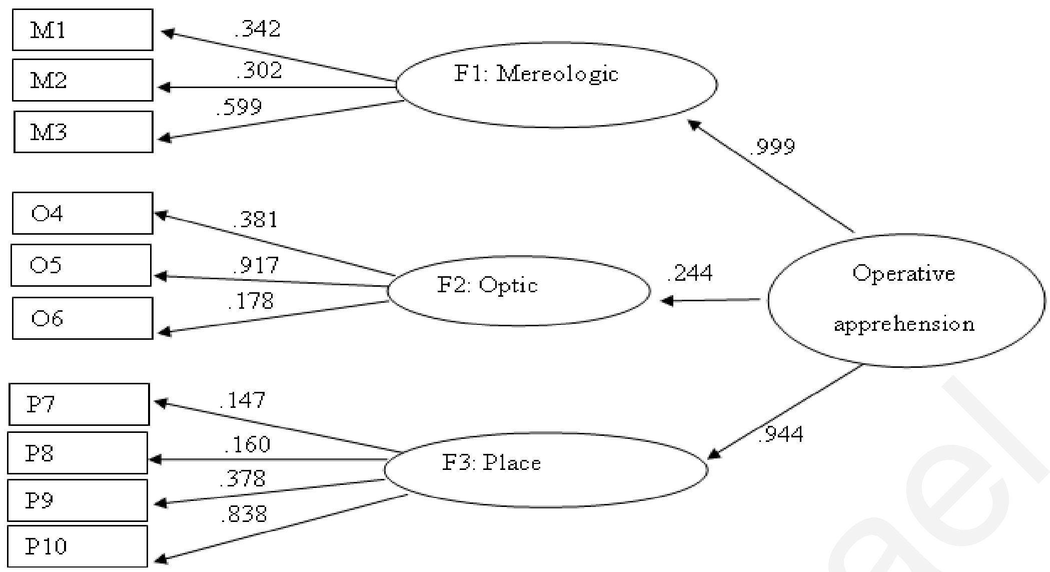


Figure 7.The CFA model of operative apprehension of primary school students

In particular, the second-order model (Figure 7) which is considered appropriate for interpreting operative apprehension, involves three first-order factors and one second-order factor. On the second-order factor that stands for operative apprehension the first-order factors F1, F2 and F3 are regressed. The first-order factor F1 refers to the tasks which correspond to the mereologic way of modifying a given figure, the first-order factor F2 refers to the optic modification tasks and the first-order factor F3 refers to the place modification tasks.

The factor loadings reveal that the mereologic and place types of modification are the primary source explaining students' operative apprehension of geometrical figures. That is, they are closely related to operative apprehension. However, the results indicate that all three ways of modifying geometrical figures have a significant effect on operative figure understanding. The results also showed that students exhibited consistency in the solution of the mereologic modification tasks and the optic modification tasks respectively, but they applied the place way of modifying geometrical figures in a rather fragmentary way.

The transition between different educational levels

The transition problem from one educational level to another is universal (Mullins & Irvin, 2000). In recent years the research community has been occupied with the difficulties that students seem to face during their transition from one educational level to another. Research findings suggest that there is a negative change in students' perceptions of school, their attitudes towards various school subjects, their motives, their self-confidence beliefs and their competence, but also a decrease in their performance during the transition from primary to secondary education (Middleton, Kaplan, & Midlgley, 2004; Mullins & Irvin, 2000; Anderson, Jacobs, Schramm, & Splittgerber, 2000; Anderman & Midgley, 1997).

Students also experience many changes in their school environment associated with the transition from elementary school to lower secondary school. The goals of elementary schools tend to be task-oriented, whereas the goals of secondary schools tend to focus on performance (Midgley, Anderman, & Hicks, 1995). Secondary school teachers tend to have many students for short periods of time; hence, the student-teacher relationship changes from elementary to secondary school (Alspaugh, 1998). Researchers have found declines in student self-perception and self-esteem associated with the transition from elementary school to intermediate-level school (Seidman, Allen, Aber, Mitchell, & Feinman, 1994).

Wanting to explore the nature of the achievement loss associated with school-to-school transitions from elementary school to lower secondary school and to upper secondary school, Alspaugh (1998) compared three groups of 16 school districts. His study showed a significant achievement loss associated with the transition from elementary school to secondary school. The transition loss in achievement was larger when students from multiple elementary schools were merged into a single lower secondary school during the transition. The students from the lower secondary schools experienced an achievement loss in the transition at 9th grade. Seidman et al. (1994) hypothesized that students may face double jeopardy if they make a transition from elementary school to middle school and then experience a second transition to high school. There also may be a relationship between the number of school-to-school transitions and high school dropout rates.

Middleton, Kaplan and Midlgley (2004) studied through the use of longitudinal survey data the change of achievement goal orientations, in a sample of school students in mathematics as they moved from sixth to seventh grade. They concluded that all goal orientations were moderately stable over time, but that individuals who feel efficacious in math while endorsing a performance-approach goal orientation may be particularly vulnerable to adopting maladaptive performance-avoid goals over time and with change in circumstances.

A change in school environment may contribute to a change in students' achievement goal orientation (Anderman & Midgley, 1997). After the transition to secondary school, students may experience an increase in the emphasis on relative ability and competition with peers (Midgley, 1993). This may be particularly true in mathema cs, a discipline oriented toward ability and achievement levels. Many secondary school reformers have suggested that relative ability should be de-emphasized, but conversations with the secondary school principals in the study of Middleton et al. (2004) suggest that most of those reform efforts have been focused on the sixth grade level and the first year of lower secondary school. Less attention has been paid to the upper grade levels in secondary school.

During the efforts for educational reform in Cyprus (Commission for Educational Reform, 2005), it seems that the transition to secondary education negatively affects the performance of students in mathematical concepts taught at both levels of education. In the case of Cyprus, students experience difficulties during the transition from primary to secondary school, which is evident in their performance in most subjects and especially in mathematics. During recent efforts for educational reform in Cyprus, it was shown that the transition to secondary education negatively affects the performance of students in mathematical concepts taught at both levels of education.

Summary

Through the literature review and the presentation of theoretical frameworks that relate to the cognitive and instructional issues of geometrical thinking, the cognitive complexity of geometry is revealed. The significance of elementary geometry is due to the fact that it mobilizes two representational registers: the one of natural language and the one of gestalt configurations (Duval, 2007).

The figure register is used in order to ‘see’ and the natural language register in order to ‘explain’ (Duval, 2000). The geometrical figure constitutes the representation that possesses a central role in geometrical activity. In geometry a “figure” merges three semiotic representations: magnitude, shape configurations and words naming the given properties. The crucial issue in the learning of geometry is the separation between magnitude and visualization (Duval, 2005). The reason is that magnitude causes visual illusions and wrong perceptual estimation of the relations between figural units.

The main part of this chapter deals with the presentation of important theoretical frameworks for the learning of geometry. These frameworks described theoretical models focusing on geometrical figures. In some of these frameworks the figure itself was not the crucial question, but the focus on it was in a side way. These were characterized as synthetic and global, whereas the rest focused directly on the figure on the discrimination of several cognitive variables. In the first category fall the different levels of van Hiele’s model of thinking in geometry and Fischbein’s theory of figural concepts. The discrimination of several cognitive variables is made in Duval’s cognitive analysis of geometrical figure apprehension and Houdement’s and Kuzniak’s suggestions for the notions of Geometrical Paradigms and Geometric Work Space. Each of these frameworks provides theoretical resources to support research into the development of geometrical reasoning in students and related aspects of visualization and construction.

The first approach presented in this review was van Hiele’s model of thinking in geometry, in which geometrical thinking is described according to different levels and its development is sequential and hierarchical. Regarding visualization, in this model it is not considered itself, but its role is defined in relation to the involvement or not of reasoning. The second approach included the theory of figural concepts proposed by Fischbein. This approach constitutes an attempt to approach geometrical figures as entities possessing not only figural, but conceptual properties as well. This model does not also emphasize the role of visualization for the heuristic functioning of geometrical figures in problem solving, creating a conflict for students between the perceptual and the conceptual aspect of figures.

On the other hand, the crucial role of visualization in solving problems in geometry was highlighted in the approaches that are based on discriminating several cognitive variables for the learning of geometry. First of all, Duval (1988) analyzed the mathematical way required in using figures in problem-solving and the way most students look at a figure. He suggested that four ways of looking at a drawing or a visual configuration, or in other words four kinds of apprehension, must be taken into account. The first one is what is called perceptual apprehension that relates to the recognition of a shape at first glance in a plane or in depth. This recognition keeps stable what is recognized and can be directly associated with naming the figure. The second kind, that is sequential apprehension, depends on the tool or the software used for the construction of figures, as there is an order of construction, taking into account the primitive or the tool. But what matters after all is what is seen when the figure is constructed. The next one is discursive apprehension, which allows deducing new properties based on the given properties. And this is because mathematical relations cannot be represented on a figure only from what is given by words of properties. The last kind of apprehension is the one that makes a figure useful for solving a problem. In fact the operative apprehension places the figure in a heuristic field, due to the fact that a given figure becomes a source for searching other sub-configurations, different from those perceived at first glance. This can be achieved through modifying a constructed or a given figure in many different ways. What is mainly involved in primary and (lower) secondary school geometrical activities is the mereologic modification, because what is often asked is to make a reconfiguration of a figure in order to find a solution.

The existence of different types of apprehension for the same figure shows the complexity of geometrical problems that may seem the most simple. This conflict between the geometric practice of figures and their cognitive style of their recognition raises difficult and crucial problems for the teaching of geometry (Duval & Godin, 2005):

– How to get students to change their look at figures.

– How to pass from a look focusing at the surfaces and their contours to a look that shows the network of lines and points underlying the different figures studied in school.

The second approach is the different types of Geometries and the notion of GWS as proposed by Houdement and Kuzniak. These researchers have taken into account the fact that Geometry is not taught in the same way at the different educational levels. Therefore, they have distinguished three paradigms: Geometry I, Geometry II and Geometry III. However, the transition from Geometry I to Geometry II can be seen through an intermediate type of Geometry, that is the Mixed Geometry GI / GII. In this model the focus on geometrical figures is not clearly through visualization or reasoning, but through the connection between them.

Through the overview of the theoretical models of van Hiele, Fischebein, Houdement and Kuzniak, the conclusion is that visualization, whose role is decisive in problem-solving in geometry, is not placed at the centre of these approaches. The approach that manages to highlight this, from those discussed in this chapter, is the one of Duval, as visualization is studied as a specific process that is a crucial factor both irreducible to perception and conceptual or discursive apprehension. The core of the framework is the coordination of registers. That means, in the case of geometry, the coordination between visualization and valid deducing of new properties (discursive apprehension). For visualization it is necessary to differentiate between discursive apprehension and perceptive apprehension. But the crucial point is operative apprehension and how to make pupils learn to see the pertinent configural change beyond factors triggering or inhibiting its visibility (Duval, 1998).

The equivocal meaning of "seeing" creates a great cognitive heterogeneity in geometrical problems that are very close mathematically or require the same knowledge. Thereafter, a categorization of problems is indispensable for interpreting students' performance and production on problems. In this research students' production on geometrical problems are going be analyzed according to the way students look at a geometrical figure in relation to the four different types of apprehension, in order to identify and suggest the factors that influence this proper way of looking at figures.

CHAPTER III

METHODOLOGY

Introduction

This research aims to study the way lower and upper secondary education students look at a geometrical figure in relation to the four different types of apprehension and to develop a theoretical model for describing the cognitive structure of the geometrical figure apprehension.

This chapter presents the methodology of the research. Actually it includes the description of the participants, the procedure that was to be followed for the fulfillment of the research and the analysis of the geometrical content of mathematics textbooks that was conducted in order to facilitate the construction of the research instrument. Next the content of the research instrument is presented and each task is analyzed according to the cognitive requirements for its solution and the expected reactions of students during their involvement with the tasks of the test. The way students' answers were scored and codified is then defined and, thus, the variables of the study are set. Also this chapter includes a description of the statistical analyses that were performed on the collected quantitative data. Furthermore, the research design and implementation of the task – based interviews with only some students as well as the way these data were analyzed are presented.

Participants

The study examines lower and upper secondary school students' geometrical figure apprehension. Therefore, the participants are Grade 9, Grade 10 and Grade 11 students. The selection of these groups of students allowed the examination of all the four types of geometrical figure apprehension, since students of these ages have the necessary teaching experiences that enable them to deal with such kind of tasks.

The changes that occur after students' transition to a next educational level and specifically from lower and upper secondary school are of a special interest in this research. For this reason Grade 9 students are selected to participate, as they are in the last grade of lower secondary school and Grade 10 and 11 students were chosen because these are the first two grades of the upper secondary school.

It should be clarified that in Grade 9 and Grade 10 the students are all attending the same mathematics course. On the other hand, in Grade 11 students have the option to continue with the teaching of a basic level of mathematics (G11a) or to attend the teaching of a more advanced level of mathematics (G11b). Besides the level of the mathematics taught in these two distinct classes, the teaching hours of mathematics in Grade 11b are more than those in Grade 11a. Therefore, the reason that students from both types of Grade 11 were selected to participate in the research is for examining the difference after students' transition from Grade 10 to each type of Grade 11.

Specifically, the research was conducted among 881 students, aged 15 to 17, of lower (Grade 9) and upper (Grade 10, Grade 11) urban and rural secondary schools, in Cyprus. More specifically the participants were 312 students from Grade 9, 304 students from Grade 10, 125 students from Grade 11a and 140 students from Grade 11b. Thus, it was feasible to compare the geometrical figure apprehension of students from lower and upper secondary school and trace the similarities and differences between them. It was also possible to see whether development occurs not only after students' change of educational level, but also after they move to a higher grade. Regarding the task – based interviews, they were conducted with 9 students. Actually there were 3 students from grade 9, 3 students from grade 10 and 3 students from grade 11, with medium or high abilities.

Procedure

The research was carried out through seven phases, described below.

Phase 1: In the first phase a relevant literature review in mathematics education in relation to geometrical thinking and geometrical abilities was conducted. Particularly, the theoretical framework about was formed by presenting the main theories about geometrical thinking. Furthermore, the relevant literature concerning students' transition from one educational level to the next one was also examined, laying emphasis to the transition from lower to upper secondary school.

Phase 2: The second phase involves the examination of the geometry curriculum and geometry textbooks used for the teaching of geometry in Cyprus, for Grade 9, Grade

10, and Grade 11. This examination provides information about the use of the geometrical figure in the teaching of geometry for the specific grades that are examined in this study and allows the discrimination of the tasks according to the type of apprehension that is expected to be mobilized during their resolution. This information was also useful for the development of the research instrument of this investigation, whose construction was done in alignment with the geometrical content students are taught.

Phase 3: The development of the research instrument for the collection of quantitative data took place in the third phase of the research. Specifically, a test was developed in order to examine students' way of looking at figures in relation to the four types of apprehension (perceptual, operative, discursive and sequential). The test was developed taking into account the relative bibliography and the examination of the curriculum and the content analysis of the mathematics textbooks.

The development of the research instrument was followed by a pilot administration for examining the construct validity of the test. After the pilot administration of the research instrument the necessary changes were made in order to reach its final form.

Phase 4: In the fourth phase the administration of the tests and the collection of quantitative data of the study took place. The test was divided into two parts and each part was administered during one teaching period of 40 minutes. The administration was mainly conducted by the mathematics teacher of each class, to whom the necessary instructions and clarification about the administration process were provided. Also, the scoring, the way of analyzing the tasks of the test and their codification were determined.

Phase 5: In this phase the task – based interviews were conducted. The interviews were conducted with only some students. In the interviews students had the chance to explain their thoughts and the way they had worked during the resolution of the tasks. The task – based interviews aimed also at identifying the difficulty level of the tasks and students' difficulties and misunderstandings in the different tasks of the test as expressed by them after their resolution.

Phase 6: The sixth phase included the analyses of the data and the organization of the results of the study. The data were analyzed using statistical packages such as SPSS, EQS, CHIC and QUEST.

The multivariate analysis of variance (MANOVA) was performed by the use of the statistical package SPSS. The Rasch model analysis was used in order to investigate the construct validity of the test and create a good interval level measure for the lower and upper secondary school students' geometrical figure apprehension. For the estimation of the Rasch models, the QUEST software (Adams & Khoo, 1996) was used. By the Confirmatory Factor Analysis (CFA), using EQS (Bentler, 1995) software, the theoretical models concerning the cognitive structure of the geometrical figure apprehension were constructed and verified. Using the implicative analysis with the software CHIC (Bodin, Coutourier, & Gras, 2000) the possible implicative relations between the tasks of the tests were traced, whereas the similarity analysis revealed groups of tasks that were handled by the students in a similar way. This phase included analysis of the qualitative data, concerning the students' answers in the task – based interviews.

Phase 7: In the last phase of the research the final results were formed and discussed, and the final conclusions were formulated. Some relevant suggestions concerning changes in the geometry curriculum and some practical recommendations about the teaching and learning of geometry are also provided.

Analysis of the geometry content of mathematics textbooks

Introduction

The aim of this section is to present and discuss the results that were obtained from the examination of the geometrical exercises and examples included in the mathematics textbooks of the last grade of lower secondary school (Grade 9) and the two first grades of upper secondary school (Grades 10 and 11) in Cyprus. Specifically, it is an attempt for classifying the exercises and the examples of these textbooks, basing on the theoretical model proposed by Duval (1995), which involves the four types of geometrical figure apprehension.

It is widely acknowledged that school textbooks reflect the aims of the curriculum (Pepin & Haggerty, 2002). In fact, the school textbooks correspond to the curriculum (Symeou – Mai, 2008), present the content that is to be taught and provide support to the teacher, in order to organize the teaching (Brändström, 2005). The wide use of textbooks in the classroom gives them the potential to influence students' learning (Pepin & Haggerty, 2004). Textbooks provide continuity and coherence for students during the lesson and prevent chaos in the classroom by keeping students occupied (Brändström, 2005). Therefore, they have a central role in the classroom, both for students and teachers.

Recently, research has turned its interest to textbooks, as far as their content and the way teachers use them are concerned (Pepin & Haggerty, 2002). Following the notion that the school textbook is the main tool for the teacher, it is reasonable to explore school textbooks as important tools for teaching.

Mathematical ability is a compound of general intelligence, visual imagery, and ability to perceive number and space configurations and retain such configurations as mental pictures (McGee, 1979). Both teachers and researchers agree that the use of visual representations is an important part of mathematics education, because such representations appear to enhance intuition and understanding in many areas of mathematics (Krutetskii, 1976). To get some idea of the representations used in a mathematics class, an easy method consists of taking a look at those in textbooks, which can be considered as "fair copies" of the ones actually made in classrooms (Parzysz, 1991).

The Analysis

In analyzing the content of mathematics textbooks regarding geometry, each exercise and example were placed in one of the four categories, which were determined according to the type of apprehension that was required in order to be solved. Representative examples and exercises from the mathematics textbooks of the three grades for each of the four types of understanding can be found in Appendix 2. While examining and classifying the exercises and examples of the specific textbooks, attention was also drawn to the theoretical content of the textbooks. It is noteworthy that in all the three grades examined, three types of apprehension are mainly involved in the different figures presented in the theoretical part; which are the perceptual, the discursive and the operative apprehension.

The percentages of exercises requiring each type of apprehension of the geometrical figure by grade are presented in Table 2. On the one hand, according to the results, the highest percentage of the exercises examined in the textbooks requires the activation of perceptual apprehension for Grades 9, 10 and 11. On the other hand, the least required type of apprehension is the sequential, for Grades 9 and 10. However, this is not the case for Grade 11, since discursive apprehension constitutes the type of apprehension that is the least expected to be mobilized in the exercises examined. As far as discursive and operative apprehension are concerned, the percentages for Grades 9 and 10 are quite similar. The situation is different for Grade 11, since there is a higher amount of exercises mobilizing operative apprehension than discursive or even sequential apprehension.

Table 2

Percentages of Exercises Requiring Each Type of Geometrical Figure Apprehension by Grade

	N	Perceptual (%)	Sequential (%)	Discursive (%)	Operative (%)
Grade 9	171	61.4	7.0	16.4	15.2
Grade 10	161	50.3	2.5	24.2	22.4
Grade 11	131	50.8	12.3	8.5	21.5

Comparing the percentages of exercises requiring each type of geometrical figure apprehension in each grade, on the one hand there is a slight fall in the amount of exercises requiring perceptual and sequential apprehension of the geometrical figure from Grade 9 to Grade 10 and, on the other hand, an increase from Grade 10 to Grade 11 is observed. There is, also, an increase from Grade 9 to Grade 10, as far as discursive and operative apprehension are concerned, followed by a decrease from Grade 10 to Grade 11, particularly as regards discursive apprehension.

Table 3 presents the exercises activating each type of apprehension of the geometrical figure for the two educational levels. According to the results, some changes occur during the transition from one level to the other. In particular, the percentage of exercises involving perceptual apprehension decreases from lower to upper secondary school. By contrast, the percentages of exercises involving discursive and operative apprehension increase from lower to upper secondary school. As for sequential apprehension, the percentages remain constant from lower to upper secondary school.

Table 3

Percentages of Exercises Requiring Each Type of Geometrical Figure Apprehension by Educational Level

Educational Level	N	Perceptual (%)	Sequential (%)	Discursive (%)	Operative (%)
Lower Secondary	171	61.4	7.0	16.4	15.2
Upper Secondary	291	50.5	6.9	17.2	22.0

The percentages of the given examples in the textbooks that require each type of apprehension of the geometrical figure for Grades 9, 10, 11 are demonstrated in Table 4. The table shows that the highest percentage of examples used in the mathematics textbooks are those involving perceptual apprehension. A small percentage of examples that require operative apprehension exist in the textbooks of the three Grades. As for discursive apprehension, a small number of examples mobilizing this type of apprehension appear only in Grade 9. It is also noticeable that examples involving sequential apprehension are totally absent from the textbooks of these three Grades.

Specifically, concerning perceptual apprehension, there is an increase in the number of examples of this type of apprehension while moving from Grade 9 to Grade 10. By contrast, this number is reduced when passing from Grade 10 to Grade 11. As far as operative apprehension is concerned, a slight decrease in the percentage of the examples appearing in the textbooks from Grade 9 to Grade 10 is observed, while this type of examples appears to be more frequently presented in the textbook of Grade 11 than in the one of Grade 10.

Table 5 presents the percentages of examples which belong in each type of apprehension of the geometrical figure for the two educational levels. Examples requiring perceptual apprehension appear more frequently in the mathematics textbooks of upper secondary school. We notice a similar situation for operative apprehension also, as there is an augmentation of a similar percentage (almost 7%) in the use of this type of examples. As mentioned above, examples of discursive apprehension are not used in high school, whereas examples of sequential apprehension are not involved at all, in both educational levels of schools.

Table 4

Percentages of Examples Requiring Each Type of Geometrical Figure Apprehension by Grade

Grade	N	Perceptual (%)	Sequential (%)	Discursive (%)	Operative (%)
Grade 9	34	79.4	0	11.8	11.9
Grade 10	57	93	0	0	7
Grade 11	54	81.5	0	0	8.5

Table 5

Percentages of Examples Requiring Each Type of Geometrical Figure Apprehension by Educational Level

Educational Level	N	Perceptual (%)	Sequential (%)	Discursive (%)	Operative (%)
Lower Sec. School	34	79.4	0.0	11.8	5.9
Upper Sec. School	111	87.4	0.0	0.0	12.6

Conclusions

The aim of the analysis of the geometry content of mathematics textbooks was the identification of the types of geometrical figure apprehension as proposed by Duval (1995) which are required in the geometry exercises and examples for Grades 9 to 11.

According to the results, the perceptual apprehension is the dominant type of apprehension required in both the exercises and the examples of the geometry content examined. Attention is also paid to the operative apprehension, even though exercises and examples of this type are found in lower percentages in textbooks. The sequential is the least demanded type of apprehension in the exercises. It is remarkable that examples which involve sequential apprehension are totally absent from the geometry textbooks of the grades examined.

The number of exercises classified in each type of geometrical figure apprehension changes from Grade 9 to Grade 11. In particular, there is a rise in the amount of exercises activating operative or discursive apprehension from Grade 9 to Grade 10, followed by a reduction in Grade 11. As far as perceptual and sequential apprehensions are concerned, the opposite situation is noticed. The number of these exercises is reduced in Grade 10, while an augmentation occurs in Grade 11. Although there is a shift in the rates of exercises of perceptual apprehension through the three Grades, this kind of exercises occupies almost half of the total number of exercises.

Consequently, we may wonder whether we should refer exclusively to a transition from lower to upper secondary school. According to the results of this analysis, students pass through two different levels; the first one being from Grade 9 to Grade 10, while the second is when passing from Grade 10 to Grade 11. Thus, attention should be paid to this fact when referring to students' problems and difficulties in order that the appropriate actions for facing these problems and difficulties may be determined.

Concerning exercises and examples, the number of examples requiring perceptual or operative apprehension corresponds to the number of exercises that require the same type of apprehension, respectively. This is not the case for sequential and discursive apprehension. Although there are exercises that involve these types of apprehension, the number of examples included in the textbooks is very small or even totally absent in some cases. Therefore, there is a lack of support to students through the examples, in order to cope with tasks of these types of geometrical figure apprehension.

By the comparison between lower and upper secondary school, the conclusion is that there is a reduction in exercises which involve perceptual apprehension, accompanied by almost a corresponding increase in exercises of operative apprehension. This is an important fact, because operative apprehension is fundamental in performing tasks. The resolution of a task requires a reorganization of the shapes which are first recognized at a glance in the figure, which in operative understanding the given figure functions as a starting point for investigating other modifications (Duval, 1999).

Besides, during the transition from lower to upper secondary school the number of examples is not in proportion to the number of exercises. Particularly, we notice that the tasks of operative apprehension decline, while the corresponding examples increase. The opposite situation is encountered regarding the perceptual apprehension, as there are more examples and fewer exercises of this type in upper secondary school.

In the light of the above, it seems that the examination of students' performance in the four types of apprehension is necessary in getting an insight into students' behaviors in relation to a relevant way of looking at geometrical figures. Such kind of research makes the discrimination of the similarities, differences, but also the problems that may occur in students' geometrical knowledge and abilities during the transition from one educational level to the next feasible. In addition, it interesting to compare the results of this analysis of the textbooks with results about students' geometrical abilities and see the effect of this geometrical content on students' performance in the geometrical figure apprehension tasks. Furthermore, it will be possible to see whether there is a discrepancy between what is intended in the teaching of geometry and what is finally achieved.

Instruments

The quantitative data of the research were collected by the use of a test for examining students' relevancy in the way of looking at a geometrical figure and their geometrical abilities, in relation to the four different types of apprehension (perceptual, operative, discursive and sequential). To eliminate the factor of students' tiredness, the test consisted of two parts, in order to be completed within two different teaching periods.

For the construction of the test Duval's (1995, 1999) apprehensions of a geometrical figure were taken into account and tasks from previous studies were either used or modified in order to correspond to the educational level of the participants. Actually, some tasks were taken from the research instrument of the preliminary study of Deliyianni, Elia, Gagatsis, Monoyiou and Panaoura (2010) that examined primary and secondary school students' apprehension of the geometrical figure and from the research of Elia, Gagatsis, Deliyianni, Monoyiou and Michael (2009) that dealt with primary and secondary school students' operative apprehension of the geometrical figure. In addition, new tasks were designed according to the research purposes of the study, the theoretical framework on which it was based and the teaching experiences of the participants. In order to align the test with the mathematics curriculum concerning geometry of Grades 9 to 11, a prior examination of the curriculum and the mathematics textbooks was conducted, as described previously in this chapter.

The test comprises four groups of tasks, each one corresponding to the four different types of geometrical figure apprehension proposed by Duval (1995, 1999). In the next section the a priori analysis of the test is presented. Each task is explained in relation to the mathematical content required and the cognitive processes that are mobilized during its resolution.

The test is presented in its full form in Appendix 1. The development of the research instrument was followed by a pilot administration to a small number of classes from each grade, by which the necessary changes were traced. After these changes the test reached its final form.

A priori analysis of the tasks

This section presents the a priori analysis of the tasks of the test. The tasks are placed in each of the four categories of tasks, according to the type of apprehension that is anticipated to be mobilized during their resolution. The characteristics of each task are described and explained below. However, cases in which students would not choose the type of apprehension anticipated were expected to occur, so it was not enough to focus just on whether the final result of students' resolution was correct or not, but it was also very important to take into account the different cognitive procedures that were related to these solutions. For this reason Duval's (2011) approach to analyzing mathematical activities was taken into account, as explained below, which refers to the mathematical way and the cognitive way of analysis. Therefore, the a priori analysis of the tasks includes the categorization of students' answers in the tasks of the test, according to the type of apprehension that is mobilized for their solutions.

Today's current notion in mathematics education is expressing a variation in tasks treatments and problems resolutions in relation to the context in which they are applied (Kuzniak & Rauscher, 2011). This issue is the main challenge and the weak point of the current research in mathematics education. The vast majority of studies are validated by reproduction *in extenso*, or much too lengthy quotations, of what very few students have told, written or drawn. We are in fact faced with a mass of raw data, whose interpretation is half clinical, half assessment and, in any case, difficult to rebuild or check. It is like a validation by testimony (Duval, 2008).

Mathematical activity has two sides that distinguish it from other forms of knowledge. The visible or conclusive side is the one of mathematical objects and valid processes used to solve a given problem. Its knowledge objects are only accessible through semiotic representations (Duval, 1999). The hidden and crucial side is the one of cognitive operations by which anyone can perform the valid processes and gain access to a mathematical object. Registers of semiotic representation and their coordination set up the cognitive architecture with which anyone can perform the cognitive operations underlying mathematical processes. That means that any cognitive operation, such as processing within a register or conversion of representation between two registers, depends on several variables. To find out what these variables are and how they interact is an important field of research in learning mathematics (Duval, 1999). These two characteristics are at the root of problems of comprehension in mathematics learning for most students (Duval, 2008).

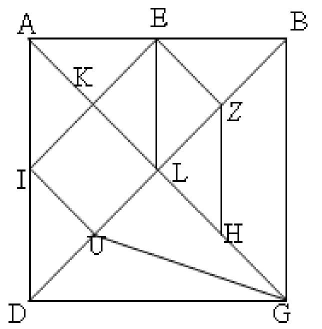
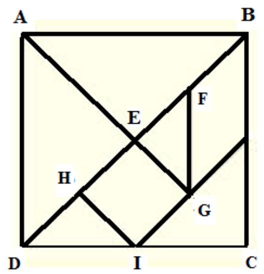
The educational issue is about the methodology and the theoretical framework relevant to both cognitive and mathematical viewpoints, if we want to analyze these problems of comprehension (Duval, 2008). In mathematics education, understanding as well as learning must be examined not only from a mathematical point of view but also from a cognitive point of view, because there may be a discrepancy in the conditions of understanding between one point of view and the other: what can appear simple from one point of view can hide a true complexity that is visible from the other (Duval, 2007).

Duval (2011) suggests there is diversity in the way mathematical activities can be analyzed. In fact he suggests two quite different points of view: the mathematical one, which focuses on the mathematical properties (definition, theorem) and methods or procedures in solving a problem, and the cognitive one, which concentrates on the role of semiotic representations, the transformations of semiotic representations and the registers involved. In concrete terms, any task or any problem that the students are asked to solve requires a double analysis, mathematical and cognitive: the cognitive variables must be taken into account in the same way as the mathematical structure for “concept construction” (Duval, 1999).

Teachers must know the cognitive variation in the geometrical figure apprehension and not confine themselves to the deceptive obviousness of perception for recognizing and using geometrical properties. They must be able to analyze the function that visualization can be performed in the context of a determined activity. We are here in front of an important field of research. But it still seems to be often neglected because most didactical studies are mainly centered on one side of the mathematical activity, as if mathematical processes were natural and cognitively transparent. There is no true understanding in mathematics for students who do not "incorporate" into their "cognitive architecture" the various registers of semiotic representations used to do mathematics, even those of visualization (Duval, 1999). In order to understand what occurs as a recurrent source of difficulties in learning mathematics we need extensive and detailed investigations about the cognitive processes involved in the mathematical way of thinking (Duval, 2004).

Perceptual Apprehension Tasks

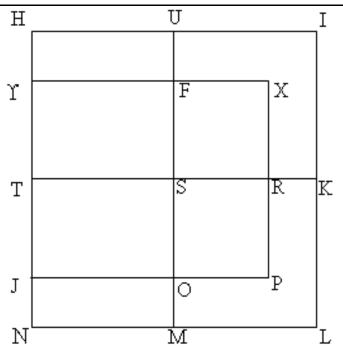
The first group of tasks examines students’ perceptual apprehension (PE) and includes tasks examining students’ ability to discriminate, recognize and name several subfigures in a complex figure. Actually the first two tasks ask students to define the type of particular subfigures that are already coded in a divided figure. Two separate tasks are used, which were administered separately, because there is a need of a number of figures that students have to recognize, in order to check their perceptual fluency. In defining “perceptual fluency” the term “fluency” from the notion of creativity is adopted. Particularly, fluency refers to the number of answers provided by students (Leikin & Lev, 2007; Torrance, 1974). In modifying this term in order to be adjusted to this examination, students’ perceptual fluency is defined based on the number of figures students are able to recognize and name.

	1. Figure KEZL is a ………….. 2. Figure IEZU is a ………….. 3. Figure EZHL is a ………….. 4. Figure IKGU is a ....……… 5. Figure LGU is a .....………… 6. Figure BEL is a …………..
	7. Figure HFGI is a …………… 8. Figure EGIH is a …………... 9. Figure DBC is a ..…………

For analysis of the data these two tasks were merged into one. The correct recognition of all the figures was concerned as the right answer from the mathematical point of view (PE1). Based on the cognitive point of view students' answers were grouped according to the number of figures they had recognized correctly. Hence four categories of answers are determined:

1) Students that recognize correctly all the 9 figures (R1)
2) Students that recognize correctly 8 or 7 figures (R2)
3) Students that recognize correctly 6 or 5 figures (R3)
4) Students that recognize correctly below 5 figures (R4)

In the third task students are asked to identify all the squares in the figure. In question 1 they are asked to denote the number of the squares they have recognized, whereas in question 2 they have to name each figure that was recognized as a square. Question 2 is used in order to check whether students have recognized figures that were, indeed, squares and did not include a false recognition of figures. From a mathematical point of view the correct solution to this task was the correct recognition of all the seven squares that are included in the given figure (PE2).

	Look at the figure carefully and answer the questions: 1. How many squares can you distinguish in the figure? 2. Name the squares in the given figure.

In fact it is possible that students can recognize all the squares, at least the two most obvious squares (FXRS and SRPO) or recognize less than seven squares, but without any false recognition. That is the reason why a different categorization of students' answers is used. Although this task is about perceptual recognition, not all the students' answers can be considered to be only perceptual recognition, because there can be cases in which someone can either recognize all the squares, can recognize correctly some but not all the squares or can have false recognition.

The case in which someone achieves the correct recognition of all the squares cannot be exclusively considered as mere perceptual recognition, because perceptively it is not possible to see and recognize all the squares. In this case, the person is considered to be at the borders of operative apprehension. On the other hand, within perception when a person can see one square, it is not possible to discriminate a different one, as the recognition of figures functions exclusively. For example, if a figure can be perceived and seen by juxtaposition, it cannot be seen simultaneously by superposition. In addition, within perception the estimation of magnitude is wrong.

Therefore, in the case of correct recognition of all the squares the person is considered to be able to go beyond the perceptual apprehension of the geometrical figure and is not far from operative apprehension. On the contrary, in the case of correct recognition of only some squares the person is not able to go out of the perceptual apprehension and remains within the limits of perceptual recognition.

To this end, three final categories of students' answers can be set:

1) The first category involves the recognition of all the squares in the figure, but without any false recognition. The answers that include all the correct squares only are regarded as successful, because, if there is even one false recognition, the answer is considered to be wrong (Rasq).

2) The second category concerned the perceptual recognition of the included squares FXRS and SRPO, which are considered to be the most obvious squares and, thus, can be recognized more easily compared to the remaining five squares (Risq).

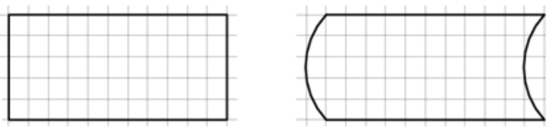
3) The third category is formed by answers including false recognition of squares, even if some squares are recognized correctly. In the case of a rotated square there are more possibilities for false recognition, because from a perceptual point of view a right angle is recognized when there are horizontal and vertical segments; otherwise an illusion in perception is possible to occur. But in the case of the particular figure given in this task, in which no square is rotated, even if one false recognition is provided by the students the answer is considered wrong (Rf).

Operative Apprehension Tasks

The second group of tasks consisted of five tasks that are chosen in examining students' operative apprehension of the geometrical figure. Actually, these tasks test students' ability to modify a geometrical figure, as the tasks ask for a reconfiguration of the given geometrical figure in order to be solved.

For the classification of the students' performance, answers showing a similar approach to a particular question were grouped together. This categorization was done according to the type of cognitive procedures by which the answers occur. Thus, three main approaches were anticipated: 1) answers from the operative apprehension (mereologic modification), 2) answers from the perceptual apprehension, and 3) answers from a different approach.

In the first task students are asked to compare the area of two figures.

Underline the right sentence and explain your answer. a) Fig. A has an equal area to Fig. B b) Fig. A has a smaller area than Fig. B c) Fig. A has a bigger area than Fig. B	 Fig. A Fig. B

Actually, students were expected to perform a reconfiguration of Figure B and realize that, when the front part of the figure is cut and moved to the back part of the figure, a rectangle is formed, similar to Figure A, as shown in the figure below.

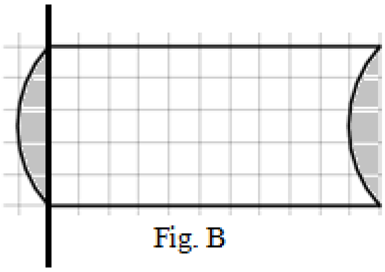


Fig. B

From a mathematical point of view the resolution of the task is considered to be successful in the case students chose answer (a) (OP1). By cognitive analysis of the task students' explanations are categorized according to the type of apprehension that is mobilized. For example, when a student used counting to find the answer, operative apprehension is not activated. On the other hand, when a student adds some new lines on the figure, it cannot be considered as only a perceptual approach, since it is an indication of modifying the figure. A reconfiguration can be made explicitly or mentally (Duval, 1999) and that is related to operative apprehension.

Thus, three categories of answers are set:

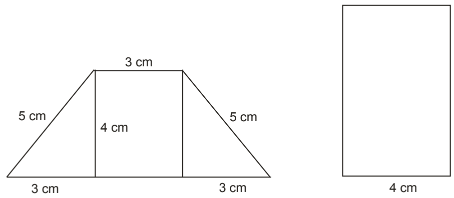
1) OP1me: This category includes right answers that occur from operative apprehension, in which the reconfiguration is explicit. The term "*explicit*" denotes a certainty that students have made a reconfiguration either by drawing extra lines in the figure, showing which parts must be "cut" and change place or by verbal description of the modifications made mentally.

2) OP1pe: In this category fall answers that occur from the perceptual apprehension, for example students may be trying to calculate the perimeter by estimating the length of each sector.

3) OP1da: In this case students' right answers that occurred from a different approach are grouped.

In the second task students are asked to find the length of the missing side of the rectangle, based on the fact that the area of the rectangle is equal to the area of the trapezium.

The trapezium and the rectangle have equal areas. Find the length of the missing side of the rectangle and explain your answer.



What is expected in this task is that students will apply a mereologic modification on the figure and make a reconfiguration. In other words, students are expected to see that if the trapezium is reconfigured by moving one of the two triangles and joining them properly, a rectangle is formed. In this way they will be able to find the answer only by visualization, without using any formulas or performing any calculations.

From a mathematical point of view, the correct answer is that the missing side equals to 6cm, independently of the way the solution is reached (OP2). However, there were cases in which students behaved in a different way during resolution of the task, to the one expected. In fact, in this task it is significant to distinguish the students that are able to recognize the proper reconfiguration from those who are not and go through the calculation of the area of the trapezium. What is important to see is that the strategy is different for those who have operative apprehension and those who don’t have and, hence, use a more complex strategy. Consequently, answers are discriminated into three categories:

1) OP2me: This group includes right answers that occur from operative apprehension and the mereologic modification of the figure is explicit.

2) OP2pe: In this case right answers that occur from perceptual apprehension are grouped. Students recognized the trapezium as a whole figure and answers are linked to the use of the formula of the area of the trapezium.

3) Op2da: In this case students’ right answers that occurred from a different approach are grouped.

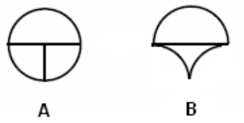
In the third task concerning operative apprehension of the geometrical figure students are asked to compare the perimeter of two figures. In this task it was anticipated that the students’ answers would occur by the application of a mereologic modification of the figures, i.e. students should make a reconfiguration of the figures by dividing then into parts and moving these parts in order to get two identical figures and check the equivalence of their perimeter.

Look at the figures and circle the correct answer. Then explain your answer.

a) Figure A has a bigger perimeter than figure B.

b) Figure A has an equal perimeter to figure B.

c) Figure A has a smaller perimeter than figure B.



The correct answer from the mathematical point of view is the choice of answer (b) since the perimeter of the two figures is the same (OP3).

From the cognitive point of view, there are two categories of answers:

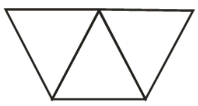
1) OP3me: Right answers that occur from operative apprehension and in which the mereologic modification of the figure is explicit are grouped. As already mentioned, students' reconfiguration can be explicit by the addition of extra lines on the figures or by the verbal explanation of the reconfiguration that they make.

2) OP3pe: Right answers that occur from perceptual apprehension are included in this category. Such a solution is based on perceptual apprehension and, thus, on an effort to calculate the perimeter by estimating the length of each sector.

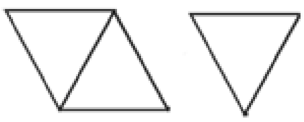
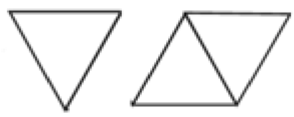
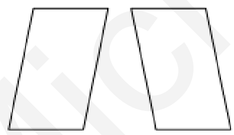
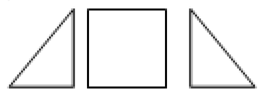
3) OP3da: this group includes right answers that occurred from a different approach.

4) The fourth task of this group demands, also, a mereologic modification of the figure. In this case students were asked to divide the given figure into different parts and combine these parts in different possible ways.

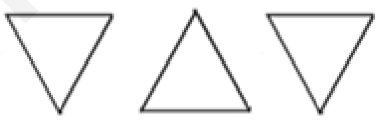
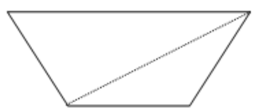
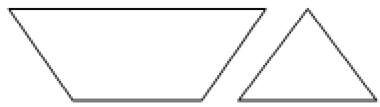
Draw the shapes you think that were combined in order to form the following figure. You can present more than one solution.



The four possible solutions that occurred from operative apprehension and reconfiguration of the figure are displayed below. So, success in this task from a mathematical point of view is determined by the number of solutions students were able to provide (OP4). Actually in this task students were expected to provide four solutions and thus they have to find four different reconfigurations for the same figure. For succeeding this students had to make many and different combinations of the subfigures included in the given figure and use each subfigure in a different way each time in order to come to a new answer. Therefore it is assumed that for the solution of this task a higher level of abilities in modifying a given figure through the involvement of the operative apprehension in necessary.

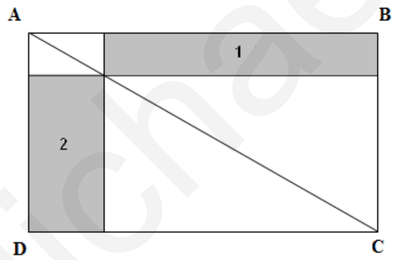
Solution 1	Solution 2	Solution 3	Solution 3
			

Besides the four expected solutions that occurred due to operative apprehension of the geometrical figure, students provided different solutions also, related to the involvement of perceptual apprehension. Actually, the first solution is related to perceptive juxtaposition (OP4pe1), solution 2 to perception of the global shape (OP4pe2) and solution 3 to perceptive superposition of the figure (OP4pe3). These mean score for these three answers were computed and express by the variable OP4pe.

Solution 1(OP4pe1)	Solution 2 (OP4pe2)	Solution 3 (OP4pe3)
		

The last task used in order to examine students' operative apprehension includes a rectangle, which is divided into different subfigures (triangles and rectangles). In figure

ABCD the diagonal AC divides the rectangle into two equal right-angled triangles (ADC and ABC). Each of these triangles includes two other right-angled triangles, which occur from the division of two rectangles included in figure ABCD respectively. Thus, from the triangles ADC and ABC equal parts are subtracted. Consequently, rectangle 1 and rectangle 2 have an equal area. For such a solution procedure operative apprehension is needed in order for someone to be able to discriminate the different reconfigurations in the figure and realize the relations between these subfigures. Students were expected to go through these reconfigurations and finally conclude that the two rectangles have the same area.

Figure ABCD is a rectangle. Look at the shadowed rectangles 1 and 2 and choose the correct answer. Then justify your choice. a. Rectangle 1 has a bigger area than rectangle 2. b. Rectangle 1 has an equal area to rectangle 2. c. Rectangle 1 has a smaller area than rectangle 2.	

From a mathematical point of view, choice (b) is considered to be the correct answer (OP5). However, students' answers were not always related to a justification occurring from operative apprehension. According to the cognitive point of view, the following categorization of students' answers was used:

1) OP5me: In this category right answers that occurred from a mereologic argument are grouped (e.g. "*I have two right-angled triangles and I take out equal parts from each triangle*").

2) OP5pe: in this case right answers that occured from a composition argument are included (e.g. students' justifications including compensatory relations between the two shadowed triangles, like "*Rectangle 1 is long and narrow, but rectangle 2 is short and wider*", "*if rectangle 1 is divided into two equal parts and these two parts are joined together, then we get rectangle 2*").

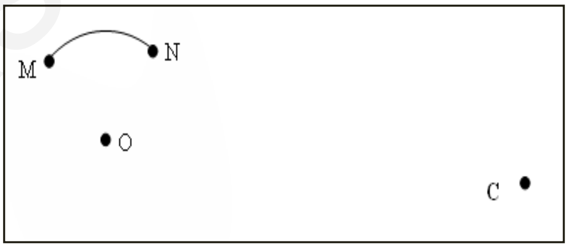
3) OP5da: This group includes right answers that occurred from the use of a different approach, as there were students who tried to measure the sides of the two rectangles and had to compare and calculate their area.

Sequential Apprehension Tasks

The third group consists of tasks examining students' sequential figure apprehension. Particularly it comprise of tasks testing students' ability to construct a figure and describe its construction.

The first task concerns students' ability to construct a figure based on given data. Students' success in this task is determined according to whether the arc they construct is, indeed, equal to the one given (SE1). Students are also asked to describe the way they worked for the construction of the geometrical figure, in order to ensure that they followed a mathematically correct procedure for their construction and that they didn't draw a random arc. Based on this description, answers in which the constructed figure seems perceptually similar to the correct one, but the procedure for its construction is wrong, were grouped as SE1ps. A last category is formed by incorrect constructions, in which students have mainly drawn a random arc (SE1ns).

Draw an arc AB with centre C, equal to arc MN with centre O. Then describe the figure's construction.



In the second sequential apprehension task students had to use the provided data properly, in order to construct a correct figure.

In a triangle ABC, point E is situated on the segment AC. The circle with diameter AE intersects side AB at point Z. Provided that AB=6, AC=6,5 , BC=2,5 and AE= $\frac{4}{5}$ AC, draw the figure using a ruler and compass.

First, they had to draw triangle ABC according to the dimensions provided for each side. Then they had to calculate the length of side AE, which is the diameter of a circle and then they had to find the midpoint of side AE and draw the circle. Finally, they had to mark the point where the circle intersects the triangle on side AB (point Z). Students' construction was taken to be correct in the case that their figure was similar to the one shown in the figure below (SE2).

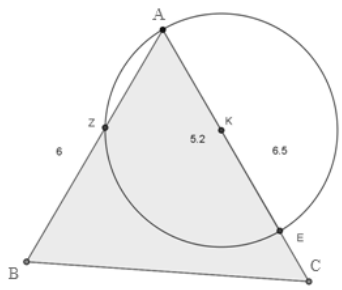


Figure 8. The correct construction in the second sequential apprehension task

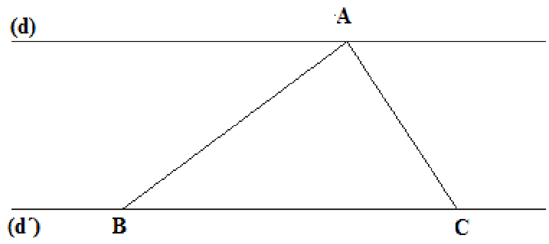
In this task it is important to define what the main difficulty in that construction is. So in the case when students do not succeed, what is the most difficult step of the construction? What was hypothesized is that the most difficult part will be related to the construction of the triangle.

Therefore, students' constructions that were not correct were categorized as shown below:

1) SE2pc: This group includes constructions that are partly correct, for example constructions in which students succeed at least the right construction of the triangle.

2) SE2ps: In this group fall constructions with no success at all and especially those who construct a wrong triangle. This can be connected with a perceptual approach: the construction can be wrong but students may produce a figure similar to the correct one. The figure can be similar, but the conditions for its construction are wrong.

The third of the sequential apprehension tasks asked students to construct a parallelogram having equal area to the area of the triangle ABC.

In the next figure lines (d) and (d′) are parallel. Construct two points M and N, so that the quadrilateral AMCN is a parallelogram with an area equal to the area of triangle ABC. Explain the way you worked for your construction.	

This task can be solved in two ways, which are both correct (SE3). In the first way students draw a line passing through the midpoint of the height of the triangle. Taking side BC as a base and the distance from BC to line e, a variety of parallelogram is created, whose area equals the area of the triangle ABC (Solution 1). In the second way students had to find the midpoint of the base BC (point O). Taking the distance between the lines d and d′ as height and the segment BO (Solution 2a) or OC (Solution 2b) as a base, a parallelogram is formed having its points M and N on line d. In this task students are also asked to provide an explanation for the way they worked for the construction of the figure. What must be taken into account is whether the relevant properties that are used are named or not and if they are named in a correct order. Besides the cases of a correct construction, for the cognitive analysis of the task the perceptual answers of students are also grouped. Thus, a variable (SE3ps) is created by grouping productions in which students construct a parallelogram by drawing only one line or joining one vertex of the triangle to the opposite parallel line. The last variable (SE3ns) consists of wrong constructions related to approaches, such as the use of irrelevant formulas or wrong measurements.

Discursive Apprehension Tasks

The fourth group of tasks comprises tasks that are considered relevant in examining students’ discursive apprehension of the geometrical figure. In these tasks the focus is on students’ inferences, which constitute the indications for the existence of discursive apprehension.

The group of the discursive apprehension tasks includes a pair of tasks, whose particularity is that they are used not only in order to examine students' discursive apprehension of the geometrical figure, but also the '*Generality of proof*' as defined by Kunimune, Fujita and Jones (2008). Specifically, these two tasks include three different types of proof: an empirical proof, a semi – empirical proof and a formal proof. Students are asked to state whether they accepted each type as a proof. Thus, this pair of task examines the degree of students' awareness concerning the discrepancy between valid reasoning and non-valid reasoning. It is significant to mention that this pair of tasks was not administered to students at the same time, since each task was included in a different part of the test. Thus, the resolution of the first task of each pair could not influence the resolution of the second task.

In the first discursive apprehension task students were asked to choose one of the four answers provided, but also to explain the way they had worked in order to find the answer. Thus, students' answers were correct in the case they proved that NH=MH (DI1).

In the triangle ABC: AB and AC are equal, line (e) is parallel to BC and AH is perpendicular to BC. Choose the right answer, based on the data provided.

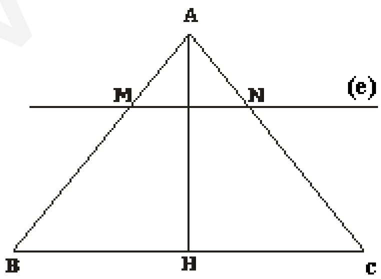
The length of NH:

a) is equal to MH

b) is bigger than MH

c) is smaller than MH

d) it cannot be determined



Explain your answer.

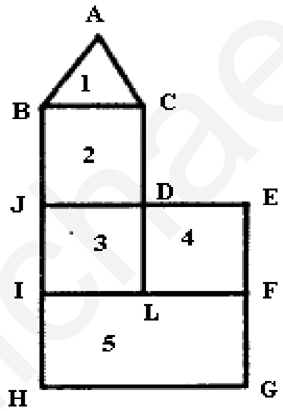
The students' inferences are decisive in the discursive apprehension tasks. Actually, the resolution of this task involves knowledge of the properties of isosceles triangles and of Thales' theorem. So, on the basis of students' justifications, the following categories were formed:

1) DI1cj: The first category comprises correct answers that occur by a correct justification.

2) DI1nj: The second category comprises correct answers for which no justification is provided.

3) DI1wj: In the case students provide a correct answer but a wrong justification they are added in the third category.

In the second discursive apprehension task students were given some information about some figures that are included in the whole figure and they had to prove that three particular sides of the figure are equal. Thus, students were expected to make an inference, based on the properties of the geometrical figures and prove the equity of the three sides (DI2).

In the given figure: • figure 1 is an equilateral triangle • figure 2 is a rectangle • figures 3 and 4 are squares • the figure that consists of figures 3, 4 and 5 is a square. Show that sides AC, LF and FG are equal.	

Besides the cognitive procedures related to the discursive apprehension and inference, the presence of the geometrical figure involves cognitive procedures that are related to another kind of geometrical figure apprehension as well. This task is resolved with inference but perceptual apprehension also intervenes, because students have to recognize each figure that is mentioned in the instructions and use its properties. Therefore, what is also important, besides inference, is transitivity. Transitivity can be described either verbally or be shown by marking the sides on the figure. For this reason students' answers are codified also for the cognitive point of view, according to the way transitivity is made explicit. Three categories of answers are, thus, discriminated:

1) DI2vr: Here fall correct answers whose success is related to the explicit visual recognition of transitivity. These can be solutions in which students draw on the figure to show the equal segments.

2) DI2vei: Solutions are included in which success can be achieved by verbal indication of transitivity.

3) DI2vrvei: this category includes answers in which both ways are used in achieving the answers – visual recognition and verbal indication of transitivity.

In the third discursive apprehension task three different students' answers are provided, which are trying to prove that the sum of the internal angles of a triangle is 180°. Student A provides a semi empirical – proof, as he/she measures the angles of a triangle and finds the sum. Student B gives an empirical proof, by drawing a triangle, cutting each of his angles and then joining them. This student realizes that a straight line is formed and, thus, proves that the sum of the interior angles of a triangle equals 180°. A formal proof is given by Student C, based on the properties of the parallel lines. The student shows that the angles A1 - B2 and A3 - C are equal because they are alternate interior angles. In this way he/she proves that angles A1, A2 and A3 form a right angle and then shows that the three interior angles of the triangle are also equal to 180°.

In this task students were asked whether they accepted each student's answer as a proof or not and then they had to explain why. Since this task is related to proof and thus is considered to be a discursive apprehension task, students were expected to make inference and proper reasoning, which would bring them to choose the answer of student C (DI3).

Read the following explanations by three students who demonstrate why the sum of the interior angles of a triangle is 180°.

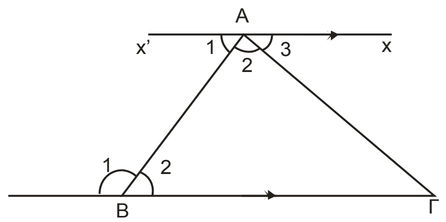
Student A: 'I measured each angle, and they are 50°, 53 °and 77°. 50+53+77=180. Therefore, the sum is 180° '.

- Do you accept the explanation of student A as a proof? Yes/ No …………

Student B: 'I drew a triangle, I cut out each angle and I put them together. They formed a straight angle. Therefore, the sum is 180 °.'

- Do you accept the explanation of student B as a proof? Yes/ No …………

Student C: Demonstration by using properties of parallel lines



xx' // BΓ $\Rightarrow \hat{A}_1 = \hat{B}_2$ and $\hat{A}_3 = \hat{\Gamma} \Rightarrow \hat{A}_1 + \hat{A}_2 + \hat{A}_3 = 180° \Rightarrow \hat{B}_2 + \hat{A}_2 + \hat{\Gamma} = 180°$

- Do you accept the explanation of student C as a proof? Yes/ No …………

However, other types of geometrical figure apprehension seem to be involved after all and influence students' choice of the acceptable type of proof. So, besides the correct answer from a mathematical point of view, it is important to approach the task also from the cognitive point of view and check the effect of other types of apprehension in students' awareness of the right reasoning.

The three solutions presented in this task, apart from corresponding to a different type of proof, also come from a different kind of geometrical figure apprehension:

1) Student A: This answer is based on measuring and computation and these approaches are not related to visualization, but rather to perception (DI3pe).

2) Student B: This answer is based on reconfiguration. The student divides the figure into parts and combines these parts in order to form a straight angle. Thus, operative apprehension of the geometrical figure is involved (DI3op).

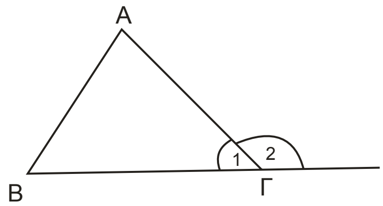
3) Student C: This answer is based on discursive apprehension because properties are used and inference is performed (DI3di).

Similar to the previous task, the discursive apprehension task 4 includes answers of three different students, who try to prove that the exterior angle of a triangle equals the sum of the angles that are opposite it. Student A gives a semi empirical proof, as he measures the angles in order to prove that they are equal. Student B gives an empirical proof, by cutting out the two interior angles and putting them together in the exterior angle to prove their equity. Student C provides a formal proof, because he is based on the use of two parallel lines. The student draws a line parallel to one side of the triangle. This parallel line divides the exterior angle into two angles. Then he proves that these two new angles are equal to the two angles that are opposite the exterior angle of the triangle, by using the properties of the parallel lines (corresponding angles/ alternate interior angles).

Similarly, as in task 3, students had to decide whether they accepted each student's answer as a proof or not and then to explain why. Students were again expected to choose the answer of student C (DI4).

Read the following explanations of three students, who explain why the exterior angle of a triangle equals the sum of the angles that are opposite it.

Student A: I measured the three angles of the triangle. A and B, which are 54° and 80° respectively. The external angle Γ2 is 134°. So we have 54°+80°=134°. Thus the external angle of a triangle equals the sum of the angles that are opposite to it.



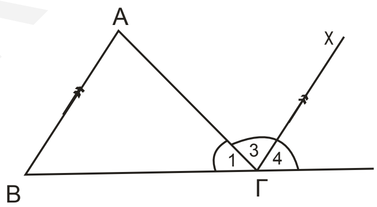
- Do you accept the explanation of student A as a proof? Yes/ No …………

Student B: I drew a triangle, I cut angles A and B and I put them together, placing them at the top of the external angle Γ2. They completed exactly external angle Γ2. Therefore, the external angle of a triangle equals the sum of the angles that are opposite to it.

-

- Do you accept the explanation of student B as a proof? Yes/ No …………

Student C: Explanation based on the use of two parallel lines.



We draw $\Gamma\chi \,//\, AB \Rightarrow \hat{A} = \hat{\Gamma}_3$ *and* $\hat{B} = \hat{\Gamma}_4 \Rightarrow \hat{A} + \hat{B} = \hat{\Gamma}_3 + \hat{\Gamma}_4 \Rightarrow \hat{A} + \hat{B} = \hat{\Gamma}_{\varepsilon\xi.}$

- Do you accept the explanation of student C as a proof? Yes/ No …………

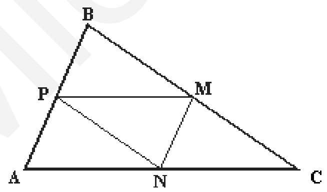
In this task the same codification of answers is used, as in the previous corresponding task. The three solutions considered by the cognitive point of view are the following:

1) Student A: This answer is based on measuring and computation and, thus, on perception (DI4pe).

2) Student B: This answer is based on reconfiguration. The student divides the figure into parts and combines them in order to compare the three angles. This answer results from the involvement of the operative apprehension of the geometrical figure (DI4op).

3) Student C: This answer is based on the discursive apprehension, as inference is made and because it includes the use of properties (DI4di).

In the fifth discursive apprehension task students were expected to make inference using Thales' theorem (intercept theorem). Thus, they were expected to prove by explaining that the segment that joints two midpoints, is parallel to the opposite segment and equal to its half. In this task the mathematically correct answers were proofs by using the intercept theorem for each of the three parallelograms (DI5).

In the triangle ABC, M, N and P are the midpoints of its sides. Prove that the quadrilaterals APMN, BMNP, and GNPM are parallelograms.	

For the cognitive analysis of the task each inference must be codified separately. What is actually needed for a proper mobilization of discursive apprehension is the following sequence of inferences for one of the parallelograms:

1) Inference 1 by using the intercept theorem for the 1st parallelogram

2) Inference 2 by using the intercept theorem

3) Inference 3 by using the definition of parallelogram from the two previous conclusions

This sequence was scored as shown in Table 6.

Table 6

Scoring for the Discursive Apprehension Task DI5

	Right	Wrong	Nothing
1. Inference 1 by using intercept theorem for the 1st parallelogram	2	1	0
2. Inference 2 by using the intercept theorem	2	1	0
3. Inference 3 by using the definition of parallelogram from the two previous conclusions	2	1	0

Thus, the variables of the cognitive analysis of this task are triplets which are considered to be one variable. The formed triplets are displayed in Table 7.

Table 7

Variables for the Discursive Apprehension Task DI5

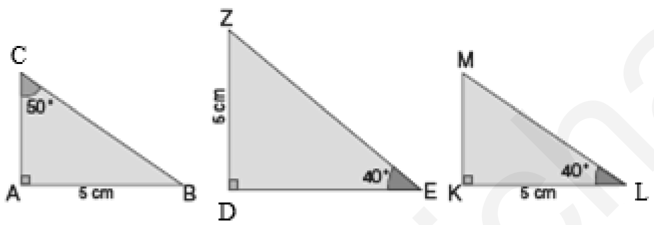
Variables		Triplets
1. DI5c	comprehension of step deduction	2-2-2
2. DI5cg	comprehension of step deduction, but a gap	2-2-0
3. DI5wa	wrong approach	1-0-0 or 2-0-0

There can be two cases of what is called a "wrong approach". The first is related to the irrelevant use of properties or of the theorem, and the reason for this is mainly perceptual apprehension. Perception overrides all of the recognition and, thus, the perception of a figure can be an obstacle to the recognition of the relevant theorem. The second concerns the case in which students just mention the except theorem, but no further reasoning occurs.

For variables 1 and 2 there is mobilization of discursive apprehension. For triplet 3 students may have no idea of reasoning and, thus, they can be completely confused. They do not realize what is to justify and there is no discursive apprehension because there are no inferences at all.

The last discursive apprehension task includes three triangles, two of which are equal. Students were asked to find the two triangles that were equal and explain their answer. Their answer was correct if they had found that the equal triangles are the first and the third (DI6). Students were also expected to make inference using the properties of triangles regarding the sum of the interior angles and also the criteria for the equity of triangles.

Find the two triangles that are equal and explain your answer.



Similarly, in the rest of the task of this group what is decisive is students' inference. Consequently, students' right answers had to be seen in relation to the justification students provided.

1) DI6ci: In this category right answers with correct inference are grouped. These are answers including the use of the properties of the triangle and the criteria for the equity of triangles are found.

2) DI6wi: This category includes right answers with wrong inference, thus answers that are not related to a proper justification.

3) DI6ni: In this category right answers with no inference, and, thus with no justification are included. If students don't give any justification, this can be because of the influence of the perceptual apprehension, as they see that the two figures are the same and they feel no need for proving it.

Reliability of the research instrument

In examining the reliability of the research instrument that was constructed in order to measure the students' geometrical figure apprehension, the Cronbach Alpha was calculated. The value of the Cronbach Alpha was higher than 0.75, therefore it can be considered satisfactory (Cronbach's Alpha = 0.803, $p < 0.05$) (Cronbach, 1990).

Variables of the research and scoring of the test

Codes serve to summarize, synthesize, and sort many observations made of the data. Coding becomes the fundamental means of developing the analysis. Researchers use codes to pull together and categorize a series of otherwise discrete events, statements, and observations which they identify in the data. At first the data may appear to be a mass of confusing, unrelated accounts. But by studying and coding the researcher begins to create order (Zhang & Wildemuth, 2009).

During correction of the students' answers in the tasks of the tests, 1 was used for scoring the correct answers and 0 was given in the cases where the answer was wrong or there was no answer. The results concerning students' answers to the tasks were codified with PE, OP, SE and DI corresponding to perceptual, operative, sequential and discursive apprehension respectively. The following table (Table 9) presents the variables that were used for the mathematical and cognitive analysis of the tasks.

Table 8

The Variables of the Study

Perceptual Apprehension tasks	
Tasks 1 – 2	
PE1	Mathematically correct answer
R1	Correct recognition of all the 9 figures
R2	Correct recognition of 8 or 7 figures
R3	Correct recognition of 6 or 5 figures
R4	Correct recognition of below 5 figures
Task 3	
PE2	Mathematically correct answer

Rasq	Recognition of all the squares
Risq	Recognition of included squares
Rf	False recognition
Operative Apprehension tasks	
Task1	
OP1	Mathematically correct answer
OP1me	Mereologic solution
OP1pe	Perceptual solution
OP1da	Different approach
Task 2	
OP2	Mathematically correct answer
OP2me	Mereologic solution
OP2pe	Perceptual solution
OP2da	Different approach
Task 3	
OP3	Mathematically correct answer
OP3me	Mereologic solution
OP3pe	Perceptual solution
OP3da	Different approach
Task 4	
OP4	Mathematically correct answer
OP4rf1	Reconfiguration – solution 1
OP4rf2	Reconfiguration – solution 2
OP4rf3	Reconfiguration – solution 3
OP4pe1	Perceptual recognition – solution 1
OP4pe2	Perceptual recognition – solution 2

OP4pe3	Perceptual recognition – solution 3
Task 5	
OP5	Mathematically correct answer
OP5me	Mereologic argument
OP5pe	Perceptual (composition) argument
OP5da	Different approach (e.g. measurement argument)
Sequential Apprehension tasks	
Task 1	
SE1	Correct construction
SE1ps	Perceptual solution
SE1ns	Constructions with no success
Task 2	
SE2	Correct construction
SE2pc	Constructions that are partly correct
SE2ps	Perceptual solution
Task 3	
SE3	Correct construction
SE3ps	Perceptual solution
SE3ns	Constructions with no success
Discursive Apprehension tasks	
Task 1	
DI1	Mathematically correct answer
DI1cj	Right answer with correct justification
DI1nj	Right answer with no justification
DI1wj	Right answer with wrong justification

Task 2	
DI2	Mathematically correct answer
DI2vr	Visual recognition of transitivity
DI2vei	Verbal indication of transitivity
DI2vrvei	Visual recognition and verbal indication of transitivity
Task 3	
DI3	Mathematically correct answer
DI3pe	Answer based on perceptual apprehension
DI3op	Answer based on operative apprehension
DI3di	Answer based on discursive apprehension
Task 4	
DI4	Mathematically correct answer
DI4pe	Answer based on perceptual apprehension
DI4op	Answer based on operative apprehension
DI4di	Answer based on discursive apprehension
Task 5	
DI5	Mathematically correct answer
DI5c	Comprehension of step deduction
DI5cg	Comprehension of step deduction, but a gap
DI5wa	Wrong approach
Task 6	
DI6	Mathematically correct answer
DI6ci	Right answer with correct inference
DI6wi	Right answer with wrong inference
DI6ni	Right answer with no inference

The research design of the task – based interviews with students

Qualitative research aims to gather an in-depth understanding of human behavior and the reasons that govern such behavior. The task – based interviews will be conducted in order to gain an insight into students' strategies, difficulties and misconceptions during the resolution of selected tasks, as students will have the chance to explain their answers verbally to the researcher.

The qualitative method investigates the *why* and *how* of decision making, hence, smaller but focused samples are more often needed than large samples (Flyvbjerg, 2006). Thus, the interviews were conducted with only some students, who were chosen on the basis of the results of the statistical analyses, according to their level of achievement. As mentioned earlier, the participants were 3 students from grade 9, grade 10 and grade 11 respectively, of medium and high abilities.

In the interviews students were given each of the selected tasks separately in a written form. They resolved each task and after its solution they had the chance to explain their thoughts and the way they had worked during performance of the tasks. So, particular questions were posed in order to help students express and explain their thinking process and solution practices. Some representative questions that were used were the following:

– How would you explain your solution to a classmate?

– How difficult was the solution of the task?

– Did you encounter any particular difficulties during the solution of the task?

– Did you use the given figure during the solution of the task?

– Was the given figure helpful or not for the solution of the task?

After the questions concerning the solution of each task, general questions were posed to students about all the tasks they were given during the interviews. In particular, the questions were about characterizing the type of the tasks, for example whether the tasks were interesting, boring, tiring or different from those usually given to students at school. Furthermore students were asked to express their opinion about the teaching of Geometry they receive at school, for example whether they were satisfied with the current way of teaching and what they would like to change. In addition, students were encouraged to express any other difficulties they encountered in geometry.

The duration of each interview was approximately 40 minutes and each student was interviewed by the researcher separately. The interviews were audio – taped.

Analysis of the data

This research design includes a combination of both quantitative and qualitative approaches. In real research work, the two approaches are not mutually exclusive and can be used in combination. As suggested by Smith (1975), "qualitative analysis deals with the forms and antecedent-consequent patterns of form, while quantitative analysis deals with duration and frequency of form".

Thus the quantitative data were collected by the administration of the test. The task – based interviews allowed collection of the qualitative data of the research. The collected data of both types were analyzed by using different techniques and statistical software. The quantitative and qualitative analyses of the data of this research are explained below in a more detailed manner.

Quantitative analysis

Structural Equation Modeling and CFA

Confirmatory factor analysis (CFA), by using the EQS program, was performed for exploring the structural organization of the various dimensions of the geometrical figure apprehension (Bentler, 1995).

Structural equation modeling (SEM) is a statistical methodology that takes a hypothesis testing (i.e. confirmatory) approach to the multivariate analysis of a structural theory bearing on some phenomenon (Byrne, 1994). This theory concerns causal relations among multiple variables (Bentler, 1988). These relations are represented by structural, namely regression equations, which can be modeled in a pictorial way to allow a better conceptualization of the theory involved.

SEM differs from the more traditional multivariate statistical techniques in at least three dimensions: First, with the use of SEM analysis of the data is approached in a confirmatory manner rather than in an exploratory way, making hypothesis testing more accessible and easier, compared with other multivariate procedures. Second, whereas SEM gives the estimates of measurement errors, the conventional multivariate methods cannot assess or correct for these parameters. Third, SEM involves not only observed but also latent (unobserved) variables, whereas the older techniques incorporate only observed measurements.

CFA is used in situations where the researcher aims to test statistically whether a hypothesized linkage pattern between the observed variables and their underlying factors exists. This a priori hypothesis draws on knowledge of related theory and past empirical work in the area of the study. In this case the knowledge comes from the theory of Duval (1995, 2005) about the geometrical figure apprehension.

CFA allows the researcher to test the hypothesis that a relationship between the observed variables and their underlying latent construct(s) exists. The researcher uses knowledge of the theory, empirical research, or both, postulates the relationship pattern a priori and then tests the hypothesis statistically. Traditional statistical methods normally utilize one statistical test to determine the significance of the analysis. However, Structural Equation Modeling, CFA specifically, relies on several statistical tests to determine the adequacy of model fit to the data. The chi-square test indicates the extent of difference between expected and observed covariance matrices. A chi-square value close to zero indicates little difference between the expected and observed covariance matrices. In addition, the probability level must be greater than 0.05 when chi-square is close to zero (Suhr, 2006).

The basic steps that a researcher follows in carrying out CFA are described below: The model is specified, based on knowledge of relevant theory and previous empirical research. Using a model-fitting program, such as EQS, the model is analyzed so that the estimates of the model's parameters with the data are derived. Then the tenability of the model is tested based on data that involve all the observed variables of the model (Byrne, 1994; Kline, 1998).

The number of levels that the latent factors are away from the observed variables determines whether a factor model is called a first-order, a second-order or a higher order model. Correspondingly, factors one level removed from the observed variables are labeled first-order factors while higher-order factors which are hypothesized to account for the variance and co-variance related to the first-order factors are termed second-order factors.

A second or a higher order factor does not have its own set of measured variables. In this study a second-order model will be considered.

QUEST software

The Rasch model analysis (Andrich, 1988) was used for determining the degree of difficulty the tasks and for creating a good interval level measure for the lower and upper school students' geometrical figure apprehension. The estimation of the Rasch models was conducted with the use of the QUEST software (Adams & Khoo, 1996).

SPSS Statistical Package

The descriptive analysis was performed by using the SPSS statistical package. The descriptive analysis provided information about the students' percentages of correct or wrong answers, the different approaches employed and the categorization of the students' justifications and wrong answers. The SPSS was used also for performing the multivariate analysis of variance (MANOVA), in order to examine the differences in the students' performances in the different types of tasks according to the educational level and according to their age.

Implicative Statistical Analysis and Hierarchical Clustering of Variables

For the analysis of the collected data, the hierarchical clustering of variables and Gras's implicative statistical method were also conducted using the computer software called C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive) (Bodin, Coutourier, & Gras, 2000). These methods of analysis determine the hierarchical similarity connections and the implicative relations of the variables respectively (Gras, 1992, 1996). For the needs of this study, similarity and implicative diagrams were produced from the application of the analyses on the sample of students.

The hierarchical clustering of variables (Lerman, 1981) is a classification method which aims to identify in a set V of variables, sections of V, less and less subtle, established in an ascending manner. These sections are represented in a hierarchically constructed diagram using a similarity statistical criterion among the variables. The similarity stems from the intersection of the set V of variables with a set E of subjects (or objects). This kind of analysis allows the researcher to study and interpret clusters of variables in terms of typology and decreasing resemblance. The clusters are established in particular levels of the diagram and can be compared with others. This aggregation may be attributed to the conceptual character of every group of variables.

In particular, the method used here is the 'likelihood linkage analysis' (LLA) (Lerman, 1991). LLA is a methodology for grouping data into significant classes and subclasses, using an algorithm of hierarchical classification. This method introduces a most original notion of statistics for measuring statistical relationships and proximities, namely the "likelihood" concept. Lerman (1991) sets up the "likelihood" notion as part of the "resemblance" notion. The flexibility of this method enables taking into account any combinatorial and logical structure of which the modality set of a given descriptive variable is provided.

The construction of the hierarchical similarity diagram is based on the following process: Two of the variables that are most similar to each other with respect to the similarity indices of the method are joined together in a group at the highest (first) similarity level. Next, this group may be linked with one variable in a lower similarity level or two other variables that are combined together and establish another group at a lower level, etc. This grouping process goes on until the similarity or the cohesion between the variables or the groups of variables becomes very weak. In this study the similarity diagrams allowed for the arrangement of the variables, which correspond to students' responses in the tasks of the tests, into groups according to their homogeneity.

The implicative statistical analysis (Gras, 1996; Gras, Peter, Briand & Philippe, 1997) aims at giving a statistical meaning to expressions like: *"if we observe variable A in a subject, then in general we observe variable B in the same subject".* Thus, the underlying principle of the implicative analysis is based on the quasi-implication: "*if A is true, then B is more or less true"*. An implicative diagram represents graphically the network of the quasi-implicative relations among the variables of the set V. In this study the implicative diagrams will contain implicative relations, indicating whether success at a specific task implies success at another task related to the former.

Qualitative data facilitate a better interpretation of the results of the statistical analyses, allowing the formation of more comprehensive conclusions. Task-based interviews were conducted with nine students, in order to triangulate the quantitative data. Thus, the students' transcripts were analyzed so that their thinking processes, misunderstandings and difficulties would be highlighted.

Analyzing qualitative data is a process that consists of three parts (Seidel, 1995) (Figure 9):

1) Noticing: it means making observations, writing field notes, tape recording interviews, gathering documents, etc. Once a record is produced, the focus of attention is turned to noticing interesting things in the record.

2) Collecting: as interesting things are noticed and named, the next step is collecting and sorting them. Sorting and sifting through the data is the primary path to analysis.

3) Thinking about interesting things: in the thinking process the things that you are collected are examined.

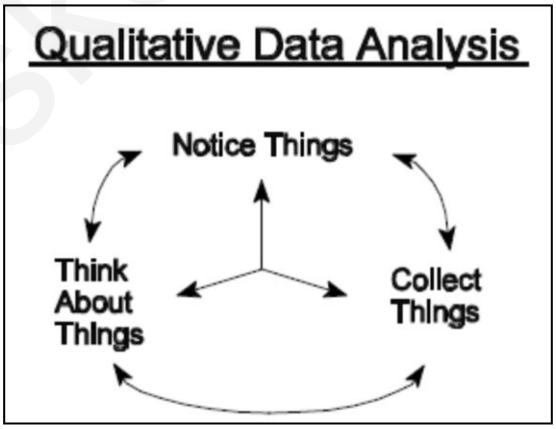


Figure 9. The process of analyzing qualitative data and the relationships among its parts

For the analysis of the qualitative data of the research the method of Qualitative Content Analysis was used. Qualitative content analysis is a valuable alternative to more traditional quantitative content analysis, when the researcher is working in an interpretive paradigm. The goal is to identify important themes or categories within a body of content, and to provide a rich description of the social reality created by those themes/categories as they are lived out in a particular setting (Zhang & Wildemuth, 2009).

Qualitative content analysis has been defined as “a research method for the subjective interpretation of the content of text data through the systematic classification process of coding and identifying themes or patterns” (Hsieh & Shannon, 2005) and as “an approach of empirical, methodological controlled analysis of texts within their context of communication, following content analytic rules and step by step models, without rash quantification” (Mayring, 2002). Patton (2002) provides another definition according to which it is “any qualitative data reduction and sense-making effort that takes a volume of qualitative material and attempts to identify core consistencies and meanings”.

Qualitative content analysis involves a process designed to condense raw data into categories or themes based on valid inference and interpretation. Through careful data preparation, coding, and interpretation, the results of qualitative content analysis can support the development of new theories and models, as well as validating existing theories and providing thick broad descriptions of particular settings or phenomena (Zhang & Wildemuth, 2009).

Zhang and Wildemuth (2009) describe the process of Qualitative content analysis in eight steps:

1) Prepare the data
2) Define the unit of analysis
3) Develop categories and a coding scheme
4) Test your coding scheme on a sample of text
5) Code all the text
6) Assess your coding consistency
7) Draw conclusions from the coded data
8) Report your methods and findings.

