THE MODELING PERSPECTIVE IN THE TEACHING AND LEARNING OF MATHEMATICAL PROBLEM SOLVING

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ABSTRACT

The purpose of the present study was the further development of the theory for the models and modeling perspective in problem solving, as it is proposed by Lesh and Doerr (2003). This new learning theory is expected to contribute in deeper understanding of the modeling processes that appear in students' work in solving real world problems. The theory also aims to trace how students' modeling abilities evolve over time. The theory is finally expected to contribute to a better understanding of modeling as a didactic means in mathematical problem solving.

Two student populations enrolled in one experimental and one control group were involved. The experimental group, consisted of four 6th grade and four 8th grade classes, was provided a curriculum of six modeling activities. Each activity lasted about four 40 minute sessions. Each activity contained a real world problem under consideration and required students to produce an explanation, method, or description for solving the problem in small groups. The control group, consisted of four 6th grade and four 8th grade classes, was given the school textbooks. The duration of the program was three months.

Using a grounded theory approach to qualitative research, through a detailed analysis of the modeling processes appeared in students' work in modeling activities, an analytical and explanatory framework has been developed. This framework takes into account students' mathematical developments and students' interactions in their groups. Before, during and after the treatment, all students in experimental and control groups were given the same assessment, involving a set of modeling problems. Student responses were evaluated towards identifying how students' modeling abilities evolved during time and what was the impact of the intervention program.

Data analysis revealed that the modeling processes significantly progressed across students' work in the modeling activities sequence. Results showed that students in both 6th and 8th grade experimental groups exhibited significant modeling processes and were involved in meaningful mathematical problem solving in a short time period. Results also showed that 8th graders constructed more complex and refined models in comparison to 6th graders. Eight graders also adopted and transformed more easily existing models, better communicated their results and reflected on their solutions.

Quantitative data analysis confirmed that there was a developmental trend in students' modeling abilities in the three categories of modeling problems, namely decision

making, system analysis and design and trouble shooting problems. Analysis also revealed that the intervention program had a significant impact on the students' modeling abilities, indicating that experimental group students in both 6th and 8th grade levels had significant better modeling abilities' rates of change compared to their counterparts. Analysis also revealed that the intervention program was more effective for students with lower initial achievement scores in the modeling abilities test.

The study resulted in a more comprehensive theoretical approach to the existing modeling procedure described in the literature (Lesh & Doerr, 2003). The new model added new modeling processes, further analyzed existing processes and incorporated other factors that influence the modeling procedure. The new model has implications for the design of model eliciting activities, for teaching mathematical modeling as a didactic means for mathematical problem solving and for teaching for improving students' modeling abilities. The study concludes that further research is needed for developing a unify theory for the models and modeling perspective. Finally, implications for researchers and teachers and directions for further research are provided.

ΠΕΡΙΛΗΨΗ

Σκοπό της παρούσας εργασίας αποτέλεσε η ανάγκη για ανάπτυξη και επέκταση της θεωρίας για τη μοντελοποίηση ως διαδικασία λύσης προβλήματος στα μαθηματικά. Ο σκοπός της παρούσας εργασίας αναλύεται σε δύο διαστάσεις. Η πρώτη διάσταση αναφέρεται στη διερεύνηση των διαδικασιών μοντελοποίησης στη λύση προβλήματος και στην εξέταση των παραγόντων που επηρεάζουν τις διαδικασίες μοντελοποίησης. Η δεύτερη διάσταση αναφέρεται στη διερεύνηση της αποτελεσματικότητας ενός παρεμβατικού προγράμματος στις δεξιότητες μοντελοποίησης των μαθητών.

Στην έρευνα συμμετείχαν δεκαέξι τμήματα Στ’ δημοτικού και Β’ γυμνασίου. Συγκεκριμένα, στην πειραματική ομάδα συμμετείχαν τέσσερα τμήματα Στ’ δημοτικού και τέσσερα τμήματα Β’ γυμνασίου. Οι μαθητές αυτοί εργάστηκαν με τις δραστηριότητες μοντελοποίησης που σχεδιάστηκαν για τις ανάγκες της παρούσας εργασίας για τρεις μήνες. Στην ομάδα ελέγχου συμμετείχαν, επίσης, τέσσερα τμήματα Στ’ δημοτικού και τέσσερα τμήματα Β’ γυμνασίου.

Η ανάλυση των αποτελεσμάτων της εργασίας οδήγησε στην ανάπτυξη ενός αναλυτικού και επεξηγηματικού πλαισίου για τις διαδικασίες μοντελοποίησης. Το επεξηγηματικό πλαίσιο παρουσιάζει μια αναλυτική και σε βάθος περιγραφή των διαδικασιών και υπό-διαδικασιών μοντελοποίησης που χρησιμοποιούν οι μαθητές στην επίλυση μαθηματικού προβλήματος. Τα αποτελέσματα της εργασίας έδειξαν, επίσης, ότι μια σειρά από άλλους παράγοντες επηρεάζουν τη διαδικασία μοντελοποίησης. Οι παράγοντες αυτοί εστιάζονται: (α) στην εμπειρία των μαθητών να εργάζονται με δραστηριότητες μοντελοποίησης, (β) στο πλαίσιο των δραστηριοτήτων μοντελοποίησης, (γ) στην αξιοποίηση τεχνολογικών εργαλείων κατά την επίλυση των προβλημάτων μοντελοποίησης, (δ) στην ηλικία των μαθητών και (ε) στις δεξιότητες μοντελοποίησης των μαθητών.

Οι μαθητές που συμμετείχαν στην έρευνα κατάφεραν να επιλύσουν με επιτυχία τα προβλήματα που δόθηκαν στο πλαίσιο των δραστηριοτήτων μοντελοποίησης. Οι μαθητές της Β’ γυμνασίου κατασκεύασαν πιο σύνθετα και πιο αποτελεσματικά μοντέλα για την επίλυση των προβλημάτων που δόθηκαν, σε σχέση με τους μαθητές της Στ’ δημοτικού. Οι μαθητές της Β’ γυμνασίου κατάφεραν, επίσης, να υιοθετήσουν και να μεταφέρουν πιο εύκολα τα μοντέλα που είχαν κατασκευάσει μεταξύ των διαφόρων δραστηριοτήτων, ενώ είχαν πιο αποτελεσματική επικοινωνία μεταξύ τους η οποία βοήθησε στη βελτίωση των προτεινομένων λύσεων. Στο πλαίσιο της ανάλυσης των δεξιοτήτων μοντελοποίησης, η

έρευνα επιβεβαίωσε την ύπαρξη μιας αναπτυξιακής τάσης από τις δεξιότητες μοντελοποίησης σε προβλήματα λήψης απόφασης στις δεξιότητες μοντελοποίησης σε προβλήματα ανάλυσης συστήματος και σχεδιασμού και στη συνέχεια στις δεξιότητες μοντελοποίησης σε προβλήματα αναγνώρισης και επίλυσης δυσλειτουργιών σε συστήματα.

Το παρεμβατικό πρόγραμμα που εφαρμόστηκε είχε σημαντική επίδραση στις δεξιότητες μοντελοποίησης των μαθητών. Συγκεκριμένα, ο ρυθμός ανάπτυξης των δεξιοτήτων μοντελοποίησης των μαθητών της Στ’ δημοτικού που συμμετείχαν στην πειραματική ομάδα ήταν δυόμιση φορές μεγαλύτερος από τον αντίστοιχο ρυθμό ανάπτυξης των μαθητών που συμμετείχαν στην ομάδα ελέγχου. Παρομοίως, ο ρυθμός ανάπτυξης των δεξιοτήτων μοντελοποίησης των μαθητών της Β’ γυμνασίου που συμμετείχαν στην πειραματική ομάδα ήταν τρεις φορές μεγαλύτερος από τον αντίστοιχο ρυθμό ανάπτυξης για τους μαθητές της ομάδας ελέγχου. Το παρεμβατικό πρόγραμμα για την ανάπτυξη των δεξιοτήτων μοντελοποίησης ήταν, επίσης, περισσότερο αποτελεσματικό για τους μαθητές με χαμηλότερες αρχικές τιμές στη μέτρηση των δεξιοτήτων μοντελοποίησης.

Η σημασία και καινοτομία της παρούσας εργασίας επικεντρώνεται στην ανάπτυξη ενός θεωρητικού μοντέλου για τις διαδικασίες μοντελοποίησης που χρησιμοποιούν οι μαθητές στην επίλυση μαθηματικού προβλήματος. Το θεωρητικό μοντέλο παρουσιάζει μια αναλυτική περιγραφή των διαδικασιών και υπό-διαδικασιών μοντελοποίησης και περιγράφει τους παράγοντες που επηρεάζουν τις διαδικασίες μοντελοποίησης. Το θεωρητικό μοντέλο έχει εφαρμογές στο σχεδιασμό δραστηριοτήτων μοντελοποίησης, στη διδασκαλία της μαθηματικής μοντελοποίησης ως μέσου για την ανάπτυξη της ικανότητας λύσης προβλήματος των μαθητών και στη διδασκαλία για ανάπτυξη των ικανοτήτων μοντελοποίησης των μαθητών.

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To Marilena, Christos, and Alexis

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CHAPTER I

THE PROBLEM

Introduction

The society has become increasingly complex and integrated, especially during the last two decades. Students face a demanding knowledge-based economy and workplace, in which they need to deal effectively with complex, dynamic and powerful systems of information and be adept with technological tools (Lesh & Zawojewski, 2007). The whole world is increasingly governed by complex systems which are present in students' everyday life. Hence, students have to deal effectively with such systems. An understanding of the world as interlocked complex systems is critical for making effective decisions (Davis & Sumara, 2006; Jacobson & Wilensky, 2006; Lesh, 2006). Educational leaders emphasize the need to develop students' abilities to deal with complex systems for success beyond school (European Research Council (NRC), 2003).

One effective medium for achieving this goal for students is mathematical modeling, a process that describes real-world situations in mathematical terms in order to gain additional understanding or predict the behaviour of these situations (National Council of Teachers of Mathematics (NCTM), 2000). Mathematical modeling can promote students' active engagement in acquiring mathematical knowledge (Blum & Niss, 1991; Schoenfeld, 1992; Lesh & English, 2005; Lesh & Sriraman, 2005). The importance of mathematical modeling as a means for optimizing increasingly complicated problems and as a venue for decision-making in the behavioural and social sciences has been documented by a number of researchers (Saaty & Alexander, 1981; Niss, 1987; English, 2002; Lesh & Zawojewski, 2007) and proposed curricular reforms by professional organizations (American Association for the Advancement of Science (AAAS), 1998; NCTM, 2000; NRC, 2001).

The significance of mathematical modeling as a means for developing students' problem solving skills has been recognized as one of the purposes of school mathematics (NCTM, 2000). Specifically, researchers documented that school mathematics needs to provide an appropriate learning environment for students to develop the fundamental components of mathematical modeling (Kaiser, Sriraman, & Blomhoj, 2006; Lesh &

Zawojewski, 2007**)**; use models, develop representational fluency, reason in mathematically diverse ways, and use sophisticated equipment and resources in solving complex real world problems (NCTM, 2000; Lesh & Heger, 2001; English, 2002). School mathematics environment needs to provide a means for students to develop the necessary modeling abilities for solving real problems. These abilities include: constructing, describing, explaining, manipulating, and predicting complex systems (such as sophisticated buying, leasing, and loan plans); working on multi-phase and multi-component projects in which planning, monitoring, and communicating are critical for success; and adapting rapidly to ever-evolving conceptual tools and resources (English, 2002; Lesh & Doerr, 2003; Gainsburg, 2006).

The introduction of mathematical modeling in school mathematics can have a positive impact on teachers' and researchers' role. Student work in thought revealing – modeling activities can help researchers and teachers attend to student thinking. Methods, approaches, models and solutions collected from students' work and classroom discussions are used to create a context in which researchers and teachers examine unnarrated information from a classroom. Since data is unnarrated, researchers and teachers are placed in a realistic position for interpreting students' thinking (Lesh & Zawojewski, 2007). The latter is crucial in improving teachers' and researchers' understanding of students' thinking, and assist teachers on building on students' prior understandings and knowledge (English, 2006).

Despite the importance of introducing modeling in school mathematics at all grade levels, as Greer, Verschaffel and Mukhopadhyay (2007) point out, mathematical modeling has traditionally been reserved for the higher secondary and tertiary levels, with the assumption that elementary school students are incapable of developing their own models and sense-making systems for dealing with complex situations (English, 2007). Greer and his colleagues (2007) continued by discussing the limited successfulness of introducing mathematical modeling even at the secondary and tertiary levels and by proposing that students should be introduced to modeling at an earlier school level. Recent research (English & Watters, 2005; English, 2006; Lesh & Zawojewski, 2007) shows that younger students can and should deal with situations that involve more than just simple counts and measures and can benefit from working with authentic modeling problems. Additionally, it has been claimed that it is important that students start from their early school years to work with authentic problems (English, 2006). It is, therefore, necessary to provide students opportunities to explore complex problems in the elementary mathematics curriculum, if we want them to be successful modellers at secondary and tertiary levels and to be successful problem solvers beyond school mathematics.

The concerns related to the successful introduction of mathematical modeling in school mathematics do not rely only on practical issues. Blum (2004) discussed that a number of epistemological questions related to mathematical modeling are still unanswered. In line with Blum, Lester (2005) acknowledged that a well established theory for models and modeling perspective is needed if mathematical modeling is to become a vehicle for improving students' problem o solving abilities. These two dimensions (theoretical and practical) are the crux of the research focus for the present study. Specifically, the purpose of the study is the further development of the current theory of the models and modeling perspective in problem solving (Lesh & Doerr, 2003).

The Problem

The traditional approach for mathematics teaching is not sufficient for empowering students with problem solving abilities and therefore can not help students develop competencies in mathematics and applications (Schoenfeld, 1987, 1991; Lesh & Zawojewski, 2007). The results of the traditional instruction in mathematics include a prevalence of mechanical or mindless solutions to word problems (Verschaffel, De Corte, & Lasure, 1994; Greer, 1997), student solutions achieved without understanding the problem (Reusser & Stebler, 1997), and a dominant problem-solving strategy of looking for key-word interpretations rather than thinking deeply about a problem (Schoenfeld, 1987; English, 2003). According to Wyndhamm and Säljö (1997), the construction of a word problem describing an everyday set of events implies that an account of a situation (formulated in everyday language) is decontextualized and subsequently recontextualized in a different setting (as a word problem in mathematics teaching). This insufficient traditional approach is also influenced by the mathematics classroom environment. Among others, Reusser and Stebler (1997) indicated that the socio-cognitive practices of traditional mathematics classroom instruction shape and also narrow students' skills of mathematization (Freudenthal, 1991). In line with previous research, results from TIMSS 2003 survey showed that students did not perform well in problem solving and mathematical applications activities; on the contrary students performed very well in calculations and one step problems (Mullis, Martin, Gonzalez, & Chrostowski, 2004).

To address the needs raised in previous paragraphs and to reform mathematics education to produce a more positive learning environment, the scientific community has consistently called for an increased emphasis in applications and mathematical modeling (Blum & Niss, 1991; Blum, 2004; English, 2006; Lesh & Zawojewski, 2007). NCTM (1989) documented that “applications and mathematical modeling to take a more central position in mathematics education at all levels” (p. 209). The movement toward inclusion of mathematical modeling in the mathematics program has been encouraging and resulted in developing a quite significant number of materials (Blum & Niss, 1991; Blum, 1993). The focus on mathematical modeling as a means for teaching problem solving changes the mathematics teaching in a number of dimensions. Mathematical modeling offers richer learning experiences to students than the standard classroom word problems (English, 2007). The use of real-world contexts that elicit the creation of useful systems or models can assist students in making the necessary connections between the specific mathematical concepts and several topic areas not only from mathematics but also from other disciplines. The implementation of sequences of modeling activities can encourage the creation of models that are applicable to a range of related situations and can promote social interactions and communication, since students work in small groups (English, 2003; Lesh & Zawojewski, 2007).

Despite the research efforts on introducing modeling as a problem solving activity in school mathematics, a number of issues prevent the successfulness of these efforts. Blum (2004) documented that the primary focus of most of the efforts was on *practice*, e.g., on developing and trying out mathematical modeling activities, writing application-oriented textbooks, implementing applications and modeling into existing curricula or developing innovative, modeling oriented curricula. As a result, there was limited focus on *theory; e*.g., examination of the modeling processes students develop in modeling activities; investigation of students’ modeling abilities and identification of difficulties and strategies activated by students when dealing with modeling problems; observation and analysis of learning and communication processes in modeling activities (Blum, 2004; Lester, 2005). A number of questions related to the issues presented above are raised by a number of researchers; how learning is activated when it is embedded in a contextual framework (Boaler, 1993, 2001), how does a teacher assess learning in the context of mathematical modeling (Moschkovich, 1998), and what is the role of communication in the processing of mathematical ideas (Yerushalmy, 1997). Additionally, questions such as what classroom didactic best supports mathematics learning, when using a modeling approach (Maull & Berry, 2001), how can teacher beliefs about mathematics education become aligned with this increase in applications and modeling (Verschaffel, De Corte, & Borghart, 1997) and what level of impact should technology play in this transition (Blum & Niss, 1991) have been documented in a number of research studies.

The problem of the unsuccessful introduction of mathematical modeling as a problem solving activity at the school level is addressed in the present study. Specifically, the study addresses the problem both at the theoretical and at the practical level. At the theoretical level, the study contributes to the current theoretical perspective on the models and modeling perspective in problem solving, by investigating modeling processes in students' work, students' modeling abilities, and by examining other factors that influence the modeling procedure. At the practical level, the study adopts the design principles for developing modeling activities and develops an intervention program for improving students' modeling abilities.

Purpose and Research Questions of the Study

The purpose of the present study is the further development of the theory for the models and modeling perspective in problem solving, as it is proposed by Lesh and Doerr (2003). This learning theory is expected to contribute in deeper understanding of the modeling processes appear in students' work in solving real world problems. The theory also aims to trace how students' modeling abilities evolve over time. The theory is finally expected to contribute to a better understanding of modeling as a didactic means in mathematical problem solving.

Specifically, the goals of the present study are to: (a) Provide an in depth description and analysis of the modeling processes and sub processes that students develop when working with modeling activities, (b) Identify possible factors that influence the modeling procedure and formulate the different modeling processes, (c) Examine students' modeling abilities in different categories of modeling problems and how are these abilities interconnected, (d) Examine the impact of an intervention program, consisting of a sequence of modeling activities, on students' modeling abilities, and (e) Utilize complementary qualitative and quantitative data analysis to triangulate the findings of the study.

In particular, the research questions for this study are:

(a) What are the modeling processes and sub processes students develop in working with modeling activities?

(b) What is the impact of the intervention program on students' modeling abilities?

(c) How modeling processes evolve across students' investigation in the sequence of the modeling activities of the intervention program?

(d) How modeling processes are differentiated between 6^{th} and 8^{th} graders?

(e) What are the similarities and differences in modeling processes between successful and unsuccessful models constructed by 6^{th} and 8^{th} graders?

(f) What are the characteristics of students' modeling abilities?

(g) Can a theoretical structure based on student modeling abilities in different categories of modeling problems be validated?

(h) How student modeling abilities are changed over time (rate of change) and what is the impact of the intervention program on the rate of change of the modeling abilities?

Significance of the Study

Burkhardt and Pollak (2006) recently raised several fundamental issues on the implementation and the place of modeling in the school curricula at micro and macro levels. In discussing implications for the future, Burkhardt and Pollak (2006) identified four major barriers to the implementation of mathematical modeling in school curricula at the macro level. These barriers include the small scale implementation of modeling in school mathematics for reasons like assessment and time needed for modeling activities, the complexity of modeling activities, the teachers' limited professional development and the inexistence of pathways between research in mathematics education and classroom practice for turning research insights into improved practice (Burkhardt & Schoenfeld, 2003).

Sriraman and Lesh (2006) further commented on the fundamental issues raised by Burkhardt and Pollak and the barriers from the perspective of the on going work of the models and modeling research group. They wrote that in comparison to the school curricula in the U.S in the 1950's and 60's, many contemporary reform based curricula were incorporating a data driven modeling approach to the teaching of mathematics in which context plays an important role in getting students interested in the material. Examples cited by Sriraman and Lesh (2006) of NSF funded high school curricula which make use of this approach were the Core Plus Mathematics Project (CPMP), and the Systemic Initiative for Montana Mathematics and Science (SIMMS). However Sriraman and Lesh were critical about the approach of implementing modeling based curricula in high schools without first being concerned about research based findings of how concept development occurs in modeling situations at earlier grades.

The present study takes a research based stance on first examining the models and modeling perspective approach in problem solving, as proposed by Lesh and Doerr (2003). This theoretical approach serves in two ways. First, it serves as a basis for developing an analytical and an explanatory framework for investigating and analyzing the modeling processes and sub processes presented in students' work in modeling activities. Second, it directs the examination of students' modeling abilities and the investigation of other factors that influence the modeling process. The analytical and explanatory framework is expected to extend and improve the current theoretical approach on modeling perspective and to serve towards developing a unify theory for the field.

Besides the significance of further developing the existing theory on the models and modeling perspective, the findings of the present study are important for a number of reasons. At a theoretical level, the study first contributes in further gaining understandings of the modeling procedure and on the evolution of students' modeling processes in problem solving. Second, the study examines how students' modeling abilities are interconnected with students' modeling processes and how these modeling abilities and processes are influenced by other factors, like the context and the principles for designing modeling activities. At a practical level, the significance of the study can be analyzed in the following three dimensions. First, the study examines the effectiveness of an intervention program consisted of a sequence of modeling activities on students' modeling processes and modeling abilities. Second, the study traces the similarities and differences between students' modeling processes at different grade levels, and third, it utilizes quantitative and qualitative data to complement its design.

The findings of the study are expected to assist other studies in the field, as well as to inform professional development and curricula. Additionally, with respect to mathematics reform movement, the study provides valuable information from the participants and data analysis for improving mathematics instruction and enhancing mathematical learning. Finally, the study provides valuable insights to curriculum designers and policy makers in Cyprus and in other countries for future reforms in elementary and secondary school mathematics curricula, including mathematical modeling in problem solving.

Limitations of the Study

There is a number of limitations resulted from the methodology employed in the present study. The study was conducted in a limited number of schools in Nicosia's district. Therefore the results are only representative of that group of classes and schools. One limitation is related to the sample of the study. Only 6th and 8th grade classes participated in the study. The decision for selecting 6th grade classes is based on the fact that the required mathematical concepts needed for working in the modeling activities are already taught to 6th graders. Additionally, 8th graders were selected since they were taught mathematics for at least one and a half years, following the secondary school mathematics curriculum.

A second set of limitations is related to the modeling activities. The choice of contexts for developing the modeling activities is appropriate to the subjects, and is written in an age-appropriate language. Additionally, the decision to do the intervention with real classes of students, rather than gathering volunteers for an after-school experiment, extended the applicability of the results. However, the facts that students had no prior experience in modeling activities and that participating classes were mixed ability in terms of students' mathematical ability, made it difficult sometimes for students to successfully work in small groups to construct models for solving the real world problems.

Another limitation is related to the time needed for working with the modeling activities. Considering that the intervention program was consisted of six modeling activities and that the implementation of each activity took four 40 minute sessions, it is obvious that the intervention was time demanding for both teachers and students. It might be better to allow students to research the problem for themselves, and gather whatever information they deemed important or necessary to model the situations without time constrains. However, this could not be done with real classes of students. Another limitation of the study is related to the use of technology. It would have been desirable to be able to incorporate computers in all modeling activities and not only in two out of the six activities. However, considering that the participating schools were equipped with computer labs with a small number of computers, it was extremely difficult that each group of students had one computer to work with in all activities.

Thesis Structure and Summary

To address the research questions presented earlier, sixteen elementary and secondary grade classes participated in this study. A sequence of six modeling activities was implemented in four 6^{th} and four 8^{th} grade classes.

The research questions for this study align to the following goals of providing an in depth analysis of students' modeling processes, identifying influencing factors of the modeling procedure, examining students' modeling abilities, and examining the impact of the intervention program. To capture student modeling processes, the primary data sources consisted of transcripts of students' interactions in their groups, their written responses in their worksheets, and their work files in the accompanying computer software. These data sources also included teachers' written notes and investigator's field notes. To examine the students' modeling abilities, the primary data sources consisted of students' responses in the three administrations of the modeling abilities test.

For the analysis of the qualitative data, the Grounded Theory approach, as described by Strauss and Corbin (1998) was used for the development of the analytical and explanatory framework. For the analysis of the quantitative data, Structure Equation Modeling (SEM) techniques were used to examine the structure of the modeling abilities in different modeling tasks and Latent Growth Modeling (LGM) techniques were used to examine how modeling abilities change over time and how the intervention program influenced students' modeling abilities (Marcoulides & Schumacker, 1996).

The next chapters describe more thoroughly the research process, analyses, and findings. Specifically, chapter two provides a review of the literature relevant to this study. Chapter three explains the methodology and presents the intervention program, the modeling activities and the test for measuring students' modeling abilities. Chapter four presents the analytical framework that guided the analysis of the case studies and the explanatory framework. Chapter four also presents the validated model for explaining students' modeling abilities in the three categories of modeling problems and students' modeling abilities growth models. Chapter five discusses the findings of the study and provides implications and suggestions for future research.

Operational Definitions

Modeling is the entire process leading from the original real problem situation to a mathematical model. The modeling process does not merely yield a simplified but true image of some part of a pre-existing reality. Rather, mathematical modeling also structures and creates a piece of reality, dependent on knowledge, intentions and interests of the problem solver (Blum & Niss, 1991).

Models are conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors. Their purposes are to construct, describe or explain other system(s) (Lesh & Doerr, 2003).

Models include both: (a) *a conceptual system* for describing or explaining the relevant mathematical objects, relations, actions, patterns, and regularities that are attributed to the problem-solving situation; and (b) *accompanying procedures* for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals. Mathematical models are distinct from other categories of models mainly because they focus on structural characteristics (rather than, for example, physical, biological, or artistic characteristics) of systems they describe.

Model development typically involves quantifying, organizing, systematizing, dimensionalizing, coordinatizing, and (in general) mathematizing objects, relations, operations, patterns, or rules that are attributed to the modeled system. Consequently, the development of sufficiently useful models typically requires a series of iterative "modeling cycles" where trial descriptions (constructions, explanations) are tested and revised repeatedly.

Problem Solving refers to the process of associating prior experiences, knowledge, information and intuition in order to determine the outcome or a solution of a situation for which the procedure for determining the outcome is not directly known (Charles, Lester, & O' Daffer, 1987).

Modeling Activities. Thought – revealing activities that require students to generate a method, explanation, prediction, description, or solution to a problem for a specific client in a context. They are designed using six principles (Lesh et al., 2000) including self – assessment, model construction, model documentation, personal meaningfulness, simplicity, and generalizability. The students work in a small group (usually three students) for a few class periods on the modeling activity. The final product is a letter to the client describing the solution to the particular problem as well as a generalizable method for solving related and structurally similar problems. Then, the students present their solutions in a formal presentation in the class. Whole class discussion assists students in further improving and refining their solutions.

Modeling Processes are the processes students develop and use during their efforts to solve a real world problem. These processes include describing the problem, manipulating the problem and building a model, connecting the mathematical model with the real problem, predicting the behavior of the real problem and verifying the solution in the context of the real problem.

Student Modeling Ability includes structuring, mathematizing, interpreting, and solving real world problems and it includes the ability to work with mathematical models: to validate the model, to analyze it critically and to assess the model and its results, to communicate the model and to observe and to control self adjustingly the modeling process (Blum, 2004).

CHAPTER II

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Introduction

During the last years, an increasing number of mathematics education researchers have focused their efforts on mathematical modeling, especially in mathematical modeling at the school level. This is evident in numerous research publications from groups of researchers in Australia (English, Galbraith and colleagues), Belgium (Verschaffel and colleagues), Denmark (Niss, Blomhøj and colleagues), Germany (Blum, Kaiser and colleagues), Netherlands (de Lange and colleagues) and the U.S (Lesh, Schoenfeld and colleagues). Among the questions that have been raised are questions related to how well students are prepared to solve real world problems that they encounter beyond school, to solve problems in their future professions, as citizens and in further learning (Blum, 2004; English, 2006). A second set of questions is related to whether students are capable of dealing with unfamiliar situations by thinking flexibly and creatively (Lesh & Doerr, 2003).

Mathematical modeling has been considered as an effective medium not only to answer questions like the ones raised above, but also to foster critical mathematics education (Skovsmose, 1994, 2000). Although the National Council of Teachers of Mathematics (NCTM, 2000) calls for purposeful activities along with skillful questioning to promote the understanding of relationships among mathematical ideas, this recommendation can be pushed further and modeling activities can be used as a way to cultivate critical thinking and critical literacy (Skovsmose, 2000; Sriraman & Lesh, 2006).

Modeling activities can assist students in using important mathematical ideas in problem solving, and can help teachers to develop an understanding of students' thinking. Mathematics education researchers need to design well structured modeling activities that provide rich opportunities for students to develop their ideas (Lesh & Doerr, 2003; Borromeo Ferri, 2006; Lesh & Zawojewski, 2007). A modeling perspective leads to the design of an instructional sequence of activities that begins by engaging students with non-routine problem situations that elicit the development of significant mathematical constructs and then extending, exploring and refining those constructs in other problem situations leading to a generalizable system (or model) that can be used in a range of contexts (Lesh & Doerr, 2003; English & Doerr, 2004). In these activities, referred to as model eliciting activities, the products that students produce go beyond short answers; they include sharable, manipulatable, modifiable, and reusable conceptual tools (e.g., models) for constructing, explaining, predicting and controlling mathematically significant systems (Lesh & Doerr, 2003; Lesh & Zawojewski, 2007). Students' descriptions, explanations, and justifications form an integral component of the models the students produce. In contrast to many of the problem situations students meet in school, modeling activities are inherently social experiences, where students work in small teams to develop a product that is explicitly sharable (Doerr & English, 2001). Numerous questions, issues, conflicts, resolutions, and revisions arise as students develop, assess, and prepare to communicate their products (English & Doerr, 2004).

In an attempt to review the related literature and provide a coherent framework for the research investigation, the literature review is organized into three major strands. The first strand discusses the emergence of mathematical modeling as a problem solving activity from the inadequacy of traditional approaches in teaching and learning mathematics. The strand situates mathematical modeling as being problem solving activity, talking about the emerging literature on modeling which comes out of the limitations in problem solving research. The second strand discusses modeling activities for students, by presenting the principles for designing and implementing modeling activities in classrooms and by discussing the products of the modeling activities for students. The second strand finally presents research findings related to the benefits for students and teachers in working with thought revealing modeling activities. Finally, the third strand provides details about issues related to the teaching and learning of mathematical modeling, like the contextual nature of modeling and assessment of mathematical modeling.

Mathematical Modeling as a Problem Solving Activity

There are a number of aspects related to the teaching and learning of modeling that present a distinct possibility for enhancing mathematics education (Sriraman, Kaiser & Blomhøj, 2006b). There are also many obstacles preventing the fruition of modeling in mathematics and especially in mathematical problem solving. The first strand of the literature review briefly discusses the inadequacy of traditional approaches in teaching mathematics and problem solving in particular. Second, the literature review focuses on problem solving in mathematics, showing the emergence of modeling as a problem solving activity and presenting two research examples of implementing modeling as a problem solving activity. Third, this strand of the literature review emphasizes on the modeling procedure at a micro level, presenting and discussing the modeling processes students develop in working with modeling activities. This strand of the literature review, finally, discusses the design principles for developing modeling activities and presents research findings related to the benefits students gain from working with modeling activities.

Inadequacy of Traditional Approaches in Problem Solving

Recent research findings in mathematics education raise important issues about students' mathematical problem solving and problem posing ability (Schoenfeld, 1992; Doerr & English, 2003; OECD, 2004; Christou, Mousoulides, Pittalis, Pitta, & Sriraman, 2005). They pointed out that students are not well prepared to solve the problems that they encounter beyond school, and to make connections between mathematics and the real world. The appropriateness of current approaches in teaching mathematics and in mathematical problem solving in particular is questioned by a number of researchers (Schoenfeld, 1987; Blum & Niss, 1991; Doerr & English, 2003; Greer, Verschaffel, & Mukhopadhyay, 2007). The inadequacy of traditional approaches is also shown worse in the case of students' work with problems that are less obviously linked to school mathematics and require students to deal with unfamiliar situations by thinking flexibly and creatively (Kaiser & Schwarz, 2006; Lesh & Doerr, 2003).

A central characteristic of traditional mathematics teaching and learning in many countries is the activity of 'solving word problems' (Schoenfeld, 1992). These word problems usually represent recontextualized forms of decontextualized descriptions of everyday life situations that serve a specific purpose; being exercises for specific types of mathematical learning, such as addition or subtraction (Wyndhamm & Saljö, 1997). This practice of word problems is not related to problem solving activity, as proposed by Polya (1962, 1973) and Schoenfeld (1992). Following Polya's (1962) interpretation, problem solving is not practising calculations. On the contrary, in problem solving students need to translate a real situation into mathematical terms, to think of possible strategies to find a way from given to goals and to finally have the opportunity to experience what mathematical concepts are related to realities.

The lack of connections between the practise of word problems in the traditional school mathematics environment and the real world, which is described above, is also identified by a significant number of recent research studies (English, 2006; Lesh & Zawojewski, 2007). Researchers pointed out that the practice of word problem solving in school mathematics hardly matches this idea of mathematical modeling and mathematization (Reusser & Stebler, 1997; Lesh & Zawojewski, 2007). As a result, students readily 'solve' unsolvable, even absurd, problems if presented in ordinary classroom contexts (Vershaffel, De Corte, & Lasure, 1994; Greer, 1997; Yoshida, Verschaffel, & De Corte, 1997).

The other consequence of using word problems as mathematical exercises, rather than opportunities to do mathematics, is the prevalence of low level cognitive processes involved. When solving stereotyped or routine word problems, students often look for key words or employ direct translation strategies (Schoenfeld, 1982, 1991). Any change in the presentation of the task affects the problem difficulty and their ability in solving these problems is influenced by contextual information (Staub & Reusser, 1995).

Hiebert, Carpenter, Fennema and Fuson (1996) pointed that traditional ways of doing mathematics focus only on mastering skills and applying them in solving routine problems; traditional approaches fail to follow reflective inquiry (Dewey, 1933) method as a way of developing students' conceptual understanding. As a result traditional approaches in teaching and learning mathematics can not prepare students for everyday life, since students are not able to transfer the specific domain-related knowledge (mathematics) and also more general problem- and solution-related skills (Hiebert et al., 1996; Romberg, Carpenter, & Kwako, 2005).

A number of curricula in the past aimed to promote problem solving ability, but they did it rather in a strict linear fashion (Gravemeijer, 1999). In these cases, students were not prepared for real world problems, because these problems are not linear. Hiebert and colleagues (1996) stressed that linear models fail to account for everyday problem solving because they are ill-suited to the dynamic conditions of the workplace. Their recommendation was to base instructional design on problem solving in out-of-school situations, rather than on problem-solving models of limited applicability.

A concluding point, referring to traditional approaches of mathematics teaching and problem solving, comes from Schoenfeld (1992), and it is related to the structures of a problem situation. Schoenfeld pointed out that even at its best, the purpose of many problems involves defining a core, and most of the times invariant structure, developing auxiliary hypotheses that act to protect the core of the problem, and then outlining goals and heuristics to solve that problem. In this process, extensions of original problems become new problems to solve, and solved problems become sub-problems for other problems in the field. This process creates restrictions on ways of understanding and responding to problems which are not given in the problem statements, and this process results in restricting students from promoting their problem solving abilities (Lesh, 2006; Blomhøj & Kjeldsen, 2006).

Problem Solving in Mathematics

Problem Solving refers to the process of associating prior experiences, knowledge, information and intuition in order to determine the outcome or a solution of a situation for which the procedure for determining the outcome is not directly known (Charles, Lester, & O' Daffer, 1987). Similar to the above definition, problem solving as proposed by Stanic and Kilpatrick (1987) and Schoenfeld (1992), is equated with Dewey's reflective thinking (Dewey, 1933). Researchers pointed out that for mastering problem solving abilities, students need to work with challenging real world problems, by using their prior experiences and learning strategies, reflecting on their solutions and connect their solutions with the real problems. In line with the above approach, National Research Council (2001) documented the necessity that mathematical problem solving needs to deal with data and observations from real world and from science. The NRC document continued pointing out that problem solving is far more than just calculation or deduction; it involves observations of patterns, testing of conjectures, and estimation of results (NRC, 2001).

A connection to mathematical problem solving comes from Freudenthal (1991), who introduced *mathematization*. By mathematization, Freudenthal referred to the structuring of reality by mathematical means. He identified mathematization as an internal part of problem solving and he stressed that it is important to link mathematics with real world problems (Burkhardt & Pollak, 2006). Following Freudenthal's interpretation, Schoenfeld (1992) argued that mathematization in problem solving is necessary for developing high level processes that influence students' mathematical abilities

(Schoenfeld, 1992; Maaβ, 2006).

Among others articulating the importance of the role of teachers in problem solving in mathematics, Lesh and Doerr (2003) recommended that teachers should put more emphasis on the use of real world based problems and on conjecturing, making their classrooms a place where students develop models for unfolding a problem, modify and refine approaches and communicate solutions (Doerr & English, 2003; Doerr, 2006; English, 2006). To succeed in improving students' problem solving abilities teachers should avoid working in a strict linear fashion, because real world problems are not linear. Therefore, developing simplistic linear models and solutions fail to account for everyday problem solving because these solutions are not suitable for the dynamic and multifaced world based problems (Roth, & McGinn, 1997).

The Modeling Perspective in Problem Solving

An interesting extension to problem solving comes from Lesh and his colleagues (Lesh & Doerr, 1998, 2003; English, 2003, 2006; Lesh & Sriraman, 2006; Lesh & Zawojewski, 2007). The proposed *models and modeling perspective* suggests that mathematical modeling can serve as a didactic means for problem solving. Lesh and Doerr argued that modeling as a problem solving activity provides an environment where students refine, transform and extend initially inadequate (but dynamically evolving) conceptual models in order to create 'successful' problem interpretations.

Modeling as problem solving activity moves beyond traditional problem solving experiences, by addressing the processes, developments and models students require in working with increasingly sophisticated systems and applying their models and solutions in a range of similar structure problems (Lesh & Lehrer, 2003; English, 2003). Zawojewski and Lesh (2003) clarified the distinction between traditional simplistic approach to problem solving and the modeling approach to problem solving by proposing the figure presented in the next page.

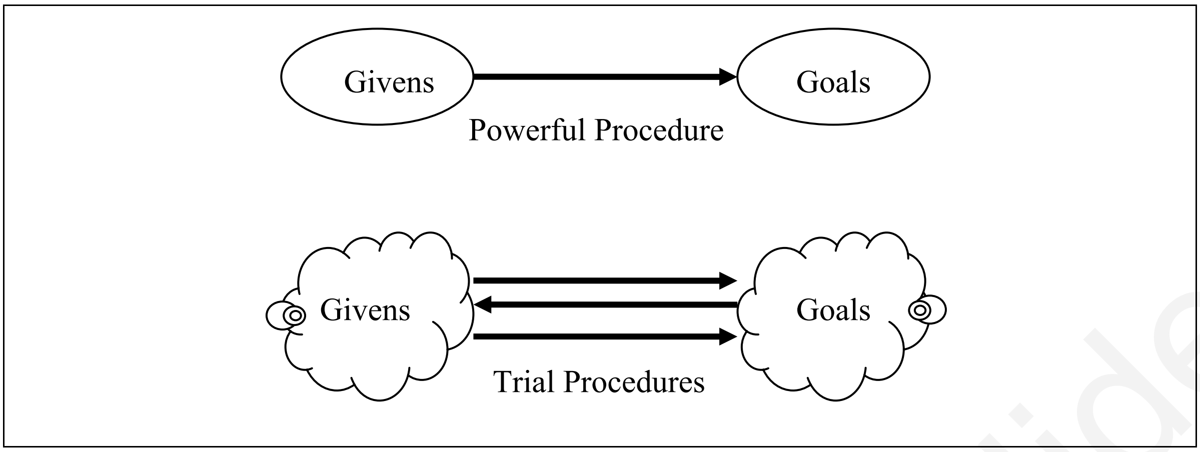


Figure 2.1. The linear and the modeling approach to problem solving.

The modeling approach to problem solving suggests that there is not a single powerful procedure between givens and goals and a set of strategies for overcoming any difficulties in this procedure. Indeed, the modeling approach indicates a number of trial procedures between givens and goals in order to succeed a solution to a real problem. Problem solving includes a number of iterative cycles, in which students move from givens to goals, go back and again moving towards goals to test their hypotheses, refine their results and to improve their solutions (Lesh, 2006; Sriraman & Lesh, 2006).

Using the modeling approach, the real problem required to be solved is not presented to students as an isolated task, but as a part of a modeling activity. Modeling activities require students to confront world based problems by understanding problem information, identifying important features and relationships between problem variables, constructing and applying appropriate representations and, finally, evaluating, justifying and communicating results as a means to further understanding the problem (Zawojewski & Lesh, 2003; Lesh & Zawojewski, 2007).

Modeling Projects in Problem Solving

Realistic Mathematics Education (RME). Realistic mathematics education (RME) is considered as one of the most successful research approaches in developing curricula for improving students' problem solving abilities using the modeling approach. In addressing the importance of mathematical modeling in transforming the classroom didactic, Gravemeijer (1997, 1999) explained that the rationale for the modeling approach in RME was twofold. He argued that the context of RME promoted students' progressive mathematization, assisted students in building on their informal experientially real knowledge and later led to algorithms, concepts and symbolic notations. Second, when modeling was constituted as an organizing activity, the students practiced this form of mathematizing (de Lange, 1987).

In extending the rationale for the modeling approach in RME, Gravemeijer (1997) reported that mathematics should enable students to solve real problems, involving a complex real-life situation that should be resolved and explained. In solving such problems, students become involved in collecting and compiling data in making decisions, in interpreting data and in discussing results and in reflecting on their solutions. Van den Heuvel-Panhuzen (2003) commented that in RME computational skills are not taught as an isolated practice, but on the contrary are used in application situations so that the use of the skill becomes meaningful to the students. She continued clarifying that curriculum developers are able to create courses that accomplish (without direct follow) the goals of RME, and, more importantly, work for the students (Gravemeijer, 1997; Van den Heuvel-Panhuzen, 2003).

In an attempt to compound the main goals of the RME program, de Lange (1987) highlighted the following: (a) Paying much attention to re-invention, that is recreating mathematical concepts and structures on the basis of students' intuitive notions, (b) carrying on at various levels of concreteness and abstraction, (c) the programming of the instruction is guided by the historical-genetic rather than the subject matter systematic method, and (d) reality bound meaningful and mathematically rich instruction (de Lange, 1987).

A number of research findings provide support to the appropriateness of RME as a modeling approach to problem solving. Verschaeffel and De Corte (1997) argued that it was possible to give middle grade students a disposition toward more realistic mathematical problem solving, by focusing on the importance of real world knowledge and sense making in the modeling and interpreting of the solution of arithmetic wor(l)d problems. Schlange (1995) claimed that RME based curriculum helped students carry concrete knowledge to more abstract levels, enabling them to become better problem solvers. Similarly, in Kief and Stewart's work (1996), it was shown that by doubling the length of the course and applying an intensive focus on applications, students could learn an equivalent amount of mathematical skill, and have improved attitudes toward math, despite the fact that they were previously unsuccessful. However, Verschaeffel and De Corte (1997) stressed that implementing RME activities in the school was not an easy task, since there was a strong need for teacher training, in accomplishing the goals of RME successfully. In line with previous findings, Van den Heuvel-Panhuzen (2003) reported that the introduction of new teaching and learning materials should be complemented by initiatives aimed at stimulating and supporting teachers to construct the proper concepts, skills, and attitudes that are needed for modeling of realistic problem situations.

Problem Based Learning (PBL). A second notable example of a modeling-like educational program is Problem Based Learning (PBL) (Gallagher, 1997). One central characteristic for PBL is the use of an ill-structured problem; such a problem changes as new information is obtained from solvers, has more information than is necessary at first, and has no single method for finding a solution. Problems representing the goals of PBL are not restricted in the field of mathematics. They are usually interdisciplinary, in an attempt to help students to learn a set of important concepts, ideas and techniques and represent a real world based problem faced by professionals in their workplace and to make connections between the mathematical concepts and the real problems. Boyce and colleagues (1997) found that the central focus for PBL's inquiry approach was to integrate student experience (including content, process and concept), and to engage students with a localized problem, and effective use of real-world resources.

According to Gallagher (1997), the goals of PBL included: (a) Fostering reasoning and problem-solving skills, (b) enhancing acquisition, retention, and use of knowledge, (c) improving students' self awareness and self directed learning skills, (d) developing students' intrinsic interest in subject matter and, subsequently, their motivation to learn, (e) developing students' capacity to see problems from multi-disciplinary viewpoints, integrating information from many different sources in an attempt to fully understand the problem, (f) facilitating the development of effective collaborative learning practices, and (g) improving flexible thought and the capacity to adapt to change (Boyce et al., 1997; Gallanger, 1997).

A number of research findings supports that problem based learning can have positive influence on students' problem solving skills. Specifically, Gallagher (1997) reported that PBL encouraged long-term retention, and Boyce and colleagues (1997) suggested that the increased motivation from students' engagement in the problems facilitated transfer of learning; they have cited research reports concluding that students were more satisfied with the PBL environment than with the traditional classroom approach. In line with previous findings, Wood and Sellers (1997) reported that students who participated in two years of problem based instruction outperformed their counterparts. Students were also found to hold different beliefs than those in the traditional program about doing mathematics that involve finding their own and/or different ways to solve problems.

Modeling Processes in Problem Solving

A number of relevant works (Blum & Niss, 1991; Lesh et al., 2003; Borromeo Ferri, 2006) have documented the different processes involved in mathematical modeling in problem solving. In an attempt to summarize previous work in modeling processes in problem solving, and to provide a theoretical framework, the present study adopts Lesh and Doerr's (2003) interpretation of the modeling procedure, incorporating the related modeling processes (see Figure 2.2).

In particular, within the problems students had to demonstrate the following processes: (a) Understand and simplify the problem. This included understanding text, diagrams, formulas or tabular information and drawing inferences from them; demonstrating understanding of relevant concepts and using information from students' background knowledge to understand the information given. (b) Manipulate the problem

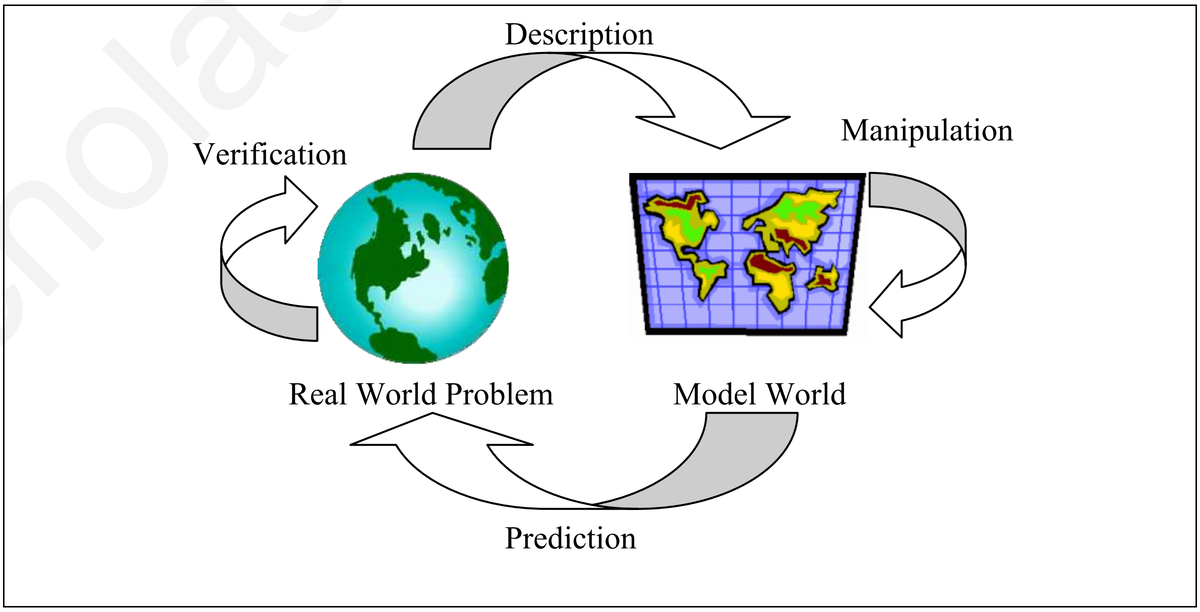


Figure 2.2. The Modeling Procedure.

and develop a mathematical model. These processes included identifying the variables and their relationships in the problem; making decisions about variable relevancy; constructing hypotheses; and retrieving, organising, considering and critically evaluating contextual information; use strategies and heuristics to mathematically elaborate on the developed model. (c) Interpreting the problem solution. This included making decisions (in the case of decision making); analysing a system or designing a system to meet certain goals (in the case of system analysis and design); and diagnosing and proposing a solution (in the case of trouble shooting tasks). (d) Verify, validate and reflect the problem solution: This included constructing and applying different modes of representations to the solution of the problem; generalize and communicate solutions; evaluating solutions from different perspectives in an attempt to restructure the solutions and making them more socially or technically acceptable; critically check and reflect on solutions and generally question the model (Blum & Kaiser, 1997; Lesh & Doerr, 2003).

Three types of problems were used in PISA 2003 study, in an attempt to identify the modeling processes in different categories of modeling problems (OECD, 2004). Specifically, the three types of problems were: *decision making*, *system analysis and design* and *trouble shooting* The modeling processes involved in these categories of problems are presented in Table 2.1 (for a detailed analysis of problem categories see OECD, 2004). As it can be noted from the Table, the modeling processes described in Figure 2.1 appear, as it is expected, in all three problem types. However, different problem types require a different mastery depth for each modeling process. As a result, different modeling abilities are necessary for success in each category of modeling problems.

As reported in PISA 2003 study, the analysis of the modeling processes students applied in problem solving resulted in three distinct performance levels. These levels provide an analytical model for describing what individual students are capable of in problem solving. An analytical model for explaining student modeling processes in problem solving and the benefits from such a model was theoretically proposed by Blum and Niss (1991). Blum and Niss (1991) suggested that an increased emphasis on modeling processes in problem solving should develop better problem-solving ability and eventually should result in fostering creative and problem solving capacities (attitudes, strategies, heuristics, techniques, etc.), open-mindedness, self-reliance and confidence.

Table 2.1.

Modeling Processes in the Three Categories of Modeling Problems

Decision Making	**System Analysis & Design**	**Trouble Shooting**
Understanding a situation where there are several alternatives and constraints and a specified task	Understanding the information that characterises a given system and the requirements associated with a specified task	Understanding the main features of a system or mechanism and its malfunctioning, and the demands of a specific task
Identifying relevant constraints	Identifying relevant parts of the system	Identifying causally related variables
Representing the possible alternatives	Representing the relationships among parts of the system	Representing the functioning of the system
Making a decision among alternatives	Analysing or designing a system that captures the relationships between parts	Diagnosing the malfunctioning of the system and/or proposing a solution
Checking and evaluating the decision	Checking and evaluating the analysis or the design of the system	Checking and evaluating the diagnosis/solution
Communicating or justifying the decision	Communicating the analysis or justifying the proposed design	Communicating or justifying the diagnosis and the solution

Modeling Activities: Design and Research

Student Models

Models are conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors (Pollak, 1970; Blum & Niss, 1991; Lesh & Doerr, 2003). Models include: (a) *a conceptual system* for describing or explaining the relevant mathematical objects, relations, actions, patterns, and regularities that are attributed to the problem-solving situation; and (b) *accompanying procedures* for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals (Lesh & Doerr, 2003; Lesh, Doerr, Carmona & Hjalmarson, 2003). Typically, this definition of model has only been used in reference to student or teacher thinking and learning (e.g., Doerr & Lesh, 2003). To provide a parallel construct at the researcher level, a design experiment carried out from a models and modeling perspective (a modeling design experiment) should be consistent with this definition. The design tested in the experiment encompasses two parts (similar to a model). Namely, the design includes theoretical assumptions (i.e., researcher-level conceptual systems about mathematical knowledge, models, teacher development, etc.) and external elements (i.e., representations of the researcher-level conceptual system in the form of interventions, curriculum, etc.) (Kaiser & Schwarz, 2006; Lesh & Doerr, 2003).

Student models in the context of the present study are mathematical models. For example, a student model for rates might include working hours and money collected (see for example in Chapter IV students' work in the University Cafeteria Activity), operations such as multiplication and division, and relationships between working hours and money collected that stand for productivity rate. In addition, models incorporate a number of external representations (e.g., graphs, tables). In constructing models, students identify, select and collect relevant data, express limitations and conditions of a model, interpret the solution in context, communicate effectively and describe situations using a variety of representation forms (Blomhøj & Kjeldsen, 2006; Jacobson & Wilensky, 2006).

In studying students' models, researchers need to develop their own models. Principles and assumptions about student-level mathematics learning and development should also apply to researcher-level models. One assumption is that researcher designs develop along multiple dimensions just as student models develop along multiple dimensions (Lesh, 2002, 2006). For example, student models may move from unstable to stable or from simple to complex. As researchers study a design, unstable early assumptions are repeatedly tested and become more well-developed and stable. Some assumptions may be revised throughout the study and eventually may stabilize at some point for the particular situation. When the models are transferred to another problem situation, they may become unstable again. Student models may be very simple at the start of their solution process and they may explain a limited part of the situation (Doerr & English, 2001; Lesh & Doerr, 2003).

Characteristics of Modeling Activities

The modeling activities are non-routine tasks because each task asks students to mathematically interpret a complex real-world situation and require them to formulate a mathematical description, procedure, or method (instead of a one-word or one-number answer as found in more traditional mathematical problems) for the purpose of making a decision for a realistic client (Lesh & Zawojewski, 2007). Groups of students are producing a description, procedure, or method and these students' solutions to the task reveal explicitly how students are thinking about the given situation (Lesh et al., 2000; Lesh & Doerr, 2003; Zawojewski, Lesh & English, 2003).

The different tools being designed and created to facilitate students' and teachers' externalization of their thinking and understandings of problem situations aim to elicit their thinking and thus researchers are referring to these tools as model eliciting activities (Lesh & Doerr, 2003; Lesh et al., 2003). Among the central characteristics of these activities are: (a) the development of a model that describes a real-life situation, (b) the developed models to encourage the solver to describe, revise, and refine their ideas and approaches, and (c) the developed models to encourage the use of a variety of representational media to explain (and document) students' conceptual systems. Modeling activities can be designed to lead to significant forms of learning because they involve mathematizing –by quantifying, dimensioning, coordinating, categorizing, algebraizing, and systematizing relevant objects, relationships, actions, patterns, and regularities (Lesh et al., 2003; English, 2006; Borromeo Ferri, 2006; Lesh & Zawojewski, 2007).

