UNIVERSITY OF CYPRUS

DEPARTMENT OF MATHEMATICS AND STATISTICS

PERTURBATION SOLUTIONS OF WEAKLY COMPRESSIBLE NEWTONIAN POISEUILLE FLOWS WITH PRESSURE-DEPENDENT VISCOSITY

Ph.D. Dissertation

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2012

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PERTURBATION SOLUTIONS OF WEAKLY COMPRESSIBLE NEWTONIAN POISEUILLE FLOWS WITH PRESSURE-DEPENDENT VISCOSITY

By

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SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

AT

UNIVERSITY OF CYPRUS

NICOSIA, CYPRUS

APRIL 2012

This thesis is dedicated to my family for their continuous support and encouragement.

Περίληψη

Μια συνήθης παραδοχή στην ανάλυση Νευτώνειων ροών είναι ότι η πυκνότητα και το ιξώδες είναι σταθερές. Η παραδοχή αυτή όμως, είναι σωστή μόνο σε συνθήκες χαμηλής πίεσης και δεν ευσταθεί σε ροές κατά τις οποίες αναπτύσσονται μεγάλες πιέσεις όπως είναι για παράδειγμα η εκβολή και η έγχυση πολυμερών, η μεταφορά κηρώδους αργού πετρελαίου, η λιπαντική ροή λεπτών υμενίων, η μικρορευστονική, και κάποιες γεωφυσικές ροές. Επιπλέον, η υπόθεση ότι η ροή είναι συμπιεστή, ενώ προκαλεί μικρές αλλαγές στης λύση της μόνιμης ροής μπορεί να επηρεάσει σημαντικά τη δυναμική της ροής λόγω εμφάνισης της παραγώγου της πυκνότητας ως προς το χρόνο στην εξίσωση συνέχειας.

Στη διατριβή αυτή θεωρούμε Νευτώνειες ροές Poiseuille στις οποίες η πυκνότητα και το ιξώδες είναι συναρτήσεις της πίεσης και έτσι δεν είναι σταθερές. Σε αυτή την περίπτωση, οι εξισώσεις κινήσεως είναι έντονα μη-γραμμικές με αποτέλεσμα η εύρεση ακριβούς λύσης να είναι δύσκολη ή ακόμα και αδύνατη. Το γεγονός αυτό επιβάλλει συχνά τη χρήση προσεγγιστικών μεθόδων όπως είναι η προσέγγιση της λύσης μέσω ασυμπτωτικών αναπτυγμάτων τα οποία υπολογίζονται με τη μέθοδο των διαταραχών.

Πριν τη επίλυση της συμπιεστής ροής Poiseuille ενός Νευτώνειου ρευστού με ιξώδες που εξαρτάται από την πίεση, επιλύσαμε τις δύο οριακές περιπτώσεις: την ασυμπίεστη ροή με μεταβλητό ιξώδες και τη συμπιεστή ροή με σταθερό ιξώδες. Στην πρώτη περίπτωση ήταν δυνατόν να βρούμε ακριβή λύση ενώ στη δεύτερη περίπτωση εφαρμόσαμε την κανονική μέθοδο των διαταραχών προσεγγίζοντας τις πρωτεύουσες εξαρτημένες μεταβλητές της ροής, δηλαδή τις δύο συνιστώσες της ταχύτητας και την πίεση με αναπτύγματα ως προς την ισόθερμή συμπιεστότητα. Στο τελευταίο μέρος της διατριβής η πυκνότητα και το ιξώδες μεταβάλλονται με την πίεση και χρησιμοποιήσαμε διπλό ανάπτυγμα μέσω της μεθόδου των διαταραχών για να βρούμε προσεγγιστική λύση.

Στην περίπτωση της ασυμπίεστης ροής επιλύσαμε την επίπεδη, την αξονοσυμμετρική και τη δακτυλιοειδή ροή Poiseuille. Υποθέτοντας ότι η ροή είναι μονοκατευθυντική και ότι το ιξώδες εξαρτάται γραμμικά από την πίεση, κατέστει δυνατή η εύρεση ακριβούς λύσεως για την ταχύτητα, η οποία είναι συνάρτηση μόνο της κατακόρυφης συντεταγμένης και για την πίεση, η οποία εξαρτάται και από τις δύο συντεταγμένες. Μελετώντας τη λύση παρατηρήσαμε ότι καθώς η εξάρτηση του ιξώδους από της πίεση αυξάνεται, το προφίλ της ταχύτητας από παραβολικό τείνει να γίνει γραμμικό. Η κλίση της πίεσης κοντά στην έξοδο

είναι η ίδια όπως στην κλασική, πλήρως ανεπτυγμένη ροή. Η πίεση αυξάνεται εκθετικά καθώς κινούμαστε αντίθετα με τη φορά της ροής, και έτσι η πίεση που χρειάζεται για να κινήσει τη ροή αυξάνεται δραματικά.

Στην περίπτωση της ροής με σταθερό ιξώδες επιλύσαμε την επίπεδη και την αξονοσυμμετρική ροή Poiseuille υποθέτοντας ότι η πυκνότητα έχει γραμμική εξάρτηση από την πίεση και ότι το ρευστό ολισθαίνει στο τοίχωμα με ταχύτητα που υπακούει στη συνθήκη του Navier. Εφαρμόσαμε κανονική μέθοδο των διαταραχών προσεγγίζοντας τις δύο συνιστώσες της ταχύτητας και την πίεση με ασυμπτωτικά αναπτύγματα ως προς την ισόθερμη συμπιεστότητα εξάγοντας έτσι προσεγγιστικές λύσεις μέχρι και τη δεύτερη τάξη. Τα αποτελέσματα έδειξαν ότι η αύξηση της ολίσθησης μειώνει της εξάρτηση της λύση από την κατακόρυφη συντεταγμένη. Επίσης, βλέπουμε ότι η ολίσθηση μειώνει την κατακόρυφη ταχύτητα και αυξάνει την οριζόντια. Καθώς κινούμαστε αντίθετα με τη φορά της ροής, η πίεση αυξάνεται καθώς η ροή γίνεται πιο συμπιεστή, αλλά αυξάνεται πιο αργά όταν η ολίσθηση στο τοίχωμα γίνεται μεγαλύτερη. Επίσης, μελετήσαμε σημαντικές ποσότητες όπως είναι ο ρυθμός ογκομετρική παροχής, η μέση πτώση της πίεσης και ο μέσος παράγοντας τριβής Darcy.

Στο τελευταίο μέρος της διατριβής υποθέσαμε ότι τόσο η πυκνότητα όσο και το ιξώδες εξαρτώνται γραμμικά από την πίεση και μελετήσαμε την επίδραση που έχουν πάνω στη ροή. Εφαρμόσαμε κανονική μέθοδο διαταραχών πάνω στις κύριες εξαρτημένες μεταβλητές της ροής προσεγγίζοντας τις με ασυμπτωτικά αναπτύγματα ως προς την ισόθερμη συμπιεστότητα και το συντελεστή εξάρτησης του ιξώδους από την πίεση. Εξάγαμε έτσι προσεγγιστικές λύσεις δεύτερης τάξης για τις περιπτώσεις της επίπεδης και της αξονοσυμμετρικής ροής. Η λύση αυτή αποτελεί γενίκευση των λύσεων που βρήκαμε στις δύο προηγούμενες περιπτώσεις. Βλέπουμε ότι η κατακόρυφη ταχύτητα είναι πάντα θετική και εξαρτάται μόνο από την κατακόρυφη συντεταγμένη. Όταν η συμπιεστότητα και ο συντελεστή εξάρτησης του ιξώδους από την πίεση είναι της ίδιας τάξης, τότε η οριζόντια ταχύτητα επηρεάζεται από τη μεταβολή του ιξώδους στη δεύτερη τάξη αλλά όχι στην πρώτη τάξη. Αντίθετα, η πίεση επηρεάζεται από τη μεταβολή της πυκνότητας και του ιξώδους τόσο στη δεύτερη όσο και στην πρώτη τάξη και η επίδραση των δύο αυτών ποσοτήτων είναι ανταγωνιστική. Η κατακόρυφη ταχύτητα δεν επηρεάζεται από την εξάρτηση του ιξώδους από την πίεση σε καμία τάξη. Επίσης, μελετήσαμε την επίδραση του λόγου του ύψους του αγωγού ως προς το μήκος και του αριθμού Reynolds πάνω στη ροή.

Abstract

A common assumption in the analysis of Newtonian flows is that both the density and the viscosity are constants. Such an assumption, however, is valid only at low processing pressures and cannot be used in important flows involving high pressures, such as polymer extrusion and injection model, waxy crude oil transport, fluid film lubrication, microfluidics, and in certain geophysical flows. Moreover, relaxing the incompressibility assumption may lead only to minor changes in the calculated steady-state solutions but may affect greatly the flow dynamics, given the density time derivative that appears in the continuity equation.

This thesis is concerned with Newtonian Poiseuille flows in which the density and the viscosity of the fluid are not constants but functions of the pressure. In this case, the non-linearity of the equations of motion is increased and the derivation of analytical solutions becomes more difficult if not ruled out opening the way to the use of approximate methods, such as the approximation of the solution by asymptotic expansions via the perturbation method.

Before tackling the compressible Poiseuille flows of a Newtonian fluid with a pressure-dependent viscosity, we solved the two limiting cases, i.e. incompressible flows with pressure-dependent viscosity and then, compressible flows with constant viscosity. In the former case it was possible to derive semi-analytical solutions whereas in the latter case a regular perturbation scheme was employed in which the primary fields, i.e. the two velocity components and the pressure where expanded in terms of the isothermal compressibility. In the last part of the thesis both the density and the viscosity of the fluid vary with pressure and we employed a double perturbation scheme in order to derive an approximate analytical solution.

In the case of incompressible flow, we considered the plane, the axisymmetric, and the annular Poiseuille flows. Assuming that the flow is unidirectional and the viscosity varies linearly with pressure, we obtained closed-form solutions for the velocity, which is a function of the transverse coordinate, and for the pressure, which is two-dimensional. It is demonstrated that as the pressure-dependence of the viscosity becomes

stronger, the velocity tends from a parabolic profile to a triangular one. The pressure gradient near the exit is the same as that of the classical fully developed flow. This increases exponentially upstream and thus the pressure required to drive the flow increases dramatically.

In the case of flow with constant viscosity, we considered the plane and axisymmetric compressible Poiseuille flows with Navier slip at the wall. A linear equation of state was employed to describe the variation of the density with pressure. We applied a regular perturbation method, perturbing the two non-zero velocity components and the pressure, using the isothermal compressibility number as the small perturbation parameter. Approximate solutions up to the second order where obtained and analysed. The results show that slip weakens the dependence of the solution on the vertical coordinate. The transverse velocity decreases and the horizontal velocity increases with slip. The pressure required to drive the flow increases slower upstream with slip but increases when the flow becomes more compressible. Important quantities such as the he volumetric flow rate, the average pressure drop and the Darcy friction factor were also studied.

In the last part of the thesis the combined effects of compressibility and viscosity pressure dependence where investigated, assuming that both the density and the viscosity vary linearly with pressure. We applied a regular perturbation method using the isothermal compressibility number and the viscosity-to-pressure coefficient as the small perturbation parameters. All the primary variables were represented by a double asymptotic expansion, and via perturbation analysis second-order approximations were obtained for both the plane and axisymmetric Poiseuille flows. The solution was then analysed in terms of the two perturbation parameters. We noted that this generalizes the solutions corresponding to the aforementioned special cases. It is demonstrated that the transverse velocity is always positive and depends only on the transverse coordinate. When the compressibility number and the viscosity-pressure coefficient are of the same order, the horizontal velocity at first-order is not affected by the viscosity but does at second order. The pressure is affected by the compressibility and the viscosity pressure-dependence at first and second-order and these effects compete each other. The transverse velocity is not affected by the viscosity's pressure dependence at any order. The effects of the aspect ratio and the Reynolds number have also been studied.

Acknowledgments

I would like to express my deepest gratitude to my supervisor, Professor Georgios Georgiou, for his guidance, advice and support in all the time of research and writing of this thesis. Without his knowledge, skills, continuous assistance and encouragement, this thesis would not have been successful.

I would also like to thank the other members of my committee, Associate Professor Christos Xenophontos, who was the chairman of the committee, Professor Christodoulos Sophocleous, Associate Professor Kostas Housiadas and Assistant Professor Marina Neophytou for their advice, comments and suggestions.

A very special thanks goes out to Assistant Professor Katerina Kaouri (Intercollege Limassol), for her valuable contribution to this thesis. I thank her for her suggestions and for the time she devoted in helping me. Her contribution was an important factor for the completion of my thesis.

Finally, I would like to thank my family and my friends, Evgenia Christodoulou especially, for their never-ending love, support and encouragement.

Stella V. Poyiadji

Nicosia, April 2012

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Chapter 1

Introduction

The equations of motion, i.e. the continuity and momentum equation, for any fluid can be written as follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.1}$$

and

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} . \tag{1.2}$$

where $\mathbf{u}$ is the velocity vector, p is the pressure, $\boldsymbol{\tau}$ is the viscous stress tensor, ρ is the density, and $\boldsymbol{g}$ is the gravitational acceleration. If the density is constant, then the continuity equation is simplified to $\nabla \cdot \mathbf{u} = 0$. Otherwise, Eqs. (1.1)-(1.2) need to be supplemented by an *equation of state*, relating the density to the pressure.

In the case of a compressible Newtonian fluid with zero bulk viscosity[1] the viscous stress tensor is given by

$$\boldsymbol{\tau} = \eta \left(2\mathbf{D} - \frac{2}{3} \mathbf{I} \nabla \cdot \mathbf{u} \right), \tag{1.3}$$

where η is the viscosity, $\mathbf{I}$ is the unit second-order tensor, and $\mathbf{D}$ is the rate of deformation tensor defined by

$$\mathbf{D} \equiv \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]. \tag{1.4}$$

In incompressible flow, Eq. (1.3) is reduced to the standard Newtonian constitutive equation:

$$\boldsymbol{\tau} = 2\eta \mathbf{D} = \eta \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]. \tag{1.5}$$

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[1] In the more general case,

$$\boldsymbol{\tau} = 2\eta \mathbf{D} + \left(\chi - \frac{2}{3} \eta \right) \mathbf{I} \nabla \cdot \mathbf{u}$$

where χ is the bulk viscosity, which is neglected in most studies.

Substituting Eq. (1.5) into the momentum equation and assuming that the viscosity is constant, one gets the Navier-Stokes equation:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\eta\nabla^2\mathbf{u}+\rho\mathbf{g}\,. \tag{1.6}$$

In the present thesis, we consider compressible Newtonian flows with pressure-dependent viscosity. In other words, both the density and the viscosity of the fluid are functions of pressure. Hence, the continuity equation (1.1) may be written as

$$\frac{\partial \rho(p)}{\partial t}+\nabla\cdot[\rho(p)\mathbf{u}]=0 \tag{1.7}$$

and the viscous stress tensor is given by

$$\boldsymbol{\tau}=\eta(p)\left(2\mathbf{D}-\frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{u}\right), \tag{1.8}$$

where η is now a known function of the pressure p. Substituting the above constitutive equation into the momentum equation (1.2) leads to the following generalization of the Navier-Stokes equation:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\eta(p)\nabla^2\mathbf{u}+2\eta'(p)\nabla p\cdot\mathbf{D}+\frac{1}{3}\eta(p)\nabla(\nabla\cdot\mathbf{u})-\frac{2}{3}\eta'(p)\nabla p\nabla\cdot\mathbf{u}+\rho\mathbf{g}\,. \tag{1.9}$$

To our knowledge, studies taking into account both the compressibility and the viscosity pressure-dependence are very scarce in the literature. The objective of this thesis is to obtain analytical solutions of the system (1.7)-(1.9) for steady two-dimensional Poiseuille flow problems. In addition to the standard incompressibility and constant-viscosity assumptions, we also relax the well-known no-slip boundary condition.

The components of Eqs. (1.7) and (1.9) in Cartesian and cylindrical coordinates are tabulated in Tables 1.1 and 1.2, respectively. This system of equations is closed by means of an equation of state and an equation describing the pressure-dependence of the viscosity. These are discussed in Sections 1.1 and 1.2, respectively. In Section 1.3, we discuss the issue of wall slip. The two-dimensionality of the Poiseuille flows of interest is a consequence of the compressibility and excludes the possibility of an exact analytical solution. Approximate analytical solutions, however, can be obtained by perturbation methods, as discussed in Section 1.4. Finally, in Section 1.5 we present the objectives, and outline the chapters of the thesis.