Summary

The present research study was conducted in seven phases. In the first phase the literature review regarding geometrical thinking and geometrical abilities was conducted, while in the second phase the geometry curriculum and the geometry textbooks used for the teaching of geometry in Cyprus, from grade 9 to grade 11 were examined. Then the research instrument of the study was developed, which comprised of four groups of tasks, each one corresponding to the perceptual, the operative, the sequential and the discursive apprehension.

The test was administered to 881 students, aged 15 to 17, of lower (Grade 9) and upper (Grade 10, Grade 11) urban and rural secondary schools, in Cyprus. In particular, the participants were 312 students from Grade 9, 304 students from Grade 10, 125 students from Grade 11a and 140 students from Grade 11b. The task – based interviews with 9 students followed, in order to triangulate the quantitative data regarding the students geometrical figure apprehension.

At a further stage the data were analyzed, with the use of different software and statistical packages. The confirmatory factor analysis (CFA), by was used for examining the structural organization of the various dimensions of the geometrical figure apprehension. The SPSS statistical package was used for the descriptive analysis of the data and for tracing the differences in the students' performances in the different types of tasks according to the educational level and according to their age, by the multivariate analysis of variance (MANOVA). The hierarchical similarity connections and the implicative relations between the variables were examined using the hierarchical clustering of variables and Gras's implicative statistical method through the computer software C.H.I.C. For the qualitative data the Qualitative Content Analysis was used.

Based on the results that occurred through the analysis of the qualitative and the quantitative data, finally the final conclusions were extracted, discussed and interpreted, in reference to the literature review and to the outcomes of previous research, and teaching implications, but recommendations for further research as well were given.

CHAPTER IV

RESULTS

Introduction

This chapter presents the results that were extracted from the statistical analyses of the data, which were performed in order to answer the research questions of the study. The chapter is organized in four subchapters; each one focusing on the four axes of investigation for this study.

The first subchapter includes the results related to the investigation of the cognitive structure of the geometrical figure apprehension in the lower and the upper secondary school. In fact, the structural model of the geometrical figure apprehension is described, through which the important role of the perceptual, the operative, the sequential and the discursive apprehension was revealed. Furthermore, the invariance of this structure is shown in relation to the students' age or educational level.

The second subchapter comprises the results answering the questions on the relationships among the four types of geometrical figure apprehension. These relations were examined with the use of the hierarchical clustering of variables and the similarity analysis of the students' answers. However, the examination of these relations was conducted not only based on students' answers considered from the mathematical point of view, but was also combined with an examination of the students' answers according to the cognitive point of view.

The next subchapter includes the description of the results that occurred regarding the comparison between the lower and the upper secondary school students' geometrical figure apprehension. This comparison was done based on the descriptive statistics of students' responses to the geometrical figure apprehension tasks from the mathematical and the cognitive point of view, on the examination of the effect of age and the educational level in the students' geometrical figure apprehension as well as the hierarchical classification of the geometrical figure apprehension tasks according to the degree of difficulty.

In the last subchapter the results of the students' mistakes and ideas about the geometrical figure apprehension are described. The students' wrong answers and mistakes in the tasks for each type of apprehension are analyzed while the similarity and implicative relations among the students' mistakes in the tasks concerning the recognition of proof and their responses to the tasks about the production of proofs were traced. These similarity and implicative relations constitute the basis of the identification of the students' geometrical paradigm. Finally, the results of the semi-structured interviews with lower and upper secondary school students are included in this subchapter.

The cognitive structure of the geometrical figure apprehension in the lower and upper secondary school

Confirmatory Factor Analysis and Development of a Structural Model of Geometrical Figure Apprehension

The exploration of the structural organization of the geometrical figure apprehension was conducted through the use of the Confirmatory Factor Analysis (CFA). A structural equation model involves two basic types of components: the variables and the processes or relations among the variables. A schematic representation of a model, which is termed path diagram, provides a visual interpretation of the relations that are hypothesized to hold among the variables under study.

A series of CFA models regarding the geometrical figure apprehension were tested and compared, in order to come to a model which would fit the data better than other models. The tenability of a model can be determined by using the following measures of goodness of fit: x^2, CFI (Comparative Fit Index) and RMSEA (Root Mean Square of Approximation). The following values of the three indices are needed to hold true in order to support an adequate fit of the model: x^2/df < 2, CFI > 0.9, RMSEA < 0.06. If the hypothesized model is not consistent with the data, the model is re-specified and the fit of the revised model with the same data is evaluated (Byrne, 1994; Kline, 1998).

Three models of the structure of the geometrical figure apprehension were tested in order to ensure that the proposed theoretical model (Figure 10) fit the data better than other models. The first-order factor structure of the geometrical figure apprehension was first investigated in order to determine whether four specific factors are needed to explain variability of performance on the tasks, or whether a single latent factor would suffice to explain this variability better. Specifically, the first model involved only one first-order factor associated with all the tasks. This model tests the assumption that a single common source of variance is sufficient to account for performance on all tasks addressed to the participants of the research. This model was the most parsimonious; however, it disregarded the related theory and past empirical work which pointed out that, different cognitive processes are needed in order to solve the tasks regarding the perceptual, the operative, the sequential and the discursive apprehension respectively. The fit of this model was poor [CFI=0. 951, $x^2(44) = 8.243$, RMSEA=0.30].

Thus, in line with the theory on which the research was based, a four-factor model was then tested where the scores representing students' performance in the four groups of tasks were prescribed to load on a separate factor. In fact the second model that was constructed and tested involved four first-order factors corresponding to the perceptual, operative, sequential and discursive apprehension. However, the fit of the second model was not acceptable [CFI= 0.411, x^2 (42) = 450.445, RMSEA= 0.105].

The last model that was examined was the one hypothesized to describe the structure of the geometrical figure apprehension for lower and upper secondary school students (Figure 11). Particularly a second order model was constructed and tested, which involved four first-order factors and one second-order factor. The four first-order factors corresponded to the perceptual, operative, sequential and discursive apprehension respectively and were regressed on the second-order factor standing for the geometrical figure apprehension. A chi-square difference test indicated a significant improvement in fit between the first and the third model [Δx^2 (7) =21.401, $p<0.005$] due to the second-order factor inclusion. Besides, the fit of the third model was acceptable [CFI= 0.971, x^2 (37) = 56.842, RMSEA= 0.25]. By comparing the second-order factor model with the theoretical first-order factor model, a small decrease of the RMSEA (i.e. from 0.030 to 0.025) and an increase of the CFI (i.e. from 0.951 to 0.971) are identified. Thus, the second-order factor model was considered preferable for both statistical reasons and reasons of parsimony (Maruyama, 1998).

It should be mentioned that particular tasks were omitted from the final model due to the fact that the factor loadings indicating their regression on the first order factors were very low. Particularly these are two of the most difficult operative apprehension tasks (OP2 and OP4) according to students' performance in them. Specifically task OP4 was the most difficult task in the category of tasks examining students' operative apprehension. In this task students had to provide four solutions by finding four different reconfigurations for the same figure. In order to succeed, such a high number of and different combinations of the subfigures included in the given figure were necessary. Task OP2 was the second most difficult operative apprehension task for the total number of students. The solutions of this task involved the mobilization of the operative apprehension, but also the involvement of the perceptual apprehension was possible combined to the use of formulas and calculations. The existence of numbers in this task can be a factor that differentiated it from the remaining operative apprehension tasks, because it may have inhibited the mobilization of the operative apprehension and reinforced the intervention of the perceptual apprehension.

Some of the tasks examining students' discursive apprehension were also omitted from the final model (DI3, DI4 and DI6). In fact these tasks can ultimately be considered a group of geometrical tasks which have indirect characteristics of geometrical proof. Particularly tasks DI3 and DI4 examine students' ability regarding the recognition of proof and not the production of a proof. In addition for the solution of task DI6 the production of a proof can be related to the perceptual apprehension, due to the given figures, which possibly influenced the mobilization of the discursive apprehension in many cases.

The final model of the study is presented in Figure 11. It is a second-order model consisting of four first-order factors and one second order factor. The four first-order factors PE.A., OP.A., SE.A. and DI.A. correspond to the perceptual, operative, sequential and discursive apprehensions respectively. The perceptual, the operative, the sequential and the discursive apprehensions are regressed on the second order factor G.F.A. that stands for the geometrical figure apprehension. The loadings of the four first order factors on the second order factor are high, indicating that all the types of apprehension influence the apprehension of geometrical figures. In fact, the factor loadings for the perceptual, the operative and the sequential apprehensions are similar. On the other hand the discursive apprehension is more strongly regressed on the second order factor, as the corresponding factor loading is the highest of all the remaining factor loadings of the first-order factors on the second order factor.

On the first-order factor PE.A, which stands for perceptual apprehension, the tasks examining the ability to recognize and name figures is regressed (PE1 and PE2). The first-order factor OP.A that represents the operative apprehension involves the tasks demanding the heuristic reconfiguration of figures and thus the mobilization of the operative apprehension (OP1, OP3 and OP5). The first-order factor SE.A consists of tasks examining students' ability to construct geometrical figures (SE1, SE2 and SE3). The proof tasks demanding inferences based on given data or based on the knowledge of theorems and definitions (DI1, DI2 and DI5) are regressed on the first order factor DI.A, which stands for the discursive apprehension of the geometrical figure. The regression of each group of tasks on a different first order factor reveals that different cognitive processes take place when each different type of apprehension is mobilized for the solution of the tasks.

To test whether a differentiation exists in the structure described above regarding the geometrical figure apprehension according to students' educational level or age, the multiple group confirmatory factor analysis was performed. The multiple-group CFA compares groups within the latent variable measurement model context, adjusting for measurement errors, correlated residuals, and so forth. Multiple-group CFA involves simultaneous CFAs in two or more groups, using separate variance – covariance matrices (or raw data) for each group. The equivalence or invariance of measurement can be tested by placing equality constraints on parameters in the groups. Equality constraints require parts of the model to be equivalent across groups (Brown, 2006). Therefore with the multiple group CFA, the invariance of the structure of the model for grades 9, 10 and 11 separately was firstly examined and then the analysis was used for testing whether the model has the same structure for the lower and upper secondary school respectively as well.

Firstly, the examination of the model was based on the hypothesis that the loadings of the observed variables on the first order factors and the loadings of the first order factors on the second order factor are equal in the models for the lower and upper secondary school respectively. The application of the multiple group CFA showed that the fit of the model under the particular constraints was acceptable [CFI= 0.940, x^2 (83) = 119.505, RMSEA= 0.31]. Thus the results are in line with the hypothesis that the same geometrical figure apprehension structure holds true for both lower and upper secondary school students (Figure 11).

In the particular model the majority of the loadings of the observed variables on the first order factors are higher for the upper secondary school students. It is also noteworthy that some loadings of the first order factors on the second order factor are higher in the group of the upper school students suggesting that the specific structural organization potency increases across the two educational levels. This is the case for the factor loadings of the operative and the discursive apprehension. On the other hand, the factor loading regarding the sequential apprehension of the lower secondary school students is lower than the factor loading regarding the sequential apprehension of the upper secondary school students. The factor loadings for the perceptual apprehension appear to be the same for the two groups of students.

Subsequently, the model was examined according to the hypothesis that the loadings of the observed variables on the first order factors and the loadings of the first order factors on the second order factor are equal in the models for the students in grade 9, the students in grade 10 and the students in grade 11. The results confirm that the structure of the geometrical figure apprehension is invariant for all the different groups of students (Figure 12), according to their age, as the model that came about from the application of the multiple group CFA is acceptable [CFI= 0.969, x^2 (129) = 161.075, RMSEA= 0.22]. For the majority of the loadings of the observed variables on the first order factors a decrease is observed from grade 9 to grade 11, whereas an increase occurs from grade 10 to grade 11. However the higher loadings of almost all the observed variables on the first order factors are found in grade 11. Regarding the loadings of the first order factors on the second order factor, these factors are higher as we move towards a higher grade of discursive apprehension. The factor loading for the discursive apprehension for grade 10 is higher than the corresponding factor loading for grade 9 and the factor loading for grade 10 is higher than the factor loading for grade 11. The opposite happens regarding sequential apprehension, as the factor loadings are lower from one grade to the next one. Specifically the highest factor loading is observed for grade 9 students. The factor loading for grade 10 students is lower than the one for 9th graders, whereas it is lower than the factor loading for grade 11 students. In addition there are fluctuations regarding the loading of the perceptual apprehension on the second order factor between the different groups of students. Specifically the factor loading of the perceptual apprehension for grade 9 is higher than for grade 10, but the factor loading for grade 10 is not higher that the factor loading for grade 11. However the factor loadings for grade 9 and grade 11 are very similar. Finally the factor loading of the operative apprehension for grade 9 is the lowest compared to the other two groups of students. For grades 10 and 11 the corresponding factor loadings are also very similar.

The results show that the structure of the geometrical figure apprehension is not influenced by students' age or educational level, as it remains invariant in the different groups of students. The fit indices of all the models that were examined and described above are presented in table 9.

Table 9

The Fit Indices of the CFA Models

CFA Model	X^2	df	X^2/df	CFI	RMSEA
Model with a first order factor for the total sample	78.243 p=.001	44	1.778	.951	0.30 (.019, .040)
Model with four first order factors for the total sample	450.445 p=.001	42	10.745	.411	.105 (.096, .114)
Model with four first order factors and a second order factor for the total sample	56.842 p=.001	37	1.536	.971	.025 (.010, .037)
Model with four first order factors and a second order factor for each educational level	119.505 p=.001	83	1.440	.940	.031 (.018, .043)
Model with four first order factors and a second order factor for each grade	161.075 p=.001	129	1.249	.969	.022 (.008, .033)

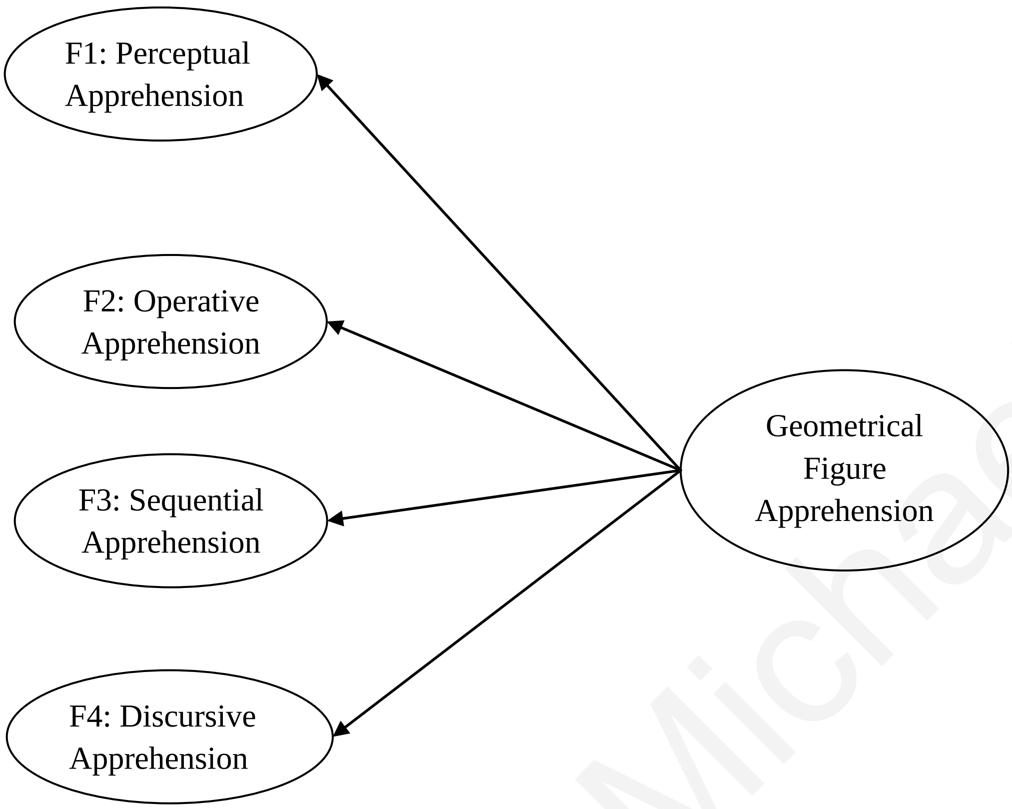
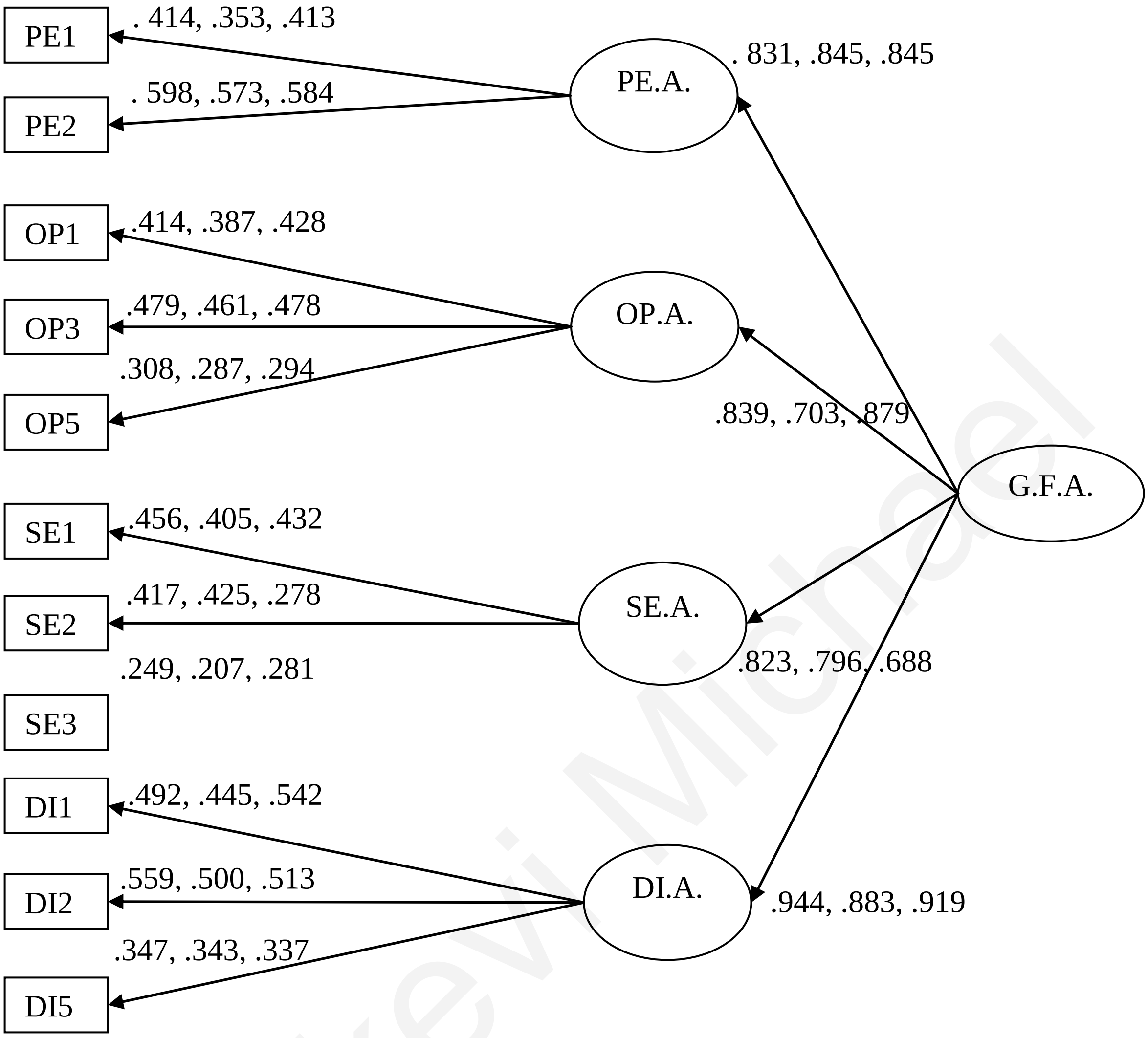
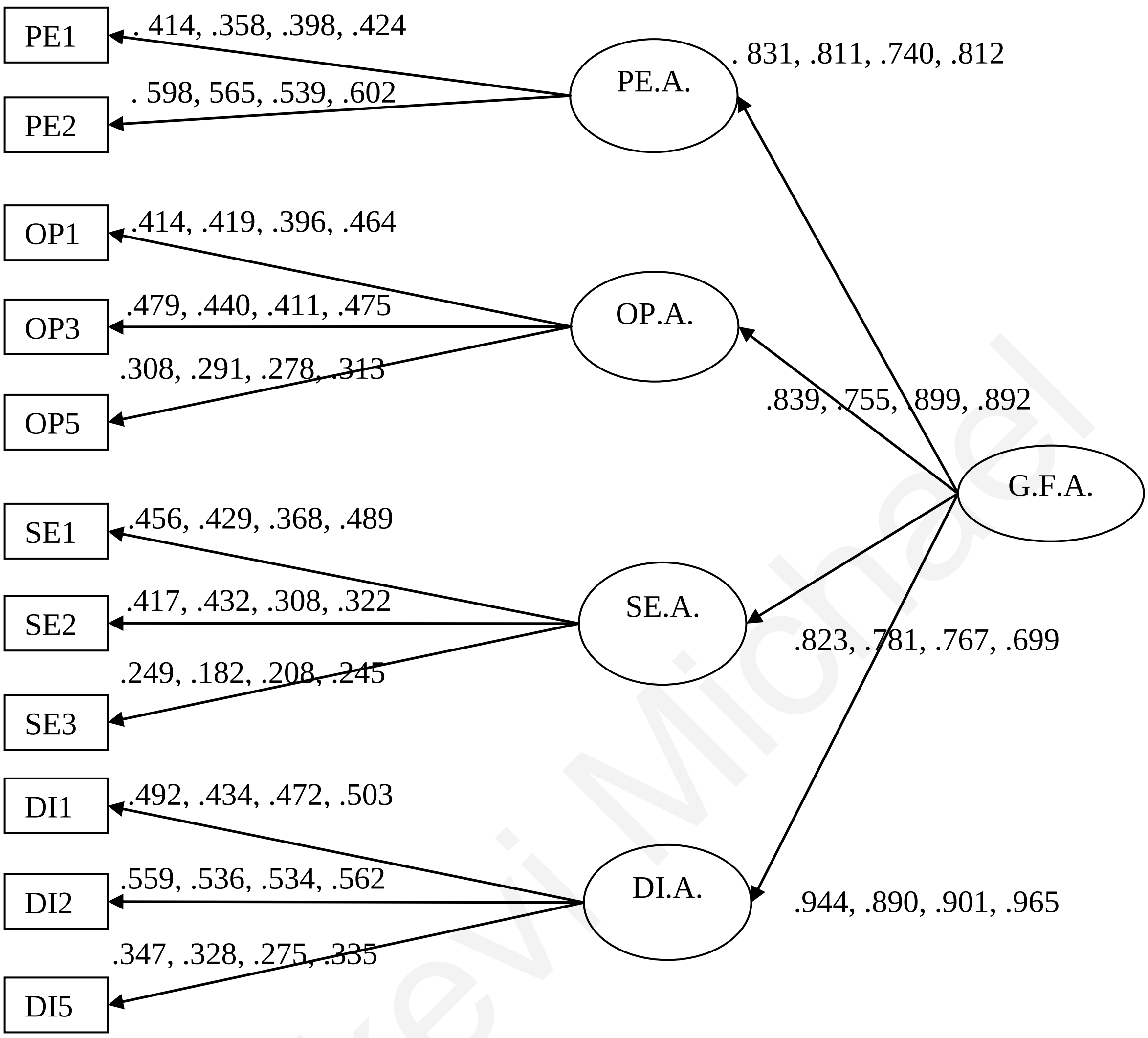


Figure 10. The proposed model of the structure of the geometrical figure apprehension



Explanation of symbols: PE1, PE2, OP1, OP2, OP3, OP4, OP5, SE1, SE2, SE3, DI1, D12, D5 = the tasks of the test, PE.A. = perceptual apprehension, OP.A. = operative apprehension, SE.A. = sequential apprehension, DI.A. = discursive apprehension, G.F.A.= geometrical figure apprehension. *Note*: The four numbers denote the factor loadings for the total sample, for lower secondary school students and for upper secondary school students respectively.

Figure 11. The CFA model for the geometrical figure apprehension for each educational level separately



Explanation of symbols: PE1, PE2, OP1, OP2, OP3, OP4, OP5, SE1, SE2, SE3, DI1, D12, D5 = the tasks of the test, PE.A. = perceptual apprehension, OP.A. = operative apprehension, SE.A. = sequential apprehension, DI.A. = discursive apprehension, G.F.A.= geometrical figure apprehension. *Note:* The four numbers denote the factor loadings for the total sample, for grade 9, for grade 10 and for grade 11 respectively.

Figure 12. The CFA model for the geometrical figure apprehension for each grade separately

The relationships between the four types of geometrical figures apprehension

In this section the data was analyzed using the computer software CHIC, with which the hierarchical clustering of variables and the implicative analysis were performed. Firstly the similarity relations among the students' answers are described from the mathematical point of view for the total of the students and for each of the groups separately, by age and by educational level, in order to trace similarities and differences between them. Next the implicative relations between the students' answers are presented in the same form.

Similarity relations among students' answers from the mathematical point of view

The similarity diagram for the total of students is displayed in Figure 13. The similarity diagram consists of three similarity clusters. The first similarity cluster is formed according to the relations between the solutions of the perceptual apprehension tasks and some of the operative and the discursive apprehension tasks. Specifically, in this first cluster there are two sub-groups which are significantly related. The first sub-group indicates the similarity between the solution of a perceptual apprehension task (PE1) in which students were asked to recognize some coded geometrical figures that were included in a divided geometrical figure and a discursive apprehension task (DI6) that asked students to find the equal triangles among three different triangles. Task DI6 was considered a discursive apprehension task, as students were asked to prove the equity of the triangles using the given properties. However the fact that the geometrical figures were given to students may have facilitated the mobilization of the perceptual apprehension, as the two equal triangles could be identified through perception. Therefore the relation between this proof task and a perceptual task can be attributed to the presence of the geometrical figure that reinforced the influence of perception in students' inferences.

In the second sub-group of this similarity cluster there is a significant similarity relation among a perceptual task (PE2), a discursive apprehension task (DI5) and two operative apprehension tasks (OP1 and OP2). The relation between task PE2 and the operative apprehension tasks can be attributed to the fact that the cognitive processes involved for the success in this task seem to exceed the borders of perceptual apprehension and to be close to operative apprehension. In fact in task PE2 students were asked to recognize seven squares in a largest square with different subfigures. Although this task regards the perceptual apprehension, the case in which someone achieves the correct recognition of all the squares cannot be exclusively considered mere perceptual recognition, because it is not possible to see and recognize all the squares perceptively. Within perception when a person can see one square, it is not possible to discriminate a different one, as the recognition of figures functions exclusively. For example if a figure can be perceived and seen by juxtaposition, it cannot be seen simultaneously by superposition. In the case of correct recognition of all the squares the person is considered to be able to go beyond the perceptual apprehension of the geometrical figure. On the contrary, in the case of the correct recognition of only some squares the person is not able to go beyond perceptual apprehension and remains within the limits of perceptual recognition. Similarly, concerning the similarity relation of task DI5 with the operative apprehension tasks, this can be explained by the fact that the mobilization of operative apprehension was necessary for identifying the necessary relations between the figural units of the given geometrical figure in order students to be able to prove. In fact, this task asked students to prove that three sub-figures of a trapezium were parallelograms, using a theorem, and by tracing relations between particular figural units. Therefore the reconfiguration of the figure was a necessary process for recognizing the proper properties that should be used for proving.

The second similarity cluster includes the solutions of the sequential apprehension tasks (SE1, SE2 and SE3), which form similarity relations with two operative apprehension tasks (OP3 and OP4) and two discursive apprehension tasks (DI1 and DI2). In fact there is a significant relation between the discursive apprehension tasks and a sequential apprehension task. However the sequential apprehension tasks are also related to the operative apprehension tasks. Thereafter in this similarity cluster significant interrelations among the three types of apprehension are revealed.

In the last similarity cluster an operative apprehension task (OP5) and two discursive apprehension tasks (DI3 and DI4) are grouped. In fact, the operative apprehension task OP5 is a task directly related to proof, due to fact that it involves the use of properties in order to make the necessary reconfiguration that leads to the right solution. This group of variables indicates once again the significant relation between operative apprehension and discursive apprehension, and specifically the role of operative apprehension not only for the production, but also for the recognition of proofs. Overall, the two last groups of variables of the similarity diagram can be considered as the groups that include the use of mathematical properties either for geometrical constructions or for inferences and geometrical proofs.

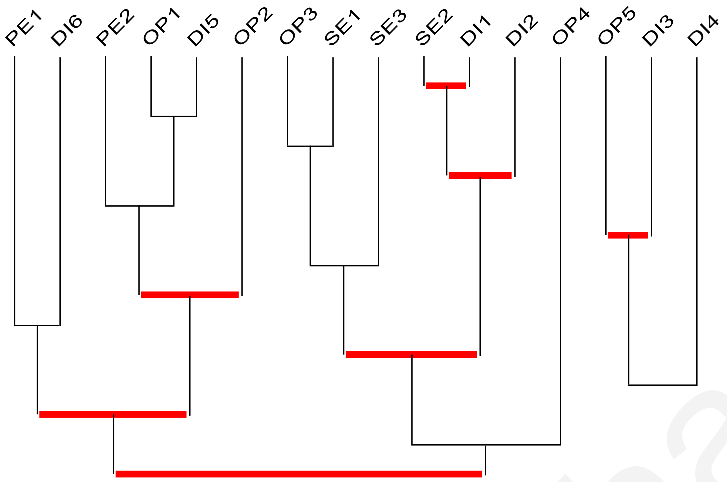


Figure 13. Similarity diagrams on the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for the total sample

Figure 14 presents the similarity diagrams for each group of students according to their grade. The similarity diagram for 9th graders comprises two similarity clusters. In fact the first cluster is divided into two subgroups of variables, which are weakly related. So these two subgroups can be almost taken as two distinct groups and therefore the similarity diagram for grade 9 can ultimately be considered to comprise three similarity clusters. The first similarity cluster can be characterized as the "perceptual group" of tasks. This is because this group includes the two perceptual apprehension tasks (PE1 and PE2) and the operative apprehension tasks OP1 and OP2, in which the role of perception is very important for their solution. In effect, in these two tasks students firstly have to recognize the parts with the same areas perceptively and then perform the mereologic modification on the given figures. In addition this group includes the DI5 and OP4 tasks, whose common characteristic is that the given geometrical figures are almost the same. In fact in these figures students have to discriminate among different triangles with the mobilization of perception and combine them and then to identify different parallelograms with the involvement of operative apprehension.

The second similarity cluster can be considered as the group of geometrical proofs and constructions, in which the sequential apprehension tasks (SE1, SE2, SE3) are related to the discursive apprehension tasks (DI2, DI2). In the sequential apprehension tasks the construction of the geometrical figures is based not only on technical constraints, but also on mathematical properties. On the other hand, the discursive apprehension tasks of this group concern the production of geometrical proofs, based on properties, axioms etc. In task DI1 students have to produce a proof using a theorem and for the solution of task DI2 the proof is based on the properties of geometrical shapes (equilateral triangle, rectangles and squares) resulting from their definitions. Finally this group includes two of the operative apprehension tasks that are related directly or indirectly to geometrical proofs. Specifically task OP5 is directly related to geometrical proofs, as the identification of the equal parts in the figure results from geometrical reasoning based on the property of the diagonal of a rectangle, which divides the rectangle in two equal triangles. Regarding the OP3task, this task is indirectly related to proof, as students are asked to show that the two figures have equal areas, either by using a formula or by modifying the figures.

The third similarity cluster is the group in which three discursive apprehension tasks are related (DI3, DI4, DI6). This group of tasks can be characterized as a group of geometrical tasks with indirect characteristics of geometrical proof. In fact in tasks DI3 and DI4 students have to choose the proper justifications and not to produce a proof themselves. Also in task DI6 the production of the proof can be based on the perception of the figure, which leads to the procedural application of the theorem on the equity of triangles. This assumption is also enhanced by the results describing the way the 9^{th} graders have actually answered in this task. The percentage of students that gave a right answer to this tasks in combination to a proper justification (12.18%) is lower than the percentage of students whose right answer was given with a wrong justification (26.92%) or no justification at all (11.22%). According to the cognitive analysis of the task, the influence of perception can be responsible for the lack of need to justify and prove or even for the wrong justification students provide.

From this diagram two important observations emerge. Firstly, perception constitutes a basic dimension of any geometrical activity, as it appears to trigger geometrical reasoning which leads to a geometrical proof and the modifications that provide a heuristic function to geometrical figures. On the other hand the tasks that do not demand an explicit geometrical proof (third similarity cluster) are not necessarily related either to perception, to construction processes or to the heuristic functioning of geometrical figures.

The similarity diagram concerning the responses of the 10^{th} graders in the tasks of the test consists of three similarity clusters. The first similarity cluster is divided into two sub-groups. The first sub-group includes the relations among a perceptual task (PE1), an operative apprehension task (OP1) and a discursive apprehension task (DI5). The relation between the task OP1 and DI5 is also found in the similarity diagram for grade 9 students, although this relation is stronger and more significant in grade 10. As explained previously, the correct inference for the proof in task DI5 is based on the mereologic modification of the figure, which allows students to see the proper reconfigurations on which the inference is based. This can justify the relation among these tasks. On the other hand, perception is crucial for the solution of these tasks, as the recognition of particular parts of the figure is prior to the application of the mereologic modification on the figure.

The other sub-group is formed by the significant relation between the discursive apprehension task DI6 and the perceptual apprehension task PE2. As previously mentioned, the solution of task DI6 can be linked to the mobilization of the perceptual apprehension, by which the two equal triangles could be recognized without the use of any properties or axioms. In grade 9 this task forms similarity relations with two other discursive apprehension tasks. However in grade 10 the similarity of this task appears to be with a perceptual task, leading to the assumption that the influence of perception in this task was greater for the 10th graders than for the 9th graders. Specifically for grade 9, the students that provide a correct answer with a right justification (8.88%) are fewer than the students that gave a right answer but wrong justification (30.59%) or no justification at all (12.50%).

Therefore, the first similarity cluster of this diagram consists of tasks corresponding to the perceptual apprehension and tasks related to the operative and the discursive apprehension, in which perception is a factor that influences the way these tasks are approached and solved by students in grade 10. So this group of tasks can be characterized as the perceptual group, because the involvement of the perceptual apprehension for the solution of these tasks appears to be intense. The second similarity cluster can be called the operative apprehension group of tasks, as it mainly includes tasks corresponding to the particular type of apprehension. Specifically, this cluster is formed by the similarity relations between the tasks OP2, OP3, OP4 and SE3.

The third similarity is mainly formed by discursive and sequential apprehension tasks. Particularly in this group the tasks D1, D2, D3 and D4 are found, which are related to all the construction tasks SE1, SE2 and SE3. Similar to grade 9, the operative apprehension task OP5 is included in this group, as it is a task which is directly related to proof, due to the fact that it involves the use of properties in order to make the necessary reconfiguration that leads to the right solution. Thus this last group of variables can be taken as the group that includes the use of mathematical properties either for geometrical constructions or for inference and geometrical proofs and therefore it could be related to the cognitive processes of geometrical constructions and geometrical reasoning.

The most important outcomes from the similarity diagram for grade 10 students is firstly that the role of perception appears once again to be essential for the success in the further mobilization of the operative and the discursive apprehension for the solution of the tasks. In addition, differently from grade 9, the three tasks that do not demand an explicit geometrical proof (DI3, DI4 and DI6) form relations with the rest proof tasks or the perceptual tasks.

The grade 11a students' responses in the tasks are also grouped in three similarity clusters in the similarity diagram. It must be mentioned that the variables corresponding to tasks SE2 and DI5 are omitted, due to the very small number of correct answers given by students (1.60% and 2.40% respectively). Regarding the first similarity cluster, it is formed by similarity relations between discursive and operative apprehension with perceptual apprehension. However, compared to the results of grades 9 and 10, these tasks are not the same. In fact, the first similarity cluster includes the significant relation between the two perceptual apprehension tasks PE1 and PE2, which are subsequently related to the operative apprehension task OP5 and the discursive apprehension task DI2. It seems that for the students in grade 11a perception was firstly activated for the solution of these two tasks. It is true that perception was necessary for discriminating the different subfigures in the divided rectangle at first glance in task OP5 and also for the recognition of the different subfigures which the given figure in task DI2 included. Therefore the perception of grade 11a students was more strongly involved and led to the necessary reconfiguration in task OP5 and the proper inference in task DI2. In addition the significant relation between the two perceptual apprehension tasks indicates the strong coherence in the mobilization of perception for these students.

In the second similarity cluster there are relations between tasks corresponding to three different types of apprehension. As a matter of fact, there are two operative apprehension tasks (OP1 and OP2), which both ask for the identification of parts with the same areas and then for the application of the mereologic modification on the given figures. These tasks are connected to the construction task SE3 and the discursive apprehension task DI3. These tasks are next linked to the significant relation between two discursive apprehension tasks, the DI1 and DI4. Although each of these two tasks belongs to different subcategories of proof, in this grade a significant relation exists between them.

This may be an indication that students start to link the cognitive processes needed for inferences and proving in geometry in a more effective way.

A similar situation applies in the last similarity cluster of the diagram, which is also created by similarity relations between operative, sequential and discursive apprehension tasks. Particularly these are the tasks OP3, OP4, SE1 and DI6. Regarding task DI6, in the two previous grades it was related either to the discursive apprehension tasks on the recognition of the proper proof (grade 9), or to perception (grade 10). However, in grade 11a these relations do not exist, but on the contrary the relation is formed with tasks that mainly demand the mobilization of the operative apprehension.

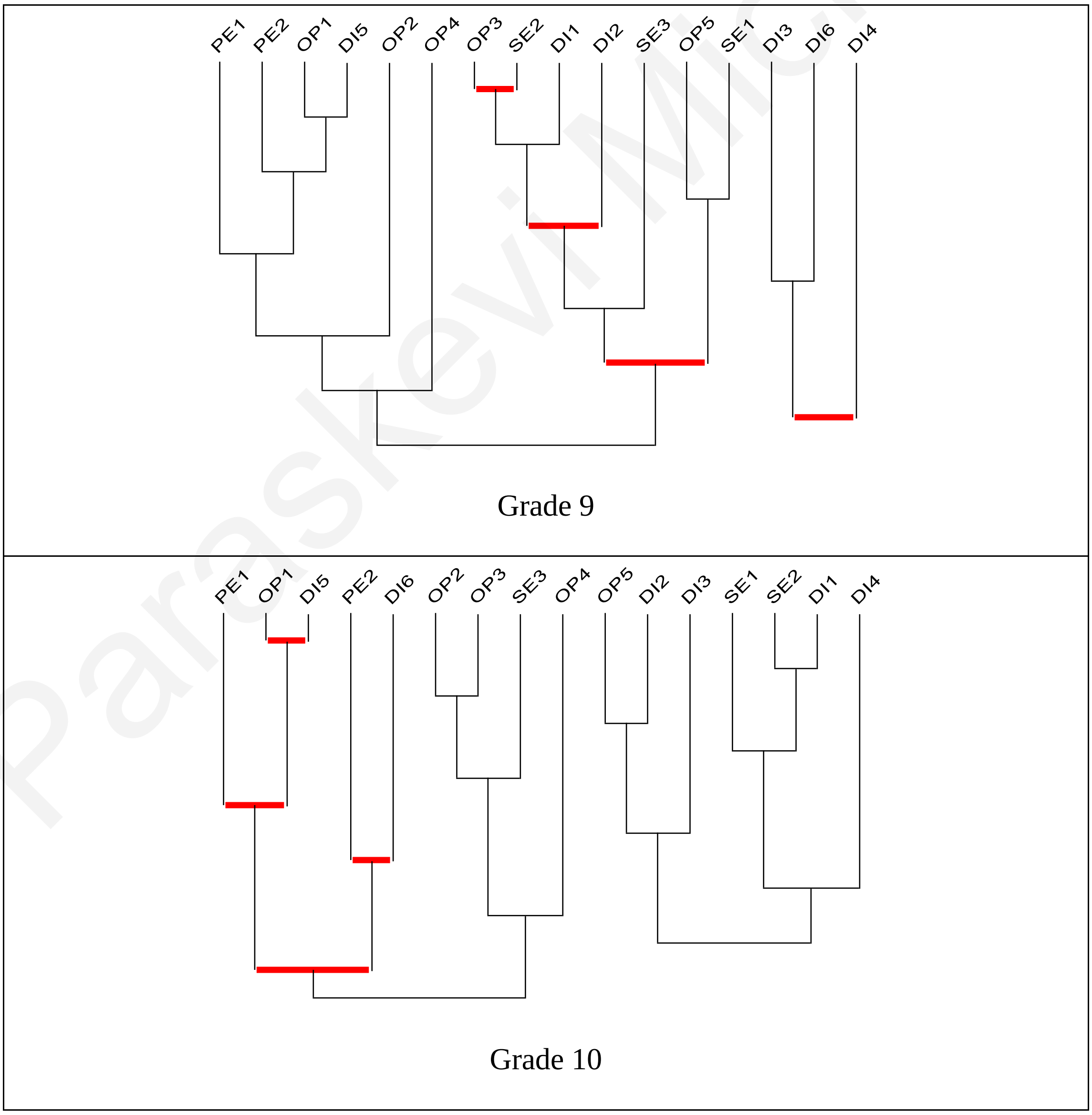
Overall, the differentiation in this grade is that the tasks with no explicit characteristics of geometrical proof (DI3, DI4 and DI6) are linked with the operative apprehension for the first time and not with the other proof tasks or the perceptual tasks, as for grades 9 and 10. Also students of this grade seem to display greater coherence regarding the use of perception for the solution of geometrical tasks. The three cognitive processes related to the discursive, the sequential and the operative apprehension are related, mostly in the second and third clusters and compartmentalization appears for the perceptual apprehension with the rest of the tasks.

The similarity diagram for the grade 11b students' answers in the tasks also consists of three groups of variables. In this diagram new relations between the tasks appear, showing differentiations in the way students of this grade confront the tasks and the different role of each type of apprehension for the solution of the tasks, in relation to the rest of the students.

Starting from the perceptual group of tasks that existed in all the previous diagrams, this group gets a new form in this case. Specifically, the first similarity cluster is created by the perceptual task PE1 and the three discursive tasks that are not directly related to a production of proof (DI3, DI4 and DI6). Thus students at this level are able to distinguish the tasks that demand the production of a proof from those that ask for the identification of the proper proof while the cognitive processes that are involved in each type of task seem to be different for these students.

The second perceptual task PE2 is included in the similarity relations between the discursive apprehension tasks that ask the production of a proof (DI1, D2 and DI5), two of the sequential apprehension task (SE1 and SE2) and one operative apprehension task (OP2), which form the second cluster. This cluster could be characterized as the proof-construction group, as the majority of the tasks correspond to the discursive and the sequential apprehension. The inclusion of the task PE2 in this group can be justified due to the fact that, as explained in the description of the tasks, those who succeed in this task seem to be able to go further than perceptual apprehension and go near the limits of operative apprehension. For grade 11b students this seems to be the case. Therefore in this cluster the cognitive processes related to construction and reasoning are mainly involved, which are also related to the cognitive processes of visualization.

The third cluster is mainly comprised by the operative apprehension tasks (OP1, OP3, OP4 and OP5), which are related to the construction task SE1. Therefore this can be called the "operative apprehension group", indicating the coherence in students' way of solving these tasks. Students display greater stability in the solution of the operative apprehension tasks, which appear to be compartmentalized for the rest of the tasks.



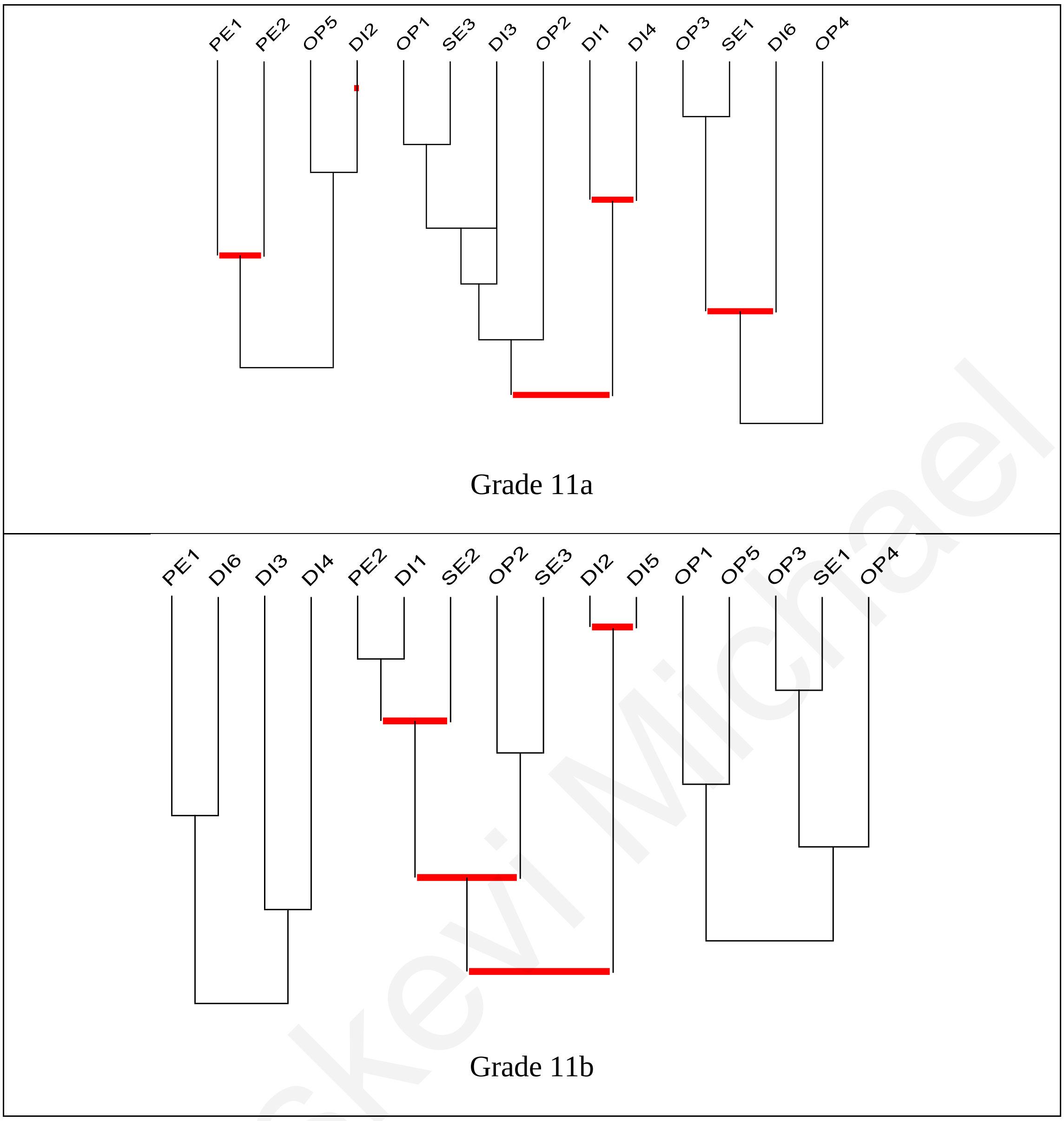


Figure 14. Similarity diagrams for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for each grade

Figure 15 is used for summarizing the invariant relations and the differences between the similarity diagrams for the different groups of students. First of all the tasks in all similarity diagrams are divided into three similarity clusters. In each diagram there is a perceptual group of variables, with some differentiations for each grade. In fact one of the similarity clusters in each diagram includes relations between the perceptual apprehension tasks and some of the operative and the discursive apprehension tasks. However this group is differentiated for each grade and the perceptual apprehension tasks are related with different tasks, showing that some tasks are confronted in a different way by the students from the different age groups.

For grade 9 students there is similarity between the solution of the perceptual apprehension tasks and the OP1, OP2, DI5 and OP4 tasks. For grade 10 students the perceptual apprehension tasks are related with the OP1, DI5 and DI6 tasks. For grade 11a students the similarity relations are between the perceptual apprehension tasks and the OP5 and DI2 tasks. For grade 11b students there is firstly a change regarding the relation between the perceptual apprehension tasks, which are not found in the same similarity cluster, as the other groups of students. In fact each of the two perceptual tasks is included into a different similarity cluster. Actually for this grade the perceptual group includes task PE1 and three discursive apprehension tasks with indirect characteristics of proof. Therefore, in this grade the perceptual group assumes a new form. This new form may be the result of the influence of teaching. On the other hand, the perceptual task PE2 is related to operative, sequential and discursive apprehension tasks.

Another similarity cluster in each diagram includes similarity relations mostly among the discursive apprehension tasks and the sequential apprehension tasks. This group can be characterized as the construction – reasoning group. Furthermore a different group existing in the diagrams of grades 10 and 11b can be named as the visualization group, which mainly involves relations between the operative apprehension tasks. For grade 10 students this group consists of the tasks OP2, OP3, OP4 and the task SE3. For grade 11b students this group is formed by the tasks OP1, OP3, OP4, OP5 and the task SE1. This group of tasks does not form any relation with the rest of the tasks, thus there is a compartmentalization in the solution of the operative apprehension tasks. In addition there is a compartmentalization of reasoning, regarding the cognitive processes involved in the recognition of proofs and the production of proofs in grades 9 and 11b. The transition from lower to upper secondary school seems to influence these students' abilities, which are correctly reformed in grade 11b, in which teaching is different.

	Perceptual group of variables	Construction – reasoning group	Compartmentalization
Grade 9	perceptual apprehension tasks and tasks – OP1 – OP2 – DI5 – OP4	Similarity relations mostly among the discursive apprehension tasks and the sequential apprehension tasks	Reasoning: between the cognitive processes involved in the recognition of proofs and the production of proofs.
Grade 10	perceptual apprehension tasks and tasks – OP1 – DI5 – DI6		Between construction – reasoning and visualization
Grade 11a	perceptual apprehension tasks and tasks – OP5 – DI2		No Compartmentalization
Grade 11b	The perceptual apprehension tasks are not found in the same similarity cluster. The perceptual group includes task PE1 and the tasks – DI3 – DI4 – DI6		1. Reasoning: between the cognitive processes involved in the recognition of proofs and the production of proofs. 2. Operative apprehension tasks.

Figure 15. Invariant relations and differences between the similarity diagrams for the different groups of students

What emerges from the comparison of the diagrams corresponding to the lower and upper secondary school students (Figure 16) is firstly that there seems to be a change in the role of proof and the influence of perception in the solution of the tasks, which differentiates the way the relations between the rest of the tasks are created for the two different educational levels.

As previously described, in the similarity diagram of the students in the lower secondary school the tasks are divided into three similarity clusters. The first similarity cluster can be considered as the "perceptual group" of tasks. The second similarity cluster is characterized as the group of geometrical proofs and constructions, whereas the last group of tasks can be named as a group of geometrical tasks with indirect characteristics of geometrical proof. Therefore in this educational level the solution of the tasks with indirect relation to proof is different from the production of a geometrical proof and no relation exists between these two types of tasks. The use of mathematical properties, axioms and theorems seems to be compartmentalized from the recognition of proofs or from the solution of tasks whose answers can be reached through the mediation of perception.

On the other hand, this is not the case for students from the upper secondary school. In fact in the similarity diagram of upper secondary school students' solutions to the tasks two similarity clusters can be identified. Both clusters include tasks from all the four types of apprehension. However some particular relations appear to be more significant than the rest of the relations. Such a relation is the one between the PE2 and DI6 tasks, indicating the influence of perception in finding the correct answer to task DI6. Overall the upper secondary school students seem to coordinate the cognitive processes related to each type of apprehension in a more effective way.

The second significant relation concerns the tasks SE2, DI1 and DI2, showing the strong relation between the use of mathematical properties in tasks of geometrical constructions and proofs. This relation remains invariant after the transition from lower to upper secondary school, as it also appears in both similarity diagrams. Another invariant relation in the similarity diagrams for the students from the two different educational levels is the one between the PE1, OP1, OP2 and DI5, showing that the involvement of perception remains the same for the solution of particular tasks, even if students are of different age groups and have gained different mathematical knowledge.

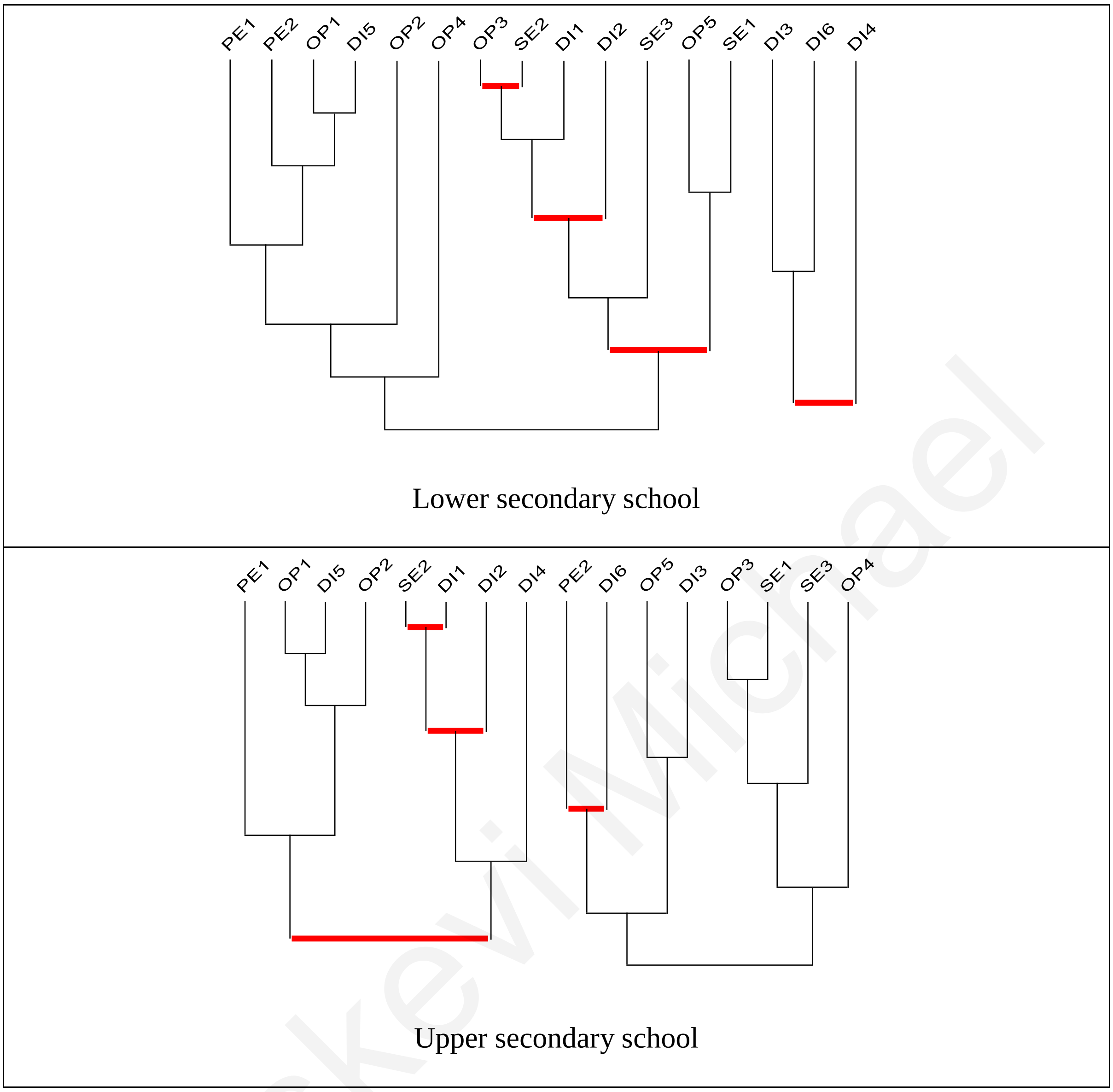


Figure 16. Similarity diagrams for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for each educational level

What is ultimately highlighted after comparing the ways the students from the different educational levels treat the same tasks, is that the mobilization of the perceptual apprehension seems to be crucial for the solution of tasks that are related to proof and the application of modifications on the geometrical figure. Therefore perception appears to be the first step towards successful use of mathematical properties for geometrical reasoning and for processes that provide a heuristic role to geometrical figures.

Implicative relations for students' answers from the mathematical point of view using the classical method

The implicative relations for the total sample are indicated in figure 17. At the top of the implicative chain there is an implication between the solution of the operative task OP4 and the discursive task DI5. Actually the figure included in the two tasks is similar and demands the same type of reconfiguration of the geometrical figure in order to be solved. So what this relation shows is that students who manage to carry out the proper reconfiguration in task OP4 also conduct the proper reconfiguration in task DI5 and thus identify the necessary properties in order to reach the right proof. Furthermore this relation makes the relation explained previously in the similarity diagram between task DI5 and the involvement of the operative apprehension clearer.

In the second part of the implicative diagram a group of variables with implicative relations is formed, comprising the sequential apprehension tasks (SE1, SE2, SE3) and some of the discursive apprehension tasks (DI2, DI3, DI4). The relations between these tasks indicate again the connection between the two types of apprehension and the relation between the technical constraints during the construction of a figure and the importance of properties represented in the figure after the construction of the figure is accomplished.

Similarly, an important relation is included in the diagram concerning the PE2 and OP2 tasks. As explained before in the similarity diagram the success in task PE2 is related to cognitive processes similar to those taking place within the mobilization of the operative apprehension. What this relation essentially shows is that students that are able to recognize all the squares in task PE2 correctly are also able to perform the mereologic modification demanded for the solution of task OP2 correctly.

The following relation in the fourth part of the implicative chain highlights the influence of perception in the solution of the discursive apprehension task DI6, as the students who succeed in this task also seem to be able to recognize all the figures in the perceptual task PE1 correctly. The lowest part of the chain reveals the implications between

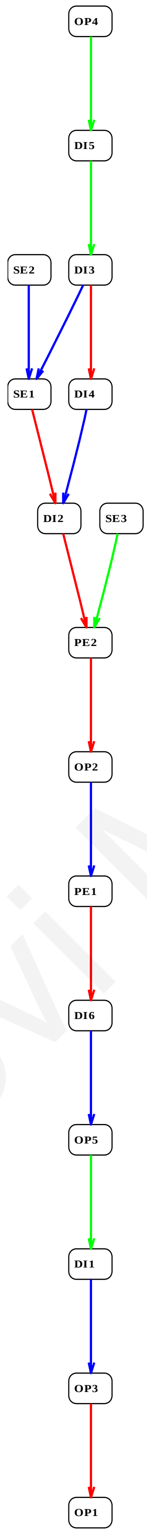


Figure 17. Implicative diagram for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for the total sample

the solution of the operative (OP5, OP3, OP1) and the discursive apprehension tasks (DI1). Therefore the relations between the ability to modify a geometrical figure and the ability to prove appear to be important. There is a hierarchy in the tasks which shows that there are implicative relations between the tasks corresponding to all the types of apprehension.

Therefore it is shown that despite the fact that each type of apprehension is different, all these different types are related.

Figure 18 presents the implicative relations for grade 9 and grade 10 respectively. At the top of the chain for grade 9 the discursive apprehension tasks are found (DI5 and DI2). In fact the production of the proof in task DI5 leads to the production of the proof in task DI2 and also in task DI6. These two tasks form another implicative relation in which a perceptual task intervenes (PE2). Therefore, although students seem to coordinate the cognitive processes to make the correct inference, perception seems to be also involved in the solution of these tasks.

The perceptual apprehension is also involved in the solution of the operative apprehension tasks. In fact an implicative chain begins with task OP4, whose solution leads to the solution of task PE2. In the implicative chain the solution of task OP2 comes next, which is also related to the solution of a perceptual task (PE1). This chain ends with the proof DI1 task, in which perception also seems to be important. Thus this implicative chain reveals the interaction between the operative and the perceptual apprehension for the correct recognition of figures and subfigures and for choosing the relevant reconfiguration which leads to the right solution.

The perceptual apprehension PE1 task is also involved in an implicative relation with the DI1task. On the contrary task PE2 forms implicative relations with operative apprehension tasks (OP4, OP2, OP3) and discursive apprehension tasks (DI2 and DI6), revealing that the 9^{th} graders that achieve the recognition of all the squares in task PE2 are also close to operative apprehension.

In this diagram the tasks concerning the constructions of geometrical figures also appear. Actually the correct solution for two of the sequential apprehension tasks (SE1 and SE2) is a prerequisite for proving correctly in task DI1. Therefore these relations highlight the importance of the proper use of mathematical properties in geometrical constructions and for geometrical proofs.

For grade 10 the first implicative chain (Figure 18) starts with the discursive apprehension DI3 task, whose solution leads to the solution of the corresponding DI4 task. These are the two tasks that concern the recognition of a proof, which students seem to face in a coherent way. The solution of these tasks as well as that of task DI5 leads to the solution of the operative apprehension task OP2.

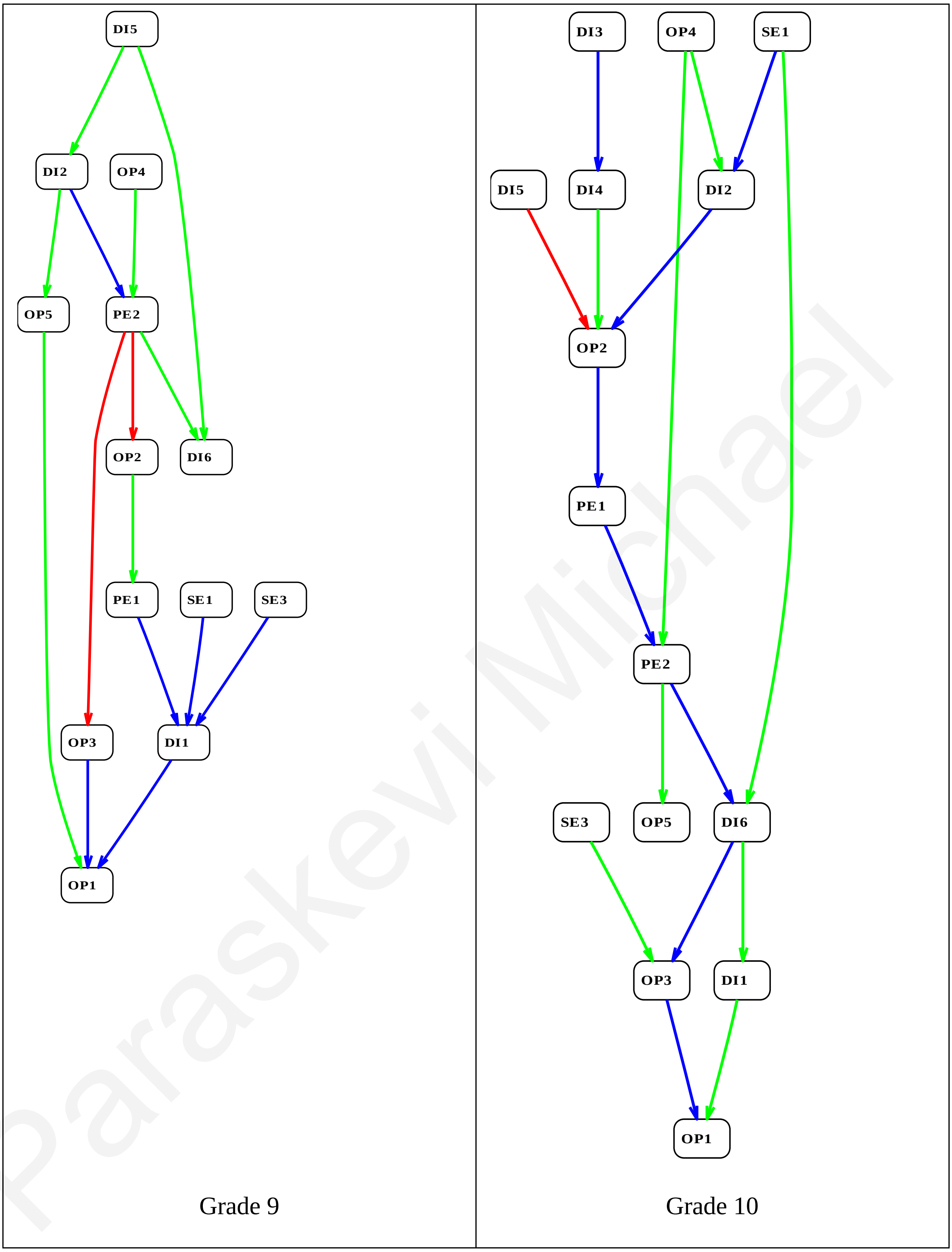
The relation between the DI5 and task OP2s highlights the importance of the involvement of the operative apprehension for the production of the necessary proof in task DI5, in which the reconfiguration of the given figure was a prerequisite for making the correct inference.

In addition, an implication exists between the two perceptual apprehension tasks (PE1 and PE2), indicating a greater level of stability regarding the use of perceptual apprehension, compared to grade 9 students. The solution of the perceptual apprehension tasks is a precondition for the solution of task DI6, whose solution seems to be influenced by perception. This relation continues with the solution of the proof DI1 task, which further leads to the solution of the operative apprehension task OP1.

The second implicative chain involves two operative apprehension tasks (OP4 and OP5) and a perceptual apprehension task (PE2). The fact that only this perceptual task is related to the operative apprehension enhances the hypothesis that for the successful solution in task PE2 perception is not enough, but also the operative apprehension seems to be necessary.

At the top of the third implicative chain a sequential apprehension task (SE1) is found. The successful construction of the geometrical figure that is required in this task leads to the production of correct proofs in the DI6 and DI1 tasks, whose solution leads to the right solution in task OP1. Task OP1 is the end of another implicative chain, which involves the sequence between task SE3 and task OP3.

What the diagrams indicate overall is that, first of all there are implicative relations between the different types of geometrical figure apprehension. Perception seems to be a presupposition for the mobilization of the operative apprehension in some tasks or intervenes in the solution of discursive apprehension tasks. In addition, grade 10 students display constancy in the recognition of proofs. Furthermore the sequential apprehension seems to be mostly related to the operative and discursive apprehension, whereas there are no relations with the perceptual apprehension.



Note: The SE2 and DI5 tasks do not appear in the diagram, because they are omitted due to the very small number of correct answers provided.

Figure 18. Implicative diagram for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for the 9th and 10th graders

The implicative relations in the diagram for grade 11a students (Figure 19) begin with a perceptual apprehension task (PE2) which forms implicative relations with a task of the same type (PE1) and two operative apprehension tasks (OP3 and OP5). These relations show that students display stability in the use of perception, but their perception is related to the operative apprehension. Thus there are indications of coordination between the perceptual and the operative apprehension for these students.

For the solution of the operative apprehension OP3 task other tasks are also preconditions. These are the DI6 and SE1 tasks, indicating the relation among the operative, the discursive and the sequential apprehension. From the construction task SE1 a new implicative chain begins, which splits into two branches and mainly involves proof tasks. The first branch continues with two tasks on the production of a proof (DI2 and DI1), whereas the second brunch includes the tasks on the recognition of proofs (DI3 and DI4). Finally the chain ends with task OP1.

In fact in this diagram two groups of variables are identified. The first group involves the tasks that are included in the first implicative chain, which are mainly operative and perceptual apprehension tasks. Therefore this group is formed by the relation between the perception of a figure and its heuristic function. The second group comprises the tasks of geometrical constructions and geometrical proofs. The common characteristic of these tasks is the importance of mathematical properties. For constructions these properties are related to the right sequence of steps that will lead to a geometrical figure which will show the relevant mathematical properties properly. In proving, the mathematical properties represented in a figure have to be used correctly for making correct inferences. So this group can be named as the "use of mathematical properties".

The implicative diagram for grade 11b students (Figure 19) is formed by implicative relations between tasks from all the types of apprehension. In the diagram different groups of tasks can be distinguished, according to the relations that occur between them. A first group shows that the correct construction of the geometrical figures in the SE1 and SE2 task leads to the solution of the perceptual PE2 task. Therefore this group indicates the relation of the geometrical constructions to perception. New relations emerge from task PE2, which bring to the solution of the PE1, OP3 and DI2 tasks. These relations show the coherence in the cognitive processes regarding the perception of geometrical figures by the students in grade 11b and that perception influences the solution of discursive and operative apprehension tasks. Another group includes the proof DI5 task and task OP4, whose common feature is the similar geometrical figure that is given. The correct solution of these two tasks appears to be a precondition for the right solution in task OP2, showing that the operative apprehension is needed to reach the right proof. The same type of relation exists in the next part of this implicative chain, in which another operative apprehension task (OP2) and discursive apprehension task (DI1) are involved. In addition there is an indication for stability in the tasks regarding the production of proofs (DI2, DI5 and DI1) as well. Therefore in this group the relation is revealed between the tasks regarding the production of a proof and the heuristic exploration of the figure. The last group concerns the relation between the discursive apprehension tasks that examine the recognition of a proof (DI3 and DI4), which reveals stability in the way students solve these tasks, as in grade 10 and 11a. On the other hand these two tasks are not related to any of the other tasks, indicating a compartmentalization regarding the cognitive processes on the recognition of proofs.

Concluding, there is coherence in the cognitive procedures related to the recognition of proofs and the production of proofs respectively. However, there is compartmentalization between the types of discursive apprehension tasks. On the other hand the tasks regarding the production of proof are related to the operative apprehension. Therefore students in grade 11b are able to coordinate the use of properties and the heuristic functioning of figures, but activate different processes for the recognition of proofs.

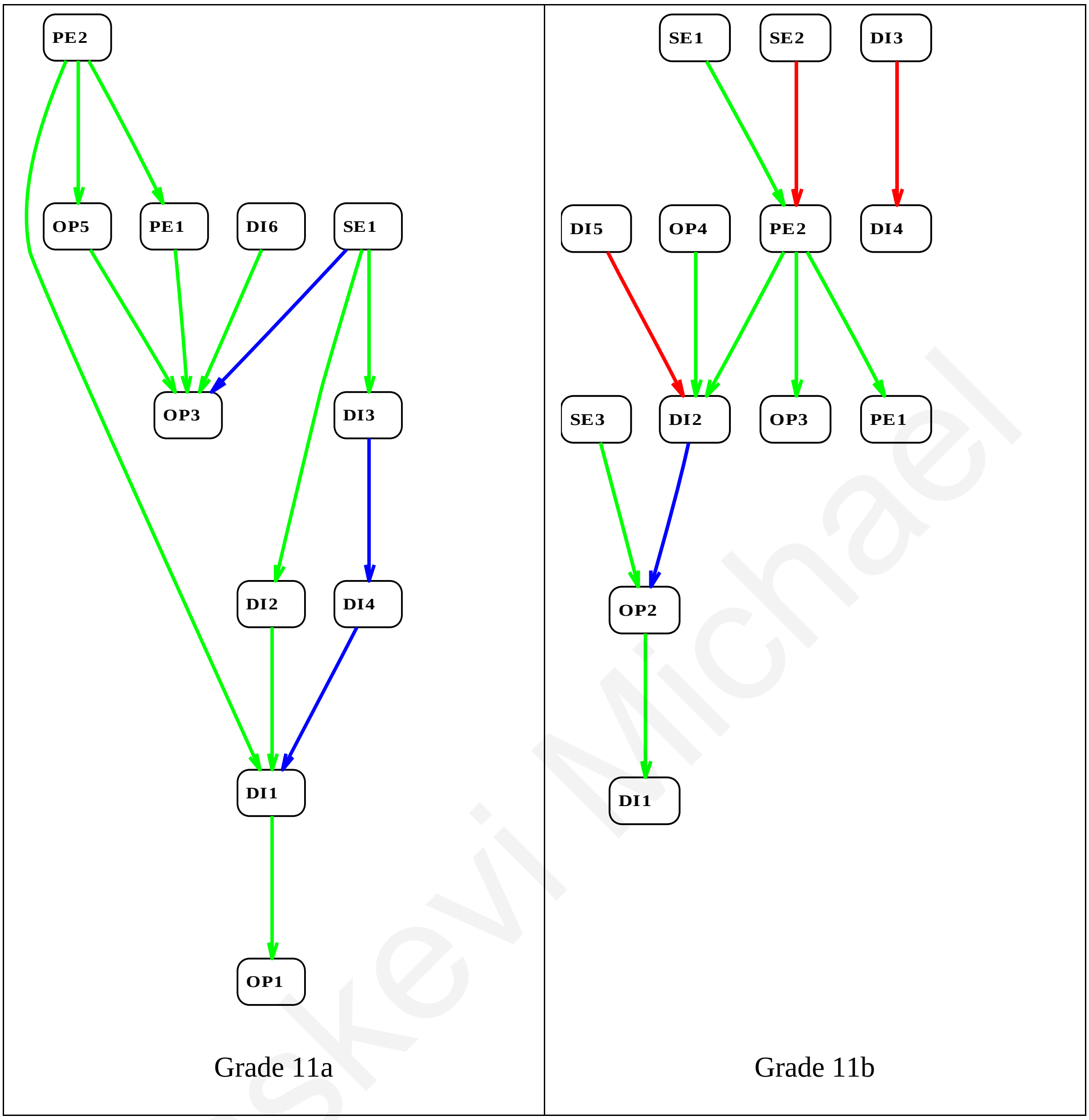


Figure 19. Implicative diagram for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for grades 11a and 11b

In the implicative diagram for the upper secondary school students (Figure 20), three groups of tasks can be identified. The first group consists of the implicative relations between the sequential apprehension tasks and the discursive apprehension tasks. Firstly, there is an implicative relation between the two tasks that examine the recognition of proof (DI3 and DI4), displaying coherence in the way students confront this type of tasks. Coherence is also displayed in the second relation, which is formed between two sequential apprehension tasks (SE2 and SE1). The third sequential apprehension task (SE3) is linked to the proof DI5 task. These three relations as well as the operative apprehension OP4 task are related to the solution of the proof DI2 task. Therefore this group can be characterized as the group of the use of mathematical properties, as the use of properties is a common characteristic of the discursive and the sequential apprehension tasks.

The second part of this diagram is a hierarchy which includes implicative relations among the OP2, PE2 and PE1 tasks. This relation shows that the solution of the operative apprehension OP2 task brings to the solution of the perceptual PE2 task, which also includes some characteristics of the operative apprehension. The solution of these tasks is a precondition for the solution of the other perceptual apprehension task (PE1), indicating coherence in the way students mobilize perception. Therefore this group can be considered as the "visualization group", in which there exists coordination between the operative and the perceptual apprehension.

The last part of the similarity diagram is formed by a group of operative apprehension tasks and proof tasks, which form implicative relations between them. The first task in this group is the discursive apprehension DI6 task, which is previously related to a perceptual apprehension task, indicating the involvement of perception for reaching the solution of this task. From this task new implicative relations occur. These are relations mostly between the operative apprehension tasks and the discursive apprehension tasks. Specifically the solution of task DI6 leads to solution of task DI1. Next the solution of task DI1 is linked to the solution of two operative tasks (OP3 and OP1). In addition the solution of the operative OP1 task is linked to the solution of task OP5. So what this group shows is the relation between the heuristic use of figures and the procedures for proof. Operative apprehension seems to very important for the identification of the relations among the different figural units of the figure, in order to trace the necessary mathematical properties for getting to the proper proof.

In both similarity diagrams the coordination between the operative and the perceptual apprehension is displayed, whereas the upper secondary school students display greater coherence regarding the perceptual apprehension than the lower secondary school students. Also in both diagrams the relations between the variables revealed the importance of the proper use of mathematical properties for the construction of geometrical figures and for the production of geometrical proofs.

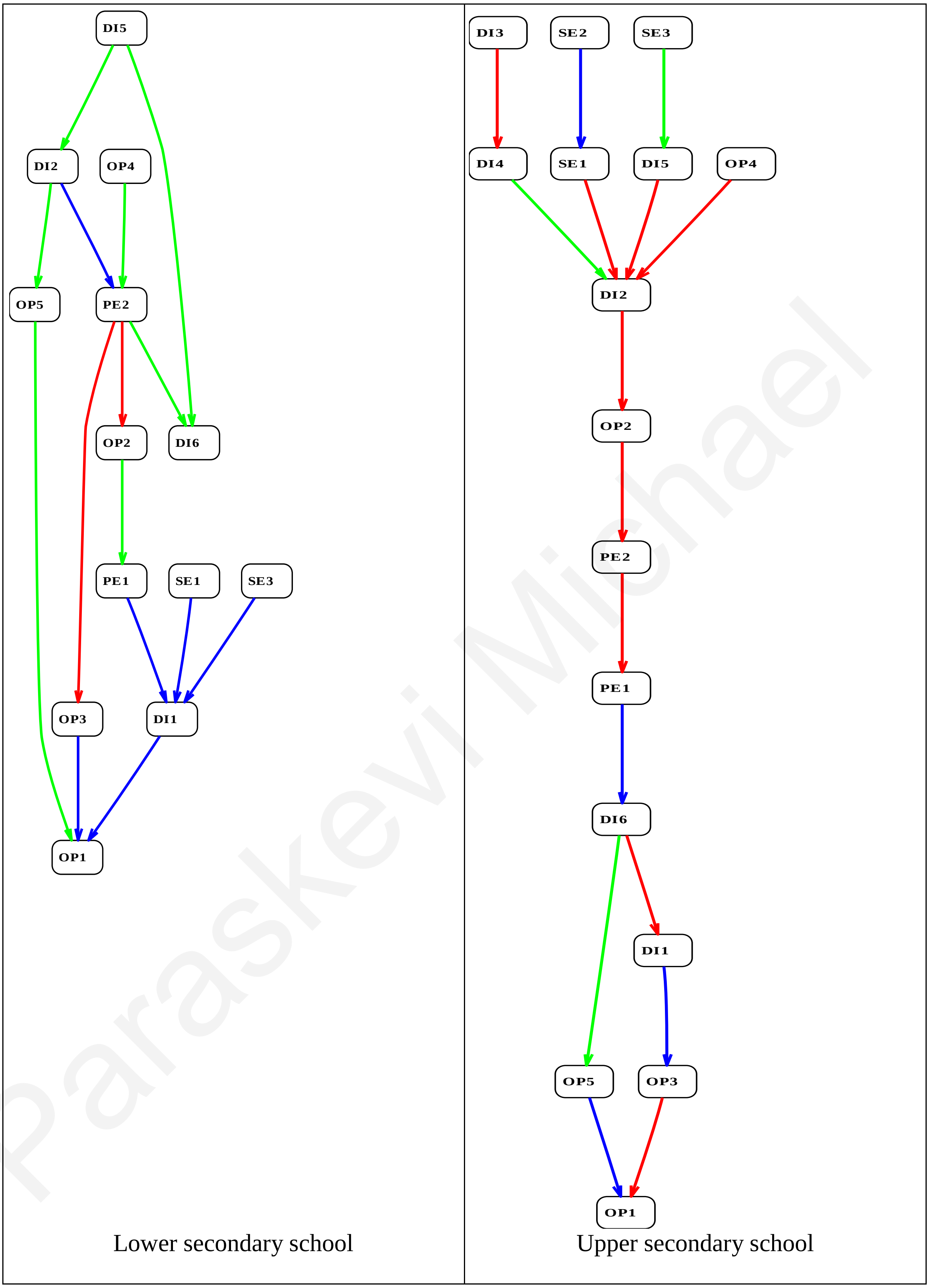


Figure 20. Implicative diagrams for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for each educational level

Implicative relations for students answers from the mathematical point of view with the entropy method

Apart from the classical in the implicative analysis, the entropy method (Gras, Peter, Briand & Philipp, 1997) was also used. The latter can better meet the objective of modeling of set inclusion or basic statistical theory of involvement and to take into account the quality of the contra positive direct involvement. But it is more severe in matters of intensity of involvement. An intensity of about 0.75 is as good as high intensities in the classical method. The entropic method is highly recommended when the number of individuals exceeds a few hundreds.