An example of a model eliciting activity for students is intended to reveal the way students are thinking about a real life situation that can be modelled through mathematics. The solution calls for a mathematical model to be used by an identified client who needs to implement the model adequately. As a result, students must clearly describe their thinking processes and justify not a single solution, but rather all (or most of) the optimal and appropriate solutions (English, 2003). Students' engagement with such mathematical tasks results in developing math concepts through the need to develop powerful math ideas in order to solve a problem. Thus, they are given a purpose (and End in View) (English & Lesh, 2003) to develop a mathematical model that best explains, predicts, or manipulates the type of real-life situation that is presented to them. In this way, model-eliciting activities allow students to document their own thinking and learning development.

Among the aims of a modeling activity are the real problem specification and understanding, the engagement in critical usage of modeling, the participation and the enhancement of communication skills (English, 2006). The modeling activities foster creative and problem solving attitudes, activities and competencies, provide the opportunity for students to practice applying mathematics that they would need as individuals in society. Modeling activities also contribute to a balanced picture of mathematics for students and assist students in acquiring and understanding the mathematical concepts presented in the activities (Battye & Challis, 1997; Pierce & Stacey, 2006; Lesh & Zawojewski, 2007).

Modeling Activities Development

One defining characteristic of a design experiment that uses the modeling perspective in developing modeling activities is that the researchers create, test, and modify a design within the context of use (Design-Based Research Collective, 2003; Lesh & Zawojewski, 2007). An example of a design experiment is the testing by researchers of a new curriculum or teaching method in a classroom (e.g., Brown, 1992; Erickson & Lehrer, 1998; Verschaffel et al., 1997). This characteristic is consistent with model-eliciting activities that ask students to develop mathematical models to explain real-life situations. The development of a design or model is also often cyclic (Lesh & Lehrer, 2003). In a typical series of cycles, the student expresses thinking in a model, tests the model, and then revises the model. For example, a student creating a consumer guide for buying snack chips developed a spreadsheet for scoring snack chips, asked someone else to test the product, and then revised the product based on testing results (Hjalmarson, 2005; Kaiser & Schwarz, 2006). The students' revisions are guided by some end-in-view that describes the functions the final product should be able to perform (English & Lesh, 2003). Similarly, for modeling design experiments, researchers should have some end-in-view for the product under development. The end-in-view should guide researcher decision-making about revisions that are made to the product from research cycle to research cycle.

An important caveat is that for design experiments using a models and modeling perspective, the assumptions and understandings of the teachers (and researchers) may change throughout the study. It is imperative to document those changes as they are made. Often, researchers are interested in the students' development of responses or in how student models change within a session or between modeling sessions (Lingefjärd, 2006; Lesh & Zawojewski, 2007). Rather than studying fixed constructs or examining snapshots of constructs in isolation, researchers may be studying changes in constructs over time and across problems and individuals. Capturing change and the effects of change can be a goal of design experiments with a models and modeling perspective. Both components of the design (theoretical assumptions and artifacts) will change just as for students' models both the internal conceptual system and the external representations change. This characteristic is another example of how the researcher-level design experiment should be consistent with the student-level (Sriraman & Lesh, 2006).

For model-eliciting activities, a crucial component is the local context that situates the task. The context guides the students' development of solutions, aids in their decision-making about whether a way of thinking (or a model) is appropriate or not, and helps them place the end-in-view in a context that is real to the students (English & Lesh, 2003). The context situates the usefulness of the design and aids development since the final product should be useful in that context. However, the products of design experiments are not generalizable to other situations (or contexts). As with model-eliciting activities where students develop a product for a particular client that is generalizable to other situations beyond the one at hand, designs should also be generalizable to other educational situations. This proviso means that the researcher needs to outline precisely the conditions under which the design was used and other possible modifications that may need to be made for different situations. In addition, although the model produced may apply specifically to the particular situation, the modified and revised models should apply to other situations and the development of other products. Again, this idea parallels the assumption that particular representations of a student model may have limited application, but the conceptual system behind the representation may be more generalizable. Related to the context for the design and modeling, throughout the design process, the researcher should be documenting the design process as an aid to the generation of theory (Collins, 1992). At the end of the design study, there should be both tangible products as well as information about why and how those products work or might work in other situations (Cobb et al., 2003; Zaritsky, Kelly, Flowers, Rogers, & O'Neill, 2003).

Collaboration is also a component of design experiments with a models and modeling perspective that parallels assumptions about student learning. Collaborators may include researchers, teachers and students proceeding along multiple levels of development similar to multi-tiered teaching experiments (Kelly & Lesh, 2000; Lesh & Kelly, 2000; Schorr & Lesh, 2003). Researchers need teachers to help design, test and implement products. Products should be developed with teachers' questions about their own practice in mind (e.g. personal meaningfulness), and researchers can provide resources to aid teacher development (Design-Based Research Collective, 2003). There may also be multiple teachers or researchers involved in the development of any product. This characteristic can aid the triangulation of interpretations about results and the generalizability of results if products have been tested in multiple contexts. Collaboration also aids the documentation of results by requiring that strategies or tools need to be communicated to other people for commenting (e.g., individual teachers develop a ways of thinking sheet or concept map to share with the group) (Koellner-Clark & Lesh, 2003; Lesh & Zawojewski, 2007).

Principles for Developing Modeling Activities

Research in the field of mathematical modeling listed a number of principles for developing modeling activities. To develop modeling activities, designers rely upon six design principles that are based on the work of the teachers and the researchers and that have subsequently been refined by Lesh and his colleagues (2000). Each principle is described and illustrated by referring to one of the modeling activities developed for the purposes of the present study, namely the Best Drug Award modeling activity (See Appendix II).

The first principle for designing a modeling activity is called the *Model Construction Principle*. This principle ensures that the solution to the case study requires the construction of an explicit description, explanation, procedure, or justified prediction for a given mathematically significant situation. Such products externalize how the students interpret the situation and also reveal the types of mathematical quantities, relationships, operations, and patterns that they take into account. In the Best Drug Award modeling activity, students are specifically asked to develop a procedure for ranking the five drugs in terms of most effective to least effective drug.

The second design principle is the *Reality Principle*. This principle could also be referred to as the meaningfulness principle, and it relates to two important characteristics of a case study. First, it requires the case study to be designed so that students can interpret the activity meaningfully from their different levels of mathematical ability and general knowledge. In the Best Drug Award modeling activity, students relate to the idea that they want to select a drug that is the most effective, and they quickly realize that lower numbers in the table indicate more effectiveness. Second, this principle requires the modeling activity to pose a problem that could happen in real life. In the Best Drug Award modeling activity, selecting a pain relief drug is a pragmatic problem.

The third design principle is the *Self-Assessment Principle*. This principle ensures that the modeling activity contains criteria the students themselves can identify and use to test and revise their current ways of thinking. Specifically, the modeling activity should include information that students can use for assessing the usefulness of their alternative solutions, for judging when and how their solutions need to be improved, and for knowing when they are finished. For the Best Drug Award modeling activity, students can carry out strategies such as finding the total or average number of minutes for each drug or counting the number of times that each drug was very effective (less than 10 minutes). Then, they can return to the data in the table to self-assess whether the results of their calculations seem to reflect what they first observe in the data. As an example, students often begin this problem by finding the average number of minutes for each drug. When they find that the averages are all nearly the same, they go back to the data and begin noticing that although the averages are almost equal, the drugs differ in the amount of time needed to be typically effective and how frequently drugs are effective. Such reflections typically lead students to revise their strategies for solving the modeling activity and to try different approaches.

The fourth principle, the *Model Documentation Principle*, ensures that while completing the modeling activity, the students are required to create some form of documentation that will reveal explicitly how they are thinking about the problem situation. Requiring external documentation of their thinking is beneficial for both the teacher and the students. First, the documentation is helpful for the teacher because it reveals how the students are interpreting and thinking about the given situation. Second, the documentation is beneficial for the students because when students externalize their thinking, it becomes easier for them to self-assess or to reflect on their thinking. In other words, externalizing their thinking helps students engage in metacognition. This principle is typically accomplished in two ways. First, students are working in groups of three; thus, they explicitly reveal their thinking when they communicate with each other to carry out processes such as planning, monitoring, and assessing their solutions. Second, the problem is stated to require students to produce explanations, procedures, or descriptions as part of their solution and to explain their solutions in written letters to the president of the association. Together, these two requirements produce documentations that reveal how students are thinking about the given situation.

The fifth principle is the *Construct Share-Ability and Re-Usability Principle*, which requires students to produce share-able and re-usable solutions. By asking the students to produce products that can be used by others beyond the immediate situation, modeling activities require students to go beyond personal ways of thinking to develop more general ways of thinking, often resulting in more powerful mathematics. In the Best Drug Award modeling activity, the students are specifically asked to develop a ranking procedure that can be shared with the Drug Association, which encourages the students to be more detailed in their explanations of their solutions. In addition, the students are specifically asked to make their procedure more general so that the president of the association may use the procedure to rank additional drugs that they may identify at a later time.

The sixth principle, the *Effective Prototype Principle*, ensures that the modeling activity will be as simple as possible yet still mathematically significant. The goal is for students to develop solutions that will provide useful prototypes for interpreting other similar situations. In the Best Drug Award modeling activity, the problem scenario is clear and simple – find the most effective drug. Furthermore, students' solutions usually provide a useful prototype for interpreting other situations. For example, some students can solve the activity by counting the number of minutes for each drug. Such a process can serve as a prototype for other situations in which frequency counts are appropriate.

The six design principles are summarized in Table 2.2, by providing questions that can be used to design a Modeling Activity.

Table 2.2

Verification Questions for the Six Design Principles

Construction Principle	Does the case study require students to develop a description, explanation, procedure, or justified prediction for interpreting a significant mathematical situation? Or does the case study merely ask them to produce a one-word or one-number answer to a question?
Reality Principle	Are students asked to make sense of the given situation based on extensions of their own personal knowledge and experiences? Could this situation and problem happen in a real life situation?
Self-Assessment Principle	Does the problem statement provide criteria that allow the students to determine themselves when their solutions need to be improved, refined, or extended?
Documentation Principle	Will completing the case study require students to produce documentation of how they are thinking about the situation?
Share-ability and Re-Usability Principle	Does the case study require students to create solutions that are shareable with others and that are modifiable for other situations?
Effective Prototype Principle	Does the solution provide a simple yet powerful metaphor for interpreting other similar mathematical situations?

Types of the Modeling Activities Products

The Model Eliciting Activities include three types of products; product as a tool, product as a construction and product as a problem.

(1) Product as a tool. Tools fulfil a functional or operational role and they include: (a) *Models* are used for ranking items, people and places; determining loan payments and may form the base of complex systems such as company's financial operations (Lesh, 2006), (b) *Descriptions and Explanations* illustrate and verify the results of an experiment or investigation or may describe why something that appears superficially correct is mathematically incorrect (Michelsen, 2006). Students can also use a developed model to describe a real situation or to explain the function elements of a real problem, (c) *Designs and Plans* are used in all walks of life, designs and plans must meet detailed and complex criteria and must incorporate appropriate mathematical and representational systems, and (d) *Assessment Instruments* are used in a wide range of contexts such as assessing learner's progress. Models as a tool normally undergo rigorous development that incorporates cycles of testing, refining and applying (Lesh & Doerr, 2003; Lesh & Zawojewski, 2007).

(2) Product as a construction. A construction normally requires students to use given criteria to develop a mathematical item. They do not define the nature of the product rather they set parameters for the design of the product (English, 2006). A construction can be in the form of: (a) Spatial Constructions, (b) Complex Artefacts. Inventions are a good example of complex artefacts. The criteria for their design frequently focus on deficits in existing artefacts or on perceived societal needs, (c) Cases make use of persuasive discourse to adopt a stance on an issue, to recommend one course of action over another, or to highlight an issue in need of attention. Cases are especially effective when they draw upon mathematical data to support their claims and (d) Assessments are the products of applying an assessment tool. Such products can serve a number of purposes and usually suggest or imply courses of action (Lesh & Doerr, 2003).

(3) Problem as a product. The ability to pose problems is becoming increasingly important in academic and vocational contexts. During modeling cycles involved in model eliciting activities students are engaged in problem posing, that is, they are repeatedly revising or refining their conception of the given problem (Kaiser & Schwarz, 2006). During the model eliciting activities, students find ways to judge strengths and weaknesses of alternative ways of thinking and whether a given response is appropriate and good enough (Lesh & Doerr, 2003; English & Lesh, 2003).

Appropriateness, Usefulness and Benefits of Modeling Activities

It is imperative that mathematics educators take students beyond the traditional classroom experiences, where problem solving rarely extends their thinking or mathematical abilities. There is a strong need to implement worthwhile modeling experiences in the elementary and middle school years if teachers are to make mathematical modeling a successful way of problem solving for students (Blum & Niss, 1991; Burkhardt & Pollak, 2006; Galbraith & Stillman, 2006; Hamilton, 2007).

Modeling activities have been found appropriate to enhance students' and teachers' capacities to engage in problem solving, thereby laying the foundation for exploring complex systems (Lesh et al., 2003). These activities are highly innovative learning experiences (English, 2003). A number of related features have emerged, indicating a number of benefits of modeling activities, both for students and teachers. Modeling activities provide a pathway in understanding how students approach a mathematical task and how their ideas develop; these activities appear to provide a strong basis for teachers to interact with students in ways that would promote their learning (Doerr, 2006). The benefits for students and teachers while working with thought revealing modeling activities are summarized in the following dimensions: (a) student mathematical literacy and conceptual understanding, (b) student social development, (c) student metacognitive strategies, and (d) teacher pedagogical approaches and teaching practices.

Mathematical Literacy and Student Conceptual Understanding

Related research in mathematical modeling indicated that student work with modeling activities assisted students to build on their existing understandings and to be successfully engaged in thought-provoking, multifaceted complex problems (Lesh & Doerr, 2003; English, 2003). Modeling activities set within authentic contexts, allow for student multiple interpretations and approaches, promoting intrinsic motivation and self regulation. As children work these activities, they engage in important mathematical processes such as describing, analyzing, coordinating, explaining, constructing, and reasoning critically as they mathematize objects, relations, patterns, or rules (Lesh & Zawojewski, 2007).

A number of related research studies showed that the use of modeling activities encouraged students to develop important mathematical ideas and processes that students normally would not meet in the traditional school curriculum (English & Watters, 2004; Zawojewski, Lesh, & English, 2003). The mathematical ideas are embedded within meaningful real-world contexts and are elicited by the students as they work the problem. Furthermore, students can access these mathematical ideas at varying levels of sophistication. Student work in modeling activities facilitates student development of generalizable conceptual systems. Students move beyond just thinking *about* their models to thinking *with* them for solving an important world based problem (Blomhøj & Kjeldsen, 2006). English (2003, 2006) reported that there was considerable evidence that students' mathematical ideas had improved after they worked in a sequence of modeling activities. Mathematical language improved but also considerable fluency with the use of tables and data were acknowledged (English, 2003, 2006). However, there was an acceptance that students needed to know basic operations to be effective in these activities.

Gravemeijer's and his colleagues (1999, 2000) related their work in connection with Freudenthal's (1971) comprehension of mathematics as an activity that involves solving problems, looking for problems, and organizing subject matter resulting from prior mathematizations or from reality. In solving modeling problems, students developmentally move from modeling situations in an informal way (model *of* the situation) to mathematize their informal modeling activity (model *for* reasoning) (Kaiser & Sriraman, 2006).

Lesh and Doerr (2000) have pointed out that modeling activities can promote students' conceptual understanding. They clarified that in modeling activities students are not simply working with ready made models. Since models are interacting systems based in more complex conceptual systems, Lesh and Doerr (2003) claimed that models must be constructed in a meaningful way. This construction leads to conceptual understanding and mathematization (Lesh & Doerr, 2003; Lesh & Sriraman, 2006). An interesting aspect of student work with modeling activities is posted by Lesh and Harel (2003). They documented that when students work in a model-eliciting activity to develop a conceptual tool that involves a construct (or conceptual system), students go through stages of development similar to general stages of development that have been investigated by developmental psychologists and mathematics educators. Lesh and Harel (2003) referred to these developmental stages as *local conceptual developments*; they consequently reported that it is reasonable to expect that processes which contribute to general cognitive development also should also contribute to progress through the modeling cycles for modeling activities (Harel & Lesh, 2003).

Lesh and Harel (2003) and Harel and Lesh (2003) further stressed the importance of modeling activities by highlighting the importance of student conceptual systems. They documented that conceptual systems are developed first as situated models that apply to particular problem solving situations. Then, these models are gradually extended to larger classes of problems as they become more sharable, more transportable, and more reusable. The aforementioned features of modeling activities helped students be successful beyond problem situations for which models were created (Lesh & Harel, 2003).

Doerr and Tripp (1999) reported that shifts in thinking that occurred during engagement with modeling tasks led to the development of mathematical models of the concepts embedded in or evoked by the experienced situation. They also attempted to identify the mechanisms that tended to give rise to shifts in learners' thinking while they were engaged with modeling tasks. These shifts in student thinking occur as the students' internal models interacted with their verbal, graphical, and technology-supported representational systems, which are shared among the group of students.

Doerr and Tripp (1999) also documented that students' representational systems also interacted with and depended on the external constructed models for solving a situation. As the learners encountered mismatches between their interpretations of the problem situation and the real problem, they tended to resolve these mismatches by modifying their interpretations and by extending their representational systems to include more carefully quantified data elements (Doerr & Tripp, 1999).

Lesh and his colleagues (2003, 2006) and English (2003) investigated the role of modeling activities with regard to student algebraic reasoning. Lesh and colleagues (2003) reported that modeling activities provide opportunities for students to explore quantitative relationships, analyze change, and identify, describe, and compare varying rates of change, as recommended in the Grades 3-5 algebra strand of the Principles and Standards for School Mathematics (NCTM, 2000). In addition, English (2003) pointed that elementary probability ideas emerging when young students linked the conditions and constrains of problems (e.g., Best Drug Award activity). The above research studies have also highlighted the contributions of these modeling activities to young students' development of mathematical description, explanation, justification, and argumentation. In modeling activities students engage in numerous questions, conjectures, arguments, conflicts, and resolutions as they work towards their final products. Furthermore, when they present their reports to the class they need to respond to questions and critical feedback from their peers (Lesh & Doerr, 2003; Zawojewski, Lesh & English, 2003; English & Watters, 2004; Lesh & Sriraman, 2006).

An important parameter in students' work in modeling activities is students' use of their informal knowledge. Researchers have observed the interplay between students' use of informal, personal knowledge and their knowledge of the key information in the problem (Zawojewski, Lesh & English, 2003). In a number of modeling activities, students' informal knowledge helped them relate to and identify the important problem information (e.g., understanding and interpreting the conditions for the solution of a problem). A number of researchers (Doerr & English, 2003; Doerr, 2006) also documented that students embellished their written reports with their informal knowledge and most importantly, many students recognized when their informal knowledge was not leading them anywhere and thus students reverted their attention to the specific task information (Lesh & Doerr, 2003; Zawojewski, Lesh & English, 2003; Doerr, 2006).

Student Social Development

The context of the modeling activities provide an interactive situation in which students learn mathematics by actively engaging their previous knowledge of related subjects and actions and discuss with their peers and teachers. This discussion results in constructing, modifying, and improving mathematical knowledge (English, 2006). English (2006) continued pointing out that when modeling activities present situations that have more than one interpretation, students use their personal interpretations, argue and debate about interpretations and their implications and socially construct meanings (Doerr & English, 2001). As a consequence, students who understand mathematics as a domain that invites interpretation and meaning construction are those most likely to become flexible and inventive mathematical problem solvers.

Modeling activities also facilitate students' social development. As students, working in groups, collaborate on constructing a model for solving a problem, they pose questions, set conjectures, engage in argumentation, and resolve issues of disagreement. In doing so, students see the different points of view and ways of thinking of their peers, which helps them to become more flexible and adoptable in their own patterns of thinking (Doerr & English, 2003). In English's (2004) study, the teachers believed that there were substantial social gains in relation to group work, social interaction, reporting and questioning skills. They also considered that by the end of the year the students at the end of the year were well better prepared to question assumptions and each others' interpretation of the data (English, 2004).

Student Metacognitive Strategies

The development of students' metacognitive and critical thinking skills is considered essential and can assist students in reflecting on their solutions in the modeling problems (Garofalo & Lester, 1985; Schoenfeld, 1992). However, there is a limited amount of literature related to metacognitive processes in mathematical modeling activity. A central characteristic of students' metacognitive skills in their work in the modeling activities is the iterative cycles of model refinement and improvement. This procedure is limited if students do not know when and how to apply the necessary cognitive and metacognitive strategies for solving the problem (Goos, Galbraith, & Renshaw, 2002). Schoenfeld (1981, 1992) found that problem-solving performance could be enhanced by focusing attention on the development and internalization of these kinds of metacognitive skills. He further pointed out that metacognitive instruction is most effective when treated in a domain-specific context. In doing so, metacognitive skills can be learned, but they cannot be acquired in isolation from content issues, and are a long-term objective.

Several metacognitive strategies have been identified in students' work in the modeling activities. For instance, Stillman and Galbraith (1998) reported a number of strategies students' use to understand the problem presented in a modeling activity. Among the strategies are re-read the problem and connect the question with provided information, reorganize problem data into graph or diagram, link data using different representations, and work with a subset of problem information. English (2006) reported that metacognitive strategies are not only important in understanding the problem; she considered important strategies that relate to the problem's question to other problems within students' experience and look for further information either cued by the problem context or the mathematical nature of the problem.

Among others, Stillman and Galbraith (1998) and Lehrer and Schauble (2003) underline the importance of the teacher's role in improving students' metacognitive strategies in mathematical modeling in problem solving. Quite important, teachers have to focus on improving students' strategy choices that are dominated by established experiences (Stillman & Galbraith, 1998; DeBellis & Goldin, 2006). Similarly, Artz and Armour-Thomas (1992) in their proposed framework for the cognitive and metacognitive aspect of students' problem solving behavior identified that teacher's intervention in students' group work can be beneficial and can promote students' metacognitive strategies.

A final aspect of metacognition in mathematical modeling is related to the possible impact intervention programs may have on students' metacognitive strategies. Tanner and Jones (1994) documented that model building activities in the classroom improve students' metacognitive competence. They specifically argued that students who participated in modeling activities significantly improved the way they approached, understood and solved a new problem; they linked the problem to their prior solving experience and reflected on their solutions. In line with previous findings, Lesh and Doerr (2000) pointed out that model building and revision in the classroom correspond to the conceptual models of students and therefore, students' work in modeling activities can promote students' metacognition. Similarly, English (2006) observed that students, working in small groups, exhibited different, but complementary metacognitive strengths. In students' work success was accompanied by a tendency to engage in a high number of organizational activities, regulation of execution activities particularly monitoring the progress of local and global plans, and verification activities especially evaluation of execution (English, 2006).

Teacher Pedagogical Approaches and Teaching Practices

A number of research studies reported on the benefits of modeling activities for teachers. Schorr and Koellner-Clark (2003) reported on a multi-tiered program in which participating teachers had opportunities to consider their approaches to teaching, make predictions about what was happening, test those predictions, and then reflect the outcomes in a collegial setting. Teachers were also able to reflect on their ideas about mathematics teaching and learning as they considered the strengths and weaknesses in their ways of thinking. Experience from teaching modeling activities provided the impetus for teachers to develop new world-views about their teaching practices. Additionally, all teachers from the above study began to reflect more deeply about their students' thinking. The teachers asked more questions and closely listened student's responses (Schorr & Koellner-Clark, 2003). Schorr and Koellner-Clark (2003) concluded by documenting that the modeling perspective is useful in considering the conditions that are necessary for generating fundamental changes in teaching practice.

According to Doerr (2006), modeling activities place new demands on teachers. By listening to students' ways of thinking the teachers' schemata (interpretations of possible ideas students might have) develop in ways that include a greater diversity of students' thinking. Listening students' ways of thinking for the purpose of understanding can enable teachers to manage the multiplicity of ideas in the class and to support the multiple developments of students' ideas. Doerr (2006) found that, in her study, the teacher supported a diversity of ideas in the classroom rather than guiding students along particular paths or trajectories. A crucial characteristic of her support was in how she shifted the role of evaluator from herself to her students. This shift was evident as the students themselves tested and rejected ideas, explored patterns of numbers and investigated relationships of possible functions (Doerr, 2006). She reported that modeling activities created a new learning environment for teacher and students. Teacher's approach to ask her students to describe and explain their thinking contributed not only to the teacher's understanding of her students' thinking, but it created a situation where the students could refine their thinking and shift to a new way of thinking about the problem. Doerr (2006) concluded that the teacher shifted the task of teaching from guiding the students along a known (to the teacher) path or trajectory to fostering a multiplicity or diversity of ideas and then engaging the students in evaluating, revising and refining their ideas (Doerr & English, 2001; Doerr, 2006).

The focus of models and modeling perspective for researchers, teachers, and students is to develop models that explain complex situations in a pragmatic and useful way. Usefulness is judged not only by fit with reality, but also by generalizability, extendability, and fit with other models (Lesh, Doerr, Carmona & Hjalmarson, 2003).

Issues Related to the Teaching and Learning of Mathematical Modeling

There are a number of aspects related to the teaching and learning of mathematical modeling (Blum & Niss, 1991; Sriraman et al., 2006a). A number of researchers have documented the importance of the role of context in modeling activities (Pace, 2000; Lingefjärd, 2006), the role of students' and teachers' affective factors on mathematical modeling, the importance of assessment (Haines & Crouch, 2007), and issues related to time and communication (Hatano, 1987; Niss, 1987).

The Role of Context in Mathematical Modeling

The role of context is very important in mathematical modeling, since modeling requires a context in which to frame the problem and apply the mathematics (Lingefjärd, 2006). There are many reasons to argue that the role of context is important and is being encouraged in teaching mathematics, and mathematical modeling in particular. Sriraman and colleagues (2006b) identified the lack of relevance as a critical factor in engaging students, and certainly, the use of contextual settings has the potential to address that lack. In line with Sriraman's findings, Pace (2000) pointed out that students need both conceptual understanding in mathematics and experience of real world situations in the chosen context before they can start working with modeling activities. In his study, Pace (2000) refers to context as a situation or activity in which concepts are introduced and applied in a meaningful manner, while Sauian (2002) argued that contextual based teaching encourages meaningful learning in an active environment.

The importance of the role of context in mathematical modeling is also supported by the studies conducted by Gravemeijer and his colleagues (Gravemeijer, 1999; Gravemeijer & Doorman, 1999). They stressed that research on the design of Realistic Mathematics Education (RME) based activities has shown that the concept of emergent models can function as a powerful design heuristic. Additionally, they concluded that the use of personalized contexts improved word problem solving by increasing the meaningfulness of contexts for students. Gravemeijer and Doorman (1999) also claimed that well-chosen context problems offer opportunities for the students to develop informal, highly context-specific solution strategies. In detail, students who studied simple problems in decontextualized contexts performed best on one-step questions, while students who studied complex problems in contextualized contexts performed best on multi-step questions (Choi & Hannafin, 1997; Gravemeijer & Doorman, 1999).

De Lange (1987) lists four functions of realistic context problems: enabling concept formation, facilitating model formation, providing a wider range of utility, and more interesting practice problems. In line with de Lange's functions of realistic context problems, DaPueto and Parenti (1999) reported that using a realistic context in mathematical problem solving can facilitate collaboration, interaction and contribution of students who have different styles of exploration, understanding and use of concepts, and different levels of formalized knowledge. They also argued that using a realistic context in problem solving can provide a more balanced development of reflective learning and experiential learning, and can facilitate the design and restructuring of the schemata through which knowledge is organized. In line with DaPueto and Parenti's findings, McNair (2000) suggested that students' everyday activities will motivate learning and development as they interact with and solve problems in their real environment. To accomplish this, he suggested keeping the curriculum close to the students' experiential awareness and using real based context for mathematical problem solving.

Research has also indicated a number of possible limitations related to the use of contexts in the teaching and learning of mathematical modeling in problem solving (Galbraith & Stillman, 2006). The choice of context may be a turn-off to particular students or be so familiar as to inhibit some approaches judged as unreasonable (Choi & Hannafin, 1997). Galbraith and Stillman (2006) pointed out that if the situation and experience used are not familiar to the students, then instruction may degenerate into the presentation of abstract ideas related to the context. In line with previous findings, Hamilton (2007 in press) reported experimental results suggesting that a sparse context better activate understanding of mathematical knowledge structures than real contexts, which engaged everyday knowledge more readily.

The Role of Affective Factor

The importance of the affective domain in mathematical modeling is stressed in a number of research studies (Gravemeijer, 1997;Verschaffel et al., 1997; Lesh & Doerr, 2003).

Yoshida, Verschaffel and DeCorte (1997) reported that mathematics education is now focussed to a broader perspective on student learning outcomes, including their attitudes and beliefs and their capacity to apply their knowledge and skills in authentic, non-routine problem situations. The importance of considering students' conceptions is also pointed by Confrey and Doerr (1994), who argued that the use of the modeling approach to problem solving can create positive beliefs in the mathematics classroom. In line with previous findings, Bonotto and Basso (2001) reported that an activity of realistic mathematical modeling problems can create an interplay between reality and mathematics and mathematical modeling of real world based problems can enhance students sense-making. Additionally, they argued that bringing real-world situations into school mathematics is a necessary condition to foster a positive attitude towards mathematics. Similarly, Gravemeijer (1997) argued that a change in classroom culture towards a modeling perspective can foster the development of positive attitudes towards mathematical problem solving.

A significant obstacle in teaching mathematical modeling was the pre-existing beliefs among teachers, and the projection of those beliefs onto their students. Verschaffel, De Corte and Borghart (1997) reported a strong and resistant tendency among pre-service teachers to exclude real world knowledge and realistic considerations when dealing with arithmetic word problems as instructional tasks. In addition, the teachers, participated in this study, even resist change toward a more open-ended, modeling-based approach in their teaching. Teachers' pedagogical beliefs, as presented through their comments, reported that world based problems are not important and they can be sometimes harmful for teaching arithmetic word problem solving in the elementary and middle school (Verschaffel et al., 1997). In summarizing their findings, Verschaffel and his colleagues (1997) pointed that teachers' beliefs are driving teachers towards the argument that realistic solutions are beyond the intention of the problem and those problems are inappropriate for school grade level. Similarly, Doerr (2006) documented that negative teacher attitudes and beliefs about the role and significance of including real world knowledge in the interpretation and solution of school arithmetic word problems might have a negative impact on their teaching practices. As a result, teachers do not assist their students in making connections between the different concepts in mathematics and in promoting problem solving skills (Doerr, 2006).

The demand on teachers to adopt the modeling approach in their mathematics teaching and their resulting beliefs are articulated by Blum and Niss (1991). Researchers stressed that problem solving is more demanding for teachers because a number of additional qualifications are necessary, and assessment students' achievements is more difficult (English & Doerr, 2003; Doerr, 2006). Moreover, many teachers do not feel able to deal with non routine problems and very often teachers simply either do not know enough problem examples suitable for instruction, or they do not have the necessary time and skills to adapt them to the actual class and to prepare their teaching (Blum & Niss, 1991).

Assessment in Mathematical Modeling

One of the evaluation standards included in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989, 2000) concerns 'Problem Solving', while many of the specific details related directly to the modeling approach in problem solving. The standard equated the assessment of problem solving with being able to formulate problems, apply a variety of strategies to solve problems, verify and interpret results, and generalize solutions (NCTM, 1989, p. 209). Furthermore, students being able to ask questions, use given information, and make conjectures (p. 209), as well as generalizing a problem and its solution to other similar problems.

Among others, Niss (1987) pointed out that assessment of modeling could be problematic, since application and modeling qualifications are difficult to assess, let alone to test, by traditional evaluation tools. Niss (1987, 1993) further clarified that there is a need to move away from conventional modes and traditional practices of mathematics assessment. It can be accepted that assessment of mathematical modeling has to be exercised as an intricately variety of components in a complex structure. This implies that assessment takes time and cannot be standardised. It does not imply that assessment cannot be exercised on a sound foundation of reflection and reasoning and articulate criteria and be subject to clear communication. It also does not imply that assessment cannot be summative (Niss, 1993; English, 2002; Garcia et al., 2006).

Considering the difficulties of assessing students' performance in working with modeling activities, a number of different types of instruments being used to evaluate students' modeling abilities and understanding of models are found in a review of the literature. Haines and Crouch (2001) and Crouch and Haines (2004) used a multiple choice format, in developing several test questions related to modeling. One set of questions focused on formulating a mathematical model, with respect to the mechanical and interpretive aspects within mathematical modeling. The other set of questions concentrated on the interface between mathematics and reality, requiring a strong appreciation of shifting paradigms between the real world and the world of mathematics.

At almost the opposite end of using multiple choice format for an assessment method, other researchers supported the use of written tests for selected aspects of modeling. Houston (1994) suggested that students' abilities to read and comprehend an existing model could be adequately assessed in this manner, while Kitchen (1993) proposed questions requiring students to set up a model, interpret a solution, or criticize a model based on the data. Finally, Smith and Thatcher (1989) designed tests so that different questions focused on different parts of a single model, and were written in such a way that it was not necessary to answer all of the preceding questions in order to complete any specific section of the test.

Modeling activities naturally involve the use of technological tools. The use of technology such as computer programs and graphing calculators naturally affects the evaluation situation and also what it is meant by assessment (Webb 1992). Clarke (1996) argued that when a technological tool is employed for the completion of a mathematics task, then the same tool should be available in the assessment setting. In addition to connecting teaching and assessment, the technology can also offer a better possibility of documentation, visualization, and reporting.

Probably the most prevalent approach to assessing student work in modeling is the use of an analytic scoring scale, one that assigns point values to various dimensions of the modeling work. Bell and colleagues (1992) described general considerations for assessing students through the use of extended tasks, and provided examples of scoring rubrics in general categories of identification, implementation and interpretation, to use in evaluating written work. Another aspect of assessment of modeling relates to students' projects and presentations. Jones, Rich and Day (1996) and Money and Stephens (1994) developed multiple rating scales to assess models made in student's projects. An interesting contribution was proposed by Haines and colleagues (1993, 2001), focusing on the analysis of video recordings of oral presentations made by students, and used a measurement scale for judging the quality of the presentations. Similarly, in her dissertation, Hjalmarson (2005) investigated the presentation tools designed by teachers in assessing their students' presentations.

Self assessment is also important, as it helps students to understand and evaluate the task which has been undertaken. Clearly, students who cannot recognise high-quality work produced by their peers (or by themselves) have little claim to soundly-based knowledge (Sriraman & Lesh, 2006). Evaluation may be seen as a substantial part of the didactical contract being negotiated between student and teacher (Brousseau, 1997). Through this interplay, the students can learn to identify the criteria for qualitatively good performance.

A quite different approach was proposed by Bell and colleagues (1992), focusing on the choice of the task to be a critical one when planning (metacognitive) and content knowledge (cognitive) are competing resources. Stillman's (2001) proposed for a balance in the importance of the two types of thinking processes (strategic and technical load) when assessing students in mathematical modeling. She focused on the open-endedness of the task, presenting the way the task is structured and presented to the student – 'low-scaffolding' (little organization of task in the description) versus 'high-scaffolding' (highly structured presentation of the task). The variation between the two presentations had several influences, in modeling assessment. More specifically, she concluded that high levels of task scaffolding should be used in learning situations with students who have low aptitude for modeling in order to increase the chance of their short term success. In assessment situations, the use of high scaffolding of tasks would be expected when students are new to modeling (Stillman, 2001; Kaiser & Sriraman, 2006).

Time and Communication

One of the issues identified in the literature review is a concern for the amount of time required to implement mathematical modeling in teaching school mathematics. Niss (1987) described that the issue of time is an important condition, since modeling activities are time consuming. In an attempt to compound the reasons for the time required to implement modeling, Hatano (1997) provided a plausible explanation, indicating that modeling is a kind of understanding through comprehension activity. Understanding through comprehension activity may offer multiple interpretations simultaneously, and requires their plausibility to be monitored carefully. Comprehension activity proceeds by deriving and testing predictions from each of the interpretations being considered, so it must be a time- and effort-consuming process (Sriraman et al., 2006a).

Modeling activities need considerable more time to be implemented, compared to traditional problem solving activities. Since modeling activities are thought revealing tasks, requiring students to express their own thinking in interpreting real problems, then time is needed not only to allow students to build their own understandings, but also to interact with other students in creating shareable models (Lesh & Zawojewski, 2007).

A second issue about the teaching of modeling relates to communication practices in mathematics. In line with NCTM' s call for an increased emphasis on communication in the mathematics classroom, Eid (1997) included the following features of modeling as promoting communication, an essential component of mathematics (NCTM, 2000): (a) Students often speak loudly, and share and present their results to class, (b) modeling tasks often require students to discuss with peers, clarify ideas and hypotheses, describe mathematical facts in their owns words, and (c) there is enough time for discussion of different ways or methods to obtain a solution to a task. Bauersfeld (1993) felt that shifting the focus of attention for instruction towards process instead of product, would help develop students' ability to communicate. Additionally, Burkhardt and Pollak (2006) maintained that the discussions in mathematics classroom are necessary for an extended clearing up of the individual understandings and for the negotiation of a sufficiently-shared interpretation for an acceptable solution.

Other Theoretical Perspectives on Concept Development relevant for the Present Study

Mason (1996) argues that one develops an awareness of generality by distinguishing between *looking through* and *looking at*, which in turn leads to "primal abstraction and concretization experiences" (p.65). *Looking through* is analogous to generalizing through the particular whereas *looking at* is analogous to specializing (identifying) a particular case in the general. *Looking through* entails recognizing the attribute of invariance in an implied domain of generality.

This *looking through* process is also referred to as reflective abstraction (Dubinsky, 1991). A far-reaching kind of reflective activity is that which leads to mathematical generalization or delineating structural similarities across different situations (Skemp, 1986, p.55; Sriraman, 2004). A concrete case of how this reflective activity works is provided in the realm of learning how indices work (Skemp, 1986), and how mathematical principles are extracted from representations of different problem situations (Sriraman, 2004). This process of mathematical generalization is a sophisticated and powerful activity. Sophisticated because it involves reflecting on the form of the method while temporarily ignoring its content. This argument easily applies to mathematical modeling, especially when students start to reflect on prior models and modeling experiences when they confront a new modeling situation. It is powerful because it makes conscious, controlled, and accurate reconstructions of one's existing schemas - not only in response to the demands of assimilation of new situations as they are encountered but ahead of these demands (Skemp, 1986, p.58).

Mithelmore (1993) expands on Skemp's ideas by proposing a model of conceptual development in mathematics consisting of two phases: *abstract-general* and *abstract-apart*. When abstraction is linked to a large number of diverse situations, it is called an *abstract-general* concept. The links are crucial because they indicate "that the learner is aware that whatever properties or relations a concept summarizes are present in a large number of other situations" (Mitchelmore, 1993, p.49). This is analogous to a professional mathematician using proof techniques from his or her particular area to solve problems in a different area. *Abstract-apart* concepts are those in which an abstraction is developed only in reference to a few situations from which it originated, and hence it never becomes linked to any situations other than those from which it originated. Using Skemp's (1986) terminology *abstract-general* concepts facilitate relational understanding, whereas *abstract-apart* concepts lead to instrumental understanding.

Mitchelmore (1993) argues that most mathematics teaching tends to produce *abstract-apart* concepts, and calls for teachers to guide students to extend the range of situations to which a given abstraction can apply. This will facilitate the growth of conceptual knowledge, "as that which is characterized most clearly as knowledge that is rich in relationships…as a connected web of knowledge. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network" (Hiebert, 1986).

Summary

Research in mathematics education documented that mathematical modeling can serve as a didactic means for improving students' problem solving skills. Mathematical modeling can also promote students' active engagement in acquiring mathematical knowledge. School mathematics environment needs to provide a means for students to develop the necessary modeling abilities for solving real problems. These modeling abilities are essential for students who face a demanding knowledge-based economy and workplace, in which they need to deal effectively with complex, dynamic and powerful systems of information (Lesh & Zawojewski, 2007).

Despite the fact that traditionally mathematical modeling has been reserved for higher education, a number of recent research studies documented the importance of implementing modeling activities at the elementary school level (English & Watters, 2005; English, 2006). This is important not only because elementary school students are capable of working with modeling activities, but also because modeling needs to be introduced early in the curriculum if we want to successfully implement modeling at all school levels (Blum & Niss, 1991; English & Doerr, 2003).

However, what is still missing is an in depth investigation of modeling processes and sub processes appear in students' work in the modeling activities and an analysis of students' modeling abilities and how these are interconnected with the modeling procedure. Therefore, the study addressed important questions related to the modeling processes that appear in students' work in modeling activities and students' modeling abilities (Blum, 2004). The goal of this chapter was to provide a coherent framework for mathematical modeling as a problem solving activity in guiding the research investigation. To this end, the theoretical framework was organized in three strands; the first discussed the emergence of mathematical modeling as a problem solving activity which comes out of the limitations in problem solving research. The second strand of the literature review discussed the benefits for students and teachers in working with thought revealing modeling activities. The review also focused on describing the six design principles for developing modeling activities and on discussing the products of the modeling activities for students. The third strand provided details and discussed the contextual nature of modeling, assessment of mathematical modeling and the relation between the affective factor and mathematical modeling.

The review of the related literature served towards the purpose of the present study which was the further development of the theoretical approach for the models and modeling perspective in problem solving, as it was proposed by Lesh and Doerr (2003). The new theoretical approach in the modeling perspective in problem solving aims to contribute in further understanding the modeling processes in students' work in solving real world problems, in tracing how students' modeling abilities evolve during time and in examining the interconnectedness of students' modeling abilities and the modeling processes.

CHAPTER III

METHODOLOGY

Introduction

The purpose of the present study was to further contribute to the current theoretical perspective of the models and modeling perspective in mathematical problem solving in elementary and secondary school level, by examining modeling processes in students' work in solving real world problems, by tracing how students' modeling abilities evolve during time, and by examining the interconnectedness of students' modeling abilities and the modeling processes.

This chapter contains a description of the research design applied in this study. In particular, it first describes the setting, the participants, and the instruments for the study. The chapter also describes the intervention program that was implemented in the 6^{th} and 8^{th} grade classes which participated in the study. Specifically, in describing the intervention program, a detailed analysis of the modeling activities and their rationale is presented. The chapter also includes an outline of the type of data collected and the methods of statistical analysis employed for answering the research questions of the study.

Participants

Eight 6^{th} and eight 8^{th} grade classrooms from four different urban/suburban schools participated in the study. In detail, four elementary and four middle schools were selected. In each school, one class participated in the experimental group and another one class participated in the control group. In total 403 students participated in the study. The experimental treatment group consisted of 197 students coming from four 6^{th} grade and four 8^{th} grade classes, while 206 students coming from another four 6^{th} grade and four 8^{th} grade classes formulated the control treatment group of the study. More specifically, 101 sixth grade and 96 8^{th} grade students worked with the modeling activities (experimental group), while 104 6^{th} grade and 102 8^{th} grade students participated in the control treatment group respectively.

Instruments

Test for Measuring Modeling Abilities in Problem Solving

A test for measuring students' problem solving abilities was developed, including tasks on modeling problem situations, providing solutions for complex problems and posing problems. The test included nine tasks. A number of these tasks were modified versions of problem solving tasks from the latest PISA 2003 study (OECD, 2004). Three tasks aimed to investigate students' modeling abilities in decision making, three in system analysis and design and three in trouble shooting problems.

The tasks reflecting the *decision making* problems presented students with a situation requiring a decision and asking to choose among alternatives under a set of conditions constraining the situation (see Table 3.1). The two tasks representing decision-making category were the "Holidays", "Restaurant" and the "Energy Needs" tasks.

For example, in "Holidays" task, students were asked to calculate the shortest possible distance between two cities that were far away, given the map of the area and a table presenting the distances between cities. Students had to understand the situation provided, identify the constraints, possibly translated the way in which the information was presented, make a decision based on the alternatives under the constraints given, check and evaluate the decision, and then communicated the required answer (OECD, 2004). The factors creating difficulty in decision-making problems were the number of constraints the student has to deal with in working through the information provided and the amount of restructuring a student had to do in sorting through the information along the way to develop a solution.

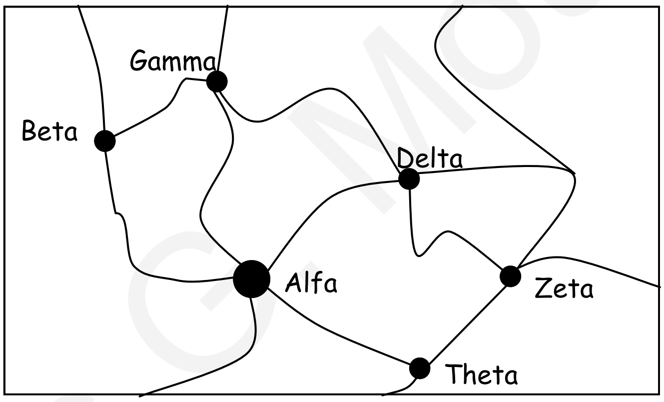
The test also included three tasks units for assessing students' modeling abilities to solve problems involving s*ystem analysis and design*. Problems of this category differ from the decision-making problems in that constrains are not obvious and not all of the possible options are given in the problem. Students' priority in the system analysis and design problems is to develop an understanding of the problem, beginning with the identification of the relationships existing between the parts of the system, or to design a system with certain relationships among its main features. The next step is to test the system with the developed model and finally, students are involved in justifying their analysis. The three tasks representing system analysis and design were the "Course Design", the "Children's Camp" and the "Cinema Outing" tasks.

Table 3.1.

Sample Problems from the Modeling Test.

Decision Making

This is the map of an area. The table below indicates the actual distances between the various towns.



Alfa						
Beta	55					
Gamma	50	30				
Delta	30	85	55			
Zeta	55		100	45		
Theta	30	85	80	60	25	
	Alfa	**Beta**	**Gamma**	**Delta**	**Zeta**	**Theta**

Calculate the shortest possible distance between towns **Zeta** and **Beta**.

System Analysis

A college is offering the following 9 subjects for a 3-year-study. Each subject can be taught in one year. Each student can take 3 subjects every year in order to complete the 3-year study.

No	Code for subject	Subject and level of subject
1	M1	Mechanical Studies Level 1
2	M2	Mechanical Studies Level 2
3	E1	Electrical Studies Level 1
4	E2	Electrical Studies Level 2
5	BM1	Business Management Level 1
6	BM2	Business Management Level 2
7	BM3	Business Management Level 3
8	IT1	Information Technology Level 1
9	IT2	Information Technology Level 2

Regulations : (a) A student can take a subject of level 2 or 3 only if he/she has already completed the lower level(s) of the subject on the year before. **(b)** A student can take **Electrical Studies Level 1** if he has/she completed **Mechanical Studies Level 1**. A student can take **Electrical Studies Level 2** if he/she has completed **Mechanical Studies Level 2.**

What subjects should the college offer for each of the 3 years of study?

Trouble-shooting

The cardiology department at a local hospital employs 5 doctors. Every doctor can work from Monday to Friday and examine 10 patients per day. In a whole year (365 days, 52 weeks) a cardiologist can have 25 days for holiday, 26 days off for attending seminars and the weekends. **Can the 5 cardiologists deal with the 12000 patients that are expected to arrive at the hospital during the following year? If not, what do you suggest that the hospital can do? Explain your answer.**

The third category of tasks is drawn from the topic of *trouble shooting*. The tasks representing trouble shooting were the “Hospital Management”, “Irrigation system” and the “Veterinary Clinic”. Trouble-shooting problems assess students’ actions when confronted with the need to specify the conditions under which a system is running properly or when they face a system of a mechanism that is underperforming in some way. To solve such problems, the student must be able to understand the main features of the system and the actions or responses that are expected of each of these features. Based on this understanding, the student must then be able to identify the causal-response relationships between interrelated parts and the role that such links play in the overall function of the mechanism or system of interest. Finally, students may need to communicate their solution in writing or through a diagram to explain their thinking and their recommended course of action. Such problems are complicated by the number of interrelated variables involved and the varied number of representations and translations that one might have to make in understanding the system or mechanism from directions or instruction booklets.

Intervention Program

The design of the intervention program for improving students’ modeling abilities in solving real world problems was based on the models and modeling perspective as proposed by Lesh and his colleagues (Lesh et al., 2000; Lesh & Doerr, 2003). The intervention program was completed in twenty seven 40 minute sessions. The duration of the intervention program was three months. The implementation of each modeling activity lasted four sessions and the modeling test administration was administered three times in three sessions. Teachers that participated in the experimental group implemented approximately one modeling activity per two weeks.

Modeling Activities

Two sequences of a total of six model eliciting activities were developed for the purposes of the study. There are several factors taken into consideration in developing modeling activities. One of these factors was the choice of context. The context of the modeling activities needed to be reasonably familiar to the students, so as to avoid distracting cognitive resources away from the effort to mathematize the situation in order to solve the problem presented. The context also needed to be interesting enough to actively engage students in the activity. A second factor was the level of complexity to incorporate into the modeling activity. It needed to be realistic, with the ability to explore the problem situation in multiple depth levels, so that the participating students could be introduced to some subtle metacognitive processes that govern modeling. However, the level of complexity needed to take into consideration students' background mathematical knowledge, and prior experiences in working with modeling activities. A third factor that was taken into consideration in developing the modeling activities was the condition that students explored and solved the modeling problems in groups. This setting was adopted to facilitate a social construction of meaning on the task, and to provide a first layer of scaffolding for metacognitive processes (prior to any assistance from the teacher or the investigator).

The developed modeling activities were non-routine tasks because each task asked students to mathematically interpret a complex real-world situation and required them to formulate a mathematical description, procedure, or method for the purpose of making a decision for a realistic client. Because groups of students were producing a description, procedure, or method, students' solutions to the task revealed explicitly how they were thinking about the given situation (Lesh, et al., 2000; Lesh & Doerr, 2003; Zawojewski, Lesh, & English, 2003).

Each modeling activity consisted of four components:

(a) Newspaper article: Students read the newspaper article to become familiar with the context of the modeling activity.

(b) Readiness questions: Students answered the reading comprehension questions about the newspaper article to become even more familiar with the context of the modeling.activity

(c) Modeling Process: In groups of three, students worked in the modeling activity for about 70 to 80 minutes. The purpose was to develop a model – solution for solving the real problem presented in the modeling activity. As the students developed their models, the researcher and the classroom teachers served as facilitators and observers and tried to avoid offering questions or comments that might steer the students toward a particular solution.

(d) Process of sharing solutions: Each group wrote their solution in a letter to an imaginary client. The purpose for including an imaginary client was to encourage students to document and explain their solutions using a variety of representational media. Then, each group presented their solution to the class. Whole class discussion was intermingled with these presentations to discuss the different solutions, the mathematics involved, and the effectiveness of the different solutions in meeting the needs of the client.

A short description of the six modeling activities that were developed for the purposes of the study is presented in Table 3.2. Table 3.2 provides the title of each of these activities along with a short explanation of the solution product to be developed by the students and the primary mathematical idea addressed. The six modeling activities that were developed for the purposes of the study can be found in Appendix II.

Table 3.2

Description of the Six Modeling Activities

Title	**Student Solution Product**	**Mathematical Idea**
Best Drug Award Activity	Students are asked to develop a procedure for ranking five drugs based on information about the number of minutes each drug needs to act for 30 cases.	Statistical concepts such as average and frequency (i.e. average number of minutes needs to act or number of cases acts fast).

Where to Live Activity	Students are asked to develop a procedure for helping a girl choosing a city to move on. The selection will be made from eight cities, based on both qualitative and quantitative information about the number of schools, restaurants, playing yards, quality of roads, budget available for city's improvements etc.	Weighting and ranking (i.e. ranking the players according to different types of data and weighting such rankings).
University Cafeteria Activity	Students are asked to develop a description of why six out of nine employees should be rehired by a university cafeteria manager based on information about the hours worked, and the money collected by the employees, in different semesters and different time periods during the previous year.	Proportional reasoning (i.e. pounds per hour).
Carpet Design Activity	Students are asked to develop a procedure for designing patterns of different 2D shapes needed for a carpet and to calculate special types and colours of fabrics to make the carpet attractive.	Geometric reasoning (i.e. area and perimeter for different shapes).
Car Painting Activity	Students are asked to develop a procedure for measuring the amount of paint needed for a brand new car. Students are given information about the outer and interior dimensions of the car.	Geometric reasoning (area and perimeter of different 2D shapes, e.g., triangle, square, rectangle, rhombus).
New House Activity	Students are asked to design a top view and a model of a new house. Students are given partial information about the building plot and the different rooms in the house.	Geometric reasoning (area of different figures, area and volume of solids).

The Best Drug Award Activity

The purpose of the "Best Drug Award" activity was to find the best pain relief drug among four different drugs. An imaginary client, the president of the drug companies association asked from the students to assist him in finding the best pain relief drug. Specifically, students had to develop a procedure for ranking the four drugs based on information about the number of minutes each drug needs to act for 20 different cases. The related mathematical concepts appear in this activity were statistical concepts such as average and frequency (i.e., average number of minutes needs to act or number of cases acts fast).

Table 3.3

The Four Drugs and the Corresponding Reaction Times

Kanatol	**Saracetamol**	**Ralpol**	**Kefapol**
20	10	12	10
18	19	14	12
19	13	15	17
22	11	15	17
15	11	7	17
14	12	9	19
23	10	9	22
12	9	8	22
11	8	8	21
10	8	15	10
7	14	19	7
9	13	10	7
10	12	10	7
17	17	23	19
13	11	24	18
12	11	23	14
14	13	10	12
14	20	8	10
8	25	17	10
9	13	19	10

During the first part of the activity, students were given a newspaper article discussing the contribution of Ian Fleming in the field of medicine. Following, students had to answer some readiness questions in order to get familiar with the context of the modeling activity. At the modeling stage, students were presented with the problem of the modeling activity. According to the activity, the president of the drug industries association announced a competition for finding the most effective pain relief drug. Students were given a table presenting the four candidate drugs, and the number of minutes each drug needed to act in 20 different cases. The information given is presented in Table 3.3 above.

After constructing a model for finding the best drug, students were asked to write a letter to the president of the drug industries association, giving their decision. In their letter students had to describe how they reached their results and give explanations on why the recommended drug was the best in terms of its effectiveness. Finally, at the stage of presentation each group of students presented their results to the rest of the class and whole class discussion followed. The discussion focused on the key mathematical ideas and processes students used for constructing their models.