Table 1.1: *Components of the equations of motion in Cartesian coordinates when both the density and the viscosity are pressure dependent.*

Continuity equation

$$\frac{\partial\left(\rho u_x\right)}{\partial x}+\frac{\partial\left(\rho u_y\right)}{\partial y}+\frac{\partial\left(\rho u_z\right)}{\partial z}=0$$

x-momentum equation

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\right)=-\frac{\partial p}{\partial x}+\eta\left(p\right)\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

$$\eta'\left(p\right)\left[2\frac{\partial p}{\partial x}\frac{\partial u_x}{\partial x}+\frac{\partial p}{\partial y}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+\frac{\partial p}{\partial z}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)-\frac{2}{3}\frac{\partial p}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)\right]$$

$$+\frac{1}{3}\eta\left(p\right)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\rho g_x$$

y-momentum equation

$$\rho\left(\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}\right)=-\frac{\partial p}{\partial y}+\eta\left(p\right)\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)$$

$$+\eta'\left(p\right)\left[\frac{\partial p}{\partial x}\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)+2\frac{\partial p}{\partial y}\frac{\partial u_y}{\partial y}+\frac{\partial p}{\partial z}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)-\frac{2}{3}\frac{\partial p}{\partial y}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)\right]$$

$$+\frac{1}{3}\eta\left(p\right)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\rho g_y$$

z-momentum equation

$$\rho\left(\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right)$$

$$+\eta'\left(p\right)\left[\frac{\partial p}{\partial x}\left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right)+2\frac{\partial p}{\partial z}\frac{\partial u_z}{\partial z}-\frac{2}{3}\frac{\partial p}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)\right]$$

$$+\frac{1}{3}\eta\left(p\right)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)+\rho g_z$$

Table 1.2: *Components of the equations of motion in cylindrical coordinates when both the density and the viscosity are pressure dependent.*

Continuity equation

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho u_r\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right)+\frac{\partial u_z}{\partial z}=0$$

r- momentum equation

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+u_\theta\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}\right)+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}$$

$$+\eta\left(p\right)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]$$

$$\eta'\left(p\right)\left[2\frac{\partial p}{\partial r}\left(\frac{\partial u_r}{\partial r}-\frac{1}{3}\left(\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)\right)+\frac{1}{r}\frac{\partial p}{\partial \theta}\left(\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}-u_\theta\right)+\frac{\partial u_\theta}{\partial r}\right)\right.$$

$$\left.+\frac{\partial p}{\partial z}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)\right]+\frac{1}{3}\eta\left(p\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)+\rho g_r$$

θ- momentum equation

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+u_\theta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)+u_z\frac{\partial u_\theta}{\partial z}\right)=$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta}+\eta\left(p\right)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_\theta\right)\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right]$$

$$+\eta'\left(p\right)\left[\frac{\partial p}{\partial r}\left(\frac{\partial u_\theta}{\partial r}+\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}-u_\theta\right)\right)+\frac{2}{r}\frac{\partial p}{\partial \theta}\left(\frac{1}{r}\left(u_r+\frac{\partial u_\theta}{\partial \theta}\right)-\left(\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)\right)\right.$$

$$\left.+\frac{\partial p}{\partial z}\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)\right]+\frac{1}{3}\eta\left(p\right)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)+\rho g_\theta$$

z- momentum equation

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+u_\theta\frac{1}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta\left(p\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right]$$

$$+\eta'\left(p\right)\left[\frac{\partial p}{\partial r}\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right)+\frac{1}{r}\frac{\partial p}{\partial \theta}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right)+2\frac{\partial p}{\partial z}\left(\frac{\partial u_z}{\partial z}-\frac{1}{3}\left(\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)\right)\right]$$

$$+\frac{1}{3}\eta\left(p\right)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\left(ru_r\right)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)+\rho g_z$$

1.1 Weak compressibility

This thesis is concerned with flows of liquids, which are usually considered as incompressible. Such an assumption is valid if the applied pressures are not high and/or the flow is steady. At high processing pressures the effects of non-zero compressibility may be magnified and the presence of the density time-derivative in the continuity equation may affect dramatically the flow dynamics even for small values of the compressibility parameter (Georgiou and Crochet, 1994a).

Laminar Poiseuille flows of weakly compressible gases have been of special interest in the past few decades and have been studied extensively as a result to their importance in many processes which involve gas flows in long capillaries or at high speeds (Venerus, 2006). Gas flows in long capillaries at high speeds are observed in micro-electro- mechanical systems (MEMS) where the gas is forced to flow in microchannels or microtubes, thus causing the appearance of compressibility effects (Arkillic and Schmidt 1997, Guo and Wu, 1997, Ansumal and Karlin, 2005, Cai et al., 2007).

Compressibility effects in liquid flows become important at high processing pressures, i.e. in flows in relatively long tubes. Waxy crude oil transport (Vinay et al, 2006), polymer extrusion (Georgiou, 2003, Georgiou and Crochet, 1994a, Tang and Kalyon, 2008a, Hadjikyriakos et al, 1992, Piau and Kissi, 1994) and polymer injection molding (Kwon, 1996) are important cases of liquid flows in long tubes were the effects of compressibility cannot be neglected.

In flows of weakly compressible materials, two equations of state are usually employed at low pressures:

(a) the linear equation of state:

$$\rho = \rho_0 \left[1 + \beta \left(p - p_0 \right) \right], \tag{1.10}$$

where $\beta = -\left(\partial V / \partial p \right)_{p_0, T} / V_0$ is the isothermal compressibility which is assumed to be constant, V is the specific volume, ρ_0 and p_0 are respectively the density and the specific volume at the reference pressure ρ_0 and T is the temperature which is assumed to be constant too; and

(b) the exponential equation of state:

$$\rho = \rho_0 e^{\beta \left(p - p_o \right)}. \tag{1.11}$$

For small β and low pressures the linear equation is a very good approximation of the exponential equation.

The Mach number, Ma, is defined as the ratio of a characteristic velocity, V_0, of the flow to the speed, σ, of sound in the fluid:

$$Ma \equiv \frac{V_0}{\sigma}, \tag{1.12}$$

where

$$\sigma = \left[\gamma \left(\frac{\partial p}{\partial \rho} \right)_T \right]^{1/2} = \left(\frac{\gamma}{\beta \rho_0} \right)^{1/2}, \tag{1.13}$$

γ being the heat capacity ratio or adiabatic index. For weakly compressible flows, $Ma \ll 1$, usually Ma<0.3.

In the literature, one can find various numerical solutions for weakly compressible Poiseuille flows for Newtonian fluids (Guo, 1997, Georgiou and Crochet, 1994a, Georgiou and Crochet, 1994b, Guo and Wu, 1998) which is the case that we focus on in the thesis, for generalised Newtonian fluids (Cawkwell and Charles, 1989, Golay and Helluy, 1998, Keshtiban et al, 2005, Mitsoulis et al., 2007, Silva and Coupez, 2002, Tang and Kalyon, 2008a) such as the Carreau fluid (Georgiou, 2003), for the Bingham plastic (Vinay et al., 2006) and for viscoelastic fluids (Belblidia et al., 2006).

In flows of liquids such as polymer melts, the combination of compressibility with nonmonotonic slip laws relating the wall shear stress to the slip velocity (Hadjikyriakos and Dealy, 1992) is reported to be the cause of the stick-slip polymer extrusion instability. The stick-slip polymer extrusion instability refers to the sustained pressure and flow rate oscillations observed under constant throughput. The compressibility-slip combination effect is confirmed by experimental observations and numerical simulations. Dubbeldam and Molenaar (2003) used one-dimensional phenomenological models to describe the pressure and flow rate oscillations and thus verifying the compressibility-slip effect while Taliadorou et al. (2007) developed numerical simulations for the stick-slip extrusion instability in the case of the time-dependent, compressible extrusion of a Carreau fluid, assuming that nonmonotonic slip occurs along the wall and employing the nonmonotonic slip law that was observed in the experiment of Hadjikyriakos and Dealy (1992a, 1992b). Tang and Kalyon (2008a, 2008b) developed a mathematical model describing the time-dependent pressure-driven flow of compressible polymeric liquids subject to pressure-dependent slip in the simple shear flow. They reported that undamped periodic pressure oscillations in pressure and mean velocity are observed when the boundary condition changes from weak to strong slip.

Taliadorou et al. (2008) reported extrusion simulations showing that the combination of strong compressibility with inertia may lead to stabled steady-state free surface oscillations, similar to those observed experimentally with liquid foams. Mitsoulis and Hadjikiriakos (2009) carried out steady flow simulations of polytetrafluoroethylene (PTFE) paste extrusion under severe slip taking into account the significant compressibility of the pastes.

Perturbation and other approximate solutions have been reported for the weakly compressible Poiseuille flow of a Newtonian fluid, mainly under the assumption of ideal gas flow. In Prud'homme et al. (1986) the flow of an ideal gas in a long tube under the assumptions of zero radial velocity component, zero pressure gradient and no gravity, is approximated using a double perturbation expansion and taking the radius to length ratio and the relative pressure drop as the perturbation parameters. Van den Berg et al. (1993) and Zohar (2002) used a one-dimensional perturbation method for radial symmetric flow and two lumped perturbation parameters to approach the compressible laminar flow in a capillary and the subsonic gas flow in microtubes and channels with wall slip.

Venerus (2006) and Venerus and Bugajsky (2010) derived perturbation solutions in terms of the compressibility for the axisymmetric and the plane isothermal Poiseuille flow of a weakly, compressible Newtonian liquid respectively, using the steamfunction/vorticity formulation and a linear relation of the density to pressure. Recently, Taliadorou et al. (2009) obtained perturbation solutions of the plane and axisymmetric Poiseuille flows of a weakly compressible Newtonian fluid. In their methodology, the perturbation is performed on the primary flow variables, i.e. on the velocity components and the pressure. Housiadas and collaborators (2011, 2012) extended the primary-variable perturbation method to derive solutions of the plane and axisymmetric Poiseuille flows of a weakly compressible Oldroyd-B fluid.

1.2 Pressure-dependence of viscosity

The viscosity of typical liquids begins to increase substantially with pressure when pressures of the order of 1000 atm are reached (Renardy, 2003). Fluids with pressure-dependent viscosity are also referred to as piezoviscous fluids (Suslov and Tran, 2008). In such fluids, the dependence of the viscosity on pressure may be several orders of magnitude stronger than that of density (Dowson and Higginson, 1966; Renardy, 2003; Rajagopal, 2006; Roux, 2008). This is the case, for example, in fluid film lubrication, in polymer extrusion, and in injection molding where the pressure can be very high leading in large variations in the viscosity while the variation in density is insignificant (Szeri, 1998; Denn, 2008).

The idea of a fluid with pressure-dependent viscosity was introduced by Stokes in his seminal 1845 paper on the constitutive response in fluids (Stokes, 1845). Barus (1893) was the first to propose an exponential isothermal equation of state for the viscosity of the form

$$\eta(p) = \eta_0 \, e^{\beta p}, \tag{1.14}$$

where η is the viscosity, p is the pressure, η_0 is the viscosity at atmospheric pressure, and β is the pressure-viscosity coefficient (which is temperature dependent)[2]. Even though Eq. (1.14) is extensively used, it is valid as a reasonable approximation only at moderate pressures. Barus (1893) himself used a more general equation of state of the form

$$\eta(p) = \eta e^{\beta p + Kp^2} \tag{1.15}$$

which describes a stronger pressure-dependency at low pressures (Goubert et al., 2001). Denn (2008) notes that, to a first approximation, the viscosity of polymer melts can be written as follows

$$\eta = \eta_0 \, e^{-\alpha(T - T_0)} \, e^{\beta p}, \tag{1.16}$$

where η_0, being the viscosity at atmospheric pressure (p=0) and the reference temperature T_0, may depend on shear rate.

Bair et al. (2001) noted that the experimental data show that at high pressures, Eq. (1.14) is not valid and that the use of more accurate models or experimental data is necessary. The experiments of Kottke (2004) showed that the accuracy of Eq. (1.14) at negative pressure

[2] The pressure coefficient of the viscosity should not be confused with the isothermal compressibility introduced in section 1.1.

(i.e. cavitation) is unknown. Rajagopal (2006) pointed out that Eq. (1.14) works well up to 500 MPa, that is up to moderate pressures, and needs to be modified. Other equations proposed in the literature in order to describe experimental observations on the pressure-dependence of the viscosity have been reviewed by Málek and Rajagopal (2007). Other useful reviews of experimental studies on the viscosity pressure dependence and the values of the pressure-dependence coefficient[3] are those of Binding et al. (1998), Goubert et al. (2001), and Carreras et al. (2006).

The pressure-dependence of the viscosity becomes important in processes involving high pressures, such as polymer processing, fluid film lubrication, microfluidics, and in geophysics. Due to these applications, fluids with pressure-dependent viscosity have received an increasing attention recently. Relevant references are provided and discussed in Chapter 2.

The pressure-dependence of the viscosity has been analyzed mathematically by Renardy (1986, 2003), Gazzola (1997), Malek et al. (2002a, 2002b), Hron et al. (2003), Huilgol and You (2006), Bulíček et al. (2007), Málek and Rajagopal (2007) and others. In these analyses, some other convenient expressions were used for the viscosity pressure dependence, such as

$$\eta(p) = \beta p \tag{1.17}$$

used by Hron et al. (2003), and

$$\eta(p) = \eta_0 (1 + \beta p) \tag{1.18}$$

employed by Renardy (2003). Suslov and Tran (2008) pointed out that the linear constitutive equation (1.17) does not guarantee positive definiteness of the viscosity which requires the pressure to remain positive. This problem is not encountered when using the exponential constitutive equation (1.14) or in flows where the pressure remains positive.

1.3 Wall slip

The idea of wall slip dates back to 1761 when Euler (1761) assumed that common liquids slip over solid surfaces exhibiting Coulomb friction. Slip at the wall occurs in many flows of complex fluids, such as suspensions, emulsions, polymer melts and solutions, miscellar solutions, and foams, leading to very interesting phenomena and instabilities. These

[3] The values of the pressure-dependence coefficient β are in the range between 7 and 45 10^{-3}/MPa (Carreras et al., 2006).

important implications of slip have been reviewed by various researchers (Denn, 2001; Hatzikiriakos and Migler, 2004). In order to better understand and simulate slip effects, it is necessary to have realistic slip velocity models. In a recent review, Hatzikiriakos (2012) classified slip models into static (weak slip) and dynamic ones and pointed out that the former are not valid in transient flows, since slip relaxation effects might become important, leading to delayed slip and other phenomena.

The experimental data show that the slip velocity is in general a function of the wall shear stress, the wall normal stress (which includes pressure), the temperature, the molecular weight and its distribution, and the fluid/wall interface, e.g. the interaction between the fluid and the solid surface and surface roughness [see Denn (2001) and references therein]. Neto et al. (2005) reviewed experimental studies of wall slip of Newtonian liquids and discussed the effects of surface roughness, wettability, and the presence of gaseous layers. More recently, Sochi (2011) reviewed slip at fluid-solid interfaces from different perspectives, such as slip factors, mechanisms, and measurement, and discussed, in particular, slip with non-Newtonian behavior, i.e. yield stress, viscoelasticity, and time dependency. In this thesis we focus on the effects of wall shear stress on the steady-state slip velocity. Therefore, we discuss only static slip models and refer the reader to the review of Hatzikiriakos (2012) for dynamic slip models.

Let $\mathbf{u}_s$ denote the velocity of a solid wall and $\mathbf{u}$ denote the velocity of a fluid. The slip velocity $\mathbf{u}_w$ is defined as the difference between the tangential velocity of the fluid and the tangential velocity of the solid wall, i.e.

$$\mathbf{u}_w \equiv (\mathbf{u}-\mathbf{u}_s) - \left[(\mathbf{u}-\mathbf{u}_s)\cdot\mathbf{n}\right]\mathbf{n}, \tag{1.19}$$

where the subscript w denotes a tangential component and $\mathbf{n}$ is the unit outward vector. An alternative way to define the slip velocity is the following (Silliman and Scriven, 1980)

$$\mathbf{u}_w \equiv \mathbf{I}_s \cdot (\mathbf{u}-\mathbf{u}_s), \tag{1.20}$$

where

$$\mathbf{I}_s \equiv \mathbf{I} - \mathbf{nn} \tag{1.21}$$

is the second-order surface identity tensor (i.e., the geometric tensor that projects vectors onto the tangent plane to the wall surface), $\mathbf{I}$ is the conventional identity tensor, and $\mathbf{nn}$ the surface normal dyadic. In the case of a fixed wall, $\mathbf{u}_s = \mathbf{0}$. In the case of no-slip, $\mathbf{u}_w = \mathbf{0}$, or equivalently, $\mathbf{u} = \mathbf{u}_s$.