In the implicative diagram for the total sample there are implicative relations mostly between the tasks of the operative, the discursive and the sequential apprehension. The operative apprehension tasks and the discursive apprehension tasks are situated at the bottom of the implicative chains, indicating the importance of these kinds of tasks for the solution of the rest of the tasks. In fact the solution of some of the operative apprehension tasks (OP1, OP3 and OP5) is a prerequisite for the solution of other operative apprehension tasks (OP4), for discursive apprehension tasks (DI2, DI3, DI5) and for the sequential apprehension tasks (SE1, SE2, SE3). This is also the case for the discursive apprehension tasks (DI1 and DI6), whose solution seems to be a prerequisite for the solution of the rest of the discursive apprehension tasks that are included in the diagram (DI2, DI3, DI5), the operative apprehension tasks (OP1, OP3, OP5) and the three sequential apprehension tasks (SE1, SE2, SE3).

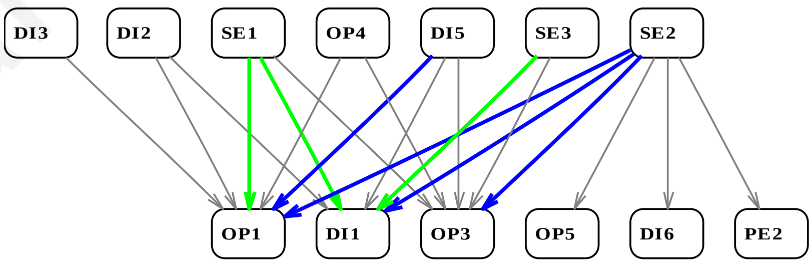


Figure 21. Implicative diagram for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for the total sample

What is noteworthy is the presence of the perceptual apprehension PE2 task at the bottom of the chain together with the operative and the discursive apprehension tasks. Based on the cognitive analysis of the particular task, the correct answer (the correct recognition of all the squares) does not remain only within the limits of the perceptual apprehension, but this ability goes towards the borders of operative apprehension. Therefore task PE2 is situated at the bottom of the implicative tasks along with the operative apprehension tasks, forming an implicative relation with the sequential apprehension SE2 task. In addition the tasks on the recognition of proof (DI3 and DI4) are not included in the implicative relations.

In the implicative diagram for the 9th graders (Figure 22) the implicative relations are mainly formed between the operative apprehension tasks and the discursive apprehension tasks and on the other hand between the discursive apprehension tasks and the sequential apprehension tasks. Particularly, the operative apprehension OP1 and OP3 tasks are situated at the bottom of the implicative chains. These tasks form implicative relations with a task of the same type of apprehension (OP4), a perceptual task (PE2), a discursive apprehension task (DI5) and a sequential apprehension task (SE2). Concerning the discursive apprehension task which is also found at the bottom of the implicative chains (DI1), the implications are created only with tasks of the same category (DI2) and construction tasks (SE1, SE2 and SE3). From the three discursive apprehension tasks that appear in this implicative diagram, task DI5 is the only one which is not related to the other discursive apprehension tasks. On the contrary the rest of the discursive apprehension tasks are related. Therefore the mobilization of the operative apprehension was crucial for the production of a proof in the particular task.

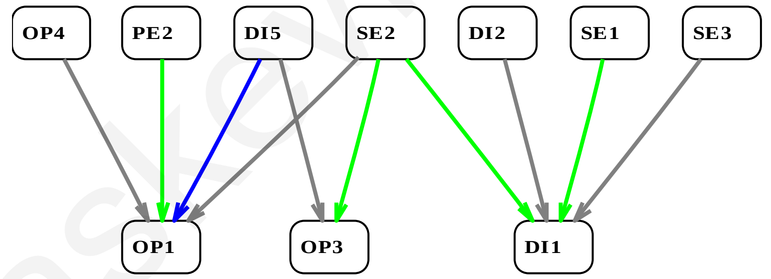
In the implicative diagram for grade 10 students (Figure 22), the implications are formed between variables corresponding to tasks from the three types of geometrical figure apprehension, as the perceptual apprehension tasks are not present in the implicative relations. In this diagram the first implicative chain is formed by the relations between two operative apprehension tasks (OP1 and OP2) and a discursive apprehension task (DI5). This relation between tasks OP1 and DI5 remain constant compared to the corresponding results about grade 9 students. This indicates that the 10th graders also mobilize the operative apprehension for the solution of this discursive apprehension task. Similar to the 9th graders' diagram, implicative relations exist among the sequential apprehension tasks (SE1, SE2 and SE3) and the tasks that are included in the category of the operative (OP5 and OP3) and the discursive apprehension tasks (DI1 and DI2). Compared to the corresponding relation for these tasks in the implicative diagram of grade 9, the relation between task SE2 with a discursive and an operative apprehension task still exists. The relation remains constant also regarding task SE1, which in both grades is only related to the discursive apprehension DI1 task. The situation is different between the two grades regarding task SE3. In grade 9 this task is only related to a discursive apprehension task (DI1), whereas in grade 10 this task is related to an operative apprehension task (OP3).

Coming to the discursive apprehension tasks, implications are formed between them (DI1 and DI2), with the operative apprehension tasks and with the sequential apprehension tasks. In fact, in grade 10 more implications appear between the discursive apprehension and the operative apprehension tasks (DI5-OP1, DI1-OP4, DI2-OP3). Compared to grade 9, task DI5 still remains with no further relations with the other discursive apprehension tasks, whereas the remaining tasks of this type (DI1 and DI2) are both related with discursive and operative apprehension tasks. Consequently, the existence of more relations among the operative apprehension, proof and constructions indicates that the role of the operative apprehension seems to be even more important for the 10th graders.

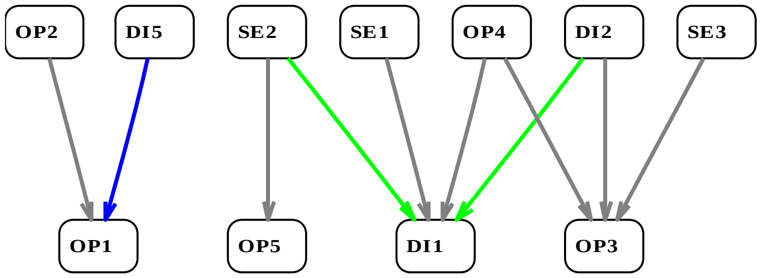
The implicative diagram for the students in grade 11a (Figure 22) includes fewer implicative relations. As previously mentioned, tasks SE2 and DI5 are omitted from this analysis due to the very small number of correct answers provided from the students. Two small implicative chains are clearly distinguished. The first concerns the implicative relation between task OP1 and tasks DI2 and SE3. So this chain includes implications among the operative, the discursive and the sequential apprehension. The second implicative chain is formed by the relation between tasks SE1 and OP3, indicating another relation between the ability to construct a geometrical figure and to modify it. Compared to the relations including task SE1 in the two previous grades, task SE1 is related to an operative apprehension task for the first time. Concerning task SE3, it continues to form implicative relation with an operative apprehension task, as in grade 10.

The implicative diagram for the students from grade 11b (Figure 22) contains more implicative relations between the variables, compared to the diagram for grade 11a students. This diagram is formed by implications between tasks from all the types of geometrical figure apprehension. First of all the perceptual apprehension PE2 task is included in an implicative chain with a sequential apprehension task (SE2) and a discursive apprehension task (DI1). Task PE2 appears again only in the 9th graders implicative diagram, though forming a relation with an operative apprehension task (OP1). Thus the role of perception seems to be different in the two tasks, as it appears to influence the rest of the tasks in a different way.

Regarding the sequential apprehension tasks, they relate to the operative and the discursive apprehension tasks. Specifically, task SE1 is linked to operative apprehension tasks (OP1 and OP3) only, as in grade 11a, whereas tasks SE2 and SE3 are both related to discursive and operative apprehension tasks. The difference in this diagram is that no relations are found between the operative and the discursive apprehension tasks. The operative apprehension tasks are mostly related to the sequential apprehension tasks or with each other. As far as the discursive apprehension tasks are concerned, they are mostly related with each other or to the sequential apprehension tasks. An important observation about task DI5 is that it is involved with the rest of the discursive apprehension tasks (DI1 and DI2) for the first time, and not with operative apprehension tasks, as in grades 9 and 10. This indicates the different treatment of this task by grade 11b students, who seem to coordinate the cognitive processes involved in the procedure of proving in geometry. A last observation about this diagram is that the solution of task DI1 turns to be crucial for the solution of tasks from the other types of apprehension, as it is involved in many implicative relations and it is situated at the bottom of these implicative chains.



Grade 9



Grade 10

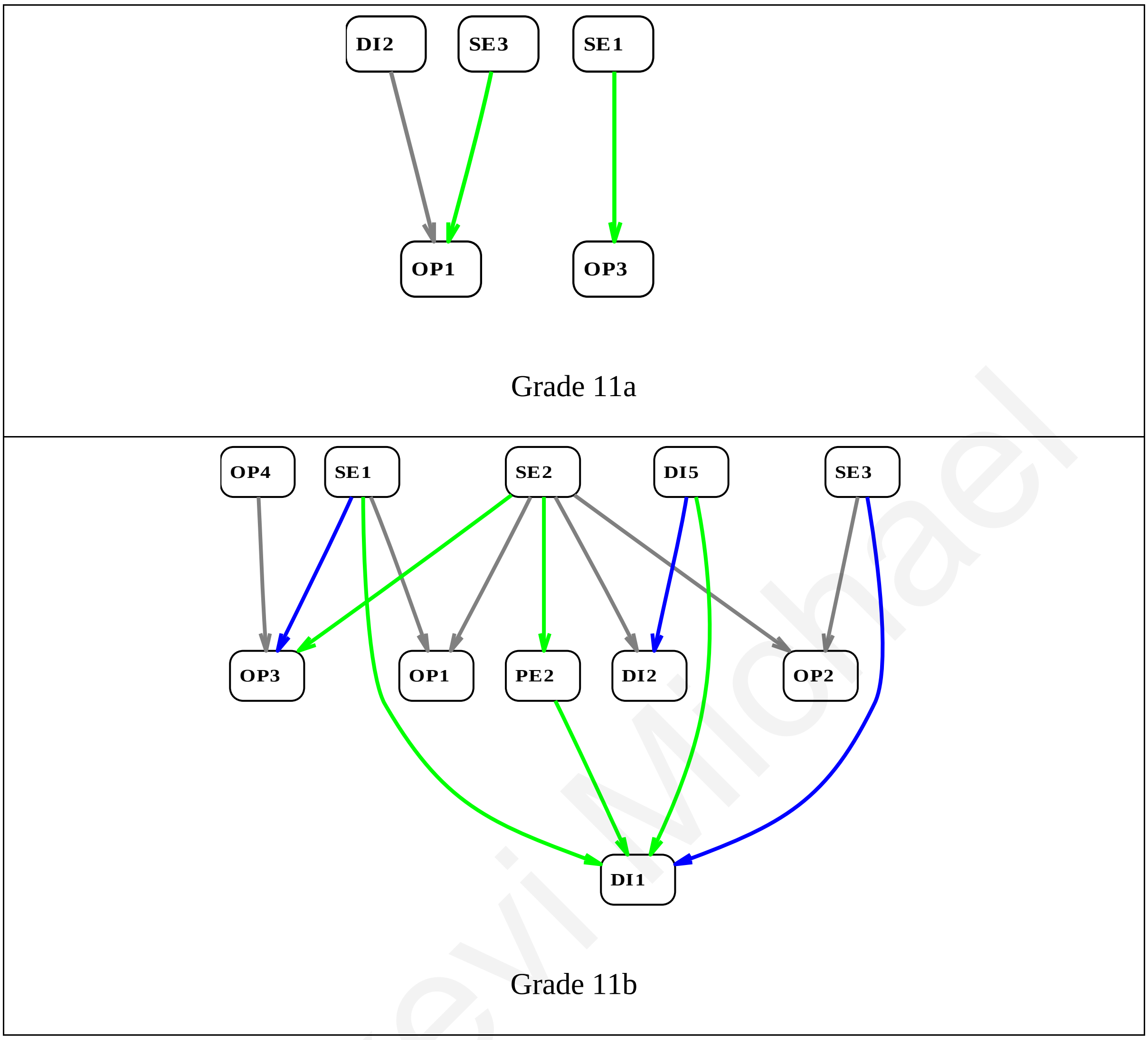


Figure 22. Implicative diagram for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for each grade

In the implicative diagram for the upper secondary school students (Figure 23), all the types of apprehension are involved in the implicative relations. In fact most of the relations are among the operative, the discursive and the sequential apprehension tasks. The perceptual apprehension task that appears in these implications (PE2) forms just one relation with the sequential apprehension SE2 task. This is a first difference compared to the implicative diagram for the lower secondary school students, in which the perceptual apprehension task is related to an operative apprehension task. Therefore a change in the role of perception in the upper secondary school is observed, as it seems to influence the sequential apprehension of the geometrical figure.

Regarding the sequential apprehension tasks, there are changes in the tasks they are related with. For example task SE1 creates implicative relations with tasks OP1 and DI1.

Regarding the results of the lower secondary school students, this task is only related to the discursive apprehension (task DI1). The second task (SE2) is the only one which is related to tasks from the other three types of apprehension (OP5, PE2, DI2). The same task is related to two operative apprehension tasks (OP1, OP3) and a discursive apprehension task (DI1) in the lower secondary school. Lastly, task SE3 is involved in implicative relations with the operative apprehension OP3 task and the proof DI1 task. The relation with task DI1 remains constant in the implicative diagrams for the two different educational levels. Another observation which is common for the diagrams of the two educational levels is that in both cases the same tasks are situated at the bottom of the implicative chains. These are tasks OP1, OP3 and DI1, whose solution seems to be decisive for the solution of other tasks that correspond to the different types of apprehension.

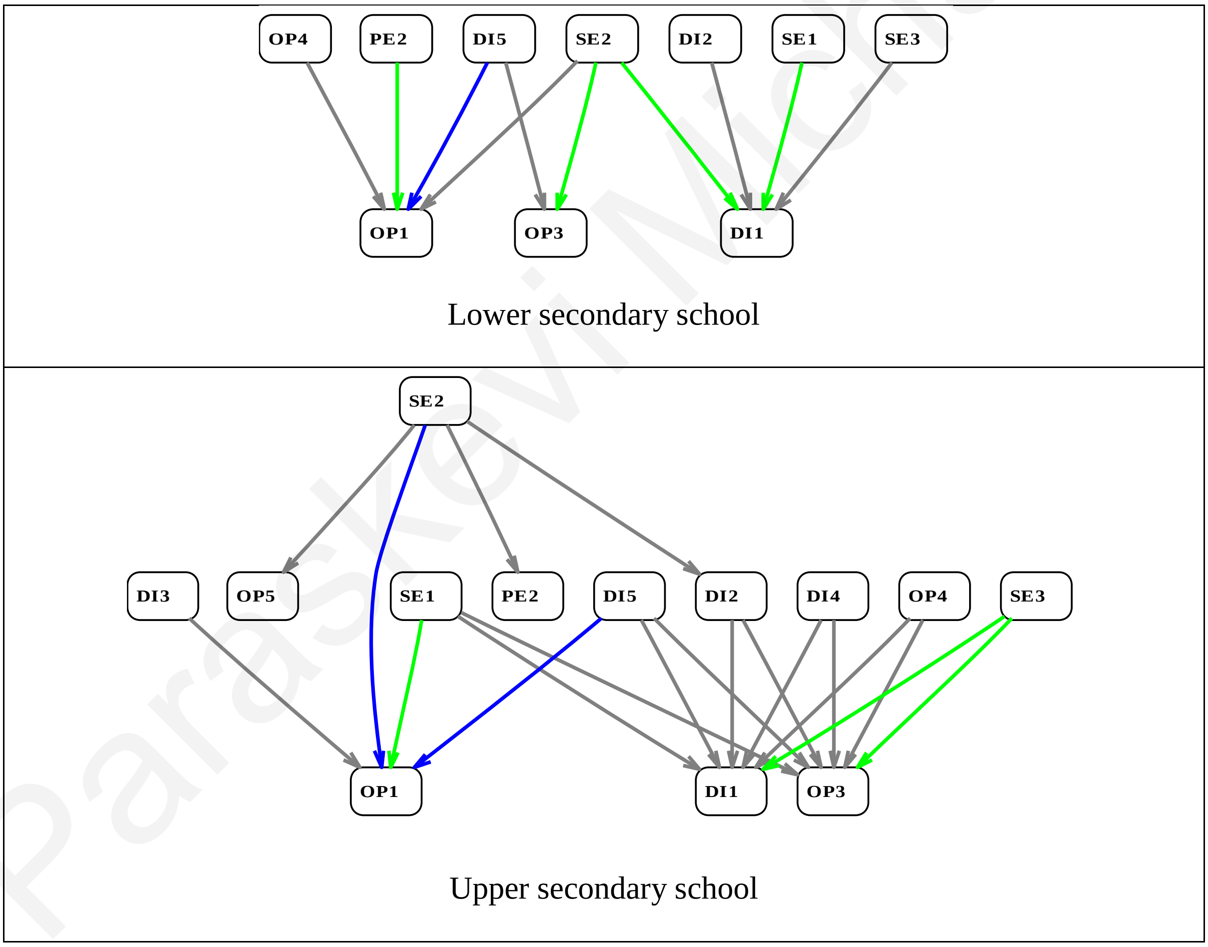


Figure 23. Implicative diagrams for the students' responses from the mathematical point of view in all the geometrical figure apprehension tasks for each educational level

Comparing the two diagrams for each educational level, what surfaces is the important role of the operative apprehension and the discursive apprehension for the solution of geometrical tasks. Furthermore, the different types of apprehension of geometrical figures seem to be more coordinated in the upper secondary school than in the lower secondary school.

Similarity relations among students' answers from the cognitive point of view

In the a priori analysis of the tasks the students' answers were not only viewed through the mathematical point of view. A cognitive approach was used as well for codifying the students' answers, mainly according to the specific type of apprehension that was mobilized for reaching the answer. In this section the data was analyzed using the hierarchical clustering of variables, by which the similarity relations among these variables were revealed. The results are presented separately for each type of apprehension.

Figure 24 presents the similarity relations among all students' answers in the perceptual apprehension tasks. In fact, there are three similarity clusters in this diagram. The first similarity cluster is formed by the significant similarity relation between the correct recognition of all the figures in task PE1 (R1) and the correct recognition of all the squares in task PE2 (Rasq). The second similarity cluster is created by the significant similarity relation formed between the recognition of the two most obvious squares (Risq) in the PE2 and the recognition of almost all the figures (R2) in task PE1. In the last similarity cluster the false recognition in task PE2 (Rf) forms similarity relations with the recognition of 5-6 figures (R3) and the recognition of less than 5 figures (R4) in task PE1. There is a weak similarity relation between the second and the third similarity cluster.

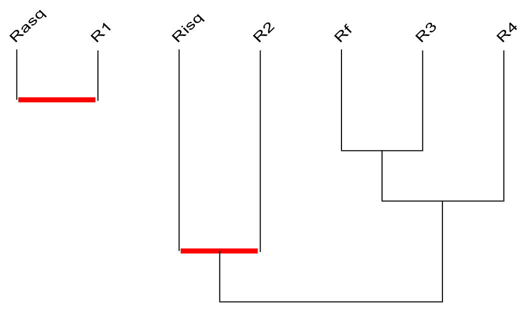
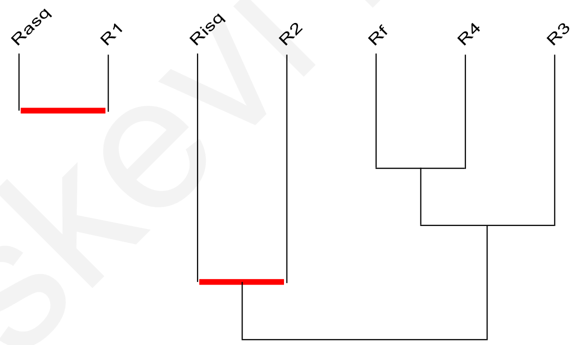


Figure 24. Similarity diagram for the students' responses from the cognitive point of view in the perceptual apprehension tasks for the total sample

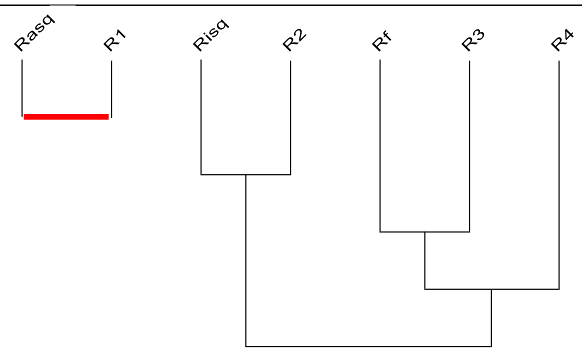
Generally the similarity diagram for the total of the students indicates that the students, who are able to recognize all the squares in task PE2 correctly, can also recognize all the figures in the PE1task. The students that recognize only the isolated squares in task

PE2 are also able to recognize almost all the figures in task PE1. Regarding the students who carry out false recognition in task PE2, they are able to recognize a fewer number of figures (6 and less than 6) in task PE1.

The similarity diagrams for each group of students are presented in figure 25. Particularly the similarity diagrams for grades 9, 10 and 11b students' responses to the perceptual apprehension tasks are the same as the similarity diagram for the total of students. The differentiation is that in grade 10 the similarity relation in the second similarity cluster is not significant, whereas for grade 11b students this is the case for the first similarity cluster. Regarding the similarity diagram of grade 11a students, the first similarity cluster is formed by the same variables, as in the rest of the similarity diagrams. The second similarity cluster also includes the recognition of the two most obvious squares (Risq) in this case, which are significantly related to the recognition of less than 5 figures (R4) and less related to the recognition of 5-6 figures (R3) in task PE1. Finally the false recognition in task PE2 (Rf) is linked to the recognition of almost all the figures (R2). Therefore the way the students in grade 11a deal with the perceptual apprehension tasks is quite different from the rest of the students.



Grade 9



Grade 10

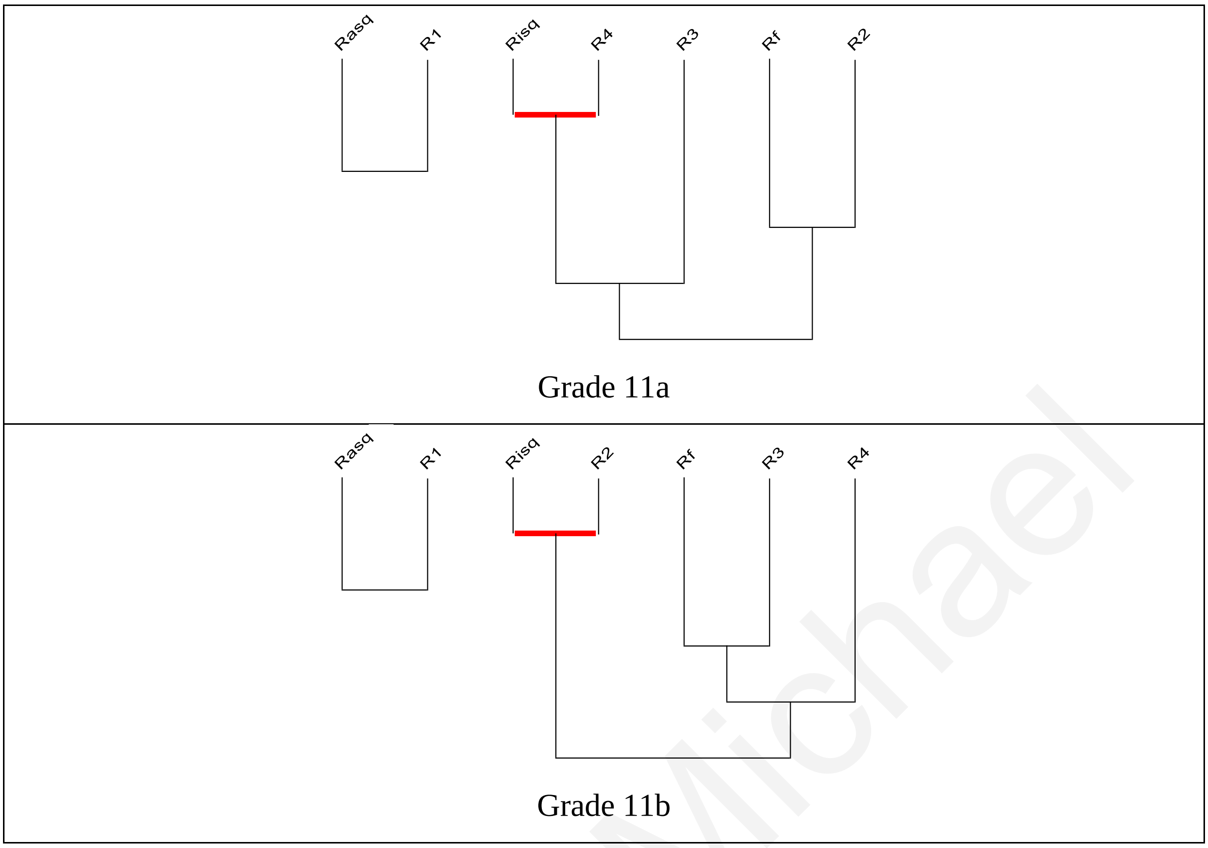


Figure 25. Similarity diagrams for the students' responses from the cognitive point of view in the perceptual apprehension tasks for each grade

The similarity relations among all students' answers in the operative apprehension tasks, on the basis of the cognitive point of view, are indicated in figure 26. In fact there are three similarity clusters in this diagram. In the first similarity cluster two subgroups are distinguished. The first subgroup includes mainly the solutions which occurred through the mobilization of the operative apprehension (OP1me, OP2me, OP3me and OP5me). This subgroup also includes two solutions that are related to the involvement of perceptual apprehension (OP2pe and OP5pe). Actually there is a significant similarity relation between perceptual solution in task OP2 (OP2pe) and the use of the mereologic modification for solving task OP5 (OP5me). This relation reveals the strong influence of perception for the solution of task OP2, which was also indicated in the implicative diagram for the total number of students.

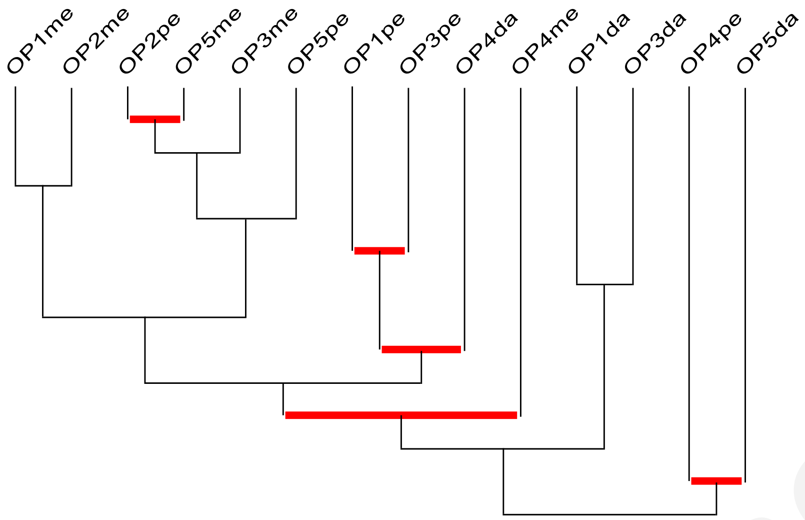


Figure 26. Similarity diagram for the students' responses from the cognitive point of view in the operative apprehension tasks for the total sample

The second subgroup includes the significant relation between the perceptual answers in tasks OP1 and OP3 (OP1pe and OP3pe), which are also significantly related to the use of a different approach in task OP4 (OP4da). These two subgroups are also significantly related to the use of the mereologic modification for the solution of task OP4 (OP4me). Therefore the first similarity cluster comprises mainly the use of the mereologic approach for the solution of the operative apprehension tasks, which seems to be used coherently enough. However indication of coherence also appears in the intervention of perception.

The second similarity cluster regards the similarity relation between the use of a different approach for solving tasks OP1 and OP3 (OP1da and OP3da). Finally the significant similarity relation between the perceptual solution in task OP4 and the use of a different approach in task OP5 (OP4pe and OP5da) creates the last similarity cluster.

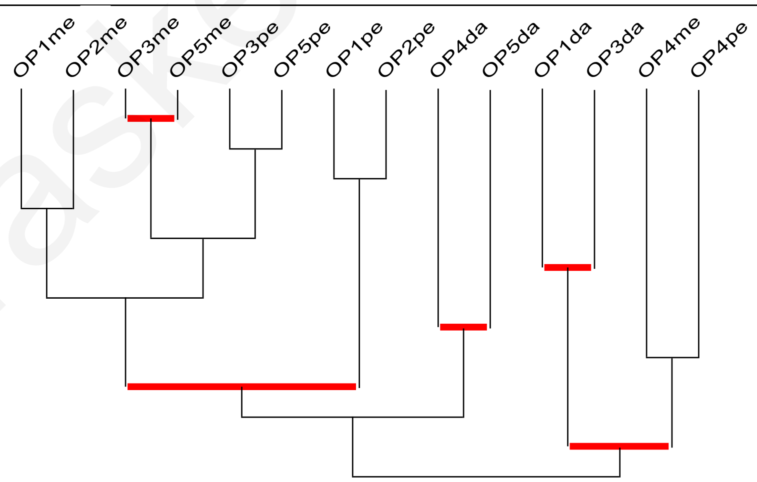
Consequently the relations among all the students' answers reveal similarity mostly between the use of the mereologic approach and the involvement of perception. However less similarity relations appear between the perceptual solutions and the use of a different approach. Coherence is displayed mostly for the use of the mereologic approach, as most of those answers are located in the same similarity cluster. Less coherence exists for the perceptual solutions and the use of the different approach.

The similarity diagram for the 9th graders (Figure 27) comprises three similarity clusters. The first cluster includes almost all the answers that occurred with the use of the mereologic approach (OP1me, OP2me, OP3me and OP5me) and almost all the perceptual solutions (OP1pe, OP2pe, OP3pe and OP5pe). Specifically there is a significant relation between the use of the mereologic approach in tasks OP3 and OP5 (OP3me and OP5me) and another significant relation between the perceptual solutions in tasks OP1 and OP2 (OP1pe and OP2pe) with the rest of the aforementioned variables. The similarity relation in the use of a different approach for the solution of tasks OP4 and OP5 (OP5da and OP5da) forms the second similarity cluster. In the last similarity cluster two subgroups appear which are significantly related to each other. The first subgroup is created by the significant relation between the use of a different approach in tasks OP1 and OP3 (OP1da and OP1da), whereas the second subgroup regards the relation between the use of the mereologic modification and the perceptual answer in task OP4 (OP4me and OP4pe). The 9th graders display consistency in each type of solution in the operative apprehension tasks, as there are similarity relations among the solutions of the same kind. In addition, the majority of the similarity relations are between the solutions which are related to the operative and the perceptual apprehension.

The 10th graders' solutions are grouped into two similarity clusters in figure 27. In the first cluster two subgroups are formed. In the first subgroup almost all the solutions occurring from the use of the mereologic modification (OP1me, OP2me, OP3me and OP5me) are significantly related to the perceptual solution in task OP5 (OP5pe). The second subgroup includes similarity relations between the perceptual solutions in tasks OP1 and OP2 (OP1pe and OP2pe), the use of the mereologic modification in task OP5 (OP5me) and the use of the different approach in task OP3 (OP3da). Actually the relation between the variables OP2pe and OP5me is significant. The perceptual solution in task OP4 (OP4pe) is related to the rest of the variables included in the first similarity cluster. The second similarity cluster also includes two subgroups of variables, which are significantly related. In the first one the use of the mereologic modification in the fourth task (OP4me) is significantly related to the perceptual solution in the fifth task (OP5pe) and the use of a different approach in the first task (OP1da). The second subgroup is formed by the similarity relation between the perceptual solution and the use of a different approach in the third and forth tasks respectively (OP3pe and OP4da). Thus the first similarity cluster mostly includes the solution involving either the operative or the perceptual apprehension, whereas the second one mainly includes the use of the different approach. Coherence in the way the 10th graders solve the operative apprehension tasks occurs mainly for the use of the mereologic modification. Compared to the solutions of the

9th graders, in the similarity diagram for the 10th graders a similarity relation appears between the use of the mereologic modification and the use of a different approach, which does not exist for the 9th graders.

The similarity diagram regarding the answers from students in grade 11a (Figure 27) is formed by three groups of variables. Specifically the first similarity cluster includes the majority of the answers related to the use of the mereologic modification (OP1me, OP2me, OP3me and OP5me) and a solution related to perception (OP5pe). Three other solutions related to perception (OP1pe, OP2pe and OP3pe) and two answers related to the use of a different approach (OP1da and OP5da) form the second similarity cluster, in which most of these variables are significantly related. In the last similarity cluster there are variables from the three types of solutions. In fact the use of a different approach in the third task (OP3da) is related to the use of the mereologic modification in the fourth task (OP4me). These variables are significantly related to the perceptual answer in the fourth task (OP4pe). Finally these three variables are significantly related to the use of a different approach in the fourth task (OP4da). In this sense, the similarity diagram for the students in grade 11a indicates stability mainly for the use of the mereologic modification. Similarly to the results of the 10th graders there are significant similarity relations between the use of the mereologic modification and the use of a different approach or the involvement of perception.



Grade 9

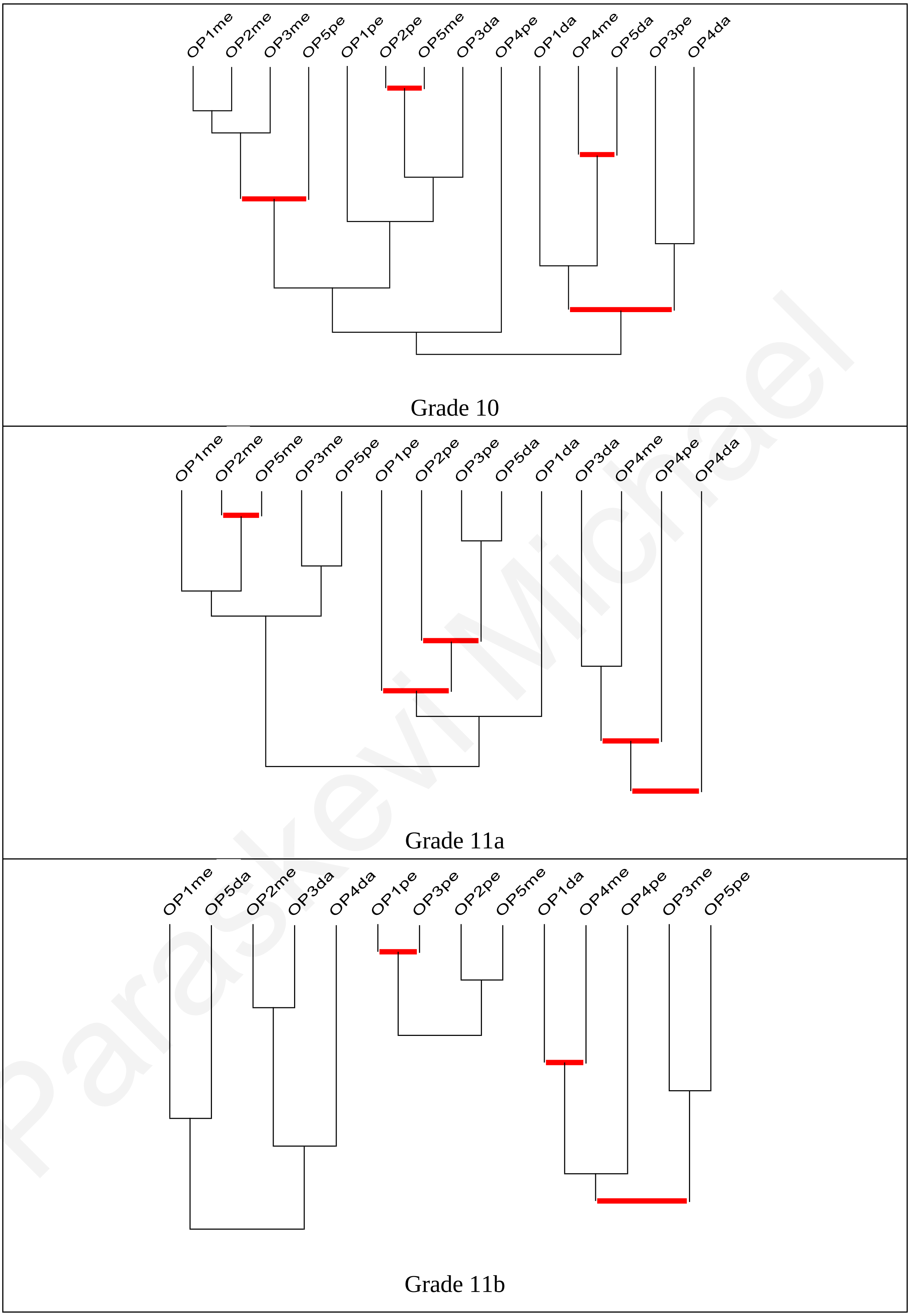


Figure 27. Similarity diagrams for the students' responses from the cognitive point of view in the operative apprehension tasks for each grade

The last similarity diagram (Figure 27) concerns the students' solutions in grade 11b in the operative apprehension tasks. These solutions are discriminated into three distinct similarity clusters. The first similarity cluster is created by the relations between the use of the mereologic modification in the two first operative apprehension tasks (OP1me and OP2me) and the use of the different approach in the rest of the tasks (OP3da, OP4da and OP5da). The next similarity cluster includes three perceptual solutions (OP1pe, OP2pe and OP3pe) and a solution related to the use of the mereologic modification in the fifth task (OP5me). In fact the perceptual solutions in the first and third task are significantly related (OP1pe and OP3pe). In the last similarity cluster the use of the different approach in task OP1 (OP1da), the use of the modification in tasks OP3 and OP4 (OP3me and OP4me) and the two perceptual solutions (OP4pe and OP5pe) are significantly related. Differently from the rest of the students, there are more similarity relations between the mobilization of the operative apprehension and the use of the different approach for the solution of the tasks in grade 11b. The coherence displayed in the previous results is not evident in this diagram. On the other hand greater coherence seems to exist in the mobilization of the perceptual apprehension for solving the tasks.

The similarity relations between the variables occurring from the cognitive analysis of the students' answers in the sequential apprehension tasks are displayed in the figure 28. The similarity diagram includes students' correct (SE1, SE2, SE3) or partly correct (SE2pc) constructions and answers that occurred with the involvement of the perceptual apprehension (SE1ps, SE2ps and SE3ps). All students' solutions are distinguished into two similarity clusters. The first similarity cluster includes the similarity relations between the correct constructions in the three sequential apprehension tasks (SE1, SE2 and SE3) and the partly correct construction in the second task (SE2pc), with these relations being significant. The other similarity cluster is formed by the similarity relations between the variables corresponding to the perceptual solutions (SE1ps, SE2ps and SE3ps). The similarity relation between the two similarity clusters is very low, which allows one to consider the two clusters as distinct. Therefore compartmentalization appears between the mobilization of the sequential apprehension, which leads to the correct constructions, and the involvement of the perceptual apprehension, which is responsible for solutions that look similar to the correct ones, but the procedure was not followed properly. In addition there seems to be coherence in the way each type of apprehension is mobilized, as the solutions occurring from each type of apprehension are grouped into separate clusters.

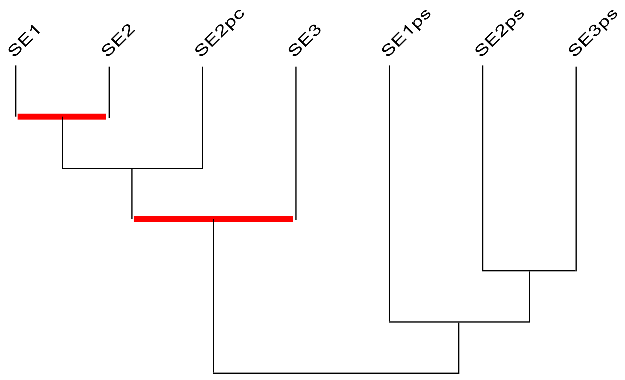


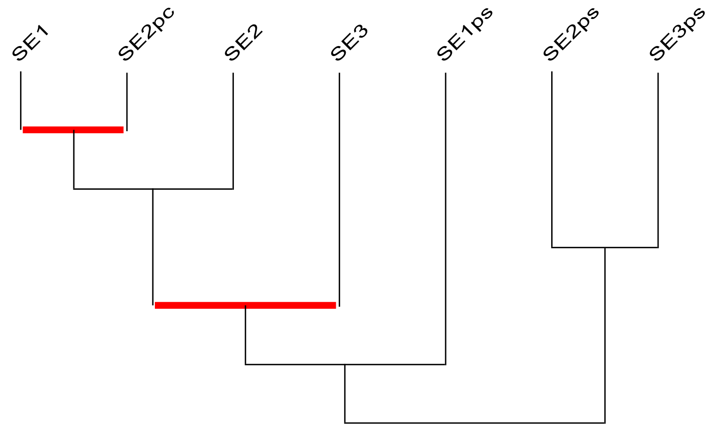
Figure 28. Similarity diagram for the students' responses from the cognitive point of view in the sequential apprehension tasks for the total sample

In the similarity diagram for the 9th graders (Figure 29) two similarity clusters are also formed, though they do not include exactly the same variables as in the diagram for the total sample. The first similarity cluster includes all the correct solutions of the sequential apprehension tasks (SE1, SE2 and SE3), the partly correct solution in task SE2 (SE2pc) and one of the perceptual solutions (SE1ps). There is a significant similarity relation between the correct construction in task SE1 and the partly correct solution in task SE2 (SE2pc) (*explain why*). Also the correct solution in task SE3 is significantly related with the variables SE1, SE2 and SE2pc. The second similarity cluster is formed by the perceptual solutions in tasks SE2 and SE3 (SE2ps and SE3ps). The similarity relations between the variables indicate coherence in the mobilization of the sequential and the perceptual apprehension, but there is no compartmentalization between the two types of apprehension, because a relation appears between the two types of solutions. Therefore the 9th graders seem to be able to mobilize the sequential apprehension, but the perceptual apprehension influences this procedure, as it intervenes in the sequence of steps for constructing a figure and leads to a solution in which the necessary steps were not followed.

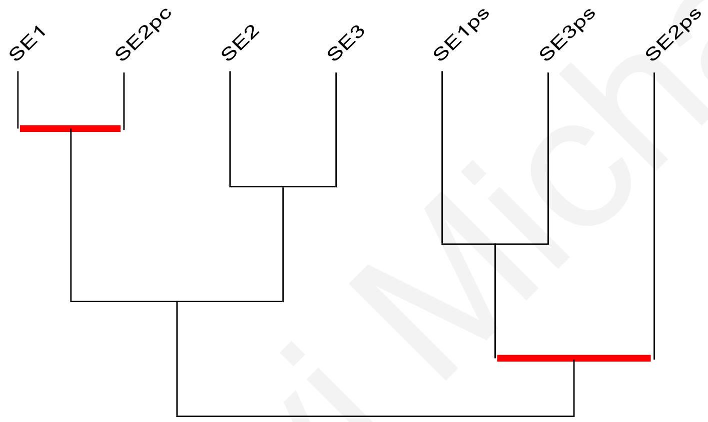
The similarity diagram for the 10th graders (Figure 29) also includes two similarity clusters. As in the similarity diagram for the total number of the students, the first cluster involves the correct and partly correct solutions (SE1, SE2, SE3 and SE2pc), whereas in the second cluster all the perceptual solutions (SE1ps, SE2ps and SE3ps) are found. Same as in the similarity diagram of the 9th graders, there is a significant similarity relation between the correct construction in task SE1 and the partly correct solution in task SE2

(SE2pc) (*explain…*), which form a subgroup in the first cluster. A second subgroup is formed by the relation between the correct solutions in the second and third task (SE2, SE3). In addition, the variables of the second similarity cluster that correspond to the perceptual solutions are significantly related. In this case too, the similarity relation between the two similarity clusters is very low, thus the two clusters can be considered as distinct. The separation of the two types of solutions into two distinct similarity clusters reveals the compartmentalization regarding the cognitive processes involved during the mobilization of the sequential and the perceptual apprehension. Also coherence is indicated in the involvement of each type of apprehension in the solution of the sequential apprehension tasks, as each type of solution forms a separate similarity cluster.

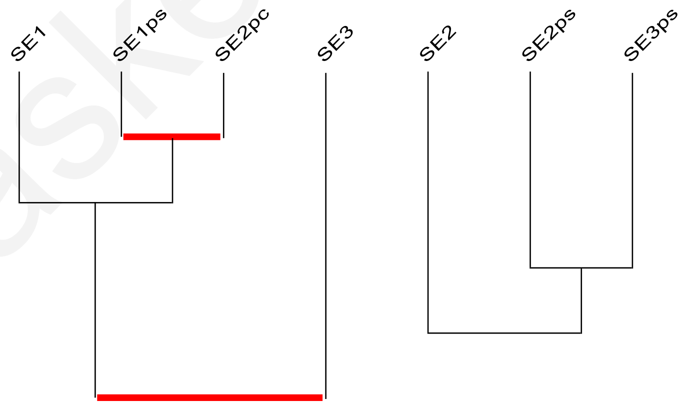
The variables for grade 11a are also distributed into two similarity clusters (Figure 29). In fact new relations appear in this similarity diagram. The first cluster includes a significant similarity relation between the variables SE1ps and SE2pc. The fact that a perceptual solution is related to a partly correct construction can be indicative of the influence of perception in the solution of task SE2. It is possible that perception intervened during the construction process and influenced the operation of the sequential apprehension. Therefore the students only achieved the construction of a part of the figure (only the triangle in this case) and then continued the procedure without following the steps in the required way, due to the involvement of perceptual apprehension. The variables SE1ps and SE2pc are also related to the variable corresponding to the successful construction in task SE1. The correct construction in task SE3 is significantly related to the three previous variables. The second similarity cluster comprises the variables SE2, SE2ps and SE3ps. What emerges in this diagram is that there is no coherence regarding the mobilization of the sequential and the perceptual apprehension. The correct constructions are related to the perceptual solution, showing that the proper mobilization of the sequential apprehension is not achieved in all cases, giving space for perception to intervene and influence the construction process. Compared to the other groups of students, in grade 11a the most relations between the correct solutions and the perceptual solutions appear.



Grade 9



Grade 10



Grade 11a

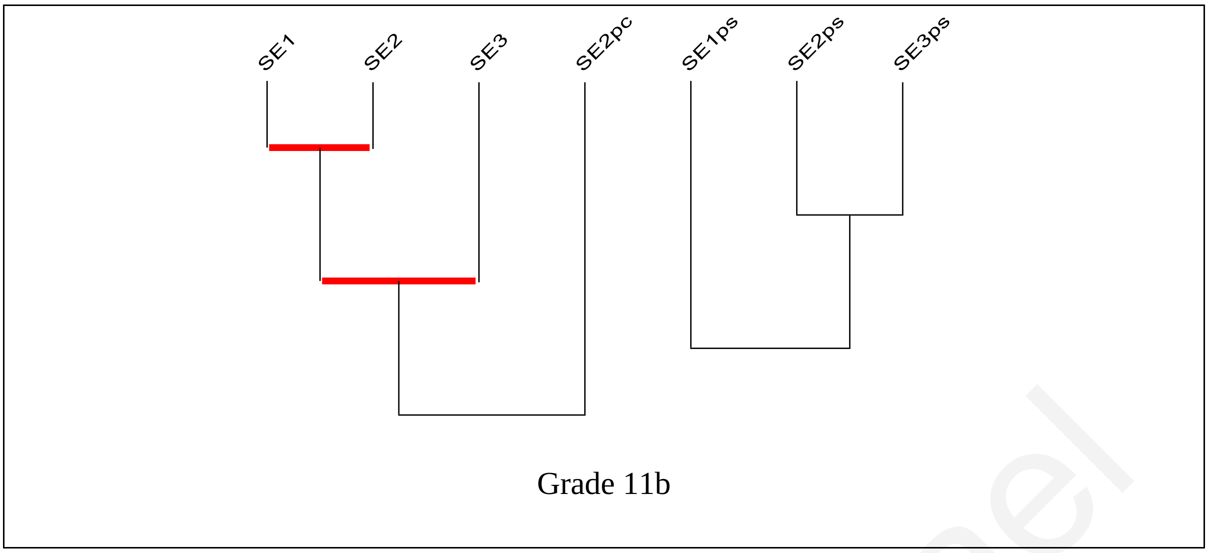
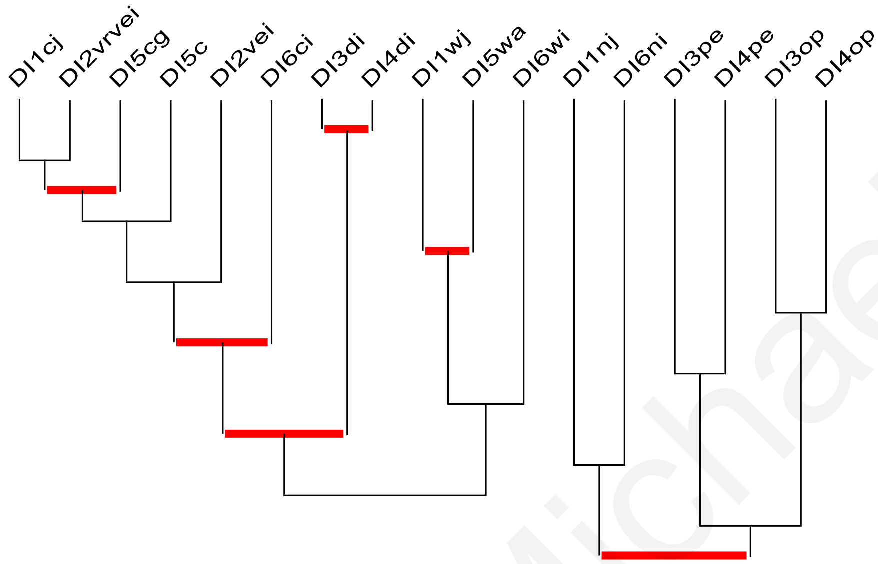


Figure 29. Similarity diagrams for the students' responses from the cognitive point of view in the sequential apprehension tasks for each grade

The last similarity diagram indicates the similarity relations between grade 11b students' answers in the sequential apprehension tasks (Figure 29). The variables are grouped in two similarity clusters in this case as well. As in some of the previous diagrams (total sample and grade10), the first similarity cluster includes all the correct or partly correct answers (SE1, SE2, SE3 and SE3pc), whereas the second is formed by all the perceptual solutions (SE1ps, SE2ps and SE3ps). In the first similarity cluster the relations among the correct solutions of the three sequential apprehension tasks are significant. This significance of the relations among these three tasks appears for the first time in grade 11b students and is indicative of the strong coherence in the way the students in this grade construct a geometrical figure, following the right sequence of steps. In addition the grouping of all the perceptual solutions in the same similarity cluster shows that there is also coherence as regards the involvement of perception in the construction process. The fact that the two similarity clusters are not related is indicative of the phenomenon of compartmentalization between the cognitive processes related to the sequential and the perceptual apprehension.

The similarity relations between the variables occurring from the cognitive analysis of all students answers in the discursive apprehension tasks are presented in figure 30. These variables form two similarity clusters and each one includes two subgroups. The first similarity cluster includes the variables related to the comprehension of proof and the variables that represent answers in which the justification was wrong. The second similarity cluster includes the answers with no justification and solutions related to the perceptual and the operative apprehension.



Note: the variable DI2vr was not included in the analysis, because of low frequency

Figure 30. Similarity diagram for the students' responses from the cognitive point of view in the discursive apprehension tasks for the total sample

Specifically, the first subgroup of the first similarity cluster includes a significant relation between the correct justification in task DI1 (DI1cj), the combination of the verbal indication and the visual recognition of transitivity in task DI2 (DI2vrvei) and the comprehension of proof with a gap (DI5cg). The variables corresponding to the comprehension of proof in task DI5 (DI5) and the verbal indication of transitivity in the second task (DI2vei) are also related to these variables. A significant similarity relation appears between the aforementioned variables and the correct inference in task DI6 (DI6ci). Then there is a significant relation between the students' answers in the tasks on the recognition of proof which occur from discursive apprehension (DI3di and DI4di). This relation reveals stability in the way the students answer in this pair of tasks. All the variables of this subgroup are significantly related, indicating coherence in the way students mobilize the discursive apprehension. Regarding the second subgroup of the first similarity cluster, it includes a significant similarity relation between the wrong justification in the first task (DI1wj) and the wrong answer in the fifth task (DI5wa). These variables are subsequently related to the wrong inference in the sixth task (DI6wi). Thus this subgroup comprises variables related to the wrong answers and inference in the proof tasks.

The second cluster, first of all, includes the similarity relation between the two variables corresponding to answers without justification (DI1nj and DI6ni). The next similarity relation is between the perceptual answers in the tasks on the recognition of proof (DI3pe and DI4pe). These variables are also related to the answers occurring from the operative apprehension in the tasks about the recognition of proof (DI3op and DI4op). These relations again show the stability in the way the students answer in the tasks regarding the recognition of proof. In addition all the variables of the second similarity cluster are significantly related. Therefore this cluster includes answers that are not related to the discursive apprehension, but on the other hand these answers are related to the perceptual and the operative apprehension.

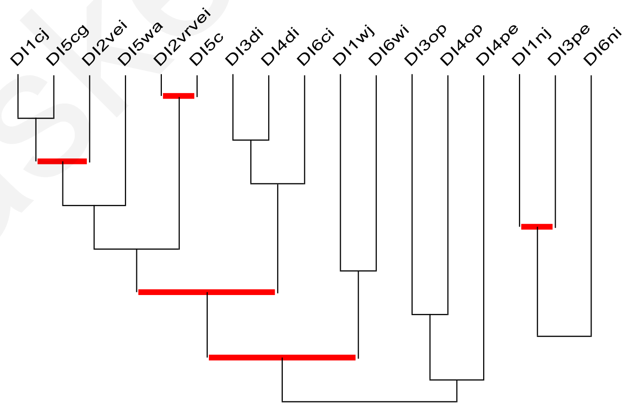
Generally, the first cluster is related to the discursive apprehension, even if wrong answers are included. These answers are related to the discursive apprehension, because an effort for inference seems to have been made, albeit unsuccessfully. The second cluster includes answers that are not related to the discursive apprehension, but on the other hand these answers are related to the perceptual and the operative apprehension. The cases in which students gave a wrong justification but their answers were correct appears to be related to the discursive apprehension, but on the contrary the cases in which no justification was provided is not related to the discursive apprehension, but can be alternatively attributed to the influence of the operative and the perceptual apprehension. In addition students display consistency in the type of apprehension that is mobilized for solving the tasks regarding the recognition of proof.

The similarity diagram for the 9th graders (Figure 31) includes three similarity clusters. The first similarity cluster includes four subgroups. In the first subgroup there is a relation between the correct justification in the first task (DI1cj) and the indication of the comprehension of proof but with a gap in the fifth task (DI5cg). These two variables are next significantly related to the verbal indication of transitivity in the second task (DI2vei). In addition there is a relation between the three variables and the wrong answer in the fifth task (DI5wa). In the second subgroup there is a significant relation between the combination of the verbal indication and the visual recognition of transitivity in task DI2 (DI2vrvei) and the comprehension of proof but with a gap in the fifth task (DI5c). The third subgroup is formed by the similarity relations between the students’ answers in the tasks on the recognition of proof which occur form the discursive apprehension (DI3di and

DI4di) and the correct inference in task DI6 (DI6ci). Therefore this group comprises mainly the students' answers that are related to the discursive apprehension, as in this group the variables regarding correct justifications and inference or variables indicating comprehension of proof are gathered. But this cluster also involves a relation between these variables and a variable related to a wrong answer. Also the verbal indication and the visual recognition of transitivity in task DI2 are related to the comprehension of proof. In essence, this cluster indicates that the behavior of students towards the way they solve the discursive apprehension tasks is coherent enough. The last subgroup comprises the significant relation between the wrong justification in the first task (DI1wj) and the wrong inference in the sixth task (DI6wi). This subgroup is significantly related to all the previous subgroups, which are mainly related to the mobilization of the discursive apprehension. The existence of this relation reveals that although the justifications prove be wrong, the procedure for solving these proof tasks is related to the involvement of the discursive apprehension. In the second similarity cluster relations are found between the students' answers in tasks DI3 and DI4, but specifically those related to the operative apprehension (DI3op and DI4op) and to the perceptual apprehension (DI4pe). The third similarity cluster includes the variables corresponding to the non provision of justification in task DI1 (DI1nj) and to no inference in task DI6 (DI6ni), which are related to the perceptual solution in the DI3 (DI3pe).

In the similarity diagram for the 10th graders (Figure 31) the variables are distributed into five similarity clusters. The first similarity comprises mainly the answers in which the inference and the justification were correct and answers displaying comprehension of proof. Specifically, this cluster includes two subgroups of variables. In the first subgroup the correct justification in task DI1 (DI1cj), the combination of the verbal indication and the visual recognition of transitivity in task DI2 (DI2vrvei), the verbal indication of transitivity in the second task (DI2vei) and the comprehension of proof displayed in task DI5 (DI5c) are related. In fact the relation between the last two variables is significant. The other subgroup contains the significant relation between the comprehension of proof with a gap in task DI5 (DI5cg) and the correct inference in task DI6 (DI6ci), which are next related to the wrong inference in the same task (DI6wi) Thus in this cluster the first subgroup only contains variables that indicate a good function of the discursive apprehension, whereas in the second subgroup there are relations between variables that show comprehension of proof but also the opposite. Therefore coherence is displayed by the first subgroup, but this is not the case for the second subgroup of variables. The first similarity cluster is significantly related to the second similarity cluster which is formed by the relation between the answers in the tasks on the recognition of proof which occur through the discursive apprehension (DI3di and DI4di). These two clusters are further significantly related with the next similarity cluster, formed by the relation between the wrong justification and the wrong answer in the DI1(DI1wj) and DI5 (DI5wa) tasks respectively. As in the previous diagrams, there are so far significant relations between the variables that are related to the mobilization of the discursive apprehension, either expressing correct or incorrect answers. The fourth similarity cluster is created by the relation between two variables corresponding to answers where no justification and no inference were displayed (DI1nj and DI6ni). In the last similarity cluster the relations regarding students' answers in the DI3 and DI4 tasks are found, but specifically those related to the perceptual (DI3pe and DI4pe) and the operative (DI3op and DI4op) apprehension. In contrast to the 9^{th} graders' responses in these two tasks, the 10^{th} graders display consistency in their answers, as the same type of apprehension is mobilized for the solution of each task. Therefore the 10^{th} graders seem to be more stable regarding the recognition of proof than the 9^{th} graders. Generally the 10^{th} grades appear to solve the proof tasks involving the discursive apprehension which leads to correct answers, but in some cases these correct answers are not accompanied by the right justification. The absence of justification and inference are not related to the discursive apprehension, but different types of apprehension seem to be related to these results.

The answers of the students in grade 11a are distributed into two distinct similarity clusters (Figure 31). In the first similarity cluster three subgroups are formed. The variables DI2vr, and DI3op were omitted from the analysis, due to low frequency. The first similarity cluster includes variables indicating the proper involvement of the discursive apprehension. In fact the relation between the correct justification in task DI1 (DI1cj) and the comprehension of proof in task DI5 (DI5c) and the significant relation between the answers in the pair of tasks on the recognition of proof related to the discursive apprehension (DI3di and DI4di) are found. Actually all the variables are significantly related. The second subgroup includes the significant relation between the combination of the verbal indication and the visual recognition of transitivity in task DI2 (DI2vrvei) and the comprehension of proof but with a gap in the fifth task (DI5cg). This subgroup forms a significant relation with the previous subgroup. The last subgroup is created by the relation between the non existence of justification in the first task (DI1nj) and no indication of inference in the sixth task (DI6ni), a relation which also appears in the similarity diagram for the 10th graders. Therefore the first two subgroups of this first similarity cluster seem to represent discursive apprehension. However, this is the first time that relations are found between variables related to the mobilization of the discursive apprehension and the non existence of justification and inference, which were so far related to the perceptual and the discursive apprehension. The second similarity cluster is formed by two subgroups. In the first one the wrong justification and wrong answer in tasks DI1 (DI1wj) and DI5 (DI5wa) are related to the verbal indication of transitivity in the second task (DI2vei) and the correct inference in task DI6 (DI6ci). The other subgroup includes the two perceptual solutions in the pair of tasks examining the recognition of proof (DI3pe and DI4pe), the wrong inference in the sixth task (DI6wi) and the solutions in task DI4 relating to the operative apprehension. It is interesting that, unlike the rest of the variables regarding the wrong answers or inference, the wrong inference in the sixth task (DI6wi) is related to the activation of the perceptual apprehension in tasks DI3 and DI4. So the incorrect inference in this task can be attributed to the influence of perception in the proving process, which inhibited the right functioning of the discursive apprehension and thus the proper inference. Therefore the influence of the perceptual apprehension on the discursive apprehension was revealed, indicating that it may lead students to wrong inferences and justifications.



Grade 9

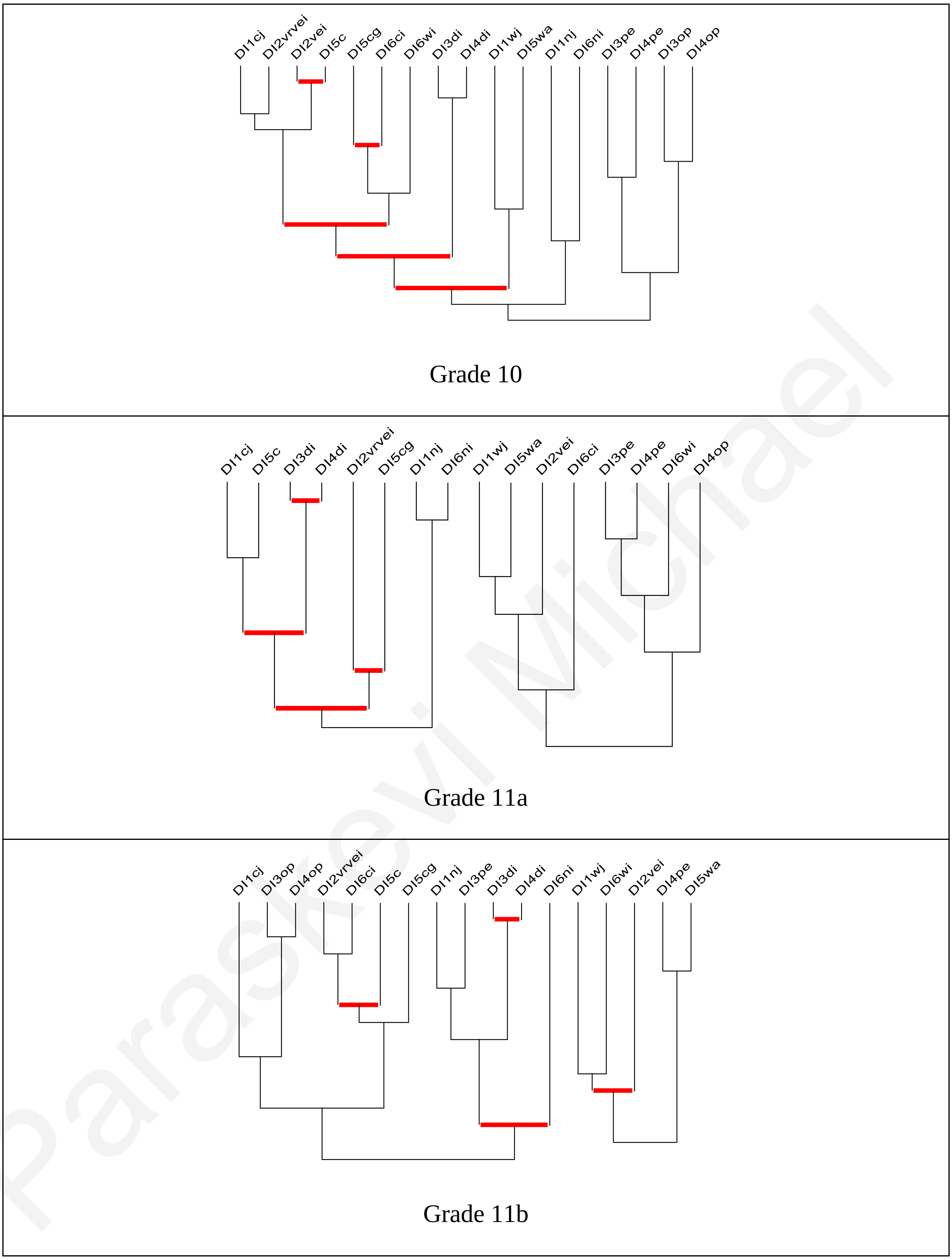


Figure 31. Similarity diagrams for the students' responses from the cognitive point of view in the discursive apprehension tasks for each grade

In the similarity diagram for the students' answers in grade 11b (Figure 31) in the discursive apprehension tasks according to the cognitive point of view, the answers are placed into three similarity clusters. Actually the two first groups of variables are related, but this similarity relation is very low; therefore, they are considered as two distinct clusters. In the first similarity cluster the variables create two subgroups. In the first the solutions occurring from the involvement of the operative apprehension for the solution of the tasks about the recognition of proof (DI3op and DI4op) are related to the correct justification in the first task (DI1cj). In the second subgroup the verbal indication and the visual recognition of transitivity in task DI2 (DI2vrvei), the correct inference in the task DI6 (DI6ci) and the comprehension of proof in the task DI5 are significantly related (DI5c). The comprehension of proof with a gap in the same task is also related to the aforementioned variables. Thus in this cluster the variables indicating the involvement of the discursive and the operative apprehension are related for the first time. In the second cluster the absence of justification in the first task (DI1nj) is related to the perceptual solution in the DI3 (DI3pe). This relation indicates that the non provision of justification in task DI1 can be related to the influence of the perceptual apprehension. Thus students seem to have given their answers without actually making any inference and using any theorem, but only through elaborating the given figure perceptively. In this cluster the significant relation between the answers through the discursive apprehension in the pair of tasks on the recognition of proof (DI3di and DI4di) is also found. Finally the two previous relations are significantly related to the absence of inference in task DI6 (DI6ni). So in this grade the absence of justification and inference is related to the discursive and the perceptual apprehension. In the last similarity cluster the wrong justification in the first task (DI1wj) and the wrong inference in the sixth task (DI6wi) are related. These two variables are also significantly related to the verbal indication of transitivity in task DI2 (DI2vei). Then, these variables form a relation with the perceptual answer in task DI4 (DI4pe) and the wrong answer in task DI5 (DI5wa). The relation between the last two variables can be indicative of the influence of perception for the incorrect answer in task DI5. In fact, for the solution of task DI5 the involvement of the operative apprehension was necessary, in order for students to discriminate the proper reconfigurations and proceed to proving. Therefore the wrong answer in this task can be the result of the influence of the perceptual apprehension, which did not allow modifications on the given figure. Thus the necessary reconfigurations were not available to students and they were not able to continue with the proving procedure.

The data which came about from the cognitive analysis of the tasks of the test were also analyzed using the implicative analysis, through which the implicative relations between the variables were revealed. The implicative analysis was performed separately for each type of apprehension and the outcome for each analysis is described below.

The implicative relations between the variables regarding the analysis of all students' responses in the operative apprehension tasks from the cognitive point of view are indicated in figure 32. In effect, students' answers in these tasks were categorized according to the type of apprehension that was involved. In fact there were answers which occurred from the mobilization of the operative apprehension and thus the use of the mereologic modification on the given figure, answers that involved the activation of the perceptual apprehension and answers in which students used a different approach, mainly related to calculations and measurements.

In the implicative diagram for the total sample (Figure 32) two groups of variables can be distinguished. The first group is formed by implicative relations among the three kinds of approaches. In fact the use of a different approach in task OP1 (OP1da), the mobilization of perception in the same task (OP1pe) and the use of the mereologic modification in tasks OP4 (OP4me) and OP5 (OP5me) form an implicative relation with the involvement of perception in the solution of task OP2 (OP2pe). The fact that task OP2 is involved in these relations between the aforementioned variables shows the strong influence of perception on the solution of the particular tasks, because it appears that although all three kinds of approaches are used by the students for the solution of other tasks (OP1, OP4, OP5), the solution of task OP2 comes mainly through the intervention of perception. This is also revealed by the percentages of students that answer through perception, which are higher than the percentages for those who use the mereologic modification for solving this task. This could be attributed to the presence of numbers in this task, which may have caused difficulties mobilizing the operative apprehension and on the other hand facilitated the intervention of the perceptual apprehension. Therefore the first group of variables shows that the perceptual apprehension is the type of apprehension that is mobilized in a more effective way for the solution of task OP2.

The use of the mereologic modification for the solution of task OP5 (OP5me) is also involved in the implicative relations that form the second group of variables in the implicative diagram. In fact this group of variables only includes the answers that occurred from the activation of the operative apprehension. Particularly the implications are formed by the use of the mereologic modification for the solution of tasks OP1, OP2, OP3 and OP5 (OP1me, OP2me, OP3me and OP5me respectively). This group reveals a greater consistency in the use of the mereologic modification for the solution of the operative apprehension tasks, in relation to the rest of the approaches.

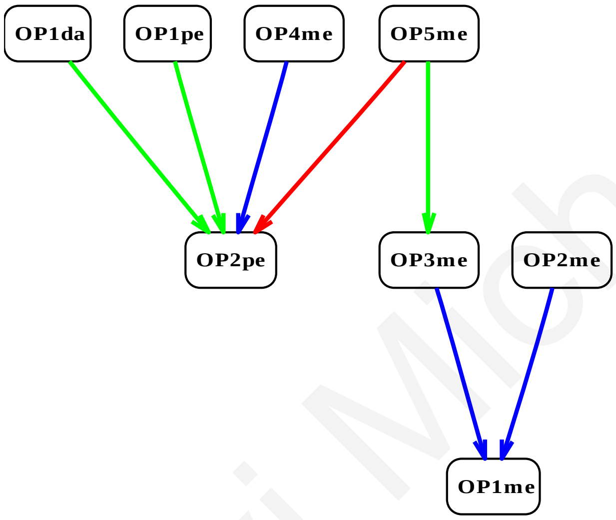
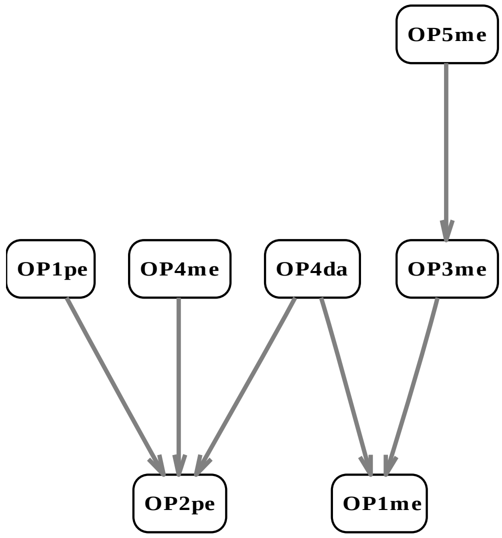
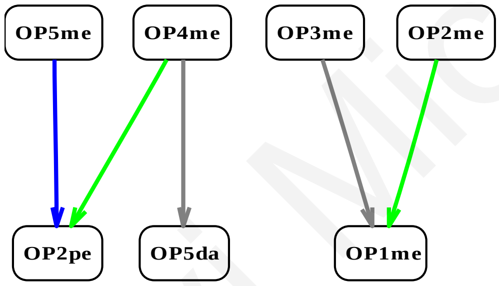


Figure 32. Implicative diagram for the students' responses from the cognitive point of view in the operative apprehension tasks for the total sample

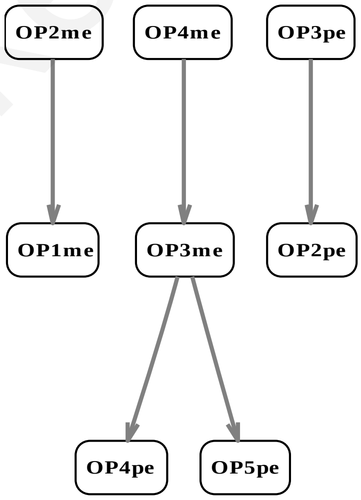
The diagram above presents the way all the students answer in the tasks that examine the operative apprehension of the geometrical figure. However it is also important to trace the particularities of each group of students in the way they confront the specific tasks. The implicative diagrams for each group of students appear in figure 33.



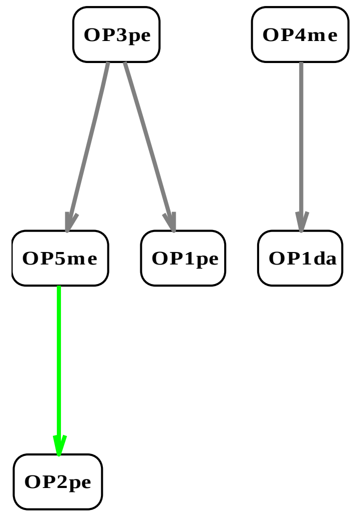
Grade 9



Grade 10



Grade 11a



Grade 11b

Figure 33. Implicative diagrams for the students' responses from the cognitive point of view in the operative apprehension tasks for each grade

In the diagram for grade 9 students (Figure 33) there are implicative relations between variables that correspond to the use of the three types of approaches. In fact there is a group similar to the one formed in the implicative diagram for all students. In this case this group includes the involvement of perception in task OP1 (OP1pe) and the use of the mereologic modification and the different approach in the fourth task (OP4me and OP4da) respectively, which form an implicative relation with the perceptual solution in the task OP2. Therefore in this grade applies what holds true for the total sample as regards the solution of task OP2. Another similar relation is the use of the mereologic modification in tasks OP1 (OP1me), OP3 (OP3me) and OP5 (OP5me). So there is coherence in the use of the mereologic modification in these three tasks.

For the 10th graders two distinct implicative chains are distinguished (Figure 33). The first implicative chain includes implicative relations among the use of the mereologic modification, the perceptual approach and the use of a different approach. Particularly the use of the mereologic modification in task OP5 (OP5me) is related to a perceptual answer in task OP2 (OP2pe). The perceptual answer in task OP2 (OP2pe) is also related to the use of the mereologic solution in task OP4 (OP4me). Similarly to grade 9, the solution of task OP2 is related to answers occurring from operative apprehension. The use of the mereologic modification in other tasks does not necessarily lead to the adoption of the same method for the solution of task OP2, but on the contrary the influence of perception cannot be inhibited. This variable OP4me is also linked to the use of the different approach in task OP5 (OP5da). Thus, what this chain indicates is that there seems to be no consistency in the use of each type of approach in grade 10. On the other hand, the second implicative chain is an indicator of consistency in the use of the mereologic approach in tasks OP1, OP2 and OP3 (OP1me, OP2me and OP3me).

The implicative diagram for grade 11a students' answers includes three implicative chains (Figure 33). The first implicative chain includes the relation between the use of the mereologic modification in tasks OP1 and OP2 (OP1me and OP2me). The second implicative chain is created by the variable corresponding to the use of the mereologic approach in task OP4 (OP4me), the use of the mereologic modification in the third task (OP3me) and the perceptual answers in the fourth and fifth tasks (OP4pe and OP5pe). The last implicative chain is formed by the answers occurring from the perceptual apprehension in tasks OP2 and OP3 (OP2pe and OP3pe). The implications between the variables provide indications of consistency in the use of the mereologic modification and the activation of the perceptual apprehension. In this diagram there are relations between the operative and the perceptual apprehension, but the use of the different approach is not present in any relations. In addition, in the implicative diagram for grade 11a the relation including the perceptual answer in task OP2 (OP2pe) does not appear, as in grades 9 and 10.

The implicative diagram for grade 11b students' answers (Figure 33) indicates two implicative chains. The first implicative chain is formed by an implicative relation between the perceptual answers in the third and the first tasks (OP3pe and OP1pe). The perceptual answer in the third task is also related to the use of the mereologic modification in task OP5 (OP5me), which is next related to the answers occurring from the perceptual apprehension in task OP2 (OP2pe). The second implicative chain includes the relation between the use of the mereologic approach in task OP4 (OP4me) and the use of a different approach in task OP1 (OP1da).

The comparison between the implicative relations regarding the cognitive analysis of the students' solutions in the operative apprehension tasks shows that there are differentiations in the way the different groups of students solve the particular tasks. In all the groups of students there are implicative relations indicating stability in the involvement of the operative apprehension for the solution of some of the tasks. In grades 9, 10 and 11b there are relations among the variables corresponding to the three types of solutions, whereas these relations are not observed in grades 11a in which the variables corresponding to the use of a different approach do not appear in the similarity diagrams.

The cognitive analysis of students' answers in the perceptual apprehension tasks was based on the number of figures students have recognized correctly. The implicative relations among the variables corresponding to all students' answers in the perceptual apprehension tasks are presented in figure 34. In fact there are three implicative chains in the diagram. The first implicative chain is formed by the implicative relations between the recognition of 5-6 figures (R3) and the recognition of less than 5 figures (R4) with the false recognition in task PE2 (Rf). The second implicative chain includes the relation between the recognition of the two most obvious squares (Risq) in task PE2 and the recognition of almost all the figures (R2) in task PE1. The last implicative relation between the variables corresponding to the correct recognition of all the figures in task PE1 (R1) and the correct recognition of all the squares in task PE2 (Rasq) forms the third implicative chain.

