DEPARTMENT OF EDUCATION

# TRANSFORMATIONAL GEOMETRY ABILITY, ITS RELATION TO INDIVIDUAL DIFFERENCES, AND THE IMPACT OF TWO INTERACTIVE DYNAMIC VISUALISATIONS

Xenia Xistouri

Submitted at the Department of Education

as partial fulfilment of the requirements for the acquisition

of a Doctoral Title in Mathematics Education

University of Cyprus

April, 2013

**UNIVERSITY OF CYPRUS**

**DEPARTMENT OF EDUCATION**

**Doctoral Dissertation Approval**

This document certifies that this doctoral dissertation prepared

By Xenia Xistouri

Titled Transformational geometry ability, its relation to individual differences, and the impact of two interactive dynamic visualisations

Fulfils the regulations of the University and corresponds to the criteria of quality and originality of the University for the acquisition of a Doctoral Title in Mathematics Education.

The dissertation was presented in public and to a five member inquiry committee and was approved on the 18th of April, 2013.

Research Supervisor: Demetra Pitta-Pantazi, Associate Professor
Department of Education, University of Cyprus

Inquiry Committee: Constantinos Christou, Professor
Department of Education, University of Cyprus
Athanasios Gagatsis, Professor
Department of Education, University of Cyprus
Charalambos Sakonidis, Professor
Department of Elementary Education, Democritus University of Thrace
Despina Potari, Associate Professor
Department of Mathematics, University of Athens

……………………………… ………………………………
Demetra Pitta-Pantazi Constantinos Christou

……………………………… ………………………………
Athanasios Gagatsis Charalambos Sakonidis

………………………………
Despina Potari

# ABSTRACT

The purpose of this study was to develop a theoretical model for the structure and development of ability in Euclidean transformational geometry concepts, to examine its relation to spatial ability and cognitive style, and to investigate the impact of two interactive dynamic visualisations on transformational geometry ability and spatial ability.

Five hundred and seven students participated in this study. Two tests and a self-report questionnaire were administered. One test measured ability in transformational geometry concepts, one test measured spatial ability, and the self-report questionnaire measured cognitive style (object-spatial imagery, and verbal dimensions). Clinical interviews with 40 students were also conducted to investigate and analyse the conceptions and strategies of students at different levels of ability. Additionally, two instructional interventions with two different types of interactive dynamic visualisations – discrete and continuous – were conducted in order to investigate their impact on students' transformational geometry and spatial abilities. Seventy-nine sixth grade students participated in the instructional interventions. Following the interventions, post-test measurements were obtained for ability in transformational geometry and spatial ability.

The results of the study showed that ability in transformational geometry consists of three factors: (a) translation ability, i.e., the ability to solve tasks related to the transformational geometry concept of translation; (b) reflection ability, i.e., the ability to solve tasks related to the transformational geometry concept of reflection; and (c) rotation ability, i.e., the ability to solve tasks related to the transformational geometry concept of rotation. Each of the three factors that refer to the abilities in transformational geometry concepts can be analysed into four similar cognitive factors: (i) recognition of image, (ii) recognition of transformation, (iii) identification of parameters, and (iv) construction of image. A hierarchical relation was also found between the three factors of ability in the geometrical concepts of translation, reflection, and rotation.

The results of the study showed the existence of four distinct levels of ability in transformational geometry concepts. The characteristics of the levels were analysed from a cognitive perspective of the individuals' conceptions and strategies. Students of the first level had low abilities in all the transformational geometry concepts. Their main characteristic was the conception of the figures that represent a geometric transformation as a holistic tangible object, and the use of holistic strategies; thus we named the level "holistic image conception". Students of the second level had average abilities in the

concept of translation, and low abilities in the concepts of reflection and of rotation. Their main characteristic was their conception of geometric transformations as processes of physical motion that can be applied on geometrical figures as objects over the plane, which serves as a background, using holistic strategies; thus we named this level “motion of an object”. Students of the third level had average abilities in all the transformational geometry concepts. Their main characteristic was a more abstract conception of geometric transformations as functions that can be applied on specific parts of a shape, which serve as objects, and the use of combinations of holistic strategies for visualising the shape as an object and analytic approaches by focusing on specific components of the image; thus, we named this level “mapping of an object”. Students of the fourth level had high abilities in all transformational geometry concepts. Their main characteristic was a conceptual understanding of geometric transformations as one-to-one mapping, and a flexibility to decompose the images into points in the plane and apply analytic strategies; thus, we named this level “mapping of the plane”.

The results of the study showed that there is a connection between ability in transformational geometry concepts and spatial ability, and that the two abilities can be considered as distinct dimensions of a more general spatial ability. They also confirmed the multidimensional construct of spatial ability, as a synthesis of three distinct factors: (a) spatial visualisation, (b) spatial relations, and (c) spatial orientation. The results also indicated a negative relation between ability in transformational geometry concepts and the verbal dimension cognitive style.

Finally, the findings of the study suggest that teaching transformational geometry concepts with the implementation of a continuous dynamic visualisation, in comparison to a discrete dynamic visualisation, has more positive impact on students’ ability in transformational geometry concepts and their spatial ability, regardless of individual differences in their spatial ability and cognitive style.

ΠΕΡΙΛΗΨΗ

Ο σκοπός της παρούσας εργασίας ήταν η ανάπτυξη ενός θεωρητικού μοντέλου για τη δομή κι εξέλιξη της ικανότητας στις έννοιες της γεωμετρίας των μετασχηματισμών, η εξέταση της σχέσης της με την αντίληψη των εννοιών του χώρου και με το γνωστικό στυλ, και η διερεύνηση της επίδρασης δύο δυναμικών οπτικοποιήσεων στην ικανότητα στις έννοιες της γεωμετρίας των μετασχηματισμών και στην αντίληψη των εννοιών του χώρου.

Στην έρευνα συμμετείχαν 507 μαθητές, στους οποίους χορηγήθηκαν δύο δοκίμια κι ένα ερωτηματολόγιο αυτό-αναφοράς. Το πρώτο δοκίμιο μετρούσε την ικανότητα στις έννοιες της γεωμετρίας των μετασχηματισμών και το δεύτερο την ικανότητα αντίληψης των εννοιών του χώρου. Το ερωτηματολόγιο αφορούσε στη μέτρηση τριών διαστάσεων γνωστικού στυλ (εικονικών-χωρικών και λεκτικών αναπαραστάσεων). Στη συνέχεια, πραγματοποιήθηκαν κλινικές συνεντεύξεις με 40 μαθητές για τη διερεύνηση κι ανάλυση των αντιλήψεων και των στρατηγικών των μαθητών που ομαδοποιήθηκαν σε διαφορετικά επίπεδα ικανότητας. Ακολούθως, 79 μαθητές της έκτης τάξης δημοτικού συμμετείχαν στη διεξαγωγή εκπαιδευτικών παρεμβάσεων με δύο διαφορετικού τύπου οπτικοποιήσεις – διακριτές και συνεχείς – με σκοπό τη διερεύνηση των επιδράσεων τους στις ικανότητες των μαθητών, για τις οποίες λήφθηκαν μετρήσεις με δεύτερη χορήγηση των δοκιμίων.

Σύμφωνα με τα αποτελέσματα της εργασίας, η ικανότητα στις έννοιες της γεωμετρίας των μετασχηματισμών αποτελείται από τρεις παράγοντες: (α) την ικανότητα στη μεταφορά, που αναφέρεται στην ικανότητα επίλυσης έργων που σχετίζονται με την έννοια της μεταφοράς, (β) την ικανότητα στην ανάκλαση, που αναφέρεται στην ικανότητα επίλυσης έργων που σχετίζονται με την έννοια της ανάκλασης, και (γ) την ικανότητα στην περιστροφή, που αναφέρεται στην ικανότητα επίλυσης έργων που σχετίζονται με την έννοια της περιστροφής. Ο κάθε παράγοντας αποτελείται από τέσσερις όμοιες γνωστικές ικανότητες: (1) την αναγνώριση εικόνας, (2) την αναγνώριση μετασχηματισμού, (3) τον προσδιορισμό παραμέτρων, και (4) την κατασκευή εικόνας. Επιπλέον, η σχέση ανάμεσα στις ικανότητες στις έννοιες της γεωμετρίας των μετασχηματισμών είναι ιεραρχική.

Τα αποτελέσματα της εργασίας έδειξαν τη διάκριση τεσσάρων επιπέδων ικανότητας στις έννοιες της γεωμετρίας των μετασχηματισμών. Η ανάλυση των χαρακτηριστικών σε κάθε επίπεδο επικεντρώθηκε στις αντιλήψεις και τις στρατηγικές των ατόμων από γνωστική σκοπιά. Οι μαθητές του πρώτου επιπέδου είχαν χαμηλή επίδοση σε όλες τις έννοιες της γεωμετρίας των μετασχηματισμών. Ονομάσαμε το επίπεδο «ολιστική

εικονική αντίληψη», γιατί το κύριο χαρακτηριστικό των μαθητών ήταν η αντίληψη της σχηματικής αναπαράστασης του γεωμετρικού μετασχηματισμού ως ολιστικό, απτό αντικείμενο, καθώς και η χρήση ολιστικών στρατηγικών. Οι μαθητές του δεύτερου επιπέδου είχαν ικανοποιητική επίδοση στην έννοια της μεταφοράς, και χαμηλή επίδοση στις έννοιες της ανάκλασης και της περιστροφής. Ονομάσαμε το επίπεδο «κίνηση αντικειμένου», γιατί το κύριο χαρακτηριστικό των μαθητών ήταν η αντίληψη των γεωμετρικών μετασχηματισμών ως μια διαδικασία φυσικής κίνησης που εφαρμόζεται στα σχήματα ωσάν να είναι αντικείμενα που κινούνται στο επίπεδο, το οποίο εξυπηρετεί ως φόντο, με τη χρήση ολιστικών στρατηγικών. Οι μαθητές του τρίτου επιπέδου είχαν ικανοποιητική επίδοση σε όλες τις έννοιες της γεωμετρίας των μετασχηματισμών. Ονομάσαμε το επίπεδο «χαρτογράφηση αντικειμένου», γιατί το κύριο χαρακτηριστικό των μαθητών ήταν μια πιο αφηρημένη αντίληψη των γεωμετρικών μετασχηματισμών ως συνάρτηση που εφαρμόζεται σε συγκεκριμένα μέρη του σχήματος, τα οποία λειτουργούν ως αντικείμενα, και η χρήση συνδυασμού ολιστικών στρατηγικών στο σχήμα και αναλυτικών στρατηγικών στα μέρη του. Οι μαθητές του τέταρτου επιπέδου είχαν ψηλή επίδοση σε όλες τις έννοιες της γεωμετρίας των μετασχηματισμών. Ονομάσαμε το επίπεδο «χαρτογράφηση επιπέδου», γιατί το κύριο χαρακτηριστικό των μαθητών ήταν η εννοιολογική αντίληψη των γεωμετρικών μετασχηματισμών ως χαρτογράφηση των σημείων του επιπέδου ένα-προς-ένα, με τη χρήση αναλυτικών στρατηγικών.

Τα αποτελέσματα της εργασίας έδειξαν ότι υπάρχει σχέση μεταξύ της ικανότητας στις έννοιες της γεωμετρίας των μετασχηματισμών και της ικανότητας αντίληψης των εννοιών του χώρου, καθώς οι δύο ικανότητες μπορούν να θεωρηθούν διακριτές διαστάσεις μιας ευρύτερης ικανότητας αντίληψης χωρικών σχέσεων. Επιβεβαιώνουν επίσης την πολυδιάστατη οντότητα της ικανότητας αντίληψης των εννοιών του χώρου, η οποία αποτελείται από τρεις παράγοντες: (α) την οπτικοποίηση των εννοιών του χώρου, (β) τις σχέσεις των εννοιών του χώρου, και (γ) τον προσανατολισμό στο χώρο. Τα αποτελέσματα έδειξαν επίσης αρνητική σχέση μεταξύ της ικανότητας στις έννοιες της γεωμετρίας των μετασχηματισμών και της λεκτικής διάστασης του γνωστικού στυλ.

Τα αποτελέσματα δείχνουν επίσης ότι η διδασκαλία στις έννοιες της γεωμετρίας των μετασχηματισμών με τη χρήση συνεχούς δυναμικής οπτικοποίησης, σε σύγκριση με τη χρήση διακριτής δυναμικής οπτικοποίησης, φέρει καλύτερα αποτελέσματα στην ικανότητα των μαθητών στις έννοιες της γεωμετρίας των μετασχηματισμών και στην ικανότητα αντίληψης των εννοιών του χώρου, ανεξάρτητα από την ικανότητας αντίληψης των εννοιών του χώρου και το γνωστικό στυλ των μαθητών.

# ACKNOWLEDGMENTS

I would like to thank the members of my advisory committee, Prof. A. Gagatsis, Prof. C. Christou, and Dr D. Pitta-Pantazi, for their valuable suggestions for the improvement of this study. I would like to especially thank my supervisor, Dr Demetra Pitta-Pantazi, for all the advice and knowledge that she has given me over the last 13 years of collaboration, and for all the skills that she has helped me develop in that time. I owe a lot of my first research experience and my passion for research to her overwhelming enthusiasm and support.

I would also like to thank Prof. George Philippou and Dr Nicholas Mousoulides who, in their way, have inspired me to carry on my studies. Their advice has been valuable for my academic course. Special thanks also go to my colleague Dr Marios Pittalis, for his help and advice in many aspects of this study.

I could not thank enough my friends and colleague teachers for their immense assistance in fulfilling this research, for opening the doors to their schools and their classrooms and for sharing my concerns for the reliable implementation of the research procedure. I also thank the students that participated in this study for their time, effort, understanding, and co-operation. I cannot only but hope that this experience has had some positive impact in everyone’s life. Additionally, I would very much like to thank the Cyprus Research Promotion Foundation for giving me the opportunity to fulfil my doctoral studies and improve my research skills, by providing funding from the Republic of Cyprus and the European Regional Development Fund, for the implementation of this research.

Most importantly, I would like to thank my family and my friends for supporting me in every way throughout this journey, especially during the difficult times of disappointment. I have not enough words to thank my parents for their love and support throughout my studies. My sister and my brother-in-law for their love and support, and for sharing my thoughts and worries; I thank them for their eagerness to help in every possible way. I thank my nephew for always putting a smile on my face.

I also thank all of my friends for their understanding and caring. Special thanks to Anna, Maria, Skevi, and Chara, for their enthusiasm, their help, and companionship. I could not thank enough my best friend Nicoletta Nicolaou for sharing my thoughts and worries, and for supporting me and helping me all the way and in every way – I would have not done this without her. Most of all, I would like to thank my husband Constantinos, my partner in life, for being there for me, for being understanding, for putting up with me during the most stressful times, and for reminding me the importance of maintaining your dignity.

“At times our own light goes out and is rekindled by a spark from another person. Each of us has cause to think with deep gratitude of those who have lighted the flame within us.”
Albert Schweitzer

To my family and friends,

for their sparks that kept

this flame lighting

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# CHAPTER I

# THE PROBLEM

## Introduction

The growing emphasis on geometry teaching during the last few decades has led to a modification of its traditionally Euclidian-based content by introducing new types of geometry, such as transformational geometry (Jones, 2002). The National Council of Teachers of Mathematics (NCTM) (2000) suggests that transformational geometry is currently one of the topics that have been advocated as an important part of the K-12 geometry curriculum. According to NCTM's *Principals and Standards for School Mathematics* (2000, p.41), "Instructional programmes from kindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyse mathematical situations". In mathematics education, transformational geometry is also considered important in supporting children's development of geometric and spatial thinking (Clements & Battista, 1992; Hollebrands, 2003; Jones & Mooney, 2003) and it is related to a variety of activities in academic and every-day life, such as geometrical constructions, art, architecture, carpentry, electronics, mechanics, clothing design, geography, navigation, and route following (Boulter & Kirby, 1994; Silver, 1987).

The beginning of research in transformational geometry was marked by its inclusion in mathematics curricula in the early '70s (Jones, 2002). This raised a lot of interest regarding the teaching and learning of geometric transformations. However, research in transformational geometry decreased substantially in the '90s, leaving unanswered questions regarding the cognitive development of geometric transformations (Boulter & Kirby, 1994). The most recent attempts of research in transformational geometry during the last two decades seem to emphasise more on understanding the development of learning transformational geometry concepts. For this purpose, a variety of learning theories from the field of mathematics education have been used by researchers (Hollebrands, 2003; Molina, 1990; Portnoy, Grundmeier, & Graham, 2006; Soon, 1989; Thaqi, Giménez, & Rosich, 2011; Yanik, 2006). Substantial research in geometry teaching is based on the theory of van Hiele (1986). This theory is a useful and widely acceptable tool for the teaching and learning of geometry, as it explains the development of geometrical thinking for students of primary, secondary, and higher education (de Villiers, 2010). Many research attempts have focused on the van Hiele theory to describe and define the characteristics of different levels of significant geometric concepts (Battista, 1999; Gutierrez, Jaine, & Fortuny, 1991). In the field of transformational geometry, Soon (1989) and Molina (1990) investigated the consistency between the van Hiele model of geometrical understanding and the understanding of transformational geometry concepts in secondary and higher level students respectively.

Another fundamental element of both geometrical and mathematical thinking is imagery and visualisation (Bishop, 1989; Presmeg, 2006a). In particular, transformational geometry involves much visualisation and spatial cognition (Battista, 1990; Bishop, 1989; Boulter & Kirby, 1994; Clements & Battista, 1992; Dixon, 1995; Kirby & Boulter, 1999). Spatial ability is generally considered valuable in generating and understanding abstract representations of spatial relations. It is also an important factor that contributes significantly in mathematical problem solving (Hegarty & Waller, 2006), geometric understanding (Bishop, 1980, 1989) and transformational geometry ability (Boulter, 1992; Dixon, 1995).

Many studies in mathematics education have emphasised the existence of different types of imagery and visual thinking in mathematics and their relation to ability (Bishop, 1983; Presmeg, 1986a, 1986b). Recent cognitive psychology studies provide evidence for the existence of two generally different types of visualisation abilities (Kozhevnikov, Hegarty, & Mayer, 2002; Kozhevnikov, Kosslyn, & Shephard, 2005). These are the object visualisation ability and the spatial visualisation ability. The former refers to the processing of visual information concerning the appearance of objects and scenes in terms of their pictorial properties and the latter to the processing of visual information concerning spatial relations between objects or their parts, and performing mental spatial transformations and manipulations. Blazhenkova and Kozhevnikov (2009) extended this distinction to the individual differences field of cognitive styles and re-conceptualised the popular verbaliser - visualiser dimension of cognitive style (Paivio, 1971; Richardson, 1977).

The evolution of new technologies has led people to be increasingly more dependent on visual images and stimuli (Lowrie, 2002). New technologies emphasise more on visual abilities as the user examines, decodes, transforms, and produces information through visual images (Kirby & Boulter, 1999). Dynamic geometry software (DGS) is a powerful tool, which develops students' spatial thinking by engaging them into interactive activities (Jones, 2000). DGS have been successfully used in geometry teaching due to their interactive features and the possibility of direct interaction with geometrical objects.

It is also believed that they promote students' thinking to the higher levels of the van Hiele model (Guven, 2012; Raquel, 2002). A substantial amount of recent studies on transformational geometry tend to build their teaching intervention methods on DGS (Dixon, 1995; Flanagan, 2001; Harper, 2002; Hollebrands, 2003; Hoong & Khoh, 2003; Yanik, 2006). However, recent studies raise attention to the fact that there are different types of learners (Hegarty, 2004a; Kozhevnikov, 2007; Kozhevnikov et al., 2005) and different types of interactive-dynamic visualisations (Hegarty, 2004b; Smith, Gerretson, Olkun, Yuan, Dogbey, & Erdem, 2009), which may influence the learning procedure.

## The Problem

Research on transformational geometry began about forty years ago when it was innovatively included in the mathematics curricula of that time (Jones, 2002). Those early studies focused on providing evidence for suggesting that teaching geometric transformations in elementary and high school education was feasible and could have positive effects on students' learning of mathematics (Kort, 1971; Usiskin, 1972; Williford, 1972) and on configurations influencing students' ability in transformational geometry (Schultz, 1978; Thomas, 1978). Later studies focused on more cognitive aspects, such as students' abilities, conceptions, and errors (Bell & Birks, 1990; Grenier, 1985; Kidder, 1976; Moyer, 1978), and strategies for solving transformational geometry problems (Boulter & Kirby, 1994; Naidoo, 2010). For the last twenty years, although research in transformational geometry started to decrease substantially, focus has shifted towards investigating possible hierarchies that can describe students' acquisition of transformational geometry concepts (Soon, 1989; Molina, 1990; Yanik & Flores, 2009).

Despite substantial research concerning the learning and teaching of geometry, only few studies focus on students' understanding of transformational geometry concepts (Hollebrands, 2003).There have been several suggestions recently that transformational geometry in mathematics education receives limited attention (Hollebrands, 2003; Yanik, 2011; Yanik & Flores, 2009). Some possible reasons for this occurrence are transformational geometry's underemphasis in mathematics curricula, minimal textbook content, teacher unfamiliarity with the concepts and uncertainty with the topic, or absolute deletion from an overcrowded curriculum (Boulter & Kirby, 1994). There are also several suggestions that literature in this area seems to lack an appropriate theoretical framework for the learning and teaching of transformational geometry concepts (Boulter & Kirby, 1994; Yanik & Flores, 2009).

Another significant part of the research regarding transformational geometry focused on investigating the relation of transformational geometry ability to spatial ability (Boulter, 1992; Clements & Battista, 1992; Dixon, 1995; Kirby & Boulter, 1999). Although there is strong evidence that such a relation exists, its nature is still not clear. Some studies support that spatial ability is an important pre-requisite for understanding transformational geometry (Kirby & Boulter, 1999), while other studies support that engaging students with transformational geometry activities can have a positive impact on developing their spatial ability (DelGrande, 1986; Dixon, 1995; Smith et al., 2009). However, some studies suggest that the impact of transformational geometry activities on students' spatial ability is not significant (Boulter, 1992; Williford, 1972). It is assumed that these two abilities may interact, but whether and how they do so is still unclear.

A growing body of work is currently advocating the potential of styles to impact on performance in education (Evans & Cools, 2011; Evans & Waring, 2011) and the predictive power of cognitive style over an individual's behaviour and success on complex tasks in real-life, academic, and educational settings (Blazhenkova, Becker, & Kozhevnikov, 2011; Sternberg & Zhang, 2001). Even though there is evidence that certain styles, such as the spatial-imagery cognitive style proposed by Kozhevnikov et al. (2005), are related to mathematical (Blazhenkova & Kozhevnikov, 2009; Kozhevnikov et al., 2002; Kozhevnikov et al., 2005) and geometrical abilities (Anderson, Casey, Thompson, Burrage, Pezaris, & Kosslyn, 2008), there do not seem to be any studies relating cognitive style to transformational geometry ability. Investigating such a relationship would be important for identifying those that may need greater support in learning transformational geometry concepts (Evans & Cools, 2011) and perhaps even for the design of appropriate instructional environments to accommodate different types of learners. Recent research evidence suggests that the task mode of presentation can have significant influence on performance according to cognitive style (Höffler, Prechtl, & Nerdel, 2010; Pitta-Pantazi & Christou, 2009b; Riding & Douglas, 1993). However, there are contradictory results in research regarding whether the type of instructional material should match or complement one's cognitive style (Pitta-Pantazi & Christou, 2009a; Thomas & McKay, 2010). Moreover, Pitta-Pantazi and Christou (2009a) suggest that not all cognitive style types of learners benefit equally from learning with dynamic geometry.

An extensive and important part of research on transformational geometry focuses on the positive effects of instructional methods for teaching transformational geometry with technological aids (Dixon, 1995; Edwards, 1990; Hollebrands, 2003; Yanik & Flores, 2009). The development of computer technologies and, more recently, of interactive DGS offers unlimited possibilities for the dynamic visualisation of geometric concepts in general (Pritchard, 2003), and for transformational geometry concepts in particular. Previous studies have emphasised on the potential of computer software environments in the teaching of transformational geometry (Dixon, 1995; Edwards, 1989; Hollebrands, 2003), as well as on the possibility of developing learners' spatial ability through interactive software for teaching transformations (Dixon, 1995; Smith et al., 2009). This has a serious impact on the role of individual differences in learning geometry. Research in the field of educational psychology and mathematics education raised serious questions on the effectiveness of such dynamic displays for different types of people and different types of learning (Hegarty 2004a; 2004b; Kirby & Boulter, 1999). There seem to be some concerns on whether people with low spatial ability may encounter problems in learning from dynamic environments (Hegarty, 2004a, 2004b; Kirby & Boulter, 1999; Smith et al., 2009; Sweller, van Merrienboer, & Pass, 1998) and it seems unclear whether all types of learners can learn transformational geometry concepts through dynamic displays with the same ease. Hegarty (2004b) draws attention to the fact that it might be simplistic to generalise results from studies with one dynamic display given that there are many different types of dynamic displays. It seems that whether to use or not to use technology in education is not the only important question anymore. It is also important to know the relative effectiveness of different types of tools, for different types of learning and content (Hegarty, 2004b).

## Aim

The aim of the study was to develop a theoretical model for the learning and teaching of transformational geometry concepts (translations, reflections, and rotations), to examine its relation to individual differences in spatial ability and cognitive style, and to investigate the impact of two interactive dynamic visualisations on the spatial and transformational geometry abilities of different types of students. The specific purposes of the study were the following:

(a) to investigate the components that synthesise students' ability in transformational geometry and the structure of this ability, (b) to investigate students' ability in transformational geometry concepts, (c) to identify and describe students' levels of ability in transformational geometry concepts, (d) to investigate the relation between students' ability in transformational geometry and their spatial ability, (e) to investigate the relation between students' ability in transformational geometry concepts and their cognitive style, (f) to investigate the impact of transformational geometry instruction with two different interactive dynamic visualisations on students' transformational geometry and spatial abilities, and (g) to investigate the interactions of students' level of spatial ability and cognitive style with different types of interactive dynamic geometry visualisations.

## Questions

The main research questions of this study, as they resulted from the aim of the study, were the following:

(1) Which components synthesise 9- to 14-year-old students' ability in transformational geometry concepts (translations, reflections, and rotations) and what is the structure of this ability?

(2) Is the structure of students' ability in transformational geometry concepts the same or different in relation to age?

(3) What are the levels of ability of 9- to 14-year-old students' development of transformational geometry concepts and in what way can these levels be described?

(4) What are the abilities, conceptions, strategies, and common errors of students at different levels?

(5) What is the relation between students' spatial ability and their transformational geometry ability?

(6) What is the relation between students' cognitive style and their transformational geometry ability?

(7) What is the impact of transformational geometry instruction on students' transformational geometry and spatial abilities when two different interactive dynamic visualisations (continuous and discrete) are used?

(8) What is the impact of the interactions between the type of dynamic visualisation (continuous and discrete) and students' individual differences in level of spatial ability and cognitive style on their transformational geometry and spatial abilities?

## Significance and Originality

The lack of a theory for the structure and the development of primary and secondary students' understanding of transformational geometry concepts and the decrease of research in this domain for the past two decades, stress the importance for systematic research in this field. Therefore, this study attempts to fill in the gap between the lack of contemporary research studies in transformational geometry concepts and researchers' suggestions for the need of systematic research of students' ability in transformational geometry concepts (NCTM, 2000). The importance of this study lies on the fact that it aspires to investigate (a) the structure of students' ability in transformational geometry concepts (translations, reflections, and rotations), which is considered important in mathematics education given its contribution in developing students' spatial thinking (Clements & Battista, 1992; Hollebrands, 2003; Jones & Mooney, 2003); and (b) to investigate students' levels of ability in transformational geometry concepts.

The originality of the study lies, firstly, on the development of a unified theoretical framework for the learning and teaching of transformational geometry concepts. This framework will consider and combine ideas from early and recent research in the areas of psychology and mathematics education and refer to the main transformational geometry concepts most often included in mathematics curricula: translations, reflections, and rotations. More specifically, a unified model of learning and teaching transformations will be developed, which will describe the components and levels of ability of primary and secondary school students' understanding of transformational geometry concepts. Secondly, this model will relate to students' personal traits regarding different types of spatial ability and cognitive style, and investigate interactions that may have impact in understanding transformational geometry concepts. Thirdly, it will investigate and compare the impact of two different computer environments, one of discrete and one of continuous dynamic visualisation, on students' development of transformational geometry and spatial abilities. Finally, it will explore the interactions between learner characteristics (spatial ability, cognitive style) and computer learning environment characteristics (discrete/dynamic).

The results of this study are expected to contribute to the teaching and assessment of transformational geometry in primary and secondary education, as well to the improvement of mathematical thinking. This valuable practical contribution can guide students' thinking of transformational geometry concepts in multiple ways and educators' instructional designs for approaching and investigating transformational geometry concepts and strategies. Moreover, it may guide curriculum developers how to structure transformational geometry mathematics curricula. It may also guide practitioners into following non-traditional approaches for active teaching of transformational geometry through interactive dynamic visualisations, with respect to students' individual characteristics.

## Limitations

One of the main limitations of this study is that the sample was not randomly selected. The selection of the subjects was opportunistic and the schools and classes that participated in the study were selected on the basis of existing personal co-operation with the school principals or classroom teachers. However, there were efforts to include equal numbers of students from each grade, and to include schools from both urban and rural areas in the sample in order to incorporate students from various socio-economical backgrounds.

Another limitation of the study is the fact that no longitudinal measurements of students abilities were obtained, in order to be able to study the developmental models of students' behaviour and investigate the manner in which a student traverses from one level of development to the subsequent. The study was limited to describing the students' levels of ability.

Finally, another limitation is related to the process of test administration. Even though spatial ability measurements in cognitive psychology are usually performed individually, in this study the measurements were performed in groups due to time limitations. Therefore, it was impossible to explain the guidelines and requests of each task and to provide examples on an individual base. This fact places some limitations in the reliability of the spatial ability measurement test.

## Structure

The following chapters describe the theoretical framework, the methodology, the results, and the conclusions of the study. Particularly, the second chapter reviews the relevant literature that constitutes the theoretical framework for designing this research with reference to main theories of geometry learning, studies on transformational geometry concepts, theories on imagery and visualisation in mathematics education and psychology, cognitive psychology studies about spatial ability and cognitive styles, theories on learning with dynamic visualisations, and how spatial abilities and cognitive style relate to performance in mathematics and to learning from dynamic visualisations.

The third chapter describes the methodological procedure with reference to the subjects, the instruments, the proposed models, the procedure of the quantitative data collection, the design of clinical interviews, the design of the teaching experiment, and the statistical analysis procedures. The fourth chapter presents the results that emerged from the analysis of the quantitative and qualitative data, with reference to the validation of the proposed models, the identification of the levels of ability and the description of the characteristics of the students at each level, the relations of ability in transformational geometry concepts to individual characteristics, and the impact of individual characteristics and the types of dynamic visualisations on the learning of transformational geometry concepts and the development of spatial ability.

Finally, the fifth chapter discusses the results and attempts to describe a unified theoretical model of the quantitative and qualitative results; it also communicates the conclusions of the study, and discusses instructional implications and suggestions for further research.

## Definitions of Concepts

### *Transformational geometry concepts*

A geometric transformation is defined as a "one-to-one and onto mapping of all points in the plane to all points in the plane" (Martin, 1982, p. 1). That is, under a particular geometric transformation, every point in the plane is mapped to a unique point in the plane and every point in the plane has a corresponding pre-image point. While there are many different kinds of transformations, this study investigates translations, reflections, and rotations.

There are many transformational groups with different properties remaining invariant: topological, projective, affine, similarity, equi-affine, and Euclidean. Translations, reflections, and rotations are part of the Euclidean transformations and they are often referred to as *rigid transformations* or *isometries*. A rigid transformation or isometry is a transformation that preserves relative distances and angles of all points in the plane (Yanik & Flores, 2009). In this study any reference to transformational geometry concepts or to geometric transformations relates to the three rigid transformations of translation, reflection, and rotation, as they have been presented in previous research studies (Edwards, 1990; Edwards & Zazkis, 1993; Flanagan, 2001; Hollebrands, 2003; Kidder, 1976; Molina, 1990; Moyer, 1978; Schultz, 1978; Yanik, 2006).

The following definitions are used in this study, adopted by Wesslén and Fernandez (2005). For examples, see Figure 1.1.

*Translation* is the rigid geometric transformation that moves the plane and all its points the same distance and direction. The information needed is the distance and direction the points are moved. This can be given verbally or represented by a vector, sometimes referred to as *displacement vector*. In translation, no point remains in the same location.

*Reflection* is the rigid geometric transformation that mirrors the whole plane and its points. There is only one piece of information needed to uniquely determine the reflection: the line it is mirrored about. This is called the *mirror line* or *line of reflection* and all points on this line are unaffected by the reflection.

*Rotation* is the rigid transformation that rotates the plane and all the points contained within it. A full description of a rotation requires two pieces of information: the angle by which the plane is rotated through, and the point it is rotated about - referred to as *centre of rotation*. This is the only point not affected by the rotation.

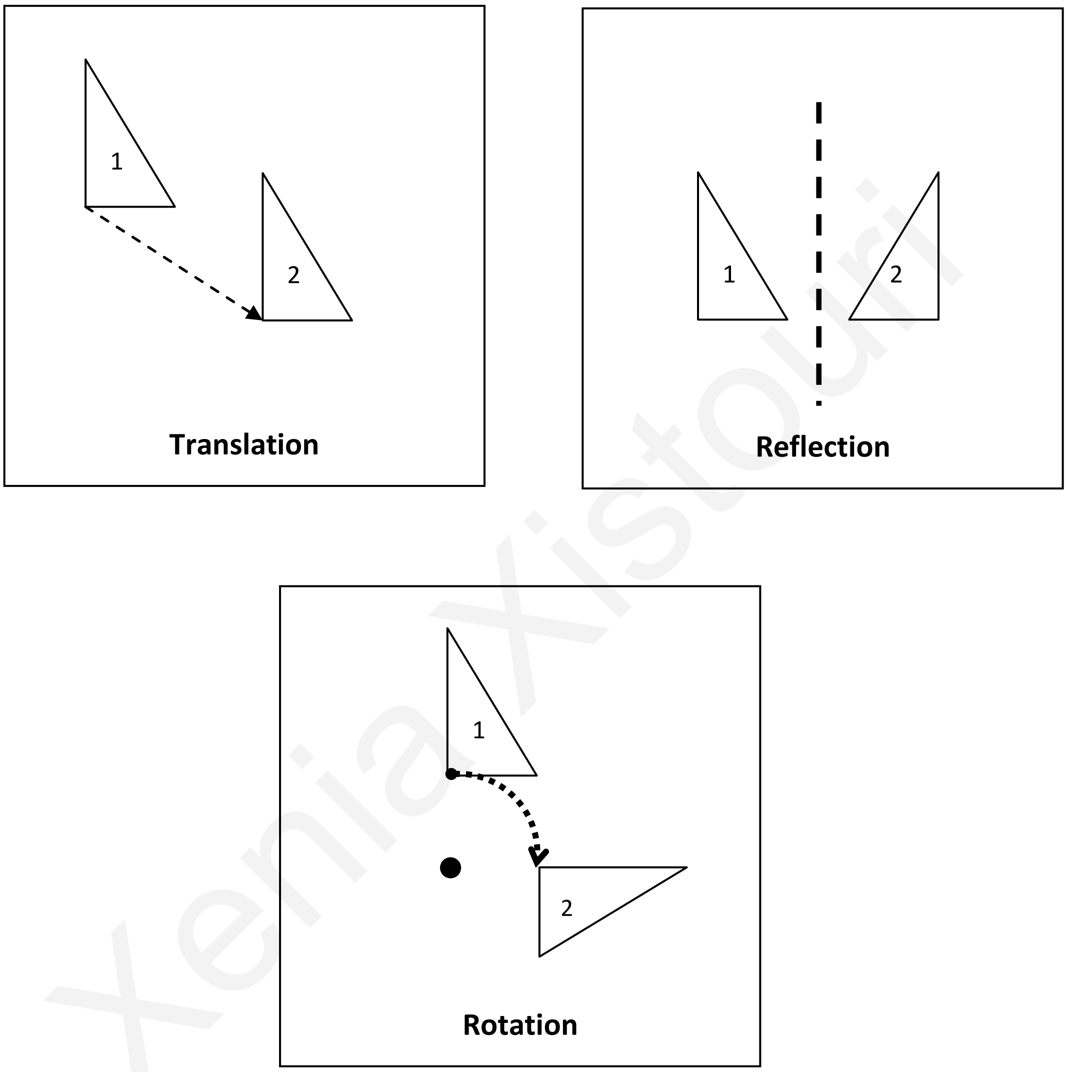



*Figure 1.1.* Rigid geometric transformations of the plane.

*Spatial ability*

There does not seem to be a unified framework of reference for spatial ability. Spatial ability is usually defined as spatial cognition, spatial intelligence, or spatial reasoning. Visual-spatial abilities refer to the process of visual and spatial information, which is information that is visual in nature, but possesses spatial attributes (Halpern & Collaer, 2005). Linn and Petersen (1985, p. 1482) define spatial ability as a skill referring to "representing, transforming, generating, and recalling symbolic, non-linguistic information".

In this study, spatial ability refers to the multifaceted ability of perceiving, processing, creating, recalling, and storing mental images that represent and signify visual-spatial information or concretise abstract spatial relations. We also regard this ability as factorial (Hegarty & Waller, 2004, 2006), comprising of (a) spatial visualisation, (b) spatial orientation, and (c) spatial relations (Lohman, 1988, 2000).

For these factors, the following definitions are used in this study, adopted by Caroll (1993).

(a) *Spatial visualisation* refers to the ability to comprehend imaginary movements in a three-dimensional space or the ability to mentally manipulate objects.

(b) *Spatial orientation* refers to one's ability to remain unconfused by the changes in the orientation of visual stimuli.

(c) *Spatial relations* refers to the ability that is defined by the speed in manipulating simple visual patterns, such as mental rotations, and describes the ability to mentally rotate a spatial object fast and correctly.

### *Cognitive style*

There are many different theories in the cognitive styles field and even though the need for a unified theory is explicitly put forward by leading researchers (Kozhevnikov, 2007; Sternberg & Zhang, 2001), a common conceptual framework and wording for cognitive style is still absent from literature. Cognitive styles are sometimes referred to as learning styles, thinking styles, intellectual styles, or approaches to learning. Messick (1994) defines cognitive style as part of the personal learning styles profile which refers specifically to an individual's habitual or typical way of perceiving, remembering, thinking, and problem solving.

In this study, we refer to cognitive style as individuals' way of acquiring knowledge and processing information. It is related to mental behaviours which individuals apply habitually when they are solving problems. In general, it affects the way in which information is obtained, sorted, and utilised. This study also regards that cognitive style is multi-dimensional (in contrast to bipolar). The Object-Spatial Imagery and Verbal model proposed by Kozhevnikov et al. (2005) was adopted in this study. According to this model, cognitive style consists of three subsystems: (a) the object imagery, (b) the spatial imagery, and (c) the verbal.

According to Kozhevnikov et al. (2005) the three subsystems can be described based on the information that each system processes:

(a) The *object imagery* system processes the visual appearance of objects and scenes in terms of their shape, colour information, and texture.

(b) The *spatial imagery* system processes object location, movement, spatial relationships and transformations, and other spatial attributes of processing.

(c) The *verbal* system processes verbal information.

There are many different terms used for dynamic visualisation in mathematics education, such as "dynamic imagery" (Presmeg, 1986), "dynamic representation" (Tall & West, 1986), "dynamic entity" (Harel & Sowder, 1997), and "dynamic visualisation" (Goldenberg, 1995). Although these terms share some common meaning, they do not always refer to the same thing.

Zazkis, Dubinsky, and Dautermann (1996) claim that visualisation is an act of construction of transformations between external media and an individual's mind. They also claim that the case of dynamic visualisation is also such an act, but this act constitutes moving pictures in the mind, or on some external medium which the learner identifies with object(s) or process(es) in her or his mind. In other words, the peculiar property of dynamic visualisation is that individuals who possess this ability can reason about the essential properties of moving, shrinking, and rotating figures, which appear either on the screen or, in their mind.

In this study we refer to dynamic visualisations as the external medium visualisation tools that appear on a computer screen.

# CHAPTER II

# LITERATURE REVIEW

## Introduction

Research on the development of students' ability in transformational geometry concepts has received limited attention within mathematics education. There are also suggestions that it significantly declined during the last two decades, leaving unanswered questions, especially in primary school education (Thaqi et al., 2011; Yanik & Flores, 2009). Moreover, there are several claims that transformational geometry is underemphasised in the mathematical curricula of many countries (Boulter & Kirby, 1994; Hollebrands, 2003; Kirby & Boulter, 1999). Nevertheless, it is considered important in supporting children's development of geometric and spatial thinking (Clements & Battista, 1992; Dixon, 1995; Hollebrands, 2003; Smith et al., 2009) and it is also useful in many other academic areas (Boulter & Kirby, 1994; Flanagan, 2001). The development of spatial thinking has been emphasised for many years by many researchers in mathematics education (Battista, 2007; Bishop, 1980, 1983; Clements & Battista, 1992; Presmeg, 1986a, 2006b) and by national unions such as the NCTM (1989, 1995, 2000).

The theoretical framework of this study brings together research and theories from the areas of cognitive psychology and mathematics education. Specifically:

(a) From the areas of cognitive psychology: theories of learning and teaching; imagery, visualisation, and visual image processing; spatial ability; cognitive styles research; and theories of learning with dynamic visualisations.

(b) From the areas of mathematics education: theories of learning and teaching geometry; transformational geometry research; imagery, visualisation, and visual image processing; and learning geometry with dynamic visualisations.

Figure 2.1 presents the structure of the theoretical framework of the study. This study has three parts. The first part focuses on cognitive development in mathematics and attempts to relate the findings of two research fields, mathematics education and cognitive psychology, to describe primary and secondary school students' ability in transformational geometry concepts. The second part focuses on individual differences in imagery and visualisation, and attempts to relate ability in transformational geometry concepts to spatial ability and its sub-components. This part also investigates the relation between ability in transformational geometry concepts and cognitive style. The third part focuses on learning with new technologies and attempts to relate the findings of the fields of cognitive psychology and educational technology in mathematics education, and to investigate the impact of technological tools on students' transformational geometry and spatial abilities, with respect to their individual characteristics.

From the field of mathematics education, this study builds on significant theories of learning and geometrical understanding, such as the theories of Piaget and van Hiele. The van Hiele model is more extensively described, with reference to the levels of students' geometrical thinking and the geometry teaching phases. The study relates these theories to the development of abilities in transformational geometry concepts (translations, reflections, and rotations) as they emerge from different kinds of studies regarding transformational geometry, in order to construct a theoretical model for the development of transformational geometry ability. The model also utilises information from studies regarding configurations that influence the difficulty of transformational geometry tasks. Finally, the study exploits research studies about the potential of technological tools, such as DGS, micro-worlds and virtual manipulatives, to enhance students' learning of geometrical concepts. We draw on theoretical frameworks that discriminate between different types of educational software in mathematics education, with emphasis on the different types of dynamic visualisations.

From the field of cognitive psychology, we use research studies for the definition and nature of spatial ability, which is supposed to relate to students' transformational geometry ability and mathematical ability in general. We emphasise on research studies that discriminate different types of spatial ability and on studies that refer to the visualisation and mental images that students need to develop for understanding mathematical concepts. We also exploit studies for the definition and nature of cognitive styles which literature extensively suggests that are related to mathematical abilities. We draw on studies that consider cognitive style as multi-dimensional and which discriminate between different dimensions of cognitive styles. Moreover, we use theories from this field, about learning with the use of technological tools, and specifically dynamic visualisations.

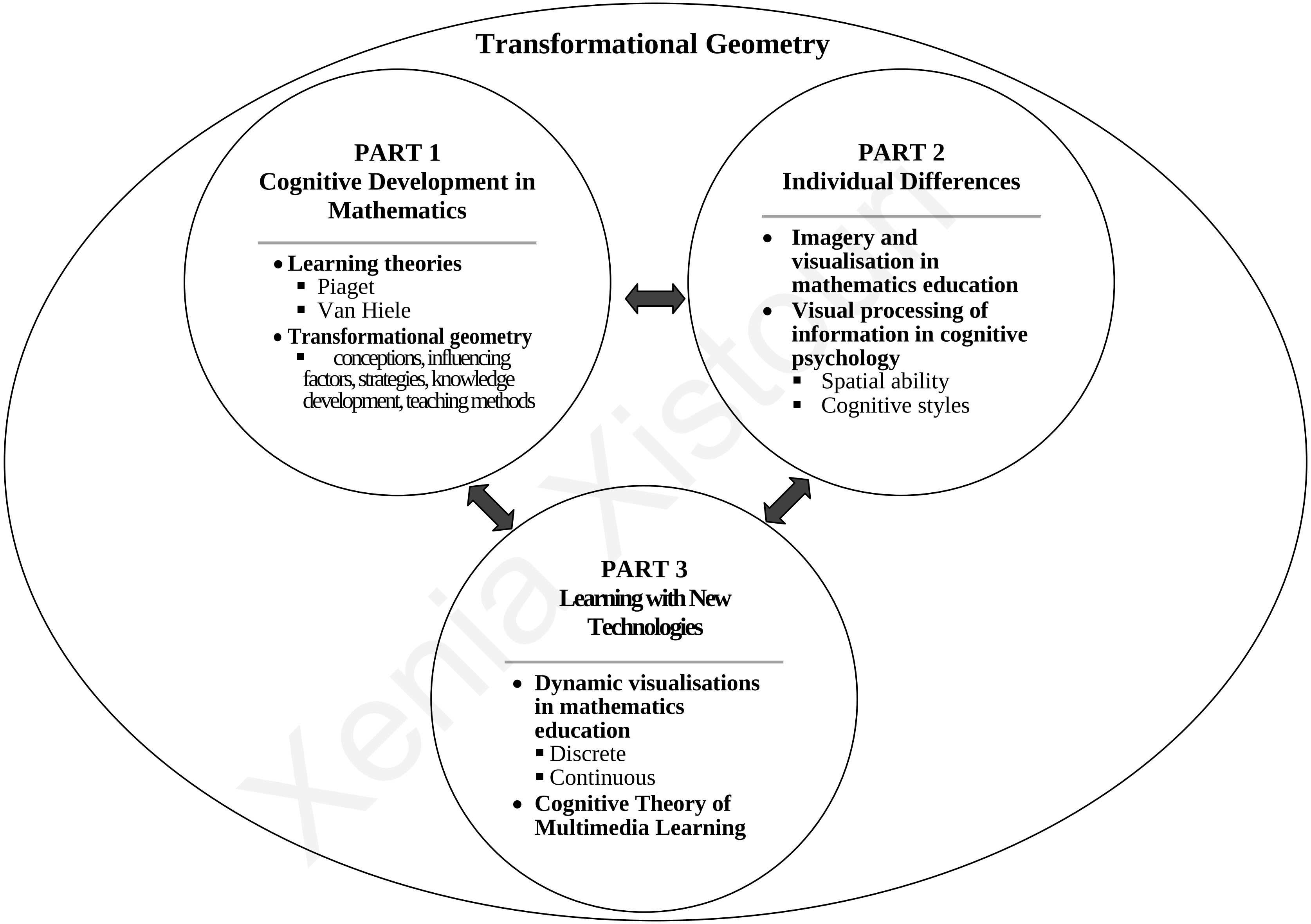





*Figure 2.1.* The structure of the theoretical framework.

## Cognitive Theories of Geometrical Development

### *Piaget's theory of spatial development*

In his first studies regarding the acquisition of spatial knowledge, Piaget's (Piaget & Inhelder, 1967) approach was an aspect of his more general study of cognitive development. He proposed that spatial knowledge is developed as children interact with their environment and the development of their spatial knowledge follows the sequence from simple, topological-like relations (proximity, order, surrounding etc.) to relatively complex projective- and Euclidean-like systems of relations. In other words, children progressively differentiate between geometric properties.

The theory was based on the influence of age development, to the transformations of real space to mental representations, and to the characteristics of real objects remaining invariant during these transformations. The developmental progression is as follows:

(1) the child first notes global properties that are independent of size and shape; these are the topological properties.

(2) the child becomes able to predict how an object will appear when viewed from different perspectives; these are the projective properties.

(3) the child learns the geometrical properties that relate to size, distance and shape, thus leading to differences between shapes based on spatial properties, such as the size of angles and the number of parallel sides; these are the Euclidean properties.

Thus, children can comprehend a spatial concept, such as the triangle, at a number of different levels:

(1) recognise and label an instance of a triangle, but without being able to explain why or to describe its properties,

(2) recognise the triangular shape in different contexts,

(3) recognise instances of the concept, describe what they have in common and how they differ from four-sided shapes, etc.

There are also other key aspects of this development of spatial knowledge. First, the distinction is made between the perception and representation of spatial knowledge. Perception is the knowledge of objects that one obtains as a result of direct interaction with them. This knowledge begins when children first become interested in their world during the sensori-motor stage. Representation refers to the child's capacity to reason about the spatial properties of an object when it is no longer present and to think about spatial concepts without referring back to specific objects. This is perceived as a form of mental imagery, whose development is thought that it begins towards the end of the sensori-motor stage (around the age of two) and becomes refined towards the beginning of the concrete-operational stage.

A second aspect of Piaget's theory is the distinction between two components of thought: the figural and the operative. The figurative aspect refers to fixed states; the child describes the spatial concept the way it is experienced. The operative aspect refers to mental operations that the child may apply to a spatial concept, for example, transforming between different states of a concept, and predicting outcomes of an intended transformation. A child that sees a square and identifies it as such is using figurative knowledge. A child that can reason that rotating a square would result in a diamond is using operative knowledge.

Various points in Piaget's model have received criticism. One point is that the distinction between perception and visual imagery would seem to be less clear in child development than Piaget would suggest. As noted by Dickson, Brown, and Gibson (1984), two-year old children show evidence of elementary mental visualisation. Another important point is that the use of terms as in the case of topological properties and Euclidean properties of objects are not mathematically precise (Clements & Battista, 1992; Darke, 1982).



*van Hiele's theory of spatial knowledge*

The second theory of spatial development is the van Hiele theory. The van Hiele theory originated in the respective doctoral dissertations of Dina van Hiele-Geldof and her husband Pierre van Hiele at the University of Utrecht, Netherlands in 1957. Based on their pedagogical experience and their teaching experiments, the van Hieles proposed a psychological/pedagogical theory of thought levels in geometry. For many researchers, such as Schoenfeld (1986), this model of thought levels provides a useful empirically-based description of what are likely to be relatively stable, qualitatively different, states or levels of understanding in learners. The van Hieles also proposed an accompanying model of teaching specifying five sequential phases of instruction, which they suggested were a means of enhancing students' thinking from one thought level to the next (this will be discussed in a subsequent section).

*Levels of geometrical thinking*

The general characteristics of each developmental level can be described as follows:

Level 1 - Visualisation: Students recognise geometric shapes as a whole (Shaughnessy & Burger, 1985). They can identify, name, and compare geometric shapes, such as triangles, squares, and rectangles, only in their visible form (Fuys, Geddes, & Tischler, 1988). No attention is given to the properties of these shapes (Mayberry, 1983). A figure is perceived as a whole recognisable by its visible form and only in some standard orientation. Students at this level "make use of imprecise qualities to compare drawings and to identify, characterise, and sort shapes" (Burger & Shaughnessy, 1986, p.43). Descriptions are based purely on visual appeal.

Level 2 - Analysis: Students at this level are able to reason about a geometric shape in terms of its properties. They see geometric shapes as collections of properties and they can recognise and name properties of geometric figures, without yet understanding the relationships between these properties and between different figures (Hoffer, 1981; van Hiele, 1986).

Level 3 – Abstraction (or Informal Deduction): Students can logically order the sequence of properties of figures previously identified and start to perceive the relationships between these properties and between different figures (Pegg, 1995). They can formulate definitions of simple geometric shapes using properties that are already known to them and can understand class inclusions (Mayberry, 1983; van Hiele, 1999). Moreover, they can make simple inferences.

Level 4 – Deduction: At this level, deduction becomes meaningful. Students understand the significance of deduction and the role of postulates, axioms, theorems, and proof (Hoffer, 1981). Students at this level should be able to justify the steps of a proof and also construct their own proofs (Pegg, 1995).

Level 5 – Rigor: Students can reason formally about mathematical systems. The necessity for rigor is understood and abstract deductions can be made (Usiskin, 1982). The students are able to analyse various deductive systems like establishing theorems in different axiomatic systems. Non-Euclidean geometries can be studied and different systems can be compared (Feza & Webb, 2005; Mayberry, 1983).

Clements and Battista (1992) argue that many school children exhibit thinking about geometric concepts in a more primitive manner, which is probably a prerequisite to van Hiele's Level 1. They, therefore, propose the existence of level 0, which they call "pre-recognition". Students at this level can distinguish between curvilinear and rectilinear shapes, but not among shapes in the same class. In this study, the existence of level 0 was taken into consideration.

According to Usiskin (1982), the important characteristics of this theory are the following: (1) fixed order, meaning that the order in which students progress through the thought levels is invariant, therefore a student cannot be at one level without having passed through the previous level, (2) adjacency, meaning that, at every level, what was intrinsic in the preceding level becomes extrinsic in the current level, (3) distinction, meaning that each level has its own linguistic symbols and own network of relationships connecting those symbols, and (4) separation, in the sense that two persons who reason at different levels cannot understand each other.

*Criticism on the van Hiele model*

As a result of its wide applicability the van Hiele theory has received criticism from mathematics education researchers worldwide. The van Hiele model has served as a basis for many research studies in mathematics education that tried to describe the development of significant mathematical concepts (Battista, 1999, 2011; de Villiers, 2010; Gutierrez, Jaine, & Fortuny, 1991; Molina, 1990). However, the majority of the studies mostly focused on the development of concepts related to the understanding of shapes (Battista, 2011; Gutierez, 1992).

Support for van Hiele's model of spatial learning has been provided by many investigators (Hoffer, 1983). However, studies that investigate the applicability of the theory in many geometrical topics suggest that there are some inconsistencies between the theory and observed behaviour. Studies have found that many children reason at multiple levels, or intermediate levels, which somewhat contradicts the van Hiele theory (Burger & Shaughnessy, 1986). In fact, there have been suggestions that students' geometrical reasoning develops from each level to the next through "overlapping waves" (Battista, 2011; Clements & Battista, 2001; Gutierrez, Jaime, & Fortuny, 1991; Siegler, 1996, 2005).

According to this view, the development of geometric reasoning is that students develop several van Hiele levels simultaneously.

Another point is that students advance through the levels at different rates for different concepts, depending on their exposure to the subject. They may, therefore, reason at one level for certain shapes and another level for other shapes (Mayberry, 1983). Some studies indicate that people exhibit behaviours indicative of different van Hiele levels on different subtopics of geometry, or even on different kinds of tasks (Clements & Battista, 1992, 2001; Denis, 1987; Gutierez & Jaime, 1998; Mayberry, 1983; Orton, 2004).

*The van Hiele model in transformational geometry*

There are very few research attempts to apply the van Hiele model in transformational geometry. Soon (1989) and Molina (1990) attempted to transfer the van Hiele model of geometrical understanding to the field of transformational geometry. Specifically, these studies focused on investigating the consistency between the van Hiele model of geometrical understanding and the understanding of transformational geometry concepts.

Nasser (1989) carried out a study with 24 secondary school students to investigate the validity of the van Hiele levels in transformational geometry and their relation to traditional geometry levels. She described the levels as: (1) Basic level, when students recognise and identify the transformations, (2) Level 1, when students identify and analyse the properties of the transformations as mirror-line (for reflection), and centre and angle (for rotation), (3) Level 2, when students recognise combinations and inverse transformations, (4) Level 3, when students understand the significance of deduction, the converse of a theorem and the necessary and sufficient conditions, and (5) Level 4, when students make formal demonstrations of properties and establish transformations in different systems. The results of the study indicated a possible hierarchy, and that the students were at higher levels of transformational geometry in comparison to traditional geometry. The results of this research are very superficial and cannot be generalised, since the authors themselves admit that they are based on a small sample that had only experienced instruction with transformational geometry. Moreover, the description of the levels is very epigrammatic. There is need for further clarification.

Soon (1989) examined whether the hierarchical levels of female secondary seniors learning concepts in transformational geometry are consistent with the van Hiele theory for geometrical understanding. Textbook material related to transformation geometry in

Singapore was also examined to determine whether they were consistent with van Hiele based levels. The subjects solved 31 problems, which were separated into levels based on the van Hiele levels of learning geometry as they relate to concepts in transformational geometry. The response patterns of the students seemed to support a possible hierarchy of the levels, consistent with the van Hiele theory. Soon described the levels as follows: (1) the Basic Level, which required students to visually identify and discriminate the transformations, (2) Level 1, where questions focused on a knowledge of the properties of each transformation, and what happens to each figure after being transformed, (3) Level 2, in which students were required to interrelate the properties of the different transformations by dealing with compositions of transformations and the use of matrices in transformations, and (4) Level 3, which required the use of transformations in proofs. Soon (1989) also found that the textbooks did not provide students with opportunities to explore and conjecture about transformations. However, according to the researcher these findings cannot be generalised, since they are based on a small, female-only sample. This is a main weakness of this study, considering that gender differences in geometry performance have been reported in favour of the males (Battista, 1990).

Both studies described above, seem to support their arguments for a hierarchy, based on the decrease in the number of students that were assigned at each pre-defined level. Since criticism in the van Hiele model suggests that the levels are discontinued and students may be at a transitional level, this method seems insufficient to confirm a hierarchy.

Molina's study (1990) also investigated and validated the applicability of the van Hiele model to transformational geometry, based on specific limitations, by comparing the van Hiele level of college students' responses to the transformational tasks with (a) the van Hiele levels of the tasks, (b) their own responses to translation, reflection, and rotation tasks, and (c) assigned van Hiele level of the students using the van Hiele Geometry Test (Usiskin, 1982) and the Burger and Shaughnessy Interview Instrument (1986). Molina (1990) described each level based on the types of tasks rather than students' reasoning. According to Molina (1990), the characteristics of each level are the following: (1) Level 0, where tasks are primarily visual and involve transformations but with no understanding of the properties or definition, (2) Level 1, where tasks involve a knowledge and understanding of the transformation properties, which includes orientation of the figure and knowledge about what remains invariant or changes under transformation, (3) Level 2, where tasks involve knowledge and understanding of the definition of the transformation and the ability to apply it to perform a transformation (here the definition includes the preservation of distance), and (4) Level 3, where tasks involve the use of deduction including the use of properties and definitions of transformations. Although this study provides a good description of the levels and good examples of tasks for each level, the assignment of the tasks to the levels seems to be arbitrary and no other criterion is used to determine that the tasks and behaviour of the students do form a hierarchy.

Although all of the above studies indicate that it is possible to describe a hierarchy of students' levels of understanding transformational geometry concepts, they are more focused on testing and confirming the application of the van Hiele model in transformational geometry understanding, and they arbitrarily define the levels *a-priori*, according to their subjective criteria. However, there seems to be some degree of agreement between the three studies regarding the abilities that each level should refer to. Another issue that these studies share is the fact that they describe their levels based on small-scale samples of groups of students who are at the same age. This makes it rather difficult to talk about development. Moreover, they are not concerned about the structure of ability in transformational geometry concepts and the components that synthesise this ability.

## Transformational Geometry

### *Studies in transformational geometry*



Although research in transformational geometry concepts only began about four decades ago, there is a great variety of studies investigating different aspects and variables that are important in the learning and understanding of these concepts. Apart from the early studies that focus on providing evidence that teaching geometric transformations in primary and secondary school education is feasible and may have positive effects on students' learning of mathematics (Usiskin, 1972; Williford, 1972), the rest of the research studies in the area of transformational geometry can be broadly categorised to the following types:

1) studies investigating students' conceptions and difficulties in transformational geometry concepts (Edward & Zazkis, 1993; Kidder, 1976; Law, 1991; Moyer, 1978);

2) studies investigating factors that influence students' ability in transformational geometry (Schultz & Austin, 1983; Thomas, 1978);

3) studies investigating students' strategies in transformational geometry tasks (Bansilal & Naidoo, 2012; Boulter & Kirby, 1994; Edwards, 1990; Naidoo, 2010);

4) studies investigating students' development of transformational geometry understanding (Flanagan, 2001; Molina, 1990; Nasser, 1989; Portnoy et al., 2006; Soon, 1989; Yanik, 2006); and

5) studies on the effects of technological tools on students' learning of transformational geometry concepts (Dixon, 1995; Edwards, 1991; Flanagan, 2001; Guven, 2012; Johnson-Gentile et al., 1994; Kirby & Boulter, 1999; Yanik, 2006).

It should be noted that, even though many of these studies used different types of geometric transformations and different types of tasks to conduct their research and draw conclusions concerning students' transformational geometry ability, we did not come across any studies that explicitly refer to or describe a model of the factors that contribute to the ability in transformational geometry concepts. Drawing on Kidder's (1976) assumption that performing transformations is a multi-faceted mental operation, this study collected tasks from previous research studies in order to form a theoretical model of the factors that synthesise students' ability in transformational geometry concepts.

The findings of the research studies that will be exploited for the structure of the unified model for students' ability in transformational geometry concepts are presented in the following paragraphs, according to their type of study.



*Students' conceptions and difficulties in transformational geometry concepts*

A number of studies have focused on investigating individuals' understanding and difficulties in transformational geometry concepts, at all levels of education (Bell & Birks, 1990; Edwards, 1990, 2003; Edwards & Zazkis, 1993; Flanagan, 2001; Glass, 2001; Harper, 2003; Hollebrands, 2003; Jung, 2002; Kidder, 1976; Moyer, 1978; Portnoy et al., 2006; Thaqi et al., 2011; Yanik, 2006, 2011). Some of these used technological tools as means of extracting learners' conceptions about transformational geometry concepts (Edwards, 1990, 2003; Edwards & Zazkis, 1993; Flanagan, 2001; Glass, 2001; Hollebrands, 2003; Jung, 2002; Yanik, 2006, 2011).

The first studies examined children's understanding of transformations through the Piagetian framework and, more precisely, through the operational and formal thinking framework (Kidder, 1976; Lesh, 1976; Moyer, 1974; Schultz, 1978). One of the first attempts was performed by Moyer (1974, 1978). Moyer (1978) argued against Piaget's position that children's spontaneous intuitive structures are built in close correspondence with structures that mathematicians have developed. He investigated the compatibility between the mathematical organisation of transformations and the cognitive structure of four- to eight- year-old children. Influenced by Piaget's experiments, Moyer interviewed children while solving nine tasks of translations, reflections, and rotations of marked circles. He found "a growth in more complex, projective- and Euclidean-like structures as grade level increases" (Moyer, 1978, p. 89) in the kind of cues children used to relate the before and after the transformation positions of the circles. The main conclusion from Moyer's (1978) study is that "mathematical and cognitive structures are not always in total accord" (p. 90). According to Moyer, children do not classify geometric transformations as translations, reflections, and rotations, and therefore the relative difficulty between the three is meaningless. This is because the difficulty does not lie within the mathematical nature, but within the cognitive nature of transformation. According to Moyer (1978), children perform scanning procedures to compare the figure to its image, and what determines the difficulty is the degree of the discrepancy between the two images.

Another attempt to study elementary students' understanding and difficulties in geometric transformations was performed by Kidder (1976, 1977). Kidder investigated the ability of 9-, 11-, and 13-year-olds to perform single transformations, compositions of transformations, and inverse transformations. He spent 10 to 15 minutes with each student instructing them in an "operational" definition of each of the three transformations through demonstrations of the transformations and discussion of student attempts at performing the transformations. Rigid figures were transformed according to operations indicated by wire models (lines and/or arrows) for each transformation by both the investigator and the student. The transformation test immediately followed the session. Students were asked to reproduce rigid motions of a triangle by building the images out of small sticks. Although Kidder (1976) stated that he patterned his investigation after Piagetian tasks, he found, contrary to Piaget's studies, that "the data did not support the experimental hypothesis that adolescents could perform Euclidean transformations, compositions of transformations, and inverse transformations at the representational level" (p.49). The most common type of error was in what Kidder referred to as "conserving length". By this, he referred to the ability of maintaining the lengths of the sides of a figure when constructing the congruent image. Kidder proposed that the ability to perform such complex tasks as geometric transformations requires attainment of the formal-operational level of thought, and that transformational tasks involve maintaining invariance of numerous distances and angles at one time. Edwards (1989) criticised that the training method, brief amount of practice time, and the rather awkward medium used to test the children, cast some doubt on the validity and generality of Kidder's conclusions, since later studies showed that middle and secondary age students were capable of carrying out a range of transformational tasks.

The largest study concerning students' understanding of transformational geometry concepts was part of an assessment of mathematics learning in British school children directed by Hart (1981). It was carried out by the Concepts in Secondary Mathematics and Science group of Chelsea College in London. The book *Children's Understanding of Mathematics: 11-16* (Hart, 1981) presented the results of large-scale testing of school children that studied a curriculum which includes transformational geometry over the course of three to four years. In this book, Kuchemann summarised and described the findings on the topics of reflections and rotations, as well as translations. However, assessment on translations was mainly based on vectors and matrices, which are different than the approach followed in this study, therefore that part of the report will be omitted.

In the rotations and reflections research, a total of 1026 13- to 15-year-olds were given a 52 item paper and pencil test. The test consisted of three parts: single reflections, single rotations, and combinations of reflections and rotations. For reflections the basic task was to sketch the result of a reflection over a mirror line, shown in various orientations on either a grid or plain background. In the rotations test students were asked to sketch the images of various figures after counter clockwise rotations of a quarter-turn. They were also asked to find centres of rotation. In the final section two types of questions addressed composite transformations. In the first task, children had to find an unknown transformation which, followed by a rotation, moved a shape onto its image. In the second task, students were asked to draw the image of a given shape after applying two sequential transformations, then to draw a mirror line or centre of rotation for the equivalent single transformation. Nearly all students experienced some success performing single reflections and rotations; however, most students had a difficult time performing combinations of transformations. The results of this assessment are summarised below:

*The results of the testing showed some increase in facility with these tasks with age and year in school. Furthermore, for the single transformations, an encouraging aspect of the results... is that nearly all the children tested had some understanding of reflection and rotation, which means a basis exists for studying the transformations in secondary school. At the same time, control over these transformations posed substantial conceptual problems to most children (which were far more severe than had originally been predicted).*

(Hart, 1981, p. 157)

Table 2.1.

*Characteristics of Motion and Mapping Understanding of Geometric Transformations* (Edwards, 2003)

| **Motion understanding** | **Mapping understanding** |
|---|---|
| The plane is an empty, invisible background | The plane is a set of points |
| Geometric figures sit *on* the plane | Geometric figures are subsets of points on the plane |
| Transformations are physical motions | Transformations are mappings of all the points |
| of geometric figures on top of the plane | of the plane |
| The “objects” of geometry are points, lines, | The “objects” of geometry are groups of transformations. |
| circles, triangles etc. | |

Based on Hart’s (1981) findings and tasks, Edwards (1990) conducted a study about children’s learning of transformational geometry concepts in a computer microworld. Edward’s study extensively investigated secondary school students’ understanding of geometric transformations during instruction. Having similar observations to Moyer (1974), that learners’ conceptualisation of transformational geometry is quite different than contemporary mathematicians’, Edwards proposed a different theory for comprehending students’ understanding of transformations. Based on the theory of embodied cognition (Lakoff & Nunez, 2000), Edwards suggests that learners can interpret transformations in two distinct ways: motion and mapping. In the “motion” understanding, all points in the plane can be carried onto the other points on the plane. That is, one may consider the plane as a background and manipulate geometric figures on top of the plane (Edwards, 2003). On the other hand, in the “mapping” understanding, motion is not an essential part of a transformation. In this conception, a transformation can be considered as a special function that maps all points in the plane to other points while preserving some properties and changing others (Edwards, 2003). The plane consists of an infinite number of points and geometric figures are considered as sets of points, which are a subset of the plane rather than as separate entities sitting on the plane (Edwards, 2003). Therefore, one needs to apply transformations to all points in the plane rather than a single object. The characteristics of each conceptualisation are presented in contrast in Table 2.1.

Many studies adopted this view (or part of it) for interpreting learners’ understanding of transformations (Edwards, 2003; Glass, 2001; Hollebrands, 2003; Yanik & Flores, 2009). Studies (e.g., Edwards, 2003; Glass, 2001; Hollebrands, 2003) revealed that elementary and high school students held a predominantly motion conception of transformations. Edwards (2003) found that when learning initial concepts within the domain of transformation geometry, students of various ages (e.g., middle school students, high school students, and college students) “had the same initial expectations of how transformations would work, and they made the same kinds of errors” (p. 4). For instance, in her several studies (e.g., Edwards, 1990, 1991, 1997, 2003; Edwards & Zazkis, 1993) Edwards revealed that students of different ages conceived transformations as motion. Edwards further discusses that learners’ general understandings of transformations arise from their embodied experience in the physical world. She describes how “the embodied, natural understanding of motion that the learners brought to the experience is the source of their misconceptions. These ‘misconceptions’ are in actuality, conceptions that are adaptive and functional outside the context of formal mathematics” (p. 9).

Hollebrands (2003) proposed that in order to have a mapping understanding of transformations, one needs to understand four fundamental concepts: (1) the domain of transformations as all points in the plane, (2) parameters (e.g., vector, reflection line, centre of rotation, and angle of rotation) that define transformations, (3) the relations and properties of transformations, and (4) transformations as being one-to-one mappings of points in the plane onto points in the plane. She found that the development of these understandings by students was important in their thinking about transformations as mappings rather than motions.

Glass (2001) worked with five eighth-grade students and explored their understandings of transformations (i.e., translations, reflections, and rotations) using *The Geometer's Sketchpad* [GSP] (Jackiw, 1991). Analysis revealed that students used (i) motion, (ii) end result, and (iii) property based reasons in identifying the transformation types being represented. The first category – motion – was based on the movement of the image from the location of the pre-image. According to Glass, students using this category showed an operational understanding of transformations. The second category - end result of motion - focused on comparisons between the pre-image and image characteristics that are direct results of the image movement. Glass stated that the use of end result-based reasons demonstrated a slightly more structural understanding of transformation than reasons from the first category "because the emphasis has shifted slightly from the movement of the image to characteristics of the image in comparison to the pre-image" (p. 173). The third category was reasoning related to properties of transformations. According to Glass, students most often used the first category, "motion", for transformation types followed by the second and third category.

Other researchers attempted to use different mathematics education frameworks to interpret students' understanding of transformations. Hollebrands (2003) studied four high school students' understanding of geometric transformations (i.e., translations, rotations, reflections, and dilations) in a technological environment in which the GSP and the TI-92 calculator were used. A seven-week instructional unit on geometric transformations was implemented and data were gathered through whole-class discussions and personal interviews. Students' conceptions of transformations as functions were analysed using the APOS theory and were informed by an analysis of students' interpretations and uses of representations of geometrical objects using the constructs of *drawing* and *figure* (Hollebrands, 2003). The results suggest that students were able to think about transformations as functions, provide information on how students reasoned about transformations in terms of figure versus drawing, and how this integration of understandings may have promoted students from working at a process conception to working at an object conception. This change was obvious when students began to reason about properties and behaviours of transformations without relying only on the drawing's appearance. The importance of Hollebrands's study is that it provides an alternative framework for the analysis of students' developing understandings of transformations along with some insight on how it can be influenced by particular dynamic geometry tools.

A similar study by Portnoy, Grundmeier, and Graham (2006), explored 19 prospective middle and high school teachers' views of geometric transformations in the context of an integrative approach to the teaching of transformational geometry in relation to linear algebra. Data were collected through interviews and were analysed based on the interiorisation-condensation-reification conceptual framework (Sfard, 1991), with some influence from the work of Tall concerning the process/object duality of geometric transformations and the conceived/perceived duality of geometric objects. In contrast to Hollebrands's (2003) research, Portnoy et al. (2006) suggest that student teachers viewed the transformations operationally and the geometric objects as "perceived". These two views seemed to inhibit the students from understanding and constructing geometric proofs in transformational geometry. The results of this study may support the view by Edwards (1990) and Hollebrands (2003) that technology supports the understanding of students when learning transformational geometry and there is a need to investigate how particular technological tools may enhance students' developing understanding of geometry.

Yanik (2006) investigated prospective teachers knowledge of rigid geometric transformations (i.e., translations, reflections, and rotations) based on APOS theory also. The study revealed that the prospective teachers had incomplete understanding of geometric transformations. Specifically, the prospective teachers had difficulties in recognising, describing, executing, and representing transformations (Yanik, 2006, 2011). Emphasising on translations, Yanik and Flores (2009) suggest that the prospective teachers conceived translations as undefined motion and geometric figures as separate from the plane. Results of the study showed that everyday experiences (e.g., walking and moving) influenced prospective teachers' reasoning about transformations. Another study with prospective mathematics teachers revealed a variety of conceptions about geometric translations: (1) translation as rotational motion, (2) translation as translational motion, and (3) translation as mapping (Yanik, 2011).

Finally, other smaller-scale attempts were performed to study conceptions and difficulties in transformational geometry concepts. One of these was a study by Malone, Boase-Jelinek, Lamb, and Leong (2004), who found that about 40% of school students misperceived on very simple tasks of reflections. A *misperception* is defined as "the act of perceiving via a single sensory modality (e.g., seeing in mathematics or hearing in music) something that is different from reality or an imagined reality (e.g., visualising a rotated shape in maths)" (Malone et al., 2004, p. 1). Similarly, Thaqi et al. (2011) found that prospective teachers conceived transformations as motion and their comments about transformations were based on their intuitions.

### *Factors influencing ability in transformational geometry tasks*

In a chapter dedicated to research issues concerning transformational geometry in primary school by Lesh (1976), one of the most important questions he addressed was the investigation of factors that influence the difficulty of tasks, such as the transformation group, the complexity of the shape, the size of the transformation, and the type of the transformation. He discussed that transformational geometry tasks that may be operationally isomorphic (in the Piagetian sense) are often different in their degree of difficulty. This means that if the transformation task involves a configuration that is too difficult for the child to deal with, the operational complexity is likely to increase. This is because the students' ability to perform a transformation is not enough on its own; an additional ability is required to interpret the configuration of the transformation.

There are several studies in mathematics education that investigated whether and how such factors can differentiate the difficulty of a transformation task and have impact on students' performance (Grenier, 1985; Kidder, 1976; Lesh, 1976; Moyer, 1974; Schultz, 1978; Thomas, 1978). Many of these studies were based on Piaget's theory of cognitive development concerning the development of geometrical thinking and were influenced by the methodology and findings of his studies. According to Piaget (Piaget & Inhelder, 1971) there are both operative (of the transformation) and figurative (of the figure) attributes that can influence a child's ability to perform a transformation.

Thomas (1978) conducted a study where students of primary and secondary school were interviewed while solving Piagetian-like tasks. The variables that this study examined were all operative and focused on whether children realise: (1) the invariance of length, i.e., the measures of the shape, (2) the invariance of incidence, i.e., the preservation of incidence and distance, and (3) the transformation of orientation in the plane. Her results suggest that (i) the conservation of length was more difficult in the case of translation rather than in the case of reflection or rotation; and that (ii) no differences were found in performance between reflections over a vertical line and reflections over a horizontal line, or between clockwise and counter-clockwise rotations (Thomas, 1978). The most important outcomes of this study were that the difficulty level of a transformation task is affected by the attributes of the configuration used, and that the difficulty varies both over the geometric transformations and over age group.

The above findings are in accord with a study performed by Schultz (1977). Schultz investigated the influence of operative and figurative attributes as factors influencing the difficulty of rigid transformations. The operative attributes were the direction (horizontal, vertical, and diagonal) and size of the displacement, and the figurative attributes were the familiarity and size of the configuration. According to the findings of this study it is over-simplistic to compare the difficulty of transformations within an isometry. In other words, one cannot say that all reflections are easier or harder than rotations. The various attributes have differing effects on difficulty of the different isometries and at different ages (Schultz, 1977, 1978; Schultz & Austin, 1983). For example, one can say that diagonal reflections are more difficult than diagonal rotations at the age of seven, but the same does not apply for horizontal reflections with rotations (Schultz, 1977). Moreover, the operative attributes must be in co-ordination with the figurative attributes, since there are characteristics of the figure that are influential enough to interfere with the operation of the transformation. However, the operative attributes are more influential and interfere more with understanding than the figurative attributes.

This study was similar in its attention to task variables to a research conducted by Grenier (1985). Grenier's work with reflections examined in more depth the factors that seem to influence students' ability to make freehand drawings of transformations. Her work, which took place in the context of actual classroom instruction, is valuable in highlighting cognitive and perceptual factors that can affect how students think and carry out reflections. Grenier found that freehand constructions of reflections performed by 11- to 14-year-olds were influenced by such variables as the orientation of the mirror line, the use of squared or plain paper and the complexity of the figure. Similar variables influenced the students' ability to recognise and draw the axis of symmetry of various figures. However, her study was limited to reflections.

All of these studies provide valuable information for the acquisition of students' ability in geometric transformations. However, even though they provide some insight into the factors that may influence the difficulty level of geometric transformations and different age groups, they do not provide a sequence or a developmental structure for this ability. Moreover, although these studies provided a variety of activities and indications for factors that influence students' ability to perform a transformation, they did not study students' strategies and their variations in relation to these factors.

*Students' strategies in transformational geometry tasks*

Investigating students' strategies is important, as it can enlighten both successful processes for comprehending concepts, as well as serious misconceptions that can be avoided with proper instruction. In the field of transformational geometry few studies have investigated students' strategies while solving transformational geometry tasks (Bansilal & Naidoo, 2012; Bellin, 1980; Boulter & Kirby, 1994; Naidoo, 2010). Moreover, some of these studies have related students' strategies to performance in geometric transformation tasks.

One of the most important studies in this category is the one by Boulter and Kirby (1994). This study analysed elementary school students' strategies based on the holistic-analytic distinction, when solving geometric transformation tasks. Students' strategies were classified as holistic when the task was solved with the visualisation of the whole shape as an entity and as analytic when the shape was visually fragmented and transformed piece by piece. The results of this study indicated that some students showed preference for either the holistic or the analytic processing and that use of analytic strategies was associated with success. What is interesting in these researchers' findings is that some items were more likely to elicit holistic (or analytic) strategies. Boulter and Kirby (1994) also described how the analytic strategy was more beneficial for some tasks and the holistic strategy for other tasks, while some tasks could be successfully solved by either strategy. It is possible that strategy choice and its efficiency depend on the type of task. They concluded that "both question type and individual preferences seem to contribute to strategy selection" (Boulter & Kirby, 1994, p. 302). Previous studies (Kirby, 1990; Kirby & Schofield, 1991) implied that the two kinds of strategies may depend upon distinct spatial abilities. Specifically, the holistic strategy was associated with "spatial orientation" ability and the analytic strategy with "spatial visualisation" ability. However, this has not been empirically investigated.

On the contrary, a recent study by Naidoo (2010) suggested that learners who have a visual (versus analytic) understanding could be better in understanding the effects of transformations on figures as a whole rather than focusing on isolated points. According to Boulter and Kirby (1994), the successful use of analytic strategies was more likely to relate to success in transformational geometry tasks. Bansilal and Naidoo (2012) suggested that perhaps a facility in moving between visual and analytic representations is a sign of deeper understanding of transformational geometry concepts. Boulter and Kirby (1994) cautiously noted that the use of analytic strategies may reflect either greater school learning or greater willingness to use such strategies. They did not rule out the possibility that the use of analytic strategies reflects greater cognitive development, since in some domains children seem to proceed from holistic to analytic strategies (Boulter & Kirby, 1994). Therefore, the researchers conjectured that the "preferences [in strategy] may be dictated by the level of cognitive development. The roles of strategy instruction and development levels in transformational geometry problem solving have yet to be determined" (Boulter & Kirby, 1994, p. 303).

While Boulter and Kirby (1994) encourage cognitive development and strategic flexibility to students, Bellin (1980), in a study with six-year olds undertaking transformations activities, found successful pattern of strategy use and concluded that strategy use, rather than presumed cognitive structure, explained the results. In a study with four- to six-year-old children solving spatial problems with shapes, Mansfield and Scott (1990) report a range of strategies. Translations were commonly used, but only high-performing students used rotations and reflections. In general, older children solved more problems than younger children; however, this was mainly a result of persistence rather than use of more efficient or varied strategies (Mansfield & Scott, 1990).

Although the attempts for describing students' strategies so far have provided good information, there seems to be a need for deeper analysis and greater precision (Boulter & Kirby, 1994). It seems that the results of existing studies are not so clear regarding the kind of strategies that are more successful for students and under which circumstances, i.e., type of tasks. Perhaps such research would exploit a model that is not bipolar and it would consider both students' preferences and tasks' demands simultaneously. Further research is required to clarify this matter.



*Students' development of transformational geometry understanding*

One of the first debates in the transformational geometry field of research was the structure of development for learning geometric transformations. According to Piaget (Piaget & Inhelder, 1971), children learn transformations in the order translations, reflections, and rotations, and that the anticipatory level precedes the representational level of understanding. Kidder (1976), Lesh (1976), Moyer (1978), and Schultz (1978), in a more general context, have argued against assuming that the mathematical structure is always in accord with the child's cognitive structure. Therefore, one cannot be absolute that children conceive the transformations as translations, reflections, and rotations in the mathematical sense. It seems possible that they may be using some entirely different system of relations to describe geometric transformations (Lesh, 1976). This can be an explanation why researchers disagree on the order of difficulty in learning geometric transformations. While Moyer (1978) suggested that translation is at least as easy as reflection, even though reflection can be considered mathematically primitive, Schultz and Austin (1983) suggest that translations seem to be the easiest transformations for students and that the direction of the transformation influences the relative difficulty of rotations or reflections.

Lesh (1976) suggested that the appropriate way to analyse the transformational geometry tasks is not to focus on translations, reflections, and rotations. Moyer (1978) suggested that one important property may be related to “up-down” and “left-right” changes, and Kidder (1976) suggested orientation in the sense of vertical, horizontal, and diagonal. Towards that direction, Schultz (1978) tried to include properties of the transformation and of the figure to further investigate the development of students’ understanding of transformations. Her results, however, do not lead to a conclusion on what the differences are in the way that children conceive these tasks and are still very much adherent to the translations-reflections-rotations structure. One of the main disadvantages of these studies is that, since the experiments were influenced in a large extent by the methodology of Piaget’s experiments, they only investigated students’ ability in a single type of task, that of finding the image or a point on it.

The first attempts that used a variety of tasks to study the development of transformational geometry concepts as a sequence of levels were based on the van Hiele model of geometric understanding. As described earlier in this chapter, Molina (1990), Nasser (1989), and Soon (1989) tried to investigate the development of transformational geometry concepts based on the van Hiele model. These investigations did not only include tasks of executing and identifying transformations (Edwards, 1990; Hart, 1981), or performing and inversing transformations (Kidder, 1976). Moreover, these studies refer to a variety of tasks matched to each level, including visual recognition of the three rigid transformations, understanding the properties of the transformations, relating the properties of the transformations and tasks of formal definition and deductive proof. However, these studies seem to be more concerned with confirming the applicability of the van Hiele theory in another domain of geometry, that of transformational geometry, rather than understanding the development of transformational geometry concepts itself. It is as if they are trying to indirectly describe the development of geometric transformations. This attempt may have some problems, especially if one considers that the van Hiele model has been criticised for the fact that people exhibit behaviours that refer to different van Hiele levels on different subtopics of geometry. Moreover, the studies are still highly influenced by the translation-reflection-rotation mathematical distinction. Even though they do not refer to the acquisition of geometric transformations in this order, they study their development in parallel (thus arbitrarily assuming that all three geometric transformations develop in the same way and pace) and focus more on the mathematical structure rather than the learner's cognitive structures (Lesh, 1976).

Law (1991) also looked at the hierarchy of the acquisition of the concepts of transformational geometry in an attempt to determine how pre-service elementary school teachers construct the concepts of translation, reflection, and rotation. Eighteen pre-service primary school teachers at the second college level mathematics class on geometry received lectures about the three transformations. Each student was interviewed following the coverage of the material. "Based on his understanding of the concept, the researcher conjectured that students learn the concept in the order of learning definition of transformations first, and then single-point movement, then figure movement, and finally identification of transformations" (Law, 1991, pp. 72-73). Furthermore, Law admits to implicitly assume that a student cannot understand any concept without understanding the previous concept in the given order. Unfortunately, the examples that Law used during his interview were limited in number (eight questions) and the fact that each subject received one interview does not seem sufficient to develop a hierarchy.

The most recent attempt for describing the development of knowledge and understanding of transformations was performed by Yanik and Flores (2009). This study focused only on creating a learning hierarchy for translations based on a case study, and it is perhaps a step towards understanding geometric transformations development not as mathematical entities that are developed in sequence or in parallel, but as structures that can develop independently. Again, a variety of tasks were used, which were organised into the categories of recognising, describing, performing, and representing a translation. The findings of this study indicated that the development of a prospective teacher's thinking about translations specifically, and of other rigid transformations started by (i) referring about transformations as undefined motions of a single object, followed by (ii) using transformations as defined motions of a single object, (iii) understanding about transformations as defined motions of all points on the plane, and (iv) mapping of the plane onto itself. The researchers provided a diagram of a theoretical model which displays the transitional understanding of learners as they progress from a motion understanding towards a mapping conceptualization of rigid transformations, and shows the connections between these two understandings (see Figure 2.2). Yanik and Flores (2009) guided research into revising and improving this theoretical model for other subjects and for other concepts. They also noted that it is important to investigate what factors may be involved in conceiving transformations as mapping.

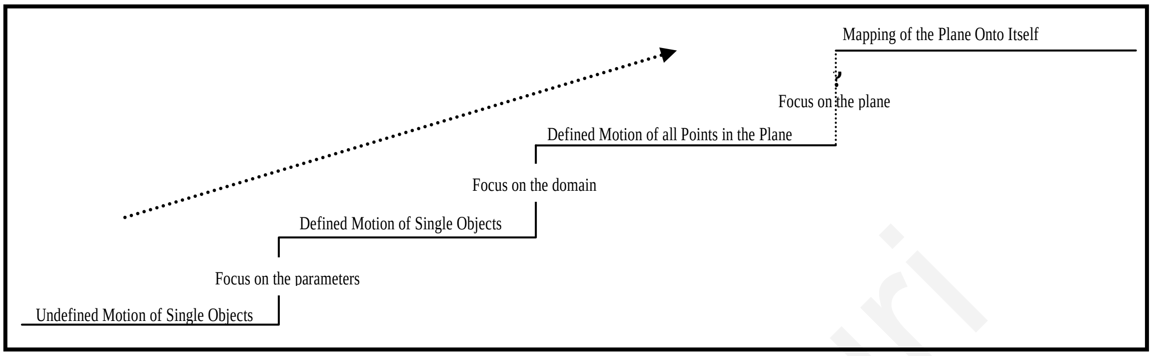



*Figure 2.2.* A hypothetical model for translation learning trajectory (Yanik & Flores, 2009).

*The effects of technological tools on students' learning of transformational geometry*

A large proportion of studies in the field of transformational geometry have involved computers in the presentation of transformations, either as a means of investigating students' conceptions or for improving learning outcomes or for both (Edwards, 1991; Edwards & Zazkis, 1993; Ernest, 1986; Guven, 2012; Hollebrands, 2003; Johnson-Gentile, 1990; Pleet, 1990; Yanik & Flores, 2009). This part will focus on the role of technology for improving learning outcomes. These studies involved teaching with (a) microworlds (e.g., Edwards, 1990; Johnson-Gentile, 1990; Ludwig, 1986), (b) virtual manipulatives or games (e.g., Ernest, 1986; Sedig, 2008), and (c) DGS (e.g., Dixon, 1995; Flanagan, 2001; Guven, 2012; Yanik, 2006).

Edwards (1990, 1991) conducted an investigation involving the use of TGEO microworld, a Logo-based computer software, by small groups of students that had no previous instruction on transformational geometry. Her studies investigated "the learning of a small group of children who interacted over a short period of time with a computer microworld dealing with transformation geometry" (Edwards, 1991, pp. 122-123). The findings suggested that the microworld and the associated activities were effective in assisting the students to construct a working knowledge of transformations and that the 12 students in grades six through eight who made up this study were able to carry out any transformation on the computer without error or hesitation. Moreover, they improved their performance in a written test, which was administered as a pre- and post-test with tasks similar to Hart's large-scale study.

In another classroom-based study carried out by Ludwig (1986), a middle-school class was investigated over a period of 11 weeks as they worked through a Logo-based curriculum for transformational geometry. Ludwig's study attempted to compare levels of understanding in transformational geometry to those in Logo, based on schemes worked out by van Hiele and Kieren (Olson, Kieren, & Ludwig, 1987). Ludwig asked her students to programme simple transformations from scratch in Logo and analysed their progression of geometric understanding in terms of the van Hiele model. She found that, at the beginning of the project, the students showed Basic Level thinking and constructed visual images of the transformations through direct application of Logo commands using trial-and-error. As the study progressed, the students "showed movement toward Level I thinking as they began to view the transformations as composed of discrete pieces. The original and image were seen as a group of commands and were eventually represented under a single name as a sub-procedure" (Ludwig, 1986, p. v). Ludwig's study was concerned both with the learning of Logo programming and the development of geometric thinking, and she acknowledged the difficulty of separating the effects of programming skill from a conceptual understanding of the mathematical ideas.

Pleet (1990) compared the use of the Motions computer programme to the use of the Mira hands-on manipulative on eighth grade students' ability to perform transformations and mental rotations. There was no significant difference between the Motions computer programme group and the Mira hands-on manipulative group on acquisition of transformation geometry concepts or mental rotation ability.

Johnson-Gentile (1990) also investigated the effects of computer and non-computer environments on students' achievement with transformational geometry. The researcher examined the effects of a Logo-computer version and a non-computer version of a "motions" unit. The non-computer groups worked with paper and pencil, transparencies and the Mira, while the Logo groups worked with Logo on the computer. There was also a control group that did not participate in any "motions" activities. A pre-test of achievement in geometry was given to all groups. The motions unit lasted two weeks. Both treatment groups scored significantly higher than the control group on the post-test and the retention test. There was no significant difference between the Logo and non-computer groups on the post-test, but there was a significant difference between the groups on the retention test.

The Logo group scored higher on the retention test compared to the post-test and the non-computer group scored lower. There was no significant difference between the Logo and non-computer groups on interview measures, but both treatment groups scored significantly higher than the control group on the same measures.

The results of these studies suggest that engaging students with logo-based activities may have positive effects on their understanding of transformations. However, their effects are not significantly better than using traditional methods of teaching and they require pre-existing knowledge of the program language or of the geometric transformations algebraic notions. Perhaps this is one of the reasons why teaching with Logo microworlds was soon abandoned by researchers and research in transformational geometry concepts turned to evaluating the effectiveness of DGS.

Ernest (1986), who also used items from Hart’s (1981) test just like in Edward’s study (1990), conducted an experiment to determine the effects of computer gaming on the performance of 15-year-old students in transformational geometry. His study differed from the previously described studies in that his students did not need to learn a computer language in order to work with the computers. In addition, he used comparative statistics rather than qualitative methods to evaluate students’ learning. The treatment group played transformation computer games, whereas the control group played computer games having no relation to transformations. While the experimental group performed significantly better on the transformations specifically related to the game, there was no significant difference in performance on the general test. Similarly, in a study investigating the effects of game-based environments using virtual Tangrams, Sedig (2008) found that grade six students exhibited significant improvement in their knowledge of transformational geometry concepts.

The earliest work found to employ DGS in transformational geometry teaching was by Dixon (1995). Dixon (1995, 1997) studied the effects of a dynamic instructional environment, the GSP. The GSP is a highly visual and dynamic tool for exploring and discovering geometric properties. The study was classroom-based. It took place in a computer lab and investigated 241 eighth grade students’ understanding of the concepts of reflection and rotation. Dixon divided the students into a control group and an experimental group. The control group used a standard textbook, mirrors, Miras, and paper folding to explore and learn the geometric ideas. The experimental group used GSP and lessons pre-planned by Dixon. She further divided the students to control for student level of computer use and visualisation ability. After controlling for these initial differences, Dixon concluded that students experiencing the dynamic environment significantly outperformed students experiencing a traditional environment on four post-tests measuring the concepts of reflection and rotation. Hoong and Khoh (2003) also supported that instruction with GSP can significantly improve 13- to14-year-old students' transformational geometry ability, but only if instruction is based on guided-inquiry and exploration of concepts with the students' active engagement, in contrast to teacher-directed demonstrative instruction.

A more recent study by Guven (2012) used another DGS, namely *Cabri*, to examine the effects of DGS on students' learning of transformational geometry. During this study, pre- and post-tests were administered to 68 eighth grade students. The students were divided into an experimental group and a control group. The experimental group studied transformational geometry using *Cabri*, and the control group received the same instruction with dotted isometric worksheets. The findings of the study suggested that the experimental group outperformed the control group in their transformational geometry performance.

Many more studies have used the GSP to explore how instruction with DGS mediates high-school students' and pre-service teachers' learning of geometric transformations (Flanagan, 2001; Harper, 2002; Hoong & Khoh, 2003; Jung, 2002; Yanik, 2006). Even though these studies were not comparative with other instructional methods, they all provided evidence and valuable information about how an interactive dynamic visualisation environment can enhance students' learning of geometric transformations. According to Harper (2002), the GSP provided immediate visual feedback, which aided the participants to "conjecture, test, and revise their solutions". Flanagan (2001) explains that:

> *…technology afforded students opportunities to act on concrete objects and treat them as if they were the actual theoretical model. This ability to engage in activities that allowed them to act on objects and observe their behaviours, which in turn reflected mathematical properties, might have promoted the development of students' reasoning about transformations.*
>
> (Flanagan, 2001, p. 367).

Even though the findings of these studies support the potential of DGS to enhance understanding, there seem to be some doubts lately on whether dynamic content is best presented by means of dynamic visualisations (Ploetzner & Lowe, 2004). The argument of this position is the possibility of dynamic visualisations to place great information processing requirements on learners and overburden their cognitive capacities. Hegarty (2004b) draws attention to the fact that there are many different types of dynamic displays and that it might be simplistic to generalize results from studies with one dynamic display. Therefore, it is not just a simple comparison between static and dynamic displays anymore, it is also important to examine the relative effectiveness of different types of dynamic displays, for different types of learning and content (Hegarty, 2004b) and for different types of learners (Boulter & Kirby, 1999; Hegarty, 2004a; Höffler & Schwartz, 2011).

## Individual Differences in Imagery and Visualisation

### *Imagery and visualisation in mathematics*

The nature of visualisation and imagery has been a major research area in psychology for approximately a century (Kosslyn, 1996). A mental image is an internal representation that produces the experience of perception in the absence of the appropriate sensory input (Wraga & Kosslyn, 2002). There are many information processing studies that have explored how imagery might be stored in the mind. Paivio (1971) proposed the dual-coding information processing theory, where information can be stored, represented, retrieved, and processed in two ways: verbally and visually. Visual imagery refers to the ability to form mental representations of the appearance of objects and to manipulate these representations in the mind (Kosslyn, 1995). However, not all research supports the dual-coding information theory.

According to Presmeg (2006a) visual imagery (internal representation) underlies the creation of a drawing or spatial arrangement (external representation). Presmeg (1997b) defined visualisation as the processes of constructing and transforming visual mental imagery, as well as all of the inscriptions of a spatial nature that may be implicated in doing mathematics. Similar to Presmeg, Hegarty (2004a, p.1) defined visualisation as “any display that represents information in a visual-spatial medium”, and discriminated between external representations that can be viewed printed on paper (static) or shown on a computer monitor (either static or dynamic), and internal visualisation, which is a representation in an individual’s mind. A more general definition is that visualisation is the ability to "represent, transform, generate, communicate, document, and reflect on visual information" (Hershkowich, 1989, p.75).

Most researchers agree that such visual representations are important in mathematics education because they enhance an intuitive view and an understanding in many areas of mathematics (e.g., Krutetskii, 1976; Usiskin, 1987). Imagery and visualisation have played a major and continuing role in mathematics education for many years (Saads & Davis, 1997). Ben Chaim et al. (1989) indicated that visualisation is a good predictor of problem solving performance. However, the wide use of visual images by students is not always effective in problem solving and can lead to erroneous solutions (e.g., Lean & Clements, 1981; Presmeg, 1992).

In mathematics education, some investigations focused on the mental processes used in solving mathematical problems, and particularly the role of diagrams and visual-spatial images in mathematical problem solving. In these studies, students reported their solution processes after solving problems or while solving problems. On the basis of such studies, Krutetskii (1976) concluded that individuals can be classified into three groups according to how they process mathematical information. The first group consists of *analytic type,* who prefer verbal-logical rather than imagery modes when attempting to solve problems; the second group, *geometric type,* involves those who prefer to use visual imagery; and the third group, *harmonic type,* contains individuals who have no tendency one way or the other.

Following the Krutetskii model, Presmeg (1986a, 1986b, 1992, 2006a) identified five kinds of imagery used by high school students in solving mathematical problems: (a) concrete pictorial imagery (having characteristics of a picture), (b) pattern imagery (pure relationships stripped of concrete details), (c) kinesthetic imagery (involving physical movement), (d) dynamic imagery (in which the image itself is moved or transformed), and (e) memory of formulas (memory images of formulae written on a blackboard or in a notebook). These five types of imagery were identified in the transcripts of mathematical task-based interviews with 54 high school learners over a complete year when they were in Grade 12 (Presmeg, 1986b).

Presmeg (1986a, 1986b, 1992) argued that the use of concrete pictorial imagery may focus reasoning on irrelevant details that take the problem solver's attention from the main elements in the original problem representation, whereas other kinds of imagery may play a more positive role. She ascribed the most essential role in mathematical problem solving to pattern imagery, in which concrete details are disregarded and pure relationships are depicted. This kind of imagery was also identified by other researchers (Johnson, 1987; Owens, 1993).

Presmeg (1986a, 1986b, 1992, 2006a) referred to another type of visual images that may also bring attendant difficulties to learners, known as “prototypical images” or “prototypes”. The prototype image may include only some of the defining attributes of the concepts, not necessarily all of them. Moreover, it may include one or more non-defining attributes, for example the up-right position of a right triangle. Vinner and Hershkowitz (1983) indicate that children’s visual images often include only the prototypes. Individuals use the prototypical example as a model in their judgments of other instances (Hershkowitz, 1989, 1990). A prototypical example is intuitively accepted as representative of the concept. That is, it is accepted immediately, with confidence, and without the feeling that any kind of justification is required. Yet intuitively accepted cognitions may also cause obstacles as they have a “coercive impact on our interpretations and reasoning strategies” (Fischbein, 1993, p. 233).

Presmeg’s classification of imagery has provided an expanded view of visual imagery, which is both succinct and relevant to students’ processing of information during classroom activities. Interpreting figural information (diagrams) and visual processing were seen by Bishop (1983) as being two distinct processes. Students’ performance on visual tasks could result from either or both of these processes and it is not easy to provide tasks of visual processing that do not call on students’ abilities to interpret figural information (diagrams). However, studies in mathematics education did not relate spatial ability to the use of different types of imagery.

The research of Hegarty and Kozhevnikov (1999) differentiates between two different visual imagery abilities identified in cognitive psychology and neuroscience research. This research provides evidence that visual imagery is not general and undifferentiated but composed of different, relatively independent visual and spatial components. Visual imagery refers to a representation of the visual appearance of an object, such as its shape, colour, or brightness. Spatial imagery refers to a representation of the spatial relationships between parts of an object and the location of objects in space or their movement; furthermore, spatial imagery is not limited to the visual modality (i.e., one could have an auditory or haptic spatial image). They argue that a dissociation between visual and spatial imagery also exists in individual differences in imagery— some individuals are particularly good at *pictorial* imagery (i.e., constructing vivid and detailed visual images), whereas others are good at *schematic* imagery (i.e., representing the spatial relationships between objects and imagining spatial transformations), and suggest that these differences appear in their mathematical problem solving strategies. They consider spatial ability as a subset of imagery abilities, related to schematic imagery and not related to pictorial imagery. In their later studies they refer to the former as *spatial visualisation* and to the latter as *object visualisation* (Kozhevnikov et al., 2002).

Research on imagery and visualisation is fundamental for many fields of psychological studies and studies in mathematics education. Examples of these fields are spatial information processing, cognitive style, and problem solving strategies. The different types of imagery have served as a basis for many studies regarding different types of spatial abilities and different cognitive styles, which will be presented in the following sections.

### *Spatial ability*

Spatial ability has been the focus of attention for over 60 years, mostly by psychologists, but also by researchers in mathematics education. Even though many definitions have been used to describe this ability, there is still not a generally accepted definition. One of the early definitions of spatial ability given by researchers in mathematics education was "the ability to formulate mental images and to manipulate these images in the mind" (Lean & Clements, 1981, p. 267). In the field of psychology, Lohman (1993) defined spatial ability as the ability to construct, retain, retrieve, and manipulate visual images of two- and three-dimensional objects. However, the most broadly accepted definition in the field seems to be the one proposed by Linn and Petersen (1985) in their meta-analysis on spatial abilities studies. Linn and Petersen (1985, p.1482) defined spatial ability as a skill referring to "representing, transforming, generating, and recalling symbolic, nonlinguistic information".

Spatial ability was found to relate to mathematics and science achievement (e.g., Guay & McDaniel, 1977; Hegarty & Waller, 2006; Tracy, 1987, 1990), and is even argued to be essential to scientific and mathematical thinking (Clements & Battista, 1992). Furthermore, spatial ability has been linked to success in several occupations, such as piloting, mechanics, engineering drawing, and surgery (Hegarty & Waller, 2006). Therefore, it seems important to teach spatial ability in schools. The NCTM (2000) acknowledged the importance of spatial ability by including spatial skills in the US curriculum standards for primary and secondary school geometry education.

The development of spatial ability is considered directly related to geometrical thinking (Battista & Clements, 1991; Bishop, 1980, 1983; Hersckowitz, 1990). In their chapter on *Geometry and Spatial Reasoning*, Clements and Battista (1992) suggested the existence of two factors considered important for the learning of geometry. These were *Spatial Visualisation* and *Spatial Orientation.* They defined spatial visualisation as the "comprehension and performance of imagined movements of objects in two- and three-dimensional space" and spatial orientation as the "understanding and operating on the relationships between the positions of objects in space with respect to one's own position" (p. 444). They stressed the importance of spatial ability in students' construction and use of mathematical concepts, and emphasised that its role in the construction of concepts is elusive and multifaceted.

Even though research in psychology around spatial ability is much more substantial, there does not seem to be a unified theory for this concept. Moreover, there seems to be a disagreement even on the nature of spatial ability. On one hand, some researchers, such as Burton and Fogarty (2003) and Colom, Contreras, Botella, and Santacreu (2002), supported that spatial ability is a single component. They claimed that, even though there were many attempts to prove that spatial ability consists of different components, the results were not convincing. On the other hand, the most common belief was that spatial ability is multi-component (Caroll, 1993; Linn & Petersen, 1985; Lohman, 1988; McGee, 1979).

Around the middle of the 20th century, research focused on determining the factor structure of spatial ability. Several researchers (Guilford & Lacey, 1947; French, 1951; Thurstone, 1950; Zimmerman, 1954) found that large batteries of spatial tests yielded evidence of several distinct subcomponents of spatial ability. However, theorists not only argued on the number of factors composing spatial ability, but also on how to best characterise these factors. Michael, Guilford, Fruchter, and Zimmerman (1957) where the first to propose a three-factor model, naming the two factors as (a) *Spatial Visualisation*, which is related to the mental manipulation of objects, (b) *Spatial Relations and Orientation*, which is considered as the ability to understand the arrangement of elements within a visual stimulus, primarily with respect to one's body of reference, and (c) *Kinesthetic Imagery*, which is associated with left-right discrimination. Many years later, McGee (1979) proposed a different model with two factors, namely *Spatial Visualisation*, defined as the ability to imagine manipulating, rotating, twisting, or reversing objects without reference to one's self, and *Spatial Orientation*, defined as the ability to imagine the appearance of an object from different perspectives.

In their meta-analysis, Linn and Petersen (1985) devised a classification of three types of spatial ability factors: (a) *Spatial Perception*, which is the ability to determine spatial relationships between two objects, most notably between an object and the test subject, (b) *Mental Rotation*, which is the ability to determine the new position of a 2D or 3D object which has been rotated from a certain position, and (c) *Spatial Visualisation*, which is defined as the ability to manipulate information sequentially and spatially. Maier (1998) expanded upon the work of Linn and Petersen (1985), identifying two more elements of spatial ability: *Spatial Orientation*, which he defined as the ability of a person to orient his or her self physically or mentally in horizontal and vertical space, and *Spatial Relations*, which is the ability to identify and understand the horizontal and vertical orientation and resulting relationships among objects or their parts. He also refined the definition of *Spatial Visualisation* as the ability to picture an image in which internal alterations are occurring.

In 1988, Lohman distinguished three factors, based on the results of a meta-analytic study. He named the factors *Spatial Visualisation, Spatial Orientation*, and *Spatial Relations*. He explained that spatial visualisation is the most general factor; however, it is difficult to identify because the tests that define it usually have high loadings on the *general intelligence*, or overall mental ability. One important characteristic of the tests that define spatial visualisation is their complexity. Some require rotation, reflection, or folding complex figures, others require combining different figures, yet some others require multiple transformations. When defining spatial orientation, Lohman (1988) agreed with McGee's definition. He also added that it is difficult to separate spatial orientation from spatial visualisation, as both of these factors require considerable reasoning skill and subjects may solve items by mentally rotating them rather than moving an image of their self to the desired perspective. Lohman (1988) believed that the spatial relations factor is defined by the tests in which subjects are required to determine whether a given stimulus is a rotated version of a two-dimensional object or is a rotated and reflected version of it. A quick answer is expected from the examinees when taking those kinds of tests.

The most recent models seem to distinguish over five different factors for spatial ability (see for example Caroll, 1993, and Kimura, 1999). However, a significant weakness of factor analytic studies on spatial ability is that they do not provide the same results (i.e., detect the same underlying factors), which may lead to incorrect conclusions and confusion (Yilmaz, 2009). While some of the studies clearly identify a spatial orientation factor, a comprehensive analysis of previous data sets by Carroll (1993) does not suggest such a factor. Kozhenvikov and Hegarty (2001), and Hegarty and Waller (2004), argued that it is possible that spatial orientation was poorly assessed and provided empirical evidence for the dissociation between the spatial orientation and the spatial visualisation factors.

Despite the spatial ability factor model proposed by Caroll (1993) being considered as the most comprehensive (Burton & Fogarty, 2003; Hegarty & Waller, 2006; Höffler, 2010), it seems that it is not the most frequently used. Since there does not appear to be a generally accepted model of spatial ability, most studies seem to measure only specific factors from the wide range found in literature. According to Höffler's (2010) meta-analytic review on studies about the influence of spatial ability when learning with dynamic and non-dynamic visualisations, spatial visualisation and spatial relations are by far the most frequently measured factors. He also found significant effects for the spatial orientation factor, which he considered significant, even though it does not so frequently appear in studies about learning from visualisations. One reason for this may be the lack of evidence for its separation from other factors. Given the recent evidence for dissociation (Hegarty & Waller, 2004), it seems that research towards learning and spatial ability should move towards a model combining three factors: spatial visualisation, spatial relations, and spatial orientation. This combination is more closely related to the theoretical model suggested by Lohman (1988). Moreover, recent studies in mathematics education support the existence of such a model and its relation to students' understanding of three-dimensional geometry concepts (Pittalis & Christou, 2010).

Renewed attention regarding the connection between cognitive styles and spatial abilities is also observed (Blazhenkova & Kozhevnikov, 2009; Kozhevnikov, Blazhenkova, & Becker, 2010; Kyritsis, Gulliver, Morar, & Macredie, 2009; Sorby, 2009). These studies are based on individuals' processing of mental imagery and evidence of different types of visual-spatial representations. This issue will be further discussed in the following section, regarding individual differences in cognitive style.

*Spatial ability and mathematics*

The literature holds a lot of discussion about the relationship between spatial ability and mathematics. Furthermore, spatial ability has been found to be positively correlated with measures of mathematical performance (Battista, 1990; Clements & Battista, 1992; Fennema & Sherman, 1977) and recognised as being a significant factor in specific areas of mathematics, such as geometry (Bishop, 1980; Hannafin, Truxaw, Vermillion, & Liu, 2008; van Garderen, 2006). Liedtke (1995) reinforced the importance of spatial ability for mathematics, suggesting that "Spatial sense or imagery is an important part of geometry and important part of mathematics learning, since it is indispensable in giving meaning to our mathematical experience" (p.18). Tso and Liang (2002) suggested that spatial ability is an important cognitive factor in learning geometry, and that incorporating spatial visualisation and manipulation into the learning process could improve geometric learning. Wilson (1992) also stressed the importance of the ability to visualise mathematical relationships and stated that it is an essential part of many people's knowledge of mathematics and facilitates the communication of ideas about mathematics.

*Spatial ability and problem solving*

Kayhan (2005) conducted a study with 251 ninth grade students to investigate the relationships between mathematics achievement, logical thinking ability, and spatial ability. Kayhan used the Spatial Ability Test and Group Test of Logical Thinking to measure students' spatial ability and mathematical achievement. The results of the study indicated that there is a significant positive relationship between spatial ability and mathematical achievement, as well as a significant positive relationship between spatial ability and logical thinking ability.

Spatial visualisation is also considered an important component in solving many types of mathematics problems (Alias, Black, & Grey, 2002; Booth & Thomas, 1999). McLeay (2006) stressed out the importance of visualisation ability for problem solving and suggested that one way to improve students' problem-solving ability is to encourage them to use imagery and visualisation strategies. Ben-Chaim et al. (1988) proposed that visualisation offers students the flexibility of having additional strategies, which can be enriching in their problem solving range of strategies. This was evident in a study by van Garderen (2006) which investigated the relationship of using visual imagery and spatial visualisation ability in solving mathematical word problems. Students with learning disabilities, average achievers, and gifted students in sixth grade participated in this study. They were assessed on measures of mathematical problem solving, visual imagery representation, and spatial visualisation ability. The findings revealed that use of visual images was positively correlated with higher mathematical word problem solving performance and there was a significant positive correlation between each spatial visualisation measure and mathematical word problem solving performance.

Battista (1990) also investigated the role of spatial visualisation in performance and gender differences in high school geometry. The sample of the study was 145 high school students. A version of the Purdue Spatial Visualisation Test (Guay, 1977), knowledge of geometry, and geometric problem solving test were administered to students. The results indicated that spatial visualisation was an important factor in geometry achievement and geometric problem solving.

In addition to spatial visualisation, the role of spatial orientation skill in mathematics problems was also investigated. Tartre (1990) carried out a study with 97 tenth grade students to explore the role of spatial orientation skill in the solution of mathematics problems and to identify possible associated gender differences. A spatial orientation test and a mathematics achievement test were used. The students were also asked to solve mathematics problems in individual interviews. The results of the study suggested that spatial orientation ability appears to be used in specific and identifiable ways in the solution of mathematical problems. Tartre also noted that spatial skills may be a more general indicator of a particular way of organising thought in which new information is linked to previous knowledge structures to help make sense of new material.

Lean and Clements (1981) findings seem contradictory to other studies, suggesting that it is desirable to use visual processes when attempting mathematical problems. They undertook a study with 116 engineering students. Students were given a battery of mathematical and spatial tests; in addition, their preferred modes of processing mathematical information were measured. The results of the study revealed that students who preferred to process mathematical information by verbal-logical means tended to outperform the students who preferred visual process on mathematical tests. The results also showed that spatial ability and knowledge of spatial conventions did not have a large influence on the mathematical tasks. Considering that the term “preferred mode of processing” is sometimes used as a synonym for cognitive style (Evans & Warring, 2011), one could say that cognitive style has more influence on mathematical ability than spatial ability does. Since no other similar attempts have come to our attention, it seems that this issue requires further investigation.

*Spatial ability and transformational geometry*

A great amount of research has investigated the relation between spatial ability and transformational geometry ability (Boulter, 1992; Del Grande, 1986; Dixon, 1995; Kirby & Boulter, 1999; Williford, 1992). However, few of these studies investigated the impact of spatial ability on students’ transformational geometry ability. The majority of these studies viewed transformational geometry activities as a means of improving students’ spatial ability. According to Clements and Battista (1992), given that spatial thinking and transformational geometry have so much in common (i.e., they both involve construction and manipulation of mental images), “one might hypothesise that work with the latter would improve skills in the former” (p. 445).

In an early study, Williford (1972) focused on primary school students for an investigation involving transformation geometry. The purpose of this study was to ascertain information regarding second- and third-grade students’ ability to perform transformations, after being taught about transformations through a specific teaching strategy. In addition, it aimed to investigate the effects that this instruction had on the students’ spatial ability. The pre- and post-test consisted of a spatial ability test and an achievement test on congruence and transformations. After the pre-test, the students were randomly divided into experimental and control groups. The experimental groups were removed from class for 12 sessions, each lasting 25 to 30 minutes over the duration of four to five weeks. During these sessions, the control groups remained in class working on subjects that were unrelated to transformational geometry. The experimental groups were involved in demonstrations and activities related to congruence and transformations. The control groups received a lesson including an overview of the experimental group’s lessons. The lesson consisted of appropriate vocabulary and examples dealing with each of the three rigid transformations, translation, reflection, and rotation. These lessons were designed to compensate for the lack of experience with the terminology of the test. The researcher hypothesised that the control group subjects may gain knowledge of transformations through everyday experience, but may not have the vocabulary to demonstrate this knowledge on the post-test. The experimental groups performed significantly better than the control groups on the achievement post-test; however, there was no statistically significant difference between group performances on the spatial ability post-test.

Boulter (1992) examined the effects of instruction in transformational geometry on seventh- and eighth-grade students’ spatial ability and geometry performance. Two instructional conditions were examined, one involving a traditional textbook approach (Traditional group), and the other incorporating object manipulation, visual imagery, and encouraging spatial thinking (Spatial group). Pre-tests of spatial ability and geometry were administered to all subjects prior to the instructional intervention. The subjects were divided into three equal groups, two interventional and one control group. A relation was found between spatial ability and geometry performance; however, the experiences provided in the instructional programmes of this study were not effective in improving spatial ability, since improvement also appeared in the non-instructional group.

Although Boulter (1992) and Williford (1972) did not manage to find improvement of spatial abilities when teaching students transformational geometry concepts, other studies provide evidence that this is possible (Del Grande, 1986; Dixon, 1995; Smith et al., 2009; Thomas, 1983). Del Grande (1986) developed 10 spatial categories and 40 test items for measuring spatial ability that were measurable using paper-and-pencil or manipulative activities. Second grade students were tested before and after a geometry intervention, which was based on transformations. The results suggest that improvement in spatial perception scores was statistically significant.

Dixon (1995) conducted a study to investigate the effects of a dynamic instructional environment – *The Geometer's Sketchpad* (GSP) – and visualisation level, independently and interactively, on middle school students' construction of the concept of reflection and rotation. Another aspect of the study was to examine the effects of a dynamic instructional environment on students' two- and three- dimensional visualisation. After controlling for initial differences, the study came to the conclusion that students taught with the GSP significantly outperformed students taught by traditional methods on content measures of transformations and in measures of two-dimensional visualisation, but not on their three-dimensional visualisation. Improvement of spatial ability when learning transformational geometry concepts with the GSP was also found in a study by Hoong and Khoh (2003) with secondary school students.

A recent study by Smith et al. (2009) described two experiments investigating how female elementary perspective teachers in three countries were trained on spatial skills through structured activities on discrete dynamic (in contrast to continuous dynamic) transformations in interactive computer programs. The data were gathered by pre- and post- tests from an experimental and a comparison group. The experimental group took part in an intervention with computer-aided visualisation exercises, concerning guided activity on 2D and 3D transformational geometry tasks. Spatial abilities were measured by standardised measures of spatial visualisation and mental rotation. In two of the three countries the spatial visualisation scores of the experimental group improved significantly compared to the control group. The same was not observed for mental rotation. The main conclusion of the study was that “training composition (or sequences) of discrete spatial transformations is viable for spatial visualisation training” (Smith et al., 2009, p. 207). The researchers guided further research to focus on comparing the effects of discrete versus continuous spatial transformations for spatial visualisation training.

### *Cognitive styles*



Cognitive styles have been a subject of research in the cognitive psychology field for over 70 years. The idea of style reflecting a person’s typical or habitual mode of problem-solving, thinking, perceiving, and remembering was initially introduced by Allport (1937). Since then, the style construct has employed a great deal of research interest (Morgan, 1997) and many theoretical models have been postulated (see Grigorenko & Sternberg, 1995; Kagan & Kogan, 1970; Kogan, 1983; Kozhevnikov, 2007; Riding & Cheema, 1991; Sternberg, 1988). In the absence of a unified theory of cognitive styles (Evans & Waring, 2011; Kozhevnikov, 2007; Peterson, Rayner, & Armstrong, 2009), there seems to be a variety of definitions and theories in this field. One definition is that they are psychological dimensions representing consistencies in an individual’s manner of cognitive functioning, particularly with respect to acquiring and processing information (Witkin, Moore, Goodenough, & Cox, 1977). One of the most recent definitions given by Peterson et al. (2009) is that cognitive style(s) can be defined as “individual differences in processing that are integrally linked to a person’s cognitive system... they are a person’s preferred way of processing... they are partly fixed, relatively stable, and possibly innate preferences” (p. 11). Although there appear to be a variety of conceptualisations of cognitive styles, most of the researchers agree that it is a construct which is relatively stable over time.

The research interest in styles was developed partially due to traditional psychometric research on abilities and IQ failing to elucidate the processes generating individual differences. Disappointment in IQ as a construct was prominent in the 1960’s in both cognitive and developmental psychology. As a result, psychologists started looking for new ways to describe individual differences in cognitive functioning that are stable, and the stylistic approach was born (Grigorenko & Sternberg, 1995; Kozhevnikov, 2007).

The first studies in the area considered cognitive style as bipolar (or uni-dimensional). Some examples of the most popular theories of this type are: (a) the *reflection-impulsivity* dimension (Kagan, 1958), which represents a preference for making responses quickly versus pausing to decrease the number of errors in problem-solving situations, (b) the *holist-serialist* dimension (Pask, 1972), which refers to an individual's tendency to respond to a problem-solving task with either holistic or a focused "step-by-step" strategy, (c) the *field dependence-independence* dimension (Witkin, 1973), which refers to an individual's tendency for high or low dependence on the surrounding field, and (d) the *visual (imagery)-verbal* dimension (Paivio, 1971; Richardson, 1977), which refers to individuals' preference to process information by verbal versus imagery means. Recent trends in cognitive style research attempted to unite existing models of style into unifying multi-dimensional models (e.g., Riding, 1991; Riding & Cheema, 1991) or build entirely new theories (e.g., Sternberg, 1997).

Within this trend, Kozhevnikov et al. (2002, 2005) challenged the bipolar nature of the visualiser-verbaliser dimension and attempted to clarify and revise it. Paivio's (1971) early work on dual-coding theory has provided the basis for research that investigated the nature of a Verbal-Imagery dimension in cognitive processes. Since then, a number of researchers have argued that learner performance was affected by the way knowledge was represented during thinking, either by using imagery or verbally, by using words (Presmeg, 1986b; Riding & Taylor, 1976). The position of individuals along the Verbal-Imagery dimension reflects the manner in which they represent information during thinking, either as words or mental pictures. Verbalisers consider the information they read, see or listen to in words. Imagers most often use mental pictures (Riding & Rayner, 1998).

Kozhevnikov et al. (2002, 2005) provide neuroscience evidence that suggest the existence of distinct subsystems of visual information processing: (i) one that processes *object* properties which are related to the visual appearance of objects and scenes in terms of their shape, colour information, and texture, and (ii) one that processes *spatial* properties that are related to object location, movement, spatial relationships and transformations, and other spatial attributes of processing. These findings provided feedback for the re-conceptualization of the verbaliser-visualiser cognitive style single dimension by proposing two different types of visualisers: *object visualisers*, who use imagery to construct images of objects and process visual information globally and holistically as whole perceptual objects, and *spatial visualisers* who use imagery to represent spatial relations, make complex spatial transformations and process visual images analytically and sequentially, part-by-part (Kozhevnikov et al., 2005). Verbalisers, who were a homogeneous group and did not show any indications of bimodality, process and represent information verbally and tend to rely primarily on verbal-analytical non-visual strategies (Kozhevnikov et al., 2005).

The object-spatial-verbal cognitive style dimension was found to relate to preference for different academic fields, to future professional intent, and to different areas of professional specialisation (Blazhenkova & Kozhevnikov, 2009; Blazhenkova et al., 2011). Specifically, the object-imagery scale was found to relate to learning preference in visual art and to the degree of becoming a visual artist or a designer, whereas the spatial-imagery cognitive style was found to relate to learning preferences in physics, chemistry, technical drawing, math, algebra, geometry, and the degree of intention to become a scientist or a computer scientist. The verbal cognitive style dimension was found to relate to learning preferences and professional intent in literature and history (Blazhenkova et al., 2011). Moreover, in a study by Blazhenkova and Kozhevnikov (2009), visual artists report on having higher object imagery abilities, scientists report on higher spatial imagery abilities, and humanities professionals have been found to report higher verbal abilities.

In the field of education, cognitive styles were found to have a predictive power for academic achievement, beyond general intelligence or other situational factors (Blazhenkova et al., 2011; Kozhevnikov, 2007; Sternberg & Zhang, 2001). A growing body of work is currently advocating the potential of styles to impact on performance in education in various contexts (Evans & Cools, 2011; Evans & Waring, 2011). However, there is still a need for research to clarify the relationship between style constructs and observed behaviour in context (Riding, 2000; Sternberg, 2008), and the implications of styles research in educational practice (Evans & Waring, 2011; Mayer, 2011). Looking at specific cognitive styles and their impact on performance makes it possible to identify those that may need greater support (Evans & Cools, 2011; Riding, 2000; Sternberg, 2008). It is important to note that Mayer (2011) suggested that the significant implications of cognitive styles in educational practice lay in the importance of finding effective instructional methods to accommodate all individuals’ learning.



*Cognitive styles and mathematics*

Within the field of mathematics education, there are many studies that consider the relationship between an individual’s cognitive style and performance important in many mathematical topics (e.g., Alamolhodaei, 2009; Anderson, Casey, Thompson, Burrage, Pezaris, & Kosslyn, 2008; Kozhevnikov et al., 2002; Nicolaou & Xistouri, 2011; Pitta-Pantazi & Christou, 2009a, 2009b, 2009c; Tinajero & Páramo, 1998; van Garderen, 2006; Xistouri & Pitta-Pantazi, 2011a, 2011b, 2012; Zhang, 2004). However, in the field of mathematics education, the verbaliser-visualiser distinction (Paivio, 1971) was the one that attracted the most attention. However, it must be pointed out that this distinction was not referred to as "cognitive style", but as preferred type/mode of thinking or type of students (Kruteskii, 1976; Lean & Clements, 1981; Pitta & Gray, 1999; Presmeg, 1986a, 1986b). The broad idea documented by a number of researchers was that visual-spatial processes are distinct from verbal processes and that mathematics involves not only verbal processes, but also visual reasoning (Presmeg, 1986a; Sfard, 1991).

Nevertheless, the results of the relationship between visualisation and mathematical performance are unclear. Some studies found that visual-spatial memory is an important factor explaining the mathematical performance of students (Battista & Clements, 1998; Eisenberg & Dreyfus, 1991; van Garderen, 2006), while other studies suggest that students classified as visualisers do not tend to be among the most successful performers in mathematics (Pitta-Pantazi & Christou, 2009a, 2009b; Presmeg, 1986a). This apparent inconsistency can be attributed to a number of reasons. One is the use of different terms, such as visualisation, visual imagery, and spatial thinking, to describe variations of an overall issue (Gorgorio, 1998; Gutierrez, 1996). In addition, in these studies, researchers have looked at a variety of age groups and mathematical abilities, used different methodologies, and measured visualisation ability differently (Battista, 1990; Battista & Clements, 1998; Lean & Clements, 1981; Pitta-Pantazi & Christou, 2009a, 2009b; Presmeg 1986a, 1986b, 1997; Pyke, 2003).

Given the recent re-conceptualisation of the verbaliser-visualiser dimension, many researchers seem to believe that it might be able to explain some of the conflicting results concerning visualisation and mathematical performance (Anderson et al., 2008; Chrysostomou, Tsingi, Cleanthous, & Pitta-Pantazi, 2011; Kozhevnikov et al., 2002; Pitta-Pantazi & Christou, 2009c; Xistouri & Pitta-Pantazi, 2011a, 2011b, 2012). In these studies, spatial imagery was the only cognitive style dimension that was found to be important in interpreting graphs (Kozhevnikov et al., 2002), mathematical creativity (Pitta-Pantazi & Christou, 2009c), and algebraic reasoning and number sense (Chrysostomou et al., 2011).

More precisely, in the field of geometry, a study by Anderson et al. (2008) investigated the relationship between the three dimensions of cognitive style (object, spatial, and verbal) and performance on geometry problems that provided clues compatible with their cognitive style. The findings of this study suggested that spatial imagery and verbal cognitive style dimensions were important for solving geometry problems, whereas object imagery dimension was not. However, a relationship was found between females who were spatial imagers and their results of geometry performance.

In the field of transformational geometry, research suggests that the spatial imagery dimension is related to transformational geometry abilities in primary school students and prospective teachers (Xistouri & Pitta-Pantazi, 2011a, 2011b). However, there are indications that this relationship could be subjective to the type of transformation or the type of task (Boulter & Kirby, 1994). There is a need for further research to clarify the relationship of cognitive style to transformational geometry concepts by considering different types of tasks.

## Learning Mathematics with New Technologies

The importance of using technology in the teaching of mathematics has been advocated by the NCTM (1989, 2000) for many years. In *Principles and Standards for School Mathematics*, the NCTM (2000) states that "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning" (p.24). Nohda (1992) supports this statement and notes how computer environments are ideal tools to support the implementation of the curricula, especially in mathematics. He defines calculators and computers as "fast pencil" where mathematical processes can be made more useful and efficient than with paper and pencil.

Miwa (1992) also stresses the power of computers to promote inquiry, investigation and discovery –concepts that are very important in mathematics education. He also notes that computer use enables students to widen their field of vision and enrich their understanding, and motivates students to practice discovery process. In line to this, Clements and Sarama (2001) stress that the computer can offer unique opportunities for learning through exploration, creative problem solving, and self-guided instruction.

## *The teaching of geometry*

Before reviewing the use of technology for teaching geometry, some theoretical frameworks for the stages of teaching will be first described. Given that no theoretical frameworks for the teaching of transformational geometry were found in literature, this study will draw on other theoretical frameworks of teaching.

### *van Hiele Learning phases of geometry*

The van Hieles proposed a sequence of learning phases to assist pupils to improve their reasoning about spatial ideas. This has been for many years the most widely acceptable framework for guiding geometry teaching (de Villiers, 2010). This sequence provides a means for helping pupils to develop. Each learning period builds on, and extends, the preceding level, and the instruction makes explicit what was only implied at the preceding level of thought. Language plays an important role in learning and each level has a vocabulary that is used to represent the concepts and relationships. New language is introduced at each learning period to discuss the new ideas. In addition, students at a lower level are not expected to understand ideas demanding a higher level of thought. The following sequence moves students from direct instruction to understanding, independent of the teacher:

(1) Inquiry: the teacher engages students in two-way discussions about the spatial ideas to be learnt. The teacher learns how the pupils interpret the words and guides them to construct an understanding of the topic being studied.

(2) Guided orientation: the teacher sequences activities for guided exploration, leading students to become familiar with the characteristic structures.

(3) Explication: the students build on their foregoing experience to refine their comprehension of the topic being examined and express their ideas and understandings.

(4) Free orientation: the students develop their own procedures for solving longer, more complex spatial problems. This allows them to identify many of the relations between the spatial ideas being learnt.

(5) Integration: the students review their findings and form an overview. The relationships are unified into a new domain of thought.

It appears that many researchers (Ahuja, 1996; Frykholm, 1994; Lawrie, 1998; Pegg & Davey, 1991) believe that the van Hiele levels are useful in the learning of geometry. However, the research on the levels is aimed at the students, and is often an attempt to verify van Hiele's work and to draw conclusions on the implications for curriculum planning. According to van Hiele, "the transition from one level to the following is not a natural process; it takes place under the influence of a teaching-learning programme" (1986, p. 50). Teachers hold the key to this transition from one level to the next, and it can be influenced by teacher's beliefs and knowledge of geometry.

*The 5Es instructional model*

A more recent theoretical model of learning phases is the 5Es instructional model (Bybee, 1997). The 5Es is an instructional model used extensively in science curriculum, which is based on the constructivist approach to learning (Bybee, Taylor, Gardner, Scotter, Powell, Westbrook, & Landes, 2006). However, it can be applied to other academic fields as a sequential instructional approach, including mathematics (Bossé, Lee, Swinson, & Faulconer, 2010). The 5Es consists of the following phases: Engagement, Exploration, Explanation, Elaboration, and Evaluation. Each phase has a specific function and contributes to the teachers' coherent instruction and the students' constructing of a better understanding of knowledge, attitudes, and skills. The five phases are:

(1) Engagement: The teacher or a curriculum task accesses the students' prior knowledge and helps them become engaged in a new concept through the use of short activities that promote curiosity and elicit prior knowledge. The activity should make connections between past and present learning experiences, expose prior conceptions, and organise students' thinking toward the learning outcomes of current activities.

(2) Exploration: Exploration experiences provide students with a common base of activities within which current concepts (i.e., misconceptions), processes, and skills are identified and conceptual change is facilitated. The students may complete investigative activities that help them use prior knowledge to generate new ideas, explore questions and possibilities, and design and conduct a preliminary investigation.

(3) Explanation: The explanation phase focuses students' attention on a particular aspect of their engagement and exploration experiences, and provides opportunities to demonstrate their conceptual understanding, process skills, or behaviours. This phase also provides opportunities for teachers to directly introduce a concept, process, or skill. At this

phase, the students explain their understanding of the concept. An explanation from the teacher or the curriculum may guide them toward a deeper understanding, which is a critical part of this phase.

(4) Elaboration: Teachers challenge and extend students' conceptual understanding and skills. Through new experiences, the students develop deeper and broader understanding, more information, and adequate skills. At this phase, the students apply their understanding of the concept by conducting additional extensive activities.

(5) Evaluation: The evaluation phase encourages students to assess their understanding and abilities, and provides opportunities for teachers to evaluate their progress towards achieving the educational objectives.

The advantage of this model over the van Hiele phases of learning is that its objective is to provide students with experiences that make them reconsider their conceptions. Then, students "redefine, reorganise, elaborate, and change their initial concepts through self-reflection and interaction with their peers and their environment" (Bybee, 1997, p. 176). The 5Es model provides a planned sequence of instruction that places pupils at the centre of their learning experiences, encouraging them to explore, construct their own understanding of concepts, and relate those understandings to other concepts.

### *Technology and the teaching of geometry*

In recent years the limitations of traditional approaches in the teaching and learning of mathematics have been expressed (Rahim, 2002). Maragos (2004) argued regarding these limitations and stated that "in a traditional geometry course, students are told definitions and theorems and are assigned problems and proofs; they do not experience the discovery of geometric relationships, nor invent any mathematics" (p. 2). Olive (1991) also stressesed the limitations of traditional approaches and stated that an inductive approach based on experimentation, observation, data recording, and conjecturing would be much more appropriate for geometry education. Such an approach would give students the opportunity to engage in mathematics as mathematicians and not merely as passive recipients of others' mathematical knowledge. Battista (2002) agreed with Olive and pointed out the importance of providing rich student-centred learning environments that give students opportunities to develop their geometrical thinking.

Nohda (1992) also noted that in mathematics, understanding cannot be generally achieved without participation in the actual process of mathematics: in conjecture and argumentation, in exploration and reasoning, in formulating and solving, in calculation and verification. Similarly Reys, Lindquist, Lambdin, Smith, and Suydam (2007) advocated that geometry is best learned in a hands-on active manner, one that should not rely on learning about geometry by reading from a textbook.

All the limitations of traditional educational approaches led researchers to study the development of alternative ways for the teaching of geometry. Technological environments have been created and computers were introduced into geometry education as an alternative way to overcome the limitations of traditional geometry teaching. According to Aviram (2001), even though the computer is a very powerful cognitive and social tool that can enhance the learning of mathematics, it will not make its maximum contribution to mathematics teaching until we have discovered the best techniques for using it.