If $\mathbf{T}$ is the stress tensor in the fluid, then the tangential stress at the wall $\boldsymbol{\tau}_w$ is

$$\boldsymbol{\tau}_w \equiv \mathbf{T}\cdot\mathbf{n} - \left[(\mathbf{T}\cdot\mathbf{n})\cdot\mathbf{n}\right]\mathbf{n} = \mathbf{I}_s\cdot(\mathbf{T}\cdot\mathbf{n}). \tag{1.22}$$

Following Fortin et al. (1991) we consider the vector form of a slip law relating the slip velocity to the wall tangential stress:

$$\begin{cases} \mathbf{u}_w = 0, & |\boldsymbol{\tau}_w| \le \tau_c \\ \boldsymbol{\tau}_w = -\left(\dfrac{1}{\alpha} + \dfrac{\tau_c}{|\mathbf{u}_w|}\right)\mathbf{u}_w, & |\boldsymbol{\tau}_w| \ge \tau_c, \end{cases} \tag{1.23}$$

where τ_c is the slip yield stress, i.e. the stress below which no slip occurs, and α is the slip coefficient. According to slip equation (1.23), the tangential stress acts parallel to the slip velocity, but in the opposite direction. The factor $1/\alpha$ can be viewed as the friction coefficient (Fortin et al., 1991) or the momentum transfer coefficient (Silliman and Scriven, 1980). According to Lawal et al. (1993) α is in general a function of the invariants of the stress tensor. Huilgol (1998) notes that $1/\alpha$ is not constant, but depends on the magnitude of the slip velocity. Huilgol and Nguyen (2001) also assumed that Eq. (1.23) can be inverted so that there is a unique solution for $\mathbf{u}_w$ in terms of $\boldsymbol{\tau}_w$.

The one-dimensional version of Eq. (1.23) can be written as follows:

$$u_w = \begin{cases} 0, & \tau_w \le \tau_c \\ \alpha(\tau_w - \tau_c), & \tau_w \ge \tau_c. \end{cases} \tag{1.24}$$

When $\tau_c = 0$, the classical slip law is recovered:

$$u_w = \alpha\tau_w. \tag{1.25}$$

The above equation was proposed by Navier (1827). The slip coefficient α varies in general with temperature, normal stress and pressure, molecular parameters, and the characteristics of the fluid/wall interface. The inverse of α can be viewed as the friction dissipation coefficient. Obviously, for $\alpha = 0$, we have no slip, while for $\alpha \to \infty$ we get perfect slip. The slip coefficient is also defined by

$$\alpha \equiv \frac{b}{\eta}, \tag{1.26}$$

where η is the viscosity and b is the *extrapolation length*, i.e. the characteristic length equal to the distance that the velocity profile at the wall must be extrapolated to reach zero. In another classic paper, Stefan (1874) also employed a linear slip equation in order to describe his experimental data on a Newtonian glycerol solution. The linear slip equation (1.25) has been used widely for many other fluid systems, including very concentrated suspensions (Yilmazer and Kalyon, 1989; Kalyon et al., 1993) and pastes (Adams et al., 1997).

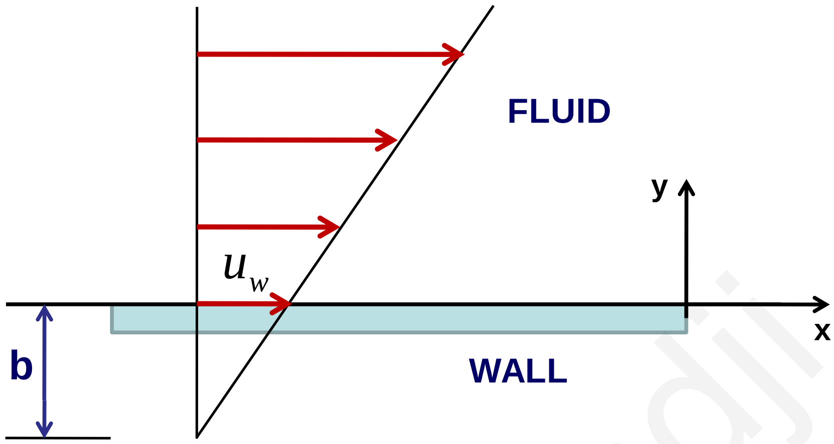


Figure 1.1: *Slip velocity and extrapolation length.*

More complex, non-linear slip equations have also been proposed. Pearson and Petrie (1965) postulated the following relationship

$$u_w = f(\tau_w)\tau_w . \tag{1.27}$$

A power-law expression,

$$u_w = \alpha \tau_w^m , \tag{1.28}$$

where m is the power-law exponent, has been widely employed by several investigators, e.g. by Cohen and Metzner (1985), who studied experimentally the occurrence of slip in aqueous and organic polymer solutions, and by Jiang et al. (1986) to describe the slip exhibited by gels used in hydraulic fracturing.

Experimental data on several fluid systems, such as linear polymers (mainly polyethylenes) (Ramamurthy, 1986; Kalika and Denn, 1987; Hatzikiriakos and Dealy, 1991), highly entangled polymers (Piau and El Kissi, 1994), pastes (Adams et al., 1997), and colloidal suspensions (Ballesta, 2008; 2011), indicate that slip occurs only when the stress exceeds a critical value τ_c, which is similar to a Coulomb friction term and can be viewed as a "wall shear", or "interfacial", or, simply, "slip" yield stress. Roquet and Saramito (2008) also used the term "yield-force" for this critical value. Hatzikiriakos and Dealy (1991) pointed out that slip model (1.29) fails to describe the slip velocity in the neighborhood of τ_c, which is critical in understanding polymer slip phenomena. They thus used the following Bingham-type equation:

$$u_w = \begin{cases} 0, & \tau_w \le \tau_c \\ \alpha \tau_w^m, & \tau_w \ge \tau_c. \end{cases} \tag{1.29}$$

The following general phenomenological slip equation

$$u_w = \begin{cases} 0, & \tau_w \le \tau_c \\ \alpha\left(\tau_w - \tau_c\right)^m, & \tau_w \ge \tau_c \end{cases} \tag{1.30}$$

has been used by various researchers in the analysis of squeeze flow of generalized Newtonian fluids with apparent wall slip (Yilmazer and Kalyon, 1989; Ji and Gotsis, 1992; Estellé and Lanos, 2007). A discussion on the validity of Eq. (1.30) as well as values of α and m for certain systems are provided by Yilmazer and Kalyon (1989). The non-monotonic slip equations proposed by Piau and El Kissi (1994) for highly entangled polymers and by Leonov (1990) for elastomers also include a critical stress threshold below which no slip occurs. These slip equations exhibit one or two stress minima.

1.4 Perturbation methods in fluid mechanics

Solving fluid mechanics problems involves the solution of a nonlinear system of partial differential equations. Due to the presence of nonlinearity for most flow problems, it is rare to find exact analytical solutions. Therefore, one can seek approximate analytical solutions to the equations of fluid flow at hand. In order to seek an approximation, one or more parameters of variables in the problem should be either small or large. These perturbation quantities, most often than not, are dimensionless parameters of the problem. The approximate solution becomes more accurate as the small perturbation quantity tends to zero (or the large perturbation quantity tends to infinity). It is therefore called an asymptotic solution.

For the sake of simplicity, let us assume we have only one perturbation parameter, denoted by ε. From a physical point of view ε can only take positive real values. Also, it is never uniquely defined; sometimes choosing the perturbation parameter ingenuously can greatly simplify the problem. As ε tends to zero, the flow approaches a limit. We can call this limit the *basic solution.* The basic solution can also be called the zero-order solution and henceforth the first perturbation term is called the first-order solution, and so on. The basic solution and the subsequent perturbation terms added to it form an *asymptotic expansion* (Holmes, 1995).

Let us now define an asymptotic expansion precisely: Firstly, we need to define an asymptotic sequence. A sequence $\{f_n\}$ is called asymptotic sequence at $\varepsilon \to 0$ if for every integer n

$$f_{n+1}(\varepsilon) = o\left(f_n(\varepsilon)\right), \text{ as } \varepsilon \to 0 \tag{1.31}$$

where the symbol o ("little oh") means that

$$\lim_{\varepsilon \to 0} \frac{f_{n+1}(\varepsilon)}{f_n(\varepsilon)} = 0 . \tag{1.32}$$

Now, let $\{f_n\}$ be an asymptotic sequence as $\varepsilon \to 0$. We say that the function f is expanded in an asymptotic series

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} a_n f_n(\varepsilon), \text{ as } \varepsilon \to 0 \tag{1.33}$$

where a_n are constants, if

$$\forall N \geq 0 \quad R_N(\varepsilon) \equiv f(\varepsilon) - \sum_{n=0}^{N} a_n f_n(\varepsilon) = o\left(f_N(\varepsilon)\right), \ \varepsilon \to 0 . \tag{1.34}$$

The series $\{f_n\}$ is called the asymptotic expansion of the function f with respect to the asymptotic sequence f_n. $R_N(\varepsilon)$ is called the remainder term of the asymptotic series. The asymptotic expansion of a function with respect to an asymptotic sequence is unique and may not converge. In a physical problem, the coefficients in an asymptotic expansion depend on space and/or time other than ε.

The usefulness of an asymptotic expansion arises from the fact that the error is, by definition, of the order of the first neglected term, and therefore tends rapidly to zero as ε is reduced. Retaining only a few terms, when ε is reasonably small, may provide a highly-accurate approximation. If the asymptotic expansion is uniformly valid (in all regions of the flow) we have a *regular perturbation problem* (in contrast to *a singular perturbation problem*, where the asymptotic expansion fails to give a good approximation in certain regions and another form of the solution must be sought). In this thesis, in Chapters 3 and 4, we solve regular perturbation problems.

Note that the above asymptotic expansion refers to a problem with a single perturbation parameter. However, it is often the case that two (or more) perturbation quantities simultaneously approach zero (or other critical value-such as infinity). In those cases we speak of a double (or multiple) asymptotic expansion. Indeed, in Chapter 4, for the problem of weakly compressible flow with viscosity that is weakly pressure-dependent, we identify two perturbation quantities, the isothermal compressibility and viscosity-pressure coefficient, and we derive approximate solutions for the flow equations as double asymptotic expansions.

1.5 Objectives and chapter content description

The objective of this thesis is to derive analytical solutions for different cases of laminar Poiseuille flows of weakly compressible Newtonian fluids with pressure-dependent viscosity with or without slip at the wall. More specifically, we consider the plane, axisymmetric, and annular Poiseuille flows (illustrated in Fig. 1.2).

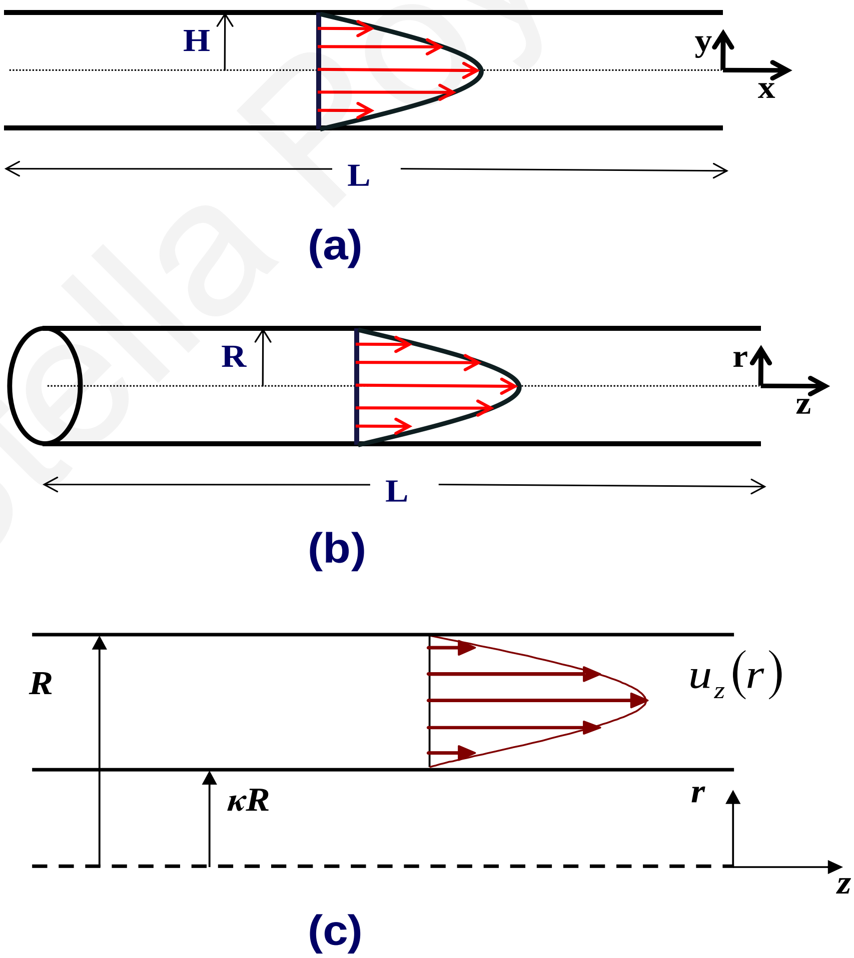


Figure 1.2: *Geometry and boundary conditions of (a) plane; (b) axisymmetric; and (c) annular Poiseuille flows.*

In Chapter 2, we derive analytical solutions for the plane, axisymmetric and annular steady, laminar Poiseuille flows of a Newtonian fluid assuming that the flow is incompressible, the velocity is one-dimensional, the viscosity increases linearly with pressure, and no-slip occurs along the wall. The solution for the velocity and the pressure is given in terms of a constant A, which is calculated numerically. The effects of the viscosity pressure-dependence on the pressure and the velocity are discussed. (The solution corresponding to Navier slip along the wall is provided in Appendix A.)

In Chapter 3, we consider both the plane, steady, laminar Poiseuille flows of a weakly compressible Newtonian fluid assuming that Navier slip occurs along the wall and that the density varies linearly with pressure. A perturbation analysis is performed in terms of the primary flow variables using the dimensionless isothermal compressibility as the perturbation parameter. Solutions up to the second-order are derived, and the combined effects of slip, compressibility, and inertia on the solutions are discussed. (The solution for the axisymmetric flow is provided in Appendix B.)

In Chapter 4, we consider the plane, steady, laminar, Poiseuille flow of a weakly compressible Newtonian fluid with a viscosity that is weakly dependent on the pressure, assuming that both the density and the viscosity vary linearly with pressure. A perturbation analysis is performed on all primary variables using the dimensionless isothermal compressibility and the dimensionless viscosity-pressure coefficient as the perturbation parameters. Perturbation solutions up to the second order in terms of the two perturbation parameters are derived and the combined effects of the compressibility and the viscosity are discussed. (The solution for the axisymmetric flow is provided in Appendix C.)

In Chapter 5, the results of this thesis are summarised and suggestions for future work are provided.

Chapter 2

Poiseuille flows with pressure-dependent viscosity

The pressure-dependence of the viscosity becomes important in flows where high pressures are encountered. Applications include many polymer processing applications, microfluidics, fluid film lubrication, as well as simulations of geophysical flows. Under the assumption of unidirectional flow, we derive analytical solutions for steady, laminar plane, round, and annular Poiseuille flow of a Newtonian liquid, the viscosity of which increases linearly with pressure. These flows may serve as prototypes in applications involving tubes with small radius-to-length ratios. It is demonstrated that, the velocity tends from a parabolic to a triangular profile as the viscosity coefficient is increased. The pressure gradient near the exit is the same as that of the classical fully-developed flow. This increases exponentially upstream and thus the pressure required to drive the flow increases dramatically[4].

2.1. *Introduction*

The viscosity of fluids, such as polymer melts and lubricants, depends strongly on temperature and to a less extent on pressure (Rajagopal, 2009). In such fluids, the dependence of the viscosity on pressure may be several orders of magnitude stronger than that of density (Rajagopal, 2009, Renardy, 2003). Denn (2008) emphasized that at a pressure of about 5 MPa, which can be reached in extrusion and in injection molding, the pressure dependence of the viscosity is expected to become important while the flow is still incompressible. Therefore, it is reasonable to study isothermal, incompressible flow of fluids with a pressure-dependent viscosity. The idea of a fluid with pressure-dependent viscosity was introduced by Stokes (1845). Barus (1893) proposed an exponential isothermal equation of state for the viscosity of the form

$$\eta(p) = \eta_0 \, e^{\lambda p} \, , \tag{2.1}$$

[4] The material of this chapter appears in Kalogirou et al. (2011).

where η is the viscosity, p is the pressure, η_0 is the viscosity at atmospheric pressure, and λ is the pressure-viscosity coefficient (which is temperature dependent). In polymer melts, λ is typically 1-5 10^{-8} Pa^{-1} (Denn, 2008). For lubricants, λ varies from 10 to 70 MPa^{-1} (Kottke, 2004). Venner and Lubrecht (2000) reported that for mineral oils λ is generally in the range between 10^{-8} and 2×10^{-8} Pa^{-1}. Carreras et al. (2006) compiled experimental values of the shear pressure coefficient λ. Even though Eq. (2.1) is extensively used, it is valid as a reasonable approximation only at moderate pressures. A compilation of other equations proposed for the pressure dependence of the viscosity and useful references on the subject has been provided by Málek and Rajagopal (2007).