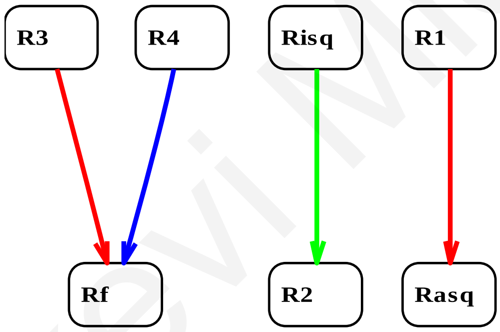
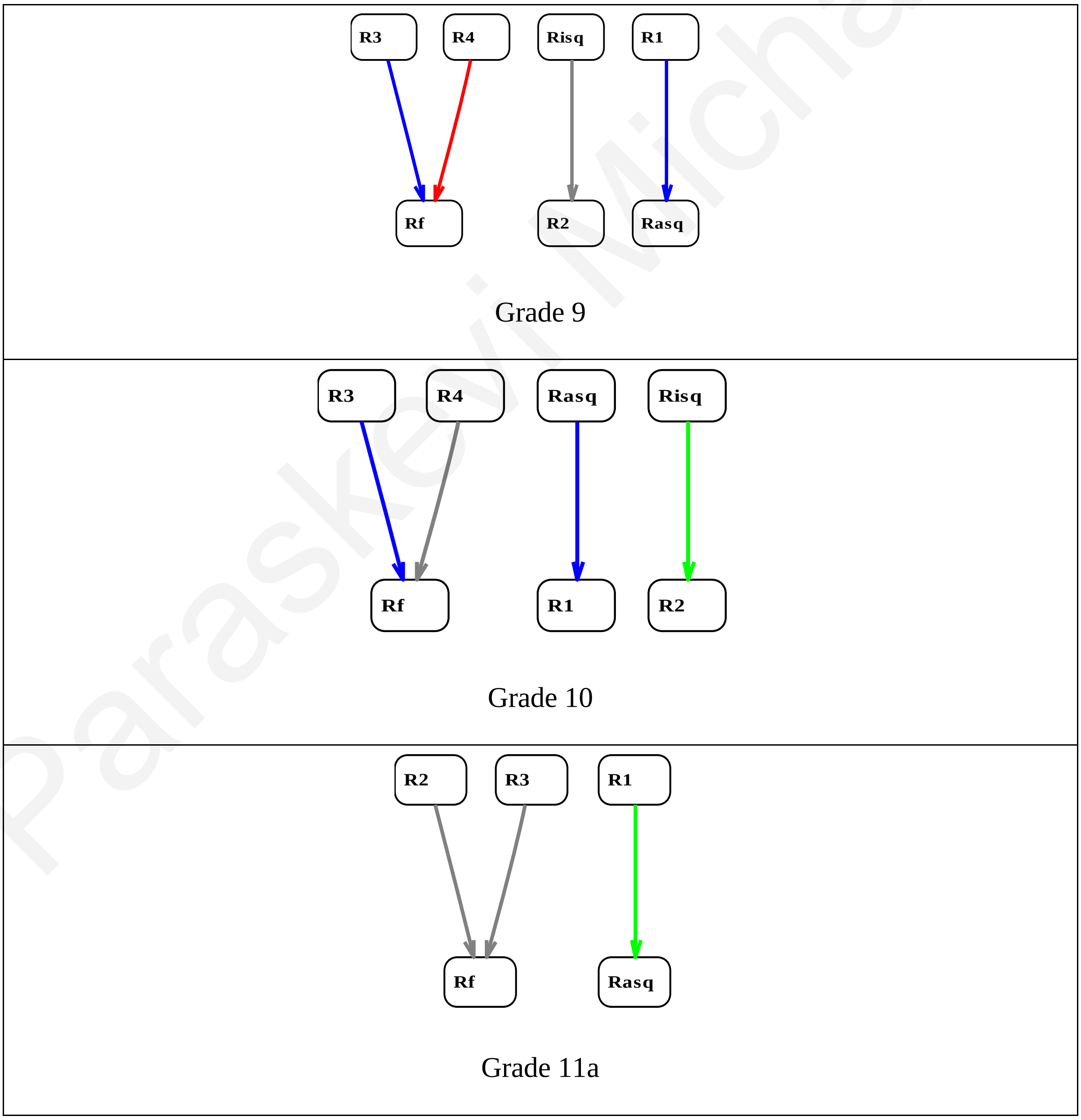


Figure 34. Implicative diagram for the students' responses from the cognitive point of view in the perceptual apprehension tasks for the total sample

Generally what the implicative diagram for the total of students indicates is that the students that recognize correctly all the squares in task PE2 also recognize all the figures in task PE1. The students that recognize only the two most obvious squares in task PE2 can recognize almost all the figures in task PE1. Finally those who carry out false recognition in task PE2 recognize a fewer number of figures (6 and below) in the PE1.

Figure 35 presents the implicative diagrams for the different groups of students, regarding their answers in the perceptual apprehension tasks. The implicative diagrams for grades 9 and 10 are the same as the implicative diagram for the total sample. However some differentiations appear in grades 11a and 11b. Specifically the false recognition of squares in task PE2 (Rf) forms implicative relations with the correct recognition of almost all the figures (R2) and the right recognition of 5-6 figures (R3). On the other hand the relation between the correct recognition of all the figures in task PE1 and the correct recognition of all the squares in task PE2 is still present in the implicative diagrams for grades 11a and 11b. Therefore the way the students deal with the perceptual apprehension tasks is the same in grade 9 and grade 10, but there is difference between these two groups and the students in grades 11a and 11b. In addition the way the students confront the perceptual apprehension tasks is the same in grade 11a and grade 11b. Thereafter the changes in the students' perceptual apprehension come after the students move to grade 11 and especially after the transition from grade 9 to grade 10.



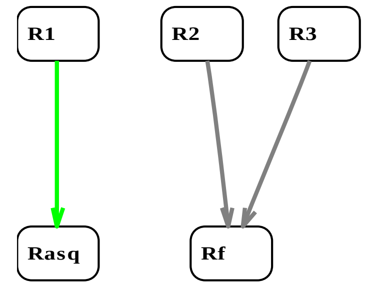


Figure 35. Implicative diagrams for the students' responses from the cognitive point of view in the perceptual apprehension tasks for each grade

The implications among all the students' solutions in the tasks examining the sequential apprehension of the geometrical figure, according to the cognitive analysis of the tasks, are displayed in figure 36. In these tasks students' answers were categorized according to whether the construction of the geometrical figure was correct (SE1, SE2, SE3), partly correct (SE2pc) or came about with the involvement of the perceptual apprehension (SE1ps, SE2ps, SE3ps). In the implicative analysis the variables related to solutions with no success at all (SE1ns, SE3ns) are also included, in order to examine the implications between the wrong answers and the rest of the variables.

In the implicative diagram for the total sample (Figure 36) two groups of variables are formed. Specifically the first implicative chain includes the variables regarding the perceptual solutions (SE2ps, SE3ps) or an unsuccessful solution (SE1ns). The second group of variables is created by the implicative relations between the correct constructions in the three sequential apprehension tasks (SE1, SE2, SE3) and the partly correct construction in the second task (SE2pc). Therefore the first implicative chain is indicative of the influence of the perceptual apprehension in the procedure of the construction of a geometrical figure. The other group of variables indicates the coherence in the activation of the sequential apprehension which leads to a correct construction of the geometrical figure or even a partly correct construction if a particular difficulty occurs at a specific step during the construction process. In addition there are no relations between the two implicative chains, showing that compartmentalization occurs regarding the cognitive processes involved in each chain.

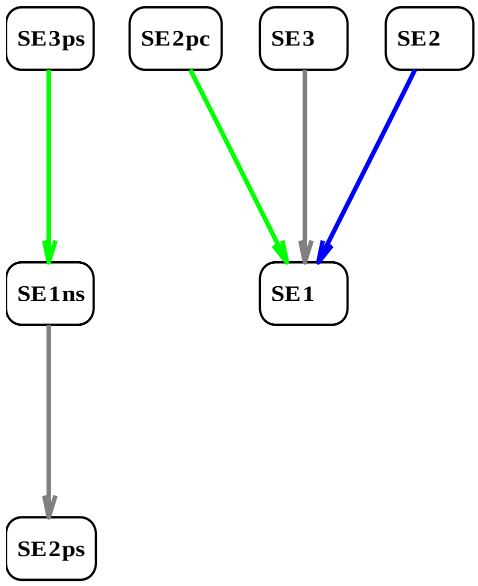


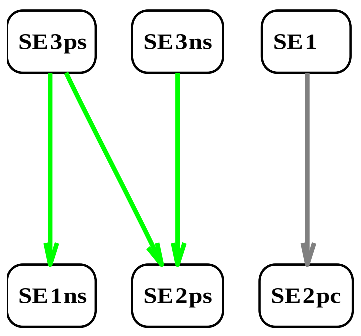
Figure 36. Implicative diagram for the students' responses from the cognitive point of view in the sequential apprehension tasks for the total sample

The implicative diagram for each group of students regarding their answers in the sequential apprehension tasks is presented in figure 37. For the 9th graders there are implicative relations between the variables that represent an unsuccessful construction or constructions that occurred after the involvement of the perceptual apprehension. In fact the perceptual solution in task SE3 (SE3ps) leads to the same behavior in task SE2 (SE2ps) and to no success in task SE1 (SE1ns). Also the incorrect construction in task SE3 (SE3ns) leads to a perceptual solution in task SE2 (SE2ps). In addition this diagram includes another distinct relation between the correct construction in task SE1 (SE1) and the partly correct construction in task SE2 (SE2pc). Therefore there are indications of coherence in the involvement of the perceptual apprehension which is also related to the unsuccessful solutions, whereas no strong consistency is found regarding the proper mobilization of the sequential apprehension for the solution of the tasks.

Weaker relations between the variables appear in the implicative diagram for grade 10 students (Figure 37). However the existence of these relations shows stability in students' solutions that occur through the involvement of the perceptual apprehension. Specifically in the first implicative chain the perceptual solution in the first sequential apprehension task (SE1ps) leads to the same type of solution in the third task (SE3ps) and this further leads to a perceptual solution in the second task (SE2ps). The second implicative chain displayed in this diagram is formed between the partly correct construction in the second task (SE2pc) and the success in task SE1. Also in this grade the correct or partly correct construction form no implicative relation with the variables corresponding to unsuccessful or perceptual answers. The second implicative relation is formed between two solutions that occur through the involvement of the perceptual apprehension (SE3ps and SE2ps). This relation indicates stability in the influence of perception during the construction process, but coherence appears regarding the proper mobilization of the sequential apprehension. This is not the case for the implicative diagram of grade 11a (Figure 37). In this grade an implicative relation is formed between the partly correct construction in task PE2 (PE2pc) and the perceptual answer in task SE1 (SE1ps).

The greatest coherence in the proper activation of the sequential apprehension and thus the successful geometrical constructions appears in grade 11b. In fact the successful construction of the geometrical figures in the three sequential apprehension tasks is related. Similarly to the rest of the implicative diagrams (Figure 37), the other group of variables includes implications between variables related to the perceptual (SE2ps, SE3ps) or unsuccessful solutions (SE1ns). Again the two groups of variables are distinct and no relations are formed between them, thus compartmentalization appears in this case as well.

What surfaces from all the diagrams of figure 37 is that there are two groups of variables, each one corresponding to the proper mobilization of the sequential apprehension or the influence of the perceptual apprehension respectively. What differs in each group of students is the degree of stability in the way these types of apprehensions are involved in the solution of the tasks. The greatest coherence regarding the mobilization of the sequential apprehension is found in grade 11b.



Grade 9

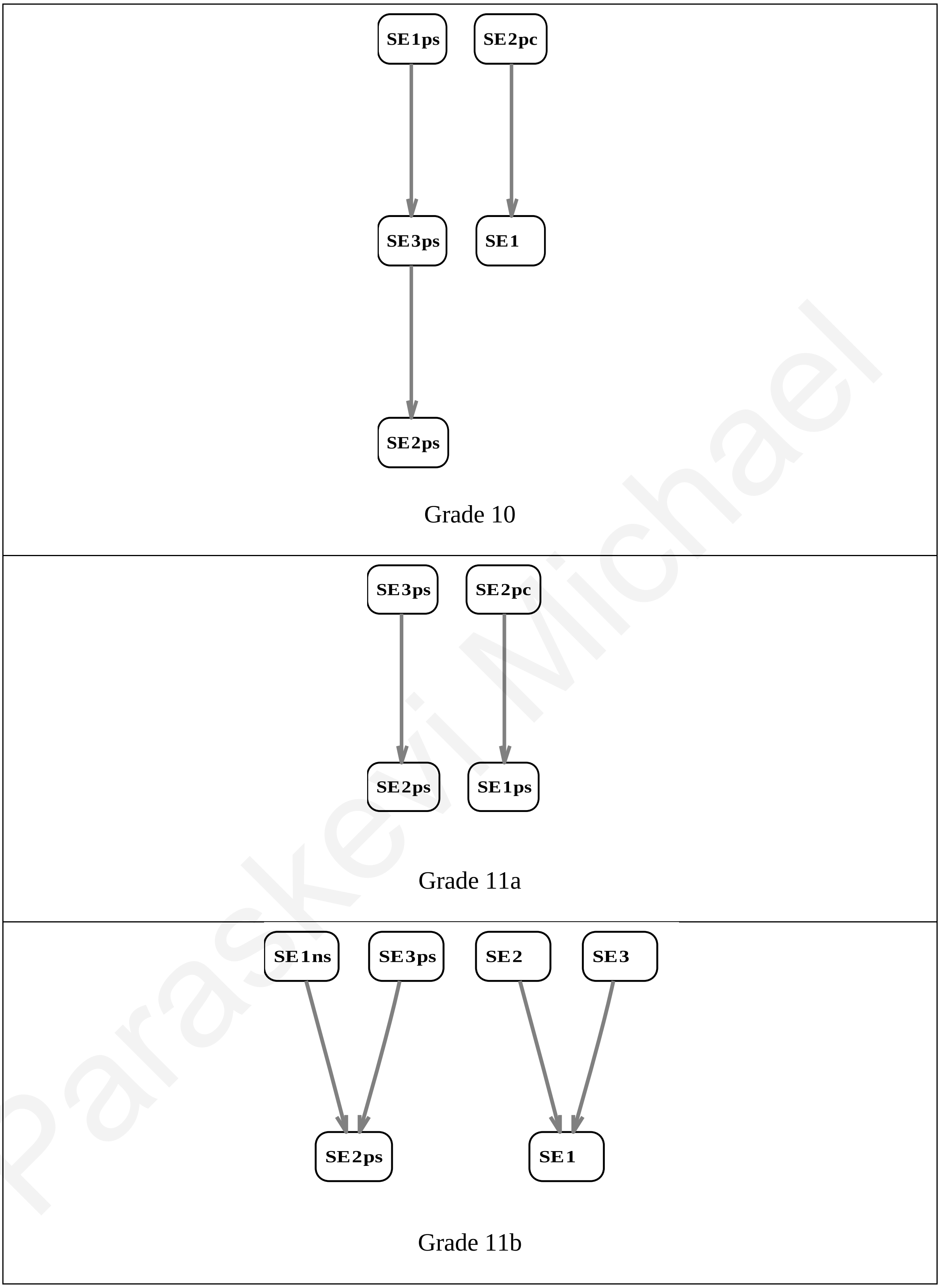


Figure 37. Implicative diagrams for the students' responses from the cognitive point of view in the sequential apprehension tasks for each grade

The consideration of students' answers from a cognitive point of view lead to categorizing these answers according to the type of apprehension involved for the solution of these tasks, to the justifications provided, to students' inferences and to the degree of the comprehension of proofs. Figure 38 includes the implicative relations between these variables and provides information about all students' ways of solving the five discursive apprehension tasks they were given. In this implicative diagram there is a large group including most of the variables. The implicative relations start with an implicative chain which includes the comprehension of proof in task DI5 (DI5c), the provision of a correct justification in task DI1 (DI1cj) and the correct inference in task DI6 (DI6ci). The former variable is also related to the comprehension of proof but with a gap in task DI5 (DI5cg).

From this variable another subgroup of variables begins which firstly includes the implicative relations among the answers that occur through the mobilization of the discursive apprehension in the two tasks testing students' abilities in the recognition of proofs (DI3di and DI4di). In this subgroup the verbal indication of the transitivity between the relations among the different figural units of the given figure in task DI2 (DI2vei) and the combination of the verbal indication and the visual recognition of transitivity in the same task (DI2vrvei) are also related to the aforementioned variables.

Another group of variables is formed at the bottom of this large group of variables. In fact, this subgroup includes the incorrect answers, justifications and inference in tasks DI5 (DI5wa), DI1 (DI1wj) and DI6 (DI6wi) respectively.

Finally this implicative diagram includes an implicative relation which is separated from the relations formed between the rest of the variables. In fact this relation regards the two tasks examining the recognition of proofs and in effect the answers that occurred through the mobilization of the perceptual apprehension (DI3pe and DI4pe).

Generally the three groups of variables included in the implicative diagram for the total number of students could be characterized as corresponding to three levels of proof abilities. The first group includes variables indicating a proper mobilization of the discursive apprehension which brings to the correct inference and justifications in the proof tasks and thus developed proof abilities. The second group involves variables that also indicate abilities regarding proofs, but of lower level, as these abilities are not merely related to the production of proof, but to the recognition of proofs or to a production of proof but not through a discursive process, but through the use of visual methods, like drawing on the figure to indicate relations between different parts of the figure. Finally the last group of variables indicates a lower level of proving abilities, as it includes variables that are related to wrong answers, justifications and inference. In addition stability is revealed in students' answers in the two tasks on the recognition of proof, regarding the answers that result from the perceptual apprehension or the discursive apprehension. Students mobilize the same kind of apprehension in order to answer these two tasks which are of the same type.

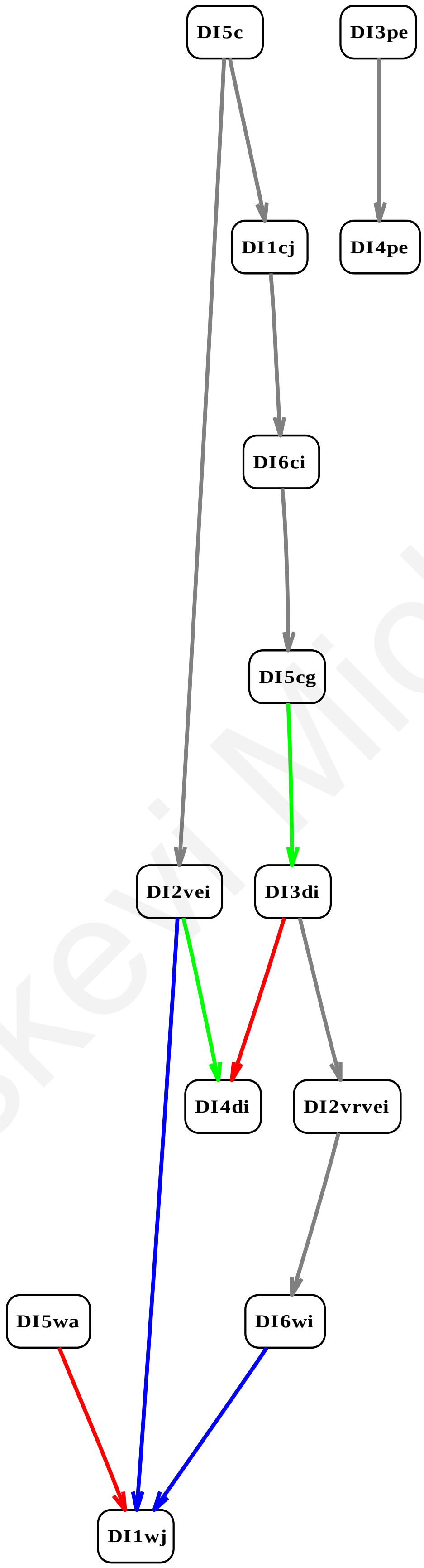


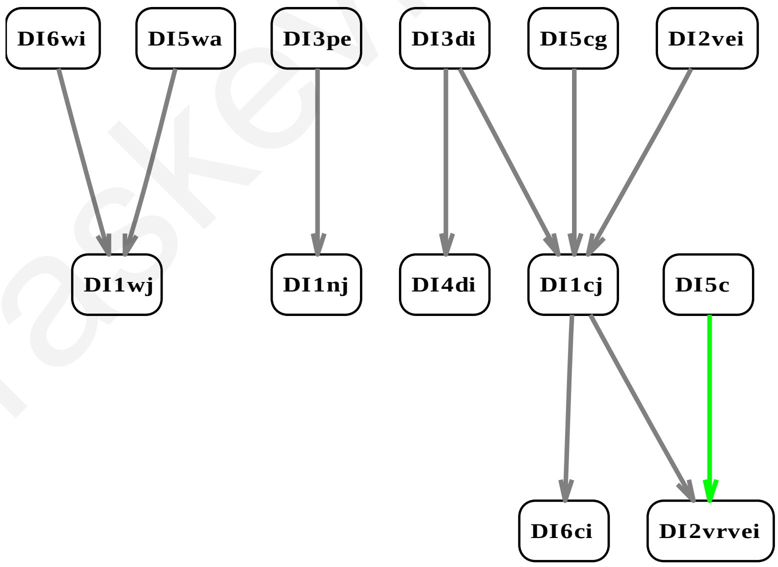
Figure 38. Implicative diagram for the students' responses from the cognitive point of view in the discursive apprehension tasks for the total sample

The implicative diagram for each group of students is included in figure 39. The variables in the implicative diagram for the 9th graders are separated in three distinct groups. The first group includes variables corresponding to a wrong answer (DI5wa), to wrong inference (DI6wi) and to a wrong justification (DI1wj). The second group of variables includes the answer that occurred though the mobilization of the perceptual apprehension in the task on the recognition of proof (DI3pe) and the absence of justification in task DI1 (DI1nj). The last group of variables includes more implicative relations between the variables. In fact this group includes the answers that are related to the mobilization of the discursive apprehension in the two tasks concerning the recognition of proof (DI3di and DI4di), the variables that show comprehension of proof or comprehension but with a gap in task DI5 (DI5c and DI5cg respectively), the correct justification and inference in tasks DI1 and DI6 respectively (DI1cj and DI6ci) and finally the verbal indication of transitivity in task DI2 (DI2vei) as well as the combination of the verbal indication and the visual recognition of transitivity in the same task (DI2vrvei).

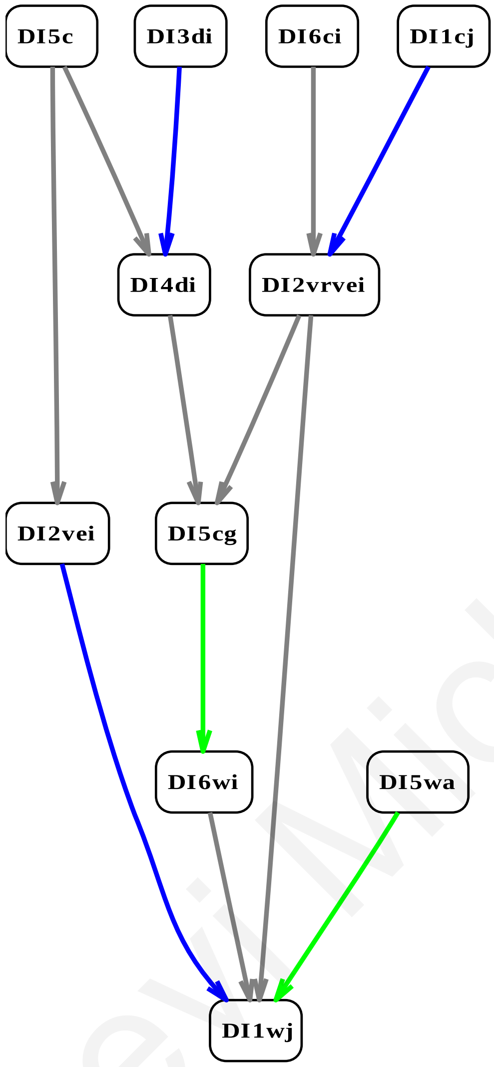
The implicative diagram of the 10th graders' responses to the discursive apprehension tasks (Figure 39) also includes three groups of variables that are related. The first group of variables includes the implicative relation among the answers that occurred through the mobilization of the discursive apprehension in the two tasks on the recognition of proof (DI3di and DI4di) and the relation with the comprehension of proof in task DI5 (DI5c). The second group of variables is formed by the implicative relations between the correct justification and inference in tasks DI1 and DI6 respectively (DI1cj and DI6ci) and the combination of the verbal indication and the visual recognition of transitivity in task DI2 (DI2vrvei). The relations present in the two aforementioned groups of variables converge at the variable representing the comprehension of proof but with the existence of a gap in task DI5 (DI5cg). From this variable new relations emerge which can be considered as the third group of variables. In fact, the last group, similarly to the previous diagram of grade 9 students, includes the variables regarding a wrong answer (DI5wa), wrong inference (DI6wi) and wrong justification (DI1wj). The variable DI1wj is also related with the verbal indication of transitivity in the second task (DI2vei).

Four distinct implicative chains are formed in the implicative diagram for grade 11a students (Figure 39). The first one involves three variables, two of which correspond to an incorrect answer (DI5wa) and to a wrong justification (DI1wj). In this implicative chain the verbal indication of transitivity in the second task (DI2vei) is also participating. The second implicative chain is formed by the relation between the answers through the mobilization of the discursive apprehension in the two tasks on the recognition of proof (DI3di and DI4di). Subsequently, the relation between the absence of inference and justifications in tasks DI1 (DI1nj) and DI6 (DI6ni) is found. The last implicative chain indicates stability in the activation of the perceptual apprehension for answering the pair of tasks about the recognition of proof (DI3pe and DI4pe). This type of answers in tasks DI3 and DI4 lead to wrong inference in task DI6 (DI6wi).

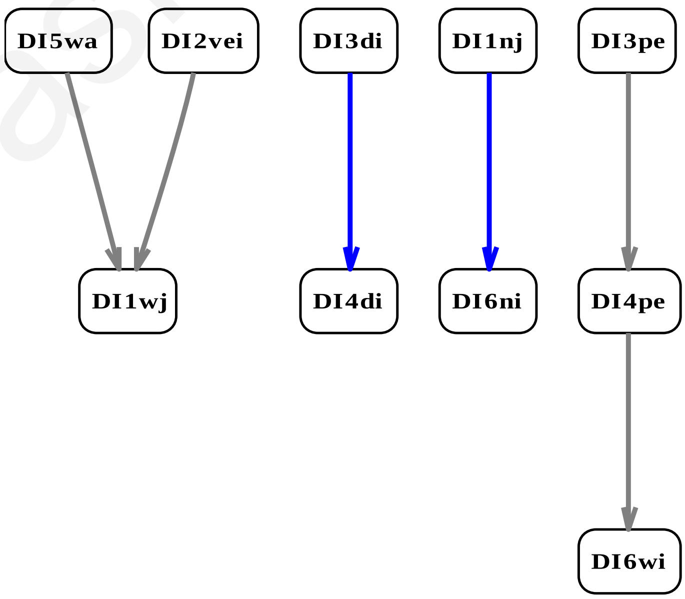
The last implicative diagram is the one comprising grade 11b students' answers (Figure 39). In the particular diagram two distinct implicative chains appear. The first one starts with the correct inference in task DI6 (DI6ci). The correct solution of this task leads either to a solution that indicates comprehension of proof in task DI5 (DI5c) or to answer showing comprehension of proof but also a gap in task DI5 (DI5cg). Surprisingly the variable DI5c is related to a wrong justification in task DI1 (DI1wj). Actually this is the first time that such a relation appears, since in all the previous diagrams there was not a relation between the comprehension of proof and a wrong answer. The second implicative chain indicates that the mobilization of either the discursive or the perceptual apprehension in task DI3 brings to the mobilization of the discursive apprehension in the similar task DI4.



Grade 9



Grade 10



Grade 11a

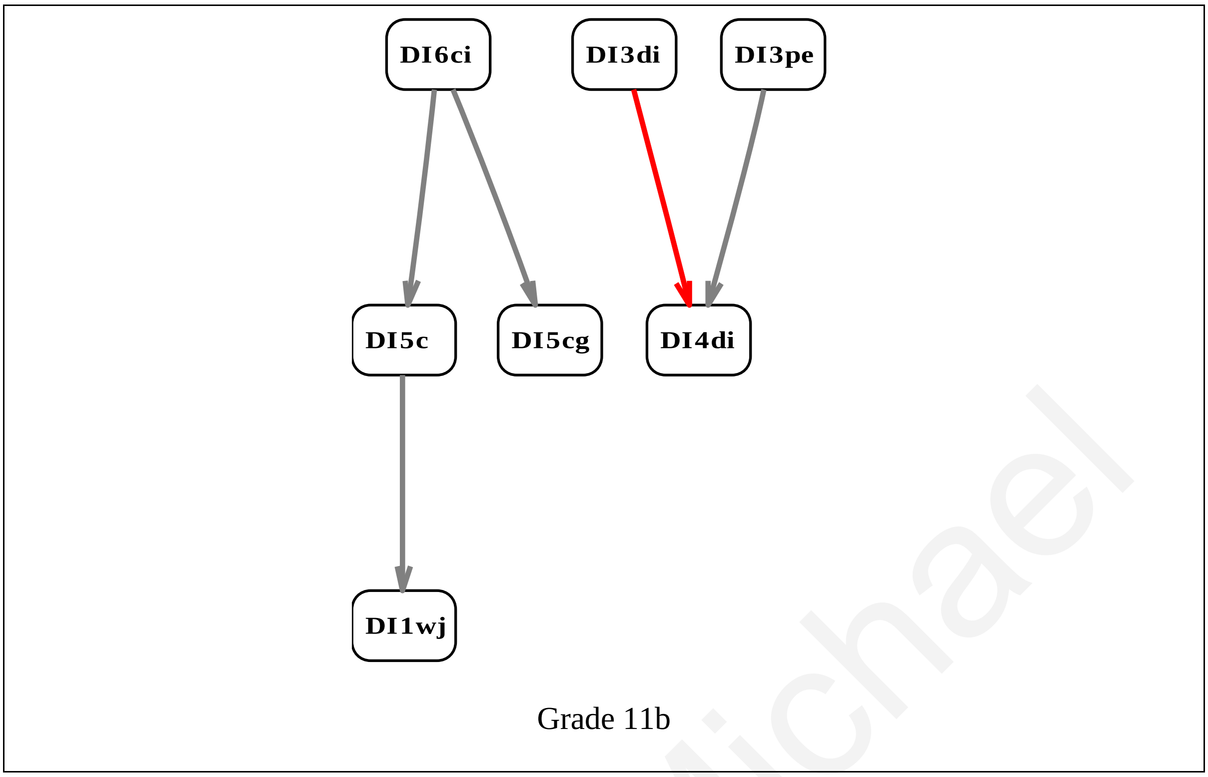


Figure 39. Implicative diagrams for the students' responses from the cognitive point of view in the discursive apprehension tasks for each grade

The comparison between the implicative diagrams for each group of students shows that in each diagram there are groups of variables indicating different levels of proving abilities, similar to what the groups of variables of the implicative diagram for the total sample express. However these groups include different relations in each grade. Therefore the groups of variables indicating possible different levels of proof abilities comprise different variables in each grade. In addition in all grades stability is displayed in tasks DI3 – DI4 regarding the discursive and the perceptual apprehension. However there is no appearance of the operative apprehension in the relations of these tasks. The three groups are more clearly distinguished in grade 9 and 10, but this is not the case for grade 11a and 11b, in which less stability appears regarding the mobilization of the discursive apprehension.

The Comparison between the Lower and the Upper Secondary School Students' Geometrical Figure Apprehension

The Effect of Age on the Geometrical Figure Apprehension

Table 10 presents the students' performance in the tasks of the test for the total sample according to grade. The results concern the percentages of right answers from a mathematical point of view. This means that what was taken into account was whether the answer was correct or not, but without taking into account the particular type of apprehension that was essentially involved in solving the tasks. This is, in fact, taken into account in a following section which deals with the cognitive analysis of the students' answers, in which the students' correct answers are examined in relation to the particular type of apprehension that was involved in reaching this answer.

Thus these are the results for students' right answers in the tasks, without taking into consideration the type of apprehension through which these answers occurred. The results revealed differences between the students' performances in the tasks for the different grades. There are cases in which the performance of students is higher as we move to a higher grade, but there are also cases in which the opposite situation occurs. Specifically, there are tasks in which students' success is higher in every next grade. These are tasks PE2, OP1, DI3 and DI4. In fact in these tasks grade 9 students performance is lower compared to grade 10 students' performance. Then the students in grade 11a are more successful than the 10^{th} graders while the same applies for students in grade 11b, compared to students in grade 11a. Therefore for these tasks there seems to be an evolution in students' performance in relation to the hierarchy of the grades they attend. It is interesting that in these tasks the tasks on the recognition of proofs are included. This may provide an indication about the way this ability develops.

On the other hand, some tasks are identified, in which students' performance is not higher in every higher grade, but instead there are fluctuations in their performances in each grade. This is actually the case for the majority of the tasks. The tasks that are included in this category are the first perceptual apprehension task (PE1), four of the operative apprehension tasks (OP2, OP3, OP4 and OP5), two of the sequential apprehension tasks (SE1 and SE2) and finally all the discursive apprehension tasks on the production of proof (DI1, DI2, DI5, DI6). In fact for these tasks students' performance is better in grade 10 compared to grade 9 and grade 11a and the highest performance is observed in grade 11b students. Thus, success in the specific tasks increases from grade 9 to grade 10, then it is reduced from grade 10 to grade 11a and next there is an augmentation in grade 11b.

Table 10

Percentages of the Students' Performance in the Tasks of the Test from the Mathematical Point of View by Grade

Tasks	All students (%)	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
PE1	35.19	21.47	45.72	20.8	55.71
PE2	43.36	36.54	43.75	44.0	57.14
OP1	71.74	62.82	71.38	79.2	85.71
OP2	37.57	32.05	35.86	26.4	63.57
OP3	62.32	54.17	62.50	59.2	82.86
OP4	11.46	10.34	11.84	8.40	15.89
OP5	60.61	53.21	66.78	57.6	66.43
SE1	16.57	9.62	18.09	12.8	32.14
SE2	5.33	1.28	3.29	1.60	22.14
SE3	8.85	8.01	7.24	7.20	15.71
DI1	60.73	57.05	61.18	44.0	82.86
DI2	29.06	21.15	25.99	24.8	57.14
DI3	15.89	9.94	15.13	21.6	25.71
DI4	19.30	14.1	18.75	22.4	29.29
DI5	12.49	9.29	15.79	2.40	21.43
DI6	53.80	50.32	51.97	46.4	72.14

Finally there is a task in which the pattern regarding the fluctuations in students' performance in each grade is rather different than the pattern described above. This is the sequential apprehension task SE3, in which the 10^{th} graders' performance is lower than the 9^{th} graders' performance, but it is equal to the performance of students in grade 11a. As previously, the students in grade 11b have the highest performance from all the other students.

In order to examine whether there are significant differences in the different types of geometrical figure apprehension among the students in each grade, a multivariate analysis of variance (MANOVA) was performed. The means and the standard deviations for students' performance in each type of the geometrical figure apprehension for each grade are presented in Table 11. In all the groups of students the highest scores are found in the operative apprehension, whereas the lowest are noticed for the sequential apprehension.

The effect of students' age is significant (Pillai's $F_{(3, 779)}=14.51$, $p<0.001$) regarding the students' performance in the geometrical figure apprehension tasks. In particular the results are indicative of the differences that exist in the mean performance of students from each grade in the perceptual apprehension tasks ($F_{(3, 779)}=21.48$, $p<0.001$), the operative apprehension tasks ($F_{(3, 779)}=25.20$, $p<0.001$), the sequential apprehension tasks ($F_{(3, 779)}=29.31$, $p<0.001$) and the discursive apprehension tasks ($F_{(3, 779)}=34.43$, $p<0.001$).

Table 11

Means and Standard Deviations of the Students' Performance in Each Type of Geometrical Figure Apprehension by Grade

	All students		Grade 9		Grade 10		Grade 11a		Grade 11b	
Types of Apprehension	$\bar{X}$	S.D	$\bar{X}$	S.D	$\bar{X}$	S.D	$\bar{X}$	S.D	$\bar{X}$	S.D
PE	0.39	0.38	0.29	0.35	0.45	0.39	0.32	0.36	0.56	0.39
OP	0.49	0.24	0.43	0.24	0.50	0.24	0.46	0.23	0.63	0.20
SE	0.10	0.19	0.06	0.15	0.10	0.18	0.07	0.14	0.23	0.28
DI	0.32	0.23	0.27	0.20	0.31	0.22	0.27	0.23	0.48	0.23

The post hoc analysis indicates the statistically significant differences among the mean performances in the tasks examining the different types of geometrical figure apprehension of the different age groups ($p<0.001$). Specifically, in the perceptual apprehension tasks (PE) the mean performance of the 9^{th} graders is statistically significantly lower than the mean performance of the students in grades 10 and 11b, whereas the difference between grade 9 and grade 11a is not statistically significant. As regards the 10^{th} graders, their mean performance in the perceptual apprehension tasks is statistically significantly higher than for the students in grade 11a and statistically significantly lower than the students in grade 11b. Also the mean performance of the students in grade 11a is statistically significantly lower than the mean performance of the students in grade 11b. Regarding the operative apprehension tasks (OP), for the 9^{th} graders the same situation stands as for the perceptual apprehension tasks. The mean performance of the 9^{th} graders is statistically significantly lower than the mean performance of the students in grades 10 and 11b, whereas the difference between grade 9 and grade 11a is not statistically significant. The mean performance of the 10^{th} graders is statistically significantly lower than the students in grade 11b, whereas there is not a statistically significant difference from the mean performance of students in grade 11a. Students' differences are the same in the sequential apprehension tasks and the discursive apprehension tasks. In fact the 9^{th} graders' mean performance is statistically significantly lower than the mean performance of students in grade 11b. However no statistically significant differences appear with the mean performance of students in grades 10 and 11a. In addition, the mean performance of students in grade 10 is statistically significantly lower than the mean performance of students in grade 11b. This is also the case for students in grade 11a.

The Effect of Educational Level on Geometrical Figure Apprehension

The students' performance in the geometrical figure apprehension tasks by educational level are displayed in table 12. The results indicate that the upper secondary school students' performance is higher than the lower secondary school students' performance in all the tasks. In addition the multivariate analysis of variance (MANOVA) was performed in order to examine whether there are significant differences between lower and upper secondary school students' performance in the different types of geometrical figure apprehension. The means and the standard deviations for each type of the geometrical figure apprehension for each grade are presented in Table 13. In all the groups of students the highest scores are indicated in the operative apprehension, whereas the lowest is noticed for the sequential apprehension. Overall, the effect of students' educational level (lower or upper secondary) is significant (Pillai's F(1, 779)=14,86, p<0.001).

Table 12

Percentages of the Students' Performance in the Tasks of the Test from the Mathematical Point of View by Educational Level

Tasks	All students (%)	Middle (%)	High (%)
PE1	43.36	36.54	47.10
PE2	35.19	21.47	42.71
OP1	71.74	62.82	76.63
OP2	37.57	32.05	40.60
OP3	62.32	54.17	66.78
OP4	11.46	10.34	12.08
OP5	60.61	53.21	64.67
SE1	16.57	9.62	20.39
SE2	5.33	1.28	7.56
SE3	8.85	8.01	9.31
DI1	60.73	57.05	62.74
DI2	29.06	21.15	33.39
DI3	15.89	9.94	19.16
DI4	19.3	14.1	22.14
DI5	12.49	9.29	14.24
DI6	53.8	50.32	55.71

In particular, the mean value of upper secondary school students' geometrical figure perceptual ability (PE) is statistically significantly higher ($F_{(3, 779)}=35.80$, $p<0.001$), than the mean value of lower secondary school students. Similarly, the mean value of upper secondary school students in the operative apprehension tasks (OP) is statistically significantly higher ($F_{(3, 779)}=32.85$, $p<0.001$) than the mean value of lower secondary school students. This is also the case for the sequential apprehension tasks (SE) ($F_{(3, 779)}=20.50$, $p<0.001$) and the discursive apprehension tasks (DI) ($F_{(3, 779)}=23.25$, $p<0.001$).

Subsequently, the results indicate that differences exist between the lower and the upper secondary school students' performance regarding the different types of geometrical figure apprehension. Specifically the higher secondary school students' performance is higher in all the types of geometrical figure apprehension compared to the lower secondary school students' performance.

Table 13

Means and Standard Deviations of the Students' Performance in Each Type of Geometrical Figure Apprehension by Educational Level

	All students		Lower		Upper	
Types of Apprehension	$\bar{X}$	S.D	$\bar{X}$	S.D	$\bar{X}$	S.D
PE	0.39	0.38	0.29	0.35	0.45	0.39
OP	0.49	0.24	0.43	0.24	0.52	0.24
SE	0.10	0.19	0.06	0.15	0.12	0.21
DI	0.32	0.23	0.27	0.20	0.35	0.24

Descriptive Analysis of the students' responses to the geometrical figure apprehension tasks from the mathematical and the cognitive point of view

Table 14 summarizes the results regarding the students' answers in the perceptual apprehension tasks, based on the cognitive analysis of the tasks. In the first perceptual task students' answers were categorized according to the number of figures students are able to recognize correctly. For the second perceptual task students' answers were discriminated based on which squares they are able to recognize. That is, whether they have recognized all the squares correctly but without making any other false recognition (Rasq), whether they have recognized the two most obvious squares only (Risq) or whether they have carried out false recognition of squares, even if some squares are recognized correctly (Rf). For these tasks the variables R1 and Rasq correspond to the variables PE1 and PE2 respectively for the consideration of the results from the mathematical point of view.

Regarding the first task PE1, most students achieve the recognition of either all the 9 figures or of 8 or 7 figures. In fact in grades 10 and 11b the higher percentages are found in the recognition of all the figures, whereas in grades 9 and 11a the majority of the students succeed the recognition of the 8 or 7 figures. The number of students that recognized all the figures correctly increases from grade 9 to grade 10 and then it is reduced in grade 11a. The highest number of such answers is found in grade 11b. The situation is the opposite concerning the recognition of the 8 or 7 figures and thus the fluctuations of these two categories of answers are almost analogous in every grade.

In the second task PE2 most of the students either recognize correctly all the squares or carry out false recognition. Specifically, in grades 9, 10 and 11a the majority of students conduct false recognition, whereas for grade 11b students most of the students achieve the correct recognition of all the squares. Regarding the recognition of the two squares, this answer is provided by a very small percentage of students in each group, which is reduced in every next grade.

Generally, in task PE1 there seems to be a progression in the recognition of figures from grade 9 to grade 11, with grade 11a being the only exception. It is also noticed that in task PE2 the number of students that achieve the right recognition of all the squares without making any other mistake increases as we move to a higher grade. As explained earlier, when students are able to recognize all the squares correctly they are also able to go beyond the perceptual apprehension of the geometrical figure and they approach the limits of the operative apprehension. On the other hand, when students are able to recognize only some squares correctly they are not able to exceed the limits of the perceptual recognition. Therefore a progression of students' perceptual apprehension can be assumed from grade 9 to grade 11, which moves towards the boarders of the operative apprehension.

Table 14

Percentages of the Students' Answers According to the Cognitive Analysis of the Perceptual Apprehension Tasks

	PE1 (%)				PE2 (%)		
Grade	9 figures	8 or 7 figures	6 or 5 figures	Less than 5 figures	All squares	Two squares	False recogn.
All Students	35.19	47.45	13.17	4.20	43.36	3.06	53.58
Gr. 9	21.47	57.05	15.71	5.77	36.54	4.17	59.29
Gr.10	45.72	39.14	11.84	3.29	43.75	3.29	52.96
Gr.11a	20.80	53.60	19.20	6.40	44.00	1.60	54.40
Gr. 11b	55.71	38.57	5.00	0.71	57.14	1.43	41.43

The results of students' performance, from the mathematical point of view, in the operative apprehension tasks are presented in table 15. The results are presented for the total sample of students and for each grade separately. Starting from the results for all the students that participated in the research, the highest performance for the total sample of students is in task OP1. Tasks OP3, OP5 and OP2 come next, whereas task OP5 seems to be the most difficult one, because it is the task in which students have the lowest performance. The performance of students in grades 9, 11a and 11b follows the same sequence as the one described for the total sample. However in grade 10 the situation is slightly different. The tasks with the highest and the lowest success are the same as for the rest of the grades and what changes is the ranking of the rest of tasks. Regarding these three tasks, the performance in task OP5 is higher than in task OP3 and the success in task OP2 is the lowest of the three.

Table 15

Percentages of Students' Answers in the Operative Apprehension Tasks According to the Mathematical Point of View by Grade

Tasks	All Students (%)	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
OP1	71.74	62.82	71.38	79.2	85.71
OP2	37.57	32.05	35.86	26.4	63.57
OP3	62.32	54.17	62.50	59.2	82.86
OP4	11.46	10.34	11.84	8.40	15.89
OP5	60.61	53.21	66.78	57.6	66.43

Based on the cognitive analysis of the tasks, the following table (Table 16) presents the students' success in the global right answer of the tasks in relation to the approach they used for getting to this answer. Thus the percentages for each approach represent only the right answers that occurred from the use of the specific approach. It should be noted that the percentages of answers in each category are not very high, because of the limited proportion of students that provided an explanation for their answer. The results are again presented for the total sample of students and for each grade separately.

Focusing on the cognitive procedures of the total sample involved in arriving at the global right answer to the tasks, the operative apprehension is not always the reason for the success in these tasks, but perception seems to be involved as well. In fact the operative apprehension gives the highest number of correct answers in tasks OP1 and OP3, compared to the other approaches. On the other hand the perceptual apprehension is related to a greater number of correct answers for the tasks OP2 and OP4. In task OP5 the two types of apprehension give almost the same amount of correct answers. It is interesting that the two tasks in which the operative apprehension gives the highest number of correct answers are the tasks in which students score higher, in relation to the rest of the operative apprehension tasks.

Table 16

Percentages of the Total of the Students' Answers According to the Type of Apprehension Involved in the Solution of the Operative Apprehension Tasks

Tasks	Global Right Answer (%)	Operative Apprehension (%)	Perceptual Apprehension (%)	Different Approach (%)
OP1	71.74	23.72	15.21	17.48
OP2	37.57	7.72	27.24	----
OP3	62.32	10.56	6.13	4.09
OP4	11.46	11.46	24.63	7.94
OP5	60.61	10.1	10.78	9.76

Besides the observations for the total sample, it is interesting to examine each grade separately, in order to see whether the different groups of students behave in a similar way. So each task is discussed separately in relation to the approach students of each age group put into work for the solution of the tasks. The results for all the tasks and for all the groups of students are displayed in table 17. Starting from task OP1, the prevalent type of apprehension involved for the correct solution of this task is the operative apprehension, through the application of a mereologic modification on the geometrical figure. This stands for the students of grades 9, 10 and 11a, whereas there is a differentiation for grade 11b. Despite the fact that a similar proportion of these students apply the mereologic modification in order to solve the task, in relation to the rest of the students, the majority of the right global answers occurs through the use of a different approach. However the amount of answers provided through the mobilization of the operative apprehension is higher than the answers that occurred through the involvement of perception. As mentioned above the performance of calculations or the focus on the global shape constitutes the different approaches used by students. The use of these kinds of approaches by the students in grade 11b can be attributed to the type of teaching these students are exposed to, since in this grade they deal with higher level (more abstract, formal) mathematics, which involve the use of formulas, the performance of calculations and less or no attention is given to visualization. The use of a different approach in grade 10 and grade 11a gives the least correct answers compared to the operative and the perceptual apprehension. On the other hand it is the use of perception that is related to the lowest amount of correct answers for grades 9 and 11b.

Concerning task OP2, it is evident that the success of students of all grades in this task can be attributed to the use of the perceptual approach of the geometrical figure, since the percentages regarding the use of this approach to solve the task correctly are greater than those concerning the application of a mereologic modification on the geometrical figure. Thus in this task students were not able to see the possible reconfigurations of the given trapezium flexibly, but they were restrained under the influence of perception and preferred to solve the task with the use of calculations.

Similarly to task OP1, in task OP3 the greatest amount of right answers occur through the application of a mereologic modification on the geometrical figures of the task. This is the case for all four different groups of students. The lowest amounts of correct global answers are related to the use of a different approach by students of grades 9, 10 and 11a, but for grade 11b it is perception that gives the least correct answers.

In the next task (OP4), the students of the four different grades give solutions that are more related to perception, rather than the mereologic modification of the figure. For these tasks the perceptual apprehension exercised a greater influence on students' behavior and seems to have decreased the flexibility in the way they were able to see and elaborate on the geometrical figure. The smallest numbers of solutions is related to the use of a different approach for grades 9, 10 and 11b, whereas for grade 11a the fewest answers come from the involvement of the operative apprehension.

In the last task (OP5), perceptual apprehension is responsible for most of the right answers for students in grade 10 and in grade 11a. On the other hand the dominant type of apprehension for grade 11b students is the operative apprehension, as most of these students' correct answers result from the mereologic modification on the geometrical figure. For grade 9 the highest percentage of students' correct answers is related to the use of a different approach.

Table 17

Percentages of the Students' Answers According to the Type of Apprehension Involved for the Solution of the Operative Apprehension Tasks by Grade

	Global Right Answer (%)				Operative Apprehension (%)				Perceptual Apprehension (%)				Different Approach (%)			
Tasks	G9	G10	G11A	G11B	G9	G10	G11A	G11B	G9	G10	G11A	G11B	G9	G10	G11A	G11B
OP1	62.82	71.38	79.20	85.71	22.12	23.68	23.2	27.86	12.18	18.09	16.0	15.0	14.42	14.47	12.0	35.71
OP2	32.05	35.86	26.40	63.57	6.09	8.55	1.60	15.0	22.44	25.99	19.2	47.86	----	----	----	----
OP3	54.17	62.50	59.20	82.86	10.90	10.53	6.40	13.57	5.13	6.91	4.80	7.86	3.53	2.63	1.60	10.71
OP4	10.34	11.84	8.40	15.89	10.34	11.84	8.40	15.89	22.33	26.54	21.6	28.33	6.87	8.14	12.14	6.13
OP5	53.21	66.78	57.6	66.43	6.73	9.21	2.40	26.43	9.62	13.49	11.2	7.14	12.82	10.53	7.20	3.57

The results of the cognitive analysis of the students' responses in the operative apprehension tasks are summarized in table 18. The tasks in which the students' performance is the highest and the lowest are the same for all the groups of students. Concerning operative apprehension, it is mainly involved in the solution of tasks OP1 and OP3. For grade 11b students the operative apprehensions is mostly related to the correct solution of task OP5. On the other hand the perceptual apprehension is the type of apprehension that is mostly mobilized for the solution of tasks OP2 and OP4 for all the students. Grade 10 and grade 11a students' greatest number of correct answers through the involvement of perception also occur in task OP5. Regarding the use of a different approach, it brings the highest number of correct answers in task OP5, for grade 9 students, and in task OP1 for the grade 11b students.

Table 18

General Presentation of the Students' Behavior in the Solution of the Operative Apprehension Tasks

Grade	Highest performance	Lowest performance	Operative apprehension	Perceptual apprehension	Different approach
Grade 9	task OP1	task OP4	tasks OP1/ OP3	tasks OP2/ OP4	task OP5
Grade 10	task OP1	task OP4	tasks OP1/ OP3	tasks OP2/ OP4/ OP5	___
Grade 11a	task OP1	task OP4	tasks OP1/ OP3	tasks OP2/ OP4/ OP5	___
Grade 11b	task OP1	task OP4	tasks OP3/ OP5	tasks OP2/ OP4	task OP1

For the sequential apprehension tasks students' answers from the cognitive point of view are divided into three categories. The correct construction involves the proper mobilization of the sequential apprehension, whereas the involvement of the perceptual apprehension also takes place in their answers, as students make constructions that are perceptually similar to the correct one, but the process followed for the construction is not correct. The last category concerns students' constructions that are partly correct or not correct at all. The answers in which the sequential apprehension was properly mobilized correspond to the right answers from the mathematical point of view (variables SE1, SE2 and SE3).

According to the results indicated in table 19, the general picture from all students' responses is that the number of students that achieved a correct construction is limited and this number is smaller than the number of the perceptual solutions or the number of partly correct or incorrect solutions. In fact grade 9, 10 and 11a students do better at task SE1, then task SE3 follows while the lowest performance in the correct constructions is observed in task SE2. This order is not the same for students in grade 11b, as they score higher in task SE1, then in task SE2 and finally in task SE3. Generally the highest percentages of correct constructions are observed in grade 11b.

For task SE1 sequential apprehension is not activated properly for most of the students. The percentages of correct solutions are higher in grade 10 than in grade 9. In grade 11a these solutions are less than in grade 10 and more than in grade 9. Especially in grades 9, 10 and 11b most of the students' constructions are not successful and these answers are more than those related to the intervention of the perceptual apprehension. The percentages of the unsuccessful constructions in this task reduce in every higher grade. Regarding the percentage of perceptual solutions, it increases from grade 9 to grade 10, it is next reduced in grade 11a and in grade 11b this percentage is the highest. In addition, in grade 11b the number of correct and incorrect constructions is close.

Regarding task SE2, the mobilization of sequential apprehension was achieved by very few students – especially in grades 9, 10 and 11b – as the corresponding percentages are extremely low. Therefore students faced great difficulties in the construction of the geometrical figure required in this task. On the other hand grade 11b students seem to have faced less difficulties, compared to the rest of the students, as the number of students who succeed in activating the sequential apprehension properly is higher. Furthermore, most of the students, in all the groups, give a solution that occurs with the influence of perceptual apprehension. Also the number of students that carried out a partly correct construction is higher than the number of correct solutions. However this is not only the case for grade 11b students, as partly correct solutions are the least. It thus appears that the number of students that are not able to follow the correct sequence of steps for the construction is higher than those who at least carried out the right construction of the triangle. Therefore, this can be an indication verifying the hypothesis that the most difficult part of this construction is the construction of the triangle.

Coming to the last sequential apprehension task SE3, sequential apprehension did not function as expected, as the percentages of the students' correct constructions are very low. In fact the number of the students' correct constructions is reduced from grade 9 to grade 11a, whereas the highest number of correct constructions appears in grade 11b. Same as in the previous task, most of the students' answers come from the intervention of perceptual apprehension, as student from all groups give answers perceptually close to the correct one, but without following the right order of the construction steps. Also the constructions with no success are more than the correct constructions in grades 9, 10 and 11b, whereas this does not stand for grade 11a students.

Table 19

Percentages of the Students' Answers According to the Type of Apprehension Involved for the Solution of the Sequential Apprehension Tasks

	Task 1 (%)			Task 2 (%)			Task 3 (%)		
Grade	SE.A.	PE.A.	N.S.	SE.A.	PE.A.	N.S.	SE.A.	PE.A.	N.S.
All	16.57	21.23	43.93	5.33	47.45	14.87	8.85	36.32	13.51
G9	9.62	19.87	50.96	1.28	50.96	15.38	8.01	42.31	9.62
G10	18.09	22.37	43.09	3.29	52.96	13.16	7.24	33.88	15.13
G11a	12.80	15.20	40.00	1.60	34.40	13.60	7.20	31.20	4.00
G11b	32.14	27.14	33.57	22.14	39.29	18.57	15.71	32.86	27.86

Explanation of symbols: SE.A. = Sequential apprehension, PE.A. = Perceptual apprehension, N.S. = No success, G9 = grade 9, G10 = grade 10, G11a= grade 11a, G11b = grade 11b.

Regarding the discursive apprehension tasks examining the students' ability of producing a proof (DI1, DI2, DI5 and DI6) from the mathematical point of view (Table 20), the highest performance of all the students is observed in task DI1. This is not the case for grade 11a students, whose highest performance is in task DI6. On the other hand all the students score lower in task DI5. Especially grade 9 and grade 11a students' scores are very low in the particular proof task. In addition, in these tasks the students' performance does not always evolve in every next grade, but there are cases in which performance is lower in the next grade. Particularly what stands for these tasks is that the score gets higher from grade 9 to grade 10, but falls from grade 10 to grade 11a. Next there is again an increase in the students' performance and in grade 11b the students have the highest percentage of correct solutions, compared to the rest of the students. The students' performances in the tasks that concern the recognition of proof (DI3 and DI4) get higher in every next grade. It seems that the higher the grade of the students, the more able to recognize a formal proof from an empirical and a semi-empirical proof they are. It can therefore be assumed that the ability of recognizing a proof develops as students grow up and attend a more advanced mathematics class.

Table 20

Percentages of the Students' Performance in the Discursive Apprehension Tasks from the Mathematical Point of View

Tasks	All Students (%)	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
DI1	60.73	57.05	61.18	44.00	82.86
DI2	29.06	21.15	25.99	24.80	57.14
DI5	12.49	9.29	15.79	2.40	21.43
DI6	53.80	50.32	51.97	46.40	72.14
DI3	15.89	9.94	15.13	21.60	25.71
DI4	19.30	14.10	18.75	22.40	29.29

This part continues with the presentation of the results that were extracted from the cognitive analysis of the tasks examining the production of proof. For the first task falling in the category of the discursive apprehension tasks examining the production of a geometrical proof, apart from the correct choice of answer, students' justification was also important. Therefore attention was paid to whether students' right answers were accompanied by a correct or wrong justification or if no justification was provided at all. So the percentages for each variable express only right answers in relation to the justification that was given. As table 21 shows, most of the students' correct answers are accompanied by a wrong justification for all the groups of students. Actually these are more than half of the right answers and this is true for all the groups of students. For grade 9, grade 10 and grade 11a the correct answers that are related to a correct justification are the least, compared to the answers with no or wrong justification. However the situation is rather different in grade 11b, in which there are more students that provide a correct justification, than those who do not give any justification. But these students are still less than those whose justification is wrong. In addition, in grade 11b there is the biggest number of correct justifications, compared to the rest of the students.

Table 21

Percentages of Students' Answers according to the Cognitive Analysis of task DI1

	All Students (%)	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Global Right Answer	60.73	57.05	61.18	44.00	82.86
DI1cj	9.42	11.54	4.93	2.40	20.71
DI1nj	13.28	15.38	12.17	9.60	14.29
DI1wj	38.02	30.13	44.08	32.00	47.86

Thus, what surfaces from these results is that, despite the fact that the students are able to provide a correct answer when a geometrical proof is asked, they are not able to express the steps they followed and their way of thinking for reaching the particular answer. It thus appears that it is not easy for students to translate their geometrical reasoning into verbal form and students appear to have difficulties writing down correctly the procedure they followed.

In the second task on the production of proofs (DI2) what was also taken into account, besides proving what was asked correctly, was the way students expresses their thinking. In fact there were students that either drew on the figure in order to show the relation between the different figural units (DI2vr), indicated these relations verbally (DI2vei) or others that used both ways for expressing their answers (DI2vrvei). The results in table 22 show the percentages of the correct answers that occurred in relation to the way students expressed their thinking procedure. What is indicated in the table is first of all that most of the students in all the grades choose to express their answers through a combination of a verbal description of the relations between the figural units and marking them on the figure.

Table 22

Percentages of the Students' Answers According to the Cognitive Analysis of task DI2

	All Students (%)	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Global Right Answer	29.06	21.15	25.99	24.80	57.14
DI2vr	7.95	0.64	0.00	0.00	0.71
DI2vei	20.20	8.65	7.24	12.80	20.00
DI2vrvei	28.38	11.86	18.42	12.00	36.43

In fact, in grade 11a the number of these answers is very close to the number of answers that are only expressed verbally. The students who give their solution only by making the visual recognition of the relation between the figural units on the given figure explicit are the least. In fact in grades 10 and 11b there are no students that reach a right answer using the particular approach, despite the fact that 7.89% and 11.20% of the students respectively have used this approach. Thereafter, it happens that just the visual recognition of transitivity in this task was not enough for students to complete their reasoning and reach the right solution of the task. The verbal expression of the proving process was necessary in order for students to follow the necessary sequence of steps to reach the proper solution correctly, whether it was used alone or combined to marking the figural units. However the combination of the two ways seems to be more effective for getting a right answer in this task, compared to the use of each approach in isolation.

For the next discursive apprehension task DI5 that examines the students' ability for producing a proof, the results are shown in table 23. The considerably low amount of students' global right answers, especially for the students in grade 9 and in grade 11a, indicates that this task was difficult enough for the students. From the cognitive analysis of this task the students' answers were categorized according to the degree of the comprehension of proof.

Table 23

Percentages of the Students' Answers according to the Cognitive Analysis of task DI5

	All Students (%)	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Global Right Answer	12.49	9.29	15.79	2.40	21.43
DI5c	8.17	5.77	5.59	5.60	21.43
DI5cg	14.76	10.26	19.40	3.20	25.00
DI5wa	10.90	8.33	13.16	15.20	7.86

In grades 9, 10 and 11a the number of students that demonstrate a real comprehension of proof (DI5c) is very limited and these are the least students compared to those with a gap in the comprehension of proof (DI5cg) and those that give a wrong answer showing no comprehension of proof (DI5wa). For grades 9 and 10 most of the students appear to have a gap in the comprehension of proof and in fact these students are more than those with no comprehension at all. As for grade 11a the highest percentage regards students with no comprehension of proof, but the percentage of those who have comprehension is higher than those who have a gap. The biggest number of students with comprehension appears in grade 11b. However there is a greater number of students that have a gap in the comprehension of proof.

Interestingly, in grade 11b all the students that reach the correct global answer are essentially those who show comprehension of proof. This is not the case for the rest of the groups of students. In grades 9 and 10 the students that give a correct answer, from a mathematical point of view, are more than those who show good comprehension of proof. On the contrary in grade 11a the students that appear to have a good comprehension of proof are more than those who just provided a correct answer from the mathematical point of view. Consequently these contradictions highlight the importance of this way of analyzing tasks, as the examination of the correctness of the answers only from a mathematical point of view does not fully enable the description of the general situation concerning students' behavior. The cognitive analysis of the tasks provides more valuable information about students' cognitive processes, their abilities and understanding and thus it is not enough to merely rely on the mathematical correctness of students' work in order to extract some results and interpretations. The combination of the two ways of examining students' answers allows for a more accurate and general description of what a researcher wishes to examine.

In task DI6 (Table 24) the students' inference was very important to examine similarly to task DI1. Consequently the students' right answers were considered in relation to the justification students provided. Specifically, most of students' correct answers in this task are accompanied by a wrong justification. In grades 9, 10 and 11b there is an observable difference between the answers with a wrong justification and the answers with a correct or no justification. In grade 11a the amount of answers with no justification is very close to the answers no justification. Another observation is that the percentage of the students' answers with no justification is higher than those with a correct justification, in grades 10 and 11b. It is possible that for the students of these grades the influence of perception was stronger, because the cases in which students do not provide any justification can be related to the influence of perceptual apprehension. And this is why the two figures that are equal can be perceptively recognized in the given figures and as a result students can decide that there is no need for making any inference and proving it.

Table 24

Percentages of the Students' Answers according to the Cognitive Analysis of task DI6

	All Students (%)	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Global Right Answer	53.80	50.32	52.49	46.40	72.14
DI6ci	10.78	12.18	8.88	8.80	13.57
DI6ni	12.71	11.22	12.50	19.20	10.71
DI6wi	30.31	26.92	30.59	18.40	47.86

The results concerning the cognitive analysis of the task examining the students' abilities on the recognition of proof are presented in table 25. It should be noted that there were students that accepted more than one answer as a proof. Thus the percentages are low, because these include only answers in which the students accepted just one type of proof. Regarding task DI3, most of the answers occur from the discursive apprehension for all the groups of students. This is also the case for task DI4, with the only exception being the students of grade 9, who give more answers through the perceptual apprehension. Even in task DI3 the number of answers that occur from the mobilization of perceptual apprehension is almost analogous to that of the answers that occur after the mobilization of discursive apprehension. It therefore seems that there is no coherence in the way the 9th graders deal with these tasks and that their ability concerning the identification of the proper proof is not well developed yet.

Table 25

Students' Answers according to the Type of Apprehension Involved in the Tasks Examining the Recognition of Proof

Tasks	All Students (%)	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Task OP3					
DI3pe	7.38	8.65	7.57	8.80	2.86
DI3op	4.31	4.81	4.61	1.60	5.00
DI3di	15.89	9.09	15.28	21.60	25.71
Task OP4					
DI4pe	14.19	17.95	14.80	13.60	5.00
DI4op	6.47	6.73	7.24	6.40	4.29
DI4di	19.30	14.10	18.94	22.40	29.29

This conclusion was also derived from the results of the hierarchical clustering of variables and the implicative analysis. Instead, perceptual apprehension is involved, which inhibits the recognition of the formal proof. The influence of perception, which appears to be stronger in grade 9, could be a factor that affects students' discursive apprehension negatively, especially as far as the recognition of proof is concerned. On the other hand operative apprehension is related to the lowest number of answers in these tasks for all the groups of students. So the influence of the particular type of apprehension does not seem to be affecting students' ability to recognize formal proof.

A local comparison between the OP4 and DI5 tasks

The discursive apprehension task DI5 is an implicit operative apprehension task which is very similar to task OP4, because the given figures in the two tasks are almost the same. An interesting question to pose is: Have students that cannot succeed in task DI5 encountered a problem with the figure? Is it possible for the students' errors to be related to their difficulties regarding the way they see the geometrical figure, which does not allow any modification on the figure? Therefore the hypothesis is that students who are not able to put the operative apprehension into work when looking at the figure in task OP4, cannot choose the right theorem or prove in task DI5 either.

In an effort to answer this question, the results concerning the way students behave in the two tasks are summarized in table 26. What this table shows is the number of wrong answers (DI5wa) while it further examines the connection among students' answers in task OP4. In fact the variables WA-ME, WA-PE, WA-DA pertain to the students that arrived at a wrong answer in task DI5 and whose answer in task OP4 was either related to the mereologic modification or to another type of answer (perception / different approach) respectively. In fact, there were students that gave two or more kinds of answers in task OP4. In this case they were distinguished from those that used only one approach. So the students that used more than one approach are not included in this analysis.

Table 26

Relations among the Students' Answers from the Cognitive Point of View in Tasks OP4 and DI5

	DI5wa (%)	OP4 (%)	
		WA – ME	WA – PE/ DA
Grade 9	8.33	3.85	65.38
	(N=312)	(N=26)	(N=26)
Grade 10	13.16	0.0	50.0
	(N=304)	(N=40)	(N=40)
Grade 11a	15.20	10.53	42.11
	(N=125)	(N=19)	(N=19)
Grade 11b	7.86	0.0	45.45
	(N=140)	(N=11)	(N=11)

Explanation of symbols: WA – ME = wrong answer in task DI5 and answer through the mereologic modification in task OP4, WA – PE/ DA = wrong answer in task DI5 and answer through the perceptual apprehension or a different approach in task OP4.

A first observation from the table is that most of the wrong answers are related to the use of the perceptual or the different approach in task OP4. This is the case for all the groups of students. In addition it is interesting that in grades 9 and 11b there are no wrong answers related to the use of the mereologic modification in task OP4. Consequently there seems to be a relation between these two tasks, providing indications for the verification of the hypothesis stated above. The results show that the students' inability to prove in task DI5 can be related to the way students look at the figure, and thus the type of apprehension they are able to mobilize. The fact that students cannot choose the necessary theorem, which leads to the proper proof, can be attributed to the lack of operative apprehension, which does not allow for the identification of the relations between the different figural units of the figure.

Hierarchical classification of the geometrical figure apprehension tasks according to the degree of difficulty

This part deals with the hierarchical classification of the geometrical figure apprehension tasks, according to the degree of difficulty, in order to see whether there are differences regarding this classification for the different groups of students. The Rasch analysis was used in order to define the degree of difficulty of the tasks. Besides the calibration of all the tasks of the test on a single scale using the Rasch models, the difficulty of the tasks was also examined through using implicative analysis. In fact in this case the difficulty of the tasks included in each of the four groups of tasks was examined separately. Therefore a hierarchy of the tasks occurred for each type of apprehension. For each group of tasks corresponding to a different type of apprehension a hierarchy of the tasks for the total number of students is presented and it is then contrasted to the hierarchy for each group of students.

Hierarchical classification of all the geometrical figure apprehension tasks according to the Rasch model

The Rasch model is a probabilistic model used widely for purposes of educational measurement around the world and for purposes of unidimensional scale-building. This model aims to describe the interaction between a person and a task, expressing the abilities of the persons and the difficulties of the tasks/questions onto a single linear scale (i.e., the logit scale). This serves the need for direct comparisons between the ability of a specific person and the difficulty of a specific question, i.e. the need to locate the two on the same psychometric continuum. Equation (1) illustrates the Rasch model in the case where a person *n* attempts to respond to question *i* which is scored on a scale from zero to *k*.

$$\log\left(\frac{P_{nik}}{P_{ni(k-1)}}\right) = B_n - D_i - F_k \qquad (1)$$

where

P_{nik} is the probability of person n being assigned on question i, the score k,

$P_{ni(k-1)}$ is the probability of person n being assigned on question i, the score $k-1$,

B_n indicates the ability of the person n,

D_i indicates the difficulty of the question i, and

F_k indicates the difficulty of score k in relation to score $k-1$.

Each distinct score on a question (marked from 0 to *k* points) may be considered as a 'step'. According to Equation (1), the interaction among a person's ability, a question's difficulty and a step's difficulty define the score that a person is most likely to obtain on the question. Equation (1) illustrates the case where all the questions on the test employ the same scale (e.g., 0 to 10 points), and it is assumed that the scale maintains the same meaning across the questions (e.g., a score of 5 has the same meaning for all questions). This is the Rating Scale model (Andrich, 1978; Wright & Masters, 1982).

In the case where each question is modeled to show its own rating scale, (e.g., when a score of 5 for one question is not equivalent to a score of 5 for another question, or when different questions have different maximum possible scores), then the equation becomes:

$$\log\left(\frac{P_{nik}}{P_{ni(k-1)}}\right) = B_n - D_i - F_{ik} \qquad (2)$$

where F_{ik} indicates the difficulty of score k in relation to score $k-1$ for question *i*. This is the Partial Credit model (Wright & Masters, 1982).

The Rasch model was considered to be more appropriate than other item response theory models. Firstly, the raw scores of the persons constituted sufficient statistic information for the estimation of their underlying ability. In addition, there was no need to award more points for correct or partly correct responses to more difficult questions and to penalize the persons for incorrect or partly-correct responses to easier questions. Finally, the nature of the open-ended questions did not encourage guessing (so a three-parameter model was not a choice). Overall, models that had weighted scores as sufficient statistics or incorporated pseudo-guessing parameters were not appropriate for the data of this research.

The Partial Credit Model (which assumes that all questions scored are either correct or incorrect) was used to analyze the data instead of the Rating Scale model. Most of the questions were scored as dichotomous (correct/incorrect) and some of the questions of the test were scored on a scale from 0 to k (not all questions have the same maximum possible score). Two statistics were selected for this study in order to evaluate model-data fit for individual items. These are the Infit Mean Square and the Outfit Mean Square (Wright & Stone, 1979). Fit statistics are used to assess whether a given person's performance (or a given item) is consistent with other persons' performances (or items) and are based on the differences between the expected and observed performances. Outfit statistics are based solely on the difference between observed and expected scores whereas in calculating infit statistics extreme persons or items are downweighted. All weighted (i.e. infit) statistics in the Rasch model actually increase the weight of targeted responses. They are called "Pearsonian" statistics because they aggregate the residuals as measures of aberrance. The two statistics are known to be able to identify general aberrance in response patterns. This is an advantage because a fit statistic that focuses only on a specific type of aberrance may not have enough power to identify other types of aberrance (Klauer, 1995).

The two statistics have been used for a long time in order to evaluate model-data fit for individual items (e.g., Wright & Masters, 1982; Smith, 1991; Smith, 2000; Wright & Mok, 2000; Lamprianou & Boyle, 2004). Both are approximately χ^2–distributed (Wright & Mok, 2000) but there is no agreement between researchers regarding the "acceptable" values (values that indicate acceptable fit). Indeed, Karabatsos (2000) suggested that the distributional properties of the two statistics might differ significantly across datasets with different characteristics such as length and item difficulty distribution. In a number of studies (e.g., Engelhard, 1992, 1994; Lunz, Wright, & Linacre, 1990) the range of acceptance for persons was set at 0.6 to 1.5. The range of acceptance for the question fit was often set between 0.7 and 1.3 but these are just rules of thumb. In the examination of the person statistics for fit to the Rasch model, the outfit square statistic is considered to provide more useful information than the infit, because a person's performances on both the easiest and the hardest items are taken into equal consideration (Andrich, 1988). Any marked difference between the calculated values for the outfit and the infit statistics is highly informative, since it indicates a tendency for a different pattern of responding to easier or harder items, when compared to items at the centre of the scale.

The Rasch model analysis (Andrich, 1988) was used in order to examine the hierarchical classification of the geometrical figure apprehension tasks according to the degree of difficulty. The estimation of the Rasch models was conducted with the use of the QUEST software (Adams & Khoo, 1996). The Rasch model also allowed the investigation of the construct validity of the geometrical figure apprehension test and the creation of a good interval level measure for the lower and upper secondary school students' geometrical figure apprehension. The findings showed that the Rasch analysis supports the conceptual design of the test, since a scale of items measuring students' geometrical figure apprehension is indicated. The students' psychometrical behavior is invariant in the four age groups examined.

The data of the research were firstly analyzed for the total number of students. The initial Rasch analysis did not indicate items that were misfitting the Rasch model, therefore no item was excluded from the scale. Next the analysis was repeated for the students in grades 9 and 10, in which the students are before and after the change of educational level respectively. The examination with the particular groups of students was chosen in order to examine whether these two groups face the test with the same coherence and whether there are differences regarding the calibration of the tasks on the scale, in relation to the fact that they belong to different educational levels.

The calibration of the items on the scale for the total sample is presented in Figure 40. The scale comprises all the tasks included in the research instrument and presents the degree of difficulty of the tasks and the students' distribution on the scale according to their abilities when dealing with geometrical figure apprehension tasks. In fact the item difficulty range is from -2.05 logits to 2.38 logits. On the other hand students' abilities are distributed between -3.32 logits and 2.51 logits. As far as relating the person ability range with the item difficulty range is concerned, it can be said that there is a satisfactory targeting of the items for the students. However the targeting of the items could be further improved by the inclusion of some tasks of less difficulty in the test. From the figure it is also noticed that most of the students are situated between -2.5 logits and 0.5 logits, whereas fewer students are found below -2.5 logits and above 0.5 logits. Hence the majority of the students are of low or medium ability regarding the geometrical figure apprehension, based on the particular tasks of the test that they were given. The tasks that are situated at the top of the scale were answered by the students of high ability, whereas the tasks found at lower parts of the scale were answered by medium and low ability students. It can therefore be said that the total sample of the students is normally distributed according to their ability regarding the items of the test.

More specifically the items with the highest difficulty are the sequential apprehension tasks (SE, SE2 and SE3) and some of the discursive apprehension tasks. In fact these are task DI5 and the two tasks which examine the students' abilities regarding the recognition of proof (DI3 and DI4). For this group of tasks the smallest number of students is found on the scale and these are the students of high achievement and thus of higher abilities concerning geometrical figure apprehension. In the middle part of the scale, where most of the students are also situated, the perceptual apprehension tasks are found together with the operative apprehension tasks OP2 and OP4 and the discursive apprehension tasks DI2 and DI6. The fact that the perceptual apprehension tasks are found near the discursive and the operative apprehension tasks on the scale, according to their degree of difficulty, could be explained by the fact that perceptual apprehension could intervene in the solution of the rest of the tasks of this group. In fact for the solution of the operative apprehension OP2 task the effect of the perceptual apprehension was possible because of the presence of number, which can more likely lead students to the performance of calculations than the reconfiguration of the figure to find the answer, and thus the neutralization role of the operative apprehension. Also for task OP4, which asked for a different solution from the students, answer only premised on the perceptual recognition of the figure were possible to occur, especially by students that cannot go beyond the perceptual recognition of figures.

Regarding the discursive apprehension tasks, the solution of task DI2 required increased perceptual abilities, as the recognition and the discrimination of the subfigures included in the given figure were important for the students to be able to use the given data in the task. Furthermore the solution of task DI6 could also be related to the perceptual apprehension of the figures, because the students could identify the two equal triangles among the three given figures, due to the visual support of the provided figure. At a lower part of the scale the easiest items are found, which are three of the operative apprehension tasks (OP1, OP2 and OP5) and a discursive apprehension task (DI1). In fact in these three tasks the students were asked to compare the area or the perimeter of two parts of the given figures. This commonality in the nature of the tasks could be the possible reason for the placement of these tasks in a similar part of the scale. However, there no tasks situated at the bottom of the scale where the easy items should be situated. This indicates that the alignment of the test to students' abilities needs further improvement, by adding tasks of less difficulty.

The hierarchy of the items on the scale was next examined for the 9th and the 10th graders separately, because these two groups of students correspond to the end of the lower secondary school and the beginning of the upper secondary school respectively. Therefore the scope was to examine whether great changes occur in the way the tasks are placed on the scale according to their difficulty after the students' transition from one educational level to a next one. In comparing the hierarchy of the items on the scale for the 9th (Figure 41) and the 10th (Figure 42) graders respectively, it can be observed that the items are situated in a very similar way on each scale. A difference is noticed regarding the placement of the two perceptual tasks on the scales, which for grade 10 students appear to have almost the same degree of difficulty. Despite the calibration of the items in the scales for the two groups of students, table 27 indicates that the degree of difficulty of the tasks is not the same for the students of the two grades. The difficulty of the tasks is higher for grade 9 students than for grade 10 students, as the scores of the 9th graders are lower than those of the 10th graders.

On the contrary difference is traced between the two models regarding the distribution of the persons on the scale. What is actually observed is that the 9th graders are distributed at a lower part of the scale, in relation to the distribution of the 10th graders. Specifically the distribution of the persons on the scale for grade 9 is between 1.5 logits and -3.5 logits, whereas for grade 10 the persons are found within the range of 2.5 and -3.5 logits. Overall the main outcome of the comparison between the Rasch models for grade 9 and grade 10 respectively is that the ranking of the tasks according to the degree of difficulty is the same for the two groups of students, but what is different is the students' abilities, which appear to be higher for the students in grade 10. This was expected because students in grade 10 are older than the students in grade 9, hence their abilities are more developed due to the richer teaching experiences they have and also due to possible cognitive development.

The summary of the item fit statistics for the total sample, for grade 9 and grade 10 students is presented in table 28. The fit statistics for the test of the items for the three Rasch models are satisfactory for the purpose of this study, since the Infit mean square and the Outfit mean square are near 1 and thus fall into the range of model acceptance. The Infit t and Outfit t are also within the range of acceptance for the fit of the items. The fit indices are also satisfactory for the persons for the three Rasch models, as the Infit mean square and the Outfit mean square are also near 1 and the Infit t and Outfit t are also within the range of acceptance. The reliability for the items and the persons is high for the items and the persons as well. Consequently the scale of items measuring students' geometrical figure apprehension that occurred from the Rasch analysis validate the test that was developed for examining lower and upper secondary students' geometrical figure apprehension.