The Where to Live Activity

The second modeling activity that was developed for the purposes of the present study was the "Where to Live" activity. The purpose of the activity was to provide opportunities for students to organize and explore data, to use statistical reasoning and to develop appropriate models for solving the problem. Additionally, the activity provided a setting for students to focus and work with the notions of ranking, selecting, aggregating ranked quantities and weighting ranks.

The purpose of the first part of the activity was to familiarize students with the context of the activity. Specifically, an article discussing a college graduate's decision to move in another city was presented to students. Questions and whole class discussion related to the factors that might influence the choice of selecting one city among others followed. During the modeling stage of the activity, the problem presented in Figure 3.1 was given to students. According to the problem, students had to decide which one was the best among six different cities. Information about the number of parks, nursery schools, school, cinemas, restaurants and shops was given. Also, information about the road quality and the budget for the next year was provided. After completing their work, each group presented their solutions to the rest of the class for questioning, comparing with others' solutions and constructive feedback. Finally, a whole class discussion focused on the key mathematical ideas and processes that were developed during the modeling activity.

Use the data in the table below to find the best city, Anastasia can live in. When you reach an answer, write a letter, explaining and documenting your results, to Anastasia.

	Parks	Nursery Schools	Schools	Cinemas	Restaurants	Shops	Road quality (%)	Next year budget*
Lakecity	2	2	7	1	3	23	45.5	Same
Relaxcity	3	1	4	3	12	16	36.8	More
Safecity	2	4	5	4	4	26	57.2	Less
Dreamcity	0	5	10	0	6	12	19.7	Less
Nicecity	3	2	8	2	5	20	25.8	Less
Livecity	4	3	7	3	8	15	76.2	More

* Next year's budget is compared to this year's budget.

Figure 3.1. The problem presented in the "Where to Live" activity.

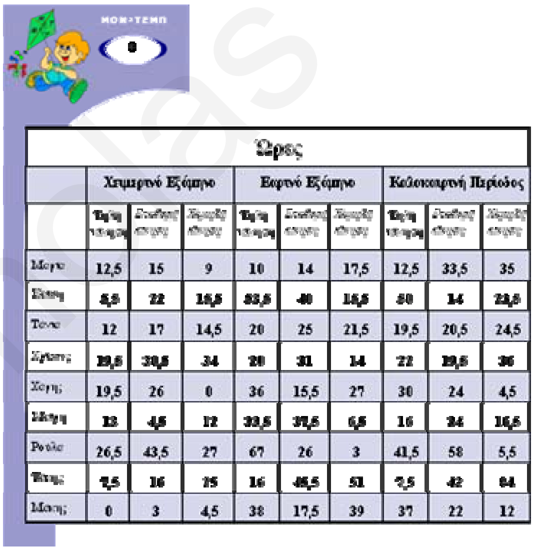
The University Cafeteria Activity

The third activity of the sequence of modeling activities focusing on statistical reasoning was the "University Cafeteria" modeling activity. The main purpose of the activity was to develop a model for ranking and selecting six among a number of nine employees who are working in a cafeteria. An imaginary client, the manager of the cafeteria, asked from the students to assist him in deciding which employees should work for the next year. The manager decided to buy automatic selling machines for the cafeteria and as a result he would not need to hire all nine employees. Specifically, students had to select three employees among nine employees that would work on a full time basis for the next year and three other that would work on a part time basis.

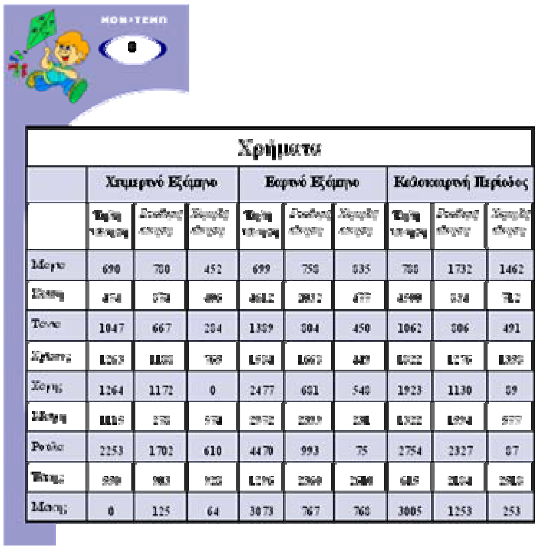
During the first part of the activity, students were given a newspaper article, discussing the impact of introducing technological tools in the working environment and how this introduction impacts the unemployment rate. Following, students had to answer some readiness questions related to the context of the article. During the modeling stage of the activity, students were given the following story related to the University cafeteria problem with the corresponding tables.

Last year, Nicholas worked as a manager in the University's cafeteria. Nine employees work in the cafeteria, serving coffees and other soft drinks. Nicholas decided to buy automatic selling machines and as a result he decided to hire only six people for next year. Nicholas needs your help do decide which workers to rehire next year.

He wants to rehire the employees who will make the most money for the company. Nicholas does not know how to compare the different employees because they worked different numbers of hours. Also, vendors worked in different time periods (busy, steady and slow) and in three different semesters (see the tables). Please evaluate how well the different vendors did last year for the cafeteria and decide which to rehire on a full-time basis and on a half time basis. Students were then asked to write a letter to



Ώρες									
	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος		
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Μαρία	12,5	15	9	10	14	17,5	12,5	33,5	35
Ελένη	4,5	22	16,5	33,5	40	16,5	50	14	22,5
[illegible]	12	17	14,5	20	25	21,5	19,5	20,5	24,5
[illegible]	19,5	30,5	34	20	31	14	22	19,5	36
[illegible]	19,5	26	0	36	15,5	27	30	24	4,5
[illegible]	13	4,5	12	33,5	37,5	6,5	16	24	16,5
[illegible]	26,5	43,5	27	67	26	3	41,5	58	5,5
[illegible]	7,5	16	25	16	45,5	51	7,5	42	84
[illegible]	0	3	4,5	38	17,5	39	37	22	12



Χρήματα									
	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος		
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Μαρία	690	780	452	699	758	835	788	1732	1462
Ελένη	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	1047	667	284	1389	804	450	1062	806	491
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	1264	1172	0	2477	601	540	1923	1130	89
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	2253	1702	610	4470	993	75	2754	2327	87
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	0	125	64	3073	767	768	3005	1253	253

Figure 3.2. The "University Cafeteria" modeling activity.

the manager of the cafeteria, explaining their results. In their letter they had to describe how they evaluated the employees and to give details, so Nicholas could check their work and understand how they reached their results. Students finally had to document in their letter their models-solution, so Nicholas could decide if their solution was a good one for him to use. Finally, students presented their solutions in their classmates and discussed the different solutions.

The Carpet Design Activity

The first activity of the second sequence of modeling activities was the “Carpet Design” activity. In this activity students were asked to develop a procedure for designing patterns of different shapes needed for a carpet and to calculate special types and colours of fabrics to make the carpet looks nice. An imaginary client, the owner of a clothing factory asked students to help him make the carpets looks attractive, and also to calculate the fabrics needed for these carpets. The mathematical idea in this activity was geometric reasoning and more specifically, the concepts of the area and the perimeter of different shapes.

The first part of the activity presented a story about Leonardo Da Vinci and his visit in Cyprus. Da Vinci was fascinated by the beauty of the Lefkara’s laces and in fact he bought a lace which he later offered in the cathedral of Milan. Readiness questions and discussion focusing on the story presented and the lace design followed. During the modeling stage of the activity, students were given a mission. They had to help an owner of a clothing factory to design attractive carpets. Part of the activity is presented in Figure 3.3.

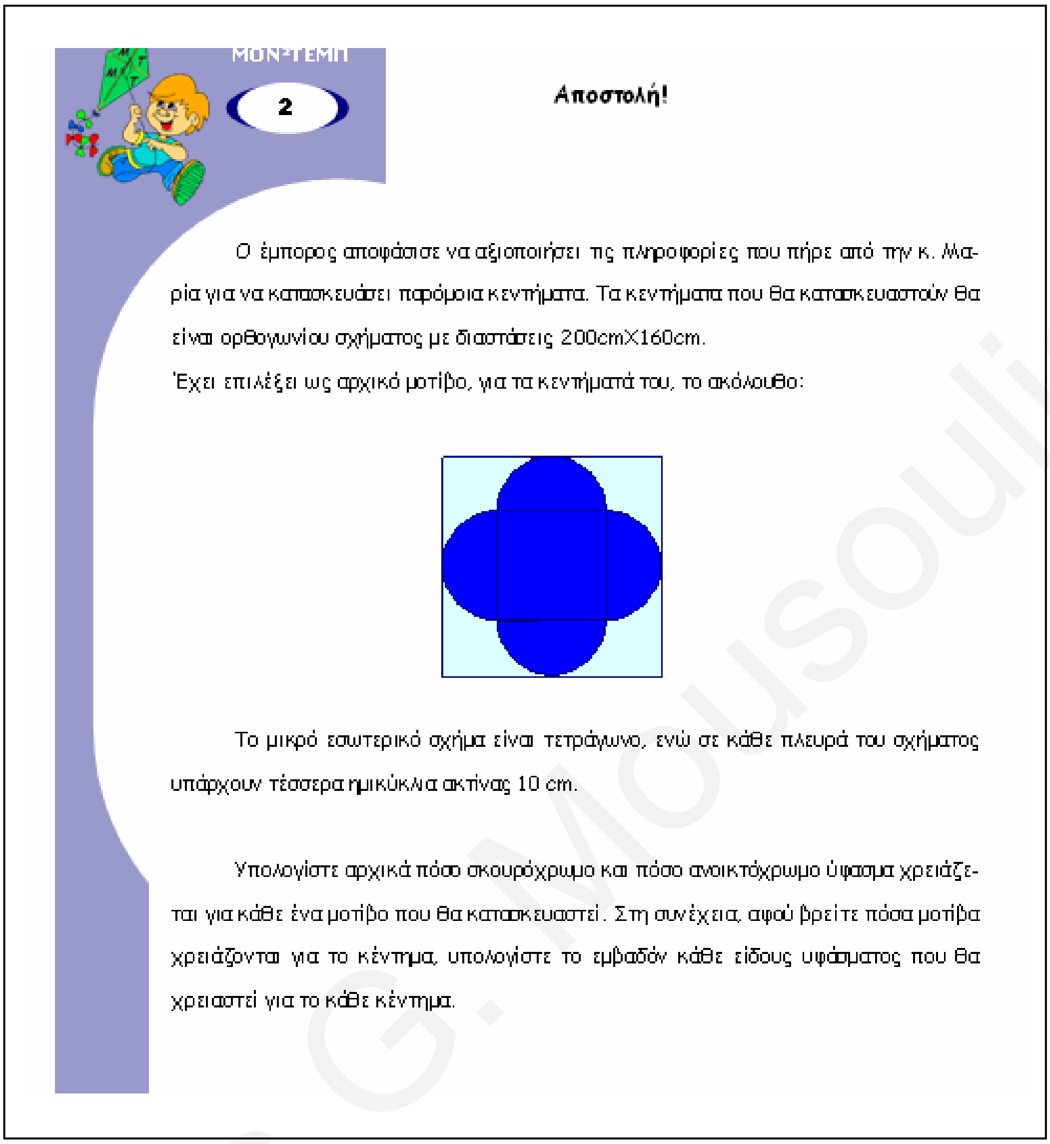


Figure 3.3. The Carpet design activity.

The Car Painting Activity

The second activity of the sequence of modeling activities focusing on geometric reasoning was the “Car Painting” modeling activity. The main purpose of the activity was to develop a method using polygons for covering and then painting a car. An imaginary client, the chairperson of a car manufacturer company, asked from the students to assist him in finding out how much paint was needed for painting a car. Specifically, students had to find the car’s total surface and then calculate the amount of paint needed.

During the first part of the activity, students were given a newspaper article, discussing the environmental changes due to fuel consumption and the decision of a number of car manufacturer companies to design solar and electricity powered cars. Following, students had to answer some readiness questions related to the context of the article. During the modeling stage of the activity, students were given a modeling problem focusing on calculating the surface of a car. The problem is presented in Figure 3.4.

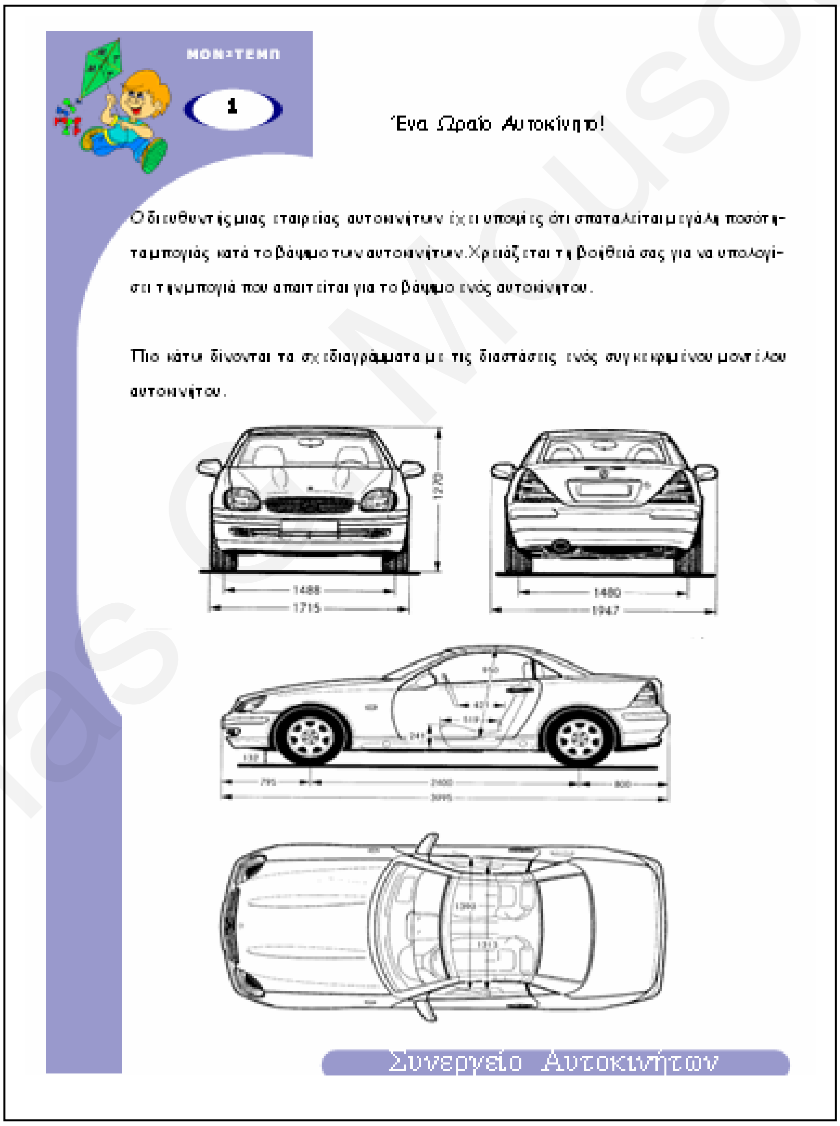


Figure 3.4. The Car Painting modeling activity.

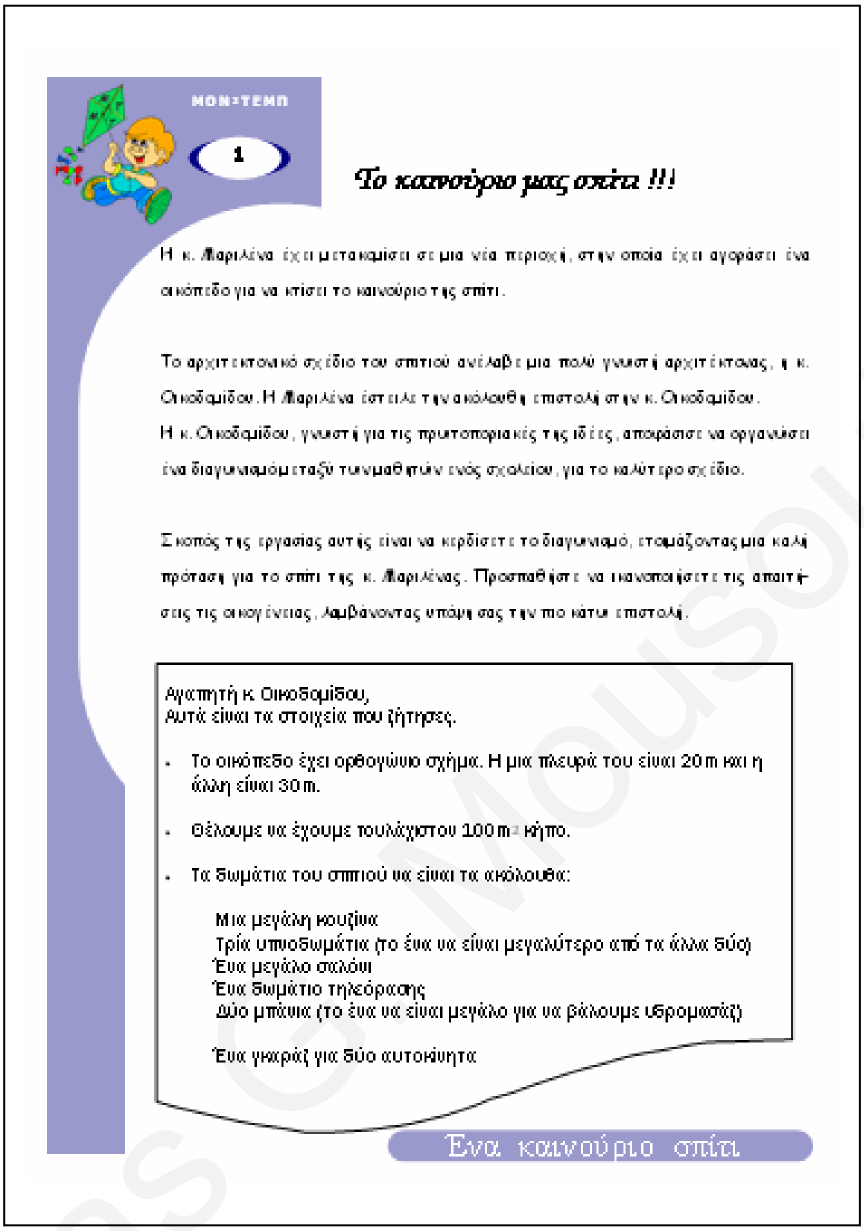
After finding a method for calculating the surface of the car and the paint needed to paint it, students had to write a letter to the chairperson of the company, describing how they found the total surface of the car and giving details so the chairperson could understand how they reached their results. Finally, students presented their solutions in their classmates and discussed the different solutions they reached.

The New House Activity

The third activity of the sequence of modeling activities focusing on geometric reasoning was the "New House" modeling activity. The main purpose of the activity was to design the floor plan of a house. An imaginary client, a lady that needs to build a new house, asked from an architect to design her house. The architect decided to run a competition among school students for selecting the best design. Students had to design the floor plan of an attractive house, following a number of regulations in order to win the competition.

During the first part of the activity, students were given a newspaper article, discussing migration in general and the reasons a number of Cypriots migrated in other countries. Following, students had to answer some readiness questions related to the context of the article. During the modeling stage of the activity, students were given a modeling problem focusing on designing the floor plan of a house, according to a number of regulations. These regulations were related to the surface and shape of the building plot, the number of rooms, and rooms' dimensions. The problem is presented in Figure 3.5.

After completing the modeling stage of the problem, students had to write a covering letter to the architect, explaining how they designed the house and to what extent their design met the necessary regulations. Finally, each group of students presented their designs in whole class presentations and discussion followed.



ΦΟΝ:ΤΕΜΠ

1

Το καινούριο μας σπίτι !!!

Η κ. Μαριλένα έχει μετακομίσει σε μια νέα περιοχή, στην οποία έχει αγοράσει ένα οικόπεδο για να κτίσει το καινούριο της σπίτι.

Το αρχιτεκτονικό σχέδιο του σπιτιού ανέλαβε μια πολύ γνωστή αρχιτέκτονας, η κ. Οικοδομίδου. Η Μαριλένα έστειλε την ακόλουθη επιστολή στην κ. Οικοδομίδου.
Η κ. Οικοδομίδου, γνωστή για τις πρωτοποριακές της ιδέες, αποφάσισε να οργανώσει ένα διαγωνισμό μεταξύ των μαθητών ενός σχολείου, για το καλύτερο σχέδιο.

Σκοπός της εργασίας αυτής είναι να κερδίσετε το διαγωνισμό, ετοιμάζοντας μια καλή πρόταση για το σπίτι της κ. Μαριλένας. Προσπαθήστε να ικανοποιήσετε τις απαιτήσεις τις οικογένειας, λαμβάνοντας υπόψη σας την πιο κάτω επιστολή.

Αγαπητή κ. Οικοδομίδου,
Αυτά είναι τα στοιχεία που ζήτησες.

- Το οικόπεδο έχει ορθογώνιο σχήμα. Η μια πλευρά του είναι 20m και η άλλη είναι 30m.
- Θέλουμε να έχουμε τουλάχιστον 100m² κήπο.
- Τα δωμάτια του σπιτιού να είναι τα ακόλουθα:

Μια μεγάλη κουζίνα
Τρία υπνοδωμάτια (το ένα να είναι μεγαλύτερο από τα άλλα δύο)
Ένα μεγάλο σαλόνι
Ένα δωμάτιο τηλεόρασης
Δύο μπάνια (το ένα να είναι μεγάλο για να βάλουμε υδρομασάζ)

Ένα γκαράζ για δύο αυτοκίνητα

Ένα καινούριο σπίτι

Figure 3.5. The New House modeling activity.

Researcher's Role

The role of the researcher in the present study consisted of four main aspects. First, the researcher supported the teachers' in their implementation of the six modeling activities for students. This included providing the modeling activities to be used in the teachers' classrooms, and as necessary, providing the supplies that the students needed. Researcher also supported the teachers' implementation efforts by sharing with them information about how other teachers have effectively implemented modeling activities and by assisting the teachers in the classroom with their first implementations of the modeling activities. Therefore, for all teachers, the researcher observed most of their implementations of the modeling activities and worked at the same time with one of more groups of students. Second, the researcher facilitated the teacher workshops. This included keeping the workshops on-task and progressing along the intended agenda, discussing the implementation of the modeling activities and suggesting possible improvements both for the context of the modeling activities as well as for their implementation. Third, the researcher videotaped the modeling activities in most of the classes and audiotaped one or more groups of students in modeling activity in each class. He finally transcribed the student work during the modeling interpreted the results using the grounded theory approach.

Teachers' Workshops

Before running the modeling activities teachers participated in one introductory six hours workshop. The teachers completed the associated first three modeling activities and then discussed the related mathematical concepts, possible student solutions, and implementation issues. After the workshop, the teachers had five-week time period in which to implement the three modeling activities within their own classrooms. The implementation of the modeling activities was scheduled in a way to avoid overlaps, so that the researcher could be present in most activity implementations.

After implementing the first three modeling activities, teachers participated in a second workshop. The duration of the second workshop was three hours. During this second workshop, teachers again completed the associated second set of three modeling activities and then discussed the related mathematical concepts, possible student solutions, and implementation issues. A third and final workshop was held after implementing the whole set of the modeling activities and the third modeling test administration. During that concluding workshop the researcher discussed with teachers issues related to the implementation of the modeling activities and possible final refinements in the modeling activities.

Procedure

Based on the literature review and especially on the design principles for developing modeling activities, six modeling activities were developed (see Appendix II). The modeling activities set, as it can be seen in Table 3.1, consisted of two sequences of modeling activities. The core mathematical ideas of the first sequence of modeling activities were statistical concepts such as average and frequency, weighting and ranking quantities. The mathematical idea behind the second modeling activities sequence was geometric thinking and especially the concepts of area and perimeter of a number of two dimensional figures.

A structure of the experimental part of the study is presented below. The study followed a three phase structure. The first phase included the following: (a) Design of the six modeling activities. The modeling activities developed for the purposes of the present study followed the design principles suggested by Lesh and his colleagues (2000). The six activities constituted two sequences of activities, related to statistical reasoning and geometrical thinking. (b) Teachers' introductory workshop. The first workshop with the teachers focused on introducing teachers to the modeling perspective in problem solving. Teacher had time for working on the six modeling activities by themselves and solving the modeling problems. Teachers also solved the modeling abilities test and provided feedback. A discussion followed and teachers provided suggestions and modifications on the six activities and the modeling test. (c) Modeling activities and modeling test modifications and corrections. The first workshop with the teachers resulted in a number of modifications and corrections on the modeling activities and the modeling test. The modeling problems presented in the "New House" and "University Cafeteria" activities were slightly modified to better correspond to students' mathematical abilities. One task from the modeling test was also modified. (d) Pilot study of two modeling activities and the modeling test. The "Best Drug Award" and the "Carpet Design" activities and the modeling test were pilot tested by the researcher in one 6th and one 8th grade class. The results of the pilot study showed that students from both grade levels successfully solved the problems presented in the modeling activities and presented their models in whole class presentations. The pilot study of the test also showed that students could complete the test within the 40 minute available time. Some minor modifications to the language of the activities and the test were finally implemented in order to make activities and test better accessible to students. The pilot test resulted to the final version of the six modeling activities and the modeling test.

The second stage of the study included the following: (a) First administration of the modeling abilities test. The modeling test was administered to all participating classes a week before the beginning of the intervention program. Specifically, the test was administered to four hundred and three students coming from sixteen classes. Four 6^{th} grade classes participated in the experimental group and four 6^{th} grade classes participated in the control group. Similarly, there were four experimental and four control 8^{th} grade classes. (b) Implementation of three modeling activities. During the next forty days after the first administration of the test, three modeling activities were implemented in the experimental group classes. Specifically, the "Best Drug Award", the "Carpet Design" and the "Where to Live" activities were implemented during the stage. At the same time, the control group students were working with their mathematics textbooks. However, the modeling activities did not replace the whole mathematics curriculum for the experimental group students. Approximately, each modeling activity was implemented in a period of ten to fifteen days. (c) Second administration of the modeling test. Similar to the first administration, the modeling test was administered to all experimental and control group students after the administration of the first three modeling activities. (d) Teachers' second workshop. During the second workshop the researcher discussed with the teachers issues related to the implementation of the first three modeling activities. The discussion resulted in a number of recommendations related to the implementation of the remaining three activities and the teachers' role in working with the activities. (e) Implementation of modeling activities. During the next forty five days, the next three modeling activities were implemented in the experimental group classes. Specifically, the "Car Painting", the "University Cafeteria" and the "New House" activities were implemented. Similarly, each modeling activity was implemented in a period of ten to fifteen days. (f) Third administration of the modeling test. The final part of the second phase of the study was the third and final administration of the modeling test to all experimental and control group students.

The third phase of the study consisted of: (a) Analysis of the data collected. The experimental design of the study resulted in a number of different sources of data. The data consisted of students' written responses in their worksheets, transcripts of students' interactions in their groups, students' responses in the modeling test, and transcripts of whole class discussions after students' presentations. (b) Building a theory driven by the models and modeling perspective, based on the findings of the present study.

Data Collection, Analysis and Statistical Techniques

Data analysis was drew upon a grounded theory approach (Strauss & Corbin, 1998) where the theory is derived from the data, and is systematically gathered and analyzed throughout the research process. This approach is consistent with modified analytic induction qualitative research methodology where data is collected and analyzed in an effort to develop a descriptive model of the phenomena studied (Bogdan & Biklen, 1998). Consistent with a models and modeling perspective, the students' records and models helped to produce a continuous trail of documentation, and these documents were used to reflect on the nature of the students' developing models and conceptual understandings (Lesh & Kelly, 2000).

Three levels of data were collected for evaluation purposes. The first level of data, to be subjected to quantitative statistical analysis, was the scores resulted from students work in the modeling processes test. A second tier of performance data came from examining specific student responses to both the modeling activities and to the assessment test. Examples of student work from experimental group provided as evidence, and all available written responses were examined. In more detail, data focused on students' written work from work-alone and group work in model formulation and application (a set of items students solved individually and in groups), students' participation in group discussions and in whole class discussions. The final source of data came from transcripts of video and audio recordings and observation notes taken by the investigator as the study unfolded. Special attention in video and audio analysis was given to student groups' presentations and argumentation on the appropriateness of their models and solutions.

Qualitative statistical analysis techniques were employed to analyse the data collection as described earlier. Additionally, quantitative techniques were used for analyzing data retrieved from the modeling test. Specifically, Mplus and SPSS software were used for the quantitative statistical analysis of the data and Atlas Ti software was used for the analysis of qualitative data of the study.

Grounded Theory

Grounded theory is an approach that "begins with an area of study and allows the theory to emerge from the data" (p. 12). There are three steps of theory building. The first step is *description*. Strauss and Corbin (1998) define description as "the use of words to convey a mental image of an event, a piece of scenery, a scene, an experience, an emotion, or a sensation" (p. 15). The second step towards theorizing is *conceptual ordering*; the "organizing of data according to a selective and specified set of properties and their dimensions" (p. 15). Finally, the researcher is prepared for *theorizing*; the process of developing theory or "a set of well-developed concepts related through statements of relationship, which together constitute an integrated framework that can be used to explain or predict phenomena" (p. 15).

According to Strauss and Corbin's recommendations (1998), a detailed line by line analysis is necessary at the beginning of a study to generate initial categories (with their properties and dimensions) and to suggest relationships among categories. This detailed analysis is made up of two types of coding, open coding and axial coding.

The process of open coding is used to identify the concepts and their properties that are discovered in data. Open coding's analytic tasks include naming concepts, defining categories, and developing categories in terms of their properties and dimensions. According to the process of open coding, concepts are labeled actions and phenomena, which provide an abstract representation of a significant event, object, or interaction. This process continues with conceptualizing. The purpose of conceptualizing is to enable the grouping of similar actions and events and create categories.

Axial coding is the process of relating categories by identifying which categories are subcategories of other categories. With axial coding, the analyst is coding around the axis of a category to add depth and structure to it (Strauss & Corbin, 1998). Axial coding includes the following tasks: (a) Laying out the properties of a category and their dimensions, a task that begins during open coding, (b) identifying the variety of conditions, actions/interactions, and consequences associated with a phenomenon, (c) relating a category to its subcategories through statements denoting how they are related to each other, and (d) looking for cues in the data that denote how major categories might relate to each other (Strauss & Corbin, 1998, p. 126).

The next step after open and axial coding is the process of selective coding. Selective coding is the process of integrating and refining the categories in order to delineate theory. Selective coding starts by deciding upon a central category, which can stand for all the products of the analysis and explain the results of the research. Strauss and Corbin (1998) provide four different mechanisms that a researcher can use to refine the theory. First, the researcher can review the theory for internal consistency and logic. Second, the researcher can fill in any poorly developed categories by identifying and describing all salient properties and dimensions associated with such categories. Third, the researcher can trim the theory, dropping ideas that seem to trail off in the data and that are not relevant to the main theory. Finally, the researcher can validate the theoretical scheme by returning to the raw data and verifying that the theory fits.

Structural Equation Modeling

This study integrates several techniques of structural equation modeling (SEM). A structural equation model is distinct from a measured variable path model in that it hypothesizes crucial variables (e.g., achievement motivations), which may not be directly observable and are better modelled as latent variables. SEM makes it possible to distinguish two different types of errors: errors of measurement in the observation of variables, and errors of prediction in structural equations. Models that involve only observed variables, as regression and path models, assume that measured variables are perfectly valid and reliable. Since all constructs under study are assumed to contain measurement errors, the SEM approach was appropriate for analyzing the quantitative data resulted from the present study.

The confirmatory factor analysis (CFA) was applied in order to assess the results of the study related to the model explaining students modeling abilities in the different categories of modeling problems. CFA is appropriate in situations where the factors of a set of variables for a given population is already known because of previous research. In the case of the present study, CFA was used to test hypotheses corresponding to students; modeling abilities in the three categories of modeling problems. Specifically, the purpose of using CFA was to investigate whether the established structure of modeling abilities as proposed by the literature fits our data.

Mplus (Muthen & Muthen, 2004), a structural equation modeling software, which is appropriate for discrete variables, was used to test for model fitting in this study. In order to evaluate model fit, three fit indices are computed: The chi-square to its degree of freedom ratio (x^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). The observed values of x^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be less than .06.

A second statistical analysis that was used to examine the effectiveness of the modeling intervention program was the Latent Growth Modeling (LGM) analysis. Latent growth modeling analysis extents the applications of Structural Equation Modeling (SEM) to the study of variables measured on at least three occasions with latent growth models, which are modifications of standard hybrid models that allow for the evaluation of change over time on both group and individual levels.

The general model for the latent growth model analysis is presented in Figure 3.6.

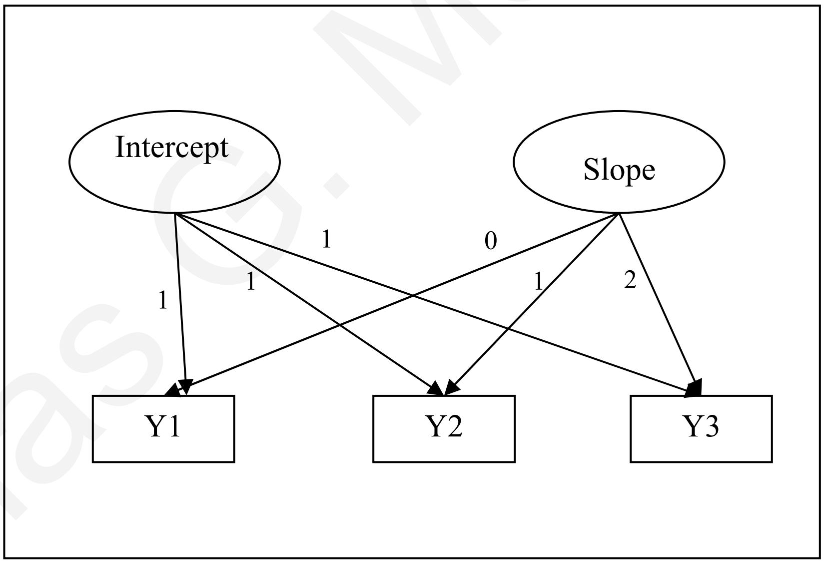


Figure 3.6. The General Model for Latent Growth Modeling.

The assessments of the modeling abilities are presented in the model as Yi, i=1-3. Specifically, Y1 refers to the first measure of student modeling achievement in the modeling test. The test was first administrated before the beginning of the intervention.

The second administration of the test, that was conducted after the implementation of the first three modeling activities, is represented in the figure as Y2. Finally, the third measure of students' modeling abilities which was administered after the completion of the intervention program is presented by Y3.

The model presented in Figure 3.6 has four essential characteristics. First, the assessments of achievement in modeling are each represented as an indicator of two underlying factors, Intercept (I) and Slope (S). As its name suggests, the Intercept (I) factor represents subjects' baseline levels of achievement in modeling. Because the intercept is a constant in growth equations, the loadings of the repeated measures indicators on the Intercept (I) factor are all fixed to equal one. In contrast, the loadings on the Slope (S) factor are fixed to constants that correspond to the times of measurement, beginning with zero for the initial measurement before working with the modeling activities and ending with two for the third measure of student achievement in the modeling test. Because these loadings are evenly spaced (i.e., 0, 1, 2), the Slope (S) factor represents linear change over time.

CHAPTER IV

DATA ANALYSIS – RESULTS

Introduction

The results of the data analysis are presented in this chapter. Specifically, the data collected from the modeling activities and from the modeling test that was administered both in control and experimental group are analysed. The analysis of the data is focused on answering the research questions presented in chapter one. The research questions can be summarized in examining the modeling processes presented in students' work in the the modeling problems, examining the similarities and differences in the modeling processes between 6th and 8th grade students, and examining modeling abilities' rate of growth and the impact of the intervention program on the modeling abilities.

As analytically presented in the previous chapter, an intervention program consisted of six modeling activities was developed and implemented in four 6th and four 8th grade classes. The test for measuring modeling abilities was administered both to the experimental and to the control groups (four 6th grade and four 8th grade classes). The test was administered three times; before the beginning of the intervention program, during the intervention program (after the implementation of the first three activities) and after the completion of the program.

The analysis of the data and the corresponding results are presented as follows. First, the analysis focused on the development of an analytical and explanatory framework, as proposed by the grounded theory approach (Strauss & Corbin, 1998). The first step is the development of the analytical framework. The analytical framework took into consideration the current theoretical perspective in using the modeling approach in problem solving and used the results from the analysis of one case study (6th grade students' work on the "Best Drug Award" modeling activity). Specifically, case study analysis is focused on the work of one group of three students. The resulted analytical framework extended the current theoretical approach by analyzing in depth the current modeling processes, and by adding new modeling processes. The analytical framework guided the analysis of the other case studies. There are six case studies in total. Three case studies referred to 6th grade students' work in the "Best Drug Award", in "Where to Live" and in "University cafeteria" activity. The other three case studies referred to the 8th grade students' work in the same activities. The analysis of the six case studies resulted in an explanatory framework, which represents the new theoretical approach in using the modeling perspective in problem solving. The explanatory framework intended to answer the following research questions:

What are the modeling processes and sub processes students develop in working with modeling activities?

How modeling processes evolve across students' investigation in the sequence of the modeling activities of the intervention program?

How modeling processes are differentiated between 6th and 8th graders?

What are the similarities and differences in modeling processes between successful and unsuccessful models constructed by 6th and 8th graders?

The data analysis of students' achievement in the modeling test is then presented in this chapter. Specifically, the analysis of students' modeling abilities resulting from the modeling test is presented as follows: (a) A preliminary analysis was conducted to examine the equivalence between the two treatment groups (experimental and control) in terms of students' achievement in the modeling test, (b) Confirmatory Factor Analysis for establishing and validating a theoretical model for student modeling abilities in the three categories of problems presented in the modeling test, (c) Latent Growth Modeling analysis for examining students' modeling abilities rate of growth. The above analyses were used to answer the following research questions:

What are the characteristics of students' modeling abilities?

Can a theoretical structure based on student modeling abilities in different categories of modeling problems be validated?

What is the impact of the intervention program on students' modeling abilities?

How student modeling abilities are changed over time (rate of change) and what is the impact of the intervention program on the modeling abilities' rate of change?

The Best Drug Award Activity

The work of one group of 6th graders in the "Best Drug Award" activity was used for developing the analytical framework. The analysis of students' work first focused on "telling the story" and on finding the key elements of this story (Miles & Huberman, 1994). These key elements are analyzed in order to describe the modeling processes and conceptual developments students presented in their work. The second step is moving from "telling a story" to "constructing a map". As Miles and Huberman (1994) suggested, to explain something satisfactory, the researcher needs to move from "telling a story" about a specified situation, problem (what happened) to constructing a "map" (formalizing the elements of the story, finding key variables), to building a theory (theorizing, how variables are connected, their causal relations).

Students' work in the "Best Drug Award" activity is presented in two dimensions. The first focused on the modeling processes presented in students' work and the second summarized students' mathematical developments. The analysis of the modeling processes is divided into stages which correspond to the steps of the modeling procedure as defined by the current theoretical approach by Lesh and Doerr (2003). Student mathematical developments are also analyzed and presented during the analysis of student work in the different steps of the modeling procedure. At the end of this analysis, student mathematical developments are summarized. The two dimensions of the analysis aim to assist the analysis in terms of identifying similarities and differences between the modeling processes and mathematical developments presented by 6th and 8th grade students.

Modeling Processes and Mathematical Developments

The analysis of the modeling processes presented in students' work is organized in stages, referring to the four steps of the modeling procedure (see Figure 4.1), namely the description of the problem, the manipulation of the problem, the prediction of the real world problem behaviour and the verification of the real problem. At the end of the analysis, students' mathematical developments are summarized and organized into categories which represent the same type of mathematical reasoning. Stages A1 to A4 refer to the modeling processes presented by students in working with the modeling activity and stages B1 to B3 refer to the students' mathematical developments.

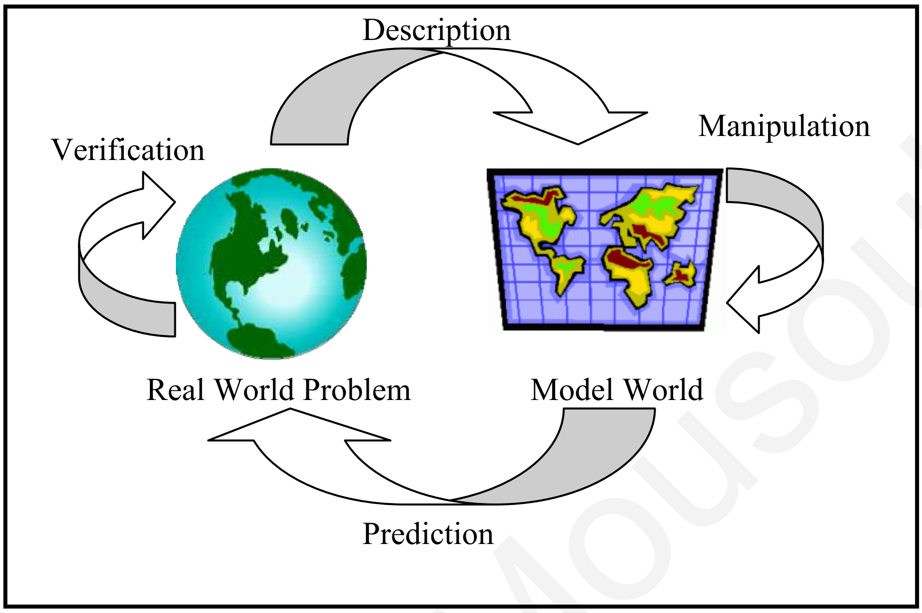


Figure 4.1. The modeling procedure in mathematical problem solving.

Modeling Processes

Stage A1: Description of the problem. In their first attempts, students started exploring the problem in obtaining more information about the four drugs and in clarifying what "the most effective drug" meant. Students observed the reaction times for each drug and discussed the smallest and biggest reaction times for drugs. The meaning of the question of the problem was also discussed. As it is presented in the following extract, each student focused on different drugs, pointing out at each time that the specific drug had small reaction times.

Helen: Look at Saracetamol. It takes less than 15 minutes to act in a number of different cases.

Alex: Kefapol also has reaction times smaller than 15 in many cases.

Alice: You are right (talking to Alex). But look over here. It takes 20 and 25 minutes (pointing to Kefapol's reaction times).

Alex, as he explored the problem situation, used a highlighter to circle small reaction times for Kefapol. Specifically, Alex discussed the meaning of "most effective drug" with Alice and Helen. For both Alex and Helen, the meaning of "most effective" was easily related with small reaction times. In an attempt to clarify the meaning of "most effective drug", Alex explained to Alice that: "It is important for a drug to act quickly. So, an effective drug is the one which has small reaction times".

In exploring the problem students tried to perceive different information about the problem than was originally apparent. For example, at the beginning, each student focused on one drug and pointed out to other students that the specific drug could probably be the best one. Later, during their first discussion, students realized that all four drugs had more or less the same characteristics in terms of small and big reaction times.

Helen: Kanatol has big reaction times. It has 22 and 23 minutes.

Alex: It is not only Kanatol that has big reaction times. Ralpol also has 23 and 24 reaction times. Look! 23 appears two times.

Alex: And Saracetamol has one 25.

Alice: Saracetamol also has 20 minutes reaction time.

Alex: Finally, all four drugs have big reaction times.

During the process of exploring the problem students used their mathematical ideas to describe the situational information of the problem. One example of early mathematizing is also presented in the previous extract, when students discussed cases in which reaction times were bigger than 20 minutes. A second example is coming from Helen, when she observed that one drug had reaction times smaller than 12 minutes. Specifically, Helen pointed out that Saracetamol had in nine cases reaction times smaller than 12 minutes. She later realized that all four drugs had reaction times smaller than 12 minutes. She found that Ralpol also had in nine cases reaction times smaller than 12 minutes!

A third characteristic of student work is related to the identification of the conditions and possible assumptions of the real world context. In the present problem, students were quite efficient to explicitly state the conditions and assumptions of the problem. Alex stated that there are four drugs and twenty cases for each one. He also pointed out that "the results presented in one row [one reaction time for each drug] are referring to the same person". He also claimed that his previous point was essential; "otherwise it would be difficult or even impossible to compare the different drugs". This identification is important, since it helped students to compare the data and to gain deep understanding of the problem.

Stage A2: Manipulation of the problem. In the second step of the modeling procedure, students link the real world problem with the mathematical structure (model) that needs to be developed in solving the problem. A central characteristic of students' work in this step of the modeling procedure is mathematizing. Mathematizing includes introducing mathematical ideas that eventually relate to the mathematical entity that is represented in the constructed model. Students try to mathematize the real problem by making a connection or a bridge between the problem and the mathematical structure of the model. Helen, for example, used the different reaction times between the four drugs to identify and rank the four drugs according to their reaction times for each row of the table. As a result, she created twenty different rankings. Her mathematization processes were different from the ones projected by Alex. Alex first followed Helen's interpretation, but when he later realized that the model was not good enough, he extended his model by assigning positive and negative points to the drugs. Specifically, Alex decided to add one point when a drug was ranked first in a row and subtract one point of the drug was the last one. He later found the total for each drug, to decide which drug was the most effective one.

Another interesting snapshot of students' work in this step of the modeling procedure was a discussion on the accuracy of the reaction times. Students discussed extensively the fact that all numbers were integers and questioned how the reaction times were rounded. One parameter in their work related to the above assumption was that it should be clarified if all data were rounded in the same way.

Therefore, it could be claimed that it is possible for a student to state many and different properties which he or she deems necessary to "unfold" the situation. The mathematical pieces generated in mathematizing may not be perceived from the modeler's view as a single mathematical object. Another characteristic of the process of mathematizing in students' work in the present modeling activity was that mathematizing some times happened implicitly, since students were quite familiar with the problem presented. As a result, students quite easily made the connections between the identified conditions/assumptions and the desired properties. This was necessary for moving into the model construction.

However, the duration and nature of mathematizing can vary and mathematizing is not always a rapid or implicit event. In fact, in the present activity students spent a considerable amount of time to clarify many things related to the concept of *average,* and how to use this statistical concept in creating a model they could use to solve the problem. In this case, more intense mathematizing happened, since students faced a demand to construct a novel and non trivial idea to serve as a property for each drug and as a property that could be compared between different drugs.

One question related to students' work is whether the problem presented in the "Best Drug Award" activity could foster the development of novel ideas and to evoke new mathematical knowledge. All students were familiar with the term "average" and it was apparent that they could apply the average formula to find the average in a number of tasks. However, in students' first attempts of mathematizing the problem, none of them used the average to find a solution. During their work, the possible use of average was discussed. Specifically, Helen pointed out that: "We can compare the drugs using their averages". However, students could not realize that either finding the total or reaction times, either finding the average for each drug would result in the same drugs' ranking. In discussing the meaning of average Alice accepted that the average was the time needed for the drug to react in most of the cases. Alex disagreed with her. He claimed that the average was not a "real" number; they could not observe the average in their data, not even in one single case. He concluded by explicitly referring to the formal definition of the average.

After connecting the real world situation with the mathematical entity, students worked with the mathematical entity, trying to organize and formulate their prior assumptions and identified properties firstly into a mathematical entity and later into a valid and appropriate model. In other words, students combined different mathematical objects into a single mathematical entity. Alex used the sum of reaction times for each drug in conjunction with his knowledge that the relation between two numbers (first and second or biggest and smaller) does not change when dividing the two numbers with the same (positive) divisor.

This step of the modeling procedure is completed when student mathematization processes result in the building of a model. Student work in the present activity showed that a number of different models were developed and presented by students in their group, making the selection among alternatives not an easy task. Among the different models constructed by students was one based on circling the smallest reaction time in each row and then on selecting the drug with the biggest number of circles.

Helen: Let's *circle in each line* the drug with the smallest reaction time.

Researcher: How will you rank the drugs?

Alice: We will count the number of circles for each drug.

As soon as students realized that their prior model did not use all provided data, they decided to find a method for ranking the drugs, using all reaction times.

Alex: We should add all reaction times for each drug.

Alice: Why should we do that?

Helen: Alex is right. By adding all numbers ... find the drug with the least sum. This one will be the most effective pain relief drug.

The final model students developed was the one based on the average concept. Part of their work is presented in the extract below.

Helen: First, we add all times and divide by ... (she was interrupted by Alice)

Alice: Four. We have four drugs.

Alex: No, this is not correct. We do not find the average like this. We need to divide by twenty, the number of cases.

Alice: You mean that we add all reaction times and divide by 20?

Alex: No, there is no reason to add all reaction times. We only add the times for each drug because we need to calculate the average for *each drug*.

Helen: We do not always divide by 20 but with the number of cases. We need one average for each drug to find the differences between the four drugs. We could also calculate one average for all drugs, but only to compare these drugs with other drugs.

In summarizing the different models that were suggested by students were the following: (a) Circle the smallest reaction time in each row. The most effective drug is the one with the biggest number of circles. (b) Circle the smallest and biggest reaction time in each row. Assign positive and negative points to circles. The most effective drug is the one with the biggest sum. (c) Sum up the reaction times for each drug. The most effective drug is the one with the smallest sum. (d) Find the average reaction time for each drug, considering all twenty cases. The most effective drug is the one with the smallest average. As a concluding point, it can be argued that selecting the best model among alternatives was challenging and promoted discussion among students. Further, this discussion enhanced student conceptual understanding in the related mathematics concept.

Stage A3: Prediction of the real problem. Student work with the mathematical entity presented in the previous session resulted in an appropriate model for solving the problem. The next step in student work was to connect the produced model with the real situation, trying to predict the behavior of the real problem. In other words, students used their model to interpret their solution and to make predictions of the behavior of the real world problem.

One of the modeling processes students developed and used during the prediction of the problem was interpretation. Like mathematizing, interpreting may be seen as a bridge between the real world and the mathematical world. Interpreting involves connecting real-world ideas with mathematical ideas. An instance of interpretation occurred as Helen reported that: “These two drugs (Kanatol and Ralpol) have almost the same effectiveness, since their averages are almost the same. Kanatol’s average is 13,85 minutes and Ralpol’s average is 13,75”. Another example of interpretation focused on examining whether the produced model aligned with the realistic situation in light of the modeling goal. Specifically, Alice explained that: “Ralpol is the second most effective drug and Kanatol the third. But they are much closed. It is better to say that these two drugs are the same in terms of their effectiveness”.

Stage A4: Verification. A number of sub-processes occurred in the step of verification. In detail, in this step students checked their solution, and validated their results. One of the characteristics of verification is cited in students’ discussion on checking that their solution could answer the question of the real world problem. Students checked whether their modeling work fitted with purpose of the activity. In her worksheet, Helen explained that: “it might be better to say that drugs whose averages are below 13,5 are quite effective, since averages of the second and third most effective drug are much closed”. This is quite important, since students not only examined the most effective drug (core question of the activity), but they also considered other results in formulating a coherent explanation for verifying their solution.

A modeling process appear in the step of verification was the communication of the results. A good practice for promoting communication both between students working in the same group as well as between one group's work and the rest of the class, was to ask from students to justify their solutions by writing a letter to an imaginary client. In the "Best Drug Award" activity students had to write a letter to the president of the Drug Industries Association, explaining and documenting their results. In this letter, Alex explained that:

> "The use of average can help your organization finding the most effective drug. Of course, you have to notice that averages are closed. My idea is that these differences are not so important, but I am not sure what would happen if I had results from more than twenty persons. Using averages is appropriate for comparing more than four drugs".

In her letter communicating her results, Helen moved a step forward, documenting that:

> "You also can use the idea of average to compare other products, like day skin cream. Be careful though, since in other cases, you might need to find the biggest and not the smallest average. For example, if you have data for how long a day skin cream lasts the best cream would be the one with the biggest average"!

Student Mathematical Developments

The presentation of students' mathematical developments is organized into three categories, which are descriptive of students' mathematical processes and developments in the "Best Drug Award" modeling activity. The central characteristic of the first category of students' mathematical developments was students' focus on subsets of information. Student consideration relied on partial data for one or more drugs. As students realized that their first approaches were insufficient, they formulated more mathematized approaches, which are presented in the second category. Following, in the third category students raised the question on the meaning of average and discussed concept's implications.

Focusing on subsets of information. As soon as students clarified the question of the problem and decoded the question into ranking the four pain-relief drugs according to their effectiveness, they focused on specific information for one or more drugs. In this first attempts, students' efforts focused on subsets of information, as they only concentrated on the smallest reaction time for each drug. The extract below shows how students perceived the solution of the problem. These initial (and insufficient) approaches encouraged students to question the appropriateness of their solutions and prompted them to search for more justifiable and generalizable solutions.

Helen: I believe that Saracetamol is the most effective [drug] since it needs the least time to act. Other drugs need more time.

Alex: Ok, but what about Kefapol? In three cases it needs fewer minutes to act.

Alice: Check over here! Kefapol's times are 17, 17 and 17 while Saracetamol's are 11, 11 and 12, respectively.

Alex: You are right, but Saracetamol has also 20 and 25 minutes reaction times.

The students engaged in debates over how to generate a comprehensive model which could handle both small and big reaction times. This first discussion led to a more systematic approach. Students used their informal knowledge to make a number of conjectures and to justify their claims:

Helen: Let's circle in each line the drug with the smallest reaction time.

Researcher: How will you rank the drugs?

Alice: We will count the number of circles for each drug.

The above process did not lead to an appropriate solution; however, this approach forced students to argue about its usefulness since there were drugs with similar reaction times. Alex suggested to circle both drugs. Based on this idea, students ranked the drugs in the following order: Saracetamol, Ralpol, Kefapol and Kanatol. The next improved model was base on Alice's suggestion to use both smallest and biggest reaction times in each row. Specifically, Alice did not agree and argued that: "it is not correct to circle only the smallest reaction time drug in each row. We should also circle the drug that has the biggest reaction time". Her comment started a new round of discussions for finding a solution taking into account both the biggest and the smallest reaction times. Their solution was to circle both the smallest and biggest reaction times and then subtracting the two numbers. The group ended with a different drug ranking: Saracetamol, Kefapol, Ralpol and Kanatol.

Stage B2: Using mathematical operations and processes. Since the group did not use a systematic approach to tackle the problem, the different rankings troubled the students. Different approaches and contradicting results in conjunction with the discussion with the teacher and the researcher led students to face the need to further mathematize their procedures. Thus, students began to use two main mathematical operations to handle the data for each drug, namely, (a) totalling the amounts of reaction times for each drug, and (b) finding and comparing drugs' averages. The core characteristic of these solutions was the adoption of more sophisticated mathematical processes. Alex's group next approach was based on the assumption that any new development should consider all reaction times and not only the best or/and worst reaction times.

Alex: We should add all reaction times for each drug.

Alice: Why should we do that?

Helen: Alex's right. By adding all numbers ... find the drug with the least sum. This one will be the most effective pain relief drug.