There are numerous experimental studies concerning the determination of the pressure dependence of the viscosity of common polymer grades, such as polyethylenes (LDPE, LLDPE, HDPE), polypropylene, polystyrene, etc. Comprehensive reviews are provided by Binding et al. (1998) and Goubert et al. (2001) who compared measurement techniques in the literature for evaluating the pressure dependence of viscosity.

As already mentioned, high pressures sufficient to cause significant change in the viscosity appear in many polymer processing operations. Driving pressures of 50 and 100 MPa are routinely required in extrusion and injection molding (Tadmor et al., 1999). The strong effect of pressure and its potential importance in plastics processing led to the development of high-pressure rheometers based on pressure driven or drag flow (Koran, 1999). Cardinaels et al. (2007) discussed different methods to obtain pressure coefficients for different polymers, such as PMMA and LDPE, from high-pressure capillary rheometer data. More recently, Park et al. (2008) also compared different experimental methods for the determination of the pressure coefficient of a styrenic polymer.

The pressure-dependence of the viscosity becomes important in other applications, such as fluid film lubrication, microfluidics, and geophysics. In fluid film lubrication studies it is essential to include the variation of the viscosity with pressure (Hamrock et al., 2004). For technological applications in elastohydrodynamic lubrication and in thrust bearing or journal bearing applications, where the lubricant is forced to flow through a very narrow region which leads to very high pressures, the reader is referred to the work of Gwynllyw et al. (1996). In the design of Micro Electro-Mechanical Systems (MEMS), the pressure-dependence of the viscosity needs to be taken into account. Experimental data for liquid flows in microtubes driven by high pressures (1-30 MPa) show that the pressure gradient is not constant, an effect attributed to the pressure-dependence of the viscosity (Cui et al., 2004, Silber-Li et al., 2006). In geophysical flows, the viscosity changes with the depth of the fluid. Convection in planetary mantles is most likely dominated by the strong variability of the mantle viscosity depending on temperature and pressure (Binding et al., 1998). In her mantle flow simulations, Georgen (2008) allowed the viscosity to vary over three orders of magnitude from 10^{19} to 10^{22} Pa s.

Mathematical issues arising in the case of incompressible Newtonian or non-Newtonian flows with a pressure-dependent viscosity have been addressed by Renardy (1986), Gazzola (1997), and Malek et al. (2002a, 2002b). The existence of flows of fluids with pressure-dependent viscosity and the associated assumptions have been discussed by Bulíček et al. (2007). The properties of such solutions are also discussed by Málek and Rajagopal (2007).

In addition to Eq. (2.1), Hron et al. (2001) also assumed the following expression for the viscosity pressure dependence:

$$\eta(p) = \lambda p \,. \tag{2.2}$$

They showed that unidirectional flows are not possible between parallel plates in the case of the former model, since a secondary flow is necessary to that end. However, unidirectional flows are possible in the latter case.

Renardy (2003) considered parallel shear flows of an incompressible Newtonian fluid allowing a general pressure dependence for the viscosity and proved that a sufficient condition for the existence of parallel pressure-driven flow in a pipe, regardless of its cross-section, is the linear dependence of the viscosity on the pressure:

$$\eta(p) = \eta_0 (1 + \lambda p) \,. \tag{2.3}$$

This condition is not necessary; Denn (1981) showed that the quadratic velocity profile in a circular pipe remains a solution if the viscosity is an exponential function of the pressure. As indicated by Renardy (2003) and also shown in the present work, the velocity profile is not parabolic in the case of linear dependence of the viscosity; it may be almost parabolic when this dependence is weak. According to Suslov and Tran (2008), the major concern of the linear constitutive equation (2.3) is that it does not guarantee positive definiteness of the viscosity which requires the pressure to remain positive. This problem is not encountered when using the exponential constitutive equation (2.1) or in flows where the pressure remains positive, such as Poiseuille flows.

It seems that Eq. (2.2) has been the most popular one in the various theoretical analyses presented in the literature. Analytical solutions have been reported by Renardy (2003) and

Vasudevaiah and Rajagopal (2005) for the round Poiseuille flow of a Newtonian fluid and by Hron et al. (2001) and Huilgol and You (2006) for the plane Poiseuille flow of a generalized Newtonian fluid. The reason for avoiding Eq. (2.1) is obvious, since this equation rules out the possibility of having analytical solutions, but Eq. (2.3) should be more preferable than Eq. (2.2), since the latter predicts a vanishing viscosity at zero pressure. Another advantage of Eq. (2.3) over Eq. (2.2) is that it involves a reference viscosity constant. However, as shown below, both equations result in the same solution for the velocity in the case of unidirectional Poiseuille flow. What is different is the pressure distribution.

In the present work, we derive and discuss analytical solutions of axisymmetric, annular, and plane Poiseuille flows of Newtonian fluids with pressure-dependent viscosity obeying Eq. (2.3).

The rest of the chapter is organized as follows: in Section 2.2 the governing equations of the flow are presented. In Section 2.3 the derivation of the analytical solution is described in the case of the round Poiseuille flow. The solutions for the other two Poiseuille flows of interest are also provided in Sections 2.4 and 2.5. In Section 2.6, the theoretical results and the effects of the viscosity pressure-dependence are discussed and finally, in Section 2.7 we provide the conclusions.

2.2 Governing equations

For an incompressible Newtonian fluid, the viscosity of which is a function of pressure, the viscous stress tensor is given by

$$\boldsymbol{\tau} = 2\eta\left(p\right)\mathbf{D}, \tag{2.4}$$

where

$$\mathbf{D} = \frac{1}{2}\left[\nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^T\right] \tag{2.5}$$

is the rate-of-deformation tensor and **u** is the velocity vector. It can be shown in this case that the Navier-Stokes equation in the absence of gravity and under the assumption of a steady flow becomes:

$$\rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \eta\left(p\right)\nabla^2\mathbf{u} + 2\eta'\left(p\right)\nabla p\cdot\mathbf{D}. \tag{2.6}$$

It should also be noted that the continuity equation for incompressible flow is

$$\nabla \cdot \mathbf{u} = 0 . \tag{2.7}$$

In this Chapter, we consider incompressible Poiseuille flows of Newtonian fluids with pressure-dependent viscosity obeying Eq. (2.3).

2.3 Axisymmetric Poiseuille flow

We consider the nondimensionalized governing equations of axisymmetric Poiseuille flow in cylindrical coordinates with the origin located at the exit of the tube. The radial coordinate, *r*, is scaled by the radius *R* and the axial coordinate, *z*, by the length *L* of the tube and the viscosity η by the reference viscosity η_0. Moreover the axial velocity u_z is scaled by the mean velocity *U* at the exit, defined by

$$U \equiv \frac{\dot{M}}{\rho H W},$$

where $\dot{M}$ is the mass flow rate and W is the unit length in the *z*-direction and finally the pressure *p* by $8\eta_0 LU / R^2$ (chosen so that the pressure at the inlet plane is equal to 1). Hence the dimensionless form of the viscosity equation is

$$\eta^* = 1 + \varepsilon p^*, \tag{2.8}$$

where stars denote dimensionless quantities and

$$\varepsilon \equiv \frac{8\lambda \eta_0 LU}{R^2} \tag{2.9}$$

is the dimensionless isothermal compressibility number. For notational convenience, stars will be dropped hereafter.

Under the assumption that the radial velocity component is zero, the continuity equation dictates that $u_z = u_z(r)$; hence, only the pressure is a function of both *r* and *z*, $p = p(r,z)$. As pointed out by Huilgol and You (2006), it is clear that as long as $\partial\eta / \partial p$ is nonzero, a pressure gradient in the flow direction induces one in the direction of the velocity gradient, unless inertia is present. The *z*- and *r*-components of the momentum equation, defined over the domain $[0,1]\times[-1,0]$, are simplified as follows:

$$-8\frac{\partial p}{\partial z} + (1+\varepsilon p)\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr} \right) + \varepsilon \frac{\partial p}{\partial r}\frac{du_z}{dr} = 0 \tag{2.10}$$

and

$$-8\frac{\partial p}{\partial r}+\varepsilon\alpha^2\frac{\partial p}{\partial z}\frac{du_z}{dr}=0, \tag{2.11}$$

where

$$\alpha\equiv\frac{R}{L} \tag{2.12}$$

is the tube aspect ratio. By eliminating $\partial p/\partial r$ from Eqs. (2.10) and (2.11) and separating variables we find that

$$\frac{\dfrac{1}{r}\dfrac{du_z}{dr}+\dfrac{d^2u_z}{dr^2}}{1-\dfrac{\varepsilon^2\alpha^2}{64}\left(\dfrac{du_z}{dr}\right)^2}=\frac{8}{1+\varepsilon p}\frac{\partial p}{\partial z}=-A, \tag{2.13}$$

where A is in general a function of r, taken here as a constant to be determined. We have thus, two differential equations to be solved for u_z and p. By solving the first equation of Eq. (2.13) for u_z and applying the symmetry boundary condition ($\partial u_z/\partial r=0$) at the axis of symmetry and the no-slip condition ($u_z=0$) at $r=1$, one finds that

$$u_z(r)=\frac{64}{A\varepsilon^2\alpha^2}\ln\left[\frac{I_0\left(\dfrac{A\varepsilon\alpha}{8}\right)}{I_0\left(\dfrac{A\varepsilon\alpha}{8}r\right)}\right], \tag{2.14}$$

where I_0 is the zero-order modified Bessel function of the first kind (Watson, 1996). The above expression has been previously derived by Renardy (2003) and Vasudevaiah and Rajagopal (2005) who employed Eq. (2.2) instead of Eq. (2.3). By integrating the other differential equation of Eq. (2.13), assuming that $p(0,0)=0$, and taking into account the velocity profile, we find that

$$p(r,z)=\frac{1}{\varepsilon}\left[I_0\left(\frac{A\varepsilon\alpha r}{8}\right)e^{-A\varepsilon z/8}-1\right]. \tag{2.15}$$

The constant A is determined by demanding that the volumetric flow rate is 2π. This leads to the following equation

$$2\int_0^1\ln\left[I_0\left(\frac{A\varepsilon\alpha r}{8}\right)\right]rdr-\ln\left[I_0\left(\frac{A\varepsilon\alpha}{8}\right)\right]+\frac{A\varepsilon^2\alpha^2}{64}=0, \tag{2.16}$$

which is easily solved for A by means of Newton's method combined with numerical integration.

If instead of Eq. (2.3), the following equation is used, as was done by Hron et al. (2001),

$$\eta(p) = \varepsilon p \tag{2.17}$$

the above procedure leads to Eq. (2.14) for the velocity and to the expression

$$p(r,z) = \frac{1}{\varepsilon} I_0\left(\frac{A\varepsilon\alpha r}{8}\right) e^{-A\varepsilon z/8} \tag{2.18}$$

for the pressure. In both cases, the pressure increases exponentially upstream, which means that an enormous pressure drop may be achieved with a tube of finite length.

2.4 Annular Poiseuille flow

Let us now consider the Poiseuille flow in an annulus of radii κR and R, where $0 < \kappa < 1$. Using the same scaling and assumptions as in the axisymmetric case, we end up with the same separated differential equations to be solved for $u_z(r)$ and $p(r,z)$. An additional dimensionless number is introduced, i.e. the radii ratio κ. With the assumption of no slip along the two walls, the following expression is obtained for the slip velocity

$$u_z(r) = \frac{64}{A\varepsilon^2\alpha^2} \ln\left\{\frac{\left[K_0(B) - K_0(B\kappa)\right] I_0(B) - \left[I_0(B) - I_0(B\kappa)\right] K_0(B)}{\left[K_0(B) - K_0(B\kappa)\right] I_0(Br) - \left[I_0(B) - I_0(B\kappa)\right] K_0(Br)}\right\}, \tag{2.19}$$

where

$$B \equiv \frac{A\varepsilon\alpha}{8} \tag{2.20}$$

and K_0 is the first order modified Bessel function of the second kind. Assuming that $p(k,0) = 0$, the pressure is found to be given by

$$p(r,z) = \frac{1}{\varepsilon} \ln\left\{\frac{\left[K_0(B) - K_0(B\kappa)\right] I_0(Br) - \left[I_0(B) - I_0(B\kappa)\right] K_0(Br)}{\left[K_0(B) - K_0(B\kappa)\right] I_0(B) - \left[I_0(B) - I_0(B\kappa)\right] K_0(B)} e^{\varepsilon Az/8} - 1\right\}. \tag{2.21}$$

Assuming that the (dimensionless) volumetric flow is equal to 2π, we find that the constant A is the root of the following equation:

$$2\int_{\kappa}^{1}\ln\left[\left(K_0(B)-K_0(B\kappa)\right)I_0(Br)-\left(I_0(B)-I_0(B\kappa)\right)K_0(Br)\right]rdr$$
$$-\left(1-\kappa^2\right)\ln\left[\left(K_0(B)-K_0(B\kappa)\right)I_0(B)-\left(I_0(B)-I_0(B\kappa)\right)K_0(B)\right]+\frac{A\varepsilon^2\alpha^2}{64}=0. \tag{2.22}$$

2.5 Plane Poiseuille flow

We consider the pressure-driven flow in a channel of half-width *H* and length *L* and work in Cartesian coordinates with the origin at the intersection of the midplane and the exit plane of the channel and the *x*-axis in the flow direction. We nondimensionalize the governing equations scaling *x* by *L*, *y* by *H*, u_x by the mean velocity *U*, and the pressure by $3\eta_0 LU/H^2$. The resulting dimensionless numbers are

$$\alpha \equiv \frac{H}{L} \text{ and } \varepsilon \equiv \frac{3\lambda\eta_0 LU}{H^2}. \tag{2.23}$$

One finds that the velocity and pressure are given by

$$u_x(y)=\frac{9}{A\varepsilon^2\alpha^2}\ln\left[\frac{\cosh\left(\frac{A\varepsilon\alpha}{3}\right)}{\cosh\left(\frac{A\varepsilon\alpha}{3}y\right)}\right] \tag{2.24}$$

and

$$p(x,y)=\frac{1}{\varepsilon}\left[\cosh\left(\frac{A\varepsilon\alpha\, y}{3}\right)e^{-A\varepsilon x/3}-1\right]. \tag{2.25}$$

The constant *A* is determined by demanding that the volumetric flow rate is equal to unity. It turns out that *A* is the root of

$$\int_0^1\ln\left[\cosh\left(\frac{A\varepsilon\alpha y}{3}\right)\right]dy-\ln\left[\cosh\left(\frac{A\varepsilon\alpha}{3}\right)\right]+\frac{A\varepsilon^2\alpha^2}{9}=0. \tag{2.26}$$

The solution (2.24) for the velocity has also been derived by Hron et al. (2001) and Huilgol and You (2006), who employed Eq. (2.2) for the pressure-dependence of the viscosity.

2.6 Results and discussion

In this section we discuss only results for the axisymmetric and annular Poiseuille flows (the results for the plane flow are similar to their axisymmetric counterparts). In order to construct solutions for the velocity and pressure for the axisymmetric Poiseuille flow, the constant A must be determined from Eq. (2.16). It turns out that the latter equation has a unique nonzero root only when the parameter

$$\alpha\varepsilon = \frac{8\lambda\eta_0 U}{R} \tag{2.27}$$

is below the critical value

$$(\alpha\varepsilon)_{crit} = 8/3 . \tag{2.28}$$

As illustrated in Fig. 2.1, at low values of $\alpha\varepsilon$, A is insensitive to $\alpha\varepsilon$; this is not the case at higher values and, as $\alpha\varepsilon$ approaches the critical value, A grows rapidly to infinity.