A number of studies have been conducted that look at the impact of technology on students' geometry achievement (Chan, Tsai, & Huang, 2006; Hollebrands, 2003). Most of these studies concluded that the use of technology in the mathematics classroom is beneficial in developing students' understanding of geometric concepts (Laborde, 2002). However, some computer applications have proved to be more successful than others and many factors influence the impact that even the most promising applications may have (Roschelle, Pea, Hoadley, Gordin, & Means, 2000).

Olkun, Altun, and Smith (2005) investigated the possible impacts of learning transformations with a virtual tangram manipulative on fourth-grade students' geometry scores and further geometric learning. The study used a pre-test – intervention – post-test experimental design. Findings revealed that students who did not have computers at home initially had lower geometry scores. Olkun (2003) also conducted a study to compare the experiences of a group of learners using computer-based representations with another group using concrete manipulative. The results of the study revealed that though both groups improved significantly, the computer group improved slightly more, with older pupils (fifth grade) benefiting more from the computer-based manipulative.

Martínez , Bárcena, and Rondriguez (2005) conducted a study on the effects of an instruction that incorporates Java applets on various learning activities in geometry. The results of the study showed that such dynamic software can facilitate some types of learning activities, such as exploration and visualisation, while enhancing others, such as proof.

Chan et al. (2006) conducted a study with third- and sixth- graders to find a way that promotes learning and van Hiele levels of geometric thought among elementary students. The study concerned the application of web-based learning with learner controlled instructional materials in a geometry course. The experimental group received instruction in a web-based learning environment, and the control group received traditional instruction in a classroom. The results observed that the learning method accounted for 19.1% of the total variation in learning effect for the third grade, and 36.5% for the sixth grade. The researchers observed that the sixth grade students' ability to learn with technology was sufficient to handle problems and promote their van Hiele levels of geometric thought.

Tutak and Birgin (2008) investigated the effects of the computer assisted instruction on students' geometry achievement at fourth grade geometry course. The experimental group was instructed by means of computer assisted teaching materials, while the control group was instructed by traditional methods. The results of this study showed that computer-assisted instruction had a significant effect on the students' geometry achievement compared to the traditional instruction at fourth grade geometry course.

The above studies, as well as the ones regarding transformational geometry and technology that were described in a previous section, highlight that computers are important tools for improving students' geometrical achievement. Given the importance of integrating computers into content area teaching, there is clearly a need for further research investigating the effects of computer in geometry education. However, some studies indicate that the effectiveness of learning with technology can be influenced by the learners' individual characteristics (Chan et al., 2006; Hatfield & Kieren, 1972). This study sought to further investigate the potentialities of technology in developing students' spatial and transformational geometry abilities, by comparing the effects of different types of software for geometry teaching.

### *Dynamic visualisations in the teaching of geometry*

According to Tall and West (1986) dynamic representations of mathematical processes may "enable the mind to manipulate them in a far more fruitful way than could ever be achieved starting from static text and pictures in a book" (p. 107). Therefore, as many authors pointed out, dynamic visualisations can be a very powerful tool to gain a greater understanding of many mathematical concepts or they can be a resource to solve mathematical problems (de Villiers, 2007; Goldenberg, Lewis, & O' Keefe, 1992; Harel & Sowder, 1998; Presmeg, 1986a; Tall &West, 1986).

Goldenberg et al. (1992) pointed out that technology can foster students' dynamic visual reasoning ability. Tall (1993) also pointed out that software environment could allow students to manipulate the picture and relate its dynamically changing state to the corresponding concepts. It, therefore, has the potential to improve understanding by minimising the cognitive burden. Software could be used for laborious constructions where the students can focus their attention on specific relationships, whilst the computer can carry out the processes which are not to be the focus of attention. Tall (1993) refers to this as the *principle of selective construction*.

In geometry, Dynamic Geometry Environments (DGE) are currently concentrating growing attention in both research and teaching, and have become one of the most widely used pieces of software in schools. DGE provide tools for constructing geometric objects. The dynamic aspect comes from the ability to drag the objects or their defining components (such as points) around the screen with a mouse. There are many suggestions that DGE facilitate visualisation (Jones, 2002). To date, most research has focused on the classical constructions available in the DGE, and has mostly focused on post-primary school students. Little research has looked at geometric transformations (translation, reflection, and rotation) in primary school (Jones, 2002; Strässer, 2003).

Dynamic visualisations seem to be a powerful tool for the learning process in mathematics and education in general. Many studies support the position that instruction with dynamic visualisations not only has positive learning outcomes, but it can also enhance the learners' spatial ability (Dixon, 1995). However, there have been some concerns raised on whether spatial ability should be considered as a prerequisite for someone to be able to learn from dynamic visualisations (Hegarty, 2004a, 2004b, 2010; Kirby & Boulter, 1999). Hegarty (2004a, 2004b) raised attention to the existence of three different possibilities on the relation between external and internal visualisations. Specifically, the use of dynamic visualisations may replace, depend on, or augment internal visualisation. Therefore, she guided research into clarifying the relationship between dynamic visualisations and internal visualisation. According to Hegarty (2004a) and Hegarty and Waller (2006), spatial visualisation ability can be thought of as a measure of internal visualisation. She added that the complexity of the spatial transformations in an external representation can influence learners' comprehension in relation to their internal visualisation skills. Therefore, she draws to the importance of studying the effects of different dynamic visualisations by considering students' individual characteristics (Hegarty, 2004b).

More recently, Höffler (2010) suggested that the role of spatial ability on learning with visualisations is still unclear and research should consider whether all learners profit equally from visualisations and how to support the ones that are less profited. According to Höffler (2010), there are suggestions that spatial ability and cognitive styles can be moderators in learning from dynamic visualisations. The role of spatial ability and cognitive styles in learning from dynamic visualisations will be taken into consideration in this study, in the context of transformational geometry.

### *Types of dynamic visualisations*

Since the inclusion of technology in mathematics education many different types of software have been developed and different taxonomies have been proposed (e.g., Alessi & Trollip, 1985; Handal & Herrington, 2003; Kurz, Middleton, & Yanik, 2005; Taylor, 1980). One of the most recent and most generally accepted is the taxonomy proposed by Handal and Harrington (2003). The categories proposed by this theory are the following:

(1) *Drill*. These types of software provide drill-and-practice activities, which are basically repetitions of the same or similar process. They reflect a traditional, behaviourist approach that focuses on mastering basic skills or reviewing material that has been previously learned. A typical drill-and-practice exercise presents learners with a question, followed by response entry, and corresponding evaluation of the question and feedback. Visuals are not necessarily required in this type of software, and interaction is minimal.

(2) *Tutorials*. Similarly to drills, tutorials also present information, while at the same time guide students through their learning processes. A tutorial typically follows a certain structure and sequence, usually starting with an introduction to the lesson and presentation of information, followed by providing a series of questions to be answered by the learner with immediate standard feedback of right-or-wrong response. Tutorials have potential in online interactive learning, as they provide many possibilities to motivate students through multimedia capabilities.

(3) *Games.* Games are goal-oriented activities that provide a multimedia simplification of reality. In these environments, learners encounter a dynamic situation to which they must respond. In a game situation the learner engages in a win or lose situation that requires the practice of skills assumed to be known or in the process of development. Hence, the artificial environment provided by the software motivates the learner through an amusing activity that indirectly provides pedagogical benefits.

(4) *Simulations*. Similar to games, simulations are also goal-oriented activities that provide multimedia simplification of reality. Simulations encourage learning within artificial situations and one of their greatest advantages is their capacity to represent and connect huge amounts of information through multimedia, thus resolving constraints related to computational difficulties, financial constraints in mimicking an activity, timeframe needed to replicate processes, or magnitude of the equipment necessary for a certain experiment.

(5) *Hypermedia***.** Hypermedia-based instruction is a more complex form of computer-assisted-instruction (Ayersman & von Minden, 1995). Hypermedia approaches combine hypertext and multimedia. Multimedia delivers content using several formats, such as text, sound, graphics, and video that work to reinforce each other (Hall, 2000). Hypertexts are learning environments in which knowledge is represented through a network of nodes of information. Nodes of information are connected through clickable buttons to other nodes, and users control navigation through the nodes. The association of nodes on such a nonlinear structure permits a learner to associate a variety of content within an exploratory context. The non-linear dynamics of hypermedia-based instruction empowers students, giving them more autonomy, responsibility, and interactivity with the software (Hall, 2000).

(6) *Tools and Open-Ended Learning Environments*. Tools are electronic processes that assist learners in carrying out tasks, such as planning, writing, calculating, drawing, composing, and communicating (Alessi & Trollip, 2001). Teachers can use these tools to assist their students in learning mathematics through higher order thinking processes rather than simply learning about the tool. According to Handal and Herrington, DGS, such as the GSP, and java-applets, such as the JAVA Gallery of Interactive On-Line Geometry, are considered examples of such tools.

The above framework, like most of its kind, is based on the different pedagogical approaches of learning and the theories of learning that guide these pedagogical approaches. Moreno-Armella, Hegedus, and Kaput (2008, pp. 102-103), proposed a different framework, viewed through a representational-semiotic perspective. Based on historic observations of the development of inscriptions in mathematics from static to dynamic, the proposed theoretical framework consists of five stages – types of inscriptions. These stages, which can still be evident in the mathematics classrooms in the 21st century, are:

*Static inert:* In this stage, the inscription is "hardened" or "fused" with the media it is presented upon or within. Even though historically this has been how ancient writing was preserved (e.g., cuneiform art, bone markings) it is also the description of many textbooks and handouts from printers in today's classroom. Early forms of writing can even include ink on parchment, especially calligraphy as an art form of writing since it was very difficult to change the writing once "fused" with the paper. In this sense, it is inert.

*Static kinaesthetic/aesthetic:* This stage began with the advance of scribable implements and the co-evolution of re-usable media to inscribe upon. Here, chalk and marker pens allow a transparent use of writing and expression, as their permanence is temporal, erased over time. Even though it is static, this type affords a more kinaesthetic inscription, since it provides flexibility to move within the media of inscription, and an aesthetic process, since colour can be used to differentiate between notations.

*Static computational:* In this stage, presentations (e.g., graph-plotting) are artefacts of a computational response to a human's action. The intentional acts of a human are computationally refined. A simple example is a calculator where the notation system (e.g., mathematical tokens, graphs, functions) is processed within the media and presented as a static representation of the user's input or interaction with the device.

*Discrete dynamic:* This stage refers to media that are less static. The user's interactions are more fluid, and the media within which notations can be expressed are more plastic and malleable. The co-action between user and environment can exist. This process of presentation and examination is discrete. For example, a spreadsheet offers an environment within which a user can work to represent a set of data by different intentional acts, e.g., "create a" list, "chart a" graph, "calculate a" regression line, or is generated through parametric inputs, e.g., a spinner or a slider alters some seed value. Both of these discretise actions and turn them into observable expressions –expressions that are co-actions between the user and the environment –yet the media is still dynamic, as it is malleable, and re-animates notations and expressions on discrete inputs.

*Continuous dynamic:* This stage builds on the previous stage by being sensitive to kinesthetic input or co-action, to make sense of physical force, or gestural interaction through space and time. Some software allows the user to navigate through continuous actions of a mouse—the perception or properties of a mathematical shape or surface through re-orientating its perspective, e.g., what does this surface look like when I click/drag and move the object? Haptic devices can detect motion through space and time, and provide feedback force on a user's input. For example, a user could perceive the steepness of a surface through a force-feedback haptic device and move it to a point of extreme value without asking the computer to calculate relative extrema.

Taking into account the potentials of tools and open-ended learning environments to provide learners with the ability to investigate concepts, such as geometrical and algebraic patterns and relationships, and also to promote higher order thinking processes, this study will only make use of such tools. Specifically, it will focus on using dynamic interactive learning environments in the form of DGS and Virtual Manipulatives (also known as Java-applets). Such tools have been found in literature to be able to support investigation and understanding of transformational geometry concepts with positive effects (Dixon, 1995; Hollebrands, 2003; Sedig, 2008; Smith, Olkun, & Middleton, 2003; Yanik, 2006). Moreover, we extend research in this domain by considering the different types of dynamic representations proposed by Moreno-Armella et al. (2008), more specifically discrete dynamic and continuous dynamic, and their potential effects on different types of learners' abilities. However, since there is a variety of both DGS tools and Virtual Maniptulatives, the selection of the software will be guided by the criteria of evaluation regarding content, instructional and pedagogical adequacy, level of interaction with user, structure-organisation, aesthetics, technical integrity, and guideline availability.



### *Dynamic visualisations and learning theories*

The rapid development of computer technology has made it relatively easy to include dynamic visualisations in multimedia learning environments. While there were many suggestions regarding the potential of dynamic visualisations to promote students' understanding and problem solving abilities, there seem to be some doubts on this matter (de Villiers, 2007; Ploetzner & Lowe, 2004; Tversky, Morrison, & Betrancourt, 2002). The argument of this position is the possibility of dynamic visualisations to place great information processing requirements on learners and overburden their cognitive capacities. Ploetzner and Lowe (2004) suggested that the effectiveness of dynamic visualisations depends on many circumstances (e.g., environments' features, such as level of interactivity, and individual characteristics, such as prior knowledge level), and guide research into finding the requirements and processes for effective design and use of dynamic visualisations. According to the researchers, "complementary theoretical, exploratory, and experimental research studies are required to produce a more sophisticated conceptualisation of this field, identify important research issues, and systematically address the complexities involved" (p. 240).

In this spirit, a large number of experimental studies began to investigate, and sometimes compare, the effects and the effectiveness of different types of visualisations in learning (Cohen & Hegarty, 2007; Keehner, Hegarty, Cohen, Khooshabeh, & Montello, 2008; Kombartzky, Ploetzner, Schlag, & Metz, 2010; Kriz & Hegarty, 2007; Münzer, Seufert, & Brunken, 2009; Park, Lee, & Kim, 2009; Smith & Olkun, 2004). A large proportion of these studies built their experiments based on the Cognitive Theory of Multimedia Learning (Mayer, 2005). According to Sorden (2005), a cognitive theory and frameworks like Mayer's Cognitive Theory of Multimedia Learning (CTML) provide empirical guidelines that may help educators to design multimedia instruction more effectively.

### *The Cognitive Theory of Multimedia Learning*

The CTML theory focuses on the effect of visuals on learning in order to develop a framework for the understanding of visuals. It aims to explain how the design of instruction should be based on what is known about how the mind works, how information is processed and how learning occurs. Effective instructional design cannot be based on intuition. Humans can only process a finite amount of information in a channel at a time and they make sense of incoming information by actively creating mental representations.

Mayer (2005) also discusses the role of three memory stores: sensory (which receives stimuli and stores it for a very short time), working (where we actively process information to create mental constructs or *schema*), and long-term (the repository of all things learned). The CTML presents the idea that the brain does not interpret a multimedia presentation of words, pictures, and auditory information in a mutually exclusive manner; rather, these elements are selected and organised dynamically to produce logical mental constructs. Furthermore, Mayer (2005) underscores the importance of learning (based upon the testing of content and demonstrating the successful transfer of knowledge) when new information is integrated with prior knowledge. Specifically, the CTML is based on three core assumptions:

(1) Dual Channel Assumption: Humans posses separate information processing channels for visually represented material and auditorily represented material. Information processing occurs in three general stages. Information enters our information processing system via either the visual or auditory processing channel. This is the input stage. The information is then processed separately but concurrently in working memory, where relevant sounds and pictures are selected and organised. Eventually the information from both channels is integrated and connected to other information already held in long term memory.

(2) Limited Capacity Assumption: Humans are limited in the amount of information that can be processed in each channel at one time. During the process of learning, people can only hold a few images and a few sounds in working memory at one time. This has been researched extensively in the field of psychology and is often referred to as cognitive load theory (Sweller, 1988). Although there is some individual variability, memory span tests have shown that average memory span is pretty small, from five to seven "chunks" of information. Because of our severely limited cognitive processing capability, we are always making decisions about the allocation of our processing resources.

(3) Active Processing Assumption: Humans actively engage in cognitive processing to construct coherent mental representations of their experiences. They do not passively collect information; on the contrary, they constantly select, organise, and integrate information with past knowledge. Active learning occurs when learners apply cognitive processes to the incoming material. The result of this processing is the creation of a mental model of the information presented. The three processes that are essential for active learning are: selecting relevant material, organising the selected material, and integrating that material into existing knowledge structures. These processes take place within our fairly limited working memory.

According to CTML, multimedia is defined as any environment where material is presented in more than one format. Any instruction that uses visual support qualifies under this definition. Mayer's theory states that in multimedia environments learners must engage in five cognitive processes:

(1) Selecting relevant words for processing in verbal working memory: the learner pays attention to relevant words in a multimedia message to create sounds in working memory.
(2) Selecting relevant images for processing in visual working memory: the learner pays attention to relevant pictures in a multimedia message to create images in working memory.
(3) Organising selected words into a verbal mental model: the learner builds connections among selected words to create a coherent verbal model in working memory.
(4) Organising selected images into a visual mental model: the learner builds connections among selected images to create a coherent pictorial model in working memory.
(5) Integrating verbal and visual models and connecting them to prior knowledge: the learner builds connections between verbal and pictorial models and with prior knowledge.

Mayer's theory also proposes ten design principles for multimedia instruction (Mayer, 2008). The principles are grouped in three types. These are:

(a) Five principles for reducing extraneous processing:

1. *Coherence Principle:* People learn better when extraneous material is excluded from a multimedia lesson.

2. *Signalling Principle:* People learn better when essential words are highlighted.

3. *Redundancy Principle:* People learn better from animation with narration than from animation with narration and text, except when the onscreen text is short, highlights the key action described in the narration, and is placed next to the portion of the graphic that it describes.

4. *Spatial Contiguity Principle:* People learn better when corresponding words and pictures are presented close together rather than far from each other on the page or screen.

5. *Temporal Contiguity Principle:* People learn better when corresponding narration and animation are presented simultaneously rather than successively (i.e., the words are spoken at the same time they are illustrated in the animation).

(b) Three principles for managing essential processing:

6. *Segmenting Principle:* People learn better when a narrated animation is presented in learner-paced segments rather than as a continuous presentation.

*7. Pre-training Principle:* People learn better from a narrated animation when they already know the names and characteristics of essential components.

*8. Modality Principle:* People learn better from graphics with spoken text rather than graphics with printed text.

(c) Two principles for fostering generative processing:

*9. Multimedia Principle:* People learn better from words and pictures than from words alone. This allows people to build connections between their verbal and pictorial models.

*10. Personalisation Principle:* People learn better from a multimedia lesson when words are in conversational style rather than formal style. If people feel as though they are engaged in a conversation, they will make more effort to understand what the other person is saying.

Recently, based on Mayer's new experiments, three more principles were added to the list (Mayer, 2009). These are:

*11. Voice Principle*: People learn better when the narration in multimedia lessons is spoken in a friendly human voice rather than a machine voice.

*12. Image Principle*: People do not necessarily learn better from a multimedia lesson when the speaker's image is added to the screen.

*13. Individual differences principle*: Design effects are stronger for low-knowledge learners than for high-knowledge learners. Design effects are stronger for high-spatial learners than for low-spatial learners.



*Criticism on the CTML*

Some criticism has been levelled on the CTML and to its principles. For example, one point is that Mayer's theory does not consider video and non-narrative audio. It is also centred on learning about physical and mechanical systems, which challenges research to investigate its applicability to other situations and topics (Reed, 2006; Zhang, Wang, Lou, Li, & Zhao, 2008). The present study will attempt to apply and investigate some of the principles of this theory in the learning of transformational geometry concepts in primary education.

*Spatial ability and learning with dynamic visualisations*

When reviewing the literature on the role of spatial ability in learning with visualisations, the picture seems to be quite inconclusive and heterogeneous. While the crucial role of spatial ability in multimedia learning seems to be undisputed (e.g., Blake, 1977; Hays, 1996; Hegarty, 2004a; Large, Beheshti, Breuleux, & Renaud, 1996; Yang, Andre, Greenbowe, & Tibell, 2003) there are, for example, disagreements regarding possible aptitude-treatment interactions. Hegarty (2004b) put forward the hypothesis that, in learning with dynamic visualisations (in contrast to non-dynamic visualisations), spatial ability might play the role of an enhancer: Learners with high spatial ability might profit from learning with animations, while learners with low spatial ability might not. Later, however, Hegarty and Kriz (2007) found no such interactions in eight studies examining a mechanical device.

Some authors pointed out the possibility of a compensating effect for low spatial ability where learners with low spatial ability might be supported by dynamic visualisations, as visualisation provides learners with an external representation of a process or procedure that helps them build an adequate mental model (Hays 1996; Höffler, 2010). This means that to construct such a mental model with the use of static visualisations alone should be more difficult. Therefore, visualisation could act as a "cognitive prosthetic" (Hegarty & Kriz, 2007) for learners with low spatial ability (Höffler & Leutner, 2011; Höffler, Sumfleth, & Leutner, 2006).

Hannafin and Scott (1998) investigated the predictive value of spatial ability on post-test performance after using a geometry program with the GSP. Although they found a non-significant association between spatial ability and achievement, they stress the need for further research in this area. Gonzalez, Thomas, and Vanyukov (2005) investigated the relationships between spatial cognitive ability as assessed by Raven's Progressive Matrices, the Visual-Span (VSPAN) test, and individuals' performance on dynamic decision making tasks as they interacted with one of three microworlds. They found a positive association between VSPAN and Raven's Progressive Matrices scores and among several performance measures in three dynamic tasks.

Hannafin, Truxaw, Vermillion, and Liu (2008) investigated the effects of student spatial ability, and type of instructional programme on geometry achievement. Sixth grade students worked through either six instructional activities in the GSP or a geometry tutorial. After controlling for mathematical ability, the findings of this study suggested that students with high spatial ability performed significantly better than low-spatial learners in both instructional treatments. Specifically, students in the GSP treatment scored only marginally higher on the post-test than learners in the tutorial condition, despite spending more time on the task.

In summary, the role of spatial ability in graphics-rich environments is not well understood. Although researchers have found that spatial ability relates to achievement in mathematics in some cases, findings vary. In addition to this, many of the studies have been relational –the researchers attempted to correlate spatial ability with some external measures of mathematical performance. Few researchers examine the effect of spatial ability in an experimental design. Furthermore, there is a rareness of studies in which researchers have examined spatial ability with primary and secondary school students in public school classrooms with specific mathematical content.

Building on the results and ideas emerging from the above studies, this study will attempt to clarify the role of spatial ability on the learning outcomes of primary school students in transformational geometry. Moreover, it will compare these outcomes for two different learning environments**:** discrete and continuous dynamic visualisations.



*Cognitive styles and learning with dynamic visualisations*

In the past decade, some studies have shown evidence of individual differences and their significance in mathematics learning using appropriate software (Parkinson & Redmond, 2002). Among these differences, cognitive styles are especially related to the manner in which information is acquired and processed. For example, Riding and Douglas (1993) found that verbalisers performed better than imagers in a text based environment. Imagers similarly tend to outperform verbalisers where the presentation mode is graphical and visual. Moreover, with regard to students' cognitive styles when using computers, Atkinson (2004) found that verbalisers had the most positive attitudes towards computers, performed the best, and achieved the greatest learning benefit, whilst the analytic/imagers gained the least from a computer aided learning environment in chemistry. However, research concerning the interaction of cognitive style and multimedia is minimal (Höffler, Prechtl, & Nerdel, 2010; Massa & Mayer, 2006).

Höffler and Schwartz (2011) conducted a study regarding the effects of cognitive style across dynamic and non-dynamic representations of a computer-based learning environment. Their results suggested that learners tending towards a visual cognitive style learned significantly better from dynamic visualisations, whereas learners towards a verbal style learned better from static pictures. The researchers emphasised the importance of considering the role of individual differences in cognitive style when choosing or developing computer-based learning environments. However, in another experiment Höffler et al. (2010) found that highly developed visualisers perform better with static images, whereas less developed visualisers perform equally well with either static or dynamic images. Höffler et al. (2010) suggest that such interactions should also be investigated in other topics.

Kyritsis et al. (2009) investigated the way in which cognitive style impacts ability to acquire spatial knowledge from a range of virtual environments, based on the field-dependence/independence cognitive style model. Their results suggested that people with different cognitive style acquire spatial knowledge differently, and show the tendency of learning faster from different virtual environments. Kyritsis et al. (2009) concluded that cognitive style plays a significant role when acquiring knowledge from a virtual environment.

Within the mathematics education field there is hardly any research on the use of dynamic geometry and its effect on learners with different cognitive styles. There have been concerns that the holistic perception of ideas and the visual form of information may raise difficulties for some learners who may find the dynamic and visual reasoning a complex process (Boulter & Kirby, 1999; Ellis & Kurniawan, 2000; Ploetzner & Lowe, 2004; Smith et al., 2009). Nevertheless, a recent attempt by Pitta-Pantazi and Christou (2009a) provides evidence that DGS seems to accommodate and enhance students' learning more in the case of learners who tend to be verbalisers. However, there is still much research required in order to clarify the relationship between different kinds of dynamic visualisations and different kinds of cognitive styles.

The lack of research regarding the effects of different features of dynamic geometry on different types of learners in mathematics education sets this field of research necessary and crucial. This study will attempt to investigate the different effects of dynamic geometry in learning transformational geometry concepts not only by considering individual differences in verbal and visual preferences, but by considering also the individual differences in object and spatial visualisation preference.

## Summary

The review of the research on transformational geometry reveals the need to search for the factors which constitute transformational geometry ability and for a sequence of hierarchical levels of ability in transformational geometry (Boulter & Kirby, 1994; Hollebrands, 2003; Yanik & Flores, 2009), which will provide for a successful sequence of learning activities in transformation geometry (Moyer, 1978). This model should be based on various tasks and a greater range of age levels (Moyer, 1978; Soon, 1989). This model should not only consider the sequence and classification of translations, reflections, and rotations, which according to Lesh (1976) and Moyer (1978) is strictly a mathematical one. Besides, research in the order of difficulty on transformations seems to be contradictory; while Moyer proposes that a translation is as easy as a reflection and rotations are the most difficult, Schultz and Austin (1983) suggest that translations are the easiest and that the difficulty of reflections and rotations depends on certain configurations. A model on transformational geometry learning should not restrict its levels based on the mathematical classification of the geometric transformations; it should take into account the variables identified by previous researchers (Bell & Birks, 1990; Grenier, 1985; Hart, 1981; Schultz & Austin, 1983), and look into the sequence of understanding within each transformation and also through all transformational geometry concepts. This means that, although the complexity of the figure may be a variable influencing performance on reflections and rotations and finding the reflection of a complex figure is considered a high-difficulty task, the corresponding task in translations might be as easy as performing the most common reflection of a simple figure over a vertical axis. A variety of task types must be accounted in this model, including recognising the image of a transformation, recognising a transformation, identifying the parameters of a given transformation and constructing the image of a transformed figure.

Given the extensive connection of transformational geometry and spatial ability it is surprising that the emphasis of the research has focused mainly on the influence of teaching transformational geometry to develop students’ spatial ability. Since it is considered that spatial ability is developed over age and through the experiences of children, one would assume that a person’s spatial ability would influence his or her understanding of transformations. Therefore, this ability should be taken into consideration in constructing a theoretical model for the understanding of transformations. Moreover, all of these studies seem to measure spatial ability as a single component. There seems to be no study seeking a relationship between transformational geometry ability and different subcomponents of spatial ability. This study will take into consideration that spatial ability is multi-faceted, and will also investigate the role of each spatial ability factor in transformational geometry ability.

It is also of great importance to examine the individual differences of students at different levels of spatial ability (high and low) and with different preferred modes of thinking, such as cognitive style, in transformational geometry ability (Kirby & Boulter, 1999). Moreover, it is important to examine the role of these differences in the development of different strategies for solving transformational tasks, as appeared in the study of Boulter and Kirby (1994). One might consider that the understanding of transformational geometry concepts might be enhanced by developing students' spatial ability. This is an important aspect in the case of low-visualisers (Kirby & Boulter, 1999), which raises the question of whether an instructional program of training on spatial abilities would assist students – and especially low-visualisers – in overcoming problems with understanding transformations. Research in the field suggests that such an instructional programme should be based on dynamic visualisations (Dixon, 1995; Hollebrands, 2003; Smith et al., 2009). However, such an instructional programme should take into consideration the existence of different types of dynamic visualisations (Hegarty, 2004a; Moreno-Armella et al., 2008; Smith el al., 2009) and how these can have a different impact on different types of learners regarding spatial ability level and cognitive style (Höffler, 2010; Mayer, 2005).

CHAPTER III

# METHODOLOGY

## Introduction

This chapter describes the methodological procedure, the subjects of the study, the assessment instruments and their procedure of development and validation. It also describes the design of the semi-structured clinical interviews and the selection procedure of the subjects that participated in the interviews. Additionally, it describes the design and the procedure of the instructional interventions. Finally, it explains the coding procedures and statistical methods which were applied for analysing the data.

One of the main issues of this study was the way of measuring students' ability in transformational geometry concepts, and specifically the type of test and the method of assessment for students' responses. In similar studies, three different methodological approaches have been used: (1) written test with multiple choice responses where each task evaluates students' ability for a specific level and students are ranked based on their overall score (Usiskin,1982), (2) clinical interviews based on specific tasks and open-ended questions (Burger & Shaughnessy, 1986) where students' responses are analysed and classified to levels based on the dominant level that characterises students' reasoning, and (c) a combination of quantitative and qualitative methods. Tashakkori and Teddlie (2002) emphasise how the use of either quantitative or qualitative methods is not adequate in modern social sciences, and guide research practice into applying complex research designs. This need for more sufficient and effective research methods has led to the development of combinatorial methodological approaches.

To overcome the aforementioned issues, the design of this research was based on the combination of both quantitative and qualitative methods, by applying both the administration of tests and questionnaires, and the conduction of clinical interviews (Kelly & Lesh, 2000; Tashakkori & Teddlie, 2002). The use of complimentary approaches for the investigation of students' ability in transformational geometry concepts enhanced the validity of this research. This was accomplished with the combination of these two methodological approaches to give responses to research questions that may not be so accurate with the use of either of the two methods. For example, this research used quantitative methods to classify students into levels of ability in transformational geometry concepts, and qualitative methods to identify the misconceptions and strategies of students at each assigned level of ability in transformational geometry concepts.

## Subjects

The subjects of the study were 9- to 14-year-old students. Overall, 507 students (266 boys and 241 girls) from primary and secondary schools participated in this study. Specifically, 91 students came from the fourth grade of primary school (9-10 years old), 115 came from the fifth grade of primary school (10-11 years old), 93 came from the sixth grade of primary school (11-12 years old), 106 came from the first grade of secondary school (12-13 years old), and 102 came from the second grade of secondary school (13-14 years old). The students came from urban and rural schools in the districts of Nicosia, Larnaca, and Famagusta. The selection of students at this age range aimed to capture the most important cognitive changes in children's development of spatial thinking, since according to Piaget (1972), this is the time when students transit from the concrete (7-11 years old) to the formal (11-16 years old and onwards) operational stage.

For the clinical interviews, 40 students were selected from the subjects, based on the results of the quantitative analyses. Specifically, four levels of transformational geometry ability emerged from the statistical analyses of the quantitative data. In order to get an insight of students' qualitative way of thinking at each level, two students were selected from each of the five grades and for each of the four levels.

For the teaching experiment, four sixth-grade classrooms were selected from the ones participating in the study. Seventy-nine students were in these classrooms, and they were divided into two groups of 40 and 39 students. The two groups of students were initially controlled for differences in their transformational geometry ability, their spatial ability, and their cognitive style.

## Instruments

Two tests were developed and one self-report questionnaire was translated for the accomplishment of this study. One test was for measuring students' ability in transformational geometry concepts, and the other test was for measuring spatial ability.

The self-report questionnaire was for measuring students' cognitive style. The tests were used both as pre-tests and post-tests to the teaching experiment.

### *Transformational geometry ability test*

For measuring students' transformational geometry ability, a written test was developed which included multiple choice and open tasks. The content of the test (see Appendix I) referred to the basic concepts of transformational geometry, with emphasis on the recognition of a transformation's image, the recognition of a transformation, the identification of a transformation's parameters, and the construction of an image under transformation, which this study considers as the four factors synthesising the ability in each of the three transformational geometry concepts. These abilities were measured for each of the three geometric transformations (translation, reflection, and rotation). The test included 48 tasks.

The tasks used for the development of this test were drawn and modified from: (a) tasks that had been used in previous research studies and were considered as suitable for measuring primary and secondary school students' ability in transformational geometry concepts, and (b) tasks that were found to be often used in instructional units of transformational geometry in mathematics curricula of foreign countries, such as the United Kingdom and the United States of America.

The modifications were mainly based on the research findings of the studies by Molina (1990) and Schultz (1978). Specifically, Molina's (1990) findings were utilised for the classification of the tasks and for their assigned level of difficulty. The findings from Schultz's (1978) study were used for selecting the figurative and operative configurations which can influence a task's level of difficulty. Following Schultz's (1978) findings, the figurative configuration that was used in the development of the tasks was the complexity of the figure, and the operative configurations were the distance and direction of the transformation.

In order to respect the figurative configuration of the complexity of the shape, the tasks of the study were primarily based on the right triangle, which is the most common figure used in transformational geometry research tasks. According to Molina (1990, p. 53), the right triangle offers "all the features necessary to study a transformation: that is, the effect on a point, a segment and a polygon… [it] provides the minimum number of points necessary to define and represent the transformation clearly." In cases where the tasks were very similar and the use of the right triangle would have made the correct response obvious, other simple quadrilaterals, or the L-shaped figure that are also common in transformational geometry tasks were mainly used, in an attempt to keep the complexity of the shape configuration as simple as possible. Squares and rectangles were avoided, since the property of symmetry would inhibit the students from distinguishing the features of a transformation, and the researchers from assessing whether the students' response took into consideration the orientation property for each transformation. However, one type of construction task with an irregular hexagon was included as a complex figure for each transformation, in order to examine the role of figure complexity in the levels of transformational geometry ability.

Regarding the operative configurations, the configuration of direction was incorporated by including tasks of the three following directions, for all three transformations, in all types of tasks: (a) horizontal left-to-right direction, (b) vertical down-to-top direction and, (c) diagonal left and down-to-right and top. According to Schultz (1978), there are eight general directions to perform a transformation, but since it was impossible to include all of them in a dissertation, some representative examples were selected. Regarding the configuration of distance, it was incorporated by including construction and identification of parameters tasks with short distance between the image and pre-image, as well as tasks with overlapping figures, for all three transformations. It should be noted here that all construction and identification of parameters tasks were given in squared paper. Recognition of transformation was given both in squared and plain paper and recognition of image was given in squared paper only in the case of translation for facilitating communication in order to use common conventional units.

Therefore, the tasks that were developed for the purposes of this research and aimed to assess students' ability in a wide spectrum of skills and abilities in transformational geometry concepts were based on the findings of relevant research studies. Analytically, the following list describes the tasks that were developed for each geometric transformation. As described earlier, the tasks were all presented in squared paper, unless mentioned otherwise in the list. The number in the parenthesis indicates their positions in Appendices I and II. Examples of each type of task for each of the three geometric transformations are presented in Table 3.1.

1. Translation tasks
    a. Recognition of a translation's image

        i. One item in vertical direction (A1)
        ii. One item in horizontal direction (A2)
        iii. One item in diagonal direction (A3)
    b. Recognition of a translation
        i. One item in vertical direction (A4)
        ii. One item in horizontal direction (A5)
        iii. One item in diagonal direction (A6)
        iv. One item with unspecified direction on plain paper (A7)
    c. Identification of a translation's parameters
        i. One item in vertical direction (A8)
        ii. One item in horizontal direction (A9)
        iii. One item in diagonal direction (A10)
        iv. One item with overlapping image in horizontal direction (A11)
    d. Construction of the image of a translation
        i. One item in vertical direction (A12)
        ii. One item in horizontal direction (A13)
        iii. One item in diagonal direction (A14)
        iv. One item with overlapping image in horizontal direction (A15)
        v. One item with complex figure in horizontal direction (A16)

2. Reflection tasks
    a. Recognition of a reflection's image
        i. One item in vertical direction on plain paper (with given horizontal line of reflection) (B1)
        ii. One item in horizontal direction on plain paper (with given vertical line of reflection) (B2)
        iii. One item in diagonal direction on plain paper (with given diagonal line of reflection) (B3)
    b. Recognition of a reflection
        i. One item in vertical direction (with no given line of reflection) (B4)
        ii. One item in horizontal direction (with no given line of reflection) (B5)
        iii. One item in diagonal direction (with no given line of reflection) (B6)
        iv. One item with unspecified line of reflection and direction on plain paper (B7)

c. Identification of a reflection’s parameters
    i. One item in vertical direction (horizontal line of reflection) (B8)
    ii. One item in horizontal direction (vertical line of reflection (B9)
    iii. One item in diagonal direction (diagonal line of reflection) (B10)
    iv. One item with overlapping image in horizontal direction (B11)
d. Construction of the image of a reflection
    i. One item in vertical direction (with given horizontal line of reflection) (B12)
    ii. One item in horizontal direction (with given vertical line of reflection) (B13)
    iii. One item in diagonal direction (with given diagonal line of reflection) (B14)
    iv. One item with overlapping image in horizontal direction (with given vertical line of reflection) (B15)
    v. One item with complex figure in horizontal direction (with given vertical line of reflection) (B16)

3. Rotation tasks
    a. Recognition of a rotation’s image
        i. One item in vertical direction on plain paper (3/4 of a turn) (C1)
        ii. One item in horizontal direction on plain paper (1/4 of a turn) (C2)
        iii. One item in diagonal direction on plain paper (1/2 of a turn) (C3)
    b. Recognition of a rotation
        i. One item in vertical direction (3/4 of a turn) (C4)
        ii. One item in horizontal direction (1/4 of a turn) (C5)
        iii. One item in diagonal direction (1/2 of a turn) (C6)
        iv. One item with unspecified direction on plain paper (C7)
    c. Identification of a rotation’s parameters
        i. One item in vertical direction (3/4 of a turn) (C8)
        ii. One item in horizontal direction (1/4 of a turn) (C9)
        iii. One item in diagonal direction (1/2 of a turn) (C10)
        iv. One item with overlapping image in horizontal direction (1/4 of a turn) (C11)
    d. Construction of the image of a rotation
        i. One item in vertical direction (3/4 of a turn) (C12)
        ii. One item in horizontal direction (1/4 of a turn) (C13)

iii. One item in diagonal direction (1/2 of a turn) (C14)

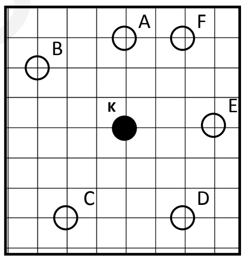
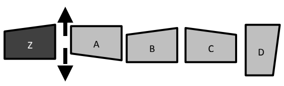
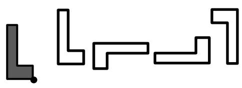
iv. One item with overlapping image in horizontal direction (1/4 of a turn) (C15)

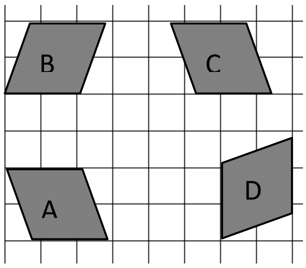
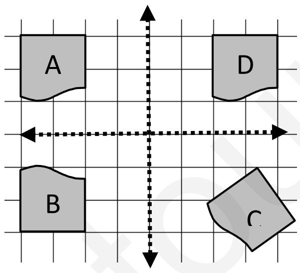
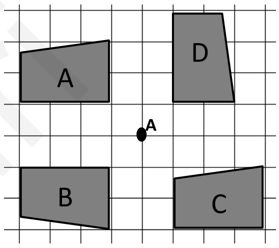
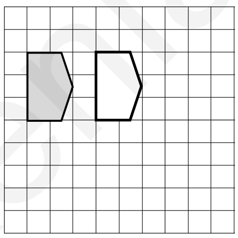
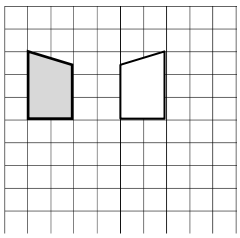
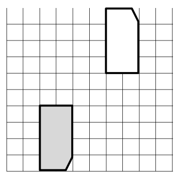
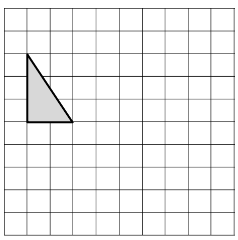
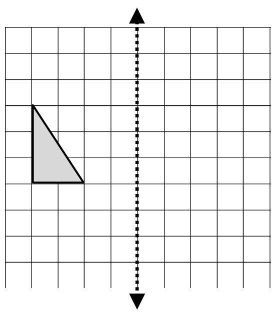
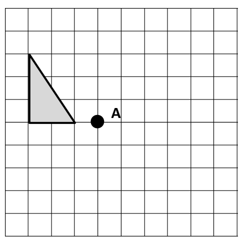
v. One item with complex figure in horizontal direction (1/4 of a turn) (C16)

Because of the large number of items in the transformational geometry concepts ability test, and due to the great amount of time required to solve them, the tasks were divided into two different parts of equal degree of difficulty and estimated time for solving requirements. This was processed in order to administer the test to the students in two separate sessions, in order to avoid children getting tired and not work efficiently and consciously throughout the whole test.

Table 3.1

*Examples of Tasks in the Transformational Geometry Ability Test*

| **Type of task** | Translation example | Reflection example | Rotation example |
|---|---|---|---|
| 1. Recognition of a transformation's image | Which of the following images is the translation of the pre-image K, when it translates 3 units up?<br>A C D E<br> | Which of the following shapes is the reflection of shape Z over a vertical line of symmetry?<br> | Which of the following shapes is the rotation of the grey figure at ¼ of a turn?<br> |

| | | | |
|---|---|---|---|
| 2. Recognition of a transformation | Which of the following pairs of shapes show a translation?<br>a) A and D<br>b) B and C<br>c) C and D<br>d) A and C<br> | Which of the following pairs of shapes show a reflection?<br>a) A and D<br>b) B and C<br>c) B and A<br>d) C and D<br> | Which of the following pairs of shapes show a rotation?<br>a) A and D<br>b) B and C<br>c) C and D<br>d) A and C<br> |
| 3. Identification of a transformation's parameters | Give the instructions for the translation of the shaded figure to the position of the white figure.<br> | Draw the line of symmetry in every case.<br> | Find the point of rotation and the fraction that shows how much the shape turned to the right.<br><br>1/4 2/4 3/4 |
| 4. Construction of an image under transformation | Translate 4 units to the <u>right</u>.<br> | Draw the reflection of each shape over the given line.<br> | Rotate the shape ¼ of a turn to the right.<br> |

Following this procedure, two primary school teachers were requested to review the tasks and comment on their level of difficulty and estimations of time required for primary school students to solve the test. The teachers' comments and suggestions for modifications were considered. Following these modifications, the test was administered to two primary school students of different age (an 11-year-old boy and a 12-year-old girl) in order to examine its applicability.

### *Spatial ability test*

For the development of the spatial ability test (see Appendix I), selected tasks from the following test were used, under certain modifications. The tests and the modifications made are described in the following list, in the order of the sections that were included in this test. Examples of items for each section are presented in Table 3.2.

1. The *Card Rotation Test* from the ETS kit (Ekstrom, French, & Harmanm, 1976). According to Hegarty and Waller (2006), this test is used for measuring Spatial Relations ability. This test requires participants to view a two-dimensional target figure and judge which of the alternative test figures are planar rotations of the target figure (as opposed to its mirror image) as quickly and as accurately as possible. The task was simplified for primary school students by modifying it into a multiple choice task with only one correct response to select over four alternatives. Four of these items were included in the pilot test.
2. The *Paper Folding Test* from the ETS kit (Ekstrom et al., 1976). According to Ekstrom et al. (1976), the Paper Folding Test measures Spatial Visualisation ability. It is believed to reflect the ability to apprehend, encode, and mentally manipulate abstract spatial forms (Lohman, 1988). Each item shows successive drawings of two or three folds made in a square sheet of paper. The final drawing shows a hole being punched in the folded paper. The participants are requested to select one of five drawings to indicate what the punched sheet would look like when fully opened. The task was simplified for primary school students by eliminating the choice of alternative answers to four. Three items were included in the pilot test.
3. The *"Where was the Photographer?"* items from Meissner's (2006) test for measuring nine-year-olds' spatial ability. These items were designed to measure

Spatial Orientation ability. The task presents the participants with a figure of three towers on an island, surrounded by five photographers in diffrent boats. Each item requires the participants to decide which of the five photographers might have taken a given picture. Three items were included in the pilot test.

4. The *Vandenberg-Kuse Mental Rotation Test*. The Vandenberg-Kuse Mental Rotation Test (Vandenberg & Kuse, 1978) measures mental rotation ability, and is considered to measure Spatial Visualisation (Hegarty & Waller, 2006). Each item requires the participants to compare two-dimensional line drawings of three-dimensional geometric figures composed of cubes, in order to indicate which two out of four figures are rotated versions of the criterion figure. The task was simplified by including only one correct response out of four alternatives. Three of these items were included in the pilot test.
5. The *Space Relations Test* from the Differential Aptitude Tests (5th Ed.) (Bennet, Seashore, & Wesman, 1972). According to Miyake, Friedman, Rettinger, Shah, and Hegarty (2001), this test measures the ability of Spatial Visualisation. In the Space Relations Test, the participants are required to select which of the four alternative figures in the right panel depicts what the pattern in the left panel would look like if folded. Four items of this test were included in the pilot test. The selection of the items was based on their low level of difficulty, in order to be suitable for primary school students.
6. The *Object Perspective Test*, developed by Hegarty and Kozhevnikov (2001), which is used to measure Spatial Orientation.  This test presents the participants with an array of seven objects. In each item, the participants are asked to imagine themselves as being at the position of one of the objects in the display (the station point) facing towards another object (which defines their imagined perspective within the array), and are asked to indicate the direction to a third (target) object. The participants are given a picture of a circle, in which the imagined station point is drawn in the centre of the circle, and the imagined direction of perspective is drawn as an arrow pointing vertically up. The participants are required to draw another arrow from the centre of the circle indicating the direction to the target object. Participants are prevented from physically rotating the test booklet, which would give them a physical perspective of the one they are required to imagine. The items of this test were simplified to be more suitable for primary school students. The modifications that were made to the paper-and-pencil version of the test were inspired from the computerised version of the test. Specifically, a picture of a head

was placed at the station point in each array, facing the direction of the required imagined perspective. Moreover, the circle was divided with arrows into circular sectors of 45°, thus requiring the students to select the arrow showing to the correct direction, instead of having to draw the arrow with approximation. Three of these items were included in the pilot test.

7. The *Cube Comparison Test* from the ETS kit (Ekstrom et al., 1976). It is believed to measure the Spatial Relations factor (Hegarty & Waller, 2004; Höffler, 2010). In the Cube Comparison Test, each item presents two drawings of cubes, with letters and numbers printed on their sides. Participants are requested to judge whether the two drawings could show the same cube. Three of these items were included in the pilot test.
8. The *Form-Board Test* from the ETS kit (Ekstrom et al., 1976). This test is suggested to measure Spatial Visualisation ability (Hegarty & Kozhevnikov, 2001). It presents the participants with a figure on top and groups of shapes below. The participants are required to select, for each group, the shapes that can be joined to form the given figure. The items were simplified for primary school students by giving only one group of shapes per figure and by indicating the correct number of shapes to join, which was always three shapes. Three of these items were included in the pilot test.
9. The *Image Perspective Test*, by Pittalis and Christou (2010). This test was developed to measure Spatial Orientation ability. The items of this test present the students with a person observing an object from its back view. The participants are required to select which of the four alternatives represent the view of the item, from the depicted observer's perspective. Four items of this test were included in the pilot test.

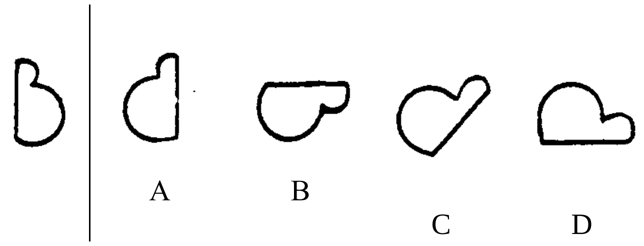
The spatial ability test was then discussed with two primary school teachers regarding the level of difficulty and time requirements for primary school students. Again, the test was administered to the two primary school students (11 and 12 years old) and to two adults to examine its applicability. No modifications were required for this test.

Table 3.2

*Examples of the Test Items that were used for the Spatial Ability Test*

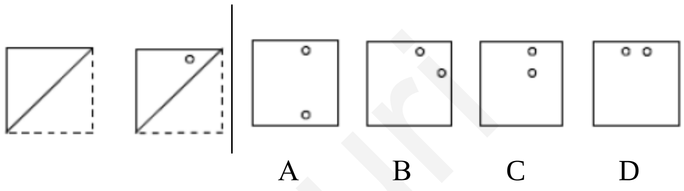
**A. Card Rotations (4 items):**

Which of the shapes after the line can be the same as the one before the line when <u>rotated</u> to the left or to the right, <u>without inverting it</u>?



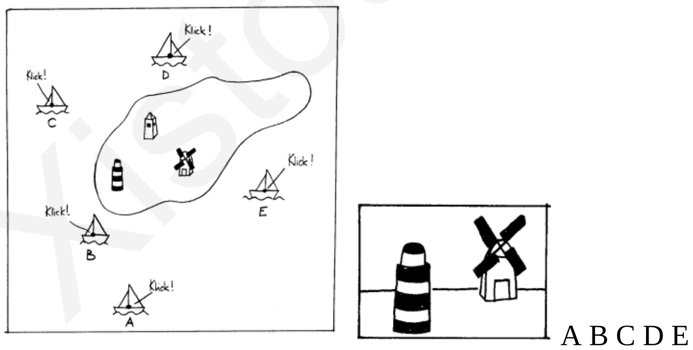

**B. Paper Folding (3 items):**

Which of the figures after the line depicts the paper when unfolded?



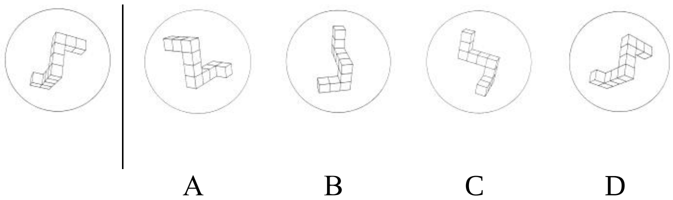

**C. Where was the Photographer? (3 items):**

In which boat was the photographer that took this picture?



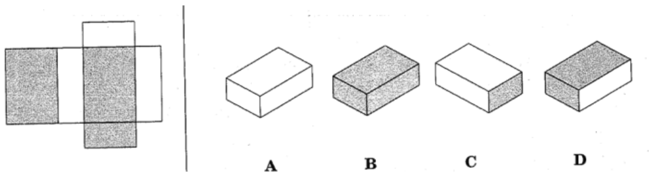

**D. Mental Rotations (4 items):**

Which two of the four pictures A, B, C, and D are rotations of the first picture?



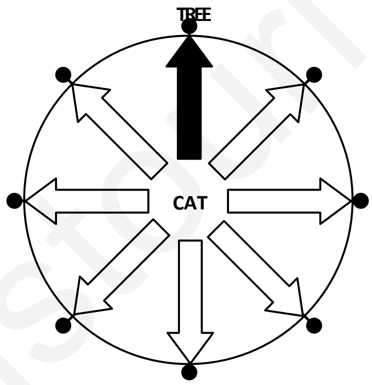

**E. Space Relations (4 items):**

When the net before the line is folded, which of the four objects after the line does it form?



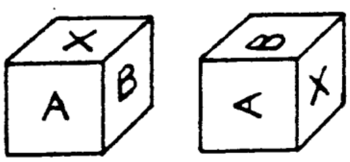
**F. Object Perspective (3 items):**

Imagine that you are standing at the position of the cat and that you are facing the tree. Find the arrow that shows towards the direction where the STOP sign will be.


**G. Cube Comparison (3 items):**

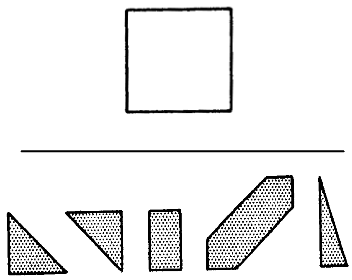
Considering that none of the cubes has the same two sides, decide whether the two shapes show the same or different cube.



SAME DIFFERENT

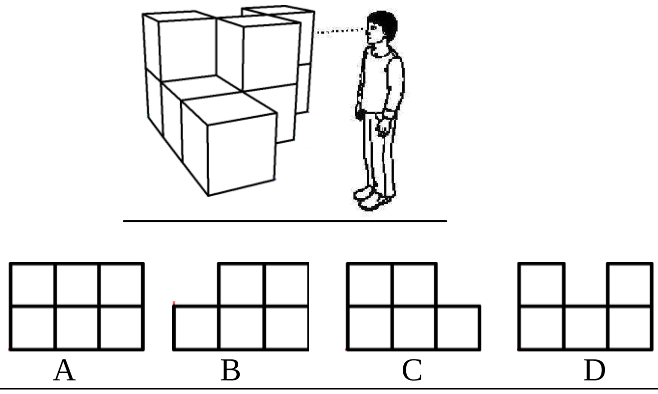
**H. Form-Board (3 items):**

Which three of the shapes that are under the line can be joined to form the shape that is above the line?



**I. Image Perspective (4 items):**

Which of the four shapes below the line, show what the observer sees from his position?




*Cognitive style questionnaire*

Regarding the cognitive style assessment, after deciding to adopt the theoretical framework of the Object-Spatial Imagery and Verbal Cognitive style, proposed by Kozhevnikov et al. (2002, 2005), the possible approaches for measuring this characteristic were evaluated. Common approaches usually include the administration of psychological tests, either in paper-and-pencil format or in computerised format (Blazhenkova & Kozhevnikov, 2009). The most practical and less time-consuming approach to assess the Object-Spatial Imagery and Verbal Cognitive Style was found to be a validated self-report questionnaire.

The self-report questionnaire was the children's Object-Spatial Imagery and Verbal Questionnaire© (c-OSIVQ) (see Appendix II), developed by Blazhenkova, Becker, and Kozhevnikov (2011). The c-OSIVQ is used to assess individual differences in spatial imagery, object imagery, and verbal cognitive style in children. For this questionnaire, the children are asked to read 45 statements and rate each item on a 5-point Likert scale with 1 indicating total disagreement and 5 indicating total agreement. Ratings 2 to 4 indicate intermediate degrees of agreement/disagreement. Fifteen of the items measure object imagery preference and experiences, fifteen items measure spatial imagery preference and experiences, and fifteen items measure verbal preference and experiences.

The items assessing object-imagery cognitive style include statements about vividness of mental imagery, photographic memory, preferences for painting with colours, ease of image maintenance, and elicit imagery (e.g., "*When reading a book, I can usually imagine clear, colourful pictures of the people and places*"). The items assessing spatial-imagery cognitive style include statements about 3D geometry, schematic mental imagery, mechanical inclination, and spatially intensive games (e.g., "*I am good at solving geometry* *problems with 3D figures*"). Verbal cognitive style items are statements referring to the speed of reading, ease of writing, fluency in expressing thoughts and ideas verbally, and storytelling (e.g., "*I am good at expressing myself in writing*") (Blazhenkova, Becker, & Kozhevnikov, 2011, p.282).

The c-OSIVQ was translated in Greek for the needs of this study. Effort was made to keep the vocabulary as simple as possible, and the meaning of the statement as close to the original English version as possible.

### *Validity of instruments*

#### *Transformational geometry ability test*

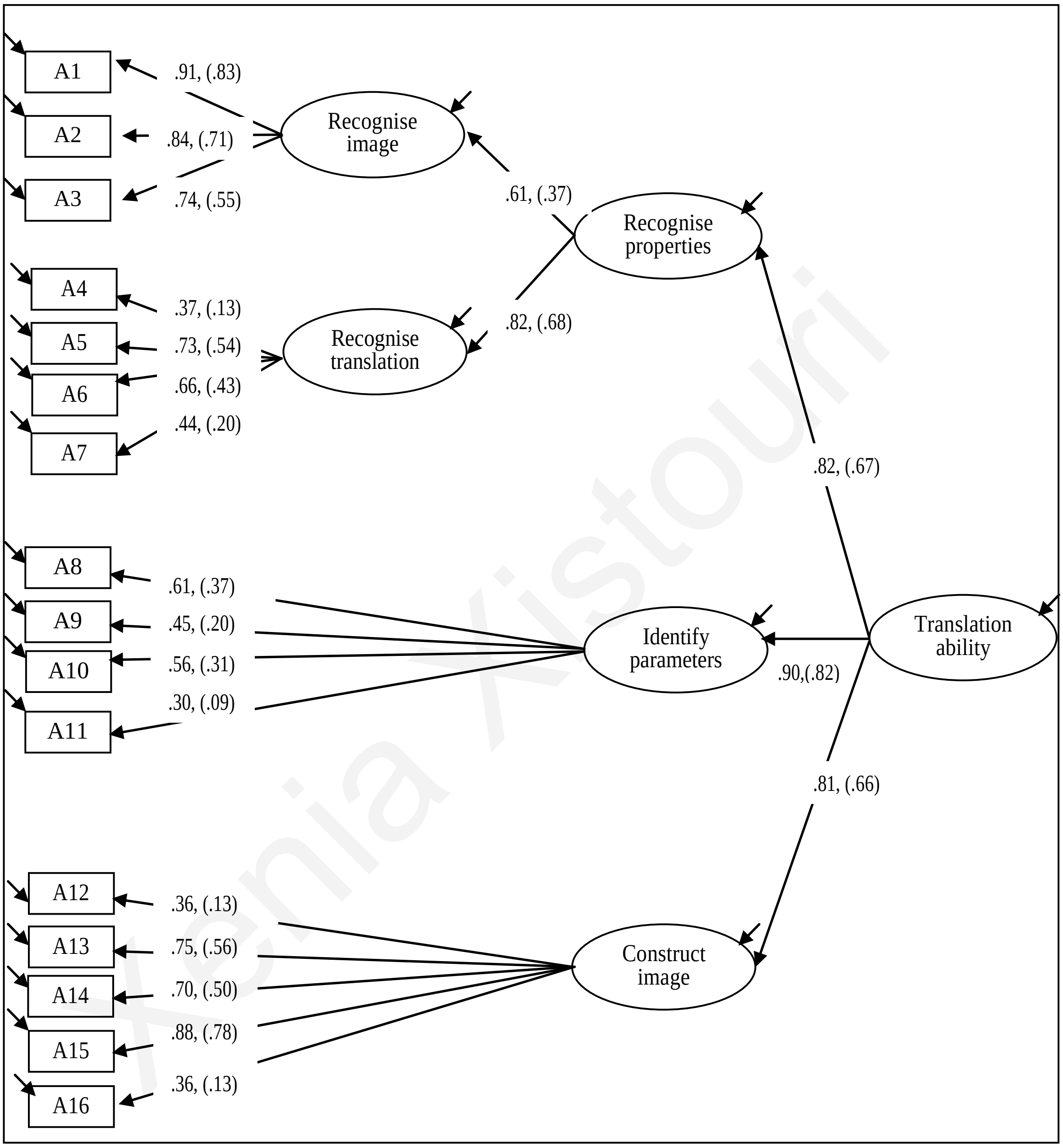
The items of the transformational geometry concepts test were developed for the aims of this research, based on textbook examples and literature review suggestions. For this reasons, it was necessary to examine the construct validity of the items to measure the theoretical factors of ability.

The transformational geometry tests were structured in three parts, each assessing one of the three concepts: translation, reflection, and rotation. In this section, the results concerning the validity for each part of the test regarding each transformation will be presented separately. This procedure was followed in the analysis, in order to assess the validity of each part of the test to measure ability in each geometric transformation, as the small number of students participating in the pilot administration would not have given reliable results for the whole test.

In the case of translation, subsequent model tests led to the model shown in Figure 3.1, which proved to have very good fit to the data ($CFI = .95$, $x^2 = 136.30$, $df = 98$, $x^2/df = 1.39$, $p < .05$, $RMSEA = .05$).

Figure 3.1 presents the suggested model of translation ability. Two of the expected factors, "Recognition of image" in translation and "Recognition of translation", seemed to constitute a second order factor, which significantly contributed to translation ability at the primary school level. This factor was named "Recognition of properties" at the primary school level, since we believe that the common characteristic shared by these tasks was the recognition of the translation properties to preserve both the orientation and size of the figure.

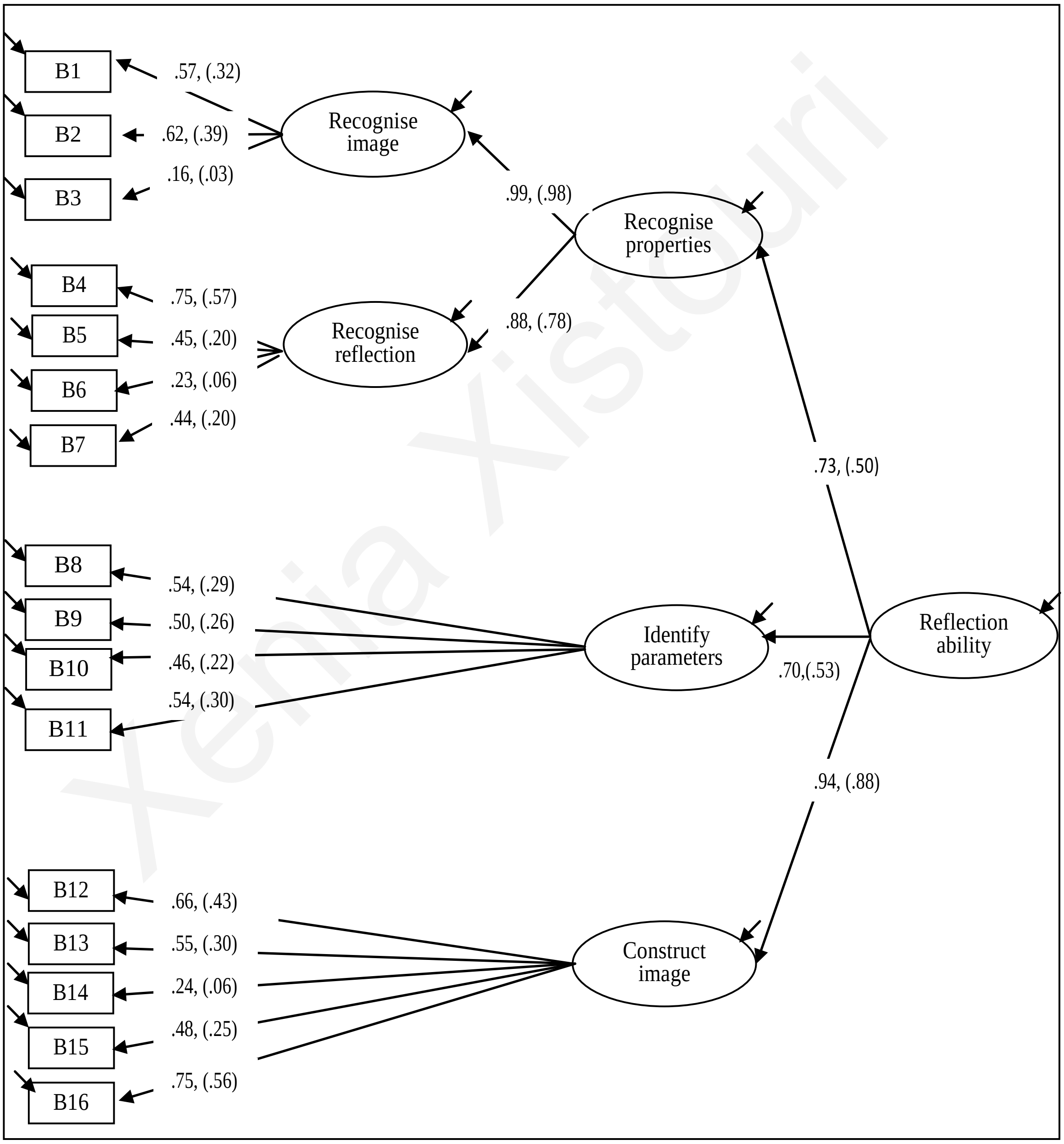
The low factor loading of some items, especially in the cases of “Construction of image” items, led to considerations for modifying these items, in order to clarify the tasks and the wording.




*Note.* The first number indicates factor loading and the number in parenthesis indicates the corresponding interpreted dispersion ($r^2$).

*Figure 3.1*. Model of construct validity for the translation part of the transformational geometry pilot test.

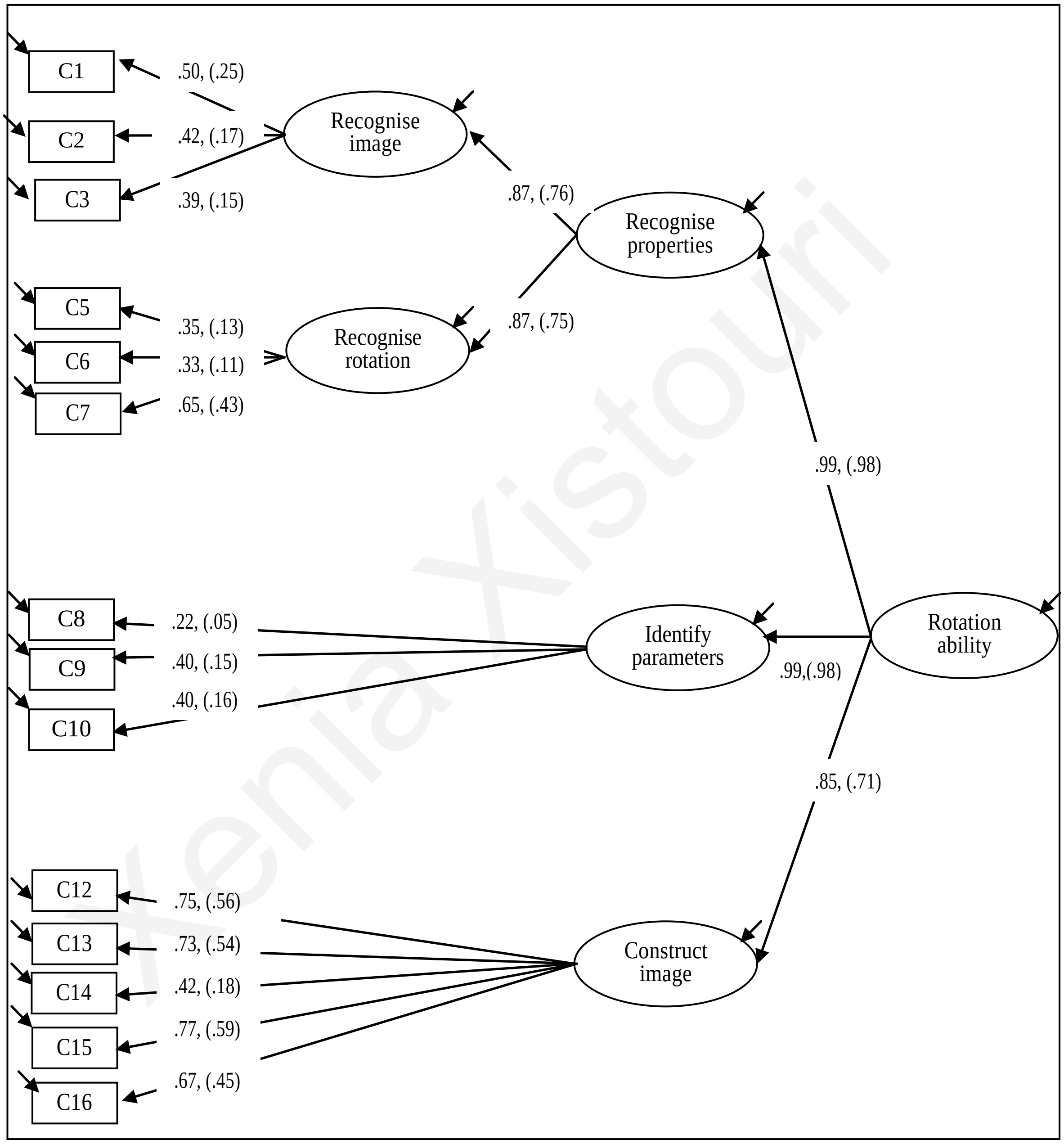
In the case of reflection, the model shown in Figure 3.2 proved to have very good fit to the data ($CFI$ = .93, $x^2$ = 126.60, $df$ = 98, $x^2/df$ = 1.29, $p$ < .05, $RMSEA$ = .04). Again, the factors of "Recognition of image" in reflection and "Recognition of reflection" seemed to constitute a second order factor, which contributed significantly to reflection ability at the primary school level. Similarly, this factor was also named "Recognition of properties" at the primary school level, since as in the case of translation, we believe that the common characteristic shared by these tasks was the recognition of the reflection properties to preserve both the size of the figure and its distance from the line of symmetry, while inverting the shape's orientation.




*Note.* The first number indicates factor loading and the number in parenthesis indicates the corresponding interpreted dispersion ($r^2$).

*Figure 3.2.* Model of construct validity for the reflection part of the transformational geometry pilot test.

In the case of rotation, the model tests indicated that the model shown in Figure 3.3 proved to have very good fit to the data (*CFI* = .95, $x^2 = 91.32$, *df* = 70, $x^2/df = 1.32$, $p < .05$, *RMSEA* = .04 ).




*Note.* The first number indicates factor loading and the number in parenthesis indicates the corresponding interpreted dispersion ($r^2$).

*Figure 3.3*. Model of construct validity for the rotation part of the transformational geometry pilot test.

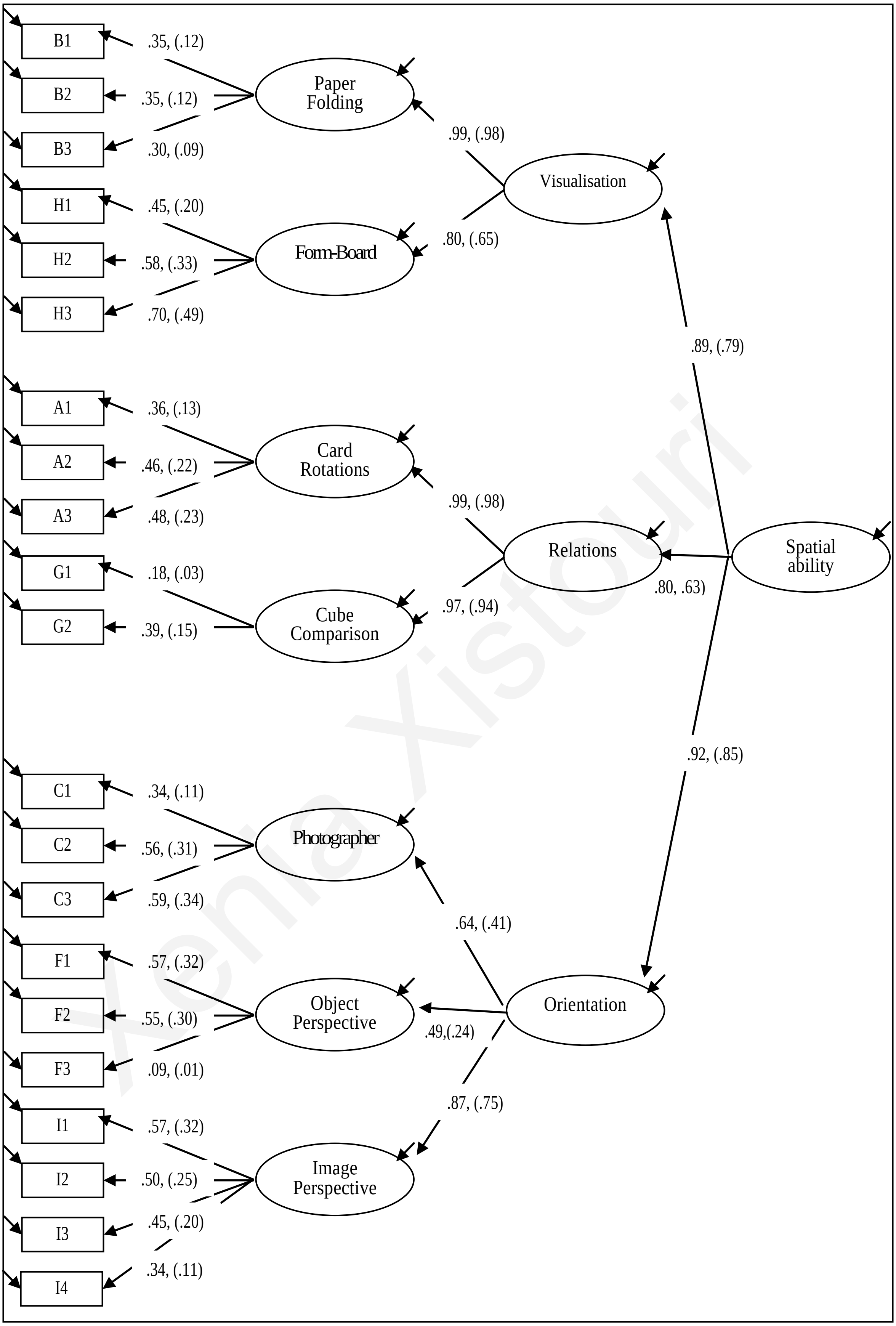
Figure 3.3 presents the suggested model of rotation ability. As in the other two geometric transformations, the same two factors "Recognition of image" in rotation and "Recognition of rotation" constituted a second order factor. This factor was the corresponding "Recognition of properties" at the primary school level in the case of rotation, as in the other two concepts. We believe that the common characteristic shared by these tasks is the recognition of the rotation properties to preserve both the size of the figure and its distance from the centre of rotation, while changing the shape's orientation and position in the plane. However, one item from the "Recognition of rotation" seemed to load better to the "Recognition of image" factor, while one item from the "Identification of parameters" factor was excluded from the analysis, because its factor loading was not statistically significant. These indications led to considerations regarding the modification of these for the final test.

*Spatial ability test*

The spatial ability test was structured by nine different types of tasks, which, according to our theoretical framework, represent three different kinds of spatial ability: spatial visualisation, spatial relations, and spatial orientation (Lohman, 1979). In the field of psychological studies in spatial ability, it is common to perform confirmatory factor analysis to examine the construct validity of tests when these are expected to measure different abilities. Therefore, confirmatory factor analysis was performed to examine the validity of the proposed model. Based on the literature review findings regarding the different kinds of spatial ability, our theoretical model proposes that the tasks from the "Paper Folding" (Ekstrom et al., 1976), the "Mental Rotation" (Vandenberg & Kuse, 1978), the "Form-Board" (Ekstrom et al., 1976), and the "Space Relations" (Bennet, Seashore, & Wesman, 1972) are features of the *Spatial Visualisation* ability factor; the "Card Rotations" and "Cube Comparisons" (Ekstrom et al., 1976) are features of the *Spatial Relations* ability factor; and the "Where was the Photographer?" (Meissner, 2006), the "Object Perspective" (Hegarty & Kozhevnikov, 2001), and the "Image Perspective Test" (Pittalis & Christou, 2010) are features of the *Spatial Orientation* ability factor.

For the validation of the spatial ability test, a pilot administration with primary school students was performed. The results from the analysis of the pilot data showed that the fit of the model to the data was satisfactory, confirming the structure of the proposed model ($CFI = .97$, $x^2 = 190.40$, $df = 182$, $x^2/df = 1.05$, $p < .05$, $RMSEA = .02$). The analysis showed that the items of the "Space Relations" and the "Mental Rotations" did not load significantly to the corresponding factors, and were therefore withdrawn from the analysis. Similarly, one item from the "Card Rotations" and one item from the "Cube Comparisons" were withdrawn because they did not load significantly to the corresponding factors. The rest of the items loaded significantly to the corresponding factors, with the exception of one "Object Perspective" item (F3) which was included for reasons of better measurement of the Spatial Orientation factor and with the intent of future modification (see Figure 3.4).

The low factor loading of some items to their corresponding factors was an indication of problems with students' understanding or with their level of difficulty for primary school students. This guided the development of the test into substituting some items, enriching some factors with more items, and rephrasing some of the written instructions. Moreover, further emphasis was given during the verbal instructions in the test administration. Additionally, for better measurement of the Spatial Visualisation factor and for increasing the construct validity of the model, the "Mental Rotation" items were replaced and modified in order to be included as a measure of this factor.





*Note.* The first number indicates factor loading and the number in parenthesis indicates the corresponding interpreted dispersion ($r^2$).

*Figure 3.4.* Model of construct validity for the spatial ability pilot test.

The confirmation of the model regarding the rest of the items indicated that they were suitable for measuring the corresponding factors. The results suggested that the validity of the test is satisfactory, even though it required improvements.

## The Proposed Models of the Study

### *The proposed model for transformational geometry ability*

Based on the literature review, ability in transformational geometry concepts is not a uni-dimensional ability (Kidder, 1976); it consists of different abilities and procedures, like the construction of an image and the identification of parameters (Edwards, 2001), and the recognition of a transformation, or of the image of a transformation. Based on the literature review findings, the validity of a model was checked according to which ability in transformational geometry concepts consists of abilities in translation, reflection, and rotation, and each of these three geometric transformations consists by four factors namely "Construction of an image", "Identification of parameters", "Recognition of the transformation", and "Recognition of the image of the transformation".

Therefore, this model consists by 12 first order factors, three second order factors, three third order factors, and one fourth order factor, and it describes ability in transformational geometry concepts based on these factors. As presented in Figure 3.5, (a) the type of tasks "Recognition of the image of translation" and "Recognition of translation" are indicators of the factor "Recognition of properties", which along with the factors "Identification of the parameters of a translation" and "Construction of a translation's image" are indicators of the factor "Translation Ability", (b) the type of tasks "Recognition of the image of reflection" and "Recognition of reflection" are indicators of the factor "Recognition of properties", which along with the factors "Identification of the parameters of a reflection" and "Construction of a reflection's image" are indicators of the factor "Reflection Ability", (c) the type of tasks "Recognition of the image of rotation" and "Recognition of rotation" are indicators of the factor "Recognition of properties", which along with the factors "Identification of the parameters of a rotation" and "Construction of a rotation's image" are indicators of the factor "Rotation Ability", and (d) the factors of

“Translation Ability”, “Reflection Ability”, and “Rotation Ability” are indicators of the “Transformational Geometry Concepts Ability” factor.

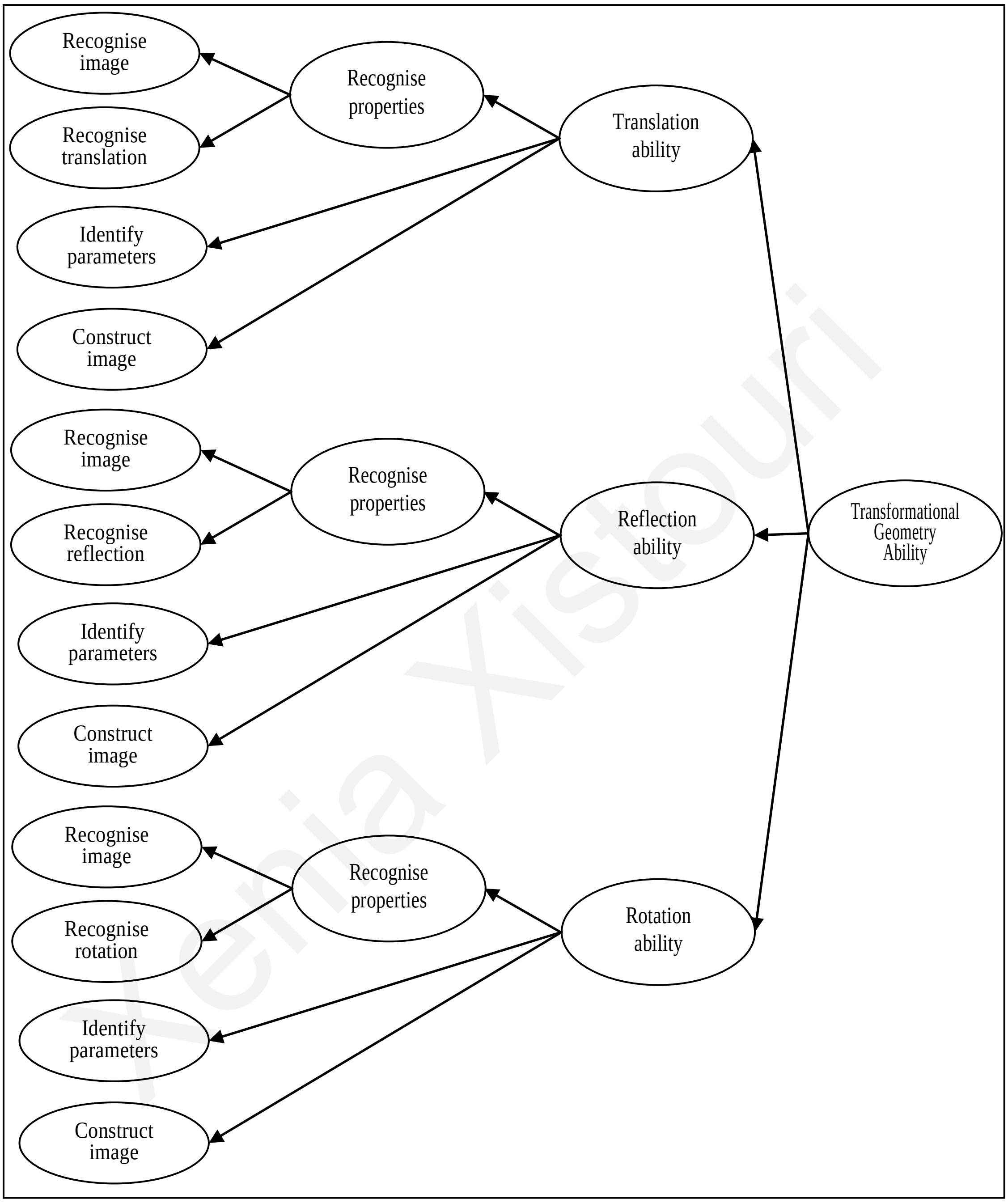



*Figure 3.5*. The suggested model of ability in transformational geometry concepts.



Therefore, the 12 described factors are different facets of transformational geometry ability that can be grouped according to the mathematical structure of the geometric transformations. The hypothesis of the model is that the 48 tasks of the test (see section “Transformational geometry ability test”) are appropriate indicators for measuring each of the 12 factors.

### *The proposed model for spatial ability*

One of the aims of this study was to find the sub-components of spatial ability and investigate their relation to the ability in transformational geometry concepts. For this reason, it was considered important to seclude these components and to confirm that they are indicators of a general spatial ability.

Following an extensive literature review regarding the sub-components of spatial ability that have been identified in the field of psychology and mathematics education, it was apparent that there is a great diversity of spatial theories and spatial ability components. However, not every component was always identified in every study. Following Höffler’s (2010) suggestion that the factors that most often appeared and were well identified in literature were spatial visualisation, spatial relations, and spatial orientation, this study will only focus on these. These three components were the ones proposed as the primary factor of spatial ability in a theoretical framework described by Lohman (1988), hence Lohman’s framework served as the theoretical model of this study.

Following literature’s suggestions regarding which psychometric tests have been found to measure these three factors of spatial ability, and after reviewing these test for their validity and their previous or potential use with children, the tests that would serve as measures of these factors were selected. As presented in Figure 3.6, the theoretical model proposes that the factor of “Spatial Visualisation” is composed by three first order sub-factors, namely “Paper Folding”, “Mental Rotation”, and “Form-Board”. The factor of “Spatial Relations” is composed by two first order sub-factors, namely “Card Rotations” and “Cube Comparisons”. The factor of “Spatial Orientation” is composed also by three first order sub-factors, namely “Where is the Photographer?”, “Object Perspective”, and “Image Perspective”. Finally, the three second order factors of “Spatial Visualisation”, “Spatial Relations”, and “Spatial Orientation” can explain students’ ability in the corresponding tests, and compose the third order factor of “Spatial Ability”.

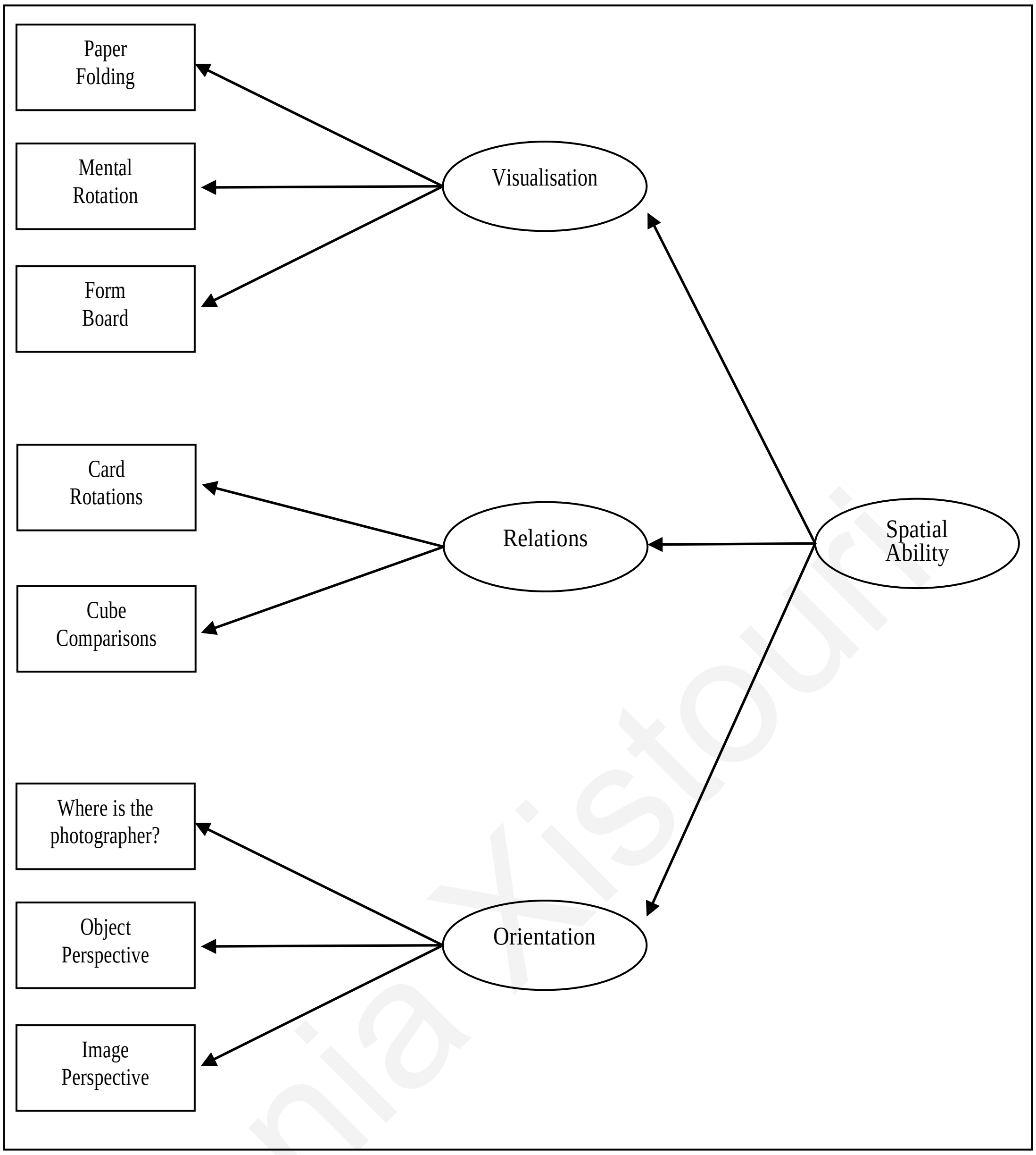



*Figure 3.6*. The suggested model for spatial ability.



The factors that are described above have been related to mathematics education. Specifically, the factors of "Spatial Visualisation" and "Spatial Orientation" have been related to mathematics education by Clements and Battista (1992). Specifically, Clements and Battista (1992) stress out the elusive and multifaceted role of spatial ability in students' construction and use of mathematical concepts, and emphasise its relation to the qualitatively different types of visual thinking. Moreover, they related spatial ability development to transformational geometry concepts development. In general, researchers in mathematics education have often considered spatial thinking or spatial abilities (in the plural sense) to be an integral part of geometry learning (Bishop, 1980, 1983; Clements, 1981; Presmeg, 2006b)

## Procedure

The procedure of the study included seven stages. The first stage included the studying of relevant research in bibliography, in order to form the theoretical framework of the research and the development of the instruments. The second stage involved the evaluation of the instruments in pre-pilot and pilot studies and their modifications. In the third stage, the final instruments were administered to the students for assessing their transformational geometry ability, their spatial ability, and their cognitive style. The fourth stage involved the implementation of the clinical interviews. In the fifth stage, the results regarding the structure of transformational geometry ability were reclaimed to design the material for the teaching experiment. The sixth stage involved the conduction of the teaching experiment and post-test measurements. Finally, in the seventh stage, all the quantitative and qualitative data were analysed and the conclusions were extracted.

In particular, regarding the first stage of the study, the following activities were accomplished:

- Research and study of the relevant literature to form the theoretical framework of the study.
- Selection, design, and development of assessment instruments to use in the study, based on the findings of previous research studies. For the development of the transformational geometry ability test and of the clinical interview, the tasks were developed for the needs of this study. For the development of the spatial ability test, tasks that have been developed in the field of psychology and mathematics education were evaluated and modified for use with children.

Regarding the second stage, the following activities were accomplished:

- Pre-pilot administrations of the transformational geometry instrument. Specifically, a pre-pilot of selected tasks in transformational geometry concepts from the studies of Edwards (1990), Dixon (1995), and Boulter (1992) were administered to two classrooms of fourth-graders and four classrooms of fifth-graders in order to examine: (a) students' ability to solve transformational geometry tasks without formal instruction, (b) their difficulties in understanding the wording of the example and of the tasks, (c) the time requirements for solving such tasks. Based on these findings, the design of the instrument was made based on the 12 components that served as the theoretical framework of this study, with extended incorporation of the configurations. A second pre-pilot administration was conducted with first- and third-year university student teachers. This administration aimed to examine: (a) which of the tasks were of high level of difficulty for student teachers in order to be excluded in a primary school student test, (b) to locate tasks whose structure yielded a variety of responses, and (c) to find difficulties in understanding the verbal elements of the tasks.

- Pre-pilot administrations of the cognitive style questionnaire. Specifically, a translated version of the adult SOIVQ with modifications for younger population was administered to the same subjects that participated in the pre-pilot administrations of the transformational geometry instrument. The aim of this procedure was to investigate: (a) primary school students' ability and time-requirements to complete the self-report questionnaire, (b) participants' difficulties in understanding the statements of the questionnaire, and (c) the possibility to relate the object-spatial-verbal dimensions of cognitive style and abilities in transformational geometry (see Xistouri & Pitta-Pantazi, 2011a; Xistouri & Pitta-Pantazi, 2011b).

- Pilot administration of the transformational geometry ability test and the spatial ability test to 166 elementary school students of the fourth, fifth and sixth grades of primary school. The administration of the tests was carried out in three separate forty-minute sessions within three weeks time. The aim of the pilot administration was to assess the validity and reliability of the instruments, to verify the level of difficulty in the tasks, and to investigate whether some tasks required modifications, replacements, or supplement.

Specifically, regarding the transformational geometry ability test, the following modifications were made: (a) the three items of the factor "Recognition of image" in translation were replaced with three separate figures of a translating rhombus, in order to clarify the requirements for each task, (b) the diagonal item of the factor "Identification of parameters" in translation was modified by increasing the horizontal distance between the images, the diagonal item of "Construction of image" in reflection was modified by reducing the distance from the line of reflection, the diagonal item of "Recognition of reflection" was modified by decreasing the distances between the shapes, and the unfamiliar shape item in "Construction of image" of a rotation was modified by changing the position of the pre-image, since in all cases the percentage of success was low, (c) the three tasks of the factors “Recognition of a translation/reflection/rotation” with unspecified direction on plain paper were modified, in order to match the structure of the other multiple choice items having one correct response out of four alternatives, and to clarify the requirements of the task, (d) the items of “Recognising the image” of a reflection and of a rotation were modified in order to have the same pre-image shape, in order to minimise the effects of figural configurations and improve factor loadings, (e) effort was made to improve the illustration and wording for each transformation example given at the beginning of each section, (f) the wording of the items in “Identification of parameters” of a rotation was clarified, (g) the horizontal and diagonal items of “Identification of parameters” of a rotation were modified by decreasing the distance between the images, and the vertical item was modified by replacing the shapes with the familiar triangles, due to low factor loadings, and (h) the overlapping item of “Identification of parameters” in rotation was modified by decreasing the distance between the shapes, due to low percentage of success and factor loading.

Regarding the spatial ability test, the following modifications were made: (a) the second item of “Card Rotations” that did not load on the factor was replaced, since students’ performance in this task appeared to be very high. This test was also supplemented with two additional items. This was done for better measurement of the second order factor of “Spatial Relations”, which is composed by two instead of three first-order factors, in order to keep some balance in the number of items for measuring the sub-component of spatial ability, (b) for the same reasons, two items of the “Cube Comparisons” test were replaced and the test was supplemented with one more item. Even though some factor loadings were low, the items remained in the test and more emphasis was given on explaining the items to the students in written and verbal instructions, (c) the “Space Relations” items were removed, since they did not form a first order factor, (d) one more item was added to the “Paper Folding” test and one item to the “Form-Board” test, to increase construct validity, (e) the “Mental Rotation” items that did not load on the same factor were all replaced with four items from a previous study by Xistouri and Pitta-Pantazi (2006) with primary school students that were tested for construct validity and had satisfactory model fit indices, and finally (f) one item was added to the “Object Perspective” test to increase construct validity. It should be noted that the removal of the “Space Relations” allowed the increase of items in the other sections, since the estimated time of reading the instructions and completing the items of this section could be covered by the additional items in the other sections of the test. Therefore, no further time requirements were imposed by the addition of new items, and this allowed the collection of more data that would enhance and clarify the theoretical model.

The third stage of the study included the following steps:

- Administration of the ability tests and self-report questionnaire to the subjects of the study. The administration of the tests was performed by the researcher. Prior to the completion of the first part of the transformational geometry ability test, an operational definition and an example for each geometric transformation were provided to the students by the researcher in speaking. Verbal instructions were also given to the students regarding the structure of the test, with emphasis on the providence of an example and the requirements of each type of task. Visual aids of paper cards and drawings on the whiteboard were used to illustrate the examples and the operational definitions of the geometric transformations. Moreover, prior to the completion of the spatial ability test, verbal instructions were explained, and one example for each part of the test was demonstrated to the students by the researcher using appropriate visual aids. For example, a paper card was used to explain "Card Rotations" to demonstrate how it can be rotated without being inversed, a square piece of paper was folded and punched for the "Paper Folding" section, a construction of Unifix cubes was used to demonstrate the rotation and different views of the object in the "Mental Rotation" section and so on. Finally, prior to the completion of the cognitive style questionnaire, the students were given verbal instructions on how to complete a self-report questionnaire, and of what the numbers on the Likert scale represent. The students were given approximately 40 minutes for each part of the transformational geometry test, 40 minutes for the spatial ability test and approximately 30 minutes for the cognitive style questionnaire. In order to avoid practice effects, half of the students received one part of the transformational geometry test first, while the other half received the other part.
- Following the administration of the instruments, the data were coded and recorded in the SPSS statistical package. The coding of the open tasks of the transformational geometry ability test was made according to the coding plan that emerged from the coding of the pilot data regarding the variety of the responses and the significance of the errors. Some basic statistical analyses were performed that

were required for proceeding with the following stages of the study, regarding the selection of the students for clinical interviews and for forming equal and controlled groups of students for the teaching experiment.

The fourth stage included the following steps:

- Selection of the subjects for participating in the clinical interview procedure, based on the results of the quantitative data analyses. Specifically, two subjects were selected from each grade level, at each level of ability.
- Performance of semi-structured clinical interviews to the selected subjects, in order to investigate the procedures, strategies, misconceptions, and visualisation processes used by students at different levels of ability in transformational geometry concepts when solving transformational geometry tasks. The structure of the clinical interview is described further on. This stage also included some preliminary analyses of the qualitative data, in order to be exploited in the design of the instructional interventions.

The fifth stage included the following activities:

- The selection of the interactive dynamic visualisations for the teaching of geometry that were used for the teaching experiment. This included the collection and evaluation of different interactive dynamic geometry instructional software and applets, and the selection of two of them for the design of the two instructional interventions.
- The design of the two instructional interventions with the same objectives, same learning principles, and similar tasks based on two different interactive dynamic visualisations. Both instructional interventions had the same structure and duration, which were 12 forty-minute sessions. Student worksheets were prepared for each session (see Appendix III). The worksheets and tasks were discussed with one researcher of mathematics education and one primary school teacher, and their comments were used for improving the instructional activities.

The sixth stage included the following activities:

- The selection of the four sixth-grade classrooms that participated in the teaching experiment. The selection of the classrooms was primarily based on the interest of the schools administration and the classroom teachers to participate in the teaching experiment. The classrooms were grouped in control of their transformational geometry and spatial abilities, in order to be assigned to either instructional intervention of the teaching experiment with two different visualisations.

- The implementation of the teaching intervention during 12 forty-minute sessions within approximately six weeks time. The instructional interventions took place in the computer lab of each school, and students were working in groups of two or three. Following the teaching experiment, these students were administered with post-tests to measure their transformational geometry concepts and spatial abilities. The transformational geometry post-test (see Appendix IV) included a selection of 36 of the pre-test items, guided by the statistical analyses of the pre-test regarding the subjects' mean performance in each item, in order to ensure that representative items from different levels of difficulty were included in the post-test. The spatial ability pre-test was also used as a post-test, without any modifications. The students were given 40 minutes to solve each test. Verbal instructions were once again explained, as in the pre-test administration, and the same visual aids and demonstrations were used for the spatial ability test.

The final stage of the study, the seventh, included the analysis of both the quantitative and qualitative data, the extraction of the conclusions, and the connections between the findings of this study and the existing literature.

## Design of Clinical Interviews

The aim of the clinical interviews was to investigate the reasoning, misconceptions, strategies, and common errors of students at different levels of ability. For this aim, the selection of the students for the clinical interviews was made according to the results of the quantitative analyses of the transformational geometry ability test.

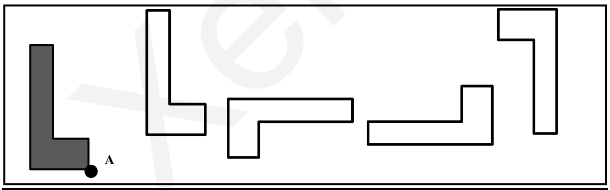
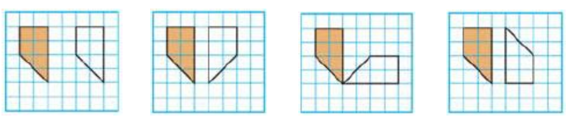
The design of the interviews was based on the four factors for each geometric transformation, as described in the suggested model in Figure 3.5. Specifically, the clinical interview instrument (see Appendix II) included tasks that referred to all 12 first-order factors of the theoretical model, which were selected from the transformational geometry test. Several open-ended questions were prepared for each task, in order to guide students into expressing their thoughts, and to allow the researcher to examine their level of understanding of transformational geometry concepts and visualisation processes.

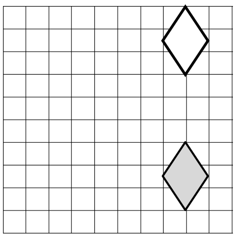
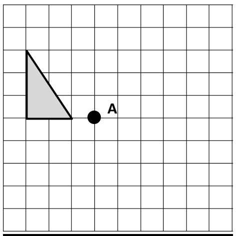
The clinical interviews were performed individually with each student during school time in a quiet room of the school building. The interviews were audio-taped. Students were given their own tests with the responses given during the test administration, and they were requested to remember and describe how they were thinking while solving some of the tasks in the test (only the ones that were included in the interview instrument), and then try to explain their thoughts in their own words, as if they were explaining to a friend about how to follow the same procedure to solve the task. The duration of each interview was approximately 35 minutes. The students were given pencil and eraser and were allowed to make modifications to their responses if they wanted. They were also given an empty piece of paper in case they needed to take notes or make a sketch of the way they thought, or of their mental images.

Table 3.3 presents some examples of the selected tasks that were included in the clinical interview instrument, with the open-ended questions.

Table 3.3

*Examples of Tasks and Questions in the Clinical Interview*

| **Task Example** | **Open-ended Questions** |
| --- | --- |
| Recognition of image | |
| Which of the following shapes is the rotation of the first shape at ¼ of a turn to the right?  | How did you reach your answer?<br>What were your thoughts for disqualifying some responses? |
| Recognition of a transformation | |
| Which of the following images shows the translation of the shaded shape?  A B C D | How did you reach your answer?<br>What were your thoughts for disqualifying some of the other responses?<br>What made you decide that this option did not show a translation? |

| | |
|---|---|
| Identification of parameters | |
| Write down the instruction in order to translate the shape from the position of the shaded image to the position of the white image.  | How did you reach your answer? What makes you sure that this is the correct answer? |
| Construction of image | |
| Draw the shape to its new position when it rotates around point A to ¼ of a turn.  | What did you think in order to draw your answer there? What makes you sure that what you drew is a rotation? What do you think changes from the original image to the new image? What do you think remains the same? |

## Teaching Experiment

For the fulfilment of the teaching experiment, two different interactive dynamic visualisations were selected. The comparison of these two technological environments was made based on the distinction between discrete and continuous dynamic visualisations (Moreno-Armella et al., 2008), and with respect to the *Segmenting Principle* of the CTML (Mayer, 2008) regarding the differences in learning from learner-paced presented information versus continuous presentation. Two instructional interventions were developed, which were considered equal regarding the potential of the interactive dynamic visualisations to develop students' ability in transformational geometry concepts. Student worksheets were developed (see Appendix III) based on the 5Es instructional model (Bybee, 1997) to guide the process of the instructional interventions. After the completion of the instructional interventions, the students who participated in the teaching experiment were administered post-tests to measure their transformational geometry and spatial abilities.

*Selection of interactive dynamic visualisations*

The aim of the selection of the interactive dynamic visualisations was to find two dynamic instructional technological environments with the maximum possibilities for exploitation in an instructional unit in transformational geometry concepts, but with different characteristics regarding the level of control and pace of their graphical representations. According to the theory of Moreno-Armella et al. (2008), there are two types of dynamic inscriptions in technological environments: discrete dynamic and continuous dynamic.

In order to evaluate the possibilities of exploitation, a large amount of instructional environments for teaching geometry were drawn from the internet. Specifically, the following technological environments (software and applets) were obtained during literature review: (1) GeoGebra, (2) Geometer Sketchpad, (3) Cabri Geométre, (4) Euclidraw, (5) Cinderella, (6) Tessellations, (7) Graphs 'n Glyphs, (8) NLVM Applets on Geometric Transformations, (9) Illuminations NCTM Applets, (10) Geometry-Measurement-Numbers: Learning Geometry and Measurement (Γεωμετρία – Μέτρηση – Αριθμοί: Μαθαίνω Γεωμετρία και Μετρώ), as well as a variety of applets available for free in various educational websites.

After reviewing these software based on their potential to meet the objectives of the instructional interventions, their level of graphical control by users, and their appropriateness of manipulation by sixth grade students, the *GeoGebra* and the *Geometry-Measurement-Numbers: Learning Geometry and Measurement* were chosen for the teaching experiment.

Specifically, the advantages of *GeoGebra* were that: (a) it is freely available on the internet, (b) it is translated in Greek and therefore it is accessible to Cypriot primary schools and students, (c) it offers continuous dynamic visualisation of geometrical concepts, (d) it is easy to navigate, (e) it offers hints and help menu in Greek, and (f) it has special and easy-to-access features for geometric transformations. The advantages of the *Geometry-Measurement-Numbers: Learning Geometry and Measurement* were that: (a) it is available to all Cypriot public schools, (b) it is in Greek and therefore it is accessible to Cypriot primary school students, (c) it offers discrete dynamic visualisation of transformational geometry concepts, (d) it is easy to navigate, and (e) it has special and easy-to-access buttons for geometric transformations. However, since the functions of the *Geometry-Measurement-Numbers: Learning Geometry and Measurement* were rather limited and seemed to be inadequate to design matching activities to the *GeoGebra's*, some activities were supplemented from the *NLVM Applets on Geometric Transformations*. It is noted that the selection of the *NLVM Applets* was based on characteristics that are similar to the characteristics of the *Geometry-Measurement-Numbers: Learning Geometry and Measurement*, and on limitations regarding the minimisation of the English language use.

*Design of instructional interventions*

The design of the instructional interventions aimed to develop students' knowledge and ability in transformational geometry concepts. The structure of the instructional unit was based on the theoretical framework of this study regarding the components that synthesise the ability in transformational geometry concepts. The activities also exploited the results of the interviews regarding strategies and misconceptions on transformational geometry.

The lessons were organised in the order of two introductory sessions, three sessions on translation concepts, three sessions on reflection concepts, three sessions on rotation concepts, and one session on making generalisations and proving statements in transformational geometry. Table 3.4 presents the structure of the instructional interventions, and the objectives that guided the development of the activities for both instructional interventions.

Table 3.4

*Structure of Instructional Interventions and Objectives for each Lesson*

| **Session** | **Topic** | **Objectives:** To develop students' ability to… |
|---|---|---|
| 1-2 | Introduction | ▪ use the software's key-features and tools<br>▪ recognise the occurrence of a transformation and meet the three different geometric transformations<br>▪ name the pre-image and image of a transformation |
| 3 | Recognition of translation's image | ▪ recognise a translation<br>▪ give their own examples of translation |
| | Recognition of translation | ▪ recognise the properties of translation to preserve the shape, size, and orientation<br>▪ recognise the property of translation to change the position<br>▪ recognise the direction of a translation |
| 4-5 | Identification of translation parameters | ▪ recognise the role of both the parameters of direction and distance in the change of position |
| | Construction of a translation's image | ▪ identify and describe the parameters of a translation<br>▪ construct the image of a translation<br>▪ identify and describe the parameters of a translation in the coordinate system<br>▪ construct the image of a translation in the coordinate system |

| | | |
|---|---|---|
| 6 | Recognition of reflection's image<br><br>Recognition of reflection | ▪ recognise a reflection<br>▪ give their own examples of reflection<br>▪ recognise the properties of reflection to preserve the shape and size<br>▪ recognise the property of reflection to change the position and inverse orientation<br>▪ recognise the direction in change of the position of an image based on the position of the line of symmetry |
| 7-8 | Identification of reflection parameters<br><br>Construction of a reflection's image | ▪ recognise the role of both the parameters of the orientation and distance of the line of symmetry in the change of position<br>▪ identify and describe the parameters of a reflection<br>▪ construct the image of a reflection<br>▪ identify and describe the parameters of a reflection in the coordinate system<br>▪ construct the image of a reflection in the coordinate system |
| 9 | Recognition of rotation's image<br><br>Recognition of rotation | ▪ recognise a rotation<br>▪ give their own examples of rotation<br>▪ recognise the properties of rotation to preserve the shape and size<br>▪ recognise the property of rotation to change the position and orientation<br>▪ recognise the direction in change of the position of an image based on the angle of rotation |

| | | |
|---|---|---|
| 10-11 | Identification of rotation parameters<br><br>Construction of a rotation’s image | ▪ recognise the role of all three parameters of the position and distance of the centre of rotation and the angle of rotation in the change of position<br>▪ identify and describe the parameters of a rotation regarding the centre and angle of rotation<br>▪ construct the image of a rotation<br>▪ identify and describe the parameters of a rotation in the coordinate system<br>▪ construct the image of a rotation in the coordinate system |
| 12 | Rules and proof in transformational geometry | ▪ apply their knowledge of transformations for discovering generalisations<br>▪ apply their knowledge of transformations to investigate and prove statements regarding the compositions of transformations |



In order to design effective learning with the interactive dynamic visualisations, the development of the instructional interventions took into consideration Mayer’s (2008) principles for fostering generative processing and principles for reducing extraneous load. Specifically, the *multimedia principle* was fundamental in the design of these instructional interventions, since students were learning from pictures (on computer screen), written text (on computer screen and handouts), and narration (on handouts and classroom discussions). Moreover, the *personalisation principle* was also fundamental, since the students were investigating in groups of two or three and were able to discuss their observations and conclusions, and also had the possibility to participate in classroom discussions. Regarding the principles for reducing extraneous load, the *coherence principle* was incorporated by minimising the number of tools and features on the screen for the students, in both instructional interventions. The *signalling principle* was incorporated by including cues regarding the important mathematical terms and symbols whenever each environment allowed it. The *redundancy principle* was applied by placing only short and important text on the screen, and by giving students handouts that were read aloud (narrated) during the instructional interventions. The *spatial continuity principle* was applied whenever possible, by having text appearing near the illustrations on the screen. Finally, regarding the *temporal continuity principle*, the researcher gave handouts of the worksheets to the students and since they were working in groups of two or three, one did the narration and the other member(s) performed the animation simultaneously. The researcher also offered group and personalised assistance, so simultaneous narration and animation was also offered in these cases while students investigated transformational geometry concepts.

Table 3.5

*Characteristics of the 5Es' Stages* (Bossé et al., 2010)

| **Stage** | **Characteristics** |
|---|---|
| Engagement | Disequilibrium; Predicting; Making Connections Between Past and Present Learning Experiences; Organise Student Thinking; Generating Curiosity |
| Exploration | Movement Toward Equilibrium; Test and Refine Predictions and Hypotheses; Communication; Mediated Investigation; Common Experiences; Patterns and Relationships |
| Explanation | Formal and Informal; Synthesise Ideas; Model Making; Clarify Concepts; Formalising Language; Demonstrating Conceptual Understanding; Multimodal |
| Elaboration/Extension | Extend/Apply Understanding; Check Peers' Understanding; Present and Defend Explanations; Draw Reasonable Conclusions; Formalising Language |
| Evaluation | Students Receive Feedback; Lead to Future Investigations; Reflection/Self-Examination; Able to Answer Open-Ended Questions; Evidence of Development of Change in Thinking/Behaviour |



Regarding the structure of each session and the development and order of the activities, this study was based on the 5Es Instructional Model: Engagement, Exploration, Explanation, Elaboration/Extension, and Evaluation (Bybee, 1997). According to Bossé et al. (2010), these five process standards are very similar to the process standards of NCTM's for mathematics education. However, they explained how these process standards can be viewed sequentially, both as instructional stages and processes, through which students learning is built. Table 3.5 summarises the activities involved in each of the five process standards/stages of the 5Es according to Bossé et al. (2010), which guided the order and type of activities in the instructional interventions of the teaching experiment.

The lessons took place at each school's computer lab. All the lessons for both instructional interventions were carried out by the researcher, in the presence of the classroom teacher or a member of the school teaching staff.

## Analysis

The design of the research combined both quantitative and qualitative data collection by using both the administration of tests and the conduction of clinical interviews (Kelly & Lesh, 2000). Therefore, different techniques were used for the analysis of each type of data. This section describes the techniques that were used for analysing the data. The statistical techniques for analysing the quantitative data are described first, followed by the description for the statistical techniques for analysing the qualitative data.

### *Statistical techniques for analysing quantitative data*

For the analysis of the quantitative data, the Mplus statistical package of structural equation analysis was mostly used. In order to answer the questions regarding the components and structure of transformational geometry ability, its relation to age, and to clarify the levels of ability, the Confirmatory Factor Analysis (CFA), the Structural Models, and the Latent Class Analysis (LCA) were performed.

For testing the fit of the proposed model, the Mplus software uses more than one fit indices to evaluate the extent to which the data fit the theoretical model under investigation. More specifically, the fit indices and their optimal values are: (a) the ratio of chi-square to its degrees of freedom, which should be less than 1.96, since a significant chi-square indicates lack of satisfactory model fit, (b) the Comparative Fit Index (CFI), the values of which should be equal to or larger than 0.90, and (c) the Root Mean Square Error of Approximation (RMSEA), with acceptable values less than or equal to 0.06 (Muthén & Muthén, 2010). Structural equation analysis also aimed to examine the validity of the instruments and the suitability of the proposed models for transformational geometry ability and spatial ability (Marcoulides & Schumacker, 1998).

The Latent Class Analysis was used to investigate the groups of students that are at different levels of ability. This analysis allows detecting groups of subjects with similar behaviour (Marcoulides & Schumacker, 1998). This analysis also offers three fit indices to evaluate the possibility of grouping the subjects into different groups: (a) the Entropy index, which needs to be the highest possible, (b) the AIC index, which needs to have the lowest value, and (c) the BIC index, which needs to have the lowest value.

The Mplus statistical package was also used, in order to investigate the relation between the subjects' transformational geometry ability and their spatial ability, as well as their cognitive style, and to investigate the impact of the instructional interventions with different interactive dynamic visualisations. Specifically, structural equation model analysis and multivariate analysis of variance were used to investigate the relations between transformational geometry ability and the individual differences factors. For examining the impact of the two different instructional interventions on the subjects, multivariate analysis of covariance was used, with the type of instructional intervention, level of spatial ability, and cognitive style group as independent variables, the students' performances in the transformational geometry pre-test and the spatial ability pre-test as covariates, and the performance differences between the pre- and post-tests as dependent variables, in order to investigate the effects of the instruction with respect to the subjects' individual differences.

*Statistical techniques for analysing qualitative data*

For a better description of the students' reasoning at different levels of ability, the clinical interviews were analysed to investigate their thinking processes, conceptions, strategies, difficulties, and visualisation processes.

The "Constant Comparative" (CC) method (Maykut & Morehouse, 1994; Miles & Huberman, 1994) was applied to analyse the qualitative data. This method of analysing qualitative data combines inductive category coding with a simultaneous comparison of all units of meaning obtained. Every time each new unit of meaning is chosen for analysis, it is first compared to all other units of meaning and then grouped with similar units of meaning. If there are no similar units of meaning, a new category is created. This process offers the possibility of continuous refinement, since initial categories can be changed, merged, or omitted, while new categories are generated and new relationships can be discovered (Maykut & Morehouse, 1994). According to Denzin and Lincoln (2000), the application of the CC method incorporates four stages: (1) inductive category coding and simultaneous comparing of units of meaning across categories, (2) refinement of categories, (3) exploration of relationships and patterns across categories, and (4) integration of data yielding an understanding of people and settings being studied.

*Data coding*

For the data coding of the transformational geometry test, three different procedures were followed according to the type of the task. Types of tasks (1) “Recognition of the image” (three tasks for each transformation), and (2) “Recognition of a transformation” (4 tasks for each transformation) were multiple choice, with four alternative responses. In these tasks, one mark was given to each correct response and zero marks were given to each incorrect response.

For the coding of the remaining tasks, partial credit was given. Specifically, in type 3, “Identification of parameters” in translation, items were graded as follows: 0 marks for incorrect response, 0.33 for finding the correct direction but false distance, 0.66 for finding the correct distance but false direction, and 1 for correct response. In the case of reflection, type 3 tasks were graded as: 0 marks for incorrect response, 0.5 for finding the orientation of the line of symmetry by in false distance from the shapes, and 1 for correct response. In the case of rotation, type 3 tasks were graded as follows: 0 marks for incorrect response, 0.33 for finding the correct centre of rotation but false angle of rotation, 0.66 for finding the correct angle of rotation but false centre of rotation, and 1 for correct response. Finally, type 4 tasks, “Construction of image”, grading was: 0 marks for incorrect or no response, 0.2 for correct image with false orientation, 0.4 for correct image in false direction, 0.6 for correct image in false distance, 0.8 for correct image with inaccuracies in the size of the shape, and 1 for correct response in all the aforementioned parameters. Items with no response were graded with zero, since students had adequate time to complete the test during the administration.

For the data coding of the spatial ability test, different procedures were used for each type of task. In the tasks of "Card Rotation", "Space Relations", "Cube Comparison", "Paper Folding", "Where is the Photographer", and "Image Perspective", correct responses were graded with one and incorrect responses were graded with zero marks. In the tasks of "Form-Board", correct responses were coded as one and incorrect responses were graded with zero marks. Partial credit of half unit was given to responses that had two out of three shapes correct. In the "Mental Rotations", correct responses were graded with one and incorrect responses were graded with zero marks. Partial credit of half unit was given when one out of two responses was correct. In the items of "Perspective Taking", correct responses were graded with one and incorrect responses were graded with zero marks. Partial credit of half unit was given when students had circled the responses that had 45° divergence to the left or to the right of the correct response. Items that had no responses were graded with zero, since students had adequate time to complete the test during the administration.

# CHAPTER IV

# DATA ANALYSIS AND RESULTS

## Introduction

This chapter presents the quantitative and qualitative results of this study, as they arose from the tests, questionnaire, and post-tests, and also from the conduction of the clinical interviews. The structure of this chapter follows the order of the aims and research questions of the study. Specifically, the components and structure of ability in transformational geometry concepts of students from the fourth grade of primary school to the second grade of secondary school are described, and its stability over age is also investigated.

Further on, the model of classifying the subjects of the study to classes of students based on their responses to the transformational geometry test is examined. The aim of this analysis is to investigate the levels of abilities of students in transformational geometry concepts from the fourth grade of primary school to the second grade of secondary school. The hierarchical structure of the components of ability in transformational geometry is also examined. Then, based on the qualitative results of the clinical interviews, the characteristics, typical errors, and strategies of the students at each level of abilities are described.

Subsequently, the structure of students' spatial ability from the fourth grade of primary school to the second grade of secondary school, and its relation to ability in transformational geometry is examined. Following that, the relationship between ability in transformational geometry concepts and cognitive style is investigated. The final part of this chapter focuses on the impact of the two interventions with different interactive dynamic visualisations on two experimental groups of sixth grade primary school students, regarding their ability in transformational geometry concepts and their spatial ability. Finally, the interactions between students' individual characteristics and interactive dynamic geometry visualisations are investigated.

This section presents the results that concern the first two aims of the research regarding the components and the structure of ability in transformational geometry concepts, and the description of students' ability in transformational geometry. Specifically, it answers the following research questions:

(1) Which components synthesise 9- to 14-year-old students' ability in transformational geometry concepts (translations, reflections, and rotations) and what is the structure of this ability?

(2) Is the structure of students' ability in transformational geometry concepts the same or different in relation to age?

In order to investigate the components and the structure of ability in transformational geometry concepts, the transformational geometry ability test was used. In this section, the descriptive results of the transformational geometry ability test are described first, followed by the testing of the proposed model for confirming the structure of ability in transformational geometry. Finally, the stability of the model in primary and secondary school students is verified.

*Descriptive results of the transformational geometry ability test*

Table 4.1 presents the descriptive information for the transformational geometry test in each of the 12 types of tasks, which constitute the first-order factors of the theoretical model. The highest means of the subjects were in the items of the "Identify parameters" factor of translation (*M* = .70), the "Recognise reflection" factor (*M* = .65), and the "Recognise image" factor of translation (*M* = .64). On the contrary, the lowest means were in the items of the "Recognise image" factor of reflection (*M* = .38), and the "Construct image" factor of rotation (*M* = .18). The range of the subjects' performance to the 12 types of tasks of the test was one, which shows that there were subjects that responded correctly to all the items of a specific type, as well as subjects that did not respond correctly to any item of a specific type. Table 4.1 also presents the values for Skewness and Kurtosis for the subjects' performance to the 12 types of tasks of the transformational geometry ability test. The values of these indices were smaller than two, which suggests that the variables of the subjects' performance for the 12 types of tasks of the transformational geometry ability test follow a normal distribution.

Table 4.1

*Descriptive Results of the Transformational Geometry Test According to Type of Task*

| Test | Mean | Standard Deviation | Range | Skewness | Kurtosis |
|---|---|---|---|---|---|
| A. Translation | | | | | |
| 1. Recognise Image | .64 | .36 | 1 | -.53 | -1.00 |
| 2. Recognise Translation | .56 | .25 | 1 | -.26 | -.32 |
| 3. Identify Parameters | .70 | .31 | 1 | -.59 | -.90 |
| 4. Construct Image | .55 | .24 | 1 | -.42 | -.03 |
| B. Reflection | | | | | |
| 1. Recognise Image | .38 | .35 | 1 | .34 | -1.18 |
| 2. Recognise Reflection | .65 | .24 | 1 | -.90 | .69 |
| 3. Identify Parameters | .45 | .30 | 1 | .08 | -.94 |
| 4. Construct Image | .45 | .31 | 1 | .16 | -1.13 |
| C. Rotation | | | | | |
| 1. Recognise Image | .48 | .41 | 1 | .07 | -1.57 |
| 2. Recognise Rotation | .45 | .28 | 1 | -.13 | -1.06 |
| 3. Identify Parameters | .44 | .30 | 1 | .25 | -.86 |
| 4. Construct Image | .18 | .22 | 1 | 1.43 | 1.80 |

Table 4.2 presents the correlations between all the items of the test measuring ability in transformational geometry concepts. The variables correspond to the 48 items of the transformational geometry ability test. Almost all correlations between the same types of tasks for each transformational geometry concept are statistically significant. For translation, items A1 to A3 correspond to the "Recognise image" factor, with the highest correlation appearing between the items A1 and A2 ($r = .44$, $p < .01$); items A4 to A7

correspond to the "Recognise translation" factor, with the highest correlation appearing between the items A5 and A6 ($r = .42$, $p < .01$); items A8 to A11 correspond to the "Identify parameters" factor, with the highest correlation appearing between the items A8 and A10 ($r = .45$, $p < .01$); and items A12 to A16 correspond to the "Construct image" factor, with the highest correlation appearing between the items A13 and A15 ($r = .63$, $p < .01$), and A14 with A15 ($r = .63$, $p < .01$). For reflection, items B1 to B3 correspond to the "Recognise image" factor, with the highest correlation appearing between the items B1 and B2 ($r = .47$, $p < .01$); items B4 to B7 correspond to the "Recognise reflection" factor, with the highest correlation appearing between the items B5 and B7 ($r = .41$, $p < .01$); items B8 to B11 correspond to the "Identify parameters" factor, with the highest correlation appearing between the items B9 and B11 ($r = .41$, $p < .01$); and items B12 to B16 correspond to the "Construct image" factor, with the highest correlation appearing between the items B12 and B16 ($r = .60$, $p < .01$). For rotation, items C1 to C3 correspond to the "Recognise image" factor, with the highest correlation appearing between the items C1 and C2 ($r = .55$, $p < .01$); items C4 to C7 correspond to the "Recognise rotation" factor, with the highest correlation appearing between the items C4 and C7 ($r = .30$, $p < .01$); items C8 to C11 correspond to the "Identify parameters" factor, with the highest correlation appearing between the items C8 and C10 ($r = .76$, $p < .01$); and items C12 to C16 correspond to the "Construct image" factor, with the highest correlation appearing between the items C12 and C13 ($r = .59$, $p < .01$). The high correlations between the items of the same factors suggest that they appear to measure the same ability.

Table 4.2

*Correlations Between Subjects' Performance in the Items of the Transformational Geometry Ability Test*

| | A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 | A11 | A12 | A13 | A14 | A15 | A16 | B1 | B2 | B3 | B4 | B5 | B6 | B7 | B8 | B9 | B10 | B11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | | | |
| A2 | .44** | 1 | | | | | | | | | | | | | | | | | | | | | | | | | |
| A3 | .28** | .30** | 1 | | | | | | | | | | | | | | | | | | | | | | | | |
| A4 | .19** | .21** | .19** | 1 | | | | | | | | | | | | | | | | | | | | | | | |
| A5 | .17** | .14** | .15** | .31** | 1 | | | | | | | | | | | | | | | | | | | | | | |
| A6 | .18** | .12** | .14** | .23** | .42** | 1 | | | | | | | | | | | | | | | | | | | | | |
| A7 | .11* | .20** | .18** | .36** | .22** | .19** | 1 | | | | | | | | | | | | | | | | | | | | |
| A8 | .20** | .23** | .21** | .17** | .27** | .21** | .22** | 1 | | | | | | | | | | | | | | | | | | | |
| A9 | .18** | .13** | .20** | .14** | .21** | .21** | .08 | .37** | 1 | | | | | | | | | | | | | | | | | | |
| A10 | .14** | .15** | .20** | .20** | .25** | .19** | .17** | .45** | .29** | 1 | | | | | | | | | | | | | | | | | |
| A11 | .23** | .19** | .19** | .13** | .25** | .26** | .19** | .42** | .30** | .27** | 1 | | | | | | | | | | | | | | | | |
| A12 | .18** | .17** | .24** | .18** | .23** | .21** | .19** | .25** | .34** | .28** | .36** | 1 | . | | | | | | | | | | | | | | |
| A13 | .22** | .18** | .22** | .21** | .27** | .27** | .14** | .28** | .31** | .29** | .24** | .33** | 1 | | | | | | | | | | | | | | |
| A14 | .22** | .14** | .19** | .23** | .33** | .31** | .18** | .33** | .27** | .31** | .25** | .33** | .57** | 1 | | | | | | | | | | | | | |
| A15 | .23** | .17** | .28** | .26** | .34** | .29** | .23** | .36** | .34** | .34** | .34** | .43** | .63** | .63** | 1 | | | | | | | | | | | | |
| A16 | .17** | .18** | .26** | .22** | .20** | .18** | .15** | .35** | .29** | .29** | .28** | .55** | .39** | .35** | .46** | 1 | | | | | | | | | | | |
| B1 | .13** | .18** | .27** | .23** | .20** | .17** | .21** | .27** | .20** | .24** | .23** | .24** | .27** | .28** | .32** | .27** | 1 | | | | | | | | | | |
| B2 | .12** | .17** | .17** | .23** | .21** | .16** | .20** | .26** | .15** | .24** | .20** | .30** | .23** | .22** | .25** | .25** | .47** | 1 | | | | | | | | | |
| B3 | .12** | .16** | .16** | .13** | .13** | .14** | .14** | .17** | .10* | .14** | .16** | .17** | .17** | .14** | .17** | .18** | .27** | .23** | 1 | | | | | | | | |
| B4 | .19** | .12** | .13** | .12** | .26** | .29** | .10* | .19** | .25** | .20** | .20** | .22** | .22** | .27** | .24** | .19** | .23** | .19** | .24** | 1 | | | | | | | |
| B5 | .09* | .10* | .09* | .11* | .32** | .25** | .08 | .21** | .19** | .18** | .26** | .20** | .19** | .23** | .25** | .21** | .14** | .19** | .16** | .35** | 1 | | | | | | |
| B6 | .07 | .09 | .06 | .14** | .17** | .16** | .10* | .16** | .16** | .25** | .15** | .21** | .10* | .12** | .18** | .22** | .26** | .17** | .16** | .14** | .16** | 1 | | | | | |
| B7 | .12** | .08 | .14** | .13** | .23** | .25** | .11* | .21** | .15** | .23** | .16** | .19** | .19** | .24** | .22** | .21** | .24** | .22** | .18** | .41** | .26** | .20** | 1 | | | | |

* $p < .05$. ** $p < .01$.

| | A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 | A11 | A12 | A13 | A14 | A15 | A16 | B1 | B2 | B3 | B4 | B5 | B6 | B7 | B8 | B9 | B10 | B11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B8 | .07 | .11* | .06 | .17** | .03 | .13** | .11* | .17** | .08 | .16** | .16** | .13** | .12** | .22** | .19** | .19** | .21** | .20** | .17** | .11* | .06 | .21** | .11* | 1 | | | |
| B9 | .04 | .12** | .10* | .16** | .19** | .11* | .14** | .19** | .09 | .16** | .12** | .14** | .14** | .23** | .20** | .21** | .11* | .20** | .09 | .14** | .18** | .08 | .06 | .21** | 1 | | |
| B10 | .13** | .06 | .11* | .16** | .08 | .07 | .09 | .12* | .13** | .11* | .15** | .17** | .05 | .20** | .13** | .22** | .17** | .23** | .10* | .13** | .16** | .12** | .12** | .16** | .19** | 1 | |
| B11 | .18** | .13** | .14** | .19** | .17** | .17** | .08 | .28** | .20** | .17** | .17** | .15** | .11* | .30** | .23** | .30** | .17** | .26** | .09* | .17** | .10* | .10* | .13** | .36** | .41** | .25** | 1 |
| B12 | .10* | .17** | .15** | .24** | .26** | .19** | .22** | .28** | .27** | .39** | .24** | .34** | .25** | .32** | .34** | .40** | .33** | .32** | .15** | .34** | .25** | .24** | .30** | .23** | .25** | .20** | .25** |
| B13 | .12** | .08 | .12** | .11* | .18** | .20** | .15** | .22** | .19** | .31** | .24** | .32** | .25** | .32** | .29** | .37** | .37** | .34** | .15** | .26** | .21** | .20** | .24** | .20** | .25** | .22** | .24** |
| B14 | .11* | .14** | .16** | .17** | .21** | .14** | .12** | .26** | .25** | .31** | .24** | .26** | .22** | .24** | .28** | .26** | .31** | .24** | .20** | .20** | .15** | .33** | .23** | .23** | .13** | .14** | .13** |
| B15 | .11* | .15** | .17** | .25** | .23** | .21** | .22** | .25** | .23** | .30** | .22** | .30** | .20** | .28** | .29** | .32** | .41** | .34** | .21** | .25** | .15** | .22** | .24** | .24** | .20** | .27** | .23** |
| B16 | .17** | .16** | .21** | .21** | .28** | .21** | .21** | .31** | .27** | .33** | .29** | .37** | .29** | .34** | .35** | .41** | .34** | .32** | .17** | .36** | .31** | .19** | .30** | .22** | .28** | .20** | .26** |
| C1 | .17** | .11* | .14** | .15** | .20** | .16** | .15** | .24** | .22** | .18** | .20** | .26** | .24** | .25** | .29** | .30** | .23** | .25** | .16** | .28** | .29** | .12** | .24** | .18** | .20** | .14** | .16** |
| C2 | .15** | .15** | .10* | .18** | .25** | .19** | .08 | .27** | .26** | .26** | .19** | .24** | .29** | .29** | .24** | .31** | .26** | .22** | .16** | .35** | .23** | .12** | .31** | .14** | .15** | .09* | .18** |
| C3 | .09* | .11* | .11* | .18** | .28** | .22** | .11* | .31** | .23** | .29** | .24** | .20** | .27** | .25** | .27** | .29** | .27** | .26** | .20** | .28** | .24** | .15** | .25** | .12** | .07 | .08 | .13** |
| C4 | .11* | .12** | .16** | .22** | .12** | .09* | .15** | .15** | .04 | .17** | .14** | .15** | .15** | .20** | .18** | .21** | .23** | .22** | .17** | .11* | .07 | .09* | .17** | .17** | .17** | .08 | .22** |
| C5 | .07 | .14** | .13** | .11* | .11* | .11* | .04 | .09* | .19** | .21** | .11* | .18** | .11* | .19** | .17** | .17** | .16** | .13** | .16** | .25** | .17** | .20** | .23** | .12** | .16** | .17** | .07 |
| C6 | .19** | .20** | .13** | .16** | .30** | .28** | .16** | .27** | .25** | .31** | .25** | .25** | .27** | .29** | .33** | .29** | .28** | .23** | .13** | .29** | .31** | .10* | .30** | .16** | .19** | .06 | .16** |
| C7 | .04 | .10* | .13** | .19** | .11* | .17** | .12** | .10* | .09 | .12** | .12* | .07 | .14** | .23** | .15** | .11* | .20** | .14** | .10* | .15** | .09 | .087 | .15** | .16** | .08 | .16** | .18** |
| C8 | .03 | .12** | .08 | .18** | .28** | .21** | .17** | .28** | .31** | .29** | .17** | .30** | .26** | .27** | .34** | .34** | .25** | .26** | .13** | .28** | .23** | .24** | .26** | .15** | .25** | .04 | .16** |
| C9 | .02 | .09 | .04 | .07 | .11* | .08 | .11* | .15** | .19** | .16** | .10* | .08 | .10* | .11* | .12** | .11* | .09* | .10* | .04 | .15** | .16** | .14** | .17** | .06 | .10* | .01 | .06 |
| C10 | .02 | .09 | .09* | .11* | .25** | .20** | .18** | .31** | .25** | .21** | .20** | .28** | .27** | .28** | .33** | .28** | .21** | .26** | .10* | .25** | .22** | .20** | .22** | .09 | .26** | .00 | .10* |
| C11 | .14** | .12** | .07 | .22** | .20** | .11* | .19** | .21** | .18** | .23** | .24** | .23** | .16** | .18** | .22** | .22** | .27** | .31** | .20** | .18** | .10* | .20** | .16** | .17** | .10* | .15** | .15** |
| C12 | .11* | .18** | .23** | .22** | .20** | .20** | .14** | .20** | .32** | .27** | .24** | .26** | .23** | .25** | .27** | .26** | .28** | .26** | .21** | .30** | .18** | .33** | .25** | .14** | .19** | .15** | .15** |
| C13 | .13** | .16** | .23** | .28** | .28** | .23** | .23** | .23** | .28** | .30** | .24** | .30** | .28** | .31** | .33** | .31** | .30** | .30** | .23** | .32** | .24** | .30** | .32** | .21** | .20** | .18** | .18** |
| C14 | .08 | .15** | .13** | .21** | .25** | .24** | .17** | .22** | .21** | .29** | .22** | .28** | .21** | .26** | .28** | .34** | .30** | .29** | .21** | .30** | .23** | .28** | .28** | .18** | .20** | .20** | .21** |
| C15 | .15** | .16** | .16** | .26** | .21** | .21** | .18** | .26** | .30** | .34** | .24** | .30** | .23** | .27** | .33** | .33** | .34** | .33** | .21** | .32** | .25** | .33** | .2** | .23** | .18** | .18** | .15** |
| C16 | .01* | .15** | .11* | .17** | .16** | .14** | .15** | .21** | .18** | .28** | .18** | .26** | .17** | .21** | .26** | .29** | .27** | .25** | .30** | .19** | .12** | .27** | .20** | .18** | .11* | .10* | .09* |

* $p < .05$. ** $p < .01$.

| | B12 | B13 | B14 | B15 | B16 | C1 | C2 | C3 | C4 | C5 | C6 | C7 | C8 | C9 | C10 | C11 | C12 | C13 | C14 | C15 | C16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B12 | 1 | | | | | | | | | | | | | | | | | | | | |
| B13 | .47** | 1 | | | | | | | | | | | | | | | | | | | |
| B14 | .46** | .33** | 1 | | | | | | | | | | | | | | | | | | |
| B15 | .44** | .48** | .34** | 1 | | | | | | | | | | | | | | | | | |
| B16 | .60** | .50** | .38** | .45** | 1 | | | | | | | | | | | | | | | | |
| C1 | .34** | .27** | .21** | .25** | .35** | 1 | | | | | | | | | | | | | | | |
| C2 | .38** | .24** | .27** | .28** | .37** | .55** | 1 | | | | | | | | | | | | | | |
| C3 | .29** | .21** | .24** | .23** | .29** | .41** | .54** | 1 | | | | | | | | | | | | | |
| C4 | .28** | .21** | .26** | .27** | .22** | .24** | .25** | .17** | 1 | | | | | | | | | | | | |
| C5 | .31** | .16** | .26** | .22** | .24** | .28** | .28** | .16** | .16** | 1 | | | | | | | | | | | |
| C6 | .36** | .31** | .27** | .25** | .35** | .33** | .35** | .34** | .17** | .28** | 1 | | | | | | | | | | |
| C7 | .17** | .19** | .13** | .19** | .17** | .10* | .11* | .14** | .30** | .00 | .10* | 1 | | | | | | | | | |
| C8 | .34** | .33** | .26** | .33** | .36** | .45** | .46** | .33** | .17** | .31** | .34** | .10* | 1 | | | | | | | | |
| C9 | .16** | .15** | .17** | .09* | .12** | .17** | .08 | .12* | .05 | .20** | .19** | .06 | .29** | 1 | | | | | | | |
| C10 | .30** | .25** | .25** | .28** | .33** | .42** | .40** | .32** | .13** | .30** | .38** | .08 | .76** | .31** | 1 | | | | | | |
| C11 | .28** | .23** | .27** | .30** | .28** | .20** | .27** | .29** | .21** | .16** | .16** | .21** | .28** | .09 | .23** | 1 | | | | | |
| C12 | .35** | .27** | .34** | .30** | .34** | .34** | .32** | .25** | .24** | .34** | .22** | .17** | .35** | .16** | .28** | .32** | 1 | | | | |
| C13 | .47** | .32** | .40** | .37** | .39** | .40** | .41** | .31** | .34** | .36** | .36** | .18** | .37** | .18** | .33** | .31** | .59** | 1 | | | |
| C14 | .37** | .32** | .33** | .38** | .39** | .30** | .28** | .23** | .30** | .25** | .29** | .20** | .29** | .14** | .27** | .30** | .42** | .45** | 1 | | |
| C15 | .40** | .32** | .44** | .34** | .34** | .37** | .36** | .34** | .23** | .32** | .40** | .14** | .36** | .17** | .33** | .30** | .45** | .50** | .36** | 1 | |
| C16 | .36** | .23** | .35** | .30** | .30** | .23** | .22** | .22** | .16** | .23** | .23** | .16** | .21** | .01 | .17** | .27** | .40** | .38** | .36** | .46** | 1 |

* $p < .05$. ** $p < .01$.