Table 27

Items Scores for Grade 9 and Grade 10 students

Tasks	Grade 9 (Maximum score=312)	Grade 10 (Maximum score=304)
PE1	114	133
PE2	67	139
OP1	196	217
OP2	100	109
OP3	169	190
OP4	108	114
OP5	166	203
SE1	30	55
SE2	4	10
SE3	25	22
DI1	178	186
DI2	66	79
DI3	31	46
DI4	44	57
DI5	29	48
DI6	157	158

Table 28

Statistics for the Geometrical Figure Apprehension Tasks for the Total Sample, the Grade 9 and the Grade 10 Students

Statistics	Total sample (N=881)	Grade 9 (N=312)	Grade 10 (N=304)
Mean			
Items*	0.00	0.00	0.00
Persons	-0.88	-1.30	-0.89
Standard deviation			
Items	1.32	1.04	1.39
Persons	1.17	1.08	1.15
Reliability			
Items	0.99	0.98	0.99
Persons	0.67	0.59	0.66
Infit mean square			
Items	1.00	0.98	1.00
Persons	1.00	1.00	1.00
Outfit mean square			
Items	1.00	1.09	1.03
Persons	1.00	0.98	1.03
Infit t			
Items	-0.02	0.20	0.01
Persons	0.03	0.04	0.03
Outfit t			
Items	0.02	0.13	-0.06
Persons	0.12	0.14	0.15

```
-----------------------------------------------------------------
   3.0  High Achievement          |     Difficult items
                                  |
                                  |
                                  |
                               X  |
                                  |      SE2
                                  |
                                  |
   2.0                         X  |
                                  |      SE3
                                  |
                                  |
                            XXXX  |
                                  |      DI5
                                  |
                          XXXXXX  |      SE1      DI3
   1.0                            |
                                  |      DI4
                          XXXXXX  |
                                  |
                                  |
                     XXXXXXXXXXX  |
                                  |      DI2
                                  |
    .0           XXXXXXXXXXXXXXX  |
                                  |      PE2      OP4
                  XXXXXXXXXXXXXX  |      OP2
                                  |
                                  |      PE1
                XXXXXXXXXXXXXXXX  |
                                  |
  -1.0                            |
               XXXXXXXXXXXXXXXXX  |      DI6
                                  |
                                  |
              XXXXXXXXXXXXXXXXXX  |      OP3      OP5      DI1
                                  |
                                  |
  -2.0          XXXXXXXXXXXXXXXX  |
                                  |      OP1
                                  |
                                  |
                   XXXXXXXXXXXXX  |
                                  |
                                  |
                                  |
  -3.0                            |
                                  |
                                  |
                         XXXXXXX  |
                                  |
                                  |
                                  |
                                  |
  -4.0 Low Achievement            |      Easy items
-----------------------------------------------------------------
   Each X represents     6 students
```

Figure 40. Scale for the geometrical figure apprehension tasks for the total sample

```
-----------------------------------------------------------------
4.0 High Achievement              | Difficult items
                                  |
                                  |
                                  |
                                  |
                                  |     SE2
                                  |
  3.0                             |
                                  |
                                  |
                                  |
                                  |
                                  |
                                  |
  2.0                             |
                                  |
                                  |
                            XX    |
                                  |     SE3
                                  |     SE1     DI3     DI5
                             X    |
  1.0                             |
                                  |     DI4
                         XXXXX    |
                                  |
                      XXXXXXXX    |
                                  |     PE2     DI2
   .0                             |
                 XXXXXXXXXXXXX    |
                                  |
                  XXXXXXXXXXXX    |     OP2
                                  |     OP4
                                  |     PE1
             XXXXXXXXXXXXXXXXX    |
 -1.0                             |
          XXXXXXXXXXXXXXXXXXXX    |
                                  |
                                  |     DI6
      XXXXXXXXXXXXXXXXXXXXXXXX    |     OP3      OP5
                                  |     DI1
                                  |
 -2.0   XXXXXXXXXXXXXXXXXXXXXX    |     OP1
                                  |
                                  |
                                  |
         XXXXXXXXXXXXXXXXXXXXX    |
                                  |
                                  |
 -3.0                             |
                                  |
                                  |
                     XXXXXXXXX    |
                                  |
                                  |
                                  |
 -4.0 Low Achievement             |     Easy items
-----------------------------------------------------------------
Each X represents     2 students
```

Figure 41. Scale for the geometrical figure apprehension tasks for grade 9 students

```
----------------------------------------------------------------
3.0    High Achievement             |  Difficult items
                                    |     SE2
                                    |
                             X      |
                                    |
                                    |
                                    |
                                    |
  2.0                               |      SE3
                                    |
                                    |
                                    |
                            XX      |
                                    |
                                    |
                       XXXXXXX      |      DI3     DI5
  1.0                               |      SE1
                                    |      DI4
                       XXXXXXX      |
                                    |
                                    |
              XXXXXXXXXXXXXXXX      |      DI2
                                    |
                                    |
   .0          XXXXXXXXXXXXXXX      |
                                    |      OP2
                                    |      OP4
            XXXXXXXXXXXXXXXXXX      |
                                    |      PE1
             XXXXXXXXXXXXXXXXX      |      PE2
                                    |
 -1.0                               |      DI6
               XXXXXXXXXXXXXXX      |
                                    |
                                    |
           XXXXXXXXXXXXXXXXXXX      |      OP3      DI1
                                    |
                                    |      OP5
                                    |
 -2.0       XXXXXXXXXXXXXXXXXX      |      OP1
                                    |
                                    |
                                    |
                    XXXXXXXXXX      |
                                    |
                                    |
                                    |
 -3.0                               |
                                    |
                                    |
                     XXXXXXXXX      |
                                    |
                                    |
                                    |
                                    |
 -4.0   Low Achievement             |  Easy items
----------------------------------------------------------------
  Each X represents    2 students
```

Figure 42. Scale for the geometrical figure apprehension tasks for grade 10 students

Examination of the hierarchy of the tasks of each type of apprehension separately according to the degree of difficulty

The figure XX presents the implicative relations among the operative apprehension tasks. The hierarchy formed in the implicative relations among the tasks reveals the difficulty of these tasks. In fact the same order of difficulty is also present in the scale of the Rasch model (Figure 43).

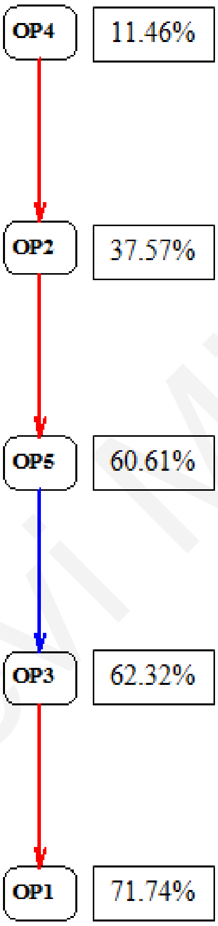


Figure 43. Implicative diagram for the students' responses from the mathematical point of view in the operative apprehension tasks for the total sample

Specifically the most difficult task from the category of tasks examining students' operative apprehension is the operative task OP4. In this task students were expected to provide four solutions and thus they had to find four different reconfigurations for the same figure. In order to do so, students had to carry out many and different combinations of the subfigures included in the given figure and use each subfigure in a different way each time in order to come to a new answer. This can be the factor that increased the difficulty of this task and led to the lowest students' performance (11.46%), compared to the rest of the operative apprehension tasks. Therefore the need to the mereologic modification on the figure more than once seems to have increased the complexity of the cognitive processes involved in the solution of the task and thus the difficulty of the task. It is assumed that for the solution of this task a higher level of abilities in modifying a given figure through the involvement of the operative apprehension seems to be necessary.

The second most difficult operative apprehension task is task OP2. As explained in the a priori analysis of the tasks, the solutions for this task could occur from the mobilization of the operative apprehension or by the involvement of the perceptual apprehension combined to the use of formulas and calculations. The existence of numbers in this task may be the reason that caused students' limited success (37.57%) in relation to the rest of the tasks of this group. In fact the presence of numbers can be a factor that inhibits the activation of operative apprehension while, on the contrary, it also creates space for the perceptual apprehension to intervene. When the chosen path for students' solutions is not through the operative apprehension, but through a different approach their solution is less short and more complex and therefore the possibility of error increase.

Next, task OP5 is found in the hierarchy of the tasks. The difficulty of task OP5 lies on the fact that, as in task OP4, students had to modify the figure more than once in order to find the relations between the different subfigures and reach the correct answer. In addition the knowledge of the properties of the different parts of the given figure was a facilitator for finding the right answer after the proper reconfiguration was conducted. Therefore the need for multiple modifications on the figure and the need for the knowledge of properties influenced the students' performance in this task.

What follows is task OP3, in which students also had to perform the mereologic modification on the given figure, but this time just once. In fact, students first had to identify the different parts in which the given figure was divided perceptively and then perform some mental modifications on the figure, in order to choose and rotate specific parts of the figure and come to the choice of the proper answer. Similar to task OP5, the knowledge of the properties of the given figure was also a factor which facilitated the correct solution of the task after the reconfiguration was performed.

Finally the easiest task in this category appears to be task OP1. In this task students had to do a simple modification on the figure without the necessity to use the properties of the given figure. They just had to identify a particular part of the figure which should be cut and moved, without involving any knowledge regarding the properties of the figure.

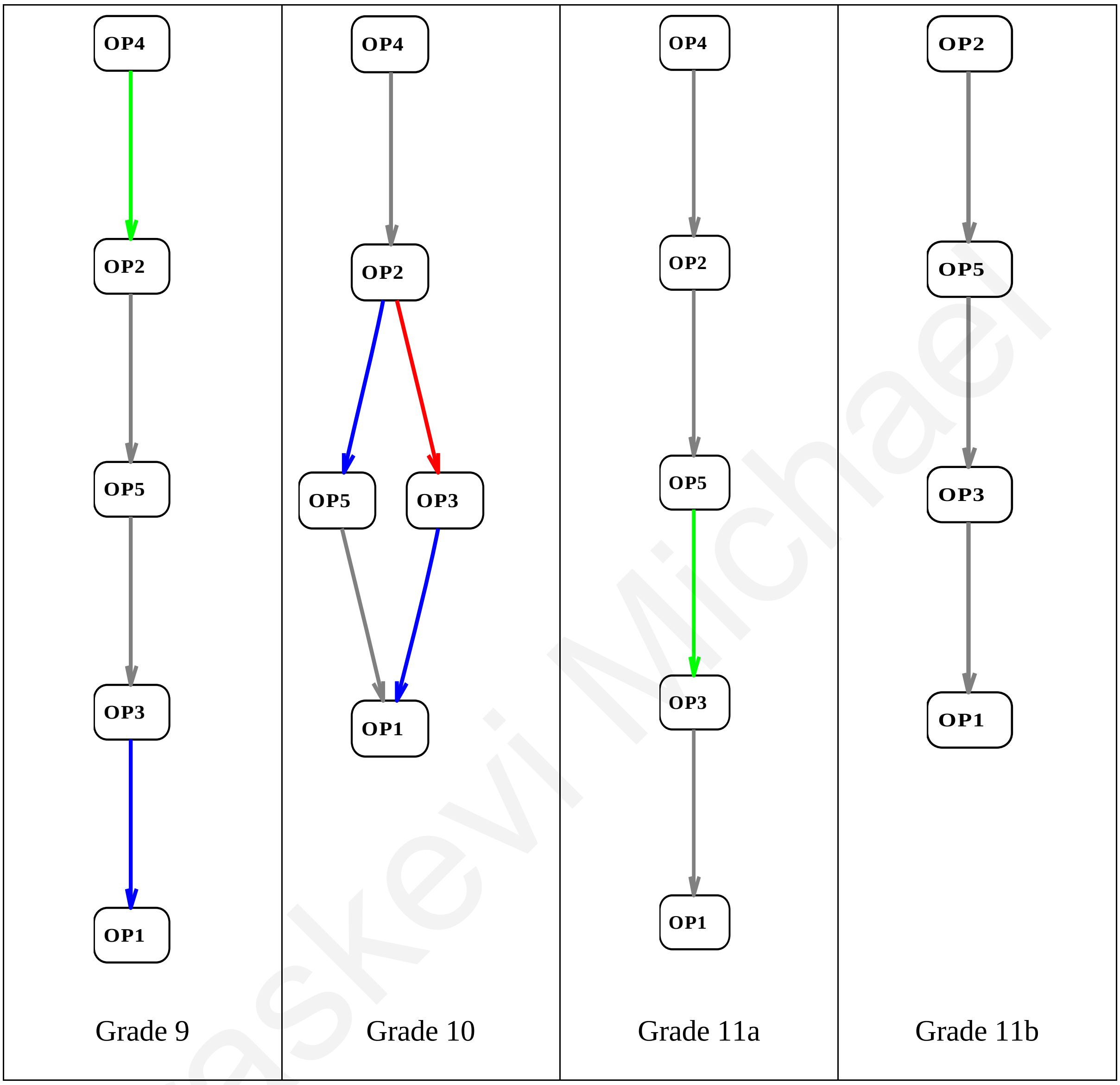


Figure 44. Implicative diagrams for the students' responses from the mathematical point of view in the operative apprehension tasks for each grade

The difficulty of the tasks was examined for students of each grade separately, in order to trace the differentiations in the ranking of the tasks for the different age groups. The implicative diagrams for each grade are presented in figure 44. First of all, there is similarity in the hierarchy of the operative apprehension tasks between grades 9 and 11a. In fact this hierarchy is the same as the one created for the total sample of students (Figure 43). Regarding grade 11b, the hierarchy formed is very similar to the one for grades 9 and 11a, with the only difference being the absence of task OP4, which does not appear in the implicative chain. A slight difference is created in grade 10, in which tasks OP3 and OP5 are situated on the same level in the implicative chain.

Based on the above, the ranking of the tasks according to their degree of difficulty seems to be almost the same for the different groups of students. What differentiates the results is the students' performance which corresponds to the different levels of ability for each group of students. Furthermore the factors that determine the difficulty of the tasks and hence the students' performance is first of all the number of modifications necessary in a figure for the solution to be reached. Another factor that influences the difficulty of the tasks is the presence of numbers in the given figure, which make the way to the solution more difficult and longer than a solution through the operative apprehension. In addition the students' geometrical knowledge is a factor that affects the difficulty of the tasks, along with the two previous factors.

The hierarchy of the sequential apprehension tasks is indicated in the following implicative diagram (Figure 45). What is firstly observed when looking at the percentages included in the diagram of the students' correct solutions is that their performance in these task is low, which shows the high degree of difficulty of these tasks for students. So what firstly surfaces is that the students' abilities regarding the construction of geometrical figures do not seem to be very developed.

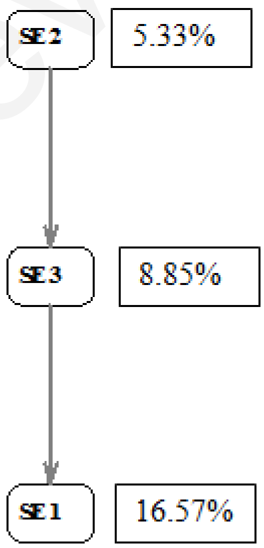


Figure 45. Implicative diagram for the students' responses from the mathematical point of view in the sequential apprehension tasks for the total sample

At the top of the implicative chain task SE2 is situated, which can be considered as the most difficult task in the group of tasks examining the sequential apprehension of geometrical figures. It seems that it was very difficult for students to use the given information properly and to follow the necessary sequence of steps, on the basis of the given data. There is a possibility that students misused the given data or either used them in an inappropriate order and thus they were unable to come to a right construction. Following the right order of steps is decisive for carrying out the process of constructing a geometrical figure successfully.

The following task, the degree of difficulty of which is somehow lower, is task SE3. In this task students had to construct a figure, but not only on the basis of the given data. The difference in this task, compared to the previous one, is that students were given a figure and they were asked to base their construction on this given figure. Therefore the fact that students were given a part of the figure, and not only some data to use in the different steps of the construction process, may be a reason that somehow facilitated the construction of the geometrical figure and brought about an increase in students' success in this task, compared to the former one.

The task with the lowest degree of difficulty among the three sequential apprehension tasks is task SE1. In this task students were given the data and the constraints for the contraction they had to do, but they were also given the figure which had to be constructed. So students knew a priori which the final result of the construction was and they already had the picture of how the correct construction of the geometrical figure should look like. The presence of the geometrical figure seems to be helpful for students, because it may have functioned as a source for providing feedback for students regarding the correctness of their construction. Therefore in this case it was important for students to use the given data appropriately and retain the mathematical properties of the figure they had to construct, on the basis of the given figure.

Consequently, the hierarchy of the tasks formed in the implicative diagram, which is in line with the one of the Rasch model, reveals that the amount of the given data in each task seems to be decisive for students' success. The first task only provided the students with some data that had to be used in a right order, while the rest of the tasks gave students either a part of the construction or even the whole figure combined to the necessary data for conducting the construction process. Therefore the amount and the type of the given information in each task appear to influence students' solutions and seem to be related to the success of their constructions.

The comparison of the ranking of the sequential apprehension tasks according to the degree of difficulty for each group of students is indicated in figure 46. The implicative diagrams for each grade reveal differences in the way the different groups of students approach sequential apprehension tasks. In effect, an implicative diagram for the students in grade 11a could not be formed. In fact the variable corresponding to task SE2 is omitted, due to the very small number of correct answers given by students (1.60%). Therefore an implicative chain for the rest of the two tasks was not possible to be formed.

What emerges for grade 9 students is that tasks SE3 and SE2 are more difficult than task SE1. A similar hierarchy is also formed for students in grade 11b, in which task SE3 is situated at the top of the implicative chain and task SE1 at the bottom of the chain. The situation is different for the10th graders, as the hierarchy of the tasks is different compared to the rest of the students. Actually for students in grade 11a the most difficult sequential apprehension task is the SE2, which is placed at the top of the implicative chain. Task SE1 is the second task in the implicative chain, whereas task SE3 is the easiest of the three, as it is found at the bottom of the hierarchy. Finally it appears that there are differentiations in the way the sequential apprehension tasks are ranked in each group of students and these hierarchies are different from the hierarchy that occurs when the total of students is examined. Thereafter the different amount of the given data in each task appears to influence the solution of the sequential apprehension tasks in each grade in a different way.

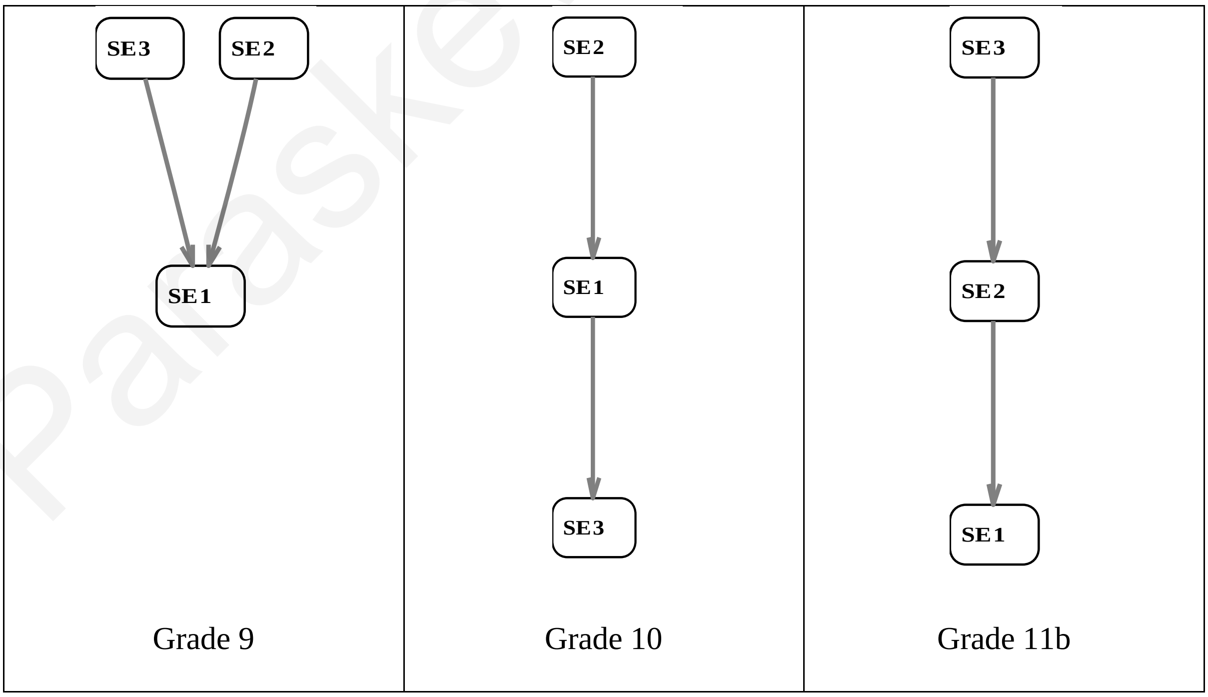


Figure 46. Implicative diagrams for the students' responses from the mathematical point of view in the sequential apprehension tasks for each grade

A hierarchy of the discursive apprehension tasks is also formed with the use of implicative analysis, the results of which are presented in the implicative diagram in figure 47.

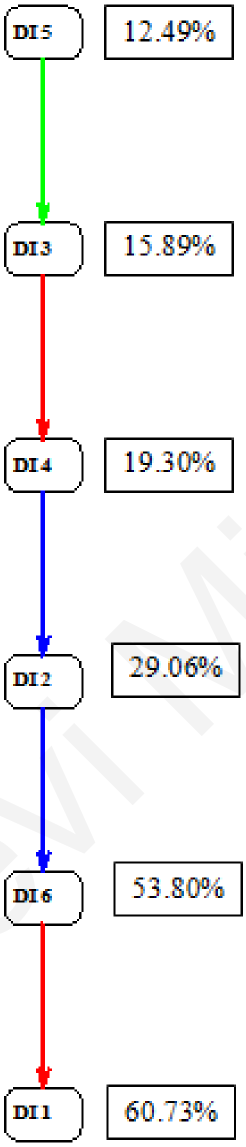


Figure 47. Implicative diagram for the students' responses from the mathematical point of view in the discursive apprehension tasks for the total sample

In this case also the hierarchy of the tasks is in line with the results of the Rasch model. At the top of the implicative chain task DI5 is found. The increased degree of difficulty of this task could be attributed to the fact, apart from discursive apprehension, that the mobilization of the operative apprehension was needed to be activated for its solution. Actually in this task the students had to make different reconfigurations on the given figure, because they firstly had to be able to see the three parallelograms related to the proofs they were asked to produce. In order for students to be able to see the three parallelograms, the different parts of the divided given figure had to be combined in more than one ways. Therefore the mobilization of operative apprehension was a crucial factor for the success in this task, as the identification of the three parallelograms was a prerequisite for the use of the necessary axioms. Besides the multiple reconfigurations of the figure for the identification of the three parallelograms, students also had to find the relations among the different figural units of the figure and then employ the knowledge of properties and axioms in order to produce the necessary proofs. Therefore, the need for increased ability in modifying a geometrical figure, combined with a discursive process seems to be the reasons for the increased difficulty of this task and the low performance observed by students (12.49%).

Next the pair of tasks examining the recognition of proof is situated in the implicative chain. Despite the fact that both tasks examine the same aspect of proof, students dealt with a different degree of difficulty in each task. Although the difference between the two tasks is not considerable, task DI3 appears to be slightly more difficult than task DI4. The explanation for that can be related to the axiom that was asked to be proved in task DI3. Task DI3 involved the proof on the sum of the internal angles of triangles, whereas the proof in task DI4 concerned the axiom on the relation between an external angle of the triangle and the sum of the internal angles that are opposite to it. Consequently students' prior knowledge of the properties of the triangles and the axioms involved in each of the tasks may have had an effect on their performance.

The following task in the implicative chain is task DI2. This task demanded good perceptual ability in order to recognize the different subfigures included in the given figure correctly and then a good knowledge of the properties of these figure. In addition students had to follow the transitivity in the relations between the different figural units of the figure correctly, which, in turn, demands the involvement of the operative apprehension as well. So in this task students appear to be more able to identify the necessary subfigures and trace the relations between the different figural units in the figure, in relation to the previous tasks.

Task DI6 on the identification of equal triangles appears at a lower part of the hierarchy, meaning that students have faced fewer difficulties in this task compared to the former ones. In fact this task did not demand any recognition or reconfiguration of the given figures. On the contrary, this was a task for which the solution only involved the use of the properties of the triangles and the knowledge of the criteria for two triangles to be equal. Thereafter in this task a discursive process was mainly needed and not any intervention of either the perceptual or the operative apprehension.

The last task in this hierarchy is task DI1. For the solution of this task the knowledge of a theorem was necessary in order for students to be able to prove correctly. Although this task also involved perceptual recognition and the ability to trace relations between particular figural units, the good knowledge of the theorem may have been the reason for students' greater success in this task, in relation to all the other tasks of this group.

Figure 48 includes the implicative diagrams for the discursive apprehension tasks for each grade separately. Comparing the different diagrams, there are small differences in the formation of implicative relations among the tasks. However the ranking of the tasks in each grade is similar and also not very different from the hierarchy that occurred for the total number of students. For all the groups of students tasks DI3, DI4 and DI5 are situated at the higher parts of the implicative chain, indicating that these tasks cause greater difficulties to students compared to the rest of the tasks. This is also the case for task DI2, which also appears to be more difficult than tasks DI1 and DI6, which are found at the bottom of all the implicative chains. Particularly task DI1 appears to be the easiest task, but this does not stand for grade 11a, as the easiest task for students in this grade is the DI6.

Overall a factor that seems to influence the discursive apprehension of geometrical figures is basically the students' geometrical knowledge. The involvement of perception and the need for multiple reconfigurations on a figure are also factors that are related to the mobilization of the discursive apprehension.

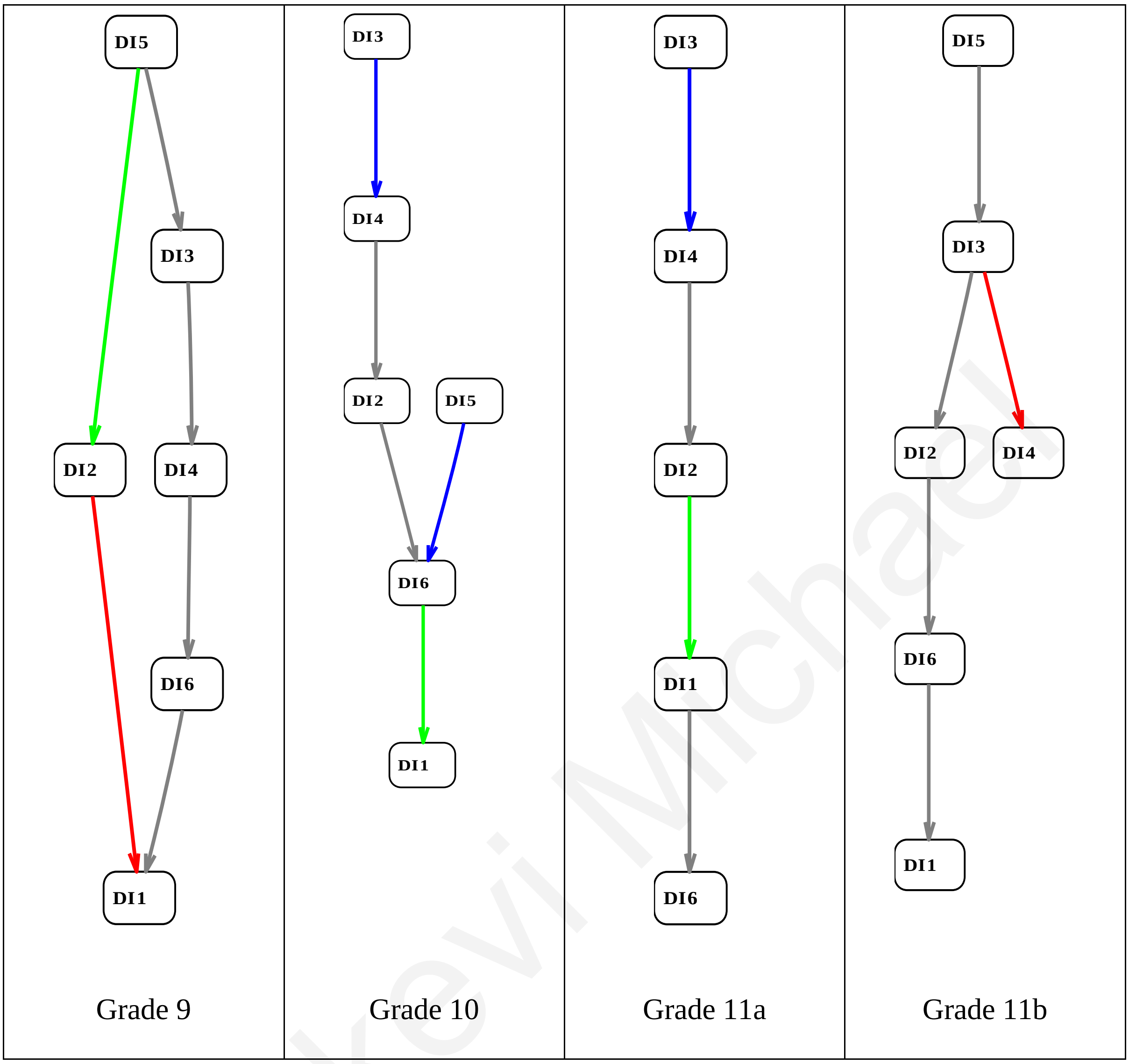


Figure 48. Implicative diagrams for the students' responses from the mathematical point of view in the discursive apprehension tasks for each grade

The implicative relation (Figure 49) between the two perceptual apprehension tasks indicates that task PE2 is more difficult than task PE1. The same result is also indicated in the Rasch model for the total of the students. Students appear to recognize the different hidden squares in the given figure in task PE2 correctly more easily than finding the name of the coded figures and defining their type in task PE1.

The examination of the implication between the two perceptual apprehension tasks for each group of students (Figure 50) shows that the ranking of the tasks is the same in grades 9, 11a and 11b, in which task PE2 is at the top of the implicative relation and task PE1 at the bottom. However the opposite happens in grade 10, in which the two tasks are placed in the opposite way.

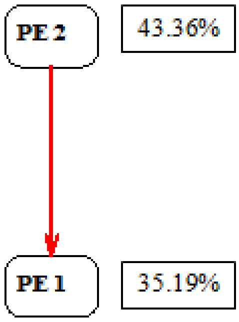


Figure 49. Implicative diagram for the students' responses from the mathematical point of view in the perceptual apprehension tasks for the total sample

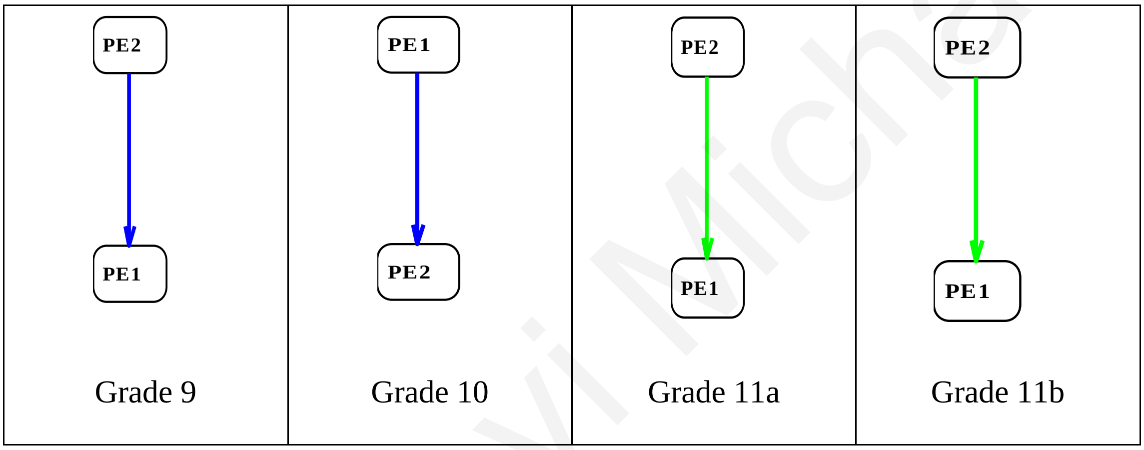


Figure 50. Implicative diagrams for the students' responses from the mathematical point of view in the perceptual apprehension tasks for each grade

The Lower and the Upper Secondary School Students' Mistakes and Ideas on the Geometrical Figure Apprehension

Analysis of wrong answers and mistakes in the tasks for each type of apprehension

This section deals with students' unsuccessful efforts during the solution of the tasks that they were given for the purposes of this study. Actually in this section students' wrong answers and the different mistakes that occurred during the solution of the tasks are presented and analyzed for each task separately and for each one of the four groups of students. First the results regarding the perceptual and the operative apprehension tasks are displayed and then the results of the sequential and finally the discursive apprehension tasks follow.

Firstly, table 29 concerns the first perceptual apprehension task and presents the percentages of students that did not achieve the recognition of each figure that was required in task PE1. Actually the table shows the percentages of wrong answers for each figure separately for all the different groups of students. A first general observation is that most of the students' mistakes appear regarding the recognition of the type of coded figures IKGU and HFGI, which are the two trapeziums. In fact this is true for all the different groups of students.

Table 29

Percentages of Wrong Answers in the Recognition of Figures in task PE1

Task PE1	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Figure KEZL	5.8	1.3	3.2	0.7
Figure IEZU	6.4	2.3	6.8	1.4
Figure EZHL	20.2	10.2	16.0	5.0
Figure IKGU	67.9	39.1	50.4	35.7
Figure LGU	7.4	5.3	8.8	2.1
Figure BEL	7.4	6.9	16.0	0.7
Figure HFGI	42.0	29.6	52.0	18.6
Figure EGIH	6.7	10.5	15.2	2.9
Figure DBC	8.0	10.9	16.0	2.1
	(N=312)	(N=304)	(N=125)	(N=140)

More specifically, the highest percentage in the false recognition of the first figure (KEZL) is found for the 9th graders. For the second figure (IEZU) most of the mistakes are made by the students in grade 11a and grade 9, whose percentages are very similar. The majority of the mistakes in the third and fourth figure (EZHL and IKGU) appear to be made by the 9th graders. In the rest of the figures (LGU, BEL, HFGI, EGIH and DBC) the biggest percentages of false recognition belong to the students from grade 11a. Furthermore, for all the figures grade 11b students have the lowest percentages of wrong recognition. Overall, in all the figures the percentages of the students' wrong recognition are the highest in grade 9 and the lowest in grade 11b. The corresponding percentages in grade 10 are lower than in grade 9 and in grade 11a. Furthermore the percentages of the students' wrong answers in grade 11a are greater than in grade 11b.

As mentioned previously, most of the students' mistakes were related to the recognition of the two trapeziums (figure IKGU and figure HFGI). Therefore the particular mistakes that came about from the recognition of these figures were further examined. The following table (table 30) shows the percentages for each type of mistake that was identified during the recognition of these figures. For the figure IKGU three were the main types of mistakes that appeared. In the first type students could not specify the type of the coded figure IKGU and their answers were that: "*The figure IKGU is nothing*". In the second type of mistake the figure IKGU was considered as "*a triangle on a square*". In the third type of mistake students have recognized the figure IKGU as a pentagon. For the figure HFGI two more types of mistakes were traced (type 4 and type 5). In particular the type 4 mistake included answers in which the figure HFGI was seen as a rectangle and in type 5 mistake, similar to type 2 mistake, the figure was taken as a right triangle on a rectangle. The small differentiation in the types of mistakes that appeared for the recognition of these two trapeziums could be attributed to the orientation of the two figures, which was different. So the different orientation of the figures can be a factor that caused the different types of mistakes in the recognition of these figures.

According to the results of table 30 for the figure IKGU, the most common mistake regarding the total number of students is the type 2 mistake. Concerning each group of students separately, this is also the case for grade 9 and grade 11b students. In grade 10 the most frequent mistake is the first one, although the percentage for this type of mistake is actually very analogous to the percentage for the occurrence of the second type of mistake. For grade 11a students the highest percentage is found for the type 1 mistake. Regarding the mistakes in the figure HFGI, for the 9th graders the percentages of students mistakes are higher in the second type than in the first type. For the 10th graders the percentages for each type of mistake are equal and this holds also for the students in grade 11b. Regarding the students in grade 11a, similarly to the students in grade 9, the percentage of mistakes in the second type is larger than in the first type. The biggest number of answers in the type 1 mistake is given by students in grade 10, while for the type 2 this is the case for the students in grade 9. It is also interesting to examine the relations between the percentages regarding the type 2 and the type 5 mistakes, as these two types of mistakes are very similar. In fact in all the groups of students the frequency of the type 2 mistake is higher than the frequency of type 5 mistake.

Table 30

Percentages of Types of Mistakes in task PE1

Mistakes	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Figure IKGU				
Type 1	5.1	6.6	13.6	0.7
Type 2	13.8	6.3	9.6	5.7
Type 3	1.6	2.3	0.8	0.7
	(N=312)	(N=304)	(N=125)	(N=140)
Figure HFGI				
Type 4	4.8	4.9	0.8	1.4
Type 5	9.9	4.9	4.0	1.4
	(N=312)	(N=304)	(N=125)	(N=140)

Explanation of symbols: Type 1 = figure IKGU is nothing, Type 2 = figure IKGU is a triangle on a square, Type 3 = Figure IKGU is a pentagon, Type 4 = Figure HFGI is rectangle, Type 5 = Figure HFGI is a right triangle on a rectangle.

Next the results regarding the wrong answers in the recognition of the seven squares that were included in the given figure in task PE2 are described. These results are displayed in table 31. For each of the seven squares the percentages of the students' wrong answers for each grade are shown separately. The figure in which most of the wrong answers occur is the square HILN, for all the groups of students. This was the largest square which included all the other squares. It seems that students focused on finding the squares that were included in the whole figure and paid less attention to the outline of the whole figure, in order to be able to recognize it as a square too. The least frequent students' mistake concerns the recognition of the square SRPO, with the only exception being the students in grade 11a, whose less frequent mistake regards the recognition of the figure FXRS. In fact these two figures (the figure SRPO and the figure FXRS) are the two most obvious squares for someone to recognize at first glance, because these two figures do not include any other subfigures. This can explain the students' success in the correct recognition of these figures as they were more easily traced and recognized in the whole given figure, compared to the rest of the figures. In fact there is not a big difference between the students' wrong answers in these two figures and for the 9th graders these two percentages are in fact the same.

Table 31

Percentages of Wrong Answers in the Recognition of Figures in task PE2

Task PE2	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Figure HUST	21.2	15.5	16.8	2.9
Figure UIKS	22.8	15.1	16.8	2.1
Figure SKLM	23.7	15.8	19.2	4.3
Figure TSMN	24.4	16.1	23.2	3.6
Figure SRPO	15.7	10.2	11.2	1.4
Figure FXRS	15.7	8.6	13.6	2.1
Figure HILN	39.1	28.0	27.2	13.6
	(N=312)	(N=304)	(N=125)	(N=140)

Regarding the four biggest squares HUST, UIKS, SKLM and TSMN the percentages of the students' wrong answers are very similar for each shape. A small differentiation appears in grade 11a though, in which the percentages of the figures SKLM and TSMN are higher than for the figures HUST, UIKS. Same as in the previous task, in all the figures the percentages of the students' wrong recognition are the greatest in grade 9 and the smallest in grade 11b. The relevant percentages in grade 10 are lower than in grade 9 and in grade 11a. Furthermore the percentages of the students' wrong answers in grade 11a are higher than in grade 11b.

Besides the wrong answers concerning the recognition of the seven squares, other figures, which were not squares, were also recognized by students. The following table (table 32) presents the three more frequent types of mistakes which occurred, in relation to the other types of mistakes. The type 1 mistake is related to the consideration of the figure YXPJ as a square. In the second type of mistake figure YFST was recognized as a square, while in the type 3 mistake it was the figure TSOJ that was mentioned as a recognized square. For grade 9, there are no major differences among the percentages of the three types of mistakes, especially for the first two types of mistakes, although the percentage is slightly higher for the type 3 mistake. In grade 10 the percentages for the type 2 and type 3 mistakes are almost analogous, while the greatest percentage is found in the first type of mistake. This is also the case for the students in grade 11a, even though the difference between the type 2 and type 3 mistakes is slightly bigger than in grade 10. For grade 11b students the percentages are the same for the two last types of mistake, whereas the type 1 mistake has the highest percentage of answers. Therefore the most observed mistake for the total of students is the first type of mistake and the least observed is the second type of mistake.

Examining the differences in each type of mistake among the different groups of students, students in grade 9 have the lowest percentage in the type 1 mistake. The percentages for the students in grades 10 and 11a, which are very similar, are greater than the percentage for the students in grade 9. In this type of mistake the biggest number of answers is given by the students in grade 11b. On the contrary, the students in grade 11b have the lowest percentage of wrong answers in the second type of mistake, whereas the 9^{th} graders have the highest percentage. The students in grades 10 and 11a are found in the middle, whose percentages are comparable in this case too. Conversely, in the type 3 mistake, the biggest percentage of incorrect answers is found for grade 11a students and the smallest for grade 11b students. In this case the 9^{th} and the 10^{th} graders are found in the middle, whose amount of answers is similar.

Table 32

Percentages of Types of Mistakes in task PE2

Mistakes	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Type 1	14.7	19.4	19.2	23.6
Type 2	14.4	13.2	13.6	11.4
Type 3	15.1	14.5	16.8	11.4
	(N=312)	(N=304)	(N=125)	(N=140)

Explanation of symbols: Type 1 = figure YXPJ, Type 2 = figure YFST, Type 3 = Figure TSOJ.

Regarding the group of the operative apprehension tasks, table 33 indicates the percentages of the students' wrong answers and mistakes in task OP1. This task included three choices, one of which was the correct answer.

Table 33

Percentages of Wrong Answers and Mistakes in task OP1

Type of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
No answer	6.1	4.9	7.2	1.4
Wrong Answer	31.7	23.7	13.6	12.9
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes				
Type 1	26.16	23.64	11.76	28.00
Type 2	73.75	77.69	88.24	72.33
	(N=99)	(N=72)	(N=17)	(N=18)

Explanation of symbols: Type 1 = figure A has a smaller area than figure B, Type 2 = that figure A has a smaller area than figure B.

The remaining two choices are presented in table as type 1 and type 2 mistakes. In fact, in the type 1 mistake the answer that the students have chosen was that "*figure A has a smaller area than figure B*". In the type 2 mistakes the answer "*figure A has a bigger area than figure B*" was given. The second type of mistake was more often observed for all the different groups of students, in comparison to the first type of mistake. In the first type of mistake the biggest percentage is observed for grade 11b students, while the second for students in grade 11a. The lowest percentages correspond to the students in grade 11a and the students in grade 11b for the first and the second type of mistake respectively.

The results of the students' wrong answers and mistakes in task OP2 are displayed in table 34. In this task the percentages of the cases in which no answer or a wrong answer were provided cannot be considered as low. In fact the highest percentage of cases in which the task was left unanswered is found for grade 11a students, whereas the lowest amount of such cases relates to the students in grade 11b. The percentages of the 9^{th} and the 10^{th} graders are similar in this category of answers. Most of the cases in which the answers that students gave were wrong were observed for the 9^{th} graders while the least for students in grade 11b. In this category the percentages of the students' answers in grades 10 and 11a are equal.

As regards the types of mistakes that were traced, three categories were formed for each type. The type 1 mistake had to do with the wrong use of calculations. Specifically in this category, students' wrong calculations during while using formulas were included, as well as answers including meaningless combinations of data (e.g. $4\times 7=28\div 4+7+4+7=22$). Answers including the calculation of the perimeter of the trapezium and subsequently of the rectangle and answers including the calculation of the big base of the trapezium (3+3+3=9) were also added in this type of mistake. The second type of mistake had to do with the wrong reconfiguration of the given figure and therefore with an unsuccessful mobilization of the operative apprehension. Finally in the last type of mistake other different less often observed mistakes were grouped.

Table 34

Percentages of Wrong Answers and Mistakes in task OP2

Type of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
No answer	36.5	36.2	45.6	15.7
Wrong Answer	31.4	28.0	28.0	20.7
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes				
Type 1	68.77	63.66	65.71	72.41
Type 2	6.05	1.07	5.71	0.00
Type 3	20.38	34.33	28.57	17.38
	(N=98)	(N=85)	(N=35)	(N=29)

Explanation of symbols: Type 1 = wrong use of calculations, Type 2 = wrong reconfiguration of the given figure, Type 3 = other mistakes.

The results show that the type 1 mistake is the one mostly observed in the students' answers and this is the case for all the groups of students. For the other two types of mistakes the percentages for each group of students are lower. In the first type of mistake the biggest percentage was observed for the students in grade 11b, while the lowest percentage is noticed for grade 10 students. Regarding the second type of mistake, the greatest percentage occurred for the students in grade 9, while the smallest percentage is noticed for grade 11b students. What is also noticed is that no answers at all were given by grade 11b students. Regarding the final type of mistake the highest percentage occurred for the students in grade 10, while the lowest percentage appears again for the students in grade 11b.

For the next task (OP3) the information on the percentages of the students' wrong answers and mistakes are summarized in table 35. The table includes the percentages of no answers and wrong answers as well as the percentages for each particular type of error for every group of students. To begin with, the cases in which no answer was given are less than the cases in which the students' answer was not correct and this is the case for all the groups of students. Most of the cases in which no answer was given are observed for grade

10 students, while the fewest cases are noted for grade 11b students. Regarding the wrong answers, most of them were given by the 10^{th} graders, whereas the lowest percentage was again observed for the students in grade 11b.

Table 35

Percentages of Wrong Answers and Mistakes in task OP3

Type of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
No answer	7.1	13.5	6.4	1.4
Wrong Answer	38.1	85.9	34.4	15.7
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes				
Type 1	98.32	32.61	97.67	95.45
Type 2	1.57	2.33	2.33	4.45
	(N=119)	(N=261)	(N=43)	(N=22)

Explanation of symbols: Type 1 = figure A has a bigger perimeter the figure B, Type 2 = figure A has a smaller perimeter than figure B.

Similarly to the first task (OP1), this task included three choices as well, one of which was the correct answer. The other two choices are found in the table as type 1 and type 2 mistakes. In fact in the type 1 mistake the answer that the students have chosen was that "*figure A has a bigger perimeter than figure B*". In the type 2 mistakes the answer "*figure A has a smaller perimeter than figure B*" was given. The first type of mistake was more often observed for all the different groups of students, in comparison to the second type of mistake. In the first type of mistakes the biggest percentage is found for grade 9 students, while the lowest percentage corresponds to students in grade 10. For the second type of mistake the students in grade 11b have the biggest percentage and the students in grade 10 have the smallest percentage.

For the next operative apprehension task (OP4) no results are displayed in this section which concerns students' wrong answers and mistakes, as none of the different responses that were given by the students could be considered a mistake. On the contrary the different answers that appeared were grouped according to the type of apprehension that was mostly involved for getting this answer and were analyzed in a previous section.

The last table (table 36) regarding the operative apprehension tasks presents the percentages of the students' wrong answers and mistakes in task OP5. Similarly to the previous task this table also includes the percentages of no answers and wrong answers as well as the percentages for each particular type of error for each group of students. First of all the cases in which no answers were given are fewer than those in which the students' answer was wrong and this is true for all the groups of students. Specifically, the students in grade 11a are related to the most cases in which no answer was provided, while the least of such cases were identified for students in grade 11b. Regarding the wrong answers, the students in grade 9 have the greatest percentage of such answers, whereas the students in grade 10 are those with the least wrong answers.

Table 36

Percentages of Wrong Answers and Mistakes in task OP5

Type of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
No answer	14.4	8.9	16.8	5.7
Wrong Answer	32.4	24.3	25.6	27.9
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes				
Type 1	41.70	46.01	50.00	64.26
Type 2	58.38	52.58	43.75	35.90
	(N=101)	(N=74)	(N=32)	(N=39)

Explanation of symbols: Type 1 = rectangle 1 has a bigger area than rectangle 2, Type 2 = rectangle 1 has a smaller area than rectangle 2.

In this task three choices were given to students as well. The two wrong choices are presented as type 1 and type 2 mistakes. Actually in the type 1 mistake the answer was that "*rectangle 1 has a bigger area than rectangle 2*". In the type 2 mistakes the answer "*rectangle 1 has a smaller area than rectangle 2*." was given. For students in grades 9 and 10 the type 2 mistake is more frequently observed than the type 1 mistake, whereas the opposite occurs for the students in grades 11a and 11b. In the first type of mistake the highest percentage is observed in grade 11b and in the second type this is the case for grade 9. The lowest percentage in the type 1 mistake is found for the 9th graders, while for the second type of mistake the lowest percentage is noticed for the students in grade 11b.

Moving to the tasks examining the students' sequential apprehension, the results regarding the students' wrong answers and mistakes in task SE1 are summarized in table 37. First of all the correct construction of the geometrical figure that was required in this task was not achieved by a large number of students, therefore the percentages of students' wrong constructions are considerably high, for all the groups of students. The lowest percentage of wrong constructions is observed for grade 9 students, whereas the highest percentage is found for grade 10 and grade 11a students, whose percentages are almost the same. However, the percentage of the cases in which no answer was provided by the students in this task is lower than the percentages of the students' incorrect constructions. This is true for all the different groups of students. The students in grade 11a possess the greatest amount of no answers, whereas the lowest amount of such answers is found for grade 11b students.

The students' incorrect answers were classified into three categories, according to the type of mistake that occurs in these answers. The first type of mistake included constructions in which students have drawn a random arc, when trying to construct a figure that would look similar to the given figure. In these cases students drew a ratio of random length and then draw an arc corresponding to this random ratio. Hence in these cases the sequence of the proper steps for the construction of the figure was not taken into account by students at all, but on the contrary their focus was on making a construction that would perceptively look similar to the given figure. In the second type of mistake the students made an effort to follow some steps for the construction of the figure. However, the procedure they followed was wrong, as it included false steps. For example there were cases in which students have mentioned that:

"*I measured the two ratios OM and ON in the given figure, I tried to keep the slope of the ruler stable and then I drew the two new ratios. Then I drew the points A and B and next I joined them in order to form an arc*".

In addition, into this category fall answers which were based on finding the middle point of the given arc. In specific the students noted that:

"*I have measured the ratio from the point O to the middle point of the arc and then I followed the same procedure for the required arc AB*".

The last type of mistake included answers in which different other mistakes were observed, which had low frequency and could not be considered a separate category.

Table 37

Percentages of Wrong Answers and Mistakes in task SE1

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Wrong construction	68.6	58.6	58.4	57.9
No answer	21.8	23.4	28.8	10.0
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes in constructions				
Type 1	70.56	83.17	55.82	40.79
Type 2	19.10	23.57	20.55	48.22
Type 3	17.20	24.08	17.81	17.11
	(N=214)	(N=178)	(N=73)	(N=81)
Correct description of procedure	8.0	11.5	2.4	28.6
Wrong description of procedure	38.5	33.2	28.8	42.9
No description of procedure	53.5	55.3	68.8	28.6
	(N=312)	(N=304)	(N=125)	(N=140)

Explanation of symbols: Type 1 = constructions with a random arc drawn, Type 2 = wrong procedure, Type 3 = other mistake.

According to table 37 the most frequent type of mistake is type 3 mistake, but this is not the case for grade 11a students, for whom type 1 mistake is more frequent. The highest percentages in type 1 mistake is observed in grade 10 and the least observed cases related to this type of mistake concern grade 11b students. On the contrary the students in grade 11b make type 2 mistake more frequently, as they have the highest percentage of such answers. For the third type of mistake it is the students in grade 10 that gave the most answers including it. The least answers of this type are found for grade 11b students.

This task also involved the description of the procedure that students have followed for constructing the geometrical figure they were asked. Besides the difficulties that students have faced with the construction of the task, students also encountered difficulties with describing the procedure they have followed and this is obvious in the percentages of students that formulated a correct description of the construction of the figure. In fact the percentages of the students that described the construction procedure correctly are low in all the groups of students, but especially for the three first groups of students. These low percentages are related to the low percentages of students that carry out the correct construction of the figure (see table 10). In all the cases the number of students that put together a correct description is lower than the number of those that conducted a correct construction of the figure. This shows that even though the students are able to follow the sequence of steps for the construction of a figure correctly, they are not always able to produce a written description of this sequence. It should also be mentioned that the highest percentage of correct descriptions is given by grade 11b students, whereas the lowest percentage is found in grade 11a.

Nevertheless, there were cases in which students tried to provide a description of the construction procedure, though unsuccessfully. The greatest percentage of wrong descriptions is observed for grade 11b students and the smallest percentage of wrong descriptions is found in grade 11a. Finally there were also cases in which students did not give any description of the procedure they have followed. The majority of such cases is found in grade 11a and the least such cases are found in grade 11b.

The next table (Table 38) illustrates the results regarding the percentages of students' wrong answers and mistakes in task SE2. Similarly to the previous task, the construction of the geometrical figure that was required in this task was also difficult for students. In fact, according to the results of table 38, the difficulties are greater in this task, than in the previous one, as students score lower in this task. Therefore, the percentages representing the students' wrong or no answers are not very low. More than half of the students' answers in grades 9, 10 and 11b are unsuccessful, with the majority of these answers found for the 9^{th} graders. The lowest percentage of wrong constructions is found in grade 11a, which is lower than 50%. Concerning the cases in which no answer was provided, most of these cases are observed in grade 11a. The percentages are lower in grades 9 and 10, in which the difference is not very significant. The lowest percentage is observed for grade 11b students.

Table 38

Percentages of Wrong Answers and Mistakes in task SE2

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Wrong constructions	66.3	65.5	48.0	57.9
No answer	34.4	31.3	50.4	20.0
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes in constructions				
Type 1	15.52	36.82	55.00	20.91
Type 2	26.98	21.08	8.33	32.15
Type 3	61.34	75.47	48.33	65.33
	(N=207)	(N=199)	(N=60)	(N=81)

Explanation of symbols: Type 1 = wrong construction of the triangle, Type 2 = wrong construction of the circle, Type 3 = other mistake.

Observing the wrong constructions drawn by the students, three types of mistakes were identified. Thus students' incorrect solutions were distributed into three categories, each one corresponding to a different type of mistake. In fact, the type 1 mistake is related to the construction of the triangle. There were cases in which students have drawn side AB of the triangle to be greater that side AC, which was opposite to the given data. There were also cases in which the students have drawn an isosceles triangle with side AB being equal to side AC. The second type of mistake is related to the construction of the circle. Cases were observed in which the construction of the diameter of the circle was wrong. For example there were students that drew the diameter AE as a ratio. In other cases students drew a random point Z for the circle, without using the given information. In the third type of mistake, different other mistakes whose appearance was less frequent were grouped.

In the third type of mistake most of the students in grades 9, 10 and 11b are found, whereas students in grade 11a mostly make the type 1 mistake. Actually the students in grade 11a have the highest percentage of answers in the type 1 mistake. In the type 2 mistake the greatest percentage is observed for grade 11b students and for the last type of mistake the 10^{th} graders are those with the biggest percentage of such answers.

For the last sequential apprehension task, the percentages of students' incorrect answers and errors are displayed in table 39. This task also reflects what was previously mentioned about the increased difficulties that students have encountered for the solution of the task, which is highlighted by the percentages of wrong or no constructions. The greatest percentage of wrong constructions is found for grade 11a students, with this percentage being similar to the percentage of the wrong answer given by the students in grade 11b. The lowest percentage of wrong constructions is found for grade 11a students. On the contrary grade 11a students have the biggest percentage of no answers in this task, while the lowest amount of no answers belongs to grade 11b students.

As far as the students' wrong constructions are concerned, four types of mistakes were traced, according to which four groups of wrong answers were formed. In particular in the type 1 mistake, answers in which students have drawn a random parallelogram were included. In the type 2 mistake constructions in which the area of the constructed parallelogram was double the area of the triangle were grouped. The third type of mistake comprised the construction of a different figure, for example a rectangle or a trapezium. In these cases students usually joined points B and C with the two edges of the given line d respectively, in order to construct a closed figure. The type 4 mistake includes all the other different mistakes which appear less frequently and thus were grouped together.

For the first three groups of students the most frequent type of mistake is the first, while for grade 11b students it is type 4 mistake that appears the most. In the type 1 mistake the highest percentage is in grade 11a. In addition, in the type 1 mistake the percentage of the 9th and the 10th graders is very similar. Grade 11b students have the smallest percentage in this type of mistake. The appearance of the type 2 mistake in the students' answers was less frequent, as the percentages in this category are lower, with the lowest percentage being for grade 11b. Regarding the type 3 mistake, the percentages are low in this case too. Most of these mistakes are made by grade 11b students, whereas very few are the mistakes of this type that committed by grade 11a students. For the last type of mistakes the percentages are low for the three first groups of students, but mostly for grade 9 and grade 11a students. The highest number of mistakes falling into this category regards grade 11b students.

Coming to the results of the description of the construction procedure, which was also asked in the task, it is obvious that the students that succeeded in giving a correct description are very few. In fact no student has managed to do this in grade 11a. On the other hand the greatest percentage of correct descriptions was provided by grade 11b students. However this is also the case for the wrong descriptions of the construction procedure in the same grade. The biggest percentages are found in the category representing the non provision of a description of the construction procedure. In this category high percentages appear for all the groups of students, with the highest being found in grade 11a, which is above 90%. The least number of no descriptions of the construction procedure is found for grade 11b students.

Table 39

Percentages of Wrong Answers and Mistakes in task SE3

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Wrong constructions	54.8	44.4	39.2	53.6
No answer	37.2	48.4	53.6	30.7
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes in constructions				
Type 1	57.84	58.55	65.31	49.28
Type 2	13.50	11.93	12.24	5.41
Type 3	5.84	5.85	2.04	6.72
Type 4	17.52	34.00	10.20	51.89
	(N=171)	(N=135)	(N=49)	(N=75)
Correct description of procedure	3.5	5.6	0.0	11.4
Wrong description of procedure	19.9	15.8	8.0	31.4
No description of procedure	76.6	78.6	92.0	57.1
	(N=312)	(N=304)	(N=125)	(N=140)

Explanation of symbols: Type 1 = constructing a random parallelogram, Type 2 = the area of the constructed parallelogram was double the area of the triangle, Type 3 = construction of a different figure, Type 4 = other mistake.

Regarding the discursive apprehension tasks, the students' wrong answers and mistakes in task DI1 are shown in table 40. In this task students had to choose the correct answer among four possible answers that were given in the task. The results show that regarding the group of students that did not reach the correct solution of this task, the percentages of those who did not provide an answer are greater than those who gave a wrong answer and this is the case for all the four groups of students. In fact, in grade 11a the percentage of students that were not able to give an answer is similar to the percentage of students that answered correctly in this task. In all the grades, the answer that the comparison between the segments NH and MH cannot be determined is chosen by the highest number of students, compared to the rest of the wrong answers. These numbers can be related to a lack of knowledge of the relevant theorem that was necessary for the solution of the task. This could also be the case when justifying the significant number of students that were not able to provide any answer.

Table 40

Percentages of Wrong Answers and Mistakes in task DI1

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
No answers	31.65	32.22	41.2	13.14
Wrong answers:	11.3 (N=312)	6.6 (N=304)	14.8 (N=125)	4.0 (N=140)
Mistakes				
M1: NH is bigger than MH	8.91	15.20	0.00	14.00
M2: NH is smaller than MH	23.18	10.64	5.26	0.00
M3: The comparison between NH and MH cannot be determined.	68.64	74.48	92.11	86.00
	(N=35)	(N=20)	(N=19)	(N=7)

Besides choosing an answer, students also had to explain and justify their choice. The results in table 41 show that the students who provided a correct justification are less than those who gave a wrong or no justification. Especially in grades 10 and 11a the percentages of correct justifications are very low, whereas the highest percentages of such answers are found for grade 11b students. For grade 9 and 11a the majority of the students did not give a justification for their answers, whereas for grades 10 and 11b the greatest percentages are for students whose justification is wrong.

Students' incorrect justifications were discriminated according to the mistakes they included and four types of mistakes were determined. The first type of mistake was related to the parallelism of lines. In particular students' justifications were based on drawing a triangle MHN, which they considered an isosceles. They explained that "*NH is equal to MH because the distance of the two points M and N on line (e) from the same point H on the parallel segment BC is the same*". The second type of mistake concerned the students' justifications that were based on the comparison of triangles. Actually the students compared the triangles MHO and HON, using that MO = ON (the point O is the intersection point of segments AH and MN), but without justifying it.

Table 41

Percentages of Types of Justification and Mistakes in Justifications in task DI1

Types of justification	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Correct justification	11.5	4.9	2.4	20.7
Wrong justification	35.6	50.0	40.0	51.4
No justification	52.9	45.1	57.6	27.9
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes in justification				
Type 1	10.96	8.60	18.00	15.17
Type 2	7.03	14.60	10.00	11.08
Type 3	30.64	33.40	12.00	30.53
Type 4	47.78	44.60	66.00	42.97
	(N=111)	(N=152)	(N=50)	(N=72)

Explanation of symbols: Type 1 = justification based on the parallelism of lines, Type 2 = justification based on the comparison of triangles, Type 3 = justification based on the reproduction of the given data, Type 4 = other mistakes.

There was also another kind of answer in which the comparison of triangles was used. In this type of answer students drew the triangles MBH and NHC and then compared them using the fact that that angle AHC is 90° as a criterion. The third type of mistake was related to the given data. There were students whose answer was actually a reproduction of the given information in the task. There were also students that characterized the given data as insufficient. Specifically they noted that "*the comparison between NH and MH cannot be determined because there is data missing*". Finally the last category of students' mistakes included other different errors that occurred in their justifications, such as the provision of equations without justifying them.

Observing table 41, most of the students' wrong justifications are included in the fourth type of mistake, for all the different groups of students. In the type 1 mistake the students from grades 11a and 11b have more answers than the 9^{th} and 10^{th} graders. The highest percentage of justifications in type 2 and type 3 mistakes are found for grade 10 students, whereas for type 4 mistake this is the case for grade 11a students.

Table 42 presents the percentages of students' wrong answers and mistakes in task DI2. The highest percentages of wrong answers are found in grade 10, while the lowest are in grade 11a. On the contrary the largest amount of no answers is given in grade 11a, whereas the smallest amount is found in grade 11b. Regarding students' justifications in this task, similarly to students' correct answers, the percentages of the correct justification were again not very high, especially for grades 9, 10 and 11a. The greatest percentage of such answers concerns grade 11b students, whose correct answers are slightly over 50% of their total answers. In fact the majority of students in grades 9, 10 and 11a did not provide any justification while the amount of wrong justifications is less than the amount of no justifications. This situation is different for grade 11b students, for whom the percentage of wrong justifications is equal to the percentage related to the absence of justification.

Students' incorrect justifications were discriminated according to three types of mistakes that were observed. In the task students were asked to prove the equity among three segments of the given figure. But there were answers in which students proved only the equity between the two of the three segments, therefore their answers were incomplete and were categorized as wrong. Thus the first type of mistake only proved that FG= LF, the second type that AC= LF and the last type was related to only proving that AC = FG. In effect, table 42 shows that the greatest amount of mistakes in justifications are found in the type 1 mistake for students in grades 9 and l0 and in the second type for students in grades 11a and 11b.

Table 42

Percentages of Wrong Answers and Mistakes in task DI2

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Wrong answer	21.2	30.9	14.4	20.7
No answer	57.7	43.1	60.8	22.1
	(N=312)	(N=304)	(N=125)	(N=140)
Correct justification	20.8	26.0	24.8	57.1
Wrong justification	21.2	30.9	14.4	22.1
No justification	58.0	43.1	60.8	22.1
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes in justification				
Type 1	45.38	41.40	36.11	21.23
Type 2	22.69	28.78	40.28	64.13
Type 3	17.96	20.37	23.61	13.55
	(N=66)	(N=94)	(N=44)	(N=31)

Explanation of symbols: Type 1 = prove only that FG= LF, Type 2 = prove only that AC= LF, Type 3 = prove only that AC = FG.

The results of students' answers in the tasks examining the identification of the formal proof are indicated in tables 43 and 44 respectively. Specifically the results of students' answers for task DI3 are included in table 43. Most of the students gave an incorrect answer in this task and the percentages of students' no answers are not very high. In particular, there were students that accepted only one answer as a proof, but there were also cases in which students accepted two answers or even all the three answers as formal proofs. Thus students' mistakes were categorized in six types. Students that accepted only one type of proof were included in the first two types of mistakes. In fact answers in which the answer of student A was accepted were included in the type 1 mistake, whereas answers in which the answer of student B was accepted were included in the type 2 mistake. The next three types of mistakes were related to answers in which two types of proofs were accepted. Specifically the type 3 mistake included answers that accepted the proofs of students A and B, the type 4 mistake included answers that accepted the proofs of students A and C and in the type 5 mistake, answers that accepted the proofs of students B and C were categorized. Finally in the type 6 mistake the acceptance of the three types of proofs was included.

Table 43

Percentages of Wrong Answers and Mistakes in task DI3

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Wrong answers	80.2	79.4	65.6	70.1
No answers	9.9	5.5	12.8	4.2
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes				
Type 1	9.66	8.96	11.22	3.90
Type 2	5.33	5.42	2.04	6.73
Type 3	5.00	6.60	9.18	5.79
Type 4	23.87	31.81	34.69	20.19
Type 5	13.55	15.55	8.16	19.25
Type 6	31.64	25.22	18.37	38.50
	(N=250)	(N=241)	(N=82)	(N=98)

Explanation of symbols: Type 1 = answer of student A , Type 2 = answer of student B , Type 3 = answer of students A and B, Type 4 = answer of students A and C , Type 5 = answer of students B and C , Type 6 = answer of students A, B and C.

Table 43 indicates that the majority of grade 9 and grade 11 students commit the type 6 mistakes. For grades 10 and 11a the majority of the students' answers is found in the type 4 mistake. On the other hand the lowest percentages are found in the type 3 mistake for grade 9 and 11b students, but for students in grades 10 and 11b this is the case for the type 2 mistake. Concerning the mistakes in which one type of proof was accepted, grade 9, 10 and 11a students' percentages are higher in type 1 mistake than in type 2. For grade 11b students the opposite situation applies. Regarding the students' mistakes in which more than one type of proof was accepted, most of these mistakes occur in the type

4 and type 6 mistakes. In fact for grade 9 and grade 11b students the biggest amount of such mistakes are in type 6. For grade 10 and 11a students the greatest amount of such mistakes is included in the type 4 mistake.

Regarding the corresponding DI4 task, the results are displayed in table 44. The amount of wrong or no answers is similar to the previous task. The numbers indicate that the majority of grade 9 students make the type 1 mistake, the majority of grade 10 students the type 4 and 6 mistakes, most of the students in grade 11a make the fourth and fifth types of mistakes with exactly the same number and finally the type 6 mistake is the most frequent for grade 11b students. This situation is different from the results of the previous task. On the contrary the results of the mistakes in which one type of proof or more than one types of proof were are very similar. Specifically, as regards the mistakes in which one type of proof was accepted, all groups of students' percentages are higher in type 1 mistake. As for the students' mistakes in which more than one type of proof were accepted, 7 most of these answers were found in the type 4 and type 6 mistakes. In particular for grade 9 and grade 10 students the largest amount of such mistakes is found in type 4. For grade 11a students the greatest amount of such mistakes is included in the type 4 and 5 mistake and for grade 11a students in the type 6 mistake.

Comparing the way the different groups of students answer in the two tasks when examining the recognition of proof, there are differentiations in students' mistakes in this pair of tasks. In particular in the first three types of mistakes the percentages are higher in task DI4 than in task DI3 for the 9th graders. This is also the case for students in the rest of the grades, but a differentiation also exists in grade 11b, in which the percentage of the type 2 mistake is lower in task DI4 than in task DI3. For the type 4 mistake the amount of such mistakes is lower in task DI4 than in task DI3 for grade 9, 10 and 11a students, whereas the opposite happens for grade 11b students. Regarding the fifth type of mistake, this mistake is more frequent in task DI3 than in task DI4 for students in grades 9, 10 and 11b, but this is not true for students in grade 11a, whose type 5 mistakes are more in task DI4 than in task DI3. Finally the last type of mistake is less observed in task DI4 than in task DI3 for all the groups of students.

Table 44

Percentages of Wrong Answers and Mistakes in task DI4

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Wrong answers	75.6	72.7	68.8	66.4
No answers	10.3	8.55	8.8	4.3
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes				
Type 1	23.66	20.36	19.77	7.53
Type 2	8.86	9.90	9.30	6.47
Type 3	15.20	12.24	12.79	10.69
Type 4	22.47	23.52	20.93	24.69
Type 5	11.90	10.87	20.93	12.95
Type 6	17.85	23.11	16.28	37.63
	(N=236)	(N=221)	(N=86)	(N=93)

Explanation of symbols: Type 1 = answer of student A , Type 2 = answer of student B , Type 3 = answer of students A and B, Type 4 = answer of students A and C , Type 5 = answer of students B and C , Type 6 = answer of students A, B and C.

Students' mistakes in task DI5 are summarized in table 45. In this task three types of mistakes are distinguished. The type 1 mistake is related to the use of the necessary theorem for the solution of the task. Students made irrelevant or no use of the theorem. For example students did not use the theorem for justifying the equal sides and the parallel lines in the figures they had to prove that they were parallelograms. The type 2 mistake included answers in which students did not make proper use of the necessary criteria in order to prove that a figure is a parallelogram. In these cases students either used the criterion of one pair of parallel lines in order to prove that the figure was a parallelogram or did not use any of the criteria and merely mentioned that "*there are three trapeziums that are parallelograms*". In the last type of mistake other different mistakes with less frequency were grouped. One example is answers in which students considered the medians as heights as well, indicating a lack of knowledge of the properties of the figures.

For the students in grade 9 and 11a the cases in which no answer was given are more than the cases in which the answer was wrong, while the opposite happened for the students in grades 10 and 11b. The majority of wrong answers is given by grade 10 students and the minority of such answers concerns grade 9 and grade 11a students. Most of the cases in which no answer was given are found in grade 11a and the least in grade 11b. Regarding the three types of mistakes, the results show that the type 2 mistake is the most frequent for all the students. More specifically the highest percentage of type 1 mistake is found in grade 11a. The type 2 mistake is mostly made by grade 9 students and the type 3 mistake by the 10th graders.

Table 45

Percentages of Mistakes in task DI5

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Wrong answers	34.6	46.1	34.4	42.9
No answers	56.11	38.11	63.2	35.67
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes				
Type 1	54.60	33.66	34.88	53.43
Type 2	4.62	11.29	23.26	26.60
Type 3	40.73	55.15	41.86	20.07
	(N=108)	(N=140)	(N=43)	(N=60)

Explanation of symbols: Type 1 = irrelevant use of the theorem, Type 2 = irrelevant use of criteria, Type 3 = other mistake.

The results regarding the students' answers in the last task examining the discursive apprehension are included in table 46. The results show that the percentages of the cases in which students did not give any answer in this task are greater than the cases in which their answer was wrong and in fact this is true for students in grades 9, 10 and 11a. On the other hand this is not the case for grade 11 b students, whose wrong answers are more than the cases they did not give an answer. Furthermore in grade 11b the least cases in which no answer was provided are found, compared to the rest of the groups of students, while the greatest number of such cases is found for grade 11a students.

Among the students' incorrect answers three kinds of mistakes were found. The first mistake (M1) was that students answered that the triangle ABC is equal to the triangle DEZ. In the second mistake (M2) the triangle DEZ was considered equal to the triangle KLM. The last type of wrong answers (M3) included the equity between the three given triangles, which shows a misuse of the instructions given in the task, as it is clarified that students have to find two triangles that are equal. This last type of mistake is also indicative of a lack in the students' theoretical background regarding the order of the criteria for the equity of triangles. In this case the students identify that all the triangles have angles of 50˚ and 40˚ and a side of 5cm, but they do not combine these information with the correct sequence of the criteria for choosing the proper triangles that are indeed equal. The most observable mistake among the three for all the groups of students is the second mistake. It seems that students focused on the indication concerning the angles of 40˚ and the side of 5cm which were present in both triangles, but did not consider the criteria for the equity of triangles (angle – side – angle) in the right order. Therefore this mistake can be indicative of a lack in students' theoretical background. The remaining two mistakes were more rarely found in students' answers, as indicated by the low percentages for all the groups of students.

In task DI6 students were also asked to justify their answer. The percentages of students that succeeded in providing a correct justification for their answer are not high in all the different groups of students. The highest amount of correct justifications is provided by grade 11b students and then the students in grade 9 follow. For grade 10 and 11a students the percentages of correct justifications are almost equal. The cases in which a wrong justification or no justification was provided are more than the cases in which students succeeded in justifying their answers correctly. For grade 9 and grade 11a students the percentages of answers with no justification are greater than the percentages of answers in which a wrong justification was provided. On the contrary for grade 10 and 11b students the percentages of answers in which no justification was given are lower than the percentages of answers in which the justification was wrong.

Table 46

Percentages of Wrong Answers and Mistakes in task DI6

Types of answers	Grade 9 (%)	Grade 10 (%)	Grade 11a (%)	Grade 11b (%)
Wrong answers	13.5	20.4	11.2	15
No answers	36.2	27.6	42.4	12.9
	(N=312)	(N=304)	(N=125)	(N=140)
Types of mistakes				
M1: ABC=DEZ	7.43	9.81	28.57	14.00
M2: DEZ = KLM	80.97	77.47	64.29	62.00
M3: ABC=DEZ= KLM	11.89	12.75	7.14	24.00
	(N=42)	(N=62)	(N=14)	(N=21)
Correct justification	12.5	8.9	8.8	13.6
Wrong justification	42.6	53.0	35.2	62.1
No justification	44.9	38.2	56.0	24.3
	(N=312)	(N=304)	(N=125)	(N=140)
Mistakes in justification				
Type 1	26.27	34.18	18.18	36.85
Type 2	6.80	2.45	0.00	6.92
Type 3	41.99	27.95	52.27	28.64
	(N=133)	(N=161)	(N=44)	(N=87)

Explanation of symbols: Type 1 = irrelevant use of criteria, Type 2 = visual observation, Type 3 = other mistake.

The students' incorrect justifications were classified into three categories according to the type of error that appeared in each. Specifically the first type of mistake is linked to the use of the criteria for proving the equity of triangles. In fact there were answers in which students referred to the proper relations between the triangles, but without using the proper criteria. There were also cases in which the students used four criteria and in a wrong sequence (angle – angle – angle – side). The category of the type 2 mistakes was formed by justifications which were mostly based on the visual observation of the figures. In such answers students justified their answer by mentioning that "*the triangles are equal because the figures look similar*". There were also answers in which the equal sides in the triangles were only mentioned, but not justified. Therefore this type of mistakes shows the influence of the students' visual perception of the figures, and hence the effect of the operative apprehension. The third type of mistakes comprised answers which included less systematic mistakes, which were grouped as other mistakes. One example of such cases is the use of formulas for calculating the areas of the triangles in order to trace the two equal triangles. The percentages in each type of mistake show that the first and the third types of mistakes appear more frequently in relation to the type 2 mistake. The highest percentage of the type 1 and type 2 mistake is observed in grade 11b. The greatest percentage for the third type of mistake is found in grade 11a. For grades 9 and 11a the most frequent is the third type of mistake. Furthermore the type of mistake that appears the most in grades 10 and 11b is the first.

The relations between the students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

In this part the students' mistakes in the tasks examining their abilities regarding the recognition of proof (DI3 – DI4) are examined in relation to their behavior in the discursive apprehension tasks, which examine their ability regarding the production of proofs (DI1, DI1, DI5 and DI6). Therefore the variables corresponding to the six types of mistakes in tasks DI3 and DI4 respectively were analyzed together with the variables that occurred from the cognitive analysis of the discursive apprehension tasks.

The results of the hierarchical clustering of variables and the implicative analysis of the data that occurred through the analysis of the students' answers in the discursive apprehension tasks showed that there were relations between the correct recognition of the formal proof in tasks DI3 and DI4 and the proper reasoning in the rest of the tasks. Therefore what is hypothesized in this case, in which the students' errors are also taken into account, is that the appearance of the six types of mistakes in tasks DI3 and DI4 will be related to a non proper reasoning and to the non provision of a justification in the rest of the tasks of this group.

The hierarchical clustering of variables and the implicative analyses were used in order to trace the relations explained above, through which a similarity and an implicative diagram occurred for each different groups of students respectively. When describing the diagrams the emphasis will be placed on the variables corresponding to the six types of mistakes in each of the tasks on the recognition of proofs and especially on the relations that are created between them and the students' answers in the rest of the proof tasks. In addition, the relations between the students' answers in the discursive apprehension tasks as analyzed from the cognitive point of view were described in a previous section. At the end of this part the type of geometrical paradigm in which the students' reasoning takes place is identified, based on the examination of the aforementioned relations.

Similarity relations between the students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

Figure 51 presents the similarity relations between the variables for grade 9 students. In this similarity diagram there are three similarity clusters. In the first similarity cluster the correct recognition of proof in tasks DI3 and DI4 is related to the correct inference in task DI6 (DI6ci), which form the first subgroup of this cluster. In the second subgroup of this cluster the sixth type of mistake in both tasks (DI3t6 and DI4t6), which included the acceptance of the three types of proofs in tasks DI3 and DI4, is related with variables indicating proper reasoning (DI1cj, DI5c, DI2vrvei) but also a comprehension of proof but with a gap (DI5cg). In addition the two variables showing the appearance of the sixth type of mistake are strongly and significantly related to each other, indicating the stability in the students' choices in these two similar tasks. In the second similarity cluster of this diagram the type 2 mistake in both tasks (DI3t2 and DI4t2) and the type 5 mistake in task DI4 (DI4t5) are related to the non proper reasoning in tasks DI1 (DI1wj) and DI6 (DI6wi). These two types of mistakes are involved in the same relations due to the fact that they both involve the choice of student B. In the last similarity cluster the type 1, type 3, type 4 and type 5 mistakes are included, which are related to the absence of inference (DI6ni) and justifications (DI1nj).

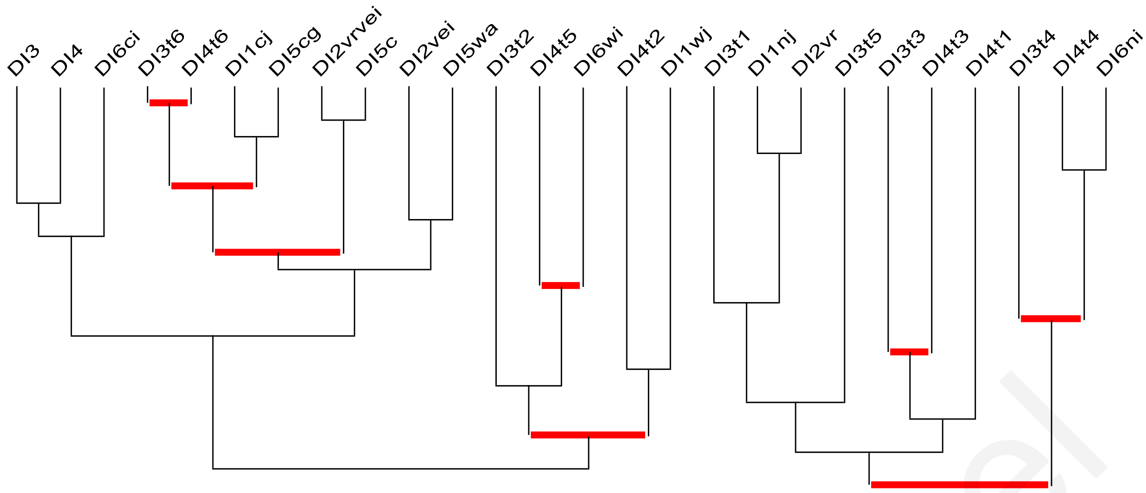


Figure 51. Similarity diagram for grade 9 students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

The similarity relations between the variables for grade 10 students are displayed in figure 52. In this similarity diagrams there are also three similarity clusters. The correct recognition of proof in tasks DI3 and DI4 in the first subgroup of the first similarity cluster are in this case also related to variables indicating proper reasoning and comprehension of proof (DI1cj and DI5c). In the second subgroup of the same similarity cluster the sixth type of mistake in both tasks (DI3t6 and DI4t6) are linked to proper reasoning (DI6ci) but also to the comprehension of proof with a gap (DI5cg). In these relations the type 5 mistake is also present (DI4t5), however with weaker relations, and it also related to a wrong reasoning (DI6wi). Similarly to the previous diagram, in the second similarity cluster of this diagram the type 2 mistakes in both tasks (DI3t2 and DI4t2) and the type 5 mistake in the task DI3 (DI3t5) are linked to incorrect reasoning (DI1wj and DI5wa). In the last similarity cluster, as in grade 9, the type 1, type 3, type 4 and type 5 mistakes are included, which are related to the absence of inference (DI6ni) and justifications (DI1nj).