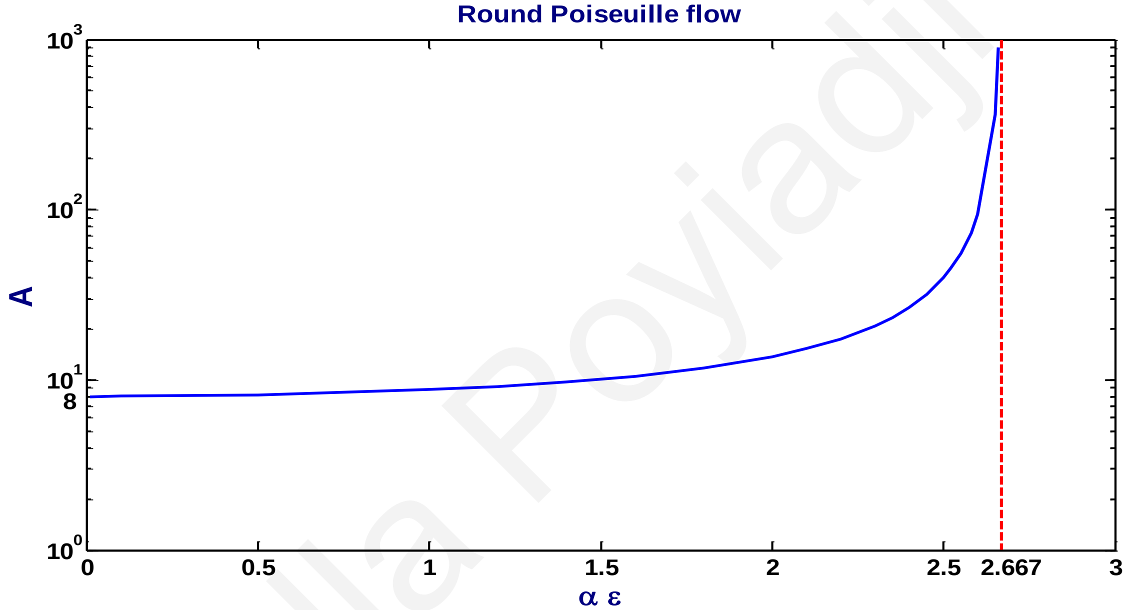


Figure 2.1: *The constant A as a function of the parameter αε in axisymmetric Poiseuille flow.*

In Fig. 2.2, the calculated velocity profiles for various values of the parameter $\alpha\varepsilon$ are shown. For $\alpha\varepsilon < 0.1$ the velocity has the parabolic profile for incompressible flow and then gradually tends to a linear profile:

$$u_{z,crit} = 3(1-r) . \tag{2.29}$$

Let us point out that $(\alpha\varepsilon)_{crit}$ can be calculated analytically as the value zeroing the denominator of the left-hand side of Eq. (2.13).

The velocity profiles of Fig. 2.2 suggest that in the two-dimensional flow the axial velocity is expected to change from a parabolic to a more triangular profile as we move upstream.

The velocity profiles of Fig. 2.2 are essentially the same as those obtained by Renardy (2003) and Vasudevaiah and Rajagopal (2005) for a Newtonian fluid obeying Eq. (2.2) instead.

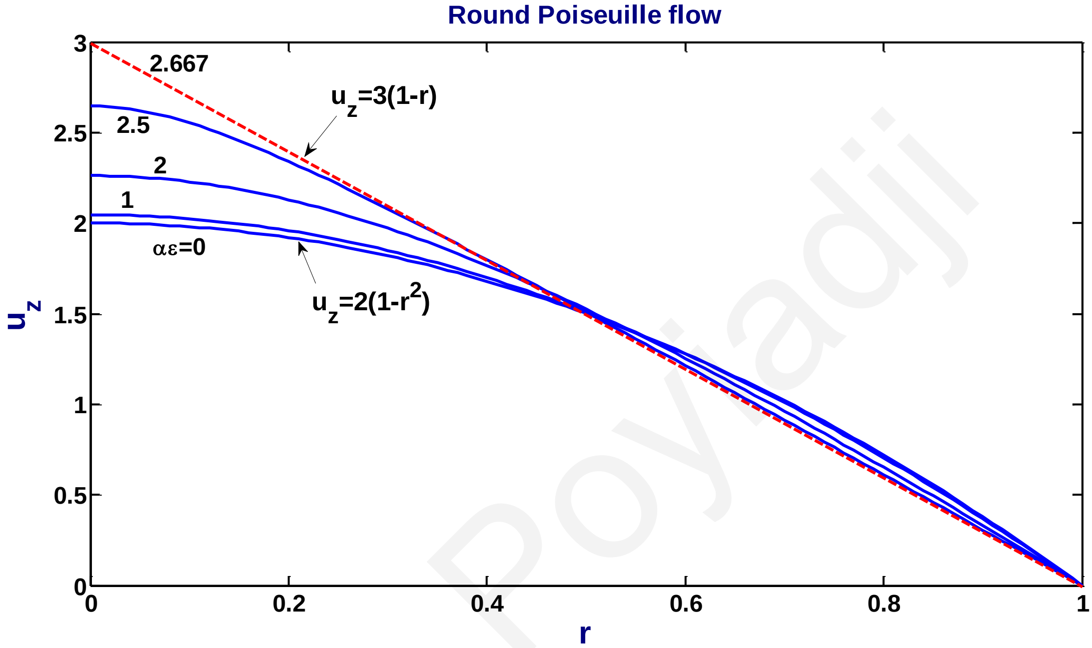
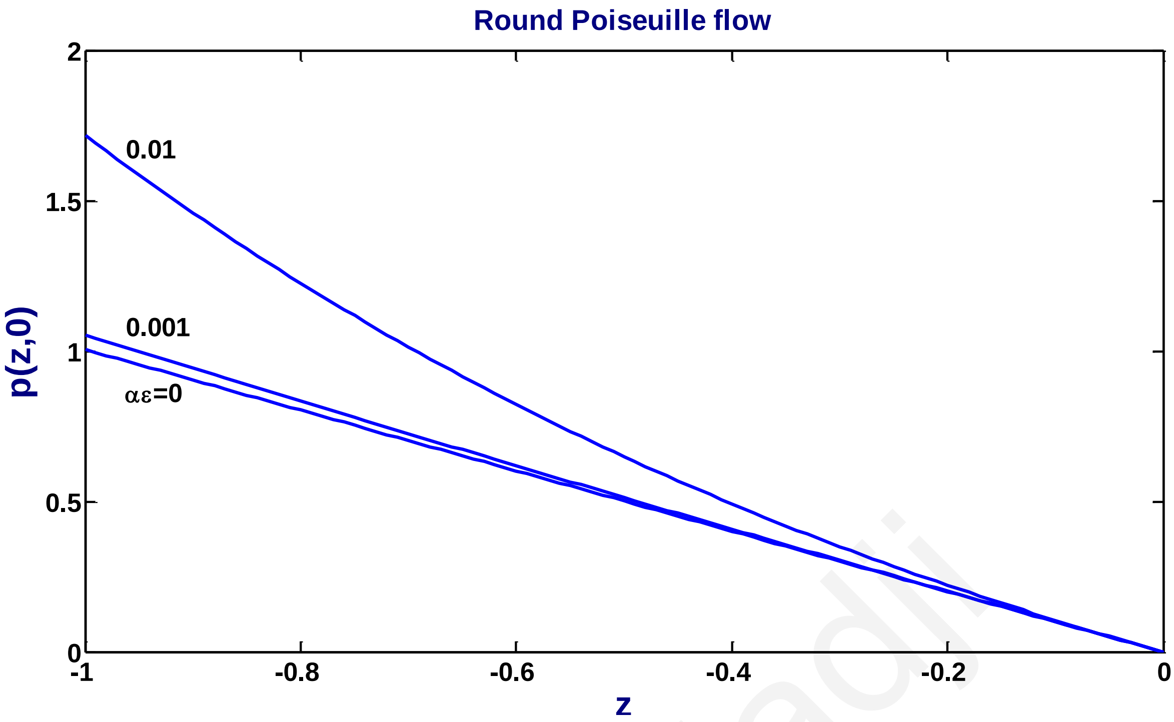
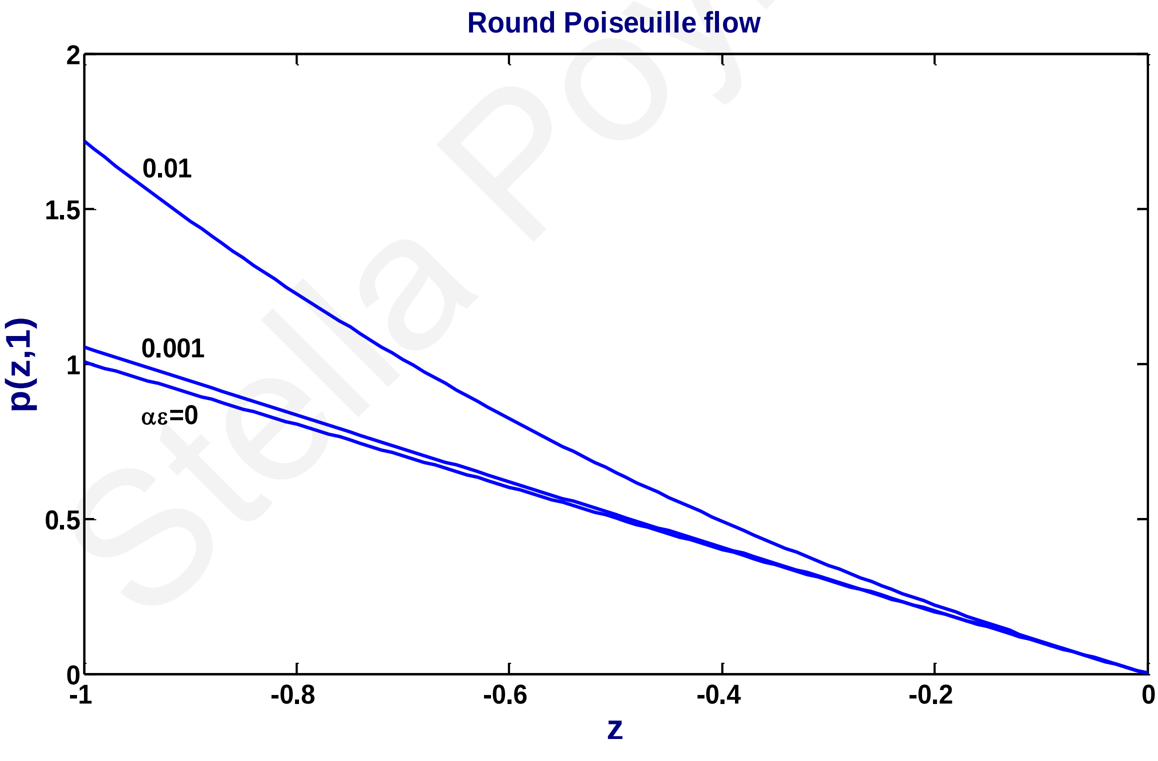


Figure 2.2: *Velocity profiles in axisymmetric Poiseuille flow for various values of the parameter* $\alpha\varepsilon$.

The pressure distributions obtained with $\alpha = 0.01$ and different values of $\alpha\varepsilon$ along the wall and the axis of symmetry are shown in Fig. 2.3. We observe that the pressure distribution remains linear only near the exit and that as the parameter $\alpha\varepsilon$ increases, the pressure upstream as well as the pressure gradient increase exponentially with the length of the tube. Clearly, the pressure required to drive the flow increases rapidly with the length of the tube.



(α)



(b)

Figure 2.3: *Pressure distribution along (a) the axis of symmetry and (b) the wall for α=0.01 and various values of $\alpha\varepsilon$; axisymmetric Poiseuille flow.*

Assuming that this is given by $\Delta P = p(0,-1)$ and that $A \simeq 8$ is a reasonable approximation for sufficiently small values of *αε*, e.g. for very long tubes, one gets

$$\Delta P \approx \frac{1}{\varepsilon}\left(e^{\varepsilon}-1\right). \tag{2.30}$$

Now, if it is also assumed that *ε* is small, Eq. (2.30) gives

$$\Delta P \approx 1+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{6}+O\left(\varepsilon^3\right). \tag{2.31}$$

The above expression can be viewed as a correction factor for the Hagen-Poiseuille formula and can be used in measuring the viscosity from viscometric data obtained using capillaries of different length.

In Fig. 2.4, we show the pressure distributions along the inlet and outlet planes of the tube. We observe that the pressure starts deviating from the linear profile at sufficiently high values of *αε*. At the inlet plane the pressure seems to be insensitive to *r*, i.e. the relative deviations are negligible. This is not the case at the outlet plane where larger deviations are observed when moving from the axis of symmetry to the wall. However, the absolute value of pressure is essentially zero. These results are also illustrated in Fig. 2.5 where the pressure contours for a short $(\alpha = 0.1)$ and a long $(\alpha = 0.01)$ tube are plotted. For small values of *α*, the contours appear to be vertical; the bending of the contours is more clearly shown for bigger values of *α*, i.e. in shorter tubes.

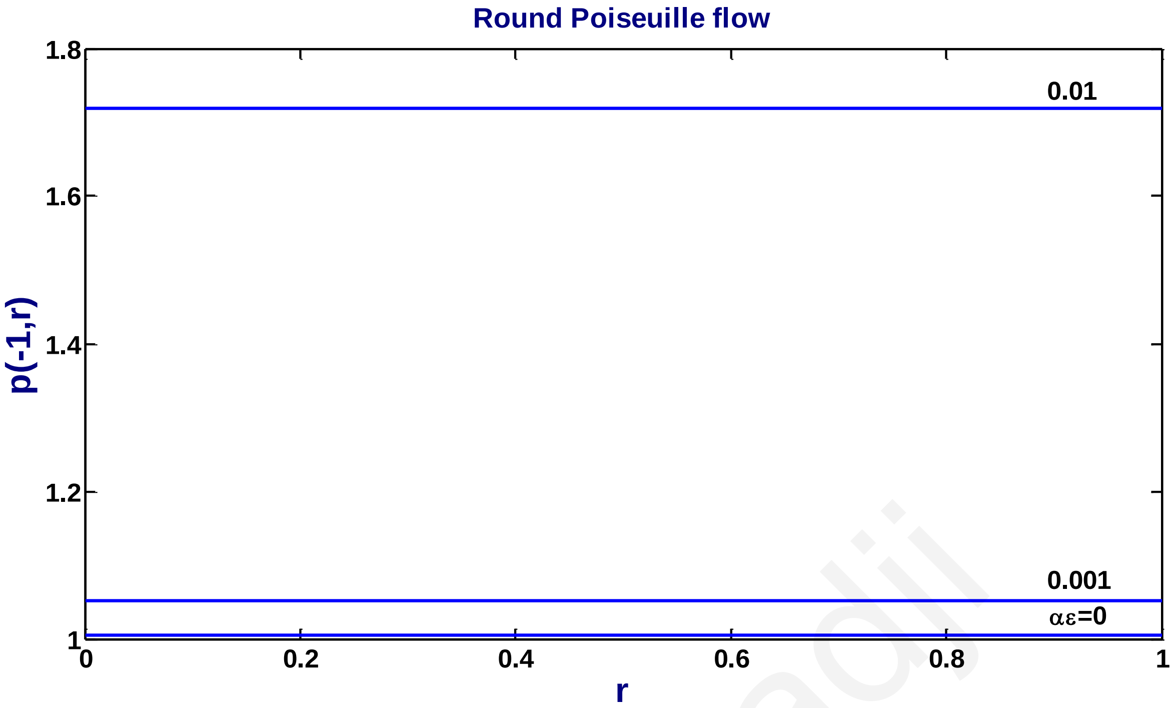
In the case of annular Poiseuille flow, we have chosen to show results for $\kappa = 0.1$. In this case, the parameter *A* is a unique nonzero root of Eq. (2.26) when *αε* is below the critical value 1.782, as illustrated in Fig. 2.6. It is easily shown that in general

$$\left(\alpha\varepsilon\right)_{crit} = 2\left(1+\kappa\right)\left(1-\kappa\right)^2 \tag{2.32}$$

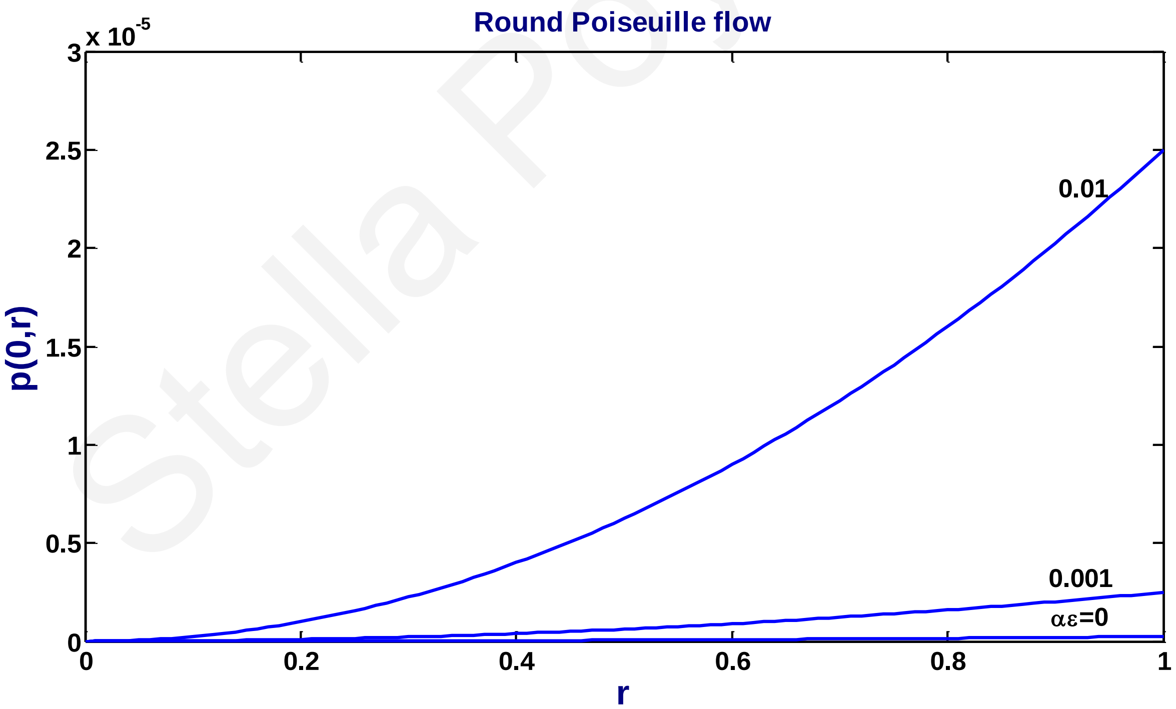
and

$$u_{z,crit} = \begin{cases} \dfrac{4(r-\kappa)}{(1+\kappa)(1-\kappa)^2}, & \kappa \le r \le (\kappa+1)/2 \\ \dfrac{4(1-r)}{(1+\kappa)(1-\kappa)^2}, & (\kappa+1)/2 \le r \le 1. \end{cases} \tag{2.33}$$

In Fig. 2.7, the velocity profiles for various values of the parameter *αε* are shown. We notice that for $\alpha\varepsilon < 0.1$ the velocity has the parabolic profile for incompressible flow which steadily tends to the triangular profile described by Eq. (2.30) as *αε* approaches the critical value.