Table 4.3 presents the correlations between the performances of the subjects in the eight types of tasks in the transformational geometry test. The highest correlations appear between factors “Construct image” in reflection and “Construct image” in rotation ($r$ = .62, $p$ < .01), factors “Recognise rotation” and “Construct image” in rotation ($r$ = .56, $p$ < .01), and “Identify parameters” in translation and “Construct image” in translation ($r$ = .55, $p$ < .01). The rest of the correlations are between .20 and .55, with the exception of the correlation between factors “Recognise image” in translation and “Identify parameters” in rotation ($r$ = .14, $p$ < .01). The fact that all correlations between the subjects’ performance in the 12 types of transformational geometry tasks were statistically significant suggests that they measure the same ability.

Table 4.3

*Correlations Between the First Order Factors of Transformational Geometry Ability*

| | Tr 1 | Tr 2 | Tr 3 | Tr 4 | Re 1 | Re 2 | Re 3 | Re 4 | Ro 1 | Ro 2 | Ro 3 | Ro 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Tr 1 | 1 | | | | | | | | | | | |
| Tr 2 | **.32**** | 1 | | | | | | | | | | |
| Tr 3 | **.34**** | **.40**** | 1 | | | | | | | | | |
| Tr 4 | **.36**** | **.45**** | **.55**** | 1 | | | | | | | | |
| Re 1 | .30** | .36** | .36** | .41** | 1 | | | | | | | |
| Re 2 | .21** | .39** | .42** | .41** | **.40**** | 1 | | | | | | |
| Re 3 | .19** | .27** | .30** | .33** | **.33**** | **.27**** | 1 | | | | | |
| Re 4 | .25** | .40** | .50** | .53** | **.51**** | **.50**** | **.44**** | 1 | | | | |
| Ro 1 | .20** | .32** | .42** | .42** | .37** | .45** | .24** | .46** | 1 | | | |
| Ro 2 | .27** | .36** | .38** | .41** | .38** | .43** | .34** | .52** | **.46**** | 1 | | |
| Ro 3 | .14** | .35** | .43** | .42** | .36** | .42** | .21** | .47** | **.51**** | **.44**** | 1 | |
| Ro 4 | .27** | .41** | .46** | .49** | .49** | .52** | .33** | .62** | **.50**** | **.56**** | **.49**** | 1 |

*Note*. Codes Tr1-Tr4 correspond to the factors of translation in the order of “Recognise image” (Tr1), “Recognise translation” (Tr2), “Identify parameters” (Tr3), and “Construct image” (Tr4). The same coding applies for Re1-Re4 in reflection and for Ro1-Ro4 in rotation.

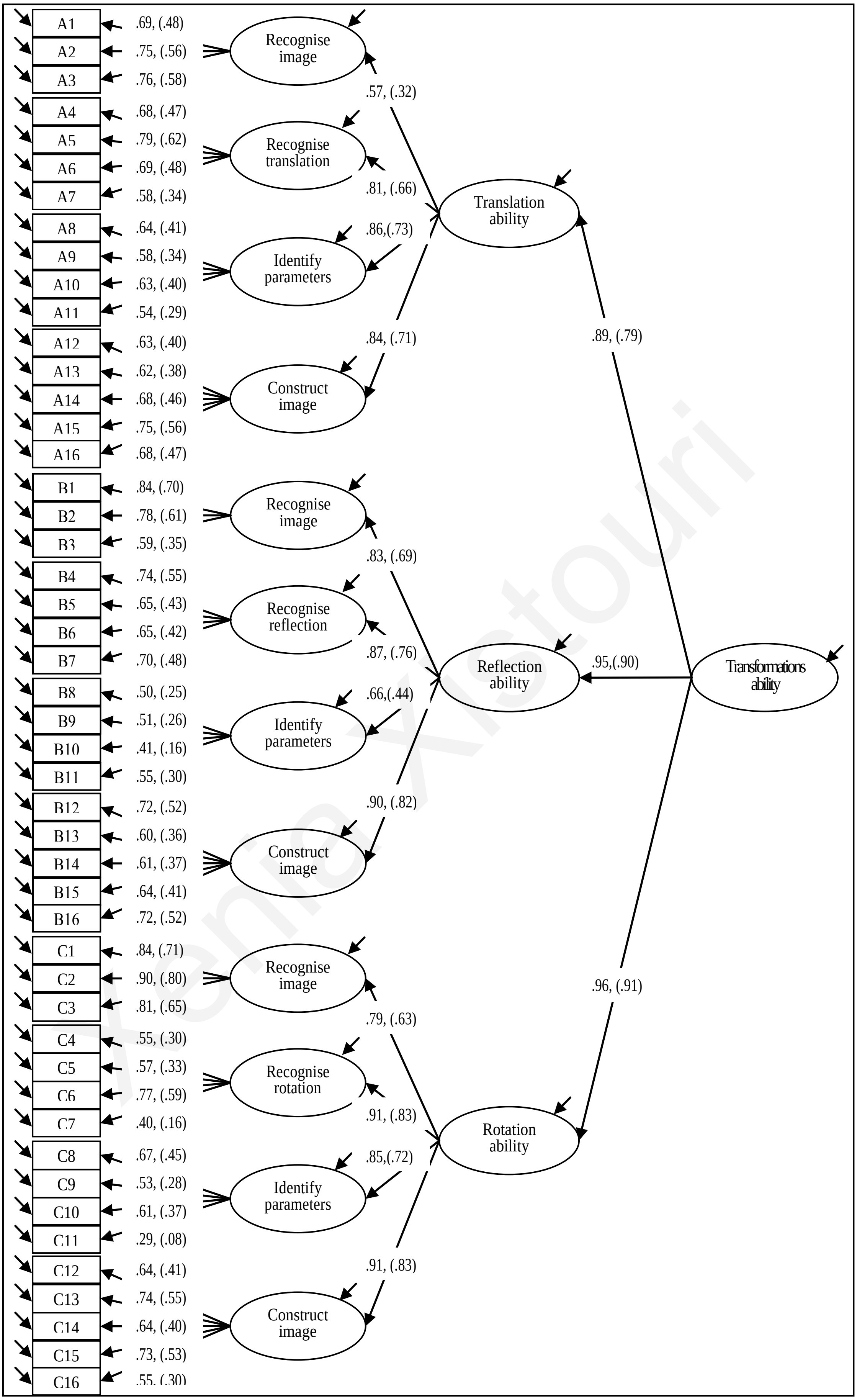
** $p < .01$.

The reliability coefficient of the transformational geometry test items was Cronbach's Alpha = .93, which is considered excellent (Kline, 1999). The reliability indices for the three factors of ability in each geometric transformation were very good ($\alpha_{Tr}$ = .82, $\alpha_{Re}$ = .83, $\alpha_{Ro}$ = .85), and the reliability indices of the twelve factors were also at satisfactory levels ($\alpha_{Tr1}$ = .61, $\alpha_{Re1}$ = .60, $\alpha_{Ro1}$ = .75, $\alpha_{Tr2}$ = .62, $\alpha_{Re2}$ = .58, $\alpha_{Ro2}$ = .50, $\alpha_{Tr3}$ = .65, $\alpha_{Re3}$ = .56, $\alpha_{Ro3}$ = .67, $\alpha_{Tr4}$ = .81, $\alpha_{Re4}$ = .80, $\alpha_{Ro4}$ = .79).

*The structure of ability in transformational geometry concepts*

In order to investigate the structure of transformational geometry ability the validity of the theoretical model was tested, which suggests that transformational geometry ability is synthesised by students' ability in (i) translation, (ii) reflection, and (iii) rotation, and that each of these three abilities is synthesised by the following four factors: (1) recognition or properties (factors Translation 1, Reflection 1, and Rotation 1), (2) recognition of transformation (factors Translation 2, Reflection 2, and Rotation 2), (3) identification of parameters (factors Translation 3, Reflection 3, and Rotation 3), and (4) construction of image (factors Translation 4, Reflection 4, and Rotation 4).

The results of the confirmatory factor analysis showed that the data of the research fitted the theoretical model at a satisfactory level ($CFI = .97$, $x^2 = 1844.52$, $df = 1067$, $x^2/df = 1.73$, $p > .05$, $RMSEA = .04$) and hence the theoretical model of 12 first-order factors, three second-order factors, and one third-order factor can describe transformational geometry ability. The factor loadings of all the items to their corresponding factors are statistically significant, as shown in Figure 4.1. The distinct nature of the 12 factors of the model is confirmed by the fact that all observable variables load only on one first-order factor. The fitting of the data to the structure of the theoretical model confirms that the items used were in fact suitable instruments for measuring ability in the 12 salient factors. Moreover, the loading of these 12 factors only to their corresponding geometric transformation ability factor denotes that they do measure ability in the three geometric transformations.




*Note*. The first number indicates factor loading and the number in parenthesis indicates the corresponding interpreted dispersion ($r^2$).

*Figure 4.1*. The model of ability in transformational geometry concepts.

The results of the analysis showed that the interpreted dispersion of the items was relatively high (see Figure 4.1), indicating that they interpret the dispersion of the factors of the model. Specifically, the factor loadings of all the first and second order factors to the corresponding higher order factors were statistically significant and very high. The factor of performance in the items of “Identify parameters” in translation had the highest ability for predicting the “Translation ability” factor ($r^2 = .73$). The factor of performance in the items of “Construct image” in reflection had the highest ability for predicting the “Reflection ability” factor ($r^2 = .82$). The factors of performance in the items of “Recognise rotation” and “Construct image” had the highest ability for predicting the “Rotation ability” factor ($r^2 = .83$, and $r^2 = .83$, respectively). The structure of the suggested model also showed that factors of “Translation ability”, “Reflection ability”, and “Rotation ability” had almost the same ability to predict the subjects’ performance in transformational geometry ($r^2_{\text{Translation Ability}} = .79$, $r^2_{\text{Reflection Ability}} = .90$, $r^2_{\text{Rotation Ability}} = .91$).

*Subjects’ ability in transformational geometry concepts*

Table 4.4 presents the descriptive results for the subjects of the study in their performance in transformational geometry concepts. The subjects’ performance was higher than .49 only in the “Translation ability” factor ($M_{\text{Translation}} = .60$), indicating an average performance. In the factor of “Reflection ability”, the subjects’ performance was just below .49 ($M_{\text{Reflection}} = .48$), and in the factor of “Rotation ability” the subjects’ performance was lower than .49 ($M_{\text{Rotation}} = .38$), indicating low performance. In the general factor of “Transformational geometry ability”, the subjects’ performance was exactly .49, indicating an average performance. The large values of standard deviations and ranges indicate that there is great diversity in the subjects’ performance. The values of Skewness and Kurtosis are lower than two, which suggests that the distributions of the subjects’ performance in transformational geometry concepts are normal.

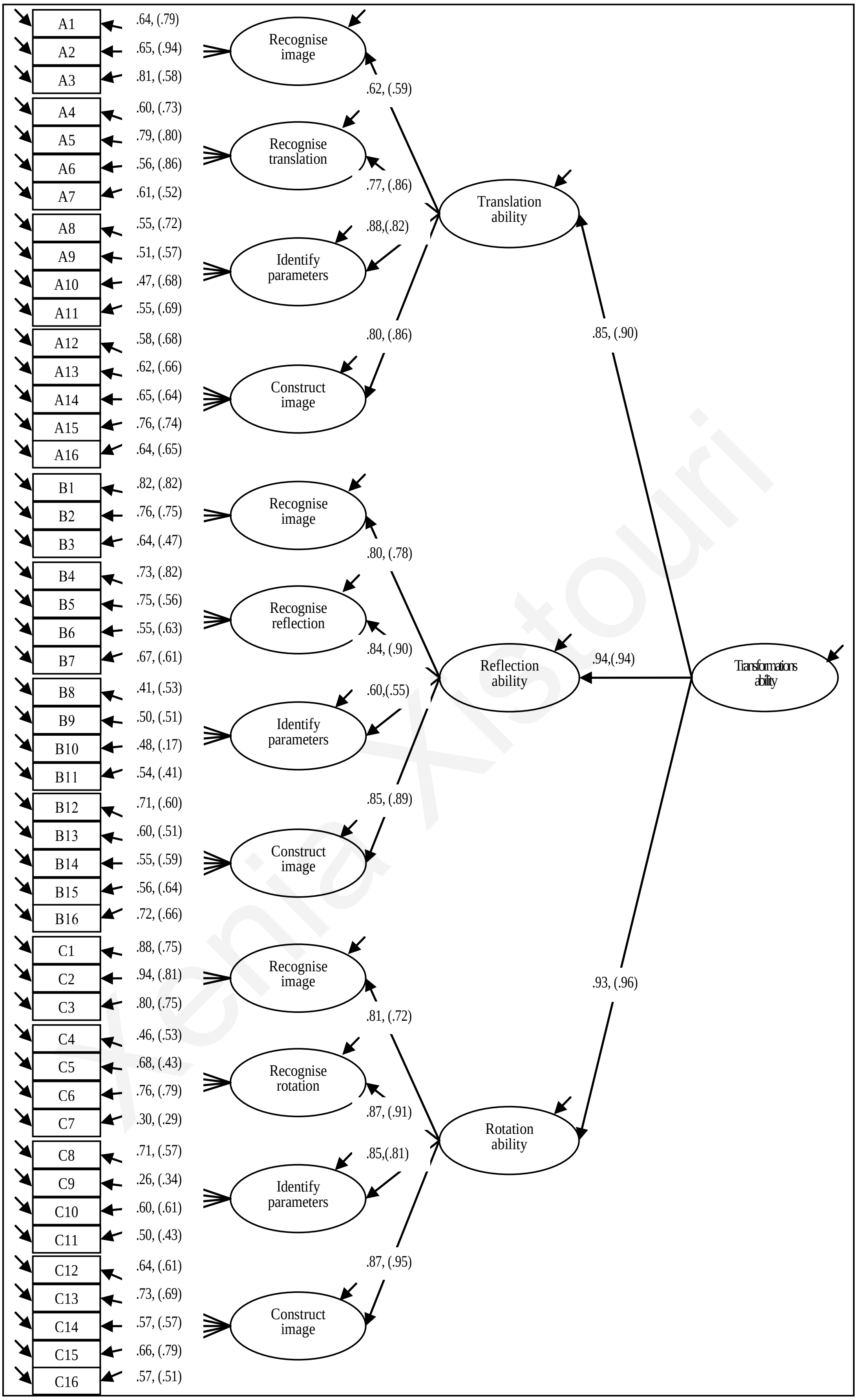
Table 4.4

*Descriptive Results of the Subjects' Performance in Transformational Geometry Concepts*

| Factor | Mean | Standard Deviation | Range | Skewness | Kurtosis |
|---|---|---|---|---|---|
| Translation | .60 | .22 | .96 | -.17 | -.59 |
| Reflection | .48 | .23 | 1 | .20 | -.72 |
| Rotation | .38 | .24 | 1 | .33 | -.76 |
| Transformational geometry | .49 | .20 | .92 | .24 | -.61 |

*Examining the stability of the model for ability in transformational geometry concepts*

To examine the stability of the structure of the proposed model for ability in transformational geometry concepts, the validity of the model was tested for both groups of primary school students and for secondary school students, thus confirming its stability at different age groups. The results of the confirmatory factor analysis suggest that the fitting of the data from primary school students and secondary school students to the proposed model was satisfactory ($CFI = .96$, $x^2 = 1495.78$, $df = 1067$, $x^2/df = 1.40$, $p > .05$, $RMSEA = .04$). All the items had statistically significant factor loadings to the corresponding factors, as presented in Figure 4.2. These results suggest that the structure of the model is stable for both populations.




*Note*. The first number indicates factor loading for primary school students and the number in the parenthesis the factor loading for secondary school students.

*Figure 4.2*. The model of ability in transformational geometry concepts for primary and secondary school.

### *Subjects' ability in transformational geometry concepts by grade level*

In order to investigate whether there are any differences in the performances of the subjects from different grade levels in transformational geometry concepts, a multivariate analysis of variance (MANOVA) was performed. Table 4.5 presents the means and standard deviations of the students at each grade level, in their performance in transformational geometry concepts.

Table 4.5

*Means and Standard Deviations of the Subjects' Performance in Transformational Geometry Concepts by Grade Level*

| | Primary School | | | | | | Secondary School | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Factor | Grade 4 | | Grade 5 | | Grade 6 | | Grade 1 | | Grade 2 | |
| | *M* | *SD* | *M* | *SD* | *M* | *SD* | *M* | *SD* | *M* | *SD* |
| Translation | .51 | .21 | .55 | .20 | .62 | .21 | .66 | .22 | .67 | .20 |
| Reflection | .31 | .17 | .42 | .20 | .47 | .21 | .57 | .20 | .62 | .21 |
| Rotation | .20 | .18 | .32 | .22 | .42 | .22 | .46 | .21 | .51 | .22 |
| Transformational geometry | .34 | .14 | .43 | .18 | .50 | .19 | .56 | .18 | .60 | .19 |

The results of the analysis showed that there are statistically significant differences between the students of the five grade levels that participated in the study (Pillai's $F_{(4,495)}$ = 12.38, $p < .01$). As presented in Table 4.6, there are statistically significant differences between the subjects of the five grade levels in their abilities in the three transformational geometry concepts, and in their overall transformational geometry ability.

Table 4.6

*Results of the Multiple Variance Analysis for Transformational Geometry Ability by Grade*

| Source | Type III Sum of Squares | Degrees of Freedom | Mean Square | *F* | Significance |
|---|---|---|---|---|---|
| Translation | 1.82 | 4 | .45 | 10.41 | .00** |
| Reflection | 5.80 | 4 | 1.45 | 36.66 | .00** |
| Rotation | 5.68 | 4 | 1.42 | 31.26 | .00** |
| Transformational geometry | 4.15 | 4 | 1.04 | 33.72 | .00** |

** $p < .01$.

Post-hoc analyses were performed to reveal existing statistically significant differences in ability in transformational geometry concepts between the different grade levels. As shown in Table 4.7, in the factor of "Translation ability", students' performance in primary school Grade 6, and secondary school Grades 1 and 2 was significantly higher than students' performance in primary school Grade 4 ($p < .05$, $p < .01$, and $p < .01$, respectively). Secondary school students of Grades 1 and 2 also had significantly higher performance than primary school Grade 5 students in "Translation ability" factor ($p < .01$, and $p < .01$, respectively). In the factor of "Reflection ability", students' performance in primary school Grades 5 and 6, and secondary school Grades 1 and 2 was significantly higher than students' performance in primary school Grade 4 (all differences significant at $p < .01$). Secondary school students of Grades 1 and 2 also had significantly higher performance than primary school Grade 5 students ($p < .01$, and $p < .01$, respectively), and significantly higher performance than primary school Grade 6 students in "Reflection ability" factor ($p < .05$, and $p < .01$, respectively). In the factor of "Rotation ability", students' performance in primary school Grades 5 and 6, and secondary school Grades 1 and 2 was significantly higher than students' performance in primary school Grade 4 (all differences significant at $p < .01$). Primary school students of Grade 6, and secondary school students of Grades 1 and 2 also had significantly higher performance than primary school students of Grade 5 in "Rotation ability" factor ($p < .05$, $p < .01$, and $p < .01$, respectively).

Table 4.7

*Comparison of the Subjects' Performance in Transformational Geometry Concepts at the Five Grade Levels*

| Dependent Variable | School Grade A | School Grade B | Post-hoc Significance |
|---|---|---|---|
| Translation | Grade 4 (Primary) | Grade 6 (Primary) | .02* |
| | | Grade 1 (Secondary) | .00** |
| | | Grade 2 (Secondary) | .00** |
| | Grade 5 (Primary) | Grade 1 (Secondary) | .01** |
| | | Grade 2 (Secondary) | .00** |
| Reflection | Grade 4 (Primary) | Grade 5 (Primary) | .01** |
| | | Grade 6 (Primary) | .00** |
| | | Grade 1 (Secondary) | .00** |
| | | Grade 2 (Secondary) | .00** |
| | Grade 5 (Primary) | Grade 1 (Secondary) | .00** |
| | | Grade 2 (Secondary) | .00** |
| | Grade 6 (Primary) | Grade 1 (Secondary) | .02* |
| | | Grade 2 (Secondary) | .00** |
| Rotation | Grade 4 (Primary) | Grade 5 (Primary) | .01** |
| | | Grade 6 (Primary) | .00** |
| | | Grade 1 (Secondary) | .00** |
| | | Grade 2 (Secondary) | .00** |
| | Grade 5 (Primary) | Grade 6 (Primary) | .02* |
| | | Grade 1 (Secondary) | .00** |
| | | Grade 2 (Secondary) | .00** |

| | | | |
|---|---|---|---|
| Transformational geometry | Grade 4 (Primary) | Grade 5 (Primary) | .02* |
| | | Grade 6 (Primary) | .00** |
| | | Grade 1 (Secondary) | .00** |
| | | Grade 2 (Secondary) | .00** |
| | Grade 5 (Primary) | Grade 6 (Primary) | .05* |
| | | Grade 1 (Secondary) | .00** |
| | | Grade 2 (Secondary) | .00** |
| | Grade 6 (Primary) | Grade 2 (Secondary) | .01** |

* $p < .05$. ** $p < .01$.

In the general factor of "Transformational geometry ability", students' performance in primary school Grades 5 and 6, and secondary school Grades 1 and 2 was significantly higher than students' performance in primary school Grade 4 ($p < .05$, $p < .01$, $p < .01$, and $p < .01$, respectively). Primary school students of Grade 6 and secondary school students of Grades 1 and 2 also had significantly higher performance than primary school students of Grade 5 in "Transformational geometry ability" factor ($p < .05$, $p < .01$, and $p < .01$, respectively). Finally, secondary school students of Grade 2 had significantly higher performance than primary school students of Grade 6 ($p < .01$).

This section presents the results that concern the third aim of the research regarding the description of students' levels of ability in transformational geometry. Specifically, it answers the following research questions:

(3) What are the levels of ability of 9- to 14-year-old students' development of transformational geometry concepts and in what way can these levels be described?

(4) What are the abilities, conceptions, strategies, and common errors of students at different levels?

In order to investigate and describe the levels of ability of students' development of transformational geometry concepts, their responses in the 48 items of the transformational geometry ability test were used. In this section, the statistical procedures for classifying students based on their ability in the transformational geometry test is described first, followed by the description of the quantitative characteristics of classes of subjects in the factors of transformational geometry ability, and the verification of the development of transformational geometry concepts. Finally, the characteristics of the students at each level of ability are described, complemented by the qualitative information of the clinical interviews.



*Classes of students in transformational geometry concepts*

In order to examine whether there are subgroups of similar behaviour towards their ability in transformational geometry concepts, latent class analysis was performed, based on students' abilities in the 48 items of the transformational geometry concepts ability test. The validity of four consecutive models was tested, according to which the subjects of the study could be divided into two, three, four, or five groups of similar behaviour to the transformational geometry concepts ability test. The results of the analysis showed that the best model with the biggest value of entropy and smallest values in the AIC and BIC indices was the solution with four groups of subjects (*entropy* = .95, *AIC* = 16881.77, *BIC* = 17791.07), as presented in Table 4.8. The average latent class probabilities of the subjects in each category for belonging in the category that the analysis placed them based on the solution of four classes, was quite satisfactory (see Table 4.9).

Table 4.8

*Fit Indices for Models with Different Number of Classes*

| Indices | *Entropy* | *AIC* | *BIC* |
|---|---|---|---|
| 2 classes model | .94 | 18672.53 | 19179.56 |
| 3 classes model | .93 | 17581.28 | 18289.44 |
| 4 classes model | .95 | 16881.77 | 17791.06 |
| 5 classes model | .95 | 17558.84 | 18669.27 |

Table 4.9

*Average Latent Class Probabilities*

| Latent Class Probabilities | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| Class 1 Subjects | .98 | .02 | .00 | .00 |
| Class 2 Subjects | .00 | .97 | .03 | .00 |
| Class 3 Subjects | .00 | .02 | .97 | .01 |
| Class 4 Subjects | .00 | .00 | .02 | .98 |



*Descriptive results of the four classes of subjects in transformational geometry ability*

According to the latent class analysis, the percentage of subjects that fall within Class 1 was 13%. In Class 2 falls 32% of the subjects, 37% falls in Class 3, and in Class 4 falls 18% (see Table 4.10). The percentage of students that falls within each category varies according to grade level. As shown in Table 4.10, 26.1% of primary school Grade 4 students fall within Class 1, 54.5% are in Class 2, 17.0% are in Class 3, and only 2.3% are in Class 4. As for primary school students at Grade 5, 17.9% is in Class 1, 42.0% in Class 2, 32.1% in Class 3, and 8.0% in Class 4. The percentages for primary school students at

Grade 6 were 16.3% for Class 1, 23.9% for Class 2, 42.4% for Class 3, and 17.4% for Class 4. The percentage of secondary school Grade 1 students in Class 1 is limited to the comparatively smaller value of 2.9%, with 23.8% in Class 2, 46.7% in Class 3, and 26.7% in Class 4. Finally, 3.3% of secondary school Grade 2 students fall in Class 1, 13.2% falls in Class 2, 46.2% falls in Class 3, and 37.4% falls in Class 4.

Table 4.10

*Percentages of Students in the Four Classes of Subjects*

| | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| Primary Grade 4 | **26.1%** | **54.5%** | 17.0% | 2.3% |
| Primary Grade 5 | 17.9% | **42.0%** | **32.1%** | 8.0% |
| Primary Grade 6 | 16.3% | **23.9%** | **42.4%** | 17.4% |
| Secondary Grade 1 | 2.9% | 23.8% | **46.7%** | **26.7%** |
| Secondary Grade 2 | 3.3% | 13.2% | **46.2%** | **37.4%** |
| Sum | 13% | 32% | 37% | 18% |



Based on these percentages, it seems that the majority of the primary school Grade 4 students are in Class 1 and Class 2. For primary school Grade 5 students, the majority seems to shift to Class 2 and Class 3, and similarly for primary school Grade 6 students. In secondary school Grade1 the majority of the students seem to shift towards Class 3 and Class 4. Finally, in secondary school Grade 2, the majority of the students are in Class 3 and Class 4.

The results of the multiple analyses of variances showed that there were statistically significant differences in students' ability in transformational geometry concepts (Pillai's $F_{(3,484)} = 67.36$, $p < .05$). Table 4.11 presents the means of performance of the subjects in the four classes in the three transformational geometry concepts (translation, reflection, and rotation), and in the general ability of transformational geometry. The mean performance of each class in the three transformational geometry concepts was significantly higher than the corresponding mean of the previous class, except in the case of performance in rotation between Class 1 and Class 2. The mean performance of Class 1 in the three transformational geometry concepts was lower than .49 ($M_{\text{Translation}} = .41$,

$M_{Reflection}$ = .21, and $M_{Rotation}$ = .19). The mean performance of Class 2 in the three transformational geometry concepts was equal to .49 for translation ($M_{Translation}$ = .49), and lower than .49 in reflection and in rotation ($M_{Reflection}$ = .34, and $M_{Rotation}$ = .19), but significantly better in translation and reflection, compared to Class 1. The mean performance of Class 3 was higher than .49 in translation and reflection ($M_{Translation}$ = .66, and $M_{Reflection}$ = .55), and lower than .49 in the case of rotation ($M_{Rotation}$ = .43). The mean performance of Class 4 in the three transformational geometry concepts was higher than .67 ($M_{Translation}$ = .85, $M_{Reflection}$ = .80, and $M_{Rotation}$ = .69).

Table 4.11

*Means and Standard Deviations of Performance in the Transformational Geometry Concepts for All Classes*

| Class | Translation | | Reflection | | Rotation | | General Performance | |
|---|---|---|---|---|---|---|---|---|
| | Mean | *SD* | Mean | *SD* | Mean | *SD* | Mean | *SD* |
| Class 1 | .41 | .18 | .21 | .12 | .19 | .14 | .27 | .12 |
| Class 2 | .49 | .16 | .34 | .12 | .19 | .12 | .34 | .08 |
| Class 3 | .66 | .14 | .55 | .12 | .43 | .13 | .55 | .08 |
| Class 4 | .85 | .13 | .80 | .11 | .69 | .14 | .78 | .08 |



Moreover, statistically significant differences were also found in the four first-order factors of ability in translation (Pillai's $F_{(3,484)} = 27.53$, $p < .05$), in the four first-order factors of ability in reflection (Pillai's $F_{(3,484)} = 72.74$, $p < .05$), and in the four first-order factors of ability in rotation (Pillai's $F_{(3,484)} = 48.02$, $p < .05$) between the four classes. For Class 1, Class 3, and Class 4, the mean performance of each class was significantly higher than the corresponding mean of the previous class in every factor. Class 2 appeared to have statistically significant differences from Class 1 only in the factors "Recognise image" in translation, and "Recognise reflection". In both cases, the mean differences of Class 2 are not statistically significant from the means of Class 3. Class 1 had a mean of performance that was higher than .49 only in one factor, in the "Identify parameters" in translation ($M_{Tr3}$ = .50) (see Table 4.12). Class 2 had a mean of performance higher than .49 in two factors, namely “Recognise image” and “Identify parameters” in translation ($M_{Tr1}$ = .59, $M_{Tr3}$ = .55) (see Table 4.12), and a mean of performance equal to .67 in the factor “Recognise reflection” ($M_{Re2}$ = .67) (see Table 4.13). Class 3 had a mean performance which was lower than .49 in three factors, one in reflection ($M_{Re1}$ = .44) (see Table 4.13), and two in rotation ($M_{Ro3}$ = .49, $M_{Ro4}$ = .18) (see Table 4.14), equal to .49 in one factor, namely “Indentify parameters” in reflection ($M_{Re3}$ = .49), and higher than .67 in two factors, namely “Identify parameters” in translation ($M_{Tr3}$ = .76) (see Table 4.13) and “Recognise reflection” ($M_{Re2}$ = .72) (see Table 4.13). Class 4 had a mean of performance which was higher than .67 in all the 12 first-order factors, except for “Construct image” in rotation ($M_{Ro4}$ = .52) (see Table 4.14).

Table 4.12

*Means and Standard Deviations of Performance in the Translation Factors for All Classes*

| Class | Recognise Image | | Recognise Translation | | Identify Parameters | | Construct Image | |
|---|---|---|---|---|---|---|---|---|
| | Mean | *SD* | Mean | *SD* | Mean | *SD* | Mean | *SD* |
| Class 1 | .42 | .37 | .37 | .24 | .50 | .31 | .37 | .23 |
| Class 2 | .59 | .33 | .44 | .21 | .55 | .30 | .42 | .21 |
| Class 3 | .64 | .36 | .60 | .18 | .76 | .28 | .62 | .14 |
| Class 4 | .86 | .26 | .82 | .17 | .94 | .15 | .79 | .18 |

Table 4.13

*Means and Standard Deviations of Performance in the Reflection Factors for All Classes*

| Class | Recognise Image | | Recognise Reflection | | Identify Parameters | | Construct Image | |
|---|---|---|---|---|---|---|---|---|
| | Mean | *SD* | Mean | *SD* | Mean | *SD* | Mean | *SD* |
| Class 1 | .13 | .24 | .23 | .21 | .29 | .25 | .18 | .22 |
| Class 2 | .20 | .26 | .67 | .16 | .28 | .24 | .22 | .18 |
| Class 3 | .44 | .32 | .72 | .16 | .49 | .25 | .55 | .20 |
| Class 4 | .75 | .26 | .80 | .16 | .79 | .19 | .85 | .15 |

Table 4.14

*Means and Standard Deviations of Performance in the Rotation Factors for All Classes*

| Class | Recognise Image | | Recognise Rotation | | Identify Parameters | | Construct Image | |
|---|---|---|---|---|---|---|---|---|
| | Mean | *SD* | Mean | *SD* | Mean | *SD* | Mean | *SD* |
| Class 1 | .23 | .33 | .32 | .25 | .22 | .21 | .04 | .11 |
| Class 2 | .22 | .30 | .25 | .23 | .28 | .22 | .04 | .07 |
| Class 3 | .60 | .36 | .55 | .21 | .49 | .26 | .18 | .14 |
| Class 4 | .87 | .27 | .71 | .19 | .77 | .23 | .52 | .20 |

Table 4.15 summarises the characteristics of the four classes of the subjects. The performance of a class of subjects is characterised as high when it is equal or higher than .67, as average when it is between .49 and .67, and as low when it is lower than .49. The subjects of Class 1 had average performance only in the factor “Identify parameters” in translation (Tr3), whereas their performance in the other factors was low. The subjects of Class 2 had high performance only in the factor “Recognise reflection” (Re2), average performance in factors “Recognise image” in translation (Tr1) and “Identify parameters” in translation (Tr3), and low performance in all the other factors. The subjects of Class 3 had

high performance in factors “Identify parameters” in translation (Tr3) and “Recognise reflection” (Re2), average performance in factors “Recognise image” in translation (Tr1), “Identify parameters” in translation (Tr3), “Construct image” in translation (Tr4), “Identify parameters” in reflection (Re3), “Construct image” in reflection (Re4), “Recognise image” in rotation (Ro1), “Recognise rotation” (Ro2), and “Identify parameters” in rotation (Ro3), and low performance in factors of “Recognise image” in reflection (Re1) and “Construct image” in rotation (Ro4). The subjects of Class 4 had a high performance in all the factors, except from factor “Construct image” in rotation (Ro4), where they had average performance.

Table 4.15

*Characteristics of the Four Classes in Transformational Geometry Concepts*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | Re2 | Tr3, Re2 | Tr1, Tr2, Tr3, Tr4, Re1, Re2, Re3, Re4, Ro1, Ro2, Ro3 |
| Average Performance ($.49 \leq M < .67$) | Tr3 | Tr1, Tr3 | Tr1, Tr2, Tr4, Re3, Re4, Ro1, Ro2, Ro3 | Ro4 |
| Low Performance ($M < .49$) | Tr1, Tr2, Tr4, Re1, Re2, Re3, Re4, Ro1, Ro2, Ro3, Ro4 | Tr2, Tr4, Re1, Re3, Re4, Ro1, Ro2, Ro3, Ro4 | Re1, Ro4 | |

*Note.* Codes Tr1-Tr4 correspond to the factors of translation in the order of “Recognise image” (Tr1), “Recognise translation” (Tr2), “Identify parameters” (Tr3), and “Construct image” (Tr4). The same coding applies for Re1-Re4 in reflection and for Ro1-Ro4 in rotation.

The results of the latent class analysis revealed the existence of four classes of subjects. Class 1 had average performance in only one translation factor, namely "Identify parameters". Class 2 had high performance in one reflection factor, namely "Recognise reflection", and average performance in two translation factors. Class 3 had high performance in one translation factor and one reflection factor, however not the corresponding factors of the two geometric transformations. Class 3 also had average performance in three translation factors, two reflection factors, and three rotation factors; however, not the corresponding factors of the three geometric transformations. Also, it is noted that for Class 3, the only geometric transformation with all factors above low performance is translation. Finally, Class 4 had high performance in all factors of the three geometric transformations, except from the rotation factor of "Construct image", where performance was average.

In order to confirm the development of transformational geometry concepts as proposed in literature, two types of theoretical models were tested: (i) model 1 is based on the mathematical structures of the transformational geometry concepts of translation, reflection, and rotation, and assumes that each geometric transformation is synthesised by the four cognitive types of tasks, and that the subjects' development follows a progression of the three different types of geometric transformations, in the order of translation, reflection, and rotation (see Figure 4.4); and (ii) model 2 is based on the cognitive structures of the four types of tasks of transformational geometry concepts, namely "Recognise image", "Recognise transformation", "Identify parameters", and "Construct image", and that the subjects' development follows a progression of the four different types of tasks for every geometric transformation, in that order (see Figure 4.5).

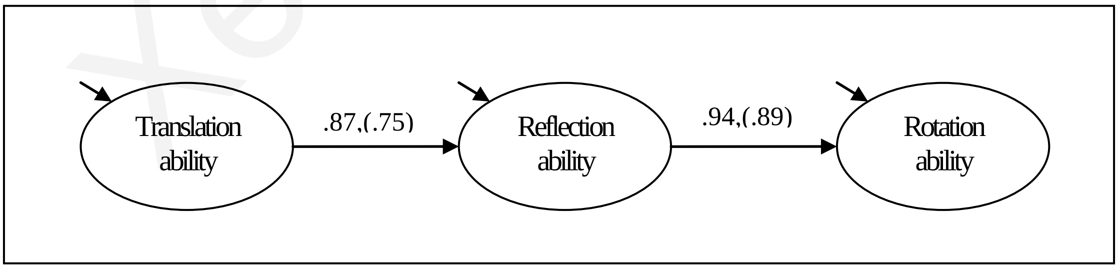
The results of the structural equation model analysis showed that the best model to describe the development of students' ability was the first model (see Table 4.16), according to which translation, reflection, and rotation are all synthesised by the corresponding factors of "Recognise image", "Recognise transformation", "Identify parameters", and "Construct image", and that performance in translation can predict performance in reflection, and performance in reflection can predict performance in rotation.

Table 4.16

*Fit Indices of Models for the Development of Ability in Transformational Geometry*

| Model | Fit Indices | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | *CFI* | $x^2$ | *df* | $x^2/df$ | *p* | *RMSEA* | *AIC* | *BIC* |
| Model 1 | .98 | 87.89 | 52 | 1.69 | < .05 | .04 | 225.34 | 335.02 |
| Model 2 | .95 | 150.11 | 51 | 2.94 | < .05 | .06 | 313.55 | 478.08 |

The regression coefficients were statistically significant. Specifically, the regression coefficient of translation performance on reflection performance was .89 ($z = 8.45$, $p < .05$), and the regression coefficient of reflection performance on rotation performance was .94 ($z = 11.73$, $p < .05$) (see Figure 4.4). Based on this model, it seems that students first acquire knowledge on concepts related to translation, followed by knowledge on concepts related to reflection, and finally on concepts related to rotation. Even though the order and pace of acquiring the cognitive aspects within each geometric transformation seem to vary among the three geometric transformations (see Table 4.15), the hierarchical model of learning transformational geometry concepts in the order of the mathematical structures of translation, reflection, and rotations provides a better description for the development of transformational geometry ability.




* p < .05.

*Figure 4.3*. Relations between the factors of transformational geometry ability based on the mathematical structures of transformational geometry (Model 1).

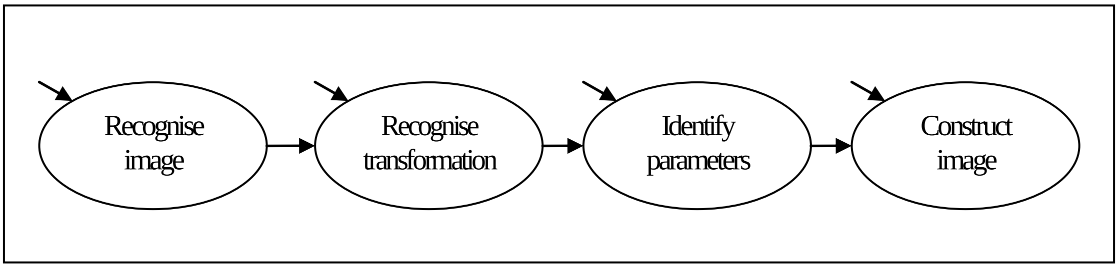



*Figure 4.4*. Relations between the factors of transformational geometry ability based on the cognitive structures of transformational geometry (Model 2).

*Description of students' levels of abilities*

The students who were classified in Class 1 are considered to be at the first level of abilities in transformational geometry concepts. These students can only perform satisfactorily in identifying the parameters of a translation. The students who were classified in Class 2 are considered to be at the second level of abilities in transformational geometry concepts. These students can perform well in recognising a reflection, and satisfactorily in recognising the image of a translation and in identifying the parameters of a translation. The students who were classified in Class 3 are considered to be at the third level of abilities in transformational geometry concepts. These students can perform well in recognising a reflection and in identifying the parameters of a translation, and satisfactorily in recognising the image of a translation, recognising a translation, and in constructing the image of a translation. They can also perform satisfactorily in recognising the parameters of a reflection and in constructing the image of a reflection, and also in recognising the image of a rotation, recognising a rotation, and identifying the parameters of a rotation. Finally, the students who were classified in Class 4 are considered to be at the fourth level of abilities in transformational geometry concepts. These students can perform well in every factor of ability in transformational geometry, and satisfactorily in constructing the image of a rotation. Constructing the image of a rotation seems to be a factor that is too difficult for students between nine and 14 years old, and cannot be fully achieved at this age range.

*Students' levels of ability in the factor of Recognise image in translation*

Table 4.17 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Recognise image" in translation. Students of Level 1 had low performance in all the items. Specifically, less than 50% of the students in Level 1 selected the correct response in the items of this factor. Particularly, in item A1, where students had to find the image of a figure translated four units upwards, the majority of the students selected the correct response (34%). However, 23% selected the response that was in a larger distance. In item A2, where students had to find the image of a figure translated to the right, 45% responded correctly, but another percentage of 27% selected the response that was diagonally translated. The largest percentage of correct responses was in item A3, where 47% responded correctly in finding the image of a figure translated diagonally.

Students of Level 2 had average performance in all items. Over 50% of the students in Level 2 selected the correct response in the items of this factor. Particularly, the erroneous responses that students of Level 2 selected for items A1 and A2 where the same as the students of Level 1, but with lower percentages (22% for bigger distance in item A1, and 21% for diagonal response in item A2). The percentages of correct responses were 58%, 60%, and 59% for items A1, A2, and A3, respectively.

Students of Level 3 had average performance in items A1 and A2, with 61% and 64% selecting the correct response respectively. However, a percentage of 30% of the students selected the response that was placed in a larger distance in item A1, and a percentage of 27% selected the response that was diagonally placed in item A2. Students of Level 3 also had a high performance in item A3, where 68% selected the correct response. However, in item A3, a percentage of 20% selected another diagonal response, with a larger distance.

Students of Level 4 had high performance in all the items, with correct response percentages of being 80%, 89%, and 89% for items A1, A2, and A3, respectively. The most common types of errors were the same as in the other levels; nevertheless, their percentages were lower.

Table 4.17

*Characteristics of the Four Classes in Factor Tr1: Recognise Image of Translation*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance (M ≥ .67) | | | A3 | A1, A2, A3 |
| Average Performance (.49 ≤ *M* < .67) | | A1, A2, A3 | A1, A2 | |
| Low Performance (*M* < .49) | A1, A2, A3 | | | |

*Note*. A1 = Item in vertical direction, A2 = Item in horizontal direction, A3 = Item in diagonal direction.



*Students' levels of ability in the factor of Recognise translation*

Table 4.18 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Recognise translation". Students of Level 1 had low performance in the items A4, A5, and A6. Specifically, less than 50% of the students of Level 1 selected the correct response in these items (48%, 44%, and 42%, respectively). The most common type of error in all three items was to select the response that presented a reflection (28%, 31%, and 27%, respectively), thus confusing translation with reflection. Students at Level 1 appear to have average performance in item A7, where they had to find the image of a translation with unspecified direction. The percentage of correct responses here was 64%. However, 19% of the students of this level were unable to respond to this item.

Students of Level 2 had low performance in item A5, with 47% of the students selecting the correct response, and 35% of the students selecting the erroneous response of reflection. However, students of Level 2 had average performance in items A4, A6, and A7. Over 50% of the students of Level 2 selected the correct response in these three items

(58%, 52%, and 62%, respectively). Again, as in Level 1, the most common erroneous response was that of selecting the response that was showing a reflection (26% for A4, and 25% for A6). In item A7, the most common type of error was to select the response which depicted a reflection in vertical line (26%).

Students of Level 3 had average performance in item A6, with 52% of the students selecting the correct response, and 25% selecting the erroneous response showing a reflection. They also had a high performance in item A4, A5, and A7, where 79%, 78%, and 80% of the students selected the correct response, respectively. The percentages of students that selected erroneous responses were lower, but the selection of reflection was again the most common type of error. Students of Level 4 had high performance in all the items, with percentages of correct response being 98%, 94%, 91%, and 92% for items A4, A5, A6, and A7, respectively.

Table 4.18

*Characteristics of the Four Classes in Factor Tr2: Recognise Translation*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | | A4, A5, A7 | A4, A5, A6, A7 |
| Average Performance ($.49 \leq M < .67$) | A7 | A4, A6, A7 | A6 | |
| Low Performance ($M < .49$) | A4, A5, A6 | A5 | | |

*Note*. A4 = Item in vertical direction, A5 = Item in horizontal direction, A6 = Item in diagonal direction, A7 = Item with unspecified direction.

*Students' levels of ability in the factor of Identify parameters in translation*

Table 4.19 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Identify parameters" of a translation. Students of Level 1 had low performance in items A8, A10, and A11. Specifically, less than 50% of the students in Level 1 responded correctly in these items (8%, 2%, and 20%, respectively). The most common type of error in all three items was to give an erroneous measurement of the distance between the pre-image and the image (48%, 27%, and 16%, respectively), which suggests poor understanding regarding matching corresponding points for mapping in translation. It is also important to note that these items had high percentages of no response or completely incorrect response (42%, 66%, and 61%, respectively), which suggests students' uncertainty on how to approach the task. Students at Level 1 appear to have average performance in item A9, where they had to find the parameters for a translation in horizontal direction. The percentage of correct responses in this item was 17%, which is still considerably less than 50%. Nevertheless, the percentage of the students in Level 1 who were unable to respond correctly to at least one of the parameters of this item, or did not respond at all, was 23%, and the percentage of students that measured the distance incorrectly was 56%.

Students of Level 2 had low performance in items A10 and A11, with 1% and 30% responding correctly to both parameters, respectively. The most common error again was regarding the distance (31% and 12%, respectively). In item A10, two thirds of the students at this level (67%) were unable to give a correct or partially correct response, and in item A11, half of the students (50%) were unable to give a correct or partially correct response. However, students of Level 2 had average performance in items A8 and A9. Specifically, 5% responded correctly in item A8 and 11% responded correctly in item A9. As in the other two items of this factor, the most common error was regarding the distance measure (71% and 75%, respectively).

Students of Level 3 had low performance in item A10, with 11% of the students responding correctly, and 50% measured the distance incorrectly. They also had average performance in item A11, where 48% of the students responded correctly and 12% responded with erroneous distance measure. Students of Level 3 also had high performance in items A8 and A9, with percentages of correct responses being 14% and

12%, respectively. In both items, the highest percentages of students were able to give a partially correct response with errors in distance measure (81% and 85%, respectively).

Students of Level 4 had high performance in all the items of this factor, with percentages of correct response being 45%, 54%, 43%, and 89% for items A8, A9, A10, and A11, respectively. In the cases of the items A8, A9, and A10, the highest percentages of erroneous responses were again regarding the distance, and were 53%, 45%, and 42%, respectively.

Table 4.19

*Characteristics of the Four Classes in Factor Tr3: Identify Parameters of Translation*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | | A8, A9 | A8, A9, A10, A11 |
| Average Performance ($.49 \leq M < .67$) | A9 | A8, A9 | A11 | |
| Low Performance ($M < .49$) | A8, A10, A11 | A10, A11 | A10 | |

*Note*. A8 = Item in vertical direction, A9 = Item in horizontal direction, A10 = Item in diagonal direction, A11 = Item with overlapping image in horizontal direction.

*Students' levels of ability in the factor of constructing image in translation*

Table 4.20 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Construct image" of a translation. Students of Level 1 had low performance in all the items of this factor. Specifically, less than 50% of the students in Level 1 responded correctly in these items. In item A12, constructing the image of a translation in vertical direction, only 9% of the students of Level 1 responded correctly, with the most common type of error being the false measurement of the distance between the image and pre-image (44%). In item A13 of constructing the image in horizontal direction, almost a fifth of the students of Level 1 responded correctly (19%), with the false measurement of the distance between the pre-image and the image being again the most common type of error (47%). For item A14, constructing the image of a translation in diagonal direction, a percentage of 2% of the students of Level 1 responded correctly, whereas 53% of the students were not able to give a correct or partially correct response to this item. The most common type of error was again in the distance (39%). In the case of constructing the image of a translation when it is overlapping the image (item A15), only 5% of students at this level responded correctly, with the majority of 53% responding with error in the measurement of the distance. Finally, for item A16, constructing the image of a complex figure, only 3% of the students of Level 1 responded correctly, whereas 62% were unable to give a correct or partially correct response to this item. The most common type of error was, as in the other items of this factor, false measurement of distance between the pre-image and the image (30%).

Students of Level 2 had low performance in all the items of this factor, except for item A13, where they had average performance. In items A12, A14, A15, and A16, the percentages of correct response were 4%, 8%, 5%, and 1% respectively. The most common error again was regarding the distance (60%, 51%, 62%, and 58%, respectively). In item A13, 10% of the students responded correctly, whereas 56% measured incorrectly the distance.

Students of Level 3 had average performance in all the items of this factor, except from item A13, where they had high performance. In items A12, A14, A15, and A16 the percentages of correct responses were 11%, 21%, 18%, and 9% respectively. The most common type of error again was regarding the distance (79%, 46%, 76%, and 78%, respectively). In item A13, 28% of the students responded correctly, whereas 66% of the students measured the distance incorrectly.

Students of Level 4 had high performance in all the items of this factor, with percentages of correct responses being 52%, 52%, 45%, 61%, and 49% for items A12, A13, A14, A15, and A16, respectively. The highest percentages of erroneous responses were again regarding the distance and were 47%, 44%, 48%, 36%, and 46%, respectively.

Table 4.20

*Characteristics of the Four Classes in Factor Tr4: Construct Image of Translation*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | | A13 | A12, A13, A14, A15, A16 |
| Average Performance ($.49 \leq M < .67$) | | A13 | A12, A14, A15, A16 | |
| Low Performance ($M < .49$) | A12, A13, A14, A15, A16 | A12, A14, A15, A16 | | |

*Note.* A12 = Item in vertical direction, A13 = Item in horizontal direction, A14 = Item in diagonal direction, A15 = Item with overlapping image in horizontal direction, A16 = Item with complex figure in horizontal direction.

*Students' levels of ability in the factor of Recognise image in reflection*

Table 4.21 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Recognise image" in reflection. Students of Level 1 had low performance in all items. Specifically, less than 50% of the students in Level 1 selected the correct response in the items of this factor. Particularly, in item B1, where students had to find the image of a figure reflected over a horizontal line, the majority of the students selected the incorrect responses which presented the translation image of the figure (41%), whereas only 16% of the students selected the correct response. In item B2, where students had to find the image of a figure reflected over a vertical line, again only 13% of the students responded correctly, and another percentage of 33% selected the response that had the same orientation and would be the image of a translation. The percentage of correct response in item A3 was also low (11%), whereas the majority of the students of Level 1 again selected the image that would be the result of a translation (41%).

Students of Level 2 also had low performance in all the items. Particularly, a percentage of 15% of the students selected the correct response in item B1, with the majority (49%) selecting the figure that would be the result of a translation. In item B2, one third of the students (33%) selected the correct response, whereas 38% of the students selected the response that showed the image of a translation. In item B3, a percentage of 12% responded correctly and the majority (35%) selected the response that showed the image of a translation.

Students of Level 3 had average performance in items B1 and B2, with 51% and 58% of the students selecting the correct response, respectively. About a quarter of the students (23%) selected the translation image in both items. Students of Level 3 also had low performance in item B3, where 23% selected the correct response. However, a percentage of 45% again selected the response that showed the result of a translation.

Students of Level 4 had high performance in items B1 and B2, with percentages of correct responses being 87%, and 90% respectively. The most common types of errors were the same as in the other levels; however, their percentages were lower. In the case of item B3, where the students of Level 4 had average performance, the percentage was just below 50% (47%), with the most common error being the selection of the image showing a reflection over a vertical line.

Table 4.21

*Characteristics of the Four Classes in Factor Re1: Recognise Image of Reflection*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance (M ≥ .67) | | | | B1, B2 |
| Average Performance (.49 ≤ *M* < .67) | | | B1, B2 | B3 |
| Low Performance (*M* < .49) | B1, B2, B3 | B1, B2, B3 | B3 | |

*Note.* B1 = Item in vertical direction, B2 = Item in horizontal direction, B3 = Item in diagonal direction.

*Students' levels of ability in the factor of Recognise reflection*

Table 4.22 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Recognise reflection". Students of Level 1 had low performance in items B4, B6, and B7. Specifically, less than 50% of the students in Level 1 selected the correct response in these items (33%, 5%, and 20%, respectively). The most common type of error in all three items was to select the response that presented a translation (41%, 30%, and 45%, respectively), thus confusing reflection with translation. Note that similar difficulties appeared in the translation section: when students were asked to recognise the translation, they confused it with reflection. Students of Level 1 appeared to have average performance in item B5, where they had to find a reflection in a vertical line. The percentage of correct responses here was 58%. However, a percentage of 16% of the students at this level failed to respond to this item, and a percentage of 14% selected the erroneous response of a half turn rotation.

Students of Level 2 also had low performance in items B4, B6, and B7, and average performance in item B5. In item B4, a percentage of 38% selected the correct response and the same percentage (38%) selected the erroneous response of a translation. Similarly in item B6, only 5% of the students selected the correct response, and 42% selected the erroneous response of a translation and in item B7, only 15% of the students selected the correct response and 54% of the students selected the erroneous response of a reflection. As for item B5, where students had to find the example showing a reflection in a vertical line, more than 50% of the students selected the correct response (53%), and 17% of the students selected the erroneous response of a half turn rotation.

Students of Level 3 had low performance in items B6 and B7, with 11% and 42% selecting the correct response, respectively. The most common erroneous response for item B6 was the quarter turn rotation (46%), and for item B7 were the translation (30%) and the quarter rotation (25%). Students of Level 3 had average performance in item B4, with almost two thirds (63%) of the students selecting the correct response, and a quarter of the students (26%) selecting the erroneous response which showed a translation. They also had a high performance in item B5, where 78% of the students selected the correct response, and the most common erroneous answer being the half turn rotation (12%).

Students of Level 4 had high performance in items B4, B5, and B7. The percentages of correct responses were 94%, 96%, and 78% respectively. The most common errors were the same as for Level 3, but in lower percentages. In item B6, the students of Level 4 had average performance, since nearly half of the students (47%) selected the correct response, and the rest of the students either selected the quarter turn rotation (25%), or the translation example (23%).

Table 4.22

*Characteristics of the Four Classes in Factor Re2: Recognise Reflection*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance (M ≥. 67) | | | B5 | B4, B5, B7 |
| Average Performance ($.49 \leq M < .67$) | B5 | B5 | B4 | B6 |
| Low Performance ($M < .49$) | B4, B6, B7 | B4, B6, B7 | B6, B7 | |

*Note.* B4 = Item in vertical direction, B5 = Item in horizontal direction, B6 = Item in diagonal direction, B7 = Item with unspecified direction.



*Students' levels of ability in the factor of Identify parameters in reflection*

Table 4.23 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Identify parameters" of a reflection. Students of Level 1 had low performance in items B8, B10, and B11. Specifically, less than 10% of the students in Level 1 responded correctly in these items (0%, 5%, and 9%, respectively). These items had high percentages of inability to respond. Specifically, in item B8, where students had to find the horizontal line of symmetry between two figures, a percentage of 81% of the students responded with either a non-horizontal line of reflection, or did not respond at all, whereas 19% of the students at Level 1 placed the horizontal line of reflection, but in false distance from the shapes. The results are very similar for items B10 and B11, with percentages of inability to respond being 84% and 78%, respectively, and partially correct at 11% and 13%, respectively. Students of Level 1 appeared to have average performance in item B9, where they had to find the vertical line of reflection. The percentage of correct responses in this item was 52%. However, the percentage of students of Level 1 who were unable to give a correct or partially correct response to this item was 42%.

Students of Level 2 had low performance in items B10 and B11, with 5% and 9% of the students responding correctly, respectively. There was again a high percentage of inability to give a correct or partially correct response (84% and 78%, respectively), and smaller percentages of students that found the correct orientation of the line, but with false distance from the shapes (11% and 13%, respectively). However, students of Level 2 had high performance in items B8 and B9. Specifically, all the subjects of Level 2 (100%) responded correctly in item B8, and 84% responded correctly in item B9.

Students of Level 3 had high performance in items B8 and B9, with 97% and 93% of the students responding correctly. They also had average performance in item B10, finding a diagonal line of reflection, where 19% of the students responded correctly. A percentage of 51% of the students was able to find the correct orientation of the diagonal line of reflection, but did not place it in the correct distance from the shapes. Students of Level 3 seemed to have low performance in item B11, finding the vertical line of reflection for overlapping figures, where 38% of the students responded correctly, but a larger percentage of 41% were unable to give a correct or partially correct response to the item.

Students of Level 4 had high performance in the items B8 and B9 of this factor, with percentages of correct responses being 99% and 100%, respectively. In the items B10 and B11, the students had average performance. Specifically, a percentage of 36% of the students of Level 4 were able to correctly find the diagonal line of reflection in item B10, and half of the students at this level placed it in the wrong distance from the shapes (51%). For item B11, a percentage of 54% of the students placed the vertical line correctly between the overlapping shapes, whereas a high percentage of 33% of the students either failed to respond, or drew a line with different orientation.

Table 4.23

*Characteristics of the Four Classes in Factor Re3: Identify Parameters of Reflection*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance (M ≥ .67) | | B8, B9 | B8, B9 | B8, B9 |
| Average Performance (.49 ≤ *M* < .67) | B9 | | B10 | B10, B11 |
| Low Performance (*M* < .49) | B8, B10, B11 | B10, B11 | B11 | |

*Note.* B8 = Item in vertical direction, B9 = Item in horizontal direction, B10 = Item in diagonal direction, B11 = Item with overlapping image in horizontal direction.



*Students' levels of ability in the factor of Construct image in reflection*

Table 4.24 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Construct image" of a reflection. Students of Level 1 had low performance in all of the items of this factor. Specifically, less than 10% of the students in Level 1 responded correctly in these items. In item B12, constructing the image of a reflection in vertical direction, only 8% of students of Level 1 responded correctly, with the most common type of error being the false orientation of the image (31%). In item B13, constructing the image of a reflection in horizontal direction, about a quarter of the students of Level 1 responded correctly (23%), with false orientation being the most common type of error again (13%). For constructing the image of a translation in diagonal direction (item B14), the majority of the students of Level 1 (91%) were not able to give a correct or partially correct response, and none of the students responded correctly. In the case of constructing the image of a translation when it is overlapping the image (item B15), only 3% of students at this level responded correctly, with the majority of 73% being unable to respond to this item. Finally, for item B16, constructing the image of a complex figure, a percentage of 16% of the students of Level 1 responded correctly, and 16% of the students gave a response with a false orientation of the image, whereas 64% were unable to give a correct or partially correct response to this item. It is important to note that students of Level 1 had more than 50% failure in each of the items of this factor.

Students of Level 2 also had low performance in all the items of this factor. In the items B12, B13, B14, B15, and B16 the percentages of correct responses were 10%, 25%, 1%, 3%, and 12%, respectively. In items B12 and B16, the most common error was regarding the orientation of the image (40% and 27%, respectively). In items B13 and B15, a smaller percentage of the students (21% and 18%, respectively) responded with errors in the distance of the image from the line of reflection, which was the most common type of error in these items. In B14 and B15, a substantial percentage (92% and 62%, respectively), was unable to give a correct or partially correct response to these items.

Students of Level 3 had high performance in the items B13 and B16 of this factor. Specifically, a percentage of 67% of the students of Level 3 responded correctly to the item B13, with false distance from the line of symmetry being the most common type of error, whereas 60% of the students of Level 3 responded correctly to the item B16, with orientation of the image being the most common type of error (15%). Students of Level 3 had average performance in items B12 and B15, where the percentages of correct responses were 46% and 32%, respectively. In item B12, the most common type of error was regarding the orientation of the image (39%), whereas in item B15 the most common type of error was the distance of the figure from the line (20%). In item B14, the students of Level 3 had low performance, since only 3% of the students were able to correctly find the image of a figure in a diagonal line, and the highest percentage (76%) of the students at this level were unable to give a correct or partially correct response to this item. However, a percentage of 11% of the students at this level responded to this item with a correct image regarding orientation and distance measure, but in a false direction.

Finally, students of Level 4 had high performance in all the items of this factor, except for item B14. The percentages of correct responses were 91% for item B12, 94% for item B13, 75% for item B15, and 89% for item B16. The percentage of correct responses in item B14 for the students of Level 4 was 38%, with the most common type of erroneous response being the false direction (21%).

Table 4.24

*Characteristics of the Four Classes in Factor Re4: Construct Image of Reflection*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | | B13, B16 | B12, B13, B15, B16 |
| Average Performance ($.49 \leq M < .67$) | | | B12, B15 | B14 |
| Low Performance ($M < .49$) | B12, B13, B14, B15, B16 | B12, B13, B14, B15, B16 | B14 | |

*Note*. B12 = Item in vertical direction, B13 = Item in horizontal direction, B14 = Item in diagonal direction, B15 = Item with overlapping image in horizontal direction, B16 = Item with complex figure in horizontal direction.



*Students' levels of ability in the factor of Recognise image in rotation*

Table 4.25 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Recognise image" in rotation. Students of Level 1 had low performance in all of the items of this factor. Specifically, less than 50% of the students in Level 1 selected the correct response in the items of this factor. Particularly, in item C1, where students had to find the image of rotation in vertical direction (three quarter turn), the majority of the students selected the incorrect response, which presented the translation image of the figure (33%), whereas only 22% of the students selected the correct response. In item C2, where students had to find the image of a horizontal rotation (quarter turn), again only 23% of the students of this level responded correctly, and another 33% of the students selected the response that had the same orientation and would be the image of a translation. A quarter of the students of Level 1 selected the correct response in item C3

(25%), whereas the majority selected the image that would be the result of a three quarter turn rotation (30%).

Students of Level 2 also had low performance in all of the items. Particularly, a percentage of 23% of the students of this level selected the correct response in item C1, and the same percentage of students selected the erroneous response that showed the result of a translation. In item C2, 19% of the students at this level selected the correct response, whereas 27% selected the response that showed the translation of the figure, and another percentage selected the erroneous response that showed the half turn rotation of the figure (24%). In item C3, almost a quarter of the students at Level 2 (23%) responded correctly, and another quarter of the students at Level 2 selected the response that showed the image of a translation (25%).

Students of Level 3 had average performance in items C1 and C3, with 61% and 53% selecting the correct responses, respectively. In item C3, about a fifth (20%) of the students selected the erroneous response of a quarter turn. Students of Level 3 also had high performance in item C2, where 68% selected the correct response, and 11% of the students at this level selected the response that showed the result of a translation. Finally, students of Level 4 had high performance in all the items of this factor, with percentages of correct response being 84% for item B1, 91% for item C2, and 84% for item C3. The common types of errors were the same as in Level 3; however, their percentages were lower.

Table 4.25

*Characteristics of the Four Classes in Factor Ro1: Recognise Image of Rotation*

| Performance Levels | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance (M ≥ .67) | | | C2 | C1, C2, C3 |
| Average Performance (.49 ≤ *M* < .67) | | | C1, C3 | |
| Low Performance (*M* < .49) | C1, C2, C3 | C1, C2, C3 | | |

*Note.* C1 = Item in vertical direction, C2 = Item in horizontal direction, C3 = Item in diagonal direction.



*Students' levels of ability in the factor of Recognise rotation*

Table 4.26 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Recognise rotation". Students of Level 1 had low performance in all of the items of this factor. Specifically, less than 50% of the students in Level 1 selected the correct response in these items, which were 9% for item C4, 20% for item C5, 20% for item C6, and 36% for item C7. The most common type of error in items C4 and C6 was to select the erroneous response that showed a reflection (25% and 30%, respectively), and in item C5 to select the response that showed a translation (25%). In item C7, the most common type of error was to select the erroneous response that showed the enlargement of the figure (23%).

Students of Level 2 also had low performance in all the items of this factor. In item C4, recognising a rotation in vertical direction, a percentage of 29% selected the correct response, and a percentage of 25% selected the erroneous response of a combination of rotation and reflection. In item C5, recognising a rotation in horizontal direction, a percentage of 15% of the students selected the correct response, whereas a quarter (25%) selected the erroneous response showing a reflection, and another quarter (26%) selected the erroneous response showing a translation. In item C6, one fifth of the students selected the correct response (20%), whereas a percentage of 24% selected the erroneous response that showed a combination of a reflection and translation, and a slightly smaller percentage (18%) selected the erroneous response that showed a translation. As for item C7, half of the students of Level 2 selected the correct response (50%), whereas 19% of the students selected the erroneous response that showed a reflection.

Students of Level 3 had low performance in items C4 and C5, with 43% and 35% selecting the correct responses, respectively. The most common erroneous response for C4 was, as in Level 2, the combination of rotation and reflection (27%), and for item C5 the reflection (29%) and the translation (21%) responses. Students of Level 3 had average performance in item C6, with almost 53% of the students selecting the correct response, and 22% selecting the erroneous response that showed a reflection. The students of this level also had average performance in item C7, where 62% of the students selected the correct response, and the most common erroneous response was to select the reflection image (14%). Students of Level 4 had high performance in all the items of this factor. The percentages of correct responses were 71% for item C4, 72% for item C5, 87% for item C6, and 78% for item C7. The most common errors were again the same as for Level 3, but in lower percentages.

Table 4.26

*Characteristics of the Four Classes in Factor Ro2: Recognise Rotation*

| Performance Levels | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | | | C4, C5, C6, C7 |
| Average Performance ($.49 \leq M < .67$) | | C7 | C6, C7 | |
| Low Performance ($M < .49$) | C4, C5, C6, C7 | C4, C5, C6 | C4, C5 | |

*Note*. C4 = Item in vertical direction, C5 = Item in horizontal direction, C6 = Item in diagonal direction, C7 = Item with unspecified direction.



*Students' levels of ability in the factor of Identify parameters in rotation*

Table 4.27 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Identify parameters" of a rotation. Students of Level 1 had low performance in all of the items of this factor. Specifically, in item C8, a percentage of 17% of the students of Level 1 responded correctly, and an equal percentage responded with the correct angle of rotation, but with erroneous centre of rotation. However, more than half of the students were unable to give a correct or partially correct response. In item C9, only 2% responded correctly and 36% were able to find the correct angle of rotation, but not the correct centre of rotation. Again more than half of the students of Level 1 (66%) were unable to give a correct or partially correct response to this item. In item C10, a percentage of 33% of the students of Level 1 responded correctly, and 19% were able to find the correct angle of rotation, but not the correct centre of rotation. Finally, in item C11, only 2% of the students of Level 1 responded correctly, and 30% were able to find the correct angle of rotation, but not the correct centre of rotation. As in items C8 and C9, again more than half of the students of Level 1 (67%) were unable to respond partially correct to item C11.

Students of Level 2 also had low performance in all the items of this factor. Specifically, in item C8, identifying the parameters of a rotation in vertical direction, a percentage of 10% of the students of Level 2 responded correctly, with 16% being able to respond correctly only to the angle of rotation (16%), and the majority of the students being unable to respond to this item (62%). In item C9, identifying the parameters of a rotation in horizontal direction, only 3% of the students responded correctly, and 31% of the students were able to find the correct angle of rotation, but not the correct centre of rotation. Again the majority of the students at Level 1 (66%) were unable to give a correct or partially correct response to this item. In item C10, a percentage of 11% of students at Level 1 responded correctly, and 25% were able to find the correct angle of rotation, but not the correct centre of rotation. Finally, in item C11, only 3% of students of Level 2 responded correctly, and 23% were able to find the correct angle of rotation, but not the correct centre of rotation. As in Level 1, again more than half of the students at Level 2 (67%) where unable to give a correct or partially correct response to this item.

Students of Level 3 had high performance in items C8 and C10, with percentages of 42% and 53% of the students responding correctly, respectively. In both items, 33% and 27%, respectively, of the students of this level were able to find the correct angle of rotation, but not the correct centre of rotation. They also had average performance in item C9, where only 14% of the students at Level 3 responded correctly, but a larger percentage of 46% gave a response that was partially correct regarding the angle of rotation. Students of Level 3 had low performance in item C11, identifying the parameters of rotation with overlapping image, where only 2% of the students responded correctly. A high percentage of 41% was at least able to find the correct angle of rotation, but did not place the centre of rotation correctly.

Students of Level 4 had high performance in the items C8 and C10 of this factor, with percentages of correct responses being 70%, and 72%, respectively. In the items C9 and C11, the students had average performance. Specifically, a percentage of 36% of the students of Level 4 were able to respond correctly to item C9, whereas 39% of the students at this level were only able to find the correct angle of rotation, and not the centre of rotation. For item C11, a percentage of 15% of the students were able to correctly find both the centre and angle of rotation, whereas almost half of the students (48%) could only find the correct angle of rotation.

Table 4.27

*Characteristics of the Four Classes in Factor Ro3: Identify Parameters of Rotation*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | | C8, C10 | C8, C10 |
| Average Performance ($.49 \leq M < .67$) | | | C9 | C9, C11 |
| Low Performance ($M < .49$) | C8, C9, C10, C11 | C8, C9, C10, C11 | C11 | |

*Note*. C8 = Item in vertical direction, C9 = Item in horizontal direction, C10 = Item in diagonal direction, C11 = Item with overlapping image in horizontal direction.



*Students' levels of ability in the factor of Construct image in rotation*

Table 4.28 presents the overall performance of the subjects in the four levels of ability in the items of the factor "Construct image" of a rotation. Students of Level 1, Level 2, and Level 3 had low performance in all the items of this factor. The only thing that slightly differentiates the students at these three levels is a minor but stable increase over the three levels in the percentages of students that are able to find the correct position of the image, but with false orientation in the case of the items C12 and C14, and the ability to find the correct image but with false distance from the centre of rotation in the case of item C13, which was the rotation item with horizontal direction. In the case of item C15, a small percentage of the students of Level 3 were able to find the correct image, but in the wrong direction. Regarding item C16, almost all of the students in the three levels were unable to give a correct or partially correct response, except for 8% of the students of Level 3, who were able to construct the image but in false direction, and 5% of the students of Level 3, who were able to construct the image, but with false orientation.

Students of Level 4 had average performance in all the items of this factor, except for item C16, which was constructing the image of a complex figure, where performance was low. Students of Level 4 were able to find the image in item C12 (29%), which was constructing the image of a rotation in vertical direction, sometimes with some errors in the distance from the centre of rotation (26%), or towards the wrong direction (24%). In item C13, constructing the image of a rotation in horizontal direction, a percentage of 35% of the students of this level were able to construct the correct image, whereas a high percentage were able to construct the image with false distance from the centre of rotation (33%), or towards the wrong direction (15%). In item C14, constructing the image of a rotation with diagonal direction, the majority of the students were able to construct the image correctly (37%), or with some errors regarding the distance from the centre of rotation (24%), or the orientation of the image (23%). In item C15, a percentage of 27% of the students of Level 4 were able to construct the image correctly, or with some errors regarding the distance from the centre of rotation (36%). Finally, in item C16, where the students of Level 4 had low performance, only 19% of the students were able to construct the image correctly, and 16% of the students were able to construct the image on false direction.

Table 4.28

*Characteristics of the Four Classes in Factor Ro4: Construct Image of Rotation*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | | | |
| Average Performance ($.49 \leq M < .67$) | | | | C12, C13, C14, C15 |
| Low Performance ($M < .49$) | C12, C13, C14, C15, C16 | C12, C13, C14, C15, C16 | C12, C13, C14, C15, C16 | C16 |

*Note*. C12 = Item in vertical direction, C13 = Item in horizontal direction, C14 = Item in diagonal direction, C15 = Item with overlapping image in horizontal direction, C16 = Item with complex figure in horizontal direction.

*Description of students' conceptions and strategies in the four levels of abilities*

In order to provide an insight into students' reasoning in transformational geometry concepts, the qualitative results of the clinical interviews were analysed to explore their conceptions of geometric transformations and their strategies for approaching the tasks of the transformational geometry test. Specifically, the conceptions, strategies, and typical errors of the students at the four levels of abilities in transformational geometry concepts are described in detail in this section, using appropriate vignettes or comments from the interviews of the selected students.

*Description of students' conceptions and strategies in translation ability*

Table 4.29 summarises the characteristics of the students in the four levels of abilities in the four factors of translation: Recognise image, Recognise translation, Identify parameters, and Construct image. The description of the four levels was based on the responses of the students in the transformational geometry test and the tasks of the clinical interviews. As shown in Table 4.29, students who are at Level 1 are unable to complete the tasks of translation, as their attention is focused only on one property and/or one parameter of translation, mainly direction. The students who are at Level 2 still focus in the parameter of direction, but they begin to understand the property of translation to conserve orientation. In Level 3, the students begin to understand the interaction of two parameters in translation and they are able to identify them in vertical and horizontal translation, but they are not able to apply them at the same time. Students of Level 4 are able to recognise the properties of translation to conserve the shape, size, and orientation of the pre-image, and to identify both the parameters of direction and distance of translation. Moreover, they are able to apply them in order to construct the image in any given parameters and for any configuration.

Table 4.29

*Description of Students' Four Levels of Abilities in the Factors of Translation*

| Type of task | Level 1 | Level 2 | Level 3 | Level 4 |
|---|---|---|---|---|
| 1. Recognise image | Use only distance as criterion | Use mainly distance as criterion, and direction when in cognitive conflict | Use mainly direction as criterion, and distance when in cognitive conflict | Recognise all images using both direction and distance parameters |
| 2. Recognise translation | Use only the shape of the pre-image as criterion | Use orientation of the image as criterion | Compare the orientation of the images | Recognise translation based on the properties of geometric transformations |
| 3. Identify parameters | Find the correct direction for horizontal translation without overlapping | Find the correct direction for vertical and horizontal translation without overlapping | Identify both parameters of direction and distance for vertical and horizontal translation without overlapping<br>Find the correct direction for overlapping translation | Identify both parameters of direction and distance in all translation examples |
| 4. Construct image | Construct overlapping image with correct direction and correct orientation | Construct all images with correct direction and correct orientation | Construct all images with correct direction and correct orientation | Construct all images with all parameters correct and all properties applied |

Level 1

Regarding the factor "Recognise image" in translation, the students of Level 1 first focused on the process of counting from the pre-image to all the alternative images, in order to find the one that had the distance that was stated in the instructions (see Figure 4.5). This implies that the students of this level may conceive translation as a process of counting. However, even when their counting approaches failed them, many students did not think of taking into consideration the parameter of direction. Even when they were counting the unit squares, the students often made mistakes, sometimes because they counted the squares of the area between the two images (see Figure 4.6), instead of the distance, and sometimes because they would regard the square units that were overshadowed by the images as “halves”, which led them to a false sum of unit squares.

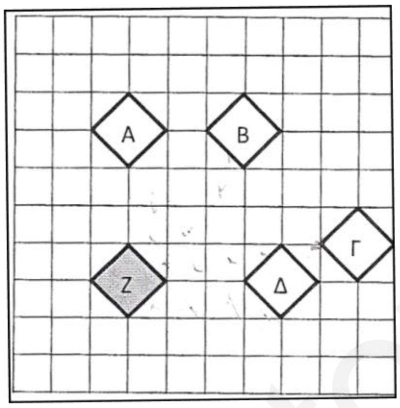



*Figure 4.5*. Level 1 student strategy for finding image by counting towards all direction in translation.

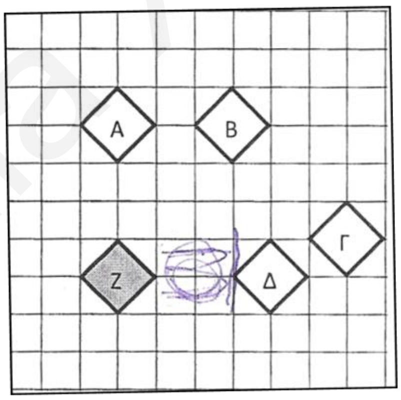



*Figure 4.6*. Level 1 student strategy for finding image by measuring the area between shapes in translation.

In the factor of “Recognise translation”, students of Level 1 were trying to find the images that, in their words, were “the same” or “did not have many differences”. However, the expression of “same” often had different and/or multiple meanings for the students. Moreover, they did not have the appropriate knowledge or vocabulary to describe that the images needed to have the same shape, size, and orientation. For some students, it was more important that the shapes were next to the other, as in one image being to the right-hand side of the other image. That served as a significant criterion for translation, which often misled these students to select the erroneous response of reflection. Two of the 10 students referred to counting the unit squares between the shapes to make sure that it was a translation, while two other students referred to counting the squares of the area or the lengths of the sides of the images to make sure that they were “the same”. Even though some students did make some informal reference to orientation, their expression refers to a prototypical way that a figure should be oriented. Some students referred to this as the “straight way”, sometimes as opposed to reflection which was the “opposite way”.

In the factor of “Identify parameters”, students of Level 1 began to count units for distance from one shape to the other, starting from a random point of the pre-image and ending to another random point of the image. This approach resulted to erroneous responses regarding the distance between the pre-image and the image. One reason for this is that the students of this level conceived the figures as holistic objects, and could not focus on their parts in order to match them. Moreover, they sometimes confused the properties of the pre-image to the properties of translation. For example, two students confused the distance between the two shapes with the concept of area, and they measured two rows of unit squares that were between the pre-image and image, thus doubling the number of the actual answer. It is possible that the students may had visualised the pre-image moving and covering the area during its motion, even though none of the students explicitly described this procedure.

Another student confused the characteristics of the shape to the parameters of translation. Specifically, the student assumed that, because the side of the rhombus where he or she started to count unit squares from was diagonally oriented in reference to the figure, that the direction of the translation was also diagonal. Another student assumed that the direction of the translation would be to the right-hand side, because the pre-image and the image were placed on the right-hand side of the square figure that represented the plane. Nevertheless, even though the students at this level made many errors by focusing on irrelevant information, such as the shape characteristics and the area of or between the shapes, the common approach at this level was to see the two shapes (the pre-image and the image) as two objects of a drawing, and count the visible unit squares between them, without realising that the beginning and end point should be corresponding.

Regarding the factor of “Construct image”, students of Level 1 were not able to construct the correct image of a translation to any direction in the correct distance by applying the properties of translation. The students of this level generally did not seem to use any visualisation strategies in order to construct the image of a translation. Instead, their main strategy was to count the number of unit squares given in the instruction, starting from a random part of the shape. Even though this strategy may appear to be analytic in some way, students of Level 1 were deeply influenced by the characteristics of the figure as a drawing, and could not realise the conventions of space that were represented in the figure. This was evident in some cases where they got confused. For example, when they had to count a unit square that was partially overshadowed by the pre-image, they either skipped it in counting, because it was not considered as a whole unit and it could not be counted, or constructed the image starting from the next unit square, to ensure that the number of unit squares was not smaller than the number given in the instructions. These strategies often resulted to erroneous measure of the distance between the pre-image and the image. One student of Level 1 characteristically explained that she or he needed to clearly see the number of unit squares between the two shapes, in order to know that the answer was correct.

Students' holistic approach in Level 1 was also evident in the way they conceived the figure of the pre-image. Almost every student of Level 1 expressed the importance of putting effort in making the triangle the same, with "the same" being a key phrase that was repeated several times. This effort was evident in two students' persistence to use a ruler in order to make sure that the shape is exactly the same, and in one student's approach of counting the area of the pre-image to confirm that he or she had constructed "the same shape". One student characteristically mentioned trying to make the figure "look the same, or at least as much possible the same".



Level 2

Regarding the factor of "Recognise image", the first thing that students of Level 2 thought of doing was counting unit squares, regardless of direction. Specifically, they tried out all the possibilities, in order to select the image that had the number of unit square distance from the pre-image that was the same with the number given in the instructions. Some of them did not even take into consideration the parameter of direction, if they found an answer that was consistent with the number of unit square distance parameter. Some of them were only able to realise their error when they were prompted to verify their answer. One student held the conception that the pre-image and image needed to be "on the same line", suggesting that translation could only be vertical or horizontal, and immediately rejected the diagonal options. However, this student could not decide between the other two options, because he or she counted the unit squares of the area between the pre-image and the alternative images, including the "half overshadowed boxes", thus measuring a distance that was more than the one that was suggested in the instructions. Therefore, the student decided that none of the responses was correct.

Several students began counting from a specific vertex of the pre-image. This approach allowed four of them to find the correct response, when they stopped counting at the corresponding vertex of the image. However, two others saw the shapes and units in a more holistic way. Hence, when they stopped measuring the distance, they rejected the correct response that was drawn in the units before the end of the distance, for which the corresponding vertex matched, because it visually appeared to have only two unit squares between the two whole shapes. Thus, they selected an erroneous response, by matching a non-corresponding vertex.

In the factor of "Recognise translation", similar to students at Level 1, the students' of this level basic criterion was that the images should have "the same shape". However, some of them often used the phrase "they are different shapes", in order to refer to the orientation, while some others mentioned that the shapes should be "on the same side" or "looking towards the same way". None of them could use the formal terms of orientation or direction. Many students also felt the need to apply the procedure of counting the unit squares between the pre-image and the image, in order to reassure themselves that the example was showing a translation. One particular student was very confused between the example of translation and rotation, because, as he or she said, "translation means: to another place", and the change of place was more obvious in the example of rotation. For two other students, the option of rotation was rejected, because they had the conception that in translation "the shapes cannot be touching".

What differentiated the students of this level to those of the previous level was that some students began to use visualisation for motion strategies. One student described how he or she mentally lifted and moved the image over the pre-image, in order to confirm that the shapes were the same. Two more students said that the image had to be "the opposite" of the pre-image, in order to have an evident change. Some students only had the criterion of position in mind, and could not select a response, because there was an evident change of place in every example, regardless of the orientation. These misconceptions are what caused difficulties to the students of this level to recognise the option which was showing a translation.

Regarding the factor of “Identify parameters” in translation, almost all of the students of this level who were interviewed performed exactly the same strategy, of counting unit squares from the top of the pre-image to the bottom of the image. Hence, there were fewer errors regarding direction. Some of them explicitly said that it was obvious that there are only three unit squares between the shapes, while others tried to make sure by counting the units in both columns of squares that were between the images, in order to confirm that the number of units was the same. When asked, some of the students had a strong intuition that the distance between the two images would always be the same, regardless of where one begins counting units. However, none of them attempted to verify this position during the interviews.

In the factor of “Construct image” in translation, some of students of Level 2 began to count unit squares from a specific side of the pre-image. However, this side was most frequently the base of the right triangle, and many students considered the base as being the whole row of squares above the segment. This sometimes created problems, because the students of Level 2, as in Level 1, got confused by units that were overshadowed by the pre-image. Therefore, in many cases, they would count the overshadowed square as half. However, even though students at this level began to realise that a shape can be analysed into smaller parts, i.e., segments, and began to count from those, they still did not realise that the beginning and ending point of counting units should be corresponding. Hence, most students began counting from the end of the base segment, up to the distance measure of translation given in the instructions, and then began drawing the beginning of the segment, following the same direction as when counting. This resulted again to a false measure of distance between the two images.

In several cases, there were students of this level who would find the measure of distance correctly. However, these students seemed to have more problems with the orientation of the figure, since they drew a reflection of the image. One possible reason for this was that it might have been an attempt to match the point of the segment where they began counting from, to the corresponding point of the image. However, they did not seem to realise that the other points did not match, and were not in the same distance. When asked about the different distances between corresponding points of the images, some of the students answered that it did not matter, as long as the counting was correct and the image was the same as the pre-image, disregarding the role of orientation. Similar to students of Level 1, students of Level 2 also had a strong intuition to count the area or measure the sides with a ruler, in order to make sure that the triangle was “the same”.

Level 3

Students of Level 3 mostly began their reasoning for approaching the "Recognise image" of translation task, by referring to the direction. This differentiates this group of students from the other two groups, who first began to count towards every option, without considering direction. Hence, the students of Level 3 first excluded the options that were in other directions than the one given in the instructions. After that, they counted the unit squares to find the distance and confirm the answer. However, the students of this level had some problems with counting the correct number of unit squares for distance. Specifically, half of the students of Level 3 that were interviewed were uncertain about the existence of a correct option, because they would often find an option which agreed with the direction of the response, but not with the distance. These students mostly began their counting by focusing on a side of the pre-image, and either (i) counted to a non-corresponding side of one option, thus finding a smaller distance, or (ii) counted to the corresponding side of another image, including the halves of the overshadowed unit squares in their sum, thus finding a larger distance. While for four of these students direction was a stronger criterion than distance, and led them to choose the correct response, for one student, distance seemed to be a stronger criterion, and this student chose the response where the whole amount of unit squares between the two images was clearly visible, even though the response was slightly in the wrong direction (see Figure 4.7).

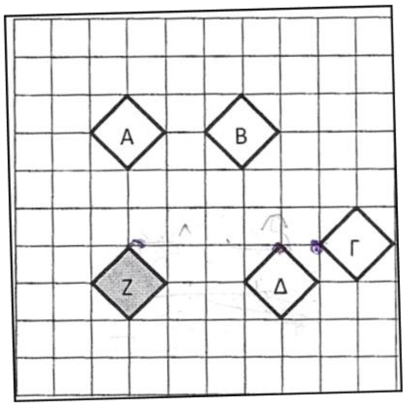



*Figure 4.7*. Level 3 student strategy for finding image in translation.

In the factor of "Recognise translation", students of Level 3 did refer to the need to have the same shape, but not with so much emphasis as the students of the preceding levels. At this level, they seemed to be more concerned with orientation, and they reflected on the matter that the sides of the pre-image and of the image need to be “on the same side”. One student also tried to verify his or her answer by counting the unit squares to confirm that every corresponding side had the same number of unit square distance between them. Moreover, even though none of the students used the word “orientation”, there were some attempts by some students to refer to the concept with informal terms, such as the word “posture”. Students of Level 3 also began to exclude options because they were able to recognise the other geometric transformations, mainly based on the concept of orientation. Therefore, options that depicted the image as “opposite” or “turned” were excluded. This seems to be a significant reason explaining why the percentage of students that confused translation with reflection decreased substantially from Level 2 to Level 3. Moreover, three out of 10 students explicitly explained that in translation, you only change the position of an image, without changing the shape, and without turning it around or over.

Regarding the factor of “Identifying parameters” in translation, students of Level 3 tried to focus on one side of the pre-image to start counting to the image. However, four of them failed to count up to the corresponding side, and stopped counting at the first line they met in their process of counting. This difficulty might have been caused by the properties of the shape, since a rhombus has four equal sides and two lines of symmetry, a fact which may have made it more difficult for these students to distinguish the corresponding side. However, these students did not consider the fact that their starting point was between the boundaries of a unit square, and counted that square as a whole unit for the distance. This suggests that, even though the students of this level may have accepted some of the configurations regarding the representation of the geometric space, and realised that the squares are not objects, they still had problems with understanding the definition of translation as one-to-one mapping of points. This was also evident in the majority of the responses of the students of Level 3 during the interview, who explicitly explained that they counted from one vertex of the pre-image to the closest vertex of the image. Of course, the closest vertex was not the corresponding one. Two of the students mentioned that they realised that the distance between the two images depends on the starting and ending point of counting, but the best way is to count from a vertex of the pre-image to the closest one of the image. It is also noted that students of Level 3 did not have any problems to understand the direction of the translation.

In contrast to the approaches of students of Level 1 and Level 2 in the items of the factor of "Constructing image", students of Level 3 began to consider the different possibilities of starting to count from different parts of the pre-image, and how this might affect their responses. Some began to focus on specific segments in order to perform the translation and construct the image, while few others began to focus on specific points, i.e., vertices. This type of strategy is described in literature as analytic (Boulter & Kirby, 1994). Hence, after counting the correct number of unit squares for distance from one segment or vertex, the students of this level described how they first drew the corresponding segment or vertex, and then reconstructed the shape by counting units for the lengths of the sides of the shape. It is noted that, apart from one student, all students of Level 3 that were interviewed were not confused by the unit squares that were overshadowed by the images. It is assumed that they have better understanding of the geometrical space, and that they are more flexible into adjusting to the configurations of the representation.

A certain student at this level described the use of both an analytic and holistic strategy at the same time, by referring to the process of counting the unit squares from one vertex to the other, and then visualising the pre-image moving from one position to the other. However, this approach was not as convenient in the item of overlapping figures, since even though the student applied the analytic strategy correctly and translated the vertex, his or her visual reasoning and conceptions about translation were contradictory. Hence, the student decided that a shape cannot be overlapping with the other, and moved the whole image one more unit to the same direction as given in the instructions, to avoid the overlap. This was characteristic of the majority of the responses of Level 3 students, even when they did not explicitly refer to the use of visual strategies during the interview. Even though they could analytically find the correct position for the image, they had strong conceptions that it had to be placed elsewhere, to avoid overlapping (see Figure 4.8).

Another important difference of the students at Level 3, in contrast to the students at the other levels, is their vocabulary. The vocabulary that students at Level 3 used was quite different from the vocabulary of students at Level 2, since they refer to the specific parts of the images and the procedure, by using more formal mathematical terms, such as base, measure, equal sides, dimensions, and angles.

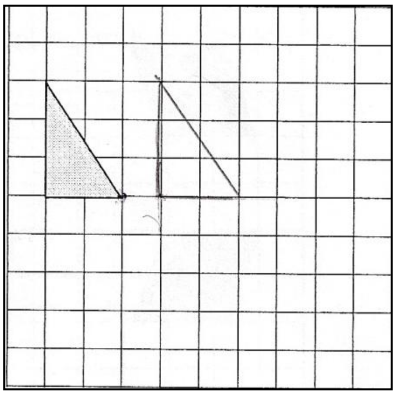


*Figure 4.8*. Level 3 student strategy for overlapping figure construction in translation.

Level 4

For the first factor, in order to recognise the image of a translation, the most common approach for students of Level 4 was first to exclude the options that were in different directions than the one stated in the instructions, and then count the unit squares to measure the distance from the pre-image towards the given direction for verification. What differentiated the students of this level from the students of the previous levels was that they started to count from a vertex of the pre-image and only stopped counting at the corresponding vertex of the image. Hence, they measured the distance correctly and were able to verify their answer by using two parameters (direction and distance), and selected the correct option.

Regarding the factor of “Recognise translation”, students of Level 4, as the students of Level 3, were able to exclude options by recognising that they represented other geometric transformations, i.e., reflection and rotation. This explains why more than 90% of the students of Level 4 were able to recognise the translation example correctly in all of the items. Moreover, at this level, the students had more advanced vocabulary about transformations, and instead of using expressions like “opposite” or “turned”, they used the terms “reflection” and rotation” more often, especially the older students. However, despite their knowledge of the names of the transformations though, they still did not use the appropriate word for orientation. They knew that the shape must not be inverted as in reflection, or rotated as in rotation; however they could only explain that in translation the image has to be “the same” as the pre-image. After a lot of thinking, one student reluctantly referred to it as “trend”, and another as “same side/other side”

In “Identifying parameters” of translation, all of the students started to count from one vertex of the pre-image, usually the one on the top of the shape, up to the corresponding vertex of the image, following the correct direction. They were confident that the answer would be the same regardless of the starting point of counting, as long as one starts and ends in corresponding vertices, and they did not require verifying their answer.

Regarding the factor of "Construct image" in translation, the students of this level followed an analytic strategy, similar to the strategy that was described by the students of Level 3. What differentiated students of Level 4 from students of Level 3 was that all students of Level 4 began counting from points, i.e., vertices of the pre-image, instead of segments, i.e., sides of the pre-image. After that, they constructed the image from the translated point, by measuring the sides in unit squares. One student described how he or she translated each vertex point to the same distance, and then connected the image points to construct the image of the triangle, without measuring the length of its sides. What is very characteristic at this level is that all of the students who were interviewed knew that every point of the image must have the same distance from its corresponding point in the pre-image, and commented that you can follow the same procedure from any point of the pre-image figure to find its corresponding. Moreover, some of these students mentioned that, even though they measured the distance from one vertex to its image and constructed the image based on the lengths and angles of the image, they would measure the distance starting from another vertex or even from two more vertices for verification. It is also worth noting that some of the students of this level did not refer to a counting procedure of units one by one. Instead, they saw the distance as a sum of four units.

One student mentioned that, after translating a vertex, he or she visualised the whole image, based on the image of the vertex, before constructing it based on the lengths of sides of the pre-image. Moreover, the visualisation abilities and the conceptions of geometric space and translation allowed the majority of the students of this level to visualise the overlap of the image and the pre-image. One student actually described a mental image of the translation pre-image as "transparent". Specifically, the students at Level 4 were confident that every part of the image must have the same distance from its corresponding part in the pre-image, even if that meant overlapping with other parts of the pre-image. There was only one minor exception of the primary school students of Grade 4, who were reluctant to accept the fact that the images could overlap, and were unable to construct the correct image.

*Description of students' conceptions and strategies in reflection ability*

Table 4.30 summarises characteristics of the students in the four levels of abilities in the four factors of reflection: Recognise image, Recognise reflection, Identify parameters, and Construct image. The description of the four levels was based on the responses of the students in the tasks of the clinical interviews. As shown in Table 4.30, the students who are at Level 1 focus on the shape, and are only able to recognise and identify the parameters of refection in vertical line, which is mostly performed visually, and reasoning is mostly based on prototypical images. In Level 2, students' ability to recognise the image and the transformation improves, in comparison to the previous level, because the students begin to focus on the orientation of the image, and they are also able to identify the parameters both in vertical and in horizontal line. Students of Level 3 are able to recognise the image of a reflection and a reflection in a horizontal and a vertical line, and also to identify the parameters in such cases of reflection. However, they have difficulties with diagonal reflection and they are only able to construct images correctly in a vertical line of reflection, with or without overlapping. Finally, in Level 4, the students are able to recognise the image, recognise the reflection, and construct the image in all cases of reflection, except from diagonal. They are also able to identify the parameters of horizontal and vertical reflection, as well as to find the correct orientation of the line in a diagonal and in an overlapping reflection; however, they are not able to place the line of reflection in the correct distance from the images.

Table 4.30

*Description of Students' Four Levels of Abilities in the Factors of Reflection*

| Type of task | Level 1 | Level 2 | Level 3 | Level 4 |
|---|---|---|---|---|
| 1. Recognise image | Use only the shape of the pre-image as criterion | Use only the shape of the pre-image as criterion | Use orientation of the image as criterion<br>Recognise image in vertical and horizontal line | Recognise images using orientation of the image as criterion<br>Confuse diagonal reflection with reflection in vertical line |
| 2. Recognise reflection | Visually recognise reflection in vertical line | Use orientation of the image as criterion | Compare the orientation of the images<br>Recognise reflection in horizontal and vertical line | Recognise reflection based on the properties of geometric transformations<br>Confuse diagonal reflection with rotation |
| 3. Identify parameters | Identify a vertical line of reflection | Identify a vertical and a horizontal line of reflection | Identify a vertical and a horizontal line of reflection<br>Find the orientation of the line in diagonal reflection | Identify a vertical and a horizontal line of reflection<br>Find the orientation of the line in diagonal and overlapping reflection |
| 4. Construct image | | | Construct images in vertical line with and without overlapping<br>Construct image in horizontal and diagonal reflection with correct direction and correct distance | Construct all images with all parameters correct and all properties applied, except for diagonal reflection |

Level 1

For “Recognise image” in reflection, students of Level 1 frequently chose the image of translation, instead of reflection. This was based on their criterion that the shape had to be the same. As some of them said, even though they did notice that the “cut-off angle” of the shape was “at a different place” in every image, they were searching for the option in which the shape would be exactly the same as the pre-image. One of the interviewed students selected the image that was a reflection in a horizontal line, instead of a vertical line, which was the example given to the students during the interview. Only one of the interviewed students of Level 1 had selected the correct response, based on the criterion that if you join the two images together, you will get a single whole figure.

In the factor of “Recognise reflection”, most of the students of Level 1 selected the correct response. However, they did not seem to have a conceptual understanding of reflection. Many of their approaches seemed to be based mostly on the holistic perception of the figure, and were mostly related to conceptions emerging from prototypical images of reflection in vertical line. More particularly, some students’ warrants imply strong prototypical conceptions, which in the case of reflection in vertical line allowed them to find the correct response. For example, two students suggested that the images need to be one next to the other in order to be a reflection, whereas another student selected the figure of reflection because “it can be divided”. Another student argued that the selected (correct) figure was a reflection, because the images must have a column of squares between them.

Another kind of approach in Level 1 included the mental folding of the whole figure. Specifically, two students explained that if they folded the figure in their mind, they would understand that the images match. One of them continued to say that she or he had tried the folding approach for all the examples and realised that the shapes did not match in the other options. A similar approach was related to the joining of the images together, as one. It is possible that what these students had in mind was the mental folding that is usually taught in primary school mathematics for teaching symmetry; however, they did not completely understand how this method works in order to match the two shapes, and they had a global knowledge of “joining shapes together” in some way. Two students seemed to have the conception that, in order to have a reflection, one must be able to join the two images into a single figure. Another student was sure that there was more than one correct option (the student considered the example of translation as a reflection), based on the criterion that the two shapes can be joined. Specifically, this student eventually selected the figure of reflection, because "it looked more correct", which is probably also based on prototypical images about reflection.

Regarding the factor of "Identifying parameters" in reflection, students of Level 1 had a variety of approaches; however, none of them seemed to be successful. What was characteristic at this level was that the students had a general conception of reflection as something that divides an object into halves. It was also characteristic that they primarily used a holistic visual perception of the images. For example, many students had a strong conception that a reflection line must be drawn over the shapes and split them into two halves, which led them to erroneous responses (see Figure 4.9). Moreover, two of these students also tried to verify their answer by visualising the folding of the image in the line they had drawn, to see if the two shapes, the pre-image and the image, matched. This misconception probably emerged from the teaching approaches for the property of symmetry in shapes and figures.

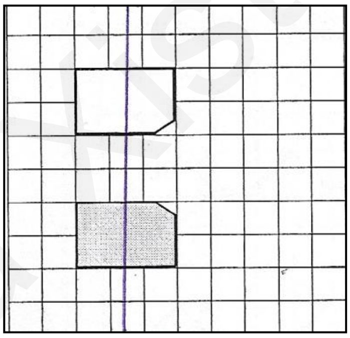


*Figure 4.9.* Level 1 student response for identifying parameters in reflection.

Four students drew the horizontal line somewhere between the shapes. However, two of them explained that what was important to them was to have one shape on one side of the line and the other shape on the other side of the line, whereas two other students said that they just needed to find the middle of "the box", meaning by that the square image that represented the geometrical plane. In addition to this, one more student explained that, in order to find the answer, she or he needed to "halve the box". In the case of finding the diagonal line of reflection, this student used the same reasoning, suggesting that "to split the box, we must make two same triangles".

One student at this level had the conception that, since the pre-image and the image were vertically placed one below the other, the line of reflection also had to be vertical, and that the same applied for diagonal placement. Another student explained it was not possible to find an answer, because the figure was not showing a reflection, since the two shapes cannot be united to form a single figure, i.e., as two right triangles can be united into a rectangle. In the case of finding the diagonal line of reflection, the student thought that in order to have a reflection, the shapes should have been attached to each other, and since they were not, the student could only draw the line of symmetry for each of the two figures. Moreover, this student had the conception that the image had been rotated and translated to a different place, and this made it more difficult to find the line of reflection.

The only attempt for a more analytic approach was performed by one student, who decided to count the unit squares between the pre-image and the image, and divide them to find the place of the line. However, this student was counting the area between the two images, instead of the distance.

In the factor of "Construct image" in reflection, students of Level 1 had some general knowledge about same place and same shape; however they did not really understand what it meant. What seemed to be characteristic at this level was that, even though in some cases the students appeared to use analytic strategies of counting, or in other cases used visual strategies with mental images, both cases seem to be driven by a concrete conception of the elements that represent the reflection, i.e., the pre-image, the line of reflection, and the unit squares. Particularly, many of the students tried to count unit squares for the distance, starting from the hypotenuse of the triangle, "the big line" as they referred to it, and counted the area of unit squares up to the line of reflection, in order to count the units for the same area on the other side of the line and construct the image of the triangle as a whole. Some other students would begin counting the distance from the pre-image to the line, but this was still problematic, since they started from a random point of the triangle, and considered that their answer would be the distance for the whole triangle; they would then try to construct the image of the triangle, with emphasis on looking the same, but without matching the corresponding parts for which they measured the distance. This suggests that they have a holistic conception of the triangle, which inhibits them from performing a reflection with accuracy. Measuring distance was more complicated in the case of the diagonal line of reflection, since students not only started from a random point of the triangle, but they also counted towards a random direction. These approaches resulted to failure for many of the students to construct the image within the correct distance from the line, but also to construct an image that maintained the properties of the pre-image, i.e., a right triangle.

The rest of the students seemed to rely strongly on holistic approaches for finding the image, but these were not always successful methods. Even though the students at this level tried to visualise the answer, based on holistic approaches again, they were still not able to find the correct image, because their mental images were not really facilitating them. For example, two students described how they imagined folding the paper either to “see the image” or to confirm that the imagined image matched the pre-image. None of these students was able to construct at least a partially correct response with this approach.

Another visual strategy that was described by two students was the conception that the image had to be on the “opposite side”, meaning after the line. According to one of these students, if the line was straight (meaning vertical), the triangle needed to be straight, and if the line was slanted, the image needed to be slanted as well. Note that this could be an indication of intuitive knowledge, since this rule applies for the special occasion of line segments that are parallel to the line of reflection. Nevertheless, whether the image “seemed to be in the opposite” was a strong criterion for students at this level, and the fact that the shape was not exactly the same was not a problem, as long as their criterion was met. In such cases, the diagonal line created more problems for some students who simply wanted to construct the image “on the opposite side”, because there was not enough space to draw it. This was a common problem for many students at this level, irrespective of the strategy they used. To overcome this obstacle, they decided to draw the image at the nearest possible place where there was enough space to draw the image as it should be. In other words, they behaved as if they were constructing the image for a vertical line of reflection, and changed its place to where there was enough area to keep the image within the boundaries of the figure representing the plane.

Another example of a visual approach was when a student described how he or she tried to visualise the triangle reversing, so that it would face the “other way”, as if looking in a mirror. Similarly, another student described imagining a mirror on the line of reflection to help him or her find the image. It appears that this student may have a holistic and concrete conception about the line of reflection. Both of these students admitted though that this was only convenient for finding how the image would look like, and not where it should be placed. However, one of the two students was not able to find the correct orientation of the image by following this way, and neither of them was able to find the correct distance, even for the case of the vertical line of symmetry. Another student described imagining the triangle lifting up from the paper, turning over the line, and becoming the image. This strategy was successful for this student only in the vertical line of reflection.

The approaches described above can explain why 91% of the students who were classified in Level 1 were unable to give a partially correct response for the diagonal case of constructing image in reflection.

Level 2

For “Recognise image” in reflection, students of Level 2, similarly to students of Level 1, also chose the image of translation instead of reflection, but not as often as the students of Level 1. Specifically, approximately half of the students selected the image of translation, and they explained that they were searching for a shape that was the same as the pre-image. The students who selected the correct response used holistic approaches, apart from one student whose reasoning was that the “cut-off angle” must be on the right-hand side. Two students described that they had visualised the shape turning over to the other side, and one explained that she or he visualised the shapes in a folding paper, in order to see if they matched. Another student tried to find the opposite shape, because when “you look in the mirror, your left-hand side becomes your right-hand side”.

In the factor of “Recognise reflection”, most of the students of Level 2 selected the correct response. What makes these students think differently from students of Level 1 was that students of Level 2 focused on the shapes and their orientation, in order to decide whether they show a reflection or not. Specifically, half of the students that were interviewed had selected the correct response, based on the conception that the one shape must be facing the left-hand side and the other one must be facing the right-hand side. Similarly, two more students explained that their answers were correct, because the two shapes were opposite, and one of them added that they were also on the same row. Another student described a holistic strategy, imagining the whole figure of the pre-image lifting above the figure of the plane, and turned over to confirm it fitted the other shape. Two students confused reflection with rotation. One student thought that in reflection the shape must be turned around, whereas the other student believed that the pre-image and image in reflection should be attached.

For “Identifying parameters”, students of Level 2 had high performance in finding the horizontal line of reflection, and low in finding the diagonal line of reflection. A very important reason for this is that most of them knew the importance of placing the line “in the middle". Even though the approaches of the students at this level were quite similar to those of students of Level 1, their conceptions facilitated them more to find the correct response.

Specifically, the basic arguments at this level were regarding the conception that the line must be in "the middle of the shapes", or in "the middle of the box" (i.e., the square figure that represented the plane). Some students counted the unit squares from the line to each image or from the line to the edges of the figure that represented the plane. In the case of the horizontal line, this approach did not create any problems, since the middle of the box was also the middle of the distance between the pre-image and the image. However, in the case of the diagonal line of reflection, the middle of the box did not coincide with the middle of the images, and this increased the errors associated with the distance of the line from the two images.

As in Level 1, the basic conceptions at this level were that the two images needed to be on different sides of the line, or that they had to be opposite to each other. The first type of conception led to errors in the diagonal line of reflection, because any line – horizontal, vertical, or diagonal – could match that criterion. Another conception was that the line could not cross over the shapes, or be touching the shapes. There were only two students at this level who had the conception that the line just needed to be between the two shapes, and that distance did not matter, as long as the line did not touch any of the shapes.

In the "Construct image" factor for reflection, the approaches of students of Level 2 were quite similar to those of Level 1. What differentiated the students of Level 2 from those of Level 1 was that they began to focus more on the shape and its motion on the plane, instead of the whole figure, as well as on the orientation of the image. However, they still visualised the pre-image as a whole object, without decomposing it into smaller parts such as sides or vertices. Students of Level 2 also made many errors in the distance, because they relied mostly on holistic strategies. However, they were still more successful than the students of Level 1 into finding the correct orientation of the image. Moreover, the students of this level appeared to have many misconceptions about reflection, as well as a variety of approaches. It should be noted that the students of Level 2 tried to apply the same approaches for both the vertical and the diagonal line of reflection.

Even though some of the interviewed students of Level 2 were able to find a correct response for the vertical line of reflection, none of them was able to find the correct response for the diagonal reflection. For example, one student described imagining the shape being lifted up, as if it was drawn on a separate piece of paper, and turned over the line to the other side. After using this holistic visual strategy, the student counted the unit squares in all of the rows between the pre-image and the line of reflection to find the position of the image in the other side of the line. In a similar way, another student imagined what the pre-image would look like if it was looking in a mirror, and drew it on the other side of the line. The student tried to apply the same approach for the diagonal line of reflection, where eventually he or she could only draw the image of the triangle in a vertical line of reflection, and placed it in the upper row, because the diagonal line was "getting in the way" of drawing the image on the same horizontal line as the pre-image.

Four out of the 10 students that were interviewed found it important to count the unit squares in order to find the area of the pre-image. One of them emphasised on the half-squares, and another on not making the image "thinner" or "fatter" than the pre-image, in order to construct the image with the same area. One of these students also had the misconception that the image must be touching the line of reflection, and failed to find the correct response. The other three students had the conception that the two shapes must be "sitting on the same horizontal line". This is probably a perceptive conception emerging from prototypical images of reflection in a vertical line. They also had the conception that "one shape must be opposite to the other", or "facing the other way", or "facing towards the line of reflection". Even though these conceptions do not suggest conceptual understanding of reflection, this approach led the students to find correct responses regarding orientation. However, it is noted that none of these students mentioned the parameter of distance. Still, two of them had placed the triangle in the correct position, whereas the third student recognised the error in the distance from the line during the interview, and admitted being too pre-occupied with making the same shape looking towards the line of reflection to think about the distance.

Some similar approaches with some more elements of understanding included the conception that the image must be the opposite of the pre-image, and that whatever goes towards the right on one side, must become going towards the left for the other side of the line, with the shapes being exactly the same, and both "sitting on the same line". This student also drew the top and bottom boundaries of the pre-image and extended them through the whole figure, in order to draw the "opposite shape" within the same boundaries on the other side of the line, suggesting that the images should have the same height (see Figure 4.10). When this strategy failed for the diagonal line of reflection, this student fell back to the holistic approaches of Level 1, and tried to imagine folding a piece of paper with the images. This only helped him/her to approximately find the direction of the image, since the orientation that the student drew was still the prototypical on a vertical line of reflection.

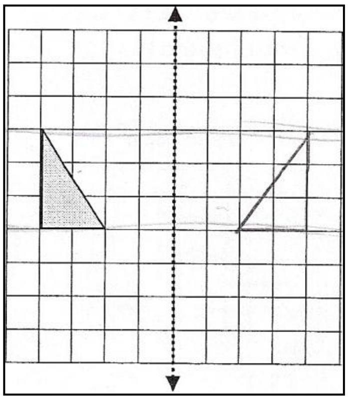


*Figure 4.10*. Level 2 student strategy for constructing image in reflection.

Nevertheless, not all students of Level 2 were able to find the correct orientation of the image. One student described having copied the shape in a way that it would have the same distances from the edges of “the box” as the pre-image had, but simply on the other side of the line. Similarly, another student had the conception that the image should be exactly the same as the pre-image, only facing the same away.

Another student described following a strategy of drawing a mental line with her or his finger, thus tracing the route that the pre-image would follow towards the line, and constructed the image on the other side of the line. Even though this student had a holistic conception and was applying a motion approach on the triangle, he or she seemed to have some knowledge, probably intuitive, that the line showing the route of this motion must be straight and perpendicular to the line of reflection.

Level 3

Even though there were still some students of Level 3 who got distracted by the “same shape” property and selected the image of translation, the students of this level were more able than those of Level 2 to “Recognise the image” of a figure in a vertical line of reflection. Specifically, most of the students of Level 3 that were interviewed, explained that they focused their attention on the “cut-off angle” and knew that it had to be on the right-hand side of the image. Then, focusing on the position of this part of the shape in the images, they excluded the other options by recognising that they were “the same” or “turned around”, probably implying the orientation of the image. In a similar way, two more students focused on a side of the pre-image, and visualised the shape turning over, to find where that side would be after the inversion. A few more students used similar holistic approaches to find the image of the given shape, either by visualising the paper folding as described earlier, or by visualising themselves drawing the opposite shape on the other side of the line.

Regarding “Recognise reflection”, most of the students of Level 3 were able to find the example of a reflection in a vertical line among other options. Based on their explanations, they seemed to be searching for the example showing the “opposite figures”. Some students tried to find the answer or verify an option by drawing a line between the shapes. What is characteristically different from the previous levels is that the students of Level 3 also began to exclude options because they were able to recognise the other transformations, based on the concept of orientation. Therefore, options that depicted the image as “moved” or “turned”, were excluded. This is a significant reason for the substantial decrease in the percentage of Level 3 students who confused translation with reflection, in comparison to the Level 2 students. However, some students were between the option of reflection and the half turn rotation, because the half turn rotation looked to them as a diagonal reflection. Even though many of the students thought that both responses were correct, they said that they selected the example of reflection because “it looked more correct”. These students were probably influenced by their prototypical images of reflection in vertical line, which probably gave them more certainty.

In the factor of “Identify parameters”, most of the students of Level 3 followed the same approach for finding the horizontal line of reflection. As they explained, they realised that the line was horizontal from the vertical placement of the shapes one below the other, or from the fact that a shape must be on each side of the line. Moreover, some of them explained that the line could not have been vertical or diagonal, because either the shapes would have to be on the same side of the line, or unequal parts of the shapes would have to be on the two sides. After that, they measured the distance between the two shapes and halved it. Some of them also focused on the positioning of the “cut-off angle” as they named it, in order to make sure that it was on the “opposite sides” of the two images.

In the case of finding the diagonal line of reflection, only half of the students were able to find the diagonal orientation of the line. The rest of the students drew either a horizontal or a vertical line, mainly based on the criterion that you must have a shape on each side of the line, even though some of them were uncertain for their response, and one of them was certain that there was no correct response. Nevertheless, some of the students that identified the orientation of the line as diagonal, were not as accurate in the distance between the two images, since many of them had the conception that the line must connect the corners of the square figure that represented the plane, which did not coincide with the line that is in the half distance between the two images. The students who managed to find the correct position of the line were the ones who followed the approach of matching two vertices of the images and halving the distance between them, which most often were the closest corresponding vertices.

In the factor of "Construct image" for reflection, students of Level 3 began to focus on parts of the shape, i.e., segments or points, which they used for following analytic strategies. They often complemented their analytic strategies with holistic strategies. For example, some students counted the unit squares from a vertex, either the top or the one that was closer to the line of reflection, then marked its position in the other side of the line, and finally either visualised the triangle flipping over to the other side, or the paper folding in the line of reflection, or simply tried to construct the "opposite" shape. One of the students explained that it was important for him or her to imagine the paper folding, only to find the orientation of the triangle's height, whether it would be to the left-hand side or the right-hand side of the base. This approach was successfully completed for both the vertical and the diagonal line of reflection, but not all of the students were able to find the correct image for the diagonal line of reflection with this approach (see Figure 4.11)

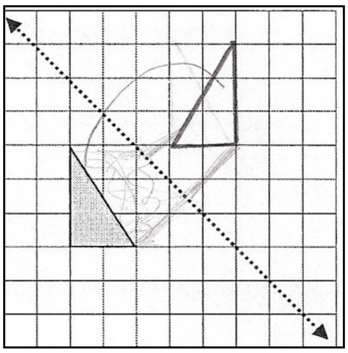


*Figure 4.11*. Level 3 student strategy for constructing image in reflection.

Most of the students at this level seemed to focus on reflecting a single side of the pre-image, instead of a vertex, and then reconstruct the shape based on its properties and lengths of the sides. For example, three students measured the distance from either the hypotenuse or the vertical side of the right triangle to the line of reflection, found its position on the other side of the line by measuring the same distance in the opposite direction, and then counted the unit squares for the other two sides of the triangle to reconstruct it based on its properties. Two of these students used the same approach to construct the image in the diagonal line, by focusing on the vertex of the right angle, and even though they reflected the vertex to the correct position, they both constructed the image of a reflection in a vertical line, starting from that point. Another student tried to count the unit squares from the hypotenuse to the line of reflection, and even though she or he was able to visualise the shape flipping over the line for the vertical line of reflection, this student was not able to visualise how the image would look like in the diagonal line of reflection, and gave up trying.

Finally, some less common strategies were expressed by two students. One student described drawing the image on the same line as the pre-image, but facing the line of reflection. This student used the same reasoning for the diagonal line of reflection, i.e., that since the top vertex of the pre-image was facing upwards to the line of reflection, then its image should be facing downwards to the line of reflection (see Figure 4.12). Also, there was one case of a fourth grader at Level 3 who used a more advanced approach: he or she measured the distance for all the vertices to construct their image, and drew the lines to connect them in order to construct the triangle. This student tried to do the same for the diagonal line, but was counting upwards instead of the direction that was perpendicular to the line of reflection. Eventually the student gave up, because the images of the vertices were incorrect, and some of them seemed to be placed within the area of the pre-image (see Figure 4.13). Eventually, the student explained that the vertices could not be connected.



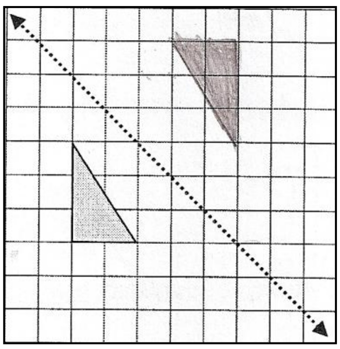


*Figure 4.12*. Level 3 student response in constructing image in diagonal reflection.

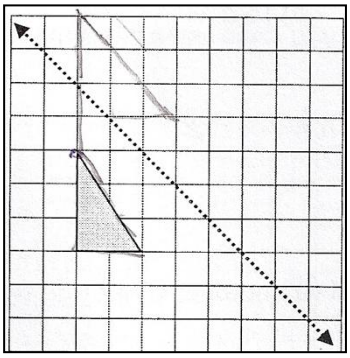


*Figure 4.13*. Level 3 student strategy for constructing image in diagonal reflection.

Level 4

For the factor of "Recognise image" in reflection, students of Level 4 had high ability to select the correct response. Moreover, they were able to explain the way that each option was related to the pre-image. They mostly approached the task by focusing on the position of the "cut-off angle", which they knew that it must be on the top right-hand side of the image. Some students described that they had visualised the pre-image flipping for verification.

Regarding the factor of "Recognise reflection", all of the students of Level 4 who were interviewed were able to find the correct response. Particularly, most of the students were able to name the geometric transformation that was represented in every option, with specific reference to the orientation of the images, even though the term "orientation" was not used by anyone. Additionally, they used more appropriate vocabulary about transformations, such as the terms "reflection" and "rotation". Only two students described that they had tried to imagine the reflection for every pre-image in every example to see whether their visual image matched the image in the figure (see Figure 4.14), whereas two of the younger students explained that they knew that the correct option was the one that would be like looking in a mirror. However, they explained that they did not require visualising a mirror to find the response.

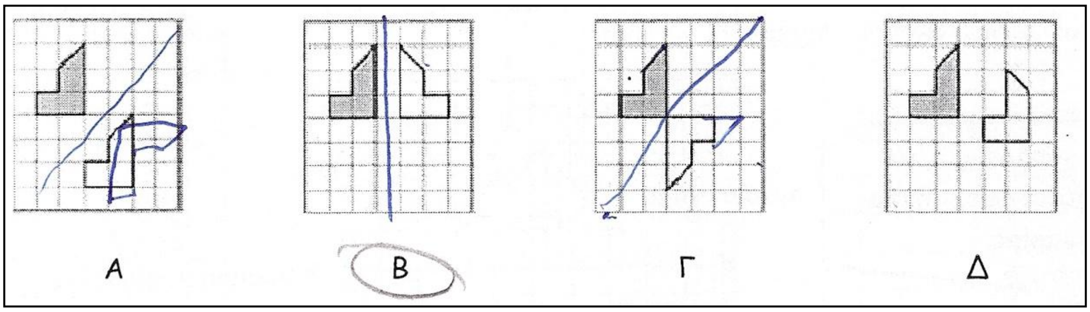



*Figure 4.14*. Level 4 student approach for recognising reflection.

For the factor of "Identifying parameters" for reflection, the students most often matched two corresponding points or vertices of the images to find their distance, and halved it (see Figure 4.15). As some of them explained, they knew that the line would be horizontal, because it had to be in the middle of the two shapes. Moreover, as in Level 3, many students also mentioned that they noticed that one angle was "cut-off", and its place in the images helped them realise the orientation of the line of reflection.

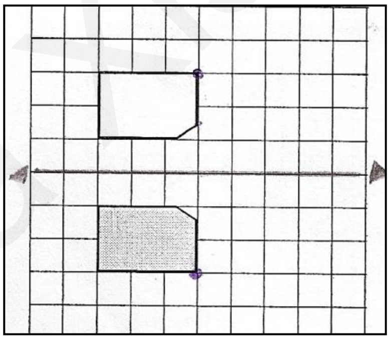


*Figure 4.15*. Level 4 student approach for identifying parameters in vertical reflection.

One student described what appears to be a visual holistic strategy of visualising the whole shape flipping over the line, in order to verify that the orientation of the diagonal line was correct. Some students described that they had visualised the image by mentally placing different lines of reflection, i.e., vertical, horizontal, and diagonal, particularly in the case of the diagonal line of reflection. Another student described how he or she connected all corresponding vertices in order to make sure that the line was diagonal (see Figure 4.16). There also were few students who made an error in the position of the line, because they had a strong intuition that the line of reflection must be exactly in the middle of the box.

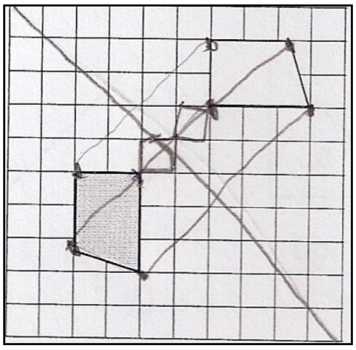


*Figure 4.16.* Level 4 student strategy for identifying parameters in diagonal reflection.

In the factor of "Construct image" for reflection, students of Level 4 that were interviewed were all successful in finding the image for the vertical line of reflection, and almost all of them for the diagonal. Their approaches were mainly underpinned by analytic strategies, and were supported by visualisation approaches of either distances, parts of the shapes, mental lines connecting vertices, or holistic images of the shapes. However, it was evident that they had a conceptual understanding of reflection as a function applied to all the points of a shape as part of the plane, as well as flexibility in manipulating the images by viewing them both holistically and analytically at the same time. Moreover, they used a much more formal vocabulary than the students of the previous levels, regarding various geometrical concepts, such as axis, angle degrees, distance, length, and right angle. Note that almost all of the students successfully followed the same approaches for both the vertical and diagonal line of reflection.

The majority of the students of Level 4 used an approach of measuring the distance from each vertex to the line of reflection, by drawing lines that were perpendicular to the line of reflection (see Figure 4.17). Then they constructed the images of the vertices by measuring the same distance on the other side of the line, and connected the vertices to construct the image. It is noted that these lines were most often imaginary, and were drawn by the students for the purpose of explaining their approach to the researcher. One of the students explained how after completing this procedure, she or he visualised the shape flipping, in order to verify that the image was correctly constructed. Another student used a different verification method, by measuring the lengths of the sides of the shapes.

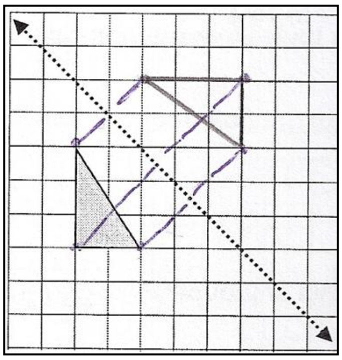


*Figure 4.17.* Level 4 student strategy for constructing image in diagonal reflection.

A similar approach used by one student was to measure the distance to the line of reflection for only two vertices (the top and the one that was closest to the line of reflection), then measured the same distance to find their images, and draw the segments to find where they met in order to construct the third vertex. This student followed the same approach for the diagonal line, explaining that once you reach the line of reflection, you must change your direction of counting unit squares for a turn of 90 degrees, “because the line is diagonal and you cannot keep measuring distance in the same direction”.

A less frequent approach described by two students was that they first visualised how and where the image would be if it was in a mirror, and then solved it “practically”, as one of them said, by measuring the distance from two vertices to the line of reflection, construct their image, and connect the vertices with lines to construct the third vertex. After that, they verified that the sides of the image had the same lengths as the pre-image. The younger students of the fourth grade of primary school described a similar approach of visualising the position and orientation of the image while measuring the distance to the line of reflection for only one vertex. This approach led one of them to some errors in the orientation of the image.

The other fourth grader described measuring the distance to the line of reflection for one vertex again, then constructing its image by measuring on the other side of the line, and constructing the image of the triangle based on the properties and dimensions of the pre-image. For the diagonal line of reflection, this student used a quite different and more complex approach. He or she described visualising a square (see Figure 4.18) which helped the student to find the position of the image. Then, the student visualised lines connecting the vertices of the pre-image triangle to the sides of the square in the other side of the line of reflection to find the orientation of the image, while at the same time reflecting the unit squares of the triangle's area one by one, following a direction parallel to the lines that were connecting the corresponding vertices.

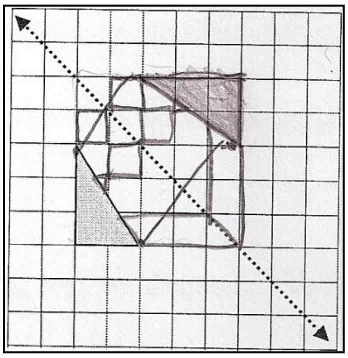


*Figure 4.18*. Level 4 fourth grade student strategy for constructing image in diagonal reflection.

*Description of students' conceptions and strategies in rotation ability*

Table 4.31 summarises the characteristics of the students in the four levels of abilities in the four factors of rotation: Recognise image, Recognise rotation, Identify parameters, and Construct image. The description of the four levels was based on the responses of the students in the tasks of the clinical interviews. As shown in Table 4.31, the students who are at Level 1 are not able to perform any task of rotation. The students who are at Level 2 are only able to visualise the rotation of an image, but this is not enough for them in order to be successful in recognising the image or an example of rotation. Students of Level 3 begin to focus on the orientation of the image, but are only able to recognise the image of a quarter rotation and to recognise the example of a half turn rotation and of a rotation with no given parameters. Moreover, they are able to identify correctly a three quarter and a half turn rotation, and to find the angle of rotation in a quarter turn and in overlapping rotation. In addition, they are able to construct the image of a quarter and three quarter rotation with correct direction and distance. Finally, students of Level 4 are able to recognise the image of a rotation based on its orientation. They are also able to recognise rotation based on the geometric transformation properties. In addition, they are able to identify a three quarter and a half turn rotation, and to find the angle of rotation in a quarter turn and in overlapping rotation, as well as construct the image of quarter, half, three quarter, and overlapping rotations with correct direction and correct distance.

Table 4.31

*Description of Students' Four Levels of Abilities in the Factors of Rotation*

| Type of task | Level 1 | Level 2 | Level 3 | Level 4 |
|---|---|---|---|---|
| 1. Recognise image | Use only the shape of the pre-image as criterion | Visualising rotational motion of the pre-image | Use orientation of the image as criterion<br>Recognise image in a quarter turn rotation | Recognise images using orientation of the image as criterion |
| 2. Recognise rotation | | Visualising rotational motion between images | Use orientation of the image as criterion<br>Recognise a half turn rotation and rotation with unspecified parameters | Recognise rotation based on the properties of geometric transformations |
| 3. Identify parameters | | | Identify a three quarter and a half turn rotation<br>Find the angle of rotation in a quarter turn and in overlapping rotation | Identify a three quarter and a half turn rotation<br>Find the angle of rotation in a quarter turn and in overlapping rotation |
| 4. Construct image | | | Construct image in quarter and three quarter rotation with correct direction and correct distance | Construct image in quarter, half, three quarter, and overlapping rotation with correct direction and correct distance |

Level 1

Regarding the factor of "Recognise image" in rotation, one third of the students of Level 1 had selected the image of translation in the task that was used for the clinical interview. The reasons for this were very explicit during the interviews, where many of the students explained that they were simply searching for "the same shape". One student used a gesture with his or her fingers representing the shape, in order to see how it would look like if rotated a quarter of a turn, and was able to find the correct response.

In the factor of "Recognise rotation", a number of students of Level 1 were not able to respond to the task, whereas many of the students who did respond had selected the option of reflection in vertical line as an example of rotation. The students that were able to select the correct option relied mainly on visualising the pre-image rotating, to see if it matched the optional images. Few of them were able to exclude options and describe the reasons for exclusion. Even in the cases where they did, their arguments were not relevant, i.e., a student explained excluding the example of translation because the shapes where far away, and the example of reflection because the shapes were joined on a side and this made it difficult for the student to visualise the rotation. Only one student at this level defined rotation as the turning of a shape upside down, and was able to focus on a part of the pre-image and its position in the image in order to decide whether it went up or down. Another student demonstrated using a gesture with her or his fingers representing the shape, and rotating it around to see if it would have the orientation of the optional images.

Regarding the factor of "Identify parameters" in rotation, the students at this level were not able to explain their thoughts, because of their insufficient understanding. Many of them referred to translating the shape, when they actually meant rotating it. The majority of the students at this level approached the task by placing the centre of rotation somewhere between the two shapes, because they thought that this was the place where it should be, and then visualised the shape turning to estimate the turn. However, the estimation was rarely accurate. Only one student focused on a part of the shape and its orientation in order to decide how many quarters of a turn the shape had performed to create the image. This student was able to find the correct fractions indicating the angle of rotations, but not the correct place for the centre of rotation.

Two of the students at this level thought that the centre of rotation had to be on the same line as the images. Regarding the angle of rotation, one student explained having selected the largest fraction, because the shapes are very far from each other, and the other explained that the fraction is always two quarters, because the figure showed two of the four possible images. Finally, two of the students believed that the centre of rotation should be in the middle of the two shapes. One of them visualised and drew a circle between the two shapes to estimate the angle of rotation from the pre-image to the image.

For "Construct image" in rotation, students of Level 1 had very low ability in completing the task. The students that were interviewed also had many difficulties in explaining their approach, because they had very little understanding of the concept of rotation and what was expected from them to do. Some of them did not even attempt to approach the task, especially in the case of the half turn. However, the descriptions of their reasoning revealed many misconceptions regarding mathematical concepts related to rotation, i.e., concepts related to circles and fractions. The majority of the students at this level though that they had to construct the image of the triangle somewhere to the right-hand side of the box, but did not know where exactly. Regarding the role of the fraction, one of them suggested that the one quarter was indicating that one had to draw one unit square to be able to find the image. The rest of the students could not explain how they would use the information of a quarter of a turn.