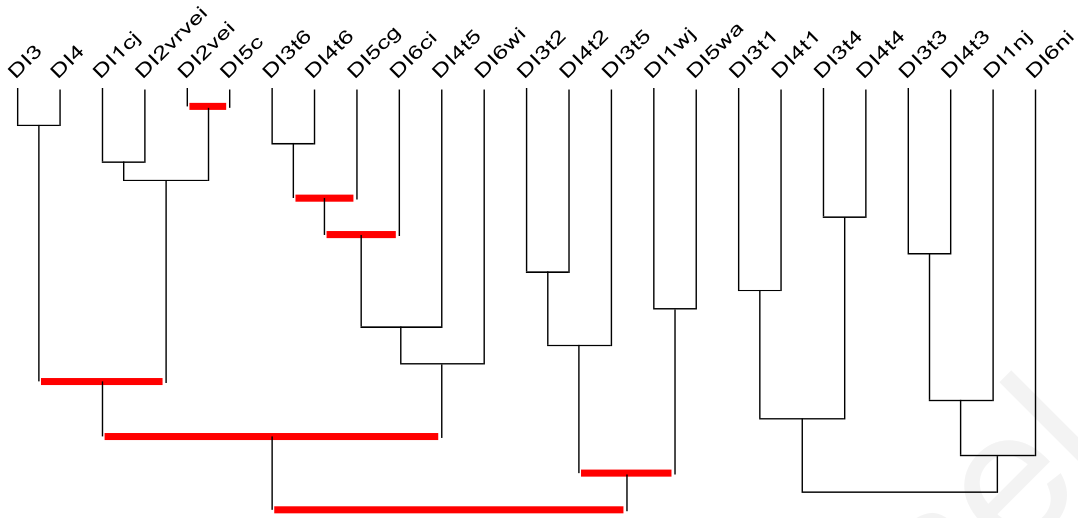


Figure 52. Similarity diagram for grade 10 students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

Therefore the formation of the three very similar groups of variables in the diagrams for the 9th and the 10th graders are indicative of three groups of students. In the first group a comprehension of proof is indicated by students, either complete or with a gap. In this group the students' answers showing a good ability to recognize a formal proof are related with answers showing that correct inference and justifications were achieved. The cases in which the students accepted all the types of proofs are also related with the aforementioned variables, but they are also related with variables indicating not a complete comprehension of proof. Hence the students that are able to distinguish a formal proof also seem able for proper reasoning in the production of proofs. On the other hand, those who accept the formal proof, as well as the other two types of proofs, seem to show comprehension of proof, which is however incomplete. What emerges from the second group of variables is that the students who accept a proof which is mostly related to the operative apprehension (student B) but also recognize the correct proof are not able to provide a proper justification, although they try to. In the last group, students' mistakes that appear are mainly related to proof which is mostly linked to the perceptual apprehension (student A). The fact that these answers are related to answers showing the students' difficulties with proving, as no inference or justifications appear, could be attributed to a less developed comprehension of proof.

The situation is not that clear for the answers of the students in grade 11a. In the similarity diagram for grade 11a students (Figure 53) there are also three similarity clusters, in which the variables form relations that are considerably different from the relations that were created in the diagrams of the ninth and the tenth graders. Specifically, in this grade in the first similarity cluster the correct recognition of proof is also related to the comprehension of proof and a correct reasoning (DI5c and DI1cj). In this cluster the type 4 mistakes in the two tasks (DI3t4 and DI4t4) are also related to the aforementioned variables. In the next similarity cluster the type 2, type 3 and type 6 mistake are related to variables expressing proper reasoning (DI6ci), while the type 3 mistake is linked to variables showing no reasoning (DI1nj, DI6ni). In the last similarity cluster the first type of mistake, which is closely related to the perceptual apprehension, forms a similarity relation with the variables showing wrong inference (DIwi) and an incomplete comprehension of proof (DI5cg). Finally the type 2, type 5 and type 6 mistakes are related to wrong justifications and answers (DI1wj and DI5wa).

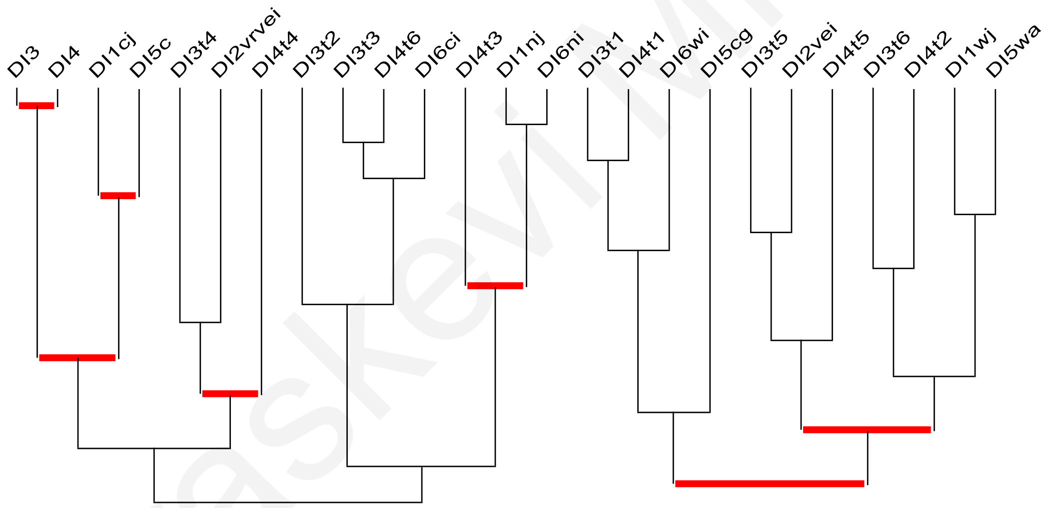


Figure 53. Similarity diagram for grade 11a students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

The relations between grade 11b students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs are displayed in the figure 54. In this diagram the correct recognition of proof (DI3, DI4) is for the first time not related to answers showing proper reasoning and good comprehension of proof. The type 1 and the type 3 mistakes, which are the types that are mostly linked to the perceptual apprehension, are linked to wrong answers (DI5wa) and justifications (DI1wj). In the second similarity cluster the type 2 and the type 5 mistakes – which mostly relate to the proof that incorporates elements of the operative apprehension – and the type 6 mistakes are related to variables that correspond to a good reasoning (DIIcj, DI6ci) and comprehension of proof (DI5c). In the last similarity cluster the type 4 and the type 5 mistakes form relations with answers showing incomplete comprehension of proof (DI6wi, DI6wj).

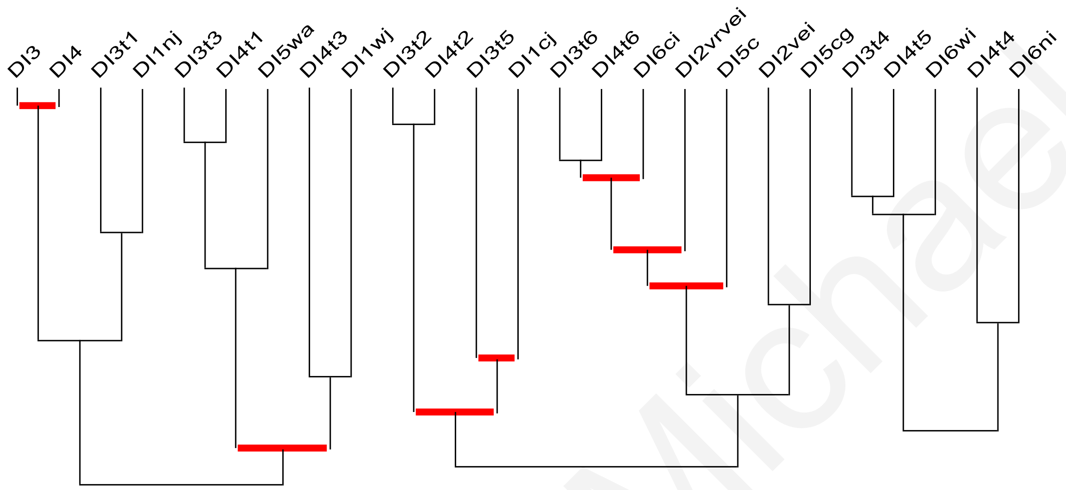


Figure 54. Similarity diagram for grade 11b students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

Implicative relations between the students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

Figure 55 displays the implicative relations between the variables for grade 9 students. First of all the correct recognition of proof in tasks DI3 and DI4 is related to correct justification (DI1cj). The type 1 mistake is involved in an implicative relation with an answer without a justification (DI1nj). Similarly the type 4 mistakes in both tasks are related to an answer in which no inference is displayed (DI6ni). Therefore these types of mistakes, whose occurrence relates to the perceptual apprehension, appear again to be related to the absence of justifications and inference, which is something that can also be attributed to the effect of perceptual apprehension. In another implicative chain the type 3 and type 5 mistakes respectively form an implicative relation with an answer in which the justification is wrong (DI6wj). Hence the types of mistakes that relate to the operative apprehension appear to be related with an effort for reasoning, though unsuccessful. Finally the type 6 mistake in each task is related to different kinds of answers. In particular the type 6 mistake in task DI4 is related to proper reasoning in task DI6 (DI6ci). On the other hand the same type mistake in task DI3 (DI3t6) is related to improper reasoning (DI6wi and DI1wj). Consequently the students that commit the type 6 mistake seem to have some comprehension of proof, however not complete or well developed. In addition students' display stability regarding the appearance of the type 4 and the type 6 mistake respectively, as well as regarding correct recognition of proof. Furthermore, these results are in line with the results that occurred through the description of the similarity relations between the students' answers that are examined in this case.

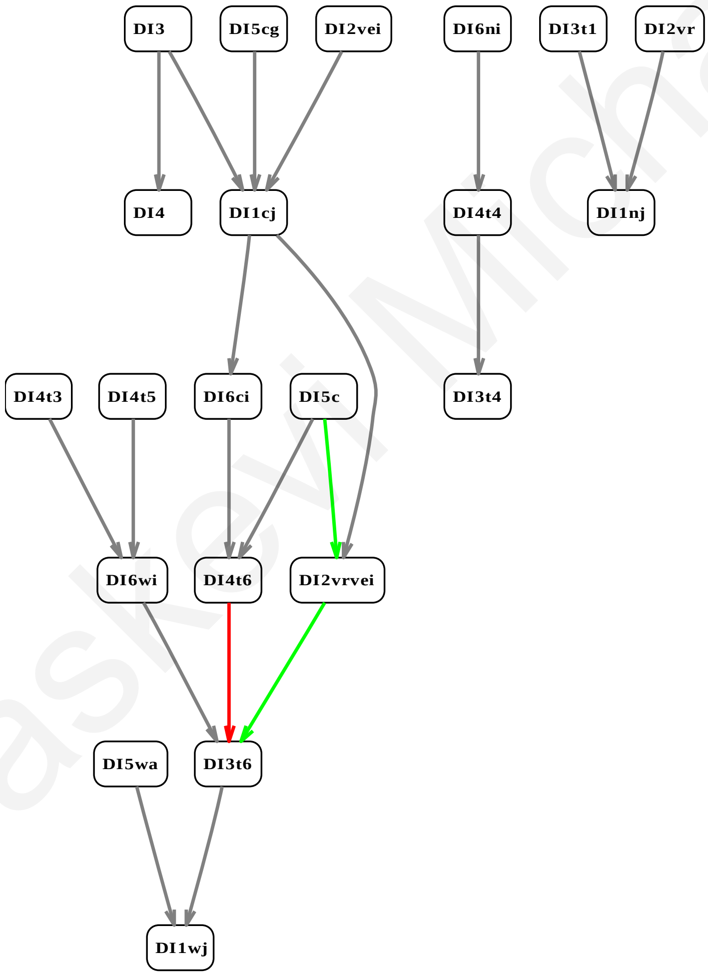


Figure 55. Implicative diagram for grade 9 students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

The implicative relations among the variables for the grade 10 students are shown in figure 56. In this diagram the students' coherence is noticed regarding the occurrence of the type 4 and the type 6 mistake respectively, but also as regards the correct recognition of proof. In particular, the type 4 mistakes and the correct recognition of proof respectively form two distinct implicative chains, in which no other variables are involved. The appearance of the type 5 and the type 6 mistake in task DI4 are related to the appearance of the type 6 mistake in task DI3. The type 6 mistake is either linked to comprehension of proof with a gap (DI5cg) or to improper reasoning (DI6wi). Finally the type 5 mistake is also related to incomplete reasoning (DI1wj). Again the students that make the type 6 mistakes appear to have some abilities for reasoning, however these abilities do not seem very complete and well developed yet.

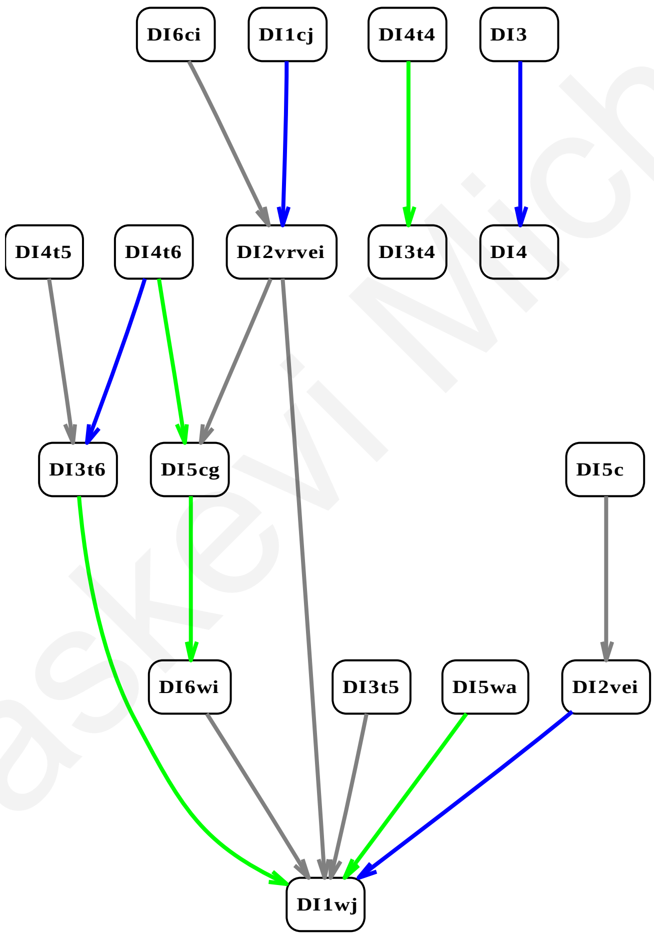


Figure 56. Implicative diagram for grade 10 students’ mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

The implicative relations among the variables for grade 11a students are displayed in figure 57. The implicative diagram indicates that the students that are able to recognize the proof in task DI3 successfully are also able to recognize the proof in task DI4 successfully. Conversely, the opposite case is not true. In particular the correct recognition of proof in task DI4 is not only related to comprehension of proof (DI5c), as previously, but also to an answer showing improper reasoning (DI1wj). This is a relation that appears for the first time in the implicative diagram, compared to the previous diagrams. As in the previous diagrams for the two first groups of students, the type 6 mistake (DI4t6) forms an implicative relation with a variable indicating good reasoning (DI6ci). However in this diagram the type 6 mistake (DI3t6) is also related to a variable indicating improper reasoning (DI1wj). Another new relation in this diagram concerns the type 1 mistake, the appearance of which is characterized by stability. This mistake is related to an answer in which wrong inference appears (DI6wi), whereas in the two previous cases this type of mistake was related to answers with no inference at all.

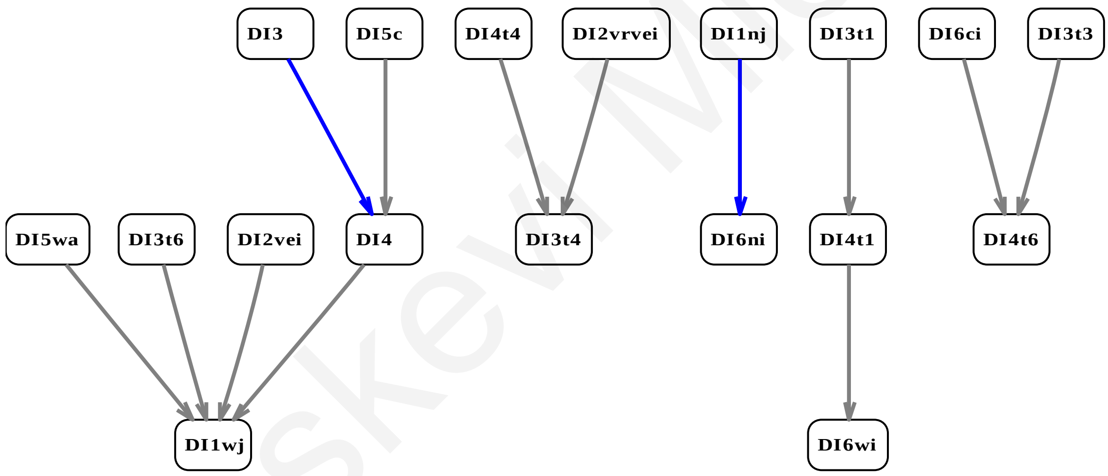


Figure 57. Implicative diagram for grade 11a students’ mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

Figure 58 presents the implicative relations among the variables for grade 11b students. The type of mistakes that are firstly noticed in this implicative diagram are the type 6 mistakes in both tasks (DI3t6 and DI4t6), for which there is stability regarding their appearance in the students’ answers. In addition these mistakes are related with answers showing proper reasoning (DI1cj and DI6ci). The second type of mistake is the type 1 mistake (DI3t1), which is for the first time related to correct recognition of proof in task DI4. In addition coherence appears regarding correct recognition of proof in the two tasks, as there is an implicative relation between the two variables DI3 and DI4. However in this grade the variable corresponding to the correct recognition of proof in task DI3 also forms a relation with a variable indicating comprehension of proof, though incomplete (DI5cg). This is in fact the first case in which such a relation appears. Another type of mistake that is included in this implicative diagram is the type 5 mistake (DI4t5) which forms an implicative relation with the type 4 mistake (DI3t4). Also the type 5 mistake is related to variables indicating incomplete reasoning (DI6wi and DI1wj).

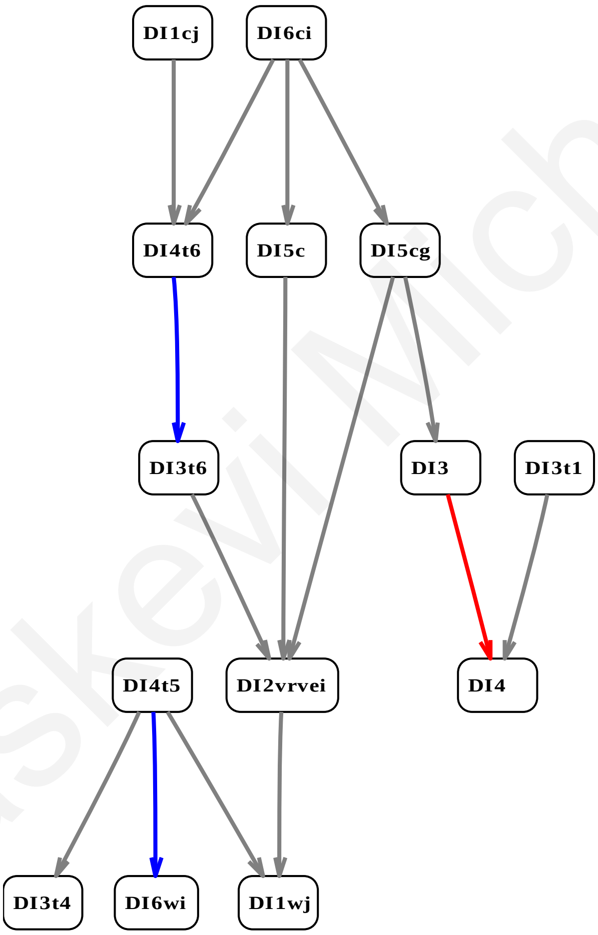


Figure 58. Implicative diagram for grade 11b students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs

Similarity relations between each type of mistakes in the tasks on the recognition of proof and the students' responses in the tasks on the production of proofs

The similarity and the implicative relations that were previously described provided indication about groups of students with different characteristics of reasoning. Therefore a need to define the characteristics of these groups more clearly emerged. For this reason each of the six types of mistakes in the tasks on the recognition of proof was examined separately in order to trace the particular relations between the appearance of this mistake and the students' abilities for proper reasoning in the tasks on the production of proof and to relate the students' behavior in the two types of tasks. In other words, how able are students for proper reasoning in the tasks on the production of proof, in relation to the type of mistake that appeared in their answer? The identification of these specific relations will provide information for defining the type of geometrical paradigm which corresponds to them. These relations will allow setting the type of geometrical paradigm the students work, based on their reasoning abilities.

Figure 59 shows the students' similarity diagrams for the type 1 mistakes in the tasks on the recognition of proof and the students' responses in the tasks on the production of proofs for each group of students. The relation between the occurrence of the type 1 mistake in the two tasks is significant for the students in grade 9 and for the students in grade in grade 11a, indicating greater stability regarding the occurrence of this mistake in the two tasks, compared to the students in grade 10. In grade 11b there is no coherence, as the two variables are found in different similarity clusters. The similarity relation between the students' particular mistakes and the variables indicating proper reasoning and comprehension of proof is extremely weak in grade 9. It could therefore be said that the two similarity clusters in the diagram for grade 9 students can be considered as distinct. This is also the case for the students in grade 10. For grade 11a students the two variables are linked to variables indicating comprehension of proof, however not complete (DI5cg). In addition these variables participate in a significant relation with the rest of the variables that correspond to proper reasoning. In grade 11b there is no direct similarity relation between the two variables corresponding to the type 1 mistake and hence no stability regarding the occurrence of this mistake. However there are significant relations between these variables and the variables indicting abilities of good reasoning.

For the type 2 mistakes (Figure 60) greater stability is displayed by students in grade 9 and students in grade 11b, than students in grade 10, as the relations between the two variables are significant in these two grades. In grade 11a there is no coherence, as the two variables are found in different similarity clusters. In grade 9 the similarity relation between the cluster containing these mistakes and the cluster that contains the answers indicating good reasoning is negligible. Hence the two clusters could be taken as distinct. Similarly in grade 10 the variables corresponding to these mistakes are distinguished from the rest of the variables. On the contrary the type 2 mistakes are significantly related to the rest of the variables for students in grades 11a and 11b.

Concerning the third type of mistakes (Figure 61), the greatest coherence in its appearance in the students' answers is only found in grade 9, in relation to the rest of the students. In grade 9 and in grade 10 there is a relation between the two mistakes and the rest of the variables, which however is not very strong. This relation is significant for grade 11a students. For grade 11b students the relation between the two mistakes and the variables related to good reasoning is negligible.

Coming to the type 4 mistake (Figure 62) there is stability in the appearance of this mistake in all the groups of students. Compartmentalization appears between these mistakes and the rest of the variables. Relations are only found in grade 11a, in which the two mistakes are related with variables indicating comprehension of proof, however incomplete (DI5cg).

Regarding the fifth type of mistake (Figure 63) stability is not noticed in the ninth and the tenth graders' answers, as no similarity relation is found between the variables corresponding to these mistakes. On the other hand indications of stability are found for students in grade 11a and grade 11b, in which the two mistakes are found in the same similarity cluster. For grade 9 students the mistake in task DI3 (DI3t5) is related to all the other variables, but the relation of the corresponding mistake in task DI4 (DI3t5) has a very weak relation with these variables. The situation is similar for students in grade 10, however the relation of the mistake (DI3t5) is significantly related to the rest of the variables. For students in grade 11a there is a relation between the two mistakes and the variables indicating proper reasoning, though not very strong. This relation is even weaker in grade 11a, almost enabling one to consider the two similarity clusters as separate.

The last type of mistake (Figure 64) appears in the students' answers with stability and this is the case for the students in grade 9, in grade 10 and in grade 11b. Specifically, this stability is greater for grade 9 and grade 11b students, due to the fact that the similarity relation that appears between the two variables is significant. No stability is displayed regarding grade 11a students, as the two variables are placed into different similarity clusters. For grades 9 and 10 the two mistakes are firstly related to the variable indicating a gap in the students' comprehension of proof (DI5cg) and subsequently these variables are related with the rest of the variables. In fact this second relation is more significant in the diagram for grade 10 students. In grade 11a the type 6 mistake in task DI3 (DI3t6) is related to the variable indicating a gap in the students' comprehension of proof (DI5cg), whereas the corresponding mistake in the other similar task (DI4t6) is significantly related to the variables indicating good comprehension of proof. As far as students in grade 11b are concerned, there are relations between the two mistakes and the rest of the variables. However the two mistakes are less related to the indications of incomplete comprehension of proof (DI5cg).

The students' mistakes were categorized in six types. According to the type of proof that each type of mistake included, the six types of mistakes were linked to a particular type of geometric paradigm. The relation between each type of proof and a particular type of geometrical paradigm is indicated in the first two columns of table 47. Specifically, in the type 1 mistake, accepting an empirical proof can be related to the GI paradigm. The empirical way for proving, which is based on measurements and thus increases the possibilities for mistakes, obviously belongs to the GI paradigm. The acceptance of the semi – empirical proof appears to have characteristics of the GI paradigm, but it is not exactly the same situation as the previous. This type of mistake is assumed to be related to a mixed type of paradigm, which still has some characteristics of the GI paradigm, namely a mixed geometry GI/GII. This is also the case for the type 3 mistake, in which an empirical and a semi – empirical proof were accepted. The type 4 can also be related to the mixed geometry GI/GII, as it comprises empirical proof (GI) as well as formal proof, which is indicative of elements of GII, since it is related to the use of axioms. As regards the type 5 mistake, it includes semi – empirical proof, which is related to GI/GII paradigm, but also formal proof, which is mainly related to the GII paradigm. Thus the type 5 mistake has more characteristics of the GII paradigm, but it cannot be exclusively considered as related to the paradigm GII. Therefore, a mixed geometry is also the case for this type of mistake, though it does not constitute the same situation as the rest of the types of mistakes corresponding to the mixed geometry paradigm. Due to the fact that the presence of the GII paradigm is more intense in the type 5 mistake, it could be considered as a mixed geometry GII/GI, as it seems to be closer to the GII paradigm and to the GI paradigm. Finally the type 6 mistake, which includes the acceptance of the three types of proofs, can be considered as relating to the mixed geometry GI/GII, and not to the mixed geometry GII/GI, because it still contains elements of the GI paradigm, which is the acceptance of the empirical proof.

The type of paradigm corresponding to each type of mistake was combined to the relations that were described above, in order to define the actual type of paradigm in which the students' reasoning takes place in the discursive apprehension tasks. The last four columns of table 47 include the type of geometric paradigm that was set for describing the students' work, for each type of mistake.

In particular the type 1 mistake, which corresponds to the GI paradigm, in grades 9 and 10 is not related to reasoning abilities. Thus the groups of students that make this type of mistake in each grade do not indicate proper reasoning in the rest of the proof tasks. So the students' work in this group appears to mostly exhibit characteristics of the GI paradigm. Based on the fact that for the students in grade 11a the type 2 mistake is related to indications of incomplete comprehension of proof, the work of these students' work could be related to the mixed geometry. The students mostly display characteristics related to the GI paradigm, however some reasoning appears too, which possesses some characteristics of GII. Therefore in this case the students' work can be related to the GI/GII paradigm. On the other hand, for grade 11b students the choice of the empirical approach is related to proper reasoning, therefore in this case a mixed type of geometry can also be assumed. However in this case there are more characteristics of the GII paradigm, in which the reasoning is based on the use of axioms and theorems. Thus the groups of students that make the type 1 mistake in grade 11b seem to work within mixed geometry GII/GI.

For the type 2 mistake, the compartmentalization that appears between this type of mistake and proper reasoning for students in grades 9 and 10 indicates that the students that make the particular mistake are not able to reason properly, hence their work keeps the characteristics of the type of paradigm that was related to this mistake, which is the GI/GII paradigm. On the contrary for students in grades 11a and 11b, these mistakes are related to proper reasoning in the rest of the proof tasks. Therefore there are more indications of the characteristics of the GII paradigm. However their work cannot be exclusively placed within the GII paradigm, due to the fact that the particular type of mistake possesses characteristics of the mixed type of geometry. Therefore the work of the particular group can be considered within the mixed geometry GII/GI.

Regarding the third type of mistake, which is also related to the GI/GII paradigm, this mistake has limited relations to the abilities of good reasoning for the different groups of students. Therefore the limited indications of proper reasoning combined to the particular type of mistake, are signs that the students' work remains within the mixed geometry GI/GII.

For the type 4 mistake, which is also indicative of the mixed geometry GI/GII, the compartmentalization that appears in grades 9, 10 and 11b between it and the indications of proper reasoning allows the assumption that the students' work is within the mixed geometry GI/GII, as no further characteristics of the GII paradigm are indicated. However, in grade 11a relations are found with the indications of incomplete comprehension of proof, though not allowing the placement of the students' work in a paradigm different from that of the rest of the students, because the presence of the characteristics of the GII paradigm is not strong.

The fifth type of mistake, which is related to the mixed geometry GII/GI, has no or limited relations to the indications of proper reasoning in the rest of the proof tasks, for all the groups of students, which allows one to place the students' work still in the same paradigm, and not within a higher type of paradigm (GII paradigm).

Finally the last type of mistake, for the first groups of students, is mostly related to incomplete abilities of reasoning; therefore the students' work can be considered again within the mixed geometry GI/GII. On the other hand, for students in grade 11b there are relations between this type of mistake and mainly the indications of proper reasoning, which are mostly related to the GII paradigm. Consequently, in this case the students' work can be considered within the mixed geometry GII/GI, due to the stronger presence of the characteristics of the GII paradigm.

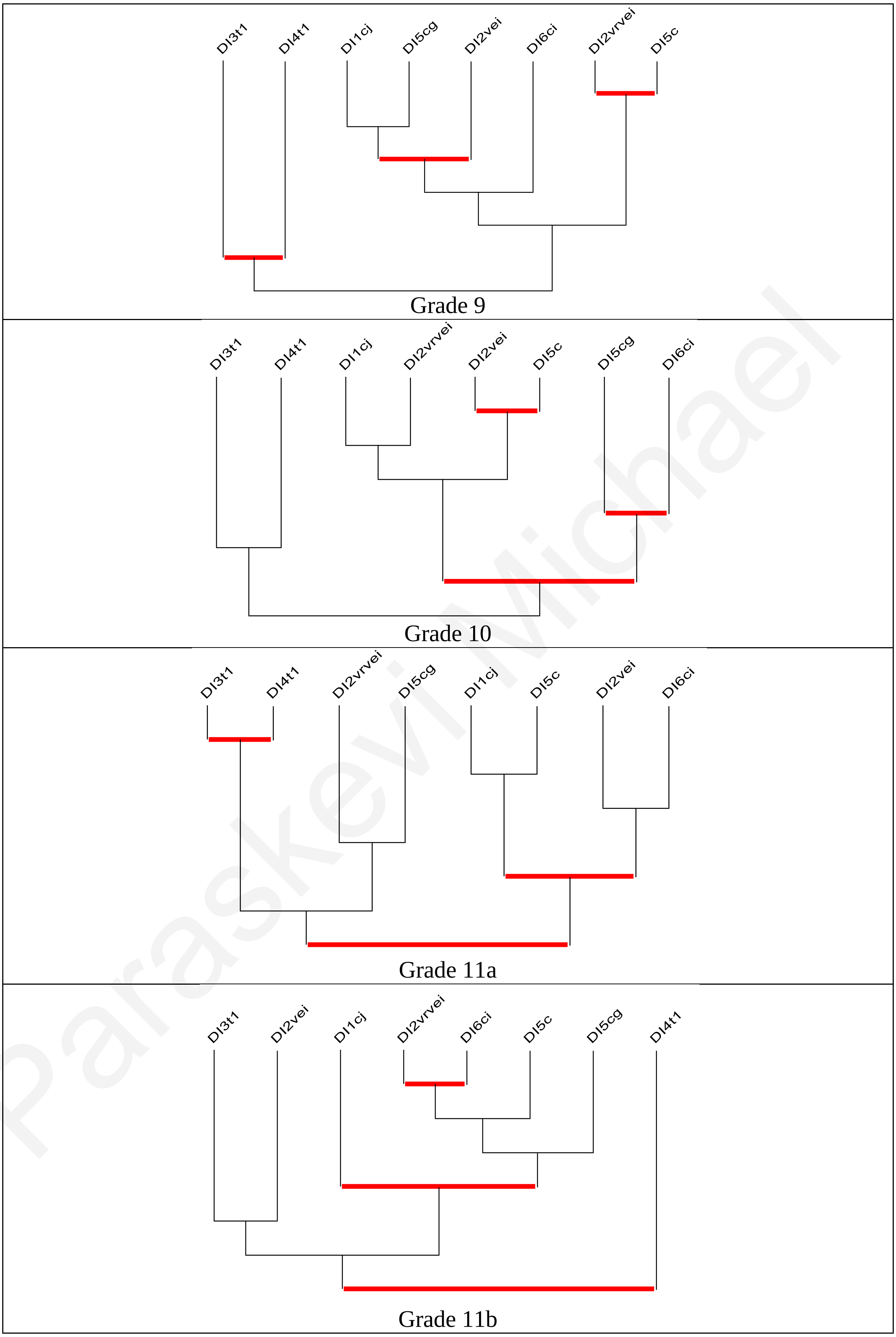


Figure 59. Similarity diagrams for the type 1 mistake in the tasks on the recognition of proof and the students' responses in the tasks on the production of proofs by grade

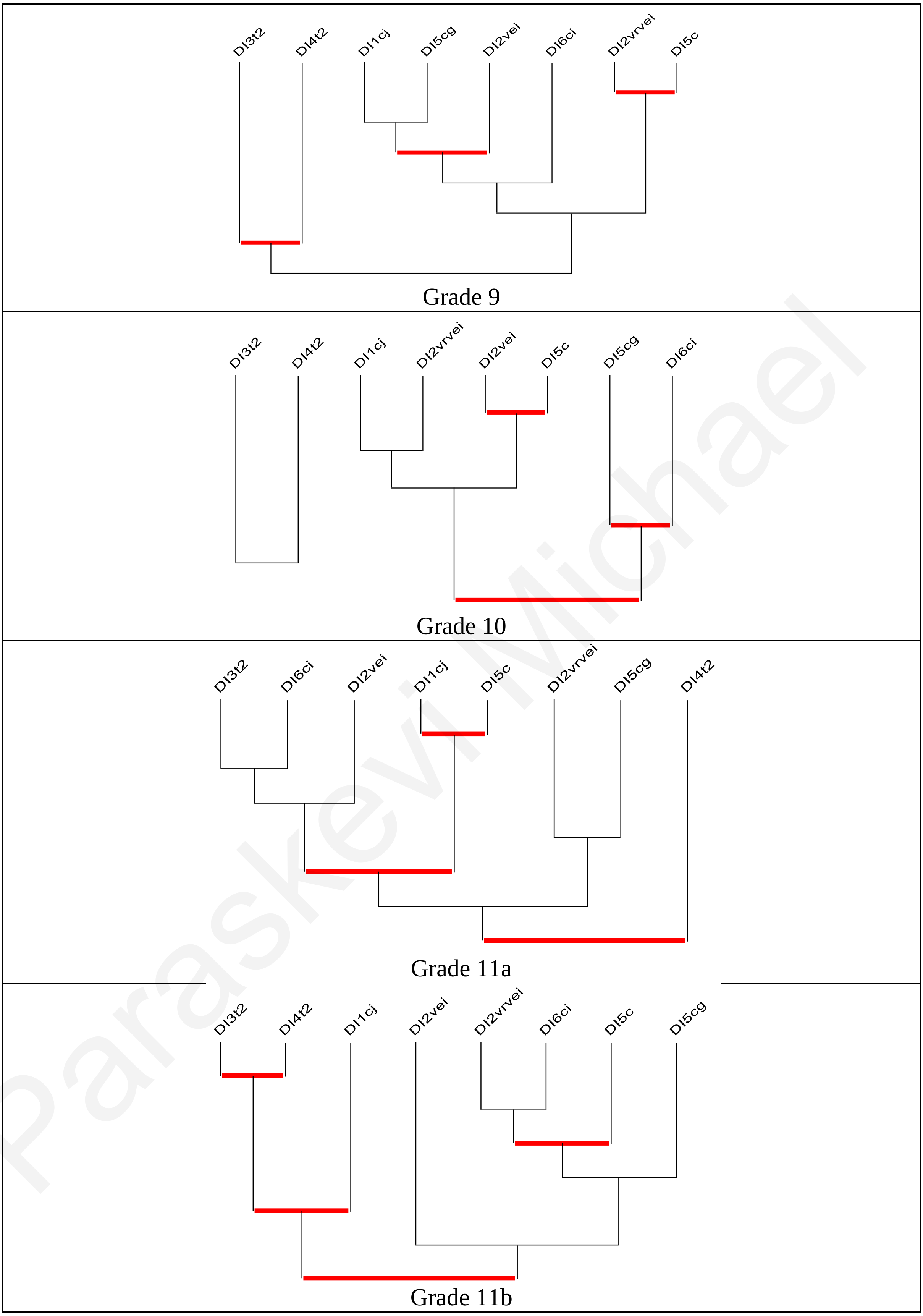


Figure 60. Similarity diagrams for the type 2 mistake in the tasks on the recognition of proof and the students' responses in the tasks on the production of proofs by grade

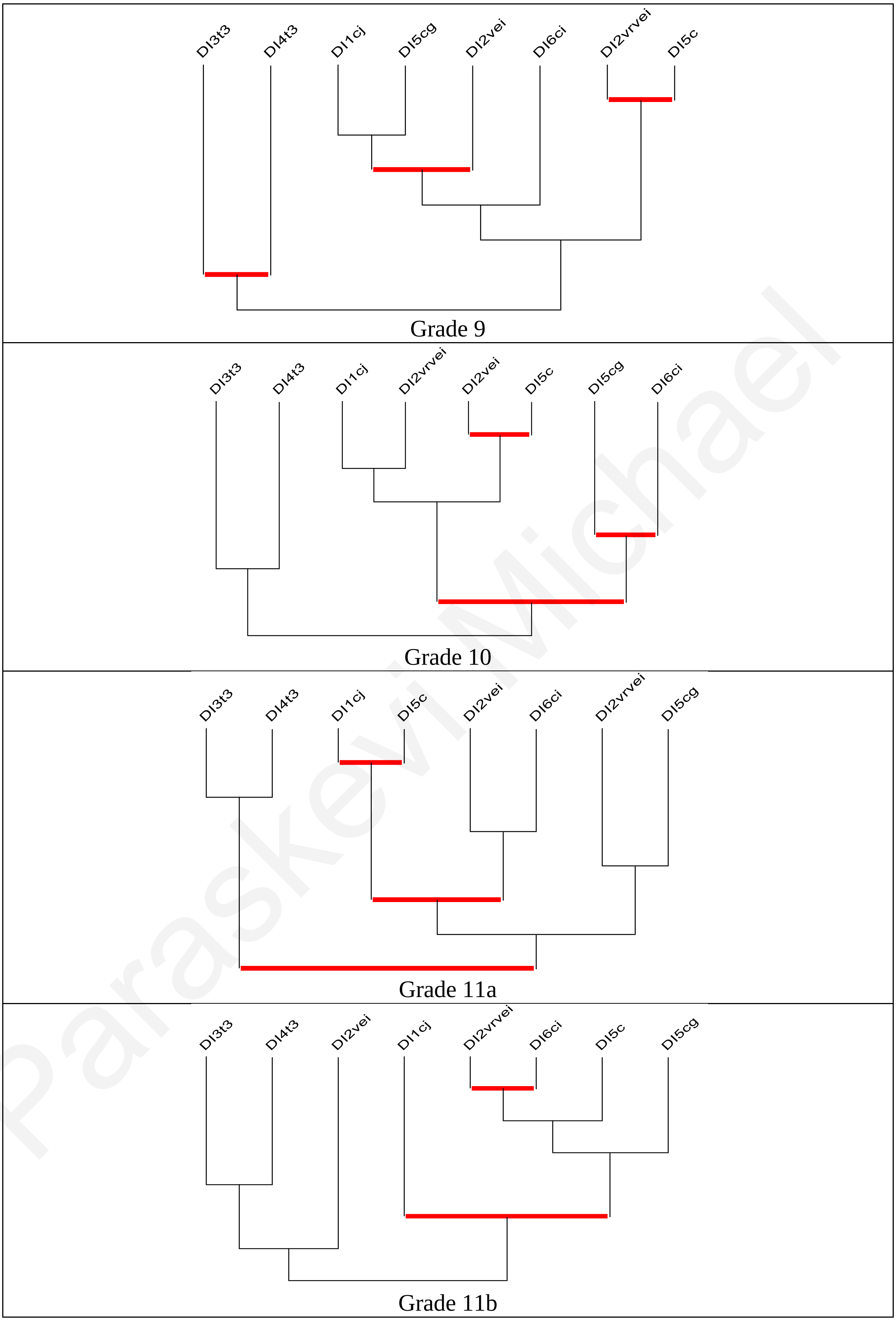


Figure 61. Similarity diagrams for the type 3 mistake in the tasks on the recognition of proof and the students' responses in the tasks on the production of proofs by grade

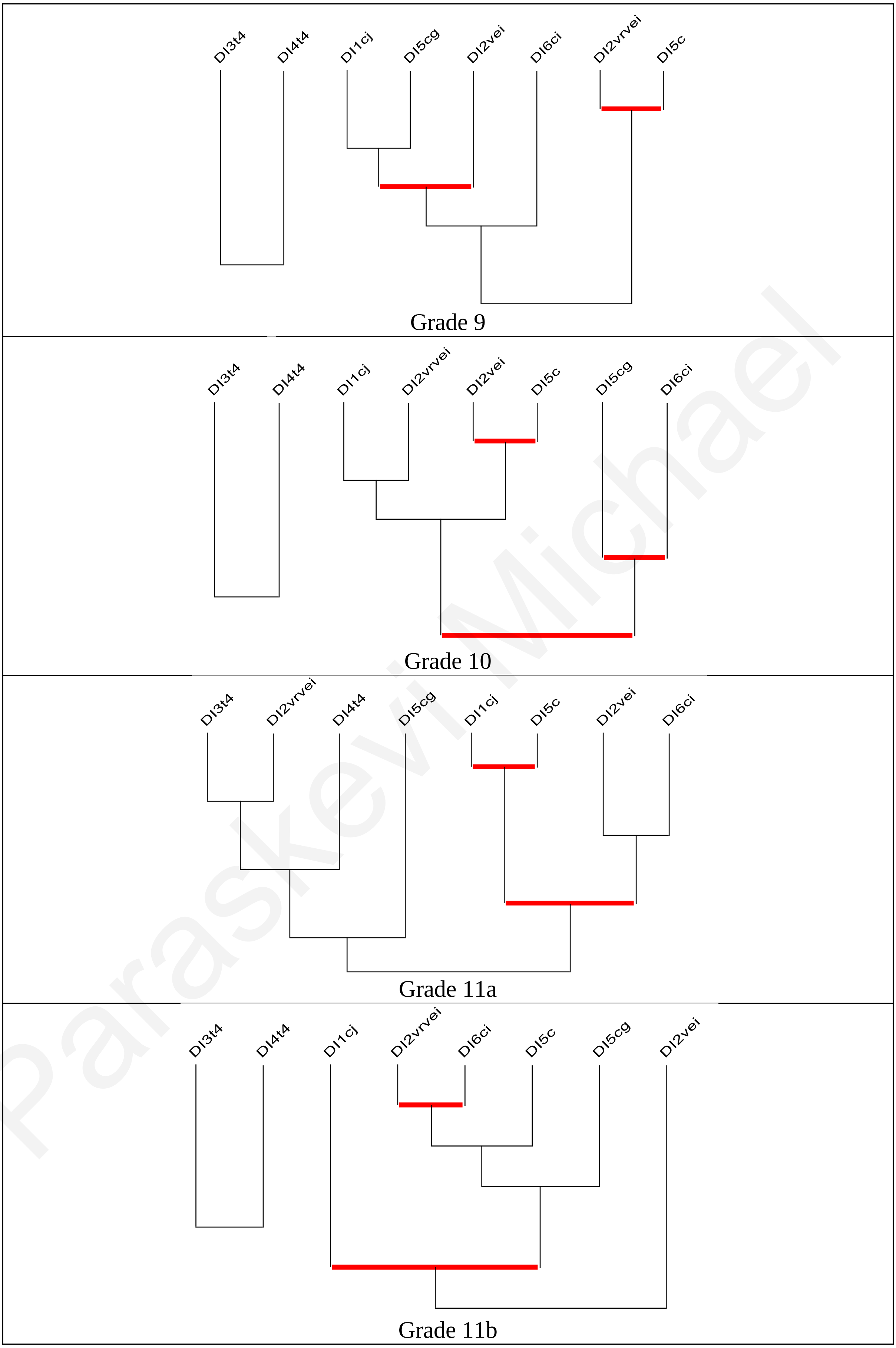


Figure 62. Similarity diagram the type 4 mistake in the tasks on the recognition of proof and the students' responses in the tasks on the production of proofs by grade

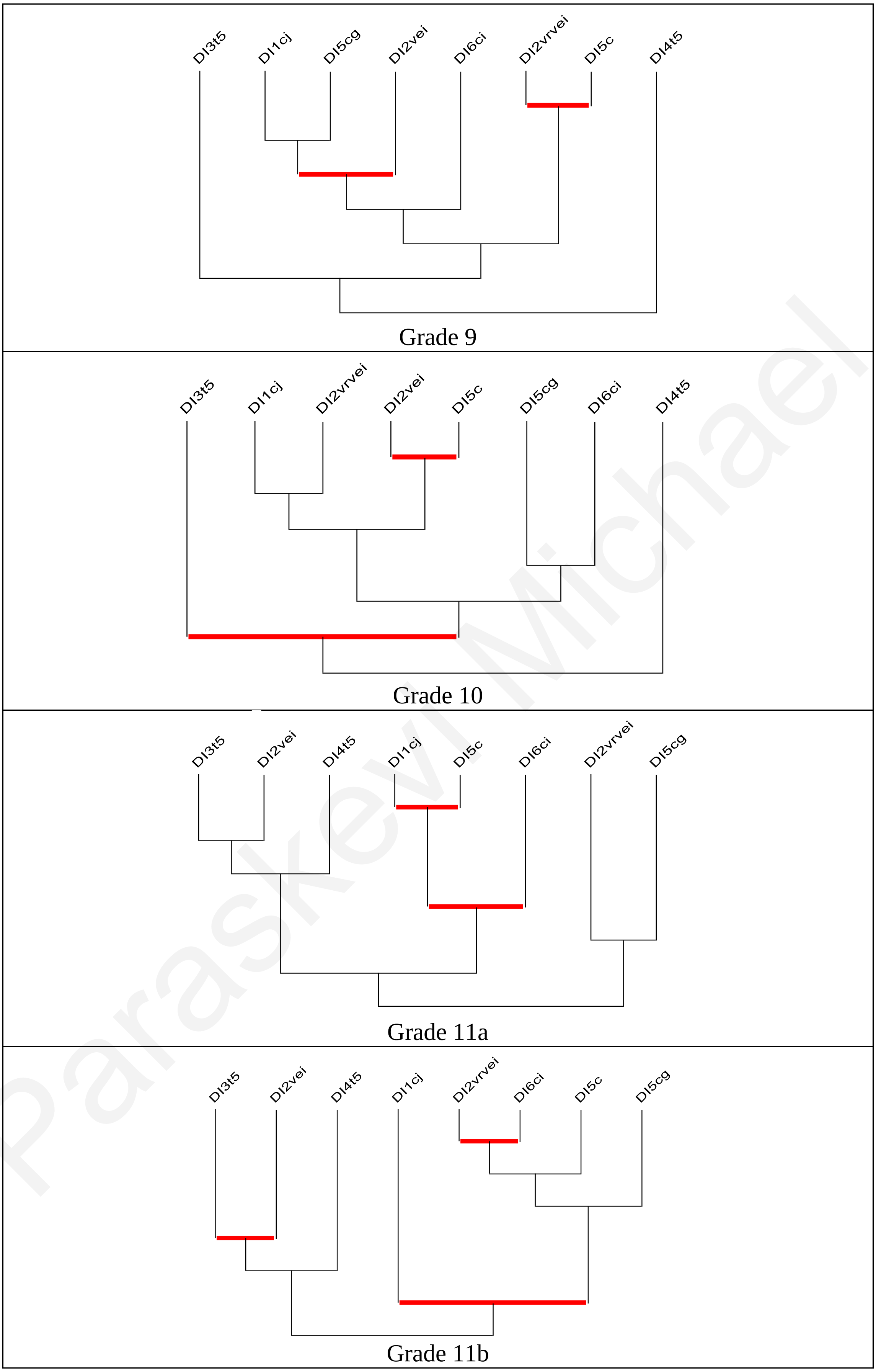


Figure 63. Similarity diagrams for the type 5 mistake in the tasks on the recognition of proof and the students' responses in the tasks on the production of proofs by grade

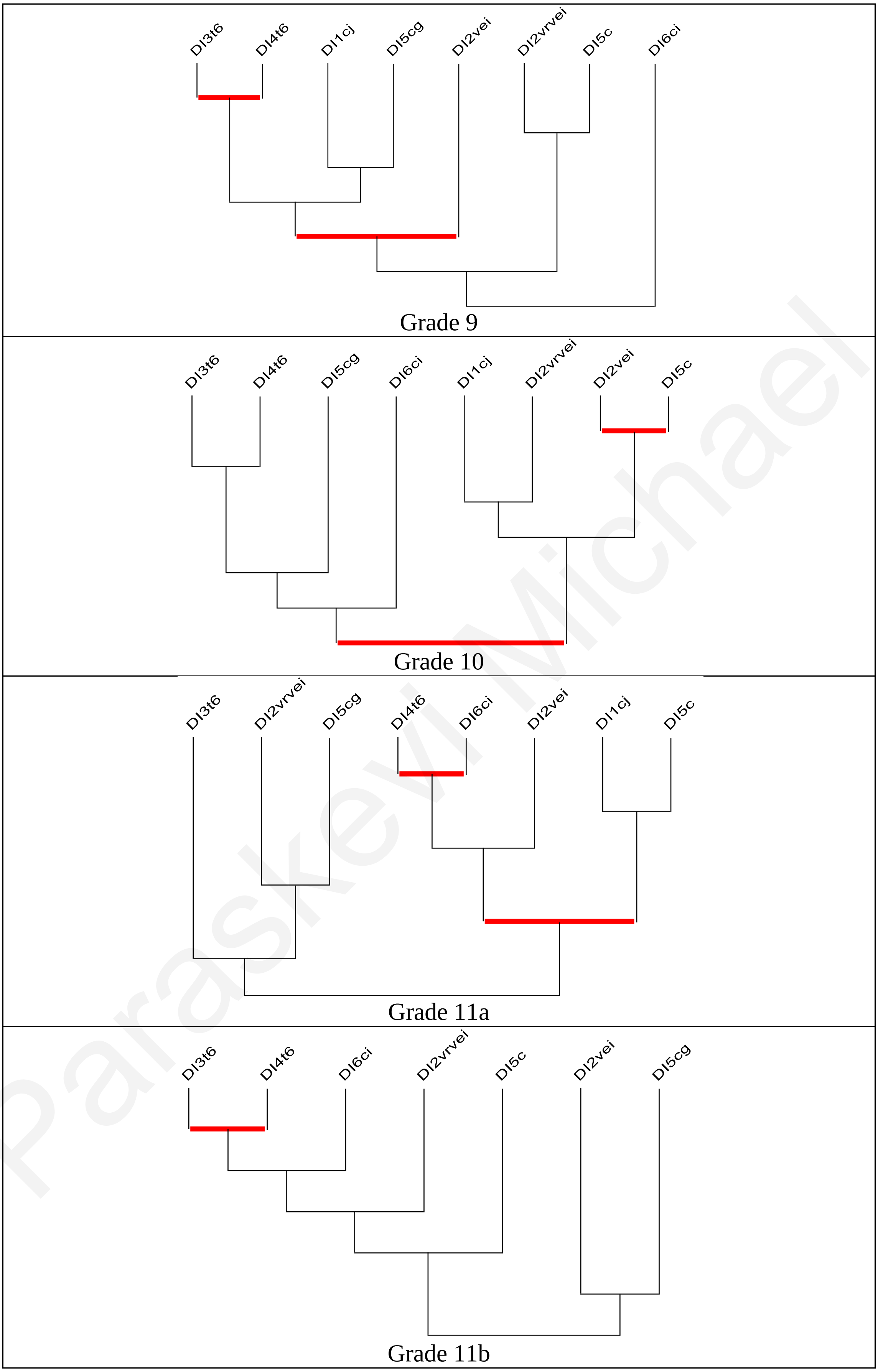


Figure 64. Similarity diagrams for the type 6 mistake in the tasks on the recognition of proof and the students' responses in the tasks on the production of proofs by grade

Table 47

The Identification of the Students' Geometrical Paradigm According to the Type of Mistake in the Tasks on the Recognition of Proof for Each Grade

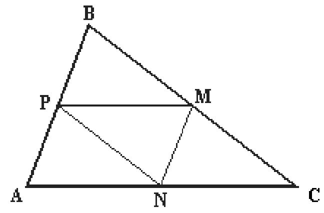
Type of mistake	Type of geometry	The students' geometrical paradigm			
		Grade 9	Grade 10	Grade 11a	Grade 11b
Type 1	GI	GI	GI	GI/GII	GII/GI
Type 2	GI/GII	GI/GII	GI/GII	GII/GI	GII/GI
Type 3	GI/GII	GI/GII	GI/GII	GI/GII	GI/GII
Type 4	GI/GII	GI/GII	GI/GII	GI/GII	GI/GII
Type 5	GII/GI	GII/GI	GII/GI	GII/GI	GII/GI
Type 6	GI/GII	GI/GII	GI/GII	GI/GII	GII/GI

Task – based interviews with lower and upper secondary school students

The task-based interviews were conducted with nine students. The students were given four tasks to solve and then questions were posed to them, in order to elicit their thinking procedure during the solution of the tasks, to trace their difficulties and mistakes and to export their ideas about the geometrical figures and the teaching of geometry.

Four tasks were selected (see Appendix 3) from the research instruments of the study, for which the students were asked to explain their answers. In particular two tasks were selected from the group of the discursive apprehension tasks and the operative apprehension tasks respectively. From the discursive apprehension, one task on the recognition of proof and one task on the production of proof were selected. From the operative apprehension tasks the two tasks which were found to be solved without using the mereologic modification were included in the tasks given to the students during the interviews, in order to extract more information about the reasons the students choose to solve these tasks using a different approach.

The first task that was given to the students was the discursive apprehension task DI5, which concerned their abilities in the production of proof.

In the triangle ABC, M, N and P are the midpoints of its sides. Prove that the quadrilaterals APMN, BMNP, and GNPM are parallelograms.	

The grade 9 students distinguished the three parallelograms they were given in the unified figure without any special difficulties. They also easily located the mean for each side, based on the data of the task. Nevertheless, it seems that the students do not know the criteria used for considering a quadrilateral as a parallelogram, nor the properties of a parallelogram very well, on which they had queries in their effort to resolve the task. Moreover, the students did not remember the relevant theorem for the resolution of the task and for this reason were not in a position to complete the task. Some students tried using different ways, for example comparing triangles or using the qualities of triangles to find a solution, albeit unsuccessfully. However, after a reference to the theorem by the researcher, the students were able to use it and provide the right proof that the figures were parallelograms. This task was ultimately characterized as difficult by grade 9 students. The difficulty of the task was connected with the fact that the students do not remember the relevant theorem. As regards the experience which these students have with such types of proof tasks, the students stated that they come across such tasks in class but not so often. They also considered the presence of a geometrical figure as helpful and said that if this figure had not been in the task, they would have tried to construct it themselves.

This task did not appear to present any special difficulties to grade 10 students. The students easily identified the subfigures in the given unified figure and provided the right proof by using the theorem. Nevertheless, two of the students showed that they were not in a position to entirely apply the theorem. The students had partly forgotten the theorem, since, as they mentioned, it was a long time since they dealt with it. The presence of the geometrical figure was characterized as helpful for the students. When asked whether they would draw a figure on their own if it was not given in the task, the students replied positively. Yet, they prefer being given a figure in a task, because *‘it is easier when it is* *provided'.* It appears that there is difficulty for the students when they are required to construct a figure themselves on the basis of the data in the task.

Grade 11 students also identify the three subfigures, which they have to prove to be parallelograms, easily. They use the relevant theorem correctly to prove that the figures they are given are in fact parallelograms. One student at first could not express the theorem completely but after guidance questions by the researcher, he gradually stated the theorem. It seems that the student knew the theorem correctly but was not in a position to entirely express it. This task presents a different degree of difficulty to grade 11 students. Some students consider the task easy or moderate. Another student characterizes the task as quite difficult, because *"you have to see, think and remember the theorem".* This student actually relates the difficulty of the task to the visualization and operative apprehension of the figure (*"you have to see"*), the proof procedures – the discursive apprehension of the figure (*"think"*) and knowledge of the theorem (*"remember the theorem"*). The fact that the geometrical figure accompanied the verbal part of the task was something that helped the students, because it facilitated visualization of the sides and the subfigures that are mentioned in the verbal part. Specifically, one student stated that the existing geometrical figure helped him to locate first the triangles and then the parallelogram in the figure.

Beyond the reinforcing role of the geometrical figure, the particular comment stressed that this student first mobilized the perceptual apprehension and then the operative apprehension of the geometrical figure. The student recognized the triangles within the whole figure through the perceptual apprehension and then, with the help of the operative apprehension, he was able to reorganize the figure in order to spot and distinguish the parallelograms, which he had to prove.

The students respond by saying that if they were asked to draw the figure in this particular task by themselves they would do it without any special difficulties. As grade 10 students, grade 11 students also state that they would prefer to be given the figure in a task. Specifically one student says that:

> Student: I think it's easier if they give it to us, because if you draw it yourself … The given figure is correct, whereas yours may not be correct.
>
> Researcher: Would the difficulty be in constructing the figure?
>
> Student: Yes, there is a difficulty.

Researcher: If you were not allowed to construct a figure and you had to use only the instruction, would you be able to resolve it, without a figure and with verbal data only?

Student: It would be difficult.

Thus, from the above extract we can see that the given figure gave assurance to the students about the correctness of its construction, as they do not feel the same certainty when they construct a geometrical figure by themselves. This is evident from the reply of a certain student as regards the difficulty presented by the construction of geometrical figures. Moreover, based on the same student's answer to the last question, it can be assumed that the absence of the geometrical figure constitutes a factor of additional difficulty in the resolution of the task. In reply to the same student another student said: *In other words shall we imagine it? It could be resolved, but with more difficulty".*

The second task given to the student was the discursive apprehension regarding the recognition of proof DI4.

Read the following explanations by three students who demonstrate why the sum of the interior angles of a triangle is 180°.

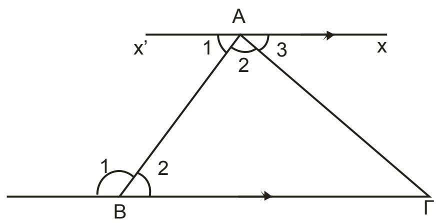
Student A: 'I measured each angle, and they are 50°, 53 °and 77°. 50+53+77=180. Therefore, the sum is 180° '.

- Do you accept the explanation of student A as a proof? Yes/ No ....

Student B: 'I drew a triangle, I cut out each angle and I put them together. They formed a straight angle. Therefore, the sum is 180 °.'

- Do you accept the explanation of student B as a proof? Yes/ No ....

Student C: Demonstration by using properties of parallel lines



We draw $\Gamma x || B\Gamma \Rightarrow \hat{A} = \hat{\Gamma}_3$ and $\hat{B} = \hat{\Gamma}_4 \Rightarrow \hat{A} + \hat{B} = \hat{\Gamma}_3 + \hat{\Gamma}_4 \Rightarrow \hat{A} + \hat{B} = \hat{\Gamma}_{\varepsilon\xi}$.

- Do you accept the explanation of student C as a proof? Yes/ No ....

This task was characterized as easy by grade 9 students. However, it is remarkable that the specific type of task is very different from the type of proof tasks given to students. The students report that the tasks they are usually given have to do with something they have to prove themselves. They add that it is the first time they come across such a type of task and that they consider it more interesting compared to the rest of the tasks they are usually given. Concerning the acceptance of the three different types of proof given to the students, in this task their answers generally showed convergence. In particular, the first proof (semi-empirical proof) was not accepted by the students, since according to their answers *"The triangle is accidental, perhaps 180 degrees come about accidentally", "We may measure with a slight error and it may not come out right".* Another student said, *"Compared to the remaining cases this is the most logical, we do it ourselves too but we may miss an angle and make a mistake".* They finally conclude that they cannot accept the first type of proof due to a weakness of generalization, since the results may come about by chance and this method may lead to errors. The second type of proof 7 (empirical proof) is not accepted by the students either, since as they state, *"It is not useful, because in a test I could not leave out, etc."* Another student remarks that she thinks it is right, but *"We cannot do it in every task".* Consequently, the students reject the empirical proof, because of the practical usefulness of this type of proof. The third proof gives more confidence to the students and they accept it. The students also agree that in this case generalization can be made and it is not proved only in a specific case of a triangle. Specifically, one student stated that *"In the third proof there was a figure that helps and there were appropriate operations to prove it. It can be done with all triangles."* In another students' answer it was stated that: *"I think it's the most correct, because it uses the criteria of parallelism. With the two parallel straight lines they use it more, it's not something of our own, like A or B".* Thus, the formal procedures that this proof contains, as well as the presence of a geometrical figure in the proof convinced the students to accept it as a proof, recognizing the informal elements of the previous proofs. At the end of the discussion of this task, a view was expressed by a student in connection to proofs. As the student said, *"We have to work on the proof until we are certain of the result."* Thus, the student stresses the necessity for reaching conclusions which we are certain and will hold for all the cases and not only for the particular case he is working on.

This task was characterized as easy to moderate in difficulty by grade 10 students. These students, too, commented on the difference of this particular task compared to the tasks they are asked to solve at school. Specifically, it was mentioned by the students that:

‘It was different to the tasks we usually do, because here there are choices, whereas in class we have to work out the proofs on our own”. The first proof proposed for this task is initially accepted by some students, who later on reject it. Specifically, these students state that it may be accepted, *“when you measure the angles and they come to 180 …”.* After intervention by and discussion with the researcher, the students understand that this is not a proof that can hold for all triangles, and they ultimately reject it. Other students reject the first proof at once, recognizing that the result could have come about by chance; therefore it is not something that can apply to all cases. Specifically, a student gives the following justification: *“It comes up to 180 degrees and the angles of the triangle come up to 180 degrees, but somebody may say that this came about accidentally. It is not always valid. I don’t accept it, because it may have come about by accident”.* Some students regard the second proof as convincing while the rest reject it. Those who are convinced by this empirical proof justify this on the basis that in this proof there is the use of the definition of a straight angle. Thus, the students are oriented towards this due to the fact that there is use of something which is valid and overlook the weakness of generalization due to the way this definition is used. The rest of the students identify the empirical dimension of this proof, too, and do not accept it, explaining that in this case also the result may be derived accidentally. The following is a characteristic extract from an interview with a grade 10 student.

> Student: This proof is almost the same as the first one, since whether I measure them and they come up to 180 ◦ or cut them and the same thing happens, it makes no difference.
>
> Researcher: Therefore, this may be accidental too?
>
> Student: It may be accidental or it may not.
>
> Researcher: Would you accept this explanation?
>
> Student: No, because it may not come out. If we did this with more than one triangle, with 2, 3, 4 triangles, and the same thing happened, yes … Then it would be more certain.
>
> Researcher: So, you wouldn’t accept it because there is only one triangle?
>
> Student: Yes, because there is one triangle and it may be coincidental.

This student finds that this proof is similar to the previous one and might not be valid, because the result may come about accidentally. Nevertheless, it seems that she could accept this particular proof if it was tried out in more triangle cases. On the one hand, she recognizes the limitations of empirical proof, but on the other hand she is prepared to be convinced regarding this approach after more trials. Thus, she would accept generalization through empirical trials. Therefore, we could say that empirical methods are not completely rejected by the students, if they could at least offer a small degree of certainty that they hold in more than one case. The third proof is different from the previous ones for the students, and they find it more accurate. The students state that this proof is the most convincing of the three. The students agree that with this proof we are more able to generalize. One factor which influences the students' view in accepting this proof is the presence of a geometrical figure. Also, the use of parallel straight lines and related axioms were a decisive factor, so that the students could feel certain about accepting this proof. More specifically, a student explains: *"I accept this, because it is always valid. As we have parallel lines, it means that angle A1 is always equal to angle B2 and A3 equal to C. Hence, A2 will always form* 180^o *when added to them, since they will always be the same. It is better proved, it does not require an addition. Because we know that the theorem is always valid, because the theorem has been proved and it, therefore, means that it will always be valid sine they will be parallel".* We note the regular use of the word "*always*" in this student's explanation. It seems, then, that the student wants to emphasize the certainty that is ensured with the use of something already proven, which is certainly valid and generalized in all cases.

The distinction of formal proof from proofs with empirical data is done in a more straightforward way by # grade 11 students. The students accept the third proof instantly, about which they express in unison that *"it is a more complete answer"*. They, however, express doubts and are considering accepting the other two proofs, which they eventually reject, because they cannot be generalized.

There are students who accept the third proof, but think that they could also accept the first proof. *"That is, it could be proved with the first one too, because the sum of the angles is 180. If you measure all the angles and find 180…"*. After a question by the researcher on what would happen if we measured a triangle and the sum of its angles were 179^o or 180^o, the student replies: *"There might be an error in the measurements"*. On the same question, another student adds: *"It is much easier to make a mistake in this way".* After this intervention, then, the first proof is rejected. Since the possibility for errors is detected in the first proof, the second proof is also rejected, because according to the students *"This student can easily make a mistake. This is not sufficient as a proof either"*.

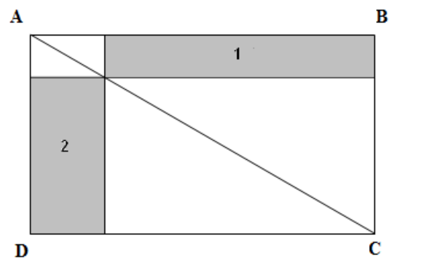
A student justifies his choice to accept the formal proof and not the empirical or semi-empirical one as follows: *"Because one of them measured the angles (1^{st} proof), the other drew a triangle which might have been completely inaccurate (2^{nd} proof), the latter proves it with two parallel straight lines (3^{rd} proof)".* Of interest is one student's answer, who at first states that all three ways could be convincing. After some thought, the student concludes that in the first proof *"it might have been accidental that he has found 180^o in the measurement"* and that the second and third proof are more convincing for him.

Although the student detects the fact that the result might be accidental in the first proof, he does not reject the second proof for which something similar may very likely be happening. What is worth noting in this case is that the acceptance of the second proof arises from the student's previous learning experiences. Specifically, he remembers that the second proof was valid from the attempts he made during his Primary Schooling *("When we did that in Primary School, the result was always 180")* as well as from the instructions of his teachers. *"If you cut a triangle and place its angles next to each other, the result will always be 180^o").* He finally states that the one which is more convincing is the third proof, due to the use of a theorem and of parallel straight lines, which the students use in class. Because the student is certain that these relationships are valid, he accepts the proof. This student, then, accepts both the formulaic and the empirical proof, but not the semi-empirical. This is due to the student's memories of applying the specific way in primary school. Although this student has been exposed to the teaching of high level mathematics and has obviously resolved proof tasks by producing formal proofs, his experience seems to be strong enough and to exert considerable influence on his views on proof. The student's exposure to and experience of formulaic proofs were not enough to distance him from empirical approximations and establish powerful apprehensions for the use of only formal proofs. In other words, the experiences regarding formulaic proofs could not transcend the experiences of empirical proofs.

The third task was the operative apprehension task OP2.

Figure ABCD is a rectangle. Look at the shadowed rectangles 1 and 2 and choose the correct answer. Then justify your choice.

a. Rectangle 1 has a bigger area than rectangle 2.
b. Rectangle 1 has an equal area to rectangle 2.
c. Rectangle 1 has a smaller area than rectangle



Grade 9 students found this task moderately difficult. *"It was neither so difficult, nor very easy".* All grade 9 students gave the correct answer; that the two shaded rectangles were equal. Nevertheless, only one student managed to make the correct reconfigurations to the figure and justify his answer correctly. The student initially detects all the subfigures which exist in the unified figure: "The diagonal line AC forms two rectangles, which it divides in two equal parts each". Subsequently, he reduces the equal parts: he takes each large triangle which the diagonal line of the rectangle forms (ABD and ADC) separately, reduces the equal triangles and what remains are the shaded rectangles. Therefore, both the perceptual and the operative apprehension were appropriately energized in this student, which led him to the correct solution. Although the rest of the students believe that the two shaded rectangles are equal, the reasons why they were led to that answer are not entirely correct. Some students tried to justify their answer by attempting to spot analogies between the two figures*: "They might be equal because one has a bigger side, therefore one is longer and the other is shorter. And one is narrower and the other is wider".*

Another student justifies her answer in the following way: *"I think they are equal, because rectangle 1 is equal to the long side of the rectangle (the big subfigure – rectangle), while the other is equal to its shortest. However, here (the small subfigure – rectangle) it is the opposite: 2 is equal to the long side of the rectangle while 1 is equal to its shortest".* This student managed to detect that the shaded figures have common sides and tried to use the relationships among the sides. It seems that this student functioned more cognitively, since she distinguished the subfigures in the unified figure and detected the common sides between them. The student appeared to proceed to an initial reconfiguration of the figure, as she detects the two rectangles-subfigures, but does not proceed to the next reconfiguration, in order to find the relationships between the figures she has detected and justify the equality, so her answer could be considered complete. Consequently, in this case we note the involvement of operative apprehension of the figure as well, but not to the desired degree. For this student the way to the solution seems to have been paved, but the student did not manage to move forward and reach the end.

To the question whether the students have faced any difficulty in breaking the unified figure into smaller parts (subfigures) and detect the relationships among them, the students primarily state that it was not that difficult. The students who managed to reorganize the figure reply: *"It was not that difficult because it gave us the lines and showed us it was its half (of the rectangle)".* The students who focused on the shaded figures only stated that *"the difficulty lay in seeing the figure"*. It seems then that the students' skill to modify the geometrical figure was not developed to a great extent, which made it difficult for them in those cases they were asked to make those modifications.

In grade 10, the students are more capable of making the appropriate modifications to the figure and justify their answer correctly. In this form too, however, there is an effort to detect analogical relationships between the two shaded figures. Most students give the correct justification, approaching the geometrical figure of the task dynamically and executing the relevant modifications to it. The students justify as follows*:" If AC is a bisector, it divides the ABCD rectangle in two triangles. From each triangle we reduce equal areas, so the same remains for both although the shaded figures do not have equal sides".* The case of a student who finds it difficult to reorganize the figure is of interest. She uses the analogy between the two figures *("In one figure, one side is longer, but the second is smaller. In the other figure one side is longer and the other is shorter")* and she starts thinking that the two figures may be equal. However, she does not consider her answer to be complete, because *"measuring by sight is no proof in mathematics"*. After guidance by the researcher, the student detects the equal triangles – subfigures, but still she does not proceed, insisting that *"it is by sight".* This student, then, seems to focus on the shaded figures at first (perceptual apprehension), then, with guidance she detects the equal triangles (1^{st} reconfiguration – operative apprehension), but she does not seem to be able to see the figure as a whole again and spot the relationships between the subfigures. She focuses on a different subfigure each time and thus, cannot move forward and reach the final solution. The student states in the end: "There is something which I do not see now…"

Through this student's answers, it has become possible to spot her views pertaining to mathematics and proof. Specifically, one could claim that the student lacks confidence due to the fact that proof comes about *"by sight"*. The student expresses her view on what proof is *("measuring by sight is no proof in mathematics")* which constitutes an indication that she has proceeded to the level of formulaic proofs, without easily accepting proofs which include empirical evidence. To a question on what she considers mathematics to be, the student states that *"Mathematics is proof"*. Hence, the active modifications of the figure in this task do not qualify as proof and are not accepted by this student.

The views of the rest of the students also emerge. They describe this task as different from the ones they are usually given. *"They mainly give us tasks with numbers, with formulas. Most of them are just theory. If you know the formulas, then you do well."*

They also mention that *"this task, which does not require formulas and calculations, is less time-consuming"* and that *"it is easier to formulate it if you observe it by sight"*. Therefore, possibly because of how they are taught, these students have related mathematics with a more algorithmic approach, with the use of theory, formulas and numbers and linked success in mathematics to the knowledge of theory. On the other hand, they exhibit a positive attitude towards tasks which are solved with different approaches, less algorithmic; in this case, on the basis of visualization.

The solution could immediately be inferred from the figure by some students in grade 11, who also gave a complete justification. Making the appropriate modifications to the figure, the students arrived at the solution by reducing equal triangles from half of the rectangle. Another approach tried by grade 11 students was the purely cognitive approach, with which there was an effort to divide shaded figure 1 in half, in order to ascertain whether these two halves correspond to figure 2. One student faced great difficulty in proceeding in the task, so guidance by the researcher was deemed necessary, in order for an attempt to be made for the student to detect the subfigures in the figure and reorganize it.

Researcher: What does the big figure break into initially?

Student: Into two triangles… Four triangles…. and two rectangles

Researcher: Let's take the four triangles, observe them carefully.

Student: They look equal.

Researcher: Can you convince me that they are equal?

Student: They have a perpendicular angle, a common side… (trying to prove based on the criteria of triangle equality).

Researcher: Where do both belong? What do these two triangles form?

Student: A rectangle. A rectangle which is divided… in half

Researcher: Then, what are these two?

Student: Equal.

Researcher: The other two?

Student: They form a rectangle which is divided in half.

Researcher: Good, which was the first step I asked you to take?

Student: To divide the big one (rectangle).

Despite everything he ascertained thus far, the student is still not able to discern the solution. Hence, the explanation for the final answer is given by the researcher. The student found it difficult to see the specific solution from the onset and when the solution was presented by the researcher, the student seemed to have been very surprised. He stated in amazement that, that was something new for him and that, *"the solution does not immediately come to you"*. The student's surprise after the presentation of the solution makes an impression on us, because we understand that the students' experience with such approaches to the resolution of geometrical tasks is very limited, although the students seem to like the specific approaches.

It is also worth referring to the case of one student, who chooses the correct answer by making different reconfigurations to the figure, albeit not managing to provide a justification as we deem complete. Specifically, the student makes the first reconfiguration to the figure; that is, he observes that the bisector divides it into two right triangles (operative apprehension). He then focuses on the shaded rectangles and finds analogical relationships between their length and width, in which case there is intervention of the perceptual apprehension of the figure. Subsequently, after being based on analogical thinking, he tries to prove the analogical relationships among the sides of the shaded figures based on the triangles, with which the shaded figures have common sides. *"If we observe the triangles, then we can see that they are analogous" (*perceptual apprehension*)*. He then detects the two small and two big triangles – subfigures, at which point he makes a second reconfiguration to the figure (operative apprehension). Then the student divides the rectangle in half again (third reconfiguration) and spots the relationships of the sides of the shaded rectangles with the triangles – subfigures (a combination of perceptual and operative apprehension). He determines that the shaded figures have common sides with the big ones and the small triangles while the two small triangles with the shaded figure constitute half of the rectangle. He can see that this is valid for both halves of the rectangle. Here, that is, the students has distinguished among all the subfigures in the unified figure, he made three reconfigurations to the figure (1st: the bisector divides the rectangle in half/ 2nd: detects and divides the two smaller rectangles/ 3rd: breaks each half of the triangle into two triangles and the shaded one). Finally, though, he seems to be focusing on each half of the rectangle separately only and cannot observe the figure as a whole again, based on the rest of the parts he has detected.

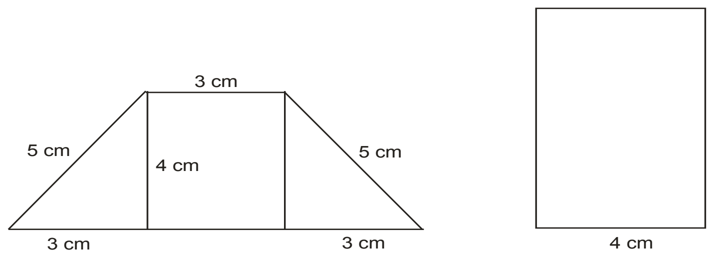
This particular student states that he has not come across such a task before. He considers it a difficult task but he adds that there are even more difficult ones, mentioning tasks with circles and figures drawn in them, which he finds difficult. In the resolution of such tasks perceptual apprehension is usually involved, in order to detect the subfigures in the circle while operative apprehension is involved to reorganize the figure and find the relationships among the different subfigures and reconfigurations of the initial figure. Consequently, the involvement of visualization and the necessity for modifications to the geometrical figure may constitute factors which bring about additional difficulty in these tasks.

To the question whether the task would have been easier if there had been numbers too, the student replies that numbers would have confused him more; he would have confused the figures more because he would have to observe the numbers and look for which other number he had to find. This student's statement is very important, since it is in agreement with everything Duval (1999) puts forward, concerning the fact that the presence of numbers in the figure hinders the activation of operative apprehension of the figure and becomes an obstacle for students trying to observe what they must in the figure, so as to reach a solution.

What is generally expressed on this task by the students is that they feel that data are missing, since they are used to working with data. They agree that it would be beneficial if there were such tasks too during the teaching of mathematics at school. *"They are more interesting in the sense that you only have to observe and think; you do not make operations, you do not need formulas, but this is something that might put strain on you."* Although the students are positive towards such tasks, they also express caution. They are more confident when they have data and calculations or the application of a theorem is required, because that is how they are used to working, mentioning that: *"Theorems are theorems, operations are another thing. We are generally used to operations though*".

The last task was the operative apprehension task OP2. Although this particular task can be solved with mainly two different approaches, the algorithmic (with the use of area formulas) or the mereologic modification (with the movement of subfigures – shift modification), the most dominant approach among grade 9 students' solutions was the algorithmic.

The trapezium and the rectangle have equal areas. Find the length of the missing side of the rectangle and explain your answer.



Nevertheless, the students were able to use the mereologic modification too, which they applied successfully after encouragement by the researcher. The application of the algorithmic approach was done in two different ways. There were students who used the relevant formula to find the area of the trapezium. Other students approached the figure somewhat differently and focused on the subfigures in which the trapezium was divided. Therefore, these students calculated the area of the two triangles and the rectangle which form the trapezium separately. After the students calculated the area of the trapezium, in one way or the other, they then used the formula of the area in the equivalent rectangle and found its unknown side.