(a)



(b)

Figure 2.4: *Pressure distribution along (a) the inlet and (b) the outlet planes for α=0.01 and various values of $\alpha\varepsilon$; axisymmetric Poiseuille flow.*

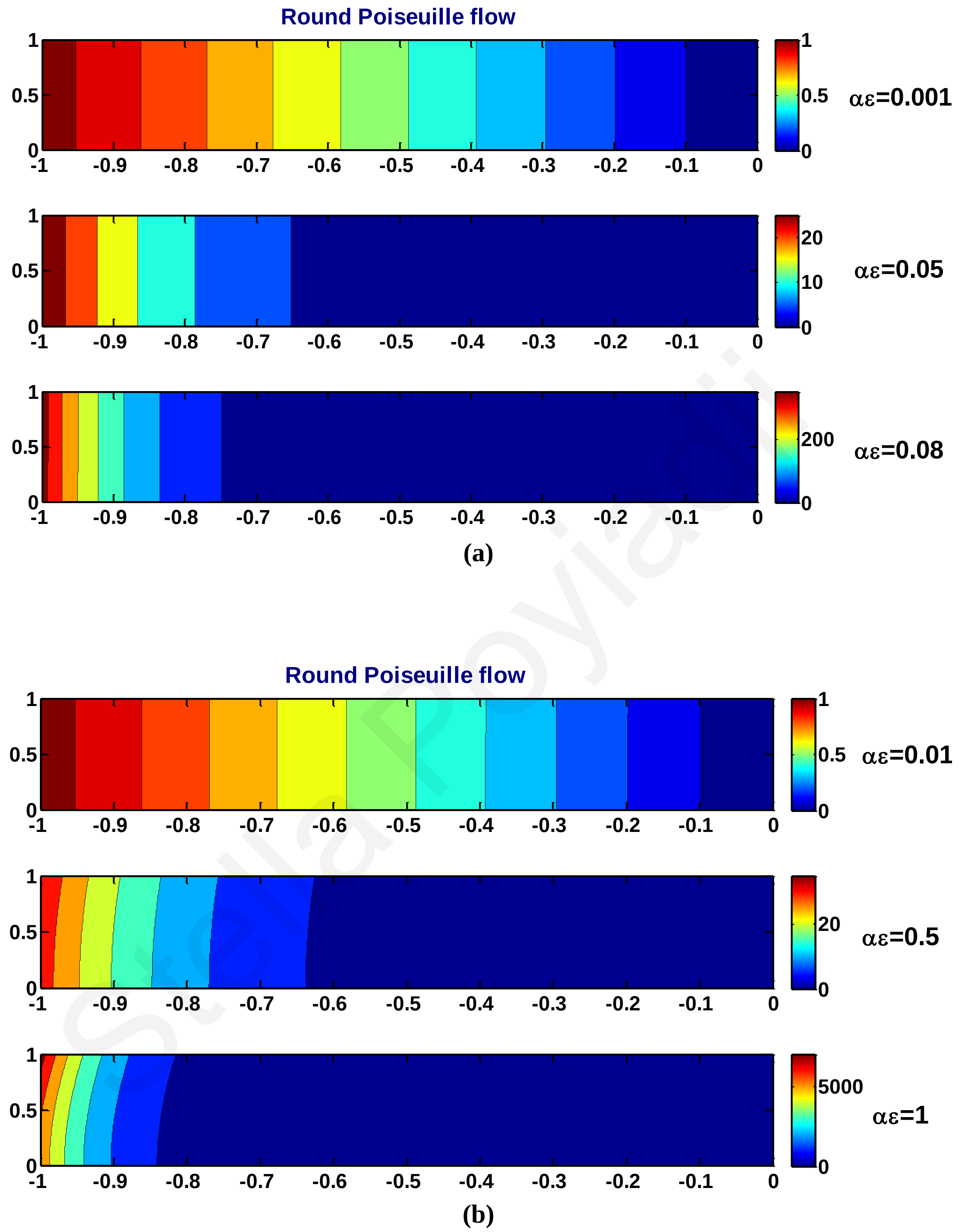


Figure 2.5: *Pressure contours for various values of $\alpha\varepsilon$ when (a) α=0.01 and (b) α=0.1; axisymmetric Poiseuille flow.*

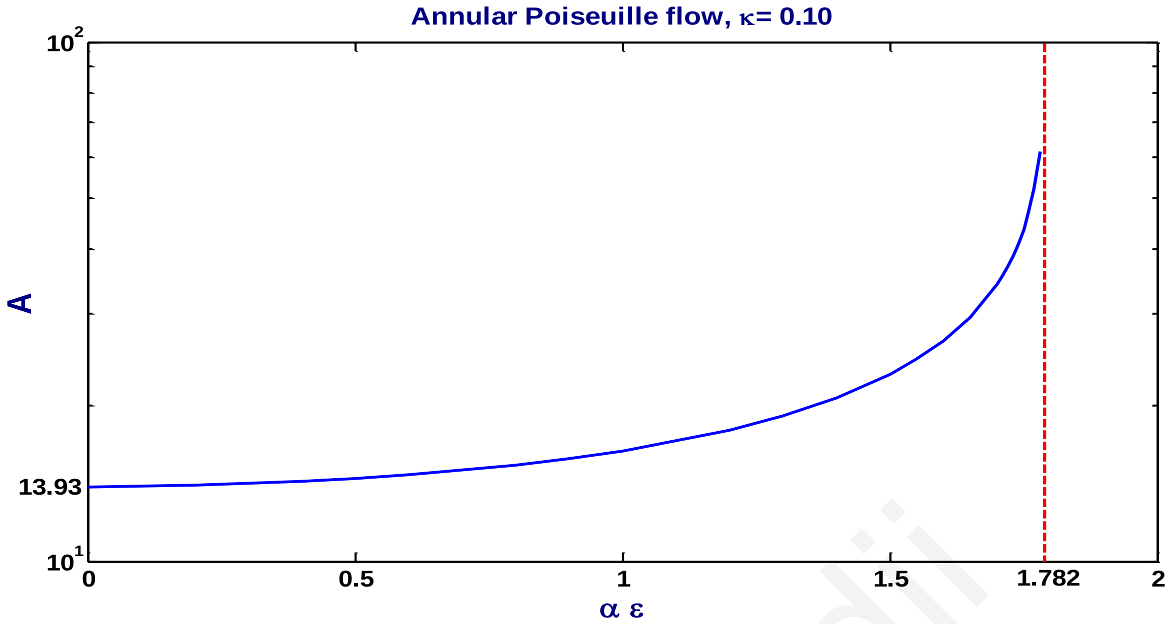


Figure 2.6: *The constant A as a function of $\alpha\varepsilon$ in annular Poiseuille flow for κ = 0.1.*

As in round Poiseuille flow, the pressure gradient is roughly constant only for low values of $\alpha\varepsilon$. As the latter parameter increases, the pressure increases faster with the distance from the exit plane. Figure 2.8 shows the pressure contours for a short $(\alpha = 0.1)$ and a long $(\alpha = 0.01)$ annulus and various values of $\alpha\varepsilon$. The vertical contours for small values of α begin to bend for bigger values of α, i.e. in shorter tubes.

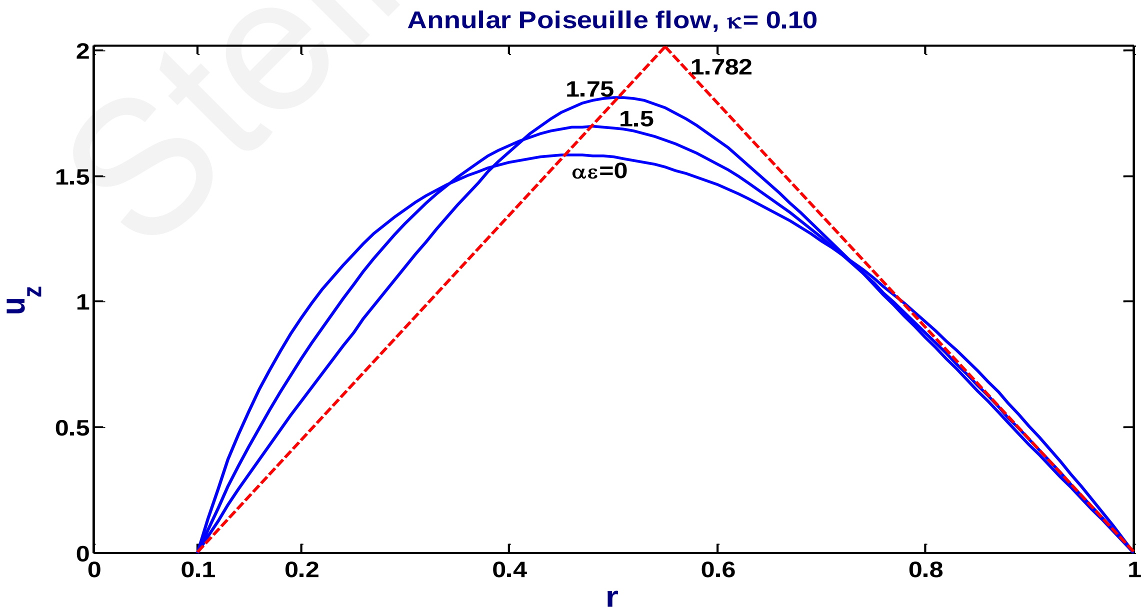


Figure 2.7: *Velocity profiles in annular Poiseuille flow for κ = 0.1 and various values of the parameter $\alpha\varepsilon$.*

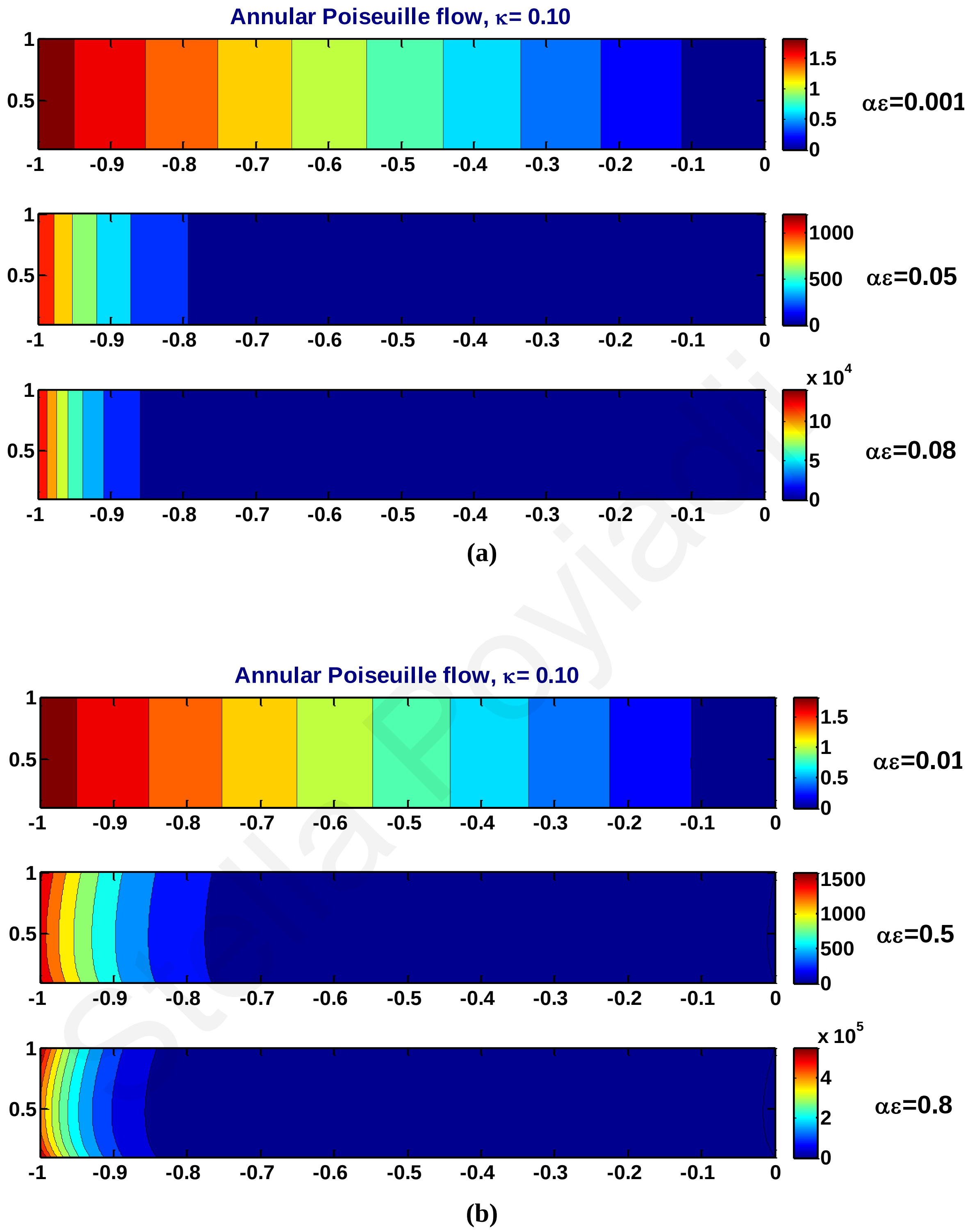


Figure 2.8: *Pressure contours for various values of αε when (a) α=0.01 and (b) α=0.1; annular Poiseuille flow for κ=0.1.*

2.7 Conclusions

Analytical solutions for the steady axisymmetric, annular, and plane Poiseuille flows of an incompressible Newtonian fluid with pressure-dependent viscosity, obeying Eq. (2.3), have been derived, under the assumption of unidirectional flow. These solutions show that as the pressure-dependence of the viscosity becomes stronger, the velocity profile, which is independent of the axial coordinate, tends from a parabolic-type to a triangular profile and the pressure, which is a function of both the axial and the radial coordinate, increases exponentially upstream. The latter result implies that the pressure required to drive the flow increases rapidly with the length of the tube.

In addition to the solution of the incompressible flow of a Newtonian fluid with pressure-dependent viscosity, the solution under the combined effect of slip at the wall with viscosity pressure dependence is presented in Appendix A, for the case of the plane flow.