One student was able to describe how he or she drew the quarter of a circle, like when doing fractions and slicing a round pizza in four pieces, where in fact the student had drawn the arc of a semi-circle connecting the hypotenuse to the centre of rotation. Then the student measured the sides of the triangle in order to construct the image with the same lengths of sides. Another student explained that he or she drew the arc of a quarter of a circle starting from the centre of rotation, and drew a random triangle at the end of it. For the half turn, one student described visualising a circle somewhere around the pre-image, and drew the image in the other half of the circle. The student then described that he or she did not know how the image would be; only that it had to be different than the pre-image.

Two students described that they visualised the triangle turning. Specifically, one student had the conception that the image had to be on the same line as the pre-image, so it needed to be touching the centre of rotation. However, only one of them was able to visualise the triangle turning to the correct direction. The other was not able to describe how much the shape had been rotated. Their responses were partially correct, with errors in the orientation of the shape, or the direction of the image, or the distance of the image from the centre of rotation. Finally, one of the students who were interviewed at this level treated the centre of rotation as a line of reflection, and performed a reflection in vertical or in diagonal line.

Level 2

In the factor of "Recognise image" in rotation, students of Level 2 all described that they visualised the shapes rotating. Some of them visualised the pre-image rotating a quarter of a turn and then searched for the image that they had visualised. This led to errors when they had visualised the pre-image rotating to the left-hand side instead of the right-hand side, and hence selected the image that was a three quarter turn to the right-hand side. Others visualised each option rotating, in order to see if they could match the pre-image. The most common error in this case was to lose track of the steps of rotation and select the option of a half-turn, instead of a quarter turn.

For "Recognising rotation", the students of this level did not confuse rotation with reflection as much as the students of Level 1. Moreover, as in Level 1, the students of this level were still unable to name the geometric transformation represented in each option. They basically relied on visualising the whole shape of the pre-image rotating for every option, to see if it matched the image.

In the factor of "Identify parameters" in rotation, students of Level 2 approached the task in the same ways as the students of Level 1, i.e., placing the centre of rotation in the middle of the shapes. The students at this level referred to counting unit squares from one shape to the other to split the distance. However, some of the students at this level placed the centre of rotation on a vertex of the pre-image. For finding the angle of rotation, most of the students described visualising the shape turning to various positions in order to estimate the turn for the given image. One student tried to draw an arc of a circle from one vertex of the pre-image to another vertex of the image, however the vertices were not the corresponding ones, and this did not help the student to find the correct response.

In the factor of "Construct image" for rotation, while ability for constructing the correct image for rotation was still low, the students of this level were more able than the students of Level 1 to find a partially correct response, regarding the direction and distance of the image from the centre of rotation. This means that, while many of these students were able to rotate the image around the centre of rotation, they had difficulties with its orientation. The students at this level had a better understanding of the relevant mathematical concepts in comparison to the students of Level 1.

Even though students of Level 2 had similar approaches to students of Level 1, and tended to visualise the pre-image rotating as a whole object on the plane, students of Level 2 often used gestures, which seemed to help them find the correct direction of the image.

For example, in the half turn rotation, one student demonstrated using his or her hands to represent the plane and the pre-image rotating in order to find the direction of the shape. One hand was still, for representing the plane, and the other hand was moving around on the top of the other. Three more students described how they drew a circle in the air with their finger to find the direction of the image. Another student demonstrated using two fingers, as if lifting the triangle from the paper and holding it to rotate, and visualising it moving with her or his fingers.

Level 3

In the factor of "Recognise image" in rotation, students of Level 3 all described that they visualised the shapes rotating. Most of them visualised the pre-image rotating a quarter of a turn and then searched for the image that they had visualised. The majority of the students at this level were able to find the correct image. Three students mentioned that they noticed the orientation of the shape's "legs", thus focusing on its parts to decide if the orientation of the optional image was the one they had visualised. One student was also able to explain the reasons for excluding each option, by referring to the geometric transformation that had created each image.

For the factor of "Recognise rotation", the majority of the students of Level 3 were able to recognise the example of rotation. Again, the most common approach among the students of this level was to visualise the shape of the pre-image, or a part of it rotating, and to focus on the orientation of a specific part in order to decide whether the image can be a rotation of the pre-image. Many students described how the "thin leg" of the shape must be on the right-hand side when it is rotated, instead of the same side as in the pre-image. Many students used arguments to exclude some responses. Specifically, they excluded the example showing a reflection, because they seemed to know that in reflection you can only have a common point between the pre-image and the image, but not a common side. They also excluded the example of translation based on the criterion of the same orientation. Some of the students were torn between the option of rotation and the option of a combination of rotation and reflection. However, only the ones that were able to focus on the orientation of the shape's "legs" were able to find the correct option from the two.

In the factor of "Identifying parameters" for rotation, the students at this level mostly described that they focused on one part of the shape, i.e., the one "leg" of the L, in order to understand the change in the orientation of the pre-image and find the fraction showing the angle of rotation. Thus, there were more correct responses regarding the angle of rotation at this level, in comparison to the previous levels. Some students said that it also helped them to visualise the shape rotating in order to find the correct angle of rotation. The students at this level had similar conceptions to the students of Level 2 regarding the position of the centre of rotation as something that needs to be somewhere between the two shapes. However, some of them described how they tried to divide the distance between two vertices or two sides of the shapes, which explains the increase in the correct responses. However, they did not measure the distance between two corresponding sides or two corresponding vertices.

Regarding the factor of "Construct image" for rotation, students of Level 3 were more able to find a partially correct response, which was more often to find the correct image with the wrong orientation. As in the other two geometric transformations, the students at this level began to focus on a part of the shape of the pre-image, either a side or a vertex, to perform the transformation. The students at this level seemed to understand the importance of the image's direction and orientation, and they began taking into consideration the parameter of distance from the centre of rotation. However, the younger students of the fourth grade seemed to have a strong conception that the image must be attached to the centre of rotation. Moreover, the students of all ages at this level seemed to have a good understanding of the relevant mathematical concepts (of fractions and circles), which facilitated their understanding and ability to perform rotation.

For example, one student described how measuring the distance from the vertex that was the closest to the centre of rotation, and constructed the image of the side that was attached to that vertex. Then, the student constructed the image by measuring the sides of the pre-image. He or she explained thinking of rotation as "going round like a clock", and that since the pre-image was "facing upwards", then the image will be facing to the right-hand side for a quarter turn, and downwards for a half turn. In a similar way, some students focused on the vertical side of the triangle and thought that if it turned a quarter of a turn, then it would become horizontal. Even though this approach was very helpful for finding the orientation of the image in most cases, it was not helpful for finding its direction around the centre of rotation. Another student described how measuring a unit square and a half distance from the hypotenuse of the triangle, and then visualising this square and a half rotating and taking the place of the one above the centre of rotation. Then the student constructed the image of the hypotenuse and the rest of the triangle based on the properties and measures of the pre-image.

Other approaches included visualising the triangle turning around the centre of rotation, and a mental circle around the triangle divided into four parts, like a fraction circle, as described by one student. The student who used this approach described how the fraction circle helped her or him understand in which quarter the image would be. Another student focused on the vertex of the right angle to draw the arc of a square of a circle (or of half a circle in the case of a half turn) in order to find the position of the image around the centre of rotation. However, what this student constructed eventually was not the image of the corresponding vertex, thus resulting to errors in the orientation of the image (see Figure 4.19). Moreover, since the student did not mention the variable of distance from the centre of rotation, the image of a half turn had errors in the distance from the centre of rotation.

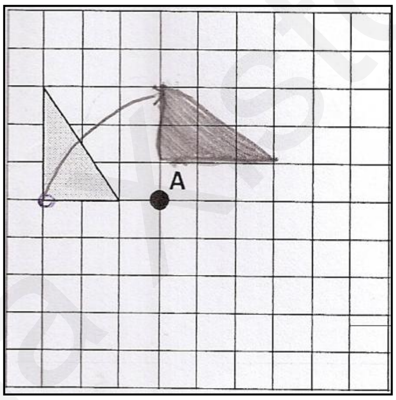



*Figure 4.19*. Level 3 student strategy for constructing image in a quarter turn rotation.

Level 4

Regarding the factor of "Recognise image" in rotation, all students of Level 4 that were interviewed were able to find the correct response. Moreover, most of them were also able to describe how each image was created, i.e., if it was a translation, a reflection, or a rotation with a different angle of rotation. While some of them described that they had visualised the whole shape rotating, some of them simply focused on a vertex of the pre-image, and visualised where it should be when the pre-image would be rotated, and searched for the option that had the vertex in the appropriate direction. However, one student characteristically claimed to have a collection of several different approaches, which he or she used almost simultaneously for verification. When asked to describe them, the student said that one would be to focus on the orientation of the point, the second to visualise the shape rotating, and the third to find the corresponding vertices.

For the factor of "Recognise rotation", all students of Level 4 who were interviewed were able to find the correct response. Even though their approaches were quite similar to those of the students of Level 3, what was different was that the students at this level were able to name the geometric transformation represented in each of the alternative options, using formal terminology; similarly to the other geometric transformation. Another difference is that, the students at this level explained that they used their visualisation strategies of rotating the figures mostly to verify their response. Their reasoning was mainly based on recognising each geometric transformation and on the orientation of the parts of the shape.

Regarding the factor of "Identify parameters" in rotation, the students at this level used similar strategies as the students of Level 3 for finding the angle of rotation, which were mostly visualising the pre-image rotating, or focusing on the parts of the shape to estimate the change in their orientation (see Figure 4.20). However, their approaches for finding the centre of rotation were much different, which explains the increase of correct responses at this level. Specifically, the majority of the students at this level visualised a circle passing through the corresponding points of the pre-image and the image, and marked the centre of this circle as the centre of rotation (see Figure 4.21). Then, they used this circle to find the angle of rotation. Two students described how they found the corresponding vertices in the two shapes and visualised where they would meet, by drawing mental perpendicular lines. Another student described extending the corresponding sides to find where they meet, in order to mark the centre of rotation. Finally, another student explained that she or he placed the centre of rotation somewhere where it would have the same distance from the two images, and then visualised the pre-image rotating in order to verify that the centre of rotation was correct, and to find the angle of rotation.

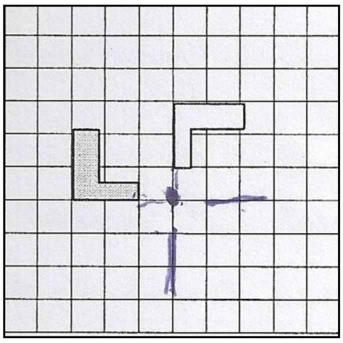


*Figure 4.20*. Level 4 student strategy for identifying parameters in rotation (1).

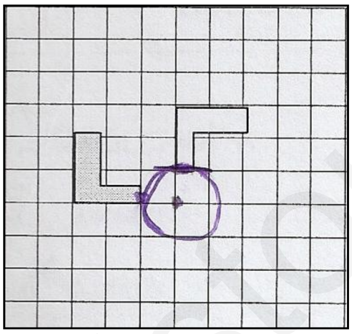


*Figure 4.21*. Level 4 student strategy for identifying parameters in rotation (2).

In the factor of “Construct image” for rotation, students of Level 4 used quite different strategies from those of the students of Level 3. Students of Level 4 were able to visualise the different parts of the shape of the pre-image and their relations more easily, and they seemed to have flexibility between the analytic and holistic strategies. Moreover, they used a more formal vocabulary regarding mathematical concepts, such as degrees of angles, perimeter, and ray of a circle. The majority of the students described that they focused on the distance of the vertex that was closest to the centre of rotation, to comment that it must always be the same around the centre of rotation. They also visualised two mental perpendicular lines crossing over the centre of rotation (see Figure 4.22). One could think that the lines may serve as diameters or rays of the circular trace of the rotation. One of the students described using these lines, referring to the four rays, to find the direction of the pre-image triangle’s base, always keeping a square unit distance from the centre of rotation. After that, the student constructed the image, knowing that the orientation of the other two sides must be to the right-hand side for the quarter turn, and downwards for the half turn. Two more students only used the lines to find the position of the image of the vertex that was closest to the centre of rotation, and then used the lengths of the pre-image's sides to construct the triangle's image. They both mentioned that they had also visualised the whole triangle rotating around the centre of rotation, to estimate its position and orientation, either before or after finding the position of the vertex.

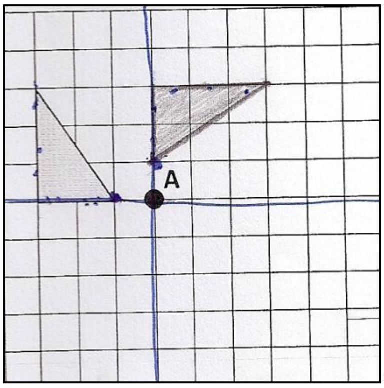



*Figure 4.22*. Level 4 student strategy for constructing image in rotation (1).

Three of the students at this level used a different approach, where they focused on the base of the triangle and visualised it rotating, not as a triangle, but only as a segment (see Figure 4.23). One of them described rotating the segment "90 or 180 degrees", keeping the same distance from the centre of rotation. After finding the position of the base, the students constructed the image of the triangle, using its lengths, and visualising its orientation. Another student described visualising a circle with the centre of rotation at the centre, and the bottom-right vertex of the pre-image being a point on the perimeter of the circle (see Figure 4.24). The student then described how visualising the triangle rotating around the centre of rotation, while at the same time describing the changes in the orientation of parts of the triangle, i.e., that the base would become vertical and the top vertex would be facing to the right-hand side.

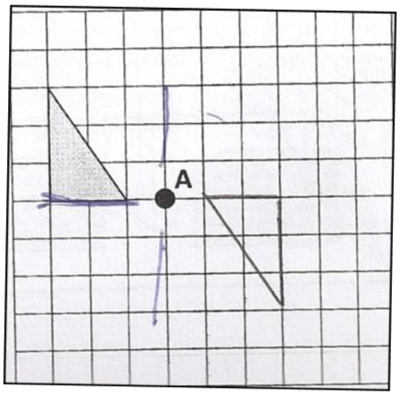



*Figure 4.23*. Level 4 student strategy for constructing image in rotation (2).

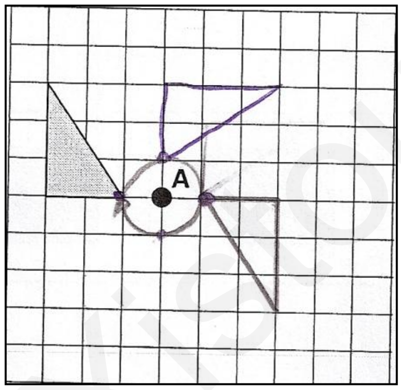



*Figure 4.24*. Level 4 student strategy for constructing image in rotation (3).



Finally, a different approach that one student described, was about constructing the image of each vertex of the pre-image on its own, by measuring its distance from the centre of rotation. Specifically, the student described how she or he found the images of the two vertices of the triangle's base, by drawing them in the same distance from the centre of rotation following an upward direction. When asked how she or he knew where the points must be after the rotation, the student said that he or she just knew where the quarter turn and the half turn position would be, because it was to the right-hand direction. After finding the second point, which was the point of the right angle vertex, the student measured the length of the side to construct the third vertex.

*Summarising description of students' conceptions and strategies in transformational geometry*

Table 4.32 summarises the characteristics of the students in the four levels of abilities in transformational geometry, as they emerge from their common abilities, conceptions, and strategies in the three transformational geometry concepts and their sub-components. The description of the four levels was based on the students' responses in the tasks of the clinical interviews. As shown in Table 4.32, we named Level 1 "Holistic image conception", because students conceive the representation of the plane and the images of the shapes as a whole realistic object. We named Level 2 "Motion of an object", because students at this level begin to use visual reasoning with motion of the shapes as holistic objects over a surface, which is the representation of the plane. We named Level 3 "Mapping of an object", since the students begin to analyse the shapes into smaller parts and map them, but without realising how the transformation affects the rest parts of the shape. Finally, we named Level 4 "Mapping of the plane", because the students at this level are fully aware that the shapes are part of the plane, and they realise that every point of the plane is affected by the geometric transformation.

Table 4.32

*Description of Students' Four Levels of Abilities in Transformational Geometry*

| Level | Name of level | Description |
|---|---|---|
| 1 | Holistic image conception | • Students visually conceive simple relations of up-down and left-right between shapes, without motion or decomposition of the shapes<br>• The focus is not on the shapes, but to their positioning in the holistic figure, based on the four directions<br>• The students see the plane and the objects as a holistic image, like a drawing<br>• No use of the properties of transformations<br>• Solution can be generated based on prototypical images<br>• Geometric transformations are seen neither as motion nor as mapping |

| | | |
|---|---|---|
| 2 | Motion of an object | • Students detach the shape from the plane and move it on top of the plane<br>• Students make movements/reversions in straight line routes<br>• Emphasis on the shape as a holistic object, without analysing it into smaller parts or decomposing it (i.e., segments, points)<br>• Emphasis on the property of orientation of the object, without considering distance |
| 3 | Mapping of an object | • Students begin to decompose the shape and focus on a single part/point and its mapping<br>• Students map a single segment or point and reconstruction of the shape's image based on its attributes<br>• Understanding of all properties: direction, orientation, and (at this level) distance<br>• Students are able to apply all the properties to construct images in straight-line displacement (i.e., translation and reflection) and visually understand circular displacement (i.e., rotation)<br>• Students begin to understand the mathematical space in the sense of:<br>  o shapes are made of segments and points which can be transformed individually<br>  o the attributes of the shapes are preserved in space |
| 4 | Mapping of the plane | • Students are able to decompose the shape into points and they understand the mapping of all the points<br>• The shape stops being perceived as a holistic object<br>• Students can apply all properties to all the points in the plane and not only to the points of the shape<br>• They begin to understand diagonal straight-line displacements<br>• They can apply properties in circular displacement and also perform circular displacements |

In Level 1, “Holistic image conception”, the students seem to visually conceive simple relations of up-down and left-right between shapes, which they use in every-day life to informally describe position, but without understanding the properties of the transformation or of the geometrical figures. The focus of the students is not on the shapes, but on their positioning in the whole image, based on the four directions experienced in the physical world. They seem to visualise the plane and the objects as a holistic picture, as a realistic drawing in the physical world, and visualise nothing more than the representation of the plane and the image as a concrete part of it, but without motion.

In Level 2, motion of an object, students begin to detach the shape from the plane and are able to visualise it moving on top of the plane. The emphasis of the students remains on the shape as a tangible object, but the students are able to visualise the dimensional deconstruction (Duval, 2011) of the representation to two separate figures: the plane and the geometrical shape. As a result of realising that they can visually manipulate the object, they begin to understand the property of orientation.

In Level 3, mapping of an object, students begin to visualise the dimensional deconstruction of the shape into smaller parts, i.e., line segments and points, and focus on the sides or vertices and their mapping. They seem to be able to map a single side or vertex and reconstruct the image of the geometrical shape based on its definition and attributes (right angles, lengths, etc.). They begin to understand all the properties of geometric transformations related to the shape, such as orientation, and also the parameters of direction and distance. Moreover, they are able to apply them in simple situations, such as constructing images in straight-line displacement, and in recognising circular displacement. Moreover, they begin to realise more general geometrical notions, such as the properties of space, in the sense that shapes are made of smaller segments and points which can change position in space, and that the attributes of a geometrical shape are preserved in space.

In Level 4, mapping of the plane, students seem to be able to deconstruct the geometrical shape into points and they understand the mapping of all the points, based on the properties and axioms of transformational geometry. The shape stops being perceived as a holistic object. The students are able to apply all axioms and properties to all the points of the shape, and apply them in performing circular displacements. Finally, they can visualise and perform the mapping of all the points in all routes and directions (straight and circular displacements), since they realise that the transformation affects all points of the plane.

This section presents the results relating to the fourth aim of the research concerning the investigation of the relation between students' ability in transformational geometry and their spatial ability. Specifically, it answers the following research question:

(5) What is the relation between students' spatial ability and their transformational geometry ability?

Before investigating the relation between transformational geometry ability and spatial ability, the structure of spatial ability was first investigated. In order to investigate the structure of spatial ability, the spatial ability measurement test was used. The descriptive information of the spatial ability test is described first, followed by the testing of the theoretical model for spatial ability that was performed with confirmatory factor analyses. Finally, the relation between spatial ability and ability in transformational geometry concepts is presented.

*Descriptive results of the spatial ability test*

Table 4.33 presents the descriptive information for the spatial ability test, in each type of task. The highest means of the subjects were in the items of the "Cube Comparisons" test ($M$ = .66), the "Where is the Photographer?" test ($M$ = .59), and the "Card Rotations" test ($M$ = .58). On the contrary, the lowest means were in the items of the "Perspective Taking" test ($M$ = .28), and the "Image Perspective" test ($M$ = .34). The range of the subjects' performance to the eight types of tasks of the test was one, which shows that there were subjects that responded correctly to all the items of a specific part of the test, as well as subjects that did not respond correctly to any item of a specific part of the test. Table 4.33 presents the values for Skewness and Kurtosis for the subjects' performance to the eight parts of the spatial ability test. The values of these indices were smaller than two, which suggests that the variables of the subjects' performance to the eight parts of the spatial ability test follow a normal distribution.

Table 4.33

*Descriptive Results of the Spatial Ability Tasks*

| Test | Mean | Standard Deviation | Range | Skewness | Kurtosis |
|---|---|---|---|---|---|
| A. Card Rotations | .58 | .33 | 1 | -.30 | -1.12 |
| B. Paper Folding | .57 | .26 | 1 | -.31 | .01 |
| C. Where is the Photographer? | .59 | .35 | 1 | -.31 | -1.14 |
| D. Mental Rotations | .54 | .25 | 1 | -.33 | -.39 |
| E. Perspective Taking | .28 | .24 | 1 | .83 | .45 |
| F. Cube Comparisons | .66 | .30 | 1 | -.86 | -.11 |
| G. Form-Board | .51 | .32 | 1 | -.17 | -1.00 |
| H. Image Perspective | .34 | .32 | 1 | .65 | -.64 |

Table 4.34 presents the correlations between the items that were used for verifying the model. The variables correspond to the 35 items of the spatial ability test. The correlations between the six items of the “Card Rotations” test were all statistically significant, with the highest correlation of this test appearing between items A4 and A5 ($r = .42$, $p < .01$). The same applies for the three items of the “Where is the Photographer” test (with the highest correlation of this test appearing between items C2 and C3, $r = .36$, $p < .01$), for the four items of the “Mental Rotations” test (with the highest correlation of this test appearing between items D1 and D2, $r = .49$, $p < .01$), the four items of the “Perspective Taking” test (with the highest correlation of this test appearing between items E3 and E4, $r = .45$, $p < .01$), the four items of the “Form-Board” test (with the highest correlation of this test appearing between items G2 and G3, $r = .49$, $p < .01$), and the four items of the “Image Perspective” test (with the highest correlation of this test appearing between items H1 and H4, $r = .34$, $p < .01$). Nine of the 10 correlations between the five items of the “Paper Folding” test were statistically significant, with the highest correlation of this test appearing between items B1 and B2, $r = .50$, $p < .01$). Nine of the 10 correlations between the five items of the “Cube Comparisons” test were statistically significant, with the highest correlation of this test appearing between items F2 and F4,

$r = .45$, $p < .01$). The significant correlations between the items of each test suggest that these items seem to measure the same ability.

Table 4.35 shows the correlations between the performances of the subjects in the eight types of tasks in the spatial ability test. The highest correlations appear between the subjects' performances in the items of the tests of "Form-Board" and "Image Perspective" ($r = .50$, $p < .01$), and the subjects' performances in the items of the tests "Card Rotations" and "Mental Rotations" ($r = .47$, $p < .01$). The fact that all correlations between the performances of the subjects in the eight types of tasks of the spatial ability test were statistically significant indicates that the eight tests seem to measure the same ability.

The reliability coefficient of the items of the spatial ability test was Cronbach's Alpha = .88, which is considered good (Klein, 1999). The reliability coefficients of the items for each type of task were also satisfactory ( $\alpha_{\text{Card Rotations}} = .77$, $\alpha_{\text{Paper Folding}} = .52$, $\alpha_{\text{Where is the Photographer?}} = .54$, $\alpha_{\text{Mental Rotations}} = .73$, $\alpha_{\text{Perspective Taking}} = .60$, $\alpha_{\text{Cube Comparisons}} = .59$, $\alpha_{\text{Form-Board}} = .76$, $\alpha_{\text{Card Rotations}} = .61$).

Table 4.34

*Correlations Between the Subjects' Performance in the Items of the Spatial Ability Test*

| | A1 | A2 | A3 | A4 | A5 | A6 | B1 | B2 | B3 | B4 | C1 | C2 | C3 | D1 | D2 | D3 | D4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A1 | 1 | | | | | | | | | | | | | | | | |
| A2 | .26** | 1 | | | | | | | | | | | | | | | |
| A3 | .25** | .42** | 1 | | | | | | | | | | | | | | |
| A4 | .37** | .39** | .35** | 1 | | | | | | | | | | | | | |
| A5 | .40** | .42** | .38** | .42** | 1 | | | | | | | | | | | | |
| A6 | .40** | .28** | .22** | .36** | .36** | 1 | | | | | | | | | | | |
| B1 | .18** | .26** | .24** | .25** | .30** | .24** | 1 | | | | | | | | | | |
| B2 | .15** | .26** | .23** | .24** | .24** | .22** | .50** | 1 | | | | | | | | | |
| B3 | .18** | .10* | .09* | .11* | .11* | .18** | .17** | .13** | 1 | | | | | | | | |
| B4 | .12** | .17** | .19** | .16** | .23** | .18** | .14** | .17** | .25** | 1 | | | | | | | |
| C1 | .12** | .10* | .13** | .12** | .15** | .07 | .18** | .14** | .08 | .18** | 1 | | | | | | |
| C2 | .24** | .30** | .23** | .23** | .25** | .24** | .27** | .23** | .08 | .17** | .23** | 1 | | | | | |
| C3 | .18** | .17** | .16** | .16** | .19** | .18** | .13** | .12** | .07 | .12** | .27** | .36** | 1 | | | | |
| D1 | .25** | .25** | .24** | .25** | .28** | .24** | .31** | .30** | .11* | .13** | .12** | .32** | .17** | 1 | | | |
| D2 | .29** | .26** | .16** | .31** | .26** | .31** | .25** | .29** | .13** | .20** | .21** | .30** | .25** | .49** | 1 | | |
| D3 | .25** | .24** | .25** | .25** | .25** | .23** | .22** | .25** | .11* | .19** | .19** | .32** | .17** | .44** | .36** | 1 | |
| D4 | .15** | .15** | .14** | .28** | .19** | .23** | .26** | .21** | .09* | .18** | .12** | .18** | .15** | .38** | .45** | .33** | 1 |

* $p < .05$. ** $p < .01$.

| | A1 | A2 | A3 | A4 | A5 | A6 | B1 | B2 | B3 | B4 | B5 | C1 | C2 | C3 | D1 | D2 | D3 | D4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| E1 | .18$^{**}$ | .14$^{**}$ | .12$^{**}$ | .14$^{**}$ | .22$^{**}$ | .19$^{**}$ | .25$^{**}$ | .25$^{**}$ | .20$^{**}$ | .21$^{**}$ | .17$^{**}$ | .20$^{**}$ | .20$^{**}$ | .13$^{**}$ | .24$^{**}$ | .29$^{**}$ | .27$^{**}$ | .19$^{**}$ |
| E2 | .13$^{**}$ | .09$^{*}$ | .14$^{**}$ | .11$^{*}$ | .12$^{**}$ | .14$^{**}$ | .08 | .05 | .12$^{**}$ | .15$^{**}$ | .16$^{**}$ | .06 | .15$^{**}$ | .15$^{**}$ | .19$^{**}$ | .24$^{**}$ | .24$^{**}$ | .13$^{**}$ |
| E3 | .05 | .05 | .11$^{*}$ | .11$^{*}$ | .11$^{*}$ | .10$^{*}$ | .10$^{*}$ | .10$^{*}$ | .09 | .15$^{**}$ | .15$^{**}$ | .00 | .08 | .12$^{**}$ | .20$^{**}$ | .15$^{**}$ | .15$^{**}$ | .09$^{*}$ |
| E4 | .13$^{**}$ | .07 | .10$^{*}$ | .10$^{*}$ | .09$^{*}$ | .17$^{**}$ | .10$^{*}$ | .15$^{**}$ | .15$^{**}$ | .12$^{**}$ | .13$^{**}$ | .13$^{**}$ | .15$^{**}$ | .07 | .17$^{**}$ | .19$^{**}$ | .18$^{**}$ | .07 |
| F1 | .09$^{*}$ | .12$^{**}$ | .19$^{**}$ | .16$^{**}$ | .07 | .20$^{**}$ | .19$^{**}$ | .27$^{**}$ | .12$^{**}$ | .07 | .15$^{**}$ | .18$^{**}$ | .18$^{**}$ | .13$^{**}$ | .22$^{**}$ | .25$^{**}$ | .25$^{**}$ | .26$^{**}$ |
| F2 | .10$^{*}$ | .10$^{*}$ | .05 | .17$^{**}$ | .11$^{*}$ | .20$^{**}$ | .15$^{**}$ | .07 | .12$^{**}$ | .04 | .07 | .08 | .07 | .08 | .12$^{**}$ | .13$^{**}$ | .16$^{**}$ | .12$^{**}$ |
| F3 | .13$^{**}$ | .11$^{*}$ | .16$^{**}$ | .11$^{*}$ | .16$^{**}$ | .16$^{**}$ | .22$^{**}$ | .22$^{**}$ | .13$^{**}$ | .09 | .07 | .15$^{**}$ | .18$^{**}$ | .06 | .21$^{**}$ | .20$^{**}$ | .17$^{**}$ | .17$^{**}$ |
| F4 | .12$^{**}$ | .04 | -.03 | .11$^{*}$ | .06 | .17$^{**}$ | .14$^{**}$ | .13$^{**}$ | .07 | .08 | .04 | .03 | .03 | .07 | .18$^{**}$ | .17$^{**}$ | .12$^{**}$ | .15$^{**}$ |
| F5 | .16$^{**}$ | .10$^{*}$ | .11$^{*}$ | .11$^{*}$ | .08 | .13$^{**}$ | .25$^{**}$ | .20$^{**}$ | .12$^{**}$ | .08 | .07 | .11$^{*}$ | .14$^{**}$ | .13$^{**}$ | .09$^{*}$ | .13$^{**}$ | .17$^{**}$ | .15$^{**}$ |
| G1 | .13$^{**}$ | .12$^{**}$ | .11$^{*}$ | .19$^{**}$ | .11$^{*}$ | .22$^{**}$ | .23$^{**}$ | .20$^{**}$ | .11$^{*}$ | .10$^{*}$ | .24$^{**}$ | .14$^{**}$ | .20$^{**}$ | .16$^{**}$ | .23$^{**}$ | .26$^{**}$ | .25$^{**}$ | .26$^{**}$ |
| G2 | .09$^{*}$ | .22$^{**}$ | .11$^{*}$ | .13$^{**}$ | .11$^{*}$ | .26$^{**}$ | .23$^{**}$ | .19$^{**}$ | .11$^{*}$ | .05 | .20$^{**}$ | .11$^{*}$ | .20$^{**}$ | .16$^{**}$ | .21$^{**}$ | .23$^{**}$ | .19$^{**}$ | .23$^{**}$ |
| G3 | .23$^{**}$ | .20$^{**}$ | .20$^{**}$ | .25$^{**}$ | .20$^{**}$ | .26$^{**}$ | .26$^{**}$ | .25$^{**}$ | .15$^{**}$ | .14$^{**}$ | .21$^{**}$ | .15$^{**}$ | .23$^{**}$ | .19$^{**}$ | .27$^{**}$ | .29$^{**}$ | .22$^{**}$ | .21$^{**}$ |
| G4 | .11$^{*}$ | .17$^{**}$ | .14$^{**}$ | .11$^{*}$ | .13$^{**}$ | .20$^{**}$ | .22$^{**}$ | .16$^{**}$ | .09$^{*}$ | .06 | .07 | .19$^{**}$ | .20$^{**}$ | .16$^{**}$ | .18$^{**}$ | .17$^{**}$ | .16$^{**}$ | .15$^{**}$ |
| H1 | .18$^{**}$ | .09 | .13$^{**}$ | .17$^{**}$ | .16$^{**}$ | .27$^{**}$ | .15$^{**}$ | .14$^{**}$ | .18$^{**}$ | .07 | .17$^{**}$ | .09$^{*}$ | .10$^{*}$ | .15$^{**}$ | .24$^{**}$ | .30$^{**}$ | .20$^{**}$ | .14$^{**}$ |
| H2 | .14$^{**}$ | .07 | .10$^{*}$ | .14$^{**}$ | .08 | .19$^{**}$ | .11$^{*}$ | .09$^{*}$ | .10$^{*}$ | .05 | .19$^{**}$ | .07 | .19$^{**}$ | .17$^{**}$ | .20$^{**}$ | .23$^{**}$ | .16$^{**}$ | .17$^{**}$ |
| H3 | .08 | .09$^{*}$ | .12$^{**}$ | .07 | .13$^{**}$ | .16$^{**}$ | .17$^{**}$ | .19$^{**}$ | .11$^{*}$ | .09 | .11$^{*}$ | .08 | .16$^{**}$ | .14$^{**}$ | .25$^{**}$ | .18$^{**}$ | .18$^{**}$ | .17$^{**}$ |
| H4 | .13$^{**}$ | .18$^{**}$ | .15$^{**}$ | .19$^{**}$ | .12$^{**}$ | .12$^{**}$ | .17$^{**}$ | .16$^{**}$ | .06 | .13$^{**}$ | .20$^{**}$ | .13$^{**}$ | .18$^{**}$ | .21$^{**}$ | .14$^{**}$ | .20$^{**}$ | .21$^{**}$ | .14$^{**}$ |

$^{*}$ $p < .05$. $^{**}$ $p < .01$.

| | E1 | E2 | E3 | E4 | F1 | F2 | F3 | F4 | G1 | G2 | G3 | G4 | H1 | H2 | H3 | H4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| E1 | 1 | | | | | | | | | | | | | | | |
| E2 | .27** | 1 | | | | | | | | | | | | | | |
| E3 | .14** | .30** | 1 | | | | | | | | | | | | | |
| E4 | .24** | .28** | .45** | 1 | | | | | | | | | | | | |
| F1 | .18** | .05 | .01 | .16** | 1 | | | | | | | | | | | |
| F2 | .24** | .10* | .08 | .12** | .34** | 1 | | | | | | | | | | |
| F3 | .13** | .10* | .16** | .14** | .12** | .24** | 1 | | | | | | | | | |
| F4 | .15** | .10* | .15** | .15** | .26** | .45** | .22** | 1 | | | | | | | | |
| G1 | .27** | .16** | .05 | .16** | .33** | .23** | .12** | .19** | 1 | | | | | | | |
| G2 | .23** | .10* | .08 | .17** | .26** | .23** | .10* | .20** | .46** | 1 | | | | | | |
| G3 | .25** | .12** | .09* | .20** | .24** | .28** | .23** | .24** | .48** | .49** | 1 | | | | | |
| G4 | .17** | .11* | .11* | .19** | .26** | .28** | .15** | .23** | .37** | .47** | .44** | 1 | | | | |
| H1 | .24** | .15** | .12** | .19** | .15** | .18** | .15** | .156** | .22** | .25** | .34** | .17** | 1 | | | |
| H2 | .17** | .16** | .16** | .15** | .15** | .09* | .08 | .10* | .31** | .31** | .29** | .31** | .32** | 1 | | |
| H3 | .12** | .13** | .15** | .17** | .15** | .04 | .07 | .13** | .20** | .21** | .26** | .23** | .23** | .31** | 1 | |
| H4 | .22** | .20** | .12** | .12** | .19** | .16** | .14** | .15** | .23** | .25** | .29** | .27** | .34** | .32** | .21** | 1 |

* $p < .05$. ** $p < .01$.

Table 4.35

*Correlations Between the Performance of the Subjects in the Types of Tasks of the Spatial Ability test*

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| A | 1 | | | | | | | |
| B | .43** | 1 | | | | | | |
| C | .36** | .31** | 1 | | | | | |
| D | .47** | .41** | .38** | 1 | | | | |
| E | .27** | .33** | .25** | .38** | 1 | | | |
| F | .26** | .32** | .23** | .36** | .29** | 1 | | |
| G | .32** | .36** | .31** | .38** | .31** | .43** | 1 | |
| H | .29** | .33** | .28** | .38** | .35** | .28** | .50** | 1 |

*Note.* A = Card Rotations, B = Paper Folding, C = Where is the photographer?, D = Mental Rotations, E = Perspective Taking, F = Cube Comparisons, G = Form-Board, H = Image Perspective.

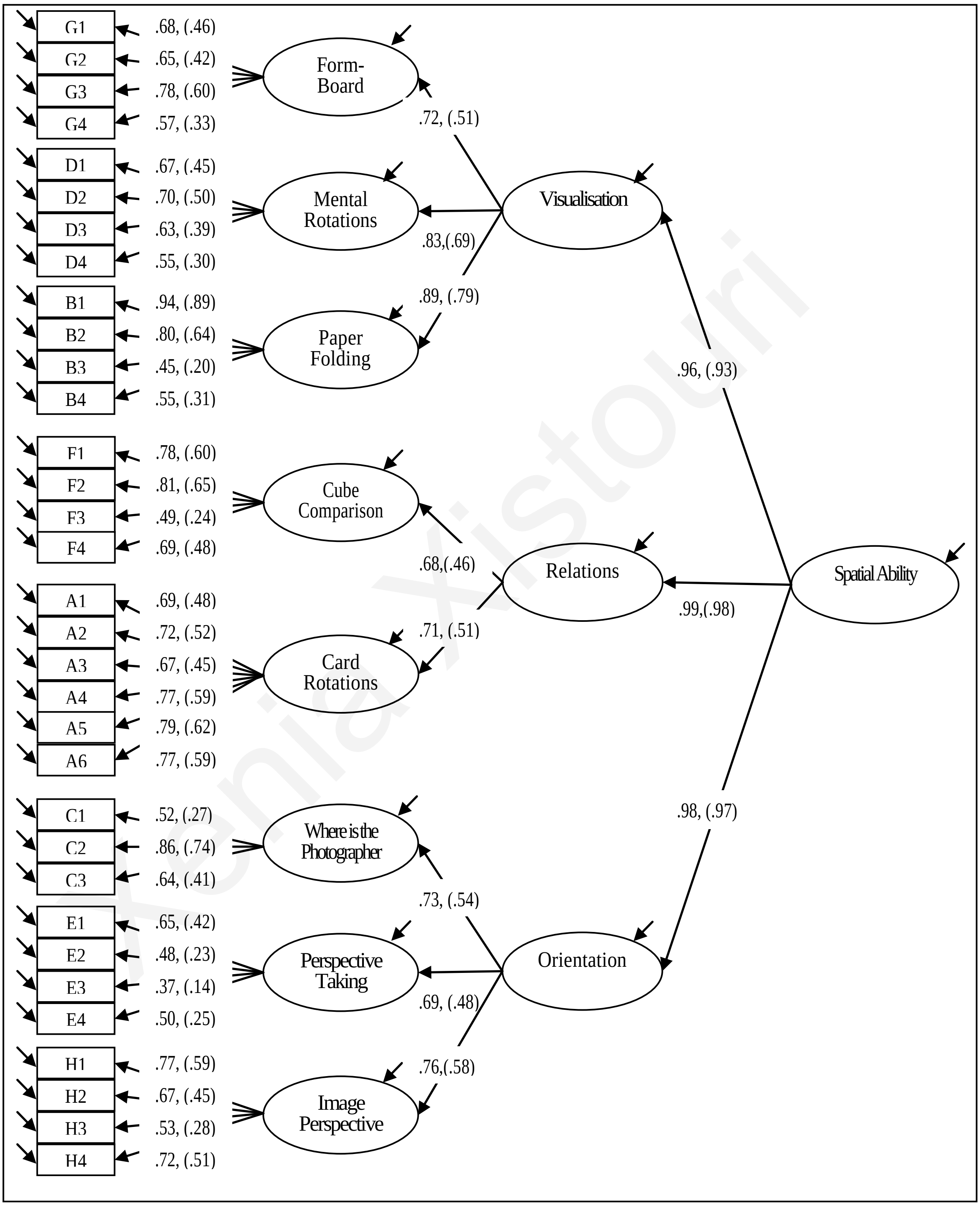
* $p < .05$. ** $p < .01$.



### *The structure of spatial ability*

In order to examine the structure of spatial ability, the validity of the study's theoretical model was tested. According to the theoretical model, spatial ability is a multi-dimensional ability which consists of three distinct factors: "Spatial Visualisation", "Spatial Relations", and "Spatial Orientation".

The result of the confirmatory factor analysis showed that the data of the study fitted the theoretical model very well, thus confirming the validity of the model to describe the structure of spatial ability ($CFI = .96$, $x^2 = 856.71$, $df = 487$, $x^2/df = 1.76$, $p < .05$, $RMSEA = .04$). All of the items had statistically significant loadings to the corresponding factors, as presented in Figure 4.25. The discrete nature of the factors is also confirmed by the fact that all of the observed variables loaded to only one first-order factor. The fitting of the data to the theoretical model of the study confirms that the items used in the test are suitable measures for the latent factors, and that the factors "Spatial Visualisation", "Spatial Relations", and "Spatial Orientation" form and predict a higher theoretical structure, which is spatial ability.




*Note.* The first number indicates factor loading and the number in parenthesis indicates the corresponding interpreted dispersion ($r^2$).

*Figure 4.25.* The model for the structure of spatial ability.

As shown in Figure 4.25, the interpreted dispersion of the items was quite high, suggesting that it can interpret the dispersion of the factors of the model. The factor loadings of all first- and second-order factors which correspond to the higher order factor of spatial ability were statistically significant and quite high. The factor of performance for the items of the "Paper Folding" test has the highest ability for predicting the "Spatial Visualisation" factor ($r^2 = .79$), the factor of performance for the items of the "Card Rotations" test has the highest ability for predicting the "Spatial Relations" factor ($r^2 = .51$), and the factor of performance for the items of the "Image Perspective" test has the highest ability for predicting the "Spatial Orientation" factor ($r^2 = .58$), respectively. The structure of the suggested model also showed that factors of "Spatial Visualisation", "Spatial Relations", and "Spatial Orientation" can predict spatial ability at the same degree ($r^2_{\text{Visualisation}} = .93$, $r^2_{\text{Relations}} = .98$, $r^2_{\text{Orientation}} = .97$, respectively).

*Subjects' performance in the spatial ability factors*

Table 4.36 presents the descriptive results for the subjects of the study in their performance in spatial ability. The subjects' performance was between .49 and .67 in the "Spatial Visualisation" factor and the "Spatial Relations" factor ($M_{\text{Spatial Visualisation}} = .54$, $M_{\text{Spatial Relations}} = .62$), indicating an average performance. In the factor of "Spatial Orientation", the subjects' performance was below .49 ($M_{\text{Spatial Orientation}} = .40$), indicating low performance. In the general factor of "Spatial ability", the subjects' performance was higher than .49, indicating an average performance ($M_{\text{Spatial Ability}} = .52$). The large values of standard deviations and ranges indicate that there is great diversity in the subjects' performance. The values of Skewness and Kurtosis are lower than two, which suggests that the distribution of the subjects' performance in the spatial ability test is normal.

Table 4.36

*Descriptive Results of the Subjects' Performance in the Spatial Ability Factors*

| Spatial Factor | Mean | Standard Deviation | Range | Skewness | Kurtosis |
|---|---|---|---|---|---|
| Visualisation | .54 | .21 | 1 | -.49 | .41 |
| Relations | .62 | .25 | 1 | -.58 | .01 |
| Orientation | .40 | .22 | 1 | .37 | -.04 |
| Spatial ability | .52 | .20 | .99 | -.40 | .80 |

*Relation between transformational geometry ability and spatial ability*

Table 4.37 presents the correlations between all the factors of transformational geometry ability and all the factors of spatial ability. There were significant correlations between all the factors of transformational geometry ability and spatial ability. Subjects' performance in all transformational geometry concepts is positively related to "Spatial Ability" ($r_{\text{Translation}}$ = .42, $r_{\text{Reflection}}$ = .51, $r_{\text{Rotation}}$ = .53, $p < .01$, respectively), as well as overall performance in the transformational geometry test ($r_{\text{Transformational Geometry}}$ = .56, $p < .01$). Moreover, performances in all the transformational geometry concepts and in overall transformational geometry appear to have the highest correlations with the "Spatial Visualisation" factor of spatial ability, and the lowest correlations with the "Spatial Relations" factor of spatial ability.

Table 4.37

*Correlations Between the Performance of the Subjects in the Transformational Geometry Factors and the Spatial Ability Factors*

| Ability | Spatial Visualisation | Spatial Relations | Spatial Orientation | Spatial Ability |
|---|---|---|---|---|
| Translation | .40** | .30** | .39** | .42** |
| Reflection | .48** | .38** | .47** | .51** |
| Rotation | .48** | .42** | .46** | .53** |
| Transformational Geometry | .53** | .42** | .50** | .56** |

** $p < .01$.

In order to investigate the relation between transformational geometry ability and spatial ability, the structure of three theoretical models was investigated. Model 1 assumes that spatial ability, as a multidimensional ability synthesised by the factors of "Spatial Visualisation", "Spatial Relations", and "Spatial Rotation", can predict performance in transformational geometry, as a multidimensional factor synthesised by the factors of "Translation ability", "Reflection ability", and "Rotation ability". Model 2 assumes that transformational geometry ability, as a multidimensional factor synthesised by the factors of "Translation ability", "Reflection ability", and "Rotation ability" can predict performance in spatial ability, as a multidimensional ability synthesised by the factors of "Spatial Visualisation", "Spatial Relations", and "Spatial Rotation". Model 3 assumes that transformational geometry ability and spatial ability are sub-factors of a more general ability, namely "General spatial abilities" (see Figure 4.26).

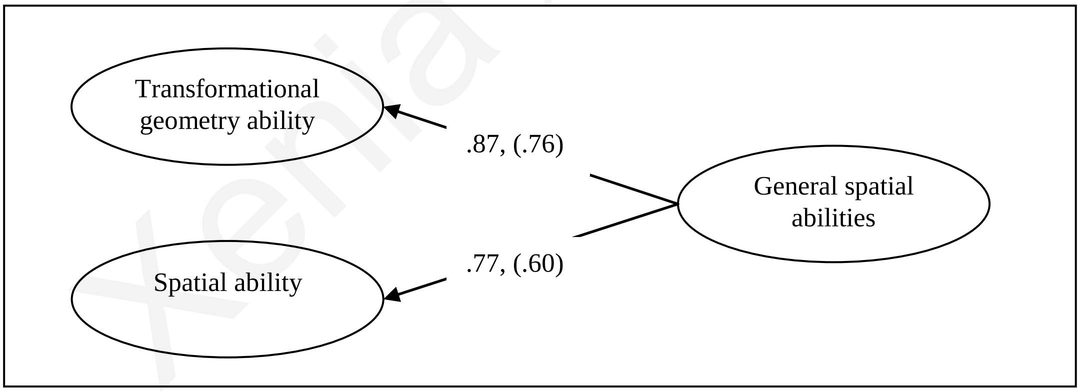
The results of the structural equation model analysis showed that the best model to describe the relation between transformational geometry ability and spatial ability was Model 3 (see Table 4.38), according to which transformational geometry ability and spatial ability are sub-factors of a more general ability, namely "General spatial abilities".

Table 4.38

*Fit Indices of Models for the Relation Between Transformational Geometry Ability and Spatial Ability*

| Model | Fit Indices | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | *CFI* | $x^2$ | *df* | $x^2/df$ | *p* | *RMSEA* | *AIC* | *BIC* |
| Model 1 | .98 | 161.09 | 87 | 1.85 | < .05 | .04 | -667.75 | -528.54 |
| Model 2 | .98 | 146.38 | 87 | 1.68 | < .05 | .04 | -682.47 | -543.25 |
| Model 3 | .98 | 136.27 | 86 | 1.58 | < .05 | .03 | -690.58 | -547.15 |

The factor loadings of the model were statistically significant. Specifically, transformational geometry ability has high ability for predicting general spatial ability ($r^2_{Transformational\ geometry}$ = .76, $p < .05$), and spatial ability has average ability for predicting general spatial ability ($r^2_{Spatial\ ability}$ = .60, $p < .05$) (see Figure 4.26). Based on this model, it seems that students develop their transformational geometry ability and their spatial ability in a more general context, and one does not precede the development of the other.




*Note.* The first number indicates factor loading and the number in parenthesis indicates the corresponding interpreted dispersion ($r^2$).

*Figure 4.26.* The model for the relation between transformational geometry ability and spatial ability.

The results of the multiple analyses of variances showed that there were statistically significant differences in students' spatial ability factors and overall spatial ability (Pillai's $F_{(3,484)} = 17.35$, $p < .01$). Table 4.39 presents the means of performance of the subjects in the four classes in the three spatial ability factors namely "Spatial Visualisation", "Spatial Relations", and "Spatial Orientation", and in the general factor of "Spatial Ability". The mean performance of each class in the three spatial ability factors was significantly higher than the corresponding mean of the previous class, except between Class 1 and Class 2, in which they do not appear to have any significant differences in their spatial ability. The mean performance of Class 1 in the spatial ability factors was lower than .49 in "Spatial Visualisation" and in "Spatial Orientation" ($M_{\text{Spatial Visualisation}} = .42$, $M_{\text{Spatial Orientation}} = .29$), and higher than .49 in "Spatial Relations" ($M_{\text{Spatial Relations}} = .53$). Similarly, the mean performance of Class 2 in the spatial ability factors was lower than .49 in "Spatial Visualisation" and in "Spatial Orientation" ($M_{\text{Spatial Visualisation}} = .47$, $M_{\text{Spatial Orientation}} = .32$), and higher than .49 in "Spatial Relations" ($M_{\text{Spatial Relations}} = .54$). The mean performance of Class 3 was higher than .49 in all three factors of spatial ability ($M_{\text{Spatial Visualisation}} = .56$, $M_{\text{Spatial Relations}} = .65$, $M_{\text{Spatial Orientation}} = .41$). The mean performance of Class 4 in the spatial ability factors was higher than .67 in "Spatial Visualisation" and in "Spatial Relations" ($M_{\text{Spatial Visualisation}} = .72$, $M_{\text{Spatial Relations}} = .78$), and between .49 and .67 in "Spatial Orientation" ($M_{\text{Spatial Orientation}} = .61$).

Table 4.39

*Means and Standard Deviations of Performance in the Spatial Ability Factors for All Classes*

| Class | Visualisation | | Relations | | Orientation | |
|---|---|---|---|---|---|---|
| | Mean | SD | Mean | SD | Mean | SD |
| Class 1 | .42 | .17 | .53 | .23 | .29 | .16 |
| Class 2 | .47 | .17 | .54 | .23 | .32 | .17 |
| Class 3 | .56 | .19 | .65 | .23 | .41 | .20 |
| Class 4 | .72 | .21 | .78 | .23 | .61 | .25 |

Table 4.40 summarises the characteristics of the four classes of the subjects. The performance of a class of subjects is characterised as high when it is equal or higher than .67, as average when it is between .49 and .67, and as low when it is lower than .49. The subjects of Class 1 had average performance only in the "Spatial Relations" factor, whereas their performance in the other two factors was low. Similarly, the subjects of Class 2 had average performance only in the "Spatial Relations" factor and low performance in the other two factors. The subjects of Class 3 had average performance in the "Spatial Visualisation" and "Spatial Relations" factors, and low performance in the "Spatial Orientation" factor. The subjects of Class 4 had high performance in the "Spatial Visualisation" and "Spatial Relations" factors, and average performance in the "Spatial Orientation" factor.

Table 4.40

*Characteristics of the Four Classes in Spatial Ability*

| Performance Level | Class 1 | Class 2 | Class 3 | Class 4 |
|---|---|---|---|---|
| High Performance ($M \geq .67$) | | | | SV, SR |
| Average Performance ($.49 \leq M < .67$) | SR | SR | SV, SR | SO |
| Low Performance ($M < .49$) | SV, SO | SV, SO | SO | |

*Note.* SV = Spatial Visualisation, SR = Spatial Relations, SO = Spatial Orientation.

This section presents the results that concern the fifth aim of the research regarding the investigation of the relation between students' ability in transformational geometry and their cognitive style. Specifically, it answers the following research question:

(6) What is the relation between students' cognitive style and their transformational geometry ability?

In order to assess cognitive style, a Greek translation of the c-OSIVQ was used. Before investigating the relation between transformational geometry ability and cognitive style, the reliability of the cognitive style assessment tool was investigated. Therefore, the descriptive information of the cognitive style questionnaire is described first and further on the investigation of the relation between transformational geometry ability and cognitive style is presented.

*Descriptive results of the cognitive style questionnaire*

Table 4.41 presents the descriptive information for the cognitive style questionnaire, in each cognitive style scale. The highest mean scores of the subjects were in the object scale items ($M$ = 3.77), and the lowest were in the spatial scale items ($M$ = 3.32). The minimum values of each scale were smaller than two, which shows that there were subjects that reported low assessment in the scales of the questionnaire, and the highest values are all 5, which shows that there were subjects that reported high assessment in the scales of the questionnaire. Table 4.41 presents the values for Skewness and Kurtosis for the subjects' scores to the three scales of the cognitive style questionnaire. The values of these indices were smaller than two, which suggests that the variables of the subjects' scores to the three scales of the cognitive style questionnaire follow a normal distribution. The reliability coefficient of the c-OSIVQ items was Cronbach's Alpha = .91, which is considered excellent (Klein, 1999). The reliability coefficients of the items for each scale of cognitive style were also satisfactory ($\alpha_{Object}$ = .82, $\alpha_{Spatial}$ = .83, $\alpha_{Verbal}$ = .84).

Table 4.41

*Descriptive Results of the Cognitive Style Scales*

| Scale | Mean | Standard Deviation | Minimum | Maximum | Skewness | Kurtosis |
|---|---|---|---|---|---|---|
| Object | 3.77 | .68 | 1.73 | 5 | -.29 | -.47 |
| Spatial | 3.32 | .77 | 1.00 | 5 | -28 | -.40 |
| Verbal | 3.50 | .73 | 1.67 | 5 | -.15 | -.62 |

*Relation between transformational geometry and cognitive style*

In order to describe the relation between transformational geometry and cognitive style, the correlations between the variables of ability in transformational geometry concepts and cognitive style scales were first investigated. Table 4.42 presents the correlations between ability in transformational geometry concepts and cognitive style scales. Statistically significant correlations appear only between the verbal scale score and abilities in reflection ($r$ = -.14, $p < .01$), rotation ($r$ = -.11, $p < .05$), and the general factor of transformational geometry ability ($r$ = -.12, $p < .01$). All the correlations are low, which suggests that there is a weak relation between the subjects' scores in the verbal scale of the cognitive style questionnaire and their performance in the transformational geometry test.

Table 4.42

*Correlations Between Ability in Transformational Geometry Concepts and the Cognitive style Scales*

| Ability | Object | Spatial | Verbal |
|---|---|---|---|
| Translation | .01 | -.03 | -.06 |
| Reflection | -.08 | .04 | -.14** |
| Rotation | -.08 | .04 | -.11* |
| Transformational Geometry | -.06 | .02 | -.12** |

* p < .05. ** p < .01.

*Classes of students in transformational geometry concepts and their cognitive style*

Table 4.43 presents the mean scores of the four classes of students in the three scales of cognitive style. The results of the multiple analyses of variances showed that there were statistically significant differences in students' cognitive style scale scores (Pillai's $F_{(3,464)} = 1.94$, $p < .05$). Specifically, the only statistically significant differences between the four classes means in the cognitive style scales appears between the verbal scale scores of Class 1 and Class 4, and between the verbal scale scores of Class 2 and Class 4 ($p < .05$ and $p < .05$, respectively).

Table 4.43

*Means and Standard Deviations of the Subjects' Scores in the Cognitive Style Scales*

| Class | Object | | Spatial | | Verbal | |
|---|---|---|---|---|---|---|
| | Mean | SD | Mean | SD | Mean | SD |
| Class 1 | 3.79 | .73 | 3.32 | .91 | 3.63 | .80 |
| Class 2 | 3.88 | .64 | 3.33 | .76 | 3.63 | .70 |
| Class 3 | 3.75 | .67 | 3.30 | .71 | 3.44 | .68 |
| Class 4 | 3.68 | .70 | 3.41 | .77 | 3.35 | .76 |

This section presents the results related to the sixth aim of the research regarding the impact of transformational geometry instruction with two different interactive dynamic visualisations on students' transformational geometry and spatial abilities. Specifically, it answers the following research question:

(7) What is the impact of transformational geometry instruction on students' transformational geometry and spatial abilities when two different interactive dynamic visualisations (continuous and discrete) are used?

In order to examine the impact of transformational geometry instruction on students' transformational geometry and spatial abilities with two different interactive dynamic visualisations, continuous and discrete, two groups with equal abilities and individual traits had to be formed. In this section, the descriptive results of the abilities and personal traits of the two groups are presented first. The mean comparisons of the students' abilities and traits are also presented. Finally, the impact of the two interventions of instruction with discrete and dynamic visualisations is presented.

*Group mean comparisons prior to the intervention*

Table 4.44 presents the means and standard deviations for the two groups, regarding their ability in transformational geometry concepts, their spatial ability, and their cognitive style. A MANOVA analysis was performed in order to compare the abilities and personal characteristics of the two groups prior to the interventions. The results suggest that the two groups did not have any statistically significant differences in their transformational geometry and spatial abilities, or in their cognitive style (Pillai's $F_{(1, 77)} = 0.26$, $p > .05$) (see Table 4.44).

Table 4.44

*Mean Comparisons of the Two Groups Prior to the Intervention*

| Performance | Group 1 | | Group 2 | | | |
|---|---|---|---|---|---|---|
| | $M_1$ | *SD* | $M_2$ | *SD* | *F* | *p* |
| Transformational Geometry | .48 | .19 | .47 | .18 | .00 | .97 |
| Spatial Ability | .52 | .16 | .52 | .11 | .02 | .90 |
| Cognitive Style | | | | | | |
| Object | 3.71 | .57 | 3.69 | .69 | .02 | .89 |
| Spatial | 3.40 | .96 | 3.24 | .66 | .67 | .42 |
| Verbal | 3.44 | .68 | 3.32 | .55 | .70 | .41 |



*The impact of dynamic visualisations*

In order to compare the impact of the two dynamic visualisation interventions on the two groups' performance in the transformational geometry post-test and in the spatial ability post-test, controlling for their pre-test scores, multivariate analysis of covariance (MANCOVA) was used. Table 4.45 presents the results of the MANCOVA of the post-test performance scores of the two groups in transformational geometry and spatial abilities.

The multiple analysis of covariance related to the post-test performance scores in the transformational geometry test indicated significant overall intervention effects, controlling for pre-test scores in the transformational geometry test and in the spatial ability test (Pillai's $F_{(1,77)} = 6.83$, $p < .05$). As shown in Table 4.45, the continuous intervention group subjects' performance was significantly higher than the discrete intervention group subjects in both transformational geometry and spatial abilities. Both effect size indices for transformational geometry ability (*partial* $\eta^2 = .08$) and for spatial ability (*partial* $\eta^2 = .13$) suggest that the effects of the continuous dynamic visualisation intervention over the discrete dynamic visualisation intervention were moderate (Cohen, 1988).

Table 4.45

*Results of the Multiple Covariance Analysis Between the Two Intervention Groups Post-test Performance in Transformational Geometry and Spatial Ability*

| | Continuous Group | | Discrete Group | | | | | |
|---|---|---|---|---|---|---|---|---|
| Ability | Mean[1] | *SE* | Mean[1] | *SE* | *df* | *F* | *p* | $\eta_p^2$ |
| Transformational geometry | .56 | .02 | .49 | .02 | 1 | 6.87 | .01* | .08 |
| Spatial ability | .63 | .02 | .55 | .02 | 1 | 10.83 | .00** | .13 |

[1] Estimated Marginal Means

* p < .05. ** p < .01.

In order to compare the differences within the groups' pre-test and post-test scores in the transformational geometry and the spatial abilities tests, paired-samples t-tests scores were performed. Table 4.46 presents the means and standard deviations of the pre-tests and post-tests of transformational geometry and spatial ability for the continuous dynamic intervention group. The results of paired samples t-tests showed statistically significant differences in the mean difference between the pre- and post-tests means of performance of the continuous dynamic visualisation intervention group. Specifically, according to Table 4.46, the students in the continuous dynamic intervention group had a significant increase in both their transformational geometry and spatial abilities means.

Table 4.46

*T-test Comparisons Between Pre-test and Post-test Performance of the Continuous Dynamic Intervention Group Subjects in Transformational Geometry and Spatial Ability*

| | Pre-test | | Post-test | | | |
|---|---|---|---|---|---|---|
| Ability | *M* | *SD* | *M* | *SD* | *t (df)* | *p* |
| Transformational geometry | .48 | .19 | .56 | .18 | -3.74(39) | .00** |
| Spatial ability | .52 | .16 | .63 | .15 | -5.76(39) | .00** |

** p = .01.

Table 4.47 presents the means and standard deviations of the pre-tests and post-tests of transformational geometry and spatial ability for the discrete dynamic intervention group. The results of paired samples t-tests showed that no statistically significant differences exist between the pre- and post-tests means of performance of the discrete dynamic visualisation intervention group.

Table 4.47

*T-test Comparisons Between Pre-test and Post-test Performance of the Discrete Dynamic Intervention Group Subjects in Transformational Geometry and Spatial Ability*

| | Pre-test | | Post-test | | | |
|---|---|---|---|---|---|---|
| Ability | *M* | *SD* | *M* | *SD* | *t (df)* | *p* |
| Transformational geometry | .47 | .18 | .49 | .18 | -.82 (38) | .42 |
| Spatial ability | .47 | .11 | .55 | .18 | -1.72 (38) | .10 |

This section presents the results concerning the seventh aim of the research regarding the impact of the interactions between students' spatial ability level, cognitive style, and types of dynamic visualisation. Specifically, it answers the following research question:

(8) What is the impact of the interactions between the type of dynamic visualisation (continuous and discrete) and students' individual differences in level of spatial ability and cognitive style on their transformational geometry and spatial abilities?

In order to investigate the impact of the interactions between dynamic visualisation type and students' individual differences for learning transformational geometry concepts, a multivariate analysis of covariance (MANCOVA) was used to assess the moderation effects of type of dynamic visualisation, spatial ability, and cognitive style, as well as their interactions on students' benefits in performance in transformational geometry ability and spatial ability, while adjusting for covariates in the students' abilities prior to the instructional interventions. Students' benefits were calculated as the difference between their post-test and pre-test scores for each ability. In the analysis, the dynamic visualisation type had two groups: continuous dynamic and discrete dynamic. The subjects were grouped in levels of spatial ability levels using median split, i.e., a student with mean score in the spatial ability pre-test that was equal to or higher than the median was classified at the high spatial ability level, while a student with a mean score in the spatial ability pre-test that was lower than the median was classified at the low spatial ability level. Therefore, in the analysis the spatial ability level had two groups: low spatial ability and high spatial ability. The subjects were also grouped for their cognitive style ratings, using median split on each of the three scale scores of object imagery, spatial imagery, and verbal, i.e., a student with mean score in the object imagery scale that was equal to or higher than the median was classified at the high object imagery level, while a student with a mean score in the object imagery scale that was lower than the median was classified at the low object imagery level. Two levels were created for each scale. Thus, after combining the three dimensions of cognitive style according to a student's level in each of them, eight groups were formed. For example, a student's cognitive style can be described as high object imagery, high spatial imagery, and low verbal, or in another case as low objet imagery, high spatial imagery, and high verbal. When considering the high and low levels for all the three dimensions, eight types of cognitive styles are created. Therefore, in the analysis the cognitive style had eight groups: (1) high object, high spatial and high verbal; (2) high object, high spatial and low verbal; (3) high object, low spatial and high verbal; (4) low object, high spatial and high verbal; (5) high object, low spatial and low verbal; (6) low object, low spatial and high verbal; (7) low object, high spatial and low verbal; (8) low object, low spatial and low verbal.

Table 4.48

*Results of the Analysis of Covariance for the Effects of Dynamic Visualisation Type and Individual Differences Interactions on the Subject's Benefits in Transformational Geometry Ability and Spatial Ability*

| | *df* | *SS* | *MS* | *F* | $\eta_p^2$ | *p* |
|---|---|---|---|---|---|---|
| Benefits in transformational geometry ability | | | | | | |
| Main Effects | | | | | | |
| Dynamic visualisation type | 1 | .10 | .10 | 6.40 | .12 | .02* |
| Spatial ability level | 1 | .04 | .04 | 2.59 | .05 | .11 |
| Cognitive style group | 7 | .13 | .02 | 1.24 | .15 | .30 |
| Interactions | | | | | | |
| Dynamic visualisation X Spatial ability level | 1 | .05 | .05 | 3.46 | .07 | .07 |
| Dynamic visualisation X Cognitive style group | 6 | .14 | .02 | 1.61 | .17 | .17 |
| Benefits in spatial ability | | | | | | |
| Main Effects | | | | | | |
| Dynamic visualisation type | 1 | .06 | .06 | 5.80 | .10 | .02* |
| Spatial ability level | 1 | .00 | .00 | .30 | .01 | .59 |
| Cognitive style group | 7 | .08 | .01 | 1.16 | .15 | .34 |
| Interactions | | | | | | |
| Dynamic visualisation X Spatial ability level | 1 | .00 | .00 | .25 | .01 | .62 |
| Dynamic visualisation X Cognitive style group | 6 | .05 | .01 | .85 | .10 | .54 |

$* p < .05$.

Table 4.48 presents the results of the multiple analysis of covariance. The main effect of dynamic visualisation type on transformational geometry ability and on spatial ability was the only significant effect after adjusting for covariates (Pillai's $F_{(2,47)} = 4.83$, $p < .05$, $\eta_p^2 = .17$). There were no other significant effects for individual differences in spatial ability and cognitive style on students' benefits from the instructional programs, nor of their interactions with the type of dynamic visualisation (see Table 4.48).

# CHAPTER V

# DISCUSSION AND CONCLUSIONS

## Introduction

According to the NCTM (2000) Standards for geometry, the application of transformations to analyse mathematical situations and the use of visualisation and spatial reasoning to solve problems are important for students. This is why many research studies have investigated the development and reasoning of students in transformational geometry concepts (Boulter & Kirby, 1994; Bansilal & Naidoo, 2012; Edwards, 1990, 1991; Edward & Zazkis, 1993; Flanagan, 2001; Guven, 2012; Hollebrands, 2003; Kidder, 1976; Law, 1991; Molina, 1990; Moyer, 1978; Nasser, 1989; Schultz & Austin, 1983; Soon, 1989; Thomas, 1978, Portnoy et al., 2006; Yanik, 2006), and have also related transformational geometry ability to spatial ability (Boulter, 1992; Del Grande, 1986; Dixon, 1995; Kirby & Boulter, 1999; Williford, 1992).

Many of the studies in the field of transformational geometry have involved technology in the presentation of transformations in order to investigate the improvement of students' learning outcomes, and even sometimes of their spatial abilities (Dixon, 1995; Edwards, 1991; Edwards & Zazkis, 1993; Ernest, 1986; Guven, 2012; Hoong & Khoh, 2003; Johnson-Gentile, 1990; Olkun, 2005; Pleet, 1990; Smith et al., 2009). The NTCM's (2000) Technology Principle suggests that technology is essential in teaching and learning mathematics, as it influences the mathematics that is taught and enhances students' learning. This principle prompts teachers to make sensible decisions about how to use technology to ensure that it is enhancing students' mathematical thinking. Researchers from both the fields of mathematics education and psychology draw attention to the fact that there are different types of technological dynamic visualisations (Hegarty, 2004a; Moreno-Armella et al., 2008; Smith el al., 2009), and also to the fact that these dynamic visualisations can have a different impact on learning. Moreover, concerns have also been raised regarding the impact that such dynamic visualisations can have on different of learners, regarding individual differences in spatial ability and in cognitive style (Höffler, 2010; Kirby & Boulter, 1999; Mayer, 2005).

The aim of the study was to develop a theoretical model for the learning and teaching of transformational geometry concepts (translations, reflections, and rotations), to examine its relation to individual differences in spatial ability and cognitive style, and to investigate the impact of two interactive dynamic visualisations on the spatial and transformational geometry abilities of different types of students. The specific purposes of the study were the following: (a) to investigate the components that synthesise students' ability in transformational geometry and the structure of this ability, (b) to investigate students' ability in transformational geometry concepts, (c) to identify and describe the students' levels of ability in transformational geometry concepts, (d) to investigate the relation between students' ability in transformational geometry and their spatial ability, (e) to investigate the relation between students' ability in transformational geometry concepts and their cognitive style, (f) to investigate the impact of transformational geometry instruction with two different interactive dynamic visualisations on students' transformational geometry and spatial abilities, and (g) to investigate the interactions of students' level of spatial ability and cognitive style with different types of interactive dynamic geometry visualisations. This chapter presents the discussion of the results to address the aims and questions of the study. Further on, it presents the statement of the conclusions, as they emerge from the discussion of the findings of the study. It also acknowledges the educational application of these findings in the field of education, and ends with suggestions for further research.

## The Components and Structure of Transformational Geometry Ability

Based on the results of this study, students' ability in transformational geometry can be analysed in three factors. These factors are: "Translation ability", "Reflection ability", and "Rotation ability". The factor of "Translation ability" refers to the ability to solve tasks related to the transformational geometry concept of translation. The factor of "Reflection ability" refers to the ability to solve tasks related to the transformational geometry concept of reflection. The factor of "Rotation ability" refers to the ability to solve tasks related to the transformational geometry concept of rotation.

Each of the three factors that refer to the abilities in transformational geometry concepts can be analysed into four similar cognitive factors. These factors are: "Recognition of image", "Recognition of transformation", "Identification of parameters", and "Construction of image". The factor "Recognition of image" in translation refers to the ability to recognise a figure's image when it is translated vertically, horizontally, or diagonally. The factor “Recognition of transformation” in translation refers to the ability to recognise an example of translation among counter-examples that represent other transformations or combinations of transformations, based on the property of translation to maintain the orientation of the image. The factor “Identification of parameters” in translation refers to the ability to identify and name the parameters of direction and distance between a pre-image and its image in translation, with respect to the operative configurations of direction, i.e., vertically, horizontally, or diagonally, and of distance i.e., overlapping images. The factor “Construction of image” in translation refers to the ability to construct the image of a figure when it is translated towards a specific direction and distance, with respect to the operative configurations of direction, i.e., vertically, horizontally, or diagonally, and of distance, i.e., overlapping images, and with respect to the figurative configuration of shape complexity.

The factor “Recognition of image” in reflection refers to the ability to recognise a figure’s image when it is reflected vertically, horizontally, or diagonally. The factor “Recognition of transformation” in reflection refers to the ability to recognise an example of reflection among counter-examples that represent other transformations or combinations of transformations, based on the property of reflection to inverse the orientation of the image. The factor “Identification of parameters” in reflection refers to the ability to identify and draw the line of the reflection in the correct direction and the correct distance between a pre-image and its image, with respect to the operative configurations of direction, i.e., vertically, horizontally, or diagonally, and of distance, i.e., overlapping images. The factor “Construction of image” in reflection refers to the ability to construct the image of a figure when it is reflected in a given line of reflection in a specific distance, with respect to the operative configurations of direction, i.e., vertically, horizontally, or diagonally, and of distance, i.e., overlapping images, and with respect to the figurative configuration of shape complexity.

The factor “Recognition of image” in rotation refers to the ability to recognise a figure’s image, based on its orientation, when it is rotated vertically, horizontally, or diagonally, using as benchmark angles the quarters of a circle. The factor “Recognition of transformation” in rotation refers to the ability to recognise an example of rotation among counter-examples that represent other transformations or combinations of transformations, based on the properties of rotation to change the orientation and the position of the image. The factor “Identification of parameters” in rotation refers to the ability to identify and draw the centre of rotation in the correct distance between a pre-image and its image, and to identify and name the correct angle of rotation using as benchmark angles the quarters of a circle, with respect to the operative configurations of direction, i.e., vertically, horizontally, or diagonally, and of distance, i.e., overlapping images. The factor “Construction of image” in rotation refers to the ability to construct the image of a figure when it is rotated around a given centre of rotation in a specific angle, using as benchmark angles the quarters of a circle, with respect to the operative configurations of direction, i.e., vertically, horizontally, or diagonally, and of distance, i.e., overlapping images, and with respect to the figurative configuration of shape complexity.

The findings of the study confirm Kidder’s (1976) proposition that transformational geometry is a multifaceted mental operation. The model for ability in transformational geometry concepts which was confirmed in this study combines both the mathematical and cognitive dimensions of ability in transformational geometry concepts suggested in previous studies (Molina, 1990; Soon, 1989). Moreover, the model takes into consideration the configurations that influence the relative difficulty of transformational geometry tasks that are suggested in literature, namely the operative configurations of transformational geometry concept, distance, and direction, and the figurative configuration of complexity of the image (Lesh, 1976; Schultz & Austin, 1983). In addition, the findings of this study suggest that the three transformational geometry concepts of translation, reflection, and rotation are synthesised by similar components, and that they also have similar structure.

The results of the study showed that the correlations between all 12 factors were statistically significant. The highest correlations within the same geometric transformation appear between the factors of “Identify parameters” and “Construct image” for the concept of translation, “Recognise image” and “Construct image” for the concepts of reflection, and “Recognise rotation” and “Construct image” for the concept of rotation. A possible reason for these differences may be that the three transformational geometry concepts follow different paths of development. It is important to note that, for all the three transformational geometry concepts of translation, reflection, and rotation, the factor of “Construction of image” had the lowest mean of performance among the students. Thus, it is possible that the cognitive ability to construct the image of a specific geometric transformation is related to different cognitive abilities for each of the three concepts.

The results of the study showed that the structure of the model of ability in transformational geometry concepts is stable for the students between nine and 14 years old. Even though the level of ability in transformational geometry concepts varies between the students of different grade levels, the dimensions of this ability are the same for all students, with respect to their educational level. Moreover, the fact that the content and the instructional methods of teaching geometry in primary and in secondary education are different does not appear to differentiate the structure of ability in transformational geometry concepts. These differences support the importance and the validity of the suggested model, since they suggest that the ability of students in the higher primary school grades and of students in lower secondary school grades in transformational geometry concepts is synthesised by the 12 discrete factors that were described above, even though they have many significant cognitive factors that may influence their ability in transformational geometry concepts.

## Students' Ability in Transformational Geometry Concepts

The results of the study showed that students from nine to 14 years old have average abilities in transformational geometry. Regarding their abilities in the three transformational geometry concepts, they have average performance in translation and low performance in reflection and in rotation. Specifically, regarding the factors of translation, the students had high performance only in the factor "Identify parameters", and average performance in the factors "Recognise image", "Recognise translation", and "Construct image". Regarding the factors of reflection, the students had average performance in the factor "Recognise reflection", and low performance in the factors "Identify parameters", "Recognise image", and "Construct image". Finally, in the factor of rotation, the students had low performance in all four factors.

The results of the study showed that there are significant differences in the transformational geometry abilities of the students of the five grades. The students of the first and second secondary school grades had significant differences from the students of the fourth and fifth primary school grades in all transformational geometry concepts, as well as in overall transformational geometry ability. Significant differences also exist between the secondary school students and the sixth grade primary school students in reflection ability and also between the students of the second grade of secondary school and the students of the sixth grade of primary school in their overall ability in transformational geometry. In primary school, there are significant differences between the students of the sixth grade and the students of the fourth grade in their abilities in translation, reflection, rotation, and overall abilities in transformational geometry, as well as between the students of the fifth grade and the students of the fourth grade in their reflection, rotation, and overall abilities in transformational geometry. The rest of the differences were not significant.

## Identification of Levels of Ability in Transformational Geometry Concepts

The results of the study indicated that there are four classes of students with different characteristics regarding their ability in transformational geometry concepts. The characteristics of each class of students are described further on, in order to highlight the differences between them.

The first class of students had low ability in the three transformational geometry concepts. Specifically, their abilities were low in all factors of the transformational geometry concepts, except for the factor of “Identify parameters” in translation, where their ability was average. The students of the second class had average ability in translation and low ability in reflection and rotation. Specifically, their abilities were high in the factor of “Recognise reflection” and average in the factor of “Recognise image” and “Identify parameters” in translation. The third class had significant differences compared to the second class. Their abilities were average in translation and reflection, and low in rotation. Specifically, the third class had high abilities in the factors of “Recognise reflection” and “Identify parameters” in translation. It also had average abilities in the factors of “Recognise image”, “Recognise translation”, and “Construct image” in translation, in the factors of “Identify parameters” and “Construct image” in reflection, and in the factors of “Recognise image”, “Recognise rotation”, and “Identify parameters” in rotation. Finally, the fourth class had high ability in all transformational geometry concepts. Specifically, their abilities were high in all the factors of the transformational geometry concepts, except for the factor of “Construct image” in rotation, where their ability was average.

The first class of students is mostly represented by students of the fourth grade of primary school, since more than a quarter of the students of this grade belong to this class. The second class is mostly represented by students of the fourth, fifth, and sixth grade of primary school, since nearly half of the students in the fourth and fifth grade and a quarter of the students in the sixth grade belong to this class. The third class is mostly represented by students in the fifth and sixth grades of primary school, and the two grades of secondary school, since one third of the students in the fifth grade of primary school and the largest percentages of the sixth grade of primary school and the secondary school classes belong to this class. The fourth class is mostly represented by the students of the first and the second secondary grade, since more than a quarter of the first grade of secondary school and more than a third of the second grade of secondary school belong in this class.

Based on the results of this study, it seems that fourth grade primary school students have different abilities than fifth grade primary school students in transformational geometry concepts, whereas fifth grade students have similar abilities to those of the students of the sixth grade. A transition seems to occur in secondary school, since the first grade secondary school students seem to have different abilities than the sixth grade primary school students; whereas the second grade secondary school students have similar abilities to those of the first grade secondary school students. Hence, it can be assumed that there are three transitional stages in students' ability in transformational geometry over the ages of nine to 14. The first stage includes the students who are at nine to 10 years old. The second stage includes the students who are at 10 to 12 years old. The third stage includes the students who are at 12 to 14 years old. These stages are consistent with the developmental stages of cognitive psychology found in literature (Piaget & Inhelder, 1967).

*Development of ability in transformational geometry concepts*

Based on the characteristics of the students in each class, the validity of two SEM models that were found in literature was investigated. The two models suggest that there is a hierarchical relationship in the development of transformational geometry concepts. The first model assumes that this hierarchy is based on the mathematical structures of transformational geometry, and follows a progression of (i) translation ability, (ii) reflection ability, and (iii) rotation ability, and is based on the research studies of Moyer (1978), Kidder (1976), and Piaget and Inhelder (1971). The second model assumes that the hierarchy is based on the cognitive structures of the students in transformational geometry concepts, as in the van Hiele levels, and follows a progression of different types of tasks, which are the same for all the transformational geometry concepts, as suggested in the research studies of Molina (1990) and Soon (1989). Thus, it would follow a progression of the factors: (i) "Recognise image" in translation, reflection, and rotation; (ii) "Recognise transformation" in translation, reflection, and rotation; (iii) "Identify parameters" in translation, reflection, and rotation; and (iv) "Construct image" in translation, reflection, and rotation.

Based on the results of the analysis, the model that best describes the development of students' ability in transformational geometry concepts is the first model, which is based on the mathematical structures. Thus, the results of the analysis suggest that students' performance in translation can significantly predict their performance in reflection. Moreover, students' performance in reflection can significantly predict their performance in rotation. Moreover, they suggest that the development of the cognitive factors within the development of each transformational geometry concept does not follow the same order, as suggested by Molina (1990) and Soon (1989). Instead, it appears that the different types of tasks follow a different order of development for each transformational geometry concept.

The results of this model are important because, firstly, they confirm that there is a hierarchical development of ability in transformational geometry concepts, and this suggests that the four classes of students that emerged from the quantitative data analysis may constitute hierarchical levels of ability in transformational geometry concepts. Secondly, the hierarchical structure of this model suggests that the mathematical structure of the tasks of transformational geometry concepts plays a more important role in the conformation of the levels of ability, rather than the cognitive structures shared by the tasks of different transformational geometry concepts. Finally, the hierarchical structure of this model suggests that abilities in the three geometric transformations do not develop simultaneously. Students first develop ability in translation, followed by ability in reflection, and finally they develop ability in rotation.



## Description of Levels of Ability in Transformational Geometry Concepts

The characteristics of the four levels of abilities are described further on, as they emerged from the quantitative and qualitative data of the study. The levels are first described based on the abilities of the students in the four levels regarding the 12 factors of transformational geometry ability, the strategies of the students at each level, their conceptions, and their typical errors. For each level, the description of abilities in the 12 factors of the model is followed by a summative description of the common characteristics of the students' approaches to the tasks of the twelve factors of the model, analysed from a cognitive perspective of the individual's reasoning and visualisation abilities.

### *Characteristics of the first level of abilities*

The students of the first level of abilities had low abilities in all the transformational geometry concepts. Specifically, the students of this level were unable to complete any type of task in transformational geometry concepts, except for one type of translation task, which regarded the identification of parameters. In particular, the students of the first level were only able to correctly identify the direction of the translation which moves a pre-image to the right. They were unable to correctly identify the direction of the translation when it was vertical, diagonal, or horizontal with overlapping figures.

The inability of the students of the first level to complete any type of task in any of the transformational geometry concepts that were described in the study may be related to their low spatial ability. Specifically, the students of the first level had low performance in spatial visualisation and spatial orientation, and average performance in spatial relations. It seems that their weakness to visualise images and manipulate them in their imagination, to visualise how an object will appear from a different point of reference, and to visualise spatial relations and spatial concepts inhibited them from succeeding in any type of transformational geometry task. Moreover, these weaknesses may also be related to the tendency of this group to prefer verbal processing of information. These students of the first level reported higher ratings in the verbal cognitive style questions, suggesting that they prefer to process information verbally, rather than visually.

The difficulties in translation of students of this level began from their inability to recognise the image of a translation. The students of the first level were not able to recognise the image of a figure in any direction. A common error of the students of the first level was that they focused more on measurement as a process of counting. They did not understand that what they were counting was actually a measure of distance. Moreover, they did not understand how distance was related to direction in translation. Therefore, they did not select the image in the correct direction. Another common difficulty of the students of the first level was to measure the correct distance between the pre-image and an optional image. Hence, the students of the first level often rejected the response that was in the correct direction, because of their errors in measuring the distance.

Apart from their difficulties regarding the parameters of direction and distance, which were evident in the type of tasks of recognition of image in translation, students' difficulties also regarded the understanding of the property of translation to maintain the orientation of the image under transformation. Specifically, in the type of tasks of recognition of translation, the students had difficulties in recognising the correct example of translation. Their difficulties arose from the fact that they had a general conception that the pre-image and the image must be the same. However, they could not explain where they expected the similarity to exist, as they lacked the appropriate vocabulary. A common error was to select the example of a reflection, which suggests that the students of the first level did not know that translation preserves the orientation of the figure. Another difficulty may be related to the position of the image in relation to the pre-image, since many students had the conception that the image had to be positioned next to the pre-image. This may also explain that, even though the students of the first level were unable to recognise the example of a translation in a specific direction, they had average performance in recognising the example of a translation when the direction was unspecified, and the two images were not placed near each other in the same figure.

Despite their average performance in the type of task of identifying parameters in translation, the students of the first level had many difficulties in measuring the correct distance between the images. Their difficulties are probably rooted in their conceptions of the figures as holistic objects. The students of the first level could not understand that translation is a function applied to all the points of the plane, and that distance must be measured from one point to its corresponding point. Hence, they were not able to measure the correct distance, because they could not match two corresponding points. The same difficulty was evident in the tasks of construction of image. The students of the first level were unable to construct the correct image in any configuration, operative or figurative. The biggest difficulty of the students at the first level was to construct the image in the correct distance from the pre-image. One of the main reasons seems to be that their low spatial ability did not allow them to analyse the pre-image using the spatial or geometrical relations between its components. As a result, they were not able to realise that the distance had to be the same between every corresponding point. Thus, a common error was to measure the distance starting from a random point of the pre-image, and construct the image at a position where the units of distance would all be visible between the two shapes. Moreover, the students of the first level were not even able to visualise the pre-image changing its position in space as a whole. Their effort was to copy the triangle correctly, as the same shape, and in the same size.

Regarding reflection, the students of the first level were unable to recognise the image of a reflection in any direction. The fact that the majority of the students of the first level selected the image that had the same orientation as the pre-image in this type of task suggests that, at this level, they did not understand that reflection inverses the orientation of a shape. The students had difficulties in understanding that a shape remains the same when its orientation changes. Nevertheless, these difficulties were not so evident in the type of task of recognition of reflection. Even though the students focused their attention on more superficial characteristics of the objects in the figure rather than the relations between the images, the majority of them were able to find the correct response. However, they were not able to explain their reasoning. This fact, as well as the fact that the students of the first level were only able to recognise the reflection in horizontal direction (i.e., in a vertical line of reflection) implies that the students of this level are sometimes able to recognise prototypical examples of reflection, without having a conceptual understanding. The students of this level also tried to visualise the folding of the paper in order to match the images. Probably emerging from the instructional approach of folding a paper when teaching symmetrical shapes or from an over-generalisation of the property of symmetry within figures, some of the students of the first level had the conception that, in order to have reflection, the two images must be the half parts of a whole figure.

The students of the first level also had difficulties in identifying the parameters of reflection. The students of this level had average performance only in finding the vertical line of reflection in horizontal direction, which again is a prototypical image for the concept of reflection. For the rest of the items, the students had difficulties in visualising the direction of the line of reflection, based on the relations between the pre-image and the image. They seemed to have difficulties in focusing on the positions of the images, as well as on the orientation of the images in order to understand the direction of the line. Moreover, they did not understand that the line of reflection needs to be in the same distance from the pre-image and the image. Their conception of halving was a significant obstacle in this task, since the students either attempted to halve the two images by drawing the line of reflection over them, or tried to halve the figure showing the representation of the reflection, as an attempt to divide an image into two halves. Thus, they had difficulties in understanding that the line of reflection is what determines the relation between the corresponding points of the images and of the plane.

The students of the first level were unable to construct any image in reflection. Their difficulties appeared to be mainly in measuring the correct distance to the line of reflection and in finding the correct orientation for the image. In the case of diagonal reflection, the students did not know that they needed to measure the distance to the line of reflection following a direction that was perpendicular to the line of reflection. In the case of vertical or horizontal line of reflection, they did measure the distance following perpendicular direction, but they did not seem to realise that this was important. Errors in the distance from the line of reflection were also a result of starting to count from a random point of the pre-image, since the students had difficulty to analyse the pre-image into components and understand that each component would have a different distance from the line of reflection. Holistic visual strategies also seemed to fail the students of this level, since due to their low spatial ability, they were not able to maintain the spatial information of the image in their memory, and failed to find the correct orientation of the image, the correct distance from the line, or both. What is characteristic at this level was the students' need to concretise the elements of the reflection. For example, the students either visualised the figure of the plane as a piece of paper folding in the real world, or the line of reflection as a mirror between the shapes.

The students of the first level had many difficulties regarding the concept of rotation. Specifically, they were unable to respond to any type of task in rotation, even partially correct. Their difficulties in recognition of the image in rotation were similar to their difficulties in reflection regarding the orientation of the image. More particularly, the students again had difficulty in understanding that changing the orientation of the image does not change its shape. In the type of task of recognition of rotation, the students of the first level had difficulties to understand that a significant property of rotation is to change the orientation of an image and its position around a fixed point. They did not have any specific strategy for approaching the task, and this was a significant reason that they often confused the examples of rotation with examples of other geometric transformations.

Regarding the identification of parameters in rotation, the students had difficulties in finding the centre of rotation, because they could not focus on the orientation or the positions of the images in order to estimate the direction of the point. Their approach was mainly to place the centre of rotation somewhere between the two shapes. They were not able to give reasons for explaining their decision for the place of the point. Regarding the angle of rotation, the students' main approach was to visualise the pre-image rotating. However, they did not understand the role of the angle of rotation. Their misconceptions regarding mathematical concepts that are relevant to rotation, such as angle and circle, inhibited them from finding the correct angle of rotation. Finally, in construction of image in rotation, the students of the first level were unable to complete any of the tasks of this type. They were not able neither to find nor express the relations between the elements of rotation. They could not understand the distance between the images and the centre of rotation, or the change of the image's position around the centre of rotation, neither the change of the orientation in the image of the shape.

The common characteristics that emerge from students' thinking in the different types of tasks in the different transformational concepts were that they seem to have a concrete, holistic conception of geometric transformations and their representation, which they conceive as a drawing. Their focus seems to be on the general picture, the figure that represents the plane, and they either observe the positioning of motionless sketches on it over the four basic directions of the real space, or imagine that they can manipulate as a tangible object, i.e., to halve or to fold, in order to solve a task. Their focus is rarely on the shapes that represent the pre-image and/or the image and the relations between them, and this is what makes it more difficult for them to approach the tasks of rotation. They do not understand the properties of geometric transformations, and they do not conceive them as a process of motion or as a concept of mapping. Because of their view of the figure as a holistic object that can be manipulated, this level was named "holistic image conception".

*Characteristics of the second level of abilities*

The students of the second level of abilities had average abilities in the concept of translation, and low abilities in the concepts of reflection and of rotation. Specifically, the students of the second level had high ability in recognition of reflection among other options, and average abilities in recognition of image and in identification of parameters for translation. However, although some of the approaches of the students at this level were quite different, the performance of the students of the second level was not very different from the students of the first level, since they performed better in only two of the 12 factors. The inability of the students of the second level to complete the majority of the tasks can be explained by the fact that they also had low performance in spatial visualisation and spatial orientation, and average performance in spatial relations, similarly to the students of the first level. Moreover, these weaknesses may again be related to the tendency of the students of the second level to prefer in verbal processing of information, in contrast to visual, more than the levels of students with higher ability in transformational geometry concepts, since they reported higher ratings in the verbal cognitive style statements of the questionnaire, in comparison to the group with the highest abilities.

Regarding their ability in translation, the students of this level had average abilities in two types of tasks, namely recognition of image and identification of parameters. Specifically, for recognition of image, the percentages of the students of the second level that were able to recognise the correct image were higher than the percentages of the first level, even though their common errors were the same. Specifically, they had similar difficulties with the students of the first level regarding the understanding of the relation between direction and distance. The students of this level also focused on counting procedures. However, the difference of the second level of students from the first level was that, even though they mainly relied on distance as a criterion of image, when they reached a cognitive conflict of two images having the same distance, the criterion of direction was then used. In the identification of parameters for translation, the students of the second level had average performance not only in the case of horizontal translation, as the students of the first level did, but also in the case of vertical translation. Their difficulties are similar to those of the first level of students. Specifically the students of the second level also made many errors in measuring the distance, because they conceived the pre-image and the image as holistic figures, and they tried to measure the distance between the figures, but not between corresponding points.



In the other two types of tasks in translation, recognition of translation, and construction of image, the students of the second level had low ability. Regarding recognition of translation, the students of the second level had average performance in recognising translation in vertical and diagonal translation, as well as in unspecified direction. They seemed to have more difficulties in the case of horizontal translation. The reason for this is that, at this level, the students began to detach the images from the general image of the plane, and used visualisation of motion over the image of the plane to solve a task. As shown from their interviews, the students had the conception that the response needed to have an evident change in the appearance of the image. Since they had difficulty to accept the change in the place of the image, they searched for a more evident change, and began to focus on the orientation of the image. For this reason, the most common type of error was to select the option of reflection.

In the type of task of construction of image in translation, the students of the second level were able to construct only the image of a horizontal translation, but with errors in the distance from the pre-image. The students of the second level had difficulties in measuring the distance, for many reasons. One reason was that they conceived the images as holistic figures, and measured the distance between random points instead of corresponding points. Moreover, their need for visualising holistic images inhibited them from counting units that were not fully visible in the figure of the plane. Sometimes they would only count the visible part of the unit for distance, thus adding half units. Similarly to the students of the first level, the students of the second level also emphasised more on drawing the shape correctly.

Regarding reflection, as mentioned above, the students had high performance in the type of task of recognition of reflection. The reason for this was that a quite high percentage of the students of the second level was able to recognise correctly the example of reflection in horizontal direction, i.e., in vertical line of reflection. In the other examples of reflection, in vertical and in diagonal direction, and in reflection with unspecified direction, the students often made the error of selecting the response of translation; however the percentages of correct responses were higher than those of the first level of students. It is presumed that what made the students of this level to have average performance in the horizontal reflection example were their experiences in symmetry instruction and their ability to recognise prototypical images of symmetry, which is usually in vertical line.

In the type of task of recognition of image in reflection, the students of the second level were unable to identify correctly the image in any direction. Similarly to the approaches of the students of the first level, they had difficulties in focusing on the effects of reflection to the orientation of image, which suggests lack of understanding of the inversion of orientation in reflection. Again, the majority of the students selected the image that had the same orientation as the pre-image in this type of task, suggesting that the students had difficulties in understanding that a shape remains the same even when its orientation changes.

Regarding the type of task of identification of parameters in reflection, the students of the second level were able to identify the line of reflection in vertical and in horizontal reflection. However, they had many difficulties in identifying the line of reflection in diagonal reflection and in reflection with overlapping images. The difficulty of the students at the second level in the diagonal reflection was, first, to find the direction of the line.

Many students had the conception that the two images needed to be on different sides of the line, and one being opposite the other. The conception of opposite often had the prototypical elements of left-right and up-down, which raised difficulties on what the opposite of a diagonal direction would be. The conception of having the images on different sides of the line was also an obstacle in finding the line of reflection between overlapping figures, since they could not separate the images. The second difficulty concerned finding the distance between the pre-image and the image in reflection, and mostly in diagonal reflection. The students at the second level knew that the line needed to be between the shapes, but they had difficulties in finding its exact position by measuring and halving the distance between two corresponding points.

In the type of task of construction of image in reflection, the students of the second level were unable to construct any image, in any direction and configuration. Their difficulties appeared to be mainly in measuring the correct distance to the line of reflection and in finding the correct orientation for the image. However, the students of the second level were more able than the students of the first level to find the correct orientation. The emphasis of the students at the second level was also on the external characteristics of the pre-image, and their aim was to construct an image with the same shape and the same dimensions. Another difficulty of the students at this level was that they tried to approach the tasks by using visualisation strategies of the images' motion over the line of reflection. However, since their spatial ability was generally low, they were not able to maintain control of the spatial relations of the elements of reflection in their mental images, and thus they made many errors in their responses, mostly in the distance of the image from the line of reflection, and also sometimes in the orientation of the image.

The students of the second level had many difficulties regarding the concept of rotation. Specifically, they were unable to respond to any type of task in rotation. Their difficulties in recognition of the image in rotation were similar to their difficulties in reflection regarding the orientation of the image. Specifically, the students again had difficulty to understand that changing the orientation of the image does not change its shape. In the type of task of recognition of rotation, the students of the second level had difficulties in understanding that a significant property of rotation is to change the orientation of an image and its position around a fixed point. Their main approach for this type of task was to visualise the shape rotating in order to find its image. However, the students had difficulty in understanding that a shape rotates both around a point, thus changing position, and around itself at the same time, thus changing orientation. This is probably the reason why many of the students of the second level selected the image of a translation, based on the criterion of the same shape, as the answer where the image changed its place by tracing an imaginary circle. In the type of task of recognition of rotation, the students had average performance in finding the example of a rotation with unspecified parameters. They were unable to recognise the examples of rotation in horizontal, vertical, or diagonal direction. Even though their approaches were again based on the visualisation of the pre-image moving to the place of the image, the students of the second level also had difficulties regarding the correct orientation of the image when rotated. Moreover, they were not able to recognise the geometric transformations in the other examples, and were not able to use appropriate vocabulary to describe the images, the shapes, or the geometric transformation.

The students of the second level also had many difficulties in the identification of parameters in rotation, in vertical, horizontal, and diagonal direction, and in overlapping figures. Specifically, most of the difficulties of the students of the second level in identification of parameters regarded finding the centre of rotation. Moreover, the students of the second level had difficulties in understanding that the centre of rotation can be positioned anywhere in the plane, and that the important parameter was to have the same distance from all the corresponding points of the two images to the centre of rotation. The students of the second level also had the erroneous conception that the centre of rotation needed to be between the pre-image and the image, which they conceived as holistic objects, and regardless of distance.

In the type of task of construction of image in rotation, the students of the second level were unable to construct any image, in any direction and configuration. The students of the second level had many difficulties in finding and expressing the relations between the elements of rotation. Even though they had difficulties in understanding the change of the image's position around the centre of rotation and the change of the orientation in the image of the shape, the students of the second level were able to realise and to find a partially correct response regarding the direction of the image around the centre of rotation, and sometimes in the correct distance. A characteristic approach of the students at the second level for solving construction of image in rotation tasks was the use of gestures, to facilitate visualisation. It is possible that this approach may be used to compensate for their inefficient spatial abilities, since the students of the second level seem to have low spatial visualisation and spatial orientation abilities.

The common characteristics that emerge from the thinking processes of the students of the second level in the different types of tasks in the different transformational concepts is that, similar to the students of the first level, they seem to have holistic conceptions of the elements of geometric transformations. However, the students of the second level are able to detach the figures of the images from the background that represents the plane, and seem to be able to use motion approaches, i.e., to visualise the images moving on top of the plane figure. However, these visualisation abilities are limited to straight line motion relations between the images, since for the circular motions of rotation, they require more concrete objects, which is why they use their hands to support the visualisation of these motions. Nevertheless, the students of the second level still conceive the figures as holistic units, and are not able to use their spatial ability to analyse the figures into smaller components, i.e., line segments and points. Finally, the students of the second level are able to understand the differences in the orientation of an image and are becoming more able to construct images of transformations with correct orientation. However, they still have difficulties with the orientation of the image in rotation, which probably relates to their conception of the image as a holistic object, and also by the use of gestures. Because of their view of the images of a geometric transformation as objects that can be manipulated around over the plane using motion, this level was named "motion of an object".

### *Characteristics of the third level of abilities*

The students of the third level of abilities had average abilities in all the transformational geometry concepts of translation, reflection, and rotation. Specifically, the students of the third level had high abilities in the types of tasks of identification of the parameters of a translation, and in recognition of reflection among other options; they also had average abilities in recognition of image, recognition of transformation, and construction of image in translation, in identification of parameters and construction of image in reflection, and in the types of tasks of recognition of image, recognition of transformation, and identification of parameters in rotation. The higher abilities of the students of the third level can be explained by the fact that they had higher ability in spatial visualisation compared to the students of the two previous levels.

Regarding their ability in translation, the students of this level had high abilities in identification of parameters. Specifically, the students of the first level were able to correctly identify both the parameters of direction and distance of a translation in vertical and in horizontal direction. Moreover, they were able to identify the direction for a horizontal translation with overlapping figures, but not the correct distance. They generally did not have any problems with understanding the direction. However, they seemed to have difficulties in understanding translation as a one-to-one mapping of points, and hence they had difficulties in measuring the distance between corresponding points.

The abilities of the students of the third level in the other types of tasks of translation, namely recognition of image, recognition of transformation, and construction of image, were average. Specifically, in recognition of image, the students were able to recognise the image in diagonal direction, and had some difficulties in recognising the image or translation in horizontal and in vertical direction. They approached the task by using the direction of the image as a criterion. Measuring the distance between the images was mostly used as a verification strategy when the students were between two options of images. However, the students of the third level also had difficulties in measuring the distance between corresponding parts of the images. In recognition of translation, the students of the third level were able to find the image of translation in any configuration, except for the diagonal translation, which was often confused with reflection. They did not have as many difficulties with the concept of orientation as the two previous levels, and they were able to refer to it, but with the use of informal vocabulary.

In construction of image in translation, the students of the third level were able to correctly construct the image of a figure in horizontal direction. However, their abilities in the construction of the image of translation in the other items of this type of task were average. The students of the third level used more analytic strategies to construct the image of a translation, and were able to decompose the figures into smaller parts, mainly line segments, and map them one by one, or map one segment and construct the image based on the relations of the segment to the other parts of the shape. Their difficulties were mainly regarding the distance between the pre-image and the image.

Regarding reflection, as mentioned above, the students had high performance in the type of task of recognition of reflection. Specifically, the students of the third level were able to recognise correctly the example of reflection in horizontal direction, i.e., in vertical line of reflection, and had average ability in recognising reflection in horizontal direction, i.e., in a horizontal line of reflection. A significant reason for this was that the students of this level compared the orientation of the images, by focusing on specific parts of the images. Thus, they were able to exclude options based on the orientation of the image. However, there were some cases where the students of the third level had difficulties in discriminating between the options of reflection and rotation, such as the case of a diagonal reflection and a half turn rotation.

The students of the third level had average ability in the types of task of identification of parameters and of construction of images in reflection. Specifically, in identification of parameters, they were able to correctly identify the parameters in vertical and in horizontal direction, by focusing on the positioning of specific parts of the images to identify the orientation, and by measuring and halving the distance between those parts to identify the position of the line. However, even though the students of the third level were able to identify the orientation of the line in a diagonal reflection, they had difficulties in identifying its distance from the images, as well as the orientation of the line in a reflection with overlapping images. The students of the third level had the conception that the images need to be on different sides of the line, which is what may have resulted to difficulties in finding the orientation of the line in the overlapping images.

In the construction of image in reflection, the students of the third level were able to construct the image correctly for horizontal direction, in both the simple and complex figure. Moreover, they had average abilities in constructing the image of a reflection in vertical direction, and of a reflection with overlapping figures. The difficulties of the students at the third level in constructing the image of a reflection were mostly regarding accuracy in the distance of the image from the line of reflection and with its orientation. The students of the third level also had many difficulties in solving the item of construction of image in diagonal reflection, since a large percentage of the students of this level were unable to solve this item. At this level, the students seemed to use a combination of analytic and holistic approaches. For example, the students of the third level often focused on a part of the pre-image to analytically find its position in the other side of the line, and then used a holistic approach of visualising the shape of the pre-image changing its position to match the part of the shape that was analytically constructed.

The students of the third level had low ability in recognition of image in reflection. Specifically, even though some of the students of this level had average abilities to recognise the image of a reflection in horizontal and in vertical direction, they were unable to recognise reflection in a diagonal line. Even though they were able to use the orientation of the image as criterion for recognising the image, they quite often confused it with the image of a translation.

Regarding ability in rotation, the students of the third level had average abilities in the types of tasks of recognition of image, recognition of rotation, and identification of parameters. Specifically, in recognition of image in rotation, the students of the third level were able to recognise the image of a rotation in horizontal direction, i.e., in a quarter turn rotation, and average abilities in recognising the image of a vertical (three quarter turn) and a diagonal (half turn) rotation. Their main approach was a combination of visualising the whole image rotating and focusing on the parts of the shape to determine the orientation of the correct image.

In the type of task of recognition of rotation, the students of the third level had some difficulties in recognising rotation in diagonal (half turn) rotation and in unspecified parameters rotation, since their abilities in this type of task were average. Moreover, they were unable to recognise rotation in vertical direction (three quarter turn) and in horizontal direction (one quarter turn). A common difficulty of the students of the third level in this type of task was to confuse rotation with reflection. The students of the third level approached this type of task by analysing the shape into parts and either visualising a specific part rotating, or visualising the whole shape rotating and focusing on a part of the image to verify its orientation. The students of the third level seemed to have an understanding that both the orientation and the position of an image change in rotation, and that only one point, the centre of rotation, can have a fixed position.

In identification of parameters in rotation, the students of the third level were able to correctly identify the parameters of a rotation in vertical direction (one quarter turn) and in diagonal direction (half turn). Their approach was to focus in the change of the orientation of a part of the pre-image, in order to find the angle of rotation. However, they had some difficulties in identifying the parameters of a rotation in horizontal direction (one quarter turn), and in rotation with overlapping image. The students' difficulties in this type of task were mostly regarding the centre of rotation, since they did not understand that the point needs to have the same distance from the corresponding vertices of the pre-image and the image, instead of being somewhere between the two images.

The students of the third level had low ability in the type of task of construction of image in rotation. Specifically, the students of the third level were only able to construct the image of a horizontal direction (one quarter turn) and a vertical direction (three quarter turn) rotation in the correct direction and in the correct distance from the centre of rotation.

Their difficulties in this type of task mainly concerned the orientation of the image. Similarly to the other types of tasks, the students of this level focused on a specific part of the pre-image, mostly a segment, in order to visualise the rotation. The students of the third level were using visual heuristics in the given figures, to help them visualise the rotational motion of the pre-image, such as circles around the centre of rotation or circular arcs tracing the motion of a vertex.

The common characteristics that emerge from the thinking processes of the students of the third level in the different types of tasks in the different transformational concepts were qualitatively very different than the characteristics of the students at the previous levels. Specifically, the students of the third level were able to decompose the figures into smaller parts of the shapes, i.e., mainly line segments and sometimes points, and focus on their mapping as objects. The students of the third level had the ability to reconstruct the shape of the pre-image as the image of it, based on the image of a single part. In other words, they were able to construct the whole image based on the properties and characteristics of the shape, starting from the image of a specific part of the shape. This was often achieved with a combination of holistic and analytic approaches, which was characteristic at this level of ability. Another characteristic of this level is that the students had better understanding of all the properties and the parameters of the three transformational geometry concepts, and they were able to apply them in the cases of straight-line displacements, i.e., in translation and reflection, but they were only able to recognise them visually in circular displacement, i.e., rotation. Hence, there was some evidence that students at this level begin to understand fundamental ideas of transformational geometry, that shapes are made of points in space and that each point can change its position, and also that the attributes of a shape are preserved in space, even when its position and orientation change under a certain transformation. Because of their tendency to use combinations of holistic approaches for visualising the shape as an object and analytic approaches by focusing on specific component of the image, this level was named "mapping of an object".



*Characteristics of the fourth level of abilities*

The students of the fourth level of abilities had high abilities in all the transformational geometry concepts of translation, reflection, and rotation. Specifically, the students of this level were able to complete any type of task in transformational geometry concepts, except for one type of rotation task, which concerned the construction of the image. The higher abilities of the students of the third level can be explained by the fact that they had higher abilities in all the spatial ability components, compared to the students of the third level. Moreover, they had reported significantly lower ratings in the verbal scale score of the cognitive style questionnaire.

Regarding their ability in translation, the students of this level had high abilities in every type of task. Specifically, regarding the identification of image in translation, the students of the fourth level were able to correctly recognise the image of a translation in any of the three given directions, by using effectively both parameters of distance and direction. Specifically, they did not have any problems in recognising the correct direction with accuracy, and they were able to measure correctly the distance between the pre-image and the image, by focusing on corresponding points. The students of the fourth level were also able to solve correctly all the items of the recognition of translation, in every direction. Specifically, not only were they able to recognise the option of translation based on the orientation of the image in relation to the pre-image, but they were also able to name each option using formal vocabulary.

In the type of task of identification of parameters, the students of the fourth level were able to identify correctly the parameters of all the items, in every direction. Particularly, the students of the fourth level measured the distance correctly. They had the knowledge that the distance would always be the same, as long as it was measured between two corresponding points. In construction of image in translation, the students of the fourth level were able to construct every image correctly, in every direction and configuration, with all parameters correct and all properties applied. The students of this level mostly applied analytic strategies for approaching this type of task, by constructing the images of the vertices of the pre-image, and connecting them to construct the image. This implies that the students of the fourth level had good understanding of translation as a function that is applied to all points of the plane, and they decomposed a shape into points in order to apply this function. This conception also allowed them to approach and solve correctly the construction of image in translation with overlapping image.

Regarding the transformational geometry concept of reflection, the students of the fourth level were able to solve every type of task, as formerly mentioned, namely recognition of image, recognition of reflection, identification of parameters, and construction of image. Specifically, in recognition of image in reflection, the students of the fourth level were able to recognise the image of a reflection in vertical direction, i.e., in horizontal line, and in horizontal direction, i.e., in vertical line. This ability is likely related to the knowledge of the students of the fourth level about how the orientation of an image changes in a geometric transformation. The students of the fourth level had some difficulties in the recognition of the image in diagonal direction, since they often confused it with reflection in horizontal direction, i.e., in a vertical line.

In the type of task of recognition of reflection, the students of the fourth level were able to recognise correctly the geometric transformation of reflection in vertical and in horizontal direction, as well as in reflection with unspecified direction. Specifically, not only were they able to recognise the option of reflection based on the orientation of the image in relation to the pre-image, but they were also able to name each option using formal vocabulary, as they did for translation. However, the students of the fourth level had some difficulties in recognising diagonal reflection, since they sometimes confused it with rotation.

In identification of parameters, the students of the fourth level were able to identify correctly the parameters in reflection in vertical and in horizontal direction. They were also able to identify the orientation of a diagonal line of reflection and of a vertical line of reflection with overlapping figures in horizontal direction. However, some of the students of this level had difficulties in identifying the correct position of the diagonal line of reflection and of the vertical line of refection with overlapping figures. Specifically, in diagonal reflection and reflection with overlapping figures, some of the students of the fourth level placed the line of reflection closer to one of the two images.

In the type of task of construction of image in reflection, the students of the fourth level were able to construct correctly the image in vertical direction and in horizontal direction, as well as for overlapping images and for complex figures, with all parameters correct and all properties applied. The students of the fourth level approached this type of task with analytic strategies, and seemed to have a conceptual understanding or reflection as a function that is applied to every point of a shape and of the plane. However, they supported these strategies with visual heuristics and mental images of either the spatial relations between the elements of the reflection or of the holistic figures. The students of the fourth level had some difficulties in constructing the image of a diagonal reflection, which seems to be the most difficult item with a high percentage of failure. The most common error of the students at this level was to construct the image of a diagonal line of reflection by measuring towards the wrong direction, i.e., not perpendicular to the line of reflection.

In the transformational geometry concept of rotation tasks, the students of the fourth level were able to solve correctly every type of task, except for construction of image. Specifically, in the type of task of recognition of image in rotation, the students of the fourth level where able to recognise the image of a rotation in every direction, i.e., vertical, horizontal, and diagonal, without any difficulties. Their main approach was to focus on the orientation of specific parts of an image, and use their spatial reasoning to compare the orientation of each part to the one of the pre-image. Thus, not only they were able to recognise the correct image, but they were also able to recognise the parameters for each option.

In the type of task of recognising rotation, the students of the fourth level were able to recognise every example of rotation in every direction, as well as with unspecified direction, without any difficulty. Specifically, as with the other two geometric transformations, the students of the fourth level were not only able to recognise the option of rotation based on the orientation of the image in relation to the pre-image, but they were also able to name the geometric transformation in each option, using the formal geometrical terms of translation, reflection, and rotation.

In identification of parameters in rotation, the students of the fourth level were able to correctly identify the parameters of vertical rotation (three quarter turn) and diagonal rotation (half turn). However, they had difficulties in identifying the correct centre of rotation for horizontal rotation (one quarter turn) and for rotation with overlapping figures. Even though they were able to identify the angle of rotation in every item by decomposing the images and focusing on comparing the orientation of corresponding line segments, they had difficulties in identifying the centre of rotation when the images had large distance between them or when they were overlapping.

In the type of task of construction of image in rotation, the students of the fourth level had some difficulties to construct the image of a rotation correctly. Specifically, in the cases of rotation in horizontal, vertical, and diagonal direction, and in rotation with overlapping figures, the students of the fourth level were able to construct the image with difficulties in maintaining the correct orientation. They knew that the image needed to have the same distance from the centre of rotation, and they knew the direction around the centre of rotation where the image would appear after the rotation, but they had difficulties in keeping track of the changes in the orientation of the figure around the centre of rotation. Moreover, they were unable to construct the image of a complex figure, which was the most difficult item in rotation. Even though they were able to visualise the correct direction of the image for the complex figure, they had many difficulties in constructing it in maintaining the characteristics of the shape and its distance from the centre of rotation, as well as finding the correct orientation of the image.

The common characteristics that emerge from the thinking processes of the students of the fourth level in the different types of tasks in the different transformational concepts were qualitatively different than the characteristics of the students at all the previous levels. Specifically, the students of the fourth level had the flexibility to view the figures both as holistic objects and as compositions of line segments and/or points in space. At this level, the students stop perceiving the shape only as a whole, and are able to decompose it into points, in order to apply the function of the geometric transformation to every point. They seem to have a very good understanding of space and its representation in the real world. Hence, they can conceptually understand geometric transformations as the mapping of every point of the plane. Moreover, they can understand and apply all the properties of geometric transformations to every point in the plane, in both straight-line and circular direction. Another significant characteristic of the students at the fourth level was their high sense of self-efficacy and certainty for their responses. This certainty stemmed from their flexibility to manipulate and coordinate their mental images and strategies. Another characteristic which probably enhanced the students’ self-efficacy was their ability and tendency to verify their responses, either by applying different approaches or by applying the same approach on different points of the images. It is also important to note that the students of the fourth level had the ability to use formal mathematical vocabulary to explain their reasoning, as well as the ability to connect their knowledge in geometric transformations with knowledge from other mathematical domains, in order to solve a problem. Hence, because of their flexibility to decompose the images into points in the plane and apply one-to-one mapping of each point of the plane, this level was named “mapping of the plane”.

## Transformational Geometry Ability and its Relation to Spatial Ability

The results of the study showed that the spatial ability of students between nine and 14 years old is multidimensional. This finding is in line with the findings of researchers in psychology (Caroll, 1993; Lohman, 1988; McGee, 1979) which suggest that spatial ability is a multidimensional construct consisting by different factors. Based on the results of the study, spatial ability can be analysed in three factors. These factors are: “Spatial Visualisation”, “Spatial Relations”, and “Spatial Orientation”. The factors of “Spatial Visualisation”, “Spatial Relations”, and “Spatial Orientation” constitute three cognitively different constructs of spatial ability and contribute almost the same to spatial ability. Specifically, “Spatial Visualisation” refers to the ability to manipulate objects in imagination, “Spatial Relations” refers to the ability to mentally rotate a spatial object fast and correctly, and “Spatial Orientation” refers to the ability to remain unconfused by the changes in the orientation of visual stimuli.

This discrimination is significant in the field of mathematics education, and more specifically to the field of transformational geometry. The reason for this is that, even though spatial ability is a cognitive ability that is involved in many mathematical and every-day activities, it is not included in any curriculum and there is no explicit instruction for its development. However, researchers in mathematics education suggest that spatial ability is closely related to transformational geometry concepts, and that instructional activities in transformational geometry can develop students’ spatial ability (Clements & Battista, 1992). Knowing that spatial ability is a multidimensional ability and identifying its dimensions can help mathematics educators and researchers to realise which dimensions of spatial ability can be developed with transformational geometry activities and in what ways.

The results of the study showed that there is a strong relation between transformational geometry ability and spatial ability. Specifically, it seems that all the sub-factors of transformational geometry ability and all the sub-factors of spatial ability are related. Moreover, the results of the analysis suggest that the two abilities can be considered as the two dimensions of a more general ability. This may suggest that the two abilities may be different aspects of the same ability. Moreover, it may explain the contradictory propositions found in literature, according to which (i) spatial ability can predict the development of ability in transformational geometry (Kirby & Boulter, 1999), and (ii) that transformational geometry activities can develop spatial ability (Clements & Battista, 1992; Dixon, 1995; Smith et al., 2009).

The results of the study showed that students from nine to 14 years old have average spatial ability. Regarding their abilities in the three spatial ability dimensions, they have average performance in “Spatial Visualisation” and in “Spatial Relations”, and low performance in “Spatial Orientation”. The results of the study also showed that significant differences seem to exist in the spatial ability of the four classes of students with different characteristics regarding their ability in transformational geometry concepts. Specifically, the first class of students had average performance in the “Spatial Relations” factor, and low performance in the “Spatial Visualisation” and in the “Spatial Orientation” factors. The low abilities of the first class of students in the spatial ability factors may explain their low abilities in the transformational geometry concepts. The second class of students also had average performance in the “Spatial Relations” factor, and low performance in the “Spatial Visualisation” and in the “Spatial Orientation” factors. This finding may explain the reason why these two classes of students did not have many significant differences in their ability in transformational geometry concepts. The third class of students had significant differences from the second class of students regarding their spatial ability. Specifically, the third class of students had average performance in the “Spatial Relations” and the “Spatial Visualisation” factors, and low performance in the “Spatial Orientation” factor. The higher performance in the “Spatial Visualisation” factor may explain why the third class of students had significant differences in their ability in transformational geometry concepts compared to the second class of students. Finally, the fourth class had high performance in the “Spatial Relations” and the “Spatial Visualisation” factors, and average performance in the “Spatial Orientation” factor. Similarly, the significantly higher performance of the fourth class of students in the spatial ability factors may explain their significantly higher performance in the transformational geometry concepts.

## Transformational Geometry Ability and its Relation to Cognitive Style

The results of the study showed that the cognitive style of students between nine and 14 years old can be measured based on their preferences and experiences. Specifically, three dimensions of cognitive style were measured for this study: spatial imagery, object imagery, and verbal. The findings of the study suggest that the mean ratings for cognitive style of students between nine and 14 years old have similar values which are near average.

The results of the study showed that a significant relation seems to exist between the verbal dimension of cognitive style and students’ ability in reflection and in rotation, as well as in their overall transformational geometry ability. The relations of the object imagery and the spatial imagery cognitive style dimensions to the ability of 9- to 14-year-old students in transformational geometry concepts were not significant. The results of the study also showed that there were only two significant differences in the cognitive style ratings of the four classes of students with different characteristics regarding their ability in transformational geometry concepts. Specifically, the verbal dimension ratings of the first and second class of students were significantly higher than the verbal dimension ratings of the fourth class of students. The rest of the ratings of the cognitive style dimensions of the four classes did not have any significant differences. This finding suggests that students' cognitive style is not strongly related to their ability in transformational geometry concepts.

## The Impact of Dynamic Visualisations on Transformational Geometry and Spatial Abilities

The results of the study showed that the instruction with a continuous dynamic visualisation in transformational geometry had better learning outcomes compared to instruction with a discrete dynamic visualisation, while controlling for initial differences.

Specifically, the students who received instruction with a continuous dynamic visualisation significantly outperformed their peers who received instruction with a discrete dynamic visualisation, in their performance in both the transformational geometry post-test and the spatial ability post-test. Moreover, it seems that the instruction with the discrete dynamic visualisation on students' transformational geometry ability and spatial ability did not have a significant impact. The findings of this study regarding the positive impact of continuous dynamic visualisation on students' ability in transformational geometry concepts and in spatial ability are consistent with the findings of similar research studies which used similar continuous dynamic geometry software (Dixon, 1995; Guven, 2012; Hoong & Khoh, 2003), and provide further evidence for the potential of continuous dynamic geometry to develop rather than restrain students ability in transformational geometry concepts (Smith et al., 2009), as well as their spatial ability, in primary school level.

The results of the study showed that the students' individual differences in spatial ability and cognitive style, and their interactions with the type of dynamic visualisation did not have a significant impact on their benefits from the instructional interventions. The only significant impact on students benefits in transformational geometry ability and in spatial ability from the instructional interventions were by the type of dynamic visualisation.

Even though several concerns were raised in literature regarding the difficulties that some learners may face in learning from dynamic visualisations due to their low spatial ability (Kirby & Boulter, 1999; Smith et al., 2009; Sweller et al. 1998) or cognitive style (Mayer, 2009), the findings of this study suggest that students with different spatial ability level and students with different cognitive styles benefit equally from dynamic visualisations, regardless of their individual traits. Similarly, in their studies of learning in mechanical devices, Hegarty and Kriz (2008) found that the interaction between spatial ability level and instruction with dynamic/non-dynamic visualisation was not significant. In the field of geometry, the findings of other studies are contradictory. While some studies suggest that high-spatial learners and visualisers benefit more from continuous dynamic instruction (Hannafin et al., 2008), other studies suggest that the benefits are more for students with verbal cognitive style (Pitta-Panazi & Christou, 2009a). The findings of this study introduce a different perspective, that students' individual characteristics in spatial ability and cognitive style do not have a significant impact in their learning with dynamic visualisations.



## Conclusions

In contrast to the existing models which describe the development of ability in transformational geometry based on either the mathematical structures of transformational geometry concepts (Kidder, 1976; Moyer, 1978; Piaget & Inhelder, 1971), or the cognitive structures of different types of tasks which are based on the implementation of theoretical models that are borrowed from other fields of geometry (Molina, 1990; Soon, 1989), the results of this study propose a model for ability in transformational geometry which combines both the mathematical and cognitive dimensions of ability in transformational geometry concepts. In addition to this, this model takes into consideration the configurations that influence the relative difficulty of transformational geometry tasks that are suggested in literature, namely the operative configurations of transformational geometry concept, distance, and direction, and the figurative configuration of complexity of the image (Lesh, 1976; Schultz & Austin, 1983). Hence, the model of this study confirms that ability in transformational geometry concepts can be described using an amalgamation of these three aspects, and that each aspect consists a different level of factors in the model for the structure of ability in transformational geometry, with the mathematical concepts of translation, reflection, and rotation having the most significant role in describing the development of ability in transformational geometry concepts. Moreover, this model was validated for its stability over different ages, specifically for its applicability in 9- to 11-year-old students and 12- to 14-year-old students.

Based on the findings regarding the dimensions of ability in transformational geometry concepts, this study also described hierarchical levels of ability. In contrast to the existing theoretical models which focused on the type of transformation (Kidder, 1976; Moyer, 1978; Piaget & Inhelder, 1971), the operative and figurative configurations (Schultz & Austin, 1983), or the type of task (Molina, 1990; Soon, 1989) to describe levels of ability in transformational geometry concepts, this study used the amalgamation of these three aspects, and focused on the individuals rather than the concepts. Particularly, the study focused on the investigation of the students' reasoning and cognitive abilities to describe the levels, by drawing on theoretical frameworks regarding conceptions of transformational geometry concepts (Edwards, 2003), and strategies for approaching transformational geometry problems (Boulter & Kirby, 1994). The results suggest that the ability of 9- to 14-year-old students in transformational geometry concepts develops over four hierarchical levels of ability. This study used and extended the theory proposed by Edwards (2003) regarding the discrimination of two qualitatively different types of understanding geometric transformations, namely motion and mapping understanding, in order to describe and name the levels. It also exploited the findings of Boulter and Kirby (1994) regarding the use of holistic and analytic strategies in solving transformational geometry tasks and their relation to performance, and related them to the cognitive development of students' transformational geometry and spatial abilities.

The first level of abilities in transformational geometry was named "holistic image conception", since the students of this level have no understanding of transformational geometry properties, and conceive the figures that represent a geometric transformation as a drawing instead of a figure, like a holistic tangible object (Duval, 2005; Hollebrands, 2003). The students of the first level do not conceive transformational geometry concepts either as physical motion or as a function of mapping, since they cannot decompose the figure to discriminate the elements of the geometric transformation, i.e., the domain, the pre-image, the parameters, the image. Their visual images are concrete pictorial (Presmeg, 1982), and their reasoning is based on procedures. For these reasons, the students of the first level seem to rely basically on holistic strategies for approaching transformational geometry problems. The results of the study suggest that the abilities of this group are probably related to their high preference for verbal processing of information, and also to their low abilities in spatial visualisation and spatial orientation.

The second level of abilities in transformational geometry was named "motion of an object", since the students of this level have an understanding of geometric transformations as processes of physical motion that can be applied on geometrical figures as objects over the plane, which serves as a background. This level is representative of the motion understanding, as described by Edwards (2003). The students of the second level still have a holistic conception of the figures as objects that can be manipulated. However, they have the ability to decompose the figure of the geometric transformation and to discriminate the elements of the geometric transformation. Even though their visual images are also concrete pictorial and they also rely on procedures, their visual reasoning allows them to recognise some prototypical images. Similarly to the students of the first level, the students of the second level also rely on procedures; however, the difference is that they conceive the geometric transformation as a procedure of motion. Their difficulties in performing these procedures correctly are likely rooted in their cognitive style preference for storing, representing, and processing information verbally. The low abilities of the students of this level in the spatial visualisation and spatial orientation components of spatial ability are also a factor with significant influence.

The third level of abilities in transformational geometry was named "mapping of an object", as an intermediate level between understanding geometric transformations as motion and as mapping. The students of this level have the tendency to use combinations of holistic approaches for visualising the shape as an object, and analytic approaches by focusing on specific component of the image. They seem to be able to decompose a figure into figural units (Duval, 2005), and they apply analytic strategies to a figural unit to find its image and solve one part of the problem, but they combine this strategy with the use of a holistic one to solve the remaining problem, or reconstruct the shape as a holistic figure based on the image of its part. Their visual images at this level become more abstract, since they can analyse a shape into figural units. The results of this study suggest that a possible reason for this transition may be their average abilities in spatial visualisation and spatial relations.

The fourth level of abilities in transformational geometry was named “mapping of the plane”, and it is corresponding to what Edwards (2003) described as mapping understanding of geometric transformations. The students of the fourth level have the flexibility to decompose the figures into points of the plane and apply every geometric transformation as a one-to-one mapping of each point of the plane. They seem to have a generic conception of the plane as a set of points, and apply the geometric transformations as functions that map each point. Their visual images do not seem to be concrete pictorial, as described in the previous levels. Instead, they have an abstract quality and a spatial nature of connecting information, such as connecting corresponding points. Their strategies are mostly analytic, which implies a development from holistic to analytic strategies. However, the students of the fourth level also have a high ability for using holistic strategies, which they sometimes utilise for verification. The results of the study suggest that the conceptions and strategies of the students of the fourth level are probably related to their low preference for verbal processing of information and to their high abilities in the spatial visualisation and spatial relations components of spatial ability, and also to their average ability in the spatial orientation component of spatial ability.

Regarding its relation to individual differences in spatial ability and cognitive style, the results of the study suggest that spatial ability is a more influential factor for determining ability in transformational geometry than cognitive style. Regarding spatial ability, it appears that ability in transformational geometry concepts is not only closely related to spatial ability and to its subcomponents, which this study confirmed to be spatial visualisation, spatial relations, and spatial orientation, as suggested by Lohman (1988). Moreover, the findings of this study suggest that the two abilities contribute to a more general spatial ability and can, thus, be considered as two dimensions of a higher ability. This new perspective can serve as an explanation for the contradictory findings appearing in literature, according to which some research studies suggest that spatial ability contributes to the development of ability in transformational geometry concepts (Kirby & Boulter, 1999), while other studies suggest that the development of transformational geometry ability contributes to the development of spatial ability (Dixon, 1995; Smith et al., 2009). In reference to cognitive style, the results of the study suggest that ability in transformational geometry concepts is negatively related to the verbal dimension of cognitive style. This suggests that individuals' higher preference for storing, representing, and processing information verbally can have a significant, even though relatively small, impact on their ability in transformational geometry concepts.

Regarding the investigation of the impact of two interactive dynamic visualisations on students' ability in transformational geometry concepts and their spatial ability, the findings of the study suggest that teaching transformational geometry concepts with the implementation of a continuous dynamic visualisation, in comparison to a discrete dynamic visualisation, has a significantly larger impact on the development of sixth grade primary school students' ability in transformational geometry concepts and spatial ability. Specifically, the findings of the study suggest that instruction of transformational geometry concepts with the use of a discrete dynamic visualisation did not have any impact on the development of sixth grade primary school students' ability in transformational geometry concepts, nor to their spatial ability, even though it was designed based on the same principles for multimedia learning (Mayer, 2005), and same instructional model of activities, the 5Es model (Bybee, 1997), as the instruction with a continuous dynamic visualisation. Moreover, no significant interactions were found between the type of dynamic visualisation and students' individual differences regarding level of spatial ability and cognitive style. The benefits in transformational geometry ability and spatial ability when learning from dynamic visualisations are more influenced by the type of dynamic visualisation, rather than students' level of spatial ability or cognitive style.

## Instructional Implications

The theoretical model of this study could serve as a significant tool for teachers and researchers in the field of mathematics education. This model acknowledges to mathematics teachers the most important dimensions of ability in transformational geometry concepts and emphasises the importance for providing students with a variety of activities that refer to the different dimensions of this ability, to facilitate its development. The results of the study also suggest the need for more emphasis in teaching transformational geometry concepts, not only for developing students' knowledge in this type of geometry, but also as a means of developing their spatial ability.

The hierarchical relationship of abilities in the three transformational geometry concepts suggests that it might be better to design instruction for developing students' ability in transformational geometry concepts based on the abilities in the mathematical structures of transformational geometry, i.e., to teach translations first, followed by reflections, followed by rotations. For this reason, mathematics teachers should emphasise more on developing students understanding and abilities in different types of tasks within one transformational geometry concept, before introducing a new one. Emphasis should be given to the geometrical concepts of orientation, congruence, and similarity for comparing figures. In addition to this, it is also important to practice students' ability to identify figural units, which according to Duval (2011) is a fundamental principle in the learning of geometry. The results of this study also underpin the role of individual differences in the development of ability in transformational geometry concepts, and especially of spatial ability. The close connection between ability in transformational geometry concepts and spatial ability suggests the need to emphasise the development of the components of spatial ability regarding spatial visualisation, spatial relations, and spatial orientation through a variety of activities not only through the mathematical curricula, but also in the general educational curricula. The results of the study also indicated the possibility that students whose cognitive style directs them to process information verbally are likely to face difficulties in transformational geometry concepts. This suggests the importance of encouraging and supporting students' visual reasoning in geometry, by using and connecting a variety of representational forms in geometry instruction, both static and dynamic.

The description of the hierarchical levels of ability in transformational geometry concepts can also serve as a tool for mathematics teachers for the identification and for the instructional treatment of students' difficulties in transformational geometry. The observation of students' behaviour in transformational geometry concepts and its comparison to the characteristics of the four levels of abilities can guide mathematics teachers into classifying their behaviour and therefore easily identify their difficulties. Moreover, the model can provide the mathematics teacher with valuable information regarding the selection of appropriate activities to facilitate the overcome of the difficulties that every student is facing. For example, if a student is facing difficulties that indicate the characteristics of the first level of thinking, the mathematics teacher should emphasise in activities which develop the ability of detaching the images from the background picture and guide the student into understanding the elements of the geometric transformation and how they can be related to motion.

The comparison of the impact of different dynamic visualisations in transformational geometry ability and in spatial ability underpins the importance of selecting appropriate instructional tools for the teaching of transformational geometry specifically, and for teaching in general. The results of this study suggest the importance of evaluating the relative results of different tools in students' abilities, and stress the importance of developing effective tools and for using technology sensibly. The model of this study can be used by mathematics teachers for guiding the selection of appropriate dynamic visualisations for the teaching of transformational geometry concepts, and for creating such activities that can promote students thinking into higher levels of abilities.

## Suggestions for Further Research

This study investigated the structure of ability in transformational geometry concepts. It has confirmed that ability in transformational geometry is synthesised by the abilities of the three transformational geometry concepts of translation, reflection, and rotation. Moreover, it has confirmed that the three transformational geometry concepts have similar structure and are each synthesised by four similar dimensions, therefore suggesting that transformational geometry ability can be analysed into 12 distinct abilities. Due to the lack of previous relevant research, the development of the theoretical model of this study was based on the analysis and synthesis of results from research studies which were implemented over the last 50 years. For this reason, it is necessary to conduct similar studies that investigate the development of students' ability in transformational geometry concepts with older students. Such studies could also consider the inclusion of other possible dimensions, which could not be included in this model, such as the ability of representation of the transformational geometry concepts, or the ability to prove theorems in transformational geometry. In addition to this, such studies could consider including or even comparing the structure of non-Euclidean transformational geometry concepts, such as enlargement, as well as in other types of transformational geometry, such as projective and affine.

In this study, the development of students' ability in transformational geometry concepts based on the identification of hierarchical levels was also investigated. Four levels were identified based on the students' abilities, conceptions, strategies, and difficulties at the age of nine to 14 years old. Similar studies can be conducted to validate the existence of these levels and investigate the possibility of higher levels, possibly in higher levels of education. Moreover, even though this study conducted two intervention programmes to compare the effects of different dynamic visualisations on students' transformational geometry and spatial abilities, it did not investigate the potential of these dynamic visualisations to influence students' thinking processes and strategies in order to promote their ability into higher levels. Further research could investigate such potential.

The findings of this study provide some evidence for a relationship between cognitive style and ability in transformational geometry. Given the lack of a unified framework for cognitive style in the existing literature (Evans & Warring, 2011), future studies could investigate the relationship between transformational geometry ability and different dimensions of cognitive style, such as the holistic-analytic dimension (Pask, 1972).

Regarding the effects of dynamic visualisations with different characteristics on students' ability in transformational geometry concepts, future studies can investigate the effects of other characteristics on students' development of ability transformational geometry concepts and of spatial ability. According to Hegarty and Kriz (2008) and Kheener et al. (2008), the level of interactivity in a dynamic visualisation may have different effects on students' learning. Such a study could investigate how providing students with less or more interactivity in a dynamic visualisation can have impact on the development of their ability in transformational geometry and their spatial ability.