This new mathematical approach, based on *finding the sums of reaction times for each drug* created a new drugs' ranking: Saracetamol, Ralpol, Kanatol and Kefapol. Students were surprised to see that this new ranking was quite different than the two previous rankings. The big numbers that students encountered while working with "sums-model" started a new round of discussion. Helen suggested that they could divide the sums by the number of the cases to find the *average.* Yet, Alice remained unconvinced and she asked for more clarifications as shown in the transcript below:

Helen: First we add all times and divide by ... (she was interrupted by Alice)

Alice: Four. We have four drugs.

Alex: No, this is not correct. We do not find the average like this. We need to divide by twenty, the number of cases.

Alice: You mean that we add all reaction times and divide by 20?

Alex: No, there is no reason to add all reaction times. You should only add times for each individual drug to calculate the drug's average.

Helen: We do not always divide by 20 but with the number of cases. We need one average for each drug to find the differences between the four drugs. We could also calculate one average for all drugs, but only to compare these drugs with other drugs.

Students' last model helped them to realize that the new ranking was the same as the previous one. This finding generated a new round of discussion addressing the "equivalence" of the two models, namely the sums and the average models. This discussion is presented in the next stage.

Stage B3: Advanced mathematical thinking. Alex was the first who realized that Helen's model would produce the same drug ranking. As soon as students performed the necessary calculations, Alex pointed out that he was expecting this. In his words: "I knew this. Since we divide the sums by the same number nothing will change in the ranking". Alice was still confused and could not understand how Alex reached that conclusion. In the following extract students are discussing with the researcher the equivalence of the two models.

Researcher: Ok. How did you know from the beginning that the new ranking would be the same like the previous one?

Alex: I knew it. We had one number for each drug [the sum of reaction times]. We divided these numbers by 20. So, the new numbers are like the previous ones.

Researcher: What do you mean "they are like the previous"?

Helen: The new numbers are correspondingly in the same order like the previous number ranking.

Alice: I do not get it.

Alex: Helen's right. I will give you an example (talking to Alice). If you have 10 and 20 then 10 is the smallest and 20 is the biggest, right? Now, if you divide both by 5, then the new numbers will be 2 and 4. You see! Number 2 is still the smallest and 4 is the biggest.

A second interesting discussion focused on the meaning of average. Alice pointed out that the average was the time needed for the drug to react in *most* of the cases. Helen did not agree with Alice. She looked into her notes, and she announced in her group that the average for Kanatol was 13,75. She asked for clarifications, since: "If average is the time needed for a drug to react in most of the cases, this number (average) should appear in many cases. In out data, 13,75 does not appear even one time”. Alex agreed with Helen that the average can not be the time needed for the drug to react in *most* of the cases, but he could not correct Helen’s interpretation. The researcher extended their discussion on the procedure they used to find the average. Students’ conclusion about average was quite impressive. Alex wrote that: “the average actually shows the reaction time of a drug if that time is *the same for all cases.* For example, Saracetamol’s reaction time is 13. This number means that if the drug needed the same time to react for all 20 patients, then this reaction time would be 13 minutes”.

A concluding remark on student “advanced mathematical thinking” was presented in the letter they wrote for explaining their results and for documenting and supporting their conclusions. In the following representative and interesting snapshot of Alex’s letter to the Chairman of the Drug Association, Alex made evident that they spent a lot of time searching for the best solution. He concluded by saying that he was sure that his solution was correct. Quite impressive was his comment on the transferability of their solution:

> “Comparing the averages is appropriate in similar competitions you will have in the future. Our solution can be used to find the most effective drug, even in cases with more than four drugs. You can also use average to compare other products, like day skin cream. Be careful though, since in other cases, you might need to find the highest and not the lowest average”.

Summary

The analytical framework that resulted from students’ work in the first case study is summarized in the Figure 4.2. The analytical framework further analyzed the existing modeling processes and added new modeling processes and sub processes. The analytical framework also documented the mathematical processes students developed in the “Best Drug Award” modeling activity.

The analytical framework is summarized in terms of the four general steps of the modeling procedure, namely: (a) Description of the problem, (b) Manipultion of the problem, (c) Prediction of the problem and (d) Verification of the problem. During the description of the problem, students explored the problem, trying to perceive different information about the problem than was originally apparent. A second process was the identification of the conditions and assumptions of the real world context. Students also specified that some information was not important, and in this way they simplified the real problem. A central characteristic of students' work in problem manipulation was mathematizing. In mathematizing students used their prior findings, such as the conditions and assumptions, to identify the necessary variables and relationships for building a model. The first modeling process that appeared in the step of predicting the behaviour of the real problem was connecting their model with the real-world situation and examining the appropriateness of the constructed model. A second modeling process was solution's validation. Students tried to validate their models in the context of the real world problem and interpreted their solution, trying to predict the behaviour of the real problem. The modeling processes that appeared in students' work during solution's verification were the verification and the communication of results. Students tried to find out whether the suggested model could answer the core question of the real world problem. Finally, by communicating students shared their suggestions and solutions and reflected on their results, trying to improve their models.

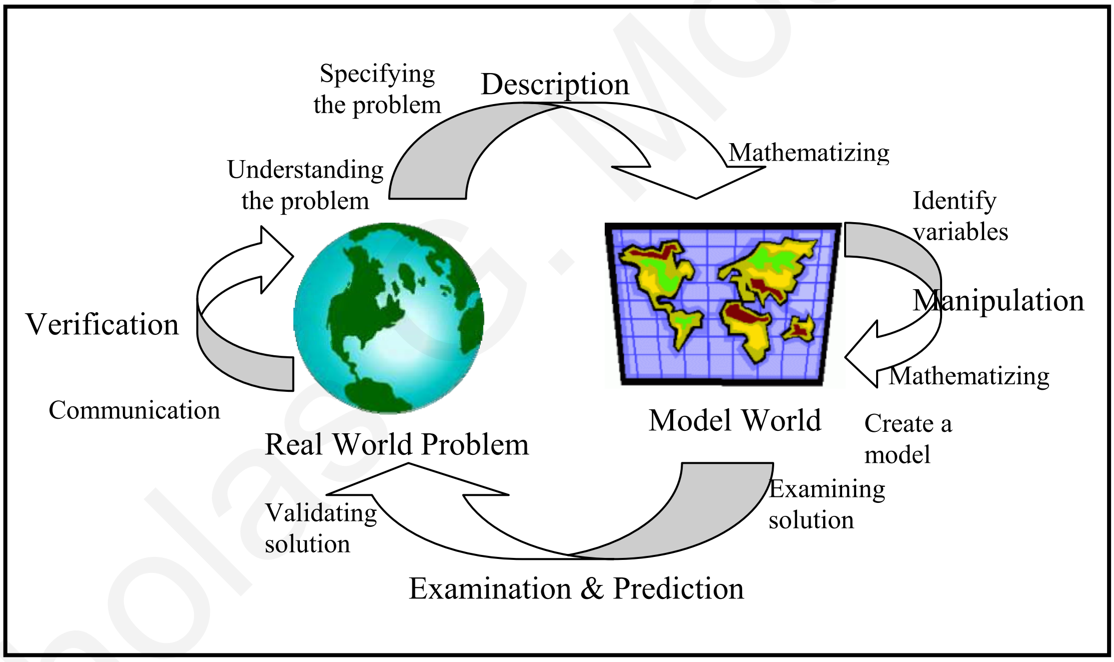


Figure 4.2. The Analytical Framework.

The analysis of the next five studies that follows used the analytical framework as a guideline for analyzing the modeling processes and students' mathematical developments. This analysis resulted in the development of the explanatory framework. In transforming the analytical into an explanatory framework, what is important to identify is how and when the elements of the analytical framework occur in the modeling processes and conceptual understandings of students in different modeling activities and if and to what extent these modeling processes are different between elementary and secondary school students.

Sixth and Eighth Grade Student Work in the Modeling Activities

Introduction

As presented earlier, the next step in the process of the grounded theory approach is focused primarily on drawing the connection between "constructing a map" (analytical framework) and "building a theory" (Miles & Huberman, 1994, p.91). Based on the outline of the analytical framework laid out, students' work in five case studies is presented. The analysed case studies of the modeling activities are summarised in Table 4.1.

All data were looked at through the step of the modeling procedure that constituted the analytical framework presented in Figure 4.2. In fact, given the large amount of data analysed, only selected extracts from different transcripts are discussed. The chosen extracts: (a) represent clear cases of student modeling processes and mathematical understandings and (b) could be compared across cases, letting similarities and differences emerge.

Table 4.1

The Case Studies Selected for the Analysis for Developing the Explanatory Framework

Case Study	**Modeling Activity**	**Student Grade**
Case 2	Best Drug Award	8^{th} grade
Case 3	Where to Live	6^{th} grade
Case 4	Where to Live	8^{th} grade
Case 5	The University Cafeteria	6^{th} grade
Case 6	The University Cafeteria	8^{th} grade

Best Drug Award Modeling Activity

The second case study presents students' work from one 8^{th} grade class in the "Best Drug Award" modeling activity. The extracts presented in this session concern students' work with respect to the modeling processes and students' mathematical understandings. The analysis of these extracts focus on the following modeling processes: understanding and specifying problem, simplifying problem, identifying variables and relationships, building a model, interpreting and evaluating solution and validating and communicating the results. A summary of students' mathematical developments follows the presentation of the modeling processes appeared in students' work.

Modeling Processes

Describing and understanding the problem. At the beginning of the problem exploration students tried to make sense of what they had to do, but they did not discuss the meaning of the core question of the problem. It seemed that it was clear for them that the most effective drug was the one with the smallest reaction time.

Mary: I don’t understand the table very well. Does it say if they are the same persons who took all the different drugs?

Chris: These are different drugs. For example, the reaction time for this (points to Sacacetamol) is 10 minutes.

Mary: Have all the persons tried these medicines at the same day?

George: No. Different persons took each drug.

Chris: I think that this is related to a person’s organism. For example, is Panatol always reacting in 2 minutes? Well, what do we need to find out?

Mary: The most effective drug.

In further exploring the problem, students tried to perceive different information about the problem than was originally apparent. Additionally, they asked the teacher for clarifications. Upon agreeing that each person tried all four medicines in different days, they reported that they were ready to find the most effective drug. After these clarifications, students easily identified the different conditions and assumptions, needed to complete the activity. Students’ successful work in finding the necessary conditions and assumptions includes moving from the original situation to the particular (real-world) problem that is to be solved.

Problem Manipulation. In their attempts to mathematize the problem, students linked the real world problem with the mathematical structure they needed to develop to solve the problem. In this case, the three students introduced a first mathematical idea that was appropriate in finding a solution to their problem. This first solution was based on finding the sum of reaction times for each drug.

Chris: I think that we must find each column’s sum.

Mary: I had in my mind the same thing!

Chris: By finding the sum for each column we will find the worst medicine.

George: I agree.

While working with the activity and finding the sum for each column (the sum for each drug), none of the three students mentioned the possibility to use average. In this case, the duration and nature of mathematizing was quite limited. Of course, a possible reason was that the specific problem was not so demanding and students easily reached an appropriate method for finding a solution. Students spent a considerable amount of time to clarify many things related to the concept of *average* and to discuss how average was related to a drug's effectiveness. In the following extract, students are discussing their results, after finding the sum of reaction times for each drug.

George: Well, the drug with the smallest sum is the most effective.

Chris: That's right… So, the most effective medicine is Caracedamol.

Mary: We find each column's sum to find the most effective medicine.

Researcher: Which medicine have you found to be the most effective?

George: Saracetamol.

The problem presented in the "Best Drug Award" modeling activity did not foster the development of novel ideas or evoke new mathematical knowledge. However, it is important to point out that although all students were familiar with the concept of average, none of the students suggested that they could use average in finding the most effective drug. In the first attempts to formulate their mathematical elements into a coherent model, students tried to combine the mathematical properties into a single model. Quite important is students' selection of mathematical concepts to represent the essential features of the realistic model and blending these selected elements into a coherent mathematical entity. The developed model was based on finding the sum of reaction times for each drug. Although all students agreed upon this model, they discussed their results; how "more effective" was the first drug than the second one was the core topic of their discussion.

Researcher: What does the difference between the sums of reaction times actually stands for?

George: You mean if we compare these two numbers (points at two sums)?

Researcher: Yes.

George: It is the difference between the two drugs … the minutes...

Mary: It is the drugs' difference in minutes.

Researcher: Let's see…the difference between Saracetamol and Ralpol is 15 minutes. What this means?

Chris: I think that this difference is for all cases. It does not say something else. I see your point … this difference can not tell us anything about the difference in these two drugs effectiveness.

The above discussion helped students to realize that they had to find an alternative model for solving the problem. The process for building a model was not easy; a number of iterative modeling cycles was necessary for developing a model. Chris suggested using the average for finding the most effective drug. In his words, "if we find the average we can say that the drug with the smallest average is the most effective one. Well, it will again be Saracetamol". It can be argued that documenting and validating that two or even more models are equivalent was a demanding task for students.

Connecting with the real-world situation: Validating and communicating results. The next step in students' work was to connect their results (models) with the actual real problem presented in the activity. Students validated their results through constant comparisons with the actual problem and they communicated their findings. A second modeling process was examining their results. Of course, as it could be seen in the extracts presented earlier, all three students were confident that Saracetamol was the most effective drug. On the contrary, none of them could provide a reasonable explanation on what the differences between the sums or reaction times meant? Students' choice to use the concept of average in finding the most effective drug resulted in new "differences" between the first and the second most effective drug. This result started a new productive and fruitful discussion, part of which is presented in the following extract.

Mary: The difference between the first and the second drug is not the same like it was before.

George: Yes. The new difference is smaller now. It is only 0,75 … less than one minute.

Chris: Look over here (points to Kanatol)… Ralpol has 13,75 and Kanatol 13,85. They are almost the same. Their difference is only 0,10.

Researcher: Ok. Let's see…in terms of their effectiveness, what does it mean that most effective drug's average is 13 and the second's average is 13,75?

Chris: The first is more effective (laughing). Well their difference is less than one minute. Of course, all differences are small. Even the last one's average is 14 minutes.

George: One minute is quite important. Especially if you have a terrible headache!

Chris: Right. Compared to 13 minutes, one minute is important. If average was 100 then one minute would not be so important.

Students compared their results with the real world problem and they tried to validate their conclusion. They also considered the modeling purpose to ensure that the real world conclusion aligned with the realistic situation in light of the modeling goal. In completing their work, students worked on communicating their results. A very important issue, considering the appropriateness of their model, was raised. Specifically, during writing their letters, Mary wrote that their solution could be also used in other situations, like in finding the best cake in a baker competition. She also worked one step forward, by explaining that: "if the owner of a bakery wants to find out the tastiest among a number of different cakes, he has to offer these cakes to twenty customers and ask them to categorize the cakes from first to last". George added that Mary's solution could be used for more than four cakes and of course, the bakery's manager could ask as many customers as he would like to. Chris agreed with him. In fact, he clarified that the result would be more valid, if more customers can participate in the competition.

It is important to mention that all students in their letters attempted to predict the behavior not only for the specific real problem but also the behavior of structurally similar problems, such as looking for the best cake or on finding the fastest racing car (Chris' suggestion in his letter). Writing letters to the president of the drug association (imaginary client) helped students in finding better ways to communicate their results through sharing their results and models.

Student Mathematical Developments

The results of this activity, with reference to the student mathematical developments are summarized below. Stages are descriptive of the mathematical understandings students presented as they worked on the “Best Drug Award” modeling activity. Students’ mathematical developments are presented in two stages. The first stage presented in 6th graders’ work, namely “focusing on subsets of information”, is not presented here, since 8th graders started their work by adopting the “sums model”. Following, in the next stage students discussed the differences between using the two models (sums and average), compared different solutions (using average or sums) and discussed the meaning of their results.

Stage 1: Using mathematical operations and processes. A central characteristic of student work was the quick adaptation of a mathematical model for finding a solution to the problem. All three students agreed upon using the “sums model” for finding the most effective drug. None of the students questioned the appropriateness of this solution. Only after researcher’s question on the appropriateness of their approach, students started to examine the possibility of the existence of a better approach. In the next extract, students are explaining why finding the sum for each drug was a good strategy for solving the problem.

Chris: Let’s add all numbers for each drug.

Researcher: Why do you think this is the best method?

Chris: Now we have twenty different numbers for each drug. If we add them all, then we will find one number for each drug. Comparing these numbers will tell us the best drug.

Researcher: Do you think you could solve the problem using another approach?

George: Why? Is this wrong?

Researcher: No. I did not say this. What I am saying is that there might be a better way to solve the problem. I am not sure either.

Chris: Our idea is very good!

Quite surprisingly students did not even mention that average could also be used to solve the problem. Only after the discussion on the sums' difference, students started thinking of the average as an alternative approach. Students easily employed the new approach and found the averages for the four drugs. Since all of them were familiar with the concept of average, they were not surprised that using the average resulted in the same drugs' ranking. The discussion of the similarities and differences between the two approaches and what the differences between averages meant, helped students in clearly presenting their ideas. In many parts of this discussion and their work, students showed significant mathematical thinking. These findings are presented in the next stage.

Stage 2: Advanced mathematical thinking. Students' discussion was quite impressive. In discussing their findings, they realized that differences between the averages could provide more information about the drugs. Additionally, students discussed issues like percentages, when they were trying to understand the importance of the one minute difference. The extract below presents a snapshot of this discussion, highlighting the important mathematical developments students presented.

Chris: Compared to 13 minutes, one minute is important. If average was 100 then one minute would not be so important. […] I agree that six seconds are not so important here (he is referring to the difference between 13,75 and 13,85).

George: Correct! One minute is about 10% … a little bit less. This is quite big.

Chris: Exactly. When we started I was surprised with the big differences. Now … with averages … it's ok. If we had more cases then the difference would be even smaller.

Researcher: Why? What would change if you had 100 cases?

Mary: We would divide their sums by 100. So…this difference (points at the difference between 260 and 275) would change to …er…

George: 0,15

Chris: This is not right. We do not know if the difference in sums would be 15 if we had 100 people.

Researcher: So, what can we say about the average?

Chris: I believe it will be the same like the one we found.

It can be observed that students' fruitful discussion helped them not only in mastering important mathematical concepts and processes but also in applying them effectively in solving a real world problem. Although, all three students were taught before concepts like average, they were eager to understand when and how average could be used and what the results actually meant. The latter is very important for the mathematics class.

Summary

Both 6th and 8th grade students easily understood the core question of the "Best Drug Award" activity and they successfully solved the problem. Both groups raised a number of questions related to the context of the problem and they documented a number of necessary assumptions for simplifying the real problem. Sixth graders spent more time on alternative models, which were focused partially on the available data. On the contrary, 8th graders used their prior mathematical knowledge related to the concept of average to find a model for solving the problem. A second difference between the two groups in favour of the 6th graders was that 6th grade students discussed extensively the meaning of the differences between the drug's averages. Students documented that when the difference between two drugs' averages was less than 0,1, then there was no difference between the drugs' effectiveness. Students in both groups explicitly documented their results in their letters to the president of the drug association. Students from both groups also extended their findings by documenting that their method (average model) could be used for other structurally similar problems.

The problem presented in the "Best Drug Award" modeling activity appeared to be interesting and easy accessible to students. As a result, all students reached an appropriate solution by linking their prior mathematical knowledge to the context of the problem. Students' solutions were quite impressive, considering that none of the students had prior experiences in working with modeling activities.

The "Where to Live" Modeling Activity

The work of the same two groups of students which worked in the "Best Drug Award" activity is presented here. The two groups of students worked on the "Where to Live" modeling activity which was also related to statistical reasoning. The third case study presents the story of the 6th grade students working on the "Where to Live" activity. The work of the 8th grade students as they worked on the same activity is presented in the fourth case study. The analysis of two groups' work is focused on the two dimensions that guided the analysis of the second case study. These two dimensions are: (a) modeling processes that appear in students' work and (b) students' mathematical developments.

Modeling Processes in 6th Grade Students' Work

Understanding and working with the real world situation. Students spent a considerable amount of time reading loudly the problem, focusing on pieces of information for each city and trying to figure out what the numbers in the table stood for. As students tried to get a better understanding of the problem, they made a number of different comparisons between couples of cities.

Helen: Limnoupoli has 2 parks, 2 nursery schools, 7 schools, 1 cinema, 3 restaurants and 23 shops. The quality of its streets is 45.5%. Iremisia has 3 parks, 1 nursery school, 4 schools, 3 cinemas…

Alice: What is the budget for next year?

Helen: Paramithoupoli neither has parks nor cinemas. I think that Paramithoupoli is the worst place for Anastasia.

Alex: Limnoupoli has 2 parks and Relaxcity has 3 parks.

Helen: I think Limnoupoli is a good place for Anastasia.

Alex: Limnoupoli has 2 parks. I think it is a quite big city. But, look…Its streets are not so good. I am not sure it is the best place for her. Asfalisia is much better than Limnoupoli. I would prefer it.

As it is shown in the above extract students made a number of comparisons between the different cities. While making these comparisons, they realised that the meaning of "budget" was not clear enough and they asked necessary clarifications from the researcher. Researcher provided the example of Nicosia and discussed with the three students the different possible meanings and importance of the budget for a city.

Researcher: Let's think of Nicosia's budget. Let's assume that this year's budget was £3M. If next year's budget will be £4M, then we can say that Nicosia's budget will be increased. Alternatively, if next year's budget will be £2,5M then this column (points to the table) will show "Less".

Alex: I see… but… these two cities (points to Asfalisia and Kifisos) have the same budget?

Researcher: What do you think?

Alex: I do not know.

Helen: No. "Less" means that both cities' budget has decreased. We do not know the budget for each one.

The above extract shows exactly why "exploring the problem" is not just a simple statement of the problem. On the contrary, in exploring the problem students tried to perceive different information about the problem than was originally apparent. The clarifications made on the budget started new explorations of the problem, since students started *observing mathematically* the provided situational information. Another example is observing the possible relationships between the different factors of the table. More specifically, when Alice commented that: "Paramithoupoli should be rejected, because its streets are so bad and it has only few shops". Alex distinguished between city's facilities and city's budget: "Paramithoupoli's budget will be increased next year. That means that there will be money for improving roads and even money for building parks and cinemas".

Another characteristic of student work within the real world situation is related to the identification of the conditions and assumptions of the real world context. In the present problem, students stated explicitly that the table did not provide all necessary information about the size of the different cities. Students assumed that all cities were of equal size. This was necessary, since according to Alex: “we should find new numbers, using proportionality”. When the researcher asked for clarifications, Alex presented the following example. He pointed out that although Paramithoupoli had ten schools and Asfalisia only five schools, if the size of Paramithoupoli was two times the size of Asfalisia, then the two cities would have the same number of students for each school. As a concluding point, it can be asserted that students managed to simplify the situation, moving from the original situation to the particular (real-world) problem that is to be solved.

Linking real-world situation with mathematical entity and working with the mathematical entity. Students found the “Where to Live” activity interesting. Their first impression was that the problem was (in student words) nice, interesting, and challenging. Soon, students realised that it was not easy to find a solution of the problem. As soon as students explored the problem and identified the necessary conditions and assumptions, they started looking for an effective way to construct a model. As it could be seen in the extract presented in the previous session, students spent a considerable amount of time discussing the differences between pairs of cities, focusing on one factor (e.g., schools, quality of roads, cinemas etc). When realizing that such strategies were not sufficient, students focused their efforts towards more mathematized strategies.

Students’ next step was to introduce their mathematical ideas that eventually related to the mathematical entity. In other words, students started mathematizing the problem; they used their prior findings to make a connection between the actual problem and the mathematical structure of the problem. In the following extract students discuss the appropriateness of finding the sum of buildings for each city.

Alex: Let’s sum the buildings for each city. This is a way to find which city is the biggest and the best one.

[Students created another column, finding the total for parks, nursery schools, schools, cinemas, restaurants and shops. They did not take into account the quality of the roads and budget].

Helen: Look. Asfalisia has 45. The second one has…40…Fantanasia.

Alice: Kifisos has also 40. Asfalisia looks the best one.

Alex: I am not sure about it. Kifisos has 20 shops. That's why Kefisos has 45. Fantanasia is the best place for Anastasia. It has forty buildings, good streets and a higher budget for next year. Kifisos' budget will be decreased and look at its roads. Quality is only 25%. By the way, Fantanasia has the chance to become even better because its budget will be increased next year.

Helen: You are right. So, what do you think we should do?

Alex: I am not sure. I feel like Fantanasia is the best city, but we need to find a way to *prove* it.

What is interesting to acknowledge is the fact that students' model (finding the sum of buildings for each city) was not "good enough" to convince (even) themselves about the appropriateness of their solution. This puzzled the students and they also started a new round of discussion searching for an alternative way to solve the problem. Quite interesting was the fact that two students recalled their primary solutions in a preceding activity, namely the "Best Drug Award" modeling activity. Helen suggested that they could select the best and the worst city in each column. The second step would be to assign +1 if a city had the biggest number in that factor and assign -1 if a city had the smallest number of buildings in that factor. All students felt quite satisfied when they realized that Fantanasia (their first choice) was again the best city. When the researcher prompted them to think about alternative solutions, all three students replied that they were convinced that their solution was good enough. According to Alice, they "proved that Fantanasia was the best city for Anastasia".

The above extracts showed that students did not consider other forms of mathematizing the situation, like finding averages for each factor, ranking factors by their importance, weighting factors etc. However, the duration and nature of mathematizing was not a rapid or implicit event. It is important to underline how the modeling activity fostered the development of students' novel ideas. These ideas were not only mathematical concepts or processes, such as average, but also more powerful ideas such as the documentation for the appropriateness of the different solutions. In students' words, it was difficult to "prove" that their solution was the best one. In other words, it was interesting that students faced the need to document in mathematical terms their solutions, so they could convince Anastasia about their (students') choice.

Connecting with the real-world situation. In connecting the mathematical entity (model) with the real world situation, students use their models to interpret their solutions and to make predictions of the behavior of the real world problem. A significant modeling process that appeared in students' work was making the connections between the mathematical entity and the real problem. These connections resulted to conclusions for the real problem. An instance of interpretation occurred as students discussed the total number of buildings in each city.

Alice: Fantanasia and Kifisos both have forty buildings.

Helen: Asfalisia has 45.

Alice: Asfalisia has five more buildings than Fantanasia. Asfalisia seems to be the best one.

Alex: I am not sure about it. Five buildings in total is not a big number. Is Asfalisia a better place than Fantanasia because it has five more buildings? Asfalisia has 26 shops. Is it important to have 26 shops?

The next extract also highlightes the process of interpretation, as students were discussing the road quality and how the road quality affects their choice.

Alice: Fantanasia is the best place for Anastasia. It has good roads … the road quality is 76,2%. Look… it is the only city with good roads. The next one is Asfalisia with 57,2%. All other cities are below 50%. Look at Paramithoupoli' roads. Quality is only 19,7%. You can not drive a car in Paramithoupoli!

Helen: You are right. It is so difficult to live in a city if the roads are not good. This factor is very important.

Alice: Well, I am not sure. Of course Paramithoupoli and Kifisos have very bad roads. But, is there a big difference between Limnoupoli's and Iremisia's road quality?

Alex: No. The road quality for both cities is quite close. We can say that these two cities are equal in this factor.

In the following extract students examine the appropriateness of their solution. It is important here to recall students' final model which was based on adding and subtracting points. For example, students added one point to Paramithoupoli, since there were more nursery schools in Paramithoupoli than in every other city (five nursery schools). Students also subtracted one point from Paramithoupoli, since there were no parks at all in this city.

Alex: We added one point to Iremisia, Paramithoupoli and Fantanasia. They will have increased budget next year. Of course, we do not know if all three increases will be equal.

Alice: Does it make any difference?

Alex: Yes. If for example Iremisia's budget increase is more than other cities, then we should add one point only to Iremisia.

Helen: You are right. But it is not only this. We also considered all buildings being of the same importance. I am not sure if this was correct. Are schools as important as shops? I do not think so.

Researcher: What's your opinion Alice?

Alice: Schools are more important.

Alex: Well, I do not agree. Schools are more important if you have children. Anastasia has just finished university. She probably does not have children. So, she might be more interested in shops and cinemas than in schools. If she was married and had children, we should consider nursery schools and schools being more important. Maybe it would be better if we added two points for cinemas, restaurants and shops and one for the other buildings.

Helen: No. I do not think this is a good idea. We do not know if Anastasia likes going to cinemas and restaurants. I believe that we keep it like it is … one point for each one [factor].

Verification and communication. During the last step in the modeling procedure, a number of different modeling processes occurred. These processes, which are presented in detail below are checking and evaluating the solution, validating and communicating the results. The next extract focused on students' attempts to check whether their solution could provide an acceptable answer for the question of the real problem.

Helen: Fantanasia is the best place for Anastasia. It is the best city.

Alice: Exactly. It has 76,2% road quality, its budget will be increased next year and there are forty buildings in Fantanasia.

Alex: You are right. Well … let's go back in the table to see again all cities.

Alice: Why?

Alex: Well, Fantanasia has four parks. All other cities have three or less parks. Paramithoupoli has no parks at all. Fantanasia's road quality is the best one. Look over here (points to road quality). The second city's road quality is only 57,2%.

Helen: Correct! But Fantanasia is not the first in nursery schools. It has three nursery schools, while Paramithoupoli and Asfalisia have five and four respectively. Of course, Fantanasia is not the worst one. It is also the third city in schools and the second in cinemas. Finally, Fantanasia is the best or among the best three cities for all factors.

Alice: Fantanasia has also eight restaurants, but only fifteen shops. Well, it would be better if Fantanasia had more shops.

Alex: Well, I believe we all agree. Fantanasia is the best city for Anastasia.

As discussed earlier in this chapter, communicating includes using the solution to predict the behavior not only of the specific problem, but also of structurally similar problems. Communicating also refers to deciding on the appropriateness of the solution. In the present activity, students expressed their solution verbally when they wrote their letters to Anastasia. Quite interestingly, Alice also copied a part of the table in her letter, in her attempt to explain to Anastasia why Fantanasia was the best city.

The results of the "Where to live" activity are presented as follows: First, consideration is given to a "microlevel analysis" of the developments displayed by students in working with the modeling activity. Following, this fine-grained microlevel analysis, a "macrolevel analysis" of the mathematization processes displayed by students is presented, to obtain the mathematical constructs students developed during their work in the modeling activity.

Microlevel analysis: Identifying and clarifying factors. Students commenced the question for finding the best place Anastasia could move on, by brainstorming on the factors presented in the provided tables, questioning the meaning and importance of these factors. Helen pointed that parks and cinemas are important: "Paramithoupoli neither has parks nor cinemas. I think that Paramithoupoli is the worst place for Anastasia". At this time, all three students did not discuss at all the meaning of increased or decreased next year budget. Additionally, students did not discuss how the budget factor was related to other factors. At this stage, students only made statements like: "No cinemas, no parks, few shops and bad roads in Paramithoupoli". At a later stage, a long discussion between the students and the researcher took place. The discussion focused on the meaning of the next year's budget. As mentioned above, the meaning of "budget" was not clear enough and the discussion gave students a clearer view of its meaning.

Beginning mathematization. After identifying and clarifying the different factors, students moved into mathematizing the problem. At the beginning, students' work focused on comparing two cities at a time. In doing so, students compared one factor every time to find out which city was "better" than the other one. "Paramithoupoli has more restaurants than Asfalisia. Paramithoupoli's streets are on the contrary worse than Asfalisia's". Students experienced difficulties in their efforts to work with factors such as next year's budget and the quality of roads. It was also difficult to combine these factors with other factors. Alex reported in his worksheet: "We added the buildings in each city. Budget is an important factor. We decided to keep this factor by itself, since we could not add it with buildings and roads". Similarly, Helen and Alice reported that: "Paramithoupoli has bad road quality and Fantanasia has good road quality".

Macrolevel analysis of mathematization processes: Categorizing and merging factors. This level of analysis is primarily focus on the mathematization processes projected by students, during their work in the modeling activity. Students categorized factors as either related to buildings (schools, restaurants, parks) and city's next year budget. An interesting strategy suggested by Alex and adopted by the other two students was assigning +1 if a city had the biggest number of buildings in a specific factor and assigning -1 if a city had the smallest number of buildings in the same factor. Students created a new column in the table for displaying their results.

Aggregating and ranking factors. In an attempt to rank the different factors, students only used a qualitative ranking; they considered some factors being more important than others, but that distinction was done only in a qualitative way. For example, Helen wrote: "Fantanasia is a better place than Paramithoupoli. Fantanasia might have fewer nursery schools, schools and shops than Paramithoupoli, but these things are not so important. Budget is important; Fantanasia's budget will increase and Paramithoupoli's budget will decrease".

Modeling Processes in 8th Grade Students' Work

The modeling processes and mathematical develoments appeared in 8th graders work are presented here. The extracts presented concern students' work with respect to the modeling processes appeared in working with the modeling activity. The analysis focus on the following modeling processes: Understanding and working with the real world situation, linking real-world situation with mathematical entity, working with the mathematical entity, building a model, connecting with the real-world situation and verification and communication of the results.

Understanding and working with the real world situation. Students spent about five minutes reading loudly parts of the problem, trying to better understand the question of the "best place". It is important to state here that one student started making quick mental calculations and she proposed almost immediately that they could exclude one city (the worst) and start working with the other cities. Of interest was another student's comment on Anastasia's studies. Chris' comment on the possible connection between Anastasia's studies and her choice of a place to live in is presented in the next extract.

Mary: Look on the first page. Which is the best place for Anastasia? I think that Paramithoupoli is the worst place for her.

Chris: It would be useful if we knew what Anastasia studied at the university.

Mary: Paramithoupoli is the worst place for Anastasia.

Chris: Wait a second. How do you know that? You have not yet seen the data on the table.

Mary: I did. Look (points to the table). There is nothing in there (she is referring to Paramithoupoli). That's why I said that it is the worst place.

Chris: It is not easy to say. We do not know what she studied at the university. What if she is a teacher? She needs a place with many schools.

George: What are these numbers?

Mary: Read the instructions. This column here is parks. There are two parks in this city.

Similar to 6th graders' work, students raised questions on the meaning of budget. The meaning of "budget" was obviously not clear enough, so students asked the researcher to make the necessary clarifications. Researcher provided them with an example for the city of Nicosia and discussed with the three students the different possible meanings and importance of city's budget. Quite important was students' reflective feedback on the meaning of budget and how this factor might be related with the other columns in the table (of course, this dimension will extensively be presented later, when students started the mathematization processes).

George: Does this refer on the level of losses or profits that each place has?

Researcher: No. It is not the balance of income and outcome. I will give you an example of a city's budget. Budget is the amount of money that each city will spend during a year.

Mary: What does this include?

Researcher: Personnel salaries, funds for improving facilities, schools, etc. If Nicosia's this year budget is £3M and next year's budget will be £4M, then we can say that Nicosia's budget has increased.

George: Asfalisia's budget will be decreased.

Mary: That's right. I think this exercise is more difficult than the previous one. It is complicated [laughing].

Chris: Budget is an important parameter of our work. But…

Researcher: What? Do you need any additional help?

Chris: If two cities have "more" in this column (refers to budget) then they will spend the same amount of money?

Researcher: Not necessarily.

Just like in presenting 6th grade students' work, it is clear that "exploring and understanding the problem" was not just a simple statement of the problem. Students felt that could not solve the problem, even after discussing in their group and with the researcher the meaning of the different factors. However, all three students were able to understand the core question of the problem and they could start working on it. Mary commented: "Let's start. If we need anything else, we can get back to the teacher".

Linking real-world situation with mathematical entity and working with the mathematical entity. Of importance is students' work towards the identification of the conditions and assumptions of the real world context of the activity. After Chris' comment on comparing the budget across the table, students claimed that the table did not provide all necessary information about the amount of the budget and the size of each city. Quite interestingly, they assumed that all cities were of equal size. Students chose to simplify the situation, moving from the original situation to the particular (real-world) problem. The latter, is of great interest, since it is among the core elements of mathematical modeling in general and modeling as a problem solving in particular.

The next step in students' work was to start observing mathematically the problem. As commented before, Chris documented that the budget factor was of great importance, since it could influence other factors of the table. Similarly to 6th grade students' work in mathematizing the problem, students acknowledged that budget is a very important factor; even though a city (they specifically discussed on Paramithoupoli's case) was the worst for a number of factors, this could change if city's authorities use increased budget to build new schools and parks. These clarifications and assumptions created a good starting point for students to begin mathematization. In the following extract, the students discuss possible ideas on narrowing the number of columns and on ranking the importance of each column/factor.

Mary: Let's find the total number of buildings for each city.

George: Which city has the bigger number of buildings?

Chris: There is a problem here. Limnoupoli is first because it has 23 shops. But I think that having 23 shops is meaningless. What I mean is that this sum depends mainly on the existence of so many shops. We can not select Limnoupoli as the best city Anastasia could live in, only because this city has many shops.

George: Anyway, let's sum the buildings in each place.

It is interesting to observe that students' initial attempt to model the problem (finding the sum of buildings for each city) was not "good enough" (at least for one student). Chris' concerns on the appropriateness of this model encouraged students to think of alternate, more appropriate solutions that could convince them. Among the ideas proposed was finding each factor's average and then categorize the cities in two columns, those above average and those below average. Chris suggested assigning +1 if a city was above the average and assigning -1 if a city was below average. This approach resulted in a more comprehensive and accepted model. However, the budget factor was excluded from the model. The results from using the average model are presented in Table 4.2 and Table 4.3.

Table 4.2

The Table of the Six Cities and the Average for Each Factor

	Parks	Nursery Schools	Schools	Cinemas	Restaurants	Shops	Road quality (%)	Next year budget
Limnoupoli	2	2	7	1	3	23	45.5	Same
Iremisia	3	1	4	3	12	16	36.8	More
Asfalisia	2	4	5	4	4	26	57.2	Less
Paramithoupoli	0	5	10	0	6	12	19.7	More
Kifisos	3	2	8	2	5	20	25.8	Less
Fanstanasia	4	3	7	3	8	15	76.2	More
Average	2,3	2,8	6,8	2,2	6,3	18,7	43,4	N/A

Table 4.3

The Points Received by Each City

	Parks	Nursery Schools	Schools	Cinemas	Restaurants	Shops	Road quality (%)	Total
Limnoupoli	-1	-1	1	-1	-1	1	1	**-1**
Iremisia	1	-1	-1	1	1	-1	-1	**-1**
Asfalisia	-1	1	-1	1	-1	1	1	**1**
Paramithoupoli	-1	1	1	-1	-1	-1	-1	**-3**
Kifisos	1	-1	1	-1	-1	1	-1	**-1**
Fanstanasia	1	1	1	1	1	-1	1	**5**

All three students were satisfied with this result. Even better, Fantanasia (their first choice) was the first city with five points. The second city, Asfalisia, had only one point! In the next extract, students are discussing their results.

Mary: Fantanasia is the best city. It is much better than every other city. Let's write Anastasia's letter now!

George: I told you that Paramithoupoli is the worst city. It has a total of -3.

Chris: Well, I am not sure George. Paramithoupoli has a total of -3 but will have an increased budget. We can not be sure that after this Paramithoupoli will still be the worst one.

Mary: Ok. But we are sure that Fantanasia is the best.

Chris: Yes, of course. It has a total of five and increased budget. No matter the importance of the budget, it will still be the best place for Anastasia.

Students were satisfied with their solution and did not consider any other forms of mathematizing the situation, like ranking factors by their importance (except budget), weighting factors etc. Of importance, was Chris' comment about Anastasia's studies. In the following extract, students are discussing with the researcher possible implications of Anastasia's studies in their selection of the best city.

Chris: If she is a teacher, it would be better to live in a place with many schools.

George: And she will not need to move in another city to work.

Researcher: You do not know what Anastasia studied at the university. But even she were a teacher, would you recommend Paramithoupoli, which has ten schools and not Fantanasia?

Chris: No. Fantanasia would still be my choice. See, there are seven schools there.

Mary: It would only be a problem if she is a teacher and Fantanasia had no schools at all or only one or two.

Researcher: Ok. Let's move on. Do you think that her studies could influence her choice … I mean your recommendation to her?

Mary: No. I still believe that Fantanasia is the best place for her.

George: I agree.

As a concluding point, it can be claimed that the modeling activity fostered the development of students' mathematization processes. Students not only developed sophisticated mathematization processes but as it can be seen later they tried to connect their findings with the real situation.

Connecting with the real-world situation. In connecting the mathematical model with the real world problem, which was Anastasia's choice of residence, students used their models to interpret their solutions and to make predictions of the behavior of the real problem. Similarly, as discussed in case study three, a significant modeling process is the interpretation of the solution. The extracts presented earlier, showed that students discussed the importance of shops and whether this factor was of less importance. In the following extract, students discuss the importance of different factors.

Chris: I think there is a problem. For example Limnoupoli has the higher sum because it has 23 shops. I think that having 23 shops is meaningless […] We can not select Limnoupoli […] only because this city has a big number of buildings, since this number is depending so much on the number of shops.

Chris: I still believe it is not correct to recommend for example Limnoupoli, because it has 23 shops. There are cities which have more schools and restaurants, places more important than shops.

Mary: What I like about Fantanasia is that the place is good for each factor. It might not be the best place for example at schools. Paramithoupoli has ten schools and Fantanasia seven. It does not have 12 restaurants but eight restaurants are okay. You can live there and be happy!

The next extract also highlights the process of interpretation, as students discuss the quality of roads and how the quality affects the choice of the best city.

George: You can not drive a car in Paramithoupoli! [Laughing]

Mary: Why?

George: Road quality is only 17%. You can only move there using a tractor.

Chris: He is right. It is difficult to live in a city if the roads are not good. It is also not safe. I guess there are more car accidents in Paramithoupoli than in other cities.

Researcher: This is an interesting conclusion.

Chris: Thanks. Can I say something else?

Researcher: Sure. Go ahead.

Chris: It is the same like night clubs and pubs. If there are so many in one place, police reports say that the crime level is too high.

An interesting result is related to students' examination of the appropriateness of their solution; as soon as they constructed a model, they used it to come back and compare their results with the actual problem question. These iterative comparisons and examinations helped them refine their solution and improve their understandings of the problem. A representative example of examination was students' decision to reject the "sum model", since "it might be a correct mathematical solution, but it does not cover the real problem situation".

Verification and communication. During verification and communication a number of modeling processes occurred; checking and evaluating the solution, validating and communicating the results. Checking, refers to students' discussion on checking whether their solution can provide an acceptable answer for the question of the real problem. The following extract shows how students are checking and evaluating their solution.

George: Fantanasia is the best city and Paramithoupoli is the worst city.

Researcher: Are you sure about this?

George: Yes. That's what we found earlier.

Chris: Exactly. Our model shows that Fantanasia is not only the best city but it is also much better than every other city. It has 76,2% road quality, its budget will be increased next year. Additionally, Fantanasia is above average in all factors.

Mary: Except in shops.

Chris: Right. But as you can see it is overall the best city.

George: It would be better if Fantanasia had more shops.

In communicating their results, all three students clearly explained in their letter to Anastasia how they worked in calculating the average for each factor and assigning positive and negative points. Students used correct mathematical language to explain their solution. Quite interestingly, Chris commented in his letter that it would be better for him if he had more information about Anastasia's studies. However, he pointed out that his final choice would not have changed.

Eighth Grade Students' Mathematical Developments

In the analysis of 8th graders' mathematical developments, first, consideration is given to a "microlevel analysis" of the developments displayed by students in working with the modeling activity. Following, this fine-grained microlevel analysis, a "macrolevel analysis" of the mathematization processes displayed by students is presented, to obtain the mathematical constructs students developed during their work in the modeling activity.

Identifying and clarifying factors. The group commenced the question for finding the best place Anastasia could move on, by brainstorming on the factors presented in the provided Table 4.2, and by questioning the meaning and importance of these factors. There was a long debate among 8th grade students to clarify what was the meaning of the next year's budget factor. Additionally, students extensively discussed how budget's increase or decrease could influence other factors: "Look at Relaxcity. There are few parks and only one nursery school … this could change, since Relaxcity's next year budget will increase […] People there can use these money to improve city's facilities".

A long discussion between the members of the group questioned the representativeness of their ideas and solutions, related to the importance of certain factors. Mary pointed out that "having parks is it not important for me…having shops and cinemas are more important". The same girl highlighted that Anastasia was a college graduate and therefore "many schools and nursery schools are not as important for her as shops, restaurants and cinemas". In line with that, Chris asked for clarifications and additional information; according to him, if Anastasia's studies were related to education, then moving in a place with many schools would be important to her.

Beginning mathematization. In their first attempts, students presented ideas such as "adding horizontally the numbers for each city" and "finding the number of buildings and facilities in each city". In doing so, students made simple calculations and compare their results: "Let's sum the total number of buildings and facilities for each city". It can be quoted here that students attempted to take into consideration road quality and budget factors: "Asfalisia's budget will be decreased next year and look at its roads. The road quality is only 25%. It can not be improved, since they will not have more money to spend on it".

Since students were not satisfied with the results from their initial mathematization processes, they started thinking and applying more sophisticated approaches. They agreed upon calculating the average for each factor and then categorize cities in two categories; those above average and those below average. Quite important was their next step in working with this model. They decided to assign positive and negative numbers to each city. So, for example, parks' average was 2,3. As a result, Limnoupoli, Asfalisia and Paramithoupoli which had 2, 2 and 0 parks respectively, were below average and therefore they were assigned -1. On the other hand, Iremisia, Kifisos and Fantanasia were assigned +1, since these cities had three or more parks.

Of importance was students' decision to exclude next year's budget from their model. Of course, there were two "sites on this coin". On one hand, this was a weakness of their model. Luckily, the best city according to that model was Fantanasia, which also had an increased budget for the next year. Therefore, according to them, their choice was valid, even after considering the budget. On the other hand, excluding next year's budget can be considered as strength of students' model. Chris explicitly explained that next year's budget is not just a factor like the rest: "This factor is very important, since it can change other factors. Therefore, it would be correct to incorporate budget with other factors".

Summary

The “Where to Live” modeling activity was more difficult than the “Best Drug Award” activity. Both groups of students understood the core question of the real problem. However, all students had difficulties to describe and understand the problem, since they could not easily use the budget factor in constructing a model for solving the problem. Sixth and eighth grade students focused on the relations between the different factors but none of the students used these relations in constructing their models. Sixth graders aggregated the different factors and created new factors such as one factor for the total number of buildings and the total number of facilities for each city. Reducing the number of factors was an effective strategy for comparing the cities.

Sixth graders successfully used the notion of average to categorize cities. They later assigned positive and negative points to each category and therefore that reached a total ranking, using all provided data. This is significant, considering that the concept of average was used and discussed in the “Best Drug Award” activity and students managed to successfully used average in the “Where to Live” activity. A difference between the two groups was the fact that 8^{th} graders had longer and more fruitful discussions on their solution’s verification in the context of the real problem. In communicating their results, both groups of students explicitly documented their constructed models in the letters they wrote to their imaginary client and tried to convince her that Fantanasia was far the best city she could move on.

The “University Cafeteria” Activity

The “University Cafeteria” was the third modeling activity in which students from both grades worked. The first part of the analysis of this activity focused on the modeling processes and mathematical developments of the 6^{th} graders’ work and the second part focused on the modeling processes and mathematical developments of the 8^{th} grade students’ work.

Modeling Processes in 6th Grade Students' Work

Understanding and working with the real world situation. Students read the problem and started discussing it. Although it was clear that they understood what the problem was asking for, they spent a few minutes looking at the two tables and making comments for the different vendors. When they decided to move ahead, they started a round of comparing couples of vendors or finding the best vendor for each column. However, one of the students used her calculator to add up the different numbers in a column. A number of interesting extracts from students' initial work are presented below.

Helen: We have to find the best vendors. Nicholas will hire these vendors next year.

Alice: He will hire six vendors.

Helen: No … Yes, he will hire six vendors, but not at the same status. Three vendors will work full time and three will work half time.

Alex: There are so many numbers. This table (points to the Hours Worked table) shows the number of hours each vendor worked. The second table shows how much money each of the vendors earned.

Helen: Look over here … the table is not simple. These columns are the same (referring to busy columns for the different semesters and summer period).

Alex: No, they are not the same. Look … the headings are not the same.

Helen: What is the difference?

Alex: I am not sure.

[Alice was adding the numbers in the first table using her calculator.]

Helen: What are you doing?

Alice: I am adding up the numbers. I found 116.

Helen: It would be better to do it in Excel.

Alex: What did you find? Let me see.

Alice: I got the total for the first column labelled “busy”.

Alex: Well, no … why did you do this? It’s wrong. You found the hours all vendors worked. How can you use this?

Alice: I do not know yet.

Group’s work focused on exploring “what they should do” rather than clarifying “what the information in the two tables meant”. Therefore, their first interpretations focused on computations (see Alice’s work) and the information that was given was treated as too complex (see their comments in the complexity of the table and *why* columns had the same headings). However, it is important to point out that all three students explicitly clarified the information presented in the two tables. Specifically, from the beginning students clearly understood that the first table referred to the hours each vendor worked and the second table referred to the money each vendor earned. In their following discussions, students tended to focus only on numbers and on table headings (Hours or Money) and they tended to ignore column labels (busy/steady/slow and different semesters).

Students’ next attempts focused on making simple comparisons between couples of vendors. These comparisons were of little importance and as a result they could not help students to reach their goal (or even to better understand the problem). Students focused only on data from one of the two tables and therefore, their results did not actually show anything useful. The following extract presents part of their discussion.

Alex: Jose and Chad both worked 19,5 hours. Wow, Robin worked many more. She worked 26,5 hours. She is first.

Alice: And Tony worked only 7,5 hours … and Kim worked 5,5.

Helen: Well, Willy did not work at all. She has a zero.

Alex: Maybe she was ill or she had a vacation.

Helen: Robin is rich (laughing). She got 2253 pounds … and … Jose and Chad … only one pound difference.

Alex: Kim and Tony earned only 550 and less than 500 pounds.

Alex: Robin worked more hours than everybody else. She also earned more money than others.

Helen: You are right. And Willy did not get anything. She did not work. I think we should not compare Willy with other vendors now. It is not fair. Ok. We can compare her with others using the other columns of the tables.

Alex: We need to consider all columns and not just one or two.

Quite early students felt the necessity to observe mathematically the cafeteria problem. Alex pointed out in the above example that they should consider data from both tables and not only one column. However, the group did not find a strategy to resolve this issue. As soon as students explicitly reported that they had to consider data from both tables but they did not know how to do it, the researcher asked them to reflect on the following two examples:

> Let's think of two employees, A and B. They are typists. They both work eight hours a day for the same company. Employer A types 100 pages per day and employer B types 120 pages per day. Which employee should the company hire next year?"

Not surprisingly, all three students agreed that the company should hire the second typist. In Helen's words, "employee B types more pages". When the researcher asked them to identify all necessary information to answer his question, Alex replied that Helen's response was incomplete. He continued by putting emphasis on the fact that "The second employer types more pages than the first employee at the same time period". Students easily answered the question of the second example, which was the following:

> Let's now think of two other employees, C and D. They are also typists. They both type 100 pages per day. Employee C works eight hours a day and employee D works six hours a day. Which employee should the company hire next year?

This students – researcher interaction assisted students in better understanding the core structure of the problem and in focusing their efforts to construct a model for solving the problem presented in the cafeteria activity. It can be concluded that students explicitly understood that they had to work with all data from both tables. Students also knew exactly what they had to do; they just did not know how to do it!

Linking real-world situation with mathematical entity and working with the mathematical entity. The “University Cafeteria” activity appeared to be challenging and interesting for students. Alice said that this (selecting the best workers) could also happen at their school’s canteen. Quite interestingly, Alex pointed out that: “This is a significant problem. Do you remember The Machine tale? […] Businessmen use more and more machines and surplus workers.” It was obvious that students clearly connected the real world situation with the context of the problem. However, the format (e.g., tables) of the provided data was not clear for all of them (see for example Alice’s actions in previous extracts). Interactions within the group facilitated students realize that they should first find a way to manipulate all data presented in the two tables. This was definitely a necessary condition for creating a satisfactory mathematical entity (model) to solve the problem.

In the following extract, students discuss possible ways of ranking the vendors in terms of the money collected by each of them. Students are also discussing the appropriateness of ranking the vendors in each column and using these simple rankings to find a total ranking for all columns. Later, when students realized that this strategy was not sufficient, they found the average amount of money collected by all vendors. It is important to underline here that students first used the strategy in “Where to Live” activity (see case study 3) and they successfully transferred the strategy in the present activity.

Alice: Let’s rank the vendors from the first to last.

Helen: For which column? There are so many.

Alice: The first one … and then we can move to other columns.

Students ranked the vendors using the first three columns in the money collected table. They ranked the nine vendors for each column. The results are presented in Table 4.4.

Helen was the first who realized that this approach was not sufficient. She pointed out that their approach could only resulted in many different rankings and therefore they could not use it to solve the problem. Her clarification was important and helped her group to start thinking of alternative methods to solve the problem.

Table 4.4

Vendors' Rankings using the Three Columns in Autumn Semester

Vendor	Ranking for Autumn Semester		
	Busy	Steady	Slow
Maria	6	6	5
Kim	8	5	6
Terry	5	7	7
Jose	3	2	2
Chad	2	3	9
Cheri	4	8	4
Robin	1	1	3
Tony	7	4	1
Willy	9	9	8

Students reflected on their work in the "Where to Live" activity and they decided to use the average notion. They found the average foreach column and grouped vendors in two categories; vendors above average and vendors below average. Alice's suggestion improved significantly students' model; she suggested finding the money average for the autumn semester and then finding the total amount of money each vendor collected during autumn semester. The results from students' average model are presented in Table 4.5. Maria for example was below average, since average was £2270 and Maria's money was only £1922. Using this classification, students selected the four vendors that were above average. However, this model was not good enough. Alex suggested further improving their model by using data from the whole money table. Students' results using data from the whole money table are presented in the Table 4.6 below. However, students did not take into consideration the different time periods (busy, steady and slow).

Table 4.5

Selecting Vendors Using the Average Model

Vendor	Money	Average	Selected
Maria	1922	Below	No
Kim	1754	Below	No
Terry	1998	Below	No
Jose	3216	Above	Yes
Chad	2436	Above	Yes
Cheri	1967	Below	No
Robin	4565	Above	Yes
Tony	2381	Above	Yes
Willy	189	Below	No

Table 4.6

Selecting Vendors Using the Data from the Money Table

Vendor	Money	Average	Selected
Maria	8196	Below	No
Kim	14921	Above	Yes
Terry	7000	Below	No
Jose	11373	Above	Yes
Chad	9284	Below	No
Cheri	11062	Below	No
Robin	15271	Above	Yes
Tony	13964	Above	Yes
Willy	9308	Below	No

An interesting dimension of students' work was their decision to use graphs in presenting their results. Specifically, students used the graphing facilities provided by Ms Excel to generate the graph presented in Figure 4.3. It can be noticed here Alex's action to draw a straight line presenting the average in their graph.