Chapter 3

Weakly compressible Poiseuille flows with Navier slip

We consider both the plane and axisymmetric steady, laminar, Poiseuille flows of a weakly compressible Newtonian fluid assuming that slip occurs along the wall following Navier's slip equation and that the density obeys a linear equation of state. A perturbation analysis is performed in terms of the primary flow variables using the dimensionless isothermal compressibility as the perturbation parameter. Solutions up to the second order are derived and compared with available analytical results. The combined effects of slip, compressibility, and inertia are discussed with emphasis on the required pressure drop and the average Darcy friction factor[5].

3.1 Introduction

In a recent paper (Taliadorou et al., 2009), second-order perturbation solutions of both the planar and axisymmetric Poiseuille flows of weakly compressible Newtonian fluids have been derived using a methodology in which the primary flow variables, i.e. the velocity components and pressure, are perturbed, a linear equation of state is employed, and compressibility serves as the perturbation parameter. The same solutions were derived by Venerus (2006) and Venerus and Bugajsky (2010) respectively, for the axisymmetric and planar flow problems using a streamfunction/vorticity formulation. Housiadas and Georgiou (2011) have recently extended the primary-variable methodology to derive perturbation solutions of the planar Poiseuille flow of a weakly compressible Oldroyd-B fluid. The aforementioned references provide useful reviews of previous perturbation and other approximate solutions of the flow problems under consideration.

The objective of the present chapter is to extend previous work for a Newtonian liquid allowing linear slip at the wall in order to study the combined effects of weak compressibility, slip and inertia. The importance of slip in a variety of macroscopic flows

[5] The material of this chapter appears in Poyiadji et al. (2012).

and processes has been emphasized in numerous studies in the past few decades (Denn, 2001; Hatzikiriakos and Migler, 2005 and references therein). Strong interest has also been recently generated due to the effects of slip in microfluidic applications (Stone et al., 2004).

In flows of liquids, such as polymer melts and waxy crude oils, compressibility may become important when the liquids are processed at high pressures, which is the case with polymer extrusion (Hatzikiriakos and Dealy, 1992; Piau and El Kissi, 1994) or with flow through long tubes (Vinay et al., 2006). The stick-slip polymer extrusion instability, referring to the sustained pressure and flow rate oscillations observed under constant throughput, is attributed to the combination of compressibility with nonmonotonic slip laws relating the wall shear stress to the slip velocity (Hatzikiriakos and Dealy, 1992), as confirmed by one-dimensional phenomenological models (Dubbeldam and Molenaar, 2003) as well numerical simulations (Taliadorou et al., 2007). Tang and Kalyon (2008a; 2008b) also developed a mathematical model describing the time-dependent pressure-driven flow of compressible polymeric liquids subject to pressure-dependent slip and reported that undamped periodic pressure oscillations in pressure and mean velocity are observed when the boundary condition changes from weak to strong slip. Taliadorou et al. (2008) reported extrusion simulations showing that severe compressibility combined with inertia may lead to stable steady-state free surface oscillations, similar to those observed experimentally with liquid foams. Mitsoulis and Hatzikiriakos (2009) carried out steady flow simulations of polytetrafluoroethylene (PTFE) paste extrusion under severe slip taking into account the significant compressibility of these pastes.

The above material flows are weakly compressible, which means that the Mach number, *Ma,* is low, i.e. *Ma*<<1. The latter number is defined as the ratio of the characteristic speed of the flow to the speed of sound in the fluid. Georgiou and Crochet (1994) pointed out that taking into account the weak compressibility of the fluid may not have an effect on the steady flow solution but changes dramatically the flow dynamics. Similarly, Felderhof and Ooms (2011) studied the flow of a viscous compressible fluid in a circular tube generated by an impulsive point source and reported that compressibility has a significant effect on the flow dynamics in confined geometries.

The combination of slip with compressibility is also very important in rarefied gas flows through microchannels and need to be taken into account in the micro-electro-mechanical systems (MEMS) technology (Beskok and Karniadakis, 1999; Zhang et al., 2009). There are of course some important differences from the liquid flow problem under consideration: (a) the continuum assumption may not be valid and slip velocity is expressed in terms of the Knudsen number *Kn* (the ratio of the mean free path of the gas to the characteristic dimension of the tube); (b) the ideal gas law is used instead of the linear equation of state; and c) the flow is non-isothermal. Arkilic et al. (1997) and, more recently, Qin et al. (2007) derived perturbation approximations for compressible gas flow in microchannels with slip at the wall using the aspect ratio as the perturbation parameter. According to conventional theory, continuum based models for channels apply as long as the Knudsen number is lower than 0.01 (Kohl et al., 2005). On the other hand, according to Venerus and Bugajsky (2010), effects of slip in microchannels can be neglected for Knudsen numbers less than 0.001 Therefore, the present analysis concerns not only flows of compressible liquids with slip at the wall but also gas flows for $0.001<Kn<0.01$.

The chapter is organized as follows: In Section 3.2, the solution of the steady, compressible plane Poiseuille flow with slip at the wall is presented; the results for the axisymmetric flow are provided in Appendix B. Both the state and slip equations are assumed to be linear. In subsection 3.2.1 the governing equations and boundary conditions for the plane flow are presented. In subsection 3.2.2, the perturbation method in terms of the primary variables with the isothermal compressibility as the perturbation parameter is outlined and a solution is derived up to the second order. Explicit analytical solutions for the two non-zero velocity components, the pressure, and the density are obtained. In subsection 3.2.3 the volumetric flow rate and the stream function are given. In Section 3.3, the results are analyzed and discussed with the emphasis given on the combined effects of slip and compressibility on the pressure drop and the Darcy friction factor and finally in Section 3.4 the conclusions are outlined.

3.2 Plane Poiseuille flow

3.2.1 Governing equations

We consider the steady, laminar plane Poiseuille flow of a Newtonian fluid in a slit of length *L* and width *2H* in Cartesian coordinates (x, y), as shown in Fig. 3.1.

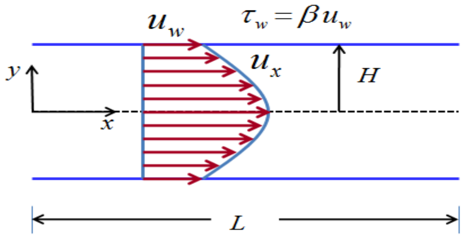


Figure 3.1: *Geometry and symbols for plane Poiseuille flow with slip along the wall.*

It is assumed that slip occurs along the wall according to a linear slip equation,

$$\tau_w = \beta u_w \tag{3.1}$$

where τ_w is the wall shear stress, β is the constant slip coefficient, and u_w is the slip velocity. The limiting case $\beta \to \infty$ corresponds to the no-slip boundary condition ($u_w \to 0$), whereas $\beta = 0$ corresponds to the theoretical case of full slip in which the velocity profile is plug.

Let us consider first the incompressible, one-dimensional flow under constant pressure gradient, $(-\partial p/\partial x)$. The velocity $u_x\left(y\right)$ is given by

$$u_x(y) = \frac{H}{\beta}\left(-\frac{\partial p}{\partial x}\right) + \frac{1}{2\eta}\left(-\frac{\partial p}{\partial x}\right)\left(H^2 - y^2\right), \tag{3.2}$$

where η is the constant viscosity. Obviously, the slip velocity is given by

$$u_w = u_x(H) = \frac{H}{\beta}\left(-\frac{\partial p}{\partial x}\right). \tag{3.3}$$

If the fluid is compressible, the flow becomes bidirectional and the two velocity components, u_x and u_y, are in general functions of both x and y. The isothermal compressibility is a measure of the ability of the material to change its volume under applied pressure at constant temperature. This is defined by

$$\kappa \equiv -\frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_{p_0, T_0}, \tag{3.4}$$

where V is the specific volume, ρ_0 and V_0 are respectively the density and the specific volume at the reference pressure, p_0, and temperature, T_0. Assuming that κ is constant, the above equation can be integrated yielding an exponential equation of state. In the present work however, we employ a linear equation of state,

$$\rho = \rho_0\left[1 + \kappa(p - p_0)\right] \tag{3.5}$$

which approximates well the exponential equation for small values of κ and for small pressures. The value of κ is of the order of 0.001 MPa^{-1} for molten polymers (Hatzikiriakos and Dealy, 1994) and increases by an order of magnitude (0.0178-0.0247 0.001 MPa^{-1}) in the case of PTFE pastes (Mitsoulis and Hatzikiriakos, 2009). Mitsoulis and Hatzikiriakos (2009) suggest that for weakly compressible flows, the values of κ range between 0 (incompressible fluids) and 0.02 MPa^{-1} (slightly to moderately compressible materials).

The linear equation of state can also be viewed as a special case of the well-established Tait equation and its variants for liquids and polymer melts (Guailly et al., 2011).

In order to nondimensionalize the governing equations and the boundary conditions of the flow, we scale x by the length of the channel L, y by the channel half-width H, the density ρ by the reference density ρ_0, the horizontal velocity, u_x, by the mean velocity at the channel exit U,

$$U \equiv \frac{\dot{M}}{\rho_0 HW},$$

where $\dot{M}$ is the mass flow rate and W is the unit length in the z-direction, and the transversal velocity, u_y, by UH/L. The Mach number is defined by

$$Ma \equiv \frac{U}{\sigma}, \tag{3.6}$$

where

$$\sigma \equiv \left[\gamma \left(\frac{\partial p}{\partial \rho} \right)_T \right]^{1/2} = \left(\frac{\gamma}{\kappa \rho_0} \right)^{1/2} \tag{3.7}$$

is the speed of sound in the fluid, γ being the heat capacity ratio or adiabatic index

($\gamma \equiv c_p / c_v$).With the above scalings, the dimensionless slip equation becomes

$$\tau_w = B u_w, \tag{3.8}$$

where all variables are now dimensionless and B is the slip number defined by

$$B \equiv \frac{\beta H}{\eta}. \tag{3.9}$$

The dimensionless velocity profile in the case of incompressible flow becomes

$$u_x(y) = \frac{3}{B+3} + \frac{3B}{2(B+3)}\left(1 - y^2\right) \tag{3.10}$$

or

$$u_x(y) = \frac{B^*}{B} + \frac{B^*}{2}\left(1 - y^2\right), \tag{3.11}$$

where

$$B^* \equiv \frac{3B}{B+3} \tag{3.12}$$

is an auxiliary slip number. In the no-slip limit, $B \to \infty$ and $B^* \to 3$. Therefore

$$u_x^{(0)}(y) = \frac{3}{2}\left(1 - y^2\right) \tag{3.13}$$

which is the standard velocity profile for incompressible flow with no slip at the wall.

By demanding that the dimensionless pressure gradient in the case of incompressible flow with no slip at the wall be equal to 1, the pressure scale should be $3\eta LU / H^2$. The dimensionless form of the equation of state (3.5) is then

$$\rho = 1 + \varepsilon p, \tag{3.14}$$

where

$$\varepsilon \equiv \frac{3\kappa\eta LU}{H^2} \tag{3.15}$$

is the dimensionless compressibility number. The Mach number takes the form

$$Ma = \sqrt{\frac{\varepsilon\alpha\, Re}{3\gamma}} \quad \Leftrightarrow \quad \varepsilon = \left(\frac{3\gamma}{\alpha Re}\right) Ma^2. \tag{3.16}$$

The present work deals with weakly compressible flows, e.g. Ma<0.3. Assuming that γ is of the order of unity, there must hold $\varepsilon\alpha Re$<0.27.

The dimensionless forms of the continuity and the *x*- and *y*-momentum equations in the case of compressible Poiseuille flow under the assumptions of zero bulk velocity and zero gravity (Taliadorou et al., 2009) are

$$\frac{\partial\left(\rho u_x\right)}{\partial x} + \frac{\partial\left(\rho u_y\right)}{\partial y} = 0 \tag{3.17}$$

$$\alpha\, Re\, \rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -3\frac{\partial p}{\partial x} + \alpha^2 \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\alpha^2}{3}\left(\frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_x}{\partial x^2}\right) \tag{3.18}$$

$$\alpha^3\, Re\, \rho\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -3\frac{\partial p}{\partial y} + \alpha^4 \frac{\partial^2 u_y}{\partial x^2} + \alpha^2 \frac{\partial^2 u_y}{\partial y^2} + \frac{\alpha^2}{3}\left(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2}\right), \tag{3.19}$$

where

$$Re \equiv \frac{\rho_0 HU}{\eta} \tag{3.20}$$

is the Reynolds number, and

$$\alpha \equiv \frac{H}{L} \tag{3.21}$$

is the aspect ratio of the channel.

As for the boundary conditions, the usual symmetry conditions are applied along the symmetry plane; along the wall u_x obeys the slip equation (3.8) while u_y vanishes. Moreover, the pressure at the upper right corner of the flow domain is set to zero and the mass flow rate at the exit plane should be equal to 1. Therefore, the conditions that close the system of the governing equations are the following:

$$\frac{\partial u_x}{\partial y}(x,0) = u_y(x,0) = 0 \tag{3.22}$$

$$-\frac{\partial u_x}{\partial y}(x,1) = B u_x(x,1) \quad \text{and} \quad u_y(x,1) = 0 \tag{3.23}$$

$$p(1,1) = 0 \tag{3.24}$$

$$\int_0^1 \rho u_x dy = 1 \tag{3.25}$$

As in Venerus (2006) and Taliadorou et al. (2009), no boundary conditions for the velocity are imposed at the entrance and exit planes ($x = 0$ and 1). The flow problem defined by Eqs. (3.14), (3.17)-(3.19) and (3.22)-(3.25) involves four dependent variables, u_x, u_y, p, and ρ, and four dimensionless numbers: ε, B, Re and α. Even though the density ρ can be eliminated by means of Eq. (3.14), it is kept in order to facilitate the derivation of the perturbation solution.

3.2.2 Perturbation solution

The present work deals with weakly compressible flows, that is the Mach number is small, typically Ma<0.3. From (3.16), it is deduced that as long as Ma is small and $\gamma/(\alpha Re)$ is of the order of unity or smaller, the compressibility number ε, is also small number that can be used as the perturbation parameter. We thus perturb all primary variables, u_x, u_y, p, and ρ, as follows:

$$\begin{aligned}
u_x &= u_x^{(0)} + \varepsilon u_x^{(1)} + \varepsilon^2 u_x^{(2)} + O\left(\varepsilon^3\right) \\
u_y &= u_y^{(0)} + \varepsilon u_y^{(1)} + \varepsilon^2 u_y^{(2)} + O\left(\varepsilon^3\right) \\
p &= p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + O\left(\varepsilon^3\right) \\
\rho &= \rho^{(0)} + \varepsilon \rho^{(1)} + \varepsilon^2 \rho^{(2)} + O\left(\varepsilon^3\right)
\end{aligned} \tag{3.26}$$

By substituting expansions (3.26) into the governing equations (3.14),(3.17)-(3.19) and also in the boundary conditions (3.22)-(3.24) and in the condition (3.25) and by collecting the terms of a given order in ε the corresponding perturbation equations and the boundary conditions are obtained. These can be found in Taliadorou et al. (2009) who present the more general case with non-zero bulk viscosity. As for the slip equation, it can easily be shown that

$$-\frac{\partial u_x^{(k)}}{\partial y}(x,1) = B u_x^{(k)}(x,1), \quad k = 0,1,2,\dots \tag{3.27}$$

where k is the order of the perturbation. In what follows, emphasis will be given only to the derivation of the second-order solution; the derivation of the leading-order solutions, which is based on the assumption that the transverse velocity u_y is zero, is straightforward and the methodology is the same as that described by Taliadorou et al. (2009). The zero-order solution is obviously the standard incompressible Poiseuille flow solution:

$$u_x^{(0)}(y) = \frac{B^*}{2B}\left(B+2-By^2\right), \quad u_y^{(0)} = 0, \quad p^{(0)}(x) = \frac{B^*}{3}(1-x), \quad \rho^{(0)} = 1. \tag{3.28}$$