# GLOSSARY

*A rigid transformation*

A transformation that preserves relative distances and angles of all points in the plane. It is called rigid, because of the inflexibility to change distances and angles, in contrast to other types of transformations. It is sometimes called *isometry*, from the Greek words ίσο (íso) and μέτρο (métro), which means equal measure.

*The domain*

The term *domain* refers to a specific meaning in geometric transformations, considered as functions of the plane (Yanik & Flores, 2009). In geometric transformations, the domain is considered as all the points in the plane (Hollebrands, 2003). This means it is not limited to a figure or a point. When transforming a figure in the plane, all points in the plane need to be considered. Students sometime consider the domain as either the labelled points belonging to the image or the vertexes of the shape, or all the points on the pre-image (Hollebrands, 2003).



*Parameters*

Parameters for transformations include translation vectors, reflection lines, points for centres of rotations, and measures distances and of angles of rotation (Hollebrands, 2003).

*Relationships and properties of transformations*

Properties can include those related to knowing that the transformation is an isometry or can be related to understanding whether a relationship, such as co-linearity, is preserved by the transformation (Hollebrands, 2003).

REFERENCES


Ahuja, O. P. (1996, November). *An investigation in the geometric understanding among elementary preservice teachers*. Paper presented at the Joint Conference of the Australian Association for Research in Education and the Educational Research Association, Singapore. Retrieved from http://www.aare.edu.au/96pap/ahujo96485.txt

Alamolhodaei, H. (2009). A working memory model applied to mathematical word problem solving. *Asia Pacific Educational Review, 10*, 183-192. doi: 10.1007/s12564-009-9023-2

Alessi, S. M., & Trollip, S. R. (2001). *Multimedia for learning: Methods and development* (3rd ed.). Boston, MA: Allyn & Bacon.

Alias, M., Black, T. R., & Gray, D. E. (2002). Effect of instructions on spatial visualisation ability in civil engineering students. *International Education Journal*, *3*(1), 1-12.

Allport, G. W. (1937). *Personality: A psychological interpretation*. NY: Holt, Rinehart, and Winston.

Anderson, K. L., Casey, M. B., Thompson, W. L., Burrage, M. S., Pezaris, E., & Kosslyn, S. M. (2008). Performance on middle school geometry problems with geometry clues matched to three different cognitive styles. *Mind, Brain, and Education*, *2*(4), 188-197. doi:10.1111/j.1751-228X.2008.00053.x

Atkinson, S. (2004). A comparison of pupil learning and achievement in computer aided learning and traditionally taught situations with special reference to cognitive style and gender issues. *Educational Psychology*, *24*(5), 659-679. doi: 10.1080/ 0144341042000262962

Aviram, A. (2001). From “computers in the classroom” to mindful radical adaptation by education systems to the emerging cyber culture. *Journal of Educational Change*, *1*(4), 331-352. doi: 10.1023/A:1010082722912

Ayersman, D., & von Minden, A. (1995). Individual differences, computers, and instruction. *Computers in Human Behavior*, *11*(3), 371-390. doi: 10.1016/0747-5632(95)80005-S

Bansilal, S., & Naidoo, J. (2012). Learners engaging with transformation geometry. *South African Journal of Education*, *32*, 26-39. Retrieved from http://www.sajournalofeducation.co.za/index.php/saje/article/view/452/291

Battista, M. T., & Clements, D. H. (1998). Finding the number of cubes in rectangular cube buildings. *Teaching Children Mathematics*, *4*(5), 258-264. Retrieved from http://www.nctm.org/publications/

Battista, M. T. (1990). Spatial visualization and gender differences in high school geometry. *Journal for Research in Mathematics Education*, *21*(1), 47-60. Retrieved from http://www.jstor.org/stable/749456

Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education*, *30*(4), 417-448. Retrieved from http://www.jstor.org/stable/749708

Battista, M. T. (2002). Learning geometry in a dynamic computer environment. *Teaching Children Mathematics*, *8*(6), 333–338. Retrieved from http://www.nctm.org/publications/

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Jr. Lester, (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 843-908). Charlotte, NC: Information Age Publishing.

Battista, M. T. (2011). Conceptualizations and issues related to learning progressions, learning trajectories, and levels of sophistication. *The Mathematics Enthusiast*, *8*(3), 507–570. Retrieved from http://www.math.umt.edu/tmme/

Battista, M. T., & Clements, D. H. (1991). *Logo geometry*. Morristown, NJ: Silver Burdett & Ginn.

Bell, A., & Birks, D. (1990). Diagnosis and response in transformation geometry. In G. Booker, P. Cobb, & T. de Mendicuti (Eds.), *Proceedings of the 14th Conference of Psychology in Mathematics Education* (Vol. 2, pp. 219-228). Mexico City, Mexico: PME-NA.

Bellin, H. (1980). Geometry structures and processing strategies in young children. In R. Karplus (Ed.), *Proceedings of the Fourth Conference of Psychology in Mathematics Education* (pp. 279-285). Berkeley, USA: PME.

Ben-Chaim, D., Lappan, G., & Houang, R. T. (1988). The effect of instructions on spatial visualisation skills of middle school boys and girls. *American Educationa1 Research Journal*, *25*(1), 51-71. Retrieved from http://www.jstor.org/stable/1163159

Ben-Chaim, D., Lappan, G., & Houang, R. T. (1989). Adolescents' ability to communicate spatial information: Analysing and affecting students' performance. *Educational Studies in Mathematics*, *20,* 121-146. Retrieved from http://www.jstor.org/stable/3482494

Bennet, G. K., Seashore, H. G., & Wesman, A. G. (1972). *Differential Aptitude Tests: Space Relations, Form T.* (5th ed.). NY: Harcourt Educational Measurement/ Psychological Corporation.

Bishop, A. J. (1980). Spatial abilities and mathematics education - A review. *Educational Studies in Mathematics*, *11*(3), 257-269. doi:10.1007/BF00697739

Bishop, A. J. (1983). Space and geometry. In R. Lesh & M. Landau (Eds.), *Acquistion of mathematical concepts and processes* (pp. 125-203). NY: Academic Press.

Blajenkova, O., Kozhevnikov, M., & Motes, M. A. (2006). Object-spatial imagery: a new self-report imagery questionnaire. *Applied Cognitive Psychology, 20*(2), 239-263. doi:10.1002/acp.1182

Blake, T. (1977). Motion in instructional media: Some subject-display mode interactions. *Perceptual and Motor Skills*, 44, 975-985. Retrieved from http://www.amsciepub.com/loi/pms

Blazhenkova, O., & Kozhevnikov, M. (2009). The new object-spatial-verbal cognitive style model: Theory and measurement. *Applied Cognitive Psychology*, *23*(5), 638-663. doi: 10.1002/acp.1473

Blazhenkova, O., & Kozhevnikov, M. (2010). Visual-object ability: A new dimension of non-verbal intelligence. *Cognition*, *117*(3), 276-301. doi:10.1016/j.cognition.2010.08.021

Blazhenkova, O., Becker, M., & Kozhevnikov, M. (2011). Object-spatial imagery and verbal cognitive styles in children and adolescents: Developmental trajectories in relation to ability. *Learning and Individual Differences*, *21*(3), 281-287. doi:10.1016/j.lindif.2010.11.012

Booth, R. D. L., & Thomas, M. O. J. (1999). Visualization in mathematics learning: Arithmetic problem solving and student difficulties. *Journal of Mathematical Behavior*, *18*, 169-190. doi: 10.1016/S0732-3123(99)00027-9

Bossé, M. J., Lee, T. D., Swinson, M., & Faulconer, J. (2010). The NCTM process standards and the five Es of science: Connecting math and science. *School Science and Mathematics*, *110*(5), 262–276. doi:10.1111/j.1949-8594.2010.00033.x

Boulter, D. R. (1992). *The effects of instruction on spatial ability and geometry performance* (Master's thesis). Retrieved from ProQuest Dissertations and Theses. (UMI MM76356)

Boulter, D. R., & Kirby, J. R. (1994). Identification of strategies used in solving transformational geometry problems. *The Journal of Educational Research*, *87*(5), 298-303. doi:10.1080/00220671.1994.9941257

Burger, W., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, *17*, 31-48. Retrieved from http://www.jstor.org/stable/749317

Burton, L. J., & Fogarty, G. J. (2003). The factor structure of visual imagery and spatial abilities. *Intelligence*, *31*(3), 289–318. Retrieved from http://www.journals.elsevier.com/intelligence/

Bybee, R. (1997). *Achieving scientific literacy*. Portsmouth, NH: Heinemann.

Bybee, R. W., Taylor, J. A., Gardner, A., Scotter, P. V., Powell, J. C., Westbrook, A., & Landes, N. (2006). *The BSCS 5E instructional model: Origins, effectiveness, and applications, 19*. CO: The Biological Science Curriculum Study.

Caroll, J. (1993). *Human cognitive abilities: A survey of factor-analytic studies*. NY: Cambridge University Press.

Chan, H., Tsai, P., & Huang, T. Y. (2006). Web-based learning in a geometry course. *Educational Technology & Society*, *9*(2), 133-140. Retrieved from http://www.ifets.info/

Chrysostomou, M., Tsingi, C., Cleanthous, E., & Pitta-Pantazi, D. (2011, February). *Cognitive styles and their relation to number sense and algebraic reasoning*. Paper presented at the Seventh Congress of the European Society for Research in Mathematics Education, Rzeszów. Retrieved from http://www.cerme7.univ.rzeszow.pl/WG/2/CERME7_WG2_Chrysostomou.pdf

Clements, D. H., & Battista, M. T. (2001). Length, perimeter, area, and volume. In L. S. Grinstein & S. I. Lipsey (Eds.), *Encyclopaedia of mathematics education* (pp. 406-410). NY: Routledge Falmer.

Clements, D. H., & Sarama, J. (2001). *Quilts and pattern block puzzles*. Salem, OR: Math Learning Center.

Clements, L., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. Grouws (Ed.). *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-464). NY: Macmillan.

Clements, M. A. (1981, April). *Spatial ability, visual imagery, and mathematical learning*. Paper presented at the American Educational Research Association Annual Meeting, Los Angeles. Retrieved from the ERIC database. (ED202696)

Cohen, C. A., & Hegarty, M. (2007). Individual differences in use of external visualisations to perform an internal visualisation task. *Applied Cognitive Psychology*, *21*(6), 701-711. doi:10.1002/acp.1344

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates.

Colom, R., Contreras, M. J., Botella, J., & Santacreu, J. (2002). Vehicles of spatial ability. *Personality and Individual Differences*, *32*, 903–912. doi: 10.1016/S0191-8869(01)00095-2

Darke, I. (1982). A review of research related to the topological primacy thesis. *Educational Studies in Mathematics*, *13*(2), 119-142. doi: 10.1007/BF00460707

de Villiers, M. (2007). Some pitfalls of dynamic geometry software. *Learning and Teaching Mathematics*, *4*, 46-52. Retrieved from http://www.amesa.org.za/

de Villiers, M. (2010, June). *Some reflections on the van Hiele theory*. Paper presented at the Fourth Congress of teachers of mathematics of the Croatian Mathematical Society, Zagreb. Retrieved from http://frink.machighway.com/~dynamicm/

Del Grande, J. J. (1986). *Can grade two children's spatial perception be improved by inserting a transformation geometry component into their mathematics program?* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI NL31402)

Denis, L. P. (1987). *Relationships between stage of cognitive development and van Hiele level of geometric thought among Puerto Rican adolescents* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 303465562)

Denzin, N. K., & Lincoln, Y. S. (2000). *Handbook of qualitative research*. Thousand Oaks, CA: Sage Publications.

Dickson, L., Brown, M., & Gibson, O. (1984). *Children Learning Mathematics*. Eastbourne, East Sussex: Holt, Rinehart and Winston.

Dixon, J. K. (1995). *English language proficiency and spatial visualization in middle school students' construction of the concepts of reflection and rotation using the "The Geometer's Sketchpad"* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 9607507)

Dixon, J. K. (1997). Computer use and visualization in students' construction of reflection and rotation concepts. *School Science and Mathematics, 97*(7), 352-358. doi:10.1111/j.1949-8594.1997.tb17376.x

Duval, R. (2005). Les conditions cognitives de l' ápprentissage de la géométrie. *Annales de Didactique et des Sciences Cognitive, 10*, 5-54. Retrieved from http://irem.u-strasbg.fr/data/publi/annales/articles/Vol10PDF/Duval.pdf

Duval, R. (2011, October). *Why figures cannot help students to see and understand in geometry? Analysis of the role and the cognitive functioning of visualization*. Abstract presented at the Symposium Mathematics Education Research at the University of Cyprus and Tel Aviv University, Nicosia.

Edwards, L. (2003, February). *The nature of mathematics as viewed from cognitive science*. Paper presented at the Third Conference of the European Society for Research in Mathematics Education, Bellaria. Retrieved from http://www.mathematik.uni-dortmund.de/~erme/CERME3/

Edwards, L. D. (1989). *Children's learning in a computer microworld for transformation geometry* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 9028812)

Edwards, L. D. (1990). The role of microworlds in the construction of conceptual entities. In G. Booker, P. Cobb, & T. de Mendicuti (Eds.), *Proceedings of the 14th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 235-242). Mexico City, Mexico: PME.

Edwards, L. D. (1991). Children's learning in a computer microworld for transformation geometry. *Journal for Research in Mathematics Education*, *22*(2), 122-137. Retrieved from http://search.proquest.com/docview/63021516?accountid=17200

Edwards, L. D. (1997). Exploring the territory before proof: Students' generalizations in a computer microworld for transformation geometry. *International Journal of Computers for Mathematical Learning*, 2, 187-215. doi: 10.1023/A:1009711521492

Edwards, L. D., & Zazkis, R. (1993). Transformation geometry: Naive ideas and formal embodiments. *Journal of Computers in Mathematics and Science Teaching*, *12*(2), 121-145.

Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmermann & S. Cunningham S. (Eds.), *Visualization in Teaching and Learning Mathematics* (pp. 25-37). Washington, DC: Mathematical Association of America.

Ekstrom, R. B., French, J. W., & Harman, H. H. (1976). *Manual for kit of factor-referenced cognitive tests*. Princeton, NJ: Educational Testing Service.

Ellis, R. D., & Kurniawan, S. H. (2000). Increasing the usability of online information for older users: A case study in participatory design. *International Journal of Human-Computer Interaction, 12*(2), 263-276. doi: 10.1207/S15327590IJHC1202_6

Ernest, P. (1986). Computer gaming for the practice of transformation geometry skills. *Educational Studies in Mathematics*, *17*, 205-207. doi: 10.1007/BF00311523

Evans, C., & Cools, E. (2009). A style perspective: the use and understanding of style differences to enhance learning. *Reflecting Education*, *5*(2), 1-18. Retrieved from http://reflectingeducation.net

Evans, C., & Cools, E. (2011). Applying styles research to educational practice. *Learning and Individual Differences*, *21*(3), 249-254. doi:10.1016/j.lindif.2010.11.009

Evans, C., & Waring, M. (2011). Applications of styles in educational instruction and assessment. In L. F. Zhang, R. Sternberg, & S. Rayner (Eds.), *The Handbook of Intellectual Styles* (pp. 297-330). NY: Springer.

Fennema, E., & Sherman, J. (1977). Sex-related differences in mathematics achievement, spatial visualization, and affective factors. *American Educational Research Journal*, *14*(1), 51-71. doi: 10.3102/00028312014001051

Feza, N., & Webb, P. (2005). Assessment standards, van Hiele levels, and grade seven learners' understandings of geometry. *Pythagoras*, *62*, 36-47. doi: 10.4102/pythagoras.v0i62.113

Fischbein, E. (1993). The theory of figurative concepts. *Educational Studies in Mathematics, 24*, 139-162. doi: 10.1007/BF01273689

Flanagan, K. (2001). *High school students' understandings of geometric transformations in the context of a technological environment* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 3020450)

French, J. W. (1951). *The description of aptitude and achievement tests in terms of rotated factors*. Chicago, IL: University of Chicago Press.

Frykholm, J. A. (1994). *External variables as predictors of van Hiele levels in algebra and geometry students*. Retrieved from the ERIC database. (ED372924)

Fuys, D., Geddes, D., & Tischler, R. (1988). The van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education. Monograph, 3*, 1-196. Retrieved from http://www.jstor.org/stable/749957

Glass, B. (2001). Making better use of computer tools in geometry. *Mathematics Teacher*, *94*, 224-229. Retrieved from http://www.nctm.org/publications/

Goldenberg, E. P. (1995). Ruminations about dynamic imagery (and a strong plea for research). In R. Sutherland & J. Mason (Eds.), *Exploiting Mental Imagery with Computers in Mathematics Education* (pp. 202–224). Berlin: Springer.

Goldenberg, E. P., Lewis, P., & O' Keefe, J. (1992). Dynamic representation and the development of a process understanding of function. In G. Harel & E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy . MAA Notes Number 25.* Washington DC: Mathematical Association of America.

Gonzalez, C., Thomas, R. P., & Vanyukov, P. (2005). The relationships between cognitive ability and dynamic decision making. *Intelligence*, *33*(2), 169–186. doi: 10.1016/j.intell.2004.10.002

Gorgorio, N. (1998). Exploring the functionality of visual and non- visual strategies in solving rotation problems. *Educational Studies in Mathematics, 35*, 207-231. doi: 10.1023/A:1003132603649

Grenier, D. (1985). Middle school pupils' conception about reflections according to a task of construction. In L. Streefland (Ed.), *Proceedings of the 9th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 183-188). Utrecht, The Netherlands: PME.

Grigorenko, E. L., & Sternberg, R. J. (1995). Thinking styles. In D. H. Saklofske & M. Zeidner (Eds.), *International Handbook of Personality and Intelligence* (pp. 205-229). New York: Plenum Press.

Guay, R. (1977). *Purdue Spatial Visualization Tests*. West Lafayette, Indiana: Purdue Research Foundation.

Guay, R. B. & McDaniel, E. D. (1977). The relationship between mathematics achievement and spatial abilities among elementary school children. *Journal for Research in Mathematics Education, 8*(3), 211–215. Retrieved from the ERIC database. (EJ161033)

Guilford, J. P., & Lacey, J. I. (1947). *Printed classification tests. AAF aviation psychology research program reports* (No. 5). Washington, DC: GPO.

Gutiérrez, A. (1992). Exploring the links between Van Hiele levels and 3-dimensional geometry. *Structural Topology*, *18*, 31-48. Retrieved from http://www.iri.upc.edu/people/ros/StructuralTopology/

Gutiérrez, A. (1996). Vizualization in 3-dimensional geometry: In search of a framework. In L. Puig & A. Guttierez (Eds.), *Proceedings of the 20th conference of the international group for the psychology of mathematics education* (Vol. 1, pp. 3-15). Valencia: Universidad de Valencia.

Gutiérrez, A., & Jaime, A. (1998). On the assessment of the van Hiele levels of reasoning. *Focus on Learning Problems in Mathematics, 20*, 27-46.

Gutiérrez, A., Jaime, A., & Fortuny, J. (1991). An alternative paradigm to evaluate the acquisition of the Van Hiele levels. *Journal for Research in Mathematics Education*, *22*(3), 237-251.

Guven, B. (2012). Using dynamic geometry software to improve eight grade students' understanding of transformation geometry. *Australasian Journal of Educational Technology*, 28(2), 364-382. Retrieved from http://www.ascilite.org.au/ajet/ajet.html

Hall, J. K. (2000). *Field dependence-independence and computer-based instruction in geography* (Doctoral dissertation). Retrieved from the Digital Library and Archives database. (URN etd-05022000-19260058)

Halpern, D., & Collaer, M. (2005). Sex differences in visuospatial abilities: More than meets the eye. In P. Shah & A. Miyake (Eds.), *The Cambridge Handbook of Visuospatial Thinking* (pp. 170-212). New York: Cambridge University Press.

Handal, B., & Herrington, A. (2003). Re-examining categories of computer-based learning in mathematics education. *Contemporary Issues in Technology and Teacher Education*, *3*(3). Retrieved from http://www.citejournal.org/vol3/iss3/mathematics/article1.cfm

Hannafin, R. D., & Scott, B. N. (1998). Identifying critical learner traits in a dynamic computer-based geometry program, *The Journal of Educational Research, 92*, 3-12. doi: 10.1080/00220679809597570

Hannafin, R. D., Truxaw, M. P., Vermillion, J. R., & Liu, Y. (2008). Effects of spatial ability and instructional program on geometry achievement. *The Journal of Educational Research*, *101*(3), 148-157. doi: 10.3200/JOER.101.3.148-157

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), Research in collegiate mathematics education III (Vol. 7, pp. 234-283). Providence, RI: American Mathematical Society.

Harper, J. (2003). Enhancing elementary pre-service teachers' knowledge of geometric transformations through the use of dynamic geometry computer software. In C. Crawford et al. (Eds.), *Proceedings of Society for Information Technology & Teacher Education International Conference* (pp. 2909-2916). Chesapeake, VA: AACE.

Harper, S. R. (2002). *Enhancing elementary pre-service teachers' knowledge of geometric transformations* (Unpublished doctoral dissertation). University of Virginia, Charlottesville, Virginia.

Hatfield, L. L., & Kieren, T. E. (1972). Computer-assisted problem solving in school mathematics. *Journal for Research in Mathematics Education*, *3*, 99-112.

Hays, T. A. (1996). Spatial abilities and the effects of computer animation on short-term and long-term comprehension. *Journal of Educational Computing Research, 14*, 139-155. doi: 10.2190/60Y9-BQG9-80HX-UEML

Hegarty, M., & Kriz, S. (2008). Effects of knowledge and spatial ability on learning from animation. In R. Lowe & W. Schnotz (Eds.). *Learning with Animation: Research and Implications for Design* (pp. 3-25). NewYork: Cambridge University Press.

Hegarty, M. (2004a). Diagrams in the mind and in the world: Relations between internal and external visualizations. In A. Blackwell, K. Mariott, & A. Shimojima (Eds.), *Diagrammatic Representation and Inference. Lecture Notes in Artificial Intelligence 2980* (pp. 1-13). Berlin: Springer-Verlag.

Hegarty, M. (2004b). Dynamic visualizations and learning: getting to the difficult questions. *Learning and Instruction*, *14*(3), 343-351. doi: 10.1016/ j.learninstruc.2004.06.007

Hegarty, M. (2010). Components of spatial intelligence. In B. H. Ross (Ed.), *The Psychology of Learning and Motivation* (Vol. 52, pp. 265-297). San Diego: Academic Press.

Hegarty, M., & Kozhevnikov, M. (1999). Types of visual-spatial representations and mathematical problem solving. *Journal of Educational Psychology, 91*(4), 684-689. Retrieved from http://www.psych.ucsb.edu/

Hegarty, M., & Waller, D. (2004). A dissociation between mental rotation and perspective-taking spatial abilities. *Intelligence*, *32*(2), 175-191. doi:10.1016/j.intell.2003.12.001

Hegarty, M., & Waller, D. (2006). Individual differences in spatial abilities. In P. Shah & A. Miyake (Eds.), *Handbook of Visuospatial Thinking* (pp. 121-169). New York: Cambridge University Press.

Hershkowitz, R. (1989). Visualization in geometry - two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61-76.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition. A Research Synthesis by the International Group for the Psychology of Mathematics Education* (pp. 70-95). New York: Cambridge University Press.

Hoffer, A. (1981). Geometry is more than proof. *Mathematics Teacher*, *74*, 11-18.

Hoffer, A. (1983). Van Hiele based research. In R. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 205-227). New York: Academic Press.

Höffler, T. N. (2010). Spatial ability: Its influence on learning with visualizations—a meta-analytic review. *Educational Psychology Review*, *22*(3), 245-269. doi:10.1007/s10648-010-9126-7

Höffler, T. N., & Leutner, D. (2011). The role of spatial ability in learning from instructional animations – Evidence for an ability-as-compensator hypothesis. *Computers in Human Behavior*, *27*(1), 209-216. doi:10.1016/j.chb.2010.07.042

Höffler, T. N., & Schwartz, R. N. (2011). Effects of pacing and cognitive style across dynamic and non-dynamic representations. *Computers & Education, 57*(2). doi:10.1016/j.compedu.2011.03.012

Höffler, T. N., Prechtl, H., & Nerdel, C. (2010). The influence of visual cognitive style when learning from instructional animations and static pictures. *Learning and Individual Differences*, *20*(5), 479-483. doi:10.1016/j.lindif.2010.03.001

Höffler, T. N., Sumfleth, E., & Leutner, D. (2006, April). *The role of spatial ability when learning from an instructional animation or a series of static pictures*. Paper presented at the Technology & Learning Symposium, New York. Retrieved from http://create.alt.ed.nyu.edu/symposium2006/ NYUSymposium2006_Hoeffler_Sumfleth_Leut ner.pdf

Hollebrands, K. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *The Journal of Mathematical Behavior*, *22*(1), 55-72. doi:10.1016/S0732-3123(03)00004-X

Hollebrands, K. (2004). Connecting research to teaching: High school students' intuitive understandings of geometric transformations. *Mathematics Teacher*, *97*(3), 207-214. Retrieved from http://www.nctm.org/publications/

Hoong, L.Y., &  Khoh, L.S. (2003). Effects of geometer's sketchpad on spatial ability and achievement in transformation geometry among secondary two students in Singapore. *The Mathematics Educator*, *7*(1), 32-48. Retrieved from http://math.nie.edu.sg/ame/matheduc/

Jackiw, N. (1991). *The Geometer's Sketchpad*. Berkeley, California: Key Curriculum Press.

Jaime, A., & Gutiérrez, A. (1995). Guidelines for teaching plane isometries in secondary school. *The Mathematics Teacher*, *88*(7), 591-597.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason.* Chicago: University of Chicago Press.

Johnson-Gentile, K. (1990). The effects of computer and non-computer environments on fifth and sixth grade students' conceptualizations of geometric motions (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 9022169)

Johnson-Gentile, K., Clements, D., & Battista, M. (1994). The effects of computer and non-computer environments on students' conceptualizations of geometric motions. *Journal of Educational Computing Research*, *11*(2), 121-140. doi: 10.2190/49EE-8PXL-YY8C-A923

Jones, K. (2002). Issues in the teaching and learning of geometry. In L. Haggarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 121-139). London, UK: Routledge Falmer.

Jones, K., & Mooney, C. (2003). Making space for geometry in primary mathematics. In I. Thompson (Ed.), *Enhancing primary mathematics teaching* (pp. 3-15). Maidenhead, Philadelphia: Open University Press.

Jung, I. (2002). *Student representation and understanding of geometric transformations with technology experience* (Doctoral Dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 0803710)

Kagan, J. (1958). The concept of identification. *Psychological Review, 65,* 296-305. doi: 10.1037/h0041313

Kagan, J., & Kogan, N. (1970). Individual variation in cognitive processes. In P. A. Mussen (Ed.), *Carmichael's manual of child psychology* (Vol. 1, pp. 1273-1365). New York: Wiley.

Kayhan, B. (2005). *Investigation of high school students' spatial ability* (Unpublished master's thesis). Middle East Technical University, Turkey.

Keehner, M., Hegarty, M., Cohen, C., Khooshabeh, P., & Montello, D. (2008). Spatial reasoning with external visualizations: What matters is what you see, not whether you interact. *Cognitive Science: A Multidisciplinary Journal*, *32*(7), 1099-1132. doi:10.1080/03640210801898177

Kelly, A. E., & Lesh, R. (2000). *Handbook of research design in mathematics and science education*. London: Lawrence Erlbaum Associates.

Kidder, F. R. (1976). Elementary and middle school children's comprehension of Euclidean transformations. *Journal for Research in Mathematics Education*, *7*(1), 40-52. Retrieved from http://www.jstor.org/stable/748764

Kidder, F. R. (1977). Euclidean transformations: elementary school spaceometry. *The Arithmetic Teacher*, *24*(3), 201-207. Retrieved from http://www.jstor.org/stable/41189226

Kimura, D. (1999). *Sex and cognition*. Cambridge, MA: MIT Press.

Kirby, J. R. (1990, July). *Spatial cognition in education*. Paper presented at the International Congress of Applied Psychology, Kyoto.

Kirby, J. R., & Boulter, D. R. (1999). Spatial ability and transformational geometry. *European Journal of Psychology of Education*, *14*, 283-294. doi:10.1007/BF03172970

Kirby, J. R., & Schofield, N. J. (1991). Spatial cognition: The case of map comprehension. In G. Evans (Ed.), *Learning and Teaching Cognitive Skills* (pp. 109-125). Melbourne, Australia: Australian Council for Educational Research.

Kline, P. (1999). *The handbook of psychological testing* (2nd ed.). London: Routledge.

Kogan, N. (1983). Stylistic variation in childhood and adolescence: Creativity, metaphor, and cognitive style. In P. A. Mussen (Ed.), *Handbook of child psychology* (Vol. 3, pp. 630-706). New York: Wiley.

Kombartzky, U., Ploetzner, R., Schlag, S., & Metz, B. (2010). Developing and evaluating a strategy for learning from animations. *Learning and Instruction*, *20*(5), 424-433. doi:10.1016/j.learninstruc.2009.05.002

Kosslyn, S. M. (1995). Mental imagery. In S. M Kosslyn & D. Osherson (Eds.), *An Invitation to Cognitive Science: Visual Cognition* (2nd ed., pp. 267-296). Cambridge, MA: MIT Press.

Kosslyn, S. M. (1996). Neural systems and psychiatric disorders. *Cognitive Neuropsychiatry*, *1*, 89-93. doi: 10.1080/135468096396569

Kozhevnikov, M. (2007). Cognitive styles in the context of modern psychology: Toward an integrated framework of cognitive style. *Psychological Bulletin*, *133*(3), 464-481. doi:10.1037/0033-2909.133.3.464

Kozhevnikov, M., & Hegarty, M. (2001). A dissociation between object manipulation spatial ability and spatial orientation ability. *Memory & Cognition*, *29*(5), 745-756. doi: 10.1.1.167.8268

Kozhevnikov, M., Blazhenkova, O., & Becker, M. (2010). Trade-off in object versus spatial visualization abilities: Restriction in the development of visual-processing resources. *Psychonomic Bulletin & Review*, *17*(1), 29-35. doi: 10.3758/PBR.17.1.29.

Kozhevnikov, M., Hegarty, M., & Mayer, R. E. (2002). Revising the visualizer-verbalizer dimension: Evidence for two types of visualizers. *Cognition and Instruction*, *20*(1), 47-77. Retrieved from http://www.jstor.org/stable/3233862

Kozhevnikov, M., Kosslyn, S., & Shephard, J. (2005). Spatial versus object visualizers: A new characterization of visual cognitive style. *Memory & cognition*, *33*(4), 710-726. doi: 10.3758/BF03195337

Kriz, S., & Hegarty, M. (2007). Top-down and bottom-up influences on learning from animations. *International Journal of Human-Computer Studies*, *65*(11), 911-930. doi:10.1016/j.ijhcs.2007.06.005

Krutetskii, V. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.

Kurz, T. L., Middleton, J. A., & Yanik, H. B. (2005). A taxonomy of software for mathematics instruction. *Contemporary Issues in Technology and Teacher Education*, *5*(2), 123-137. Retrieved from http://www.citejournal.org/vol5/iss2/mathematics/article1.cfm

Kyritsis M., Gulliver S. R., Morar S., Macredie R. (2009). Impact of cognitive style on spatial knowledge acquisition. *Proceedings of the IEEE International Conference on Multimedia & Expo* (pp. 966-969). doi: 10.1109/ICME.2009.5202657

Laborde, C. (2002). Integration of technology in the design of geometry tasks with Cabri-geometry. *International Journal of Computers for Mathematical Learning*, *6*(3), 283-317. doi: 10.1023/A:1013309728825

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Large, A., Beheshti, J., Breuleux, A., & Renaud, A. (1998). The effect of animation in enhancing descriptive and procedural texts in a multimedia learning environment. *Journal of the American Society for Information Science, 47*, 437-448. doi: 10.1002/(SICI)1097-4571

Law, C. K. (1991). *A genetic decomposition of geometric transformation* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 9132464)

Lawrie, C. (1998). An alternative assessment: The Gutierrez, Jaime and Fortuny technique. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd PME International Conference* (Vol. 3, 175-182). Stellenbosch, South Africa: University of Stellenbosch.

Lean, G., & Clements, M. A. (1981). Spatial ability, visual imagery, and mathematical performance. *Educational Studies in Mathematics*, *12*(3), 267-299. doi:10.1007/BF00311060

Lesh, R. (1976). Transformation geometry in elementary school. In J. Martin & D. A. Bradbard (Eds.), *Space and geometry: Papers from research workshop*. Retrieved from the ERIC database. (ED132033).

Liedtke, W. (1995). Developing spatial abilities in the early grades. *Teaching Children Mathematics*, *2*(1), 12-18.

Linn, M. C, & Petersen, A. C. (1985). Emergence and characterization of sex differences in spatial ability: A meta-analysis. *Child Development*, *56*(6), 1479-1498. Retrieved from http://www.jstor.org/stable/1130467

Lohman, D. (1979). *Spatial ability: a review and reanalysis of the correlational literature. (Technical report, No. 8).* Stanford University, School of Education: Aptitude Research Project.

Lohman, D. (1988). Spatial abilities as traits, processes, and knowledge. In R. Sternberg (Ed.), *Advances in the psychology of human intelligence* (Vol. 40, pp. 181-248). Hillsdale, NJ: Erlbaum.

Lohman, D. (2000). Complex information processing and intelligence. In R. Sternberg (Ed.), *Handbook of human intelligence* (2nd ed., pp. 285-340). Cambridge, MA: Cambridge University Press.

Lohman, D. F. (1993, July). *Spatial ability and g*. Paper presented at the first Spearman Seminar, University of Plymouth, England.

Maier, P.H. (1998). Spatial geometry and spatial ability: How to make solid geometry solid?" In E. Cohors-Fresenborg, H. Maier, K. Reiss, G. Toerner, & H. G. Weigand (Eds.), *Selected Papers from the Annual Conference of Didactics of Mathematics 1996* (pp. 63-75), Regensberg. Retrieved from http://www.fmd.uni-osnabrueck.de/ebooks/gdm/annual1996.html

Malone, J., Boase-Jelinek, D., Lamb, M., & Leong, S. (2004, July). *The problem of misperception in mathematical visualization*. Paper presented at the International Conference on Mathematics Education, Copenhagen. Retrieved from http://www.merga.net.au/documents/RR_MaloneEtAl.pdf

Mansfield, H., & Scott, J. (1990). Young children solving spatial problems. In G. Booker, P. Cobb, & T. de Mendicuti (Eds.), *Proceedings of the 14th PME conference* (Vol. 2, pp. 275-282). Mexico City, Mexico: PME.

Maragos, C. (2004). Building educational scenarios with "Sketchpad." Retrieved from http://hermes.di.uoa.gr/edugames/sketchpad/ScetchPad.pdf

Marcoulides, G. A., & Schumacker, R. E. (1996). *Advanced structural equation modelling: Issues and techniques*. Mahwah: Erlbaum.

Martin, G. (1982). *Transformation geometry: an introduction to symmetry*. New York: Springer.

Martinez, M., Barcena, F., & Rodriguez, S. (2005). ICT in mathematics education: geometry problem solving with applets. In A. Méndez-Vilas, B. González-Pereira, J. González, & J. A. González (Eds.), *Recent research developments in learning technologies: proceedings of the third international conference on multimedia and information and communication technologies in education* (Vol. 2, pp. 211-215). Caceres, Spain: FORMEX.

Massa, L. J., & Mayer, R. E. (2006). Testing the ATI hypothesis: Should multimedia instruction accommodate verbalizer-visualizer cognitive style? *Learning and Individual Differences, 16*, 321-336. doi: 10.1016/j.lindif.2006.10.001

Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, *14*(1), 58-69. Retrieved from http://www.jstor.org/stable/748797

Mayer, R. E. (2005). *The Cambridge handbook of multimedia learning*. Cambridge: Cambridge University Press.

Mayer, R. E. (2009). *Multimedia learning*. New York: Cambridge University Press.

Mayer, R. E. (2011). Does styles research have useful implications for educational practice? *Learning and Individual Differences, 21*(3), 319-320. doi: 10.1016/j.lindif.2010.11.016

Maykut, P., & Morehouse, R. (1994). *Beginning qualitative research : a philosophic and practical guide*. London: Falmer.

McGee, M.G. (1979). Human spatial abilities: Psychometric studies and environmental, genetic, hormonal, and neurological influences. *Psychological Bulletin, 86*(5), 889-918. doi: 10.1037/0033-2909.86.5.889

McLeay, H. (2006). Imagery, spatial ability and problem solving. *Mathematics teaching incorporating*, 195, 36-38. Retrieved from http://www.atm.org.uk/journal/

Meissner, H. (2006). Development of spatial abilities. J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 299-305). Prague, Czech Republic: PME.

Messick, S. (1994). The matter of style: Manifestations of personality in cognition, learning, and teaching. *Educational Psychologist*, *29*(3), 121-136. doi:10.1207/s15326985ep2903_2

Michael, W. B., Guilford, J. P., Fruchter, B., & Zimmerman, W. S. (1957). The description of spatial-visualization abilities. *Educational and Psychological Measurement,* 17, 185-199. doi: 10.1177/001316445701700202

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis : an expanded sourcebook.* Thousand Oaks: Sage Publications.

Miwa, T. (1992). An overview of computer use for mathematical problem solving in Japanese schools. In J. P. Becker & T. Miwa (Eds.), *Proceedings of the US-Japan Seminar on Computer Use in School Mathematics*. Retrieved from the ERIC database. (ED351203)

Miyake, A., Friedman, N. P., Rettinger, D. A., Shah, P., & Hegarty, M. (2001). How are visuospatial working memory, executive functioning, and spatial abilities related? A latent-variable analysis. *Journal of Experimental Psychology: General*, *130*(4), 621–640. doi:10.1037//0096-3445.130.4.621

Molina, D. (1990). *The applicability of the van Hiele theory to transformational geometry* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 9116934)

Moreno-Armella, L., Hegedus, S. J., & Kaput, J. J. (2008). From static to dynamic mathematics: historical and representational perspectives. *Educational Studies in Mathematics*, *68*(2), 99-111. doi:10.1007/s10649-008-9116-6

Morgan, H. (1997). *Cognitive styles and classroom learning*. Westport, CT: Praeger.

Moyer, J. C. (1974). *An investigation into the cognitive development of Euclidean transformations in young children* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 7507959)

Moyer, J. C. (1978). The relationship between the mathematical structure of Euclidean transformations and the spontaneously developed cognitive structures of young children. *Journal for Research in Mathematics Education*, *9*, 83-92. Retrieved from http://www.jstor.org/stable/748871

Münzer, S., Seufert, T., & Brünken, R. (2009). Learning from multimedia presentations: Facilitation function of animations and spatial abilities. *Learning and Individual Differences*, *19*(4), 481-485. doi:10.1016/j.lindif.2009.05.001

Muthén, L. K., & Muthén, B. O. (2010). *Mplus User's Guide* (6th ed.). Los Angeles, CA: Muthén & Muthén.

Naidoo, J. (2010). Strategies used by Grade 12 mathematics learners in transformation geometry. In V. Mudaly (Ed.), *Proceedings of the Eighteenth Annual Meeting of the Southern African Association for Research in Mathematics, Science and Technology Education* (Vol. 2, pp. 182-190). Natal: University of Kwazulu.

Nasser, L. (1989). Are the van Hiele Levels applicable to transformation geometry? In G. Vergnaud, J. Rogalski, & M. Artique (Eds.), *Proceedings of the 13th International Group for the Psychology of Mathematics Education Conference* (Vol. 3, pp. 25-32). Paris: PME.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (1995). *Mathematics anxiety [Supplemental Brochure]*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Nicolaou, A. A., & Xistouri, X. (2011). Field dependence/independence cognitive style and problem posing: An investigation with sixth grade students. *Educational Psychology, 31* (5), 611-627. doi: 10.1080/01443410.2011.586126

Nohda, N. (1992). Geometry teaching in Japanese school mathematics. *Pythagoras*, *28*, 18-25.

Olive, J. (1991). Logo programming and geometric understanding: An in-depth study. *Journal for Research in Mathematics Education, 22*(2), 90-111. Retrieved from http://www.jstor.org/stable/749587

Olkun, S. (2003). Comparing computer versus concrete manipulatives in learning 2D geometry. *Journal of Computers in Mathematics and Science Teaching*, *22*(1), 43-56. Retrieved from http://www.editlib.org/p/17806

Olkun, S., Altun, A., & Smith, G. (2005). Computers and 2D geometric learning of Turkish fourth and fifth graders. *British Journal of Educational Technology, 36*(2), 317-326. doi:10.1111/j.1467-8535.2005.00460.x

Orton, A. (2004). *Learning Mathematics: Issues, Theory and Classroom Practice*. London: Continuum International Publishing Group.

Owens, K. (1993). *Spatial thinking processes employed by primary school students engaged in mathematical problem solving* (Unpublished doctoral dissertation). Deakin University, Geelong, Victoria, Australia.

Paivio, A. (1971). *Imagery and verbal processes*. New York: Holt, Rinehart & Winston.

Park, S. I., Lee, G., & Kim, M. (2009). Do students benefit equally from interactive computer simulations regardless of prior knowledge levels? *Computers & Education*, *52* (3), 649-655. doi: 10.1016/j.compedu.2008.11.014

Parkinson, A., & Redmond, J. A. (2002). Do cognitive styles affect learning performance in different computer media. *Proceedings of the 7th annual conference on innovation and technology in computer science education, association for computing machinery* (pp. 39-43). New York: ACM.

Pask, G. (1972). A fresh look at cognition and the individual. *International Journal of Man–Machine Studies*, 4, 211–216.

Pegg, J. (1995). Learning and teaching geometry. In L. Grimison & J. Pegg (Eds.), *Teaching Secondary Mathematics: Theory into Practice* (pp. 87-103). Sydney: Harcourt and Brace.

Pegg, J., & Davey, G. (1989). Clarifying level descriptions for children's understanding of some basic 2D geometric shapes. *Mathematics Education Research Journal*, *1*(1), 16-27. doi: 10.1007/BF03217200

Peterson, E. R., Rayner, S. G., & Armstrong, S. J. (2009). Researching the psychology of cognitive style and learning style: is there really a future? *Learning and Individual Differences, 19*(4), 518-523. doi: 10.1016/j.lindif.2009.06.003

Piaget, J. (1972). Intellectual evolution from adolescence to adulthood. *Human Development*, *15(1)*, 1-12. doi: 10.1159/000271225

Piaget, J., & Inhelder, B. (1967). *A child's conception of space*. New York: Norton.

Pitta, D., & Gray, E.M. (1999). Images and their frames of reference: A perspective on cognitive development in elementary arithmetic. In O. Zaslavsky (Ed.), *Proceedings of the 23rd International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 49-56). Haifa: Israel.

Pittalis, M., & Christou, C. (2010). Types of reasoning in 3D geometry thinking and their relation with spatial ability. *Educational Studies in Mathematics*, *75*(2), 191–212. doi:10.1007/s10649-010-9251-8

Pitta-Pantazi, D., & Christou, C. (2009a). Cognitive styles, dynamic geometry and measurement performance. *Educational Studies in Mathematics*, *70*(1), 5-26. doi:10.1007/s10649-008-9139-z

Pitta-Pantazi, D., & Christou, C. (2009b). Cognitive styles, task presentation mode and mathematical performance. *Research in Mathematics Education*, *11*(2), 131-148. doi:10.1080/14794800903063331

Pitta-Pantazi, D., & Christou, C. (2009c). Mathematical creativity and cognitive styles. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 377-384). Thessaloniki, Greece: PME.

Pleet, L. (1990). *The effects of computer graphics and Mira on acquisition of transformation geometry concepts and development of mental rotation skills in grade eight* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 9130918)

Ploetzner, R., & Lowe, R. (2004). Dynamic visualisations and learning. *Learning and Instruction*, *14*(3), 235-240. doi:10.1016/j.learninstruc.2004.06.001

Portnoy, N., Grundmeier, T., & Graham, K. (2006). Students' understanding of mathematical objects in the context of transformational geometry: Implications for constructing and understanding proofs. *The Journal of Mathematical Behavior*, *25*(3), 196-207. doi:10.1016/j.jmathb.2006.09.002

Presmeg, N. (1986a). Visualisation and mathematical giftedness. *Educational Studies in Mathematics*, *17*(3), 297-311. doi:10.1007/BF00305075

Presmeg, N. (1986b). Visualization in high school mathematics. *For the Learning of Mathematics*, *6* (3), 42-46. Retrieved from http://www.jstor.org/stable/40247826

Presmeg, N. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, *23*(6), 595-610. doi: 10.1007/BF00540062

Presmeg, N. (1997a). Generalization using imagery in mathematics. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp. 299-312). Mahwah, NJ: Erlbaum.

Presmeg, N. (1997b). Reasoning with metaphors and metonymies in mathematics learning. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors and images* (pp. 267-279). Mahwah, NJ: Erlbaum.

Presmeg, N. (2006a). Research on visualization in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 205-236). Rotterdam: Sense.

Presmeg, N. (2006b). Spatial abilities research as a foundation for visualization in teaching and learning mathematics. In P. Clarkson & N. Presmeg (Eds.), *Critical Issues in Mathematics Education* (pp. 83-95). Boston, MA: Springer US.

Pritchard, C. (2003). *The changing shape of geometry: celebrating a century of geometry and geometry teaching*. New York: Cambridge University Press.

Pyke, C. L. (2003). The use of symbols, words, and diagrams as indicators of mathematical cognition: A causal model. *Journal for Research in Mathematics Education*, 34(5), 406-432. Retrieved from http://www.jstor.org/stable/30034794

Rahim, M. H. (2002). *A classroom use of the Geometer's Sketchpad in a mathematics pre-service teacher education program* [Electronic version]. Retrieved from http://math.unipa.it/~grim/Jrahaim

Reed, S. (2006). Cognitive architectures for multimedia learning. *Educational Psychologist*, *41*(2), 87-98. doi:10.1.1.88.6569&rank

Reys, R., Lindquist, M., Lambdin, D., Smith, N., & Suydam, M. (2007). *Helping children learn mathematics* (8th ed.). Hoboken  NJ: John Wiley & Sons.

Richardson, A. (1977). Verbalizer-visualizer: A cognitive style dimension. *Journal of Mental Imagery*, *1*, 109–125.

Riding, R. J. & Cheema, I. (1991). Cognitive styles: An overview and integration. *Educational Psychology*, 11(3/4), 193 – 216. doi: 10.1080/0144341910110301

Riding, R. J. (1991). *Cognitive styles analysis*. Birmingham, UK: Learning and Training Technology.

Riding, R. J. (2000). Cognitive style: A review. In R. J. Riding & S. G. Rayner (Eds.), *International Perspectives on Individual Differences* (Vol. 1, pp. 365-377). Stamford Connecticut: Ablex Publishing Corporation.

Riding, R. J., & Rayner, S. (1998). *Cognitive styles and learning strategies: Understanding style differences in learning and behaviour*. London: D. Fulton.

Riding, R. J., & Taylor, E. M. (1976). Imagery performance and prose comprehension in seven-year-old children. *Educational Studies*, 2, 21-27. doi: 10.1080/0305569760020103

Riding, R., & Douglas, G. (1993). The effect of cognitive style and mode of presentation on learning performance. *British Journal of Educational Psychology, 63*(2), 297-307. doi:10.1111/j.2044-8279.1993.tb01059.x

Roschelle J. M, Pea, R. D., Hoadley, C. M., Gordin, D. N., & Means, B. M. (2000). Changing how and what children learn in school with computer-based technologies. *The Future of Children, 10* (2), 76-101. Retrieved from http://ctl.sri.com/publications/ downloads/PackardChangingLearning.pdf

Saads, S., & Davis, G. (1997). Spatial abilities, van Hiele Levels and language use in three dimensional geometry. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 104-111). Lahti, Finland: PME.

Schoenfeld, A. H. (1986). On having and using geometric knowledge. In J. Hiebert (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp.225-264). Hillsdale, N.J: Lawrence Erlbaum Associates.

Schultz, K. A. (1978). Variables influencing the difficulty of rigid transformations during the transition between the concrete and formal operational stages of cognitive development. In R. Lesh & D. Mierkiewicz (Eds.), *Recent research concerning the development of spatial and geometric concepts* (pp. 256-165). Columbus, Ohio. Retrieved from the ERIC database. (ED159062)

Schultz, K. A., & Austin, J. D. (1983). Directional effects in transformation tasks. *Journal for Research in Mathematics Education, 14*(2), 95-101. Retrieved from http://www.jstor.org/stable/748577

Sedig, K. (2008). From play to thoughtful learning: A design strategy to engage children with mathematical representations. *Journal of Computers in Mathematics and Science Teaching, 27*(1), 65–101. Retrieved from http://www.editlib.org/d/ 23627/article_23627.pdf

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36. doi: 10.1007/BF00302715

Shaughnessy, M. J., & Burger, W. F. (1985). Spadework prior to deduction in geometry. *Mathematics Teacher*, *78*(6), 419-28. Retrieved from http://www.jstor.org/stable/27964574

Siegler, R. (1996). *Emerging minds: The process of change in children's thinking***.** New York: Oxford University Press.

Siegler, R. S. (2005). Children's learning. *American Psychologist, November*, 769-778.

Silver, E. A. (1987). Foundations of cognitive theory and research for mathematics problem solving instruction. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 33-60). Hillsdale, NJ: Lawrence Erlbaum.

Smith, G. G., & Olkun, S. (2004). How interactivity works: Priming of mental rotation. In L. Cantoni & C. McLoughlin (Eds.), *Proceedings of World Conference on Educational Multimedia, Hypermedia and Telecommunications* (pp. 4293-4302). Chesapeake, VA: AACE.

Smith, G. G., Gerretson, H., Olkun, S., Yuan, Y., Dogbey, J., & Erdem, A. (2009). Stills, not full motion, for interactive spatial training: American, Turkish and Taiwanese female pre-service teachers learn spatial visualization. *Computers & Education*, *52*(1), 201-209. doi:10.1016/j.compedu.2008.07.011

Soon, Y. P. (1989). *An investigation on van Hiele-like levels of learning transformation geometry of secondary school students in Singapore* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 8915764)

Sorby, S. A. (2009). Educational research in developing 3-D spatial skills for engineering students. *International Journal of Science Education*, *31*(3), 459-480. doi:10.1080/09500690802595839

Sorden, S. (2005). A cognitive approach to instructional design for multimedia learning. *Informing Science*, *8*, 263-279. Retrieved from http://www.informingscience.us/icarus/journals/informingscij/

Sternberg, R. J. (1988). *The triarchic mind: A new theory of human intelligence.* New York: Viking.

Sternberg, R. J. (1997). *Thinking styles*. New York: Cambridge University Press.

Sternberg, R. J. (2008). Applying psychological theories to educational practice. *American Educational Research Journal*, *45*(1), 150-165. doi: 10.3102/0002831207312910

Sternberg, R., & Zhang, L. F. (Eds.). (2001). *Perspectives on thinking, learning, and cognitive styles*. Mahwah,  NJ: Lawrence Erlbaum.

Sträßer, R. (2003). Research on dynamic geometry software (DGS) - an introduction. *Zralblatt für Didaktik der Mathematik*, 34 (3), 65. doi: 10.1007/BF02655707

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. *Cognitive Science*, *12*, 257-285. doi: 10.1016/0364-0213(88)90023-7

Sweller, J., van Merrienboer, J., & Paas, F. (1998). Cognitive architecture and instructional design. *Educational Psychology Review*, *10*(3), 251-296. doi: 10.1.1.89.9802

Tall, D. (1993). Real mathematics, rational computers and complex people. *Proceedings of the Fifth Annual International Conference on Technology in College Mathematics Teaching* (pp. 243-258). Addison-Wesley.

Tall, D., O., & West. B. (1986). Graphic insight into calculus and differential equations. In A. G. Howson & J. P. Kahane, *The Influence of Computers and Informatics on Mathematics and its Teaching*. *ICMI Study Series* (pp. 107-119). Strasbourg: Cambridge University Press.

Tartre, L. A. (1990). Spatial skill, gender, and mathematics. In E. Fennema & G. Leder (Eds.), *Mathematics and gender: Influences on teachers and students* (pp. 27–59). New York: Teachers College Press.

Tashakkori, A., & Teddlie, C. (2002). *Handbook of mixed methods*. London: SAGE.

Taylor, R. (1980). *The computer in the school: tutor, tool, tutee.* New York: Teachers College Press.

Thaqi, X., Giménez, J., & Rosich, N. (2011, February). *Geometrical transformations as viewed by prospective teachers*. Paper presented at the Seventh Conference of European Research in Mathematics Education, Rzészow. Retrieved from https://www.cerme7.univ.rzeszow.pl/WG/4/WG4_Xhevdet.pdf

Thomas, D. (1978). Students' understanding of selected transformation geometry concepts. In R. Lesh & D. Mierkiewicz (Eds.), *Recent research concerning the development of spatial and geometric concepts* (pp. 177-184). Columbus, Ohio. Retrieved from the ERIC database. (ED159062)

Thomas, F. (1983). An investigation into the relative effects of two methods of teaching geometry. *Mathematical Education for Teaching, 4*(2), 19-29.

Thomas, P. R., & McKay, J. B. (2010). Cognitive styles and instructional design in university learning. *Learning and Individual Differences*, *20*(3), 197-202. doi:10.1016/j.lindif.2010.01.002

Thurstone, L. L. (1950). *Some primary abilities in visual thinking*. Chicago, IL: University of Chicago Psychometric Lab.

Tinajero, C., & Páramo, M. F. (1998). Field dependence-independence cognitive style and academic achievement: A review of research and theory. *European Journal of Psychology of Education*, *13* (2), 227-251. doi: 10.1007/BF03173091

Tracy, D. M. (1987). Toys, spatial ability, and science and mathematics achievement: Are they related? *Sex Roles*, *17*(3/4), 115-138. doi: 10.1007/BF00287620

Tracy, D. M. (1990). Toy-playing behavior, sexrole orientation, spatial ability, and science achievement. *Journal of Research in Science Teaching*, *27*(7), 637-649. doi: 10.1002/tea.3660270704

Tso, T., & Liang, Y. N. (2002). The study of interrelationship between spatial abilities and van Hiele levels of thinking in geometry of eight-grade students. *Journal of Taiwan Normal University*, *46*(2), 1-20. Retrieved from http://www.ntnu.edu.tw/acad/epub/epub.htm

Tutak, T., & Birgin, O. (2008). *The effects of computer assisted instruction on the students' achievement in geometry* [Online version]. Retrieved from http://ietc2008.home.anadolu.edu.tr/ ietc2008/208.doc

Tversky, B., Morrison, J. B., & Betrancourt, M. (2002). Animation: can it facilitate? *International Journal of Human-Computer Studies*, *57*(4), 247-262. doi:10.1006/ijhc.2002.1017

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry*. *Final report of the Cognitive Development and Achievement in Secondary School Geometry Project.* Chicago: University of Chicago, Department of Education.

Usiskin, Z. (1987). Resolving the continuing dilemmas in school geometry. In M. Lindquist & A. Shulte (Eds.), *Learning and Teaching Geometry,K-12, 1987 Year-book* (pp. 17-31). Reston, VA: NCTM.

van Garderen, D. (2006). Spatial visualization, visual imagery, and mathematical problem solving of students with varying abilities. *Journal of Learning Disabilities*, *39*(6), 496-506. doi:10.1177/00222194060390060201

Van Hiele, P. M. (1986). *Structure and insight*. New York: Academic Press.

Van Hiele, P. M. (1999). Developing geometric thinking through activities that begin with play. *Teaching Children Mathematics*, *5*(6), 310-316.

Vandenberg, S. G., & Kuse, A. R. (1978). Mental rotations: A group test of three-dimensional spatial visualization. *Perceptual and Motor Skills*, *47*, 599-604. Retrieved from http://www.ncbi.nlm.nih.gov

Vinner, S., & Hershkowitz, R. (1983). On concept-formation in geometry. *Zentralblatt fur Didaktik der Mathematik*, *83*, 20-25.

Whitkin, H. A., Moore, C. A., Goodenough, D. R., & Cox, P. W. (1977). Field dependent and field independent cognitive styles and their educational implications. *Review of Educational Research*, *47*, 1-65. Retrieved from http://www.jstor.org/stable/1169967

Williford, H. (1972). A study of transformational geometry instruction in the primary grades. *Journal for Research in Mathematics Education*, *3*, 260-271. Retrieved from http://www.jstor.org/stable/748493

Wilson, J. W. (1992). Mathematical visualization in problem-solving facilitated by computers. In J. P. Becker & T. Miwa (Eds.), *Proceedings of the U.S.-Japan Seminar on Computer Use in School Mathematics*. Retrieved from the ERIC database. (ED 351 203)

Witkin, H. A. (1973). *The role of cognitive styles in academic performance and in teacher-student relations*. Princeton, NJ: Educational Testing Service.

Wraga, M., & Kosslyn, S. M. (2002). Imagery. *Encyclopaedia of Cognitive Science*. London: Nature Publishing Group.

Xistouri, X., & Pitta-Pantazi, D. (2006). Spatial rotation and perspective taking abilities in relation to performance in reflective symmetry tasks. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 425-432). Prague, Czech Republic: PME.

Xistouri, X., & Pitta-Pantazi, D. (2011a, February). *Elementary students' transformational geometry abilities and cognitive style.* Paper presented at the Seventh European Research Conference in Mathematics Education, Rzeszów, Poland. Retrieved from http://www.cerme7.univ.rzeszow.pl/WG/4/WG4_Xistouri_Pitta.pdf

Xistouri, X., & Pitta-Pantazi, D. (2011b). Prospective teachers' cognitive styles and ability in geometry tasks of axial reflection. In G. Vanthournout, L. Coertjens, V. Donche, D. Gijbels, C. Evans, E. Cools, & H. Pedrosa de Jesus (Eds.), *Proceedings of the 16th Annual Conference of the Education, Learning, Styles, Individual Differences Network* (pp. 433-440). Antwerp, Belgium: Universiteit Antwerpen.

Xistouri, X., & Pitta-Pantazi, D. (2012). The influence of cognitive style on children's ability to solve reflection and rotation tasks. In Z. M. Charlesworth, E. Cools, & C. Evans (Eds.*), Proceedings of the 17th Annual Conference of the Education, Learning, Styles, Individual Differences Network* (pp. 404-416). Brno: Tribun EU.

Yang, E., Andre, T., Greenbowe, T. J., & Tibell, L. (2003). Spatial ability and the impact of visualization/animation on learning electrochemistry. *International Journal of Science Education*, *25* (3), 329-349. doi: 10.1080/09500690210126784

Yanik, H. B. (2006). *Prospective elementary teachers' growth in knowledge and understanding of rigid geometric transformations* (Doctoral dissertation). Retrieved from ProQuest Dissertations and Theses database. (UMI 10.1080/ 09500690210126784)

Yanik, H. B. (2011). Prospective middle school mathematics teachers' preconceptions of geometric translations. *Educational Studies in Mathematics*, *78*(2), 231-260. doi:10.1007/s10649-011-9324-3

Yanik, H. B., & Flores, A. (2009). Understanding rigid geometric transformations: Jeff's learning path for translation. *The Journal of Mathematical Behavior*, *28*, 41-57. doi:10.1016/j.jmathb.2009.04.003

Yilmaz, H. B. (2009). On the development and measurement of spatial ability. *International Electronic Journal of Elementary Education*, *1*(2), 83-96. Retrieved from http://www.iejee.com/

Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytic strategies: a study of students' understanding of the group D4. *Journal for Research in Mathematics Education*, 27(4), 435-457.

Zhang, H., Wang, Y., Lou, Y., Li, G., & Zhao, B. (2008). Multimedia instructional design corresponded to cognitive psychology. In E. W. C. Leung et al. (Eds.), *Advances in blended learning* (pp. 155-164). Berlin, Heidelberg: Springer Berlin Heidelberg. doi:10.1007/978-3-540-89962-4_16

Zhang, L. F. (2004). Revisiting the predictive power of thinking styles for academic performance. *The Journal of Psychology, 138* (4), 351-370. doi: 10.3200/JRLP.138.4.351-370

Zimmerman, W.S. (1954). The influence of item complexity upon the factor composition of a spatial visualization test. *Educational and Psychological Measurement*, *4*, 106-119. doi: 10.1177/001316445401400109

APPENDICES

APPENDIX I

PILOT INSTRUMENTS

## ΔΟΚΙΜΙΟ 1Α

Όνομα: ................................................... Τάξη: .......

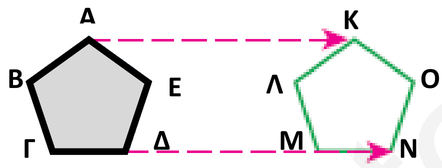
Αρ. στον κατάλογο: ...... Σχολείο: ........................................

Φύλο (βάλε ✓): Αγόρι ☐ Κορίτσι: ☐

## ΜΕΡΟΣ Α

### ΜΕΤΑΦΟΡΑ

Οι πιο κάτω ασκήσεις αναφέρονται στη **μεταφορά** (μετακίνηση) σχημάτων από μια **αρχική θέση** σε μια **νέα θέση**.




αρχική θέση σχήματος

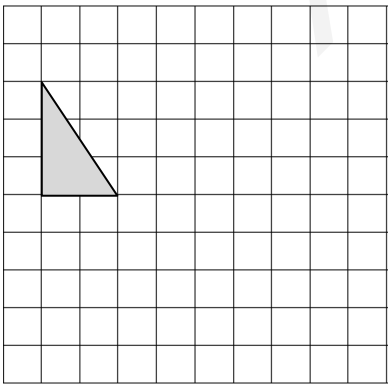
νέα θέση σχήματος

Στο διπλανό παράδειγμα, το σχήμα μεταφέρεται 3,5 cm προς τα δεξιά.

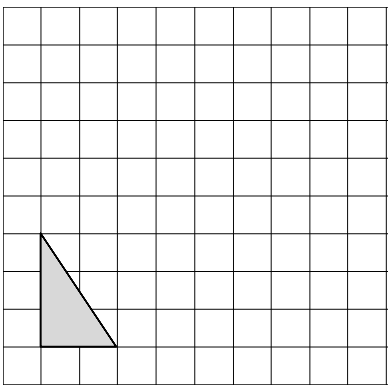
---

**Βρες τη νέα θέση του σχήματος και ζωγράφισε το, σύμφωνα με τις οδηγίες που δίνονται κάθε φορά πάνω από το σχήμα.**

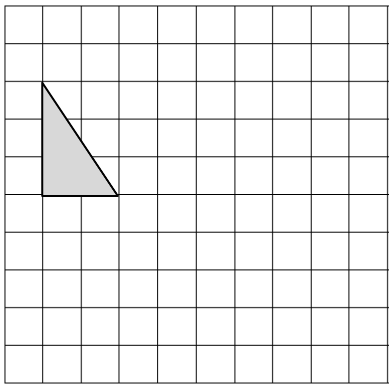
**Α13)** Όταν μεταφερθεί 4 κουτάκια προς τα δεξιά.



**Α14)** Όταν μεταφερθεί 4 κουτάκια διαγώνια προς τα πάνω και δεξιά.



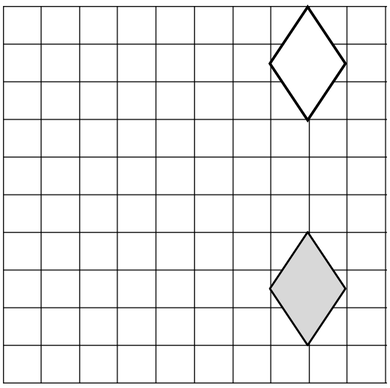
**Α15)** Όταν μεταφερθεί 1 κουτάκι προς τα δεξιά.

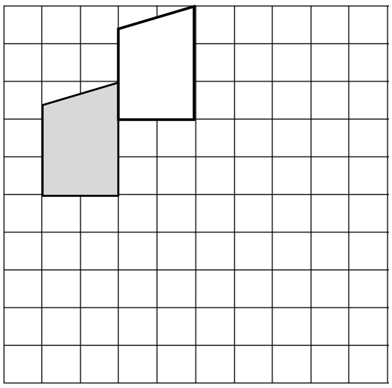




* It is noted that during test administration, the tasks were given to the students in serial numbering of each section. The numbering in the Appendix was modified for the purpose of the analysis and its presentation

**Τώρα, <u>δώσε</u> εσύ τις <u>οδηγίες</u> σε κάποιον για το πώς <u>να μεταφέρει</u> το <u>χρωματισμένο σχήμα</u> στη <u>νέα θέση</u> (όπως τις οδηγίες που σου δόθηκαν στην προηγούμενη άσκηση).**





**Α8.** Όταν μεταφερθεί :

…………………………………………………………….

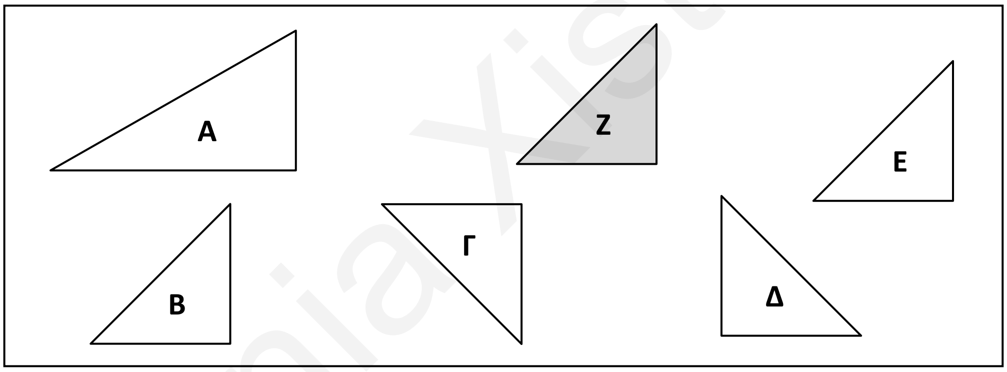
…………………………………………………………….

**Α10.** Όταν μεταφερθεί :

…………………………………………………………….

…………………………………………………………….

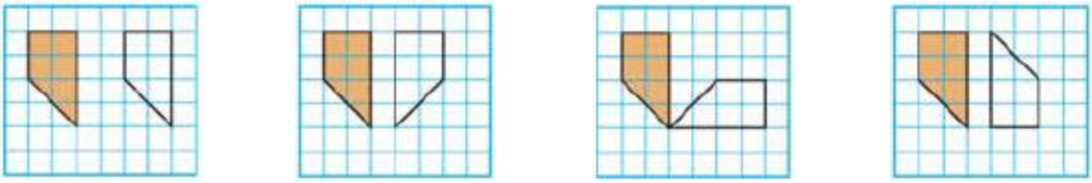
**Α7) <u>Κύκλωσε</u> όλα τα τρίγωνα που είναι <u>μεταφορές</u> του τριγώνου Ζ:**





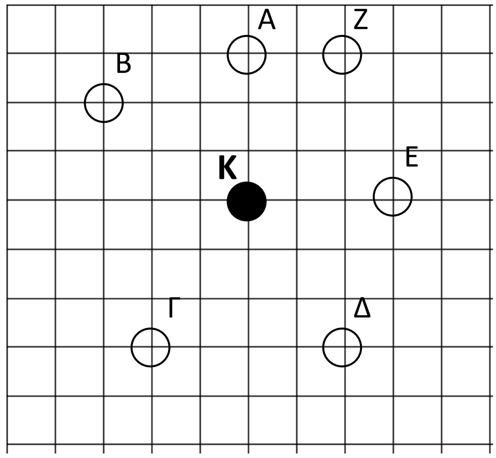

**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι <u>μόνο μία</u>. <u>Κύκλωσε</u> τη σωστή απάντηση σε κάθε περίπτωση:**

**Α5**) <u>Ποια</u> από τις πιο κάτω εικόνες παρουσιάζει τη <u>μεταφορά</u> του χρωματισμένου σχήματος;



Α Β Γ Δ

Ποιο από τα πιο κάτω αποτελεί μεταφορά του αρχικού σχεδίου Κ, όταν:




**Α3**) όταν το Κ μεταφερθεί 2 κουτάκια προς τα δεξιά και 3 προς τα πάνω;

α) Το Α β) Το Β γ) Το Δ δ) Το Ζ

**Α1**)όταν το Κ μεταφερθεί 3 κουτάκια προς τα πάνω;

α) Το Α β) Το Γ γ) Το Δ δ) Το Ε

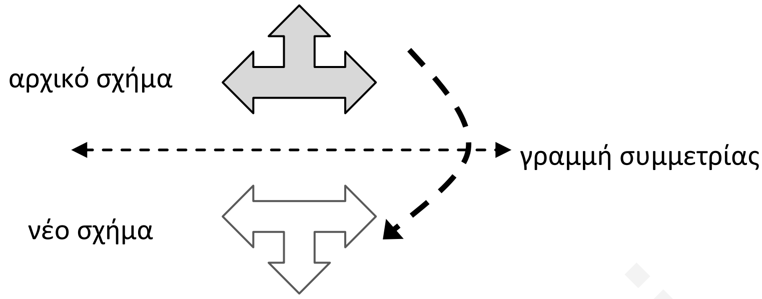
**Α2**) όταν το Κ μεταφερθεί 3 κουτάκια προς τα δεξιά;

α) Το Β β) Το Γ γ) Το Ε δ) Το Ζ

# ΜΕΡΟΣ Β

## ΑΝΑΚΛΑΣΗ

Οι πιο κάτω ασκήσεις αναφέρονται στην **ανάκλαση** (καθρέφτισμα) αντικειμένων με οριζόντια, κατακόρυφη ή διαγώνια γραμμή συμμετρίας.

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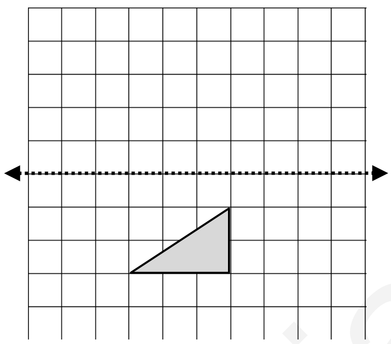
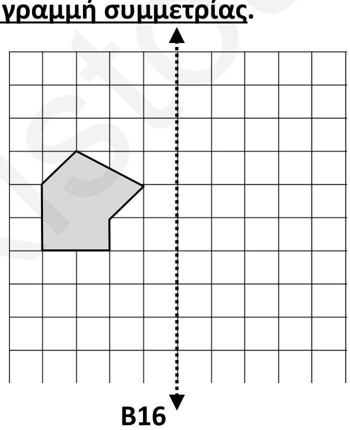
**Βρες την ανάκλαση και ζωγράφισε το συμμετρικό του κάθε αρχικού σχήματος που δίνεται, χρησιμοποιώντας κάθε φορά τη διακεκομμένη γραμμή συμμετρίας.**

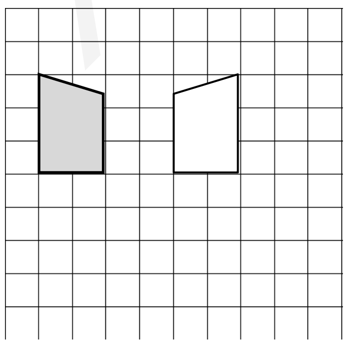


**Β12**

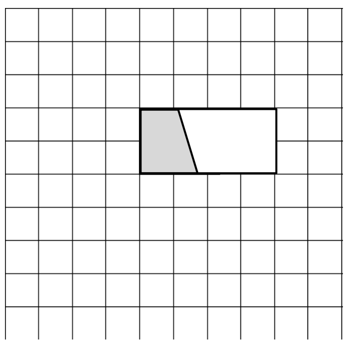


**Β16**

**Β) Τώρα, βρες και χάραξε με τη ρίγα σου τη γραμμή συμμετρίας για κάθε περίπτωση.**

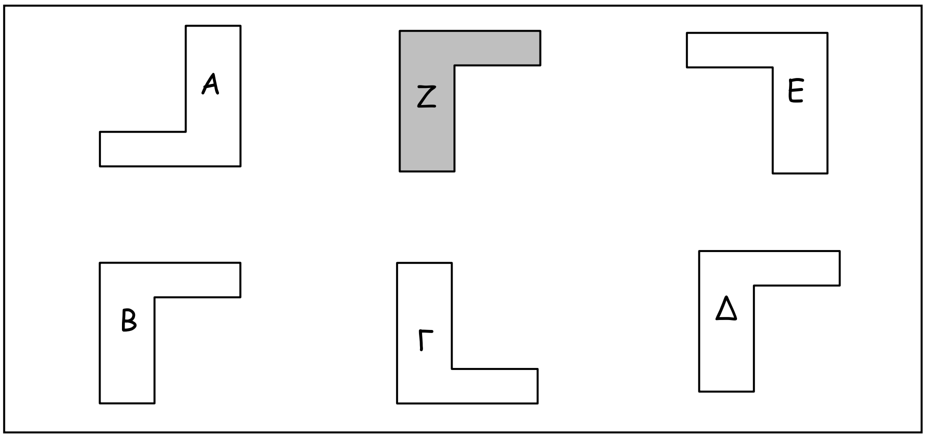


**Β9**



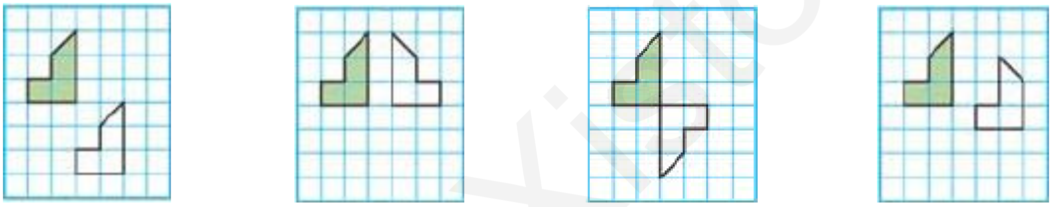
**Β11**

**B7)  <u>Κύκλωσε</u> τις επιλογές που είναι <u>ανακλάσεις</u> (συμμετρικά) του <u>αρχικού σχήματος Ζ</u>:**



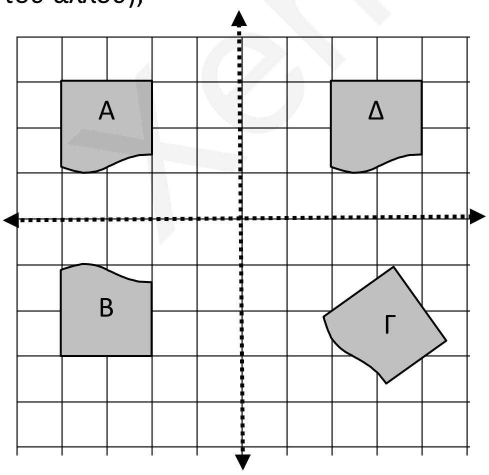

**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι <u>μόνο μία</u>. <u>Κύκλωσε</u> τη σωστή απάντηση σε κάθε περίπτωση:**

**B5**) <u>Ποια</u> από τις πιο κάτω εικόνες παρουσιάζει την <u>ανάκλαση</u> του χρωματισμένου σχήματος;



Α　　　　Β　　　　Γ　　　　Δ

**B4**) <u>Ποιο</u> από τα πιο κάτω ζευγάρια σχημάτων σχετίζονται με <u>ανάκλαση</u> (το ένα είναι συμμετρικό του άλλου);




α) Το Α με το Δ

β) Το Β με το Γ

γ) Το Β με το Α

δ) Το Γ με το Δ

# ΜΕΡΟΣ Γ

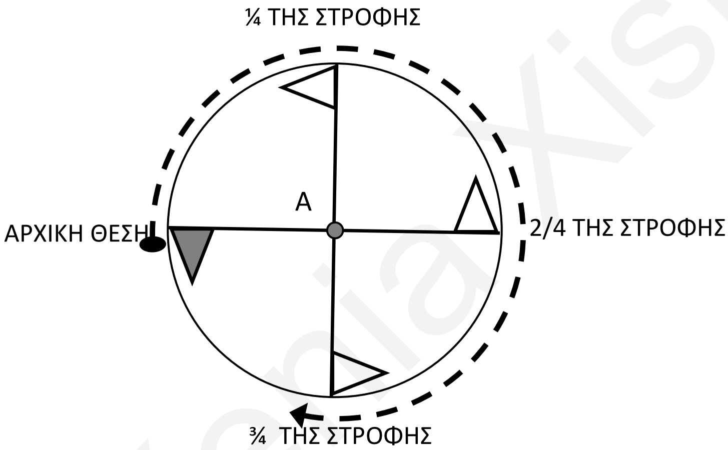
## ΠΕΡΙΣΤΡΟΦΗ

**Οι πιο κάτω ερωτήσεις αναφέρονται στην <u>περιστροφή</u> των σχημάτων, γύρω από συγκεκριμένο σημείο.**

**Τα σχήματα μπορούν να κάνουν στροφή (να γυρίσουν) προς τα δεξιά, δηλαδή όπως κινούνται οι δείκτες του ρολογιού, ή προς τα αριστερά δηλαδή αντίθετα με τους δείκτες του ρολογιού. <u>Στις πιο κάτω ασκήσεις, θεώρησε ότι όλες οι στροφές γίνονται προς τα δεξιά, όπως γυρίζουν οι δείκτες του ρολογιού</u>.**

**Μια ολόκληρη στροφή είναι 4/4 του κύκλου. Όταν η ερώτηση λέει ότι το σχήμα κάνει ¼ της στροφής, εννοεί ¼ του κύκλου. Τα 2/4 της στροφής είναι ίσα με 2/4 του κύκλου και τα ¾ της στροφής ίσα με τα ¾ του κύκλου.**

## ΠΑΡΑΔΕΙΓΜΑ

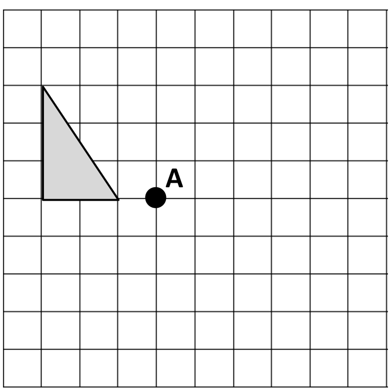



Το παράδειγμα δίπλα δείχνει την γκρίζα σημαία στις θέσεις που θα πάρει όταν κάνει στροφή προς τα δεξιά γύρω από το σημείο Α κατά $\frac{1}{4}$ της στροφής, 2/4 της στροφής και $\frac{3}{4}$ της στροφής.

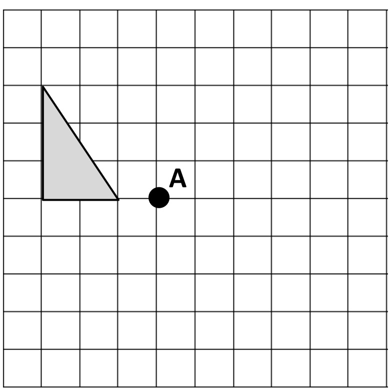
---

**<u>Ζωγράφισε</u> το σχήμα στη <u>νέα του θέση</u>, όταν κάνει <u>στροφή</u> <u>γύρω από το σημείο Α</u>, σύμφωνα με τις <u>οδηγίες</u> που δίνονται πάνω από <u>κάθε</u> <u>περίπτωση</u>.**

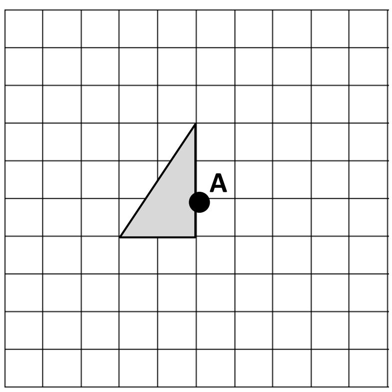
**C13)** Όταν κάνει ¼ της στροφής προς τα δεξιά .




**C12)** Όταν κάνει ¾ της στροφής προς τα δεξιά .

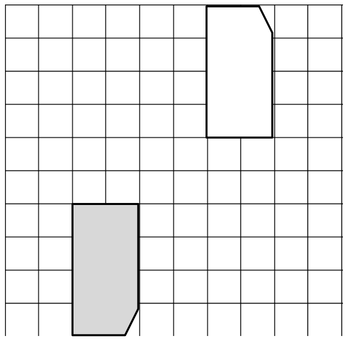



**C15)** Όταν κάνει 1/4 της στροφής προς τα δεξιά.



**Βρες και ΖΩΓΡΑΦΙΣΕ το ΣΗΜΕΙΟ Α γύρω από το οποίο έκανε στροφή το χρωματισμένο σχήμα, και στη συνέχεια κύκλωσε το κλάσμα που δείχνει πόση στροφή προς τα δεξιά έκανε το σχήμα.**

**C10**

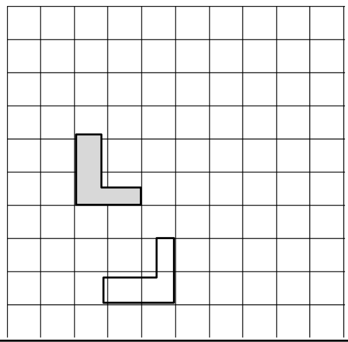


**Κύκλωσε το σωστό:**

**Παρουσιάζει στροφή προς τα δεξιά κατά:**

**1/4 2/4 3/4**

**C8**

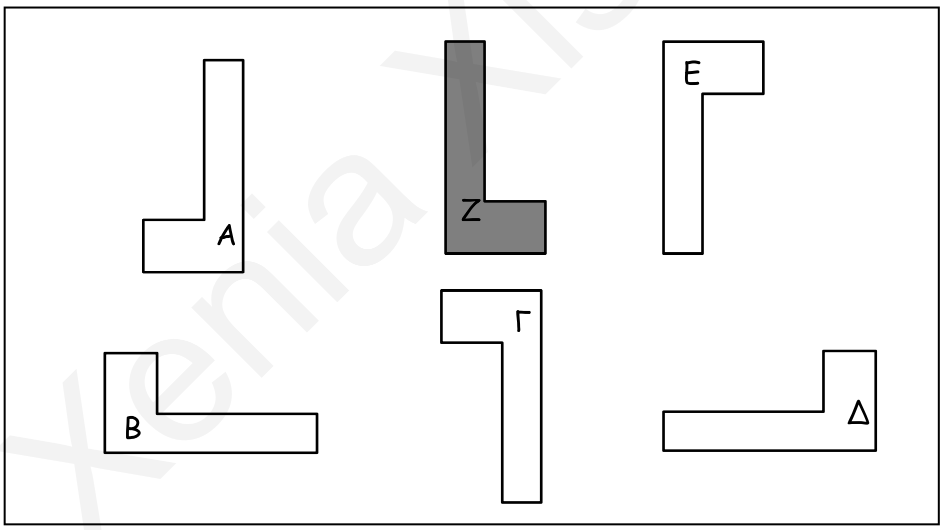


**Κύκλωσε το σωστό:**

**Παρουσιάζει στροφή προς τα δεξιά κατά:**

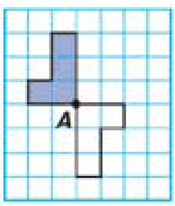
**1/4 2/4 3/4**

**C7) Κύκλωσε τις επιλογές που είναι περιστροφές του αρχικού σχήματος Ζ:**

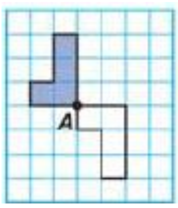



**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι μόνο μία. Κύκλωσε τη σωστή απάντηση σε κάθε περίπτωση:**

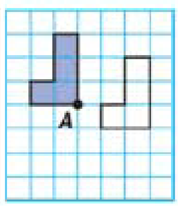
**C6**) Ποια από τις πιο κάτω εικόνες παρουσιάζει την περιστροφή του χρωματισμένου σχήματος;



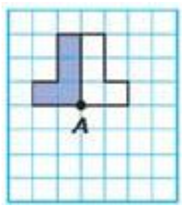

Α




Β




Γ




Δ

## ΔΟΚΙΜΙΟ 1Β

Όνομα: ........................................................ Τάξη: .......

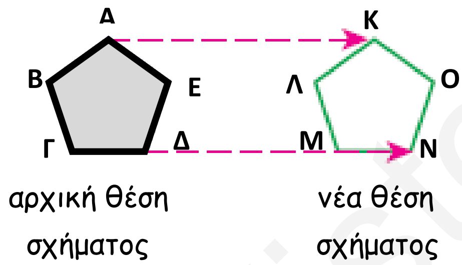
Αρ. στον κατάλογο: ...... Σχολείο: ........................................

Φύλο (βάλε ✓): Αγόρι ☐ Κορίτσι: ☐

## ΜΕΡΟΣ Α

### ΜΕΤΑΦΟΡΑ

Οι πιο κάτω ασκήσεις αναφέρονται στη **<u>μεταφορά</u>** (μετακίνηση) σχημάτων από μια **αρχική θέση** σε μια **νέα θέση**.

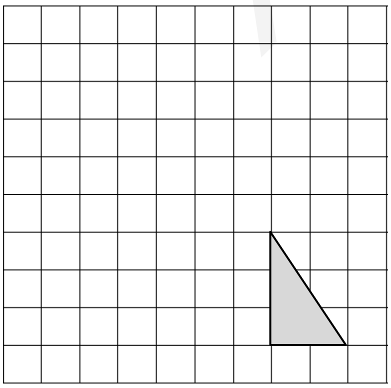



Στο διπλανό παράδειγμα, το σχήμα μεταφέρεται 3,5 cm προς τα δεξιά.

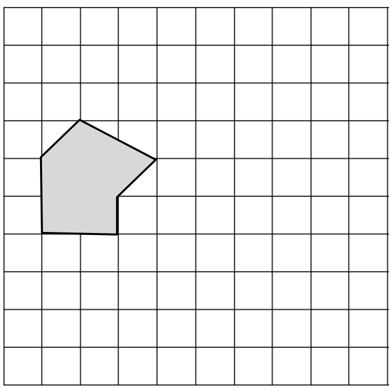


**Βρες τη <u>νέα θέση</u> του σχήματος και <u>ζωγράφισε</u> το, σύμφωνα με τις <u>οδηγίες</u> που δίνονται κάθε φορά <u>πάνω από το σχήμα</u>.**

**Α12)** Όταν μεταφερθεί <u>5</u> κουτάκια προς τα <u>πάνω</u>.

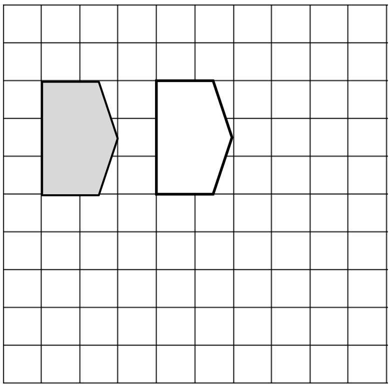


**Α16)** Όταν μεταφερθεί <u>5</u> κουτάκια προς τα <u>δεξιά</u>.



* It is noted that during test administration, the tasks were given to the students in serial numbering of each section. The numbering in the Appendix was modified for the purpose of the analysis and its presentation

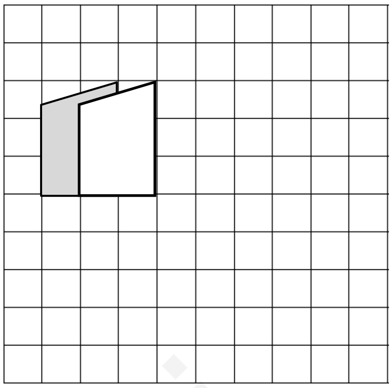
**Τώρα, δώσε εσύ τις οδηγίες σε κάποιον για το πώς να μεταφέρει το χρωματισμένο σχήμα στη νέα θέση (όπως τις οδηγίες που σου δόθηκαν στην άσκηση Α).**



**Α9.** Όταν μεταφερθεί :

……………………………………………………………..

……………………………………………………………..



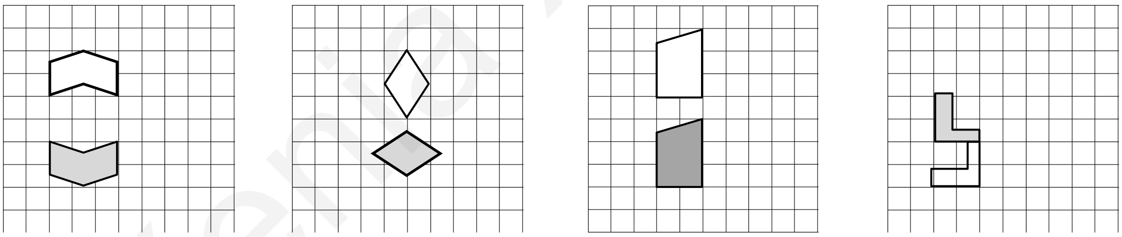
**Α11.** Όταν μεταφερθεί :

……………………………………………………………..

……………………………………………………………..

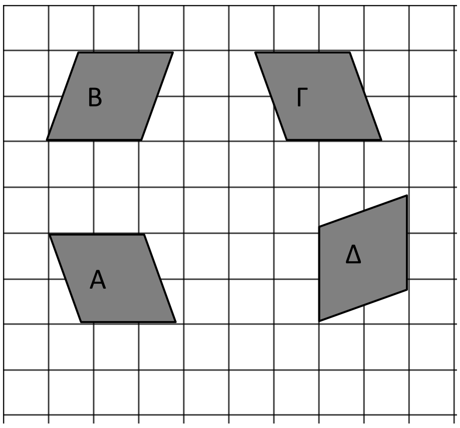
**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι μόνο μία. Κύκλωσε τη σωστή απάντηση σε κάθε περίπτωση:**

**Α4**) Ποια από τις πιο κάτω περιπτώσεις παρουσιάζει μεταφορά;



Α Β Γ Δ

**Α6**) Ποιο από τα πιο κάτω ζευγάρια σχημάτων σχετίζονται με μεταφορά (το ένα να είναι μεταφορά του άλλου);

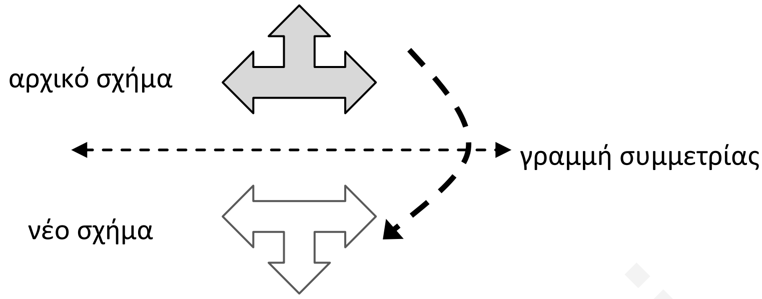



α) Το Α με το Δ
β) Το Β με το Γ
γ) Το Γ με το Δ
δ) Το Α με το Γ

## ΜΕΡΟΣ Β

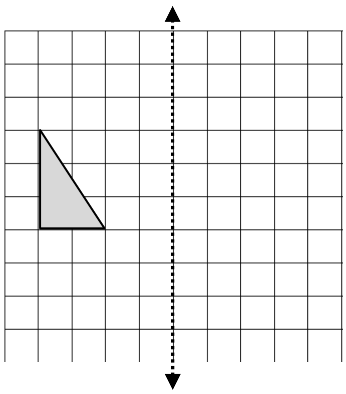
### ΑΝΑΚΛΑΣΗ

Οι πιο κάτω ασκήσεις αναφέρονται στην **<u>ανάκλαση</u>** (καθρέφτισμα) αντικειμένων με οριζόντια, κατακόρυφη ή διαγώνια <u>γραμμή συμμετρίας</u>.

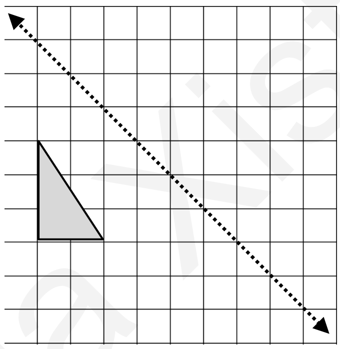



---

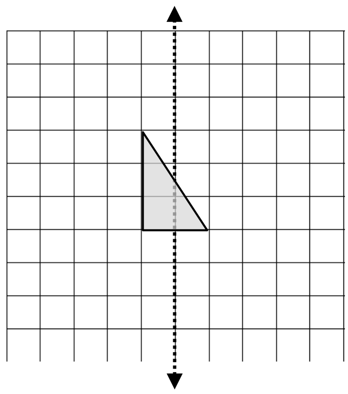
**Βρες την ανάκλαση και <u>ζωγράφισε το συμμετρικό</u> του <u>κάθε</u> αρχικού σχήματος που δίνεται, χρησιμοποιώντας κάθε φορά τη <u>διακεκομμένη γραμμή συμμετρίας</u>.**



**B13**

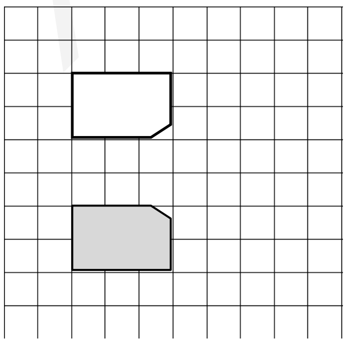


**B14**

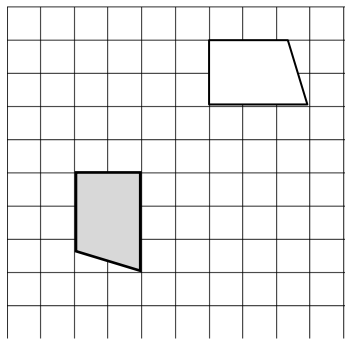


**B15**

**Τώρα, <u>βρες</u> και <u>χάραξε</u> με τη ρίγα σου τη <u>γραμμή συμμετρίας</u> για κάθε περίπτωση.**



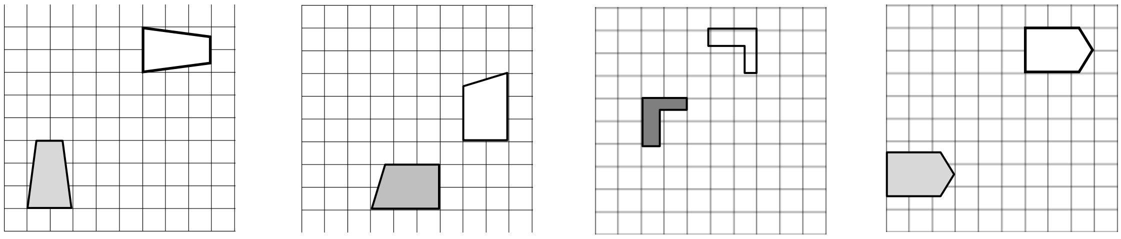
**B8**



**B10**

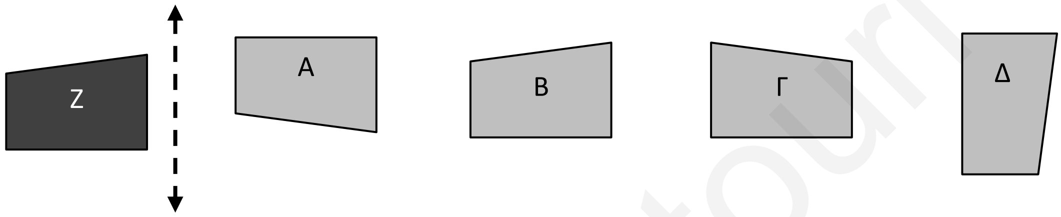
**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι <u>μόνο μία</u>. <u>Κύκλωσε</u> τη σωστή απάντηση σε κάθε περίπτωση:**

**Β6**) <u>Ποια</u> από τις πιο κάτω εικόνες παρουσιάζει <u>ανάκλαση</u>;



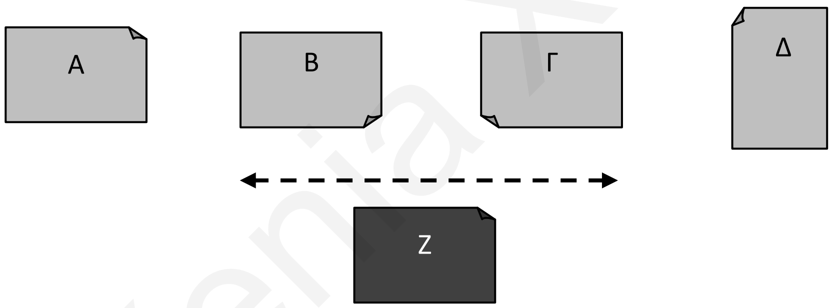
Α Β Γ Δ

**Β2**) Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Ζ σε <u>κατακόρυφη γραμμή συμμετρίας</u>;



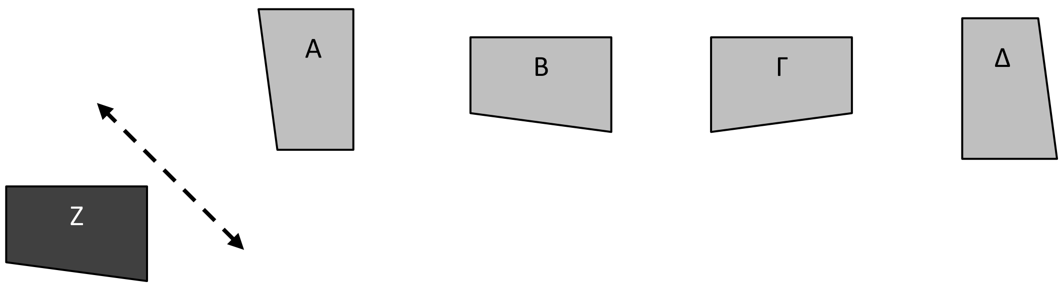

α) Το Α β) Το Β γ) Το Γ δ) Το Δ

**Β1**) Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Ζ σε <u>οριζόντια γραμμή συμμετρίας</u>;




α) Το Α β) Το Β γ) Το Γ δ) Το Δ

**Β3**) Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Ζ σε <u>διαγώνια γραμμή συμμετρίας</u>;




α) Το Α β) Το Β γ) Το Γ δ) Το Δ

# ΜΕΡΟΣ Γ

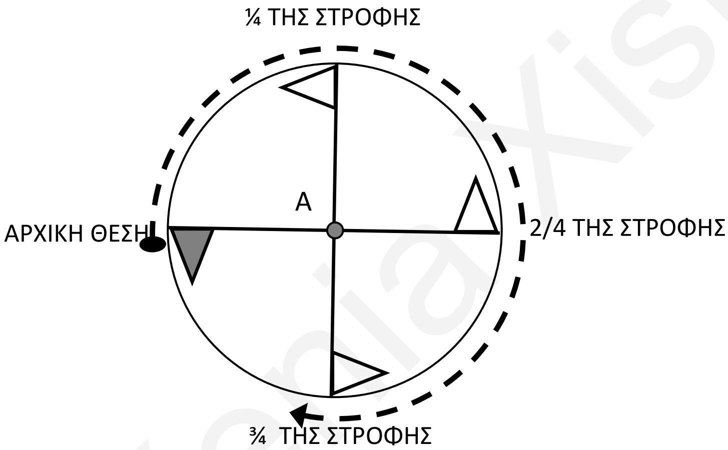
## ΠΕΡΙΣΤΡΟΦΗ

**Οι πιο κάτω ερωτήσεις αναφέρονται στην <u>περιστροφή</u> των σχημάτων, γύρω από συγκεκριμένο σημείο.**

**Τα σχήματα μπορούν να κάνουν στροφή (να γυρίσουν) προς τα δεξιά, δηλαδή όπως κινούνται οι δείκτες του ρολογιού, ή προς τα αριστερά δηλαδή αντίθετα με τους δείκτες του ρολογιού. <u>Στις πιο κάτω ασκήσεις, θεώρησε ότι όλες οι στροφές γίνονται προς τα δεξιά, όπως γυρίζουν οι δείκτες του ρολογιού</u>.**

**Μια ολόκληρη στροφή είναι 4/4 του κύκλου. Όταν η ερώτηση λέει ότι το σχήμα κάνει ¼ της στροφής, εννοεί ¼ του κύκλου. Τα 2/4 της στροφής είναι ίσα με 2/4 του κύκλου και τα ¾ της στροφής ίσα με τα ¾ του κύκλου.**

## ΠΑΡΑΔΕΙΓΜΑ

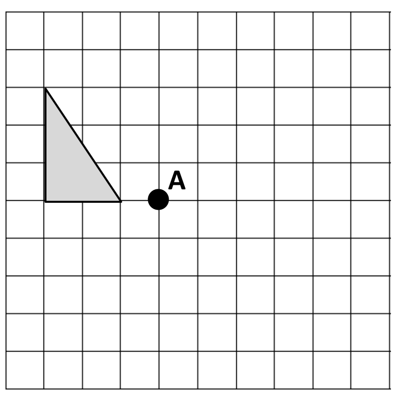



**Το παράδειγμα δίπλα δείχνει την γκρίζα σημαία στις θέσεις που θα πάρει όταν κάνει στροφή προς τα δεξιά γύρω από το σημείο Α κατά $\frac{1}{4}$ της στροφής, 2/4 της στροφής και $\frac{3}{4}$ της στροφής.**

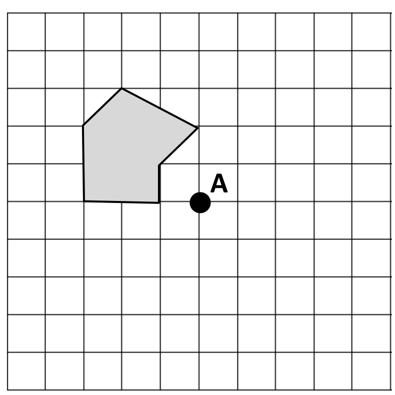
---

**<u>Ζωγράφισε</u> το σχήμα στη <u>νέα του θέση</u>, όταν κάνει <u>στροφή</u> <u>γύρω από το σημείο Α</u>, σύμφωνα με τις <u>οδηγίες</u> που δίνονται πάνω από <u>κάθε</u> <u>περίπτωση</u>.**

**C14)** Όταν κάνει 2/4 της στροφής προς τα δεξιά .

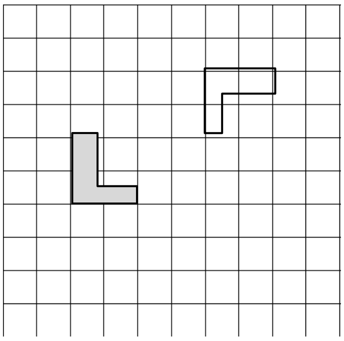



**C16)** Όταν κάνει 1/4 της στροφής προς τα δεξιά.



**<u>Βρες</u> και <u>ΖΩΓΡΑΦΙΣΕ</u> το <u>ΣΗΜΕΙΟ Α</u> <u>γύρω από το οποίο</u> <u>έκανε στροφή το χρωματισμένο σχήμα</u>, και στη συνέχεια <u>κύκλωσε το κλάσμα</u> που δείχνει πόση στροφή προς τα δεξιά έκανε το σχήμα.**

**C9**

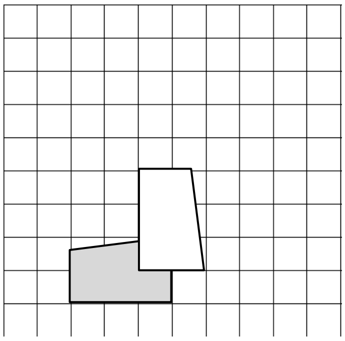


**Κύκλωσε το σωστό:**

**Παρουσιάζει στροφή προς τα δεξιά κατά:**

**1/4 2/4 3/4**

**C11**



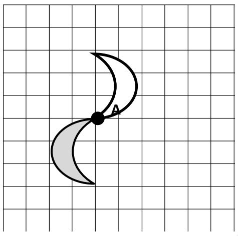
**Κύκλωσε το σωστό:**

**Παρουσιάζει στροφή προς τα δεξιά κατά:**

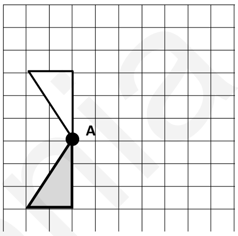
**1/4 2/4 3/4**

**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι <u>μόνο μία</u>. <u>Κύκλωσε</u> τη σωστή απάντηση σε κάθε περίπτωση:**

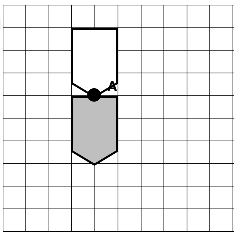
**C4**) <u>Ποια</u> από τις πιο κάτω περιπτώσεις παρουσιάζει <u>περιστροφή</u>;



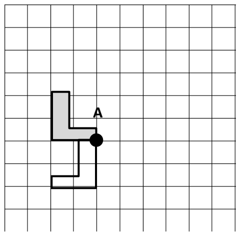

Α




Β

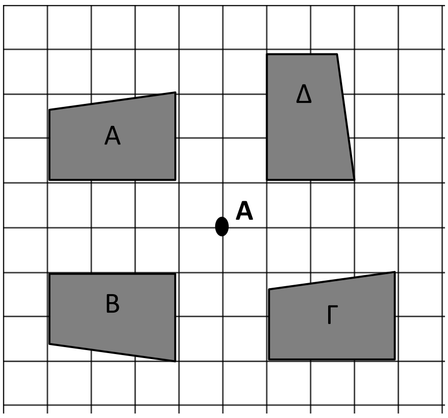



Γ




Δ

**C5**) <u>Ποιο</u> από τα πιο κάτω ζευγάρια σχημάτων σχετίζονται με <u>στροφή γύρω από το σημείο Α</u>;



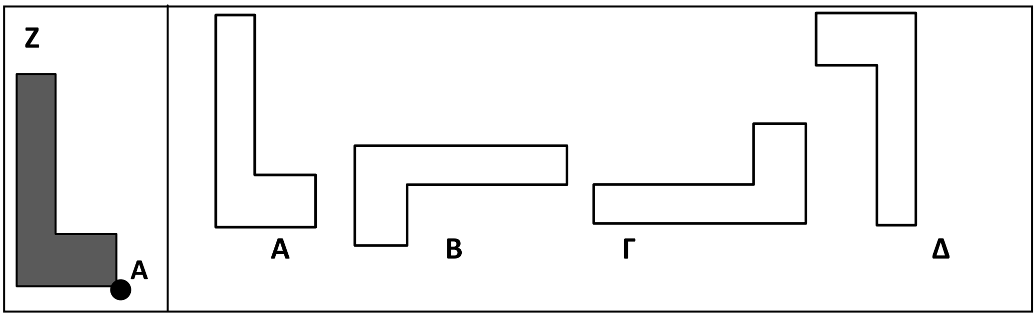

α) Το Α με το Δ

β) Το Β με το Γ

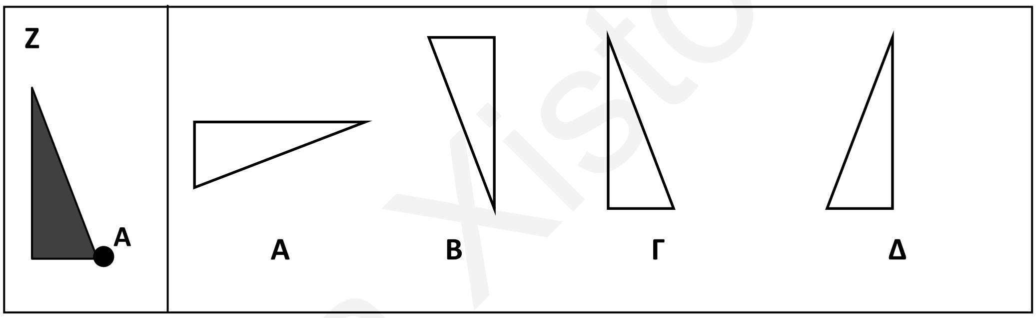
γ) Το Β με το Α

δ) Το Γ με το Α

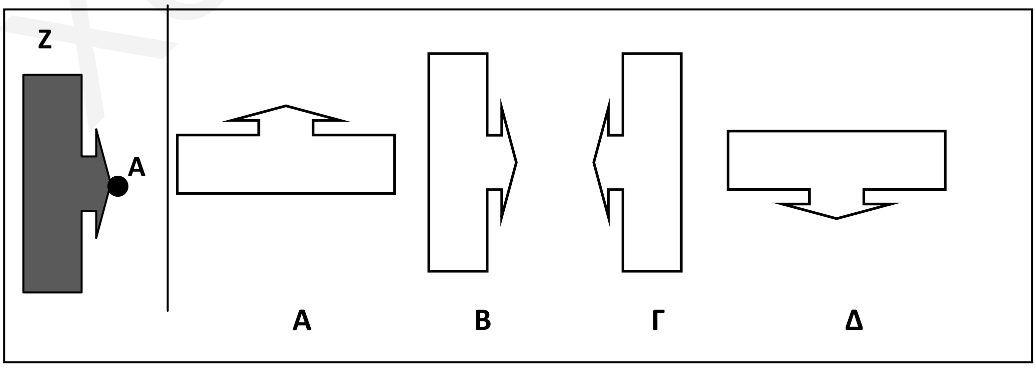
**C3**) Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά ¼ του κύκλου προς τα δεξιά; Κύκλωσε τη σωστή απάντηση.




**C2**) Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά 2/4 του κύκλου προς τα δεξιά; Κύκλωσε τη σωστή απάντηση.




**C1**) Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά ¾ του κύκλου προς τα δεξιά; Κύκλωσε τη σωστή απάντηση.



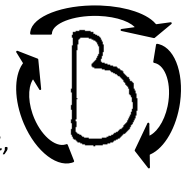
**ΔΟΚΙΜΙΟ 2**

Όνομα: ………………………………………………………………………………………………………… Τάξη: ………………….

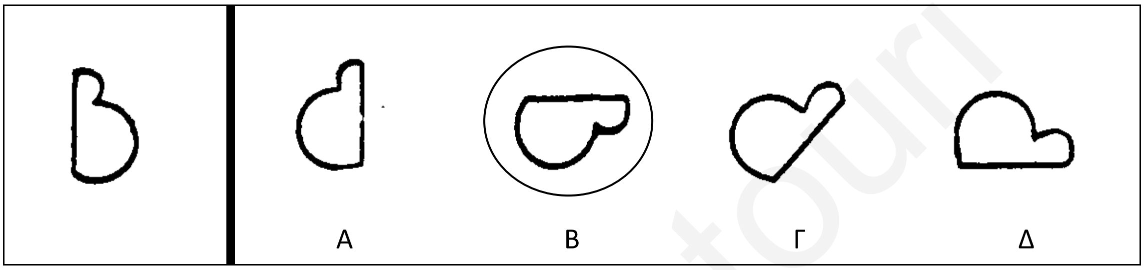
Αρ. στον κατάλογο: ………… Σχολείο: ……………………………………………………………………………………

**Μέρος Α΄:**

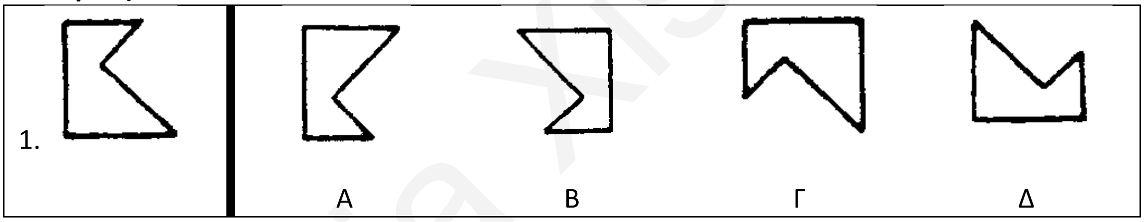
Στο μέρος αυτό δίνεται ένα σχήμα πριν από τη γραμμή και τέσσερα σχήματα μετά από τη γραμμή. Ποιο από τα σχήματα που βρίσκονται μετά από τη γραμμή μπορεί να προκύψει όταν <u>περιστρέψουμε</u> το σχήμα που βρίσκεται πριν από τη γραμμή προς τα δεξιά ή αριστερά, <u>χωρίς να το αντιστρέψουμε</u>; Να το βάλεις σε κύκλο.

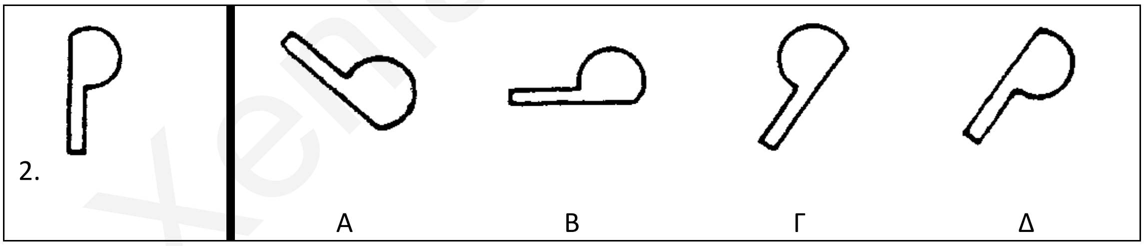


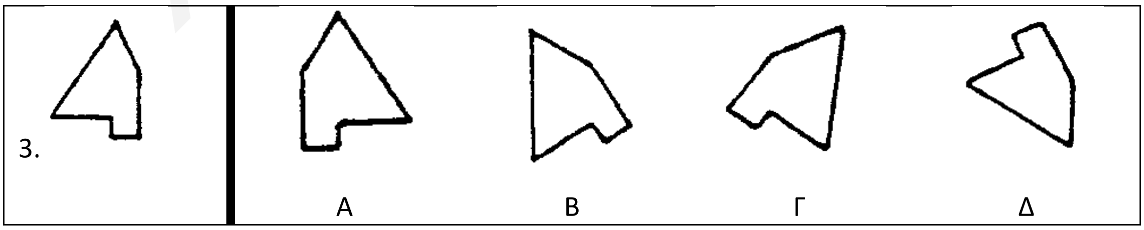
**Παράδειγμα:**

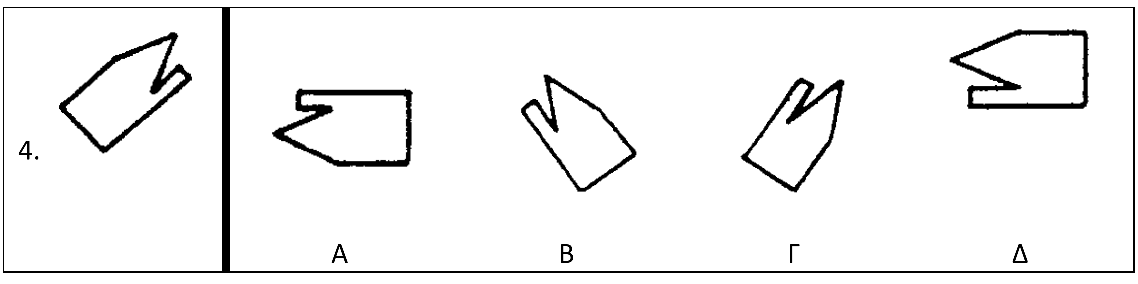



**Ασκήσεις:**



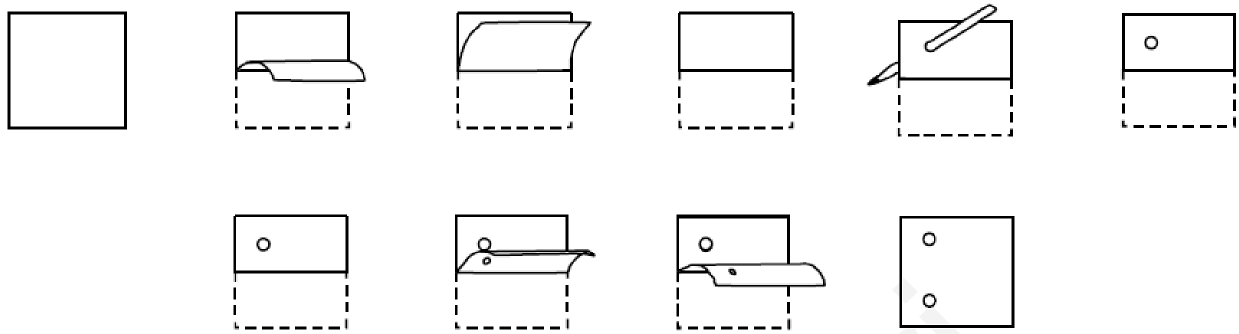



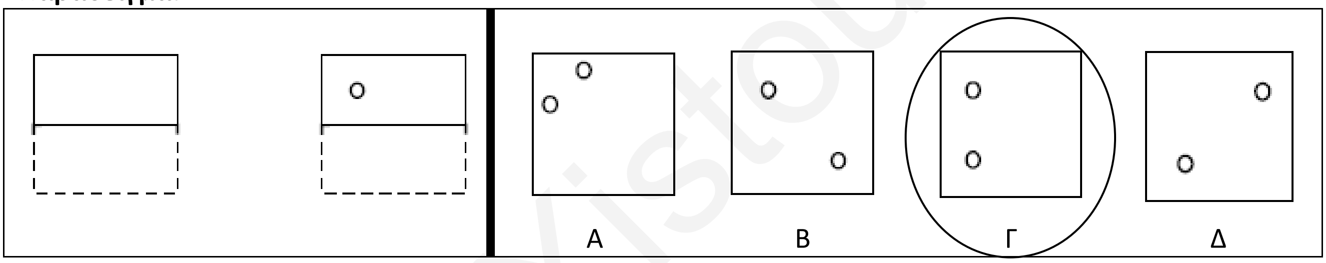



**Μέρος Β΄:**

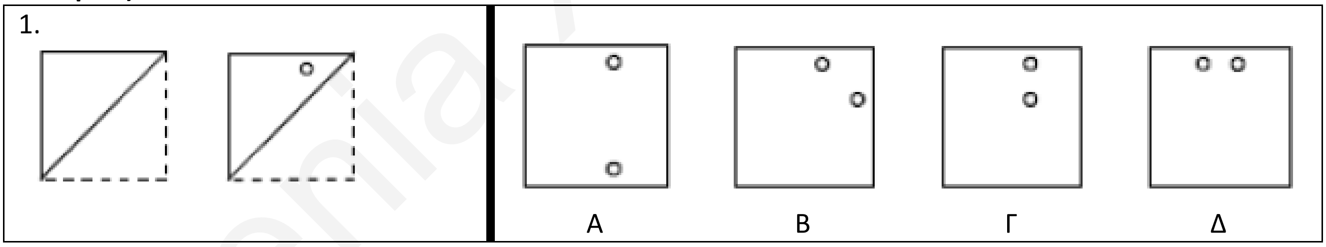
Στο μέρος αυτό παρουσιάζεται πριν από τη γραμμή ο τρόπος με τον οποίο διπλώνεται ένα τετράγωνο χαρτόνι και η θέση στην οποία ανοίγουμε μια τρύπα όταν το χαρτόνι είναι διπλωμένο. Να βάλεις σε κύκλο το σχήμα μετά από τη γραμμή που δείχνει πώς θα φαίνεται το χαρτόνι <u>όταν ανοιχθεί</u>.

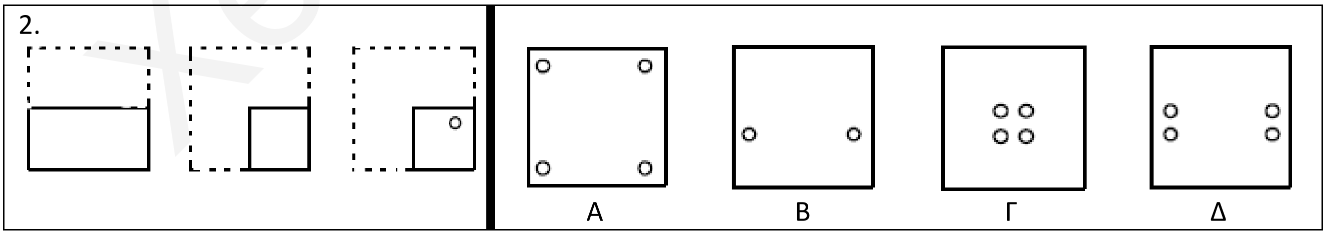


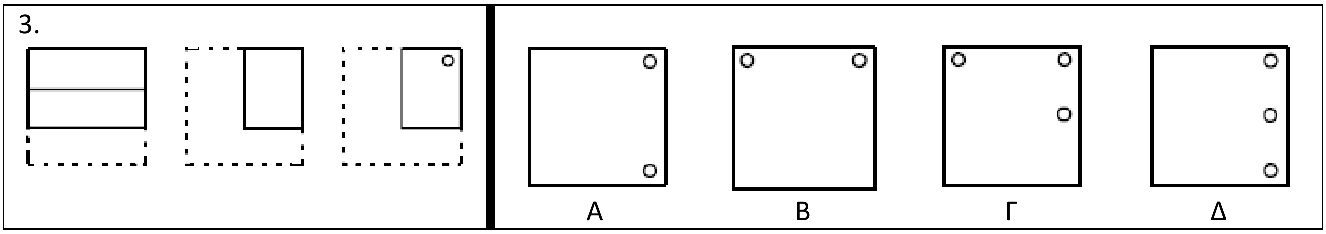
**Παράδειγμα:**




**Ασκήσεις:**

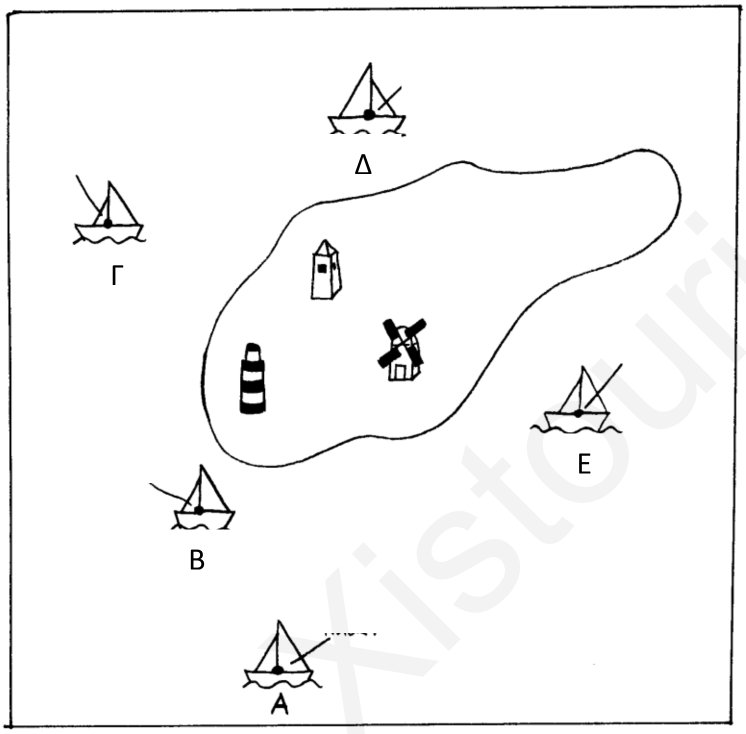


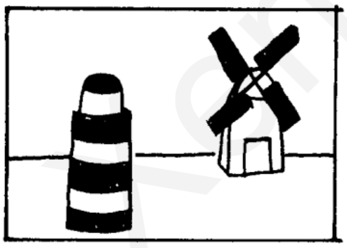
**ΜΕΡΟΣ C΄:**

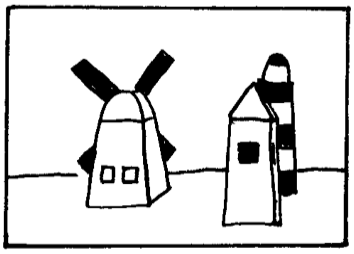
Στο μέρος αυτό παρουσιάζεται μια εικόνα από ένα τοπίο, και τρεις φωτογραφίες. Πρέπει να αποφασίσεις σε ποια βάρκα βρισκόταν ο φωτογράφος που έβγαλε την κάθε φωτογραφία. Βάλε σε κύκλο τη σωστή απάντηση για κάθε φωτογραφία.

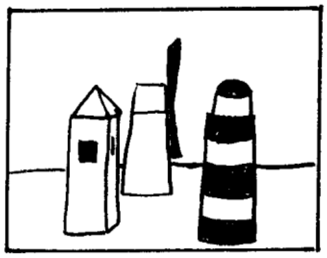





**Ο φωτογράφος ήταν στη βάρκα:**





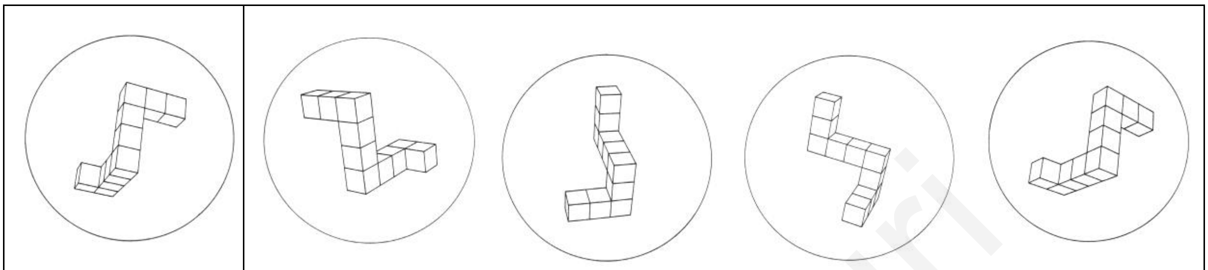


**Α Β Γ Δ Ε** **Α Β Γ Δ Ε** **Α Β Γ Δ Ε**

**ΜΕΡΟΣ D΄:**

Στο μέρος αυτό, το σχήμα που βρίσκεται πριν από τη γραμμή κρέμεται από ένα σχοινί και περιστρέφεται γύρω-γύρω. Ένας φωτογράφος τα φωτογραφίζει την ώρα που γυρίζουν. Βοήθησε τον να βρει ποια από τις Α, Β, Γ και Δ είναι φωτογραφία του κάθε σχήματος. Βάλε σε κύκλο τη σωστή φωτογραφία.

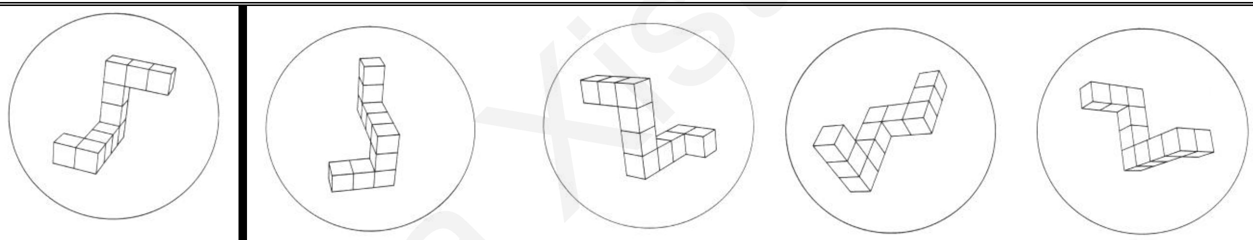
**Παράδειγμα:**



Α Β Γ Δ

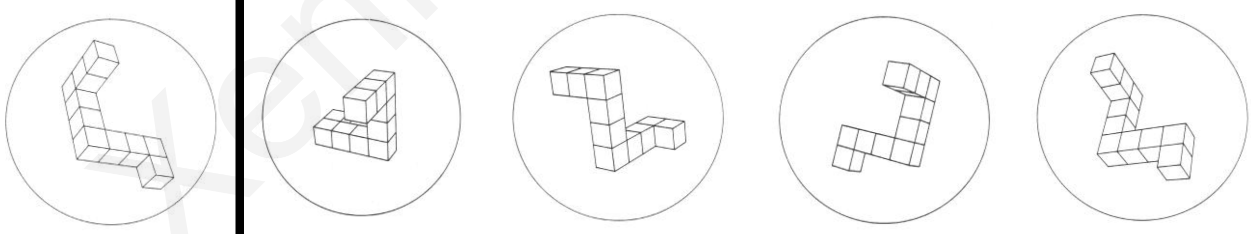
**Ασκήσεις:**

**1.**



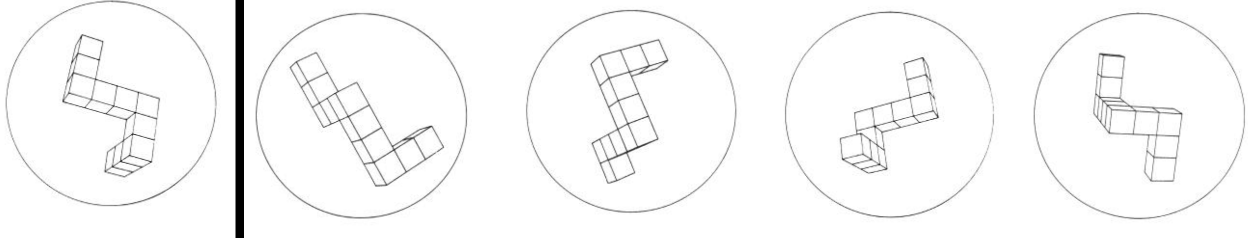
**Α** **Β** **Γ** **Δ**

**2.**



**Α** **Β** **Γ** **Δ**

**3.**

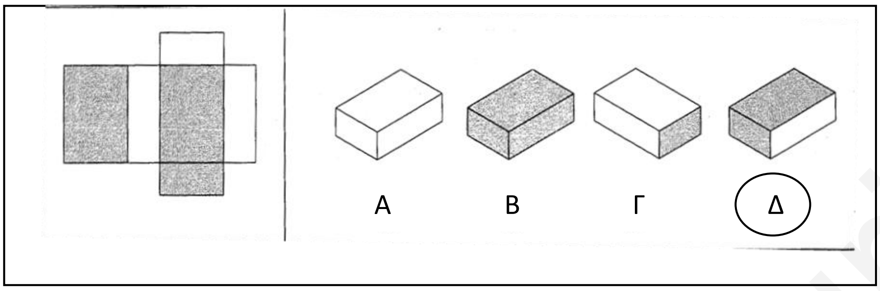


**Α** **Β** **Γ** **Δ**

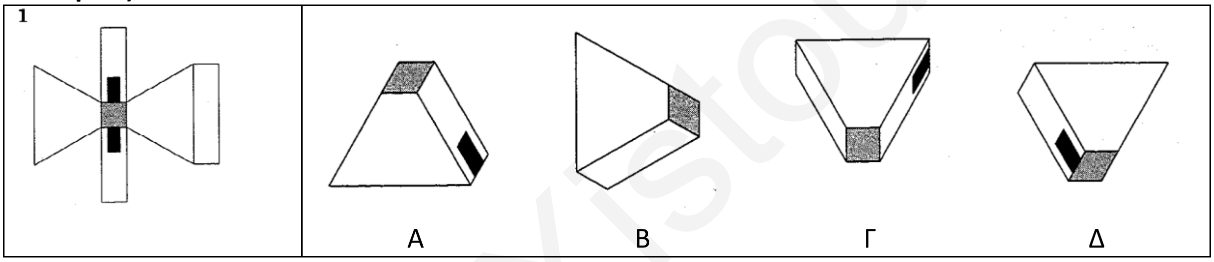
**ΜΕΡΟΣ Ε΄:**

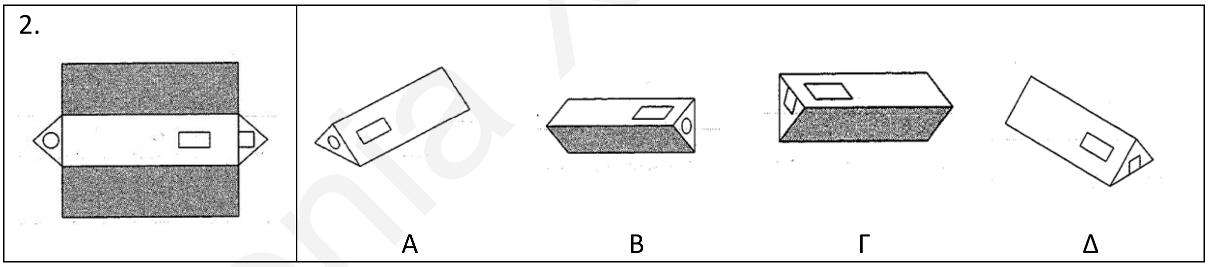
Αυτά τα προβλήματα περιλαμβάνουν αναπτύγματα γεωμετρικών σχημάτων. Αυτά τα αναπτύγματα μπορούν να διπλωθούν για να φτιαχτούν τρισδιάστατα γεωμετρικά σχήματα. Κάθε πρόβλημα παρουσιάζει πριν από τη γραμμή και μετά τη γραμμή τέσσερα τρισδιάστατα γεωμετρικά σχήματα. Θα πρέπει να διαλέξεις ένα από τα σχήματα, <u>αυτό που νομίζεις ότι φτιάχνεται από το ανάπτυγμα</u>. Να το βάλεις σε κύκλο.