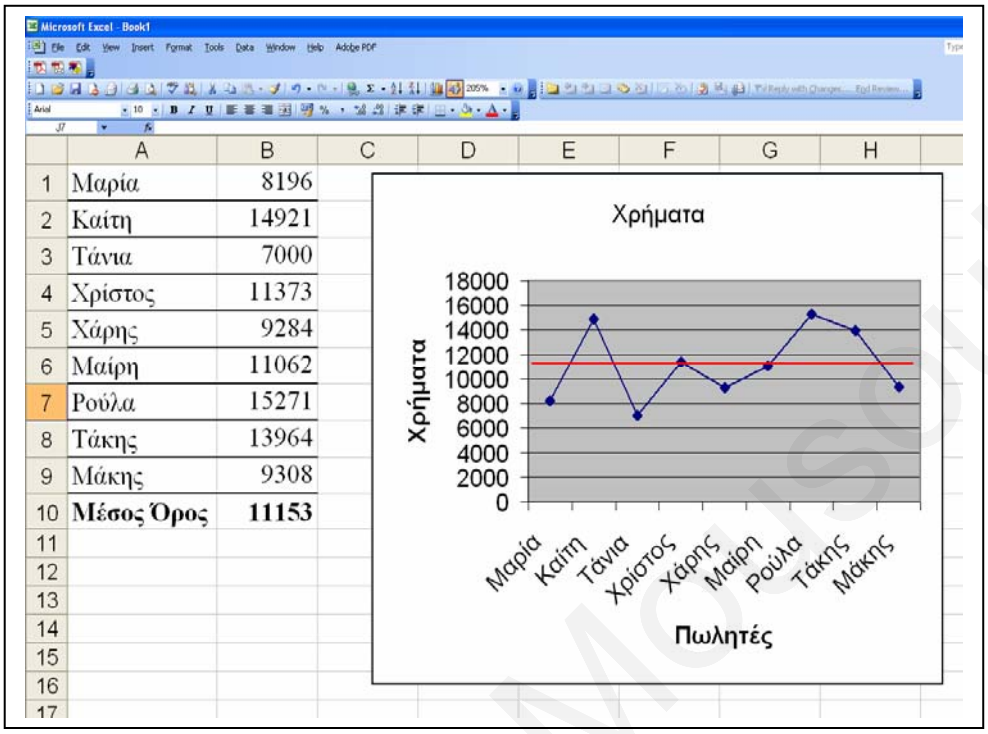


Figure 4.3. The table and the corresponding graph for the nine vendors.

Although students' work was impressive, they did not consider other forms of mathematizing the situation. It was disappointing to see that students did not realize that they had to use data from both tables. As a result, they only used data from the money collected table and totally ignored the hours each vendor worked during the last year. When the researcher encouraged them to work with the hours worked table, students used the same "average and ranking strategy" as they did with the money collected table to rank the nine vendors according to the total number of hours they worked. They finally indicated that this number corresponded to the willingness of each vendor to work and therefore they selected the first vendors. Their results are presented in Figure 4.4. It should be noted that Helen suggested to use bar chart instead of a line graph. She explained that they could use either line graph or bar chart but she suggested using bar chart for plurality reasons!

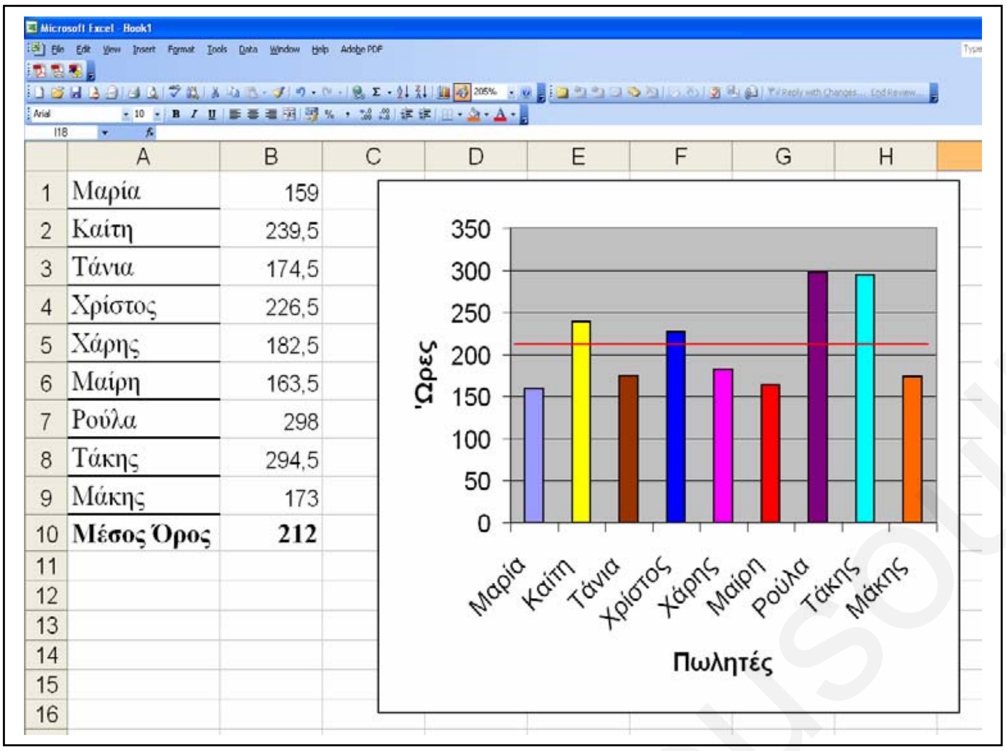
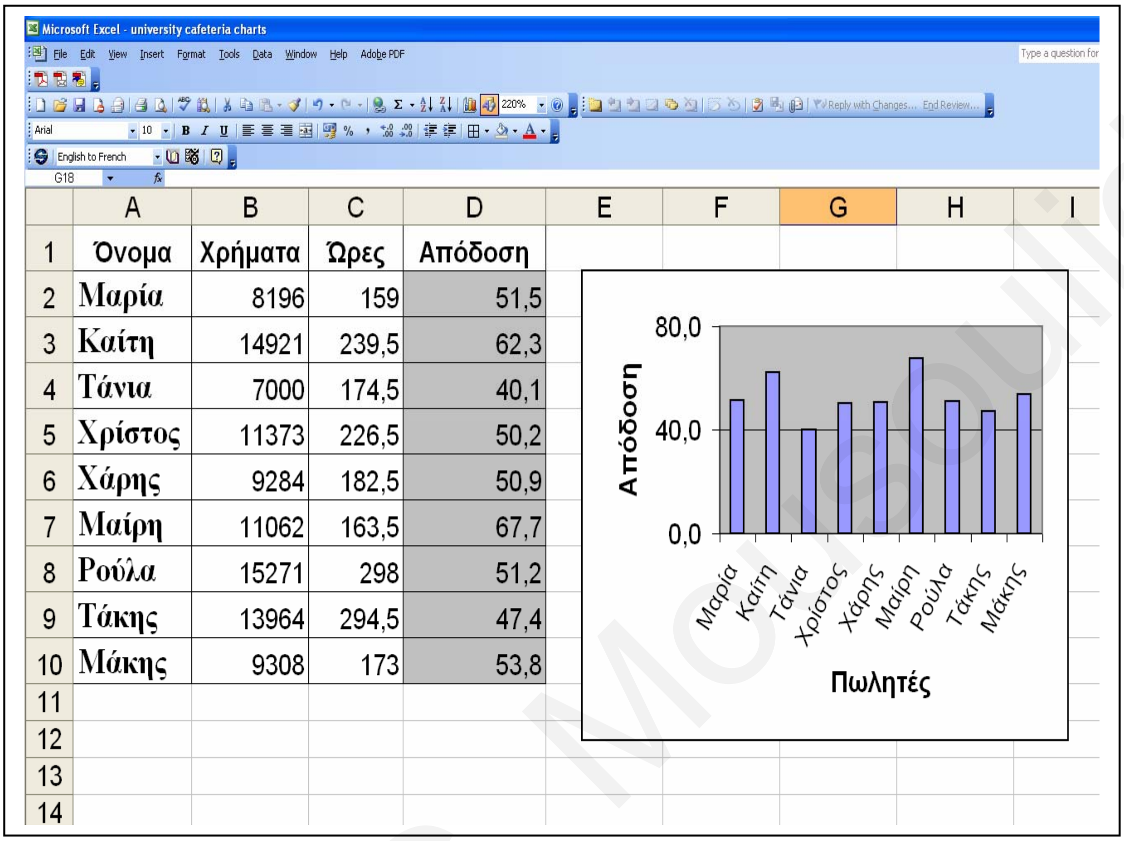


Figure 4.4. The table and bar chart for the hours the nine vendors worked.

Students' discussion with the researcher crafted students' following work. Students formulated a revised model using data from both tables. Specifically, according to Alex, they decided to use data from both tables for "finding one number for each vendor … use them [numbers] to rank the vendors". Specifically, students used their prior calculations (hours worked and money collected) to compute a "performance rate" for each vendor. Students successfully used spreadsheet's formulas to calculate the performance rate for each vendor. In the following Figure 4.5 there is a caption of students' spreadsheet, presenting the "performance rate" for each vendor and the corresponding bar chart.

Using the performance rate model students resulted in the following vendor ranking: Cheri, Kim, Willy, Maria, Robin, Chad, Jose, Tony and Terry. None of the students commented that this ranking was totally different than the previous one. Researcher prompted them to elaborate and comment on this finding. Helen replied that she expected that the new ranking would be different, since students used data from both tables. Alex indicated that the previous ranking was totally wrong and that the new ranking seemed more appropriate. It can be argued that students made implicitly comparisons between the two models. However, they only discussed their findings after researcher's question. A second issue is related to students' failure to initially propose a performance rate for each vendor. In other words, students failed to understand (at the beginning) that using data from only one table could not help them in constructing a valid model.



	A	B	C	D
1	Όνομα	Χρήματα	Ώρες	Απόδοση
2	Μαρία	8196	159	51,5
3	Καίτη	14921	239,5	62,3
4	Τάνια	7000	174,5	40,1
5	Χρίστος	11373	226,5	50,2
6	Χάρης	9284	182,5	50,9
7	Μαίρη	11062	163,5	67,7
8	Ρούλα	15271	298	51,2
9	Τάκης	13964	294,5	47,4
10	Μάκης	9308	173	53,8

Figure 4.5. The table and the corresponding bar chart for the "performance rate" for the nine vendors.

Connecting with the real-world situation. Students used the average and ranking model and later the performance rate model to interpret their solutions and to connect their work with the real problem. At first, interpretation and examination were presented in students' work. Using the average model was problematic; only four vendors were above average and therefore selected. However, students had to choose six vendors. Students attempted to modify their model and to successfully link it to the original problem. An instance of interpretation is presented in the extract below.

Alice: Ok. Who should we hire?

Alex: I am not sure about it. Look (he points to the graph). Only four vendors are above average. We need to hire six of them.

Helen: Well, no. We need to hire three full-time and three part-time. So, I select Robin. She is the first one. Kim is the second one. The third is … Tony.

Alex: Exactly, these three will work on a full time basis. I have an idea! Why do not we change the line (referring to the average line) so six people will be above it … and we will select these.

[Alex moved downwards the line so that six people were above the line]

Helen: It this correct? The line now does not refer to average.

Alex: It is difficult to do this. These vendors are so closed. Look (pointing to Chad and Willy).

Helen: But they are below average. Unless, we do not care about average.

Alex: We can not solve the problem using average line. Ok. But this is not a problem. We select Jose and Cheri and … Chad and Willy … er …

Alice: Willy is better. He earned £ 9308. Chad earned less. His amount was £ 9284.

Alex: Chad is not lucky! (Laughing)

The next extract also highlights the process of examination. Students discuss their results using data from the hours worked table. However, as presented earlier, students only used data from hours worked table as ancillary to their results from the money collected table.

Helen: Rosin is first. Kim is the second and Tony is the third one. [The ranking] is the same. Chad worked more hours than Willy. I will hire Chad.

Alice: Chad worked 9,5 hours more than Willy.

Alex: We are done then. Let’s write our letters.

The above extract further shows that students were constantly comparing their conclusions with the actual situation. Students examined the appropriateness of their results, considering the actual statement of the problem. For example, according to Helen, it was a relief (for her) to see that there was a big difference between the third and the fourth vendor. She continued pointing out that it would be unfair to select one for full time and one for part-time if their performance rates were closed.

Verification and communication. Students finally verified their models within the context of the real problem. In other words, students evaluated their solution by aligning their model with the recommended solution for the real problem. Students' discussion in the following extract highlights the aforementioned modeling processes.

Helen: Robin is the best vendor. She earned as much money as Maria and Terry earned together. She is really good.

Alice: Kim and Tony are also very good. They earned about £15000 and £14000 respectively.

Helen: These three are the best. Robin, Kim and Tony will work on a full time basis.

Alice: It was easy to find the first three. You see, there were people who earned almost the same amount of money. The difference between Jose's and Cheri's collected money is only … er … 312 pounds.

Helen: Similarly, the difference between Chad and Willy was only 24 pounds! These two were pretty closed.

Alex: And we throw out the rest of them (Laughing). No, seriously, our solution is fine. And it is better than the previous one we had.

Students' letters to the cafeteria manager demonstrated explicitly their work. All three students expressed their solution both verbally and well as pictorially; students explained how they worked with the provided data and all three students copied their spreadsheet graphs to further support their verbal clarifications. Interestingly, students' letters were both comprehensive and mathematically correct.

The results of the students' mathematical developments are presented in three cycles of increased sophistication of mathematical thinking. Each cycle is representing a shift in thinking.

Cycle 1: Focusing on subsets of information. The group commenced the "University Cafeteria" activity by scanning the money collected table to find vendors who scored highly in one or more columns (i.e., money collected in busy, steady or slow time periods, in autumn or spring semester etc.). Only limited mathematical thinking was displayed in students' unsystematic work. This is also evident in students' comments: "Jose and Chad both worked 19,5 hours. Wow, Robin worked more. She worked 26,5 hours. She is first". Students decided to use data from both tables when they failed to find a solution to the problem. However, students' approach still remained unsystematic and isolated as they did not manage to "merge" data from both tables. As a result, students still used descriptive comments: "Robin worked more hours than every other vendor. She also earned more money than others". Additionally, students frequently brought in their discussion assumptions based on their real-world knowledge.

The inexistence of any systematic approach resulted in debates on which vendors students should choose. These contradictions generated the need to mathematize their approach. The group began to use two mathematical operations to aggregate the data for each vendor, namely: (a) simply totalling the amount of money each vendor earned and how many hours each vendor worked, and (b) finding the average for each category (money, hours) and classifying vendors above and below average.

Cycle 2: Using mathematical operations. The need to mathematize their procedures was first initiated by Alex. Alex acknowledged the weakness of their approach and suggested finding the total amount of money collected by each vendor in autumn semester. He justified his decision by explaining that: "Well, it's difficult to find the best in each column. Maria is sometimes amongst the best ones and in other columns she is amongst the worst vendors". To resolve the issue and to better classify vendors, students decided to find the average and then classify vendors. At the same time, substantial discussion and argumentation took place when students interpreted their results using a line graph. However, students used data from the hours worked table only when they needed to select between two vendors.

Another dimension of students' work was the misinterpretation of the hours worked table of hours. Specifically, among two vendors who collected the same amount of money students chose the vendor who worked more hours. They analytically explained in their worksheets that: "we found the money collected average and chose for the six best vendors. If two earned the same amount of money we chose the one who worked more hours".

Cycle 3: Identifying trends and relationships. Students realized that using the average model in the money collected table was not sufficient for providing an acceptable solution. Therefore, students tried to find a relationship between money collected and hours worked for each vendor who worked in the University Cafeteria problem. The examples provided by the researcher assisted students in identifying the relationship between the two tables and in constructing a new model. The acknowledgement of these requirements led students to progress to the notion of rate. However, the notion or rate was employed in part, since students did not take into consideration the different time periods (busy, steady and slow or different semesters).

It was apparent that the students successfully transferred their findings from previous modeling activities to the context of the present activity. Interestingly, they frequently referred to what they had done in previous problems (Best Drug and Where to Live activities) and tried to make connections. These connections were appropriate and students' progression was apparent; they did not only transfer findings and conclusions from previous modeling activities but they also adopted them to investigate new mathematical concepts, like the rate concept. The latter was crucial for solving the real problem.

Modeling Processes in 8th Grade Students' Work

Similar to the analysis of the modeling processes in previous case studies, the present analysis focuses on the following modeling processes: understanding and working with the real world situation, linking real-world situation with mathematical entity, working with the mathematical entity (building a model), connecting with the real-world situation and verifying and communicating the results.

Understanding and working with the real world situation. Students' first impressions showed that they found the problem to be interesting. Students also enjoyed the fact that they were in a powerful position; they could hire someone for next year or not! Further, it was obvious that students quickly understood what they had to do. However, it was not an easy task to consider data from both tables and to clarify differences between the columns. Students reviewed both tables and started their work by focusing on specific vendors, commenting on money collected or hours worked.

Mary: Look! Robin worked more hours than anybody else (referring to autumn semester). Also here (points to the spring semester, busy column) … she is again the first.

Chris: Well, in some cases she was the first. She only worked three hours here. Wow, Tony worked 51 hours. He is the first one in this column.

Mary: She might was on vacations.

George: Also here… Robin did not work many hours (points to the show column in the summer period).

George: Robin collected more money than everybody else in the first two columns. She is not the first in the rest of the columns. She was lazy (laughs).

Chris: She might not be the first one but she earned a lot of money in almost every column. She is really a good vendor. We must hire her.

Mary: Er … We need to hire six vendors. Three of them will work full time and three part-time. I have an idea … we rank the nine vendors. The first three will work full time and fourth, fifth and sixth will work part time. That's it.

Students put much emphasis on exploring "what they should do". Students' commented on data from both tables from the beginning (in contrast to 6^{th} graders). However, similar to 6^{th} graders, students did not focus on clarifying "what the information in the two tables meant". The fact that no vendor was the best in all the columns prompted students to search for a new approach. Students' next interpretations showed explicitly that they clearly understood the core question of the problem. However, they did not distinguish the different columns and the influence that these columns might have on the question of the problem. Additionally, at the beginning students did not try to link data from the two different tables.

Students' next attempts focused on data sets. However, students tended to focus on only a small subset of the provided information or on isolated pieces of the provided data. For example, George started working with first column's data in both tables, ignoring the rest of the tables. His work resulted in two totally meaningless rankings for the first columns. He noticed soon that this was not efficient since the two rankings were different. It was clear that George's emphasis was not based on a thoughtful selection about which information was most important; it was simply the first information that came to his attention. Excert from the other two students' discussion is presented below.

Mary: I think we need to find out the number of hours each vendor worked. This can help us rank the nine vendors.

Chris: Well … it seems ok. What will we do after this?

Mary: We will know how many hours each vendor worked. We will select the first six vendors in our list.

Chris: I am not sure about this.

Mary: Why?

Chris: We will find how many hours each vendor worked. This does not mean that if someone worked many hours then she is a good vendor. Look at Robin. She is first here but she only worked three hours here. She might was sick or she was on vacations. So, we can not work like this.

Students connected the data in the two tables with the context of the problem. Apparently, this was necessary for understanding the problem. An important dimension was students' efforts to immediately mathematize the provided information for the cafeteria problem. However, this process was not easy since students had to consider and link data from both tables. Moreover, it was extremely difficult to work with the complex set of data, provided in three different semesters (autumn, spring and summer) and three different time periods for each semester (busy, steady and slow). It seemed that students knew exactly what they had to do in solving the "University

Cafeteria" problem; they just did not know how to merge data from both tables to do solve the problem!

Linking real-world situation with mathematical entity and working with the mathematical entity. Students' last fruitful discussion stressed the need for developing a new model. It was obvious that students managed to connect the real world situation with the context of the problem. However, the format (e.g., tables' columns) was not clear for all of them. Students' interactions helped them to realize that they should first find a way to manipulate all data provided in the two tables. This was definitely a necessary condition to effectively deal with the problem and to create a satisfactory mathematical entity (model) to solve the problem.

K4 =SUM(B4:J4)

	Ώρες									
	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος			Σύνολο
	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	
Μαρία	12,5	15	9	10	14	17,5	12,5	33,5	35	**159**
Καίτη	5,5	22	15,5	53,5	40	15,5	50	14	23,5	**239,5**
Τάνια	12	17	14,5	20	25	21,5	19,5	20,5	24,5	**174,5**
Χρίστος	19,5	30,5	34	20	31	14	22	19,5	36	**226,5**
Χάρης	19,5	26	0	36	15,5	27	30	24	4,5	**182,5**
Μαίρη	13	4,5	12	33,5	37,5	6,5	16	24	16,5	**163,5**
Ρούλα	26,5	43,5	27	67	26	3	41,5	58	5,5	**298**
Τάκης	7,5	16	25	16	45,5	51	7,5	42	84	**294,5**
Μάκης	0	3	4,5	38	17,5	39	37	22	12	**173**

Figure 4.6. Students calculated the total number of hours each vendor worked, using the autosum function in the spreadsheet software.

Students' first model was based on finding the total number of hours each vendor worked and the total number of money each vendor earned. As presented in Figures 4.6 and 4.7 students easily used the available tools (Autosum function) provided in the spreadsheet software to calculate the total number of hours worked and total amount of money collected for each vendor.

K18 =SUM(B18:J18)

Χρήματα

	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος			Σύνολο
	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	
Μαρία	690	780	452	699	758	835	788	1732	1462	**8196**
Καίτη	474	874	406	4612	2032	477	4500	834	712	**14921**
Τάνια	1047	667	284	1389	804	450	1062	806	491	**7000**
Χρίστος	1263	1188	765	1584	1668	449	1822	1276	1358	**11373**
Χάρης	1264	1172	0	2477	681	548	1923	1130	89	**9284**
Μαίρη	1115	278	574	2972	2399	231	1322	1594	577	**11062**
Ρούλα	2253	1702	610	4470	993	75	2754	2327	87	**15271**

Figure 4.7. Students calculated the total amount of money each vendor earned.

Students' first findings are presented in the extract below.

Chris: Now … let's see. Robin is the first one in each table. I believe Robin is in. Tony is second here (points to the total hours) and third in money. And Kim is third and second. These three should work on full time basis.

Mary: Jose is the fourth one. He will work on half time basis. Who's next?

George: It is Chad … then Terry.

Chris: Oups … Terry is the last one here (points to the money collected table).

Mary: Maria is the last one in the first table and she is penultimate in the second table. She is out.

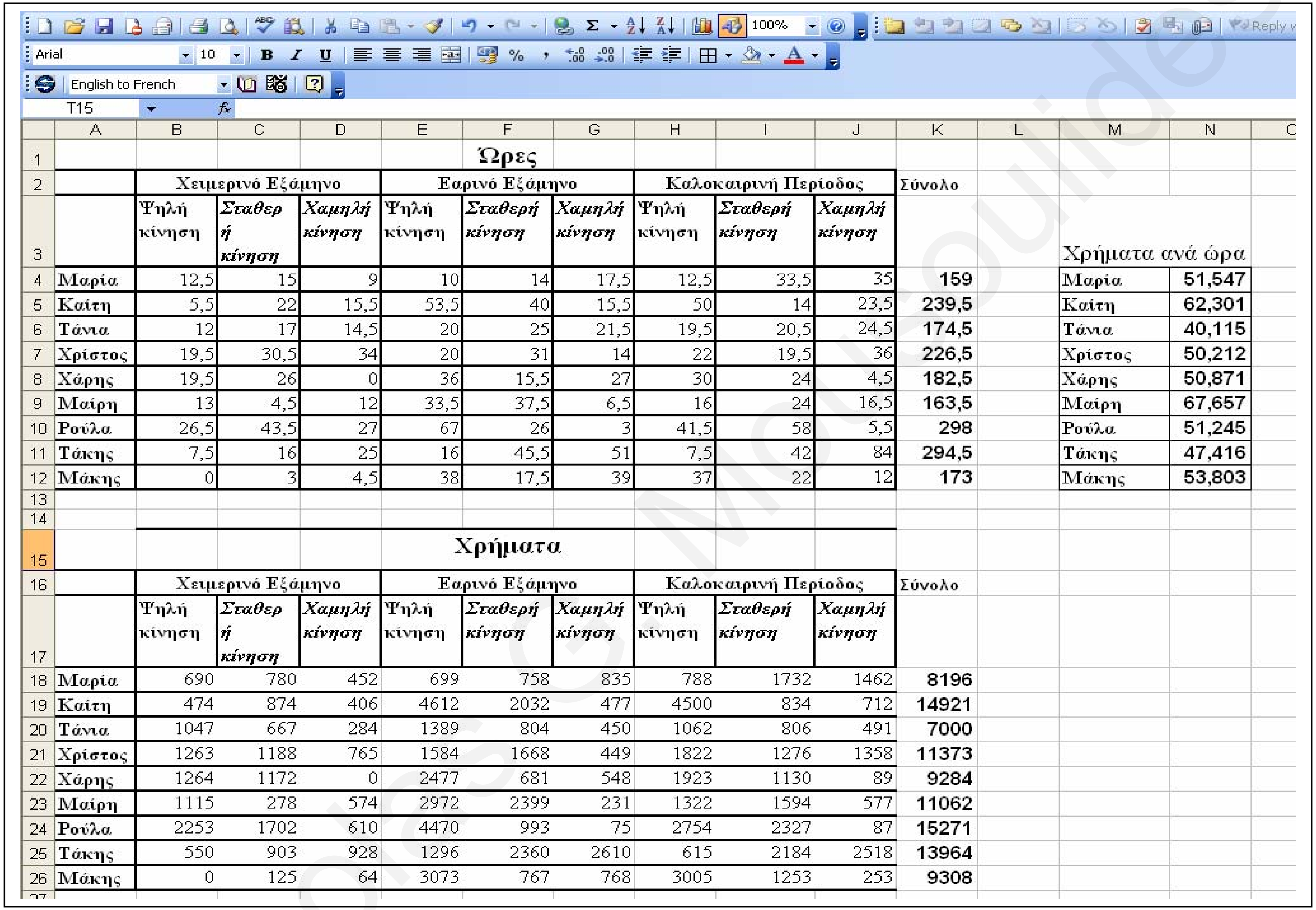
Chris: I am not sure if this is correct. Look at Terry. She got less money than everybody else but she worked more hours than Willy, Cheri and Maria. She confuses me.

George: Why?

Chris: She is not productive. Maria worked less hours and earned more money. Maria is better than Terry.

Students questioned the representativeness of their first solution. Chris raised the issue of "productiveness". This was the first step towards a new and improved model based on merging data from both tables. Students employed the notion of "rate", by calculating "how much money per hour" each vendor earned. However, it is important to bear in mind that students did not take into consideration the different semesters as well as the different time periods. The above model resulted in the ranking presented in the Table 4.7.

Students were quite surprised to realize that their findings were totally different than the results they obtained before. Specifically, using the first model Robin was ranked first in both tables. Using the new model, Robin was only fifth! This contradiction encouraged students to further explore the problem. It was apparent in students' discussion that their second model (performance rate) was much better than the first model (finding the first vendor in each table). However, students were uncertain whether the performance rate model was the best possible one. This doubt was productive; it encouraged students to examine how the different semesters might influence their results.



Ώρες

	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος			Σύνολο
	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	
Μαρία	12,5	15	9	10	14	17,5	12,5	33,5	35	159
Καιτη	5,5	22	15,5	53,5	40	15,5	50	14	23,5	239,5
Τάνια	12	17	14,5	20	25	21,5	19,5	20,5	24,5	174,5
Χρίστος	19,5	30,5	34	20	31	14	22	19,5	36	226,5
Χάρης	19,5	26	0	36	15,5	27	30	24	4,5	182,5
Μαιρη	13	4,5	12	33,5	37,5	6,5	16	24	16,5	163,5
Ρούλα	26,5	43,5	27	67	26	3	41,5	58	5,5	298
Τάκης	7,5	16	25	16	45,5	51	7,5	42	84	294,5
Μάκης	0	3	4,5	38	17,5	39	37	22	12	173

Χρήματα ανά ώρα	
Μαρία	51,547
Καιτη	62,301
Τάνια	40,115
Χρίστος	50,212
Χάρης	50,871
Μαιρη	67,657
Ρούλα	51,245
Τάκης	47,416
Μάκης	53,803

Χρήματα

	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος			Σύνολο
	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	
Μαρία	690	780	452	699	758	835	788	1732	1462	8196
Καιτη	474	874	406	4612	2032	477	4500	834	712	14921
Τάνια	1047	667	284	1389	804	450	1062	806	491	7000
Χρίστος	1263	1188	765	1584	1668	449	1822	1276	1358	11373
Χάρης	1264	1172	0	2477	681	548	1923	1130	89	9284
Μαιρη	1115	278	574	2972	2399	231	1322	1594	577	11062
Ρούλα	2253	1702	610	4470	993	75	2754	2327	87	15271
Τάκης	550	903	928	1296	2360	2610	615	2184	2518	13964
Μάκης	0	125	64	3073	767	768	3005	1253	253	9308

Figure 4.8. Students calculated “money per hour” for each vendor.

Table 4.7

The Money Per Hour for each Vendor and the Corresponding Ranking

Vendor	Money per Hour	Ranking
Maria	51,5	4
Kim	62,3	2
Terry	40,1	9
Jose	50,2	7
Chad	50,9	6
Cheri	67,7	1
Robin	51,2	5
Tony	47,4	8
Willy	53,8	3

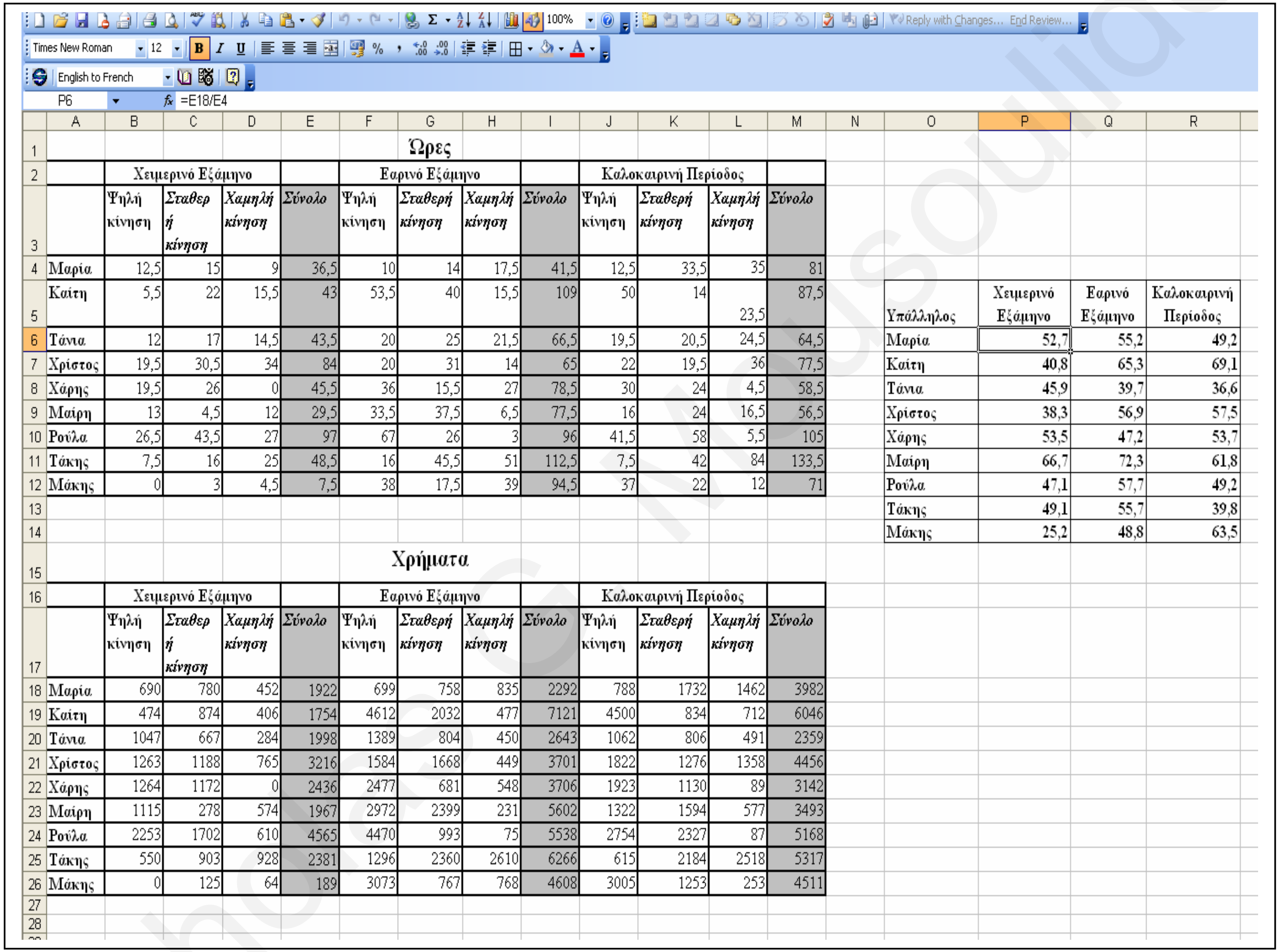
Students' next attempts focused on calculating for each vendor the total number of money collected and hours worked for each semester. Thereafter, students created a ranking for each semester and compared the three rankings. The results of students' work are presented in Figure 4.8 and the corresponding rankings are presented in Table 4.8. Due to a disagreement, students decided to work separately on two different sheets. Mary used more columns to total money and hours, while the other two students implemented a more sophisticated formula for making the necessary calculations. Mary's work is presented in Figure 4.9 and Chris' and George's solution in Figure 4.10.

Table 4.8

The "Money Per Hour" Ratio and Corresponding Rankings

	Money per Hour			Ranking		
Vendor	Autumn	Spring	Summer	Autumn	Spring	Summer
Maria	52,7	55,2	49,2	3	6	7
Kim	40,8	65,3	69,1	7	2	1
Terry	45,9	39,7	36,6	6	9	9
Jose	38,3	56,9	57,5	8	4	4
Chad	53,5	47,2	53,7	2	8	5
Cheri	66,7	72,3	61,8	1	1	3
Robin	47,1	57,7	49,2	5	3	6
Tony	49,1	55,7	39,8	4	5	8
Willy	25,2	48,8	63,5	9	7	2

As it was expected, students reached the same rankings. George suggested adding up the three numbers to create a unique ranking. This new ranking was the following: Cheri, Kim, Robin, Chad, Jose, Tony, Willy and Terry. Therefore, as Chris documented in his worksheet, Cheri, Kim and Robin would work on a full time basis, and Chad, Jose and Tony would work on a part-time basis. At that time, students reported that they solved the problem. Researcher encouraged them to compare their final with previous solutions. It came out that the two solutions were different. Specifically, according to their final model Jose would work on a half time basis but according to the previous model Jose would be fired. Similarly, Robin would work on a full time basis according to final model but she was only selected to work half time according to the previous solution. More interestingly, following last model, Willy was only seventh among the nine vendors. According to previous model Willy was third!



Ώρες

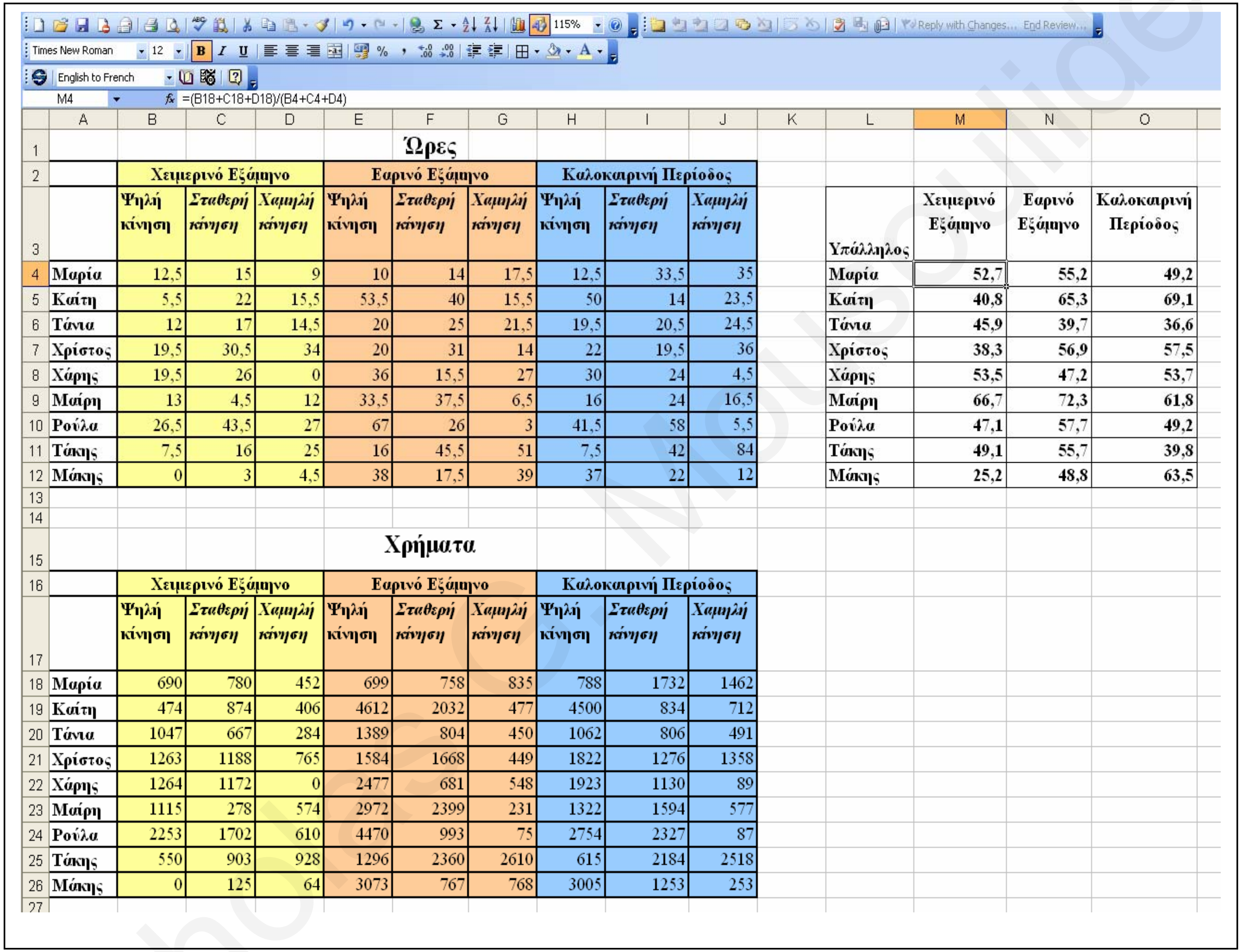
	Χειμερινό Εξάμηνο				Εαρινό Εξάμηνο				Καλοκαιρινή Περίοδος			
	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Σύνολο	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Σύνολο	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Σύνολο
Μαρία	12,5	15	9	36,5	10	14	17,5	41,5	12,5	33,5	35	81
Καιτη	5,5	22	15,5	43	53,5	40	15,5	109	50	14	23,5	87,5
Τάνια	12	17	14,5	43,5	20	25	21,5	66,5	19,5	20,5	24,5	64,5
Χρίστος	19,5	30,5	34	84	20	31	14	65	22	19,5	36	77,5
Χάρης	19,5	26	0	45,5	36	15,5	27	78,5	30	24	4,5	58,5
Μαίρη	13	4,5	12	29,5	33,5	37,5	6,5	77,5	16	24	16,5	56,5
Ρούλα	26,5	43,5	27	97	67	26	3	96	41,5	58	5,5	105
Τάκης	7,5	16	25	48,5	16	45,5	51	112,5	7,5	42	84	133,5
Μάκης	0	3	4,5	7,5	38	17,5	39	94,5	37	22	12	71

Υπάλληλος	Χειμερινό Εξάμηνο	Εαρινό Εξάμηνο	Καλοκαιρινή Περίοδος
Μαρία	52,7	55,2	49,2
Καιτη	40,8	65,3	69,1
Τάνια	45,9	39,7	36,6
Χρίστος	38,3	56,9	57,5
Χάρης	53,5	47,2	53,7
Μαίρη	66,7	72,3	61,8
Ρούλα	47,1	57,7	49,2
Τάκης	49,1	55,7	39,8
Μάκης	25,2	48,8	63,5

Χρήματα

	Χειμερινό Εξάμηνο				Εαρινό Εξάμηνο				Καλοκαιρινή Περίοδος			
	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Σύνολο	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Σύνολο	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Σύνολο
Μαρία	690	780	452	1922	699	758	835	2292	788	1732	1462	3982
Καιτη	474	874	406	1754	4612	2032	477	7121	4500	834	712	6046
Τάνια	1047	667	284	1998	1389	804	450	2643	1062	806	491	2359
Χρίστος	1263	1188	765	3216	1584	1668	449	3701	1822	1276	1358	4456
Χάρης	1264	1172	0	2436	2477	681	548	3706	1923	1130	89	3142
Μαίρη	1115	278	574	1967	2972	2399	231	5602	1322	1594	577	3493
Ρούλα	2253	1702	610	4565	4470	993	75	5538	2754	2327	87	5168
Τάκης	550	903	928	2381	1296	2360	2610	6266	615	2184	2518	5317
Μάκης	0	125	64	189	3073	767	768	4608	3005	1253	253	4511

Figure 4.9. Mary's solution using extra columns.



M4 =(B18+C18+D18)/(B4+C4+D4)

Ώρες

	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος		
	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση
Μαρία	12,5	15	9	10	14	17,5	12,5	33,5	35
Καίτη	5,5	22	15,5	53,5	40	15,5	50	14	23,5
Τάνια	12	17	14,5	20	25	21,5	19,5	20,5	24,5
Χρίστος	19,5	30,5	34	20	31	14	22	19,5	36
Χάρης	19,5	26	0	36	15,5	27	30	24	4,5
Μαίρη	13	4,5	12	33,5	37,5	6,5	16	24	16,5
Ρούλα	26,5	43,5	27	67	26	3	41,5	58	5,5
Τάκης	7,5	16	25	16	45,5	51	7,5	42	84
Μάκης	0	3	4,5	38	17,5	39	37	22	12

Υπάλληλος	Χειμερινό Εξάμηνο	Εαρινό Εξάμηνο	Καλοκαιρινή Περίοδος
Μαρία	52,7	55,2	49,2
Καίτη	40,8	65,3	69,1
Τάνια	45,9	39,7	36,6
Χρίστος	38,3	56,9	57,5
Χάρης	53,5	47,2	53,7
Μαίρη	66,7	72,3	61,8
Ρούλα	47,1	57,7	49,2
Τάκης	49,1	55,7	39,8
Μάκης	25,2	48,8	63,5

Χρήματα

	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος		
	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση	Ψηλή κίνηση	Σταθερή κίνηση	Χαμηλή κίνηση
Μαρία	690	780	452	699	758	835	788	1732	1462
Καίτη	474	874	406	4612	2032	477	4500	834	712
Τάνια	1047	667	284	1389	804	450	1062	806	491
Χρίστος	1263	1188	765	1584	1668	449	1822	1276	1358
Χάρης	1264	1172	0	2477	681	548	1923	1130	89
Μαίρη	1115	278	574	2972	2399	231	1322	1594	577
Ρούλα	2253	1702	610	4470	993	75	2754	2327	87
Τάκης	550	903	928	1296	2360	2610	615	2184	2518
Μάκης	0	125	64	3073	767	768	3005	1253	253

Figure 4.10. Chris' and George's solution using a complex formula.

This contradiction doubted students about the appropriateness of their solution. At that time, the researcher asked them to compare their solutions and individually document in their worksheets their results and possible flaws in their solutions. Mary simply reported that the solution was appropriate with no flaws. George stated that the last solution seemed appropriate. He continued documenting that they used data from both tables and that they considered all three semesters. On the contrary, Chris reported that the solution was appropriate but he identified a possible flaw. He noticed that they did not use the different time periods in their solution.

Following Chris' comment, the researcher started a discussion on the possibility that including the different time periods might change findings. All three students replied that they hoped not! Part of this discussion is presented below.

George: We can add up money and hours for each time period, as we did with the different semesters.

Researcher: Will this result on a different ranking?

George: Err…yes.

Chris: Probably yes. The last two ranking were different. Now … let's see. I do not know. We could use data for busy, steady and … summer…no no, slow. Semester? …

George: We can use both semester and time periods.

Mary: How? We can not do this.

George: Why not?

Chris: We can do the same and find how much money each person earned per hour for busy, steady and slow periods.

Mary: But we will not use different semesters if we do this.

Chris: Exactly. How can we simultaneously use semester and time periods?

Mary: Mmm. I do not know.

Students started rethinking of the necessary conditions and assumptions for solving the problem. For a few minutes students did not know what they should do. At that time Chris suggested to decide on the importance of the two different factors (semester or the time period) and based their model on this assumption. This discussion is presented in detail in the next session (connecting mathematical entity to the real world problem). Students finally decided that working with different time periods seemed more appropriate. They used this to construct a final model. Their results are presented in Figure 4.11 and the corresponding rankings in Table 4.9. Students' new model resulted in a different ranking. For example, Cheri and Kim were selected for working on a full time basis in both solutions. On the contrary, following the "semester" model Tony was only selected on a part-time basis, but following "time period" model Tony was selected to work on a full time basis.

M6 =(B18+E18+H18)/(B4+E4+H4)

Ώρες

	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος		
	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*
Μαρία	12,5	15	9	10	14	17,5	12,5	33,5	35
Καίτη	5,5	22	15,5	53,5	40	15,5	50	14	23,5
Τάνια	12	17	14,5	20	25	21,5	19,5	20,5	24,5
Χρίστος	19,5	30,5	34	20	31	14	22	19,5	36
Χάρης	19,5	26	0	36	15,5	27	30	24	4,5
Μαίρη	13	4,5	12	33,5	37,5	6,5	16	24	16,5
Ρούλα	26,5	43,5	27	67	26	3	41,5	58	5,5
Τάκης	7,5	16	25	16	45,5	51	7,5	42	84
Μάκης	0	3	4,5	38	17,5	39	37	22	12

Χρήματα

	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος		
	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*
Μαρία	690	780	452	699	758	835	788	1732	1462
Καίτη	474	874	406	4612	2032	477	4500	834	712
Τάνια	1047	667	284	1389	804	450	1062	806	491
Χρίστος	1263	1188	765	1584	1668	449	1822	1276	1358
Χάρης	1264	1172	0	2477	681	548	1923	1130	89
Μαίρη	1115	278	574	2972	2399	231	1322	1594	577
Ρούλα	2253	1702	610	4470	993	75	2754	2327	87
Τάκης	550	903	928	1296	2360	2610	615	2184	2518
Μάκης	0	125	64	3073	767	768	3005	1253	253

Υπάλληλος	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*
Μαρία	62,2	52,3	44,7
Καίτη	87,9	49,2	29,3
Τάνια	67,9	36,4	20,2
Χρίστος	75,9	51,0	30,6
Χάρης	66,2	45,5	20,2
Μαίρη	86,5	64,7	39,5
Ρούλα	70,2	39,4	21,7
Τάκης	79,4	52,6	37,9
Μάκης	81,0	50,5	19,5

Figure 4.11. Students' solutions in finding the "money per hour" ratio for each vendor for each time period.

It can be argued that students considered many forms of mathematizing the real problem; they found a method for merging the data from both tables and using the semester and time period categorizations to refine their solutions. The modeling activity fostered not only the application of students' prior mathematical developments and also their refinement. Of importance were model comparisons and the documentation of these comparisons. However, although students succeeded in finding a performance rate for each vendor, used a number of different approaches (time periods or semesters), they failed to weight these factors (e.g., deciding that busy slot was more important than slow slot).

Connecting with the real-world situation. During the whole process students connected the real world problem that was presented in the activity with the mathematical models they developed. Students' concerns related to the real problem directed most of the mathematical developments presented in the previous session. Additionally, students' discussion on their last two models and their final selection was totally influenced by the context and constrains of the real world situation.

Table 4.9

The "Money per Hour" Ratio for each Time Period and the Corresponding Rankings

	Money per Hour			Ranking		
Vendor	Busy	Steady	Slow	Busy	Steady	Slow
Maria	62,2	52,3	44,7	9	3	1
Kim	87,9	49,2	29,3	1	6	5
Terry	67,9	36,4	20,2	7	9	7
Jose	75,9	51,0	30,6	5	4	4
Chad	66,2	45,5	20,2	8	7	8
Cheri	86,5	64,7	39,5	2	1	2
Robin	70,2	39,4	21,7	6	8	6
Tony	79,4	52,6	37,9	4	2	3
Willy	81,0	50,5	19,5	3	5	9

Students regularly compared the real-world conclusion with the situation while considering the modeling purpose to ensure that the real-world conclusion aligned with the realistic situation in light of the modeling goal. An interesting snapshot of this modeling process was presented in selecting among two vendors who had collected almost equal amounts of money. This started debates between students, especially in the case that one vendor would be chosen to work for next year and the other would be fired. Specifically, Mary supported that it was unfair to fire one person because she earned only 23 pounds less than somebody else. The other two students agreed with her, mentioning that it would be better to consider other factors, like the personality of the vendors.

A second example of examination was presented when students calculated rankings based on the different semesters. Specifically, during their discussion on the different rankings for busy, steady and slow periods, Chris commented that Maria was ranked as ninth, third and first, while Kim was ranked as first, sixth and fifth accordingly. When totaling rankings, Maria was fourth and Kim was third. Chris commented that although Maria was finally selected, she was not productive during busy periods. Chris also argued that working on busy hours was very important. An example of interpretation is related to the fact that Willy and Chad did not work at all during busy and slow periods in autumn semester. George commented that they might be sick and Mary added that Willy and Chad might be on vacations. However, students did not incorporate this finding in their "money per hour" model.

The context of the activity encouraged students to relate their work (constructed models) with the real problem. Additionally, students based their selection among different models not only on mathematical factors, but also on the conditions of the real problem.

Verification and communication. Students iterative discussions on the appropriateness of their models highlighted their efforts to verify their solutions. Students' efforts focused on checking and evaluating whether their solution could answer the question of the real problem and whether their models fitted with the purpose of the modeling activity. Another instance of verification is related to students' uncertainty about the most important factor (semester or time period). It was obvious that the desired outcome was difficult to be reached and students would finally use a number of assumptions to reach a solution. Finally, quite impressive was the process of aligning in students' work; aligning included the ongoing metacognitive activity of comparing the current state of their solutions to earlier states and comparing their results. In the following extract students' discussion highlights the aforementioned modeling processes.

Chris: Cheri is definitely the best one. She is second, first and second. She is among the best in all time periods.

Mary: Yes … she was also very good when we used semesters. We will select her for sure. Tony is also very good. He is stable. He and Cheri are the best vendors.

George: Maria is first here (he points to the slow time period). Wow, she is the last one in busy slot.

Mary: She is fourth. She is good but she is not stable.

Chris: Besides this … It is more important to be productive during busy hours. Maria was also selected to work on a half time basis when we used semesters.

Chris: Robin was a surprise. See … she was third when we used data from semesters but she is only seventh now. Hmm … It's a problem.

George: The new solution is better than the previous one. Right?

Chris: Hopefully (laughing). I think yes. It is better than using data from different semesters.

Students' letters to the cafeteria's manager were impressive. Students expressed their solution both verbally and well as pictorially; students explained their work and their choice to use time periods in their final model. All three students copied screens from their spreadsheets to support their verbal clarifications.

Eighth Grade Students' Mathematical Developments

The results of the students' mathematical developments are summarised in terms of cycles of increased sophistication of mathematical thinking, with each cycle representing a shift in thinking. The analysis of students mathematical developments are presented in the following order. First, students' efforts were limited to focusing on subsets of information. Second, students started using mathematical operations (e.g., finding ratios) and third students' work was based on more sophisticated mathematical ideas, such as identifying trends and relationships among data.

Cycle 1: Focusing on subsets of information. Similarly to 6th grade group' work, 8th grade students commenced the "University Cafeteria" problem by scanning the two tables to find the vendors who scored highly in one or more columns (i.e., money collected in busy, steady or slow time periods, in autumn or spring semester, hours worked in each column etc.). This initial approach can be characterized as unsystematic, since students did not have a plan in their minds to start working with the provided data. However, they made comments like "Look! Robin worked more hours than anybody else" or "Tony might was on vacations (when Tony worked three hours)". Another finding was that students did not pay attention to the column headings. As a consequence, when students started mathematizing the problem, their first model was totally based on finding the total number of hours worked and total number of money collected by each vendor.

In contrast to 6th graders' work, 8th grade students immediately realised that they should use data from both tables. However, students' approach still remained unsystematic and limited as they only commented on specific vendors, without making any connections between the two tables. Students' next attempts to better explore and understand the problem started a new round of work during which students started focusing on data sets. However, students tended to focus on only a small subset of the provided information or focus on isolated pieces of data. An example of this unsystematic strategy was George's attempt to rank vendors using data from only the first column in each table. Very quickly, he realized that his strategy was not good enough for solving the problem; different rankings came out and it was obvious that he could not work with nine different rankings in each table!

The above results demonstrated the inefficiency of their approaches and encouraged students to mathematize their work. At that time, students began to use mathematical operations to aggregate the data for each vendor, namely simply totalling the amount of money each vendor earned and how many hours each vendor worked. They finally used the two rankings for selecting the six vendors.

Cycle 2: Using mathematical operations. Students' need to rank the nine vendors resulted in totalling the number of hours each vendor worked. Students also found the total amount of money each vendor collected in all three semesters. Students justified their decision by explaining that:

> Well, using this strategy we can find one ranking for the hours worked and one ranking for the money collected. We can then use these rankings to select the three vendors that will work on a full time basis and the three vendors that will work on a half time basis.

Of course, the above model could not help students to find an appropriate solution for the problem, since the two rankings were contradictory. For example, Terry was fifth and ninth in the two rankings. This was problematic, since only six vendors were selected. Another example that highlighted the need to develop a better model was the following. Chris commented that Terry's data were confusing. Following the first model, Terry was classified to work on a half time basis and Maria who was not selected, although Maria earned more money than Terry. Chris concluded that Maria was more productive than Terry and therefore if they had to choose between the two of them, they should hire Maria.

Cycle 3: Identifying trends and relationships. To resolve previous issue and to classify vendors, students decided to proceed in finding the money per hour ratio. At the same time, substantial discussion and argumentation took place when students tried to find a way to merge data from both tables. Students progressed to looking for a relationship between money collected and hours worked. Students quite easily identified the relationship between money and hours and they constructed a model for calculating the money per hour ratio. An important dimension was students' discussion on the "vendor's productivity". Specifically, students agreed upon selecting the most productive vendors and they defined productivity as the amount of money each vendor collected in an hour. The acknowledgement of this new parameter directed students' "performance rate" model. This model resulted in a good solution for solving the problem.