Since all the students chose the algorithmic approach to solve the problem, they were encouraged by the researcher to attempt to solve it in another way, which does not require them to use a formula or conduct a series of operations. The students were able to suggest an alternative way based on the mereologic modification and different ways were expressed. A student suggested: *"Let's break it away (the rectangle). Let's place it over the trapezium, cut off its two sides, change the two rectangles into triangles and a rectangle will appear in the middle."* This student, then, begins from the unified figure (the rectangle) and breaks it into the subfigures of the trapezium. Namely, she moves from the unified figure to its subfigures. Another student follows the opposite procedure: *"We will cut the trapezium at the rectangle and the two triangles, as we are shown. We will place the rectangle with side 4 down and we will link the two triangles. The triangles will turn and form a rectangle."* This student, then, first breaks the trapezium, uses its subfigures and with these she, then, forms the new unified figure.

To the question on which of the two ways was the shortest, the students answer that the second way, that is the mereologic modification, was the shortest and easiest.

Nevertheless, some concern was expressed about this approach: *"We feel that if we apprehend it differently, we might do something and it might not come out correct; therefore, we prefer to do the operations which will certainly show how much it is".* In this statement, we note the students' confidence about arriving at a solution through a procedure, which may entail operations, calculations, use of theorems etc.

The algorithmic approach is also the one used by grade 10 students. This approach is applied in the two different ways which appeared to be used by grade 9 students above. Namely, they either find the area of the trapezium with the specific formula, or they find the area of each subfigure of the trapezium separately. As with grade 9 students, these students were also asked to suggest a different way to solve the problem, in order for them to be encouraged to approach the figure dynamically. The students manage to suggest numerous modifications to the figure. A student states that to form the rectangle, he moves the triangle on the right which is found in the trapezium and links it to the triangle on the left. Another student states that the two triangles and the rectangle (the trapezium's subfigures) form the adjacent rectangle. Hence, the student moves both triangles upwards as if to form a rectangle. The students who managed to make these modifications to the figure stated that this way was briefer.

It is worth noting the case of a student who was the only one to answer that *"there is no other solution"* when she was asked to provide an alternative way. After guidance, this student managed to make the reconfiguration to the figure. Specifically, this student does not approach the trapezium dynamically in order to reorganize it but proceeds to the rectangle, which she breaks into the subfigures of the trapezium. She moves then from the unified figure to its subfigures. Later, the student states that she has not done anything like that before in a task, she has not solved a task in this way. Nevertheless, she prefers the algorithmic approach, as she considers it to be *"mathematically easier"*.

Since the student managed to approach the figure dynamically and suggest modifications on it, the researcher suggested they returned to the previous task, which is solved in a similar way and to which the student did not manage to give a complete answer. Therefore, since the figure in task 6 is approached dynamically too by the student, she was expected to be able to proceed with a similar strategy for the figure in task 5 as well. Indeed, this is what happened; returning to the previous task, the student manages to see the equality between the triangles formed by the two rectangles – subfigures. She explains as follows: *"We would have two big triangles; this is equal to this (from the big rectangle – subfigure), this is equal to this (from the small rectangle – subfigure), therefore* *the shaded figures are equal."* This is a probable indication that with appropriate practice, the students will be able to approach the figure dynamically. The student could not at first do it due to lack of experience (they are not taught such approaches), but with guidance and practice in a single task she was able to detect the way to solve the task which she was unable to do before. Furthermore, the student accepts this proof this time, although she did not proceed before because *"by sight is no proof in mathematics".* In task 3, the strategy of the dynamic approach is eventually accepted as proof, unlike task 5, in which the algorithmic one is considered more representative based on her view of what mathematics is. This might also be related with the fact that the dynamic approach was the only way to solve task 5; hence the student had no chance to reject this way, whereas, in task 5, it was possible to apply an alternative method, by making calculations. For grade 10 students this type of task with the specific alternative solution, that is the dynamic approach, was something different, as they state that they usually use algorithmic approaches to solve the tasks they are given.

In the solutions of grade 11 students we also observe the use of the dynamic approach by a student, apart from the algorithmic approach. The rest of the students solve the task by applying the formula of the area of the trapezium. Of interest is the case of a student who gives the correct answer to this task by applying the dynamic approach. Specifically, the student proceeds to the reconfiguration of the figure by moving the triangle on the right so as to link it to the one on the left and form a rectangle. He mentions that this is a way he has learnt to use from before: *"As we were taught in Primary School, we cut out one triangle, we place it adjacently and it forms a rectangle"*. On the researcher's comment that most students approach the problem algorithmically and give the solution with the use of the formula of the area of the trapezium, the student states that he *"observes the figures and 'cuts and sews'"*. He notes that he does not favor the use of formulas and if he used a formula, he would do so at the final stages of the procedure. This student, then, appears to prefer more practical methods to solve tasks, whenever possible, and avoids algorithmic approaches. What is noteworthy is that this behavior of his seems to have been established since primary school, as the student appears to be influenced by the approaches used during the teaching of mathematics in primary school. Moreover, this student had invoked empirical approaches used in primary school in a previous task (task 3) too. Additionally, on the specific task, he mentions that he does not often solve such tasks but he does remember them from primary school.

The rest of the students apply the algorithmic approach to solve the task, using the formula of the area of the trapezium. In this grade, finding the area of the subfigures of the trapezium separately does not occur as in previous grades. The students manage to arrive at the correct solution through the algorithmic approach. Nevertheless, it is asked of the students to try applying a different method, without the application of a formula. In that case, two different ways of reorganizing the figure are suggested. Some students recommend moving one triangle to the right or the left, in order to form a rectangle, while others suggest the movement and connection of the triangles upwards.

Despite seeing the alternative ways, the students who solved the task algorithmically still say that they prefer the algorithmic approach, since they are more confident in using it. In particular, a student, comparing the dynamic with the algorithmic approach, states: *"You do the operations and you find a solution, while in this way (dynamically) you find the same thing. It is something you imagine… if it connects in this way… the answer is the same. Alright, it is correct, though I think it is a more complete proof, to be able to prove something with operations"*. Then, apart from feeling more confident about their solution when it comes about through the use of algorithms, they also feel it is more complete and perhaps more acceptable.

After the resolution of the four geometrical tasks previously analyzed, more general questions were posed to the students. In specific, these questions pertained to the type of geometrical tasks usually given to the students in class and the types of tasks they prefer more. Questions were also formulated concerning the role and the presence of the geometrical figure in the tasks as well as the construction of a geometrical figure while the approaches used for the resolution of geometrical tasks were also discussed.

The students of grade 9 note that they come across proof tasks where the geometrical figure is paramount to the solution, but not so often. The students generally state that when there is no given geometrical figure in a task, they usually draw a figure themselves because it helps them more with the resolution of the task. Nevertheless, the students state that there were cases in which the given figure confused them. They also add that they are usually asked to draw the geometrical figure themselves in class, often in the context of a task. Apart from drawing a figure in tasks, the students note that, generally in class, they make geometrical constructions with the use of geometrical instruments, which they say that they can use with ease. As regards the type of tasks the students prefer, namely between tasks which do not include numeric data and are solved with modifications on the figure and tasks which include numeric data and are approached algorithmically, the students agree on being given a combination of both types. However, they show more preference for tasks with numeric data although tasks which are solved using the geometrical figure are described as more interesting. *"Tasks with operations are easier, but the others are more interesting. With operations there is a standard procedure to follow, while you can do the other tasks in any way you want, even if you cannot do it with your mind, you observe what the figure shows".* Therefore, it is concluded that the students apply algorithmic approaches with more certainty, while they do not feel equally confident with solutions exclusively based on visualization.

Regarding the presence of the geometrical figure in the tasks, grade 10 students state that in some cases the geometrical figures are given, but they usually have to construct them themselves. However, the students say that it is more helpful when the figure is given: *"It is easier to observe them by sight, because you are certain that the figure is correct, as there is a chance that it will not be entirely correct if you draw it yourself".* Consequently, the students prefer the task to be accompanied by the relevant geometrical figure, since, in this way, they can be certain of its correctness; this does not apply when they construct the figure themselves. As far as geometrical constructions are concerned, the students add that most of the times they have to construct a geometrical figure is in the context of a task, to help them solve it. No additional time is dedicated to the students learning and familiarizing themselves with using geometrical instruments or to asking them to make geometrical constructions without requiring them to conduct a proof.

As regards the specific tasks the students were called to solve during the interview, they comment that what was more difficult and confusing was the task in which the figure was not in accordance with the verbal data. Therefore, the role of the geometrical figure in the solution of a task emerges as paramount, since in case the figure is not appropriate, it can trigger additional difficulty and prevent the students from arriving at a solution. They, then, add that a case in which a given figure confused them has never occurred; what did happen is that they faced difficulty and confusion with constructing a figure.

The role of visualization in the solution of geometrical problems is acknowledged by the students, as they mention that *"Geometry is interesting but requires some power of observation".* The role of visualization is also emphasized in the question on the presence of a geometrical figure in the tasks, on which the students state that they would themselves draw a geometrical figure if it did not exist in a task, because its use facilitates the solution. They, therefore, consider the presence of a geometrical figure "necessary" for the solution of geometrical problems. The students comment that the tasks which could be solved with modifications on the figure were something different for them. This way is not like the usual methods the students use; as they primarily apply algorithmic approaches to solve the tasks they are given. Similarly to grade 9 students, grade 10 students state: *"The most interesting tasks are those in which we do not have to follow typical procedures, but work a little by sight and find the solution".*

Regarding the question whether the students would like something to be done differently in geometry teaching, the students answered that they would prefer less theory. It is stated that: *"Not a lot of verbal data in the tasks, just to give you the data so that you can move forward. Not to be entirely dependent on formulas and theory. To be more aimed at IQ testing and such. To have formulas, but not to be entirely dependent on them".* The students, therefore, express their wish for tasks which can be solved more with conceptual thought and less with mechanical ways and algorithmic approaches.

A question which was posed to the students was whether the tasks which include numeric data or can only be solved with modifications to the geometrical figures differ from the tasks they usually solve. Grade 11 students replied that they indeed differ and that they do not often encounter different tasks of this type. Specifically, it is noted that: *"These tasks are simpler. Usually, in the tasks we are given, you have to prove a series of things. These are simple applications, but in order to prove the ones we are given at school, you have to use a little bit of everything, make combinations."* Nevertheless, the students would prefer to be given a combination of tasks, both tasks of this type and tasks they are familiar with. Additionally, from the tasks given to the students to solve, the ones which could be solved differently from the algorithmic approach were described as the most interesting.

A student however does not entirely share the afore opinion expressed by the rest, as he states that he is unsure if such tasks (which do not include numeric data or can be solved with modifications to the geometrical figure only) should be given to the upper secondary school students. He thinks that the students that attend the higher level mathematics course should not be given such easy tasks because *"they need to go deep, to more difficult tasks, they have to learn".* He thinks it is possible to give such tasks to the students that attend the basic mathematics courses, *"to help them get higher grades"*. This student, then, expresses views, which may have come about from his teachers' opinions on the difficulty, level and aim of the tasks given to the more advanced grades of the upper secondary school.

Summary

In this chapter the results were presented according to the four main axes of investigation of this research. First of all the cognitive structure of the geometrical figure apprehension for the total of the students was presented. The geometrical figure apprehension comprises the perceptual apprehension, the operative apprehension, the sequential apprehension and the discursive apprehension. The invariant of the structure was also verified for the students of the two educational level and the students from each grade. The differentiations between the different groups of students regarding the structural model of the geometrical figure apprehension were also described.

Based on the importance of all the different types of apprehension, which emerged from the structural model describing the students' geometrical figure apprehension, the relationships among these four types of geometrical figures apprehension were further examined. This examination was conducted with the hierarchical clustering of variables and the similarity analysis of the students' answers, as analyzed from the mathematical and the cognitive point of view. Apart from the fact that the different types of apprehension are interrelated, specific relations were noticed between particular types of apprehension. The operative and the discursive apprehension were found to be strongly related. Significant relations were also found between the discursive and the sequential apprehension. On the other hand, the operative apprehension appears to have limited relations with the sequential apprehension. The role of the perceptual apprehension surfaced as very important for the mobilization of the discursive apprehension and the operative apprehension.

In the third part the students' geometrical figure apprehension was compared, either according to the different educational level or according to the different age groups of students. The results revealed that the students' geometrical figure apprehension mainly improves from one grade to the next and from one educational level to the following. In addition a hierarchical classification of the geometrical figure apprehension tasks according to their degree of difficulty emerged, according to which the sequential and the discursive apprehension task were of higher difficulty for the students than the operative and the perceptual apprehension tasks. Nonetheless, the examination of the difficulty of the tasks revealed some additional factors that determine the hierarchical classification of the tasks, for each type of apprehension. The analysis of the students' answers from the cognitive point of view also highlighted that, despite the fact some tasks can be solved more easily or more accurately through a particular type of apprehension, in some cases other types of apprehension also intervene. This intervention leads the students to giving wrong answers or even answering correctly, though those answers come about through wrong procedures. The type of apprehension which appeared to cause most of these interventions is the perceptual apprehension.

The results of the students' mistakes and ideas on the geometrical figure apprehension were described in the last subchapter. The difficulties of the students in each type of apprehension were identified, providing indications for identifying the possible reasons that are responsible for their appearance as well. Furthermore the similarity and the implicative relations among students' mistakes in the tasks on the recognition of proof and their responses in the tasks on the production of proofs were examined, in order to be able to determine the students' type of geometrical paradigm in which their geometrical work is situated. The geometrical work of the students in the lower secondary school and the first grade of the upper secondary school seems to be mostly situated within a paradigm of a mixed type geometry, possessing mostly characteristics of the Natural Geometry (GI), whereas the geometrical work of the rest of the students in the upper secondary school appears to be mostly related to a mixed type geometry which mainly comprises the characteristics of the Natural Axiomatic Geometry (GII).

CHAPTER V

DISCUSSION

Introduction

This chapter comprises the discussion of the results that emerged from this research study. In effect, this chapter presents the main results of the research, which are interpreted and related to the results of the literature review and the results of previous research investigations. The conclusions are drawn in relation to the description of the results. In addition, suggestions for further future research are made and implications for the teaching and learning of geometry are provided.

The chapter is organized in four parts, each one corresponding to the research questions of the four axes of investigation, on which performance of this research was based. The conclusions for each axis are presented in a separate part.

In the first part, the cognitive structure of the geometrical figure apprehension in the lower and upper secondary school that emerged through the analysis of the data is presented and discussed. The dimensions comprising the geometrical figure apprehension are described and attention is also paid to the similarities and differences regarding the particular verified structure among the students from the different age groups and the different educational levels.

The second part includes the discussion of the results from the examination of the relationships among the four types of geometrical figures apprehension. In fact, the way in which each type of geometrical figure apprehension contributes to the solution of geometrical tasks and the relations and effects on the rest types of apprehension are described. In addition, emphasis is given on the students' behavior while solving the geometrical tasks.

In the third part the outcomes of the comparison between the lower and upper secondary school students' geometrical figure apprehension are discussed. In particular these outcomes were extracted though the examination of the lower and the upper secondary school students' abilities in solving the geometrical problems corresponding to the four types of geometrical figure apprehension and the identification of the differences between the different groups of students. Furthermore, the discussion also focuses on the conclusions drawn regarding the predominant kinds of geometrical figure apprehension for each kind of geometrical task and at each level of teaching.

The last part includes a discussion regarding the lower and the upper secondary school students' mistakes and ideas about the geometrical figure apprehension. Specifically, the students' mistakes that were identified in their answers in the geometrical tasks, corresponding to each type of apprehension, are presented and interpreted according to the reasons that cause them. In addition, the results from the task – based interviews are also discussed in relation to the students' mistakes and ideas about the geometrical figure apprehension. The discussion of these results is also evolved in relation to the type of geometric paradigm in which the students' geometric work takes place.

The cognitive structure of geometrical figure apprehension in the lower and upper secondary school

The acquisition of mathematical knowledge and development of mathematical structures was examined by different researchers, albeit through different points of view (i.e Gray, Pinto, Pitta & Tall, 1999; Hejny, 2003). Despite the importance of studying and comprehending the way the students' knowledge of geometrical concepts is structured (Battista, 1999), there is a lack of theoretical models describing the structure of the students' geometrical abilities in the literature.

Therefore the first axis of this research was to examine the cognitive structure of the geometrical figure apprehension for the lower and upper secondary school students. Based on Duval's (1995) discrimination of four different types of apprehension of geometrical figures, the geometrical figure apprehension was expected as multidimensional and comprising the perceptual, the operative, the sequential and the discursive apprehension. Thus, the following research questions were examined:

– What is the cognitive structure of the geometrical figure apprehension in the lower and the upper secondary school?

– What are the similarities and differences between the lower and the upper secondary school students regarding the structure of their geometrical figure apprehension?

The results of the confirmatory factor analysis indicated that the structure of the lower and upper secondary school students' geometrical figure apprehension can be described by a second-order model. This second-order model consists of four first-order factors, each one corresponding to the four different types of apprehension, as suggested by Duval (1988). Hence, the four first-order factors represent the perceptual apprehension, the operative apprehension, the sequential apprehension and finally the discursive apprehension. These four types of apprehension are all regresses on a second order factor which corresponds to the geometrical figure apprehension. The high factor loadings of all the four first order factors on the second order factor reveal that all the four different types of apprehension are important. According to Duval (1988) the resolution of problems very often requires interactions between the four types of apprehension, but the same applies to the realization of the distinction between these types of apprehension. However the discursive apprehension is the type of apprehension which is more strongly regressed on the second order factor, revealing the crucial role of the discursive apprehension for the apprehension of geometrical figures.

The invariance of the structure described above regarding the geometrical figure apprehension was examined, in order to trace any possible differentiation in relation to the students' educational level or age. The structure for the geometrical figure apprehension is the same for both the lower and the upper secondary school students. What the CFA model also shows is that some loadings of the first order factors on the second order factor are higher in the group of the upper school students than in the group of the lower secondary school students, suggesting that the specific structural organization potency increases across the two educational levels. Actually this is the case for the operative and the discursive apprehension. Therefore these findings indicate that as students move to a higher educational level they seem to be more based on mathematical properties and discursive processes while their abilities about the heuristic exploration of figures are more developed. On the contrary the students' abilities in constructing geometrical figures appear to reduce after their transition to the upper secondary school. This reduction can be attributed to the emphasis given on geometrical constructions, which is usually less in the upper secondary school, as observed from the analysis of the textbooks. In addition the students' perceptual apprehension appears to remain at the same level between the two educational levels and it is thus unaffected from the students' transition from one educational level to the next one. This conclusion is in line with Duval's (2013) remark that the perceptual recognition of figures remains steadily predominant over the curriculum for most of the students.

Furthermore, the invariance of the structure of the geometrical figure apprehension for all the different groups of students, in relation to their age was also confirmed. This result indicates that the development of the students' geometrical figure apprehension is characterised by the creation of "invariant structures". Panaoura and Gagatsis (2008) define as "invariant" the group of tasks for which the relations between them remain the same for students of different ages and characterize this phenomenon as "conservation of geometric structures". In addition the potency of the model of the different age groups regarding the dimension of the discursive apprehension and the operative apprehension seems to increase as we move to a higher grade, but the opposite phenomenon is observed for the sequential apprehension. The perceptual apprehension does not correspond to any of these cases, as the loadings rise, but then fall from one grade to the next one.

Consequently, the results show that the structure of the geometrical figure apprehension is not influenced by students' age or educational level, as it remains invariant in the different groups of students. Despite the fact that in the domain of mathematics education there is no research showing the structure of geometric abilities, the results of this study confirm previous studies based on the theory of architecture of the mind as proposed by Demetriou (1998, 2004), which have shown with empirical data the organization of experience on the same field in joint structures. The description of the structure of the lower and upper secondary school students' geometrical figure apprehension is one of the contributions of this research, as this aspect was not previously studied in the field of mathematics education.

The second axis of investigation in this research concerned the identification of the way the four types of geometrical figures apprehension are related. Apart from the relations among the four types of apprehension, the effect of one type of apprehension on another type during the solution of geometrical tasks is also discussed. Therefore the interest was turned to the students' behavior when solving the geometrical figure apprehension tasks. Based on the above, the following research questions were studied:

– What are the relationships among the perceptual, the operative, the discursive and the sequential apprehension for the solution of geometric tasks?

– How do lower and upper secondary school students behave during the solution of geometrical tasks involving each type of geometrical figure apprehension?

– Is there a development from the perceptual apprehension either to the operative apprehension or to the discursive apprehension, from one grade to the next one, and mainly during the transition from lower to upper secondary school?

– Is operative apprehension closely connected to the discursive apprehension or are they independent from each other? In other words, are the abilities to solve problems by visualization closely connected to the development of deductive reasoning?

In answering these questions, indications were provided through the examination of the similarity and the implicative relations between the students' answers in the tasks. The students' answers in the tasks were examined though the mathematical point of view; that is, according to whether the students' answers were mathematically correct or not, and from the cognitive of view, which provided information about the type of apprehension that was mainly involved in reaching the right solution.

The main outcome from the similarity and the implicative diagrams is that the different types of apprehension, hence the different ways of looking at a geometrical figure are interrelated. In many cases the solutions of the tasks do not mobilize only one type of apprehension, as expected, but other types of apprehensions are involved as well, showing the interactions between the different ways of looking at figures. These results are in line with the verified structure of the lower and upper secondary school students' geometrical apprehension, in which the importance of each type of apprehension and the relation between them was shown.

More specifically, strong relations between the operative and the discursive apprehension are revealed. The similarity and the implicative relations between the students' answers as observed from the mathematical point of view reveal that the operative apprehension constitutes a basic dimension of geometrical thinking and highlight its important role for geometrical proofs and constructions. This is mostly true for students in grade 9, in grade 10 and in grade 11a. In this vein, Duval (1995) highlighted their importance, suggesting that only the operative and the discursive ways of looking at figures are the ones required in mathematics, because the operative way is the heuristic use of figures in solving problems, whereas the discursive way corresponds to the way of looking at figures according to the given properties in order to deduce new properties, but it always involves a dimensional deconstruction of the shapes recognized. Therefore the analysis of these two forms of apprehension opens the horizon not only for the classification of geometric problems, but also for a different approach to geometric activities for students.

Besides the relations that occurred between the discursive and the operative apprehension through the similarity and the implicative relations, the local comparison between an operative apprehension task and a discursive apprehension task provided more indications that the students' abilities in solving problems by visualization are closely connected to the development of deductive reasoning. In particular, the local comparison between a discursive apprehension task and an operative apprehension task was done, because the given figures in the two tasks were very similar, with the only difference being their orientation. In fact, in cases in which the cognitive development of visualization is not taken into account in the teaching of geometry, most of the students are not able to recognize the same configuration or sub-configuration if the orientation of the figure is changed (Duval 2005). So a possible question could be on the reasons that cause a variation in the students' performance when they are given similar figures for different geometrical situations. It was thus hypothesized that the students' inability for choosing the relevant theorem for proving in the discursive apprehension task is related to the fact that they are not able to mobilize the operative apprehension for applying the mereologic modification on the given figure and discriminating the necessary relations among its figural units.

The results provided evidence for verifying this hypothesis. Actually, the students' behavior in the two tasks seems to be related. The insufficient functioning of the operative apprehension is probably responsible for the students' inability to identify the necessary relations between the different figural units of the figure and as a result the students are not able to choose the necessary theorem. The heuristic use of figures is either based on seeing or some interactions between reasoning and seeing depending on the given figure (Duval, 2013). In addition the fact that the double use of some subfigures was necessary, that is, the same object may have been used simultaneously as two different objects in the same operation of comparison or reasoning, is also a possible factor that made the visibility of the proper reconfiguration more difficult (Duval, 1995). And when students are not able to trace the necessary relations they cannot detect the mathematical properties represented in the figure and relate them in order to choose the proper theorem either. Thereafter the answer to the question posed previously can answer the crucial question about the way a figure functions heuristically. The irrelevant use of properties of the theorem can be attributed to the perceptual apprehension, which overrides the recognition of the necessary relations between the different figural units and hence the perception of a figure can be an obstacle for the recognition of the relevant theorem. These conclusions are in line with the results of Duval's (1995) study with 15 – 16 years old students, which showed that the simple application of the Thales' theorem was influenced by the perceptual organization of the given figure. Consequently the operative apprehension does not function independently from the discursive apprehension, but usually the heuristic processing of figures is subordinated to the discursive apprehension and can be forgotten when the problem is solved (Duval, 1995).

On the other hand, fewer relations also occurred between the operative and the sequential apprehension, showing there is no dependence of visualization on construction. Despite the fact that a construction process results in visualization, this process is only related to the connections between the mathematical properties and the technical constraints of the used tools (Duval, 1998). It is often believed that learning how to construct geometrical figures is enough for learning visualization in mathematics. But any such a task of construction requires only a succession of local apprehensions, as the focus has to be on units and not on the final configuration. As a result a student can succeed in constructing a geometrical figure, but not being able to look at the final configurations differently from as iconic representations (Duval 1999).

Significant relations are found between the discursive and the sequential apprehension as well. The similarity and the implicative relations revealed the importance of a right construction in order for the mathematical properties to be retained and represented in a proper way in a geometrical figure. In the construction activities any figure has to be seen according to the geometrical properties that make its construction with specific tools possible (Duval, 2013). In fact in a construction process these properties take the form of technical constraints, which must be taken into account in a right sequence in order to ensure the production of a geometrical figure which will represent the mathematical properties it possesses properly. Regarding the procedure of proving, the mathematical properties occurring from the geometrical figure have to be identified and used properly in a right sequence in order for the right inference to be accomplished. Therefore these relations highlight the importance of the proper use of mathematical properties not only for geometrical proofs, but also for the geometrical constructions. However Duval (1988) differentiates radically the tasks of demonstration and the tasks of construction, explaining that discursive apprehension has a different nature from the description of a construction procedure. Although he mentions that in a construction task the figure is independent from all the statements, in the tasks given for this research the constructions were based on particular instructions. Consequently the students' constructions were not unrelated to the statements and this can justify the relations between these two types of apprehension.

The results extracted from the similarity and the implicative diagrams also revealed the influence of perceptual apprehension on discursive apprehension. For the students in grade 9 and in grade 10 the perceptual apprehension seems to possess a crucial role: the mobilization of the discursive apprehension. Perception appears to be the basis of any geometrical activity, as it appears to be a prerequisite for geometrical reasoning. However the students in grade 10 display greater stability regarding the mobilization of perceptual apprehension, compared to the students in grade 9. In addition the grade 11a students mobilize perception more coherently in relation to the rest of the students. This coherence does not appear especially for the students in grade 11a. Actually in grade 11b in the perceptual group of variables, relations are found between the perceptual apprehension and the discursive apprehension; namely, for the recognition of a proof or a proof that can be achieved based on the effect of perception. Also, relations are found between the discursive apprehension tasks in this grade, hence there is greater coordination of the cognitive processes corresponding to proving in geometry. Furthermore the involvement of the perceptual apprehension and its effects on the solution of some of the other tasks seems to be different for grade 11b students, in comparison to the rest of the students.

Relations also appear between the perceptual apprehension and the operative apprehension, as perception appears to be a prerequisite for the mobilization of operative apprehension. Hence the correct recognition of figures and subfigures in a figure is basic for proceeding to choosing the proper reconfiguration for reaching the solution of a geometrical problem. The students in grade 11b appear to have greater stability regarding the mobilization of the operative apprehension, whose relevant cognitive processes appear to be compartmentalized from the cognitive processes corresponding to the rest three types of apprehension. However the greatest coordination between perceptual and operative apprehension is found for grade 11a students. Consequently, students in grade 11a are able to coordinate the cognitive processes regarding the visualization of geometrical figures, which consists of the recognition of figures and the heuristic exploration of figures.

Regarding the three tasks that were considered in the end as tasks with indirect characteristics of proof, for the students in grade 9 the cognitive processes corresponding to the recognition of proof are compartmentalized in relation to the rest of the tasks, showing that for the grade 9 students the production of proof and the recognition of proofs involve different types of cognitive processes. In grade 10 these tasks are related to the rest of the proof tasks or the perceptual apprehension tasks. This indicates that there is a change in the way the students approach the discursive apprehension tasks in grade 10 and that their behavior in tasks that are more or less related to geometrical proofs is becoming more coherent. In grade 11a the results are indicative of a more effective coordination between the cognitive processes that are related to proving in geometry. Furthermore, differently from the students in grades 9 and 10, in grade 11a the tasks with indirect characteristics of proof are related to the operative apprehension and not linked to the rest of the proof tasks or the perceptual tasks. In grade 11b the cognitive processes related to the production of a proof are distinguished from the relevant cognitive processes related to the identification of the proper proof. This compartmentalization can be attributed to the influence of teaching, as these students' teaching involves more experiences regarding the production of proof, thus they develop this ability more and practice such cognitive processes more. Therefore, de-compartmentalization regarding the cognitive procedures of proving, which is achieved after the students move from the lower secondary school to the upper secondary school, is invalidated due to the influence of teaching.

The implicative relations regarding the students' behavior in the tasks on the recognition of proofs showed stability for the students in grade 10. This is also the case for the students in grade 11b, for whom coherence appears not only in the recognition of proofs, but also in the production of proofs. In addition for these students the production of a proof is related to the heuristic exploration of the figure, thus they seem to be keen on coordinating the use of properties and the heuristic functioning of figures, despite the fact that different processes are activated for the recognition of proofs.

Overall, in comparing the students from the two different educational levels regarding the relations that occur between the different types of apprehension, the importance of the proper use of mathematical properties for the construction of geometrical figures and for the production of geometrical proofs was highlighted in the results for both groups. The importance of the operative apprehension in identifying the necessary relations between the different figural units of a figure, which are necessary to identify the proper mathematical properties for proving were also shown. Finally the comparison between the two educational levels showed that the upper secondary school students' seem to coordinate the cognitive processes related to each type of apprehension in a more effective way. In addition it occurs once again that the role of perception seems to be a basic step towards effectively identifying and using the relevant mathematical properties for geometrical reasoning and for proceeding to further processes which give a heuristic role to geometrical figures. The comparison of the implicative relations between the lower and upper secondary school students' answers highlights first of all the different role of the perceptual apprehension and its effect on the sequential apprehension of the geometrical figure for the upper secondary school students. Furthermore the crucial role of the operative apprehension and the discursive apprehension for the solution of geometrical tasks was also indicated. Finally, the upper secondary school students appear to coordinate the different types of apprehension of geometrical figures more effectively than the lower secondary school students. This could be attributed to the students' cognitive development that occurs after students' transition to upper secondary school, in which there are changes regarding the teaching style and the mathematical content students are taught. Therefore the effect of teaching may be the reason for the appearance of more relations between the different types of geometrical figure apprehension in the upper secondary school.

The examination of the students' answers through the cognitive point of view provided more specific information about the way the students behave during the solution of geometrical tasks. In fact, the analysis of the data that came about the cognitive analysis of the students' answers gave a clearer idea of the cognitive processes that take place during the solution of the tasks corresponding to the different types of apprehension. First of all, the similarity and the implicative relations between the students' answers in the perceptual apprehension tasks, as considered from the cognitive point of view, showed that the correct recognition of all the figures in the two tasks is related. However, those students who answer what is recognized at first glance in the first task are able to define the type of less coded figures in the second task. Finally the students with false recognition of figures in the first task are able to define the type of the least coded figures in the second task. This behavior is observed in almost all the group of students, with the students in grade 11a being the exception, as some slight differentiations occur regarding the way they answer in these tasks. The formation of these relations indicates the existence of different levels of the students' perceptual abilities. In fact, the students' could be classified according to their perceptual fluency, based on the number of figures they were able to recognize. Specifically, the three groups of variables formed in the similarity and the implicative diagrams can be considered as indicative of three different groups of perceptual fluency. In the first group higher perceptual fluency appears. In the first group it seems that the students have the ability to extricate themselves from perceptual apprehension and reach the borders of operative apprehension. So the development of recognition abilities towards the abilities related to the heuristic exploration of a geometrical figure appears to be possible. The second group could be characterized by an intermediate perceptual fluency and the last group corresponds to a lower perceptual fluency. In these two groups the students appear to be restrained within the limits of perceptual apprehension.

The similarity relations between the students' answers in the operative apprehension tasks through the cognitive analysis of the tasks show that for the total of the students the use of the mereologic approach is mostly related to the involvement of perception, whereas the relations that appear between the perceptual apprehension and the use of a different approach are less. Thus, students appear to coordinate the cognitive processes corresponding to the operative and perceptual apprehension more, as perceptual apprehension has so far been revealed as basic for the mobilization of operative apprehension. On the other hand the cognitive processes related to the heuristic exploration of a geometrical figure appear to have no commonalities with the cognitive processes related to the use of approaches involving measurements, calculations and other similar strategies. These kinds of approaches occur mostly after the influence of the perceptual apprehension, which seems to block the students' flexible way of looking at the figures and hence there are no possibilities for doing any modification on the figure. Thereafter the students end up with approaches involving measurements, calculations etc for solving a geometrical task, which is more closer to their teaching experiences and the way they are used to solving such tasks. But visualization in geometry is about "seeing" within a figure and has no relation to the estimation or the measurement of magnitudes (Duval, 2013).

More specifically, the examination of each group of students separately showed that differentiations appear regarding the formation of these relations in the different groups. In particular in grade 9 the use of the mereologic modification, and thus the involvement of the operative apprehension, is mostly related to the perceptual apprehension. In grade 10, but also in grades 11a and 11b there are relations between the involvement of operative apprehension and the use of a different approach, which are not found for the grade 9 students. In fact, most of such relations were noticed for grade 11b students. It is therefore obvious that the use of each type of approach is used differently by the students of the two educational levels. The students in grade 9, hence the lower secondary students, seem to coordinate the cognitive processes corresponding to the operative and the perceptual apprehension respectively in a better way. On the other hand the rest of the groups of students, who correspond to the upper secondary school level, appear to involve the operative apprehension on the one hand, but also use approaches that are related to measurements, the use of formulas and calculations. This difference between the students of the two different educational levels can be attributed to the difference in the teaching style in these two levels. In fact the teaching of mathematics becomes more formal and encourages analytical approaches such as the use of formulas for solving mathematical tasks as students move from the lower to the upper secondary school. According to the results this phenomenon gets more intense in the higher grades of the upper secondary school.

Consequently, the formalistic and axiomatic teaching style intervenes in the development of the students' operative apprehension and of their abilities to use the heuristic function of geometrical figures for the solution of geometrical tasks. Furthermore, the implicative relations between the students' answers from the cognitive point of view in the operative apprehension tasks, for the total of the students, also revealed the strong influence of perception in the solution of the only operative apprehension task that included numbers on the given figure. Therefore, the existence of numbers is revealed to be another possible factor that affects the activation of operative apprehension and leads to the reinforcement of perception and the use of calculations. This is in accordance with Duval (1999), who explains that the recognition of a figure is independent from its magnitude or its perimeter, thus a conflict between measurements and what can be seen from the figure is possible for the students. In fact in the cases the students form a hypothesis based on measurements, the operative apprehension is neutralized and the figure stands only as a picture.

Regarding coherence in the use of the mereologic modification, the perceptual approach or a different approach for the solution of the operative apprehension tasks, the similarity relations between the students' answers showed that the lower and upper secondary school students use the mereologic modification more coherently than perception or a different approach. In examining each group of students separately, the results showed that the students in grade 9 consistently use either the mereologic modification, perception or a different approach for solving the operative apprehension tasks. In grade 10 and also in grade 11a the students appear to apply only the mereologic modification for the solution of the tasks coherently. On the other hand for the students in grade 11b coherence seems to exist regarding the involvement of the perceptual apprehension for the solution of the tasks. A different degree of coherence in the use of each type of approach was also shown through the implicative relations, as greater consistency appears in the use of the mereologic modification for the solution of the majority of the operative apprehension tasks.

Consequently, according to the degree of coherence in the way the students solve the operative apprehension tasks, it can be assumed that there are two groups of students. Specifically, there are students that appear to be able to look at a geometrical figure more flexibly and thus they can discriminate the proper reconfiguration that brings them quickly and effectively to the right solution of the tasks. These are the students that mobilize the operative apprehension for the solution of the tasks in a coherent way. It is important that this group also involves the use of the mereologic modification for solving the task that contains numbers, which shows that the students with stability in the activation of the operative apprehension do not get influenced by the presence of numbers in the tasks. Therefore it seems that the presence of numbers can have an effect on the mobilization of the operative apprehension, but not for the students with developed abilities in modifying a given figure for reaching a solution in a geometrical task. On the other hand there are students that use other approaches in the solution of each task. In some cases perception is responsible for the students' solution, whereas in other cases they either mobilize the operative apprehension or they use a different approach. However, in these cases the geometrical figure is not supportive for all students, as they are not helped in gaining an insight into the solution of the problem and according to Duval (1995) these students are thought to look at a geometrical figure in a "blind" way.

Overall, there seems to be an evolution from the perceptual apprehension to the operative apprehension from grade 9 to grade 11a, as greater consistency is displayed in each higher grade regarding the use of the mereologic approach. On the other hand there is less coherence in the operative apprehension in grade 11b, but greater coherence is found regarding the mobilization of perception. This could be attributed to the teaching style in this grade, in which the great the emphasis that is given on the procedural knowledge of theorems, algorithms etc. influences the operative apprehension and strengthens the influence of perception. This is in line with Clements and Battista (1992), who highlight that a primary cause for this performance may be the curriculum, both in what topics are treated and how they are treated. The major focus of standard elementary and middle school curricula is on recognizing and naming geometric shapes, writing the proper symbolism for simple geometric concepts, developing skills with measurement and construction tools such as a compass and protractor, and using formulas in geometric measurements.

Coming to the cognitive analysis of the students' answers in the sequential apprehension tasks the similarity relations showed that for the total of the students the mobilization of sequential apprehension, which helps students to successfully construct geometrical figures, is compartmentalized from the involvement of perceptual apprehension, by which the students construct a figure that seems similar to the correct one, though no proper sequence of steps is followed. This phenomenon of compartmentalization between the cognitive processes related to the perceptual apprehension and sequential apprehension appears in almost all the groups of students, but consistency appears concerning the way the students construct geometrical figures as well, either through the right procedure or through the influence of perceptual apprehension.

These indications about compartmentalization that exists between the cognitive processes related to sequential and perceptual apprehension respectively occurred through the formation of two distinct groups of variables. The distinction of the two groups allows the consideration of two distinct groups of students. In the first group the students do not seem to be very keen on constructing geometrical figures and thus the perceptual apprehension overrides the sequential apprehension. On the other hand, the second group of students seems to be able to mobilize the proper type of apprehension while constructing a geometrical figure and they do not permit the influence of perception to get stronger than the sequential apprehension. Furthermore the cognitive processes related to each type of apprehension seem to be different, as the phenomenon of compartmentalization appears. These results are true for all the groups of students. What changes for the different groups of students is how coherently the students mobilize each type of apprehension for the solution of the tasks.

In particular, the mobilization of either sequential or perceptual apprehension appears to be achieved coherently by the students. The implicative relations indicated greater coherence regarding the activation of the sequential apprehension, which leads to a correct construction of the geometrical figure or even a partially correct construction, if a particular difficulty occurs at a specific step during the construction process. The comparison between the four groups of students shows that the greatest coherence appears for grade 11b students, who seem to have developed their sequential apprehension more than the rest of the students. The group of students for which these observations do not stand is the grade 11a students, for whom there is no stability regarding the activation of the sequential apprehension. These students seem not to be able to mobilize the sequential apprehension in all the cases properly and hence the perceptual apprehension affects the construction process. It can thus be assumed that these students' sequential apprehension is not fully developed yet, as perception finds space to interfere in the process of construction.

Consequently, it seems that perception inhibits the activation of the sequential apprehension or even interrupts its operation and influences the sequence of steps followed in a process of constructing a geometrical figure. The students that do not succeed in mobilizing the sequential apprehension properly, but on the contrary the perceptual apprehension is involved in their solutions, end up with constructions that seem similar to the correct figure, although the sequence of steps was not followed correctly, or end up with an unsuccessful construction. Sometimes returns happen through other figures that do not belong to the intended figure. In such cases the sequential apprehension may cause a rupture to perceptual apprehension (Duval, 1995).

Finally, the similarity relations between the students' answers in the discursive apprehension tasks through cognitive analysis of the tasks were also examined. The results for the total of the students showed that in the cases in which the students made an effort for inference that resulted to be unsuccessful, the discursive apprehension seemed to have been mobilized; however it did not function sufficiently. This insufficient functioning could be attributed to a lack in the students' geometrical knowledge, but also to the students' difficulties in expressing their reasoning in verbal form. Therefore the cases in which the students' correct answers were accompanied with a wrong justification could be related to the discursive apprehension, whereas the cases in which the students did not provide any justification appear not to be related to the discursive apprehension. The occurrence of such cases could be on the other hand due to the influence of the perceptual apprehension mostly. Consequently, the mobilization and functioning of discursive apprehension can be inhibited or influenced by the intervention of mainly perceptual apprehension. This is another indication which enhances the fact that in the cases the students think that the solution was found through perceptual recognition of relations between the different parts of the figure, no further need for reasoning using their geometrical knowledge is needed. This students' conception was identified also in the study of Sharon, McCrone and Martin (2004), adding that this conception may have implications for the students' motivation to construct proofs. If examples constitute a proof or visually obvious relationships may be used as facts in a proof, why is it necessary to write formal, deductive arguments? In addition, the students' difficulties in moving away from perceived features of a figure can mislead the students as to the mathematical properties and objects represented by a drawing and can obstruct appreciation of the need for the discovery of proofs (Duval, 1995). Furthermore, the similarity and the implicative relations revealed stability in the students' answers in the tasks regarding the recognition of proof, because they appear to involve the same type of apprehension for the solution of these tasks. Therefore consistency is displayed in the type of apprehension that is mobilized for solving the tasks regarding the recognition of proof.

The separate examination for each group of students indicated that, as for the total of the students, in the cases in which grade 9 and grade 10 students' justifications were wrong, although the necessary type of apprehension was involved for the solution of the proof tasks, this was not sufficient in reaching a correct justification for the tasks. Therefore, the mistakes that appeared do not seem to be related to the non mobilization of the discursive apprehension, but probably to a lack of knowledge related to theorems and axioms or their inability to explain their reasoning. The grade 9 students seem to have well structured reasoning abilities, as they display coherence in the way they deal with the proof tasks. On the other hand the students in grade 10 display greater stability in the recognition of proof in relation to the students in grade 9. For the grade 11a students different relations appear, compared to the two previous groups of students. In fact in grade 11a the students' wrong inferences are still mostly related to the functioning of discursive apprehension, but relations appear between the absence of inference and discursive apprehension for the first time. The students in grade 11a appear to be less stable regarding the mobilization of the discursive apprehension for the solution of proof tasks in relation to the 9th and 10th grades, but they appear to answer coherently enough in the tasks regarding the recognition of proof. In grade 11b there is for the first time a relation between operative and discursive apprehension, whereas in the previous groups the operative apprehension was mainly related to the perceptual apprehension. Another difference with the rest of the grades is that in this grade the discursive and the perceptual apprehension are mainly related to the absence of inference and justifications. Furthermore the perceptual apprehension is also related to the cases in which wrong inference and justifications were given. Therefore in this grade the influence of perception in the functioning of the discursive apprehension is more clearly revealed.

This can be explained by the fact that the lag between the discursive interpretations of a figure, required in a geometric situation, and the perceptual apprehension finds partly its origins in the laws of the perceptual organization (Duval, 1988). In addition the coherence displayed previously regarding the mobilization of the discursive apprehension is not so evident in this case, but stability is shown in the role of discursive apprehension for the solution of the tasks regarding the recognition of proofs. This is also the case for the role of the operative apprehension in these tasks, but not for the perceptual apprehension.

The results of the implicative analysis about the students' answers in the discursive apprehension tasks showed the existence of three groups with different levels of proof abilities. The first group includes the students that seem to be able to mobilize the discursive apprehension properly, thus they arrive at correct inference and justifications. Therefore these students appear to have well developed proof abilities. The second group includes students that indicate proving abilities, though of lower level, as these abilities do not only concern the correct production of proofs, but also the recognition of proofs or the production of proof through the use of visual methods (i.e indicating relations between different figural units by marking them). The third group of students corresponds to reasoning abilities of a lower level, as these students provide wrong answers and justifications and make improper inference. This separation of students is evident for all the different groups of students.

Comparison between lower and upper secondary school students' geometrical figure apprehension

The third axis of investigation concerns the comparison between the lower and the upper secondary school students' geometrical figure apprehension. In relation to this axis the students' abilities in solving the geometrical tasks corresponding to the four types of geometrical figure apprehension were examined and compared. In addition the comparison also regards the type of apprehension which is mainly mobilized at each grade for the solution of the geometrical tasks. Specifically there were four questions on which these examinations were based. These questions are:

– How able are the lower and the upper secondary school students to solve geometrical problems corresponding to the perceptual, the operative, the discursive and the sequential apprehension?

– What are the differences between the four groups of students (grade 9, grade 10, grade 11a and grade 11b) with reference to their performance in the geometrical figure apprehension tasks?

– What are the predominant kinds of figure apprehension for each kind of geometrical task and at each level of teaching for the four groups of students?

– Do students mainly mobilize the same kind of figure apprehension for the solution of geometric tasks at each level of teaching for the four groups of students?

The examination on the effect of the students' age on their performance in the geometrical figure apprehension tasks indicated that the students' age is a significant factor that influences the students' performance. The comparison between the students' performance revealed differences between the mean performances of the different age groups in the different geometrical figure apprehension tasks. In effect, cases were observed in which the students' performance is higher from one grade to a next one, but there are also cases in which the opposite situation occurs.

In particular, the descriptive analysis indicated that in some of the tasks an evolution was shown regarding the students' abilities in relation to the hierarchy of the grades they attend, as in these tasks the students' performance is higher from grade 9 to grade 11. The tasks on the recognition of proofs are among these tasks, providing indications of the way this students' ability develops, which seems to be influenced by students' cognitive development and by their teaching experiences. On the contrary, for the majority of the tasks of the research instrument, fluctuations were noticed among students' performances in each grade. For these tasks what seems to happen is that the relevant abilities seem to develop after the students' transition from grade 9 to grade 10. This development is interrupted for the students that continue to grade 11a and in fact their performance does not remain stable, but it is, instead, reduced. On the contrary the students that choose to continue to grade 11b seem to further develop these abilities and this development can be mostly attributed to the influence of teaching, as these students attend mathematics courses for more teaching hours than the students from the rest of the forms and the mathematical content they are taught is of a higher level. So the teaching style and the mathematical content included in the mathematics curriculum for grades 11a appears to influence the way students' abilities regarding the different type of geometrical figure apprehension evolve. In all the groups of students, the highest scores are indicated in the operative apprehension, whereas the lowest scores are noticed for the sequential apprehension. A general remark that can be extracted from these results is that the mathematical content and the teaching style the students are exposed to seem to be factors that influence the way the students' abilities in each type of apprehension evolve.

The students' educational level also appears to be a significant factor that influences the students' performance in the tasks examining their geometrical figure apprehension. Differences were traced between the lower and the upper secondary school students' performance in the different types of geometrical figure apprehension. In particular the outcome of the comparison regarding the lower secondary school students' performance and the upper secondary school students' performance is that the upper secondary school students outperformed the lower secondary school students in all the tasks. This development in the upper secondary school students' abilities regarding the geometrical figure apprehension can be related to the students' possible conceptual development and their teaching experience, which seem to positively affect their performance. Furthermore this could be attributed to the students’ transition from one education level to a higher one. Actually there are many differences regarding the mathematical knowledge and the teaching of mathematics between the two educational levels. In addition this evolution that appears in the upper secondary school could also be related to the students’ cognitive development due to their age. In addition, for the two groups of students the highest scores are found in the operative apprehension, whereas the lowest scores are noticed in the sequential apprehension.

The Rasch model allowed the hierarchical classification of the geometrical figure apprehension tasks according to the degree of difficulty and created a good interval level measure for the lower and the upper secondary school students’ abilities regarding the geometrical figure apprehension. The hierarchy of the items that occurred through the Rasch model seems reasonable and also the psychometrical behavior of the items is constant for the four different age groups that were examined. However, there are no similar previous studies in order to test the agreement of this hierarchy with them. In particular, the Rasch model indicated that the students are normally distributed on the scale regarding their ability to carry out the items of the test. In fact the majority of the students are of low or medium ability regarding the geometrical figure apprehension. Concerning the hierarchy of the items the outcome is that the sequential apprehension tasks and some of the discursive apprehension tasks are those having the highest difficulty. The perceptual apprehension tasks, some of the operative apprehension tasks and a part of the discursive apprehension tasks are of medium difficulty for the students. The grouping of these tasks on a similar position on the scale could be related to the influence of the perceptual apprehension, whose intervention in the solution of these tasks is possible. Finally the rest of the operative apprehension tasks and a discursive apprehension task are found to be the easiest for the students. The placement of these tasks at the same level of the scale could be attributed to the similarities in the nature of the tasks.

Generally the distribution of the tasks on the scale of the Rasch model indicates that the tasks mostly demanding the mobilization of the sequential apprehension and the discursive apprehension create more difficulties for the students, in relation to the tasks involving mainly the activation of the operative or the perceptual apprehension. These results enhance the results regarding the students’ mean performances, which show that the students’ lower performance regards the solution of the sequential apprehension tasks, whereas the highest performance is observed for the solution of the operative apprehension tasks. The students' limited experience with tasks of construction, due to the limited emphasis that is given in the curriculum, seems to be ultimately related to the students' reduced performance in the sequential apprehension tasks. In fact the analysis of the students' textbooks showed that the mobilization of the sequential apprehension is the least demanded in the exercises they include and that there are no examples regarding the construction of geometrical figures.

The hierarchy of the tasks for each type of apprehension using the implicative analysis is in line with the results occurring from the Rasch model. The ranking of the tasks showed not only the degree of difficulty of the tasks for the different groups of students, but also revealed some factors that seem to be related with the degree of difficulty for each group of tasks. Starting from the operative apprehension tasks, for the different groups of students, the degree of difficulty appears to be almost invariant. A factor that causes some differentiations on the results, similar to the results of the Rasch models, is the different level of the students' abilities in each group, which corresponds to a different performance in these tasks.

As regards operative apprehension tasks, apart from the level of the students' abilities, additional factors appear to be determining the difficulty of these tasks. The number of the intermediate modifications needed to be done in a figure for reaching a solution is a factor that firstly emerges. Another factor that is assumed to be influencing the difficulty of the operative apprehension tasks is the students' geometrical knowledge, which is also necessary in some cases, besides the ability to modify a given figure. In the tasks of qualitative comparison between lengths or areas, the students have to infer within the natural language only by using the given properties without using any numerical values. However, solving geometric problems about relations between magnitudes does not primarily depend on the difficulty of the mathematical properties that have to be used, but on the cognitive complexity of visualization that is involved (Duval, 2013). The presence of numbers in the given figure seems to be an additional factor that affects the difficulty of the tasks, because it seems to lead the students to paths that are more difficult and longer than reaching a solution through the path of the operative apprehension. In fact the inclusion of numerical data in the tasks is closely related to empirical measurements (Duval, 2013).

The results of the implicative analysis of the hierarchy of the sequential apprehension tasks according to their degree of difficulty indicate that a factor that appears to influence the difficulty of these tasks is the amount, but also the type of the provided data in the tasks. The provision of only verbal instructions or of verbal data combined to a geometrical figure affects the success of students' constructions differently. Furthermore these influences are different for each group of students. Therefore in each grade the amount and the type of the provided data in the tasks appear to determine the solution of the sequential apprehension tasks differently. This is further explained by Duval (2013), who highlights that in geometrical activities at least two registers of semiotic representations are involved: natural language and figures. Therefore he distinguishes two kinds of cognitive tasks in relations to the direction of the conversion of representation to be carried out. The first kind concerns the conversion from words to configurations (from the given information of a problem or from the given instructions in a task of construction of figures), whereas the second type is the conversion from configuration to words, which is done in tasks of description for producing a sequence of instructions that are necessary for constructing some configuration. He thus stresses that in these tasks of description the cognitive complexity of the coordination between language and visualization appears.

Regarding discursive apprehension tasks, no major differences are found between the different groups of students as far as the hierarchy of the tasks is concerned. What is revealed as influencing the difficulty of the tasks, in this case, is the students' geometrical knowledge. The knowledge of theorems, of axioms and of the properties of figures defines the degree of difficulty of the tasks, but this is not only the case. The additional factors that appear to be related to the mobilization and the proper functioning of the discursive apprehension in the geometrical proof tasks are the influence of the perceptual apprehension and the necessity for the mobilization of the operative apprehension, for achieving the multiple intermediate reconfigurations of the given figure that are sometimes needed for the solution of the tasks. The identification of the influence of the two last factors justify Duval (2013), according to whom the properties of a figure (parallelism, equality, symmetry, etc.) can never be inferred neither from perceptual estimation nor from physical measurements. On the other hand, visualization and production of statements in geometry require cognitive functioning that is different from and more complex than those implemented outside geometry. This is why their development and coordination should be considered as learning objectives, as essential as the mathematical contents themselves (Duval, 2005).

Finally, for perceptual apprehension, the recognition and definition of the type of coded figures included in a whole figure seems to be more difficult for the students than discriminating among figures of a specified type in a whole figure. It thus occurs that the students have some difficulties in identifying the type of coded figures in a divided given figure, even if some of these figures are basic geometrical figures, such as triangles, rectangles and squares.

The examination of the students' answers from the cognitive point of view, besides providing information about the students' abilities, also offered valuable information about the particular types of apprehension that are involved in the solution of the different types of tasks. A first outcome of the descriptive analysis of the students' answers in the tasks examining their perceptual apprehension, from the cognitive point of view, is that there seems to be a progression regarding the students' perceptual abilities from one grade to the next one, which can be related to the effects of teaching or to cognitive development factors due to students' increment of age. It also seems that some students have the ability to abandon perceptual apprehension and reach the borders of operative apprehension. Essentially, operative apprehension is different from perceptual apprehension because perception fixes the vision of some shapes at first glance (Duval, 1999). So the development of recognition abilities towards abilities related to the heuristic exploration of a geometrical figure appears to be possible. It could thus be assumed that a progression regarding the students' ability in recognizing figures and coding them as they grow up can occur and this ability can be developed towards the operative apprehension. In such a case the students will be able to coordinate the two types of apprehension and will be able to look at a geometrical figure in a flexible way. Reconfiguration goes against the perceptual recognition of shapes, so for the students that look at the geometrical figure through perception, the figure is mostly seen as a tangible object. And when geometry is reduced to perceptual apprehension there is no real progression for the students (Duval, 1999). Thereafter, the students who can achieve a coordination of perceptual and operative apprehension will be able to recognize the different parts of a divided figure and then continue to a combination of them in different ways in order to make new reconfigurations, which will open the path for the solution of problems in a fast and easy way.

The results of the cognitive analysis of the operative apprehension tasks showed that, despite the fact that these tasks could easily be solved through the mobilization of the operative apprehension; other types of apprehension can be also involved in their solution.

In grade 9 the operative apprehension is mobilized at the same degree with the perceptual apprehension, whereas for the students in grades 10 and 11a the perceptual apprehension is the main type of apprehension mobilized in most of the tasks. The opposite holds true for the students in grade 11b. Thus it seems that in some cases it is easier for the students to see, whereas in other cases it is less easy for them to see the proper reconfiguration. But why is it easier for students to see in one task and more difficult in another task? A factor that appears to intervene and block the way the students look at geometrical figures is the predominance of perception. Perception make figures steady (Duval, 2013) and thus prevents the students from seeing different reconfigurations on the figure, which would bring them to the solution of the task very easily and without the need for measurements and calculations.

The results regarding the cognitive analysis of the tasks examining the students' sequential apprehension indicate that the students are not very keen on mobilizing the sequential apprehension. The students seem to face difficulties in following the necessary steps for constructing the geometrical figures based on the given information in each task. Therefore, they sometimes manage to follow some steps of the construction procedure correctly and thus construct correctly only a part of the figure or in other cases they are not able to follow the correct procedure at all and end up constructing a figure that appears similar to the correct one, but without maintaining the mathematical properties of the figure. In these cases perceptual apprehension takes place. According to Schoenfeld (1986), most students who have had a year of high school geometry are "naive empiricists whose approach to straightedge and compass constructions is an empirical guess-and-test loop". The students make a conjecture and then test it by examining their construction. If a construction looks sufficiently accurate, the student is satisfied that the conjecture has been verified. He also adds that in various problem sessions the students have rejected correct solutions because they did not look sufficiently accurate and have accepted incorrect solutions because they looked good".

Consequently, it seems that students activate perceptual apprehension in the construction tasks more and they face difficulties with mobilizing the necessary type of apprehension, which is sequential. This phenomenon is less intense for grade 11b students; therefore it can be assumed that these students' sequential apprehension is more developed, in relation to the rest of the students. This development could be mostly attributed to the students' teaching experiences, as observed through the content analysis of the textbooks used in this grade. In fact, the highest number of exercises mobilizing sequential apprehension was found in this grade, compared to the other group of students. Instructional strategies used in high school classrooms were found to be influencing the students' construction abilities (Schoenfeld, 1986). Although theorems and deduction were used to introduce and validate constructions, the emphasis was on constructions as procedures; that is as a skill acquisition.

As regards cognitive analysis of discursive apprehension tasks, the results, are indicative of incomplete knowledge on what the students consider as a formal proof and what the characteristics of a formal proof are. This can be further justified by Shoenfeld's (1986) ascertainment, after an experiment he conducted in the classroom, that perhaps what is needed, and what has been lacking, is an understanding of how proof really works. Students appear to be able to present some steps of their inferences, but they are not able to justify them by using the proper mathematical properties, axioms or theorems. Besides, an alternate approach suggests that the students must first understand the nature of proof, in order to develop ability with proof (Clements & Battista, 1992).

In other words students seem able to reason properly, but the difficulty concerns the expression of this reasoning in a written form. Therefore it seems that, despite the fact that the students follow a correct reasoning process, they cannot reproduce a proof verbally or in written form. The answers show that in some cases there is an effort by the students to convey their reasoning in written form, though unsuccessfully. In other cases the students appear not to try at all, possibly due to the fact that their relevant abilities are very limited and do not even allow them to make an effort to justify their answers. This is in line with Duval's (1998) discrimination between a natural discursive process – which is spontaneously performed in ordinary speech through description, explanation, argumentation – and a theoretical discursive process – which is performed through deduction. The experience of logical necessity is closely connected to this theoretical process. This can be performed in a purely symbolical register or in the natural language register. But these two registers do not pose either the same difficulty or have the same significance for the pupils. Therefore there is often a gap between the natural discursive process and the theoretical discursive process. Either way, there is no valid reasoning without language, because only propositions can be true and because there is no proposition without statements. Therefore the problem of the relationship between reasoning and explicit wording cannot be ignored, especially in mathematics education (Duval, 2007).

On the other hand, there can be cases in which the students' inference can be influenced by additional factors, such as the involvement of other types of apprehension. For the specific tasks that were given in this research, the type of apprehension that seems to be mostly involved is the perceptual, which appears to be an inhibiting factor for the mobilization and the proper functioning of the discursive apprehension. The influence of perception is also highlighted by the results on the tasks that examined the students' abilities in recognizing a formal proof. Thus the perceptual apprehension occurs again as a possible factor which causes a negative effect on the students' discursive apprehension, emphasizing that students find it useless, and sometimes absurd, to demonstrate a property that is seen on the figure (Duval, 1988). But perceptual apprehension cannot determine the mathematical properties represented in a drawing and very often it runs against the mathematical way of looking at figures. On the other hand, some mathematical properties must be given through speech (denomination and hypothesis) and others can be derived from the given properties. Therefore, it cannot be said that a mathematical property "is seen" in a figure. The absence of denomination and hypothesis in a drawing makes it an ambiguous representation and, thus, the properties that are seen are not the same for everyone (Duval, 1995). For this reason it is possible to have a gap between what the figure shows and what it represents.

Lower and upper secondary school students' mistakes and ideas about geometrical figure apprehension

In the fourth and last part of this chapter the lower and the upper secondary school students' mistakes and ideas about the geometrical figure apprehension are discussed. The students' mistakes are described and the possible factors that are related to their appearance are explored. In order to be able to further investigate the students' mistakes and in order to have the chance to gain more information on the reasons that cause them, qualitative data were also collected from the students. The task-based interviews that were conducted, by which the quantitative data were triangulated, provided information not only regarding the students' mistakes, but also on their ideas about the geometrical figure apprehension. The results of the students' mistakes and ideas allowed a discussion of the types of geometry in which the students' work takes place, based on the theoretical framework of Houdement and Kuzniak (2003). This theoretical approach allows understanding and interpretation of the students' difficulties during the solution of geometrical tasks, compared to Van Hiele's (1967) levels which do not facilitate studying and explaining these difficulties (Gutiérrez, Kuzniak, & Straesser, 2005).

In essence, this last axis of investigation comprised the following questions:

– What are the lower and upper secondary school students' mistakes in the geometrical tasks corresponding to each type of apprehension and what are the reasons that cause them?

– Do students get a right answer mainly when the perceptive apprehension is not mathematically deceptive and do they fail when an operative or discursive apprehension is needed?

– What differences exist between the four groups of students (grade 9, grade 10, grade 11a and grade 11b) in regard to their answers in the tasks of the interview?

– Are most students in the lower secondary school still working in paradigm GI or in Mixed Geometry (GI/GII) according to the educational standards?

– Are most students in the upper secondary school working in paradigm GII, Mixed Geometry (GI/GII) paradigm, or only in paradigm GI?

Results concerning students' mistakes and ideas

The presentation and analysis of students' mistakes in each group of tasks examining the different types of geometrical figure apprehension provided information about the students' difficulties and weaknesses regarding specific aspects of each type of apprehension. These results also provided indications of the relation of these mistakes and the different types of geometrical figure apprehension, whose involvement seems to make students deviate from the right path leading to successful solutions of the tasks. Thus in this part the students' mistakes in the different groups of tasks are interpreted through the specific types of apprehension that seem to be related to their appearance and the degree of their effect on the different groups of students.

Starting from the group of tasks examining the students' perceptual apprehension, the results show that in both tasks the lowest percentages of wrong recognition and wrong naming of figures is found for students in grade 11b. On the contrary the 9th graders are related to the highest number of cases of wrong recognition and wrong naming of figures. The percentages of wrong recognition for students from the two remaining groups are found between those of students in grade 9 and the students in grade 11b. In particular the mistakes made by the students in grade 10 are fewer than those by the students in grade 9 and in grade 11a and the mistakes by students in grade 11a are more than those in grade 11b. Consequently the students could be ranked according to their perceptual abilities based on the number of mistakes they have made in the recognition and the naming of the figures. Specifically the students in grade 11b could be characterized as having the highest perceptual abilities, followed by students in grade 10 and then by the students in grade 11a. The 9th graders are the students who are found to have the lowest perceptual abilities in relation to the rest of the students. This ranking of different groups of students seems to be affected by two parameters: the students' age and the teaching they receive.

The fact that the students' perceptual abilities appear to be higher from grade 9 to grade 11, which is in line with the students' evolution in these abilities from grade 9 to grade 11 as also observed in the cognitive analysis of the perceptual apprehension tasks, could possibly be attributed to the students' cognitive development due to the increase of their age, but also due to the greater number and the different type of teaching experiences as the students move to a higher grade. However, despite the fact that development seems to appear in the students' perceptual abilities from grade 9 to grade 11b, a different situation is noticed in grade 11a. In ranking the students' perceptual abilities, the grade 11a students are situated between grade 9 students and grade 10 students, instead of being situated after grade 10 students and perhaps before grade 11b students. Hence, although the students in grade 11a should be better than the younger students due to the students' possible cognitive development in recognizing and naming geometrical figures, this situation is not observed, thus creating questions about what the reason for this might be. What actually differs between the grade 11a and the grade 11b students is the teaching of mathematics, regarding the content and the amount of teaching. As previously mentioned in the part describing the participants of the research, the number of the teaching periods for the mathematics course in grade 11a is lower than the respective number of teaching periods in grade 11b. In addition the teaching of mathematics in grade 11a remains restrained to more basic mathematical concepts, whereas in grade 11b the level of teaching is higher, as more complex mathematical concepts are involved. Thereafter the teaching of mathematics in grade 11a can be a possible reason that is also influencing the development of the students' perceptual abilities, which were expected to be higher because of the students' age. So in this case the teaching of mathematics seems to be a reason that influences the development of the students' perceptual abilities, by intervening and interrupting the development of these abilities developing due to cognitive developmental factors.

Another observation that surfaces from the analysis of the students' wrong answers and mistakes is that, as far as the recognition of the coded figures is concerned, most of the students' incorrect answers occur during the recognition of the trapeziums. In fact these mistakes were mostly observed for students in grades 9 and 11a, which according to the ranking described previously, are placed on the lowest level of the scale regarding the level of students' perceptual abilities. For the students in grade 11a the most difficulties regarding the recognition of triangles are also observed. Furthermore the squares and rectangles seem to be the easiest figures to be recognized by students, as the lowest number of mistakes are noticed for these figures. According to Vitz and Todd (1971), the simplest figure to be recognized appears to be the figure with the fewest number of angles and with the least variability in angle size.

Overall, what can be assumed is that the degree of difficulty regarding recognition of figures is influenced by two factors: the type of figure and the number of subfigures it contains. Actually, concerning the type of the figure, the degree of difficulty for each type of figure is different in each grade. However, the recognition of the trapezium is the most difficult for all the groups of students, while the easiest is the recognition of the squares and rectangles. About the number of subfigures a figure includes, it appears that the more the subfigures a figure includes the greater the difficulty is in distinguishing a figure at first glance. The complexity of the figure is determined according to the number of subfigures and as this number becomes higher, the mobilization of the operative apprehension is necessary, in order for some modifications to be done in the figure, which will allow the discrimination of more figures than those distinguished at first glance.

Furthermore, the students' mistakes regarding the recognition of figures can be related to the predomination of typical examples, which creates many obstacles to the students. In fact, the position and the orientation of the figures, in the Euclidean geometry on the plane of paper, is an independent factor which does not affect the properties of the figure. But in teaching and also in the textbooks, most figures have a “regular” orientation at the drawing level. The basic lines or the total orientation of the figures is parallel and vertical on the lines, which define the limits of the drawing fields (sheet of paper, computer screen, blackboard etc.) (Lemonidis, 1997).

For the operative apprehension tasks what is observed first of all is that the students in grade 11b produce the least amount of wrong or no answers. Besides, for most of the operative apprehension tasks the cases in which the students do not provide an answer are not very high. This shows that the students’ difficulties regarding the solution of the operative apprehension tasks are not strong enough to discourage them to try to solve these tasks. However, the mistakes which appeared in the students’ solutions are mostly related to a wrong perceptual estimation of the area or the perimeter of the figure they had to compare or even to the wrong performance of calculations. Therefore, the perceptual apprehension appears to be mainly related to the students’ unsuccessful attempts to solve the tasks. The intervention of the perceptual apprehension appears as a possible factor that inhibited the further mobilization of the operative apprehension, whose involvement would lead students to the right solution of the tasks. Thus, these results underline that the visual recognition of a figure, at a glance, constitutes the first cognitive condition for solving problems in elementary geometry and when such recognition does not take place, the students are not able to solve the classical problems of comparing areas (Duval, 2013). In fact, the mistakes related to the activation of the operative apprehension appear to be the least in the students’ wrong answers and they are even totally absent in grade 11b. This is another indication that the students in the highes grade have the most developed operative apprehension of the geometrical figure, in relation to the rest of the students. This result is in line with the results of the multivariate analysis of variance (MANOVA), which showed that the average performance in the operative apprehension tasks is found for the students in grade 11b.

Nevertheless, not all students are on the same level, since there were students with limited abilities, and possibly experiences, to conduct such modifications to the geometrical figure so as to solve geometrical tasks. The students, though, report that they like such approaches, despite facing difficulties. These difficulties may have been increased if there were numeric data on the figure too, as a student states. According to Duval (1999), there might be a conflict between the students’ measurements and what they can discern on the figure. In the case the students formulate a hypothesis based on measurements, the operative apprehension of the geometrical figure is neutralized and the geometrical figure eventually functions as an image.

The students' answers regarding the solution of the operative apprehension task they were given during the interviews provided additional information about the way the students solved the tasks. Actually, similarities emerged among the students in respect to the type of approach they employ to solve the operative apprehension task. In particular, the students converge as regards the use of the algorithmic approach, although, in the specific exercise, the solution could come about through operative apprehension, by modifying the geometrical figure. For the grade 9 and the grade 10 students, the application of the algorithmic approach to find the area of the trapezium was conducted in two different ways: by using the formula of the area of the trapezium and by finding the area of each subfigure separately. The grade 11 students used the first way only, since they appear to be more familiarized with the use of formulas due to their learning experiences.

According to Duval (1999), when hypotheses include numbers as measures of sides or segments, the operative apprehension is neutralized and the figure fulfils only an illustrative or support function. However, after being encouraged, the students appeared to be able to reach a solution through modifications on the geometrical figure as well, by using different procedures. Some students began with the whole figure and divided it into subfigures, whereas others followed the opposite procedure and composed the subfigures on the whole figure. This composition was suggested in two different ways by students: either with the movement of one triangle to the right or left, or with the movement of both triangles upwards.

Although the students consider the mereologic modification a briefer approach for the solution of a problem, they prefer the use of algorithmic approaches, which are easier for them and are directly linked to their teaching experiences. The algorithmic approach is primarily used by the students, thus they feel more confident when a result comes through such a procedure. It is, however, important to mention that, in solving this exercise, evidence emerged concerning the fact that the students' learning experience is related with the activation of the operative apprehension of the geometrical figure. There was actually a student who conducted modifications to the figure due to the experiences he gained while using similar approaches in primary school, as well as a student who managed to successfully complete the exercise, because of the experience with modifications on the figure from a previous task that was solved during the interview.

Consequently, the aforementioned remarks create the need for reconsidering whether the sufficient teaching practices and the more frequent exposure of the students in using such modification approaches could improve the activation of operative apprehension of the geometrical figure in appropriate cases. On this account, Godfrey (1910) proposed that the “geometrical eye, the ability “to see geometrical properties detach themselves from a figure” would be essential to solve geometrical problems. He thus stated that learners’ geometrical eye could be developed through experimental tasks.

As regards the sequential apprehension tasks, the results revealed great difficulties regarding not only the construction of the geometrical figures, but the description of the procedure the students have followed for constructing the figures as well. Besides the cases of wrong answers in the sequential apprehension tasks, a high number of cases in which the students left the tasks unanswered were observed. This observation was less frequent for the students in grade 11b, but was more common for the rest of the students and especially for the students in grade 11a. However the cases in which no answer was given were less than the cases of incorrect constructions, showing that students make an effort to construct the figures, but the number of difficulties they face does not allow them to reach the correct construction of the figures. On the other hand, the cases in which the students managed to describe the construction procedure correctly are not very frequent for all the groups of students. Therefore, it appears that despite the fact that some students are able to follow the correct sequence of the steps for the construction of a figure, they are not always able to express this sequence of the steps in a written form.

In fact, grade 11b students appear to be more capable at describing the construction procedure correctly, compared to the rest of the students. On the contrary, the least able students at describing their construction procedure correctly are the students in grade 11a. This is indicative of the influence of teaching, which is the factor that differentiates the two groups of students, although they both belong to the same age group. The limited focus placed on the construction of geometrical figures in grade 11a appears to be related to students reduced performance in the construction and the description of this construction of geometrical figures and the difficulties they face during this procedure. On the other hand the richer teaching experiences of the students in grade 11b seem to enhance the students’ sequential apprehension, as they appear keener on constructing geometrical figures and describing the relevant procedure.

Generally the construction of geometrical figures and the description of this procedure create great obstacles for students, showing that sequential apprehension is not developed to a great extent. Mariotti (1999) mentions that the theoretical meaning of geometrical constructions, i.e. the relationship between a geometrical construction and the theorem that validates it, is very complex and certainly not immediately perceived by students. It seems that the very nature of the construction problem makes it difficult to assume a theoretical perspective. The weak abilities related to the sequential apprehension of geometrical figures can be linked to the students' teaching experiences, which are not very rich, but also to the cognitive level of the students according to their age.

In addition, the students' inability to transform the sequence of the steps to a verbal form can be explained in reference to the relation between the sequential and the discursive apprehension. In fact, the students that achieve the proper mobilization of the sequential apprehension, and thus reach a successful solution do not seem able to subsequently activate discursive apprehension, by which the description of the procedure will occur. The discursive apprehension is assumed to be related to the description of the procedure, because a construction procedure is related to the use of mathematical properties, which must be taken into account during the construction of a correct figure. In a properly constructed figure the mathematical properties of a figure should be maintained and clearly represented (Duval, 1999). Thereafter this situation could be explained as the students' weakness to move forwards from the sequential to the discursive apprehension.

But what are the factors that suspend the mobilization of the sequential apprehension and the evolution towards discursive apprehension? In observing most of the students' mistakes, it was noted that these are related to constructions in which no sequence of steps was followed, but on the contrary the students tried to draw a figure that would be perceptually similar to what the properly constructed figure should represent, and also to efforts in which the sequence of the steps was not followed correctly until the end. Such behaviors can be resulting from the intervention of the perceptual apprehension, which usually leads to such reactions. The involvement of the perceptual apprehension during the construction of geometrical figures seems to neutralize the role of the sequential apprehension and as a result a proper procedure is not put to work. Although the figure a student has constructed may appear close to a proper figure, the necessary sequence of the steps was not followed. It therefore seems that in the cases the influence of the perceptual apprehension is strong, the students do not feel the need to follow a sequence of steps and they are instead satisfied when the final result is similar to what was expected as a correct construction. So, perceptual apprehension could be considered as a factor inhibiting the students' development from sequential apprehension to discursive apprehension.

The students' difficulties in constructing geometrical figures were expressed directly by the students during the interviews. In the context of the general questions posed to the students about the teaching of geometry, the students expressed their views in respect to the presence and the construction of geometrical figures. The students of all three grades agreed that the frequency of the cases they are given a task in which the geometrical figure is determinant for its solution, is not so high. On the contrary, the cases in which the students need to construct a geometrical figure in order to solve a geometrical task are more frequent. What is generally expressed by the students is that the presence of a geometrical figure in the tasks is helpful, while the absence of a geometrical figure creates additional difficulty. Nevertheless, the students agreed that they feel more confident when the geometrical figure is given in a task, because in such cases they are certain the figure is correct. In contrast, they appear to feel uncertain about the suitability of the geometrical figure, in the cases they have to construct it themselves. As regards geometrical constructions, the students mentioned that at school they are asked to perform geometrical constructions, although no exclusive teaching time is dedicated to this. In most cases, the geometrical constructions are done for the needs of exercises, where the presence of the figure facilitates their solution. Therefore, the students appear to have the tendency to create a geometrical figure in the tasks in which it is not provided. The students also express difficulties regarding the construction of geometrical figures, which seem to be related with the students' limited learning experiences in this domain.

As for the discursive apprehension tasks, similarly to the sequential apprehension tasks, the students encountered great difficulties solving these tasks as well. This is evident when observing the high amount of the students' wrong or no answers and justifications for their answers. In most of the tasks there were many answers that were incorrect, but the number of cases in which the students did not manage to provide any solution was also high. In fact the cases of no provision of an answer appeared more frequently in relation to the cases in which the students' answers were wrong. This is also true for the cases in which the students had to justify their answer. In this case students' correct justifications are very few, as most of the students did not provide any justification or gave a wrong justification. These observations are mostly related to the students in grade 9, 10 and 11a, whereas the situation is better for students in grade 11b, who seem more capable to handle the proof tasks and provide a proper justification for their answers. On the other hand, same as for the previous group of tasks, the students in grade 11a appear to have the greatest difficulties providing an answer and justifying it, as the highest number of no answers and consequently of no justifications was observed for these students. This enhances the possibility that the influence of the students' teaching experiences and the teaching of mathematics in the specific grade can be negatively affecting the students' cognitive development regarding the discursive apprehension of geometrical figures.

Concerning students' mistakes in these tasks, these are mostly related to a lack of theoretical geometrical knowledge. From the specific tasks emerged that the students have some gaps regarding the knowledge of theorems, axioms or the properties of the geometrical figures. This lack of knowledge was a factor that made the mobilization of the discursive apprehension even harder. Despite the fact that the discursive apprehension was involved in the solution of the proof tasks, this lack of knowledge did not permit the students to reach the proper proof. Furthermore, the absence of the students' reasoning can also be related to the intervention of the perceptual apprehension during the solution of the discursive apprehension tasks. The students' real confidence in the appearance of the figure might be a reason for the absence of justification in their answers. According to (Duval, 1988), the influence of the perceptual apprehension on the discursive apprehension makes the students feel that it is useless and sometimes absurd to demonstrate a property that is seen in the figure. As Duval (1995) explains, there is a potential conflict between perceptual apprehension of a figure and mathematical perception, as the perceived features of a figure can mislead students as to the mathematical properties and objects represented by a drawing, and can obstruct appreciation of the need for the discovery of proofs. But perceptual apprehension cannot determine the mathematical properties represented in a drawing and very often it runs against the mathematical way of looking at figures. On the other hand, some mathematical properties must be given through speech (denomination and hypothesis) and others can be derived from the given properties. Therefore, it cannot be said that a mathematical property "is seen" in a figure. The absence of denomination and hypothesis in a drawing makes it an ambiguous representation and, thus, the properties that are seen are not the same for everyone (Duval, 1995). For this reason it is possible for a gap to exist between what the figure shows and what it represents. Consequently, the students' mistakes can either be attributed to a lack of geometrical knowledge or to the intervention of perceptual apprehension, which functions as an obstacle for the mobilization of discursive apprehension.

Interesting information appeared in the students’ answers in the tasks on the recognition of proof during the interviews as well. In particular, regarding the acceptance or rejection of the different types of proofs in the tasks examining the recognition of proof, the students appeared to share some common ideas, but some differences were also detected. The formal proof seems to be clearly accepted without hesitation or uncertainty by the students of all grades. As regards the specific proof, the students were convinced by the presence of the geometrical figure and the use of a formulaic procedure, through the use of a theorem and the parallelism of lines. The students assigned precision to this proof and recognized the fact that it can be generalized to all cases of triangles. With respect to the use of theorems, grade 9 students seem not to have good knowledge of theorems, attributes and criteria of different geometrical figures. Grade 10 students seem to have better knowledge of the theorems, but appear not to remember them well, due to the limited frequency of their use. As anticipated, grade 11 students have good knowledge of theorems, due to their increased learning experiences. However, they seem to lack understanding of the verbal formulation of theorems. Therefore the results of the interviews reinforce the aforementioned results concerning the students’ lack of geometrical knowledge.

Regarding the other two types of proof, grade 9 students did not accept the proof related to perceptual apprehension, neither the empirical proof, due to its practical usefulness, since it constitutes an approach which cannot be easily applied at any given moment. The proof related to the operative apprehension, hence the semi-empirical proof, was also rejected, due to the fact that the factor of chance interferes with the correctness of the results. For the same reason, this proof was neither accepted by the grade 10 nor the grade 11 students, who stressed the possibility of error when using this approach. Nevertheless, there were some students in these grades who did accept it, due to the use of the definition of the right angle and because of prior experiences which included memories of attempts which have lead to this result. Thus, it can be inferred that the students are generally able to discern the empirical dimension to a proof and therefore consider it incomplete and unconvincing, leading to its rejection. It can also be assumed that the students’ prior experiences influence the formation of their new knowledge, since the students’ new experiences with formal proof seem not to be sufficient enough to overlap the older ones which concern experiences of empirical proofs and help students reject empirical methods from their arsenal as regards proof. Additionally, it can be said that the students do not entirely reject empirical methods, if there is evidence that they could be valid in a limited number of cases.