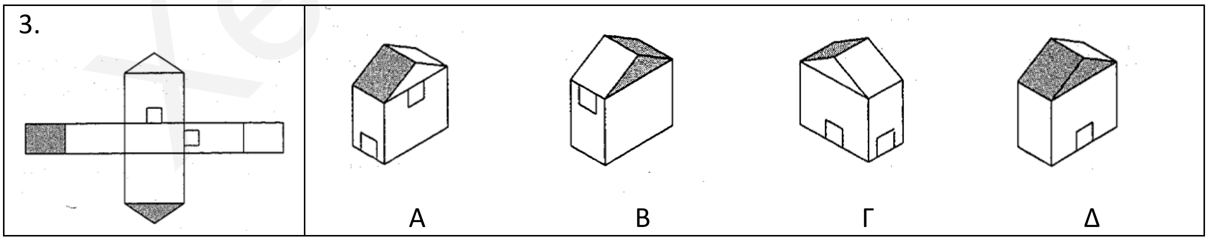
**Παράδειγμα:**

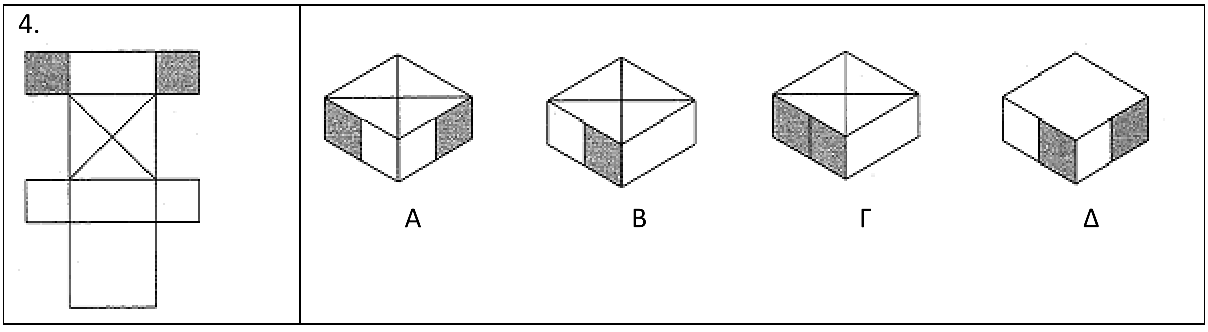



**Ασκήσεις:**



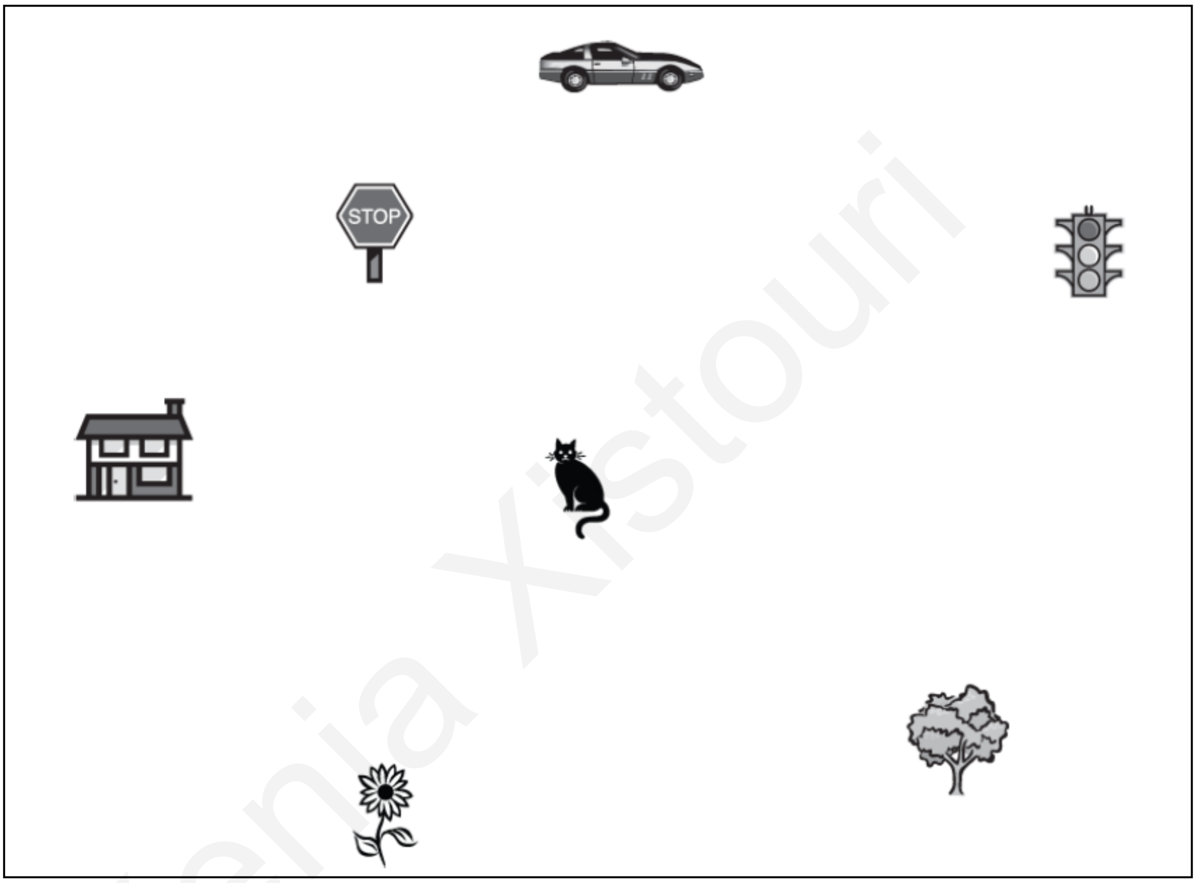



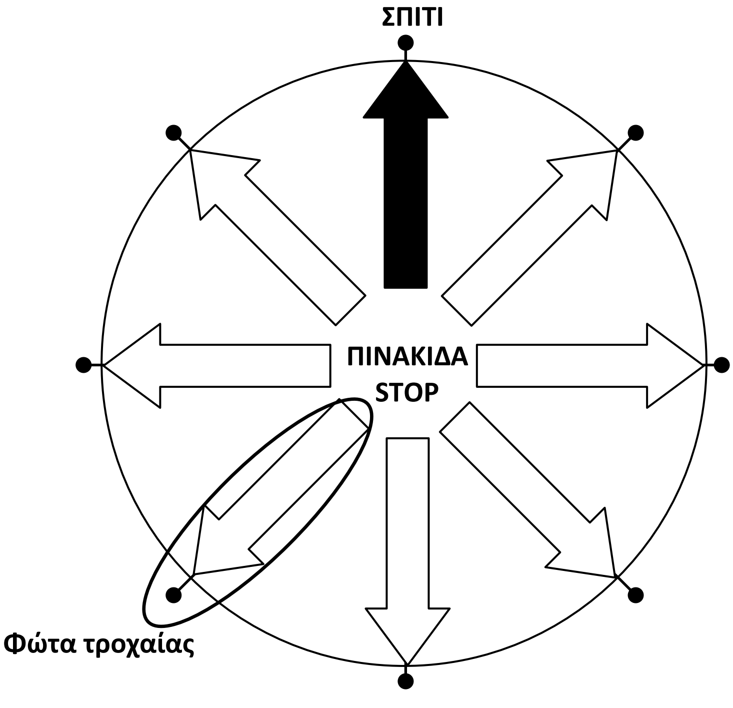
**Μέρος F΄:**

Στην πιο κάτω εικόνα παρουσιάζονται εφτά αντικείμενα. Πρέπει **<u>να φανταστείς</u>** ότι είσαι το ανθρωπάκι (  ) μέσα στην εικόνα και ότι κοιτάζεις **<u>προς την κατεύθυνση</u>** που βρίσκεται ένα από τα αντικείμενα. Κύκλωσε το κατάλληλο τόξο στην εικόνα που ακολουθεί, για να δείξεις σε ποια κατεύθυνση θα βρίσκεται το αντικείμενο που ζητά η ερώτηση.

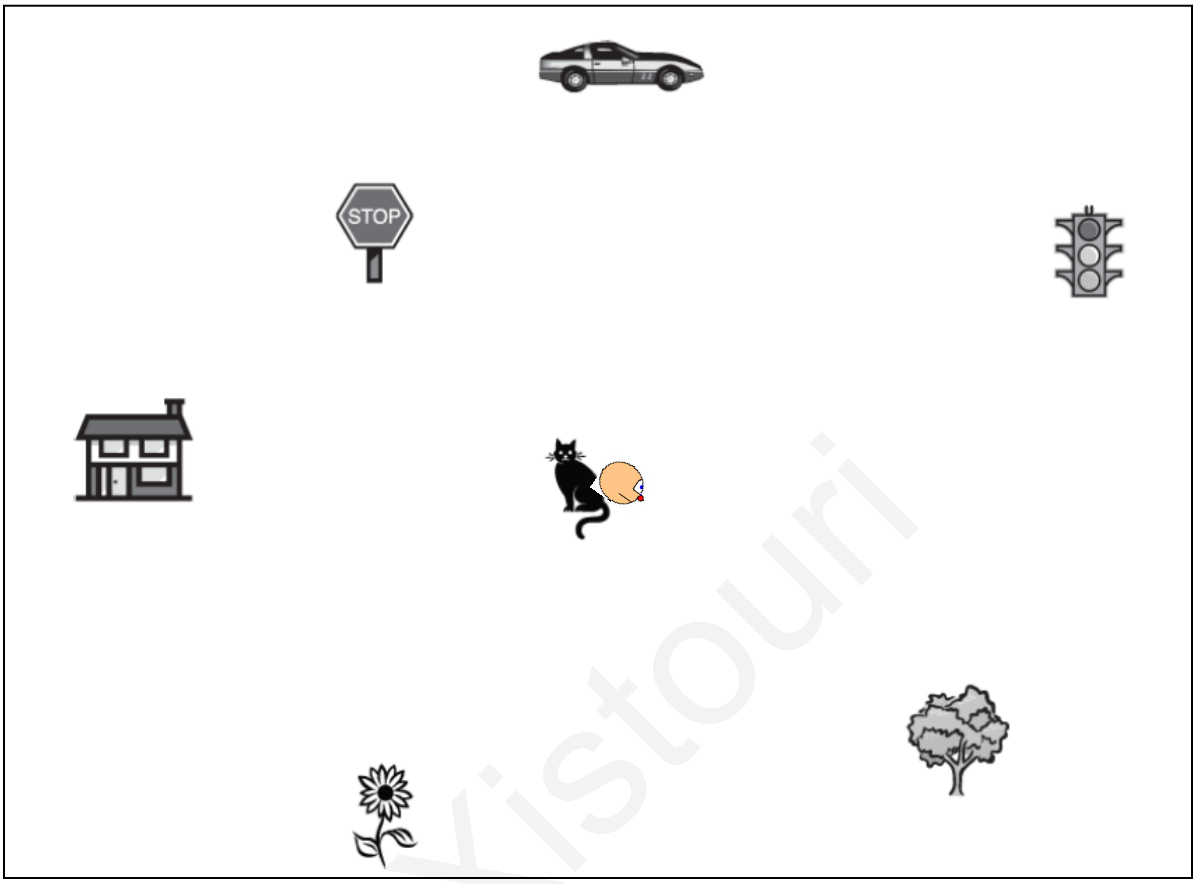
**ΠΡΟΣΕΞΕ:** Κάθε φορά που θα βρίσκεσαι σε διαφορετική θέση στην εικόνα, **<u>θα βλέπεις τα αντικείμενα από διαφορετική πλευρά</u>**. Προσπάθησε να βρεις τη **<u>νέα θέση</u>** στην οποία θα βλέπεις το αντικείμενο που ζητά η κάθε ερώτηση.




Να φανταστείς ότι βρίσκεσαι στην **πινακίδα STOP** και κοιτάζεις προς το **σπίτι**.
Βάλε σε κύκλο το τόξο που δείχνει προς ποια κατεύθυνση θα βρίσκονται τα **φώτα τροχαίας**.

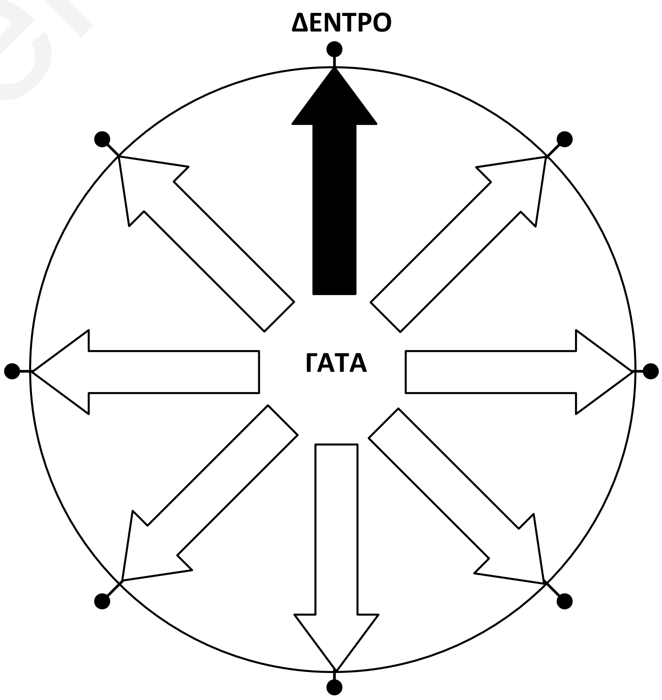


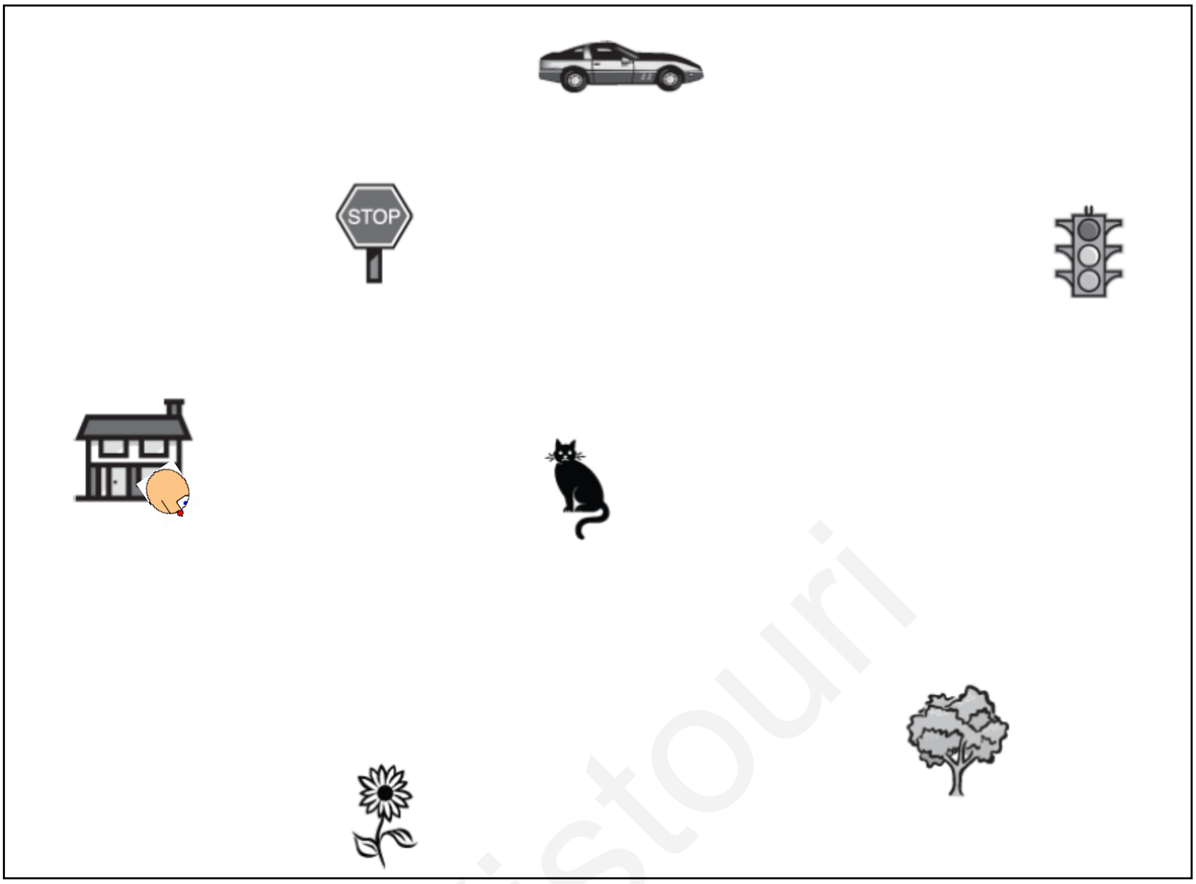

Στο παράδειγμα, όταν βρίσκεσαι στην πινακίδα STOP και κοιτάζεις προς το σπίτι, τότε τα φώτα τροχαίας βρίσκονται στην αριστερή σου μεριά, προς τα πίσω.




1. Να φανταστείς ότι βρίσκεσαι στη **γάτα** και κοιτάζεις προς το **δέντρο**.

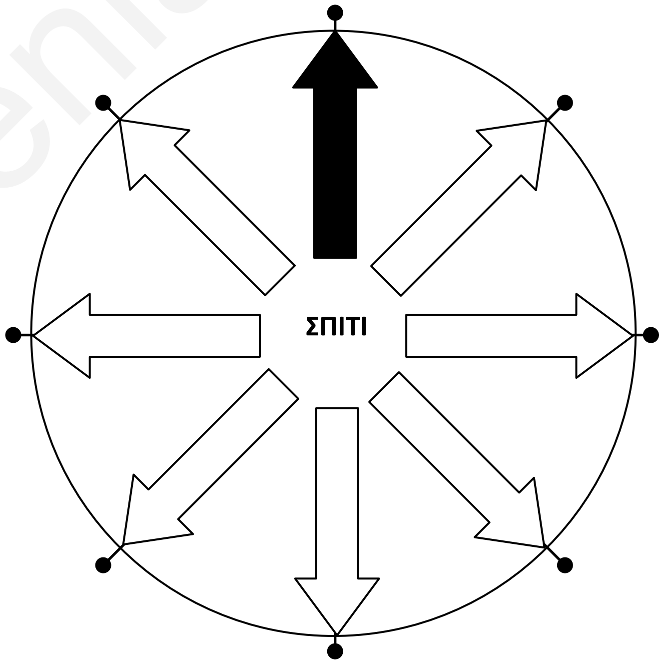
Βάλε σε κύκλο το τόξο που δείχνει προς ποια κατεύθυνση θα βρίσκεται η **πινακίδα STOP**.

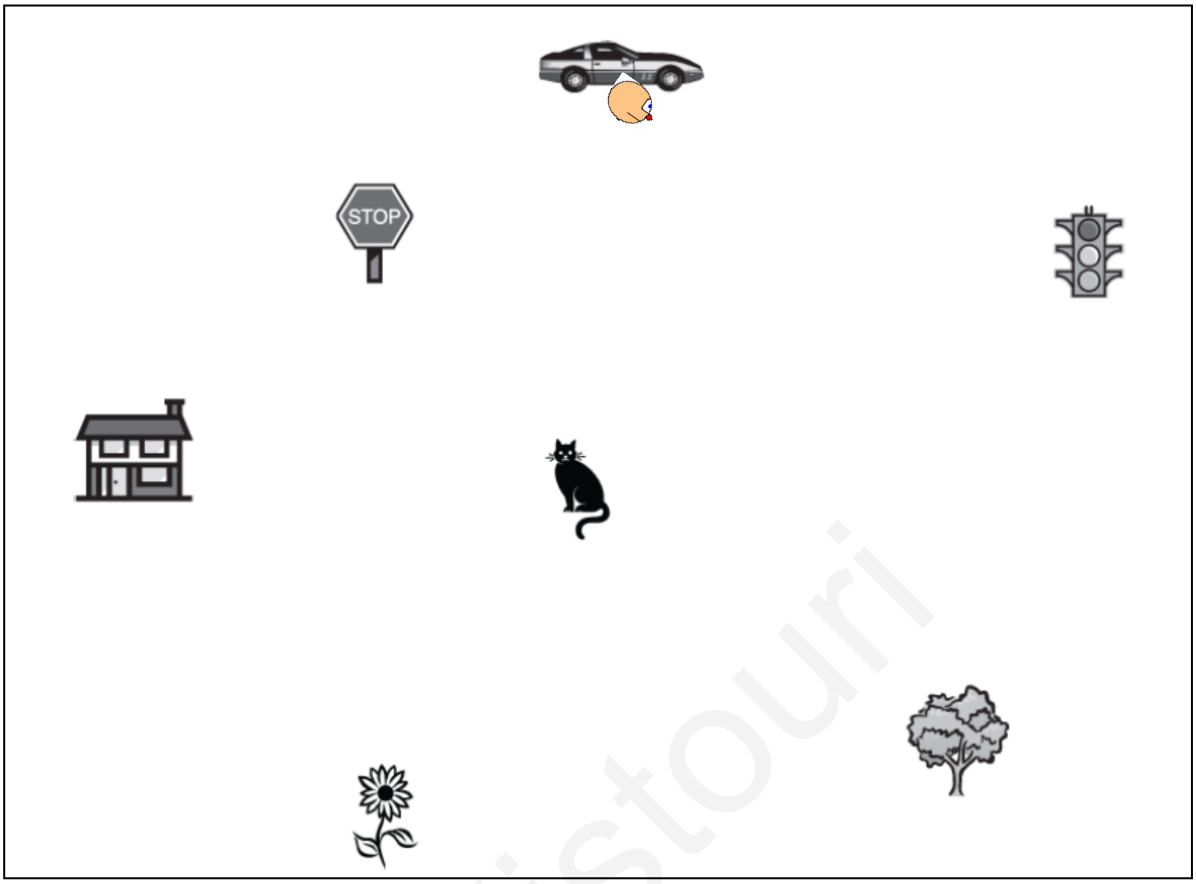





2. Να φανταστείς ότι βρίσκεσαι στο **σπίτι** και κοιτάζεις προς το **λουλούδι**.

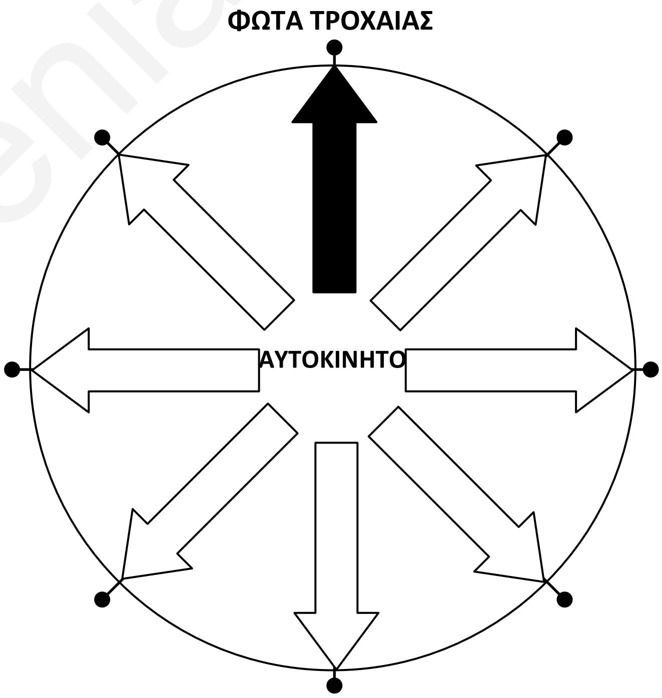
Βάλε σε κύκλο το τόξο που δείχνει προς ποια κατεύθυνση θα βρίσκεται το **αυτοκίνητο**.






3. Να φανταστείς ότι βρίσκεσαι στο **αυτοκίνητο** και κοιτάζεις προς τα **φώτα τροχαίας**.

Βάλε σε κύκλο το τόξο που δείχνει προς ποια κατεύθυνση θα βρίσκεται η **πινακίδα STOP**.

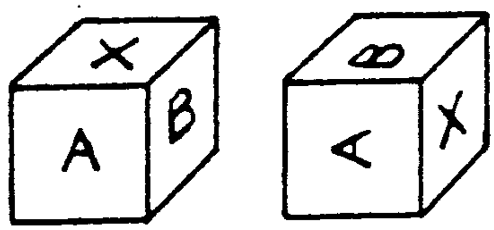


**ΜΈΡΟΣ G΄:**

Σ' αυτή την άσκηση θα δεις κάποια ζευγάρια κύβων. Σε κάθε πλευρά του κύβου υπάρχει ένα σύμβολο. Τα σύμβολα σε κάθε πλευρά είναι όλα **διαφορετικά**.

Πρέπει να κυκλώσεις τη λέξη «ΙΔΙΟΙ», αν ο πρώτος κύβος μπορεί να είναι ο ίδιος με το δεύτερο όταν περιστραφεί, και τη λέξη «ΔΙΑΦΟΡΕΤΙΚΟΙ» αν δεν μπορεί.

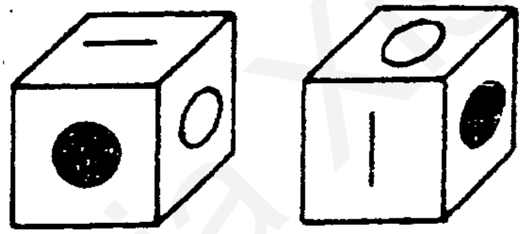
**Παράδειγμα:**



ΙΔΙΟΙ ΔΙΑΦΟΡΕΤΙΚΟΙ

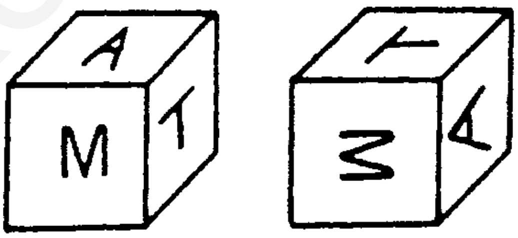
**Ασκήσεις:**

1.



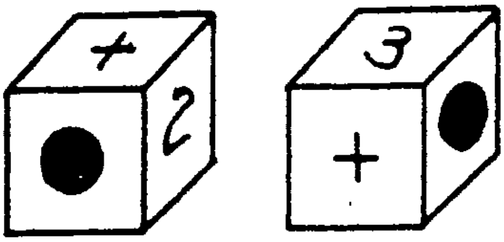
**ΙΔΙΟΙ** **ΔΙΑΦΟΡΕΤΙΚΟΙ**

2.



**ΙΔΙΟΙ** **ΔΙΑΦΟΡΕΤΙΚΟΙ**

3.

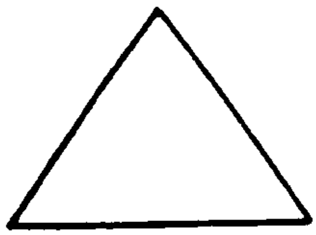


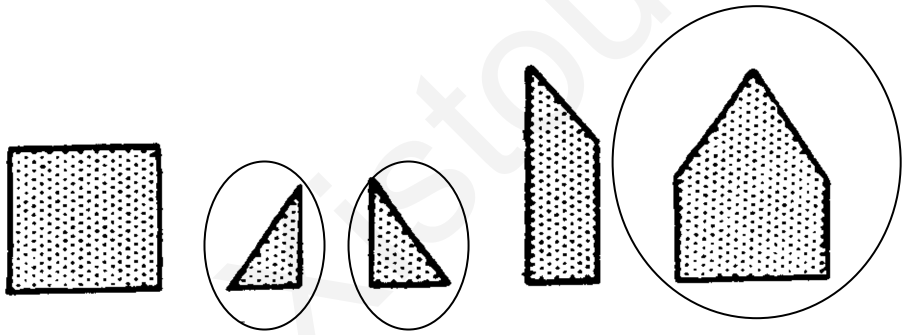
**ΙΔΙΟΙ** **ΔΙΑΦΟΡΕΤΙΚΟΙ**

**ΜΕΡΟΣ Η΄:**

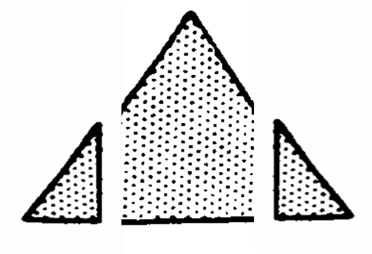
Στο μέρος αυτό υπάρχει πάνω από τη γραμμή ένα σχήμα. Να βάλεις σε κύκλο ποια από τα σχήματα που βρίσκονται κάτω από τη γραμμή πρέπει να **ΕΝΩΘΟΥΝ** ώστε να κατασκευαστεί το σχήμα που βρίσκεται πάνω από αυτή. Τα σχήματα που βρίσκονται κάτω από τη γραμμή μπορούν **μόνο να περιστραφούν**.

**Παράδειγμα:**

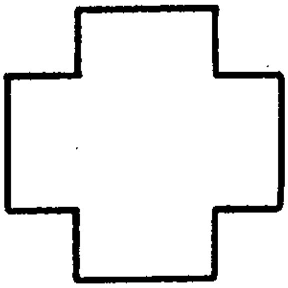




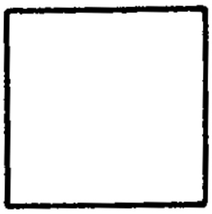
Σε αυτό το παράδειγμα χρειάζεται να ενωθούν τα τρία σχήματα που έχουν κυκλωθεί για να σχηματιστεί το τρίγωνο που βρίσκεται πάνω από τη γραμμή, όπως φαίνεται πιο κάτω:

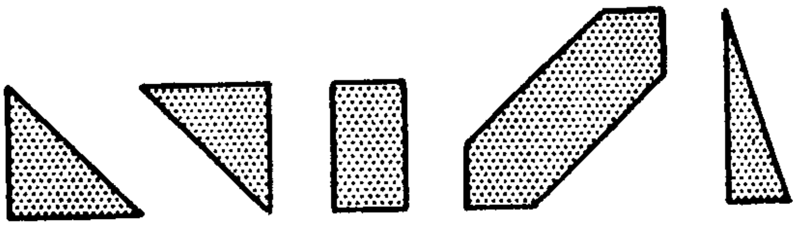


1. Για να κατασκευαστεί ο σταυρός πρέπει να ενωθούν **ΤΡΙΑ** από τα πέντε σχήματα που υπάρχουν κάτω από τη γραμμή. Βάλε τα σχήματα αυτά σε κύκλο.

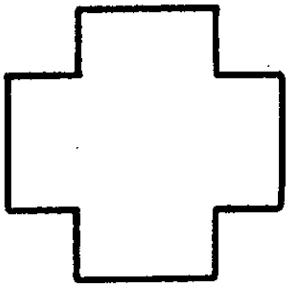


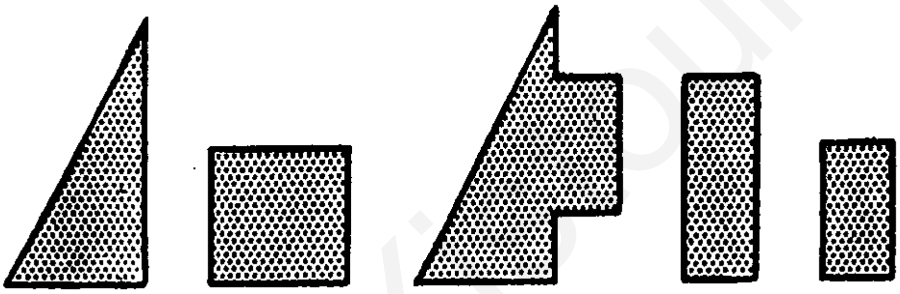
2. Για να κατασκευαστεί το τετράγωνο πρέπει να ενωθούν **ΤΡΙΑ** από τα πέντε σχήματα που υπάρχουν κάτω από τη γραμμή. Βάλε τα σχήματα αυτά σε κύκλο.





3. Για να κατασκευαστεί ο σταυρός πρέπει να ενωθούν **ΤΡΙΑ** από τα πέντε σχήματα που υπάρχουν κάτω από τη γραμμή. Βάλε τα σχήματα αυτά σε κύκλο.

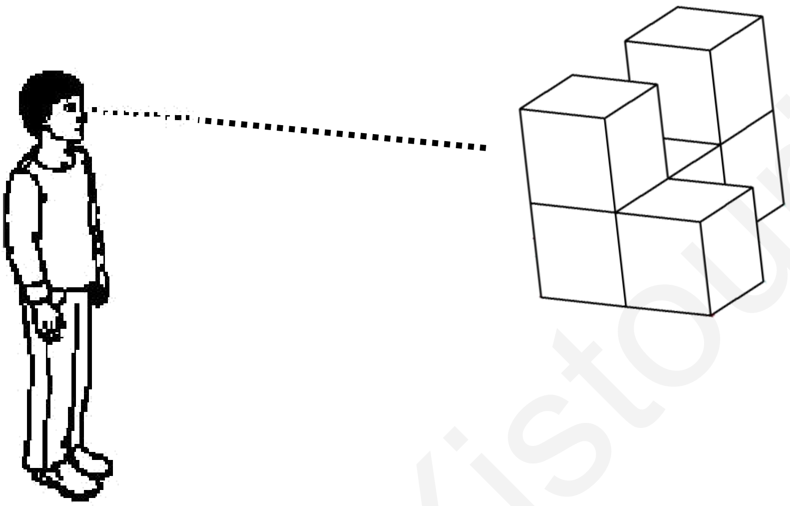


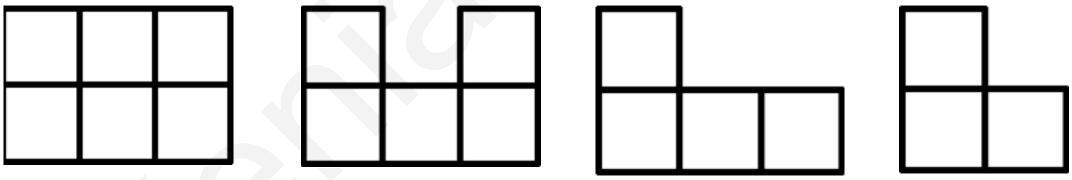


**ΜΕΡΟΣ Ι΄:**

Στο μέρος αυτό παρουσιάζεται ένας άνθρωπος να κοιτάζει προς ένα στερεό. Η διακεκομμένη γραμμή δείχνει την κατεύθυνση του βλέμματός του. Οι τέσσερις εικόνες που υπάρχουν κάτω από τη γραμμή δείχνουν ποια εικόνα θα μπορούσε να έχει μπροστά του ο άνθρωπος από τη θέση που βρίσκεται. Να βάλεις σε κύκλο τη σωστή.

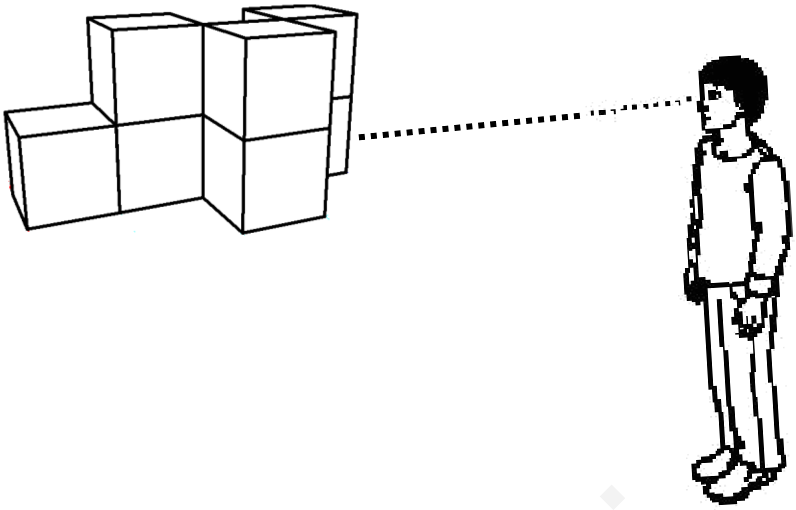
**Παράδειγμα:**

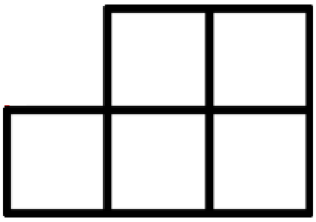




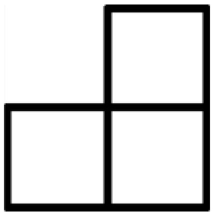
Α Β Γ Δ

1.

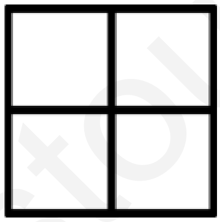




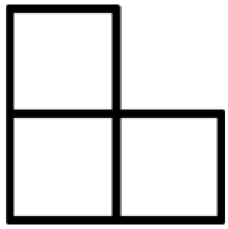
Α



Β

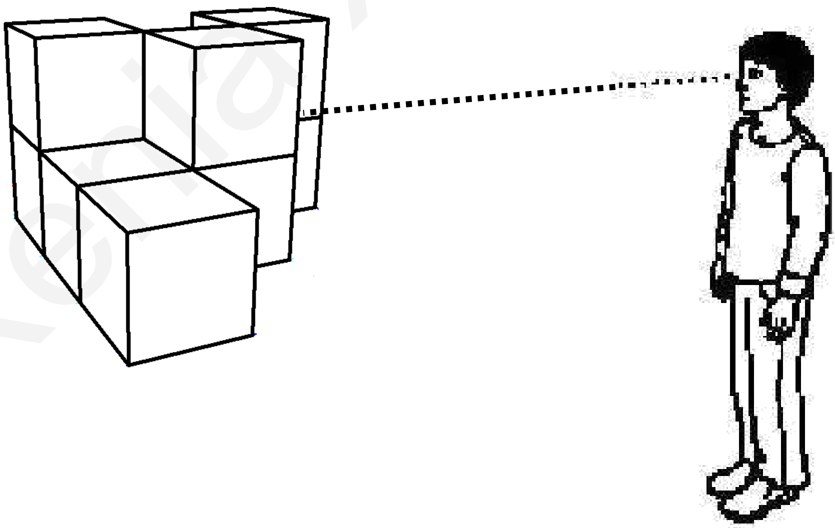


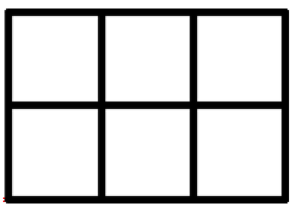
Γ



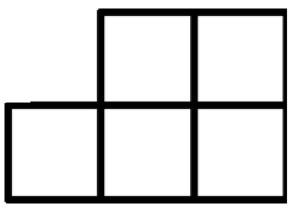
Δ

2.

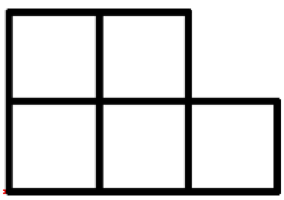




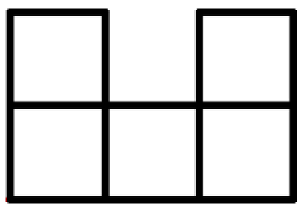
Α



Β

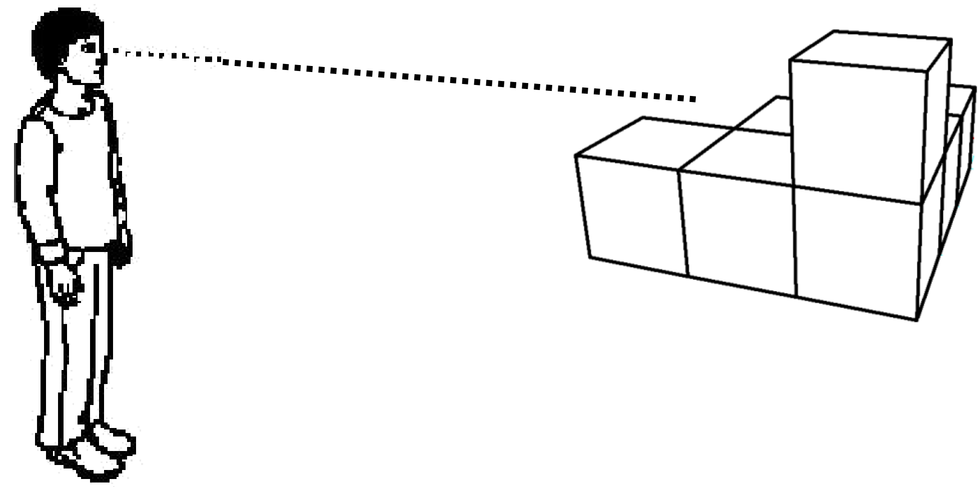


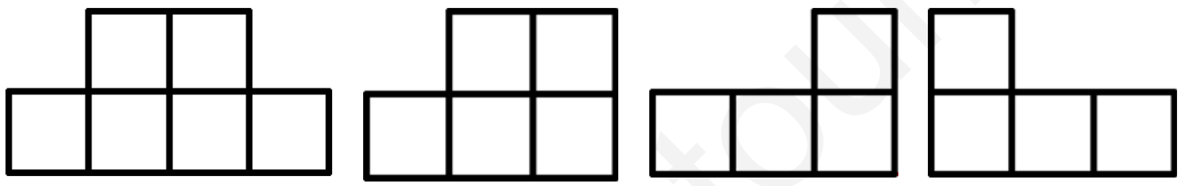
Γ



Δ

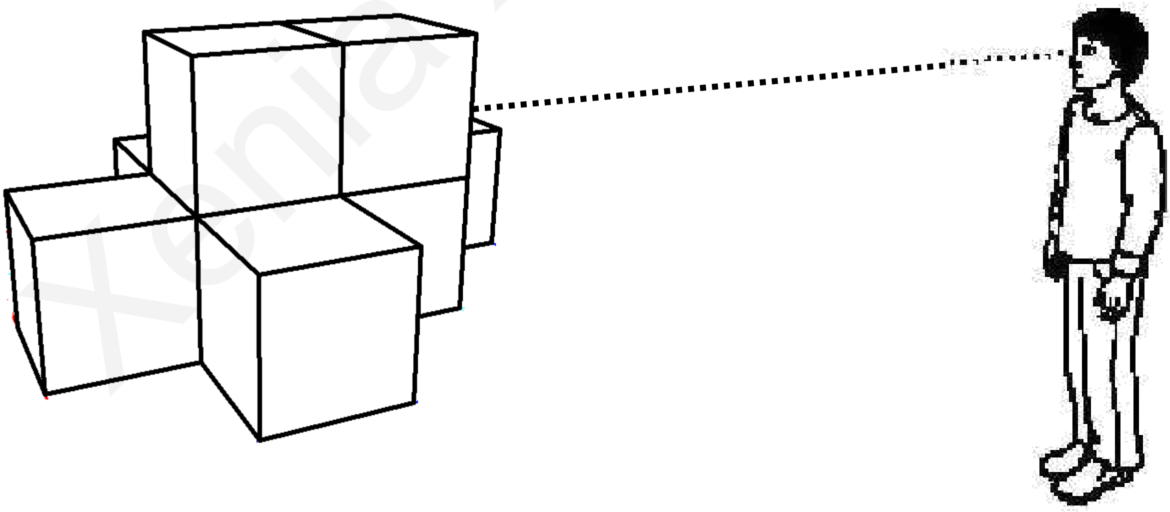
3.

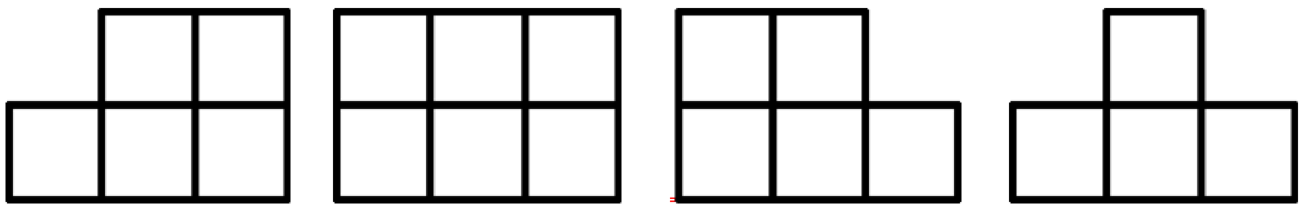




Α Β Γ Δ

4.





Α Β Γ Δ

APPENDIX II

FINAL INSTRUMENTS

# ΔΟΚΙΜΙΟ 1Α

**Όνομα:** .................................................... **Τάξη:** .......

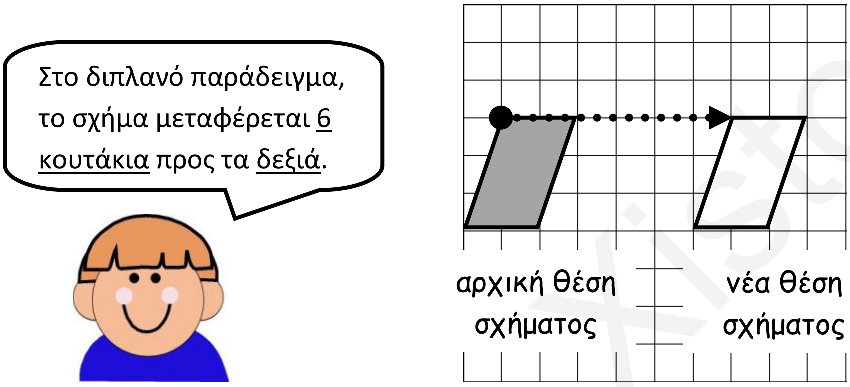
**Αρ. στον κατάλογο:** ...... **Σχολείο:** .....................................

**Φύλο (βάλε ✓):** **Αγόρι** ☐ **Κορίτσι:** ☐

## ΜΕΡΟΣ Α

### ΜΕΤΑΦΟΡΑ

Οι πιο κάτω ασκήσεις αναφέρονται στη **<u>μεταφορά</u>** (μετακίνηση) σχημάτων από μια **αρχική θέση** σε μια **νέα θέση**.

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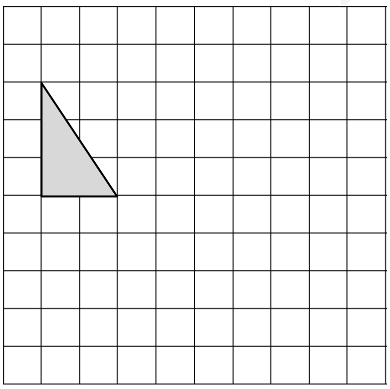
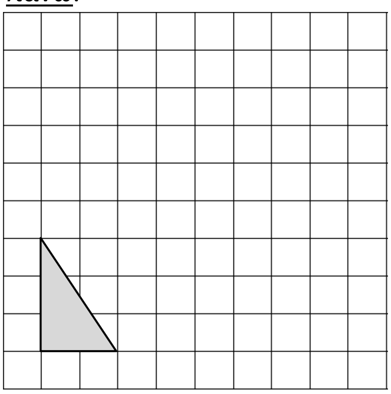
**Να μεταφέρεις και <u>να σχεδιάσεις το τρίγωνο στη νέα θέση</u> που θα βρίσκεται. Οι οδηγίες για τη μεταφορά δίνονται πάνω από το σχήμα.**

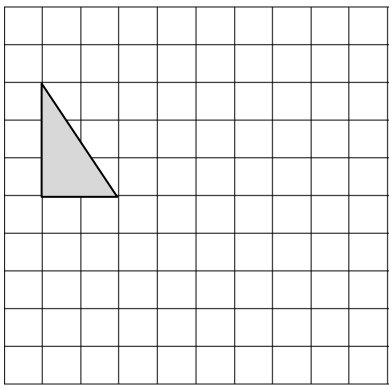
**Α13)** Να μεταφέρεις το σχήμα <u>4</u> κουτάκια προς τα <u>δεξιά</u>.



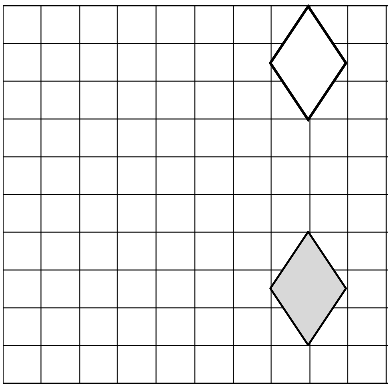
**Α14)** Να μεταφέρεις το σχήμα <u>2</u> κουτάκια προς τα <u>δεξιά</u> και <u>3</u> προς τα <u>πάνω</u>.



**Α15)** Να μεταφέρεις το σχήμα <u>1</u> κουτάκι προς τα <u>δεξιά</u>.



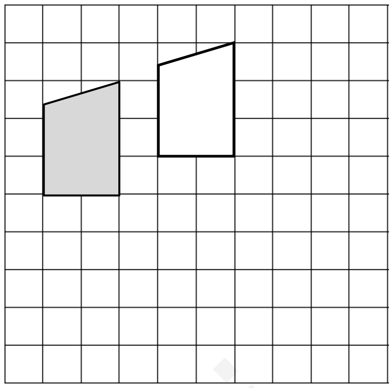
**Τώρα, <u>να γράψεις</u> εσύ τις <u>οδηγίες</u> για το πώς <u>να μεταφέρει</u> κάποιος το <u>χρωματισμένο σχήμα</u> στη <u>νέα θέση</u> (όπως τις οδηγίες που σου δόθηκαν στην προηγούμενη άσκηση).**



**Α8**. Να μεταφέρεις το σχήμα :

……………………………………………………………

……………………………………………………………



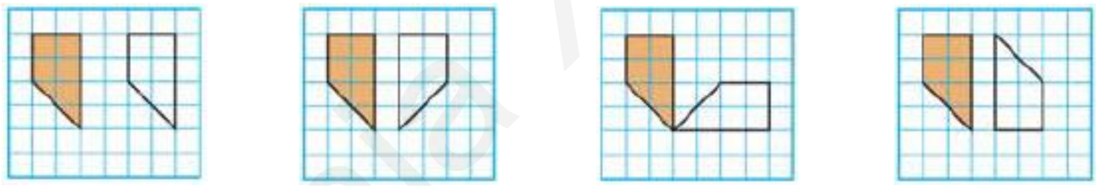
**Α10**. Να μεταφέρεις το σχήμα :

……………………………………………………………

……………………………………………………………

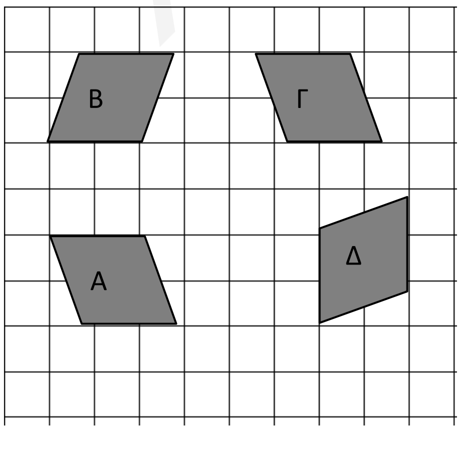
**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι <u>μόνο μία</u>. <u>Να βάλεις σε κύκλο</u> τη σωστή απάντηση σε κάθε περίπτωση:**

**Α5**) <u>Ποια</u> από τις πιο κάτω εικόνες παρουσιάζει τη <u>μεταφορά</u> του χρωματισμένου σχήματος;



Α Β Γ Δ

**Α6**) <u>Ποιο</u> από τα πιο κάτω ζευγάρια σχημάτων σχετίζονται με <u>μεταφορά</u> (το ένα να είναι μεταφορά του άλλου);




α) Το Α με το Δ

β) Το Β με το Γ

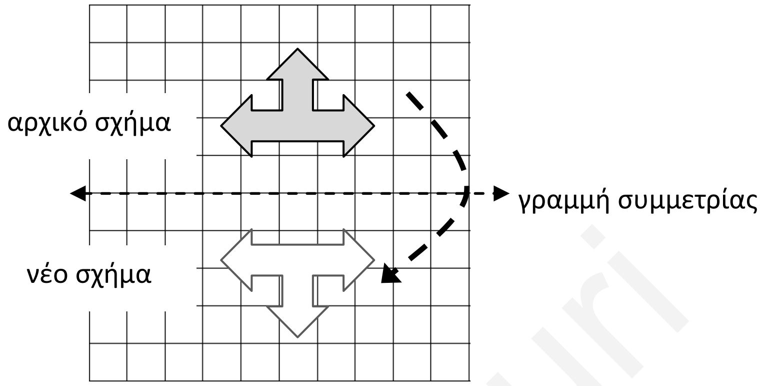
γ) Το Γ με το Δ

δ) Το Α με το Γ

# ΜΕΡΟΣ Β

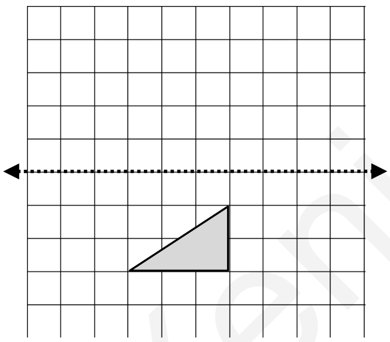
## ΑΝΑΚΛΑΣΗ

Οι πιο κάτω ασκήσεις αναφέρονται στην **<u>ανάκλαση</u>** (καθρέφτισμα) αντικειμένων με οριζόντια, κατακόρυφη ή διαγώνια <u>γραμμή συμμετρίας</u>.

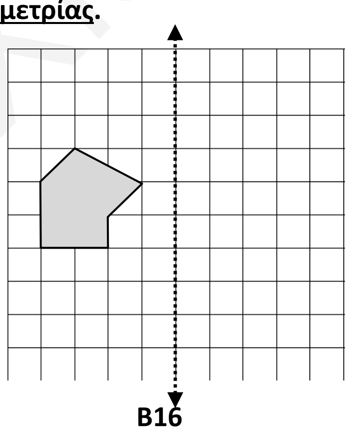




---

**Να βρεις και <u>να σχεδιάσεις το συμμετρικό</u> του <u>κάθε</u> αρχικού σχήματος όταν γίνει ανάκλαση του στη <u>διακεκομμένη γραμμή συμμετρίας</u>.**

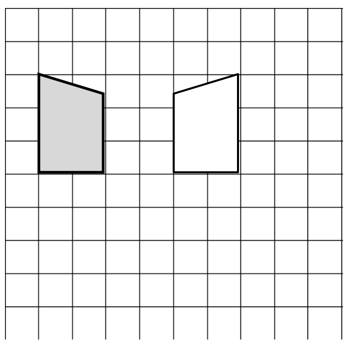


**B12**

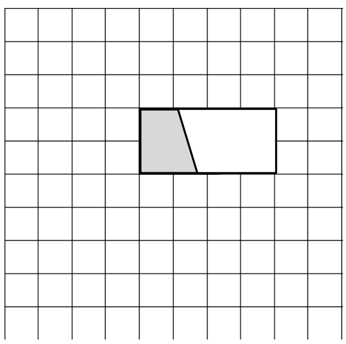


**B16**

**<u>Να βρεις</u> και <u>να χαράξεις</u> με τη ρίγα σου τη <u>γραμμή συμμετρίας</u> για κάθε περίπτωση.**



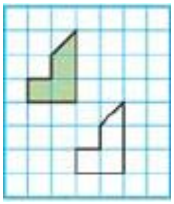
**B9**



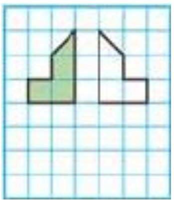
**B11**

**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι μόνο μία. Να βάλεις σε κύκλο τη σωστή απάντηση σε κάθε περίπτωση:**

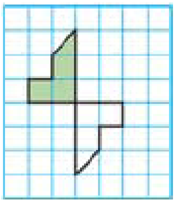
**Β5**) Ποια από τις πιο κάτω εικόνες παρουσιάζει την ανάκλαση του χρωματισμένου σχήματος;



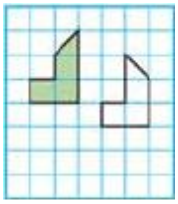
Α



Β

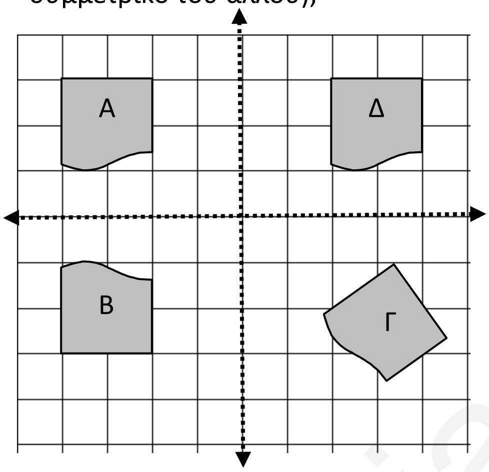


Γ



Δ

**Β4**) Ποιο από τα πιο κάτω ζευγάρια σχημάτων σχετίζονται με ανάκλαση (το ένα είναι συμμετρικό του άλλου);



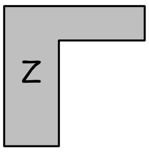

α) Το Α με το Δ

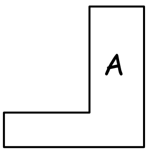
β) Το Β με το Γ

γ) Το Α με το Β

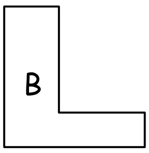
δ) Το Γ με το Δ

**Β7**) Ποιο από τα πιο κάτω σχήματα μπορεί να δημιουργήθηκε από την ανάκλαση του χρωματισμένου σχήματος Ζ;

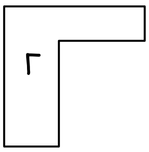


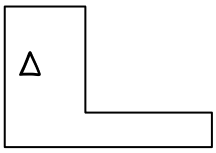

Α




Β




Γ



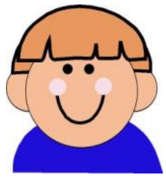

Δ

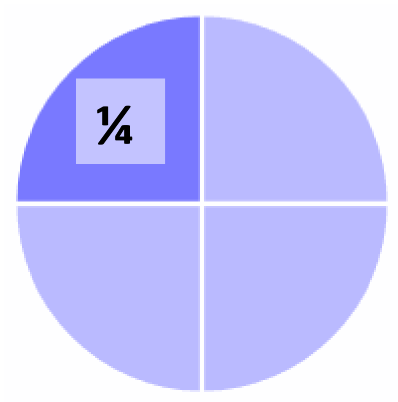
# ΜΕΡΟΣ Γ

## ΠΕΡΙΣΤΡΟΦΗ

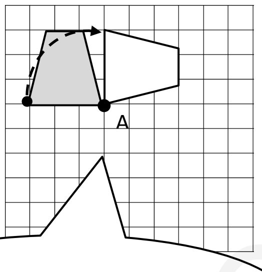
**Οι πιο κάτω ερωτήσεις αναφέρονται στην <u>περιστροφή</u> των σχημάτων, γύρω από συγκεκριμένο σημείο.**

**Τα σχήματα μπορούν να κάνουν στροφή (να γυρίσουν) προς τα δεξιά, δηλαδή όπως κινούνται οι δείκτες του ρολογιού, ή προς τα αριστερά, δηλαδή αντίθετα με τους δείκτες του ρολογιού. <u>Στις πιο κάτω ασκήσεις, θεώρησε ότι όλες οι στροφές γίνονται προς τα δεξιά, όπως γυρίζουν οι δείκτες του ρολογιού</u>.**

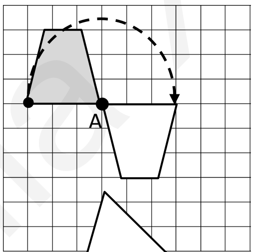




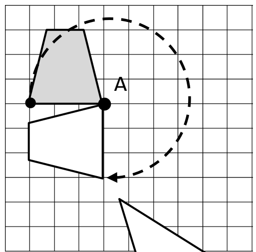

**Μια ολόκληρη στροφή είναι 4/4 του κύκλου. Όταν η ερώτηση λέει ότι το σχήμα κάνει ¼ της στροφής, εννοεί ¼ του κύκλου. Τα 2/4 της στροφής είναι ίσα με 2/4 του κύκλου και τα ¾ της στροφής ίσα με τα ¾ του κύκλου.**




**$\frac{1}{4}$ της στροφής γύρω από την τελεία *Α***




**2/4 της στροφής γύρω από την τελεία *Α***



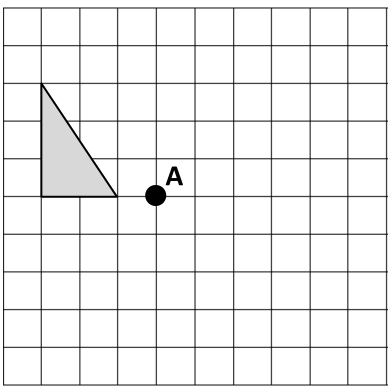

**$\frac{3}{4}$ της στροφής γύρω από την τελεία *Α***



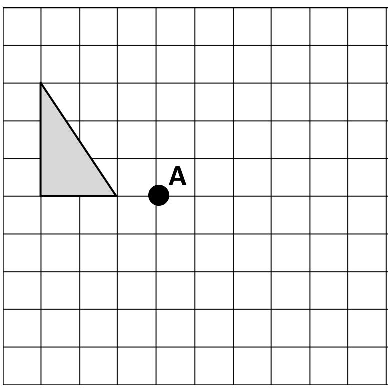
---

**<u>Να σχεδιάσεις</u> το κάθε σχήμα στη <u>νέα του θέση</u>, όταν κάνει <u>στροφή</u> (γυρίσει) <u>γύρω από την τελεία Α</u>.**

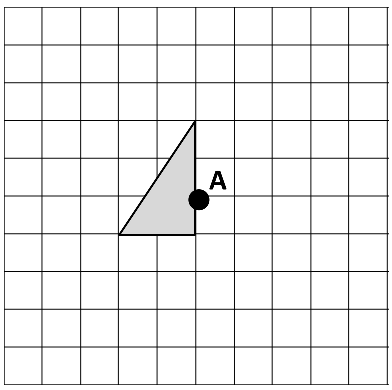
**C13)** Όταν κάνει ¼ της στροφής προς τα δεξιά .



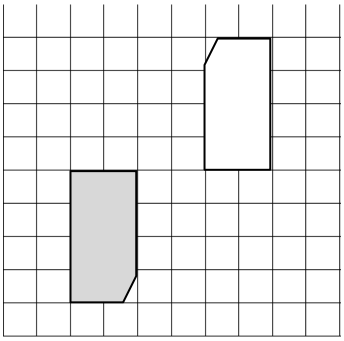

**C12)** Όταν κάνει ¾ της στροφής προς τα δεξιά.




**C15)** Όταν κάνει 1/4 της στροφής προς τα δεξιά.



**C10**



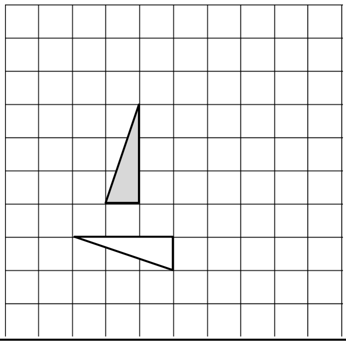
1) **<u>Να σχεδιάσεις</u>** την **<u>ΤΕΛΕΙΑ Α</u>** γύρω από την οποία έκανε στροφή το σχήμα.

2) **<u>Βάλε σε κύκλο</u>** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

1/4 2/4 3/4

**C8**



1) **<u>Να σχεδιάσεις</u>** την **<u>ΤΕΛΕΙΑ Α</u>** γύρω από την οποία έκανε στροφή το σχήμα.

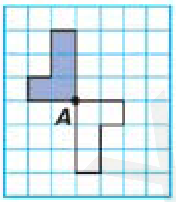
2) **<u>Βάλε σε κύκλο</u>** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

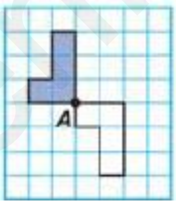
1/4 2/4 3/4

**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι <u>μόνο μία</u>. <u>Να βάλεις σε κύκλο</u> τη σωστή απάντηση σε κάθε περίπτωση:**

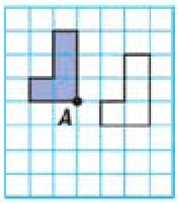
**C6**) <u>Ποια</u> από τις πιο κάτω εικόνες παρουσιάζει την <u>περιστροφή</u> του χρωματισμένου σχήματος;



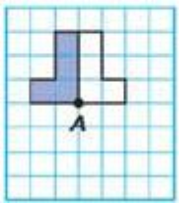

A




B

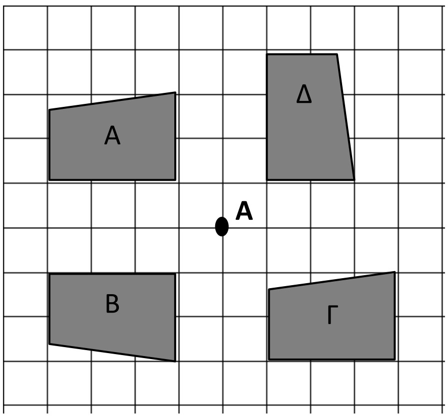



Γ




Δ

**C5**) <u>Ποιο</u> από τα πιο κάτω ζευγάρια σχημάτων σχετίζεται με <u>στροφή γύρω από το σημείο Α</u>;



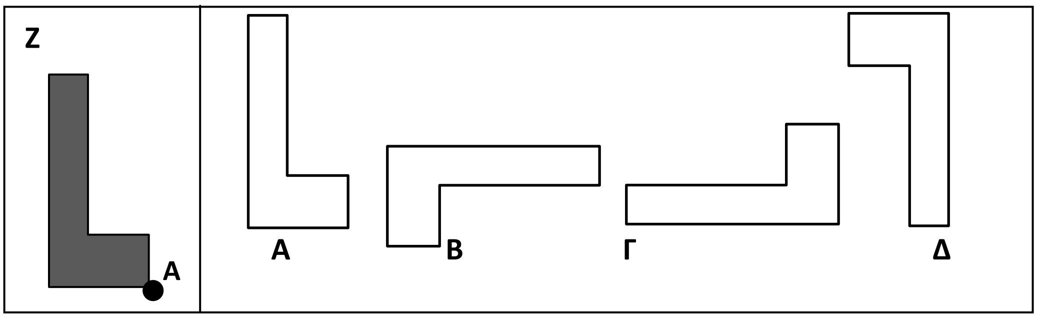

α) Το Α με το Δ

β) Το Β με το Γ

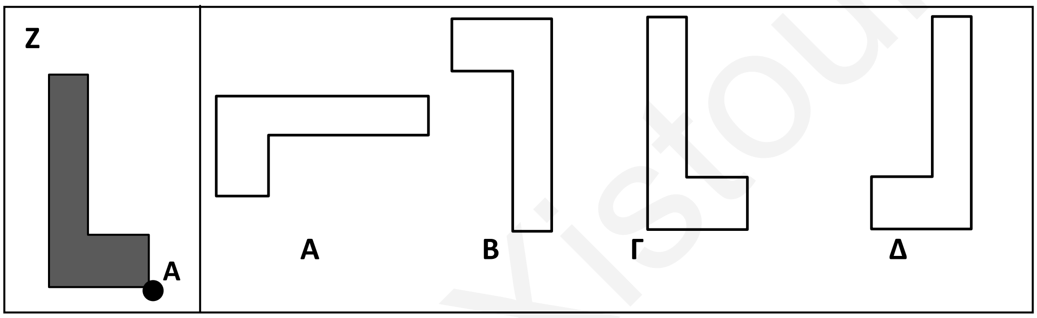
γ) Το Α με το Β

δ) Το Α με το Γ

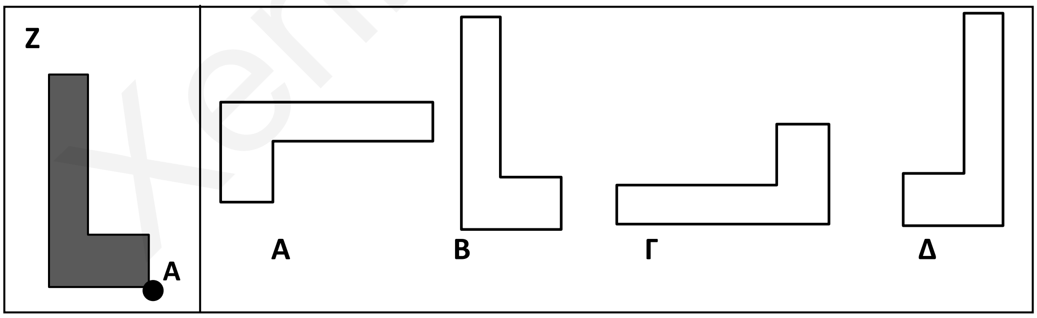
**C2**) Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά ¼ του κύκλου προς τα δεξιά;




**C3**) Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά 2/4 του κύκλου προς τα δεξιά;




**C1**) Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά ¾ του κύκλου προς τα δεξιά;



**ΔΟΚΙΜΙΟ 1Β**

**Όνομα:** ........................................................ **Τάξη:** .......

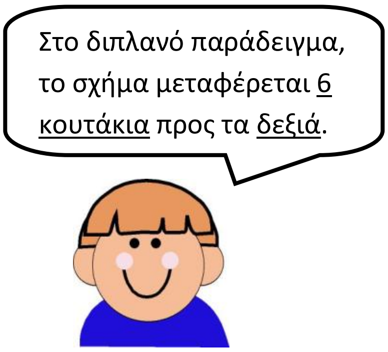
**Αρ. στον κατάλογο:** ...... **Σχολείο:** ........................................

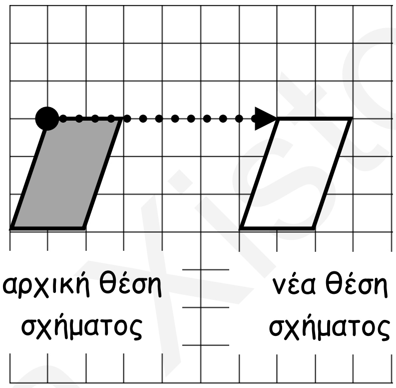
**Φύλο (βάλε ✓):** **Αγόρι** ☐ **Κορίτσι:** ☐

## ΜΕΡΟΣ Α

### ΜΕΤΑΦΟΡΑ

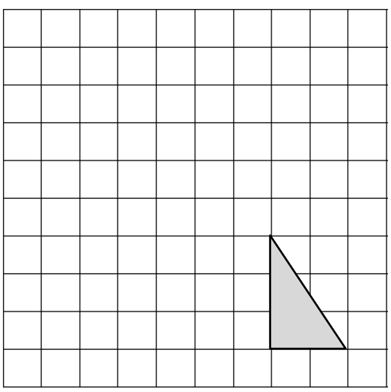
Οι πιο κάτω ασκήσεις αναφέρονται στη **<u>μεταφορά</u>** (μετακίνηση) σχημάτων από μια **αρχική θέση** σε μια **νέα θέση**.



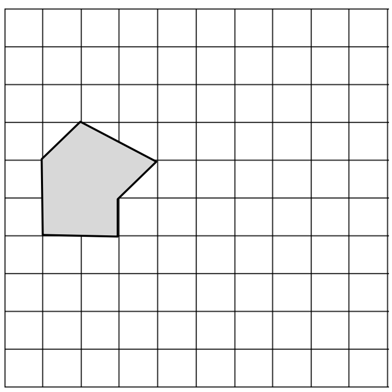




**Να κάνεις τη μεταφορά και <u>να σχεδιάσεις το τρίγωνο στη νέα θέση</u> που θα βρίσκεται. Οι οδηγίες για τη μεταφορά δίνονται πάνω από το σχήμα.**

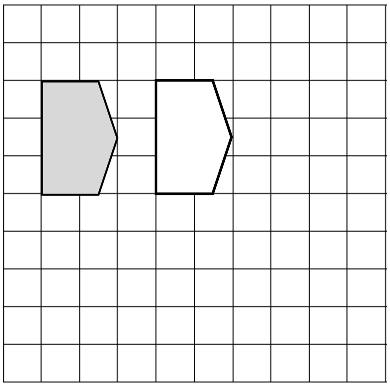
**Α12)** Να μεταφέρεις το σχήμα <u>3</u> κουτάκια προς τα <u>πάνω</u>.



**Α16)** Να μεταφέρεις το σχήμα <u>3</u> κουτάκια προς τα <u>δεξιά</u>.



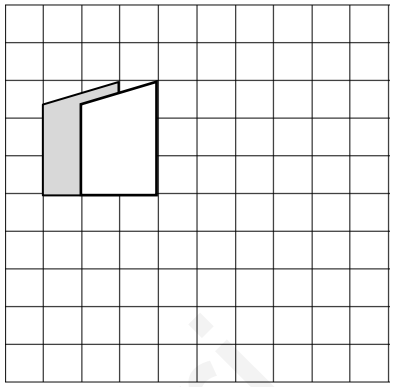
**Τώρα, να γράψεις εσύ τις οδηγίες για το πώς να μεταφέρει κάποιος το χρωματισμένο σχήμα στη νέα θέση (όπως τις οδηγίες που σου δόθηκαν στην προηγούμενη άσκηση).**



**Α9**. Να μεταφέρεις το σχήμα:

……………………………………………………….

……………………………………………………….



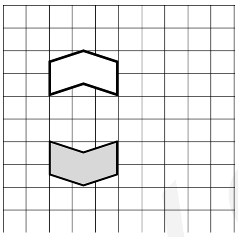
**Α11**. Να μεταφέρεις το σχήμα:

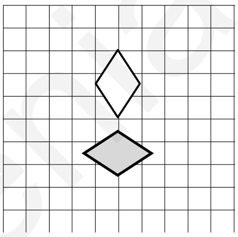
……………………………………………………….

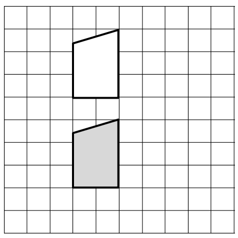
……………………………………………………….

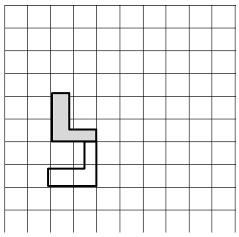
**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι μόνο μία. Να βάλεις σε κύκλο τη σωστή απάντηση σε κάθε περίπτωση:**

**Α4**) Ποια από τις πιο κάτω περιπτώσεις παρουσιάζει μεταφορά;

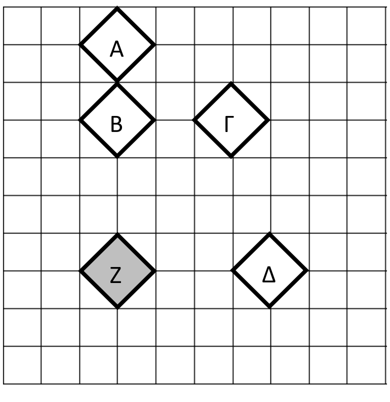

Α


Β


Γ


Δ

**Α2**) Ποιο από τα πιο κάτω αποτελεί μεταφορά του αρχικού σχεδίου Ζ, όταν μεταφερθεί 4 κουτάκια προς τα δεξιά;



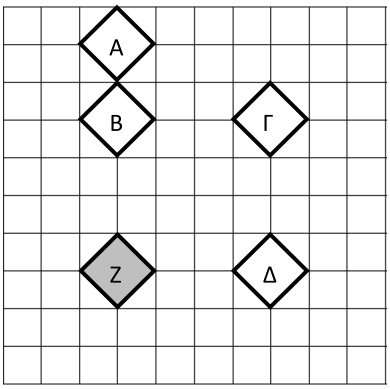

α) Το Α

β) Το Β

γ) Το Γ

δ) Το Δ

**Α1**) Ποιο από τα πιο κάτω αποτελεί μεταφορά του αρχικού σχεδίου Ζ, όταν μεταφερθεί 4 κουτάκια προς τα πάνω;



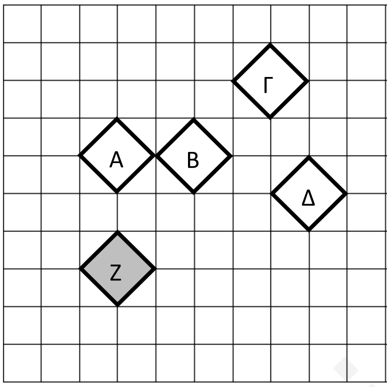

α) Το Α

β) Το Β

γ) Το Γ

δ) Το Δ

**Α3**) Ποιο από τα πιο κάτω αποτελεί μεταφορά του αρχικού σχεδίου Ζ, όταν μεταφερθεί 2 κουτάκια προς τα δεξιά και 3 προς τα πάνω;



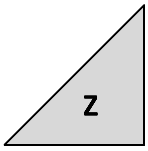

α) Το Α

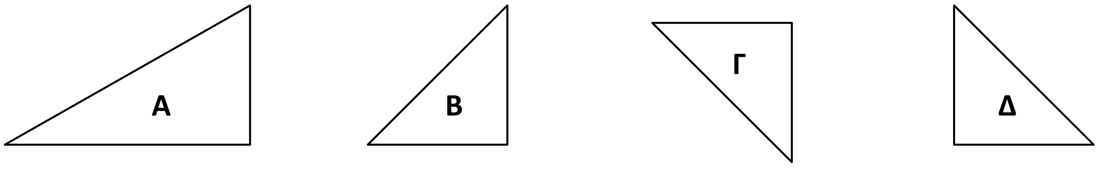
β) Το Β

γ) Το Γ

δ) Το Δ

**Α7**) Ποιο από τα πιο κάτω τρίγωνα μπορεί να δημιουργήθηκε από τη μεταφορά του χρωματισμένου τριγώνου Ζ;



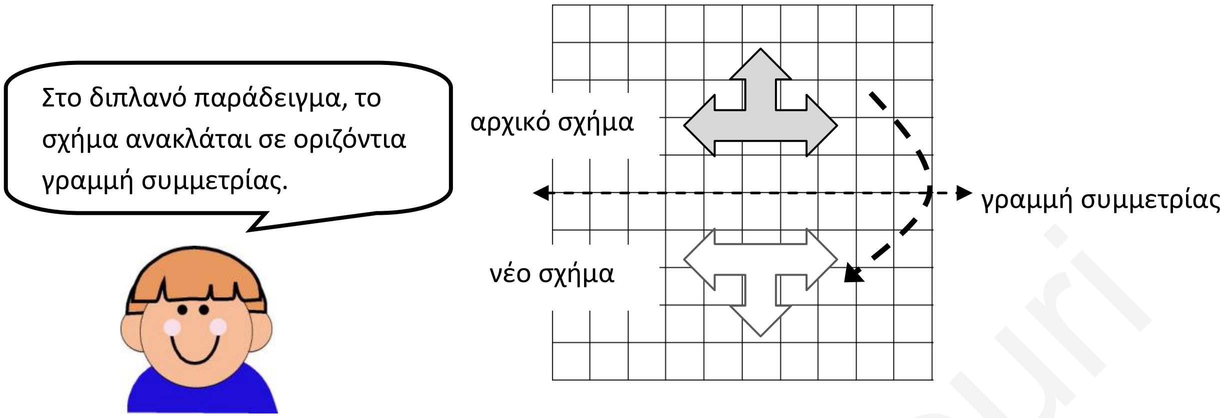




Α Β Γ Δ

# ΜΕΡΟΣ Β

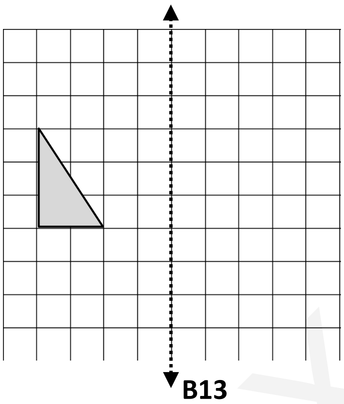
## ΑΝΑΚΛΑΣΗ

Οι πιο κάτω ασκήσεις αναφέρονται στην **ανάκλαση** (καθρέφτισμα) αντικειμένων με οριζόντια, κατακόρυφη ή διαγώνια γραμμή συμμετρίας.

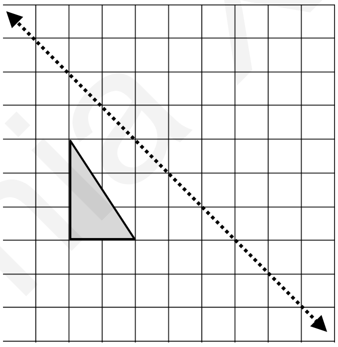



---

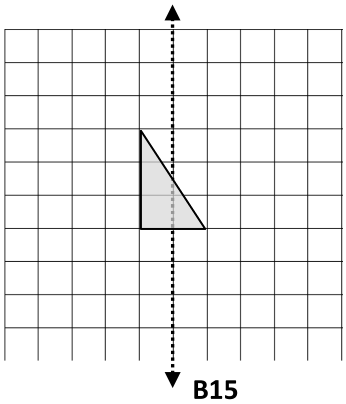
**Α) Να βρεις και να σχεδιάσεις το συμμετρικό του κάθε αρχικού σχήματος όταν γίνει ανάκλαση του στη διακεκομμένη γραμμή συμμετρίας.**



**B13**

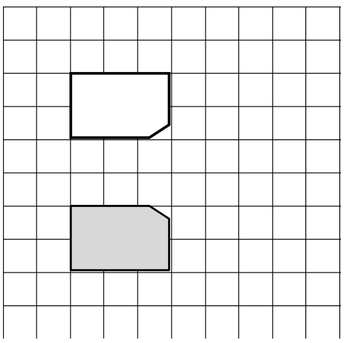


**B14**

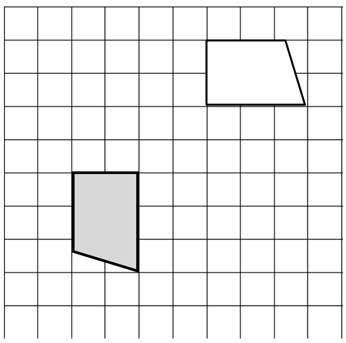


**B15**

**Να βρεις και να χαράξεις με τη ρίγα σου τη γραμμή συμμετρίας για κάθε περίπτωση.**



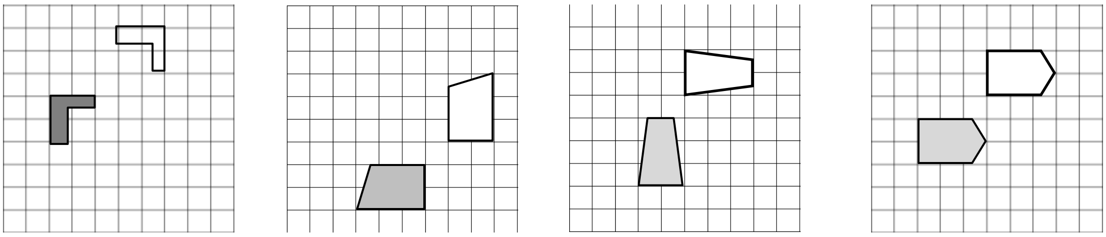
**B8**



**B10**

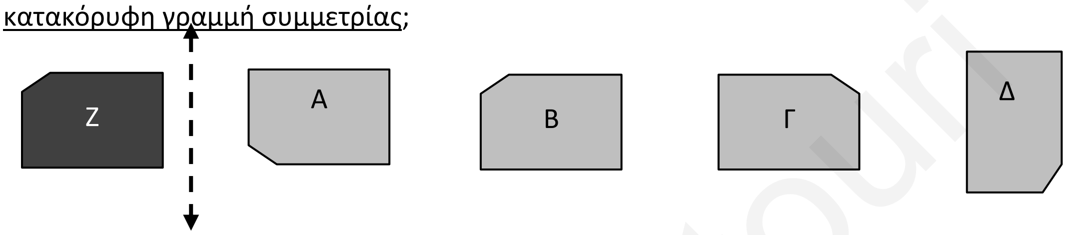
**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι μόνο μία. Να βάλεις σε κύκλο τη σωστή απάντηση σε κάθε περίπτωση:**

**Β6**) Ποια από τις πιο κάτω εικόνες παρουσιάζει ανάκλαση;



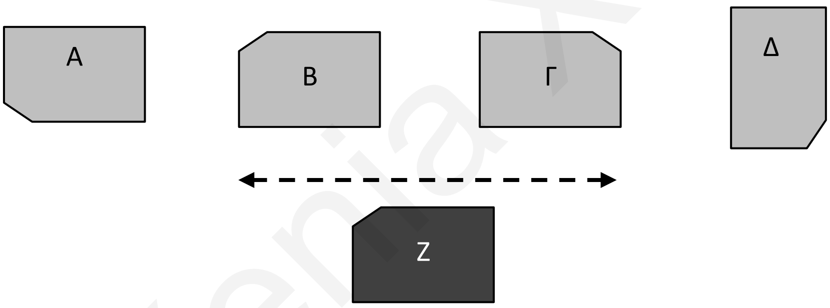
Α Β Γ Δ

**Β2**) Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Ζ σε κατακόρυφη γραμμή συμμετρίας;



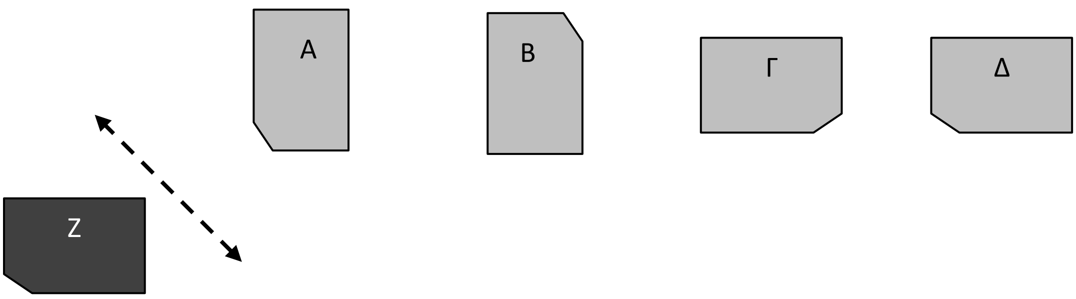

α) Το Α β) Το Β γ) Το Γ δ) Το Δ

**Β1**) Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Ζ σε οριζόντια γραμμή συμμετρίας;




α) Το Α β) Το Β γ) Το Γ δ) Το Δ

**Β3**) Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Ζ σε διαγώνια γραμμή συμμετρίας;



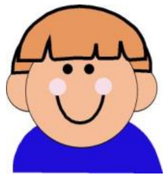

α) Το Α β) Το Β γ) Το Γ δ) Το Δ

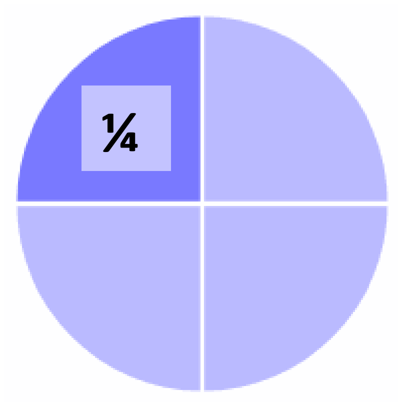
# ΜΕΡΟΣ Γ

## ΠΕΡΙΣΤΡΟΦΗ

**Οι πιο κάτω ερωτήσεις αναφέρονται στην <u>περιστροφή</u> των σχημάτων, γύρω από συγκεκριμένο σημείο.**

**Τα σχήματα μπορούν να κάνουν στροφή (να γυρίσουν) προς τα δεξιά, δηλαδή όπως κινούνται οι δείκτες του ρολογιού, ή προς τα αριστερά, δηλαδή αντίθετα με τους δείκτες του ρολογιού. <u>Στις πιο κάτω ασκήσεις, θεώρησε ότι όλες οι στροφές γίνονται προς τα δεξιά, όπως γυρίζουν οι δείκτες του ρολογιού</u>.**



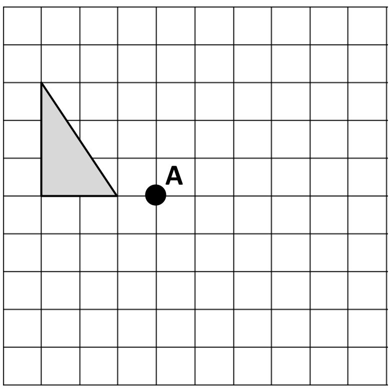



**Μια ολόκληρη στροφή είναι 4/4 του κύκλου. Όταν η ερώτηση λέει ότι το σχήμα κάνει ¼ της στροφής, εννοεί ¼ του κύκλου. Τα 2/4 της στροφής είναι ίσα με 2/4 του κύκλου και τα ¾ της στροφής ίσα με τα ¾ του κύκλου.**

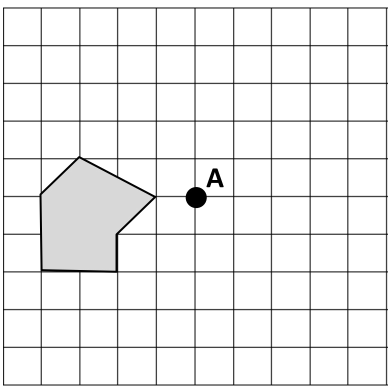

---

**<u>Να σχεδιάσεις</u> το κάθε σχήμα στη <u>νέα του θέση</u>, όταν κάνει <u>στροφή</u> (γυρίσει) <u>γύρω από την τελεία A</u>.**

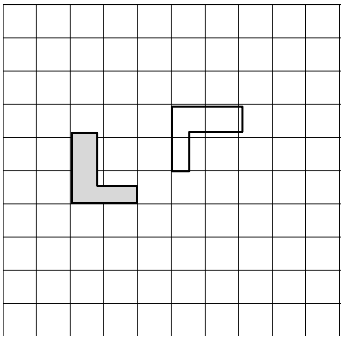
**C14)** Όταν κάνει 2/4 της στροφής προς τα δεξιά .




**C16)** Όταν κάνει 1/4 της στροφής προς τα δεξιά.



**C9**



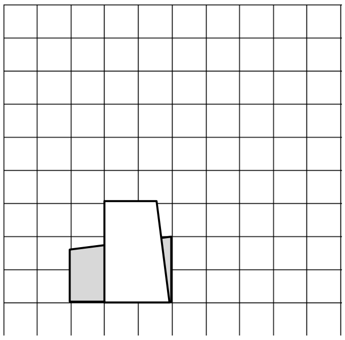
1) **<u>Να σχεδιάσεις</u>** την **<u>ΤΕΛΕΙΑ Α</u>** γύρω από την οποία έκανε στροφή το σχήμα.

2) **<u>Βάλε σε κύκλο</u>** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

1/4 2/4 3/4

**C11**



1) **<u>Να σχεδιάσεις</u>** την **<u>ΤΕΛΕΙΑ Α</u>** γύρω από την οποία έκανε στροφή το σχήμα.

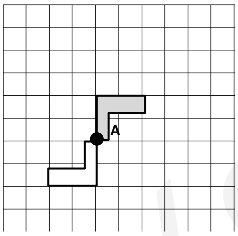
2) **<u>Βάλε σε κύκλο</u>** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

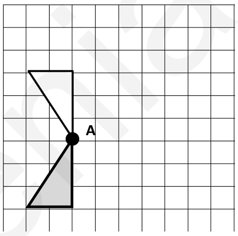
1/4 2/4 3/4

**Σε κάθε ερώτηση πιο κάτω, η σωστή απάντηση είναι <u>μόνο μία</u>. <u>Να βάλεις σε κύκλο</u> τη σωστή απάντηση σε κάθε περίπτωση:**

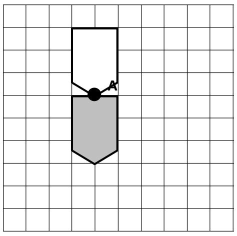
**C4**) <u>Ποια</u> από τις πιο κάτω περιπτώσεις παρουσιάζει <u>περιστροφή</u>;



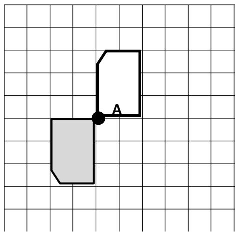

Α




Β

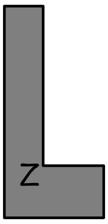


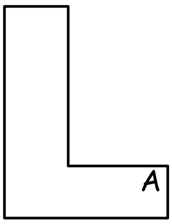

Γ



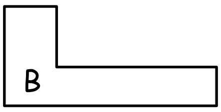

Δ

**C7**) Ποιο από τα πιο κάτω σχήματα μπορεί να δημιουργήθηκε <u>από την περιστροφή</u> του χρωματισμένου <u>σχήματος Ζ</u>;

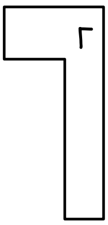


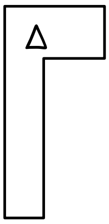

Α




Β




Γ



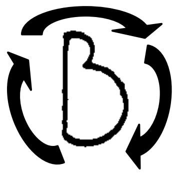

Δ

**ΔΟΚΙΜΙΟ 2**

Όνομα: ……………………………………………………………………………………………………… Τάξη: …………………..

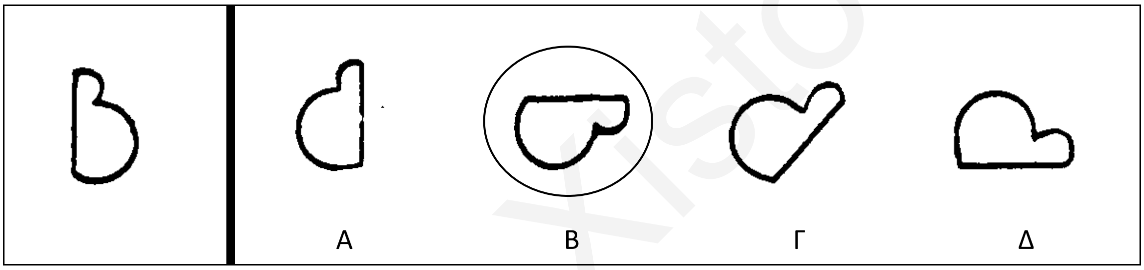
Αρ. στον κατάλογο: ………… Σχολείο: ………………………………………………………………………………………

**ΜΕΡΟΣ Α΄**



Στο μέρος αυτό δίνεται ένα σχήμα πριν από τη γραμμή και τέσσερα σχήματα μετά από τη γραμμή. **<u>Ποιο</u>** από τα σχήματα που βρίσκονται μετά από τη γραμμή μπορεί να προκύψει όταν **<u>ΠΕΡΙΣΤΡΕΨΕΙΣ</u>** το σχήμα που βρίσκεται πριν από τη γραμμή προς τα δεξιά ή αριστερά, **<u>ΧΩΡΙΣ ΝΑ ΤΟ ΑΝΤΙΣΤΡΕΨΕΙΣ</u>**; Να το βάλεις σε κύκλο.

**ΠΑΡΑΔΕΙΓΜΑ:**






**Ασκήσεις:**

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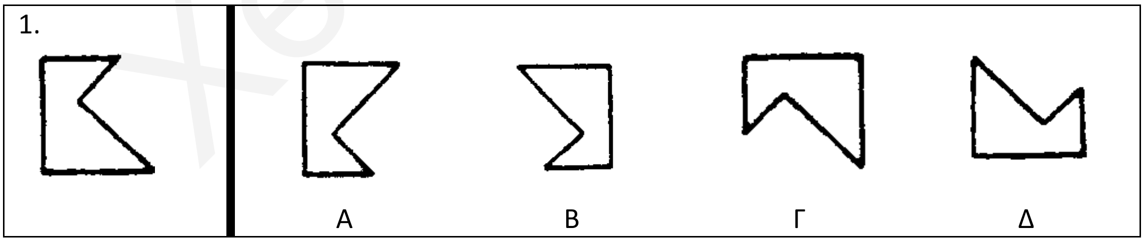
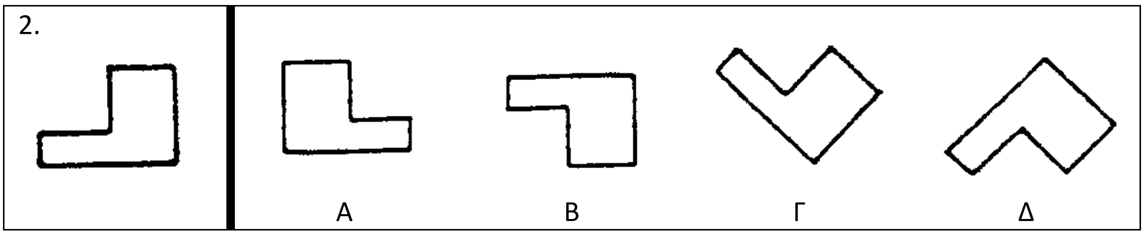
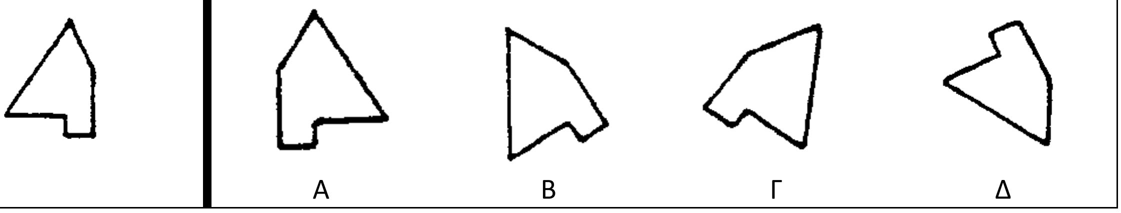


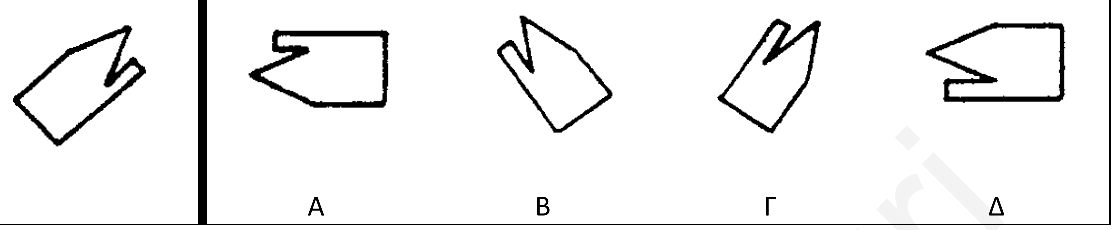

2.



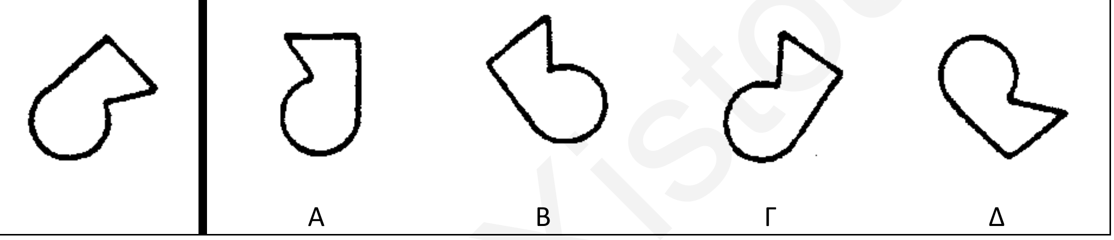
3.



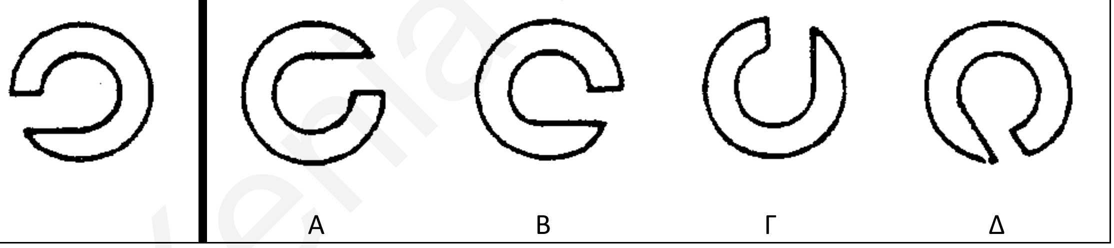

4.




5.



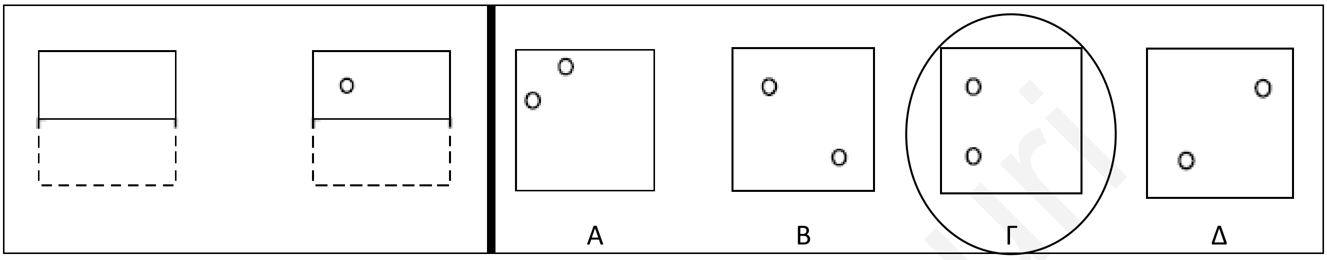

6.



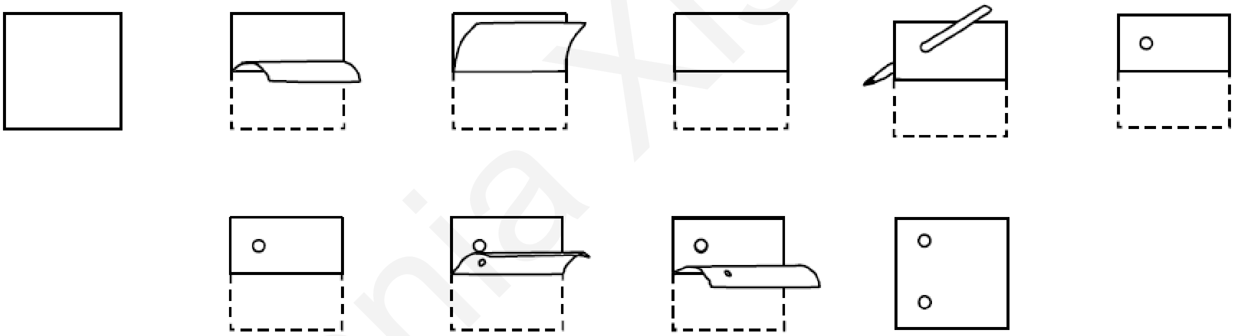
**ΜΕΡΟΣ Β΄**

Στο μέρος αυτό παρουσιάζεται πριν από τη γραμμή ο τρόπος με τον οποίο **ΔΙΠΛΩΝΕΤΑΙ** ένα τετράγωνο χαρτόνι και η θέση στην οποία **ΑΝΟΙΓΟΥΜΕ ΜΙΑ ΤΡΥΠΑ** όταν το χαρτόνι είναι διπλωμένο. Να βάλεις σε κύκλο το σχήμα μετά από τη γραμμή που δείχνει πώς θα φαίνεται το χαρτόνι **ΟΤΑΝ ΑΝΟΙΧΤΕΙ**.

**ΠΑΡΑΔΕΙΓΜΑ:**



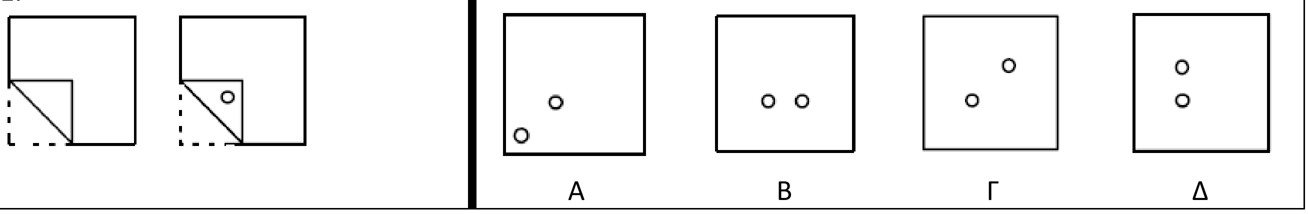

**Η σωστή απάντηση είναι η Γ, όπως φαίνεται στο πιο κάτω σχέδιο:**



**ΓΥΡΙΣΕ ΣΕΛΙΔΑ ΓΙΑ ΤΙΣ ΑΣΚΗΣΕΙΣ**

**Ασκήσεις:**

1.

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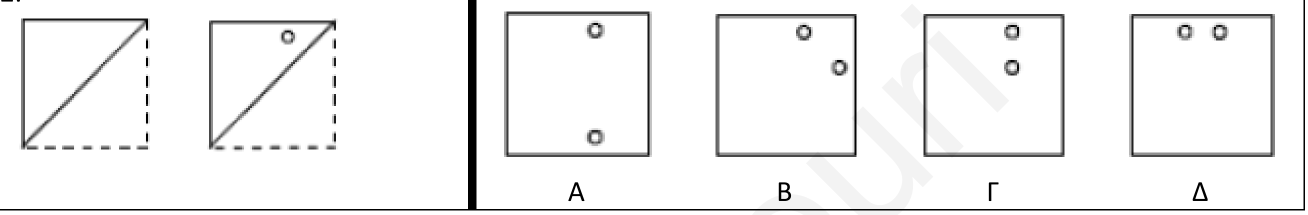


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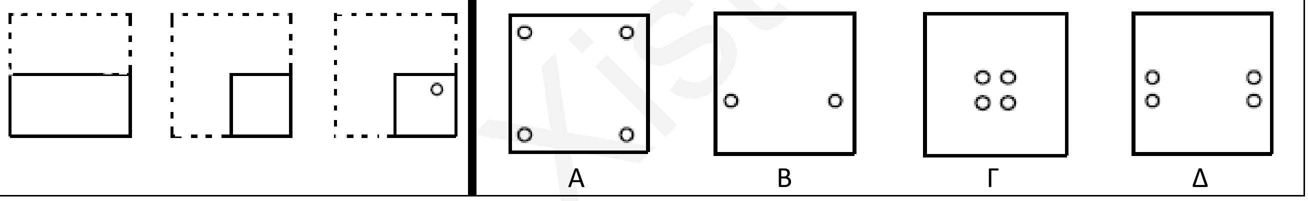


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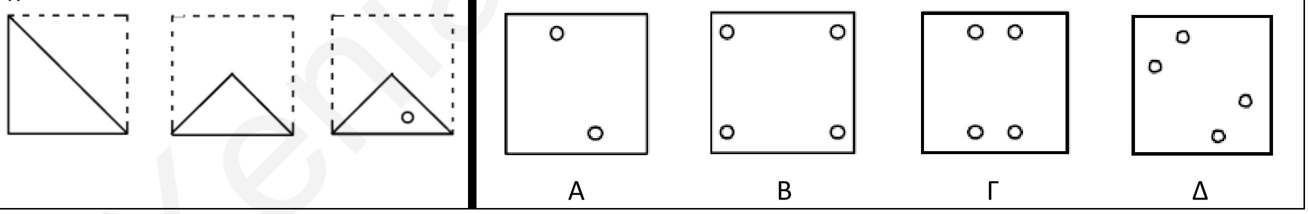
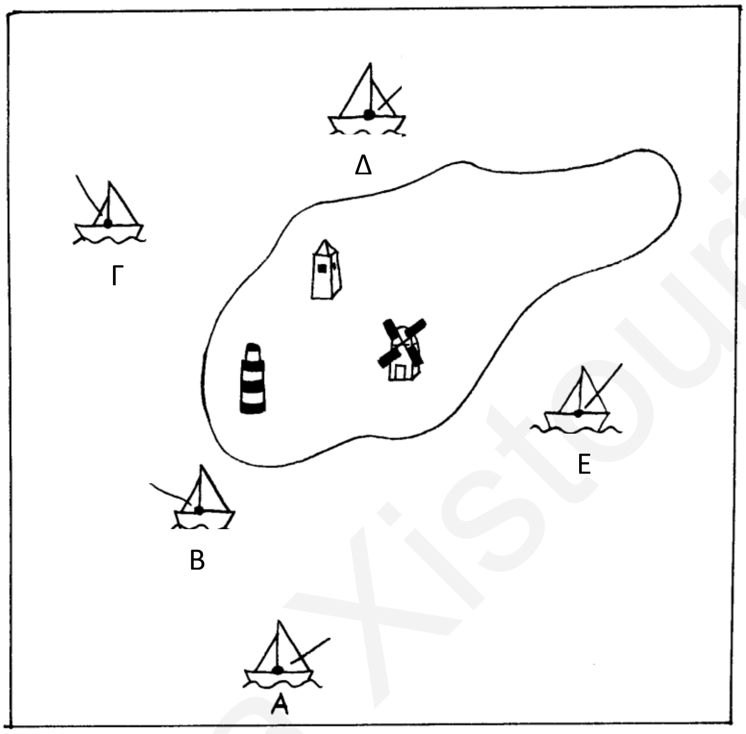


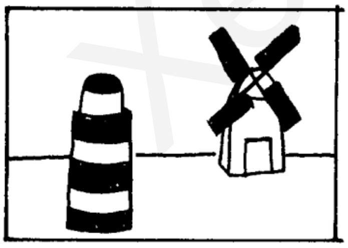
**ΜΕΡΟΣ Γ΄**

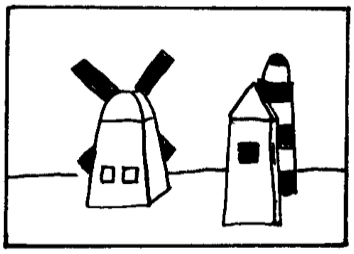
Στο μέρος αυτό παρουσιάζεται μια εικόνα από ένα τοπίο, και τρεις φωτογραφίες. Πρέπει να αποφασίσεις **ΣΕ ΠΟΙΑ ΒΑΡΚΑ ΒΡΙΣΚΟΤΑΝ Ο ΦΩΤΟΓΡΑΦΟΣ** που έβγαλε την κάθε φωτογραφία. Να βάλεις σε κύκλο τη σωστή απάντηση για κάθε φωτογραφία.

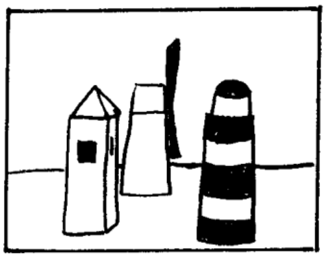





**Ο φωτογράφος ήταν στη βάρκα:**







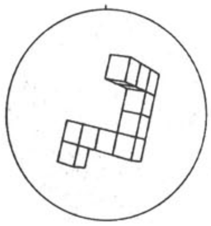
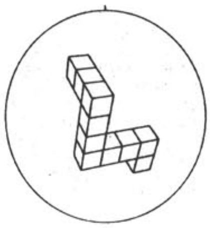
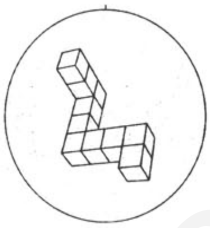
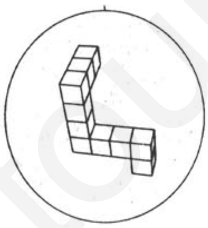
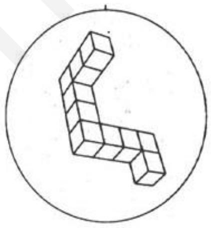
**Α Β Γ Δ Ε**  **Α Β Γ Δ Ε**  **Α Β Γ Δ Ε**

## ΜΕΡΟΣ Δ΄

Στο μέρος αυτό, το κάθε ένα από τα σχήματα που βρίσκονται αριστερά από τη γραμμή κρέμεται κατακόρυφα από ένα σχοινί και **ΠΕΡΙΣΤΡΕΦΕΤΑΙ** γύρω-γύρω. Ένας φωτογράφος **τα φωτογραφίζει** την ώρα που γυρίζουν, μόνο που μπέρδεψε τις φωτογραφίες. Βοήθησε τον να βρει **ΠΟΙΕΣ** από τις τέσσερις επιλογές Α, Β, Γ και Δ που βρίσκονται μετά από τη γραμμή είναι φωτογραφίες του σχήματος που βρίσκεται πριν από τη γραμμή.

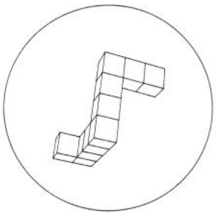
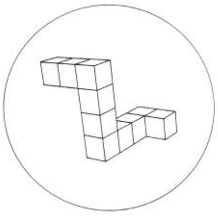
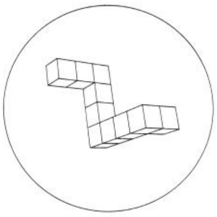
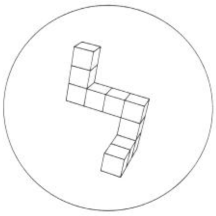
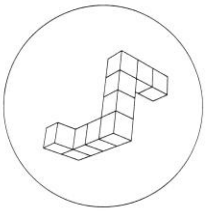
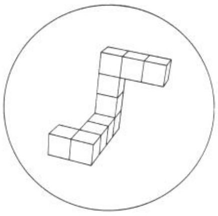
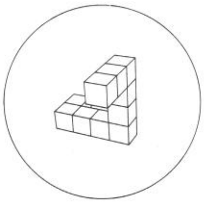
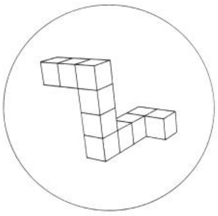
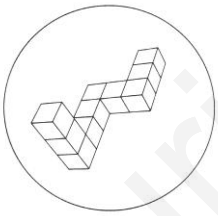
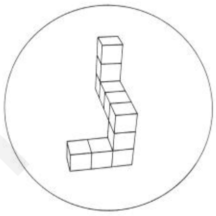
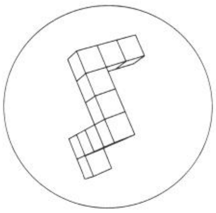
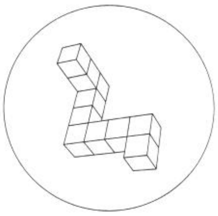
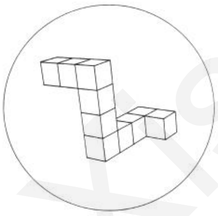
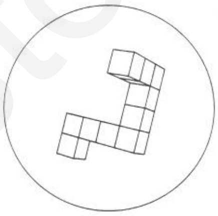
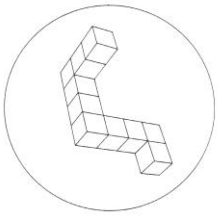
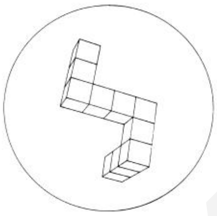
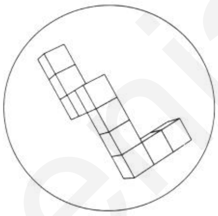
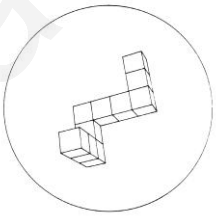
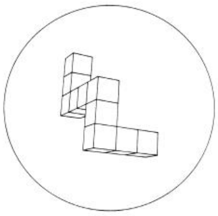
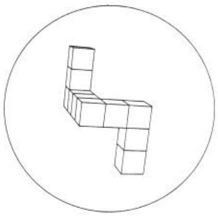
**ΠΡΟΣΕΞΕ**: Υπάρχουν **ΔΥΟ ΣΩΣΤΕΣ ΑΠΑΝΤΗΣΕΙΣ** για κάθε σχήμα. Να τις βάλεις σε κύκλο.

**ΠΑΡΑΔΕΙΓΜΑ:**

| | | | | |
|---|---|---|---|---|
|  |  |  |  |  |
| | Α | Β | Γ | Δ |

**ΓΥΡΙΣΕ ΣΕΛΙΔΑ ΓΙΑ ΤΙΣ ΑΣΚΗΣΕΙΣ**

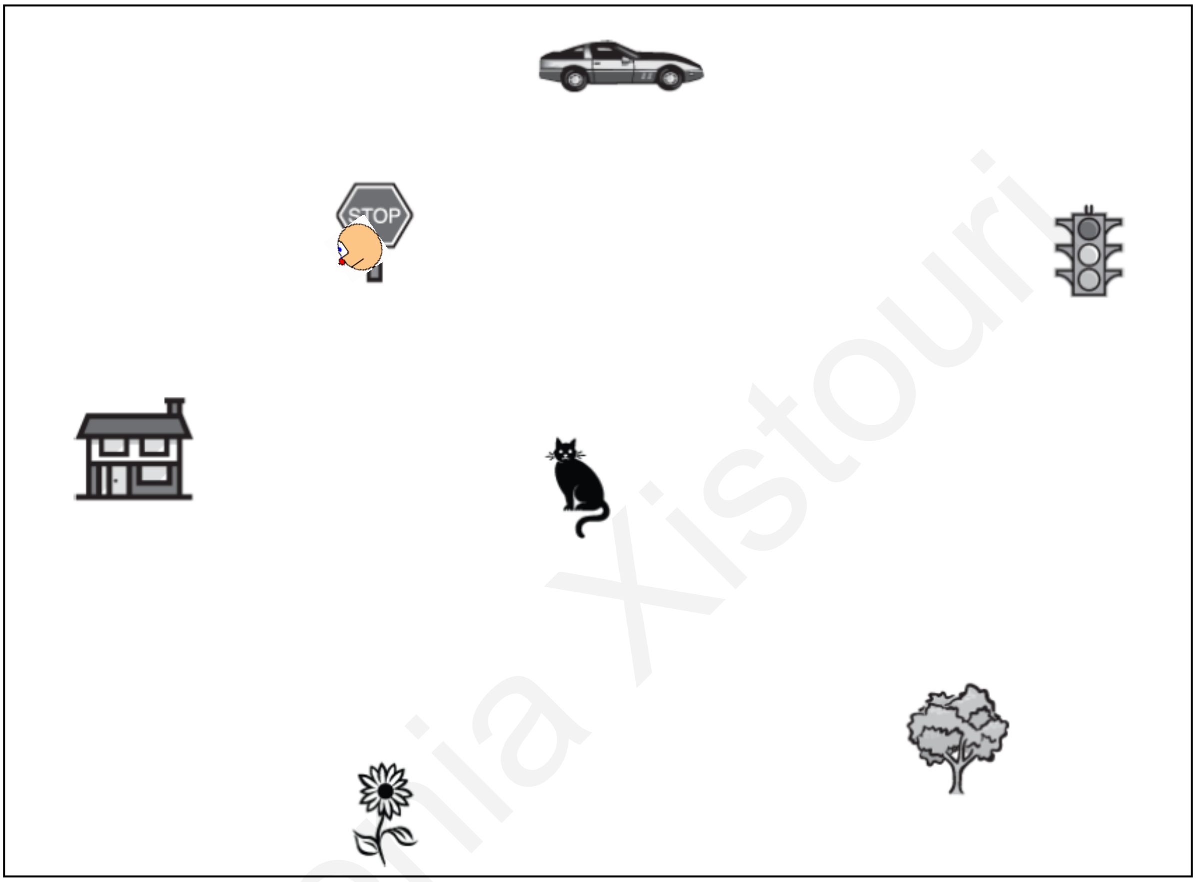
**Ασκήσεις:**

| | | | | |
|---|---|---|---|---|
| 1.  |  **Α** |  **Β** |  **Γ** |  **Δ** |
| 2.  |  **Α** |  **Β** |  **Γ** |  **Δ** |
| 3.  |  **Α** |  **Β** |  **Γ** |  **Δ** |
| 4.  |  **Α** |  **Β** |  **Γ** |  **Δ** |

**ΜΕΡΟΣ Ε΄**

Στην πιο κάτω εικόνα παρουσιάζονται εφτά αντικείμενα. **ΝΑ ΦΑΝΤΑΣΤΕΙΣ** ότι είσαι το ανθρωπάκι ( ) μέσα στην εικόνα και ότι **ΚΟΙΤΑΖΕΙΣ ΠΡΟΣ ΤΗΝ ΚΑΤΕΥΘΥΝΣΗ** που βρίσκεται το αντικείμενο που λέει η οδηγία. Να βάλεις σε κύκλο το κατάλληλο τόξο στον κύκλο που είναι κάτω από την εικόνα, για να δείξεις σε ποια κατεύθυνση θα βρίσκεται το αντικείμενο που ζητά η ερώτηση. Στο κέντρο του κύκλου είναι το σημείο στο οποίο βρίσκεσαι και το κατακόρυφο τόξο δείχνει το σημείο προς το οποίο κοιτάζεις.

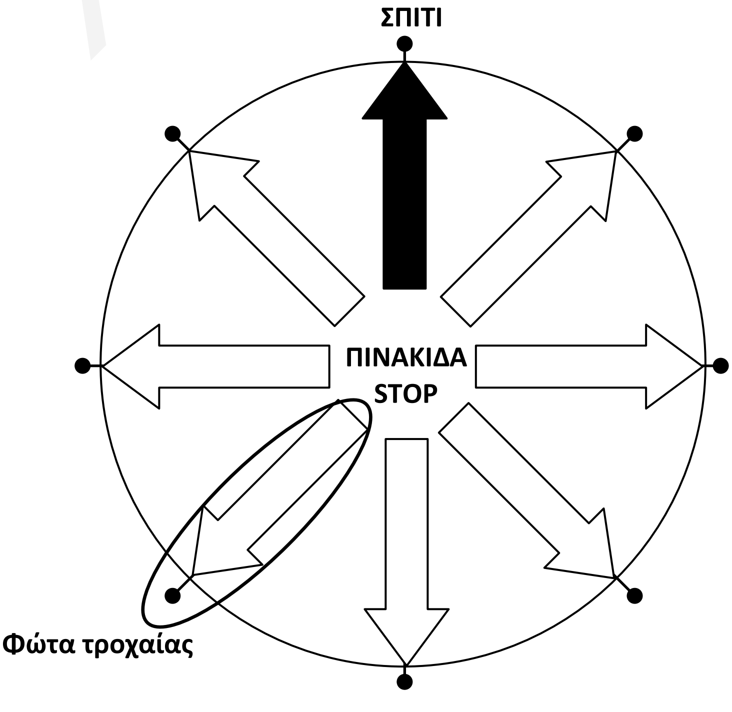
**ΠΡΟΣΕΞΕ:** Κάθε φορά που θα βρίσκεσαι σε διαφορετική θέση στην εικόνα, **θα βλέπεις τα αντικείμενα από διαφορετική πλευρά**. Πρέπει να σκεφτείς **ΠΡΟΣ ΤΑ ΠΟΥ** θα βλέπεις το αντικείμενο που ζητά η κάθε ερώτηση.






**ΠΑΡΑΔΕΙΓΜΑ:** Να φανταστείς ότι **ΒΡΙΣΚΕΣΑΙ** στην **ΠΙΝΑΚΙΔΑ STOP** και κοιτάζεις προς το **ΣΠΙΤΙ**.

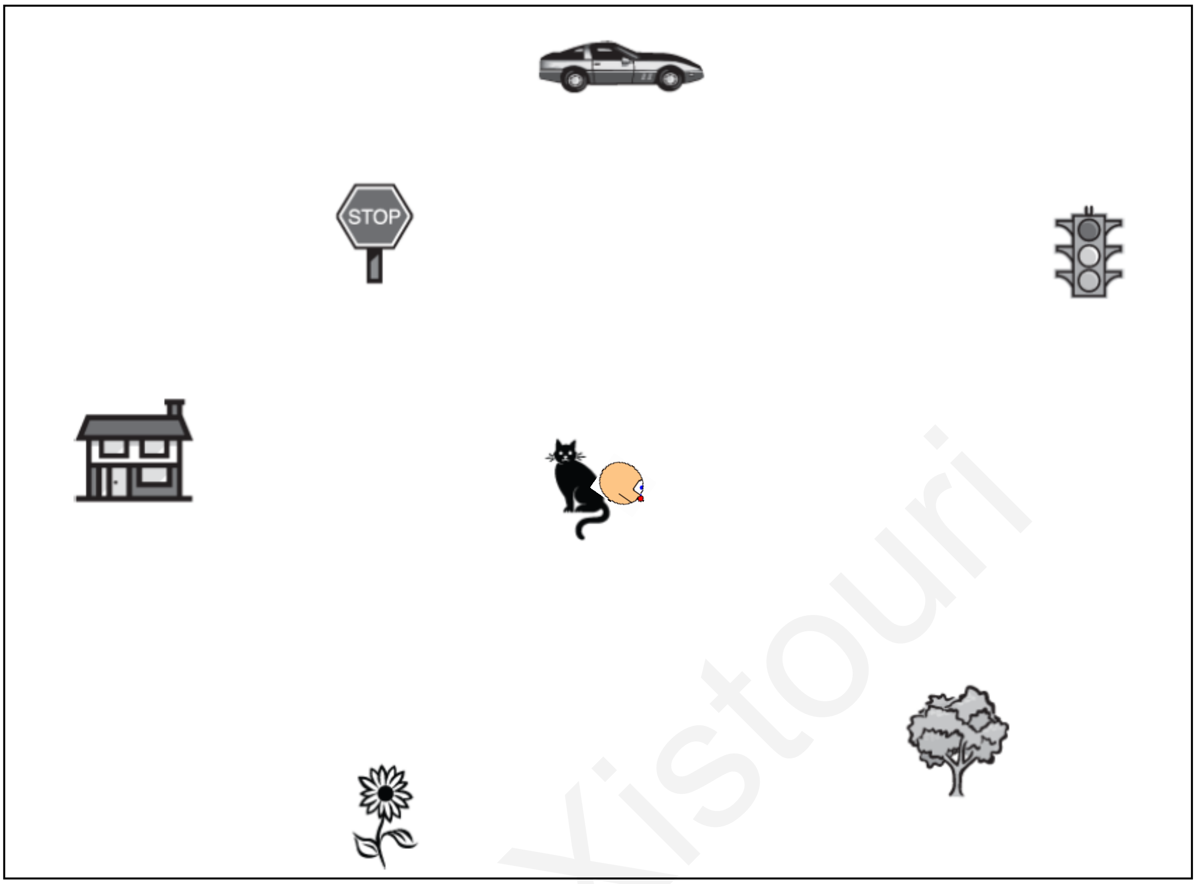
Βάλε σε κύκλο **ΤΟ ΤΟΞΟ ΠΟΥ ΔΕΙΧΝΕΙ** προς ποια κατεύθυνση θα βρίσκονται τα **ΦΩΤΑ ΤΡΟΧΑΙΑΣ**.




Στο **ΠΑΡΑΔΕΙΓΜΑ**, όταν βρίσκεσαι στην πινακίδα STOP και κοιτάζεις προς το σπίτι, τότε τα φώτα τροχαίας βρίσκονται στην αριστερή σου μεριά, προς τα πίσω.

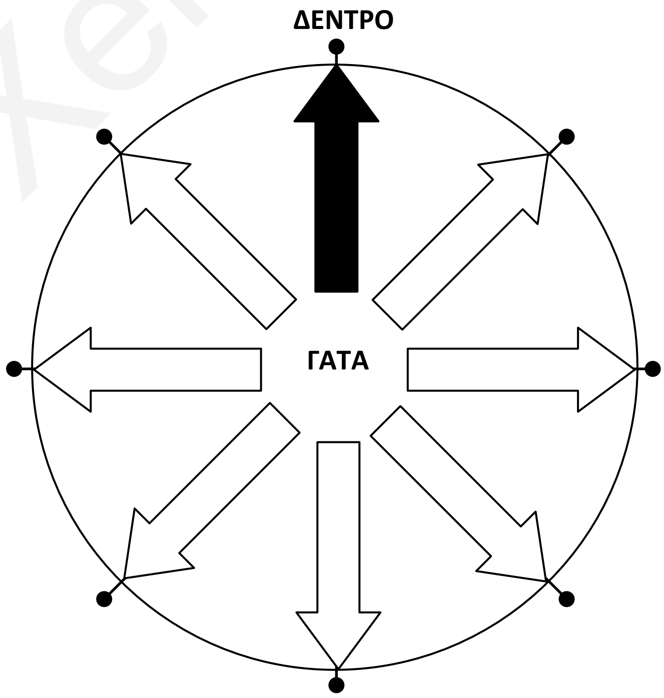
**ΓΥΡΙΣΕ ΣΕΛΙΔΑ ΓΙΑ ΤΙΣ ΑΣΚΗΣΕΙΣ**

**Ασκήσεις:**



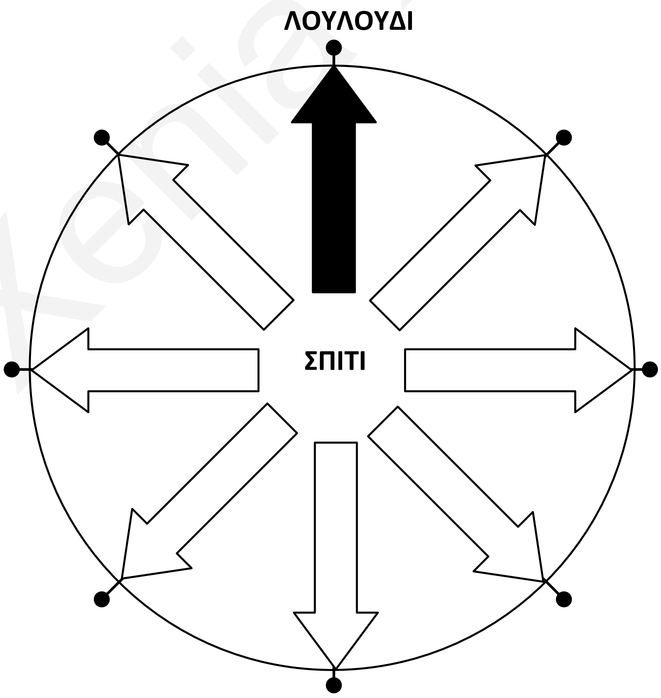
1. Να φανταστείς ότι **ΒΡΙΣΚΕΣΑΙ** στη **ΓΑΤΑ** και κοιτάζεις προς το **ΔΕΝΤΡΟ**.

Βάλε σε κύκλο **ΤΟ ΤΟΞΟ ΠΟΥ ΔΕΙΧΝΕΙ** προς ποια κατεύθυνση θα βρίσκεται η **ΠΙΝΑΚΙΔΑ STOP**.



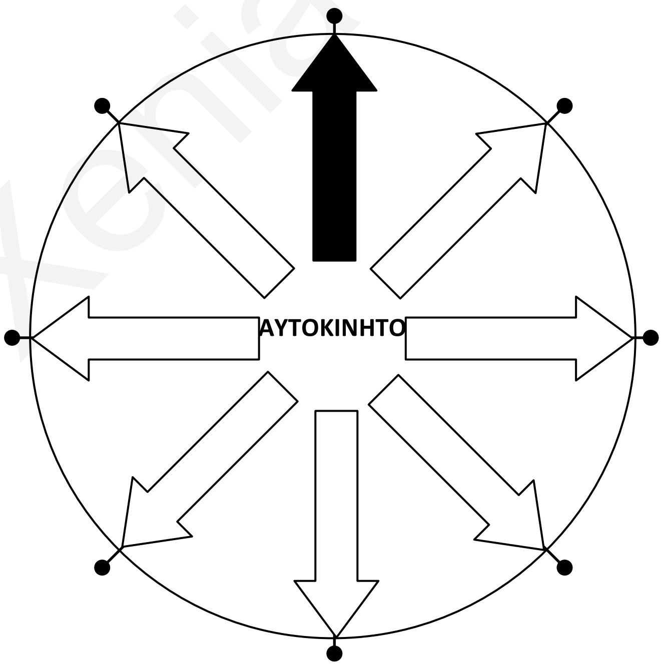

2. Να φανταστείς ότι **ΒΡΙΣΚΕΣΑΙ** στο **ΣΠΙΤΙ** και κοιτάζεις προς το **ΛΟΥΛΟΥΔΙ**.