Students were concerned that using the "performance rate" model resulted in a totally different ranking. Although it was apparent that students were satisfied with the solution, they questioned the appropriateness of their model. Consequently, students refined their last model by finding money per hour ratio for each semester. This new ranking was again different than the previous one. The discussion with the researcher that followed, encouraged students to alternatively find the money per hour ratio for the busy, steady and slow time periods. This final ranking was quite similar to the "semester ranking". However, there were differences for two vendors. Students decided to adopt this last model, since "it is reasonable to work with the different time periods instead of the different semesters. It is important that one vendor is good in all time periods, especially during busy hours". However, students failed to use their assumption in refining their model.

Summary

The "University Cafeteria" activity was challenging for students. All students understood the core question of the problem but at least 6th graders failed to use all necessary information to find a solution. Sixth graders' first models were inadequate to solve the problem and only researcher's intervention helped them overcome the difficulties. Eighth graders found easier the necessary patterns and relations and managed to solve the problem. Older students also reached more sophisticated solutions, taking into consideration different semesters and time periods variables.

The different, but all appropriate, "performance per rate" models 8th grade students constructed, helped them in predicting the behaviour of the real situation and in verifying their solutions in the context of the real problem. The fact that both final models were mathematically correct indicated that the best solution was the one resulted after the verification of the model in the framework of the real world problem. On the contrary, sixth grade students failed to verify their solution in the context of the real problem and did not successfully document their results.

Students' communication process was another difference between the two groups of students. Eighth grade students extensively discussed many of the issues that arose in their work and these discussions helped them overcome a number of difficulties. Sixth graders did not interpret their results in the context of the real problem and this was an obstacle in their efforts to solve the problem. The complexity of the "University Cafeteria" activity was a significant factor that influenced students' efforts and resulted in a number of differences between 6th and 8th graders. It can be argued that the more complex a modeling activity was, the more differences appeared between the models constructed by the two groups of students.

The Explanatory Framework

Introduction

The construction of a "theoretical model", as proposed by the grounded theory approach (Miles & Huberman, 1994), is the next result of the analysis. This theoretical model is based upon the analysis of the six case studies presented earlier. This will lead to the development of an explanatory framework, which links all modeling processes and mathematical developments of student work in the modeling activities. The case studies are related to one another and intersected, so that a model describing the modeling processes and mathematical developments will emerge and answers to the research questions related with students' modeling processes, similarities and differences between 6th and 8th grade students. Figure 4.12 sets this section in the context of the analytical process as described earlier in this chapter.

The development of the explanatory framework is based on the presentation of the modeling processes appeared in students' work in each step of the modeling procedure, based on a cross case study comparison.

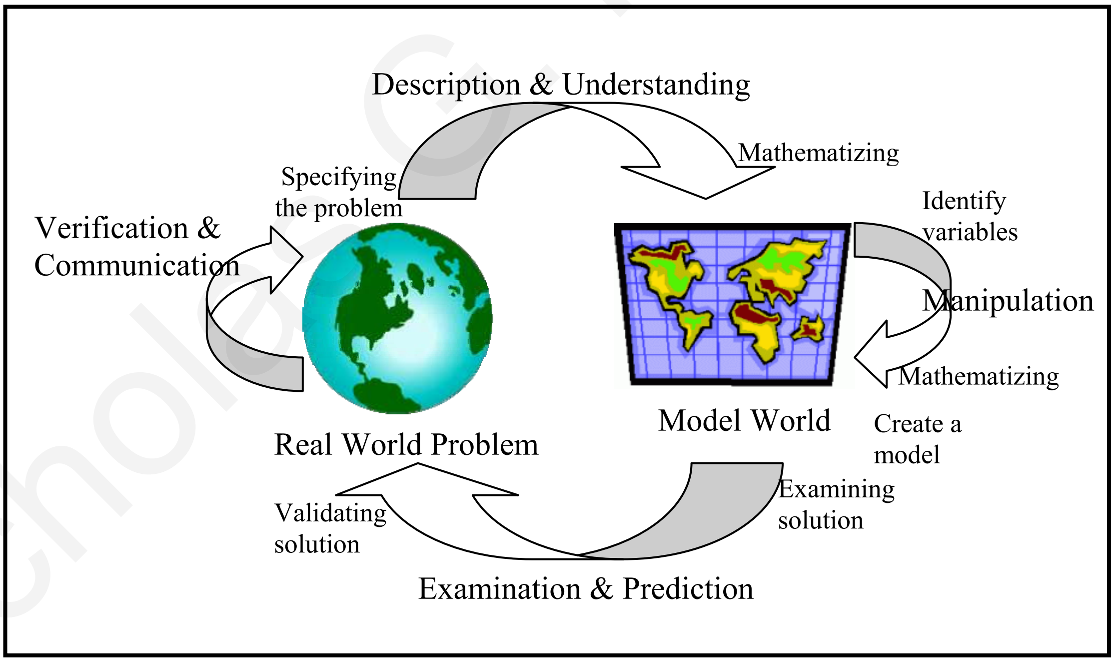


Figure 4.12. The analytical framework for the modeling procedure.

Steps of the Modeling Procedure: Findings Arise from the Case Studies

The main elements related to the modeling processes arose from the six case studies are presented here, with regard to each step of the modeling procedure. Specifically, the steps of the modeling procedure are: (a) Description of the problem, (b) Manipulation of the problem for building a model, (c) Using the model to Predict the behaviour of the real problem and (d) Verification of the solution within the context of the real world based problem. Examples and counter examples from the case studies are presented for each step of the modeling procedure to provide evidence for the assertions made here.

Description of the Problem

The analysis of the case studies showed that students exploited the content of the problems to gain deeper understanding of the problems. In most of the cases, students easily understood the core questions of the activities. However, in a number of cases students needed extra time to obtain more information about the problem and to discuss in their groups the meaning of the questions of the problems. Exploring the problem was not just a simple statement of the problem. On the contrary, during the process of exploring and specifying the problem (Dunne, 1998), students tried to perceive different information about the problem than was originally apparent. A second finding presented in all six case studies was students' early mathematical observations. Specifically, students started *observing mathematically* during the process of exploring the problem, using simple (or more complex) mathematical ideas to describe situational information.

Another characteristic of student work during the description of the modeling problem was related to the identification of the *conditions* and *assumptions* of the real world context. Students also specified that some information was not important. In this way, students simplified the situation in the sense that NCTM (1989) recommends to use "simplify" or "simplification" (p.138) in mathematical problem solving. The modeling process of simplification is also similar to Lester and Kehle's (2003) Simplifying / Problem Posing phase of problem solving. Specifically, finding conditions and assumptions includes moving from the real world problem to the particular (real-world) problem that is to be solved. The latter is essential in solving real world problems.

Helen and Alex (case study 1) easily understood that the most effective drug was the one which had the smallest reaction time. In the same case study, Alice needed more time to clarify the question of the problem. On the contrary, 8th grade students (case study 2) easily understood and clarified the core question of the problem and they immediately started discussing and questioning the data quality. They posed, for example, questions related to the persons received treatment, the duration of this experiment and the number of different cases.

Sixth grade students primarily concetrated on specific reaction times, when they started observing mathematically the problem. Similarly, 8th grade students started mathematization by finding the best drug in each row and they then calculated how many times a drug was first. A difference between sixth and eighth grade students relied on finding the necessary conditions and assumptions for solving the problem. Sixth graders did not make any assumptions. On the contrary, 8th graders raised a number of necessary assumptions for constructing a model.

In the third case study, 6th grade students had difficulties to describe and therefore understand the problem. Specifically, they failed to clarify the meaning of budget and they did not make any connections between the budget and other factors. Eighth grade students (case study 4) questioned and discussed the meaning of budget but they also did not incorporate budget in their models (at the beginning of the modeling activity). Both groups made a number of remarkable mathematical observations. Alice (6th grade student) commented that Paramithoupoli should be rejected, since its streets were so bad and there were only few shops in the city. Alex replied that Alice's suggestion was not correct, since Paramithoupoli's budget would be increased and therefore city's authorities could use this money to improve city's facilities. Similar argumentation and suggestions were also presented by 8th grade students. Helen, for example, proposed to exclude Paramithoupoli but Chris noted that an important parameter was Anastasia's studies and probably the specific city might be of interest (see case study 4).

A number of differences occurred between 6th and 8th grade students' work in the "University cafeteria" modeling activity. Sixth graders failed to understand the problem and as a result they only focused on isolated parts of the data. Students' initial models were inadequate to solve the problem and only researcher's comments and suggestions helped students improve their models. Eighth graders' work was significantly better in a number of ways. Students merged data from tables, and identified patterns and relations. However, they focused on specific vendors and they commented on either the amount of money collected or either the hours each vendor worked. Therefore, students concluded in a number of different vendor rankings and they could not provide a reasonable and justifiable solution to the problem. It was apparent that students identified the necessary variables and relationships to describe and understand the problem. However, students failed to handle and relate these understandings with the core question of the problem.

Manipulation of the Problem

A central characteristic of students' work in problem manipulation was m*athematizing.* Students used their prior findings, such as the conditions and assumptions, to acknowledge, understand and create the necessary mathematical *properties* that correspond to the conditions and assumptions situated within the real problem. The result of the mathematization process is a number of mathematical pieces related to the problem's conditions. However, these mathematical pieces were not perceived from the students' view as a single mathematical object. Additionally, in a number of models, too many pieces were also present or key pieces necessary for developing a successful model were missing.

In problems students are familiar with, like the "Best Drug Award", mathematizing happened implicitly. In this case, students easily made the connections between the identified conditions and assumptions and the desired properties and therefore mathematizing was not explicitly observed. A last characteristic appeared in all case studies was that mathematizing processes required multiple journeys between conditions and assumptions and properties and parameters, a pattern that contributes to a non-linear modeling path.

Students' work in the first case study showed that students used a number of different approaches to mathematize the problem. Specifically, Helen first ranked the four drugs in each row. On the contrary, Alex assigned positive and negative points to each drug. On the other hand, 8th grade students' mathematization processes were focused on calculating the sum or reaction times for each drug. The nature of mathematization was quite limited in the case of the 8th grade group. However, it can be argued that the specific problem was not demanding for them and therefore students easily reached an acceptable and satifactory solution.

A number of differences occurred between 6th and 8th graders' work in the "Where to Live" modeling activity. At the beginning, 6th graders found the total number of buildings in each city and they decided to exclude from their first model city's budget and road quality. Their decision was based on the fact that they could not merge quantitative and qualitative data. However, they did not consider converting the two factors, as 8th graders did. An interesting dimension in 6th graders mathematization processes was the assignment of positive and negative points to the different cities according to their ranking. It can be argued that students successfully transferred the specific method from their work in "Best Drug Award" modeling activity. Similar to 6th graders' work, 8th graders first model was based on finding the total number of buildings in each city. Of interest was students' reflection on that first model. Students questioned the appropriateness and representativeness of the model. They based their argumentation on the fact that one city was ranked among the best cities only because it had too many shops. Students concluded that finding the total number of buildings could be trivial. So, in their next attempt, they decided to use a more refined strategy; they found each factor's average and they categorized the six cities above and below average. Cities above average were assigned positive points and cities below average were assigned negative points respectively.

Both groups of students presented interesting and challenging solutions in the "University cafeteria" modeling activity. A number of differences can be tracked between 6th and 8th grade students' models. Sixth grade students' initial model was based on ranking the nine vendors in each column and then finding a ranking for autumn semester. Students proceeded on calculating the total amount of money each vendor earned when they realized that their prior solution was not efficient enough. However, even this new solution was not appropriate since it did not incorporate data from both tables. A parameter of students' work was the use of spreadsheet's graphing capabilities in constructing their model. The graphical representation of students' model made students' realized that the model was not good enough. This finding in conjunction with students' discussion with the researcher assisted students in reaching a final and appropriate model. Their last "performance rate" model could answer (to some extent) the core question of the problem.

Eighth grade students' work in the same modeling activity was quite different compared to 6th graders' work. Eighth graders specifically presented better and more sophisticated solutions and they succeeded to solve the problem. Specifically, students easily found a model based on the "performance rate" for each vendor and used this method to select the best six vendors. At that time, students' method did not consider factors like different semesters and/or time periods. However, 8th graders improved their "performance rate" model by employing first the semester dimension and second the time period dimension. These two final models were not only sophisticated but they also extensively used the whole data set, including constrains, parameters and patterns.

In an attempt to summarize students' work, it can be argued that students organized and formulated prior assumptions and identified properties first into a mathematical entity and later into a valid and appropriate model. At the beginning of this process, students worked on combining mathematical objects, properties, and parameters (that have been introduced in the description and understanding of the problem) into a single mathematical entity. The process of combining followed iterative modeling cycles and was not simple. This resulted to not sufficient or incomplete models. These models missed necessary properties and ignored important parameters of the real problem. However, in most of the cases students succeeded to improve their models and to finally reach good solutions. The process of combining was similar to reifying or encapsulating. However, combining as presented in students' work, referred to creating a mathematical object from other mathematical objects or identifying a mathematical object as containing all the required properties and parameters rather than conceiving of something as an object rather than a process.

Predict the Behaviour of the Real Problem

The first modeling process on students' work focused on connecting students' model with the real-world situation. This was the first step towards predicting the behaviour of the real problem. Students used their models to interpret their solutions and to make predictions of the behaviour of the real world problem. The processes of connecting the model with the real problem and predicting the behaviour of the real problem were aligned with with the process of interpretation. Like mathematizing, interpreting assisted students in bridging the real world (real problem) with the mathematical world (model). Additionally, interpreting helped students to make decisions and conclusions on the quality of their models. In other words, students stated conclusions about the real problem, by interpreting their model in the context of the real problem.

A second modeling process was examination. Examining means comparing the real world conclusion (that resulted in interpretation) with the real problem. In examining, students considered the modeling purpose to ensure that the real world conclusion aligned with the realistic situation, in light of the modeling goal. Examining included acknowledging the presence or absence of some characteristics of the situation in the mathematical entity (model).

Sixth grade students' models in the "Best Drug Award" modeling activity showed that students used their average model not only to find the most effective drug but also to elaborate on the differences between the four drugs. For example, Helen asserted that two or more drugs were equal in terms of their effectiveness if their averages' differences were less than 0,1 minutes. Eighth grade students focused on interpreting the different drug averages and examining the relations between the differences in drug averages and the effectiveness of each drug.

Interpreting and examining were also present in the "Where to Live" activity. Sixth graders extensively discussed how their first model (total number of building in each city) was connected to the decision they had to make for the real problem. Alex clarified that five more buildings in favour of Asfalisia did not guarantee that Asfalisia was a better place than Fantanasia. The above interpretation encouraged students to search for a better model. Eighth graders also discussed the importance of weighting the different factors and acknowledged that some factors were more important than other factors. However, they did not use factor weighting in constructing a model and as a result their model was incomplete. On the other hand, they considered parameters related to the road quality and as a result they excluded two cities. They based their conclusion on the fact that it was impossible to drive a car in a city if road quality was only 17%!

One of the reasons that 6^{th} grade students' work in the "University Cafeteria" modeling activity was not successful was due to the fact that students did not interpret their results in the context of the real problem. Of course, students' examined the appropriateness of their model and discussed issues related to the interpretation of their model. There were debates, for example, on deciding which one of the two vendors to chose, in the case they were quite closed in terms of money collected. However, as presented in the fifth case study, students did not further examine their model and therefore they did not make any attempts to align their model with the real problem, by incorporating factors like semesters or time periods. A possible reason was that students might lack necessary mathematical concepts to better mathematize the real problem and therefore to construct a better model.

Eighth grade students made significant efforts not only to interpret their model in the context of the real problem but also to examine different models. Students' first interpretations questioned the appropriateness of their model, since that model could not help them in selecting the best vendors. Interestingly, students suggested including factors like personality in selecting vendors for the cafeteria. Another dimension of models' interpretation and examination was presented in students' final discussion. Students arrived in two different but correct models. One model was based on the different semesters and the second on the different time periods. At that time, students asserted that being competent in different time periods was more important than in different semesters. Students concluded in adopting the time periods model for solving the problem. Finally, it can be argued that in contrast to students' work in previous activity, there were more differences between 6th and 8th graders in the "University cafeteria" modeling activity. A possible reason might be the high complexity level of the aforementioned modeling activity.

Verification of the Solution within the Context of the Real World

Two modeling processes were present in students' work during solution's verification. These processes included checking and evaluating the solution, validating and communicating the results. Verification was apparent in students' discussions on the appropriateness of their solution. A major issue was whether the suggested method or solution could answer the core question of the real world problem. Students discussed whether their modeling work fitted with the purpose of the modeling activity. Part of the verification was aligning. An example of aligning is considering (perhaps during mathematizing or analyzing) whether the desired outcome (model) was a single value or a range of values (e.g., recommending one or more cities in the "Where to Live" activity). Alternatively, aligning can include the ongoing metacognitive activity of comparing the current state (in this case model) to earlier states valued by Lester and Kehle (2003).

Helen's suggestion in the first modeling activity can be considered as an example of aligning. She argued that a drug can be considered as effective if its average reaction time was less than 13,5 minutes. This was quite important, since it resulted in suggesting two drugs instead of one. Similar to 6th graders' comments and verification processes, 8th grade students extensively discussed that one minute difference in drugs' reaction time was a very important factor in deciding whether one drug was effective or not. It should be noted that students not only reached a method (model) to mathematically solve a problem, but they moved within the context of the real problem to verify their results.

Sixth graders' work in "Where to Live" modeling activity resulted in a "positive and negative points" model. Applying that model resulted in selecting the Fantanasia city. Of interest was students' decision to further examine their result, within the context of the real problem. As a result, they moved back on the provided data and commented on city's each individual aspect (e.g., bulldings, road quality, and budget). This examination helped students in concluding that the city was the best possible. Eight grade students verified their solution. They acknowledged the importance of budget's increase and examined the impact of budget on other parameters. This verification was crucial on improving their model and therefore their solution.

A number of differences related to the verification of the solutions appeared between 6th and 8th grade students' work in the "University Cafeteria" activity. Sixth graders did not actually verify their solution. They compared their last model to previous ones and made comments, but they did not actually verify their approach in the context of the problem. This was a major disadvantage of their work and blocked their efforts to further improve their solutions. On the contrary, 8th graders not only reached two mathematically correct methods for solving the problem but they also based their final decision on the context of the real problem. Students finally chose the time periods method and not the semester method and documented that it was more important to base their selection upon real world related factors.

One of the most important modeling processes presented by students in all modeling activities was communication. It should be emphasized here that in all modeling activities students expressed their ideas and solutions not only verbally; students used a variety of representational media, including pictures, graphs and sketches. Students' efforts to convince the imaginary clients about the correctness of their solutions encouraged students to reflect on their models. In their letters, students documented their results about the specific problem and tried to provide solutions for structurally similar problems.

Three differences resulted in comparing 6th and 8th graders' communicating modeling processes. Eighth grade students appeared to be more flexible in using different representations and making the necessary connections between them. Students used a variety of representations in documenting and explaining their results in their letters. The second difference was again located in students' letters. Eighth graders' letters were presented in details, and were based on students' previous approaches. The third difference relied on the discussions students had in their groups. Eighth grade students extensively discussed most of the issues arose during their investigations. On the contrary, 6th graders' discussions were of less importance and in many times students just expressed their ideas, without trying to elaborate on their partners' ideas.

Summary of the Explanatory Framework

The analysis presented earlier elaborated on 6th and 8th grade students' modeling processes and constructed models in three modeling activities. This analysis leads to the interpretation of the students' modeling processes in problem solving as a progressive procedure which occurs within student work in the modeling activities.

The modeling processes occur in an iterative and cyclic way. The nature of the modeling activities fosters the cyclic and continuing development of student modeling processes and allows a progressive modeling processes development to take place. The analytical framework, as discussed, identified the general modeling processes to be observed in the modeling procedure. The analysis of the case studies which focused not only on the modeling processes but also on student mathematical developments helped to identify a number of new modeling processes and sub processes. Additionally, previous analysis concluded that there are also other factors influencing the development of the modeling processes. Among these factors, the present study identified students' modeling abilities, students' grade level, the context and complexity of modeling activities and other tools that might assist students in constructing their models. The general explanatory framework consisted of three layers is presented in Figure 4.13. Modeling processes are presented in a fine framework which means that new insights for the development of the modeling procedure now occur. The first layer of the explanatory framework, consists of modeling processes in real world problem solving is presented in Figure 4.14. Modeling abilities and influencing factors are presented in Figure 4.15.

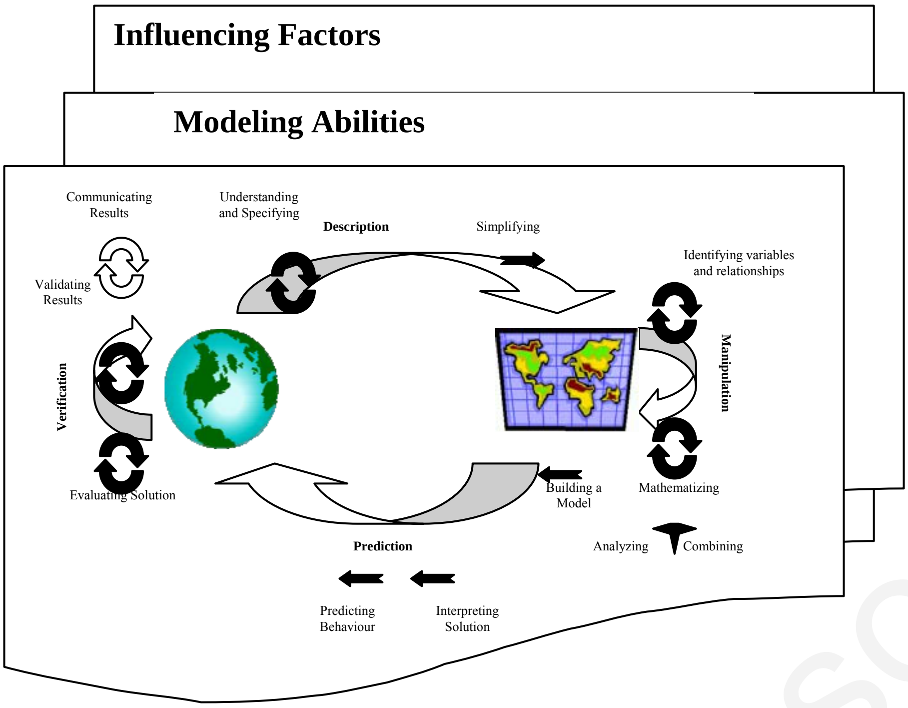


Figure 4.13. The explanatory framework consists of the modeling procedure, modeling abilities and influencing factors.

A number of modeling processes and sub processes are related to the description of the problem. Students first try to understand the core question as they start working with the real world problem. Understanding the problem is not simple yet one way road. Exploring the problem to gain understandings is not just a simple statement. Immediately, students start specifying the problem. In doing so, students try to perceive different and additional information about the problem (Dunne, 1998). What is important to observe is that students immediately start observing mathematically the problem context during the process of exploring and describing the problem. Of course, mathematization is a process that appears during manipulation of the mathematical entity. However, in most of the cases it is apparent that students can not simply describe the problem without implicitly or explicitly start mathematizing the problem.

During describing and specifying the problem, students identify a number of conditions and assumptions of the real world problem. The identification of conditions and assumptions assists students in excluding irrelevant or less important information and moves students' efforts into simplifying the situation in the sense that NCTM (1989) recommends to use "simplify" or "simplification" (p.138) in mathematical problem solving. The latter is of great importance since it actually makes the first connection between the original problem situation to the particular model world (Lester & Kehle, 2003)

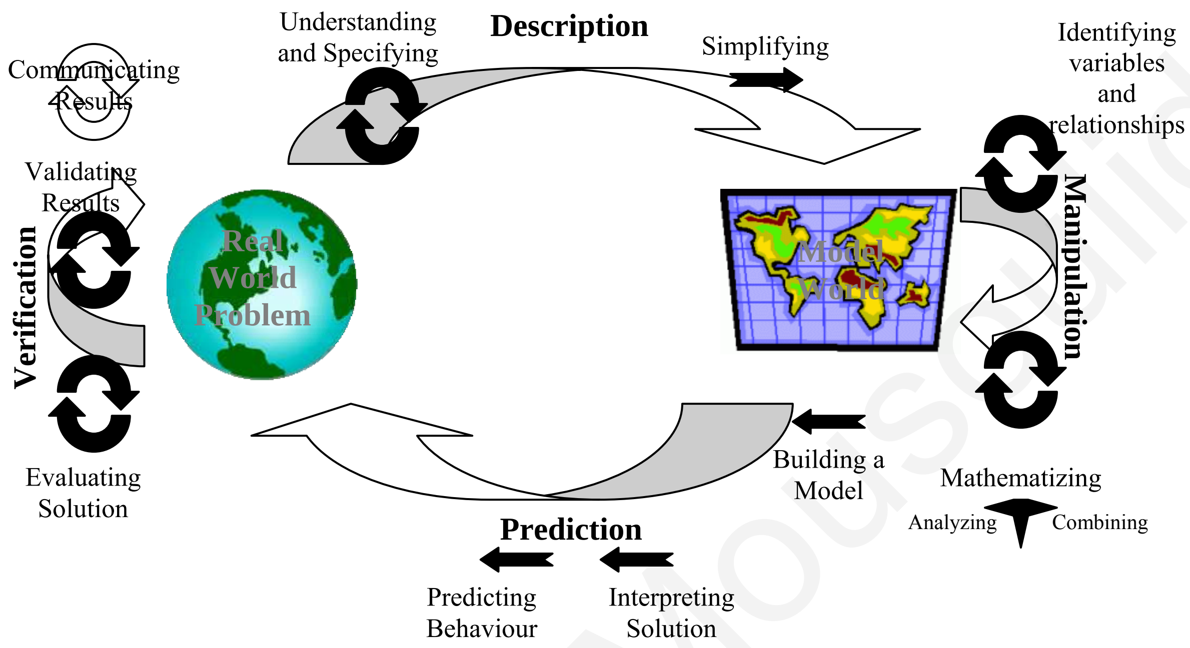


Figure 4.14. Modeling processes in mathematical problem solving.

The core characteristic of student work in the problem manipulation is mathematizing. Mathematizing includes introducing mathematical concepts and processes that relate to the mathematical model. A number of modeling sub processes appears during mathematization. Using mathematization students use the conditions and assumptions they identify during the problem description, to acknowledge and create the necessary mathematical properties in formulating the mathematical model. A second modeling process is the identification of variables and relationships within the mathematical entity. A third process relates to the construction of a model. Students analyze and combine the variables and relationships they identified to further manipulate and create a coherent and well structured mathematical model. A last characteristic of mathematizing is that it may require multiple journeys between conditions and assumptions and properties and parameters, a pattern that contributes to a non-linear modeling path. Mathematization is completed with the model construction.

During the stage of prediction students use their mathematical model to interpret the solution and to predict the behaviour of the real problem. Students' efforts first focus on connecting their model with the real world problem. One of the modeling processes is interpretation. Interpreting the solution is grouped together with other processes as in "interpret and validate the solution" (Dunne, 1998) or "interpret the solution, explain and predict" (Barnes, 1991). By interpretation students are creating a link between the mathematical world and the real world problem. In predicting the behaviour of the real problem students first examine the appropriateness of their model. Examining is comparing the real world conclusion (that resulted in interpretation) with the situation while considering the modeling purpose to ensure the real world conclusion aligns with the realistic situation in light of the modeling goal.

A number of modeling processes occurred during the step of verification. Specifically, during verification students are checking and evaluating the solution, validating and communicating their results. In evaluating their solution students check whether their solution can answer the core question of the real world problem. Students are evaluating their solution through aligning their resulting model with the question of the problem. Students are finally validating and communicating their results. Under communication, a number of sub processes are presented; the interchange of ideas, information and instructions about the mathematical entity, the solution, or the process. Communicating can help students not only to explain the solution of the problem but to also predict the behaviour of similar structurally problems and to finally elaborate on and to enrich their solution for solving more complex problems.

Influencing Factors

1. Student Modeling Awareness
2. Context of the problem
3. Grade level
4. Presence of tools

Modeling Abilities

1. Description and Organization
   Context knowledge, Problem Formulation, Conjecture, Data Identification
2. Planning and Execution
   Selection of mathematical ideas
   Tasking, Simplification, development and application of algorithms to data, Estimation, Results evaluation,
3. Verification
   Solution Refinement, Results Interpretation and Applicability, Prediction Extension

Figure 4.15. Modeling abilities and influencing factors layers.

The second and third layers of the proposed theoretical model presents the modeling abilities that influence the modeling procedure and a number of factors that also have an impact on the modeling processes presented in students' work, as students work on real world based problems. As the third layer suggests, one factor that influences the modeling procedure is student modeling awareness. Modeling awareness is considered as student prior modeling experiences, student mastery of modeling and problem solving abilities, and student understanding of related mathematical concepts. Results from the study indicated that students' work was improved as students moved through the sequence of the modeling activities. This improvement in students' work was evident in their constructed models and in the documentation of their results. As modeling awareness suggests, students' understandings of related mathematical concepts was a significant parameter. The mathematical concepts and processes that were necessary for solving the problems presented in the modeling activities were not too complex or sophisticated. However, the extent to which students approached and solved the modeling problems appeared to be related to students' mastery of the necessary mathematical statistical concepts.

The second factor that appears to influence the modeling process is the context of the modeling activities. The created models are molded and shaped by the situation in which they are created and they can not be simply problem specific knowledge. The complexity of a modeling activity can encourage students to integrate a number of mathematical ideas and processes in developing sophisticated and refined models. The "Best Drug Award" activity was not too demanding and therefore students' solutions were quite similar. The "Where to Live" and the "University Cafeteria" activities were more complex and therefore they encouraged students to create a number of different, more complex and refined models.

A third factor that influences the modeling procedure is student grade level. Results from this study showed that students from different grade levels investigated and tried to solve the problems appear in the modeling activities by using quite different approaches. There were a number of differences between the suggested solutions by 6^{th} and 8^{th} graders. A possible reason for that might be the differences in the mathematization processes presented by the two groups of students. It can be argued that eight grade students outperformed their counterparts since they used more sophisticated formulas and approaches in two out of the three activities. Considering that students in middle school are taught mathematics in a more formal and abstract way, focusing more in formulas, symbolic expressions and algorithms, it can be argued that this approach might be a possible reason that 8th graders effectively used such formulas and algorithms in solving the modeling problems.

A last factor is the presence of available tools. It was evident from student work that modeling activities naturally involve the good use of technology. Specifically, students used the available spreadsheet's capabilities and functionality in two modeling activities, not only to make calculations but also to use algebraic reasoning, and to export their results using multiple forms of representations. Further, the presence of the tools encouraged students to immediately mathematize the problem and therefore to use mathematization as a vehicle to understand and specify the problem.

The analysis of students' modeling abilities that are presented in the second layer is the focus of the following part of the analysis. Specifically, the analysis of students' modeling abilities is focused on validating a model for describing the modeling abilities across three categories of modeling problems and on examining the impact of the intervention program on students' modeling abilities.

Validating a Model for Examining Student Modeling Abilities in Problem Solving

The hypothesis implied by the fourth research question of the study, i.e., whether the modeling processes required in solving decision making, system analysis and design and trouble shooting problems constitute distinct modeling abilities in mathematical problem solving was validated using structure equation modeling techniques.

For the validation of the above theory driven model, a model that was based on three latent first order factors, which corresponded to the three distinct categories of problems, namely the decision making, the system analysis and design and the trouble shooting problems was tested and validated. The model was first validated using the data for the whole sample of the study (all 403 students) resulting from the first test administration (before implementing the intervention program). Secondly, in order to show the invariance of the model, the model was separately validated for the different students' sub groups, namely the experimental and the control group students and the 6^{th} grade and the 8^{th} grade students. The model was finally validated using the data from the second test administration for the experimental and the control group students.

Before presenting the proposed model for describing students' modeling abilities, an analysis of the correlations between the observed variables which corresponded to the nine modeling tasks that were used to validate the proposed model is conducted. The correlations between the observed variables are presented in Table 4.10. The proposed model as well as the correlations significance was tested at the 0,05 significance level.

Table 4.10

Correlations Between the Observed Variables

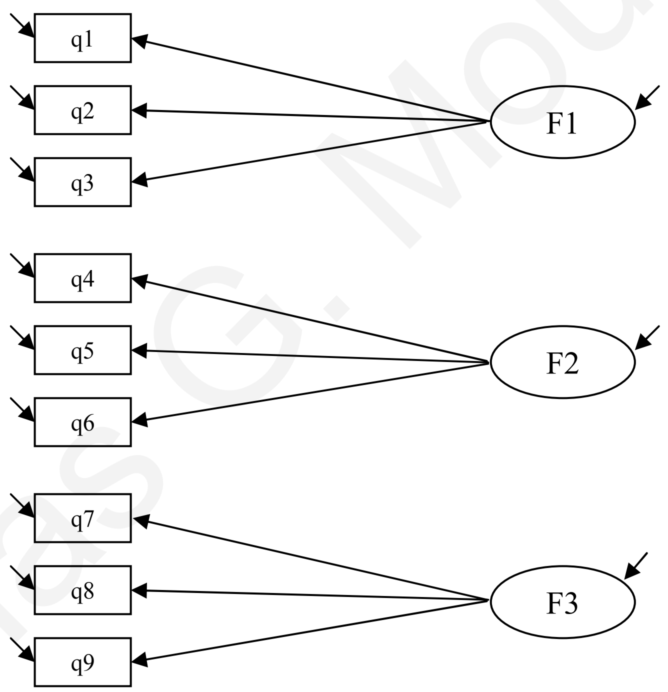
	q1	q2	q3	q4	q5	q6	q7	q8	q9
q1	1								
q2	0,262*	1							
q3	0,293*	0,506*	1						
q4	0,200*	0,130*	0,153*	1					
q5	0,284*	0,283*	0,269*	0,282*	1				
q6	0,145*	0,176*	0,190*	0,190*	0,293*	1			
q7	0,194*	0,164*	0,166*	0,121*	0,242*	0,230*	1		
q8	0,117*	0,085	0,102	0,138*	0,185*	0,485*	0,665*	1	
q9	0,172*	0,132*	0,263*	0,134*	0,263*	0,152*	0,373*	0,382*	1

* $p<0,05$

According to Table 4.11, all correlations were statistically significant, except the correlations q2-q8 and q3-q8. Specifically, the correlations between the three variables that were assumed to contribute to the decision making problems factor were 0,262 for the pair q1-q2, 0,293 for q1-q3 and 0,506 for the pair q2-q3. Similarly, the correlations between the tasks corresponding to the system analysis and decision factor were statistically significant. The correlation between the q4 and the q5 was 0,282, between q4 and q6 was 0,190 and the correlation between q5 and q6 was

0,293. The correlations between these three tasks were the highest among the correlations of the three tasks (q4, q5 and q6) and the rest of the tasks. Finally, the correlations between the three tasks that corresponded to the trouble shooting factor were again all statistically significant. Specifically, the correlation q7-q8 was 0,665, the correlation between q7-q9 was 0,373 and the correlation between q8-q9 was 0,382.

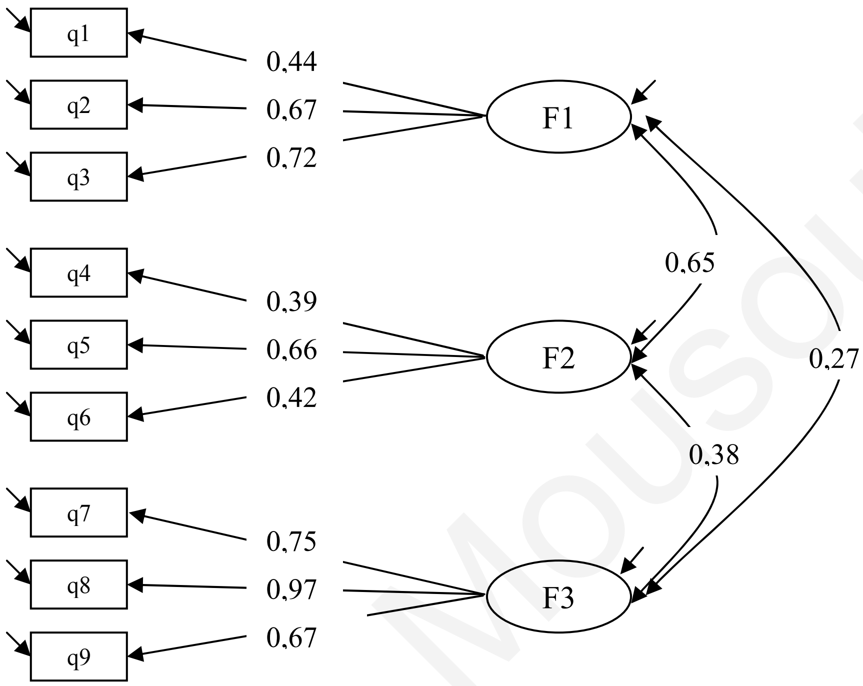
The study posited an a-priori structure of the proposed model and tested the ability of a solution based on this structure to fit the data. The proposed model, which may enable students' modeling processes to be described across three factors, namely the modeling processes involved in Decision Making problems, in System Analysis problems and in Trouble Shooting problems, is presented in Figure 4.16. The proposed model consisted of three first-order factors which corresponded to the student achievement in the three categories of problems. The first order factors corresponded to the decision making factor involving three items, the system analysis factor with three items and the trouble-shooting factor with three items.



Note: q1-q3 refer to the Decision Making Problems, q4-q6 refer to the System Analysis Problems and q7-q9 refer to the Trouble Shooting Problems.
F1=Performance in Decision Making Problems, F2=Performance in System Analysis Problems, F3=Performance in Trouble Shooting Problems

Figure 4.16. The theory driven proposed model.

Confirmatory factor analysis showed that each of the tasks employed in the present study loaded adequately (i.e., they were statistically significant since z values were greater than 1.96) on each factor, as shown in Figure 4.17.

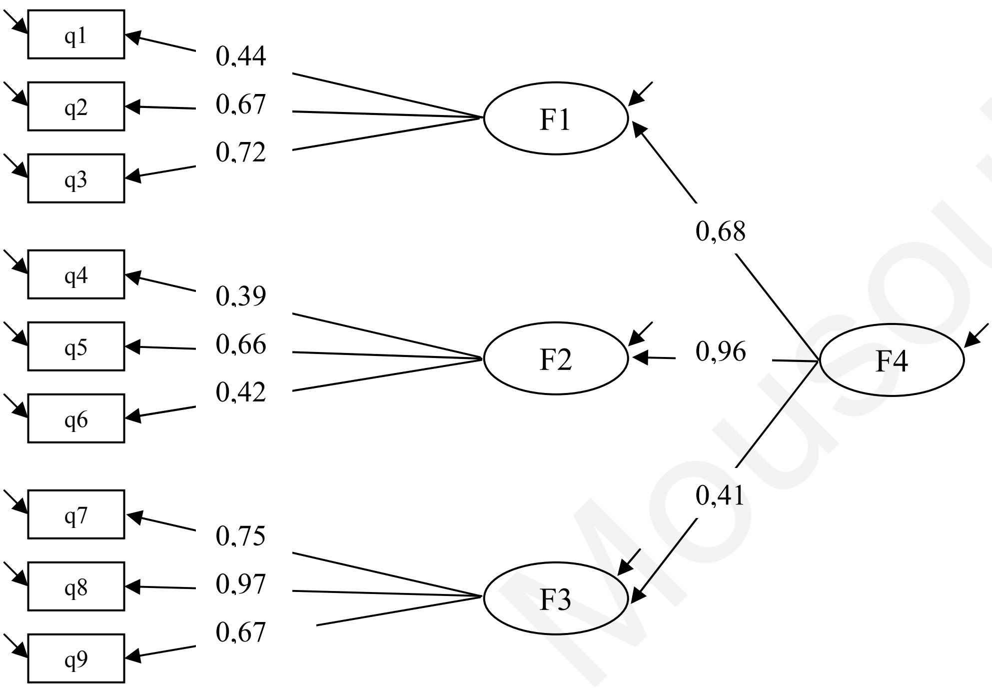


Fit Indices: x^2/df=2,07; CFI:0,971; RMSEA:0,051

Note: q1-q3 refer to the Decision Making Problems, q4-q6 refer to the System Analysis Problems and q7-q9 refer to the Trouble Shooting Problems. F1=Performance in Decision Making Problems, F2=Performance in System Analysis Problems, F3=Performance in Trouble Shooting Problems

Figure 4.17: The model consisted of three first order factors referring to Student Modeling Abilities in Decision Making, System Analysis and Design and Trouble Shooting Problems.

The analysis also showed that the observed and theoretical factor structures matched for the data set of the present study and determined the "goodness of fit" of the factor model (CFI=0,971; x^2 = 49,679; df= 24; x^2/df=2,07; p>0.15; RMSEA=0,051), indicating that modeling abilities in decision making, system analysis and design and trouble-shooting problems can represent three distinct categories of modeling abilities in mathematical problem solving. The distinctness of the factors, as shown by the fact that each item loads on only one first-order factor and all loadings are statistically significant, provides evidence that the tasks used in the test are appropriate measures of the latent factors.



Fit Indices: x^2/df=2,07; CFI:0,971; RMSEA:0,051

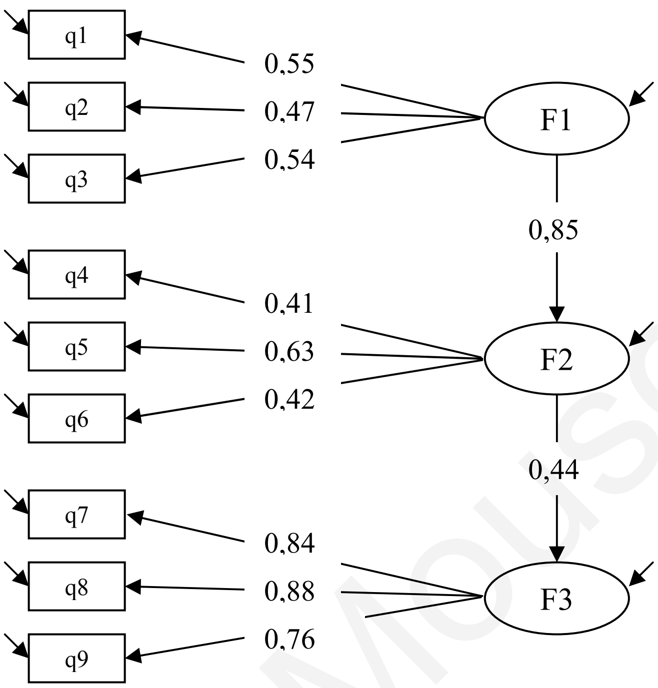
Note: q1-q3 refer to the Decision Making Problems, q4-q6 refer to the System Analysis Problems and q7-q9 refer to the Trouble Shooting Problems. F1=Performance in Decision Making Problems, F2=Performance in System Analysis Problems, F3=Performance in Trouble Shooting Problems, F4=Performance in Modeling Problems

Figure 4.18: The model consisted of three first order factors referring to Student Modeling Abilities in Decision Making, System Analysis and Design and Trouble Shooting Problems and one Second Order Factor representing Student Modeling Abilities

A second model was then tested to examine whether the data of the study could better fit the theoretical model consisted of the three first order factors which are presented in the previous model as well as one second order factor. In this new model, the decision making, system analysis and design and the trouble shooting factors were hypothesized to construct a second order factor referring to a general measure for students' "modeling abilities". This second order factor was hypothesized to account for any correlation or covariance between the first order factors. Figure 4.18 represents the model with the second order "modeling abilities" factor.

A hypothesis implied by the fourth research question was to examine whether the modeling abilities required to successful problem solving representing the aforementioned factors constitute students' abilities in the three types of modeling problems. The structure of the proposed model addresses the appropriateness of such a hypothesis, indicating that modeling abilities in decision making, system analysis and design and trouble-shooting factors formulate a general construct measuring modeling abilities in problem solving. However, it is interesting to discuss the first factor loadings from decision making and system analysis and design factors to the second order factor. These high loadings raised a hypothesis about the existence of a specific developmental trend in the modeling abilities students need to master in solving such problems. To this end, a third model was tested (Model 3), assuming the existence of direct causal relations from decision making factor to system analysis and design factor and from system analysis and design factor to the trouble shooting factor. This model implies that students firstly grasp and successfully employ the modeling abilities in the decision making tasks, secondly by further elaborating and enhancing these modeling abilities in analyzing and designing systems and finally students incorporate modeling abilities in trouble shooting problems.

The results of the analysis validated the hypothesized structural model. The fit indices of the model (see Model 3 in Table 4.11) are adequate enough to support that this model explains in a more accurate way the structure of students' modeling abilities (CFI=0,992; x^2 = 30,334; df=23; x^2/df=1,32; RMSEA=0,029). As presented in Figure 4.19, all loadings from observed variables are statistically significant and their values range from 0,41 to 0,88. Specifically, the loadings from q1, q2 and q3 problems that correspond to the decision making problems in the modeling test to the latent factor representing students' modeling abilities in decision making problems are 0,55, 0,47 and 0,54 respectively. The loadings from the problems related to system analysis and design (q4, q5 and q6) to the corresponding latent factor representing student modeling abilities in system analysis and design problems are also statistically significant and their values are 0,41, 0,61 and 0,42 respectively. Similarly, loadings from the observed variables referring to the trouble shooting problems, namely q7, q8 and q9, to the latent factor representing student modeling abilities in trouble shooting problems are 0,84, 0,88 and 0,76.



Fit Indices: x^2/df=1,32; CFI=0,992; RMSEA=0,029

Note: q1-q3 refer to the Decision Making Problems, q4-q6 refer to the System Analysis Problems and q7-q9 refer to the Trouble Shooting Problems.
F1=Performance in Decision Making Problems, F2=Performance in System Analysis Problems, F3=Performance in Trouble Shooting Problems

Figure 4.19. The model consisted of three first order factors referring to Student Modeling Abilities in Decision Making, System Analysis and Design and Trouble Shooting Problems with Causal Relations.

The model also showed that the observed and theoretical driven factor structures matched for the data set of the present study and determined the ‘goodness of fit’ of the factor model. These results reaffirm the developmental trend as described above and indicate that students are more fluent in employing modeling processes firstly on decision making problems, secondly on system analysis and design and thirdly on trouble shooting problems.

Table 4.11

Fit Indices of the Three Models

Model	CFI	x^2	Df	x^2/df	RMSEA
Three first order factors representing Decision Making, System Analysis and Design and Trouble Shooting (Model 1)	0,971	49,679	24	2,07	0,051
Three first order factors representing Decision Making, System Analysis and Design, and Trouble Shooting and one second order factor General Modeling Ability (Model 2)	0,971	49,679	24	2,07	0,051
Three first order factors representing Decision Making, System Analysis and Design and Trouble Shooting with causal relations (Model 3)	0,992	30,334	23	1,32	0,029

As it is presented in Figure 4.19, students' performance in decision making problems constitutes a strong predicting factor of their performance in system analysis and design problems. The regression coefficient of students' performance in decision making problems on their performance in system analysis and design problems was extremely high (r=0.85, z=4.84, p<0.05). Results also showed that students' performance in system analysis and design problems was a strong predictor of students' performance in trouble shooting problems (r=0.44, z=4.77, p<0.05).

Examining the Model's Invariance

The invariance of the validated model for student modeling abilities in the three categories of problems had to be tested to reaffirm the validity of the model. Measurement invariance can be thought of as operations yielding measures of the same attribute under different conditions. These different conditions include stability of measurement over time, across different populations or over different mediums of measurement administration. To test the invariance of the above model, multiple group structural equation modeling analysis was used examining the viability of the model between (a) 6^{th} grade and 8^{th} grade students, (b) experimental and control group students and (c) first and second administration of the modeling test. These analyses aimed to validate the model for student modeling abilities across different populations and across time.

The results of the three analyses showing that all models are validated (Model 4, Model 5 and Model 6 respectively) are presented in Table 4.12. All fit indices were adequate; CFI was greater than 0,9, RMSEA was less than 0,08 and x^2/df was less than 2. Specifically, the fit indices of the three models were adequate to support that the structural model (Model 3) which describes the relation between students' modeling abilities is not dependent on students' grade (x^2/df=1,65; CFI=0,962; RMSEA=0,05) or treatment group (x^2/df=1,58; CFI=0,965; RMSEA=0,05) and it is stable over time (x^2/df=1,95; CFI=0,982; RMSEA=0,05).

The multiple group analysis models tested (Model 4 and 5) assumed that factor loadings, structural variances and covariances and measurement errors were constant across groups. In the test of multiple group invariance, the fit of the two samples taken across the two groups showed that the factors structure, the correlations and the residuals were invariant when all three elements were constrained to be equal. In summary, the pattern of results from the invariance tests demonstrated that problems of the modeling test were measuring the same latent factors across the various groups of the population and confirmed that there was a developmental trend across the three modeling abilities factors.

Table 4.12

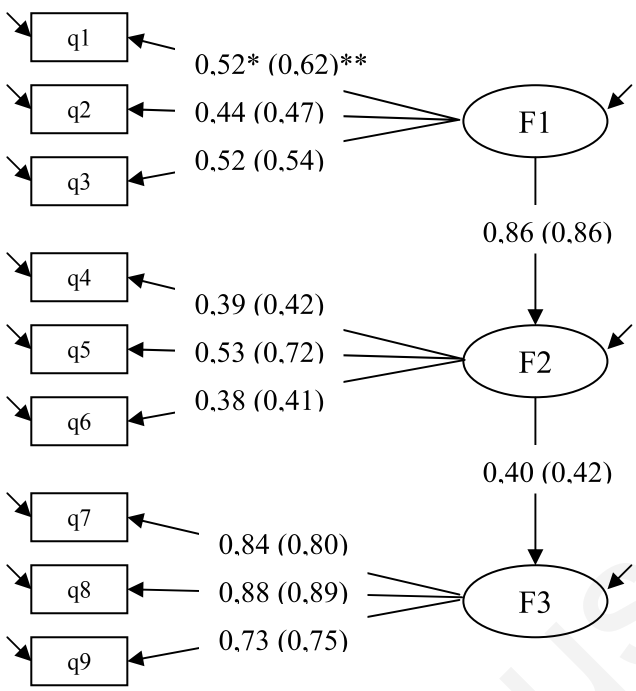
Fit Indices of the Validated Models

Model	CFI	x^2	Df	x^2/df	RMSEA
6th and 8th Grade Groups Model (Model 4)	0,962	85,710	52	1,65	0,057
Experimental and Control Group Model at the 1st Modeling Test Administration (Model 5)	0,965	85,165	54	1,58	0,053
Student Modeling Abilities at the 2nd Modeling Test administration (Model 6)	0,982	44,955	23	1,95	0,050
Experimental and Control Group Model at the 2nd Modeling Test Administration (Model 7)	0,970	88,867	52	1,70	0,059

Sixth and Eighth Grade Students' Model

The standardized solution of Model 4, presented in Figure 4.20, shows that there are only slight variations in the measures described above. There are small differences in the factor loadings for the three latent factors while the regression coefficients are almost identical.

Specifically, decision making factor loadings are slightly higher for the 8th grade group students. Similarly, the factor loadings for the system analysis and design factor are higher for the 8th grade students. Finally, factors loadings for the trouble shooting latent factor are almost identical for the two groups. The only difference between the two groups in favor of the 6th grade students is on the factor loading from the q7 problem. Of course, the most important outcome of the model is that all factor loadings and regression coefficients are statistically significant and the model appears to be invariant for the two student grades.



Fit Indices: x^2/df=1,65; CFI:0,962; RMSEA:0,05

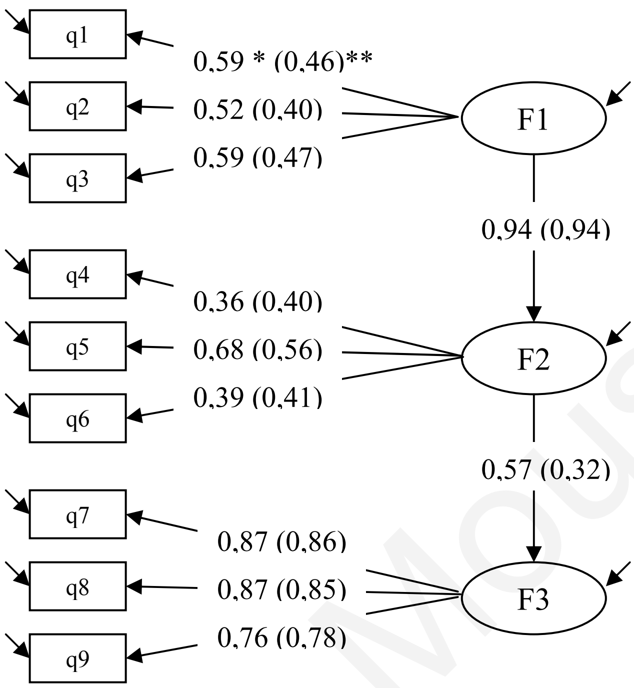
Note1: * 6^{th} grade students group and ** 8^{th} grade students group.

Note2: F1=Performance in Decision Making Problems, F2=Performance in System Analysis Problems, F3=Performance in Trouble Shooting Problems

Figure 4.20. The model for 6^{th} and 8^{th} grade students consisted of three first order factors referring to Student Modeling Abilities in Decision Making, System Analysis and Design and Trouble Shooting Problems with Causal Relations.

Experimental and Control Group Students' Model

Model 5 which is presented in Table 4.12, refers to the Student Modeling abilities model for the experimental and the control group students. Figure 4.21 presents the standardized solution of Model 5. The most important outcome of the Model 5 is that the model is invariant for the two groups, namely the experimental and the control group. Again, all factor loadings and regression coefficients are statistically significant for both groups. A second outcome of the model presented in Figure 4.22 is that there are only small variations between the factor loadings of the two groups.



Fit Indices: x^2/df=1,58; CFI:0,965; RMSEA:0,05

Note1: * experimental group students and ** control group students.

Note2: F1=Performance in Decision Making Problems, F2=Performance in System Analysis Problems, F3=Performance in Trouble Shooting Problems

Figure 4.21. The model for experimental and control group students consisted of three first order factors referring to Student Modeling Abilities in Decision Making, System Analysis and Design and Trouble Shooting Problems with Causal Relations.