Generally, as far as geometrical proofs are concerned, the students stated that for the proof tasks they are given in class, the usual procedure required is the production of a proof. This is the reason why the students described the proof tasks (about the recognition of proof) given to them during the interview as different, because they had to think and comment on given proofs. The questions posed to the students during the interviews made it possible to extract the students' views regarding proofs, as well as for mathematics in general. It, thus, occurs that when something can be proved with perceptual approaches (i.e. by recognition), this does not constitute a proof and, also, that mathematics is keenly related to proving. Another view expressed by the students connects mathematics to a more algorithmic approach, with the use of theorems, formulas and numbers. A more formalistic nature is thus assigned to mathematics and this view is perhaps formed due to the students learning experiences and the kind of mathematics they come in contact with during teaching. Thus indications about the influence of the didactical contract are provided. The notion of a didactic contract has been introduced by Brousseau (1997), to describe a system of rules, mostly implicit, associating the students and the teacher, as regards a given piece of knowledge. Therefore the students' view that success in mathematics is related to good knowledge of theory seems to be related to a didactic contract for mathematics, concerning generally mathematics in the institution: for example, the requirement of rigorous proofs (De Vleeschouwer & Gueudet, 2010).

However, although the students seem to be more accustomed to exercises in which the algorithmic approaches are used, their attitude towards exercises which are solved with different, less algorithmic approaches appears to be positive. Although exercises which can be solved with different approaches from the algorithmic ones are considered interesting, the students still prefer exercises which can be solved algorithmically, because they feel more certain about their answer through this approach. Although the students express a wish for more conceptual and less mechanical approaches in the teaching of geometry, they state that using the data in an exercise and following a procedure to reach the solution gives them more certainty about the result. On the other hand, solutions premised exclusively on visualization and on modifying a given geometrical figure, which does not include any numerical data, do not offer them the same certainty. This also seems to be related to the influence of the didactic contract, since students state that they usually work with numeric data and through them they are led to a solution. Students are, thus, cautious in respect to the use of approaches which are different from the algorithmic ones, as well as for the results coming about through them. Nevertheless, the students conclude that they wish to experience a combination of exercises, which can be either solved with an algorithmic or a non-algorithmic approach respectively.

It is also worth mentioning that the assurance that occurs through the use of algorithmic approaches is emphasized by the lower secondary school students, while the wish for different, less mechanical approaches is expressed more by the upper secondary school students. This fact could be related to the students' learning experiences, which are increased and enhanced as they move to higher grades. The students with less learning experiences and knowledge of mathematics seem to prefer the use of more standardized methods, with which they feel more confident, while students with more knowledge and experiences are more receptive to different approaches and new types of exercises.

Concluding, the influence of the didactic contract was evident in the students' answers to the exercises they were given and the questions that followed, both as regards the approaches the students use for solving exercises and the presence of a geometrical figure and numeric data in the exercises. In specific, the relationship of the didactic contract with the preference and application of the algorithmic approach to the exercises by students was detected. The didactic contract even appeared to influence the students' confidence when they have to use numeric data in exercises and when the exercises are accompanied by the required geometrical figure, so that its construction is not required of them.

Besides the descriptive analysis of the students' mistakes in the different tasks, the similarities and the implicative relations between the students' mistakes in the tasks regarding the recognition of proof and their responses in the tasks concerning the production of proofs were also examined, in order to study the students' behavior during the tasks related the recognition of proof and the rest of the tasks, so as to be able to set the students' geometrical paradigm based on these indications. The similarity diagrams for the students in grades 9 and 10 provide indications about the possible existence of three groups of students, who seem to have a different level of proof comprehension. The first group expresses good abilities of reasoning. In this group fall the students that are able to discriminate a formal proof, who also seem able to conduct proper reasoning towards the production of proofs. On the other hand, comprehension of proof, though incomplete, appears for the students who accept not only the formal proof, but also the other two types of proof. In the second group fall the students who accept the proofs which are mostly related to operative apprehension, but also recognize the correct proof. These students seem to have fewer abilities for reasoning, as they do not always arrive at proper inference. In addition, the implicative diagrams revealed that the students' efforts for reasoning that are unsuccessful are linked to their mistakes that are related to operative apprehension. The last level includes the students' mistakes that are mainly related to perceptual apprehension, which are linked to no inference or justifications in the rest of the proof tasks. Therefore, in this case the level of proof comprehension is lower than the two previous groups. Furthermore, these relations enhance the results that occurred from the cognitive analysis of the students' answers, in which no inference and no justifications were related to the influence of the perceptual apprehension, which ultimately seems to inhibit the proper mobilization of the discursive apprehension. In fact, the mathematical recognition of objects represented in figures depends only on the given properties and not on the perceptual recognition of shapes and the use of the given properties is the first step for recognizing the significant shapes for solving a problem (Duval, 2013). But the students that mostly rely on a perceptual approach of a geometrical figure and recognize perceptually what they are asked to prove, they consider that there is no need for proving or justifying something that is very easily observed in the figure.

Regarding the students in grade 11a, there is no coherence in the way they answer in the tasks on recognition of proof, as the same types of mistakes are in some cases related to answers showing a correct reasoning, whereas in other cases they form relations with answers indicating limited comprehension of proof. An indication of stability is only displayed in the choice of the proof that relates to the involvement of the perceptual apprehension, which is linked to an incomplete proof ability and also in the correct recognition of proof which is connected to answers showing good reasoning abilities. In this grade the acceptance of the three types of proof is not related exclusively to proper reasoning, but also with a gap in the comprehension of proof. This is also the case for the correct recognition of proof which is not only related to correct inference but also to wrong reasoning, differently from the two previous groups of students. The type of mistake which corresponds to perceptual apprehension relates to an answer in which there was an effort for reasoning which was unsuccessful. Therefore in this case the perceptual apprehension was not an obstacle for the students' reasoning, but it seems to be still influencing the students' inference.

For the students' mistakes in grade 11b there are more indications of coherence in the appearance of the mistakes in the recognition of proofs compared to grade 11a students. The acceptance of proofs that are related to the influence of perceptual apprehension are linked to indications of an incomplete comprehension of proof. However in this grade what differs is that this type of mistake is, for the first time, also linked with the correct recognition of proof, based on the implicative relations. The acceptance of proofs that are related to the mobilization of operative apprehension seems to be related to proper reasoning, according to the similarity diagram, but the implicative diagram also reveals relations with improper reasoning. As in grade 9 and 10 and differently from grade 11a, in grade 11b the acceptance of all the three types of proof is related to a good comprehension of proof and a correct reasoning. This also surfaces from the formation of the implicative relations in the implicative diagram. Furthermore for this group of students the correct recognition of proof seems not to be clearly indicative of complete reasoning abilities, but of reasoning abilities that have a gap.

Identification of the students' geometrical paradigm according to their answers in the discursive apprehension tasks

The identification of the students' geometrical paradigm in which their work takes place was done though the examination of the relations between each type of mistake that appeared in their answers in the tasks pertaining the recognition of proof and the answers mostly indicating proper reasoning, in the tasks examining the students' abilities regarding the production of proof. The results indicated three types of paradigm for characterizing the students' work regarding the discursive apprehension tasks.

First of all, some groups of students were related to Natural Geometry (Geometry I), as their work was based on their sensorial experience and mainly connected to the real world while their assertions were generated using arguments based on perception and experiment (Kuzniak & Rauscher, 2011). In other cases, there were groups of students who were found to reason within the type of geometry possessing mostly characteristics of the GI paradigm, but also less characteristics of the Natural Axiomatic Geometry (Geometry

II) (Houdement, 2007). Thus this type of paradigm was considered as the mixed geometry one GI/GII. Finally, a third type of paradigm was identified, which included mainly the characteristics of the GII paradigm, but some characteristics of the GI paradigm were also present, though to a lesser extent. Thereafter this type of paradigm was named as mixed geometry GII/GI.

The identification of the work of each group of students in each grade, according to the type of mistake they made, reveals a type of paradigm which enables a general characterization for the students' work in each grade. In fact in grades 9 and 10 the students appear to work within the same geometric paradigm. For these two grades the presence of the three aforementioned types of geometry were noticed, but the most frequent type of paradigm was mixed geometry GI/GII. Therefore the students' work in these two grades seems to be mostly turned to mixed geometry GI/GII, as there are characteristics of both the GI and the GII paradigms, with the first being more prevalent. However, mixed geometry GII/GI starts to appear in these two groups. For grade 11a students there are more indications of mixed geometry GII/GI, compared to the two previous groups, however the main type of paradigm for the students' work is mixed geometry GI/GII. The most indications of mixed geometry GII/GI are found for grade 11b students' work.

Relating this discrimination to Kuzniak's (1999) discrimination of types of geometry the students in grades 9 and 10 could be considered as working within what Kuzniak and Rauscher (2011) call "*Established Geometry I".* In this type of geometry the study of configurations from the real world is the goal of Geometry I, where it is allowed and even encouraged to take measurements on a figure to solve problems. Some theorems can also be used as technical tools to replace measurement by calculations: this is the case for Pythagoras' Theorem or the Intercept Theorem. The work of the students in grade 11a could be situated within the "*Established Geometry II"* (Vivier & Kuzniak, 2009), which rests on a set of properties and experiments provided by Geometry I and is intuitively useful. But then the axiomatic horizon is clearly a part of this geometry, which brings it close to Euclidean geometry.

Finally the work of the students in grade 11b could be related to the "*Fragmented Geometry II*". As in the previous case, the fragmented Geometry II is based on a set of properties and experiments issued from Geometry I. But in contrast to the previous case, this geometry is characterized by discrete blocks of hypothetico-deductive reasoning organized around properties and some basic geometric configurations. These blocks of reasoning are founded on a few properties justified by an experiment validated by measurement or by software.

Overall, there seems to be a progression in each grade from the GI paradigm towards the GII paradigm, although the students do not appear to reach the second type of paradigm. However they seem to remain somewhere between the two types of paradigms, which is a type of mixed geometry, combining characteristics of both paradigms GI and GII. These results verify Kuzniak's (2011) assumption that the constant emphasis on a transition towards Geometry II based on Geometry I can let one suppose that a mixed Geometry is possible.

What is interesting is that there seem to be two sub-types of the mixed geometry paradigm, which are defined according to the predominance of either the GI or the GII paradigm. Furthermore the students' transition from the lower to upper secondary school does not appear to influence the students' geometrical paradigm directly, because the students' paradigm remains invariant after this transition. However the change in the students' geometric paradigm appears in a higher grade of the upper secondary school, in which this change is not the same. What seems to differentiate regarding the way in which the students' transfer their work to a different type of paradigm is the difference in the teaching of mathematics between grade 11a and grade 11b, regarding the teaching style, but the mathematical content as well.

Consequently, the hypothesis that the students' transition from the lower to the upper secondary school is accompanied by a transition from the mixed geometry GI/ GII to the GII paradigm is not verified. On the contrary it was identified that the students do not reach the GII paradigm in the upper secondary school, but instead they seem to work within a type of paradigm that is similar to the GII paradigm, which still combines some characteristics of the GI paradigm. Kuzniak (2011) actually mentions that the existence of a mixed Geometry (GI / GII) is related to the persistence of the authors of the curriculum on the difference between demonstration and experimental proofs, as both forms live ceaselessly and also seem justifiable. Furthermore, the theoretical set of reference is split and does not allow the assumption that students completely transitioned to Geometry II, due to the lack of an axiomatic horizon.

CHAPTER VI

CONCLUSIONS

Introduction

The last chapter consists of the conclusions of this research study. The conclusions are presented according to the four axes of investigation of the study, emphasizing on answering the research questions that correspond to each axis. In addition, directions for further future research are suggested and implications for the teaching of geometry are provided.

Synoptic description

The synoptic description of the results is compiled in relation to the four main axes of investigation which were set in conducting this research. The first axis of investigation pertained to the examination of the cognitive structure of the geometrical figure apprehension. The main outcome of the verified structure proposed in this study is that, in order to reach the apprehension of geometrical figures all the four different types of apprehension are necessary and important. This is indicated by the high factors loadings of all the four first order factors, which correspond to the four types of apprehension and on the second order factor, which correspond to the geometrical figure apprehension. The contribution of each of the four types of apprehension is necessary in order for the students to apprehend a geometrical figure properly and to be able to use it effectively for the solution of geometrical problems.

The verified structure describing the geometrical figure apprehension remains invariant for both the lower and the upper secondary school students. It is thus important to note that, despite the fact there are differences between the teaching of geometry in the lower and the upper secondary school and the students' cognitive development changes, the formation of the basic structure of the geometrical figure apprehension is not influenced. Consequently, the results show that the students' age or educational level has no effect on the structure of the geometrical figure apprehension.

Regarding the second axis of investigation, which concerned the examination of the relationships between the four types of geometrical figures apprehension, the results showed first of all the interrelations between the different ways of looking at geometrical figures. These interactions were indicated in the students' solutions in the different tasks, in which not only one type of apprehension was mobilized, but other types of apprehensions were necessary as well. However, greater coordination of the cognitive processes corresponding to each type of figural apprehension is displayed by the upper secondary school students, compared to the lower secondary school students

Perception appears to be influencing the mobilization of the operative apprehension. However the recognition of the relevant reconfiguration is determined by particular cognitive factors. In fact operative apprehension is mainly mobilized properly when the necessary reconfiguration can be recognized perceptively. Recognizing the subfigures in a figure through the involvement of the perceptual apprehension sufficiently is crucial for the students to be able to choose the proper reconfiguration of the figure which leads to the solution of a geometrical problem further on. Therefore, the operative apprehension is very important in problem solving, but on the other hand the perceptual apprehension is very often an obstacle for the operative apprehension.

The perceptual apprehension is an obstacle for the discursive apprehension as well, as it overrides the recognition of the necessary relations between the different figural units and hence the perception of a figure can be an obstacle for the recognition of the relevant theorem for the solution of a geometrical proof task. On the contrary, the operative apprehension was found to be very important for the discursive apprehension and the improper mobilization of the operative apprehension, and thus the students' inability to modify a figure is responsible for the students' inability to recognise which theorem is relevant for a proof. In this case the students cannot identify the mathematical properties, because they are not able to recognize the necessary figural units. Therefore the abilities to solve problems by visualization are closely connected to the development of deductive reasoning.

The comparison between the lower and the upper secondary school students' geometrical figure apprehension is the third axis of investigation of this research. In particular the lower and the upper secondary school students' abilities to solve the different geometrical problems corresponding to the four types of geometrical figure apprehension were examined and the differences between the different groups of students were identified. Also, the predominant kinds of geometrical figure apprehension involved for the solution of each kind of geometrical task and at each level of teaching were found.

In comparing the students of the two educational levels, what surfaces is that the students' age and educational level are factors that significantly influence the students' performance in the geometrical figure apprehension tasks. However, no striking changes are observed regarding the students' geometrical figure apprehension through the curriculum from the lower to the upper secondary school. Concerning the difficulty of the tasks, the results indicated that the sequential apprehension tasks and some of the discursive apprehension tasks created more difficulties to the students than the perceptual apprehension tasks and the operative apprehension tasks.

Regarding the role of the different types of figural apprehension in the geometrical problem solving, the results of the perceptual apprehension showed that the students are adept enough at recognizing figures, but perceptual apprehension may also lead them to deficient recognition of figures. The role of the operative apprehension is limited in problem solving. Regarding the discursive apprehension, the students have incomplete knowledge about what is considered a formal proof and what the characteristics of a formal proof are.

The last axis of investigation concerned the students' mistakes and ideas about the geometrical figure apprehension. The students' mistakes in the tasks corresponding to each type of apprehension were identified and interpreted according to the reasons that are responsible for their appearance. The information provided through the task – based interviews revealed the reasons that cause these mistakes and the students' ideas on the geometrical tasks. Based on these results the type of geometric paradigm in which the students' geometric work takes place was identified.

The students' lack of theoretical geometrical knowledge and the insufficient development of comprehension of mathematical reasoning are the main reasons for the students' mistakes in the discursive apprehension tasks. In addition, the intervention of the perceptual apprehension during the solution of the discursive apprehension tasks and thus the students' real reliance on what is recognized in the figure at a glance is responsible for the absence of justification in their answers. Furthermore, the type of the figure and the number of subfigures it includes are factors that influence the recognition of figures.

The students' mistakes in the solutions of the operative apprehension tasks are mostly caused by a wrong perceptual estimation of the area or the perimeter of the figure or to the wrong performance of calculations. These mistakes are linked to perceptual apprehension, which in many cases inhibits the further mobilization of the operative apprehension. From the students' answers it was also extracted that, although modifying a figure is considered a briefer approach for the solution of a geometrical problem, the use of algorithms and calculations are the approaches which are mostly preferred and used by them, due to their teaching experiences. The algorithmic approaches provided the students with more certainty about the correctness of their solution. On the other hand, solutions that did not include the use of any numerical data, but are exclusively based on visualization do not provide the students with the same certainty.

According to the students' answers in the discursive apprehension tasks, the identification of the geometrical paradigm within which their work takes place was conducted. In fact the students' geometrical paradigm remains constant after their transition from the lower to the upper secondary school. However the change in the students' geometric paradigm appears in a higher grade of the upper secondary school. Despite the fact that there is a general progression from the GI paradigm towards the GII paradigm, the students do not ultimately reach the second type of paradigm, but remain between the two types of paradigms. In effect, the students' transition from Geometry I towards Geometry II revealed the existence of a mixed type of geometry. This mixed type of geometry combines characteristics of both paradigms GI and GII. More specifically, two sub-types of the mixed geometry paradigm are identified, which are defined according to the predominance of either the GI or the GII paradigm (GI/GII and GII/GI respectively).

In fact, in grades 9 and 10 the students appear to work within the same geometric paradigm, which is the mixed geometry GI/GII, having more characteristics of the GI paradigm. Grade 11a students also work within the mixed geometry GI/GII, having more characteristics of the GII paradigm compared to grade 9 and grade 10 students. Finally grade 11b students' work is situated within the mixed geometry GII/GI, in which the presence of the characteristics of the GII paradigm is the most dominant, compared to the previous cases.

Implications for teaching and suggestions for further research

The aim of this study was the examination of the different ways the students look at a geometrical figure; in other words, it set out to examine the four different types of geometrical figure apprehension. On this account, the focus was on defining the structure of the geometrical figure apprehension and tracing the relations between these types of apprehension as well as the cognitive processes that are involved during the mobilization of these different types of apprehension. A main concern was also to identify the type of geometrical paradigm in which the students' geometrical work takes place.

The results of this study have direct implications for teaching. In fact, this study revealed that solving geometrical tasks requires interactions between the four types of apprehension in most cases, but the realization of the distinction between these types of apprehension is also important. Thus, the basic problem for the teaching of geometry at lower and upper secondary schools is how to get the pupils to manage the coordination between these four different types of apprehension. Therefore, reflection on a new approach for introducing geometry in primary and secondary education is necessary; the principle of which would be that the awareness of the different ways of looking at figures is prior to the knowledge of the classical basic figures. Thereafter, the focus of teaching geometry should be firstly turned on the differentiation among the visualization processes, the reasoning processes and the construction processes, because the coordination of these kinds of processes can really occur only after this focus on their differentiation.

Based on Duval' s (1995) suggestion, according to whom the mobilization of the perceptual apprehension in combination with at least another type of apprehension is basic for allowing a drawing to function as a geometrical figure, the role of perception in the apprehension of geometrical figures is highlighted. However the mobilization of only the perceptual apprehension of figures and the strong influence of perception in a way that captures the students' glance into a steady vision of geometrical figures, according to the results of this study, seems to inhibit the further involvement of the remaining types of apprehension. Mathematical perception is not simple, as it overlaps the rest of the different apprehensions. Consequently strengthening the students' perceptual recognition abilities is an important point, but on the other hand these abilities must be developed in a way that will not trap the students into just a perceptual approach of figures.

In order to improve the students' geometrical knowledge the teaching of geometry should include teaching practices focusing on sharpening the students' perceptual apprehension. This could be achieved by using tasks in which students will be asked to decompose a whole given geometrical figure and recognize different subfigures in it. Such tasks promote an in-depth perceptual understanding, because they do not only favor the simple perceptual recognition of whole figures, but the recognition of figures in a divided figure, though this must be done at first glance. In this sense, the students' perceptual apprehension must be developed in a way that will function as a basis, and not an obstacle, to enable the students to mobilize the proper type of apprehension for reaching a solution to a geometrical situation.

The perceptual recognition of figures can be the basis for further recognition of the relevant reconfiguration in a geometrical figure, in relation to the operative apprehension, but also for identifying some parts of the figure that are related to the use of theoretical geometrical knowledge for reaching a proof, through the discursive apprehension of figures. The discursive apprehension and the operative apprehension of figures very often obscure the perceptual apprehension. Special and separate learning of operative apprehension, as well as of discursive apprehension and sequential apprehension are required, as a mathematical way of looking at figures only comes about the coordination between the separate processes of figural apprehension.

Consequently, apart from enhancing the students' perceptual apprehension, another crucial point to take into account as regards the teaching of geometry is promoting the involvement of the operative apprehension, though the use of the mereologic modification of figures for solving geometrical problems. The development of the students' abilities in reconfiguring a given figure appears to be very important, because visualization can compensate for a lack of geometrical knowledge, which does not allow students to solve a geometrical problem. The mechanistic knowledge and use of algorithms, formulas or theorems and axioms, which are usually forgotten by students if they are not used for a long period of time, stops students from reasoning properly through discursive apprehension. On the contrary through the operative apprehension the figure becomes a heuristic tool and students can reach a solution to a geometrical task by modifying a given figure. The processes of visualization are cognitively independent from any concept acquisition, thus the operative apprehension allows the solution of problems without explicitly referring to properties and reasoning, as it only constitutes a qualitative calculation on 2D/2D figural units, under the assumption of their conservation throughout reconfiguration processes. Therefore, reconfiguration simultaneously uses and goes against the perceptual recognition of shapes.

In order to analyze visualization in geometry and the everlasting conflict between the perceptual way and the mathematical way of looking at figures, the focus should be mainly turned to seeing independently from any magnitude consideration. To this end, teaching geometry should be focused on making students able to look at figures in a flexible way, as it is possible to initiate the development of skills which enhance the mobility of seeing. Thus, tasks on discriminating various figural units must be separated from the ones on magnitudes. In addition, tasks of production of reconfiguration in a figure must be given to the students, because what is ultimately important for solving problems are the tasks on the production of reconfigurations. It is well known that such tasks of production are more difficult than multiple – choice tasks that were given in this research for examining the students' operative apprehension. Such tasks must be introduced in the teaching of geometry from primary school, in order to develop the students' operative apprehension, and not only use tasks of perceptual apprehension. The development of the operative apprehension must be continued until the lower secondary school and at this level, specific tasks involving discursive apprehension must also be proposed, in order for the students to use properties and theorems in a mathematical way.

Furthermore, it has also become evident that the students' experiences regarding modifications on a geometrical figure are very limited. The students' skills are not developed in this respect; therefore they face difficulties when they need to apply such skills. Thereafter, in the future, research must focus mainly on the conditions of development of the operative apprehension for the learning of geometry. Nonetheless, the results from the students' interviews provided some evidence regarding the influence of learning experiences and of an appropriate teaching intervention for the development of such skills. This evidence could be further researched, in order to design appropriate teaching interventions, which will help the students acquire more experiences and develop their abilities in modifying geometrical figures.

Based on the above, it is essential to focus on each different way of looking at geometrical figures. Thus, emphasis should be given on all the aspects of geometrical figure apprehension concerning the teaching of geometry, in order to improve students' abilities regarding the way they look at and use a geometrical figure. Therefore, there is a need for identifying the cognitive processes and the types of apprehension that students mobilize during the resolution of geometrical tasks and examine whether this selection leads to a proper way of looking at geometrical figures. There are many factors that can inhibit or encourage the discrimination of these visual operations, which can be studied experimentally. Extensive and detailed investigations on the cognitive processes involved in the mathematical way of thinking are necessary for understanding the source of difficulties in learning geometry.

Through the descriptive analysis of the students' answers from both the mathematical and the cognitive point of view the importance of the cognitive analysis of the tasks was highlighted. A contradiction between the results of the analysis of the students' answers from the mathematical point of view and from the cognitive point of view indicated that the examination of the students' answers based only on whether they are mathematically correct or not does not provide the researcher with the necessary information for formulating a complete description of the students' cognitive behavior. On the other hand, the information on the students' cognitive processes during the solution of any problem can be more effectively approached through the cognitive analysis of the tasks. Consequently, a more accurate and general description of the students' cognitive processes can be achieved through the combination of the two ways of examining the students' answers, as it is not sufficient to focus only on the mathematical correctness of the students' answers, in order to extract some accurate results and interpretations. In this way we will be able to identify which of the students' strategies should be enhanced or which strategies create difficulties to them and lead them to mistakes. Concluding, in order to get to know efficient ways of teaching geometry in secondary schools, more research on the process of the development and the learning of visualization and reasoning are still needed.

REFERENCES

Adams, R. J., & Khoo, S. T. (1996). *Quest: The interactive test analysis system*. Camberwell, Victoria: ACER.

Alspaugh, J.W (1998). Achievement Loss Associated With the Transition to Middle School and High School. *The Journal of Educational Research*, *92* (1), 20 – 25.

Anderman, E., & Midgley, C. (1997). Changes in achievement goal orientations, perceived academic competence, and grades across the transition to middle-level schools. *Contemporary Educational Psychology, 22,* 269–298.

Anderson, L., Jacobs, J., Schramm, S., & Splittgerber, F. (2000). School transitions: beginning of the end or a new beginning? *International Journal of Educational Research, 33,* 325 – 339.

Andrich, D. (1988). A general form of Rasch's extended logistic model for partial credit scoring. *Applied Measurement in Education, 1*, 363–378.

Andrich, D. (1978). A rating formulation for ordered response categories. *Psychometrika, 43*, 561-573.

Battista, M. T. (1999). The importance of spatial structuring in geometric reasoning. *Teaching Children Mathematics, 6*(3), 170-177.

Bentler, M. P. (1995). *EQS Structural equations program manual*. Encino, CA, Multivariate Software Inc.

Bentler, P. M. (1988). Causal modeling via structural equation systems. In J. R. Nesselroade & R. B. Cattell (Eds.), *Handbook of multivariate experimental psychology* (2nd ed., pp. 317-335). New York: Plenum.

Bishop, A. (1983). Space and geometry. In R. Lesh and M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 175-203). New York: Academic Press.

Bodin, A., Coutourier, R., & Gras, R. (2000). *CHIC : Classification Hiérarchique Implicative et Cohésive*-*Version sous Windows – CHIC 1.2.* Rennes : Association pour la Recherche en Didactique des Mathématiques.

Brändström, A. (2005). Differentiated Tasks in Mathematics Textbooks: An analysis of the levels of difficulty. Licentiate Thesis, Luleå University of Technology – Department of Mathematics, Sweden.

Brown, T.A. (2006). *Confirmatory Factor Analysis for Applied Research*. New York: Guilford Press.

Brousseau, G. (1997). *Theory of didactical situations in mathematics 1970-1990*. Dordrecht: Kluwer Academic Publishers.

Bulf, C. (2009). Analyses en Termes d'Espaces de Travail Géométrique sur l' Enseignement Français de la Symétrie en Début de Collège. In A. Gagatsis, A. Kuzniak, E. Deliyianni, & L. Vivier, (Eds), *Cyprus and France Research in Mathematics Education* (pp. 51-70). Lefkosia: University of Cyprus.

Byrne, B. M. (1994). *Structural Equation Modeling with EQS and EQS/Windows: Basic concepts, applications and programming.* Thousand Oaks, CA: SAGE Publications, Inc.

Carroll, W. (1998). Middle school students' reasoning about geometric situations. *Mathematics Teaching in the Middle School, 3,* 398 – 403.

Clements, D. (2001). Teaching and Learning Geometry. In J. Kilpatrick, W.G. Martin, & D. Schifter, (Eds), *Research Companion to the NCTM Standards for Mathematics.* Reston, VA: NCTM.

Clements, D. & Battista, M. (1992). Geometry and Spatial Reasoning. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, NewYork: Macmillan Publishing Co.

Commission for educational reform (2005). Report of the commission for the educational reform. Cyprus Republic – Ministry of education and civilization. Retrieved on 14th October, 2006 from http://www.paideia.org.cy/upload/ekthesi_epitropis.pdf (In Greek).

Cronbach, L. J. (1990). *Essentials of Psychological testing* (3rd ed.). New York: Harper and Row.

De Vjeeschouwer, M. & Gueudet, G. (2011). Secondary-tertiary transition and evolutions of didactic contract: the example of duality in linear algebra, *Proceedings of the 7th congress of the European Society for Research in Mathematics Education*. Rzeszów, Poland: CERME 7.

Deliyianni, E., Elia, I., Gagatsis, A., Monoyiou, A., & Panaoura, A. (2010). A theoretical model of students' geometrical figure understanding. *Proceedings of the 6th*

Conference of the European Society for Research in Mathematics Education: Working Group 5, Geometrical Thinking (pp.696-705), Lyon, France: ERME.

Demetriou, A. (2004). Mind, intelligence, and development: A general cognitive, differential, and developmental theory of the mind. In A. Demetriou & A. Raftopoulos (Eds.), *Developmental change: Theories, models and measurement* (pp. 21-73). Cambridge: Cambridge University Press.

Demetriou, A. (1998). Cognitive development. In A. Demetriou, W. Doise, & C.F.M. van Lieshout (Eds.), *Life-Span Developmental Psychology* (pp. 179-270). New York: John Wiley & Sons Ltd.

Duval, R. (2013, in press). The first crucial point in geometry learning: visualization. *Mediterranean Journal for Research in Mathematics education.*

Duval, R. (2012). Seminar at the University of Cyprus: *What kind of activity is required for understanding and solving problem in geometry?* Nicosia: Cyprus.

Duval, R. (2011). Why figures cannot help students to see and understand in geometry? Analysis of the role and the cognitive functioning of visualization. *Abstract booklet of Symposium Mathematics Education Research at the University of Cyprus and Tel Aviv University* (pp. 22-23). Nicosia: Cyprus.

Duval, R. (2008). Eight problems for a semiotic approach in mathematics education. In L. Radford, G. Schubring, & F. Seeger, F. (eds.), *Semiotics in mathematics education: epistemology, historicity, classroom, and culture* (pp. 39 –62). Rotterdam, NL, Sense.

Duval R. (2007). Cognitive functioning and the understanding of mathematical processes of proof. *Theorems in schools* (pp. 137-161). Rotterdam/Tapei: Sense Publishers.

Duval, R. (2006). The cognitive analysis of problems of comprehension in the learning of mathematics. *Educational Studies in Mathematics, 61,* 103–131.

Duval R. (2005) Les conditions cognitives de l'apprentissage de la géométrie : développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements. *Annales de Didactique et de Sciences Cognitives, 10,* 5-53.

Duval R. (2004). A crucial issue in mathematics education: The ability to change representation register. *Proceedings of ICME10.*

Duval, R. (2000). Basic issues for research in mathematics education.. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th Conference of PME* (vol. 1. pp. 55–69). Hiroshima: Nishiki Print Co. Ltd.

Duval, R. (1999). Representation, vision and visualization: cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st Annual Meeting of the North American, Chapter of the International Group for the Psychology of Mathematics Education* (vol. I. pp. 3-26). Morelos: México.

Duval, R. (1998). Geometry from a cognitive point a view. In C. Mammana & V. Villani (eds.), *Perspectives on the Teaching of Geometry for the 21st Century* (pp. 37-52). Boston: Kluwer Academic Publishers Dordrecht.

Duval, R. (1998-2001). Figures' Representational Function in Geometry and Figure's Multiple and Parallel Entries, Working Group on Representations and Mathematics Visualization, *21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Cuernavaca: Mexico.

Duval, R. (1995). Geometrical Pictures: Kinds of representation and specific processes. In R. Sutherland & J. Mason (eds.), *Exploiting mental imagery with computers in mathematical education* (pp. 142- 157). Berlin, Springer.

Duval, R. (1994). Les différents fonctionnements d'une figure dans une démarche géométrique. *REPERE – IREM, 17,* 121 – 138.

Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la demonstration. *Educational Studies in Mathematics, 22* (3), 233–261.

Duval, R. (1988). Approche cognitive des problèmes de géométrie en termes de congruence. *Annales de Didactique et de Sciences Cognitives, 1,* 57 – 74.

Duval, R. & Godin, M. (2005). Les changements de regard nécessaires sur les figures. *Grand N,* 76, 7 – 27.

Elia, I., Gagatsis, A., Deliyianni, E., Monoyiou, A., & Michael. P. (2009). A structural model of Primary school students' operative apprehension of the geometrical figure. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, C. (Eds.). *Proceedings of the 33rd*

Conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp.1-9). Thessaloniki, Greece: PME.

Engelhard, G. (1994). Examining rater errors in the assessment of written composition with a many-faceted Rasch model. *Journal of Educational Measurement*, 31(2), 93-112.

Engelhard, G. (1992). The measurement of writing competence with a many-faceted Rasch model. *Applied Measurement in Education*, *5* (3), 171-191.

Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior*, *17*(1), 105-121.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V., & Empson, S. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, *27*, 403-434.

Fischbein, E. (1993). The Theory of Figural Concepts. *Educational Studies in Mathematics, 24*(2), 139-162.

Fischbein, E. (1963). *Conceptele Figurale* (in Rumanian). Bucuresti: Editura Academiei RPR.

Fischbein, E., & Nachlieli, T. (1998). Concepts and figures in geometrical reasoning. *International Journal of Science Education*, *20*(10), 1193- 1211.

Flyvbjerg, B. (2006). Five Misunderstandings about Case Study Research. *Qualitative Inquiry, 12* (2), 219-245.

Gras, R. (1996). *L'implication statistique. Collection associée à Recherches en Didactique des Mathématiques*. Grenoble: La Pensée Sauvage.

Gras, R. (1992). Data analysis: a method for the processing of didactic questions, Research in Didactics of Mathematics. *Selected papers for ICME 7*. Grenoble : La Pensée Sauvage.

Gras, R., Peter, P., Briand, H., & Philippé, J. (1997). Implicative Statistical Analysis. In C. Hayashi, N. Ohsumi, N. Yajima, Y. Tanaka, H. Bock, Y. Baba (Eds.), *Proceedings of the 5th Conference of the International Federation of Classification Societies* (vol. 2, pp. 412-419). Tokyo, Berlin, Heidelberg, New York: Springer-Verlag.

Gray, E., Pinto, M., Pitta, D., & Tall, D. (1999). Knowledge construction and diverging thinking in elementary and advanced mathematics. *Educational Studies in Mathematics, 38*, 111-133.

Gutiérrez, A., Kuzniak, A., & Straesser, R. (2005). Research on geometrical thinking. In M. Bosch (Ed.), *Proceedings of the IV Congress of the European Society for Research in Mathematics Education CERME 4* (pp.725-726). Sant Feliu de Guixols: CERME.

Hejny, M. (2003). Understanding and structure. In M. Mariotti (Ed.), *Proceedings of CERME3*. Bellaria, Italy. Retrieved February 24, 2010, from http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft

Houdement, C. (2007). Geometrical Working Space, a tool for Comparison. *Proceedings of CERME 5* (pp. 972–982). Larnaka, Cyprus.

Houdement, C., & Kuzniak, A. (2003). Elementary geometry split into different geometrical paradigms. In M. Mariotti (Ed.), *Proceedings of CERME 3*. Bellaria: Italy. Retrieved February 24, 2010, from http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft.

Hsieh, H.F., & Shannon, S.E. (2005). Three approaches to qualitative content analysis. *Qualitative Health Research, 15*(9), 1277-1288.

Jones, K. (2000). Critical Issues in the Design of the Geometry Curriculum. In B. Barton (Ed), *Readings in Mathematics Education* (pp. 75-90). Auckland, New Zealand: University of Auckland.

Jones, K. (1998). Theoretical Frameworks for the Learning of Geometrical Reasoning. *Proceedings of the British Society for Research into Learning Mathematics, 18* (1&2), 29-34.

Karabatsos, G. (2000). A critique of Rasch residual fit statistics. *Journal of Applied Measurement, 1*, 152-176

Klauer, K. C. (1995). The assessment of person fit. In G. H. Fischer & I. W. Molenaar (Eds.), *Rasch models: Foundations, recent developments, and applications* (pp. 97-110). New York: Academic Press.

Kline, R. B. (1998). *Principles and practice of structural equation modeling*. New York: Guilford Press.

Kunimune, S., Fujita, T., & Jones, K. (2008). "Why do we have to prove this?" Fostering students' understanding of 'proof' in geometry in lower secondary school. Retrieved March 28, 2009 from http://ocs.library.utoronto.ca/index.php/icmi/8/paper/view/877/51.

Kurina, F. (2003). Geometry – The resource of opportunities, in M. Mariotti (Ed.), *Proceedings of CERME 3*, Bellaria, Italy, Retrieved March 10, 2009 from http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft.

Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren.* Chicago: University of Chicago Press.

Kuzniak, A. (2012). Understanding the Nature of the Geometric Work Through its Development and its Transformations. *Proceedings of the 12th International Congress on Mathematical Education.* Seoul, Korea.

Kuzniak, A. (2011). Geometric Work at the End of the Compulsory Education. *Proceedings of Cerme 7* (pp. 618-628). Rzeszow, Poland: CERME.

Kuzniak, A. (2009). Un essai sur la nature du travail géométrique en fin de la scolarité obligatoire en France. In A. Gagatsis, A. Kuzniak, E. Deliyianni, & L. Vivier, L. (Eds), *Cyprus and France Research in Mathematics Education* (pp. 71-90). Lefkosia: University of Cyprus.

Kuzniak, A. (2008). Personal geometrical working space: a didactic and statistical approach. In Gras, R., Suzuki, E., Guillet, F., Spagnolo, F. (Eds), *Statistical implicative analysis: theory and applications studies in computational intelligence*. Heidelberg: Springer.

Kuzniak, A. (2006). Paradigmes et espaces de travail géométriques. Éléments d'un cadre théorique pour l'enseignement et la formation des enseignants en géométrie. *Canadian Journal of Science, Mathematics and Technology Education, 6* (2), 167-187.

Kuzniak, A. & Rauscher, J.C. (2011). How do teachers' approaches to geometric work relate to geometry students' learning difficulties? *Educational Studies in Mathematics, 77* (1), 129–147.

Kuzniak, A., & Vivier, L. (2009). A French Look on the Greek Geometrical Working Space at Secondary School Level. *Proceedings of CERME 6* (pp. 686-695). Lyon: France.

Lamprianou, I., & Boyle, B. (2004). Accuracy of measurement in the context of mathematics National Curriculum tests in England for Ethnic Minority pupils and pupils who speak English as an additional language. *Journal of Educational Measurement, 41* (3), 239-260.

Laborde, C. (1994). Enseigner la geometrie: permanences et revolutions. *Bulletin APMEP, 396*, 523-548.

Leikin, R. & Lev, M. (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo, (Eds.). *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 161-168). Seoul: PME.

Lemonidis, C. (1997). A Few Remarks Regarding the Teaching of Geometry through a Theoretical Analysis of the Geometrical Figure. *Proceedings of the 2nd World Congress of Non-linear Analysis*, *30* (4), pp. 2087-2095.

Lerman, I.C. (1991). Foundations of the likelihood linkage analysis (LLA) classification method. *Applied and Stochastic Models and Data Analysis,* 7, 63 – 76.

Lunz, M. E., Wright, B. D., & Linacre, J. M. (1990). Measuring the impact of judge severity on examination scores. *Applied Measurement in Education*, 3(4), 331-345.

Lerman, I.C. (1981). *Classification et analyse ordinale des données*. Paris: Dunod.

Mason, M. (1998). The Van Hiele levels of geometric understanding. *Professional handbook for teachers, geometry: Explorations and applications.* Boston, McDougal Inc.

Mariotti, M. A. (1995). Images and Concepts in Geometrical Reasoning. In R. Sutherland & J. Mason (Eds), *Exploiting Mental Imagery with Computers in Mathematics Education*. Berlin: Springer.

Mariotti, M. A. (1999). Geometry: dynamic intuition and theory. Retrieved February 14, 2009 from http://telearn.archives-ouvertes.fr/docs/00/19/04/93/PDF/Mariotti-m-1999.pdf.

Mayring, P. (2000). Qualitative content analysis. *Forum: Qualitative Social Research, 1*(2). Retrieved July 28, 2008, from http://217.160.35.246/fqs-texte/2-00/200mayring-e.pdf.

McGee, M. G. (1979). Human spatial abilities: Psychometric studies and environmental, genetic, hormonal, and neurological influences. *Psychological Bulletin, 86,* 889-918.

Mesquita, A. L. (1998). On conceptual obstacles linked with external representation in geometry. *Journal of mathematical behavior, 17*(2), 183-195.

Mesquita, A. L. (1996). On the utilization of encoding procedures on the treatment of geometrical problems. In L. Puig & A. Gutierrez (Eds), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (vol. III, pp. 399- 406). Valencia: PME.

Michael, P., Elia, I., Gagatsis, A. & Kalogirou, P. (2010). Primary school students' operative apprehension of geometrical figures. *Quaderni di Ricerca in Didattica Mathematica,Supplemento n.1,* N.2, 47-62.

Middleton, M., Kaplan, A., & Midgley, C. (2004). The change in middle school students' achievement goals in mathematics over time. *Social Psychology of Education, 7,* 289–311.

Midgley, C. (1993). Motivation and middle level schools. *Advances in motivation and achievement, 8,* 217–274.

Midgley, C, Anderman, E., & Hicks, L. (1995). Differences between elementary and middle school teachers and students: A goal theory approach. *Journal of Early Adolescence, 75* (1), 90-113.

Mistretta, R.M. (2000). Enhancing Geometrical Thinking. *Adolence, 35* (138), 365 – 379.

Mullins, E. R., & Irvin, J. L. (2000). Transition into middle school. *Middle School Journal, 31*(3), 57-60.

National Research Council, (2004). *Engaging schools: Fostering high school students' motivation to learn.* Washington DC: National Academies Press.

NCTM (2000). *Principles and Standards for School Mathematics.* Reston, Va: NCTM.

Panagidou, E., Tsianni, M., & Gagatsis, A. (2004). The role of didactical variables and perception of shape in the finding of the perimeter. In A. Gagatsis, A. Evangelidou, I. Elia & P. Spyrou (Eds), *Representations and Learning in Mathematics, Vol. 2: Geometry* (pp.37-50). Nicosia: Intercollege Press (in Greek).

Panaoura, G., & Gagatsis, A. (2008). Spatial abilities in relation to performance in items involving net-representations of geometrical solids. In A. Gagatsis (Ed.), *Research in Mathematics Education* (pp. 101-114). Nicosia: University of Cyprus (In greek).

Parzysz, B (1991). Representation of space and students' conceptions at high school level. *Educational Studies in Mathematics, 22*, 575-593.

Parzysz, B (1988). Knowing vs Seeing, Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics*, *19* (1), 79-92.

Patton, M.Q. (2002). *Qualitative Research and Evaluation Methods.* Thousand Oaks,

CA: Sage.

Pepin, B., & Haggarty, L. (2002). An investigation of mathematics textbooks and their use in English, French and German classrooms: Who gets an opportunity to learn what? *British Educational Research Journal, 28*, (4), 567-590.

Schoenfeld, A. H. (1986). *On having and using Geometric Knowledge in Conceptual and Procedural Knowledge the case of mathematics* (Ed. J. Hiebert). Hillsdale NJ, Erlbaum.

Seidman, E., Allen, L., Aber, J., Mitchell, C, & Feinman, J. (1994). The impact of school transitions in early adolescence on the self-system and perceived social context of poor urban youth. *Child Development, 65*, 507-522.

Seidel, J. (1995). *The Ethnograph v4.0: A User's Guide.* Amherst, MA: Qualis Research Associates.

Sharon M., McCrone, S. & Martin. T.M. (2004). Assessing high school students' understanding of geometric proof. *Canadian Journal of Science, Mathematics and Technology Education*, *4* (2), 223-242.

Smith, R. M. (2000). Fit analysis in latent trait measurement models. *Journal of Applied Measurement, 1*, 199-218.

Smith, R. M. (1991). The distributional properties of Rasch item fit statistics. *Educational and Psychological Measurement*, *51,* 541-565.

Smith, H.W. (1975). *Strategies of Social Research: The Methodological Imagination.* Englewood Cliffs, NJ: Prentice-Hall.

Suhr, D.D. (2006). Exploratory or Confirmatory Factor Analysis? *Proceedings of the Thirty-first Annual SAS Users Group International Conference*. San Francisco, California.

Symeou – Mai, T. (2008). Content analysis of mathematics textbooks in Elementary and Secondary Education: The case of functions. In A.Gagatsis, P. Damianou & A.Philippou (Eds.), *Proceedings of the 10th Conference of Mathematics Education and Science* (pp. 247 – 266). Cyprus: Cypriot Mathematics Committee.

Torrance, E. P. (1974). *The Torrance Tests of Creative Thinking.* Bensenville, IL: Scholastic Testing Servise, INC.

Van Hiele, P. (1986). *Structure and Insight: A Theory of Mathematics Education*. Orlando, FL: Academic Press.

Van Hiele, P. (1959). *Development and learning process: A study of some aspects of Piaget's psychology in relation with the didactics of mathematics*. Utrecht: University of Utrecht.

Vitz, P.C & Todd, T.C. (1971). A model of the perception of simple geometric figures. *Psychological Review, 78* (3), 207-228.

Vourgias, Ch., Peskias, A., Chrysostomou-Vourgia, S., & Gagatsis, A. (2004). The perceptual ability of geometrical shapes. In A. Gagatsis, A. Philippou, D. Pitta & P. Spyrou (Eds.), *Proceedings of the 6th Cypriot Conference on Mathematics Education and 3rd Symposium of Astronautics and Space* (pp. 429-439). Nicosia: Cyprus Mathematical Society, Department of Education-University of Cyprus (in Greek).

Wright, B. D., & Mok, M. (2000). Rasch models overview. *Journal of Applied Measurement*, *1,* 83-106.

Wright, B. D., & Masters, G. N. (1982). *Rating scale analysis*. Chicago: MESA Press.

Wright, B. D., & Stone, M. H. (1979). *Best test design.* Chicago: MESA Press.

Yakimanskaya, I.S. (1991). The development of spatial thinking in schoolchildren. *Soviet studies in Mathematics Education.* N.C.T.M.: Reston, USA.

Zhang, Y. & Wildemuth, B.M. (2009). Qualitative analysis of content. In B. Wildemuth (Ed.), *Applications of Social Research Methods to Questions in Information and Library*. Portland: Book News.

APPENDIX 1

THE INSTRUMENT OF THE STUDY

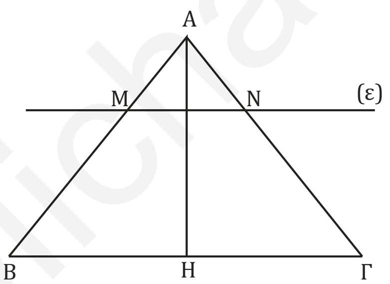
ΔΟΚΙΜΙΟ ΓΕΩΜΕΤΡΙΑΣ Α1

ΣΧΟΛΕΙΟ:…………………………………………………………………….…

ΤΑΞΗ: ……………… ΤΜΗΜΑ: ……..…… ΗΜΕΡΟΜΗΝΙΑ: ………………..

ΟΝΟΜΑΤΕΠΩΝΥΜΟ ΜΑΘΗΤΗ: ……..………………………….…………

Ερώτηση 1. Στο πιο κάτω σχήμα δίνονται ότι ΑΒ = ΑΓ,(ε) // ΒΓ και ΑΗ ⊥ ΒΓ. Να συγκρίνετε τα ευθύγραμμα τμήματα ΜΗ και ΝΗ.



α) Τα τμήματα ΜΗ και ΝΗ είναι ίσα

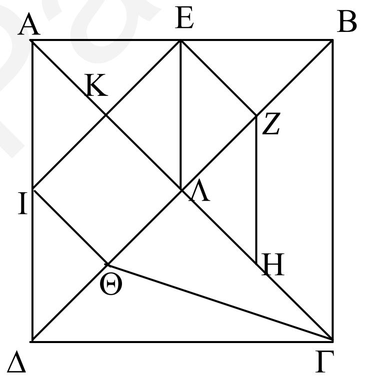
β) Το τμήμα ΜΗ είναι μεγαλύτερο από το τμήμα ΝΗ

γ) Το τμήμα ΝΗ είναι μεγαλύτερο από το τμήμα ΜΗ

δ) Δεν μπορεί να προσδιοριστεί

Εξήγησε πώς βρήκες την απάντηση.

Ερώτηση 2. Συμπλήρωσε τις παρακάτω φράσεις. Χρησιμοποίησε τις λέξεις που νομίζεις ότι ταιριάζουν καλύτερα.



Το σχήμα ΚΕΖΛ είναι ………………………….

Το σχήμα ΙΕΖΘ είναι ………………………….

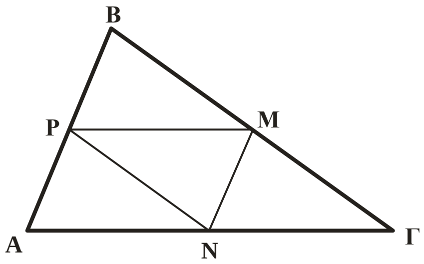
Το σχήμα ΕΖΗΛ είναι ………………………….

Το σχήμα ΙΚΓΘ είναι …………………………..

Το σχήμα ΛΓΘ είναι …………………………...

Το σχήμα ΒΕΛ είναι …………………………...

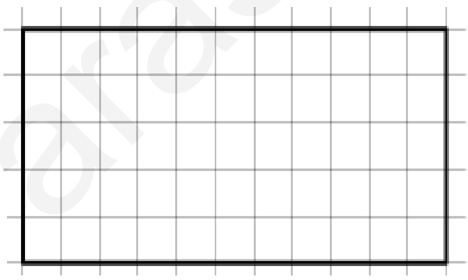
Ερώτηση 3. Τα σημεία Μ, Ν και Ρ είναι τα μέσα των πλευρών του τριγώνου ΑΒΓ. Να δείξεις ότι τα τετράπλευρα ΑΡΜΝ, ΒΜΝΡ, και ΓΝΡΜ είναι παραλληλόγραμμα.



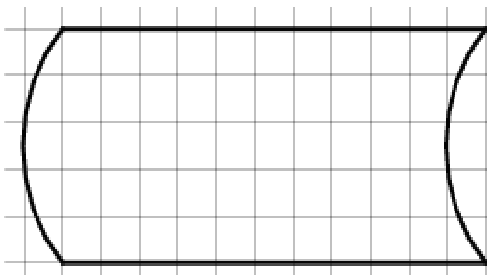
Ερώτηση 4. Σχεδίασε τόξο ΑΒ με κέντρο Γ, ίσο με το τόξο ΜΝ με κέντρο το Ο.

Εξήγησε την κατασκευή σου.

Ερώτηση 5. Υπογράμμισε τη σωστή πρόταση.



Σχήμα 1



Σχήμα 2

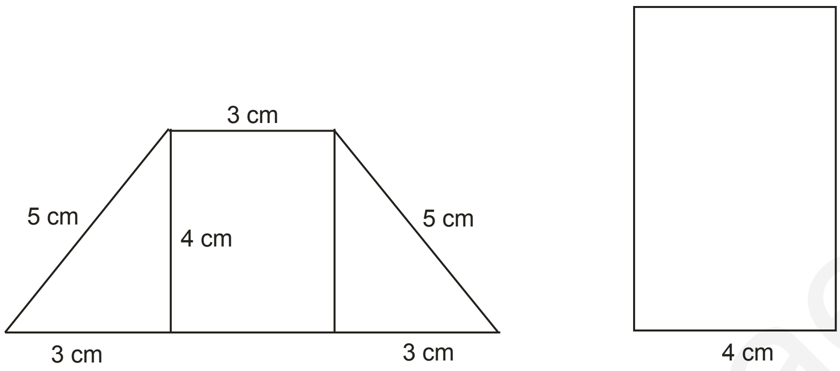
α) Το Σχήμα 1 έχει ίσο εμβαδόν με το Σχήμα 2.

β) Το Σχήμα 1 έχει μικρότερο εμβαδόν από το Σχήμα 2.

γ) Το Σχήμα 1 έχει μεγαλύτερο εμβαδόν από το Σχήμα 2.

Εξήγησε πώς βρήκες την απάντηση.

Ερώτηση 6. Το τραπέζιο και το ορθογώνιο είναι ισοδύναμα. Να βρείτε το μήκος της πλευράς του ορθογωνίου.



Επεξήγησε τον τρόπο που εργάστηκες.

Ερώτηση 7. Διάβασε τις ακόλουθες εξηγήσεις των τριών μαθητών, οι οποίοι εξηγούν γιατί το άθροισμα των εσωτερικών γωνιών ενός τριγώνου είναι 180 μοίρες.

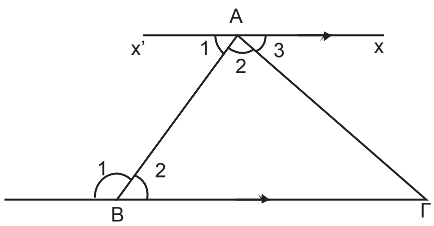
Μαθητής Α: *Μέτρησα και τις τρεις γωνιές και είναι 50°, 53° και 77°. 50°+53°+77°=180°. Άρα, το άθροισμα (των γωνιών ενός τριγώνου) είναι 180°.*

- Αποδέχεσαι την εξήγηση του μαθητή Α ως απόδειξη; **Ναι/Όχι …………**

Μαθητής Β: *Σχημάτισα ένα τρίγωνο και έκοψα κάθε γωνία και τις έβαλα μαζί. Σχημάτιζαν μια ευθεία γραμμή. Άρα, το άθροισμα τους είναι 180°.*

- Αποδέχεσαι την εξήγηση του μαθητή Β ως απόδειξη; **Ναι/Όχι …………..**

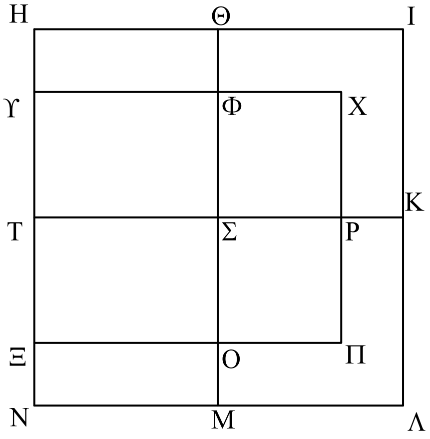
Μαθητής Γ: *Επεξήγηση με τη χρήση δύο παράλληλων ευθειών.*



$xx' \parallel \mathrm{B\Gamma} \Rightarrow \hat{A}_1 = \hat{B}_2 \quad \text{και} \quad \hat{A}_3 = \hat{\Gamma} \Rightarrow \hat{A}_1 + \hat{A}_2 + \hat{A}_3 = 180° \Rightarrow \hat{B}_2 + \hat{A}_2 + \hat{\Gamma} = 180°$

Αποδέχεσαι την εξήγηση του μαθητή Γ ως απόδειξη; **Ναι / Όχι**

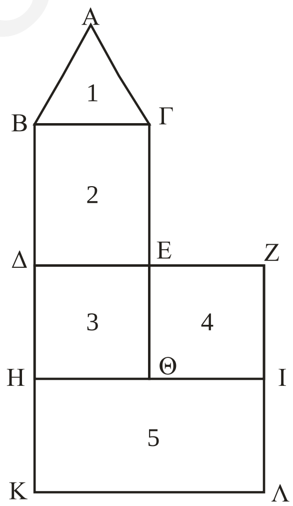
Ερώτηση 8. Μελέτησε προσεκτικά το παρακάτω σχήμα.



Α. Πόσα τετράγωνα βλέπεις στο σχήμα;

Β. Ονόμασέ τα.

Ερώτηση 9. Στο πιο κάτω σχήμα το σχήμα 1 είναι ισόπλευρο τρίγωνο, το σχήμα 2 είναι ορθογώνιο παραλληλόγραμμο, τα σχήματα 3 και 4 είναι τετράγωνα και το σχήμα που σχηματίζεται από τα 3,4 και 5 είναι τετράγωνο. Να δείξετε ότι τα ευθύγραμμα τμήματα ΑΓ, ΘΙ και ΙΛ είναι ίσα.



Εξήγησε πώς βρήκες την απάντηση.

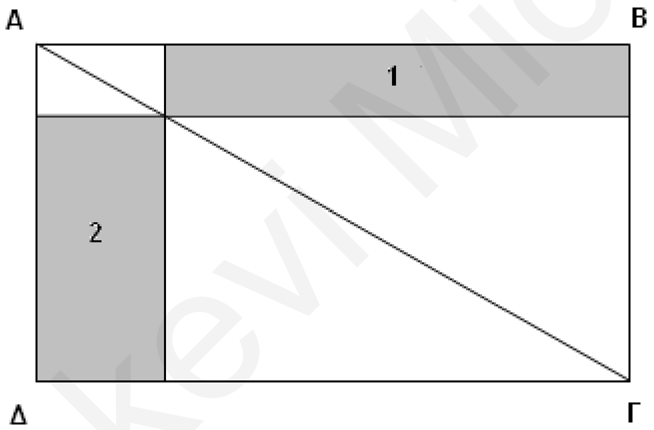
ΔΟΚΙΜΙΟ ΓΕΩΜΕΤΡΙΑΣ Α2

ΣΧΟΛΕΙΟ:……………………………………………………………………….…

ΤΑΞΗ: ……………… ΤΜΗΜΑ: ……..…… ΗΜΕΡΟΜΗΝΙΑ: ………………..

ΟΝΟΜΑΤΕΠΩΝΥΜΟ ΜΑΘΗΤΗ: ……..…………………………….…………

Ερώτηση 1. Στην πιο κάτω εικόνα, το ΑΒΓΔ είναι ορθογώνιο. Παρατήρησε τα σκιασμένα ορθογώνια 1 και 2. Βάλε σε κύκλο τη σωστή πρόταση και αιτιολόγησε την απάντησή σου



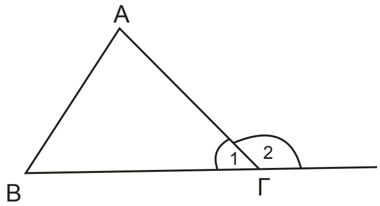
α. Το ορθογώνιο 1 έχει μεγαλύτερο εμβαδόν από το ορθογώνιο 2.

β. Το ορθογώνιο 1 έχει ίσο εμβαδόν με το ορθογώνιο 2.

γ. Το ορθογώνιο 1 έχει μικρότερο εμβαδόν από το ορθογώνιο 2.

Ερώτηση 2. Διάβασε τις ακόλουθες εξηγήσεις των τριών μαθητών, οι οποίοι εξηγούν γιατί η εξωτερική γωνία τριγώνου ισούται με το άθροισμα των απέναντι των γωνιών του τριγώνου.

Μαθητής Α: *Μέτρησα και τις τρεις γωνιές Α και Β και είναι 54°, 80° αντίστοιχα.Η εξωτερική γωνία της Γ είναι 134°. Έτσι έχουμε ότι 54°+80°=134°. Άρα, η εξωτερική γωνία ενός τριγώνου ισούται με το άθροισμα των δύο απέναντι εσωτερικών γωνιών του.*

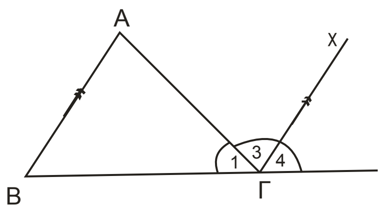


- Αποδέχεσαι την εξήγηση του μαθητή Α ως απόδειξη; **Ναι/Όχι …………**

Μαθητής Β: *Σχημάτισα ένα τρίγωνο και έκοψα τις γωνίες Α και Β και τις έβαλα μαζί στην εξωτερική γωνία στην κορυφή Γ. Συμπλήρωσαν ακριβώς την εξωτερική γωνία. Άρα, η εξωτερική γωνία ενός τριγώνου ισούται με το άθροισμα των δύο απέναντι εσωτερικών γωνιών του.*

- Αποδέχεσαι την εξήγηση του μαθητή Β ως απόδειξη; **Ναι/Όχι …………..**

Μαθητής Γ: *Επεξήγηση με τη χρήση δύο παράλληλων ευθειών.*

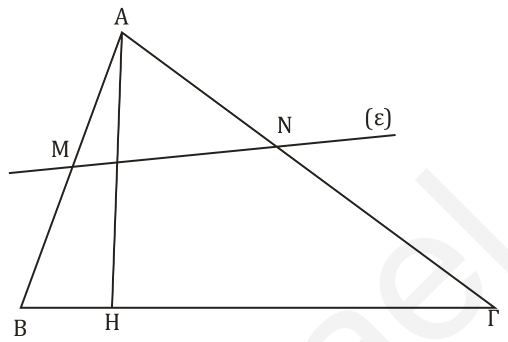


Φέρνουμε την

$Γχ \,//\, ΒΑ \Rightarrow \hat{A} = \hat{\Gamma}_3 \Rightarrow$ και $\hat{B} = \hat{\Gamma}_4 \Rightarrow \hat{A} + \hat{B} = \hat{\Gamma}_3 + \hat{\Gamma}_4 \Rightarrow \hat{A} + \hat{B} = \hat{\Gamma}_{εξ.}$

- Αποδέχεσαι την εξήγηση του μαθητή Γ ως απόδειξη; **Ναι / Όχι**

Ερώτηση 3. Στο πιο κάτω σχήμα δίνονται ότι ΑΒ = ΑΓ,(ε) // ΒΓ και ΑΗ ⊥ ΒΓ. Να συγκρίνετε τα ευθύγραμμα τμήματα ΜΗ και ΝΗ.



α) Τα τμήματα ΜΗ και ΝΗ είναι ίσα

β) Το τμήμα ΜΗ είναι μεγαλύτερο από το τμήμα ΝΗ

γ) Το τμήμα ΝΗ είναι μεγαλύτερο από το τμήμα ΜΗ

δ) Δεν μπορεί να προσδιοριστεί

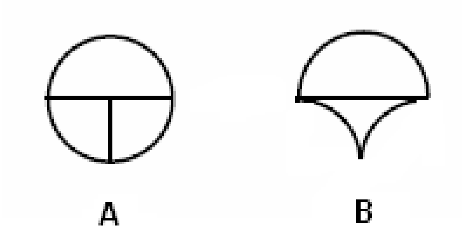
Εξήγησε πώς βρήκες την απάντηση.

Ερώτηση 4. Δίνεται τρίγωνο ΑΒΓ. Το σημείο Ε είναι ένα σημείο που ανήκει πάνω στο ευθύγραμμο τμήμα ΑΓ. Ο κύκλος με διάμετρο ΑΕ τέμνει την πλευρά ΑΒ στο σημείο Ζ.

Δίνεται ότι: **ΑΒ = 6, ΑΓ = 6.5 , ΑΕ =** $\frac{4}{5}$ **ΑΓ**

Να κατασκευάσετε το σχήμα με τη χρήση χάρακα και διαβήτη.

Ερώτηση 5. Βάλε σε κύκλο τη σωστή απάντηση.

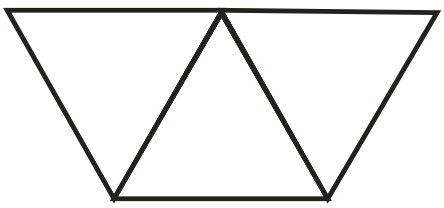


α. Το σχήμα A έχει μεγαλύτερη περίμετρο από το σχήμα B.

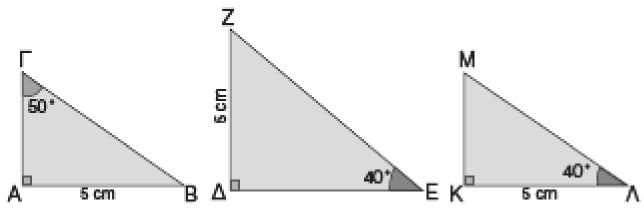
β. Το σχήμα A έχει ίση περίμετρο με το σχήμα B.

γ. Το σχήμα A έχει μικρότερη περίμετρο από το σχήμα B.

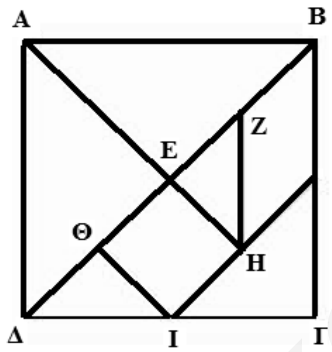
Ερώτηση 6. Σχεδίασε τα σχήματα τα οποία νομίζεις ότι συνδυάστηκαν για να προκύψει το πιο κάτω σχήμα. Μπορείς να παρουσιάσεις περισσότερες από μια λύση.



Ερώτηση 7. Βρες το ζεύγος των ίσων τριγώνων και αιτιολόγησε την απάντησή σου.



Ερώτηση 8. Συμπλήρωσε τις παρακάτω φράσεις. Χρησιμοποίησε τις λέξεις που νομίζεις ότι ταιριάζουν καλύτερα.



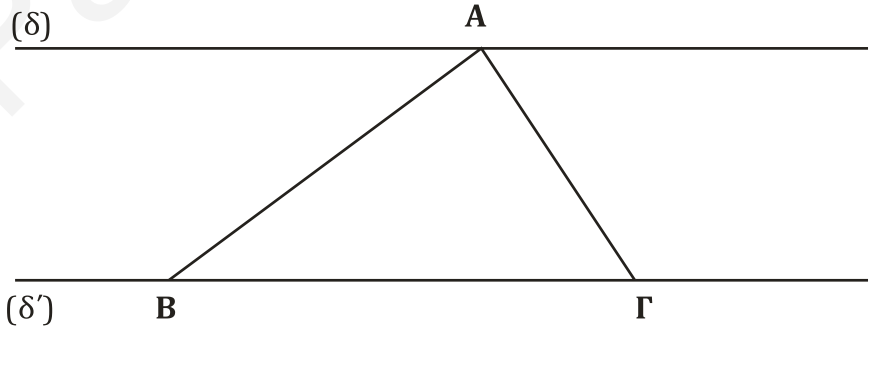
Το σχήμα ΘΖΗΙ είναι ……………………….

Το σχήμα ΕΗΙΘ είναι……………………….

Το σχήμα ΔΒΓ είναι………………………….

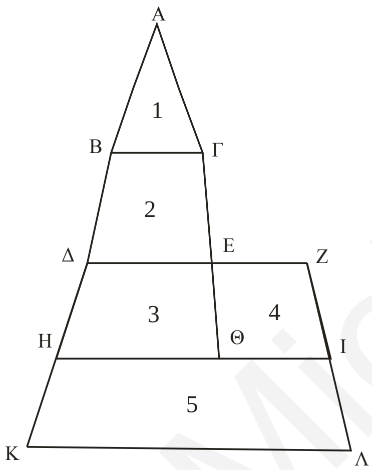
Ερώτηση 9. Οι ευθείες (δ) και (δ΄) είναι παράλληλες

Α) Να κατασκευάσετε δύο σημεία Μ και Ν έτσι ώστε το τετράπλευρο ΑΜΓΝ να είναι ένα παραλληλόγραμμο ισοδύναμο με το τρίγωνο ΑΒΓ.



Β) Να αιτιολογήστε την κατασκευή σας.

Ερώτηση 10. Στο πιο κάτω σχήμα το σχήμα 1 είναι ισόπλευρο τρίγωνο, το σχήμα 2 είναι ορθογώνιο παραλληλόγραμμο, τα σχήματα 3 και 4 είναι τετράγωνα και το σχήμα που σχηματίζεται από τα 3,4 και 5 είναι τετράγωνο. Να δείξετε ότι τα ευθύγραμμα τμήματα ΑΓ, ΘΙ και ΙΛ είναι ίσα.



Εξήγησε πώς βρήκες την απάντηση.

APPENDIX 2

GEOMETRY EXERCISES AND EXAMPLES FOR EACH TYPE OF GEOMETRICAL FIGURE APPREHENSION FROM MATHEMATICS TEXTBOOKS

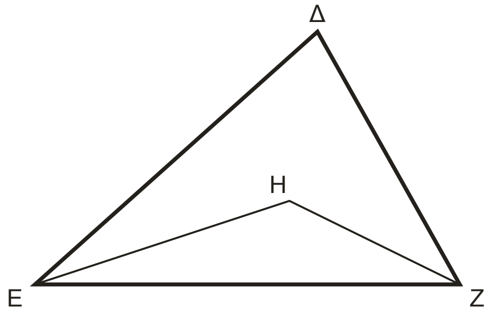
FOR GRADES 9, 10 AND 11

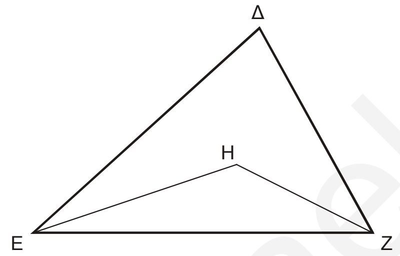
1. Perceptual Apprehension

ΔEZ is a triangle. H is the intersection point

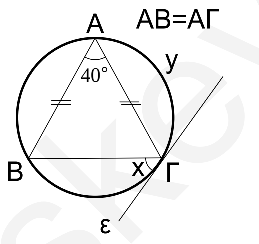
of the dichotomous of angles E and Z. If angle Δ is 80°

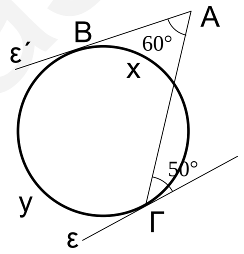
and ΔEZ is 40°, find the angle EHZ.





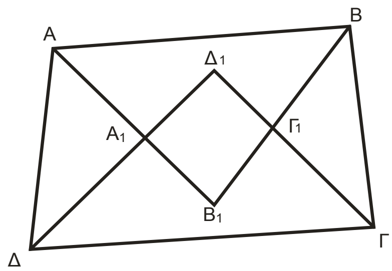
Lines ε and ε΄ are tangents. Find x and y. (Grade 10)





Example (Grade 11)

Prove that the dichotomous of the angles of a quadrilateral shape an inscribed quadrilateral.



2. Sequential Apprehension

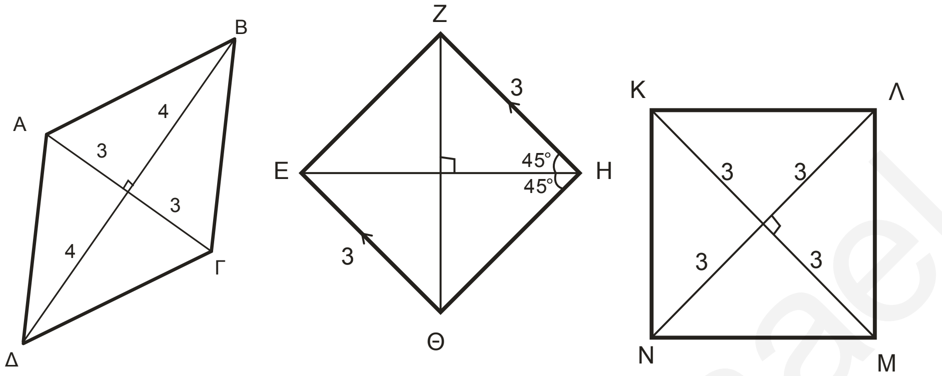
Use a compass in order to draw the dichotomous of the following angles. (Grade 9)

Draw a circle with ratio r that goes through a constant point K. How many circles of this type can we draw on a plane? At which point are their centers found? (Grade 10)

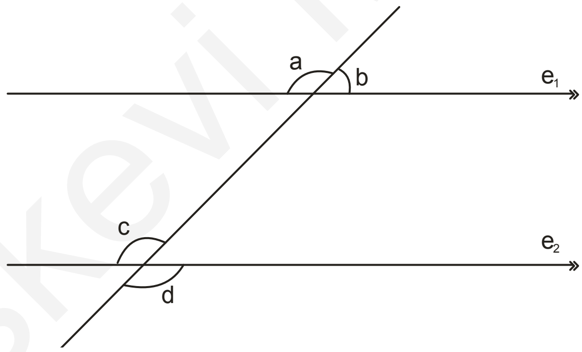
Draw a straight line that passes from a constant point O and intersects two fixed incompatible straight lines. (Grade 11)

3. Discursive Apprehension

Which of the following shapes are squares and why? (Grade 9)



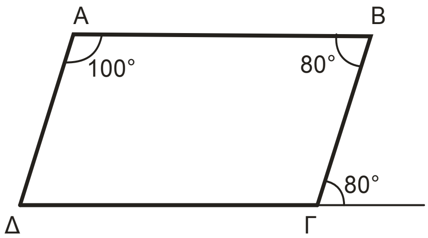
How can we call angles a and b? What is the relation between them? (Grade 10)



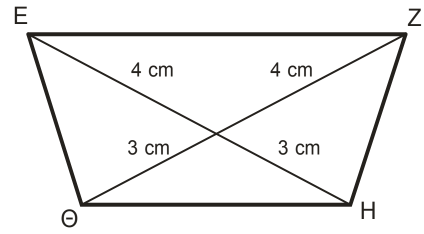
Example (Grade 9)

Are the following shapes parallelograms?

(α)

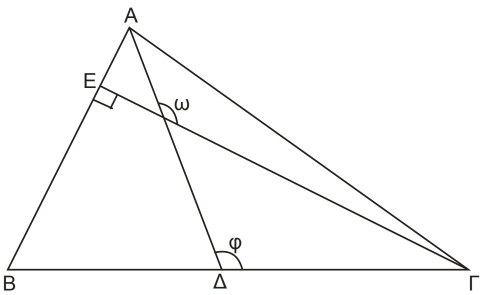


(β)

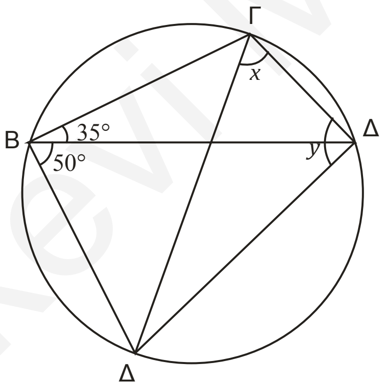


4. Operative Apprehension

ΓE is an altitude and AΔ is a dichotomous of the triangle ABΓ. If angle EΓB= 30° and BAΔ= 40°, calculate the angles B, ω, φ. (Grade 9)

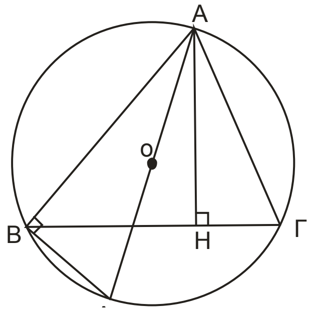


Find the value of x and y. (Grade 10)



Example (Grade 10)

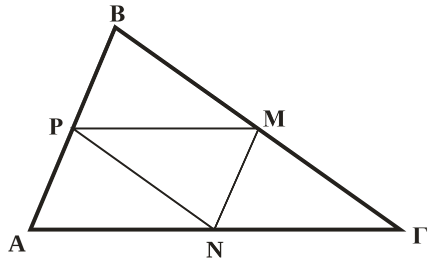
ABΓ is a triangle and $υ_α$ = AH. Prove that βγ = 2R$υ_{α,}$ where R is the radio of the circle in which the triangle ABΓ is inscribed.



APPENDIX 3

THE TASKS OF THE SEMI – STRUCTURED INTERVIEWS

1. **Τα σημεία Μ, Ν και Ρ είναι τα μέσα των πλευρών του τριγώνου ΑΒΓ. Να δείξεις ότι τα τετράπλευρα ΑΡΜΝ, ΒΜΝΡ, και ΓΝΡΜ είναι παραλληλόγραμμα.**



2. **Διάβασε τις ακόλουθες εξηγήσεις των τριών μαθητών, οι οποίοι εξηγούν γιατί το άθροισμα των εσωτερικών γωνιών ενός τριγώνου είναι 180 μοίρες.**

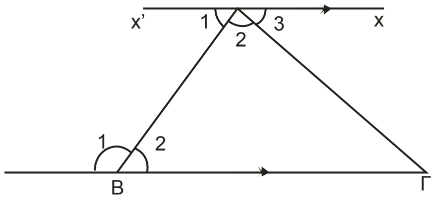
Μαθητής Α: *Μέτρησα και τις τρεις γωνιές και είναι 50°, 53° και 77°. 50°+53°+77°=180°. Άρα, το άθροισμα (των γωνιών ενός τριγώνου) είναι 180°.*

- Αποδέχεσαι την εξήγηση του μαθητή Α ως απόδειξη; **Ναι/Όχι …………**

Μαθητής Β: *Σχημάτισα ένα τρίγωνο και έκοψα κάθε γωνία και τις έβαλα μαζί. Σχημάτιζαν μια ευθεία γραμμή. Άρα, το άθροισμα τους είναι 180°.*

Αποδέχεσαι την εξήγηση του μαθητή Β ως απόδειξη; **Ναι/Όχι …………..**

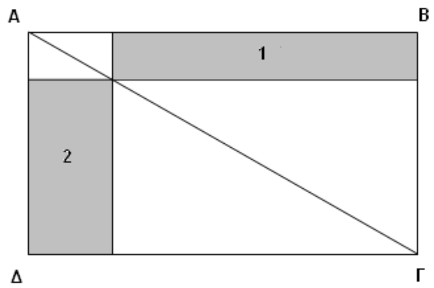
Μαθητής Γ: *Επεξήγηση με τη χρήση δύο παράλληλων ευθειών.*



$xx'||B\Gamma \Rightarrow \hat{A}_1 = \hat{B}_2$ και $\hat{A}_3 = \hat{\Gamma} \Rightarrow \hat{A}_1 + \hat{A}_2 + \hat{A}_3 = 180° \Rightarrow \hat{B}_2 + \hat{A}_2 + \hat{\Gamma} = 180°$

Αποδέχεσαι την εξήγηση του μαθητή Γ ως απόδειξη; **Ναι / Όχι**

3. Στην πιο κάτω εικόνα, το ΑΒΓΔ είναι <u>ορθογώνιο</u>. Παρατήρησε τα σκιασμένα ορθογώνια 1 και 2. Βάλε σε κύκλο τη σωστή πρόταση και <u>αιτιολόγησε</u> την απάντησή σου.



α. Το ορθογώνιο 1 έχει μεγαλύτερο εμβαδόν από το ορθογώνιο 2.

β. Το ορθογώνιο 1 έχει ίσο εμβαδόν με το ορθογώνιο 2.

γ. Το ορθογώνιο 1 έχει μικρότερο εμβαδόν από το ορθογώνιο 2.

4. Το τραπέζιο και το ορθογώνιο είναι ισοδύναμα. Να βρείτε το μήκος της πλευράς του ορθογωνίου. Επεξήγησε τον τρόπο που εργάστηκες.