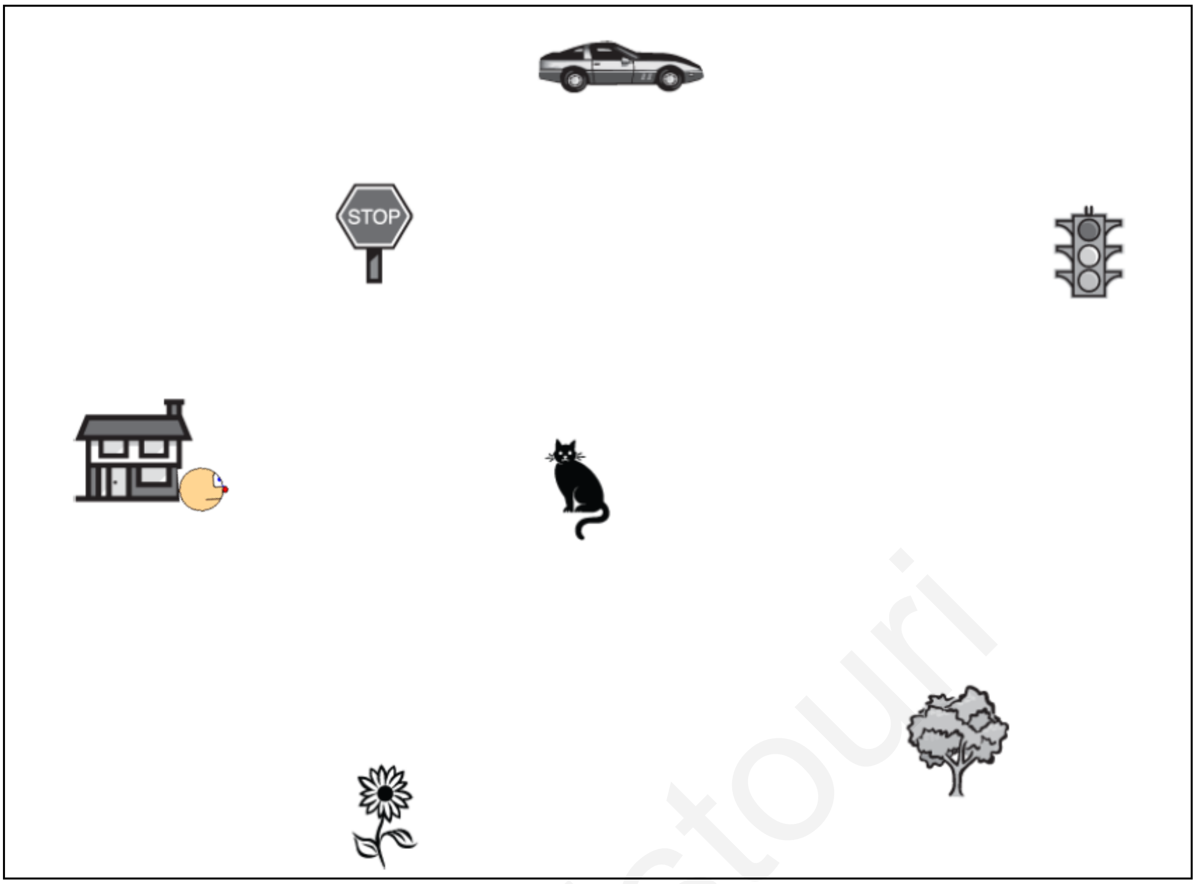
Βάλε σε κύκλο **ΤΟ ΤΟΞΟ ΠΟΥ ΔΕΙΧΝΕΙ** προς ποια κατεύθυνση θα βρίσκεται το **ΑΥΤΟΚΙΝΗΤΟ**.




3. Να φανταστείς ότι **ΒΡΙΣΚΕΣΑΙ** στο **ΑΥΤΟΚΙΝΗΤΟ** και κοιτάζεις προς τα **ΦΩΤΑ ΤΡΟΧΑΙΑΣ**.

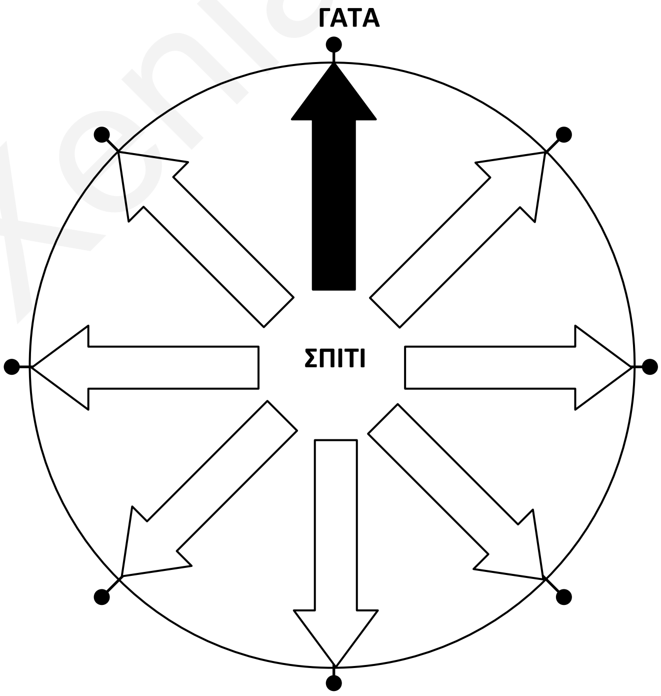
Βάλε σε κύκλο **ΤΟ ΤΟΞΟ ΠΟΥ ΔΕΙΧΝΕΙ** προς ποια κατεύθυνση θα βρίσκεται το **ΛΟΥΛΟΥΔΙ**.






4. Να φανταστείς ότι **ΒΡΙΣΚΕΣΑΙ** στο **ΣΠΙΤΙ** και κοιτάζεις προς τη **ΓΑΤΑ**.

Βάλε σε κύκλο **ΤΟ ΤΟΞΟ ΠΟΥ ΔΕΙΧΝΕΙ** προς ποια κατεύθυνση θα βρίσκεται το **ΛΟΥΛΟΥΔΙ**.

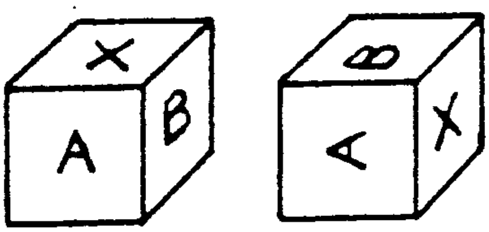


## ΜΕΡΟΣ Στ΄

Σ’ αυτή την άσκηση θα πρέπει κάθε φορά **ΝΑ ΣΥΓΚΡΙΝΕΙΣ** δύο κύβους. Σε **ΚΑΘΕ ΠΛΕΥΡΑ** του κάθε κύβου υπάρχει **ΕΝΑ ΔΙΑΦΟΡΕΤΙΚΟ ΣΥΜΒΟΛΟ**.

Πρέπει **ΝΑ ΦΑΝΤΑΣΤΕΙΣ ΟΤΙ ΠΕΡΙΣΤΡΕΦΕΙΣ** τον ένα από τους δύο κύβους, και να συγκρίνεις τα σύμβολα τους για να αποφασίσεις αν οι δύο κύβοι **ΜΠΟΡΕΙ** να είναι οι **ΙΔΙΟΙ**. Να κυκλώσεις τη λέξη «ΙΔΙΟΙ», αν ο πρώτος κύβος μπορεί να είναι ο ίδιος με το δεύτερο όταν περιστραφεί, και τη λέξη «ΔΙΑΦΟΡΕΤΙΚΟΙ» αν δεν μπορεί.

**ΠΑΡΑΔΕΙΓΜΑ:**

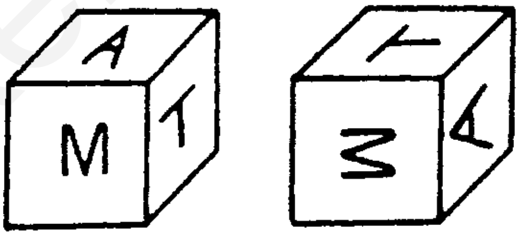



**ΙΔΙΟΙ** **ΔΙΑΦΟΡΕΤΙΚΟΙ**

Στο πιο πάνω **ΠΑΡΑΔΕΙΓΜΑ** οι κύβοι είναι **ΔΙΑΦΟΡΕΤΙΚΟΙ** γιατί όταν **ΠΕΡΙΣΤΡΕΨΟΥΜΕ** τον **<u>δεύτερο κύβο προς τα ΔΕΞΙΑ</u>** για να έχουμε το Α και το Β στην ίδια θέση με τον πρώτο κύβο, το Χ θα βρίσκεται στην **ΚΑΤΩ** πλευρά, ενώ αν ήταν οι ίδιοι θα έπρεπε να βρίσκεται στην **ΠΑΝΩ**.

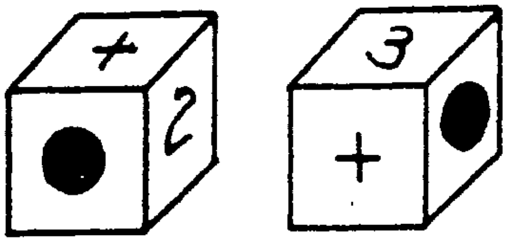
**Ασκήσεις:**

1.



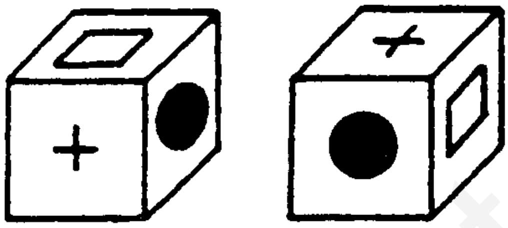

**ΙΔΙΟΙ** **ΔΙΑΦΟΡΕΤΙΚΟΙ**

2.



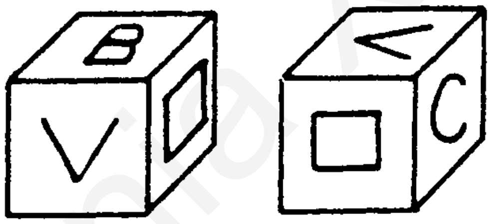

ΙΔΙΟΙ ΔΙΑΦΟΡΕΤΙΚΟΙ

3.



ΙΔΙΟΙ ΔΙΑΦΟΡΕΤΙΚΟΙ

4.

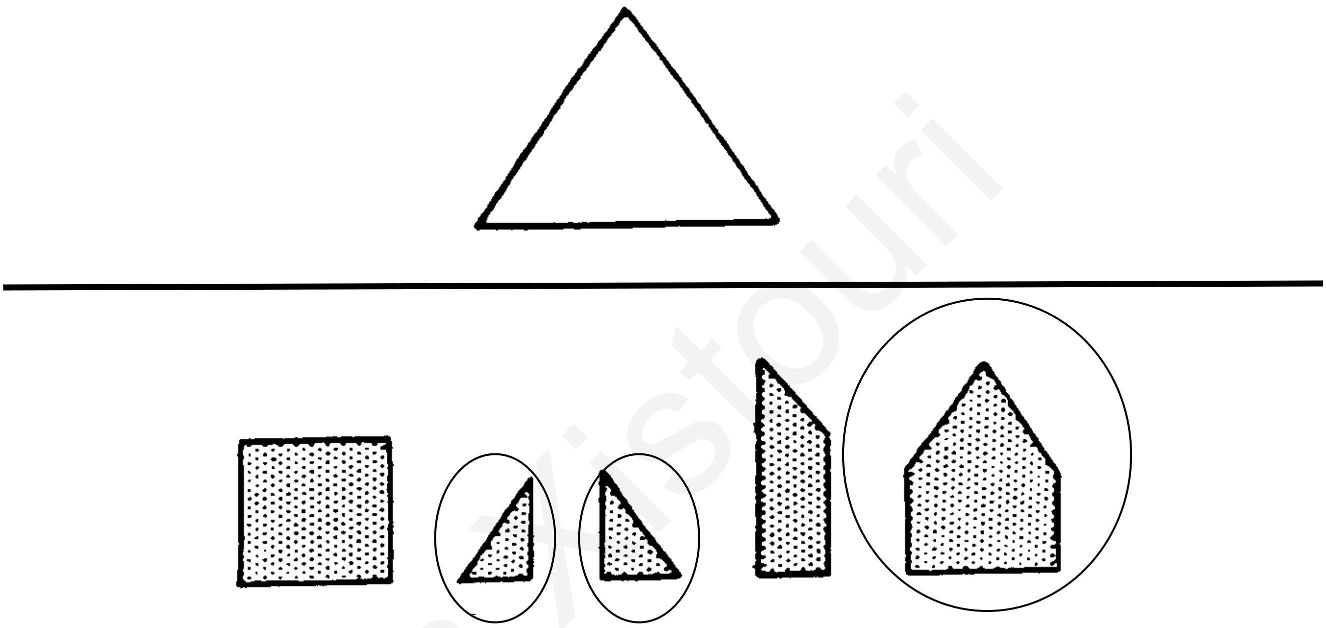



ΙΔΙΟΙ ΔΙΑΦΟΡΕΤΙΚΟΙ

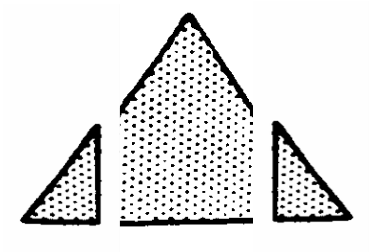
**ΜΕΡΟΣ Ζ΄**

Στο μέρος αυτό υπάρχει πάνω από τη γραμμή ένα σχήμα. Να βάλεις σε κύκλο ποια από τα σχήματα που βρίσκονται κάτω από τη γραμμή πρέπει να **<u>ΕΝΩΘΟΥΝ</u>** ώστε να κατασκευαστεί το σχήμα που βρίσκεται πάνω από αυτή. Τα σχήματα που βρίσκονται κάτω από τη γραμμή μπορούν **<u>μόνο να περιστραφούν</u>**.

**ΠΑΡΑΔΕΙΓΜΑ:**



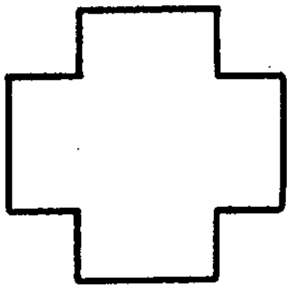
Η σωστή απάντηση σε αυτό το **ΠΑΡΑΔΕΙΓΜΑ** όταν ενωθούν τα τρία σχήματα που έχουν κυκλωθεί για να σχηματιστεί το τρίγωνο, φαίνεται πιο κάτω:

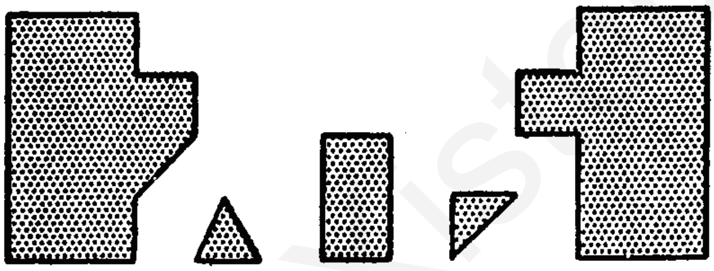


**ΓΥΡΙΣΕ ΣΕΛΙΔΑ ΓΙΑ ΤΙΣ ΑΣΚΗΣΕΙΣ**

**Ασκήσεις:**

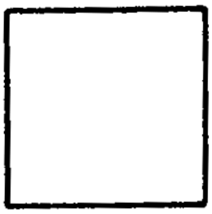
1. Για να κατασκευαστεί ο σταυρός πρέπει να ενωθούν **ΤΡΙΑ** από τα πέντε σχήματα που υπάρχουν κάτω από τη γραμμή. Να βάλεις τα σχήματα αυτά σε κύκλο.

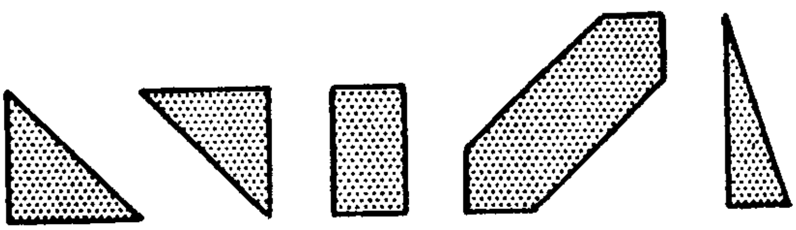




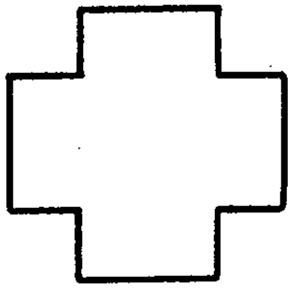


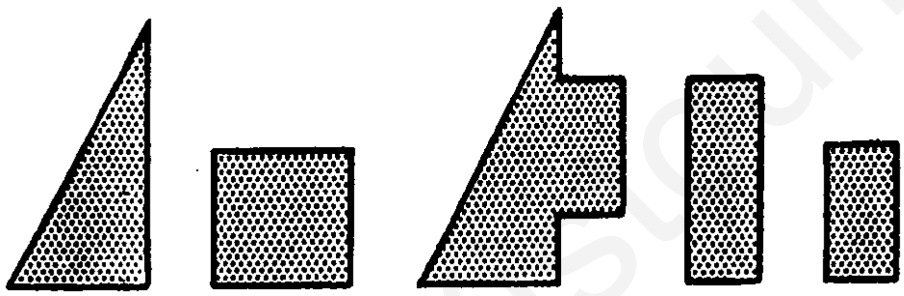
2. Για να κατασκευαστεί το τετράγωνο πρέπει να ενωθούν **ΤΡΙΑ** από τα πέντε σχήματα που υπάρχουν κάτω από τη γραμμή. Να βάλεις τα σχήματα αυτά σε κύκλο.



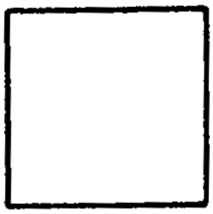


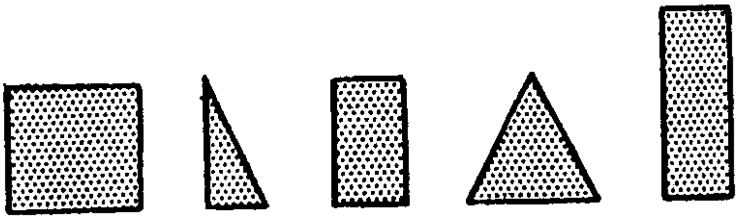
3. Για να κατασκευαστεί ο σταυρός πρέπει να ενωθούν **ΤΡΙΑ** από τα πέντε σχήματα που υπάρχουν κάτω από τη γραμμή. Να βάλεις τα σχήματα αυτά σε κύκλο.





4. Για να κατασκευαστεί το τετράγωνο πρέπει να ενωθούν **ΤΡΙΑ** από τα πέντε σχήματα που υπάρχουν κάτω από τη γραμμή. Να βάλεις τα σχήματα αυτά σε κύκλο.

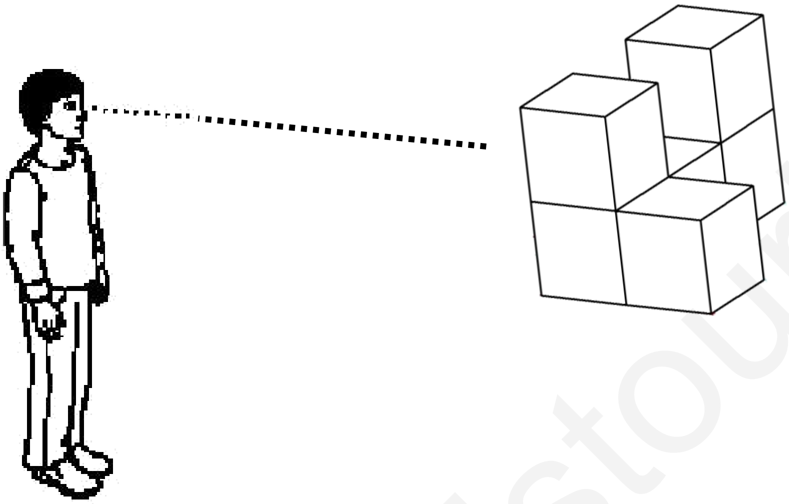


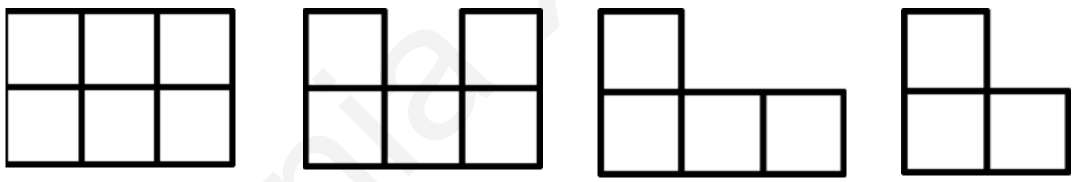


**ΜΕΡΟΣ Η΄**

Στο μέρος αυτό παρουσιάζεται ένα παιδί να **ΚΟΙΤΑΖΕΙ** προς ένα στερεό. Η διακεκομμένη γραμμή δείχνει την **ΚΑΤΕΥΘΥΝΣΗ** του βλέμματός του. Οι τέσσερις εικόνες που υπάρχουν κάτω από τη γραμμή δείχνουν **ΠΟΙΑ ΕΙΚΟΝΑ** θα μπορούσε να έχει μπροστά του ο άνθρωπος **ΑΠΟ ΤΗ ΘΕΣΗ ΠΟΥ ΒΡΙΣΚΕΤΑΙ**. Να βάλεις σε κύκλο τη σωστή.

**ΠΑΡΑΔΕΙΓΜΑ:**





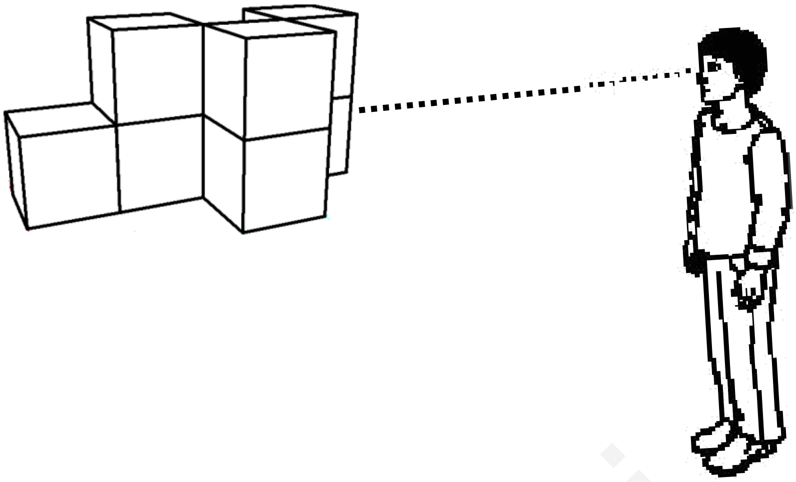
Α Β Γ Δ

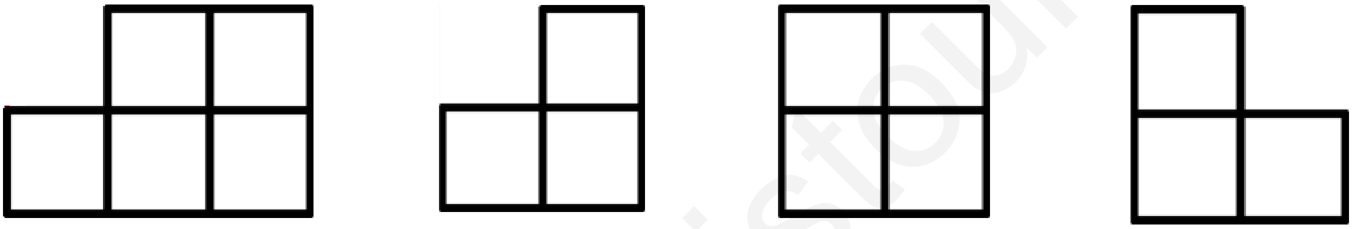


**ΓΥΡΙΣΕ ΣΕΛΙΔΑ ΓΙΑ ΤΙΣ ΑΣΚΗΣΕΙΣ**

**Ασκήσεις:**

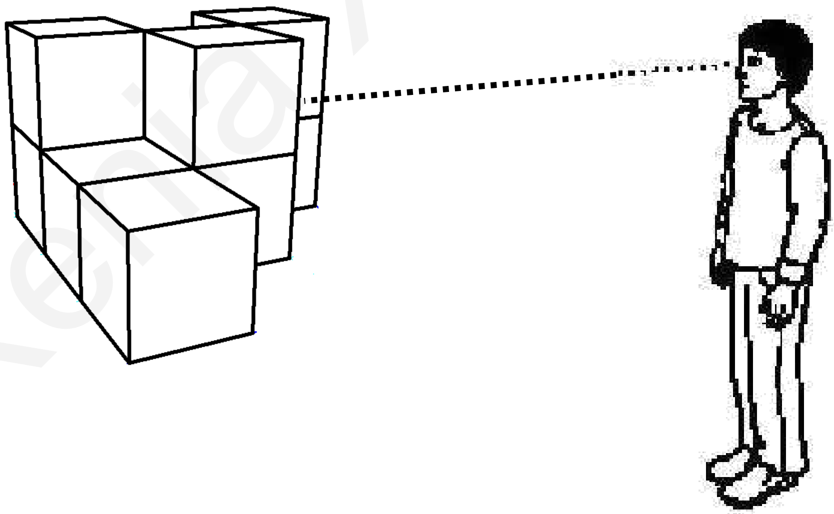
1.

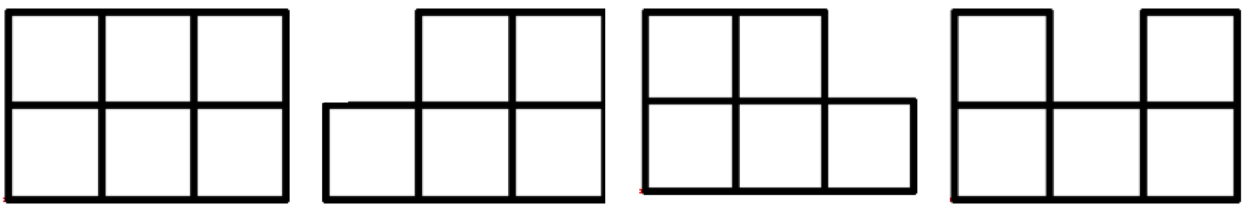




Α Β Γ Δ

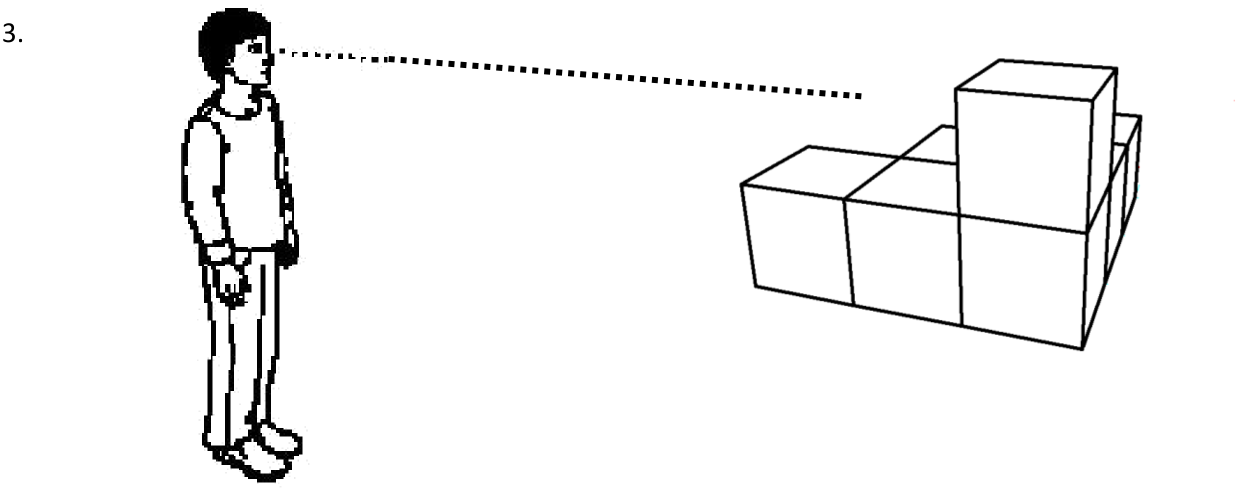
2.

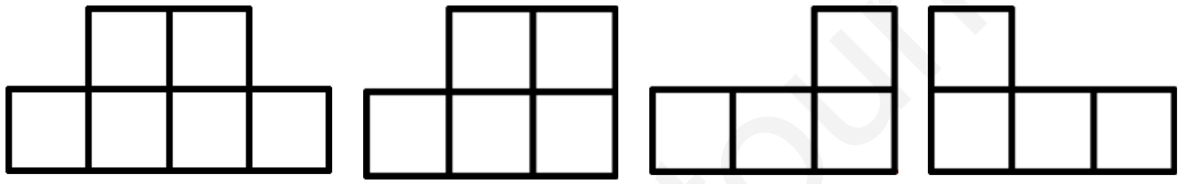




Α Β Γ Δ

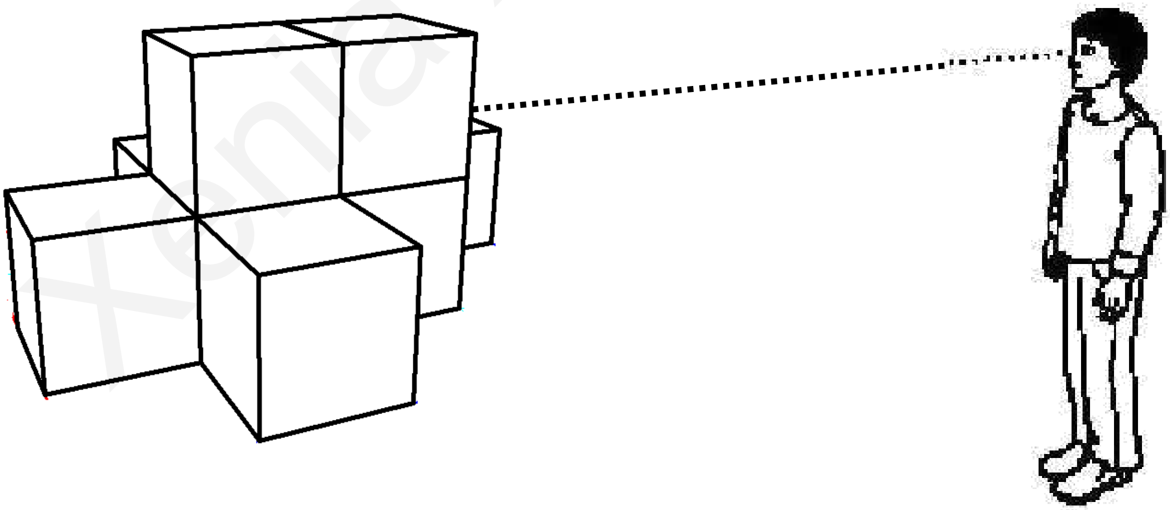
3.

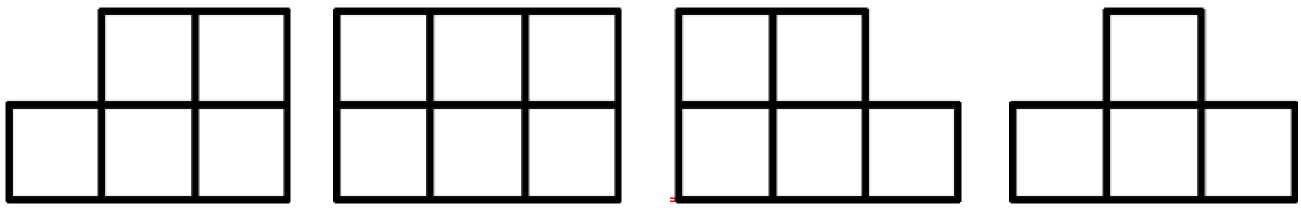




Α Β Γ Δ

4.





Α Β Γ Δ

# ΕΡΩΤΗΜΑΤΟΛΟΓΙΟ

**Όνομα: ............................................... Τάξη: .......**
**Αρ. στον κατάλογο: ... Σχολείο: ...................................**

**<u>Ποιο είναι το γνωστικό σου στυλ;</u>**

Διάβασε τις πιο κάτω προτάσεις και δήλωσε πόσο συμφωνείς ή διαφωνείς με αυτό που λέει η πρόταση, κυκλώνοντας ένα αριθμό από το 1 μέχρι το 5. Να κυκλώσεις τον αριθμό «1» για να δείξεις ότι **διαφωνείς** απόλυτα με την πρόταση και τον αριθμό «5» για να δείξεις ότι **συμφωνείς** απόλυτα με την πρόταση που σε περιγράφει. Να χρησιμοποιήσεις τις ενδιάμεσες τιμές 2, 3 και 4 για να δείξεις τον ενδιάμεσο βαθμό στον οποίο σε περιγράφουν οι προτάσεις.

Δεν υπάρχουν σωστές ή λανθασμένες απαντήσεις σε αυτό το ερωτηματολόγιο, γι' αυτό προσπάθησε να είσαι όσο πιο ειλικρινής γίνεται.

Είναι πολύ σημαντικό να απαντήσεις σε όλες τις ερωτήσεις αυτού του ερωτηματολογίου.

Σε ευχαριστούμε πολύ για τη συνεργασία!

| | | **Διαφωνώ απόλυτα** ↓ | | | | **Συμφωνώ απόλυτα** ↓ |
|---|---|---|---|---|---|---|
| 1 | Όταν ζωγραφίζω, μου αρέσει να επιλέγω συγκεκριμένες αποχρώσεις χρωμάτων και να αναμιγνύω χρώματα. | **1** | **2** | **3** | **4** | **5** |
| 2 | Αν κλείσω τα μάτια μου, μπορώ εύκολα να φανταστώ μια εικόνα, μια σκηνή ή ένα πρόσωπο που έχω ξαναδεί. | **1** | **2** | **3** | **4** | **5** |
| 3 | Οι ικανότητές μου στο να σχεδιάζω σκίτσα θα μου επέτρεπαν να γίνω καλός αρχιτέκτονας. | **1** | **2** | **3** | **4** | **5** |
| 4 | Καταλαβαίνω πώς είναι κατασκευασμένο το εσωτερικό ενός ηλεκτρονικού υπολογιστή. | **1** | **2** | **3** | **4** | **5** |
| 5 | Είμαι καλός/ή στο να επιδιορθώνω συσκευές. | **1** | **2** | **3** | **4** | **5** |
| 6 | Όταν φαντάζομαι το πρόσωπο ενός φίλου μου, έχω μια εντελώς ξεκάθαρη και φωτεινή εικόνα του. | **1** | **2** | **3** | **4** | **5** |
| 7 | Το να γράφω είναι εύκολο για εμένα και απολαμβάνω να το κάνω. | **1** | **2** | **3** | **4** | **5** |
| 8 | Είναι εύκολο για εμένα να καταλάβω πώς να συνδέσω διάφορες συσκευές με καλώδια. | **1** | **2** | **3** | **4** | **5** |
| 9 | Είναι εύκολο για μένα να μαθαίνω νέες γλώσσες. | **1** | **2** | **3** | **4** | **5** |
| 10 | Μπορώ να συνδέσω δύο ηλεκτρονικές συσκευές (για παράδειγμα, μια τηλεόραση και μια συσκευή DVD), ακόμα και χωρίς το βιβλιαράκι με τις οδηγίες. | **1** | **2** | **3** | **4** | **5** |
| 11 | Είμαι πολύ καλός/ή σε παιχνίδια κατασκευών (παιχνίδια στα οποία κάνεις κατασκευές με τουβλάκια και χαρτί). | **1** | **2** | **3** | **4** | **5** |
| 12 | Όταν γράφω κάνω ελάχιστα ή καθόλου γραμματικά ή ορθογραφικά λάθη. | **1** | **2** | **3** | **4** | **5** |

| | | | | | | |
|---|---|---|---|---|---|---|
| 13 | Διαβάζω πολλά μυθιστορήματα. | **1** | **2** | **3** | **4** | **5** |
| 14 | Είμαι καλός στο να εκφράζομαι γραπτώς. | **1** | **2** | **3** | **4** | **5** |
| 15 | Οι λεκτικές μου ικανότητες θα μου επέτρεπαν να γίνω ένας καλός συγγραφέας. | **1** | **2** | **3** | **4** | **5** |
| 16 | Έχω καλή οπτική μνήμη. | **1** | **2** | **3** | **4** | **5** |
| 17 | Είμαι καλύτερος στο να σχεδιάζω σχηματικά σκίτσα, όπως αρχιτεκτονικά σχέδια, αντί να ζωγραφίζω με χρώματα. | **1** | **2** | **3** | **4** | **5** |
| 18 | Είμαι καλός στο να παίζω παιχνίδια που απαιτούν αίσθηση του χώρου (π.χ. Λέγκο, Τέτρις και Οριγκάμι). | **1** | **2** | **3** | **4** | **5** |
| 19 | Οι οπτικές μου εικόνες είναι πολύ ζωηρές. | **1** | **2** | **3** | **4** | **5** |
| 20 | Μου αρέσει να φαντάζομαι διαφορετικές εικόνες/σκηνές με χρώματα και λεπτομέρειες. | **1** | **2** | **3** | **4** | **5** |
| 21 | Προσέχω πότε οι φίλοι μου φοράνε καινούρια ρούχα. | **1** | **2** | **3** | **4** | **5** |
| 22 | Είμαι γρήγορος στην ανάγνωση. | **1** | **2** | **3** | **4** | **5** |
| 23 | Οι οπτικές μου εικόνες είναι πολύ χρωματιστές και φωτεινές. | **1** | **2** | **3** | **4** | **5** |
| 24 | Είμαι πολύ περίεργος για το πώς κατασκευάζονται διάφορες μηχανές. | **1** | **2** | **3** | **4** | **5** |
| 25 | Είναι εύκολο για μένα να λύνω προβλήματα γεωμετρίας. | **1** | **2** | **3** | **4** | **5** |
| 26 | Όταν διαβάζω ένα βιβλίο του σχολείου, προτιμώ έγχρωμες εικονογραφήσεις. | **1** | **2** | **3** | **4** | **5** |
| 27 | Θυμάμαι με ευκολία τους στίχους ποιημάτων και τραγουδιών. | **1** | **2** | **3** | **4** | **5** |
| 28 | Είμαι καλός στο να παίζω τρισδιάστατα βιντεοπαιχνίδια δράσης (για παράδειγμα σκοποβολή, προσομοιώσεις οδήγησης αεροπλάνου/αυτοκινήτου, ή διαφυγής του παίκτη από λαβύρινθο). | **1** | **2** | **3** | **4** | **5** |
| 29 | Είναι εύκολο για μένα να καταλαβαίνω πώς να συνδέσω μια καινούρια συσκευή στον ηλεκτρονικό υπολογιστή. | **1** | **2** | **3** | **4** | **5** |
| 30 | Μου αρέσει να διαβάζω ποίηση. | **1** | **2** | **3** | **4** | **5** |
| 31 | Μπορώ με ευκολία να φανταστώ και να περιστρέψω τρισδιάστατα σχήματα στο μυαλό μου. | **1** | **2** | **3** | **4** | **5** |
| 32 | Είμαι καλός/ή στο να λύνω προβλήματα γεωμετρίας με τρισδιάστατα σχήματα. | **1** | **2** | **3** | **4** | **5** |
| 33 | Όταν ακούω τις ιστορίες των φίλων μου, συνήθως φαντάζομαι ζωηρές και έγχρωμες εικόνες, λες και οι ιστορίες συνέβησαν σε μένα. | **1** | **2** | **3** | **4** | **5** |
| 34 | Γράφω πολύ περίπλοκες εκθέσεις/ιστορίες. | **1** | **2** | **3** | **4** | **5** |
| 35 | Λέω ιστορίες καλύτερα από πολλούς άλλους ανθρώπους. | **1** | **2** | **3** | **4** | **5** |

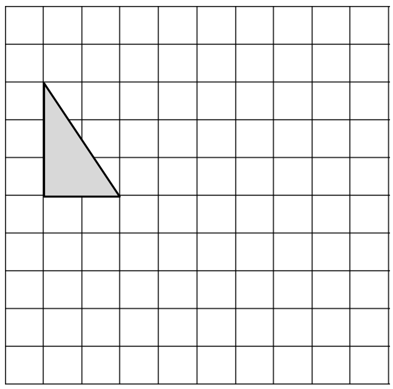
| | | | | | | |
|---|---|---|---|---|---|---|
| 36 | Όταν διαβάζω ένα βιβλίο, μπορώ συνήθως να φανταστώ ξεκάθαρες χρωματιστές εικόνες των ανθρώπων και των τοπίων. | 1 | 2 | 3 | 4 | 5 |
| 37 | Όταν διαβάζω ένα βιβλίο, φαντάζομαι τα πάντα πολύ φωτεινά και καθαρά, όπως σε μια ταινία. | 1 | 2 | 3 | 4 | 5 |
| 38 | Οι οπτικές μου εικόνες μοιάζουν με έγχρωμες φωτογραφίες, ή εικόνες πραγματικών αντικειμένων και σκηνών. | 1 | 2 | 3 | 4 | 5 |
| 39 | Είμαι γρήγορος στο γράψιμο. | 1 | 2 | 3 | 4 | 5 |
| 40 | Μου αρέσει να συναρμολογώ και να αποσυναρμολογώ διάφορες συσκευές. | 1 | 2 | 3 | 4 | 5 |
| 41 | Έχω φωτογραφική φαντασία, δηλαδή, μπορώ να φαντάζομαι πράγματα ξεκάθαρα και με λεπτομέρεια όπως στις φωτογραφίες. | 1 | 2 | 3 | 4 | 5 |
| 42 | Μου αρέσουν τα γλωσσικά παιχνίδια (παιχνίδια με λέξεις, προτάσεις και γράμματα, για παράδειγμα Σκράμπλ ή Κρεμάλα). | 1 | 2 | 3 | 4 | 5 |
| 43 | Οι οπτικές μου εικόνες είναι στο μυαλό μου συνεχώς. Είναι απλά εκεί. | 1 | 2 | 3 | 4 | 5 |
| 44 | Συνήθως, οι εκθέσεις μου είναι αρκετά μεγάλες. | 1 | 2 | 3 | 4 | 5 |
| 45 | Όταν γράφω, μου αρέσει να σκέφτομαι διαφορετικούς τρόπους για να διατυπώσω ή να εκφράσω ιδέες. | 1 | 2 | 3 | 4 | 5 |

# ΗΜΙΔΟΜΗΜΕΝΗ ΣΥΝΕΝΤΕΥΞΗ

## ΕΡΓΟ Α1

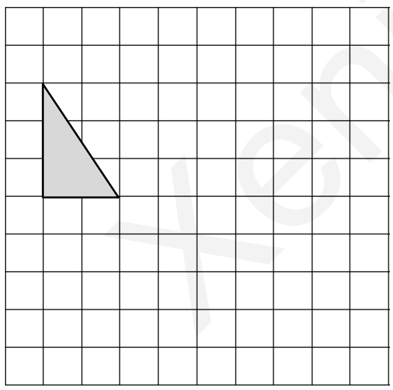
Βρες τη νέα θέση του σχήματος και σχεδίασε το, σύμφωνα με τις οδηγίες.

**1)** Όταν το τρίγωνο μεταφερθεί 4 κουτάκια προς τα δεξιά.





**2)** Όταν μεταφερθεί 1 κουτάκι προς τα δεξιά.



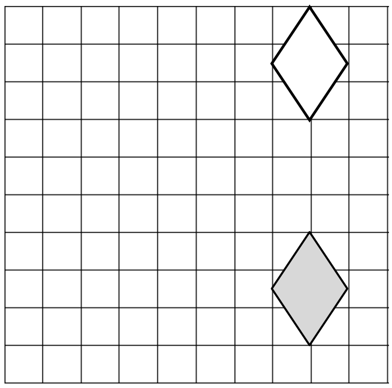
Τι σκέφτηκες για να σχεδιάσεις την απάντησή σου εκεί;

Πώς είσαι σίγουρος/η ότι αυτό που έκανες είναι μεταφορά;

Τι είναι αυτό που αλλάζει ανάμεσα στο αρχικό τρίγωνο και στο νέο τρίγωνο όταν γίνεται μεταφορά; Τι είναι αυτό που δεν αλλάζει;

**ΕΡΓΟ Α2**

Γράψε τι πρέπει να κάνει κάποιος για να μεταφέρει το χρωματισμένο σχήμα στη νέα θέση.



**Μετάφερε το χρωματισμένο σχήμα**……………………….

……………………………………………………………………………….

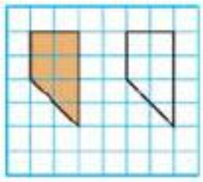
……………………………………………………………………………….

Τι σκέφτηκες για να βρεις την απάντηση;

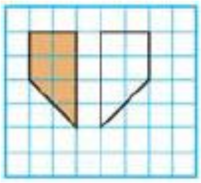
Πώς είσαι σίγουρος ότι αυτή είναι η σωστή απάντηση;

## ΕΡΓΟ Α3

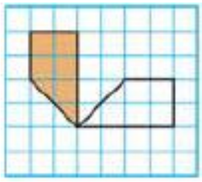
Ποια από τις πιο κάτω εικόνες παρουσιάζει τη μεταφορά του χρωματισμένου σχήματος;



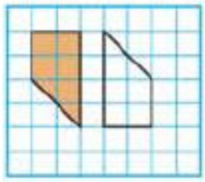
Α



Β



Γ



Δ

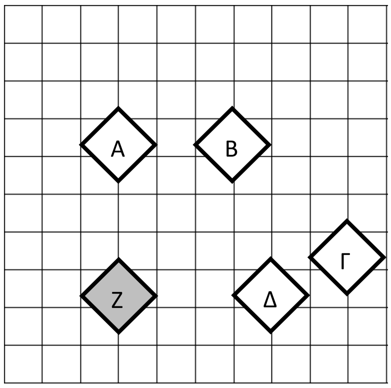
Τι σκέφτηκες για να καταλήξεις σε αυτή την απάντηση;

Τι σκέψεις έκανες για να επιλέξεις ή για να αποκλείσεις κάποια απάντηση;

Γιατί αποφάσισες ότι αυτή η επιλογή δεν παρουσιάζει μεταφορά;

## ΕΡΓΟ Α4

Ποιο από τα πιο κάτω αποτελεί μεταφορά του αρχικού σχεδίου Ζ, όταν μεταφερθεί 4 κουτάκια προς τα δεξιά;




α) Το Α

β) Το Β

γ) Το Γ

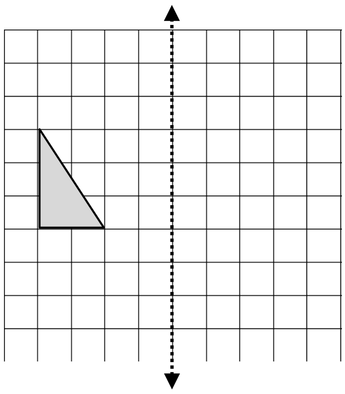
δ) Το Δ

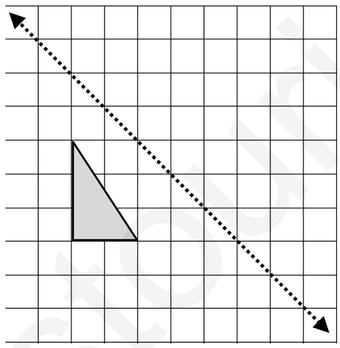
Τι σκέφτηκες για να καταλήξεις σε αυτή την απάντηση;

Τι σκέψεις έκανες για να επιλέξεις ή για να αποκλείσεις κάποια απάντηση;

**ΕΡΓΟ Β1**

Βρες την ανάκλαση και σχεδίασε το συμμετρικό του κάθε αρχικού σχήματος που δίνεται, χρησιμοποιώντας κάθε φορά τη διακεκομμένη γραμμή συμμετρίας.





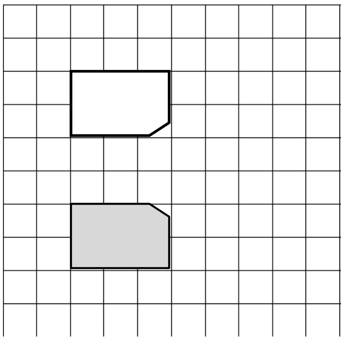
Τι σκέφτηκες για να σχεδιάσεις την απάντησή σου εκεί;

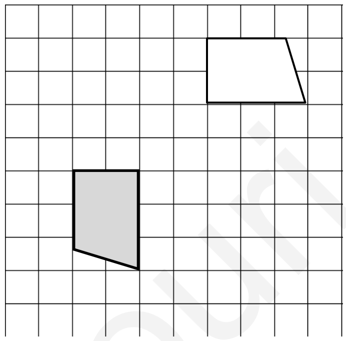
Πώς είσαι σίγουρος/η ότι αυτό που έκανες είναι ανάκλαση;

Τι είναι αυτό που αλλάζει ανάμεσα στο αρχικό τρίγωνο και στο νέο τρίγωνο όταν γίνεται μεταφορά; Τι είναι αυτό που δεν αλλάζει;

**ΕΡΓΟ Β2**

Βρες και <u>χάραξε</u> με τη ρίγα σου τη <u>γραμμή συμμετρίας</u> για κάθε περίπτωση.



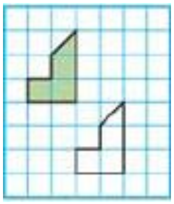


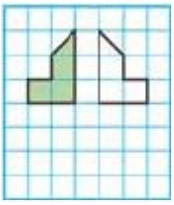
Τι σκέφτηκες για να βρεις την απάντηση;

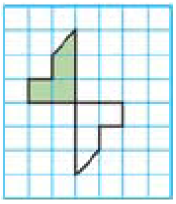
Πώς είσαι σίγουρος ότι αυτή είναι η σωστή απάντηση;

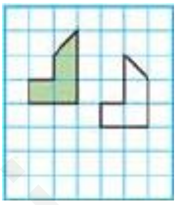
**ΕΡΓΟ Β3**

Ποια από τις πιο κάτω εικόνες παρουσιάζει την ανάκλαση του χρωματισμένου σχήματος;









Α  Β  Γ  Δ

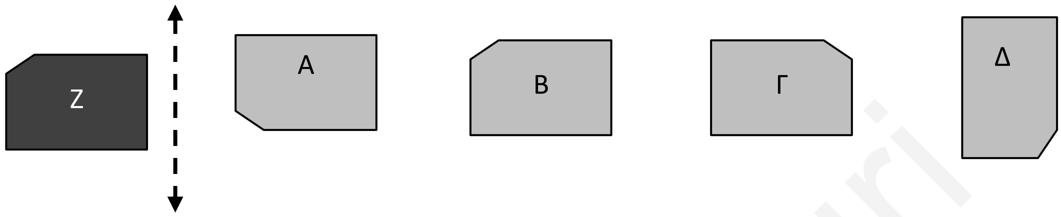
Τι σκέφτηκες για να καταλήξεις σε αυτή την απάντηση;

Τι σκέψεις έκανες για να επιλέξεις ή για να αποκλείσεις κάποια απάντηση;

Γιατί αποφάσισες ότι αυτή η επιλογή δεν παρουσιάζει ανάκλαση;

**ΕΡΓΟ Β4**

Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Ζ ως προς κατακόρυφη γραμμή συμμετρίας;



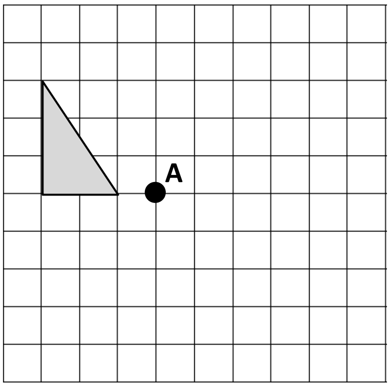

Τι σκέφτηκες για να καταλήξεις σε αυτή την απάντηση;

Τι σκέψεις έκανες  για να επιλέξεις ή για να αποκλείσεις κάποια απάντηση;

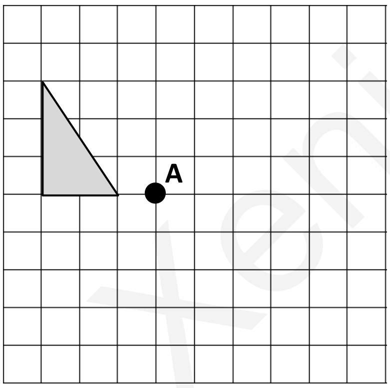
**ΕΡΓΟ Γ1**

Σχεδίασε το σχήμα στη νέα του θέση, όταν κάνει στροφή (γυρίσει) γύρω από το σημείο Α, σύμφωνα με τις οδηγίες που δίνονται πάνω από κάθε περίπτωση.

**1)** Όταν κάνει ¼ της στροφής προς τα δεξιά .




**2)** Όταν κάνει 2/4 της στροφής προς τα δεξιά .



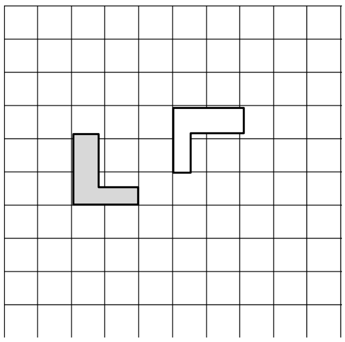

Τι σκέφτηκες για να σχεδιάσεις την απάντησή σου εκεί;

Πώς είσαι σίγουρος/η ότι αυτό που έκανες είναι ανάκλαση;

Τι είναι αυτό που αλλάζει ανάμεσα στο αρχικό τρίγωνο και στο νέο τρίγωνο όταν γίνεται μεταφορά; Τι είναι αυτό που δεν αλλάζει;

**ΕΡΓΟ Γ2**

1. Βρες και ΣΧΕΔΙΑΣΕ το ΣΗΜΕΙΟ Α γύρω από το οποίο έκανε στροφή το χρωματισμένο σχήμα, και στη συνέχεια
2. Βάλε σε κύκλο το κλάσμα που δείχνει πόση στροφή προς τα δεξιά έκανε το σχήμα.

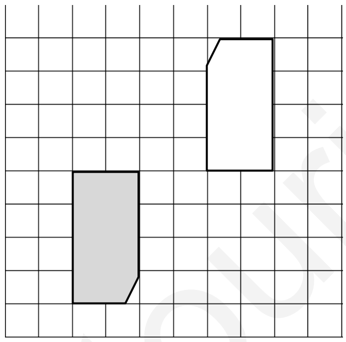


1) **Να σχεδιάσεις** την **ΤΕΛΕΙΑ Α** γύρω από την οποία έκανε στροφή το σχήμα.

2) **Βάλε σε κύκλο** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

1/4 2/4 3/4



1) **Να σχεδιάσεις** την **ΤΕΛΕΙΑ Α** γύρω από την οποία έκανε στροφή το σχήμα.

2) **Βάλε σε κύκλο** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

1/4 2/4 3/4



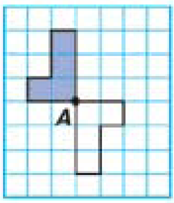
Τι σκέφτηκες πρώτα για να βρεις το σημείο Α;

Τι σκέφτηκες για να αποφασίσεις πόση στροφή έκανε το σχήμα;

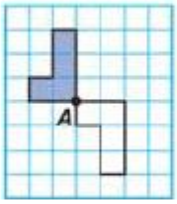
Πώς είσαι σίγουρος ότι αυτή είναι η σωστή απάντηση;

**ΕΡΓΟ Γ3**

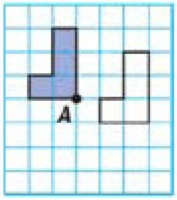
Ποια από τις πιο κάτω εικόνες παρουσιάζει την περιστροφή του χρωματισμένου σχήματος;



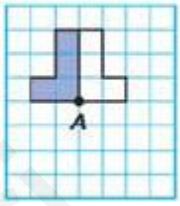

Α




Β




Γ




Δ

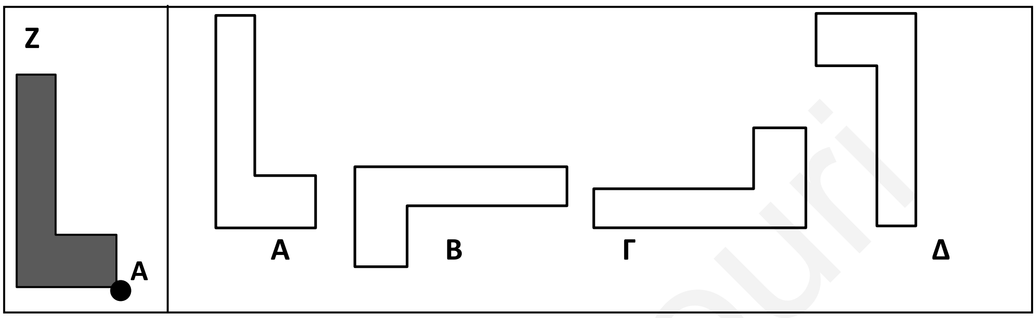
Τι σκέφτηκες για να καταλήξεις σε αυτή την απάντηση;

Τι σκέψεις έκανες για να επιλέξεις ή για να αποκλείσεις κάποια απάντηση;

Γιατί αποφάσισες ότι αυτή η επιλογή δεν παρουσιάζει περιστροφή;

**ΕΡΓΟ Γ4**

Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά ¼ του κύκλου προς τα δεξιά;




Τι σκέφτηκες για να καταλήξεις σε αυτή την απάντηση;

Τι σκέψεις έκανες για να επιλέξεις ή για να αποκλείσεις κάποια απάντηση;

APPENDIX III

STUDENT WORKSHEETS FOR INSTRUCTIONAL INTERVENTIONS

INSTRUCTIONAL INTERVETION WITH DISCRETE DYNAMIC VISUALISATION WORKSHEETS

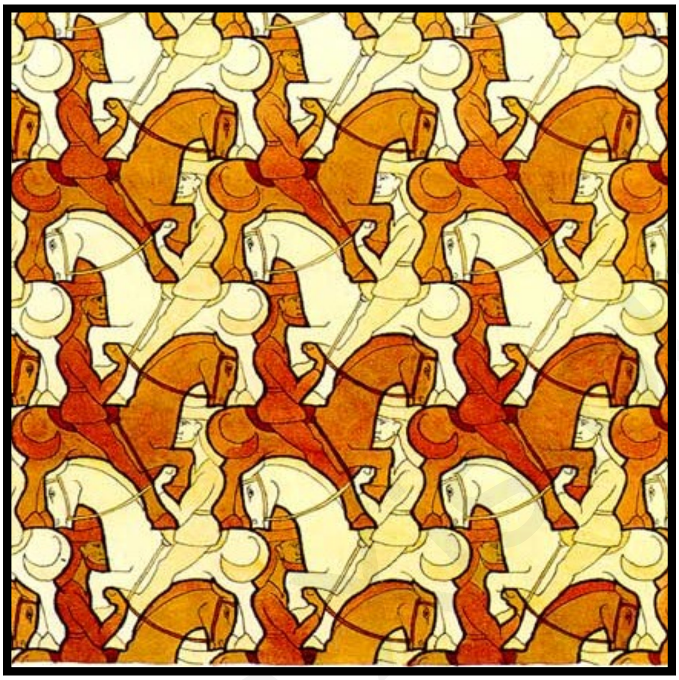
# ΜΑΘΗΜΑ 1: ΕΙΣΑΓΩΓΗ ΣΤΟΥΣ ΓΕΩΜΕΤΡΙΚΟΥΣ ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΥΣ

**Δραστηριότητα 1**

*Εξερευνώντας το σχηματισμό των ψηφιδωτών.*

Ένα Ολλανδός ζωγράφος, ο M.C. Escher ( 1898-1972 ), έφτιαχνε ζωγραφικούς πίνακες χρησιμοποιώντας διάφορα μοτίβα, τα οποία είναι γνωστά ως ψηφιδωτά.

Να παρατηρήσεις τους πιο κάτω πίνακες και να προσπαθήσεις να περιγράψεις με λεπτομέρειες τα διαφορετικά μοτίβα που παρατηρείς. Να γράψεις λίγα λόγια ξεχωριστά για κάθε πίνακα.



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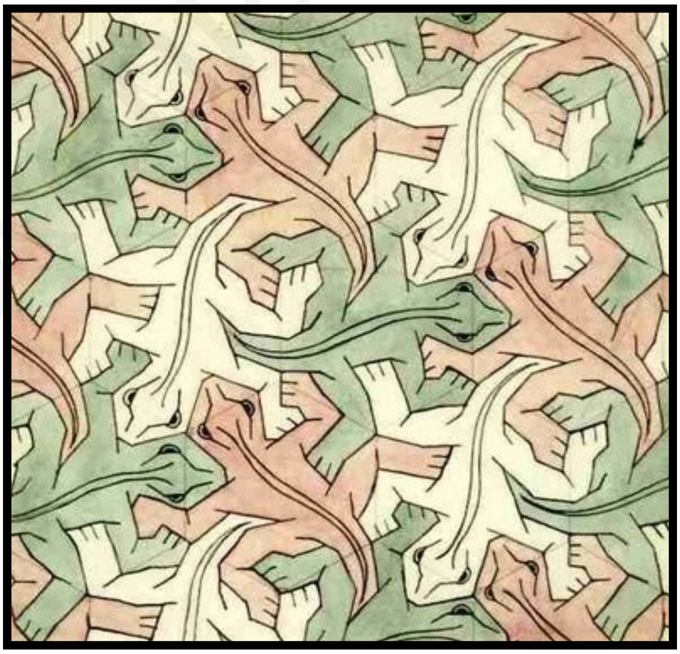

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Ο Escher μετακινούσε τις φιγούρες σύμφωνα με τους κανόνες των γεωμετρικών μετασχηματισμών της μεταφοράς, της ανάκλασης και της περιστροφής, τους οποίους θα γνωρίσουμε στις επόμενες εργασίες.

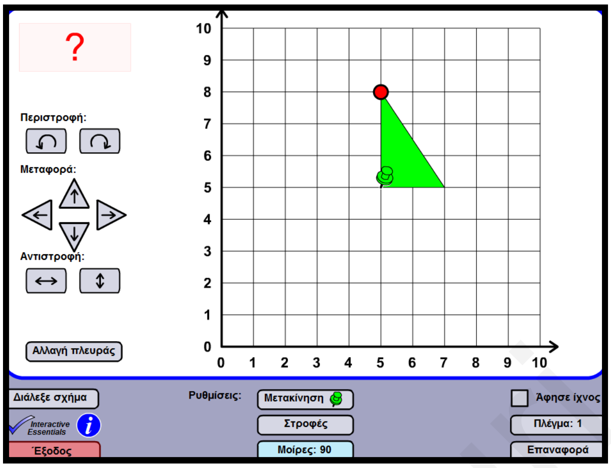
Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

## Δραστηριότητα 2

*Διερευνώντας τη μεταφορά.*

1. Να επιλέξεις το λογισμικό Μαθαίνω Γεωμετρία και Μετρώ.
2. Από το μενού δραστηριοτήτων, να επιλέξεις το Περιστροφή, Μεταφορά, Αντιστροφή, ώστε να ανοίξει το παράθυρο που βλέπεις στην Εικόνα 1.

Εικόνα 1

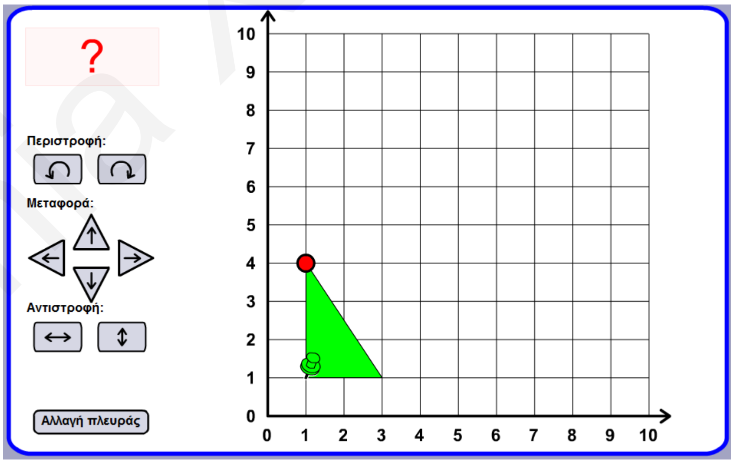



3. Χρησιμοποιώντας **<u>μόνο</u>** τα εικονίδια 

προσπάθησε να τοποθετήσεις το τρίγωνο στη θέση που βλέπεις στην Εικόνα 2.

Κάθε φορά που θες να αρχίσεις μια καινούρια προσπάθεια, να επιλέγεις το εικονίδιο


Εικόνα 2




4. Να σχεδιάσεις στο πιο κάτω ορθογώνιο <u>με τη σειρά</u> τα εικονίδια  που επέλεξες για να βρεις την απάντηση.

5. Να απαντήσεις τις πιο κάτω ερωτήσεις:

➢Τι συμβαίνει στο τρίγωνο κάθε φορά που επιλέγεις το εικονίδιο ⚠ ;

……………………………………………………………………………………

➢Τι συμβαίνει στο τρίγωνο κάθε φορά που επιλέγεις το εικονίδιο ▷ ;

……………………………………………………………………………………

6. Να διαβάσεις το πιο κάτω κείμενο.

Μια φιγούρα μπορεί να **απεικονιστεί σε μια νέα θέση**. Αυτό στα μαθηματικά ονομάζεται ***μεταφορά***. .
Η αρχική φιγούρα ονομάζεται ***πρότυπο***. Το αποτέλεσμα της μεταφοράς ονομάζεται ***εικόνα***.

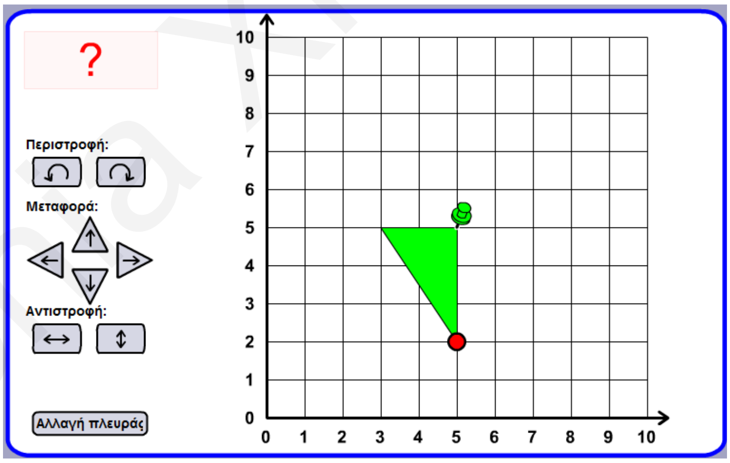
**Δραστηριότητα 3**

*Διερευνώντας την ανάκλαση.*

1. Να επιλέξεις το εικονίδιο **Επαναφορά** ώστε το τρίγωνο να τοποθετηθεί στη θέση που φαίνεται στην Εικόνα 1.

2. Να χρησιμοποιήσεις <u>**μόνο**</u> τα εικονίδια ↔ ↕ για να τοποθετήσεις το τρίγωνο στη θέση που βλέπεις στην Εικόνα 3.

Εικόνα 3




4. Να σχεδιάσεις στο πιο κάτω ορθογώνιο με τη σειρά τα εικονίδια ↔ ↕ που επέλεξες για να βρεις την απάντηση.

Κάθε φορά που θες να αρχίσεις μια καινούρια προσπάθεια, να επιλέγεις το εικονίδιο **Επαναφορά**

5. Να απαντήσεις τις πιο κάτω ερωτήσεις:

➢Τι συμβαίνει στο τρίγωνο κάθε φορά που επιλέγεις το εικονίδιο ↔ ;

……………………………………………………………………………………………

➢Τι συμβαίνει στο τρίγωνο κάθε φορά που επιλέγεις το εικονίδιο ↕ ;

……………………………………………………………………………………………

7. Να διαβάσεις το πιο κάτω κείμενο.

Μια φιγούρα μπορεί **να αντιστραφεί** ως προς ένα άξονα συμμετρίας που λειτουργεί ως καθρέφτης. Αυτό στα μαθηματικά ονομάζεται ***ανάκλαση***. Όπως και στους άλλους μετασχηματισμούς, η αρχική φιγούρα ονομάζεται ***πρότυπο***. Το αποτέλεσμα της ανάκλασης ονομάζεται ***εικόνα***.

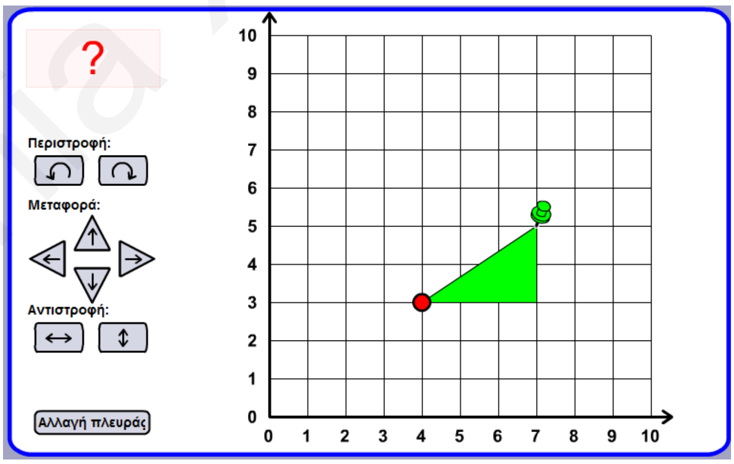
**Δραστηριότητα 4**

*Διερευνώντας την περιστροφή.*

1. Να επιλέξεις το εικονίδιο **Επαναφορά** ώστε το τρίγωνο να πάει στη θέση που φαίνεται στην Εικόνα 1.

2. Να χρησιμοποιήσεις **μόνο** τα εικονίδια ↺ ↻ και **Μετακίνηση** για να τοποθετήσεις το τρίγωνο στη θέση που βλέπεις στην Εικόνα 4.

Εικόνα 4




3. Να σχεδιάσεις στο πιο κάτω ορθογώνιο με τη σειρά τα εικονίδια ↺ ↻ και **Μετακίνηση** που επέλεξες για να βρεις την απάντηση.

4. Να απαντήσεις τις πιο κάτω ερωτήσεις:

➢Τι συμβαίνει στο τρίγωνο κάθε φορά που επιλέγεις το εικονίδιο  ;

……………………………………………………………………………………………

➢Τι συμβαίνει στο τρίγωνο κάθε φορά που επιλέγεις το εικονίδιο  ;

……………………………………………………………………………………………

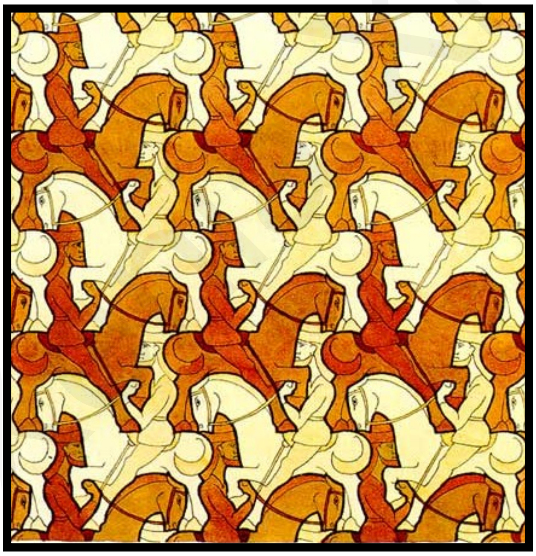
5. Να διαβάσεις το πιο κάτω κείμενο.

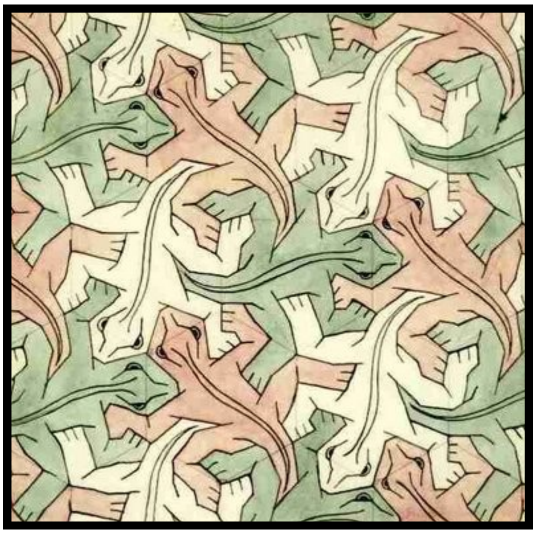
Μια φιγούρα μπορεί να **<u>κάνει στροφή</u>** γύρω από ένα σημείο. Αυτό στα μαθηματικά ονομάζεται ***<u>περιστροφή</u>***.
Η αρχική φιγούρα ονομάζεται ***<u>πρότυπο</u>***. Το αποτέλεσμα της περιστροφής ονομάζεται ***<u>εικόνα</u>***.

**Δραστηριότητα 5**

*Παρατηρώντας τους μετασχηματισμούς στους πίνακες του Escher.*

1. Οι γεωμετρικοί μετασχηματισμοί της Μεταφοράς, της Ανάκλασης και της Περιστροφής είναι οι κανόνες που εφάρμοζε στην τέχνη του ο Escher. Μπορείς να εντοπίσεις παραδείγματα στους πίνακες του;





2. Να γράψεις ποιος από τους τρεις γεωμετρικούς μετασχηματισμούς της μεταφοράς, της ανάκλασης και της περιστροφής μπορεί να περιγράψει καλύτερα τη σχέση ανάμεσα:

α ) στο άσπρο άλογο με ένα καφέ άλογο; ……………………………………

β ) στο άσπρο άλογο με ένα άσπρο άλογο: ……………………………………

γ ) στην άσπρη σαύρα με μια χρωματιστή σαύρα: ……………………………..

δ ) στην άσπρη σαύρα με μια άσπρη σαύρα: …………………………………

**Δραστηριότητα 6**

*Ανακαλύπτοντας το μετασχηματισμό.*

ΠΡΟΣΟΧΗ!!

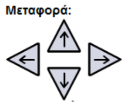
Σε αυτή τη δραστηριότητα, θα πρέπει να απεικονίσεις το τρίγωνο σε μια νέα θέση, **χωρίς να βλέπουν τα μέλη της ομάδας σου τα εικονίδια που επιλέγεις**.

Στη συνέχεια, θα πρέπει να ανακαλύψουν το μετασχηματισμό και τα εικονίδια που επέλεξες.

1. Να επιλέξεις το εικονίδιο **Επαναφορά** ώστε το τρίγωνο να πάει στη θέση που φαίνεται στην Εικόνα 1.

2. Να επιλέξεις έναν από τους τρεις μετασχηματισμούς που γνώρισες σε αυτό το μάθημα (μεταφορά, ανάκλαση, περιστροφή).

3. Να προσπαθήσεις με **3 επιλογές εικονιδίων ενός μόνο γεωμετρικού μετασχηματισμού** να απεικονίσεις το τρίγωνο σε νέα θέση:

Για τη μεταφορά: 


Για την ανάκλαση: ↔ ↕ **Μετακίνηση**

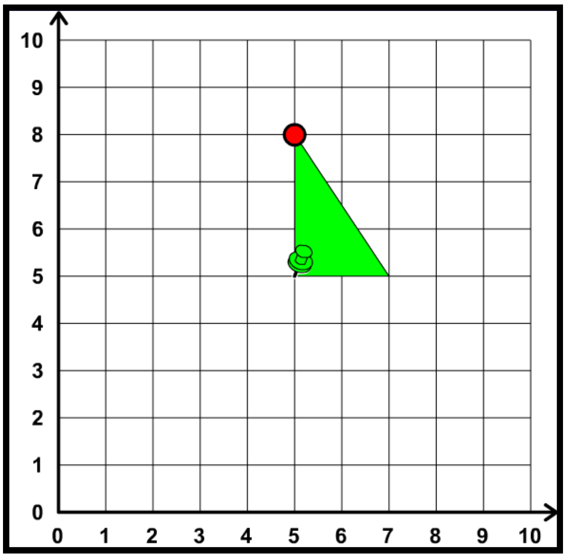
Για την περιστροφή: ↶ ↷ **Μετακίνηση** **Στροφές**

4. Να σχεδιάσεις τα εικονίδια που επέλεξες για την απεικόνισή σου.
**Προσοχή!** Μόνο **3 εικονίδια** επιτρέπεται να επιλέξεις.

*Να κρύψεις τις σημειώσεις σου από την ομάδα σου!*

5. Να σχεδιάσεις στην Εικόνα 5 την απεικόνιση του τριγώνου στη νέα θέση.

Εικόνα 5




6. Να ζητήσεις από τα μέλη της ομάδας σου να ανακαλύψουν το μετασχηματισμό και τα εικονίδια που επέλεξες.

# ΜΑΘΗΜΑ 2: ΜΕΤΑΦΟΡΑ

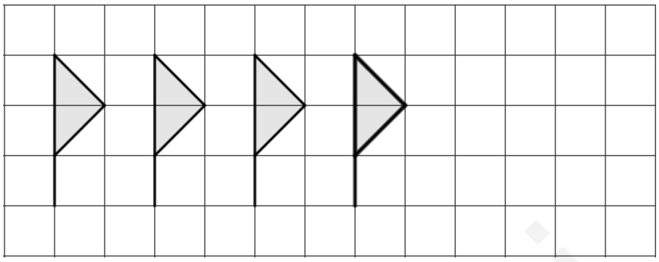
1. Η Στέλλα φτιάχνει μια κορνίζα για το δωμάτιο της. Θα φτιάξει ένα μοτίβο μεταφοράς.

**Δραστηριότητα 1**

*Εξερευνώντας τους κανόνες της μεταφοράς.*

2. Παρατήρησε το μοτίβο της Στέλλας στην Εικόνα 1, και προσπάθησε να σχεδιάσεις <u>τα επόμενα 2 σχήματα</u>.

Εικόνα 1



3. Ποιους κανόνες πρέπει να εφαρμόσει η Στέλλα για να μεταφέρει σωστά το σχήμα;

..........................................................................................................................................

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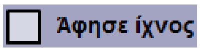
**Δραστηριότητα 2**

*Διερευνώντας το σχήμα της εικόνας.*

1. Να επιλέξεις το λογισμικό Μαθαίνω Γεωμετρία και Μετρώ.

2. Από το μενού δραστηριοτήτων, να επιλέξεις το Περιστροφή, Μεταφορά, Αντιστροφή.

3. Να επιλέξεις το κουτάκι με την επιλογή 


4. Χρησιμοποιώντας τα εικονίδια 

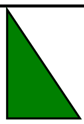
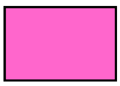
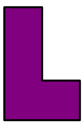
να μεταφέρεις το τρίγωνο **<u>τρία κουτάκια προς τα δεξιά</u>**.

5. **<u>Να σχεδιάσεις</u>** στον Πίνακα 1 την εικόνα του τριγώνου.

6. Να επιλέξεις το σχήμα Β του Πίνακα 1, επιλέγοντας το εικονίδιο 


7. Να κάνεις το ίδιο για τα σχήματα Β και Γ και για ένα σχήμα δικής σου επιλογής.

Πίνακας 1

| Σχήμα | Πρότυπο | Εικόνα |
|---|---|---|
| Α |  | |
| Β |  | |
| Γ |  | |
| Δ (δικό σου σχήμα) | | |



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

8. **<u>Να γράψεις</u>** τις παρατηρήσεις σου σχετικά με το τι **<u>αλλάζει</u>** / **<u>δεν αλλάζει</u>** όταν ένα σχήμα μεταφέρεται:

*Όταν ένα σχήμα μεταφέρεται, τότε η θέση του ...................................*
*ενώ η μορφή του .................................*

## Δραστηριότητα 3

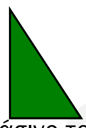
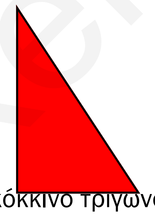
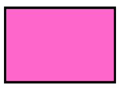
*Διερευνώντας το μέγεθος της εικόνας*

1. Να επιλέξεις μια φορά το εικονίδιο **Πλέγμα: 1** έτσι ώστε να εμφανιστεί η επιλογή **Πλέγμα: 2**

2. Να επιλέξεις με τη σειρά τα σχήματα που παρουσιάζονται στον Πίνακα 2, επιλέγοντας το κουμπί **Διάλεξε σχήμα**

3. Με τη βοήθεια του πλέγματος, **<u>να μετρήσεις τα κουτάκια</u>** και να συμπληρώσεις στον Πίνακα 2:
α) στις πρώτες δύο στήλες, να γράψεις πόσο είναι το **<u>ύψος</u>** και η **<u>βάση</u>** του πρότυπου.
β) στις δύο τελευταίες στήλες, να γράψεις πόσο είναι το **<u>ύψος</u>** και η **<u>βάση</u>** της εικόνας, όταν μεταφέρεις το σχήμα 5 θέσεις προς τα δεξιά.

Πίνακας 2

| | Πρότυπο | | Εικόνα | |
|---|---|---|---|---|
| | Ύψος | Βάση | Ύψος | Βάση |
|  (πράσινο τρίγωνο) | | | | |
|  (κόκκινο τρίγωνο) | | | | |
|  (ροζ ορθογώνιο) | | | | |
|  (μπλε ορθογώνιο) | | | | |

4. **<u>Να γράψεις</u>** τις παρατηρήσεις σου σχετικά με το τι **<u>αλλάζει</u>** / **<u>δεν αλλάζει</u>** στο **<u>μέγεθος των πλευρών</u>** του σχήματος όταν αυτό μεταφέρεται:

| |
|---|
| *Όταν ένα σχήμα μεταφέρεται, τότε το μέγεθος των πλευρών του ..............................................................* |

**Δραστηριότητα 4**

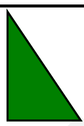
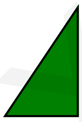
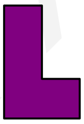
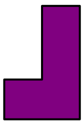
*Διερευνώντας τον προσανατολισμό της εικόνας.*

1. Να επιλέξεις με τη σειρά τα σχήματα που παρουσιάζονται στον Πίνακα 3, επιλέγοντας το εικονίδιο Διάλεξε σχήμα

2. Να μεταφέρεις το κάθε σχήμα προς όποια κατεύθυνση θέλεις, και όσα κουτάκια θέλεις.

3. **<u>Να σχεδιάσεις</u>** στον Πίνακα 3 την εικόνα για κάθε σχήμα.

Πίνακας 3

| Σχήμα | Πρότυπο | Εικόνα |
|---|---|---|
| Α |  | |
| Β |  | |
| Γ |  | |
| Δ |  | |

4. **<u>Να γράψεις</u>** τις παρατηρήσεις σου σχετικά με το τι **<u>αλλάζει</u>** / **<u>δεν αλλάζει</u>** στον **<u>προσανατολισμό</u>** (δηλαδή την κατεύθυνση στην οποία βρίσκεται η κάθε πλευρά/κορυφή στο σχήμα) του σχήματος όταν αυτό μεταφέρεται:

| |
|---|
| *Όταν ένα σχήμα μεταφέρεται, τότε ο προσανατολισμός του ..............................................................* |

**Δραστηριότητα 5**

*Διατυπώνοντας ένα γενικό συμπέρασμα.*

1. **Να γράψεις ένα γενικό συμπέρασμα** για τις παρατηρήσεις σου στις προηγούμενες δραστηριότητες.

Να αναφερθείς στη **μορφή**, στο **μέγεθος** και τον **προσανατολισμό** της εικόνας ενός σχήματος που μεταφέρεται.

**ΓΕΝΙΚΟ ΣΥΜΠΕΡΑΣΜΑ**

..........................................................................................................................

..........................................................................................................................

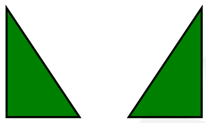
..........................................................................................................................

**Δραστηριότητα 6**

*Αναγνωρίζοντας το παράδειγμα της μεταφοράς.*

1. **Να παρατηρήσεις** τα πιο κάτω παραδείγματα και **να αποφασίσεις** αν παρουσιάζουν ή όχι μεταφορά. Μπορείς να δοκιμάσεις να τα εφαρμόζεις στο λογισμικό Μαθαίνω Γεωμετρία και Μετρώ για να βεβαιωθείς.

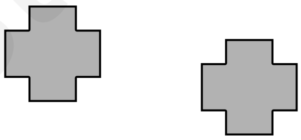
2. **Να βάλεις σε κύκλο** την απάντησή σου και **να γράψεις για ποιο λόγο** την επέλεξες.



Είναι / Δεν είναι μεταφορά,

γιατί ……………………

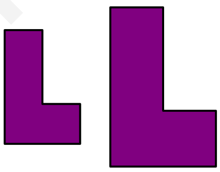
…………………………..



Είναι / Δεν είναι μεταφορά,

γιατί ……………………

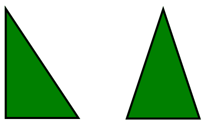
…………………………..



Είναι / Δεν είναι μεταφορά,

γιατί ……………………

…………………………..



Είναι / Δεν είναι μεταφορά,

γιατί ……………………

…………………………..

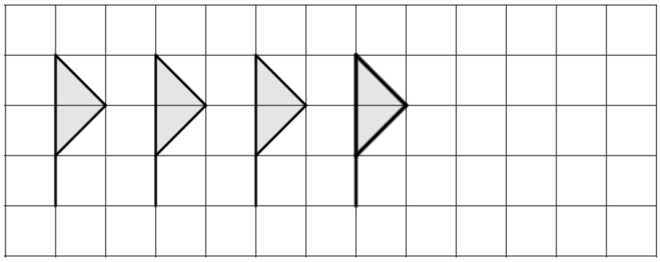
# ΜΑΘΗΜΑ 3: ΜΕΤΑΦΟΡΑ (συνέχεια)

**Δραστηριότητα 1**

*Εξερευνώντας τις παραμέτρους της μεταφοράς.*

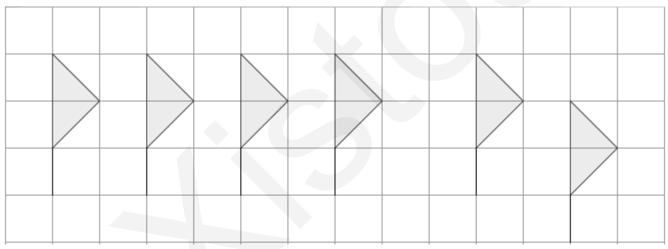
1. Η Στέλλα φτιάχνει μια κορνίζα για το δωμάτιό της. Θα φτιάξει ένα μοτίβο μεταφοράς, όπως την Εικόνα 1.

Εικόνα 1



2. Ζήτησε τη βοήθεια του μικρού αδερφού της για να το τελειώσει. Του είπε να μεταφέρει το σχήμα κάθε φορά 2 κουτάκια προς τα δεξιά. Ο αδερφός της συνέχισε όπως την Εικόνα 2.

Εικόνα 2



3. Ποια λάθη έκανε ο αδερφός της Στέλλας στο μοτίβο;

..................................................................................................................

..................................................................................................................



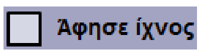
**Δραστηριότητα 2**

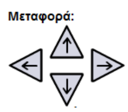
*Διερευνώντας την απόσταση ανάμεσα στο πρότυπο και την εικόνα.*

***<u>Να επιλέξεις με γρήγορο ρυθμό</u>*** *το εικονίδιο τρεις φορές για να αφήσεις* ***<u>μόνο ένα ίχνος</u>***

1. Να επιλέξεις το λογισμικό Μαθαίνω Γεωμετρία και Μετρώ.

2. Από το μενού δραστηριοτήτων, να επιλέξεις το Περιστροφή, Μεταφορά, Αντιστροφή.

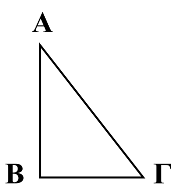
3. Να επιλέξεις το κουτάκι με την επιλογή 


4. Χρησιμοποιώντας τα εικονίδια 

να μεταφέρεις το τρίγωνο **<u>τρεις θέσεις προς τα δεξιά</u>**.

5. Να θεωρήσεις ότι οι τρεις κορυφές του τριγώνου ονομάζονται Α, Β και Γ όπως στην Εικόνα 3:

Εικόνα 3

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6. **<u>Να μετρήσεις</u>** τα κουτάκια για να βρεις την **<u>απόσταση</u>** ανάμεσα στις **<u>αντίστοιχες κορυφές</u>** και να συμπληρώσεις τον Πίνακα 1

Πίνακας 1

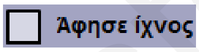
| Απόσταση | Κουτάκια |
|---|---|
| Από την κορυφή Α **του προτύπου** μέχρι την κορυφή Α΄ **της εικόνας** | |
| Από την κορυφή Β **του προτύπου** μέχρι την κορυφή Β΄ **της εικόνας** | |
| Από την κορυφή Γ **του προτύπου** μέχρι την κορυφή Γ΄ **της εικόνας** | |

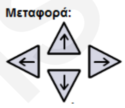
7. Να μελετήσεις τον πίνακα και να γράψεις τις παρατηρήσεις σου.

.......................................................................................................................

.......................................................................................................................

8. Να επιλέξεις το εικονίδιο 


9. Να επιλέξεις το εικονίδιο 


10. Χρησιμοποιώντας τα εικονί- 

δια να μεταφέρεις το τρίγωνο **μια θέση προς τα δεξιά.**

11. **<u>Να γράψεις</u>** τις παρατηρήσεις σου για **<u>τη θέση</u>** που εμφανίζεται η εικόνα.

.......................................................................................................................

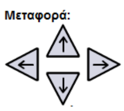
**Δραστηριότητα 3**

*Διερευνώντας την κατεύθυνση της μεταφοράς.*

1. Να επιλέξεις το εικονίδιο 


2. Να επιλέξεις την καρτέλα με το ερωτηματικό 

ώστε **<u>να εμφανιστούν</u>** οι συντεταγμένες (5,8) του σημείου που φαίνεται με κόκκινο χρώμα στο τρίγωνο.

3. Να επιλέξεις τα εικονίδια 

που χρειάζεται ώστε οι συντεταγμένες του σημείου **<u>να γίνουν (5,6)</u>**.

4. Να γράψεις ποιο εικονίδιο επέλεξες και πόσες φορές το επέλεξες;

.......................................................................................................................

5. Να επιλέξεις το εικονίδιο 

6. Να συμπληρώσεις στον Πίνακα 2 αυτό που λείπει κάθε φορά.

*Να θυμάσαι να επιλέγεις κάθε φορά την ΕΠΑΝΑΦΟΡΑ για να αρχίζεις από τη θέση (5,8)*

Πίνακας 2

| **Συντεταγμένες κόκκινου σημείου** | **Αριθμός θέσεων και κατεύθυνση** |
|---|---|
| | μια θέση δεξιά |
| (2,8) | |
| | τρεις θέσεις κάτω |
| (5,10) | |
| | μια θέση δεξιά και τρεις κάτω |
| (3,5) | |
| (3,9) | |
| (7,9) | |

7. Να γράψεις τις παρατηρήσεις σου .

............................................................................................................................

............................................................................................................................

**Δραστηριότητα 4**

*Βρίσκοντας τη σχέση ανάμεσα στην κατεύθυνση και τις συντεταγμένες ενός σημείου*

1. Με βάση τις παρατηρήσεις σου στον πιο πάνω πίνακα, **να απαντήσεις** στις πιο κάτω ερωτήσεις:

α) Σε ποια κατεύθυνση πρέπει να μεταφέρεις το αντικείμενο ώστε οι **τιμές του χ** να μένουν πάντα οι **ίδιες** και να **αλλάζουν** μόνο οι **τιμές του ψ**;

............................................................................................................................

β) Σε ποια κατεύθυνση πρέπει να μεταφέρεις το αντικείμενο ώστε οι **τιμές του ψ** να μένουν πάντα οι **ίδιες** και να **αλλάζουν** μόνο οι **τιμές του χ**;

............................................................................................................................

3. Όταν το τρίγωνο μεταφερθεί προς τα **πάνω και δεξιά**, τι πρέπει να ισχύει για τις συντεταγμένες του κόκκινου σημείου;

..............................................................................................................................

4. Εάν το τρίγωνο μεταφερθεί κάποιο **α** αριθμό θέσεων προς τα δεξιά και κάποιο **β** αριθμό θέσεων προς τα πάνω, τότε πώς θα αλλάξουν οι συντεταγμένες **της εικόνας** ενός σημείου (χ, ψ);

..............................................................................................................................

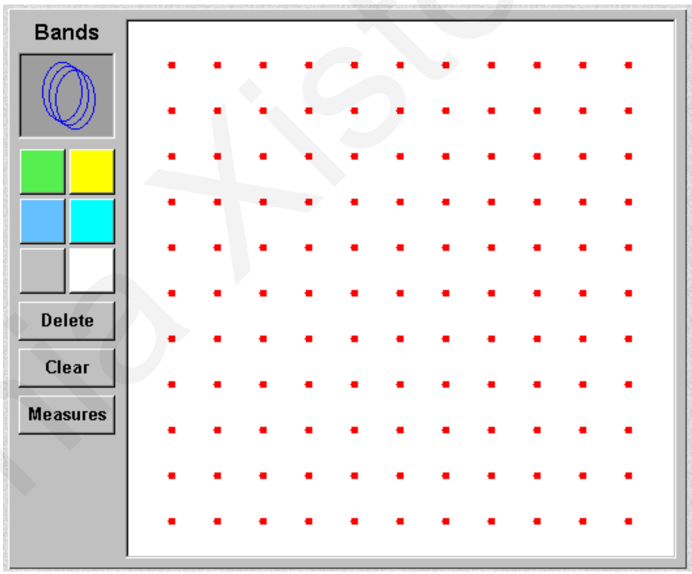
5. Μπορείς να χρησιμοποιήσεις τα σύμβολα α, β, χ, και ψ για να συμβολίσεις μέσα στην παρένθεση τις νέες συντεταγμένες;

( , )

**Δραστηριότητα 5**

*Κατασκευάζοντας την εικόνα στη μεταφορά.*

1. Να ανοίξεις το λογισμικό Geoboard στην ηλεκτρονική διεύθυνση
**http://nlvm.usu.edu/en/nav/frames_asid_172_g_2_t_3.html**

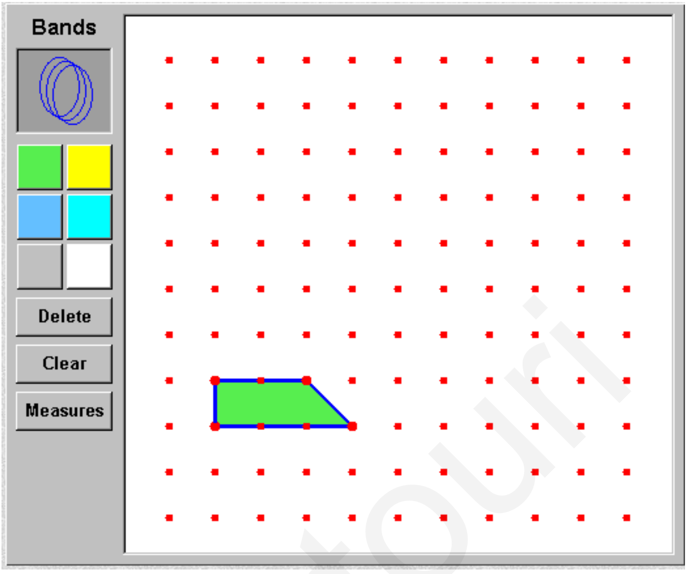


2. Να χρησιμοποιήσεις ένα από τα εικονικά λαστιχάκια (κάτω από τη λέξη Bands) και **<u>να κατασκευάσεις</u>** το σχήμα που φαίνεται στην Εικόνα 4.

* **<u>Να το χρωματίσεις</u>** πράσινο. Αυτό είναι το πρότυπο σου.

** Για να χρωματίσεις ένα σχήμα, να επιλέξεις πρώτα το χρώμα και μετά το εσωτερικό του σχήματος.*

Εικόνα 4




3. **<u>Να κατασκευάσεις την εικόνα</u>** του σχήματος για **<u>κάθε μια</u>** από τις πιο κάτω περιπτώσεις:

3α) **<u>Να κατασκευάσεις</u>** με ένα νέο λαστιχάκι **<u>την εικόνα</u>** του όταν μεταφερθεί **<u>πέντε θέσεις δεξιά</u>**. Να το χρωματίσεις **<u>κίτρινο</u>**.

3β) **<u>Να κατασκευάσεις</u>** με ένα νέο λαστιχάκι **<u>την εικόνα</u>** του όταν μεταφερθεί **<u>εφτά θέσεις πάνω</u>**. Να το χρωματίσεις **<u>γαλάζιο</u>**.

3γ) **<u>Να κατασκευάσεις</u>** με ένα νέο λαστιχάκι **<u>την εικόνα</u>** του όταν μεταφερθεί **<u>τέσσερις θέσεις δεξιά και τρεις πάνω</u>**. Να το χρωματίσεις **<u>γκρίζο</u>**.

4. Να εκτυπώσεις την εργασία σου.

# ΜΑΘΗΜΑ 4: ΑΝΑΚΛΑΣΗ

1. Να επιλέξεις το λογισμικό Μαθαίνω Γεωμετρία και Μετρώ.

## Δραστηριότητα 1

*Εξερευνώντας τους κανόνες της ανάκλασης.*

2. Από το μενού δραστηριοτήτων, να επιλέξεις το «Συμμετρία».

3. <u>Να επιλέξεις</u> με τη σειρά τα εικονίδια: [Τετράγωνα] [☑ Συμμετρία]

4. <u>Να σχεδιάσεις</u> ακόμα τρία σημεία αριστερά από τη γραμμή συμμετρίας.

5. <u>Να γράψεις</u> τις παρατηρήσεις σου σχετικά με τη θέση που εμφανίζονται τα σημεία στα δεξιά της γραμμής.

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6. <u>Να γράψεις</u> πού συναντάς το φαινόμενο αυτό στην καθημερινή ζωή;

.....................................................................................................................

7. Να επιλέξεις το εικονίδιο [Έξοδος]

## Δραστηριότητα 2

*Διερευνώντας το σχήμα της εικόνας.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Συμμετρία».

2. Να επιλέξεις το εικονίδιο [☑ Συμμετρία]

3. <u>Να επιλέξεις</u> κουτάκια στην <u>αριστερή</u> μεριά της γραμμής για να αλλάξεις το σχήμα και <u>να παρατηρήσεις</u> τις αλλαγές που συμβαίνουν στα δεξιά της γραμμής. Μπορείς να χρησιμοποιήσεις όποια χρώματα θέλεις.

4. Να συγκρίνεις το <u>πρότυπο</u> και την <u>εικόνα</u> όσον αφορά το <u>σχήμα</u> και τη <u>θέση</u> τους και να γράψεις τις παρατηρήσεις σου.

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5. Να επιλέξεις το εικονίδιο [Έξοδος]

## Δραστηριότητα 3

*Διερευνώντας το μέγεθος της εικόνας.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Συμμετρία».

2. Να επιλέξεις με τη σειρά τα εικονίδια [Πλέγμα: 3] [Καθαρισμός] [☑ Συμμετρία]

3. Να κατασκευάσεις τέσσερα διαφορετικά ορθογώνια, χρωματίζοντας κάθε φορά όσα κουτάκια θέλεις.

4. Να μετρήσεις το <u>μήκος</u> και το <u>πλάτος</u> του <u>προτύπου</u> και της <u>εικόνας</u>.
Να συμπληρώσεις τον Πίνακα 1.



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

Πίνακας 1

| | Πρότυπο | | Εικόνα | |
|---|---|---|---|---|
| | Μήκος | Πλάτος | Μήκος | Πλάτος |
| Ορθογώνιο 1 | | | | |
| Ορθογώνιο 2 | | | | |
| Ορθογώνιο 3 | | | | |
| Ορθογώνιο 4 | | | | |

5. Να παρατηρήσεις τον πίνακα και **<u>να συγκρίνεις</u>** το πρότυπο και την εικόνα όσον αφορά το **<u>μέγεθος των πλευρών</u>** και **<u>να γράψεις</u>** τις παρατηρήσεις σου.

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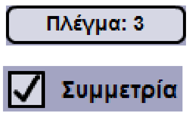
6. Να επιλέξεις το εικονίδιο 


**Δραστηριότητα 4**

*Διερευνώντας τον προσανατολισμό της εικόνας.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Συμμετρία».

2. Να επιλέξεις με τη σειρά τα εικονίδια 


3. Να επιλέξεις το **<u>πράσινο</u>** χρώμα και να χρωματίσεις **<u>ένα κουτάκι</u>** στην **<u>αριστερή μεριά</u>** του προτύπου.
Σε ποια μεριά της εικόνας εμφανίζεται το πράσινο κουτάκι;

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4. Να επιλέξεις το **<u>κίτρινο</u>** χρώμα και να χρωματίσεις **<u>ένα κουτάκι</u>** στη **<u>δεξιά μεριά</u>** του προτύπου.
Σε ποια μεριά της εικόνας εμφανίζεται το κίτρινο κουτάκι;

..............................................................................................................................

5. Να επιλέξεις το **<u>μπλε</u>** χρώμα και να χρωματίσεις **<u>ένα κουτάκι</u>** στην **<u>πάνω μεριά</u>** του προτύπου.
Σε ποια μεριά της εικόνας εμφανίζεται το μπλε κουτάκι;

..............................................................................................................................

6. Να επιλέξεις το **<u>πορτοκαλί</u>** χρώμα και να χρωματίσεις **<u>ένα κουτάκι</u>** στην **<u>κάτω μεριά</u>** του προτύπου.
Σε ποια μεριά της εικόνας εμφανίζεται το πορτοκαλί κουτάκι;

..............................................................................................................................

7. Νομίζεις ότι οι ίδιες αλλαγές θα συμβαίνουν και όταν ο άξονας συμμετρίας είναι **οριζόντιος**; Αν όχι, **τι διαφορετικό θα συμβαίνει** στον προσανατολισμό της εικόνας;

……………………………………………………………………………………………………

……………………………………………………………………………………………………

**Δραστηριότητα 5**

*Διατυπώνοντας ένα γενικό συμπέρασμα.*

1. **Να γράψεις ένα γενικό συμπέρασμα** για τις παρατηρήσεις σου στις προηγούμενες δραστηριότητες.

Να αναφερθείς στη **μορφή**, στο **μέγεθος** και τον **προσανατολισμό** της εικόνας ενός σχήματος που ανακλάται.

**ΓΕΝΙΚΟ ΣΥΜΠΕΡΑΣΜΑ**

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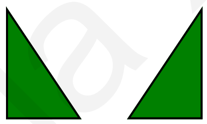
**Δραστηριότητα 6**

*Αναγνωρίζοντας το παράδειγμα της ανάκλασης.*

1. **Να παρατηρήσεις** τα πιο κάτω παραδείγματα και **να αποφασίσεις** αν παρουσιάζουν ανάκλαση, μεταφορά ή κανένα από τα δύο. Μπορείς να δοκιμάσεις να τα εφαρμόζεις στο λογισμικό Μαθαίνω Γεωμετρία και Μετρώ για να βεβαιωθείς.

2. **Να βάλεις σε κύκλο** την απάντησή σου και **να γράψεις για ποιο λόγο** την επέλεξες.

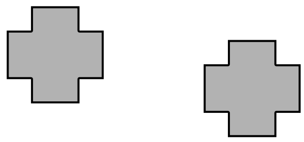


Είναι ανάκλαση/ μεταφορά/

κανένα από τα δύο γιατί

……………………..……

…………………………..

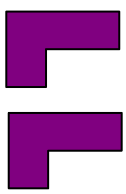


Είναι ανάκλαση/ μεταφορά/

κανένα από τα δύο γιατί

……………………..……

…………………………..

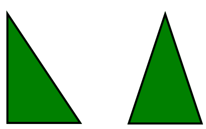


Είναι ανάκλαση/ μεταφορά/

κανένα από τα δύο γιατί

……………………..……

…………………………..



Είναι ανάκλαση/ μεταφορά/

κανένα από τα δύο γιατί

……………………..……

…………………………..

# ΜΑΘΗΜΑ 5: ΑΝΑΚΛΑΣΗ (συνέχεια)

**Δραστηριότητα 1**

*Εξερευνώντας τις παραμέτρους της ανάκλασης.*

1. Να επιλέξεις το λογισμικό Μαθαίνω Γεωμετρία και Μετρώ.

2. Από το μενού δραστηριοτήτων, να επιλέξεις το «Συμμετρία».

3. Να επιλέξεις με τη σειρά τα εικονίδια: Πλέγμα: 3 / Καθαρισμός / ☑ Συμμετρία

4. **<u>Να χρωματίσεις</u>** τετραγωνάκια σε όποιο χρώμα θέλεις, και **<u>να παρατηρήσεις</u>** σε ποια θέση εμφανίζεται η εικόνα τους.

5. Να εξερευνήσεις τις πιο κάτω περιπτώσεις:

α) Να χρωματίσεις ένα **<u>μπλε</u>** τετραγωνάκι, που η εικόνα του να έχει τη **<u>μεγαλύτερη απόσταση</u>** από το πρότυπο.

Πότε έχεις τη **<u>μεγαλύτερη απόσταση</u>** ανάμεσα στο πρότυπο και την εικόνα;

.................................................................................................................................

β) Να χρωματίσεις ένα **<u>πράσινο</u>** τετραγωνάκι, που η εικόνα του να έχει τη **<u>μικρότερη απόσταση</u>** από το πρότυπο.

Πότε έχεις τη **<u>μικρότερη απόσταση</u>** ανάμεσα στο πρότυπο και την εικόνα;

.................................................................................................................................

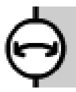
6. Να επιλέξεις το εικονίδιο Έξοδος

**Δραστηριότητα 2**

*Διερευνώντας την κατεύθυνση της εικόνας σε σχέση με το πρότυπο.*

1. Από το μενού δραστηριοτήτων να επιλέξεις το «Συμμετρία».

2. Να επιλέξεις με τη σειρά τα εικονίδια Καθαρισμός / ☑ Συμμετρία

3. Να χρησιμοποιήσεις το Εικονίδιο  για **<u>να περιστρέψεις</u>** τον άξονα συμμετρίας προς διάφορες κατευθύνσεις.

4. **<u>Να εξερευνήσεις</u>** ποιες από τις περιπτώσεις που περιγράφονται στον Πίνακα 1 μπορεί να συμβαίνουν και **<u>να σημειώσεις</u>** ✓ στην κατάλληλη στήλη.

Να θυμάσαι στο τέλος κάθε προσπάθεια να επιλέγεις το εικονίδιο Καθαρισμός



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

Πίνακας 1

| ΠΕΡΙΠΤΩΣΗ | Μπορεί να συμβεί | Δεν μπορεί να συμβεί |
|---|---|---|
| Να είναι η εικόνα δεξιά από το πρότυπο | | |
| Να είναι η εικόνα πάνω από το πρότυπο | | |
| Να είναι η εικόνα πάνω και δεξιά από το πρότυπο | | |
| Να είναι η εικόνα αριστερά από το πρότυπο | | |
| Να είναι η εικόνα κάτω από το πρότυπο | | |
| Να είναι η εικόνα κάτω και αριστερά από το πρότυπο | | |
| Να είναι η εικόνα κάτω και δεξιά από το πρότυπο | | |
| Να βρίσκονται και το πρότυπο και η εικόνα αριστερά από τον άξονα συμμετρίας | | |

5. Τι είναι αυτό που καθορίζει την κατεύθυνση της εικόνας;

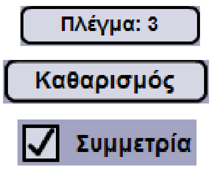
.............................................................................................................

6. Να επιλέξεις το εικονίδιο 


**Δραστηριότητα 3**

*Διερευνώντας την απόσταση της εικόνας από το πρότυπο.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Συμμετρία».

2. Να επιλέξεις με τη σειρά τα εικονίδια 


3. **Να σχηματίσεις** ένα δικό σου σχήμα χρωματίζοντας **8 κουτάκια**.

4. **Να μετρήσεις** την απόσταση που έχει το κάθε κουτάκι **του προτύπου** από τον **άξονα συμμετρίας** και να την καταγράψεις στον Πίνακα 2. Να κάνεις το ίδιο και για **τα αντίστοιχα** κουτάκια της **εικόνας**.

Πίνακας 2

| Κουτάκι | Απόσταση προτύπου από άξονα συμμετρίας | Απόσταση εικόνας από άξονα συμμετρίας |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |

5. Να συγκρίνεις τις **αποστάσεις** ανάμεσα στον **άξονα συμμετρίας** και στα **αντίστοιχα κουτάκια** και **να γράψεις** τις παρατηρήσεις σου.

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....................................................................................................................

6. Να επιλέξεις το εικονίδιο Έξοδος

## Δραστηριότητα 4

*Βρίσκοντας τη σχέση ανάμεσα στην κατεύθυνση και τις συντεταγμένες ενός σημείου.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Συμμετρία».

2. Να επιλέξεις με τη σειρά τα εικονίδια Τετράγωνα, Πλέγμα: 3, Καθαρισμός, ☑ Συμμετρία

3. Να επιλέξεις το εικονίδιο ☑ Κλίμακα

4. **Να τοποθετήσεις** στο Πλέγμα 3 τα **σημεία** που αναγράφονται στην **πρώτη στήλη** του Πίνακα 3.

**Να γράψεις** στη **δεύτερη στήλη** του Πίνακα 3 τις **συντεταγμένες** όπου εμφανίζεται η **εικόνα** του κάθε σημείου.

Πίνακας 3

| Συντεταγμένες προτύπου | Συντεταγμένες εικόνας |
|---|---|
| (2,4) | |
| (2,5) | |
| (2,7) | |
| (3,4) | |
| (3,6) | |
| (4,4) | |

5. Να εξερευνήσεις τη **σχέση** ανάμεσα στις **συντεταγμένες** μιας κορυφής και στις **συνταγμένες** της **εικόνας** της με **κατακόρυφο άξονα** συμμετρίας .

**Να περιγράψεις** τη σχέση που ανακάλυψες.

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6. Να επιλέξεις το εικονίδιο Καθαρισμός

7. Με βάση τις παρατηρήσεις σου στην Ερώτηση 5, ποια νομίζεις ότι θα είναι η **σχέση** ανάμεσα στις **συντεταγμένες** ενός σημείου και στις **συνταγμένες** της **εικόνας** του με **οριζόντιο** άξονα συμμετρίας; **Να γράψεις** την υπόθεση σου.

....................................................................................................................

8. <u>**Να επιλέξεις**</u> το εικονίδιο  για να κάνεις τον άξονα συμμετρίας <u>**οριζόντιο**</u>.

<u>**Να τοποθετήσεις**</u> δικά σου σημεία σε <u>**διάφορα σημεία**</u> του πλέγματος και για <u>**να ελέγξεις την υπόθεσή**</u> σου στην Ερώτηση 7.

<u>**Να περιγράψεις**</u> τη σχέση που ανακάλυψες.

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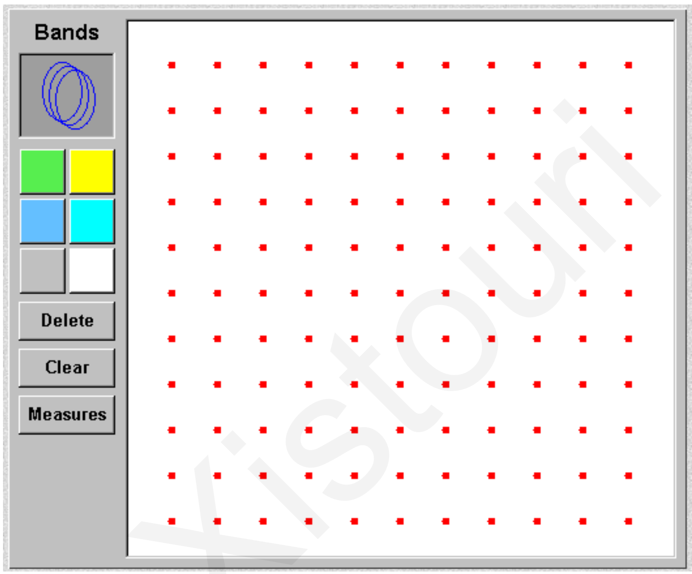
## Δραστηριότητα 5

*Κατασκευάζοντας την εικόνα στην ανάκλαση.*

1. Να ανοίξεις το λογισμικό Geoboard στην ηλεκτρονική διεύθυνση
   http://nlvm.usu.edu/en/nav/frames_asid_172_g_2_t_3.html

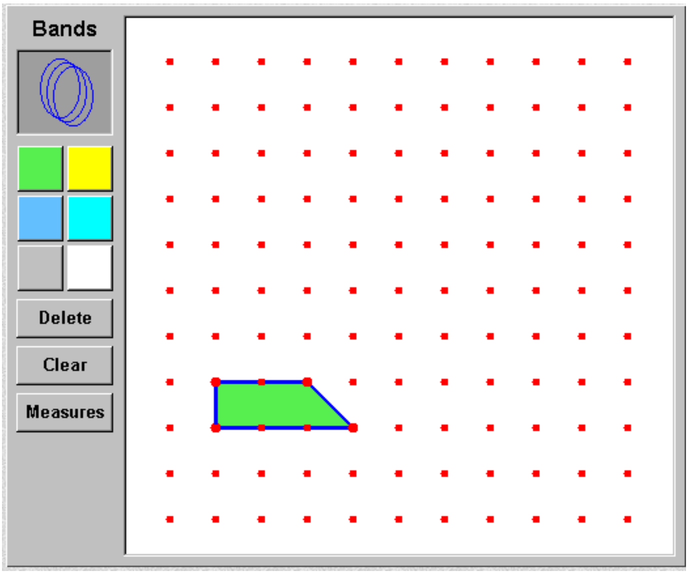



2. Να χρησιμοποιήσεις ένα από τα εικονικά λαστιχάκια (κάτω από τη λέξη Bands) και <u>**να κατασκευάσεις**</u> το σχήμα που φαίνεται στην Εικόνα 1.

* <u>**Να το χρωματίσεις**</u> πράσινο. Αυτό είναι το πρότυπό σου.

** Για να χρωματίσεις ένα σχήμα, να επιλέξεις πρώτα το χρώμα και μετά το εσωτερικό του σχήματος.*

Εικόνα 1



3. <u>**Να κατασκευάσεις την εικόνα**</u> του σχήματος για <u>**κάθε μια**</u> από τις πιο κάτω περιπτώσεις:

Μπορείς να χρησιμοποιήσεις ένα <u>**νέο λαστιχάκι**</u> για να κατασκευάζεις κάθε φορά τον <u>**άξονα συμμετρίας**</u>.

α) <u>**Να κατασκευάσεις**</u> με ένα νέο λαστιχάκι <u>**την εικόνα**</u> του όταν ανακλαστεί σε <u>**κατακόρυφο άξονα συμμετρίας**</u>. Να το χρωματίσεις <u>**Να κατασκευάσεις**</u> με ένα νέο λαστιχάκι <u>**την εικόνα**</u> του.

β) <u>**Να κατασκευάσεις**</u> με ένα νέο λαστιχάκι <u>**την εικόνα**</u> του όταν ανακλαστεί σε <u>**οριζόντιο άξονα συμμετρίας**</u>. Να το χρωματίσεις <u>**γαλάζιο**</u>.

γ) <u>**Να κατασκευάσεις**</u> με ένα νέο λαστιχάκι <u>**την εικόνα**</u> του όταν ανακλαστεί σε <u>**διαγώνιο άξονα συμμετρίας**</u>. Να το χρωματίσεις <u>**γκρίζο**</u>.

4. Να εκτυπώσεις την εργασία σου.

# ΜΑΘΗΜΑ 6: ΠΕΡΙΣΤΡΟΦΗ

**Δραστηριότητα 1**

*Εξερευνώντας τους κανόνες της περιστροφής.*

1. Να επιλέξεις το λογισμικό Μαθαίνω Γεωμετρία και Μετρώ.

2. Από το μενού δραστηριοτήτων, να επιλέξεις το «Περιστροφή».

3. Να επιλέξεις το εικονίδιο ☑ Γραμμές-οδηγοί

4. Να επιλέξεις το εικονίδιο [Διάλεξε σχήμα] και στη συνέχεια να **επιλέξεις τον μοβ κύκλο**.

5. Να επιλέξεις τη γραμμή-οδηγό **από τον πράσινο βραχίονα** και να την περιστρέψεις **προς τα δεξιά**.
**Να γράψεις** τις παρατηρήσεις σου.
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.............................................................................................................................

6. Πού συναντάς το φαινόμενο αυτό στην καθημερινή ζωή;
.............................................................................................................................

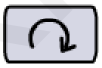
7. Να επιλέξεις το εικονίδιο [Έξοδος]

**Δραστηριότητα 2**

*Διερευνώντας το σχήμα της εικόνας.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το Περιστροφή, Μεταφορά, Αντιστροφή.

2. Να επιλέξεις το κουτάκι με την επιλογή ☐ Άφησε ίχνος

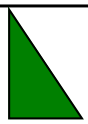
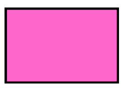
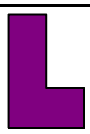
3. Να επιλέξεις το εικονίδιο για  να περιστρέψεις το τρίγωνο **μια φορά προς τα δεξιά.**

4. **Να σχεδιάσεις** στον Πίνακα 1 την εικόνα του τριγώνου.

5. Να επιλέξεις το σχήμα Β του Πίνακα 1, με το εικονίδιο [Διάλεξε σχήμα]

6. Να κάνεις το ίδιο για τα σχήματα Β και Γ και για ένα σχήμα δικής σου επιλογής.

Πίνακας 1

| Σχήμα | Πρότυπο | Εικόνα |
|---|---|---|
| Α |  | |
| Β |  | |
| Γ |  | |
| Δ (δικό σου σχήμα) | | |

Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

8. **<u>Να γράψεις</u>** τις παρατηρήσεις σου σχετικά με το τι **<u>αλλάζει</u>** / **<u>δεν αλλάζει</u>** όταν ένα σχήμα μεταφέρεται:

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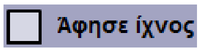
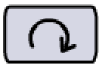
9. Να επιλέξεις το εικονίδιο 


## Δραστηριότητα 3

*Διερευνώντας το μέγεθος της εικόνας.*

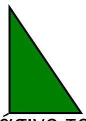
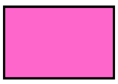
1. Από το μενού δραστηριοτήτων, να επιλέξεις το Περιστροφή, Μεταφορά, Αντιστροφή.

2. Να επιλέξεις το κουτάκι με την επιλογή 


3. Να επιλέξεις το εικονίδιο  για να περιστρέψεις το τρίγωνο **<u>μια φορά προς τα δεξιά</u>**.

4. Με τη βοήθεια του πλέγματος, **<u>να μετρήσεις τα κουτάκια</u>** και να συμπληρώσεις στον Πίνακα 2:
α) στις πρώτες δύο στήλες, να γράψεις πόσο είναι το **<u>ύψος</u>** και η **<u>βάση</u>** του πρότυπου.
β) στις δύο τελευταίες στήλες, να γράψεις πόσο είναι το **<u>ύψος</u>** και η **<u>βάση</u>** της εικόνας, όταν κάνει 1/4 της στροφής. Να συμπληρώσεις τον Πίνακα 2.

Πίνακας 2

| | Πρότυπο | | Εικόνα | |
|---|---|---|---|---|
| | Ύψος | Βάση | Ύψος | Βάση |
|  (πράσινο τρίγωνο) | | | | |
|  (κόκκινο τρίγωνο) | | | | |
|  (ροζ ορθογώνιο) | | | | |
|  (μπλε ορθογώνιο) | | | | |

5. **<u>Να συγκρίνεις</u>** το **<u>πρότυπο</u>** και την **<u>εικόνα</u>** στην περιστροφή όσον αφορά το **<u>μέγεθος των πλευρών</u>** της και το **<u>μέγεθος των γωνιών</u>** της και **<u>να γράψεις</u>** τις παρατηρήσεις σου.

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## Δραστηριότητα 4

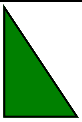
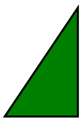
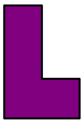
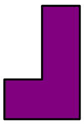
*Διερευνώντας τον προσανατολισμό της εικόνας.*

1. Να επιλέξεις με τη σειρά τα σχήματα που παρουσιάζονται στον Πίνακα 3, επιλέγοντας το εικονίδιο [Διάλεξε σχήμα]

2. Να περιστρέψεις το κάθε σχήμα κατά 1/4 στροφή προς τα δεξιά, επιλέγοντας μια φορά το εικονίδιο 

3. **Να σχεδιάσεις** στην τρίτη στήλη του Πίνακα 3 τη εικόνα για κάθε σχήμα.

Πίνακας 3

| Σχήμα | Πρότυπο | Εικόνα 1/4 στροφή | Εικόνα 1/2 στροφή | Εικόνα 3/4 στροφή |
|---|---|---|---|---|
| Α |  | | | |
| Β |  | | | |
| Γ |  | | | |
| Δ |  | | | |

4. Να επιλέξεις με τη σειρά τα εικονίδια [Επαναφορά] [Στροφές: ½]

5. Να επαναλάβεις τα Βήματα 1 και 2 για τη **στροφή 1/2** και να συμπληρώσεις την **τέταρτη στήλη** του Πίνακα 3.

6. Να επιλέξεις με τη σειρά τα εικονίδια [Επαναφορά] [Στροφές: ¾]

7. Να επαναλάβεις τα Βήματα 1 και 2 για τη **στροφή 3/4** και να συμπληρώσεις την **τελευταία στήλη** του Πίνακα 3.

8. **Να γράψεις** τις παρατηρήσεις σου σχετικά με το τι **αλλάζει** / **δεν αλλάζει** στον **προσανατολισμό** του σχήματος όταν αυτό περιστρέφεται σε διάφορες θέσεις.

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9. Νομίζεις ότι οι ίδιες αλλαγές θα συμβαίνουν και όταν το **σημείο περιστροφής** βρίσκεται **στο πάνω μέρος** του σχήματος; Αν όχι, **τι διαφορετικό** νομίζεις ότι θα συμβαίνει στον προσανατολισμό της εικόνας;

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10. Να ελέγξεις την υπόθεση σου επιλέγοντας το εικονίδιο [Μετακίνηση] μέχρι **να μετακινηθεί** το σημείο περιστροφής που ορίζει η **πινέζα** στο **πάνω** μέρος του σχήματος.

Να επαναλάβεις το Βήμα 2 για τα σχήματα του Πίνακα 3.

Τι παρατηρείς; Γιατί νομίζεις ότι συμβαίνει αυτό;

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**Δραστηριότητα 5**

*Διατυπώνοντας ένα γενικό συμπέρασμα.*

1. <u>**Να γράψεις ένα γενικό συμπέρασμα**</u> για τις παρατηρήσεις σου στις προηγούμενες δραστηριότητες.

Να αναφερθείς στη <u>**μορφή**</u>, στο <u>**μέγεθος**</u> και τον <u>**προσανατολισμό**</u> της εικόνας ενός σχήματος που περιστρέφεται.

**ΓΕΝΙΚΟ ΣΥΜΠΕΡΑΣΜΑ**

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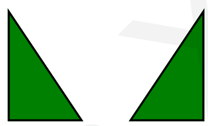
**Δραστηριότητα 6**

*Αναγνωρίζοντας τα παραδείγματα της περιστροφής.*

1. <u>**Να παρατηρήσεις**</u> τα πιο κάτω παραδείγματα και <u>**να αποφασίσεις**</u> αν παρουσιάζουν μεταφορά, ανάκλαση ή περιστροφή. Μπορείς να δοκιμάσεις να τα εφαρμόζεις στο λογισμικό Μαθαίνω Γεωμετρία και Μετρώ για να βεβαιωθείς.

2. <u>**Να βάλεις σε κύκλο**</u> την απάντησή σου και <u>**να γράψεις για ποιο λόγο**</u> την επέλεξες.



Είναι περιστροφή/ ανάκλαση/ μεταφορά γιατί

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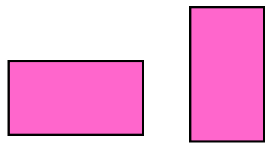

Είναι περιστροφή/ ανάκλαση/ μεταφορά γιατί

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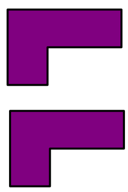

Είναι περιστροφή/ ανάκλαση/ μεταφορά γιατί

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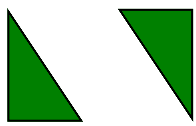

Είναι περιστροφή/ ανάκλαση/ μεταφορά γιατί

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# ΜΑΘΗΜΑ 7: ΠΕΡΙΣΤΡΟΦΗ (συνέχεια)

**Δραστηριότητα 1**

*Εξερευνώντας τις παραμέτρους της περιστροφής.*

1. Να επιλέξεις το λογισμικό Μαθαίνω Γεωμετρία και Μετρώ.

2. Από το μενού δραστηριοτήτων, να επιλέξεις το «Περιστροφή, Μεταφορά, Αντιστροφή».

3. Να επιλέξεις το εικονίδιο **Διάλεξε σχήμα** και να επιλέξεις τον μπλε σταυρό.

4. Έχοντας επιλεγμένο το εργαλείο **Άφησε ίχνος** για να μπορείς να βλέπεις το πρότυπο, να περιστρέψεις το σταυρό προς όποια κατεύθυνση θέλεις.

5. Με το εργαλείο **Μετακίνηση** <u>να αλλάξεις τη θέση</u> του <u>σημείου περιστροφής</u>. Να περιστρέψεις και πάλι το σχήμα και <u>να συγκρίνεις</u> τη <u>θέση</u> της εικόνας σε σχέση με πρότυπο.

<u>Να γράψεις</u> τις παρατηρήσεις σου.

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Κάθε φορά που θες να αρχίσεις μια καινούρια προσπάθεια, να επιλέγεις το εικονίδιο **Επαναφορά**

6. Να εξερευνήσεις τις πιο κάτω περιπτώσεις:

α) Πότε έχεις τη <u>μεγαλύτερη απόσταση</u> ανάμεσα στο πρότυπο και την εικόνα;

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β) Πότε έχεις τη <u>μικρότερη απόσταση</u> ανάμεσα στο πρότυπο και την εικόνα;

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7. Να επιλέξεις το εικονίδιο **Έξοδος**



**Δραστηριότητα 2**

*Διερευνώντας την κατεύθυνση της εικόνας σε σχέση με το πρότυπο στην περιστροφή.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Περιστροφή, Μεταφορά, Αντιστροφή».

2. Να επιλέξεις το εργαλείο **Άφησε ίχνος**

3. Να επιλέξεις το εργαλείο **Μοίρες** και να γράψεις στο παράθυρο που εμφανίζεται τον αριθμό 210 μοίρες. Αυτή είναι η <u>**γωνία περιστροφής**</u>.

Να επιλέξεις το εργαλείο 

Τι αλλάζει στο σχήμα;

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Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

4. Να συμπληρώσεις στον **Πίνακα 1** την **<u>κατεύθυνση</u>** που βρίσκεται η εικόνα σε σχέση με το **<u>σημείο περιστροφής</u>**, όταν η **<u>γωνία περιστροφής</u>** έχει την **<u>τιμή</u>** που αναφέρεται στην πρώτη στήλη.

*Να θυμάσαι να επιλέγεις κάθε φορά την ΕΠΑΝΑΦΟΡΑ για να αρχίζεις από την ίδια θέση και ΑΦΗΣΕ ΙΧΝΟΣ για να βλέπεις το πρότυπο.*

Πίνακας 1

| Γωνία Περιστροφής | Κατεύθυνση προτύπου |
|---|---|
| 45° | *κάτω δεξιά* |
| 60° | |
| 90° | |
| 180° | |
| 210° | |
| 270° | |
| 315° | |

5. Να επιλέξεις **<u>μια φορά</u>** το εικονίδιο 

για να αλλάξεις τη **<u>θέση</u>** του σημείου περιστροφής.

6. Να ελέγξεις κατά πόσο εξακολουθούν να ισχύουν οι ίδιες απαντήσεις που έδωσες στον Πίνακα 1, για τις ίδιες γωνίες περιστροφής.

Ισχύουν οι ίδιες απαντήσεις; ..........................................

7. Ποια νομίζεις ότι είναι τα δύο πράγματα που μπορούν να καθορίσουν την **<u>κατεύθυνση</u>** που θα έχει η **<u>εικόνα</u>** ενός σχήματος σε σχέση με το **<u>πρότυπο</u>**;

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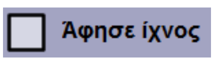
8. Να επιλέξεις το εικονίδιο 


**Δραστηριότητα 3**

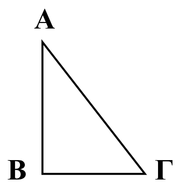
*Διερευνώντας την απόσταση της εικόνας από το σημείο περιστροφής.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Περιστροφή, Μεταφορά, Αντιστροφή».

2. Να επιλέξεις το εικονίδιο 

3. Να θεωρήσεις ότι οι κορυφές του τριγώνου ονομάζονται Α, Β και Γ όπως φαίνεται στην Εικόνα 1.

Εικόνα 1




4. Να επιλέξεις το εικονίδιο  για να περιστρέψεις το τρίγωνο με τη σειρά στις θέσεις <u>**1/4**</u>, <u>**1/2**</u> και <u>**3/4**</u> .

5. Να μετρήσεις τα κουτάκια για να βρεις την απόσταση ανάμεσα <u>**στην κάθε κορυφή**</u> και το <u>**σημείο περιστροφής**</u> (πινέζα) και να συμπληρώσεις τον Πίνακα 2.

Πίνακας 2

| | Στροφή 1/4 | Στροφή 1/2 | Στροφή 3/4 |
|---|---|---|---|
| Απόσταση | Κουτάκια | Κουτάκια | Κουτάκια |
| Από την κορυφή Α΄ **του προτύπου** μέχρι **το σημείο περιστροφής** | | | |
| Από την κορυφή Β΄ **του προτύπου** μέχρι **το σημείο περιστροφής** | | | |
| Από την κορυφή Γ΄ **του προτύπου** μέχρι **το σημείο περιστροφής** | | | |

6. Να συγκρίνεις τις <u>**αποστάσεις**</u> ανάμεσα σημείο περιστροφής και στις <u>**αντίστοιχες κορυφές**</u> και να γράψεις τις παρατηρήσεις σου.

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9. Να επιλέξεις το εικονίδιο 


**Δραστηριότητα 4**

*Βρίσκοντας τη σχέση ανάμεσα στην περιστροφή και τις συντεταγμένες.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Περιστροφή, Μεταφορά, Αντιστροφή».

2. Να επιλέξεις την καρτέλα με το ερωτηματικό 

ώστε <u>**να εμφανιστούν**</u> οι συντεταγμένες (5,8) του σημείου που φαίνεται με κόκκινο χρώμα στο τρίγωνο.

3. Να επιλέξεις το εικονίδιο  μια φορά για να περιστρέψεις το τρίγωνο κατά <u>**1/4 της στροφής**</u>. Να συμπληρώσεις τις συντεταγμένες της εικόνας <u>**του κόκκινου σημείου**</u> στον Πίνακα 3.

Πίνακας 3

| Συντεταγμένες προτύπου | Συντεταγμένες εικόνας |
|---|---|
| (5,8) | |
| | |
| | |
| | |

4. Να επαναλάβεις το Βήμα 3 για διαφορετικά σχήματα. Να συμπληρώσεις στον πίνακα τις <u>συντεταγμένες του προτύπου</u> και τις <u>συντεταγμένες της εικόνας</u> για το κόκκινο σημείο.

5. Να εξερευνήσεις τη <u>σχέση</u> ανάμεσα στις <u>συντεταγμένες</u> μιας κορυφής και στις <u>συνταγμένες</u> της <u>εικόνας</u> της <u>όταν κάνει 1/4 στροφή</u>.
Να περιγράψεις τη σχέση που ανακάλυψες.

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6. Να επιλέξεις το εικονίδιο ώστε να γίνει 

Να επαναλάβεις το Βήμα 2 για τα ίδια σχήματα και να εξερευνήσεις τη <u>σχέση</u> ανάμεσα στις <u>συντεταγμένες</u> μιας κορυφής και στις <u>συνταγμένες</u> της <u>εικόνας όταν κάνει στροφή 1/2</u>.

7. Να συμπληρώσεις τον Πίνακα 4, όπως έκανες στον Πίνακα 3.

Πίνακας 4

| Συντεταγμένες προτύπου | Συντεταγμένες εικόνας |
|---|---|
| (5,8) | |
| | |
| | |
| | |

8. Να εξερευνήσεις τη <u>σχέση</u> ανάμεσα στις <u>συντεταγμένες</u> μιας κορυφής και στις <u>συνταγμένες</u> της <u>εικόνας</u> της <u>όταν κάνει 1/2 στροφή</u>.
Να περιγράψεις τη σχέση που ανακάλυψες.