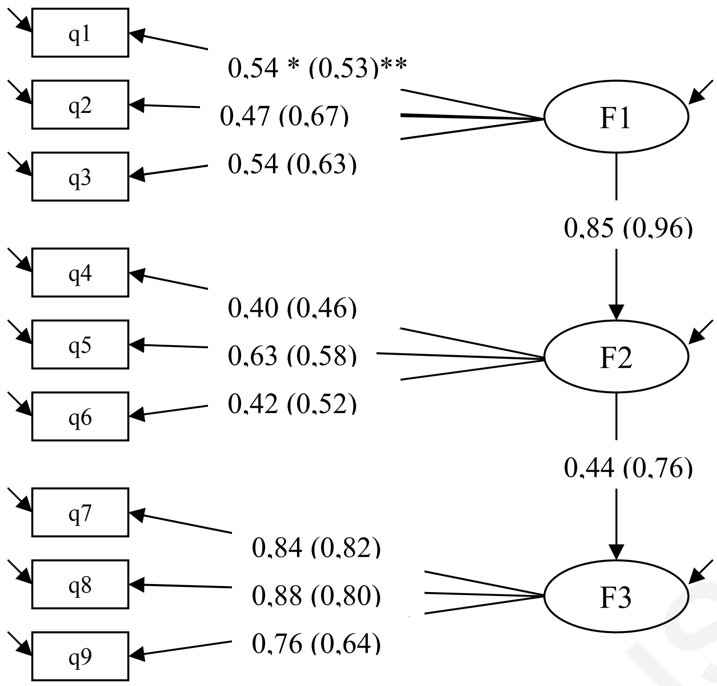
Specifically, the factor loadings for the decision making factor are slightly higher for the experimental group students. On the contrary, loadings for the system analysis and design factor are higher for the control group students, for two problems (q4 and q6). Finally, loadings for the trouble shooting latent factors are almost identical for the two groups. The regression coefficient from the decision making factor to the system analysis and design factor are identical for the experimental and the control group. There is, however, a difference between the two groups in regards to the regression from the system analysis and design factor to the trouble shooting factor. As it can be seen in Figure 4.21 the coefficient was 0,57 for the experimental group and 0,32 for the control group respectively.

Student Modeling Abilities Model at the Second Test Administration

As discussed earlier, model invariance should be tested not only across different populations but also over time (Chan & Schmitt, 2000). To this end, the invariance of the model over time was tested, by examining the fit of the data from the second measurement to the hypothesized structure. The results shows that the model fits the data, and fitting indices are adequate to provide evidence that supports the relation implies in it (CFI=0,982; χ^2/df=1,95; RMSEA=0,05) and give strong support to the assumption that the hypothesized model is stable over time. Figure 4.22 presents the standardized solution of the initial model (Model 3) and the model for students' modeling abilities using data from the second measurement (Model 6).

According to the model presented in Figure 4.22, there are only small variations in the three latent factors loadings. Specifically, factor loadings were improved in the second measurement for the decision making factor. Similarly, two out of the three system analysis and design factor loadings were higher at the second measurement. On the contrary, trouble shooting factor loadings at the second measurement were lower than the corresponding loadings at the first measurement.

What it is important to discuss are regression coefficients. Both regressions were improved. More importantly, the regression coefficient of the system analysis and design factor (F2) on trouble shooting factor (F3) was 0.44 (z=4.72, p<0.05) at the first measurement and 0.76 (z=6.48, p<0.05) at the second measurement respectively.



1st Measurement Fit Indices: x^2/df=1,32; CFI=0,992; RMSEA=0,03
2nd Measurement Fit Indices: x^2/df=1,95; CFI=0,982; RMSEA=0,05

Note: * 1st measurement loadings and ** 2nd measurement loadings

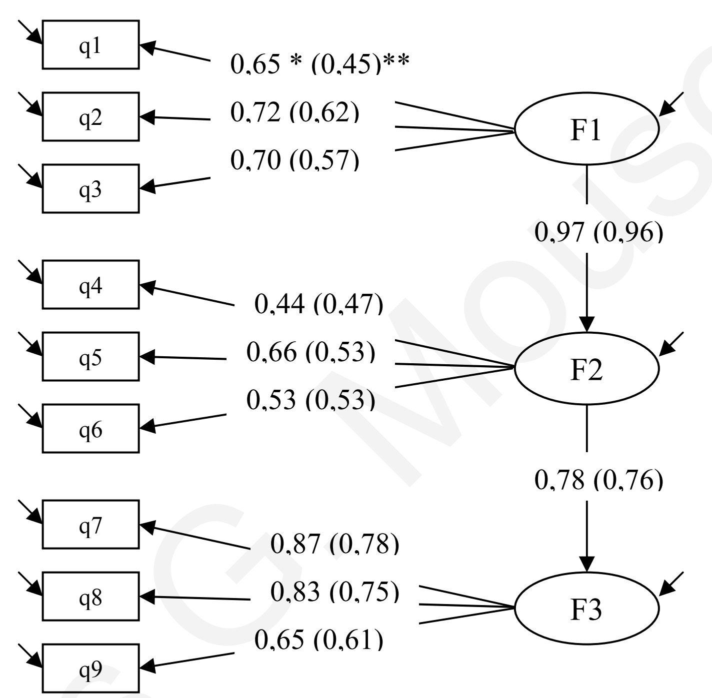
Note2: F1=Performance in Decision Making Problems, F2=Performance in System Analysis Problems, F3=Performance in Trouble Shooting Problems

Figure 4.22. The model for the 1st and the 2nd Measurement, consisted of three first order factors referring to Student Modeling Abilities in Decision Making, System Analysis and Design and Trouble Shooting Problems with Causal Relations.

Experimental and Control Group Student Model at the Second Test Administration

Multiple group analysis was conducted to examine the effect of the intervention program on the structure of the model. Specifically, the fit of the data from the second measurement to the hypothesized model was examined while tracing differences between the experimental and the control group students was attempted. The multiple group analysis model tested (Model 7), assumed that factor loadings, structural variances and covariances and measurement errors were constant across the experimental and the control groups.

The descriptive-fit measures indicated support for the hypothesized structure for the two groups (CFI=0,970; χ^2/df=1,70; RMSEA=0,06). The fit of the model was very good and the values of the estimates were high in all cases for both groups, suggesting that the causal architecture between the three latent



Fit Indices: x^2/df=1,70; CFI:0,970; RMSEA:0,06

Note1: * experimental group students and ** control group students

Note2: F1=Performance in Decision Making Problems, F2=Performance in System Analysis Problems, F3=Performance in Trouble Shooting Problems

Figure 4.23. The model for the experimental and control group students for the 2nd Measurement, consisted of three first order factors referring to Student Modeling Abilities in Decision Making, System Analysis and Design and Trouble Shooting Problems with Causal Relations.

factors accurately captures the data. Specifically, as shown in Figure 4.23 the analysis showed that each of the problems used in measuring student modeling abilities in decision making, in system analysis and design and in trouble shooting problems loaded adequately on each of the three hypothesized factors (F1, F2 and F3). Factor loadings were large and statistically significant for both groups and the patterns of correlations were logical and consistent.

As it is presented in Figure 4.23, factor loadings for all three factors were slightly higher for the experimental group students. A second outcome of the model was that regression coefficients were almost identical, showing that the structure elements of the model hold true for both groups, even after the intervention program. The validation of the multiple group analysis structural model provides support to the claim that the hypothesized model developed on the data of the first measurement is invariant regardless of the treatment of the students. It was shown that students' participation in the intervention program did not result to a change in the relation between students modeling abilities in the three types of problems or the predictive validity of the three factors. The causal relation between F1 and F2 was almost identical for the two groups ($r_{experimental}$=0.97 and $r_{control}$=0.96) and close to the regression coefficients of the first measurement ($r_{experimental}$=0.94 and $r_{control}$=0.94). The causal relation between F2 and F3 was also almost identical for the two groups ($r_{experimental}$=0.78 and $r_{control}$=0.76) but much stronger than the first measurement ($r_{experimental}$=0.57 and $r_{control}$=0.32).

Effectiveness of the Intervention Program and Student Modeling Abilities Growth

To examine the effectiveness of the intervention program and to examine students' modeling abilities growth, a Latent Growth Modeling (LGM) analysis was conducted. Specifically, the analysis was used to answer the following research questions:

What is the impact of the intervention program on students' modeling abilities?

How student modeling abilities are changed over time (rate of change) and what is the impact of the intervention program on the modeling abilities' rate of change?

Comparison of the Experimental and Control Group Initial Achievement

Prior to conducting data analysis to answer the research questions and to examine the differences between the two treatment groups, an alpha coefficient of 0,924 was determined, demonstrating the instrument's reliability as a scale.

Analysis of Variance test (ANOVA) was conduced to examine whether there were statistically significant differences between the two treatment groups in the two different grades prior to the implementation of the intervention program. Two ANOVA tests were conducted, one for the 6th grade students and one for the 8th grade students. In the ANOVA test, students' achievement in the modeling test was used as dependent variable and student group as the independent variable. The results for the 6th grade students ($F_{(1, 195)}$=.427; p>0.05) showed that there were no initial significant differences between the experimental and the control group. Similarly, the results for the 8th grade students ($F_{(1, 204)}$=2,252; p>0.05) showed that there were no initial significant differences between the experimental and the control group. Table 4.10 presents the means and the standard deviations of the experimental and the control group students in the three categories of problems.

Sixth grade students' performance in the modeling test was equal in the two groups ($\overline{X}_{\text{experimental}}$ = 0,685; $SD_{\text{experimental}}$ =0,237; $\overline{X}_{\text{control}}$ = 0,707; SD_{control} = 0,234). Specifically, control group students' mean achievement score was higher than the respective mean score for the experimental group students. However, this difference was not significant. Eighth grade students' performance in the modeling test was also equal between the experimental and the control group. Specifically, the mean achievement score for the 8th grade control group students ($\overline{X}_{\text{control}}$ = 0,673; SD_{control} = 0,253) was higher than the respective mean score for the experimental group students ($\overline{X}_{\text{experimental}}$ = 0,604; $SD_{\text{experimental}}$ =0,297). However, the difference between the two groups was not significant.

Table 4.13

Means and Standard Deviations of Experimental and Control Group Student Achievement in the Three Categories of Modeling Problems.

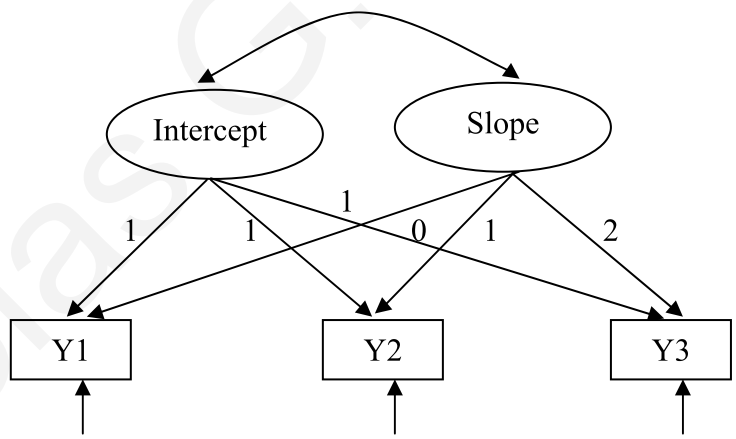
	6th Grade (n=197)				8th Grade (n=206)			
	Experimental		Control		Experimental		Control	
	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Achievement in the Modeling Test	0,685	0,237	0,707	0,234	0,604	0,297	0,673	0,253

As a concluding remark it is important to stress out that analysis of students' achievement during the first test administration showed that there were no statistically significant differences between the experimental and the control group students not only in the whole test in general but also in all three problem types. The equivalence between the two groups is essential in order to examine the effectiveness of the intervention program. Since students' initial achievement in the modeling test was equivalent between the two groups and the fact that from each school one class participated in the experimental and one in the control group, allowed the investigator to assume that possible differences in students' achievement at the second and third test administration will be a result of the implementation of the intervention program.

Latent Growth Modeling

Latent growth modeling (LGM) was used to analyze the longitudinal data for this study. As presented in the third chapter, the test for measuring student modeling abilities was administered three times, both for the experimental and the control group students. The first test administration was conducted before the beginning of the intervention program. The second administration was conducted after the implementation of the first three modeling activities. Finally, the test was administered for a third time after the implementation of the whole set of the modeling activities. The latent growth modeling method was used since it is appropriate in not only describing each individual's growth trajectory, but also capturing individual differences in these trajectories over time. The resulted are expected to contribute in answering the following research question: Can working with modeling activities enhance students' achievement in mathematical modeling? In other words, by examining students' growth in the modeling abilities for the two groups the investigator can examine the effectiveness of the intervention program and observe the patterns of students' modeling abilities change over time.

The hypothesis implied was that students' achievement in the modeling test would develop in a linear way across the three test administrations. To this end, the investigator validated the model fit of a model assuming that the loadings on the rate of change - Slope (S) latent factor are fixed to constants that correspond to the times of measurement, beginning with zero for the initial measurement before working with the modeling activities and ending with two for the third measure of student achievement in the modeling test. Because these loadings are evenly spaced (i.e., 0, 1, 2), the Slope (S) factor represents linear change over time (see Figure 4.24).

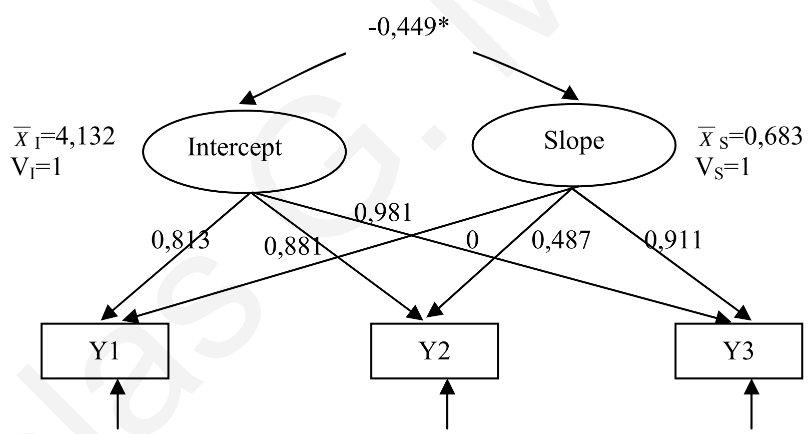


Note: Y1, Y2, and Y3, indicate modeling achievement in first, second and third test administration.

Figure 4.24. A hypothetical latent growth model of student schievement in modeling.

The latent growth model of 6th grade students' achievement in modeling is presented in Figure 4.25. The model has good model fit indices. Specifically, the fit indices for the 6th grade experimental group were $x^2/df=1,625$; CFI=0,996; and RMSEA=0,05. The values of the estimates were high in all cases, suggesting that the linear growth curve model (0, 1, 2) captures the data for the 6th experimental group students.

As it is presented in Figure 4.25, the mean value of the intercept latent factor for the 6th grade intervention group was 4,132 and the mean value of the slope latent factor was 0,683 respectively. The slope was positive and statistically significant, indicating that student achievement in modeling improved significantly during the intervention program. Furthermore, the latent factor of intercept is negatively and significantly associated with the latent factor of slope (r = -0,449), indicating that 6th grade students with lower initial intercept values are likely to have bigger rates of change (in other words have bigger slopes). This result gives strong support to the assumption that the students with low achievement in the first measurement of the modeling test benefited more from the intervention program, than their classmates who had better initial achievement levels.



Fit Indices: $x^2/df = 1,625$; CFI = 0,996; RMSEA = 0,05
* p<0,05

Figure 4.25. The latent growth model for the 6th grade experimental group.

The latent growth model of 8th grade students' achievement in modeling is presented in Figure 4.26. The results showed that the shape of the growth curve for the 8th grade experimental group was linear. In other words, the linear model of growth for the 8th grade experimental group fitted the data adequately. The values of the estimates were high in all cases, suggesting that the linear growth curve model (0, 1, 2) captures the data for the 8th graders. Specifically, the fit indices for the 8th grade experimental group were $x^2/df=1,993$, CFI=0,994, and RMSEA=0,07. An important finding of the latent growth modeling analysis was that the slope was positive and statistically significant (S=2,071, Z=17,565).

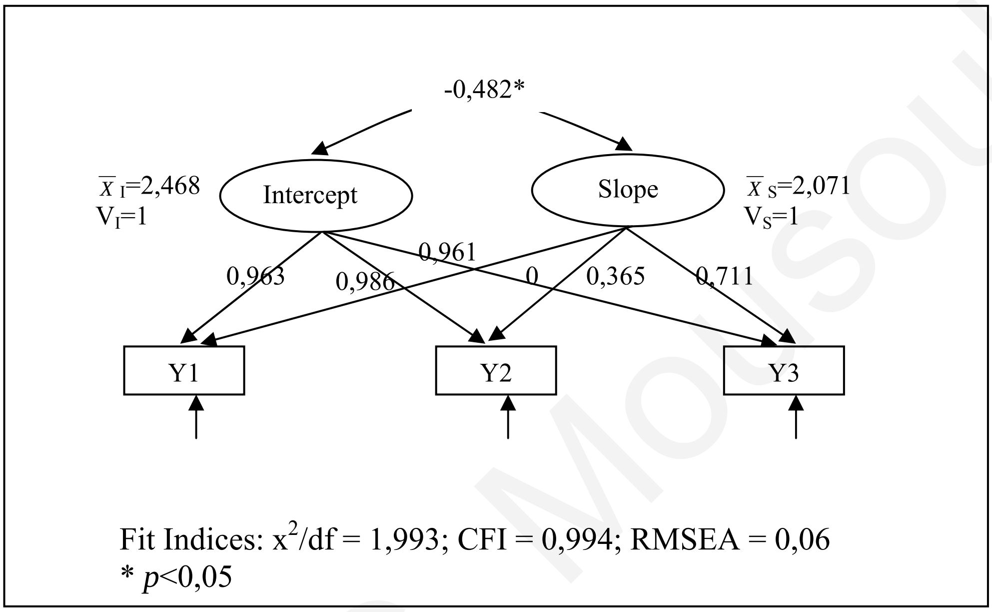


Figure 4.26. The latent growth model for the 8th grade experimental group.

The mean value of the intercept latent factor for the 8th grade intervention group students was 2,468. Furthermore, similarly to the 6th grade students' model, intercept is negatively and significantly associated with slope (r = -0,482), indicating that 8th grade students with lower initial intercept values (low achievement at the 1st test administration) are likely to have bigger rate of change. This result supports the assumption that students with low achievement in the first measurement of the modeling test benefited more from the intervention program than their classmates who had better initial achievement levels.

To examine the effectiveness of the intervention program the investigator also tested data fit to the latent growth model for 6th grade control group students' achievement in modeling. The results showed that the shape of the growth curve for the 6th grade control group was also linear (0, 1, 2). Specifically, the fit indices for the 6th grade control group were x^2/df=1,611; CFI=0,998; and RMSEA=0,05. Similarly to the respective model for the 6th grade experimental group, the slope for the above model was positive and statistically significant (S=0,242, *Z*=2,137).

The mean value of the intercept latent factor for the 6th grade control group was 4,055 (see Figure 4.27). Furthermore, and in contrast to the 6th grade students' model, the intercept is not significantly associated with slope. This result indicates that 6th grade students with lower initial intercept values (low achievement at the 1st test administration) are likely to also have lower rate of change. In other words, students who did not perform well in the test at the first measurement they tend to have similar results in the next two measurements. A second finding related to the growth model for the 6th grade control students was the fact that the slope was quite low (S=0,242), implying that students who followed the traditional curriculum did not had any significant impact on students achievement in modeling, as it was measured by the test.

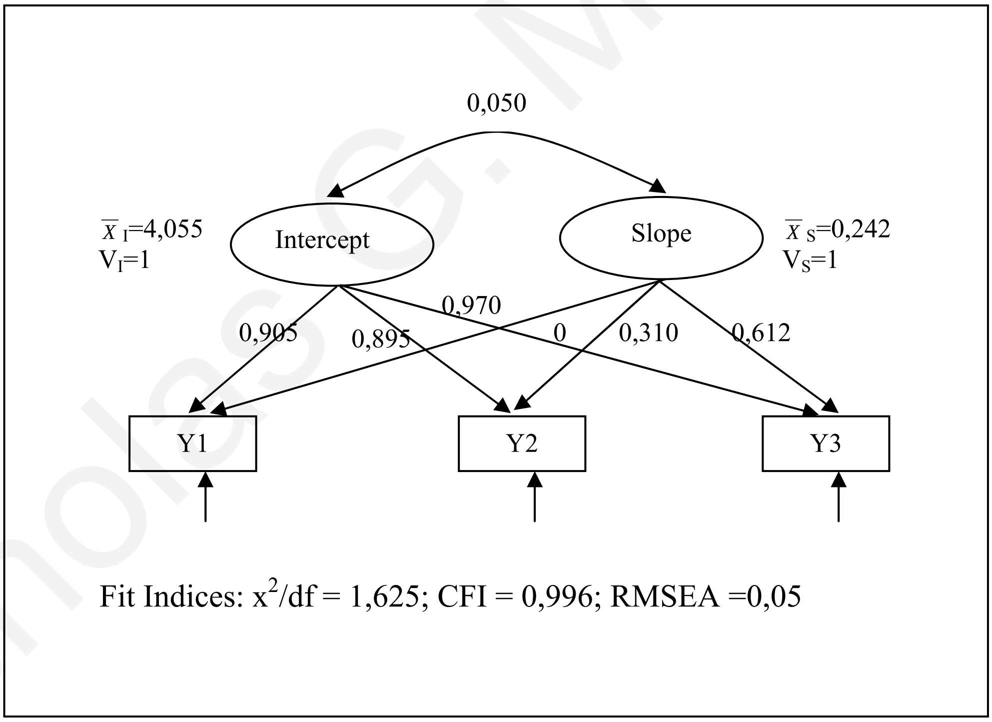


Figure 4.27. The latent growth model for the 6th grade control group.

The latent growth model for the 8th grade control group students is presented in Figure 4.28. The results showed that the shape of the growth curve for the 8th grade control group was also linear (0, 1, 2). Specifically, the fit indices for the 8th grade control group were x^2/df = 1,751; CFI = 0,995; and RMSEA = 0,05. Quite importantly, the slope for the above model was positive and statistically significant (S=0,647, Z=8,055).

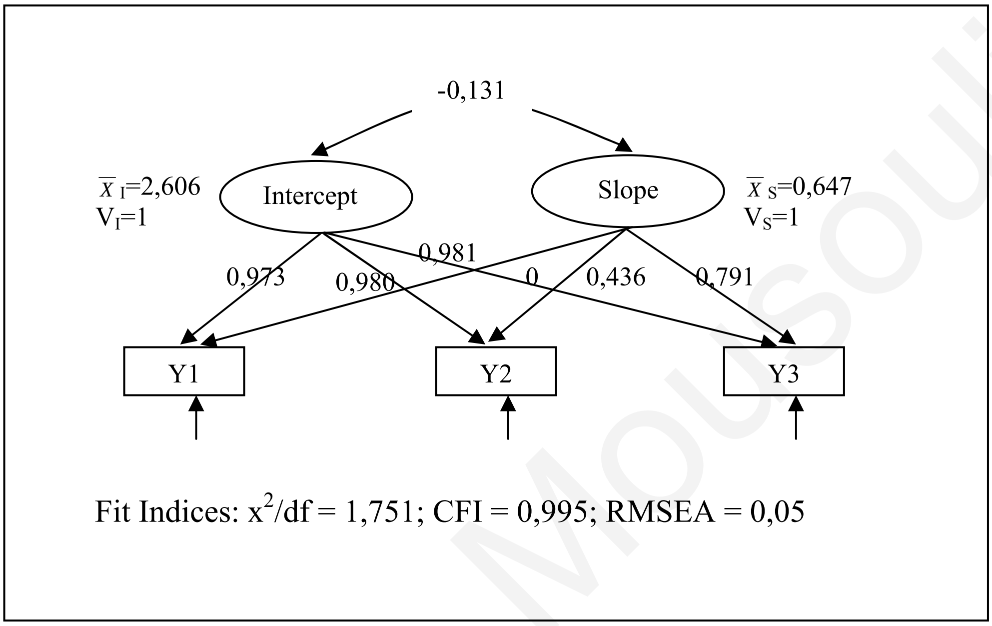


Figure 4.28. The latent growth model for the 8th grade control group.

It is important to observe that in the above latent growth model for the 8th grade control group students was the inexistence of a significant correlation between the intercept and the slope latent factors. This indicates, in contrast to the corresponding results for the experimental group, that there is no relation between students' initial measurement in the modeling test and students' rate of change. The fit indices, slopes and z scores for the four latent growth models are summarized in the Table 4.14 below.

Table 4.14

The Latent Growth Models for the Four Groups

Group	Function	$\frac{x^2}{df}$	CFI	RMSEA	Slope (S)	Z-score
6th Grade experimental	0, 1, 2	1,625	0,996	0,05	0,683	8,187
8th Grade Experimental	0, 1, 2	1,993	0,994	0,06	2,071	17,565
6th Grade Control	0, 1, 2	1,611	0,998	0,05	0,242	2,137
8th Grade Control	0, 1, 2	1,751	0,995	0,05	0,647	8,055

The standardized solutions for the two models for 6th and 8th grade experimental group students indicated that independently of the age group, students benefited from the intervention program. As presented earlier, the slopes for both experimental groups were positive and statistically significant. This implies that students had a positive rate of change in their achievement in modeling. However, the rate of change (slope) for the 8th grade experimental group was higher than the corresponding slope for the 6th grade group. This implies that the intervention program was more effective in improving 8th grade experimental group students' modeling abilities.

A last comment is related to the fact that intercept was negatively associated with slope for both experimental groups. The relations were r=-0,494 and r=-0,482 for 6th and 8th grade respectively. This result indicates that students with initial low achievement levels (performance at the first test administration) had better rates of change (slopes) in their performance and benefited the most compared to students who had better achievement levels at the first measurement. The observed trajectories for the 6th and the 8th grade experimental groups are presented in Figure 4.29.

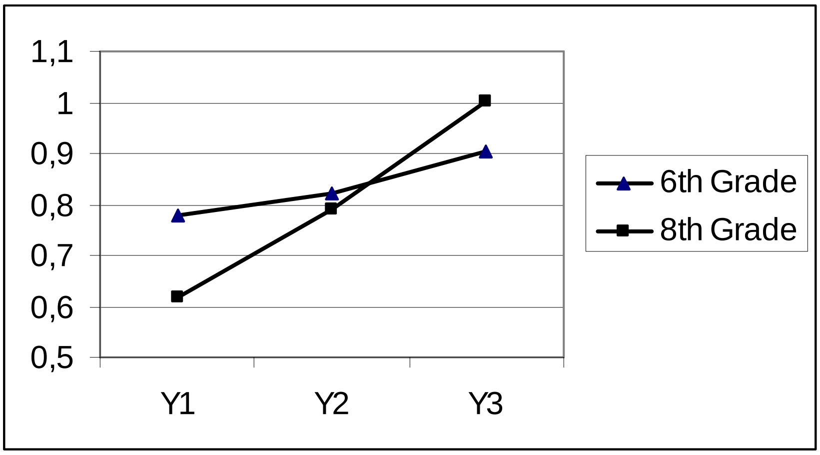


Figure 4.29. Observed trajectories for the 6th and the 8th grade experimental groups.

Sixth grade students not only significantly improve their results compared to their counterparts (6th grade control group), but they also follow a significant growth in their achievement in modeling (Figure 4.29). In other words, the intervention program appears to be very successful for 6th graders participated in the program. As presented in the Table 4.14, the slope for the experimental group was S=0,683 and the corresponding slope for the control group was S=0,242. This is a significant finding; considering that the two groups of students were equal at the beginning of the intervention program, it can be implied that the intervention program was successful for the 6th graders, since the slope for the experimental group was about three times the slope for the control group. A final result considering the growth models for the two 6th grade groups is related to the two latent factors. Specifically, intercept and slope were significantly negative related in the experimental group's model. On the contrary, there was no relation in the control group's model.

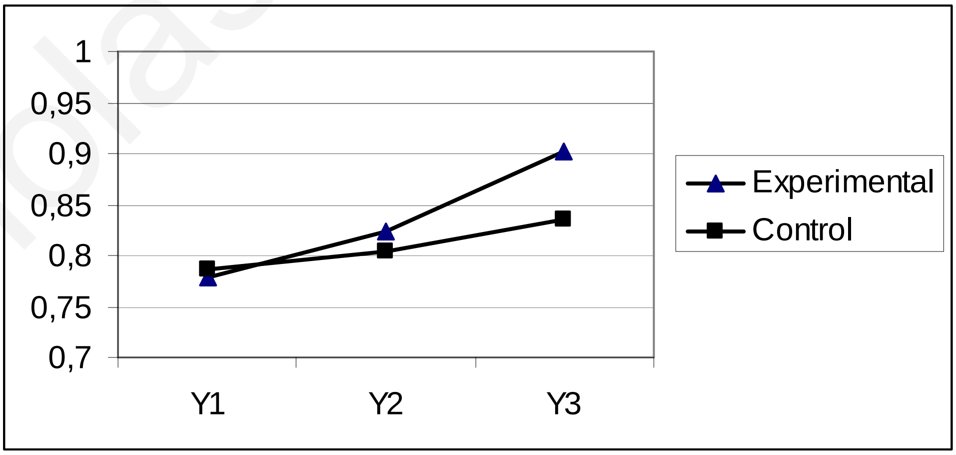


Figure 4.30. Observed Trajectories for the 6th grade experimental and control groups.

The results for the two 8th grade groups' models are quite similar to the results presented earlier for the 6th grade groups. One important finding was the high mean value for the slope in the experimental group ($\overline{X}_S$=2,071). This can be considered as an indicator of the program's effectiveness. Of course, it should be noted that students in the control group also had positive rate of change ($\overline{X}_S$=0,647). However, it can be seen that students that participated in the intervention program had more than three times bigger slope than the control group students. Another interesting result was that intercept and slope were negative related in the experimental group's model. On the contrary, there was no relation in the control group's model. This implies that the intervention program helped more students with low initial achievement in the modeling test. These students appeared to have better values for slope, comparing to other students in the experimental group.

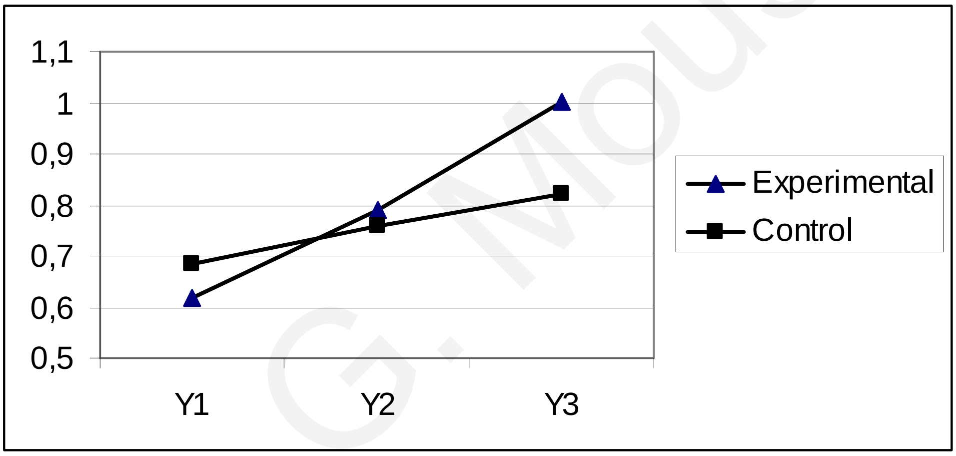


Figure 4.31. Observed Trajectories for the 8th grade experimental and control groups.

CHAPTER V

DISCUSSION AND CONCLUSIONS

Introduction

A number of research studies document the significance of mathematical modeling as a problem solving activity (Lesh & Doerr, 2003; English, 2006; Maaβ, 2006). The review of the related literature reinforces that viewpoint, while poignantly suggesting that the position understates the truth. Mathematical modeling as a didactic means for enhancing mathematical understanding and improving student mathematical problem solving skills can be quite powerful within the framework of appropriately designed and implemented modeling activities (Lesh & Zawojewski, 2007; Lesh & Sriraman, 2005). The complexity in thinking and reasoning required by the interplay between the mathematical world and the "real" world, along with the cyclic and dynamic nature of modeling, has the potential to assist students in developing conceptual understanding of mathematical concepts (Lester, 2005).

Among the key features of the models and modeling framework is the use of a variety of representational media to express the models that have been developed by students, the modeling framework's direction toward solving real world based problems, and its situated-ness (i.e., models are created for a specific purpose in a specific situation). Another feature is that models are developed so that they are modifiable, adaptable and reuseable (Lesh & Sriraman, 2005). The results of the present study provided further support to the above features of the modeling framework. Students not only constructed models for solving the problems presented in the modeling activities, but they also successfully adopted their existing models to solve more complex problems. For example, students adopted the average method they had developed in the "Best Drug Award" activity to solve the "Where to Live" activity. Not surprisingly, there were differences between 6th and 8th graders' ability to reuse their models.

Modeling is a fairly recent branch of mathematics education in general and of teaching mathematical problem solving in particular. Modeling is rich in potential to meet the challenges of today's classrooms and experiencing prolific growth as society and economy ask for school graduates able to be effective problem solvers. However, it is still not extensively researched, and there are many areas of mathematical modeling as a means for teaching and learning mathematics that need further investigation. The theoretical framework presented in the second chapter was meant to help the investigator to interpret the observations and data generated by the investigation. The findings resulted from this study extended the theoretical framework for the models and modeling perspective and provided valuable insights into the modeling processes development.

The models and modeling perspective has been considered as a "prime example of a conceptual framework that has been very useful for the mathematics education research" (Lester, 2005). The models and modeling perspective is not yet a coherent theory for mathematics education. Instead, it is a system of thinking about problems of mathematics learning that integrates ideas from a variety of theories. In line to what Lesh and Sriraman (2005) and Lester (2005) have argued that the development of theory is absolutely essential in order for significant advances to be made in the collective and individual thinking of the mathematics education research community with regard to the modeling perspective. The core purpose of the present study is situated in the above demand for developing a theory – conceptual framework for models and modeling perspective, especially for the school level.

It is obvious that not everything related to models and modeling can be incorporated into a single, even grand theory. A grand theory of everything related to models and modeling can not be developed and even worse, efforts to develop one are very likely to keep the mathematics education community away from making progress toward the goals of the research field (Lester, 2005). Therefore, the purpose of the present study was not to contribute towards the development of a grand theory. On the contrary, the centerpiece of the study was on developing a smaller, more focused theory of the models and modeling as a problem solving activity for teaching, learning and development at the elementary and secondary school level. This position was followed throughout the study, by making use of the conceptual framework of the models and modeling perspective (Lesh & Doerr, 2003) to design and to conduct the study's inquiry.

The purpose of the present study was to develop a conceptual framework – theory for the modeling perspective as a problem solving activity at the school level which can be analyzed along the following three dimensions: (a) Examine the characteristics of the modeling processes, and how they develop as students work in a sequence of modeling activities, (b) examine whether students' work with thought-revealing, modeling activities can enhance students' modeling abilities, and (c) examine similarities and differences between elementary and secondary school students' modeling abilities and how these abilities might be influenced by working with modeling activities. The extent to which the purpose of the study is accomplished arises from the results presented in previous chapters. In the current chapter the results of the study are discussed under the viewing angle of the purpose of the study and more specifically under the research questions of the study. First, the discussion focuses on students' conceptual development. Results concerning the modeling processes and students' modeling abilities are then discussed. Third, the findings related to the new learning theory are discussed. Finally, conclusions are drawn based on the discussion of results and a number of implications for mathematics teaching and learning are presented.

Students' Conceptual Development

What is evident from the results of the study is that modeling activities that do not rely on direct instruction methods or procedures help students to get involved in an authentic process that is both mathematical and scientific in nature. Such engagement aligns closely with expectations outlined by national standards and by national science and mathematics organizations (NCTM, 2000; NRC, 2001; AAAS, 1998). To solve the real problems, students need to make connections between mathematics and the real world; this has been and will continue to be at the forefront of most major reform efforts (NCTM, 2000). This was apparent in all modeling activities. Students made appropriate connections between the real problems and the mathematical models and they were also involved in iterative cycles between the real and the mathematical world in further improving their solutions. Exploring the nature of students' thinking with regard to the constructed models, provides rich learning trajectories that could help students link the real world with more abstract, mathematical models in a far more conceptual way. Linking specific properties and relations that characterize a model to more abstract properties and characteristics can help students to develop abstract – general and abstract – apart relations (Skemp, 1986; Mithelmore, 1993).

The above process facilitates the growth of conceptual knowledge, "as that which is characterized most clearly as knowledge that is rich in relationships… as a connected web of knowledge. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network" (Hiebert, 1986). It also provides students with another facet of the nature of mathematical thinking and learning and enhances mathematical empowerment they otherwise would not obtain through direct instruction.

The modeling activities described in the study challenged the students to produce valid interpretations of a non-structured environment, which involved significant forms of learning. The students' interpretations required an intertwining of their existing (mathematical as well as extra-mathematical knowledge) with new knowledge into a complex conceptual system (a model), that in order to function well needed to be shaped and integrated to fit this new situation. The findings of the study agree with Lesh and English (2005) when they claim that attention should be shifted from asking what kind of computations students can perform correctly to asking what kind of situations they can describe productively. This study does not recommend that all teaching in mathematics should consist of modeling, but in the light of the results, the study proposes that a shift of paradigm is needed when teaching mathematics. Instead of focusing on teaching students to do complex mathematics using simple thinking, the focus should be shifted towards teaching students complex thinking using simple mathematics (Iversen & Larson, 2006).

A final parameter of students' work in the modeling activities was reflective thinking. Specifically, during their modeling work students were engaged in reflection on their own models as well as on other students' models. Similar to what Wood and Turner-Vorbeck (2000) describe for encouraging students to engage in reflection, the context of the modeling activities encouraged students to rethink on their approach in order to offer more information to other students and teacher. The context of the activities also placed the students in situations where they had to deal with vagueness, complexity, or ambiguity. Students finally had to resolve conflicts or disagreements through critical examination. This reflection promoted students' conceptual understandings and assisted them in constructing better and more refined solutions to the problems.

Modeling Processes in Problem Solving

Summary of the Characteristics of Modeling Processes

The first research question examines the modeling processes students presented in their work on thought-revealing, modeling activities. Respectively, the second research question focuses on examining how these modeling processes evolve when students move from model eliciting activities (Best Drug Award) to model exploration (Where to Live) and model adaptation (University Cafeteria) activities.

Data analysis revealed that: (a) working in authentic thought-revealing problems enhanced students' modeling processes; significant modeling processes were present and students effectively employed these processes in solving the provided problems and (b) as students moved from model eliciting activities to model exploration and model adaptation activities, they successfully transferred and modified their models by effectively turning to account the necessary modeling processes. A final comment concerns the mathematical developments students presented in relation to their modeling processes. It was apparent that students gradually employed more sophisticated and complex mathematical concepts and processes in solving the problems.

Discussion of Modeling Processes Within Each Modeling Activity

Understanding the Problem

It appeared that understanding the problem was not a simple neither a straightforward process for students. Students understood the core questions of the problems quite easily, in almost all modeling activities. However, this was not sufficient; more information and understandings were required for providing a comprehensive solution. As a result, students needed more time to explore and understand the problems. Despite the Best Drug Award activity, which was an introductory and quite simple activity, students tried to perceive different information about the provided problems than was originally apparent. This was necessary, since students' first understandings of the problems were not sufficient enough and could not help students move from the real world problem to the mathematical entity. It can be argued from the case studies that students managed to identify the necessary conditions and assumptions for understanding the problem, but not in all cases studies. Specifically, this was one of the differences between 6th and 8th grade students' work. Eighth graders could easier identify the necessary conditions for understanding and therefore for creating a model to solve the problem. On the contrary, 6th graders spent considerable more time to do this. Additionally, when problems were more difficult, the difference between 6th and 8th graders was even more distinct.

As described in detail in Chapter Four, there were occasions in which both 6th and 8th grade students iteratively tried to understand the problem appeared in the "University Cafeteria" activity. The first occasion occurred during 6th graders' initial interpretation of the provided data. The students originally felt that there was no reason to work with the hours worked table, since the money collected table provided all necessary data for solving the problem. It can be argued that it was not easy for students to realize that data from both tables were necessary and important; on the contrary, students started working with data from the first table when they realized that their first model was insufficient and could not help them to solve the problem. Thus, being presented with an inappropriate model and later conflicting information led the students to revise their interpretation and understanding of the problem. The second occasion refers to the work of 8th grade students. Students successfully worked with data from both tables and they quite easily reached and used a "performance rate" model to solve the problem. However, 8th graders also failed to use data related to different semesters and different time periods.

What is important to discuss here is the fact that every time students felt the need to refine their model they came back to reflect on the provided data, trying to identify what was missing from their initial understanding. This finding underlines the importance of the role of the context in the modeling activities. Specifically, the context of a modeling activity and the level of difficulty of the core question need to be demanding and interesting for students. This can increase students' interest to solve the problem and can allow iterative cycles of interpretation and model refinement.

A modeling process that appeared in all case studies during the process of understanding the problem was mathematization. Mathematization processes are presented and discussed by a number of researchers (Borromeo, 2006; Kaiser, 2006). Unlike previous research, the present study shows that the process of mathematization does not appear only when formulating the mathematical entity (which is the next step in the modeling procedure). On the contrary, students start mathematizing the problem during their attempts to understand it. It is obvious from the results of the study that mathematization is necessary for understanding a real world problem. If students do not use, even implicitly, mathematization during understanding and specifying the problem then their efforts might not be successful. In that case, understanding and structuring the problem might need more time. However, the immediate employment of mathematization processes may have negative results. As shown in the "University Cafeteria", in their attempts to understand the problem, 6th grade students started mathematizing their data and they limited their focus on partial information. As a result, their first model was totally inappropriate for solving the problem.

Manipulation of the Problem

The centerpiece of the manipulation of the problem is mathematizing. By mathematizing students manage to make relations between their findings in understanding the problem and building a model. A sub process that appeared in student work was linking the identified conditions and assumptions with the necessary mathematical properties for constructing a model. Based upon the work done by the students, this sub process could not be adopted easily. As a result, there were cases in which students identified important conditions of a problem, they could not link them with properties of their developed model. An example comes from 6th graders' work in the "Where to Live" activity. During their discussion students explicitly realized that the importance of some factors was greater than other factors. However, when students constructed their models they did not consider this important finding and their solution was not the best possible. A possible reason for this inconsistency might be students' lack of the necessary mathematical concepts and processes for making the necessary links between conditions and mathematical properties. Sixth graders' work in the aforementioned activity, for example, and both groups' work in the "University's cafeteria" activity showed that students could not upgrade their model from using a formula like y = a + b + c (city's ranking = schools + parks + shops) to a formula like y = ax + bz + cw (city's ranking = schools*3 + parks*2 + shops*1).

Concerning differences between the two groups of students, it can be argued that eight grade students outperformed their counterparts since they used more sophisticated formulas and approaches in two out of the three activities. Considering that students in middle school are taught mathematics in a more formal and abstract way, focusing more in formulas, symbolic expressions and algorithms, it can be argued that this approach might be a possible reason that 8th graders effectively used such formulas and algorithms in solving the modeling problems.

The above modeling processes that appear in manipulating the problem result in a final (for this step) modeling sub process which emphasizes on comprehending the structure of the model. This sub process is not only important for accomplishing the problem manipulation; it helps students to organize the mathematical properties in a mathematical entity (model) and makes it easier for students to use their model in predicting the behaviour of the real problem (next modeling step). If students do not explicitly comprehend the structure of their model, then it is more probable that they will not transfer successfully their model in predicting the behaviour of the real problem. Therefore, students will not be able to identify possible flaws in and to improve their model.

Predict the Behaviour of the Real Problem

Case studies analysis showed that students mainly used two modeling processes, namely interpretation and examination. These processes are used to determine and describe limitations of the model behaviour. In other words, by interpretation students transfer the produced model to the context of the real problem and by examination students compare the real world conclusion (that resulted in interpretation) with the real problem.

The core question that needs to be answered is whether and to what extent the model/solution addresses the real problem. There are several examples from students' work that show students' awareness of the need to examine their models in the framework of the real problems and how this examination had lead to model refinement. One example comes from 6th grade students work in the "Where to Live" activity. Students had a first intuition arising from examining the provided data, that Fantanasia was the best city. When their "total sum" model ranked another city as the best one, students questioned the appropriateness of their model. By examining, they realized that their first model was not appropriate and they successfully described the changes that needed to be made, in order to refine and improve their model.

Results from the present study showed that in more than half of the cases students examined their models and tried to improve them. There were cases in which students failed to recognize that their models were not sufficient and in that cases a discussion with the researcher and the classroom teacher was necessary to help students overcome this difficulty. This may have been the result of the students' lack of prior knowledge in dealing with problems that do not have one straightforward solution.

It is important to underline that appropriate discussion and explorations helped students in both groups to productively reevaluate their models and to use the necessary mathematical concepts to construct a revised model. However, a substantial amount of time was spent in interactions within the groups and with the researcher and this might be problematic for implementing modeling activities. No significant differences between 6th and 8th graders' work were found with regard to this modeling process. Both groups were engaged in testing and refining their models. However, one difference between the two groups in favor of the 8th graders was that eighth grade students explicitly determined whether a change in a model was a significant one and could result in model refinement. On the contrary, there were cases that 6th graders (see "Drug Award" activity for an example) produced a number of models and they could not easily adopt the best possible model.

Verification of the Solution

In verifying their solutions, students moved from the model world into the real world and tried to align their model with the real situation. This process is not a straightforward task and needs a number of modeling cycles to be achieved. Transcripts from students' work presented in previous chapter showed that both groups of students were engaged in model/solution verification. What is important to underline here, which is somehow contradictory to previous research, is that the solution verification was strongly related to students' first interpretations of data and to the context of the problem. Students did not extensively discuss the meaning and the context of the problem, and as a result they did not question or verify their models. In more complex problems, such as the "University cafeteria" problem, students were not so confident that their model was the best possible and as a result they were attentive in verifying their solution.

Communication is an important modeling process. It needs to be addressed here that communicating is presented not only in the current step of the modeling procedure but during the whole modeling procedure. For the modeling activities presented in the study, communicating is not limited to discuss the solution to solve the real problem, but also in predicting the behaviour of structurally similar problems and for improving the shareability of the solutions. Towards improving communicating modeling process, students were asked in all activities to write a letter to an imaginary client, explaining and documenting their results. This can be considered as a successful approach; in most of the cases students tried to justify their results and got involved in expressing their solutions using verbal and symbolic forms.

Another characteristic of communication was the fact that it is related and influenced by the context of the problem and students' understanding of the problem. If the modeling activity is too simple, then students easily manage to reach a solution and therefore do not get involved in fruitful communication. Additionally, if an activity is too complex then students can not easily communicate their results and they just reproduce the same solution in different forms. For example, 6th grade students produced a number of bar charts in the spreadsheet software in their attempts to solve the "University Cafeteria" problem. In their letters, two of them just reproduced by interpreting verbally their graphs, without discussing their graphs and rankings.

Significant differences appear between the two groups of students in relation to their communicating modeling processes. It was clearly presented that 8th graders explicitly justified their solutions. Additionally, they linked their suggestions with their previous work and provided a pathway moving from less to more efficient models. Sixth graders' argumentations were also significant, although in many cases students did not clearly explained how they reached their solutions. A second difference in favor of 8th grade students was the fact that they actually discussed in their group. Of course, 6th grade students also had fruitful discussions in solving the activities. However, older students' discussions appeared to be more precise, and students really improved their ideas through interaction. In this direction, they managed to solve conflicts that arose and overcome related problems.

The study had demonstrated that communication is an essential part of the modeling procedure. In cases where communication was not appropriate, students failed to improve their models and therefore failed to reach a good model for solving the real problem. It can be argued that the role of communication in the modeling procedure is not simple and further research towards this direction is needed.

Results from the present study showed that students' work with thought-revealing, modeling activities had an impact on students' modeling abilities. For the purposes of this study a sequence of modeling activities was developed and implemented in a number of elementary and secondary school classes. As it was clearly presented in Chapter Four, students who participated in the intervention program significantly improved their modeling abilities and outperformed their counterparts who attended at the same time the regular curriculum in mathematics.

The latent growth models presented in Chapter Four showed that the impact of the intervention program on students' modeling abilities was significant. Specifically, experimental group students' rate of change was three times bigger then the respective rate of change for the control group students. Considering that modeling abilities can enhance problem solving skills, the above finding indicates that it is essential to incorporate modeling activities in the mathematics curricula. These findings are important for a number of reasons.

Modeling abilities are teachable and can be improved with appropriate modeling activities. Even though the implemented intervention program consisted of only six modeling activities, it was evident that students' work with these activities had a significant impact on their modeling abilities. Specifically, there were no significant differences between the experimental and control groups before running the intervention program. The analysis conducted during and after the completion of the program showed that the experimental group students significantly improved their results in the modeling test, in contrast to control group students' results in the same test.

There is a relationship between the school level and students' modeling abilities. The study's findings showed that 8^{th} grade students who participated in the experimental group improved at a greater degree, compared to the 6^{th} grade experimental group students their modeling abilities. In other words, although participating in the experimental group was beneficial for all students, secondary school students' achievement slope was bigger than elementary students' one.

The negative relation between students' initial achievement scores and rate of change implies that although the intervention program was successful for all participating students, the program was more effective for students with low initial achievement in the modeling abilities test. In other words, students with low modeling abilities benefited the most from the intervention program. It can be argued that the context of the modeling activities, social interactions between the students and the teacher, and the absence of direct instruction created a safe environment for those students to present their ideas, to discuss possible solutions and to finally improve their abilities in solving modeling problems.

The confirmatory factor analyses presented in the previous chapter underlined a significant difference in favour of the experimental group students. Specifically, the structural model that was theoretically proposed, fitted to the data for both treatment groups without significant differences before the intervention. Testing the same model for students' data after the intervention program resulted in an important difference between the two groups. Interestingly, it appeared that the experimental group students could transfer easier their achieved modeling abilities from simple to more complex tasks. On the contrary, control group students could do this transfer to a smaller degree. In other words, experimental group students could fluently transfer their modeling abilities from decision making tasks into the more complex system analysis tasks. Consequently, they could use their modeling abilities for solving system analysis tasks into the most demanding and complex trouble shooting tasks.

The above findings imply that teachers can introduce their students to mathematical modeling using tasks like the ones presented in the administered modeling test. Second, the implication of the modeling tasks needs to follow the developmental trend came out from the analysis of the present study. Specifically, students need to start working with decision making tasks in order to develop their modeling processes. Further, students can move on working with system analysis and design tasks. These tasks are more demanding and students need not only to apply prior modeling processes (as the ones they developed in decision making problems) but also to extend and refine these processes to cover the more demanding system analysis and design tasks. In other words, in system analysis and design tasks students need to first analyse the system represented in a specific task and then, based on this analysis, make a decision. Finally, in trouble shooting tasks students need to analyse the presented system and also to identify which part or process of the system is not functioning properly. This identification will lead to proper decision making and trouble shooting the problem.

Student Mathematical Understandings

In line with findings from previous research, the present study showed that student work with mathematical modeling activities had an impact on student mathematical understandings (English, 2006). As this study has illustrated, modeling activities for students develop important mathematical ideas and processes that would be left largely untapped in more traditional classroom activities. The authentic context of the problems allowed multiple interpretations and approaches and as students worked on these activities they got engaged in important mathematical processes such as describing, analyzing, coordinating, explaining, constructing, and reasoning critically as they mathematize objects, relations, patterns, or rules.

In the modeling activities sequence that was presented and analyzed in the present study, students focused on the mathematical ideas of average, ranking, weighting ranks, and selecting and aggregating ranked quantities. Students were also involved in transforming ranked quantities (from quantitative to qualitative and vice versa and transforming from one form of quantitative into another) and finally in generating relationships between and among the above quantities to define descriptive and explanatory relationships. Students also projected a considerable fluency with the use of tables and data. Additionally, as explicitly presented in the "University cafeteria" activity, students were well advanced in the use of multiple format graphs (bar charts, line graphs) to represent and discuss their data.

An important parameter of students' mathematical development relies on the modeling activities sequence. In other words, if modeling activities are presented as isolated parts, then students will not have the opportunity to transfer their mathematical developments from one situation to another and therefore their development will be narrowed. Specifically, a significant change in mathematical thinking occurred when students realized that they should use the models they had developed in "Where to Live" activity for solving the "University Cafeteria" activity. Having made this connection, students tried to transfer the model they had developed for ranking buildings and facilities for the first problem in the context of the second problem (Lesh & Harel, 2003). Quite interestingly, when students realized that transferring the previous model without any adaptations was not good enough, they tried to elaborate on and improve their previous model to solve the new problem. As English and Lesh (2003) reported, these qualitative changes in mathematical thinking can be seen by noticing changes in the mathematical objects and relations that the students attended to, and in their changing notions of the purpose of a given modeling problem.

As a concluding remark, it is important to stress out that the development of powerful mathematical ideas and processes occurs in the presence of diversity, selection and communication. To take advantage of this, modeling activities need to be designed in such a way as to ensure that a diversity of mathematical ideas is expressed and that productive ideas and processes are selected and preserved in the conceptual tools (models) students need to develop.

Building a Theory for Models and Modeling Perspective in Problem Solving

Need to Develop a Theory

To consolidate the significance of the findings of the present study and to identify the necessary characteristics that make the models and modeling perspective going beyond the traditional problem solving research, the investigator will situate the core of the research presented in this study towards the need arose from and Lester's (2005) and Lesh's (2006) work which calls for preparing students to successfully develop the necessary problem solving skills to work collaboratively in solving complex real world problems. In particular, this study investigated how students' work in modeling activities sequences had an impact on student modeling abilities, on students' mathematical understandings and finally, the study investigated similarities and differences in the above two dimensions between elementary and secondary school students.

Previous research in problem solving failed to provide precise answers on how to improve student problem solving abilities. More precisely research did not succeed to change shortcomings that are related to student work which continually introduce new terms to recycle old discredited ideas – without any perceivable value added, continually embellishing ideas that haven't worked - rather than going back to re-examine foundation-level assumptions and do not develop tools to document and assess the constructs they claim to be important (Lesh & Sriraman, 2005). Similarly, Schoenfeld (1993) concluded that teaching students to use general problem-solving strategies generally had not been successful. Among his recommendations were the development and teaching of more *specific problem-solving strategies* (modeling processes) and the development and study of successful ways to improve students' views of the nature of mathematical problem solving.

Towards this dimension, the present study moves from Polya – style problem solving heuristics which are likely to have descriptive power and are more like names for large categories of abilities and processes to go beyond and make such processes being prescriptive ones. Instead of using general heuristics, the models and modeling perspective suggests to use longer lists of more restricted but also more clearly specified and applied modeling processes. Under the models and modeling perspective, the focus is to understand how these modeling processes and mathematical understandings develop.

Towards a Theory for the Models and Modeling Perspective

The aim of the study was to contribute in a theory development for the models and modeling perspective in problem solving at the school level. Different stories of different students, from different grade levels working on a modeling activities sequence were explored. Through these stories an analytical and explanatory framework emerged to describe and interpret the development of student modeling processes in problem solving. Additionally, measures of students' modeling abilities examined the effectiveness of the intervention program and explored differences between the different grade levels.