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9. Με βάση τις παρατηρήσεις σου στις ερωτήσεις 6 και 8, ποια νομίζεις ότι θα είναι η **<u>σχέση</u>** ανάμεσα στις **<u>συντεταγμένες</u>** μιας κορυφής και στις **<u>συνταγμένες</u>** της **<u>εικόνας</u>** της **<u>όταν κάνει 3/4 στροφή</u>**;

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10. Να ελέγξεις την υπόθεσή σου και **<u>να περιγράψεις τη σχέση</u>** που ανακάλυψες.

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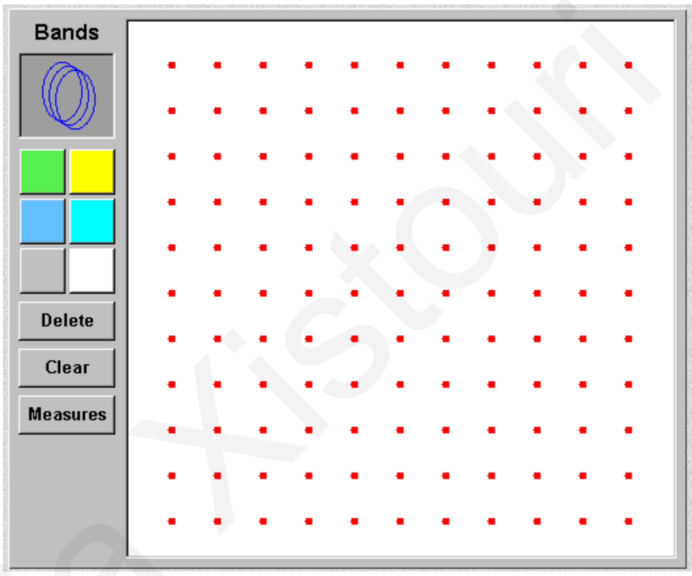
**Δραστηριότητα 5**

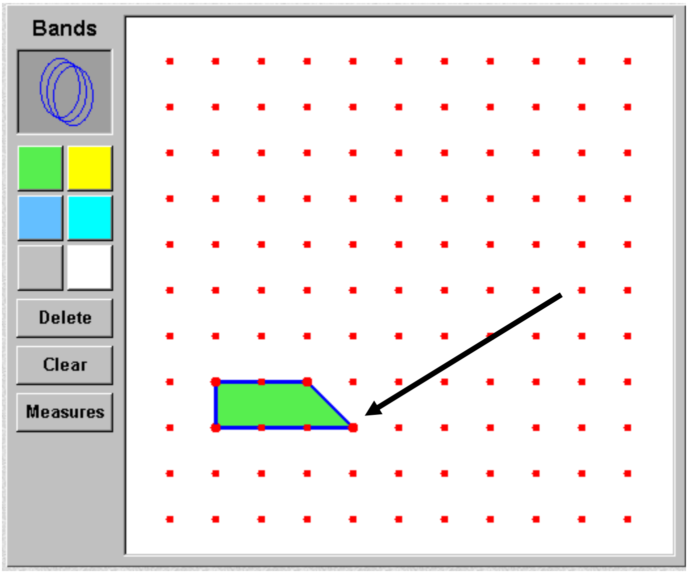
*Κατασκευάζοντας την εικόνα στην περιστροφή.*

1. Να ανοίξεις το λογισμικό Geoboard στην ηλεκτρονική διεύθυνση
   **http://nlvm.usu.edu/en/nav/frames_asid_172_g_2_t_3.html**




2. Να χρησιμοποιήσεις ένα από τα εικονικά λαστιχάκια (κάτω από τη λέξη Bands) και να κατασκευάσεις το σχήμα που φαίνεται στην Εικόνα 2.
   Να το χρωματίσεις πράσινο. Αυτό είναι το πρότυπο σου.

Εικόνα 2



3. Να χρησιμοποιήσεις για **σημείο περιστροφής** την κουκκίδα στην οποία ακουμπά η **κορυφή της οξείας γωνίας** του σχήματος και υποδεικνύεται από το **τόξο** στην Εικόνα 2.

**Να κατασκευάσεις την εικόνα** του σχήματος για **κάθε μια** από τις πιο κάτω περιπτώσεις:

3α) Να κατασκευάσεις με ένα νέο λαστιχάκι την εικόνα του όταν περιστραφεί κατά **1/4 της στροφής προς τα δεξιά**. Να το χρωματίσεις **κίτρινο.**

3β) Να κατασκευάσεις με ένα νέο λαστιχάκι την εικόνα του όταν περιστραφεί **1/2 της στροφής προς τα δεξιά** . Να το χρωματίσεις **γαλάζιο**.

3γ) Να κατασκευάσεις με ένα νέο λαστιχάκι την εικόνα του όταν περιστραφεί **3/4 της στροφής προς τα δεξιά**. Να το χρωματίσεις **γκρίζο**.

4. Να εκτυπώσεις την εργασία σου.

# ΜΑΘΗΜΑ 8: ΣΥΜΜΕΤΡΙΑ ΚΑΙ ΣΥΝΔΥΑΣΜΟΙ ΜΕΤΑΣΧΗΜΑΤΙΣΜΩΝ

**Δραστηριότητα 1**

*Εξερευνώντας την αξονική συμμετρία.*

1. Να επιλέξεις το λογισμικό Μαθαίνω Γεωμετρία και Μετρώ.

2. Από το μενού δραστηριοτήτων, να επιλέξεις το «Περιστροφή, Μεταφορά, Αντιστροφή».

3. Να επιλέξεις με τη σειρά **όλα** τα σχήματα που προσφέρει το λογισμικό, επιλέγοντας το εικονίδιο [Διάλεξε σχήμα]

4. Για κάθε σχήμα, να βρεις **όλες τις πιθανές εικόνες** που μπορεί να σχηματίσει κάποιος με **ανάκλαση**, επιλέγοντας τα εικονίδια [↔] [↕] [Μετακίνηση]

5. Να διακρίνεις στον πιο κάτω πίνακα τα σχήματα που δοκίμασες σε αυτά που οι πιθανές εικόνες τους δεν έχει **καμιά διαφορά** από το πρότυπο και σε αυτά που **κάποιες ή όλες** οι εικόνες τους έχουν.

Πίνακας 1

| Σχήματα που οι εικόνες του δεν έχουν καμιά διαφορά με το πρότυπο | Σχήματα που κάποιες ή όλες οι εικόνες του έχουν διαφορές από το πρότυπο |
|---|---|
| | |

6. Πότε νομίζεις ότι συμβαίνει σε ένα σχήμα οι εικόνες του να μην έχουν καμία διαφορά με το πρότυπο;

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7. Όταν η εικόνα ενός προτύπου στην ανάκλαση δεν έχει καμία διαφορά με το πρότυπο ως προς το μέγεθος, το σχήμα και τον προσανατολισμό, τότε το σχήμα έχει ***αξονική συμμετρία***.

8. Να βρεις και να γράψεις τη **σχέση** ανάμεσα στον **αριθμό αξόνων** συμμετρίας ενός σχήματος και τον **αριθμό** των **εικόνων** που δεν έχουν διαφορά από το πρότυπο.

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9. Να επιλέξεις το εικονίδιο [Έξοδος]



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

## Δραστηριότητα 2

*Εξερευνώντας την περιστροφική συμμετρία.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Περιστροφή».

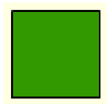
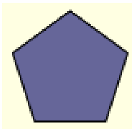
2. Να επιλέξεις με τη σειρά τα σχήματα που παρουσιάζονται στον Πίνακα 2, επιλέγοντας το εικονίδιο **Διάλεξε σχήμα**

3. Για κάθε σχήμα, να βρεις **όλες τις πιθανές εικόνες** που μπορεί να σχηματίσει κάποιος με περιστροφή, επιλέγοντας τα εικονίδιο  που είναι μέσα στο σχήμα.

4. Να προσπαθήσεις να βρεις για κάθε σχήμα όλες τις περιπτώσεις που η εικόνα **συμπίπτει ακριβώς** με το πρότυπο.
Όταν η εικόνα ενός προτύπου στην περιστροφή συμπίπτει ακριβώς με το πρότυπο τότε το σχήμα έχει *περιστροφική συμμετρία*.

5. Να συμπληρώσεις στον Πίνακα 2 τον **αριθμό των πλευρών** του κάθε κανονικού πολυγώνου και σε **πόσες** θέσεις έχει **περιστροφική συμμετρία**.

Πίνακας 2

| |  |  |  |
|---|---|---|---|
| Αριθμός πλευρών κανονικού πολυγώνου | | | |
| Αριθμός θέσεων περιστροφικής συμμετρίας | | | |

6. Να βρεις και να γράψεις τη **σχέση** ανάμεσα στον **αριθμό των πλευρών** ενός κανονικού πολυγώνου και τον **αριθμό των θέσεων** περιστροφικής συμμετρίας που έχει.

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**Έξοδος**

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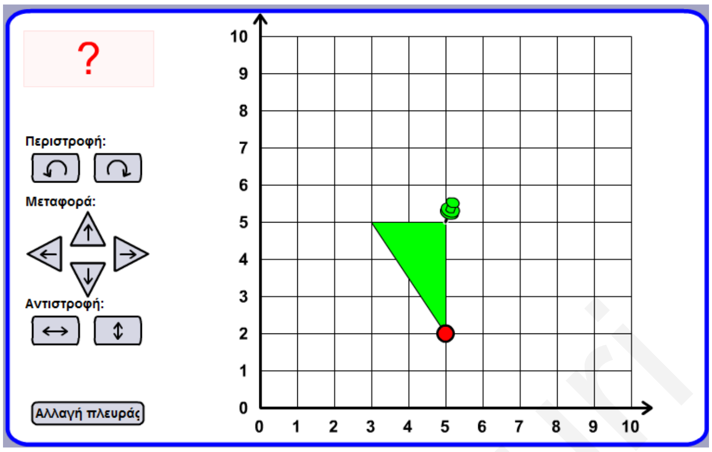
7. Να επιλέξεις το εικονίδιο

**Δραστηριότητα 3**

*Ανάκλαση σε δύο παράλληλες ευθείες.*

1. Από το μενού δραστηριοτήτων, να επιλέξεις το «Περιστροφή, Μεταφορά, Αντιστροφή».

2. Να επιλέξεις **όποια εικονίδια** χρειάζεσαι, ώστε να βρεις **δύο διαφορετικούς τρόπους** για να κατασκευάσεις την εικόνα του τριγώνου, όπως παρουσιάζεται πιο κάτω.




3. Να περιγράψεις ποιο/ποιους γεωμετρικούς μετασχηματισμούς χρησιμοποίησες την πρώτη φορά και ποιο/ποιους την δεύτερη φορά.

1η φορά: ....................................................................................................

2η φορά: ....................................................................................................

4. Με βάση τις παρατηρήσεις σου, να περιγράψεις **ποιος μοναδικός** μετασχηματισμός είναι **ισοδύναμος** με το αποτέλεσμα δύο άλλων διαδοχικών μετασχηματισμών.

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5. Να διερευνήσεις κατά πόσο το ίδιο φαινόμενο συμβαίνει και σε άλλες περιπτώσεις. **Να καταγράψεις** τις περιπτώσεις που βρίσκεις όπου το αποτέλεσμα **δύο διαδοχικών μετασχηματισμών** είναι **ισοδύναμο** με το αποτέλεσμα **ενός μετασχηματισμού**.

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INSTRUCTIONAL INTERVETION WITH CONTINUOUS DYNAMIC VISUALISATION WORKSHEETS

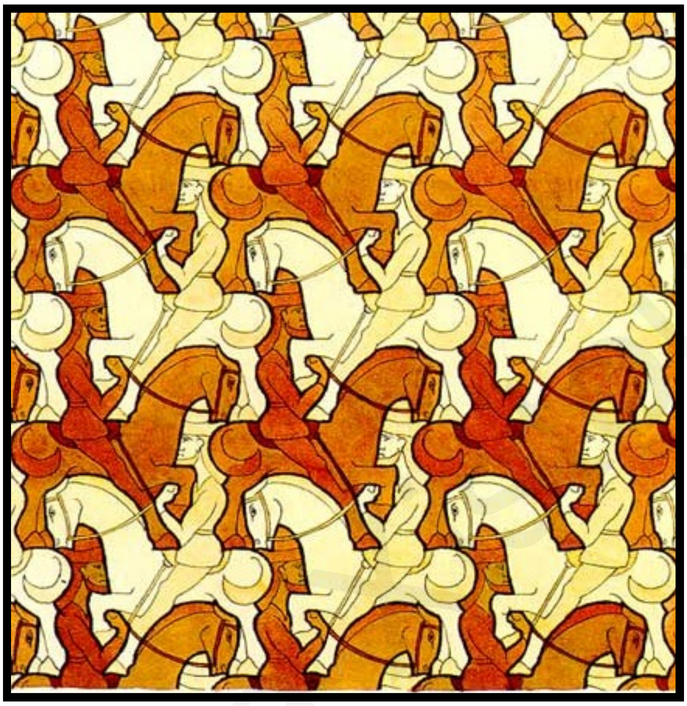
# ΜΑΘΗΜΑ 1: ΕΙΣΑΓΩΓΗ ΣΤΟΥΣ ΓΕΩΜΕΤΡΙΚΟΥΣ ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΥΣ

**Δραστηριότητα 1**

*Εξερευνώντας το σχηματισμό των ψηφιδωτών.*

Ένα Ολλανδός ζωγράφος, ο M.C. Escher (1898-1972), έφτιαχνε ζωγραφικούς πίνακες χρησιμοποιώντας διάφορα μοτίβα, τα οποία είναι γνωστά ως ψηφιδωτά.

1. Να ανοίξεις το αρχείο Δραστηριότητα1.ggb.
2. Να παρατηρήσεις τους πιο κάτω πίνακες και να περιγράψεις με λεπτομέρειες τα διαφορετικά μοτίβα που παρατηρείς. Να γράψεις λίγα λόγια ξεχωριστά για κάθε πίνακα.



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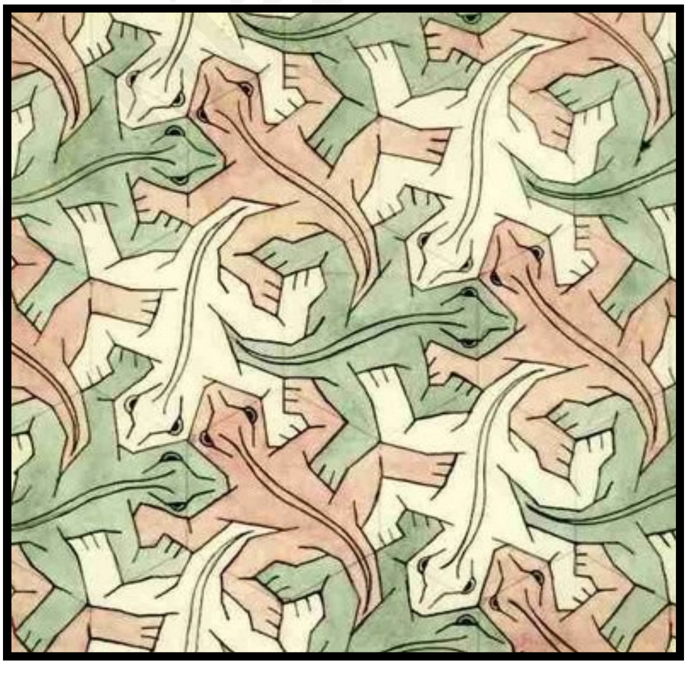

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Ο Escher μετακινούσε τις φιγούρες σύμφωνα με τους κανόνες των γεωμετρικών μετασχηματισμών της μεταφοράς, της ανάκλασης και της περιστροφής, τους οποίους θα γνωρίσουμε στις επόμενες εργασίες.

Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

**Δραστηριότητα 2**

*Διερευνώντας τη μεταφορά.*

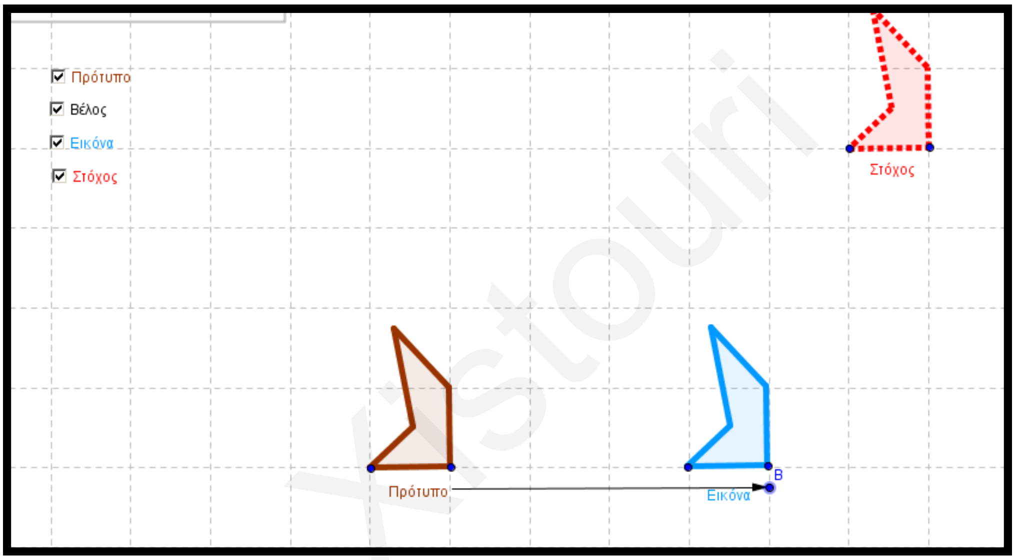
1. Να διαβάσεις το πιο κάτω κείμενο.

Μια φιγούρα μπορεί να **απεικονιστεί σε μια νέα θέση**. Αυτό στα μαθηματικά ονομάζεται ***μεταφορά***.
Η αρχική φιγούρα ονομάζεται ***πρότυπο***. Το αποτέλεσμα της μεταφοράς ονομάζεται ***εικόνα***.

2. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb

3. Να επιλέξεις τα κουτάκια δίπλα από τις λέξεις Πρότυπο, Βέλος, Εικόνα και Στόχος και να παρατηρήσεις τι εμφανίζεται/εξαφανίζεται κάθε φορά.

Εικόνα 1






Κάθε φορά που θες να αρχίσεις μια καινούρια προσπάθεια, να επιλέγεις το κουμπί της αναίρεσης

4. Να επιλέξεις **όλα** τα τετραγωνάκια ώστε να εμφανίζονται όλα τα αντικείμενα.
Να προσπαθήσεις να τοποθετήσεις την **εικόνα** στη θέση του **στόχου**.

5. Τι ενέργειες έκανες για να μπορέσεις να πετύχεις το στόχο;
Να τις περιγράψεις πιο κάτω.

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6. Τι παρατηρείς για τη σχέση του προτύπου και της εικόνας;

…………………………………………………………………………………………

7. Απάντησε στις πιο κάτω ερωτήσεις:

➢Τι παθαίνει η **εικόνα** όταν μετακινείς το **πρότυπο** προς τα **πάνω**;

…………………………………………………………………………………………

➢Τι παθαίνει η **εικόνα** όταν μετακινείς την **άκρη του βέλους** προς τα **δεξιά**;

…………………………………………………………………………………………

## Δραστηριότητα 4

*Διερευνώντας την περιστροφή.*

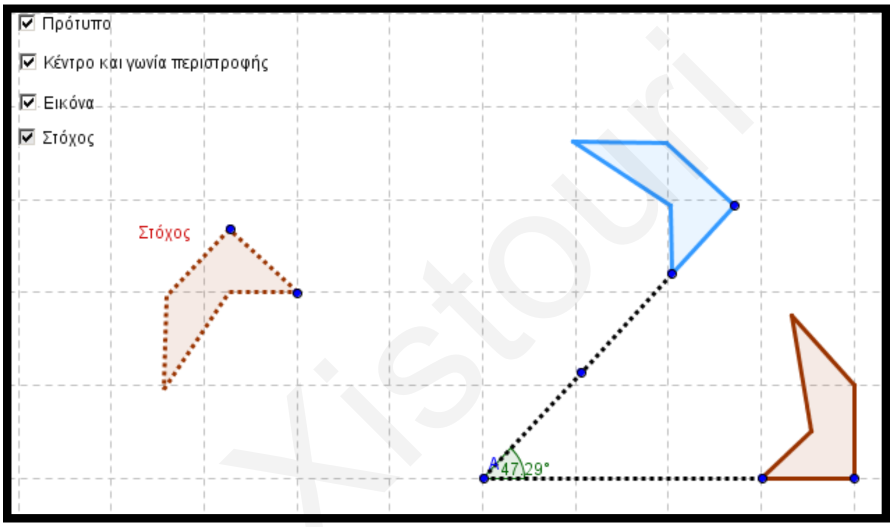
1. Να διαβάσεις το πιο κάτω κείμενο.

Μια φιγούρα μπορεί να **κάνει στροφή** γύρω από ένα σημείο. Αυτό στα μαθηματικά ονομάζεται ***περιστροφή***.
Η αρχική φιγούρα ονομάζεται ***πρότυπο***. Το αποτέλεσμα της περιστροφής ονομάζεται ***εικόνα***.

2. Να ανοίξεις το αρχείο Δραστηριότητα4.ggb

3. Να επιλέξεις τα κουτάκια δίπλα από τις λέξεις Πρότυπο, Κέντρο και γωνία περιστροφής, Εικόνα και Στόχος και να παρατηρήσεις τι εμφανίζεται/ εξαφανίζεται κάθε φορά.

Εικόνα 3




🖈 Κάθε φορά που θες να αρχίσεις μια καινούρια προσπάθεια, να επιλέγεις το κουμπί της αναίρεσης

4. Να επιλέξεις όλα τα τετραγωνάκια ώστε να εμφανίζονται όλα τα αντικείμενα.
Να προσπαθήσεις να τοποθετήσεις την εικόνα στη θέση του στόχου.

5. Τι ενέργειες έκανες για να μπορέσεις να πετύχεις το στόχο;
Να τις περιγράψεις πιο κάτω.

………………………………………………………………………………………………

………………………………………………………………………………………………

6. Τι παρατηρείς για τη σχέση του προτύπου και της εικόνας;

………………………………………………………………………………………………

7. Απάντησε στις πιο κάτω ερωτήσεις:

➢Τι παθαίνει η **εικόνα** όταν **μεγαλώνεις τη γωνία περιστροφής**;

………………………………………………………………………………………………

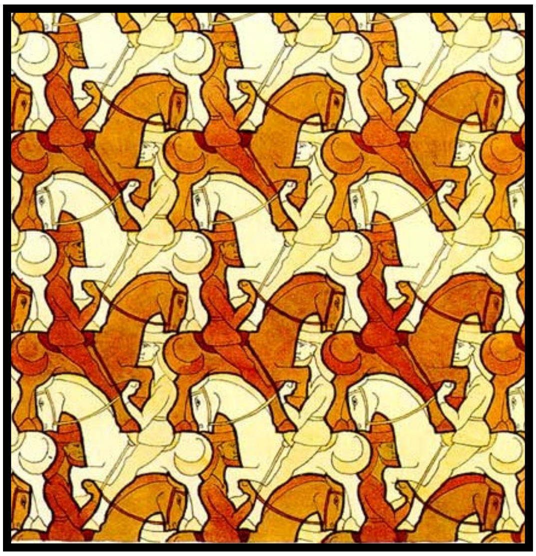
➢Τι παθαίνει η **εικόνα** όταν μετακινείς το **πρότυπο πιο κοντά στο κέντρο περιστροφής**;

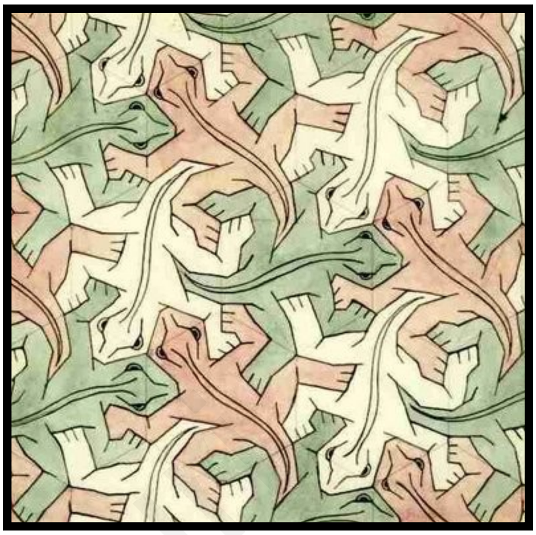
………………………………………………………………………………………………

**Δραστηριότητα 5**

*Παρατηρώντας τους μετασχηματισμούς στους πίνακες του Escher.*

1. Οι γεωμετρικοί μετασχηματισμοί της Μεταφοράς, της Ανάκλασης και της Περιστροφής είναι οι κανόνες που εφάρμοζε στην τέχνη του ο Escher. Μπορείς να εντοπίσεις παραδείγματα στους πίνακες του;





2. Να γράψεις ποιος από τους τρεις γεωμετρικούς μετασχηματισμούς της μεταφοράς, της ανάκλασης και της περιστροφής μπορεί να περιγράψει καλύτερα τη σχέση ανάμεσα:

α ) στο άσπρο άλογο με ένα καφέ άλογο; ……………………………………

β ) στο άσπρο άλογο με ένα άσπρο άλογο: ……………………………………

γ ) στην άσπρη σαύρα με μια χρωματιστή σαύρα: …………………………..

δ ) στην άσπρη σαύρα με μια άσπρη σαύρα: ………………………………

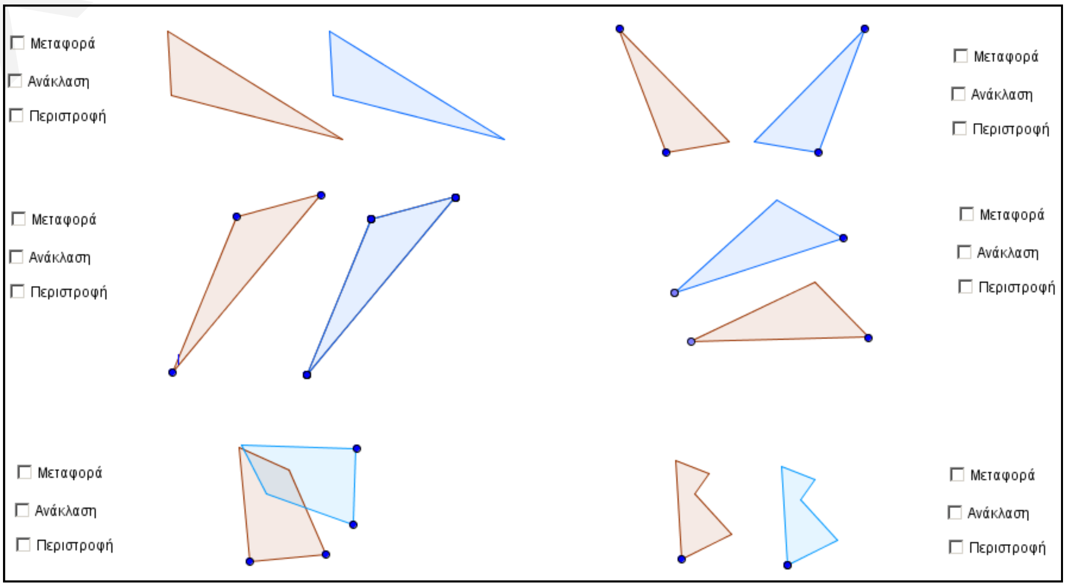
**Δραστηριότητα 6**

*Ανακαλύπτοντας το γεωμετρικό μετασχηματισμό.*

1. Να ανοίξεις το αρχείο Δραστηριότητα6.ggb.

2. Να επιλέξεις το είδος του μετασχηματισμού που περιγράφει καλύτερα το κάθε παράδειγμα, επιλέγοντας το αντίστοιχο κουτάκι.

3. Να εκτυπώσεις την εργασία σου.



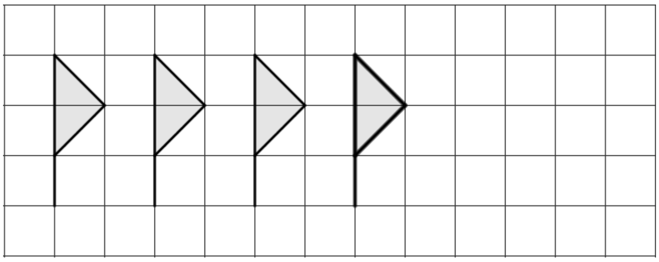
# ΜΑΘΗΜΑ 2: ΜΕΤΑΦΟΡΑ

**Δραστηριότητα 1**

*Εξερευνώντας τους κανόνες της μεταφοράς.*

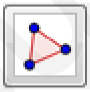
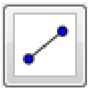
1. Η Στέλλα φτιάχνει μια κορνίζα για το δωμάτιο της. Θα φτιάξει ένα μοτίβο μεταφοράς.

Εικόνα 1



2. Να ανοίξεις το αρχείο Δραστηριότητα1.ggb.

3. Να παρατηρήσεις το μοτίβο της Στέλλας στην Εικόνα 1. Να ακολουθήσεις τις πιο κάτω οδηγίες για να σχεδιάσεις τα επόμενα 2 σχήματα στο GeoGebra.

4. Να χρησιμοποιήσεις το εργαλείο «Πολύγωνο»  για να κατασκευάσεις τα τρίγωνα και το εργαλείο «Τμήμα μεταξύ δύο σημείων»  για να κατασκευάσεις τα ευθύγραμμα τμήματα.

5. Ποιους κανόνες πρέπει να εφαρμόσει η Στέλλα για να μεταφέρει σωστά το σχήμα;

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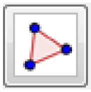
**Δραστηριότητα 2**

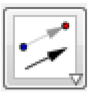
*Διερευνώντας το σχήμα της εικόνας.*

1. Να χρησιμοποιήσεις τα εργαλεία του GeoGebra για να μεταφέρεις ένα δικό σου πολύγωνο, ακολουθώντας τα πιο κάτω βήματα.

2. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb.

3. Να κατασκευάσεις ένα πολύγωνο με 5 κορυφές, χρησιμοποιώντας το εργαλείο «Πολύγωνο».  Αυτό το σχήμα θα είναι το **<u>πρότυπο</u>** σου.

4. Να επιλέξεις το εργαλείο «Διάνυσμα μεταξύ δύο σημείων»  για να κατασκευάσεις ένα διάνυσμα.

5. Να επιλέξεις το εργαλείο «Μεταφορά αντικειμένου με διάνυσμα» 

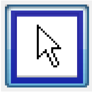
Στη συνέχεια να επιλέξεις **<u>πρώτα</u>** το πολύγωνο και **<u>στη συνέχεια</u>** το διάνυσμα.

Το νέο σχήμα που εμφανίζεται είναι η **<u>εικόνα</u>** του προτύπου σου.



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

ΔΕΣΜΗ 2009-2010 Ίδρυμα Προώθησης Έρευνας

5. Με το εργαλείο «Μετακίνηση»  να επιλέξεις και **<u>να σύρεις</u>** οποιαδήποτε **<u>κορυφή</u>** ή **<u>πλευρά</u>** του **<u>προτύπου</u>** σου.

Να κάνεις το ίδιο και με άλλες πλευρές/κορυφές.

**<u>Να γράψεις τις παρατηρήσεις σου</u>** για τη σχέση που έχουν το πρότυπο και η εικόνα όσον αφορά το **<u>σχήμα</u>** και τη **<u>θέση</u>** τους.

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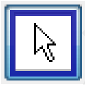
**Δραστηριότητα 3**

*Διερευνώντας το μέγεθος της εικόνας.*

1. Να ανοίξεις το αρχείο Δραστηριότητα3.ggb.

2. Με τη βοήθεια του εργαλείου «Μετακίνηση»  να επιλέξεις και **<u>να σύρεις</u>** την **<u>κορυφή Γ</u>** του προτύπου σου.

**<u>Να γράψεις τις παρατηρήσεις σου</u>** για τη σχέση που έχουν το πρότυπο και η εικόνα όσον αφορά το **<u>μέγεθος των πλευρών</u>** της και το **<u>μέγεθος των γωνιών</u>** της.

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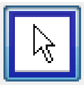
**Δραστηριότητα 4**

*Διερευνώντας τον προσανατολισμό της εικόνας.*

1. Να ανοίξεις το αρχείο Δραστηριότητα4.ggb. 

2. Με τη βοήθεια του εργαλείου «Μετακίνηση» να επιλέξεις και **<u>να σύρεις</u>** την **<u>κορυφή Β</u>** του προτύπου, έτσι ώστε να περιστραφεί ολόκληρο το σχήμα.

**<u>Να γράψεις τις παρατηρήσεις σου</u>** για τη σχέση που έχουν το πρότυπο και η εικόνα όσον αφορά τον **<u>προσανατολισμό</u>** της (δηλαδή την κατεύθυνση στην οποία βρίσκεται η κάθε πλευρά/κορυφή στο σχήμα).

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**Δραστηριότητα 5**

*Διατυπώνοντας ένα γενικό συμπέρασμα.*

1.**<u>Να γράψεις ένα γενικό συμπέρασμα</u>** για τις παρατηρήσεις σου στις προηγούμενες δραστηριότητες.

Να αναφερθείς στη **<u>μορφή</u>**, στο **<u>μέγεθος</u>** και τον **<u>προσανατολισμό</u>** της εικόνας ενός σχήματος που μεταφέρεται.

**ΓΕΝΙΚΟ ΣΥΜΠΕΡΑΣΜΑ**

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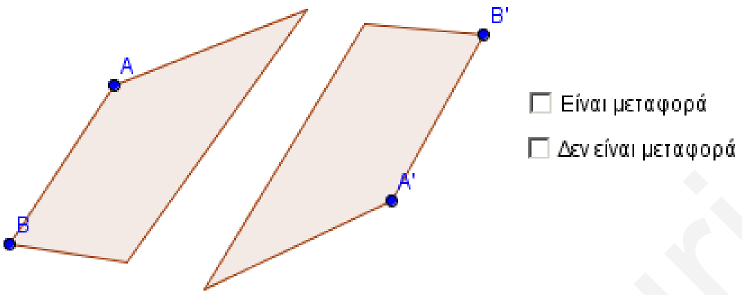
**Δραστηριότητα 6**

*Αναγνωρίζοντας τα παραδείγματα της μεταφοράς.*

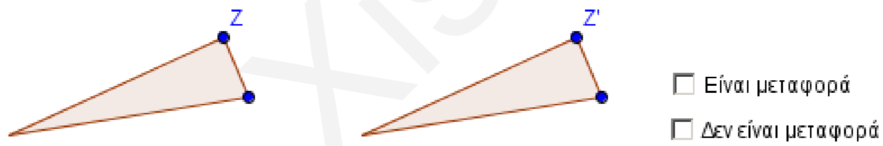
1. Να ανοίξεις το αρχείο Δραστηριότητα5.ggb.

2. Να παρατηρήσεις τα παραδείγματα και **να αποφασίσεις** εάν παρουσιάζουν ή όχι μεταφορά.
Μπορείς να δοκιμάσεις να επιλέξεις το πρότυπο και να το μετακινήσεις για να βεβαιωθείς.

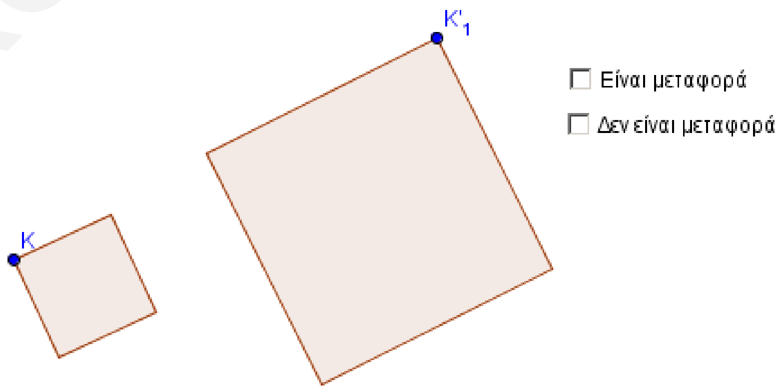
3. **Να σημειώσεις** το κουτάκι με την απάντησή σου και **να γράψεις** για ποιο λόγο την επέλεξες.




Γιατί: ......................................................................................




Γιατί: ......................................................................................

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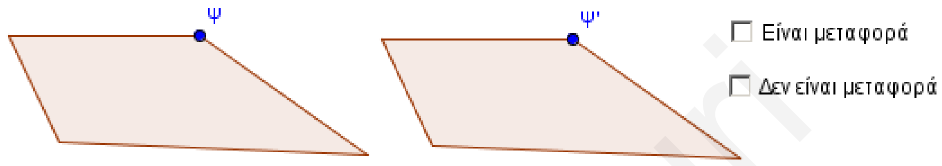
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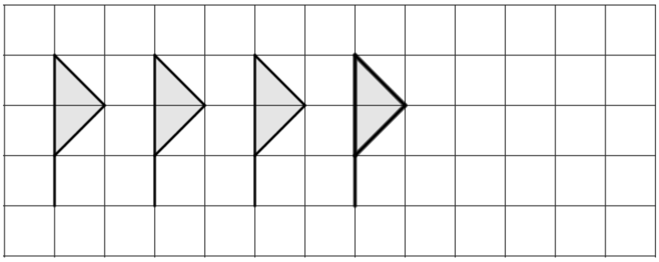
# ΜΑΘΗΜΑ 3: ΜΕΤΑΦΟΡΑ (ΜΕΡΟΣ Β΄)

## Δραστηριότητα 1

*Εξερευνώντας τις παραμέτρους της μεταφοράς.*

1. Η Στέλλα φτιάχνει μια κορνίζα για το δωμάτιό της. Θα φτιάξει ένα μοτίβο μεταφοράς, όπως την Εικόνα 1.

Εικόνα 1



2. Να ανοίξεις το αρχείο Δραστηριότητα1.ggb.

3. Να κατασκευάσεις τα επόμενα δύο σχήματα του μοτίβου χρησιμοποιώντας το διάνυσμα ΑΒ και **<u>το εργαλείο</u>** «Μεταφορά αντικειμένου με διάνυσμα»

4. Να περιγράψεις **<u>τον τρόπου που εργάστηκες</u>** για να συμπληρώσεις το μοτίβο:

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## Δραστηριότητα 2

*Διερευνώντας την κατεύθυνση στη μεταφορά.*

1. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb.

2. Να σύρεις **<u>την κορυφή</u> του διανύσματος** έτσι ώστε να αλλάξεις την κατεύθυνση του γύρω από το πρότυπο.

3. Να τοποθετήσεις το διάνυσμα στη θέση ώστε η **<u>γωνία</u>** που σχηματίζει το διάνυσμα με τον οριζόντιο άξονα **<u>να είναι 30˚</u>**.

Να γράψεις τις αλλαγές που παρατηρείς στο σχήμα.

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4. **<u>Να σύρεις</u>** την **<u>κορυφή</u>** του διανύσματος έτσι ώστε να τοποθετήσεις την εικόνα σε **<u>διάφορες θέσεις</u>** γύρω από το πρότυπο.

Αυτή τη φορά, να παρατηρήσεις τον **<u>προσανατολισμό</u>** (δηλαδή την κατεύθυνση που βρίσκεται η κάθε κορυφή Α΄ , Β΄ και Γ΄ ) στο σχήμα της εικόνας και του προτύπου. Να γράψεις τις παρατηρήσεις σου.

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Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

5. Να συμπληρώσεις στον **Πίνακα 1** την **κατεύθυνση** που βρίσκεται η εικόνα σε σχέση με το πρότυπο, όταν η **γωνία** που σχηματίζει το διάνυσμα με τον οριζόντιο άξονα έχει την **τιμή** που αναφέρεται στην πρώτη στήλη.

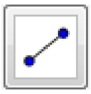
Πίνακας 1

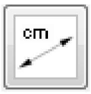
| **Γωνία Διανύσματος με Οριζόντιο άξονα** | **Κατεύθυνση προτύπου** |
|---|---|
| $0^{\circ}$ | *δεξιά* |
| $60^{\circ}$ | |
| | *πάνω* |
| $100^{\circ}$ | |
| $120^{\circ}$ | |
| | *αριστερά* |
| $200^{\circ}$ | |
| $250^{\circ}$ | |
| $270^{\circ}$ | |
| $310^{\circ}$ | |

**Δραστηριότητα 3**

*Διερευνώντας την απόσταση ανάμεσα στο πρότυπο και την εικόνα.*

1. Να ανοίξεις το αρχείο Δραστηριότητα3.ggb.

2. Να χρησιμοποιήσεις το εργαλείο «Τμήμα μεταξύ δύο σημείων»  για να δημιουργήσεις <u>**ευθύγραμμα τμήματα**</u> που να συνδέουν τις <u>**αντίστοιχες κορυφές**</u> (πχ. την Α με την Α΄ ) του πρότυπου και της εικόνας.

3. Να επιλέξεις το εργαλείο «Απόσταση ή μήκος»  και στη συνέχεια να επιλέξεις <u>**τα ευθύγραμμα τμήματα**</u> που κατασκεύασες για να μετρήσεις την <u>**απόσταση**</u> ανάμεσα στις αντίστοιχες κορυφές.

Να σύρεις την κορυφή Ε του διανύσματος και να παρατηρήσεις τις αλλαγές.

Να συγκρίνεις τις <u>**αποστάσεις**</u> ανάμεσα στις <u>**αντίστοιχες κορυφές**</u> και να γράψεις τις παρατηρήσεις σου.

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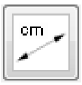
4. Να επιλέξεις πάλι το εργαλείο «Απόσταση ή μήκος»  και στη συνέχεια να επιλέξεις τα <u>**σημεία**</u> του διανύσματος **Δ** και **Ε** για να υπολογίσεις το μήκος του.

5. **Να επιλέξεις το σημείο Ε** και να το σύρεις για **να αλλάξεις το μήκος** του διανύσματος.

Να συγκρίνεις τις **αποστάσεις** ανάμεσα στο **πρότυπο** και στην **εικόνα** με το μήκος του **διανύσματος** και να γράψεις τις παρατηρήσεις σου.

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6. **Να σύρεις** το διάνυσμα τόσο ώστε το μήκος του **να είναι ίσο με 1** cm.

Τι παρατηρείς για την εικόνα;

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**Δραστηριότητα 4**

*Βρίσκοντας τη σχέση ανάμεσα στην κατεύθυνση και τις συντεταγμένες ενός σημείου.*

1. Να ανοίξεις το αρχείο Δραστηριότητα4.ggb.

2. Να παρατηρήσεις τα σχήματα και το διάνυσμα και **να περιγράψεις τη μεταφορά** του προτύπου ΑΒΓ (πόσο μεταφέρθηκε και προς τα πού).

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3. Να σύρεις την κορυφή του διανύσματος ή οποιαδήποτε από τις κορυφές του τριγώνου και να παρατηρήσεις τις **αλλαγές στις συντεταγμένες**

Να βρεις μια **σχέση** ανάμεσα στις συντεταγμένες ενός σημείου και στις συντεταγμένες της εικόνας του όταν αυτό μεταφέρεται.

Να την περιγράψεις:

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4. Να απαντήσεις στις πιο κάτω ερωτήσεις:

α) Σε ποια κατεύθυνση μπορείς να σύρεις την κορυφή του διανύσματος έτσι ώστε τα σημεία του προτύπου και οι αντίστοιχες εικόνες τους να έχουν πάντα τις **ίδιες** συντεταγμένες **χ** και **διαφορετικές ψ**;

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β) Σε ποια κατεύθυνση μπορείς να σύρεις την κορυφή του διανύσματος έτσι ώστε τα σημεία του προτύπου και οι αντίστοιχες εικόνες τους να έχουν πάντα τις **ίδιες** συντεταγμένες **ψ** και **διαφορετικές χ**;

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γ) Να επιλέξεις το κουτάκι για να εμφανίσεις τις συντεταγμένες του Δ.

Όταν το διάνυσμα μεταφέρει το πρότυπο προς τα **πάνω και δεξιά**, τι πρέπει να συμβεί στις συντεταγμένες του Δ;

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δ) Εάν το τρίγωνο μεταφερθεί κάποιο **α** αριθμό θέσεων προς τα δεξιά και κάποιο **β** αριθμό θέσεων προς τα πάνω, τότε πώς θα αλλάξουν οι συντεταγμένες **της εικόνας** ενός σημείου (χ, ψ);

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ε) Μπορείς να χρησιμοποιήσεις τα σύμβολα **α, β, χ, και ψ** για να συμβολίσεις **μέσα στην παρένθεση** τις νέες συντεταγμένες;

( , )

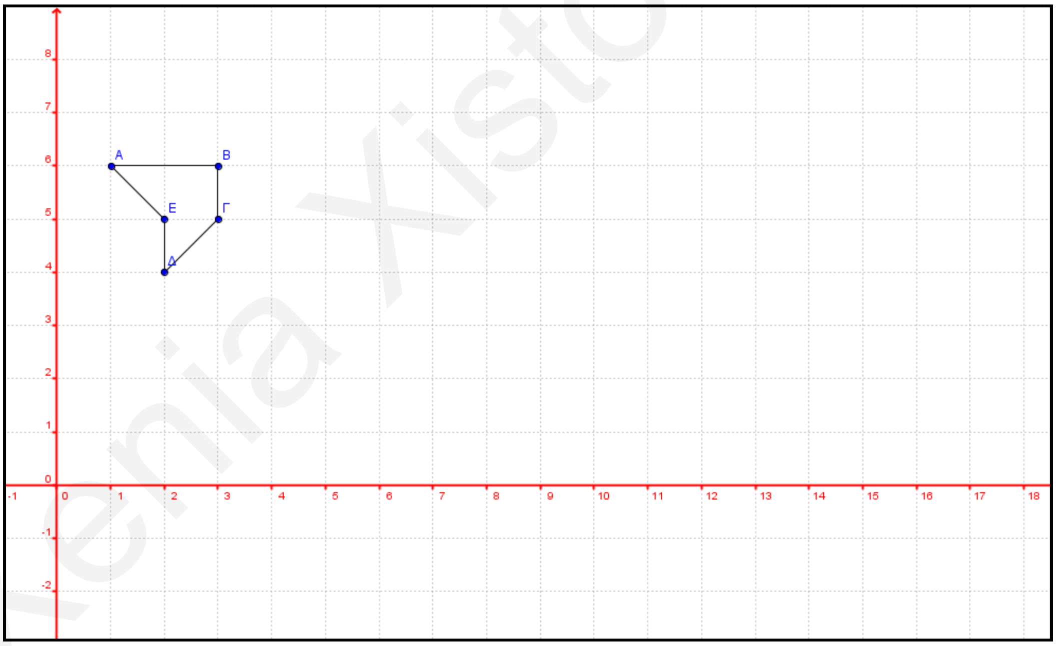
## Δραστηριότητα 5

*Κατασκευάζοντας την εικόνα στη μεταφορά.*

1. Να ανοίξεις το αρχείο Δραστηριότητα5.ggb.

2. Να χρησιμοποιήσεις τα εργαλεία που έχεις για να μεταφέρεις το σχήμα ακριβώς **πέντε θέσεις δεξιά και δύο κάτω**.

3. **Να σχεδιάσεις** την απάντησή σου στον πιο κάτω χώρο.



# ΜΑΘΗΜΑ 4: ΑΝΑΚΛΑΣΗ

**Δραστηριότητα 1**

*Εξερευνώντας τους κανόνες της ανάκλασης.*

1. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb.

2. Να επιλέξεις το εργαλείο «Ευθεία που περνά από δύο σημεία» και να κατασκευάσεις μια ευθεία ΑΒ με το εργαλείο .

3. Να επιλέξεις το εργαλείο «Κατασκευή σημείου» και να κατασκευάσεις ένα σημείο Γ στα δεξιά του ευθύγραμμου τμήματος.

4. Να επιλέξεις το εργαλείο «Συμμετρία αντικειμένου ως προς ευθεία»
   Στη συνέχεια να επιλέξεις πρώτα το σημείο Γ και στη συνέχεια την ευθεία.

5. Να κάνεις δεξί κλικ στο σημείο Γ και να επιλέξεις την επιλογή «Ίχνος ενεργό».
   Να κάνεις το ίδιο για το σημείο Γ΄ .

6. Να σύρεις το σημείο Γ ώστε να γράψεις το όνομά σου.
   Τι ίχνος αφήνει το σημείο Γ΄ ;

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7. Πού συναντάς το φαινόμενο αυτό στην καθημερινή ζωή;

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**Δραστηριότητα 2**

*Διερευνώντας το σχήμα της εικόνας.*

1. Να χρησιμοποιήσεις τα εργαλεία του GeoGebra για να ανακλάσεις ένα δικό σου πολύγωνο, ακολουθώντας τα πιο κάτω βήματα.

2. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb.

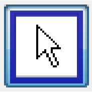
3. Να κατασκευάσεις ένα <u>**πολύγωνο**</u> με 5 κορυφές, χρησιμοποιώντας το εργαλείο «Πολύγωνο». Αυτό το σχήμα θα είναι το <u>**πρότυπο**</u> σου.

4. Να επιλέξεις το εργαλείο «Ευθεία που περνά από δύο σημεία» και να κατασκευάσεις μια ευθεία.

4. Να επιλέξεις το εργαλείο «Συμμετρία αντικειμένου ως προς ευθεία»
   Στη συνέχεια να επιλέξεις <u>**πρώτα**</u> το πολύγωνο και στη <u>**συνέχεια**</u> την ευθεία.
   Το νέο σχήμα που εμφανίζεται είναι η <u>**εικόνα**</u> του προτύπου σου.



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

5. Με το εργαλείο «Μετακίνηση»  να επιλέξεις και **να σύρεις** οποιαδήποτε **κορυφή** ή **πλευρά** του **προτύπου** σου.

Να κάνεις το ίδιο και με άλλες πλευρές/κορυφές.

**Να γράψεις τις παρατηρήσεις σου** για τη σχέση που έχουν το πρότυπο και η εικόνα όσον αφορά το **σχήμα** και τη **θέση** τους.

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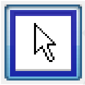
**Δραστηριότητα 3**

*Διερευνώντας το μέγεθος της εικόνας.*

1. Να ανοίξεις το αρχείο Δραστηριότητα3.ggb.

2. Με τη βοήθεια του εργαλείου «Μετακίνηση»  να επιλέξεις και **να σύρεις** την **κορυφή Γ** του προτύπου σου.

**Να γράψεις τις παρατηρήσεις σου** για τη σχέση που έχουν το πρότυπο και η εικόνα όσον αφορά το **μέγεθος των πλευρών** της και το **μέγεθος των γωνιών** της.

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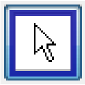
**Δραστηριότητα 4**

*Διερευνώντας τον προσανατολισμό της εικόνας.*

1. Να ανοίξεις το αρχείο Δραστηριότητα4.ggb.

2. Με τη βοήθεια του εργαλείου «Μετακίνηση»  να επιλέξεις και **να σύρεις** την **κορυφή Β** του προτύπου, έτσι ώστε να περιστραφεί ολόκληρο το σχήμα.

**Να γράψεις τις παρατηρήσεις σου** για τη σχέση που έχουν το πρότυπο και η εικόνα όσον αφορά τον **προσανατολισμό** της (δηλαδή την κατεύθυνση στην οποία βρίσκεται η κάθε πλευρά/κορυφή στο σχήμα).

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**Δραστηριότητα 5**

*Διατυπώνοντας ένα γενικό συμπέρασμα.*

1. **Να γράψεις ένα γενικό συμπέρασμα** για τις παρατηρήσεις σου στις προηγούμενες δραστηριότητες.

Να αναφερθείς στη **μορφή**, στο **μέγεθος** και τον **προσανατολισμό** της εικόνας ενός σχήματος που μεταφέρεται.

**ΓΕΝΙΚΟ ΣΥΜΠΕΡΑΣΜΑ**

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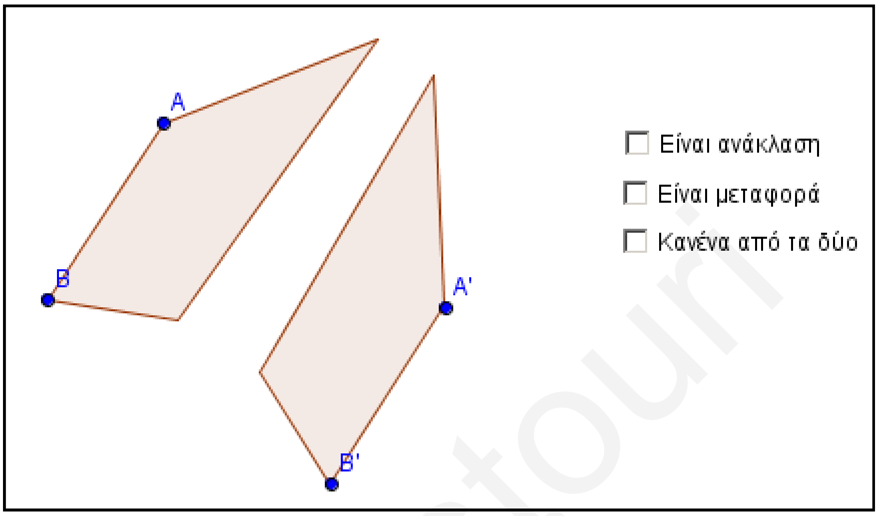
## Δραστηριότητα 6

*Αναγνωρίζοντας το παράδειγμα της ανάκλασης.*

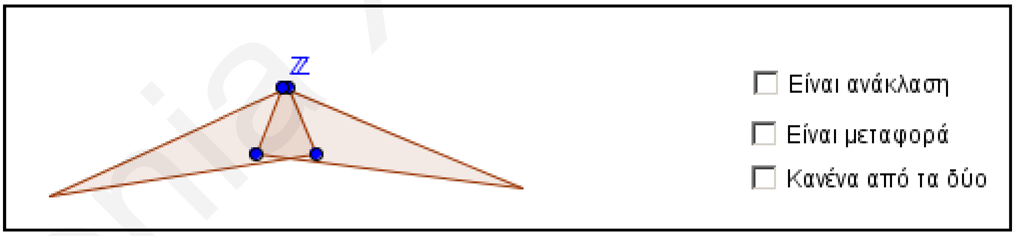
1. Να ανοίξεις το αρχείο Δραστηριότητα5.ggb.

2. Να παρατηρήσεις τα παραδείγματα και **<u>να αποφασίσεις</u>** εάν παρουσιάζουν ανάκλαση ή μεταφορά.
Μπορείς να δοκιμάσεις να επιλέξεις το πρότυπο και να το μετακινήσεις για να βεβαιωθείς.

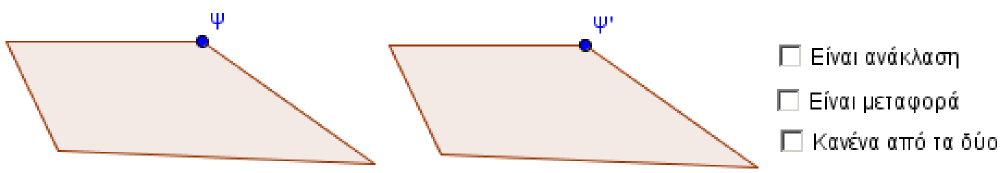
3. **<u>Να σημειώσεις</u>** το κουτάκι με την απάντησή σου και **<u>να γράψεις</u>** για ποιο λόγο την επέλεξες.




Γιατί: ..........................................................................




Γιατί: ..........................................................................



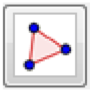

Γιατί: ..........................................................................

# ΜΑΘΗΜΑ 5: ΑΝΑΚΛΑΣΗ (συνέχεια)

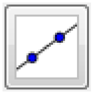
1. Να ανοίξεις το αρχείο Δραστηριότητα1.ggb.

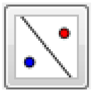
**Δραστηριότητα 1**

*Εξερευνώντας τις παραμέτρους της ανάκλασης.*

2. <u>Να κατασκευάσεις</u> ένα πολύγωνο, χρησιμοποιώντας το εργαλείο  «Πολύγωνο». Αυτό το σχήμα θα είναι το πρότυπο σου.

3. Να χρησιμοποιήσεις τα εργαλεία του GeoGebra για <u>να ανακλάσεις</u> το πολύγωνο σου. Να ακολουθήσεις τα πιο κάτω βήματα.

4. Να επιλέξεις το εργαλείο «Ευθεία που περνά από δύο σημεία»  και να κατασκευάσεις μια ευθεία.

5. Να επιλέξεις το εργαλείο «Συμμετρία αντικειμένου ως προς ευθεία»  Στη συνέχεια να επιλέξεις πρώτα το πολύγωνο και στη συνέχεια την ευθεία. Το νέο σχήμα που εμφανίζεται είναι η <u>εικόνα</u> του προτύπου σου.

6. Με το εργαλείο «Μετακίνηση»  <u>να σύρεις</u> και <u>να περιστρέψεις</u> την ευθεία γύρω από το πρότυπο. Εξακολουθεί να ισχύει η ανάκλαση;

<u>Να γράψεις</u> τις παρατηρήσεις σου.

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7. <u>Να εξερευνήσεις</u> τις πιο κάτω περιπτώσεις:

α) Πότε έχεις τη <u>μικρότερη απόσταση</u> ανάμεσα στο πρότυπο και την εικόνα;

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β) Πότε έχεις τη <u>μεγαλύτερη απόσταση</u> ανάμεσα στο πρότυπο και την εικόνα;

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γ) Πότε η εικόνα <u>επικαλύπτει</u> το πρότυπο;

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**Δραστηριότητα 2**

*Διερευνώντας την κατεύθυνση της εικόνας σε σχέση με το πρότυπο.*

1. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb.

2. <u>Να εξερευνήσεις</u> ποιες από τις περιπτώσεις που περιγράφονται στον Πίνακα 1 μπορεί να συμβαίνουν και <u>να σημειώσεις</u> ✓ στην κατάλληλη στήλη.



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

ΔΕΣΜΗ 2009-2010

Ίδρυμα Προώθησης Έρευνας

Πίνακας 1

| ΠΕΡΙΠΤΩΣΗ | ΝΑΙ | ΟΧΙ |
|---|---|---|
| Να είναι η εικόνα δεξιά από το πρότυπο | | |
| Να είναι η εικόνα πάνω από το πρότυπο | | |
| Να είναι η εικόνα πάνω και δεξιά από το πρότυπο | | |
| Να είναι η εικόνα αριστερά από το πρότυπο | | |
| Να είναι η εικόνα κάτω από το πρότυπο | | |
| Να είναι η εικόνα κάτω και αριστερά από το πρότυπο | | |
| Να είναι η εικόνα κάτω και δεξιά από το πρότυπο | | |
| Να βρίσκονται και το πρότυπο και η εικόνα αριστερά από τον άξονα συμμετρίας | | |

3. Τι ήταν αυτό που άλλαξες για να ελέγξεις τις πιο πάνω περιπτώσεις;

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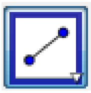
**Δραστηριότητα 3**

*Διερευνώντας την απόσταση της εικόνας από το πρότυπο.*

1. Να ανοίξεις το αρχείο Δραστηριότητα3.ggb.

2. Να χρησιμοποιήσεις το εργαλείο «Τμήμα μεταξύ δύο σημείων»  για να δημιουργήσεις <u>ευθύγραμμα τμήματα</u> που να συνδέουν τις αντίστοιχες κορυφές (πχ. την Α με την Α΄ ) του πρότυπου και της εικόνας.

3. Να σύρεις τον <u>άξονα συμμετρίας</u> ή το <u>πρότυπο</u> σε διάφορες θέσεις και να παρατηρήσεις τη σχέση ανάμεσα στα ευθύγραμμα τμήματα και στον άξονα συμμετρίας.

Ποια είναι <u>η σχέση του άξονα συμμετρίας με ένα ευθύγραμμο τμήμα</u> που συνδέει μια κορυφή με την εικόνα της;

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*Να επιλέγεις κάθε φορά πρώτα την ευθεία και στη συνέχεια το σημείο που θέλεις να μετρήσεις την απόσταση*

4. Να επιλέξεις το εργαλείο «Απόσταση ή μήκος».

Να μετρήσεις την απόσταση που έχει ο άξονας συμμετρίας από τα σημεία Α, Α΄ ,Β, Β΄ ,Γ, Γ΄ .

Να σύρεις τον άξονα συμμετρίας και να παρατηρήσεις τις αλλαγές στις αποστάσεις.

Να συγκρίνεις τις <u>αποστάσεις</u> ανάμεσα στον άξονα συμμετρίας και στις <u>αντίστοιχες κορυφές</u> και να γράψεις τις παρατηρήσεις σου.

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**Δραστηριότητα 4**

*Βρίσκοντας τη σχέση ανάμεσα στην κατεύθυνση και τις συντεταγμένες ενός σημείου.*

1. Να ανοίξεις το αρχείο Δραστηριότητα4α.ggb.

2. **Να σύρεις** τις κορυφές του προτύπου σε διάφορα σημεία του πλέγματος και να εξερευνήσεις τη **σχέση** ανάμεσα στις **συντεταγμένες** μιας κορυφής και στις **συνταγμένες** της **εικόνας** της με άξονα συμμετρίας τον **κατακόρυφο άξονα**.

**Να περιγράψεις** τη σχέση που ανακάλυψες.

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3. Να ανοίξεις το αρχείο Δραστηριότητα4β.ggb.

4. Με βάση τις παρατηρήσεις σου στην ερώτηση 2, ποια νομίζεις ότι θα είναι η **σχέση** ανάμεσα στις συντεταγμένες **μιας κορυφής** και στις συνταγμένες της **εικόνας** της με άξονα συμμετρίας τον **οριζόντιο άξονα**;

5. **Να επιλέξεις** το κουτάκι δίπλα από τη λέξη «Συντεταγμένες» ώστε **να εμφανιστούν** οι συντεταγμένες των κορυφών της εικόνας.

**Να σύρεις** τις κορυφές του προτύπου σε **διάφορα σημεία** του πλέγματος και για **να ελέγξεις την υπόθεσή** σου.

**Να περιγράψεις** τη σχέση που ανακάλυψες.

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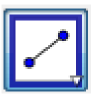
**Δραστηριότητα 5**

*Κατασκευάζοντας την εικόνα στην ανάκλαση.*

1. Να ανοίξεις το αρχείο Δραστηριότητα5.ggb.

2. **Να φανταστείς** ότι το GeoGebra **δεν έχει** εργαλείο για την ανάκλαση. Πώς θα κατασκεύαζες την εικόνα ενός σημείου σε κατακόρυφο άξονα συμμετρίας;

3. **Να βρεις** την εικόνα ενός σημείου κι ενός ευθύγραμμου τμήματος, χρησιμοποιώντας τα εργαλεία «Νέο σημείο» 

και «Τμήμα μεταξύ δύο σημείων» 

**Να περιγράψεις** τον τρόπο που εργάστηκες:

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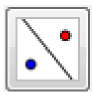
4. **Να χρησιμοποιήσεις** το εργαλείο «Συμμετρία αντικειμένου ως προς ευθεία» για **να ελέγξεις την απάντησή** σου. 

5. Να εκτυπώσεις την εργασία σου.

# ΜΑΘΗΜΑ 6: ΠΕΡΙΣΤΡΟΦΗ

**Δραστηριότητα 1**

*Εξερευνώντας τους κανόνες της περιστροφής.*

1. Να ανοίξεις το αρχείο Δραστηριότητα1.ggb.

2. Να επιλέξεις το εργαλείο «Τμήμα με δοσμένο μήκος από σημείο» και να κατασκευάσεις ένα ευθύγραμμο τμήμα με **μήκος 4 εκατοστά**.

3. Να κάνεις δεξί κλικ στο **σημείο Β** και **να επιλέξεις** την επιλογή «Ίχνος ενεργό».
   Να κάνεις το **ίδιο** για το σημείο Α.

4. Να επιλέξεις το εργαλείο «Στροφή γύρω από σημείο» και να επιλέξεις το **σημείο Α**.

5. Να σύρεις **το σημείο Β** προς τα **δεξιά**. Και να παρατηρήσεις τι συμβαίνει.
   Να δοκιμάσεις να το σύρεις και προς τα **αριστερά**.
   Να γράψεις τις παρατηρήσεις σου. Τι ίχνος αφήνει το σημείο Β΄ ;

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6. Πού συναντάς το φαινόμενο αυτό στην καθημερινή ζωή;

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**Δραστηριότητα 2**

*Διερευνώντας το σχήμα της εικόνας.*

1. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb.

2. Να κατασκευάσεις ένα πολύγωνο, χρησιμοποιώντας το εργαλείο Πολύγωνο. Αυτό το σχήμα θα είναι το **πρότυπο** σου.

3. Να χρησιμοποιήσεις τα εργαλεία του GeoGebra για να περιστρέψεις το πολύγωνο σου.
   Να επιλέξεις το εργαλείο «Νέο σημείο» και να κατασκευάσεις ένα σημείο κοντά, αλλά όχι πάνω στο πολύγωνο. Αυτό θα είναι το σημείο περιστροφής.

4. Να επιλέξεις το εργαλείο «Στροφή αντικειμένου γύρω από σημείο κατά γωνία»
   Στη συνέχεια να επιλέξεις πρώτα το πολύγωνο και στη συνέχεια το σημείο.

5. Στο παράθυρο που εμφανίζεται **να γράψεις** ένα αριθμό για τη γωνία περιστροφής, που να είναι **μικρότερος** από 360. Να επιλέξεις το OK
   Το νέο σχήμα που εμφανίζεται είναι η **εικόνα** του προτύπου σου.



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

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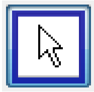
6. Με το εργαλείο «Μετακίνηση»  να επιλέξεις και **<u>να σύρεις</u>** οποιαδήποτε **<u>κορυφή</u>** ή **<u>πλευρά</u>** του **<u>προτύπου</u>** σου.

Να δοκιμάσεις να κάνεις το ίδιο και με άλλες πλευρές/κορυφές.

Να συγκρίνεις το **<u>πρότυπο</u>** και την <u>εικόνα</u> όσον αφορά το **<u>σχήμα</u>** τους και **<u>να γράψεις</u>** τις παρατηρήσεις σου.

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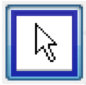
**Δραστηριότητα 3**

*Διερευνώντας το μέγεθος της εικόνας.*

1. Να ανοίξεις το αρχείο Δραστηριότητα3.ggb.

2. Με τη βοήθεια του εργαλείου «Μετακίνηση»  να επιλέξεις και **<u>να σύρεις</u>** την **<u>κορυφή Γ</u>** του προτύπου σου.

Να συγκρίνεις το **<u>πρότυπο</u>** και την **<u>εικόνα</u>** όσον αφορά το **<u>μέγεθος των πλευρών</u>** της και το **<u>μέγεθος των γωνιών</u>** της και να γράψεις τις παρατηρήσεις σου.

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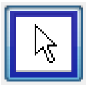
**Δραστηριότητα 4**

*Διερευνώντας τον προσανατολισμό της εικόνας.*

1. Να ανοίξεις το αρχείο Δραστηριότητα4.ggb.

2. Με τη βοήθεια του εργαλείου «Μετακίνηση»  να επιλέξεις και **<u>να σύρεις</u>** την **<u>κορυφή Β</u>** του προτύπου, έτσι ώστε να περιστραφεί ολόκληρο το σχήμα.

Να συγκρίνεις το πρότυπο και την εικόνα όσον αφορά τον <u>προσανατολισμό</u> της (δηλαδή τη θέση στην οποία βρίσκεται η κάθε πλευρά/κορυφή στο σχήμα). Να γράψεις τις παρατηρήσεις σου.

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**Δραστηριότητα 5**

*Διατυπώνοντας ένα γενικό συμπέρασμα.*

1. **<u>Να γράψεις ένα γενικό συμπέρασμα</u>** για τις παρατηρήσεις σου στις προηγούμενες δραστηριότητες.

Να αναφερθείς στη **<u>μορφή</u>**, στο **<u>μέγεθος</u>** και τον **<u>προσανατολισμό</u>** της εικόνας ενός σχήματος που μεταφέρεται.

**ΓΕΝΙΚΟ ΣΥΜΠΕΡΑΣΜΑ**

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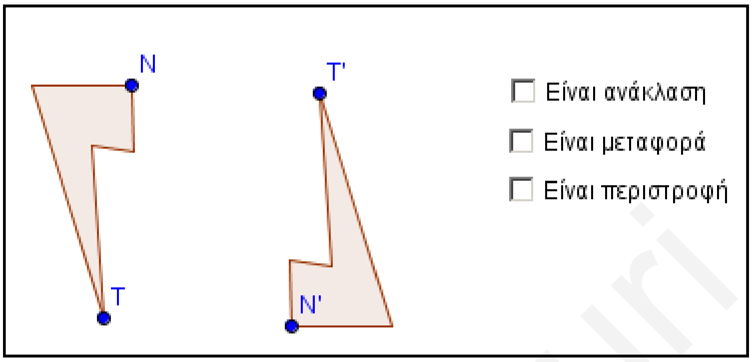
**Δραστηριότητα 5**

*Αναγνωρίζοντας τα παραδείγματα της περιστροφής.*

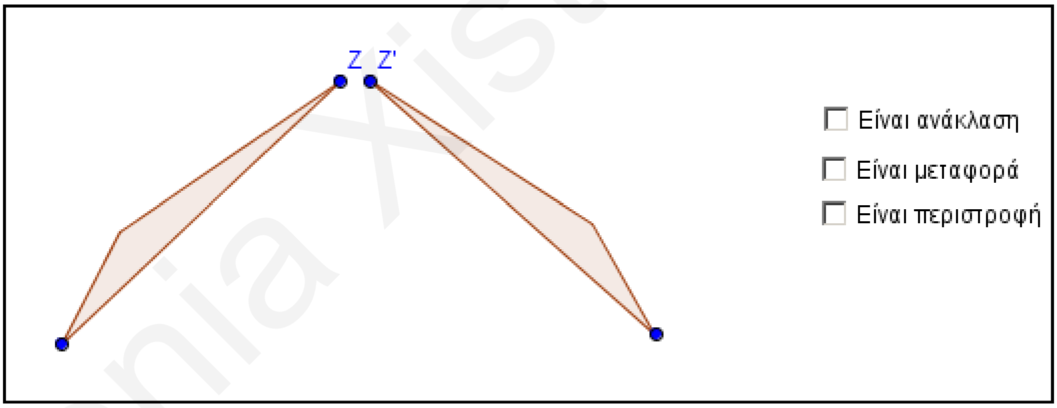
1. Να ανοίξεις το αρχείο Δραστηριότητα5.ggb.

2. Να παρατηρήσεις τα παραδείγματα και **<u>να αποφασίσεις</u>** εάν παρουσιάζουν ή ανάκλαση, μεταφορά ή περιστροφή.
Μπορείς να δοκιμάσεις να επιλέξεις το πρότυπο και να το μετακινήσεις για να βεβαιωθείς.

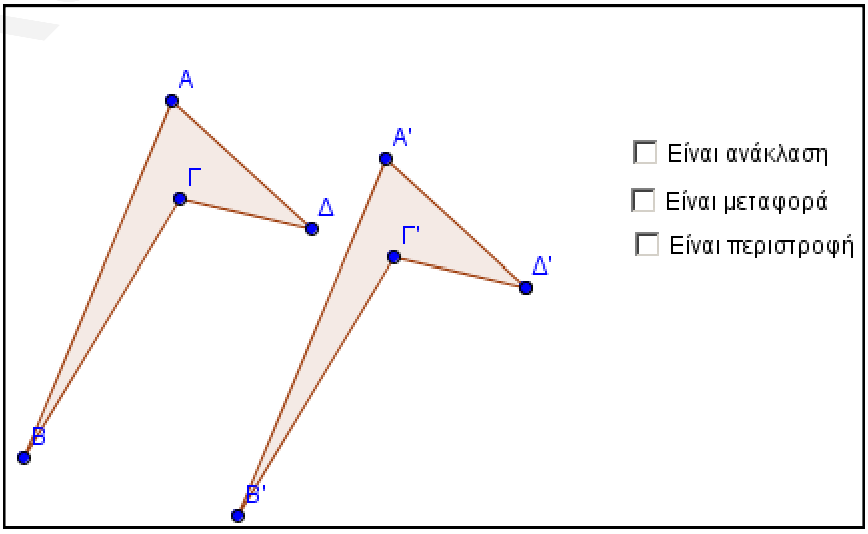
3. **<u>Να σημειώσεις</u>** το κουτάκι με την απάντησή σου και **<u>να γράψεις</u>** για ποιο λόγο την επέλεξες.




Γιατί: ..........................................................................




Γιατί: ..........................................................................




Γιατί: ..........................................................................

# ΜΑΘΗΜΑ 7: ΠΕΡΙΣΤΡΟΦΗ (συνέχεια)

**Δραστηριότητα 1**

*Εξερευνώντας τις παραμέτρους της περιστροφής.*

1. Να χρησιμοποιήσεις τα εργαλεία του GeoGebra για να περιστρέψεις ένα δικό σου πολύγωνο.

2. Να ανοίξεις το αρχείο Δραστηριότητα1.ggb.

3. Να κατασκευάσεις ένα πολύγωνο, χρησιμοποιώντας το εργαλείο «Πολύγωνο». Αυτό το σχήμα θα είναι το **<u>πρότυπο</u>** σου.

4. Να επιλέξεις το εργαλείο «Νέο σημείο» και να κατασκευάσεις ένα σημείο **<u>κοντά</u>**, αλλά **<u>όχι πάνω</u>** στο πολύγωνο. Αυτό θα είναι το **<u>σημείο περιστροφής</u>**.

5. Να επιλέξεις το εργαλείο «Στροφή αντικειμένου γύρω από σημείο κατά γωνία»

Στη συνέχεια να επιλέξεις **<u>πρώτα</u>** το πολύγωνο και **<u>στη συνέχεια</u>** το σημείο.

5. Στο παράθυρο που εμφανίζεται **<u>να γράψεις</u>** για γωνία περιστροφής, τον αριθμό **<u>160</u>**. Να επιλέξεις το OK

Το νέο σχήμα που εμφανίζεται είναι η **<u>εικόνα</u>** του προτύπου σου.

6. Με το εργαλείο «Μετακίνηση» **να σύρεις** το σημείο περιστροφής. Να κάνεις το ίδιο για το πολύγωνο.

Να γράψεις τις παρατηρήσεις σου. Εξακολουθεί να ισχύει η περιστροφή;

.....................................................................................................

.....................................................................................................

7. Να εξερευνήσεις τις πιο κάτω περιπτώσεις:

- Πότε **<u>μικραίνει</u>** η απόσταση ανάμεσα στο πρότυπο και την εικόνα;

.....................................................................................................

- Πότε **<u>μεγαλώνει</u>** η απόσταση ανάμεσα στο πρότυπο και την εικόνα;

.....................................................................................................

- Πότε η εικόνα **<u>επικαλύπτει</u>** το πρότυπο;

.....................................................................................................



Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

**Δραστηριότητα 2**

*Διερευνώντας την κατεύθυνση της εικόνας σε σχέση με το πρότυπο στην περιστροφή.*

1. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb.

2. Να σύρεις τον <u>**κόκκινο δρομέα**</u> για να <u>**αλλάξεις**</u> τη γωνία περιστροφής του τριγώνου **ΧΩΡΙΣ ΝΑ ΜΕΤΑΚΙΝΗΣΕΙΣ ΤΟ ΠΡΟΤΥΠΟ.**

3. Να τοποθετήσεις το δρομέα στη θέση ώστε η <u>**γωνία περιστροφής να είναι 140°**</u>. Τι αλλάζει στο σχήμα;

..........................................................................................................................

4. Να συμπληρώσεις στον **Πίνακα 1** την <u>**κατεύθυνση**</u> που βρίσκεται η εικόνα σε σχέση με το <u>**σημείο περιστροφής**</u>, όταν η <u>**γωνία περιστροφής**</u> έχει την <u>**τιμή**</u> που αναφέρεται στην πρώτη στήλη.

Πίνακας 1

| **Γωνία Περιστροφής** | **Κατεύθυνση προτύπου** |
|---|---|
| 40° | *πάνω αριστερά* |
| 60° | |
| 90° | |
| 180° | |
| 210° | |
| 270° | |
| 320° | |

5. Σε ποιες <u>**δύο γωνίες περιστροφής**</u> η εικόνα μπορεί να έχει την <u>**ίδια θέση**</u>;

Γωνία: ............................................. και Γωνία: .............................................

6. Να σύρεις το πρότυπο ώστε να βρίσκεται <u>**κάτω**</u> από το <u>**σημείο περιστροφής**</u>.

7. Να ελέγξεις κατά πόσο εξακολουθούν να ισχύουν οι ίδιες απαντήσεις που έδωσες στον Πίνακα 1, για τις <u>**ίδιες γωνίες**</u> περιστροφής.

Ισχύουν οι ίδιες απαντήσεις; .............................................

8. Ποια νομίζεις ότι είναι τα δύο πράγματα που μπορούν να καθορίσουν την <u>**κατεύθυνση**</u> που θα έχει η <u>**εικόνα**</u> ενός σχήματος σε σχέση με το <u>**πρότυπο**</u>;

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9. Τι <u>**διαφορά**</u> θα έχουν οι απαντήσεις που έδωσες στον Πίνακα 1, εάν αντί προς τα δεξιά, περιστρέφεις την εικόνα <u>**προς τα αριστερά**</u>;

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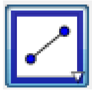
..........................................................................................................................

## Δραστηριότητα 3α

*Διερευνώντας την απόσταση της εικόνας από το σημείο περιστροφής.*

*Να επιλέγεις κάθε φορά πρώτα το σημείο περιστροφής Δ και στη συνέχεια την κορυφή από την οποία που θέλεις να μετρήσεις την απόσταση*

1. Να ανοίξεις το αρχείο Δραστηριότητα3α.ggb.

2. Να χρησιμοποιήσεις το εργαλείο «Τμήμα μεταξύ δύο σημείων»  για να δημιουργήσεις ευθύγραμμα τμήματα που **να συνδέουν** το σημείο περιστροφής Δ με την **κάθε κορυφή** Α, Β και Γ του πρότυπου και με την κάθε κορυφή Α΄ , Β΄ και Γ΄ της εικόνας.

3. Να επιλέξεις το εργαλείο «Απόσταση ή μήκος».
   Να μετρήσεις το **μήκος** των ευθύγραμμων τμημάτων ΑΔ, Α΄ Δ, ΒΔ, Β΄ Δ, ΓΔ και Γ΄ Δ για να βρεις τις αποστάσεις του σημείου περιστροφής Δ από τις κορυφές.

4. Να σύρεις το σημείο περιστροφής και να παρατηρήσεις τις αποστάσεις.
   Να συγκρίνεις τις **αποστάσεις** ανάμεσα σημείο περιστροφής και στις **αντίστοιχες κορυφές** και να γράψεις τις παρατηρήσεις σου.

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5. Να σύρεις τον **κόκκινο δρομέα** για αλλάξεις την τιμή της γωνίας περιστροφής και να παρατηρήσεις τις αλλαγές στις αποστάσεις.
   Να συγκρίνεις τις **αποστάσεις** ανάμεσα σημείο περιστροφής και στις **αντίστοιχες κορυφές** και να γράψεις τις παρατηρήσεις σου.

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## Δραστηριότητα 3β

*Διερευνώντας το ρόλο της γωνίας περιστροφής.*

1. Να ανοίξεις το αρχείο Δραστηριότητα3β.ggb.

2. Να επιλέξεις τα κουτάκια για **να εμφανίσεις** τις γωνίες ΑΔΑ', ΒΔΒ' και ΓΔΓ' με τη σειρά. Να γράψεις τις παρατηρήσεις σου.

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3. Να σύρεις το **σημείο περιστροφής** και να παρατηρήσεις τις **αλλαγές** στα μεγέθη των τριών γωνιών.
   Να συγκρίνεις τις **γωνίες** που σχηματίζονται ανάμεσα στο σημείο περιστροφής και στις **αντίστοιχες κορυφές** και να γράψεις τις παρατηρήσεις σου.

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4. Να σύρεις τον **κόκκινο δρομέα** για αλλάξεις την τιμή της **γωνίας περιστροφής** και να παρατηρήσεις τις **αλλαγές** στα **μεγέθη** των τριών γωνιών.
   Να συγκρίνεις τις **γωνίες** που σχηματίζονται ανάμεσα στο σημείο περιστροφής και στις **αντίστοιχες κορυφές** και να γράψεις τις παρατηρήσεις σου.

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## Δραστηριότητα 4

*Βρίσκοντας τη σχέση ανάμεσα στην περιστροφή και τις συντεταγμένες.*

1. Να ανοίξεις το αρχείο Δραστηριότητα4.ggb.

2. **<u>Να σύρεις</u>** τις κορυφές του προτύπου σε διάφορα σημεία του πλέγματος και να εξερευνήσεις τη **<u>σχέση</u>** ανάμεσα στις **<u>συντεταγμένες</u>** μιας κορυφής και στις **<u>συνταγμένες</u>** της **<u>εικόνας με γωνία περιστροφής 190°</u>** γύρω από το σημείο (0,0).

Να περιγράψεις τη σχέση που ανακάλυψες.

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3. Να σύρεις τον **<u>κόκκινο δρομέα</u>** ώστε η γωνία περιστροφής να γίνει 180°.

Να επαναλάβεις το Βήμα 2 για να εξερευνήσεις τη **<u>σχέση</u>** ανάμεσα στις **<u>συντεταγμένες</u>** μιας κορυφής και στις **<u>συνταγμένες</u>** της **<u>εικόνας με γωνία περιστροφής 190°</u>**.

Να περιγράψεις τη σχέση που ανακάλυψες.

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4. Με βάση τις παρατηρήσεις σου στις ερωτήσεις 2 και 3, ποια νομίζεις ότι θα είναι η **<u>σχέση</u>** ανάμεσα στις **<u>συντεταγμένες</u>** μιας κορυφής και στις **<u>συνταγμένες</u>** της **<u>εικόνας</u>** της **<u>με γωνία περιστροφής 270°</u>**;

5. **<u>Να σύρεις</u>** τον **<u>κόκκινο δρομέα</u>** ώστε η γωνία περιστροφής να γίνει 270° .

**<u>Να σύρεις</u>** τις κορυφές του προτύπου σε διάφορα σημεία του πλέγματος και για να ελέγξεις την υπόθεσή σου.

Να περιγράψεις τη σχέση που ανακάλυψες.

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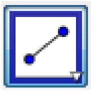
## Δραστηριότητα 5

*Κατασκευάζοντας την εικόνα στην περιστροφή.*

1. Να ανοίξεις το αρχείο Δραστηριότητα5.ggb.

2. Να φανταστείς ότι το GeoGebra **<u>δεν έχει</u>** εργαλείο για την περιστροφή. Πώς θα κατασκεύαζες την εικόνα ενός σημείου με περιστροφή **<u>κατά γωνία 180°</u>**;

3. Να δοκιμάσεις να βρεις την εικόνα ενός σημείου κι ενός ευθύγραμμου τμήματος όταν περιστραφούν **<u>κατά 90°</u> (ορθή γωνία)**, χρησιμοποιώντας τα εργαλεία «Νέο σημείο» 

και «Τμήμα μεταξύ δύο σημείων» 

**<u>Να περιγράψεις</u>** τον τρόπο που εργάστηκες:

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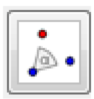
..........................................................................................................................

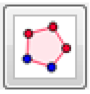
4. Να χρησιμοποιήσεις το εργαλείο «Στροφή αντικειμένου γύρω από σημείο κατά γωνία» για **<u>να ελέγξεις</u>** την απάντησή σου. 

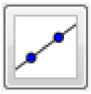
# ΜΑΘΗΜΑ 8: ΣΥΜΜΕΤΡΙΑ ΚΑΙ ΣΥΝΔΥΑΣΜΟΙ ΜΕΤΑΣΧΗΜΑΤΙΣΜΩΝ

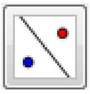
**Δραστηριότητα 1**

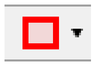
*Εξερευνώντας την αξονική συμμετρία.*

1. Να ανοίξεις το αρχείο Δραστηριότητα1.ggb.

2. Να επιλέξεις το εργαλείο «Κανονικό πολύγωνο»  για να κατασκευάσεις ένα **<u>ισόπλευρο τρίγωνο</u>**, ένα **<u>τετράγωνο</u>**, ένα **<u>κανονικό πεντάγωνο</u>** ή ένα **<u>κανονικό εξάγωνο</u>**.

3. Να επιλέξεις το εργαλείο «Ευθεία που περνά από δύο σημεία»  και να κατασκευάσεις μια **<u>ευθεία</u>** δίπλα στο πολύγωνο σου.

4. Να επιλέξεις το εργαλείο «Συμμετρία αντικειμένου ως προς ευθεία»  για να ανακλάσεις το πολύγωνο (να επιλέξεις **<u>πρώτα</u>** το πολύγωνο και **<u>μετά</u>** την ευθεία).

5. Να επιλέξεις την εικόνα και στη συνέχεια να επιλέξεις το εργαλείο  για να της δώσεις ένα διαφορετικό **<u>χρώμα</u>**.

6. Να σύρεις και να **<u>περιστρέψεις</u>** τον **<u>άξονα</u>** συμμετρίας έτσι ώστε η εικόνα να εφαρμόζει ακριβώς πάνω από το πρότυπο.

Για ποιο λόγο νομίζεις ότι συμβαίνει αυτό;

.................................................................................................................................

7. Τι παρατηρείς για τη θέση που έβαλες τον άξονα συμμετρίας σε σχέση με το σχήμα;

.................................................................................................................................

8. Όταν η εικόνα ενός προτύπου **<u>συμπίπτει ακριβώς</u>** με το πρότυπο, τότε το σχήμα έχει **<u>αξονική συμμετρία</u>**.

Να σύρεις/περιστρέψεις την ευθεία μέχρι να βρεις όλους τους άξονες συμμετρίας που έχει το πολύγωνο σου.

9. Να συμπληρώσεις τον Πίνακα 1.

Πινάκας 1

| Αριθμός πλευρών κανονικού πολυγώνου | 3 | 4 | 5 | 6 | ... | ν |
|---|---|---|---|---|---|---|
| Αριθμός αξόνων συμμετρίας | | | | | | |

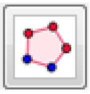


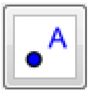
Ετοιμάστηκε στα πλαίσια ερευνητικού προγράμματος που χρηματοδοτείται από το Ίδρυμα Προώθησης Έρευνας Κύπρου (ΔΕΣΜΗ 2009-2010) και συγχρηματοδοτείται από την Κυπριακή Δημοκρατία και το Ευρωπαϊκό Ταμείο Περιφερειακής Ανάπτυξης της ΕΕ (Επιχορήγηση: ΠΕΝΕΚ/0609/57).

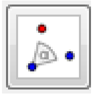
## Δραστηριότητα 3

*Εξερευνώντας την περιστροφική συμμετρία.*

1. Να ανοίξεις το αρχείο Δραστηριότητα2.ggb.

2. Να επιλέξεις το εργαλείο «Κανονικό πολύγωνο»  για να κατασκευάσεις ένα **ισόπλευρο τρίγωνο**, ένα **τετράγωνο**, ένα **κανονικό πεντάγωνο** ή ένα **κανονικό εξάγωνο**.

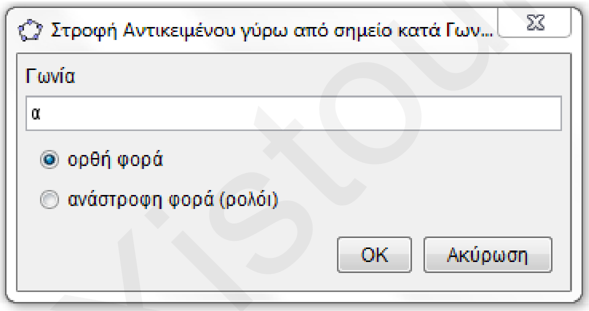
3. Να χρησιμοποιήσεις το εργαλείο «Νέο σημείο»  για να κατασκευάσεις ένα σημείο περιστροφής.

4. Να επιλέξεις το εργαλείο «Στροφή αντικειμένου γύρω από σημείο κατά γωνία» 

Στη συνέχεια να επιλέξεις **πρώτα** το πολύγωνο και **στη συνέχεια** το σημείο.

5. Στο παράθυρο που εμφανίζεται να σβήσεις τον αριθμό 45 και να γράψεις στη θέση του το **γράμμα α**, όπως φαίνεται στην Εικόνα 1.

Εικόνα 1




6. Να επιλέξεις την εικόνα και **στη συνέχεια** να επιλέξεις το εργαλείο  για να της δώσεις ένα διαφορετικό **χρώμα**.

7. Να μετακινήσεις το σημείο περιστροφής ώστε να είναι **μέσα** στο πρότυπο.

8. Να σύρεις **το σημείο** καθώς και τον **κόκκινο δρομέα** της Γωνίας α για να περιστρέψεις το πολύγωνο σου, έτσι ώστε η εικόνα **να εφαρμόζει ακριβώς** πάνω από το πρότυπο.

Για ποιο λόγο νομίζεις ότι συμβαίνει αυτό;

..............................................................................................................................

9. Τι παρατηρείς για τη **θέση** που έβαλες το σημείο περιστροφής σε σχέση με το σχήμα;

..............................................................................................................................

10. Σε ποια τιμή της γωνίας περιστροφής (Γωνία α) η εικόνα **συμπίπτει ακριβώς** με το πρότυπο;

Είδος πολυγώνου: ................................. Γωνία περιστροφής: .......................

11. Όταν η εικόνα ενός προτύπου **συμπίπτει ακριβώς** με το πρότυπο, τότε το σχήμα έχει **περιστροφική συμμετρία**.

Συνέχισε να περιστρέφεις το πολύγωνο με τη βοήθεια του κόκκινου δρομέα για να βρεις όλες τις θέσεις που το πολύγωνο έχει περιστροφική συμμετρία.

12. Να συμπληρώσεις τον Πίνακα 2.

Πίνακας 2

| Αριθμός πλευρών κανονικού πολυγώνου | 3 | 4 | 5 | 6 | ... | ν |
|---|---|---|---|---|---|---|
| Αριθμός θέσεων περιστροφικής συμμετρίας | | | | | | |

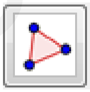
13. Να συγκρίνεις τις απαντήσεις του Πίνακα 1 με αυτές του Πίνακα 2. Τι παρατηρείς;

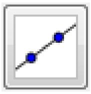
.................................................................................................................................

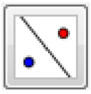
## Δραστηριότητα 3

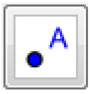
*Εξερευνώντας τα αποτελέσματα της ανάκλασης σε δύο παράλληλες ευθείες.*

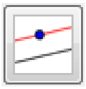
1. Να ανοίξεις το αρχείο Δραστηριότητα3.ggb.

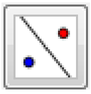
2. Να χρησιμοποιήσεις το εργαλείο «Πολύγωνο»  για να κατασκευάσεις ένα **ακανόνιστο πολύγωνο** ΑΒΓΔ με **4 κορυφές**.

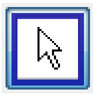
3. Να χρησιμοποιήσεις το εργαλείο «Ευθεία που περνά από δύο σημεία»  για να κατασκευάσεις μια ευθεία ΕΖ **κοντά** αλά **όχι πάνω** στο πολύγωνο ΑΒΓΔ.

4. Να επιλέξεις το εργαλείο «Συμμετρία αντικειμένου ως προς ευθεία»  για να ανακλάσεις το πολύγωνο στην ευθεία ΕΖ. Το πολύγωνο Α΄ Β΄ Γ΄ Δ΄ είναι η **εικόνα** του πολυγώνου σου.

5. Να επιλέξεις το εργαλείο «Νέο σημείο»  και να κατασκευάσεις ένα σημείο Η κοντά αλλά όχι πάνω στο πολύγωνο Α΄ Β΄ Γ΄ Δ΄ .

6. Να επιλέξεις το εργαλείο «Παράλληλη γραμμή»  και **να κατασκευάσεις** μια ευθεία που να περνά από το σημείο Η και να είναι **παράλληλη** στην ευθεία ΕΖ.

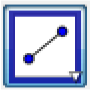
7. Να επιλέξεις το εργαλείο «Συμμετρία αντικειμένου ως προς ευθεία»  για **να ανακλάσεις** το πολύγωνο **στην ευθεία που περνά από το Η**. Το πολύγωνο Α΄ ΄ Β΄ ΄ Γ΄ ΄ Δ΄ ΄ είναι η **δεύτερη εικόνα** του πολυγώνου σου.

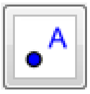
8. Να επιλέξεις το εργαλείο «Μετακίνηση»  και να σύρεις το πρότυπο κι τις δύο ευθείες και να παρατηρήσεις τις **αλλαγές** στις δύο εικόνες.

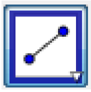
9. Να απαντήσεις στην πιο κάτω ερώτηση:

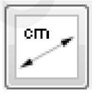
**Δύο ανακλάσεις** μετακινούν το πρότυπο σου από τη θέση που βρίσκεται στη θέση της δεύτερης εικόνας Α΄΄ Β΄΄ Γ΄΄ Δ΄΄. Ποιος **μοναδικός μετασχηματισμός** νομίζεις ότι θα μπορούσε να αντικαταστήσει τις δύο ανακλάσεις και να έχει το **ίδιο αποτέλεσμα**;

.................................................................................................................

10. Να χρησιμοποιήσεις το εργαλείο «Τμήμα μεταξύ δύο σημείων»  για να κατασκευάσεις ένα ευθύγραμμο τμήμα που να συνδέει τα σημεία Α και Α΄΄.

11. Να χρησιμοποιήσεις το εργαλείο «Νέο σημείο»  για να κατασκευάσεις τα σημεία Θ και Ι, εκεί που τέμνει η ΑΑ΄΄ τις ευθείες ΕΖ και Η αντίστοιχα.

12. Να χρησιμοποιήσεις το εργαλείο «Τμήμα μεταξύ δύο σημείων»  για να ενώσεις τα σημεία Θ και Ι.

13. Να επιλέξεις το εργαλείο «Απόσταση ή μήκος»  και να μετρήσεις τα μήκος των ευθύγραμμων τμημάτων ΑΑ΄΄ και ΘΙ.

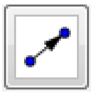
14. Να σύρεις τη μια από τις δύο ευθείες και να παρατηρήσεις τις δύο αποστάσεις. Ποια είναι η σχέση τους;

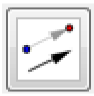
.................................................................................................................

15. Η ΘΙ είναι η απόσταση ανάμεσα στις δύο ευθείες. Γιατί νομίζεις ότι συμβαίνει αυτό που παρατήρησες;

.................................................................................................................

.................................................................................................................

16. Να χρησιμοποιήσεις το εργαλείο "Διάνυσμα μεταξύ δύο σημείων»  και να κατασκευάσεις ένα **διάνυσμα** που να ξεκινά από το **σημείο Α** και να τελειώνει στο σημείο Α΄΄.

17. Να χρησιμοποιήσεις το εργαλείο «Μεταφορά αντικειμένου με διάνυσμα»  για να μεταφέρεις το πρότυπο ΑΒΓΔ με το βοήθεια του διανύσματος ΑΑ΄΄. Τι παρατηρείς;

.................................................................................................................

18. Με βάση τις παρατηρήσεις σου στο Βήμα 17, ποιος μοναδικός μετασχηματισμός είναι ισοδύναμος με το αποτέλεσμα δύο διαδοχικών ανακλάσεων σε παράλληλες ευθείες;

.................................................................................................................

**Δραστηριότητα 3**

*Διερευνώντας την ανάκλαση.*

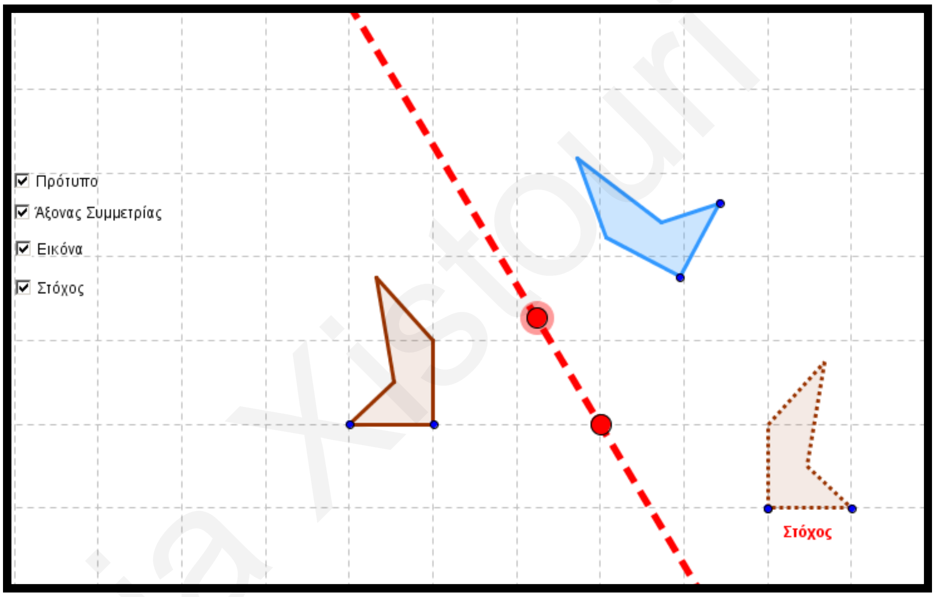
1. Να διαβάσεις το πιο κάτω κείμενο.

> Μια φιγούρα μπορεί να **αντιστραφεί** ως προς ένα άξονα συμμετρίας που λειτουργεί σαν καθρέφτης. Αυτό στα μαθηματικά ονομάζεται ***ανάκλαση***. Όπως και στους άλλους μετασχηματισμούς, η αρχική φιγούρα ονομάζεται ***πρότυπο***. Το αποτέλεσμα της ανάκλασης ονομάζεται ***εικόνα***.

2. Να ανοίξεις το αρχείο Δραστηριότητα3.ggb

3. Να επιλέξεις τα εικονίδια δίπλα από τις λέξεις Πρότυπο, Άξονας Συμμετρίας, Εικόνα και Στόχος και να παρατηρήσεις τι εμφανίζεται/εξαφανίζεται κάθε φορά.

Εικόνα 2




Κάθε φορά που θες να αρχίσεις μια καινούρια προσπάθεια, να επιλέγεις το κουμπί της αναίρεσης

4. Να επιλέξεις όλα τα τετραγωνάκια ώστε να εμφανίζονται όλα τα αντικείμενα.
Να προσπαθήσεις να τοποθετήσεις την εικόνα στη θέση του στόχου.

5. Τι ενέργειες έκανες για να μπορέσεις να πετύχεις το στόχο;
Να τις περιγράψεις πιο κάτω.

……………………………………………………………………………………

……………………………………………………………………………………

6. Τι παρατηρείς για τη σχέση του προτύπου και της εικόνας;

……………………………………………………………………………………

7. Απάντησε στις πιο κάτω ερωτήσεις:

- Τι παθαίνει η **εικόνα** όταν μετακινείς το **πρότυπο προς τα πάνω**;

……………………………………………………………………………………

- Τι παθαίνει η **εικόνα** όταν μετακινείς τον **άξονα συμμετρίας προς τα δεξιά**;

……………………………………………………………………………………

APPENDIX IV

TRANSFORMATIONAL GEOMETRY POST-TEST

## ΔΟΚΙΜΙΟ 1

**Όνομα:** ........................................................ **Τάξη:** .......

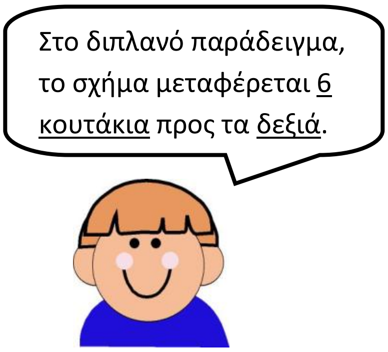
**Αρ. στον κατάλογο:** ...... **Σχολείο:** ........................................

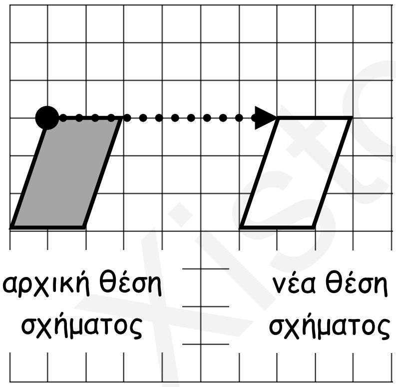
**Φύλο (βάλε ✓):** **Αγόρι** ☐ **Κορίτσι:** ☐

## ΜΕΡΟΣ Α

### ΜΕΤΑΦΟΡΑ

Οι πιο κάτω ασκήσεις αναφέρονται στη **μεταφορά** (μετακίνηση) σχημάτων από μια **αρχική θέση** σε μια **νέα θέση**.

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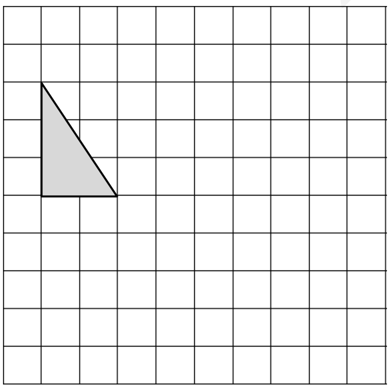
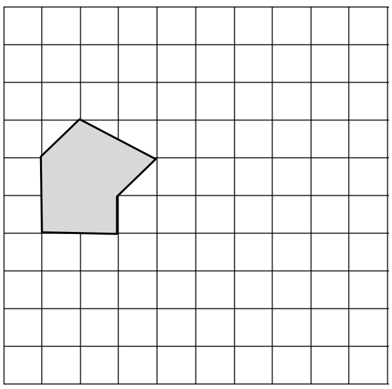
**A) Να μεταφέρεις και να σχεδιάσεις το τρίγωνο στη νέα θέση που θα βρίσκεται.**
**Οι οδηγίες για τη μεταφορά δίνονται πάνω από το σχήμα.**

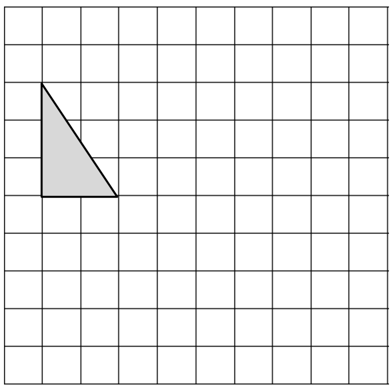
**1)** Να μεταφέρεις το σχήμα 4 κουτάκια προς τα δεξιά.



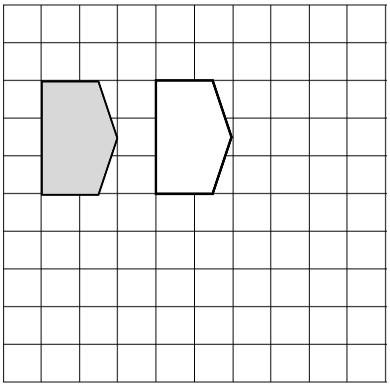
**2)** Να μεταφέρεις το σχήμα 3 κουτάκια προς τα δεξιά.



**3)** Να μεταφέρεις το σχήμα 1 κουτάκι προς τα δεξιά.



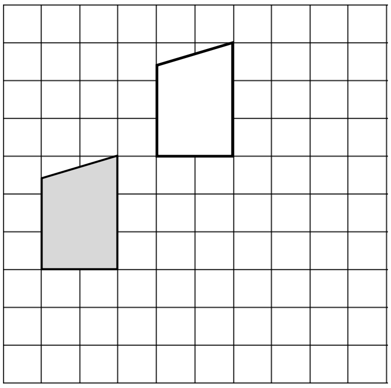
**Β) Τώρα, <u>να γράψεις</u> εσύ τις <u>οδηγίες</u> για το πώς <u>να μεταφέρει</u> κάποιος το <u>χρωματισμένο σχήμα</u> στη <u>νέα θέση</u> (όπως τις οδηγίες που σου δόθηκαν στην προηγούμενη άσκηση).**



**1. Να μεταφέρεις το σχήμα**:

……..............................................................

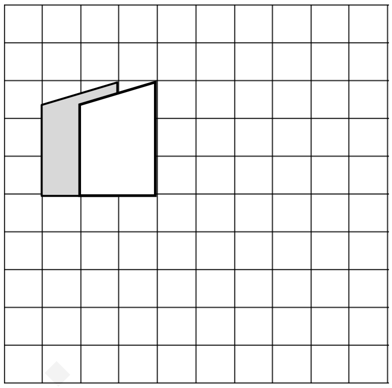
.....................................................................



**2. Να μεταφέρεις το σχήμα** :

...................................................................

.....................................................................



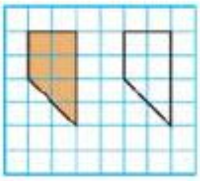
**3. Να μεταφέρεις το σχήμα**:

......................................................................

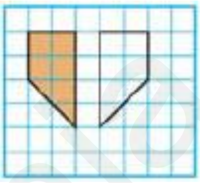
.......................................................................

**Γ) <u>Να βάλεις σε κύκλο</u> τη σωστή απάντηση σε κάθε περίπτωση.**
**Η σωστή απάντηση σε κάθε ερώτηση πιο κάτω είναι <u>μόνο μία</u>.**

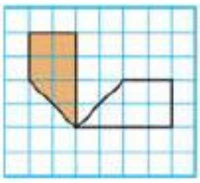
1) <u>Ποια</u> από τις πιο κάτω εικόνες παρουσιάζει τη <u>μεταφορά</u> του χρωματισμένου σχήματος;



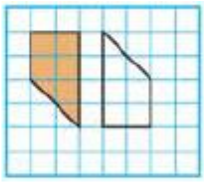
Α



Β

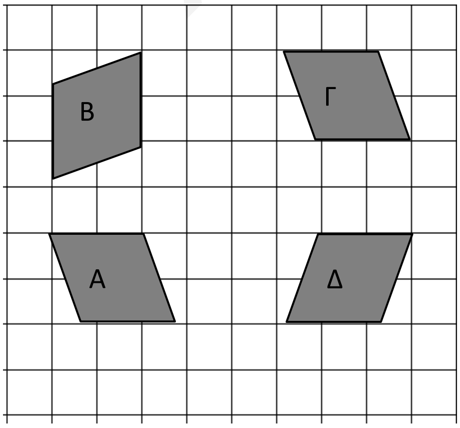


Γ



Δ

2) <u>Ποιο</u> από τα πιο κάτω ζευγάρια σχημάτων σχετίζονται με <u>μεταφορά</u> (το ένα να είναι μεταφορά του άλλου);



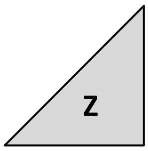

α) Το Α με το Δ

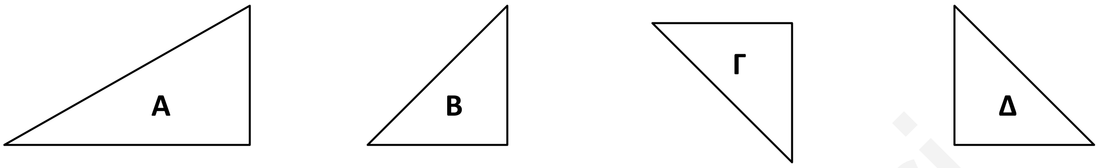
β) Το Β με το Γ

γ) Το Γ με το Δ

δ) Το Α με το Γ

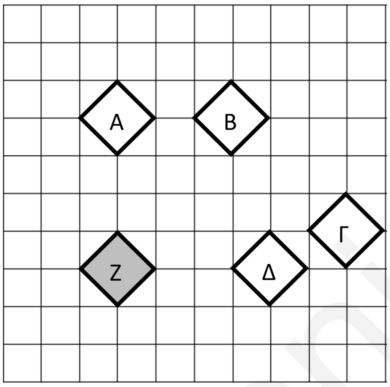
3) Ποιο από τα πιο κάτω τρίγωνα μπορεί να δημιουργήθηκε από τη μεταφορά του χρωματισμένου τριγώνου Ζ;




Α Β Γ Δ

4α) Ποιο από τα πιο κάτω αποτελεί μεταφορά του αρχικού σχεδίου Ζ, όταν μεταφερθεί 4 κουτάκια προς τα δεξιά;



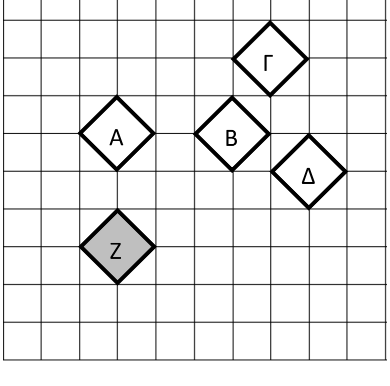

α) Το Α

β) Το Β

γ) Το Γ

δ) Το Δ

4β) Ποιο από τα πιο κάτω αποτελεί μεταφορά του αρχικού σχεδίου Ζ, όταν μεταφερθεί 3 κουτάκια διαγώνια προς τα πάνω και δεξιά;




α) Το Α

β) Το Β

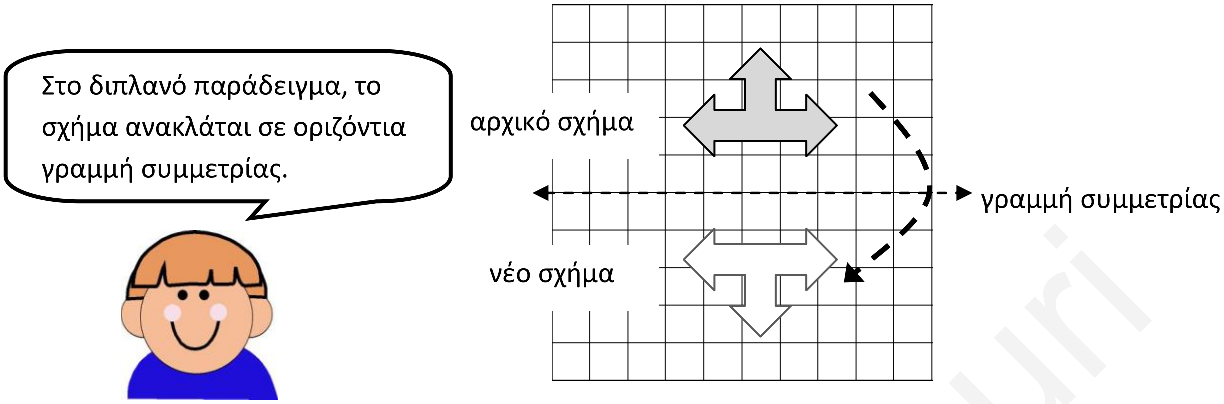
γ) Το Γ

δ) Το Δ

# ΜΕΡΟΣ Β

## ΑΝΑΚΛΑΣΗ

Οι πιο κάτω ασκήσεις αναφέρονται στην **ανάκλαση** (καθρέφτισμα) αντικειμένων με οριζόντια, κατακόρυφη ή διαγώνια γραμμή συμμετρίας.

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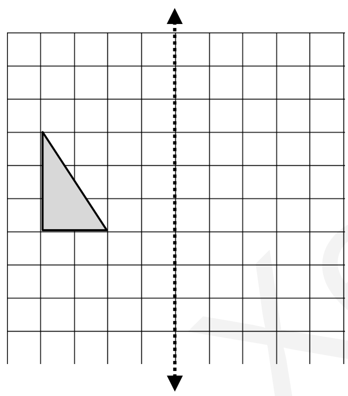
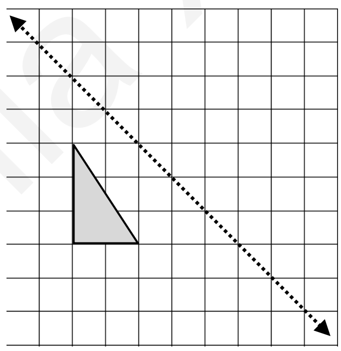
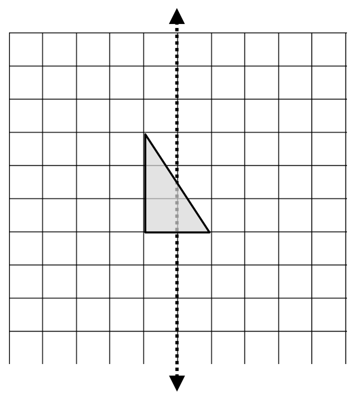
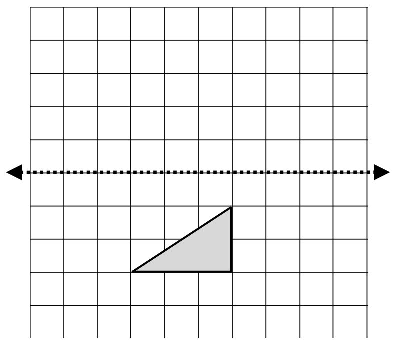
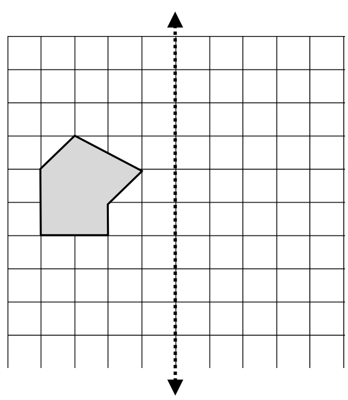
**Α) Να σχεδιάσεις το συμμετρικό του κάθε αρχικού σχήματος όταν γίνει ανάκλαση του στη διακεκομμένη γραμμή συμμετρίας.**



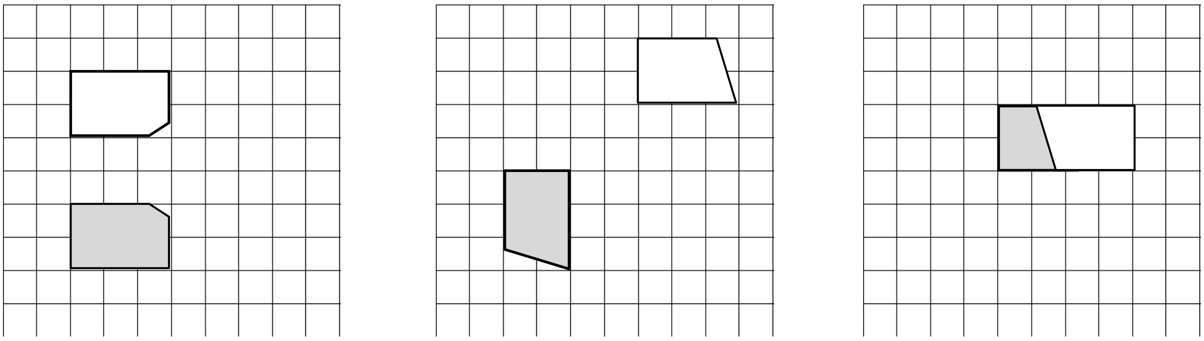






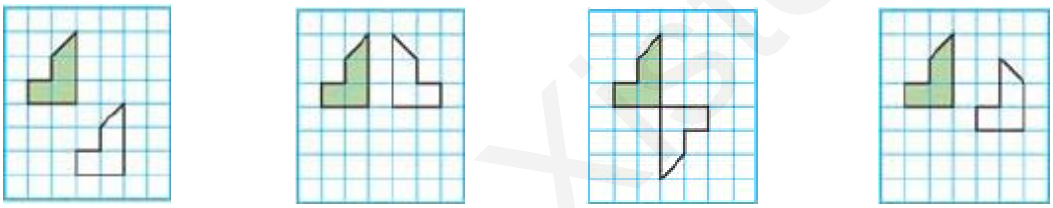


**Β) <u>Να βρεις</u> και <u>να σχεδιάσεις</u> με τη ρίγα σου τη <u>γραμμή συμμετρίας</u> για κάθε περίπτωση.**



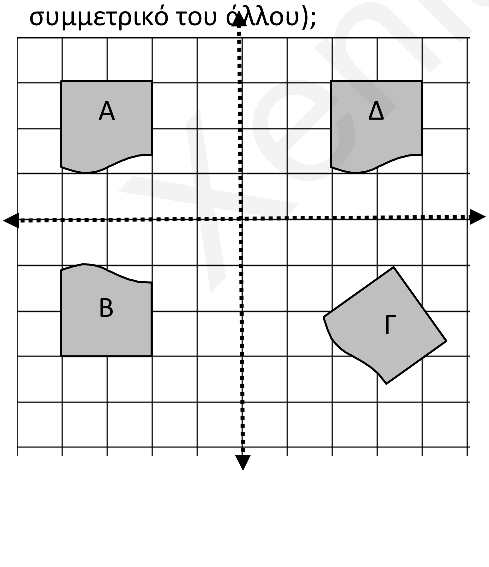
**Γ) <u>Να βάλεις σε κύκλο</u> τη σωστή απάντηση σε κάθε περίπτωση.**
**Η σωστή απάντηση σε κάθε ερώτηση πιο κάτω είναι <u>μόνο μία</u>.**

1) <u>Ποια</u> από τις πιο κάτω εικόνες παρουσιάζει την <u>ανάκλαση</u> του χρωματισμένου σχήματος;



Α Β Γ Δ

2) <u>Ποιο</u> από τα πιο κάτω ζευγάρια σχημάτων σχετίζονται με <u>ανάκλαση</u> (το ένα είναι συμμετρικό του άλλου);



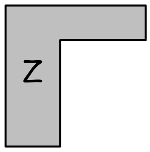

α) Το Α με το Δ

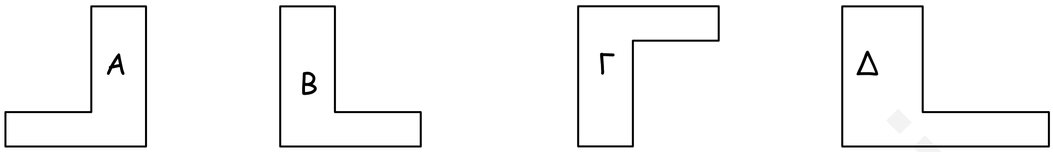
β) Το Β με το Γ

γ) Το Α με το Β

δ) Το Γ με το Δ

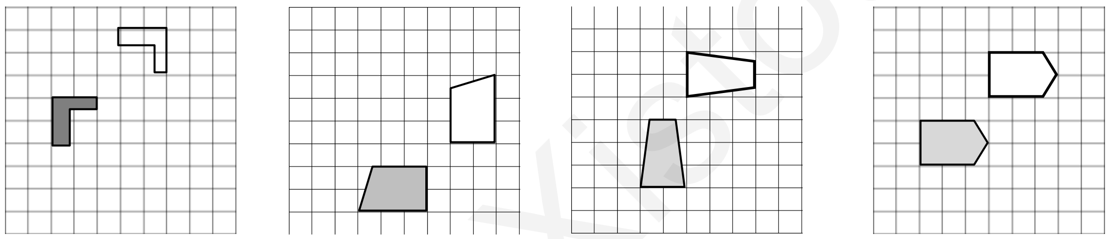
3) Ποιο από τα πιο κάτω σχήματα μπορεί να δημιουργήθηκε από την ανάκλαση του χρωματισμένου σχήματος Z;



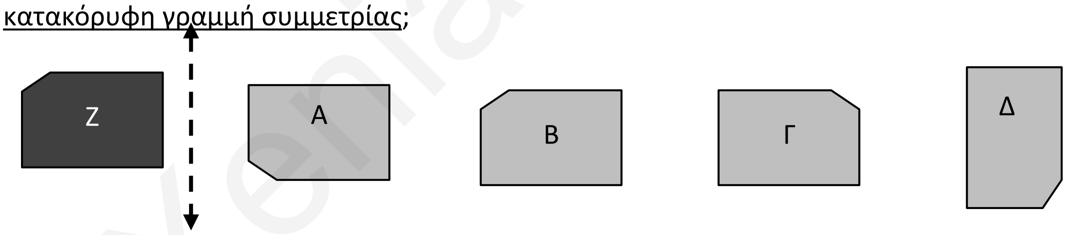

A B Γ Δ

4) Ποια από τις πιο κάτω εικόνες παρουσιάζει ανάκλαση;



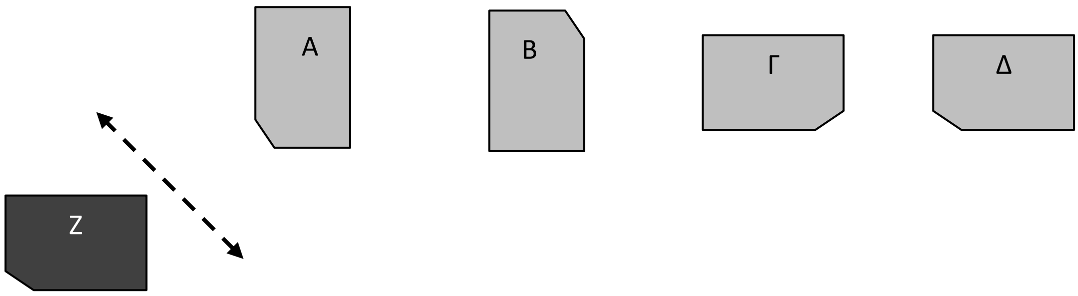
A B Γ Δ

5α) Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Z ως προς κατακόρυφη γραμμή συμμετρίας;




α) Το Α β) Το Β γ) Το Γ δ) Το Δ

5β) Ποιο από τα πιο κάτω σχήματα είναι συμμετρικό του αρχικού σχήματος Z ως προς διαγώνια γραμμή συμμετρίας;



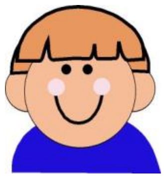

α) Το Α β) Το Β γ) Το Γ δ) Το Δ

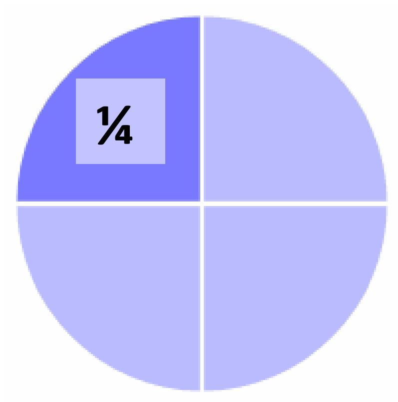
## ΜΕΡΟΣ Γ

### ΠΕΡΙΣΤΡΟΦΗ

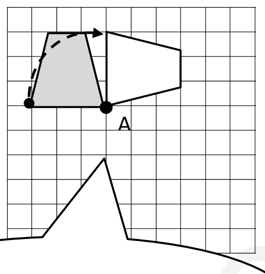
**Οι πιο κάτω ερωτήσεις αναφέρονται στην <u>περιστροφή</u> των σχημάτων, γύρω από συγκεκριμένο σημείο.**

**Τα σχήματα μπορούν να κάνουν στροφή (να γυρίσουν) προς τα δεξιά, δηλαδή όπως κινούνται οι δείκτες του ρολογιού, ή προς τα αριστερά, δηλαδή αντίθετα με τους δείκτες του ρολογιού. <u>Στις πιο κάτω ασκήσεις, θεώρησε ότι όλες οι στροφές γίνονται προς τα δεξιά, όπως γυρίζουν οι δείκτες του ρολογιού</u>.**

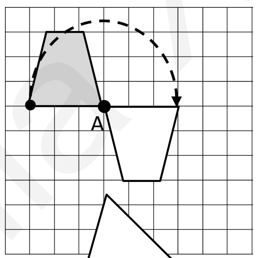




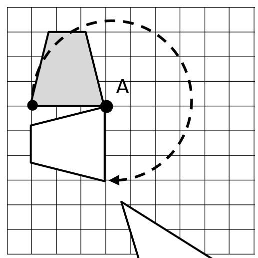

**Μια ολόκληρη στροφή είναι 4/4 του κύκλου. Όταν η ερώτηση λέει ότι το σχήμα κάνει ¼ της στροφής, εννοεί ¼ του κύκλου. Τα 2/4 της στροφής είναι ίσα με 2/4 του κύκλου και τα ¾ της στροφής ίσα με τα ¾ του κύκλου.**




$\frac{1}{4}$ της στροφής γύρω από την τελεία *A*




2/4 της στροφής γύρω από την τελεία *A*




$\frac{3}{4}$ της στροφής γύρω από την τελεία *A*

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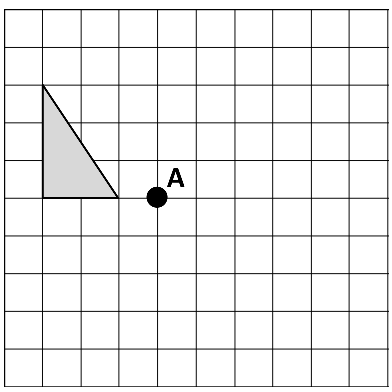
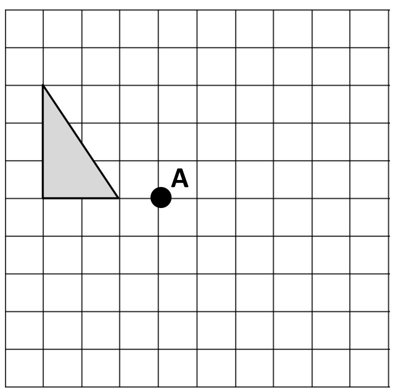
**Α) <u>Να σχεδιάσεις</u> το κάθε σχήμα στη <u>νέα του θέση</u>, όταν κάνει <u>στροφή</u> (περιστραφεί) <u>γύρω από την τελεία Α</u>.**

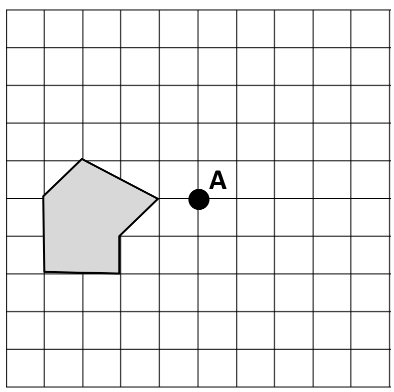
**1)** Όταν κάνει 2/4 της στροφής προς τα δεξιά .



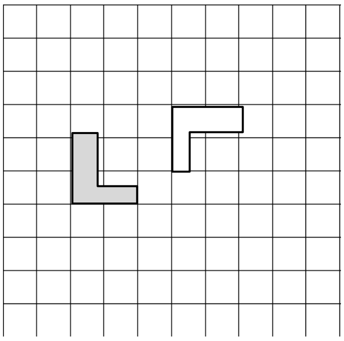

**2)** Όταν κάνει ¾ της στροφής προς τα δεξιά.




**3)** Όταν κάνει 1/4 της στροφής προς τα δεξιά.



**Β)**

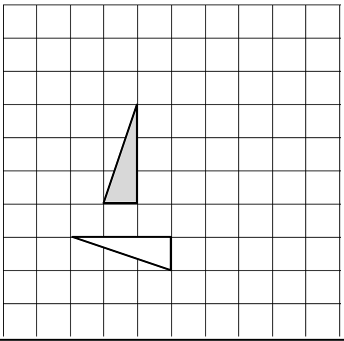


1) **<u>Να σχεδιάσεις</u>** την **<u>ΤΕΛΕΙΑ Α</u>** γύρω από την οποία έκανε στροφή το σχήμα.

2) **<u>Βάλε σε κύκλο</u>** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

1/4 2/4 3/4

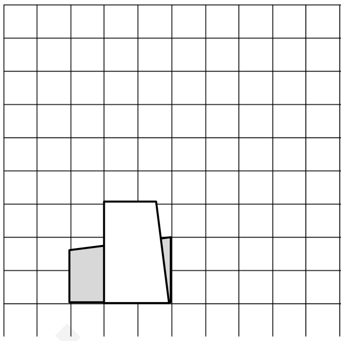


1) **<u>Να σχεδιάσεις</u>** την **<u>ΤΕΛΕΙΑ Α</u>** γύρω από την οποία έκανε στροφή το σχήμα.

2) **<u>Βάλε σε κύκλο</u>** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

1/4 2/4 3/4



1) **<u>Να σχεδιάσεις</u>** την **<u>ΤΕΛΕΙΑ Α</u>** γύρω από την οποία έκανε στροφή το σχήμα.

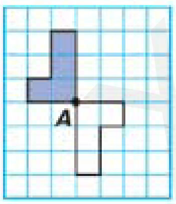
2) **<u>Βάλε σε κύκλο</u>** το σωστό:

Έκανε στροφή προς τα δεξιά κατά:

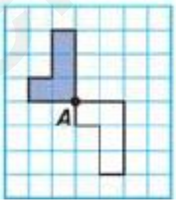
1/4 2/4 3/4

**Δ) <u>Να βάλεις σε κύκλο</u> τη σωστή απάντηση σε κάθε περίπτωση.**
**Η σωστή απάντηση σε κάθε ερώτηση πιο κάτω είναι <u>μόνο μία</u>.**

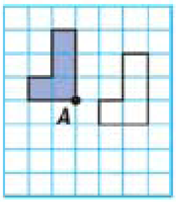
1) <u>Ποια</u> από τις πιο κάτω εικόνες παρουσιάζει την <u>περιστροφή</u> του χρωματισμένου σχήματος;



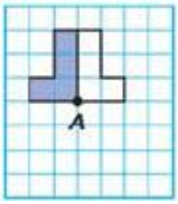

Α




Β

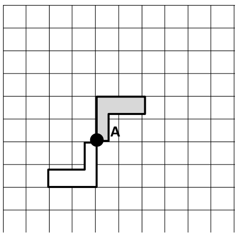



Γ

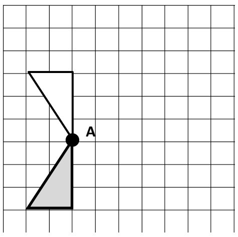



Δ

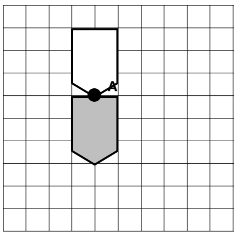
2) <u>Ποια</u> από τις πιο κάτω περιπτώσεις παρουσιάζει <u>περιστροφή</u>;



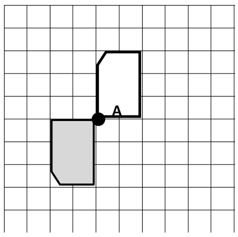

Α




Β

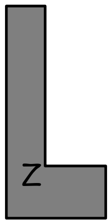


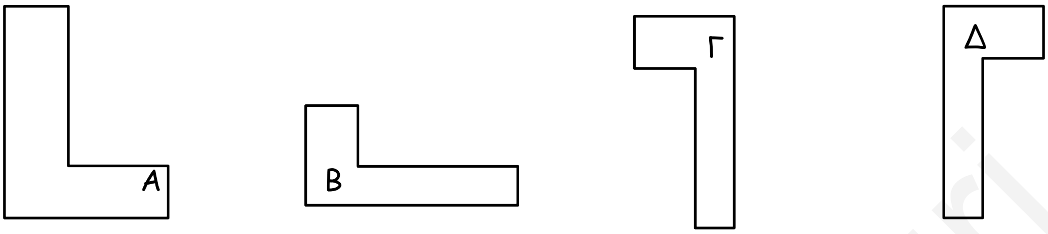

Γ




Δ

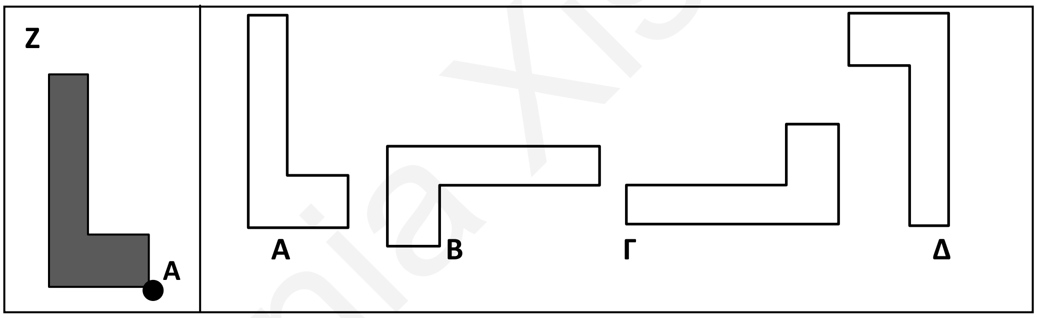
3) Ποιο από τα πιο κάτω σχήματα μπορεί να δημιουργήθηκε από την περιστροφή του χρωματισμένου σχήματος Ζ;




Α Β Γ Δ

3α) Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά ¼ του κύκλου προς τα δεξιά;




3β) Ποιο από τα πιο κάτω σχήματα παρουσιάζει την περιστροφή του σχήματος Ζ κατά 2/4 του κύκλου προς τα δεξιά;