As expected, modeling is not easy yet straightforward task. The results from this study and especially the explanatory framework presented in Chapter Four provide an in depth presentation of the modeling processes appear in students' work in problem solving. Additionally, research findings focus on a number of influencing factors that interact with the modeling procedure and with students' modeling abilities. A major significant factor is student modeling abilities, which are presented as an additional layer. As presented in Figure 5.1, there are two additional layers which influence and formulate the modeling procedure (front layer).

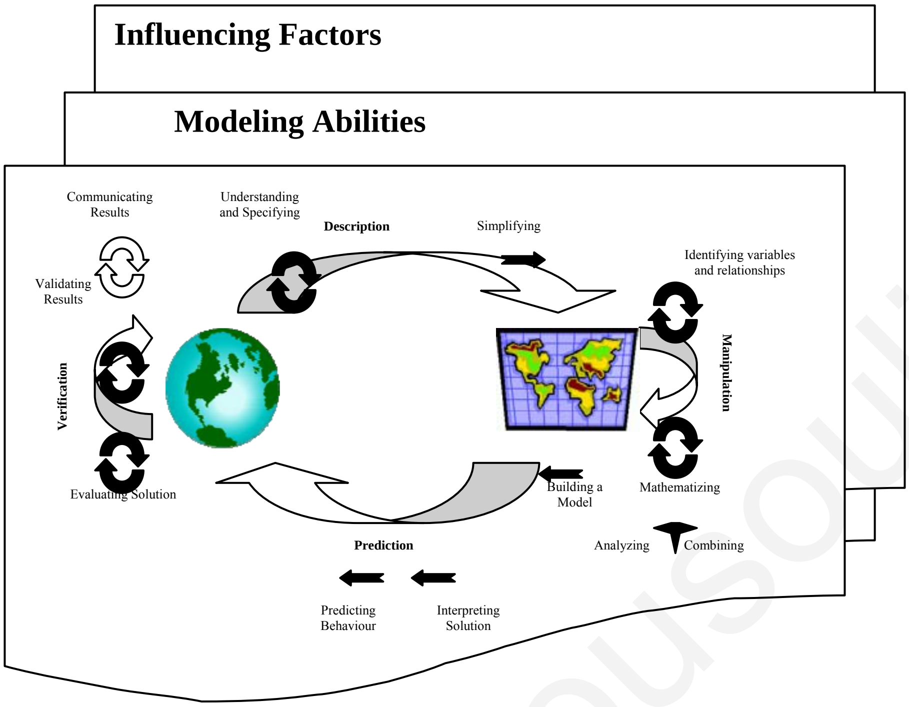


Figure 5.1. The modeling procedure with influencing factors and modeling abilities.

The external factors identified in this study were: Student Modeling Awareness, Context of the problem, Grade level and Presence of tools. A major part of the modeling awareness is student modeling abilities, which are presented in the third layer of the Figure 5.1.

Real world based problems providing the context of the modeling activities are not one way road from given to goals. They are goal directed activities in which adaptations need to be made in existing ways of thinking about givens, goals, and possible solution steps. As a result, products can not usually be short answers but need to involve models (conceptual tools) that can provide an acceptable answer to the specific problem, can be adopted and used in other situations (reuseable) and can also be used by other students (shareable).

As the third layer suggests, the created models are molded and shaped by the situation in which they are created (Context of the problem). However, they can not be simply problem specific knowledge since they need to be adoptable, shareable and reuseable. The constructed models are also influenced by "student modeling awareness". In realistically complex problem situations useful ways of thinking (towards solving the problem) usually must integrate a number of mathematical ideas and processes. This does not fit into a single mathematical textbook topic area or even into a single discipline. Therefore, the student constructed models are much influenced by students' experiences and prior knowledge on modeling problems.

What is claimed is that students' modeling awareness is influenced the quality and effectiveness of the provided models. What is also presented in the third layer is that the whole modeling procedure is influenced by student grade level. Results from this study showed that students from different grade levels investigated and tried to solve the problems appear in the modeling activities by using quite different approaches. Among the significant differences was the bigger number of modeling cycles that older students involved in and the more frequent use of symbolic forms used again by older students.

A parameter that appears to influence the modeling procedure is the presence of available tools. It was evident from student work that modeling activities naturally involve the good use of technology. Specifically, as clearly presented in the fifth and sixth case studies students effectively employed the available spreadsheet's capabilities and functionality, not only to make calculations quickly but also to export their data using multiple forms of representations and connecting these representations to construct a model and suggest a solution for the problem. Further, the presence of the spreadsheet encouraged students to immediately mathematize the problem and therefore to use mathematization as a vehicle to understand and specify the problem.

An important factor is student modeling awareness. In this study modeling awareness is considered as the following: Student prior modeling experiences, student mastery of modeling and problem solving skills (an analytical list of these skills is presented in the second layer), and student understandings of related mathematical concepts and processes. The development of models for solving a real problem is not an all or nothing process. It is assumed that models (conceptual systems) develop along a variety of dimensions (e.g., concrete-abstract, simple-complex, internal-external, situated-decontextualized, intuitive-formal). The procedure and the extent to which these models will be developed and refined are influenced by the above sub factors. Modeling and problem solving skills (strategies) can help students develop adaptations to conceptual systems that they do have and apply these conceptual systems in solving real life situations. These modeling skills are closely linked to student representational fluency. This representational fluency is enhanced as students construct models, expressed using spoken language, written symbols, diagrams, experience-based metaphors, or technical tools.

Recommendations, Implications and Directions for Future Research

This chapter provided a summary of the findings associated with each research question and emphasized on the study's contribution toward developing a theory for the models and modeling perspective. This session will summarize the implications and the directions for future research provided in this chapter. The implications are organized in three categories, implications that related with the design of the study and the modeling activities, implications for teachers, and possible implications that are founded in the results of the study and are in need of further research.

One of the study's implications is related to conceptual change. Specifically, the results of the study confirm all the theories of how conceptual change occurs in mathematical situations, that is when students are able to adapt their existing knowledge to link it to other situations. This conforms and provides empirical evidence to Skemp's (1986) and Hiebert's (1986) theories of mathematical thinking and learning and triangulates the results of the study.

The basic design for this study has tremendous potential to reveal details about students' understanding in modeling. The first set of implications is related with variances of the basic experiment presented in the thesis. The experimental design presented in the study can be tried with different age groups (e.g., younger students at 4th or 5th grade and older students at 9th or 10th grade), different mathematical ability levels (e.g., formulate groups of students based on their achievement in mathematics), and using different types of contexts (e.g., to explore concepts in probabilities). Finally, although the sample size of the student population participated in the study was not small (eight classes participated in the experimental and eight classes in the control treatment group), it would be ideal to repeat the experiment with a larger sample size for the population. Additionally, the basic design for this study referring to the multiple measures of the modeling abilities test could be slightly adjusted to consider a forth measure, in order to monitor the conservation of modeling abilities among the students.

The amount of information gathered for each school, classroom and student was not sufficient to be able to investigate results which depend on students' backgrounds. However, working within the perspective of socio-cultural research, the investigator is aware of the implications of students' backgrounds (at the level of the classroom culture and experience) on their behaviors when working with the given modeling activities. Further research is needed in order to explore the influence of these factors upon students' modeling abilities and the modeling processes appear in students' work. To this end, further research is needed to extend the number of influencing factors appear in the third layer of the proposed theoretical model.

The range of models constructed by students in the study suggests a number of implications for instruction. Given the number of different models constructed by the students, teachers need to be aware that modeling activities will produce diverse strategies and thinking on the part of the students. In addition, teachers should expect a wide array of models that vary from weak and poorly justified models to powerful and well-justified models.

The role of the teacher with respect to the development of student modeling processes needs to be studied in more detail. In particular, the role of the teacher in the interaction with the groups of students in the process of constructing and communicating a model for solving a real world problem is to be investigated. An important question to be answered is whether modeling activities can become a shared space of communication and interaction between students and teacher. Other possibilities of creating a shared space of communication to foster the development of the modeling abilities and a productive interaction between students (not necessarily in the same group) should be investigated. Another example is to consider the use of electronic whiteboards, that provide a medium available to the whole classroom for interaction, discussion and reflection, in order to see the different affordances of these environments in the development of student modeling processes and mathematical understandings.

Another important dimension related to the introduction of mathematical modeling in teaching and learning mathematics is related to the students' affective domain. Quite predictably, students who are beginning to explore modeling often refer to modeling as a pretty interesting and enjoyable activity. Additionally, in most of the cases students feel quite confident to start working on a modeling activity and to create their own models for solving a real world problem. Further research is needed towards this direction to examine, first what is the relation between students' achievement in modeling and their cognitive and metacognitive strategies, and their beliefs and attitudes towards mathematics. Second, it is interesting to investigate whether student work in modeling activities might have a positive impact in their beliefs system.

The last recommendation involves the perceived need for metacognitive scaffolding. With the current level of the available technological tools for mathematics teaching and learning (e.g., dynamic geometry tools, spreadsheets, simulation tools for probabilities and statistics), it is conceivable that instructional technology would be appropriate to create a support structure for modeling. It was not among the aims of the present study to examine the contribution of technological tools in student modeling explorations. However, as presented in the third modeling activity the role of instructional technology is very promising and integrated software tools can be customized to present modeling situations for students to explore in a computerized learning environment. A recommendation towards this direction would be the development of an e-learning platform to support student modeling explorations. This e-learning environment will provide students not only with the appropriate tools for exploring the problem situations but also with a shared virtual space to present and discuss their models, reflect on other student models (from different sites). The examination of the effective introduction of instructional technological tools in modeling activities is a final recommendation for further work.

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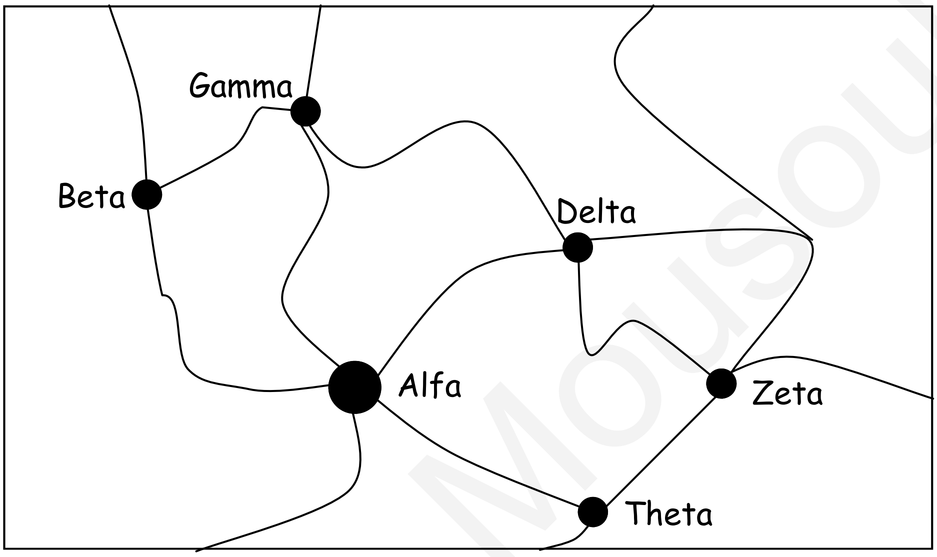
APPENDIX

MODELING ABILITIES TEST

Name: . . . . Class: . . . .

School: . . . . Date : . . . .

1. This is the map of an area. The table below indicates the actual distances between the various towns.



Alfa						
Beta	55					
Gamm	50	30				
Delta	30	85	55			
Zeta	55		100	45		
Theta	30	85	80	60	25	
	Alfa	**Beta**	**Gamm**	**Delta**	**Zeta**	**Theta**

Calculate the shortest possible distance between towns **Zeta** and **Beta**.

Zoe lives in Alfa city. She wants to visit Beta and Gamma. She can only travel **up to 30 kilometres** in one day, but can break her journey by camping overnight anywhere between towns.

Zoe will stay for **two nights** in each town, so that she can spend one whole day sightseeing in each town.

Show Zoe's itinerary by completing the following table to indicate where she stays each night.

Day	Overnight Stay
1	Camp-site between Alfa and Beta.
2	
3	
4	
5	
6	
7	

2. The table below indicates the number of calories that a person needs to burn every day, depending on the job that he or she has and the level of activity.

Age (years)	Level of activity	*MALES* Daily energy need	*FEMALES* Daily energy need
18 to 29	Light	2132	1672
	Medium	2216	1756
	Heavy	2884	1964
30 to 59	Light	2090	1714
	Medium	2424	1798
	Heavy	2842	1958
60 and up	Light	1756	1500
	Medium	2048	1598
	Heavy	2382	1756

The level of activity is closely related to the profession. Some examples are given below.

Light	Medium	Heavy
Salesmen Office employees Housemaids	Teacher Travelling salesmen Nurse	Builder Athlete

Mr. David is a 45 year-old teacher. What is the number of calories that he should burn **daily**?

……………………………………

Mrs. Crystal is a 30 year-old housemaid. What is the number of calories that he should burn **weekly**?

……………………………………

3. Gianna is a **19-year old** athlete. One evening, some of Gianna's friends invited her out for dinner at a restaurant. Here is the menu.

Menu		**Calories**
Soups:	Tomato Soup	70
	Cream of Mushroom Soup	120
Main Courses:	Mexican Chicken	180
	Caribbean Ginger Chicken	160
Salads:	Potato Salad	125
	Couscous Salad	95
Deserts:	Apple and Raspberry Crumble	240
	Carrot Cake	115
Milk shakes:	Chocolate	320

The restaurant also has a Day Menu.

Day Menu
£12
Tomato Soup
Caribbean Ginger Chicken
Carrot Cake

Before dinner on that day Gianna's total intake of energy had been **1700 calories.**

Decide whether the special "Day Menu" will allow Jane to stay within ±100 calories of her recommended energy needs (see Table in previous page). Show your work.

4. A college is offering the following 9 subjects for a 3-year-study. Each subject can be taught in one year.

No	Code for subject	Subject and level of subject
1	M1	Mechanical Studies Level 1
2	M2	Mechanical Studies Level 2
3	E1	Electrical Studies Level 1
4	E2	Electrical Studies Level 2
5	BM1	Business Management Level 1
6	BM2	Business Management Level 2
7	BM3	Business Management Level 3
8	IT1	Information Technology Level 1
9	IT2	Information Technology Level 2

Each student can take 3 subjects every year in order to complete the 3-year study.

Regulations

- A student can take a subject of level 2 or 3 only if he/she has already completed the lower level(s) of the subject on the year before.
- A student can take **Electrical Studies Level 1** if he has/she completed **Mechanical Studies Level 1**. A student can take **Electrical Studies Level 2** if he/she has completed **Mechanical Studies Level 2.**

What subjects should the college offer for each of the 3 years of study?
Give your answer by completing the table below.

	Subject 1	Subject 2	Subject 3
Year 1			
Year 2			
Year 3			

5. 33 children (20 girls and 13 boys) and 7 teachers (3 male and 4 female adults) attended a camping trip.

TABLE 1: Teachers

Ms Margaret
Miss Caroline
Miss Janine
Mr Michael
Mr Mary
Mr Steve
Mr Nicholas

TABLE 2: Dorms

Dorm	Number of beds
Red	12
Blue	10
Green	8
Yellow	6
White	6

Camp Rules
1. Boys and girls should be allocated to separate dorms.

2. There should be at least one adult person sleeping on each dorm.

3. The male adults can only be allocated to the boys' dorms and the female teachers can be allocated only to a girls' dorm.

How should the 33 children and the 7 teachers be allocated to the dorms? Give your answer by completing the table below.

Name	No of boys	No of girls	Teachers
Red			
Blue			
Green			
Yellow			
White			

6. The cardiology department at a local hospital employs 5 doctors. Every doctor can work from Monday to Friday and examine 10 patients per day. In a whole year (365 days, 52 weeks) a cardiologist can have 25 days for holiday, 26 days off for attending seminars and the weekends.

How many days does a cardiologist work?

Can the 5 cardiologists deal with the 12000 patients that are expected to arrive at the hospital during the following year? If not, what will you suggest to the hospital's authorities?

7. Isaac, a 15-year-old, wants to organise a cinema outing with two friends, who are of the same age, during the vacation from 24th March to Sunday to 1st April. Isaac asks his friends for suitable dates and times for the outing.

Fred: *"I've to stay home on Monday and Wednesday afternoons for music practice between 2:30 and 3:30"*

Stanley: *"I've to visit grandmother on Sundays, so it can't be Sundays. I have seen Pokamin and don't want to see it again."*

Isaac's parents insist that he only goes to movies suitable for his age and does not walk home. They will fetch the boys home at any time up to 10 p.m.

<table>
<tr><td colspan="2">EUROPE Cinema

Advance Booking Number: 22 123456
All films £ 5
Films showing from Fri 23rd March for two weeks:</td></tr>
<tr><td>Children on the Net

113 mins

14:00 (Mon-Fri only) 21:35 (Sat/Sun only)
Suitable only for persons of 12 years and over</td><td>Pokamin

105 mins

14:00 (Mon-Fri only) 21:35 (Sat/Sun only)
General viewing, but some scenes may be unsuitable for young children</td></tr>
<tr><td>Enigma

164 mins
15:00 (Mon-Fri only) 18:00 (Sat/Sun only)
Suitable only for persons of 12 years and over</td><td>King of the Wild

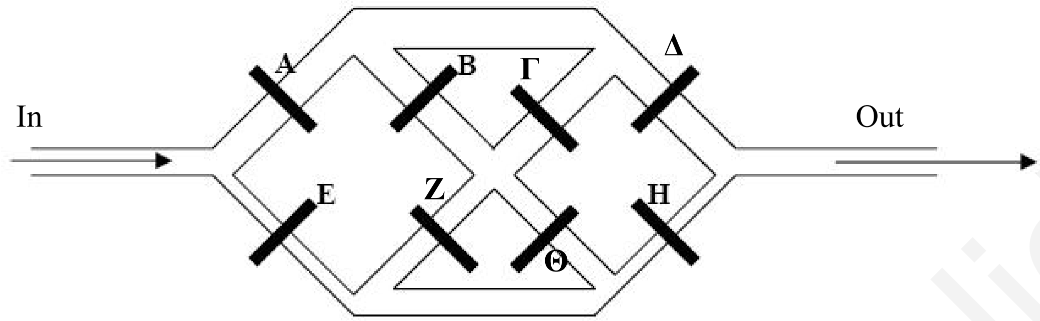
117 mins
14:35 (Mon-Fri only) 18:50 (Sat/Sun only)
Suitable for persons of all ages</td></tr>
</table>

Taking into account the information Isaac found on the movies, and the information he got from his friends, which of the four movies should Isaac and the boys consider watching?

Circle "Yes" or "No" for each movie.

Movie	**Should the three boys consider watching the movie?**
Children on the Net	Yes / No
Pokamin	Yes / No
Enigma	Yes / No
King of the Wild	Yes / No

8. The diagram below indicates how water comes through a system of pipes. Find 5 different ways of getting the water out, opening some of the "doors" every time (for example A,B,Γ,Δ).



1.

2.

3.

4.

5.

Decide for each problem case below whether the water will flow through from **In** to **Out**. Circle "Yes" or "No" in each case.

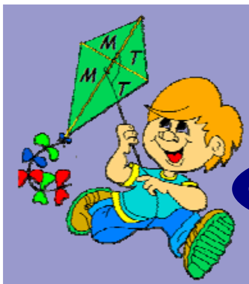
Problem Case	Will water flow through from In to Out?
Gates **A** and **Δ** are stuck closed. All other gates are working properly	Yes / No
Gates **A** and **H** are stuck closed. All other gates are working properly	Yes / No
Gates **H** and **Δ** are stuck closed. All other gates are working properly	Yes / No

9. A vet has to start injecting medication to a sick dog. 240 ml of medicine should be given to the dog over a period of 2 hours. The bottle containing the medicine can provide with 5 drops per 1ml. how many drops per minute should be injected to the dog for the 2-hour period?

Το Καλύτερο Φάρμακο!

Ο Φλέμινγκ ήταν ένας Άγγλος γιατρός και βακτηριολόγος, ευεργέτης της ανθρωπότητας. Το 1945 κατάφερε να πάρει το Νόμπελ χημείας και να μείνει γνωστός στον τομέα της ιατρικής ως ο εφευρέτης της πενικιλίνης Η πενικιλίνη είναι μία ουσία που παρασκευάστηκε με την ιδιότητα να εμποδίζει την ανάπτυξη μικροβίων στον οργανισμό μας και ταυτόχρονα την καταπολέμηση διάφορων ασθενειών. Η πενικιλίνη αποτέλεσε τη βάση για την ανακάλυψη των φαρμάκων τα οποία, πολλές φορές, είναι υπεύθυνα και για την αποτροπή θανάτου σε περιπτώσεις όπου μιλάμε για πολύ σοβαρές ασθένειες.

Με το πέρασμα των χρόνων και την τεράστια ανάπτυξη της τεχνολογίας, η φαρμακευτική επιστήμη σε συνδυασμό με την ιατρική έχει φτάσει σε εκπληκτικά επίπεδα έτσι ώστε για κάθε ενόχληση να μας παρέχεται πληθώρα επιλογών φαρμάκων. Αυτό έχει ως αποτέλεσμα την απαλλαγή της ανθρωπότητας από τη μάστιγα των θανατηφόρων ασθενειών και της παιδικής θνησιμότητας αλλά ταυτόχρονα και την παράταση του ορίου ζωής.



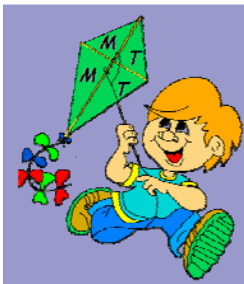
ΜΟΝ²ΤΕΜΠ

2

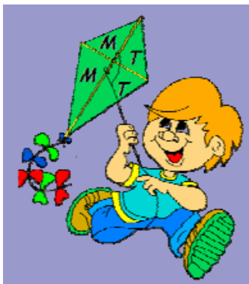
Η επιτροπή φαρμακοβιομηχανιών Κύπρου θέλει να βραβεύσει τη βιομηχανία με το πιο αποτελεσματικό παυσίπονο και να της απονέμει το χρηματικό ποσό των £10000. Για τον πιο πάνω διαγωνισμό έχουν λάβει μέρος 4 φαρμακοβιομηχανίες.

Στον πίνακα αναγράφονται τα λεπτά που χρειάστηκε το κάθε παυσίπονο για να δράσει, σε 20 διαφορετικές περιπτώσεις.

ΚΑΝΑΤΟΛ	ΣΑΡΑΣΕΤΑΜΟΛ	ΡΑΛΠΟΛ	ΚΕΦΑΠΟΛ
20	10	12	10
18	19	14	12
19	13	15	17
22	11	15	17
15	11	7	17
14	12	9	19
23	10	9	22
12	9	8	22
11	8	8	21
10	8	15	10
7	14	19	7
9	13	10	7
10	12	10	7
17	17	23	19
13	11	24	18
12	11	23	14
14	13	10	12
14	20	8	10
8	25	17	10
9	13	19	10



Δείξετε εδώ τον τρόπο με τον οποίο εργαστήκατε. Μπορείτε να γράψετε και κάτι το οποίο δεν θεωρείτε σημαντικό, αλλά το σκεφτήκατε καθώς προσπαθούσατε να βρείτε τη λύση.

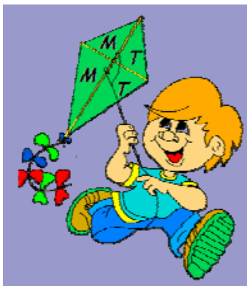


Γράψετε μια επιστολή προς τον πρόεδρο της επιτροπής, κατατάσσοντας τα παυσίπονα σε σειρά αποτελεσματικότητας. Να εξηγήσετε ποια φαρμακοβιομηχανία πρέπει να πάρει το βραβείο και να καταγράψετε το κριτήριο που χρησιμοποιήσατε για την απόφασή σας.

Το ιδανικό μέρος για να ζήσεις!

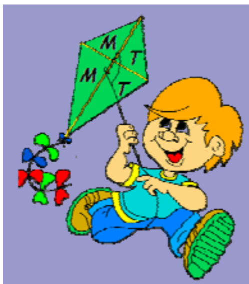
Η Αναστασία τελειώνει φέτος τις σπουδές της και ψάχνει διάφορα μέρη όπου μπορεί να ζήσει και να δουλέψει τον επόμενο χρόνο. Η ποιότητα ζωής και οι ανέσεις είναι πολύ σημαντικά για αυτή. Έχει περιορίσει τις επιλογές των πόλεων που θα μπορούσε να ζήσει, σε αυτές που φαίνονται στον πιο κάτω πίνακα. Κάποιος της είπε ότι η Φαντανάσια είναι μια από τις ιδανικότερες πόλεις. Θα πρέπει η Αναστασία να επιλέξει τη Φαντανάσια ή θα πρέπει να επιλέξει κάποια άλλη πόλη;

Βοηθήστε λοιπόν, την Αναστασία να επιλέξει ποιο μέρος θα ήταν το ιδανικό για να ζήσει. Στην προσπάθειά σας να ενημερώσετε την Αναστασία, θα πρέπει να της στείλετε ένα γράμμα, όπου θα πρέπει να δικαιολογήσετε την επιλογή σας, περιγράφοντας τη διαδικασία με την οποία αξιοποιήσατε τα δεδομένα του πίνακα.



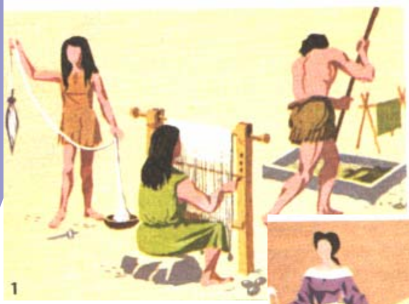
	Πάρκα	Παιδικοί σταθμοί	Σχολεία	Κινημα-τογράφοι	Εστια-τόρια	Καταστή-ματα	Ποιότητα δρόμων (ποσοστό %)	Προϋπολο-γισμός επόμενης χρονιάς
Λιμνούπολη	2	2	7	1	3	23	45.5	—
Ηρεμισία	3	1	4	3	12	16	36.8	πάνω
Ασφαλισία	2	4	5	4	4	26	57.2	κάτω
Παραμυ-θούπολη	0	5	10	0	6	12	19.7	πάνω
Κεφισός	3	2	8	2	5	20	25.8	κάτω
Φαντανάσια	4	3	7	3	8	15	76.2	πάνω

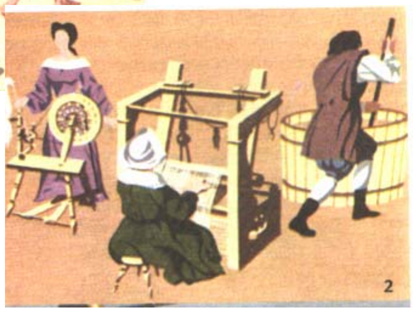
Μπορείτε να χρησιμοποιήσετε τη σελίδα αυτή για να κάνετε τους υπολογισμούς σας. Σε κάθε βήμα όμως μην ξεχνάτε να καταγράφετε και τον τρόπο σκέψης σας…



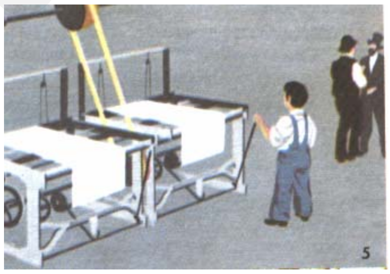
Να γράψετε μια επιστολή προς την Αναστασία, αναφέροντάς της ποια πόλη θα έπρεπε να επιλέξει για να ζήσει. Να εξηγήσετε τον τρόπο με τον οποίο εργαστήκατε για να καταλήξετε στην απόφασή σας.

Από τον άνθρωπο στη μηχανή...



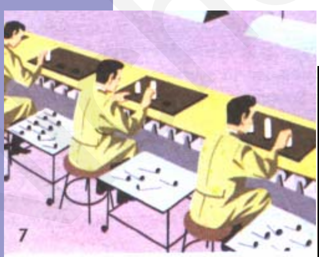


Από τη νεολιθική εποχή μέχρι και το 18ο αιώνα το σύστημα της οικονομικής παραγωγής στηριζόταν σχεδόν ολοκληρωτικά στον ανθρώπινο παράγοντα, μιας και τα τεχνολογικά μέσα ήταν ελάχιστα και σχεδόν ανύπαρκτα. Κύριο βοήθημα για τους εργάτες αποτελούσε ο τροχός, στην απλούστερη, όμως μορφή του.

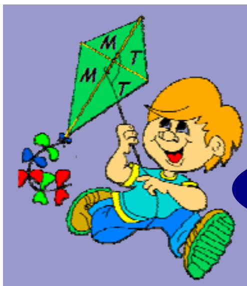


Στη συνέχεια, ο εργάτης ειδικεύεται σε μια μόνο φάση της παραγωγής. Γίνεται δηλαδή, αυτό που λέμε διαχωρισμός και καταμερισμός των εργασιών μεταξύ των εργατών για βελτίωση της ποιότητας των προϊόντων.

Με την πάροδο των χρόνων, οι διάφορες φάσεις της παραγωγής συγκεντρώθηκαν στα εργοστάσια, με αποτέλεσμα να χρειάζονται λιγότερα ανθρώπινα χέρια σε σύγκριση με τα προηγούμενα χρόνια. Ακολούθως, με την αλματώδη εξέλιξη της τεχνολογίας και τη δημιουργία πολύπλοκων μηχανημάτων, η σύγχρονη βιομηχανία κατόρθωσε να παράγει σε επαρκείς ποσότητες ποιοτικά προϊόντα και μάλιστα σε μικρότερο χρονικό διάστημα.



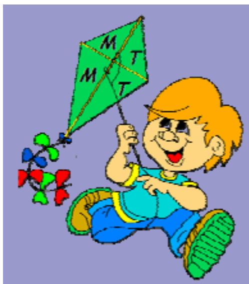
Στις μέρες μας, η εφαρμογή του αυτοματισμού στην παραγωγή προϊόντων και παροχή υπηρεσιών έχει οδηγήσει σε αύξηση της ανεργίας, αφού οι μηχανές έχουν αντικαταστήσει σε μεγάλο βαθμό τον ανθρώπινο παράγοντα. Η δραστηριότητα του ανθρώπου επικεντρώνεται κυρίως στον έλεγχο των παραγωγικών διαδικασιών.



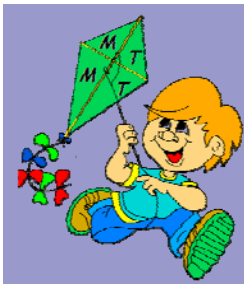
Ο κ. Νικόλας είναι διευθυντής στα κυλικεία του πανεπιστημίου της Φαντανάσιας. Για να αυξήσει την ποιότητα της εξυπηρέτησης που προσφέρουν τοακυλικεία αποφάσισε να αγοράσει αυτόματες μηχανές πώλησης αναψυκτικών και γλυκών. Ως αποτέλεσμα, αποφάσισε να μην επαναπροσλάβει όλους τους υπαλλήλους που απασχολούσε κατά την περσινή περίοδο.

Την περσινή χρονιά εργάζονταν στα κυλικεία εννιά πωλητές. Ο κ. Νικόλας αποφάσισε να εργοδοτήσει για τη φετινή χρονιά τρία άτομα με πλήρη απασχόληση και τρία άτομα με μερική απασχόληση.

Χρειάζεται τη βοήθειά σας για να αποφασίσει ποιοι είναι οι πιο αποδοτικοί υπάλληλοι, ώστε να επαναπροσλάβει τους καλύτερους. Στους πίνακες που βρίσκονται στη διπλανή σελίδα παρουσιάζονται οι ώρες που είχε εργαστεί ο κάθε υπάλληλος και τα χρήματα που εισέπραξε.

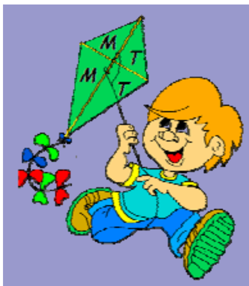


Ώρες									
	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος		
	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*
Μαρία	12,5	15	9	10	14	17,5	12,5	33,5	35
Καίτη	5,5	22	15,5	53,5	40	15,5	50	14	23,5
Τάνια	12	17	14,5	20	25	21,5	19,5	20,5	24,5
Χρίστος	19,5	30,5	34	20	31	14	22	19,5	36
Χάρης	19,5	26	0	36	15,5	27	30	24	4,5
Μαίρη	13	4,5	12	33,5	37,5	6,5	16	24	16,5
Ρούλα	26,5	43,5	27	67	26	3	41,5	58	5,5
Τάκης	7,5	16	25	16	45,5	51	7,5	42	84
Μάκης	0	3	4,5	38	17,5	39	37	22	12



Χρήματα

	Χειμερινό Εξάμηνο			Εαρινό Εξάμηνο			Καλοκαιρινή Περίοδος		
	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*	Ψηλή κίνηση	*Σταθερή κίνηση*	*Χαμηλή κίνηση*
Μαρία	690	780	452	699	758	835	788	1732	1462
Καίτη	474	874	406	4612	2032	477	4500	834	712
Τάνια	1047	667	284	1389	804	450	1062	806	491
Χρίστος	1263	1188	765	1584	1668	449	1822	1276	1358
Χάρης	1264	1172	0	2477	681	548	1923	1130	89
Μαίρη	1115	278	574	2972	2399	231	1322	1594	577
Ρούλα	2253	1702	610	4470	993	75	2754	2327	87
Τάκης	550	903	928	1296	2360	2610	615	2184	2518
Μάκης	0	125	64	3073	767	768	3005	1253	253



Να δείξετε στη σελίδα αυτή με ποιο τρόπο θα επιλέξετε τους καλύτερους υπαλλήλους. Να θυμηθείτε ότι χρειάζεται να επιλέξετε τρεις υπαλλήλους για πλήρη εργοδότηση και τρεις υπαλλήλους για μερική απασχόληση.

Στην εργασία σας να φαίνεται ο τρόπος με τον οποίο χρησιμοποιήσατε τα δεδομένα από τους πίνακες.

Να γράψετε ένα γράμμα στον κ. Νικόλα, στο οποίο να αναφέρετε τα αποτελέσματά σας. Εξηγήστε στο κ. Νικόλα τον τρόπο με τον οποίο εργαστήκατε και τεκμηριώστε γιατί η λύση που προτείνετε είναι η καλύτερη δυνατή.

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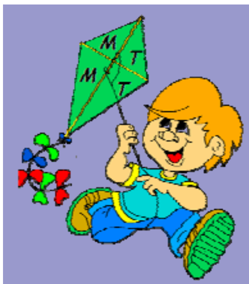
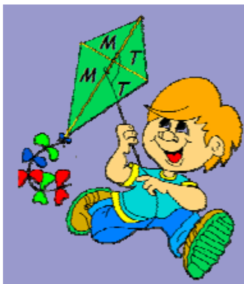


Το Λευκαρίτικο κέντημα

Σε μια επικλινή πλαγιά από ασβεστοχώματα, ανάμεσα στους ποταμούς Πεντάσχοινο και Μαρώνι, απλώνεται συμμαζεμένο το χωριό Λεύκαρα. Τα πετρόχτιστα σπίτια, οι κεραμιδένιες στέγες και οι παραδοσιακές καμάρες καθηλώνουν τον επισκέπτη. Ωστόσο, σήμα κατατεθέν για το χωριό αποτελούν τα γνωστά κεντήματα, τα ξακουστά «Λευκαρίτικα».

Τη μοναδική αυτή ομορφιά είχε την ευκαιρία να νιώσει από κοντά, πριν λίγες μέρες, ένας μεγάλος παραγωγός κεντημάτων στον κόσμο, ο οποίος και θέλησε να μάθει την αξιοζήλευτη αυτή τέχνη των κατοίκων του χωριού. Η κ. Μαρία, η οποία ασχολείται πολλά χρόνια με το Λευκαρίτικο κέντημα του ανάφερε τα ακόλουθα:

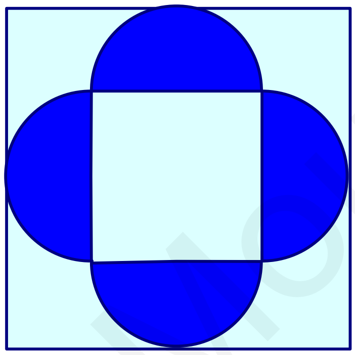
«Η κεντητική τέχνη στα Λεύκαρα, αρχικά, ήταν απλή. Αργότερα όμως, η Βενετσιάνικη επίδραση βοήθησε στη δημιουργία ενός πιο σύνθετου και πιο λεπτοδουλεμένου κεντήματος. Λέγεται ότι, ο Λεονάρδος Ντα Βίντσι αγόρασε μια Λευκαρίτικη δαντέλα και αργότερα τη χάρισε στον καθεδρικό ναό του Μιλάνου. Ο πλούτος και η ποικιλία των σχεδίων είναι μεγάλος, ωστόσο το κοινό χαρακτηριστικών των κεντημάτων εναπόκειται στην επανάληψη κάποιου μοτίβου. Η επανάληψη αυτή, επιτρέπει τη συνοχή των κεντημάτων και την αβίαστη ένωση των επιμέρους κομματιών για την ολοκλήρωση του, ιδιαίτερα εάν αναφερόμαστε σε μεγάλα σε εμβαδόν κεντήματα. Στις άκρες του κεντήματος συνήθως, τοποθετείται μία δαντέλα που δίνει μία επιπρόσθετη γοητεία στο όλο δημιούργημα».



Αποστολή!

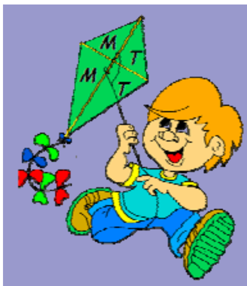
Ο έμπορος αποφάσισε να αξιοποιήσει τις πληροφορίες που πήρε από την κ. Μαρία για να κατασκευάσει παρόμοια κεντήματα. Τα κεντήματα που θα κατασκευαστούν θα είναι ορθογωνίου σχήματος με διαστάσεις 200cmX160cm.

Έχει επιλέξει ως αρχικό μοτίβο, για τα κεντήματά του, το ακόλουθο:



Το μικρό εσωτερικό σχήμα είναι τετράγωνο, ενώ σε κάθε πλευρά του σχήματος υπάρχουν τέσσερα ημικύκλια ακτίνας 10 cm.

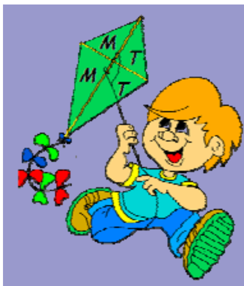
Υπολογίστε αρχικά πόσο σκουρόχρωμο και πόσο ανοικτόχρωμο ύφασμα χρειάζεται για κάθε ένα μοτίβο που θα κατασκευαστεί. Στη συνέχεια, αφού βρείτε πόσα μοτίβα χρειάζονται για το κέντημα, υπολογίστε το εμβαδόν κάθε είδους υφάσματος που θα χρειαστεί για το κάθε κέντημα.



Υπολογίστε το εμβαδόν του ανοικτόχρωμου και του σκουρόχρωμου υφάσματος για κάθε μοτίβο.

(Βοήθεια: Από τι σχήματα αποτελείται το κάθε είδος υφάσματος;)

Σχεδιάστε πιο κάτω το κέντημά σας. Σε αυτό πρέπει να φαίνεται ο αριθμός και οι διαστάσεις των επιμέρους κομματιών του κεντήματος.



ΜΟΝ²ΤΕΜΠ

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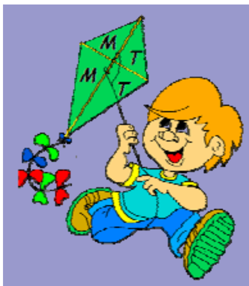
Αποφασίσαμε να τοποθετήσουμε δαντέλα γύρω από τα ημικύκλια, έτσι ώστε να φαίνεται πιο όμορφο το κέντημά μας. Πόση δαντέλα θα χρειαστούμε για ολόκληρο το κέντημα;



Θα μπορούσατε να εισηγηθείτε ένα διαφορετικό μοτίβο για να κατασκευαστούν εξίσου ωραία κεντήματα;

Αφού σχεδιάσετε το μοτίβο σας, δείξετε πώς θα φαίνεται το κέντημα που θα κατασκευαστεί.

Το μοτίβο και το κέντημα που έχετε σχεδιάσει θα δοθούν σε συμμαθητές σας.

Να γράψετε μερικές ερωτήσεις που θα μπορούσαν να απαντήσουν οι συμμαθητές σας σχετικά με το μοτίβο και το κέντημα.

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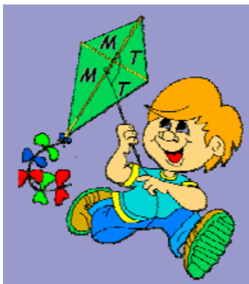
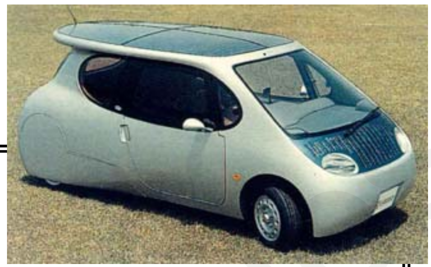


Το νέο μας αυτοκίνητο!



Γνωρίζεις ότι…

- Η χρήση των ορυκτών καυσίμων και της πυρηνικής ενέργειας επηρεάζει τις κλιματικές συνθήκες του πλανήτη μας;
- Σημαντική έκταση των επιφανειακών υδάτων του πλανήτη μας έχει ρυπανθεί από πυρηνικά απόβλητα;
- Το 1952 στο Λονδίνο, χιλιάδες άνθρωποι πέθαναν από άπνοια, η οποία οφειλόταν στα αέρια απόβλητα των εργοστασίων της πόλης;
- Το 1982, πολλά δάση της Κεντρικής Ευρώπης καταστράφηκαν από την όξινη βροχή λόγω της καύσης των υδρογονανθράκων σε μονάδες παραγωγής ηλεκτρικής ενέργειας;
- Στο διάστημα 1989-1995 παρατηρήθηκε μεγάλη αύξηση της θερμοκρασίας κατά 0,3—0,6 βαθμούς Κελσίου λόγω του φαινομένου του θερμοκηπίου;

Πολλές εταιρείες παραγωγής αυτοκινήτων στον κόσμο έχουν ευαισθητοποιηθεί περιβαλλοντικά. Αποφάσισαν να κατασκευάσουν ένα αυτοκίνητο το οποίο θα χρησιμοποιεί ανανεώσιμες πηγές ενέργειας, όπως την ηλιακή ενέργεια.

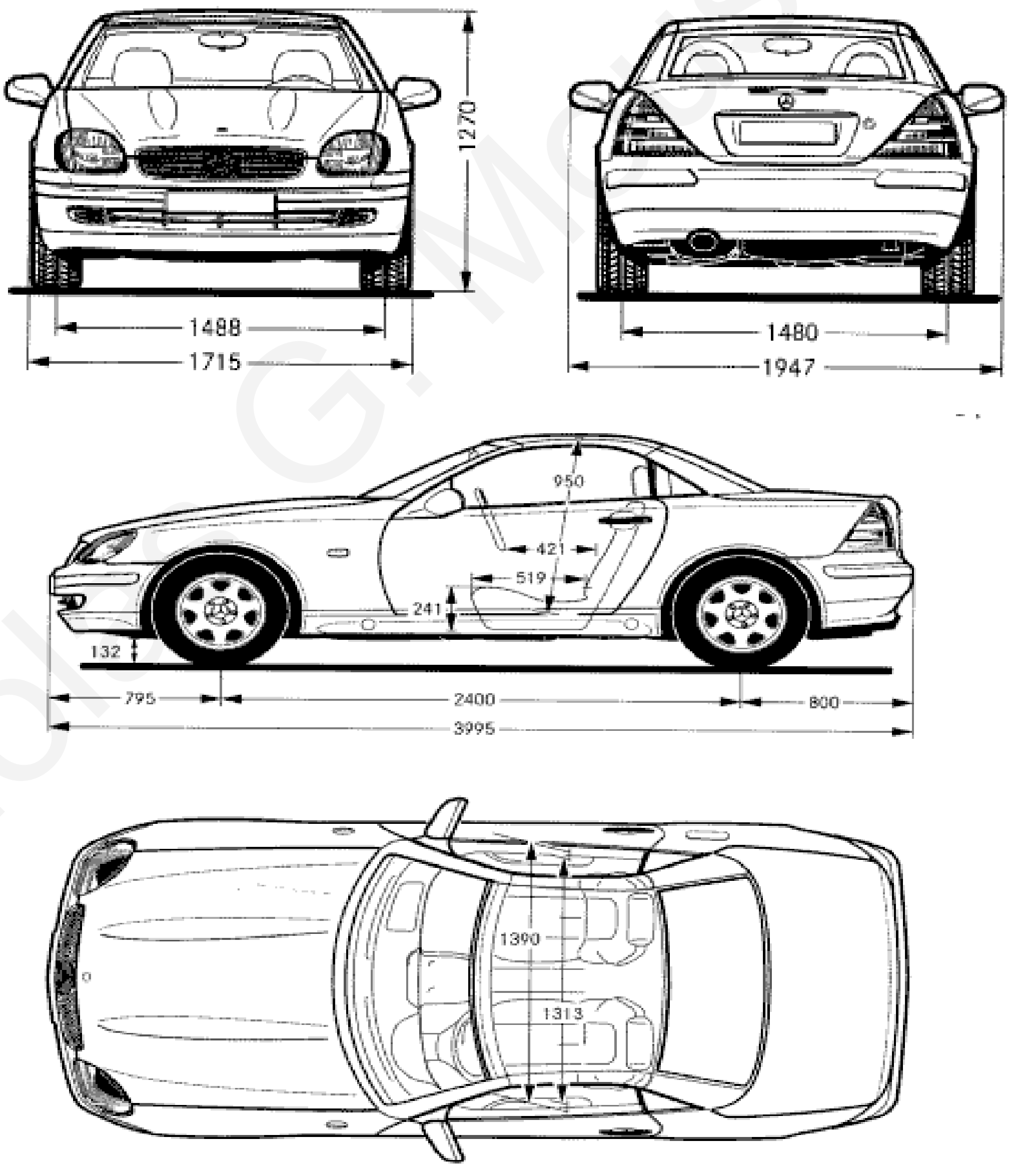
Αρκετές εταιρείες αυτοκινήτων έχουν ξεκινήσει τη διάθεση αυτοκινήτων που κινούνται με εναλλακτικές πηγές ενέργειας, όπως με ηλιακή ενέργεια, ηλεκτρική ενέργεια και ενέργεια που προέρχεται από το υδρογόνο.

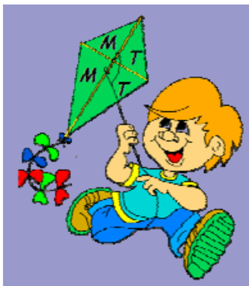
Τα αυτοκίνητα αυτά δεν στερούνται οτιδήποτε σε σχέση με τα βενζινοκίνητα αυτοκίνητα. Περιλαμβάνουν διάφορες ανέσεις όπως: ραδιοκασετόφωνο, ηλεκτρικά παράθυρα, κεντρικό σύστημα κλειδώματος, συναγερμό κτλ.

Ένα Ωραίο Αυτοκίνητο!

Ο διευθυντής μιας εταιρείας αυτοκινήτων έχει υποψίες ότι σπαταλείται μεγάλη ποσότητα μπογιάς κατά το βάψιμο των αυτοκινήτων. Χρειάζεται τη βοήθειά σας για να υπολογίσει την μπογιά που απαιτείται για το βάψιμο ενός αυτοκίνητου.

Πιο κάτω δίνονται τα σχεδιαγράμματα με τις διαστάσεις ενός συγκεκριμένου μοντέλου αυτοκινήτου.

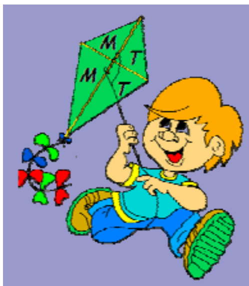




Να υπολογίσετε το εμβαδόν της επιφάνειας του αυτοκινήτου που θα βαφτεί, αφού μελετήσετε τα διαγράμματα της προηγούμενης σελίδας. Να εξηγήσετε, παράλληλα, πώς εργαστήκατε για να καταλήξετε στο τελικό σας αποτέλεσμα.

Να υπολογίσετε πόσα λίτρα μπογιάς απαιτούνται για το βάψιμο ενός αυτοκινήτου, αν το πάχος της μπογιάς που χρειάζεται είναι 0,1 mm.

Ο διευθυντής έχει ζητήσει από τους εργάτες να περιορίσουν τις απώλειες στο 30% της μπογιάς που απαιτείται για το βάψιμο ενός αυτοκινήτου. Ποια είναι η μέγιστη ποσότητα μπογιάς που μπορεί να χρησιμοποιήσει ένας εργάτης για το βάψιμο ενός αυτοκινήτου, έτσι ώστε να μη δεχτεί παρατήρηση από το διευθυντή του εργοστασίου;



ΜΟΝ^{2}ΤΕΜΠ

3

Γράψετε μια επιστολή προς το διευθυντή της αυτοκινητοβιομηχανίας. Στην επιστολή σας να εξηγήσετε αναλυτικά τον τρόπο με τον οποίο υπολογίσατε την επιφάνεια του αυτοκινήτου.

Μετανάστευση

Η μετανάστευση των ανθρώπων είναι πανάρχαιο φαινόμενο και καθορίζεται από διαφορετικούς παράγοντες. Στα προϊστορικά ακόμη χρόνια, οι διάφορες ανθρώπινες φυλές ήταν αναγκασμένες να μεταναστεύουν από τον έναν τόπο στον άλλο, προσπαθώντας να επιβιώσουν. Από τα κρύα κλίματα πήγαιναν στα πιο ζεστά, από τα ορεινά στα πεδινά, από τα φτωχά σε καρπούς και κυνήγι στα περισσότερο πλούσια.

Η μετανάστευση, που άρχισε στις αρχές του αιώνα μας έπαιξε καθοριστικό ρόλο στην όλη εξέλιξη της ανθρώπινης κοινωνίας. Ειδικότερα, μετά την Τουρκική εισβολή, πολλοί Κύπριοι μετανάστευσαν στην Ελλάδα, στην Αγγλία, στις Ηνωμένες Πολιτείες και στην Αυστραλία.

Οι κυριότεροι λόγοι που ώθησαν τους Κύπριους να μεταναστεύσουν ήταν η προσπάθεια για εύρεση καλύτερων συνθηκών ζωής και εργασίας, καθώς επίσης και ασφάλειας.

Στις μέρες μας, παρόλο που το φαινόμενο αυτό έχει περιοριστεί, οι ευκαιρίες που υπάρχουν για εργοδότηση και κατοικία στις χώρες της ενωμένης Ευρώπης έχουν πολλαπλασιαστεί και αρκετοί Κύπριοι επιλέγουν να εργαστούν σε μια άλλη ευρωπαϊκή χώρα. Παράλληλα, μεγάλος αριθμός Ευρωπαίων επιλέγει την χώρα μας για μόνιμη διαμονή και εργασία.

Ερωτήσεις

1. Ποιοι λόγοι ωθούν τους Κύπριους να μεταναστεύουν σε άλλες χώρες;

2. Σε ποιες χώρες μεταναστεύουν οι Κύπριοι;

3. Για ποιους λόγους μετάναστευαν τα αρχαία χρόνια οι άνθρωποι;

Το καινούριο μας σπίτι !!!

Η κ. Μαριλένα έχει μετακομίσει σε μια νέα περιοχή, στην οποία έχει αγοράσει ένα οικόπεδο για να κτίσει το καινούριο της σπίτι.

Το αρχιτεκτονικό σχέδιο του σπιτιού ανέλαβε μια πολύ γνωστή αρχιτέκτονας, η κ. Οικοδομίδου. Η Μαριλένα έστειλε την ακόλουθη επιστολή στην κ. Οικοδομίδου.
Η κ. Οικοδομίδου, γνωστή για τις πρωτοποριακές της ιδέες, αποφάσισε να οργανώσει ένα διαγωνισμό μεταξύ των μαθητών ενός σχολείου, για το καλύτερο σχέδιο.

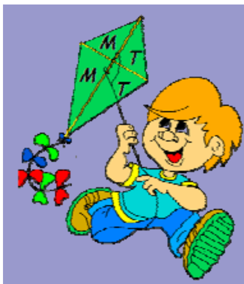
Σκοπός της εργασίας αυτής είναι να κερδίσετε το διαγωνισμό, ετοιμάζοντας μια καλή πρόταση για το σπίτι της κ. Μαριλένας. Προσπαθήστε να ικανοποιήσετε τις απαιτήσεις τις οικογένειας, λαμβάνοντας υπόψη σας την πιο κάτω επιστολή.

Αγαπητή κ. Οικοδομίδου,
Αυτά είναι τα στοιχεία που ζήτησες.

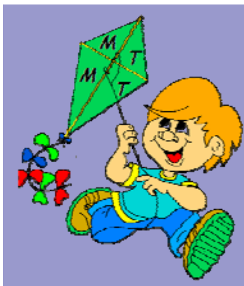
- Το οικόπεδο έχει ορθογώνιο σχήμα. Η μια πλευρά του είναι 20m και η άλλη είναι 30m.
- Θέλουμε να έχουμε τουλάχιστον 100m² κήπο.
- Τα δωμάτια του σπιτιού να είναι τα ακόλουθα:

  Μια μεγάλη κουζίνα
  Τρία υπνοδωμάτια (το ένα να είναι μεγαλύτερο από τα άλλα δύο)
  Ένα μεγάλο σαλόνι
  Ένα δωμάτιο τηλεόρασης
  Δύο μπάνια (το ένα να είναι μεγάλο για να βάλουμε υδρομασάζ)

  Ένα γκαράζ για δύο αυτοκίνητα



Να κατασκευάσετε την κάτοψη του σπιτιού της κ. Μαριλένας, με τη βοήθεια του λογισμικού EucliDraw. Αφού ολοκληρώσετε την κατασκευή σας, να την αντιγράψετε στο πιο κάτω ορθογώνιο. Να θυμάστε ότι η κατασκευή πρέπει να είναι όχι μόνο λειτουργική, αλλά και αισθητικά ωραία.



Να γράψετε ένα γράμμα στην κ. Οικοδομίδου, επεξηγώντας τον τρόπο που εργαστήκατε και κατά πόσο ικανοποιούνται οι απαιτήσεις της πελάτισσάς της.

Να εξηγήσετε, επίσης, τα στοιχεία εκείνα που κάνουν την δική σας πρόταση να ξεχωρίζει και να διεκδικεί με αξιώσεις το βραβείο.

Αν θέλετε να δείξετε κάτι