The first-order solution is as follows:

$$u_x^{(1)} = -\frac{B^{*2}}{6B}\left(B+2-By^2\right)(1-x)$$
$$-\frac{\alpha Re B^{*4}}{7560B^2}\left[5B^2+45B+98-3\left(11B^2+77B+140\right)y^2+35\left(B^2+5B+6\right)y^4-7B\left(B+1\right)y^6\right] \tag{3.29}$$

$$u_y^{(1)} = 0 \tag{3.30}$$

$$p^{(1)} = -\frac{B^{*2}}{18}(1-x)^2 + \frac{\alpha\, Re\, B^{*4}}{315B^3}\left(2B^3+14B^2+35B+35\right)(1-x) + \frac{\alpha^2 B^{*2}}{54}\left(1-y^2\right) \tag{3.31}$$

$$\rho^{(1)}(x) = \frac{B^*}{3}(1-x) \tag{3.32}$$

At second order, the assumption for zero transverse velocity is relaxed. Based on symmetry arguments, it is assumed that $u_y^{(2)}$ is an odd function of the form

$$u_y^{(2)} = A_1 y + A_2 y^3 + A_3 y^5 + A_4 y^7, \tag{3.33}$$

where A_1, A_2, A_3, and A_4 are unknown constants. By following the procedure outlined below it can be shown that all higher-order terms in (3.33) must be zero. For simplicity, we employ here only the non-zero terms. Also, from the boundary condition $u_y(x,1)=0$, we get

$$A_1 + A_2 + A_3 + A_4 = 0. \tag{3.34}$$

From the state equation we have $\rho^{(2)} = p^{(1)}$ and hence

$$\rho^{(2)} = -\frac{B^{*2}}{18}(1-x)^2 + \frac{\alpha\, Re\, B^{*4}}{315B^3}\left(2B^3+14B^2+35B+35\right)(1-x) + \frac{\alpha^2 B^{*2}}{54}\left(1-y^2\right). \tag{3.35}$$

Integrating the second-order continuity equation with respect to *x* gives

$$u_x^{(2)} = -\rho^{(1)}u_x^{(1)} - \rho^{(2)}u_x^{(0)} + (1-x)\frac{\partial u_y^{(2)}}{\partial y} + F(y), \tag{3.36}$$

with $F(y)$ being an unknown function. Substituting all the known quantities into the above equation, we get

$$\begin{aligned}
u_x^{(2)} = {} & \frac{B^{*3}}{12B}\left(B+2-By^2\right)(1-x)^2 + \left(A_1 + 3A_2y^2 + 5A_3y^4 + 7A_4y^6\right)(1-x) \\
& -\frac{\alpha Re B^{*5}}{22680B^4}\Big[67B^4 + 603B^3 + 2170B^2 + 3780B + 2520 \\
& \qquad -\left(39B^4 + 273B^3 + 840B^2 + 1260B\right)y^2 \\
& \qquad -\left(35B^4 + 175B^3 + 210B^2\right)y^4 + \left(7B^3 + 21B^2\right)y^6\Big](1-x) \\
& -\frac{\alpha^2 B^{*3}}{108B}\left[B+2-2(B+1)y^2 + y^4\right] + F(y).
\end{aligned} \tag{3.37}$$

Applying now the second-order slip condition (3.27) we get the following equation

$$(1-x)\left[\left(A_1+3A_2+5A_3+7A_4\right) + \frac{1}{B}\left(6A_2+20A_3+42A_4\right)\right] + F(1) + \frac{1}{B}F'(1) + \frac{\alpha^2 B^{*3}}{27B^2} = 0$$

which is satisfied for any *x* in $[0,1]$ only if

$$F'(1) + BF(1) = -\frac{\alpha^2 B^{*3}}{27B} \tag{3.38}$$

or

$$BA_1 + 3(B+2)A_2 + 5(B+4)A_3 + 7(B+6)A_4 = 0. \tag{3.39}$$

Integrating now the *y*-component of the second-order momentum equation with respect to *y* and substituting all known quantities gives

$$
\begin{aligned}
p^{(2)} &= -\frac{\alpha^2 B^{*3}}{54B}\left(B+2-By^2\right)(1-x)+\frac{\alpha^2}{3}\left(A_1+3A_2y^2+5A_3y^4+7A_4y^6\right)\\
&+\frac{\alpha^3 ReB^{*5}}{204120B^4}\Big[67B^4+603B^3+2170B^2+3780B+2520\\
&\qquad -\left(39B^4+273B^3+840B^2+1260B\right)y^2\\
&\qquad -\left(35B^4+175B^3+210B^2\right)y^4+\left(7B^4+21B^3\right)y^6\Big]+G(x),
\end{aligned}
\tag{3.40}
$$

where $G(x)$ is a second unknown function to be determined.

Substituting all the known quantities in the second-order x-momentum equation and after some rearrangement we get

$$
\begin{aligned}
&-\frac{\alpha ReB^{*}}{2B}\left(B+2-By^2\right)\left(A_1+3A_2y^2+5A_3y^4+7A_4y^6\right)-\alpha ReB^{*}\left(A_1y^2+A_2y^4+A_3y^6+A_4y^8\right)\\
&+\frac{\alpha^2 Re^2 B^{*6}}{22680B^5}\Big[31B^5+341B^4+1594B^3+3962B^2+5040B+2520\\
&-\left(34B^5+306B^4+1288B^3+2940B^2+2520B\right)y^2-\left(32B^5+224B^4+350B^3-210B^2\right)y^4\\
&+\left(42B^5+210B^4+252B^3\right)y^6-\left(7B^5+21B^4\right)y^8\Big]-\frac{\alpha^2 B^{*3}}{54B}\left(11B+20-15By^2\right)-F''(y)\\
&=-\frac{B^{*3}}{6}(1-x)^2-3G'(x)+\left(6A_2+60A_3y^2+210A_4y^4\right)(1-x)\\
&+\frac{\alpha ReB^{*5}}{1890B^3}\left[59B^3+413B^2+980B+840-70(B^3+5B^2+6B)y^2+35(B^3+3B^2)y^4\right](1-x).
\end{aligned}
\tag{3.41}
$$

In order to be able to separate variables, we demand that the terms involving both $(1-x)$ and y are scalar multiples of $(1-x)$. This is equivalent to setting

$$
A_3=\frac{\alpha ReB^{*5}}{1620B^2}\left(B^2+5B+6\right) \text{ and } A_4=-\frac{\alpha ReB^{*5}}{11340B}\left(B+3\right). \tag{3.42}
$$

Solving the system of Eqs. (3.34) and (3.39) for A_1 and A_2 we find:

$$
A_1=\frac{\alpha ReB^{*5}}{11340B^2}\left(5B^2+45B+98\right) \text{ and } A_2=-\frac{\alpha ReB^{*5}}{11340B^2}\left(11B^2+77B+140\right). \tag{3.43}
$$

Therefore, $u_y^{(2)}$ is given by

$$
u_y^{(2)}=\frac{\alpha ReB^{*5}}{11340B^2}\left[\left(5B^2+45B+98\right)y-\left(11B^2+77B+140\right)y^3+7\left(B^2+5B+6\right)y^5-\left(B^2+3B\right)y^7\right] \tag{3.44}
$$

or

$$u_y^{(2)} = \frac{\alpha Re B^{*5}}{11340 B^2} y\left(1-y^2\right)\left[5B^2+45B+98-2\left(3B^2+16B+21\right)y^2+\left(B^2+3B\right)y^4\right]. \tag{3.45}$$

In order to complete the derivation of the second-order solution we now need to determine the unknown functions $F(y)$ and $G(x)$. Substituting $u_y^{(2)}$ into Eq. (3.41) and separating variables we get the following two ODEs:

$$\begin{aligned}
&-\frac{\alpha^2 Re^2 B^{*6}}{11340 B^5}\Big[-\left(13B^5+143B^4+703B^3+1883B^2+2520B+1260\right) \\
&\qquad +\left(3B^5+27B^4+252B^3+1050B^2+1260B\right)y^2+\left(39B^5+273B^4+525B^3+105B^2\right)y^4 \\
&\qquad -\left(35B^5+175B^4+210B^3\right)y^6+6\left(B^5+3B^4\right)y^8\Big]-\frac{\alpha^2 B^{*3}}{54B}\left(11B+20-15By^2\right)-F''(y) \\
&\qquad =-\frac{B^{*3}}{6}(1-x)^2+\frac{4\alpha Re B^{*5}}{315 B^3}\left(2B^3+14B^2+35B+35\right)(1-x)-3G'(x)=A,
\end{aligned} \tag{3.46}$$

where A is an unknown constant. Solving the first ODE of Eq. (3.46) for $F(y)$ we get:

$$\begin{aligned}
F(y) &= -\frac{\alpha^2 Re^2 B^{*6}}{151200 B^5}\Big[-60\left(13B^5+143B^4+703B^3+1883B^2+2520B+1260\right)y^2 \\
&\quad +30(B^5+9B^4+84B^3+350B^2+420B)y^4+12\left(13B^5+91B^4+175B^3+35B^2\right)y^6 \\
&\quad -\frac{\alpha^2 B^{*3}}{216B}\left[2\left(11B+20\right)y^2-5By^4\right]-\frac{1}{2}Ay^2+c_1 y+c_2,
\end{aligned} \tag{3.47}$$

where c_1 and c_2 are unknown constants.

Condition $\partial u_x(x,0)/\partial y=0$ in Eq. (3.22), gives $F'(0)=0$ and thus $c_1=0$. Applying conditions (3.25) and (3.38) and solving the resulting system for A and c_2 we find that

$$\begin{aligned}
A &= -\frac{B^{*4}\alpha^2}{81B^2}\left(4B^2+19B+27\right)+\frac{\alpha^2 Re^2 B^{*7}}{9823275 B^6}\big(3044B^6+42616B^5+267036B^4 \\
&\quad +951720B^3+1964655B^2+2182950B+1091475\big)
\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}
c_2 &= \frac{\alpha^2 B^{*4}}{648B^2}\left(B^2+7B+4\right)-\frac{\alpha^2 Re^2 B^{*7}}{314344800 B^5}\big(2193B^5+35088B^4+221641B^3 \\
&\quad +731346B^2+1409100B+1358280\big).
\end{aligned} \tag{3.49}$$

Hence,

$$F(y)=\frac{\alpha^2 B^{*3}}{648B}\left[B^2+7B+4-6\left(B^2+5B+2\right)y^2+5\left(B^2+3B\right)y^4\right]$$
$$-\frac{\alpha^2 Re^2 B^{*7}}{314344800B^5}\Big[2193B^5+35088B^4+221641B^3+731346B^2+1409100B+1358280$$
$$-4\left(2839B^5+39746B^4+239316B^3+803880B^2+1576575B+1455300\right)y^2 \qquad (3.50)$$
$$+2310\left(B^5+12B^4+111B^3+602B^2+1470B+1260\right)y^4+924\left(13B^5+130B^4+448B^3\right.$$
$$\left.+560B^2+105B\right)y^6-5775\left(B^5+8B^4+21B^3+18B^2\right)y^8+616\left(B^5+6B^4+9B^3\right)y^{10}\Big].$$

Integrating for $G(x)$ the second ODE in Eq. (3.46) we get:

$$G(x)=\frac{B^{*3}}{54}(1-x)^3-\frac{2\alpha ReB^{*5}}{315B^3}\left(2B^3+14B^2+35B+35\right)(1-x)^2+\frac{A}{3}(1-x)+c_3, \qquad (3.51)$$

where the unknown constant c_3 is determined from condition (3.24):

$$c_3=-\frac{\alpha^3 ReB^{*5}}{25515B^4}\left(-2B^3+56B^3+315B+315\right), \qquad (3.52)$$

with which the derivation of the second-order solution is completed.

In summary, the perturbation solution of the flow problem up to second order is:

$$u_x(x,y)=\frac{B^*}{2B}\left(B+2-By^2\right)+\varepsilon\Bigg[-\frac{B^{*2}}{6B}\left(B+2-By^2\right)(1-x)$$
$$-\frac{\alpha ReB^{*4}}{7560B^2}\left[5B^2+45B+98-3\left(11B^2+77B+140\right)y^2+35\left(B^2+5B+6\right)y^4-7\left(B^2+3B\right)y^6\right]\Bigg]$$
$$+\varepsilon^2\Bigg[\frac{B^{*3}}{12B}\left(B+2-By^2\right)(1-x)^2-\frac{\alpha ReB^{*5}}{7560B^4}\Big[19B^4+171B^3+658B^2+1260B+840$$
$$+3\left(3B^4+21B^3-140B\right)y^2-35(B^4+5B^3+6B^2)y^4+7\left(B^4+3B^3\right)y^6\Big](1-x)$$
$$-\frac{\alpha^2 B^{*4}}{648B^2}\left[B^2+3B+8+2\left(B^2+7B\right)y^2-3\left(B^2+3B\right)y^4\right]$$
$$-\frac{\alpha^2 Re^2 B^{*7}}{314344800B^5}\Big[2193B^5+35088B^4+221641B^3+731346B^2+1409100B+1358280$$
$$-4\left(2839B^5+39746B^4+239316B^3+803880B^2+1576575B+1455300\right)y^2$$
$$+2310\left(B^5+12B^4+111B^3+602B^2+1470B+1260\right)y^4+924\left(13B^5+130B^4+448B^3\right.$$
$$\left.+560B^2+105B\right)y^6-5775\left(B^5+8B^4+21B^3+18B^2\right)y^8+616\left(B^5+6B^4+9B^3\right)y^{10}\Big]+O\left(\varepsilon^3\right)$$
$$(3.53)$$

$$u_y(y)=\varepsilon^2\frac{\alpha ReB^{*5}}{11340B^2}y(1-y^2)\left[5B^2+45B+98-2\left(3B^2+16B+21\right)y^2+\left(B^2+3B\right)y^4\right]+O\left(\varepsilon^3\right)$$
$$(3.54)$$

$$p(x,y)=\frac{B^*}{3}(1-x)+\varepsilon\left[-\frac{B^{*2}}{18}(1-x)^2+\frac{\alpha ReB^{*4}}{315B^3}\left(2B^3+14B^2+35B+35\right)(1-x)+\frac{\alpha^2B^{*2}}{54}\left(1-y^2\right)\right]$$
$$+\varepsilon^2\left[\frac{B^{*3}}{54}(1-x)^3-\frac{\alpha ReB^{*5}}{945B^3}\left(4B^3+28B^3+70B+70\right)(1-x)^2\right.$$
$$-\frac{\alpha^2B^{*4}}{486B^2}\left[11B^2+53B+72-3(B^2+3B)y^2\right](1-x)+\frac{\alpha^2Re^2B^{*7}}{29469825B^6}\left(3044B^6+42616B^5\right.$$
$$\left.+267036B^4+951720B^3+1964655B^2+2182950B+1091475\right)(1-x)$$
$$+\frac{\alpha^3ReB^{*5}}{204120B^3}\left[97B^3+889B^2+2310B+1260-3\left(79B^3+553B^2+1120B+420\right)y^2\right.$$
$$\left.\left.+175\left(B^3+5B^2+6B\right)y^4-35\left(B^3+3B^2\right)y^6\right]\right]+O\left(\varepsilon^3\right) \tag{3.55}$$

$$\rho=1+\varepsilon\frac{B^*}{3}(1-x)+\varepsilon^2\left[-\frac{B^{*2}}{18}(1-x)^2+\frac{\alpha ReB^{*4}}{315B^3}\left(2B^3+14B^2+35B+35\right)(1-x)\right.$$
$$\left.+\frac{\alpha^2B^{*2}}{54}\left(1-y^2\right)\right]+O\left(\varepsilon^3\right) \tag{3.56}$$

Letting $B\to\infty$ we get the solution obtained by Taliadorou et al. (2009) and Venerus and Bugajsky (2010) for flow with no-slip at the wall. The perturbation solution for the axisymmetric flow is given in Appendix B.

3.2.3 Volumetric flow rate and stream function

The volumetric flow rate,

$$Q(x)\equiv\int_0^1 u_x(x,y)\,dy \tag{3.57}$$

is given by

$$Q(x)=1-\varepsilon\frac{B^*}{3}(1-x)+\varepsilon^2\left[\frac{B^{*2}}{6}(1-x)^2-\frac{\alpha ReB^{*4}}{315B^3}\left(2B^3+14B^2+35B+35\right)(1-x)\right.$$
$$\left.-\frac{\alpha^2B^{*3}}{405B}(2B+5)\right]+O\left(\varepsilon^3\right). \tag{3.58}$$

The streamfunction $\psi(x,y)$, defined by

$$\frac{\partial\psi}{\partial x}\equiv\rho u_y \quad\text{and}\quad \frac{\partial\psi}{\partial y}\equiv-\rho u_x$$

is found to be as follows:

$$
\begin{aligned}
\psi(x,y) = & -\frac{B^*}{6B}\left[3(B+2)y - By^3\right] \\
& + \varepsilon \frac{\alpha Re B^{*4}}{7560B^2}\left[\left(5B^2+45B+98\right)y - \left(11B^2+77B+140\right)y^3 + 7\left(B^2+5B+6\right)y^5 - \left(B^2+3B\right)y^7\right] \\
& + \varepsilon^2\left[-\frac{\alpha Re B^{*5}}{11340B^2}\left[\left(5B^2+45B+98\right)y - \left(11B^2+77B+140\right)y^3\right.\right. \\
& \qquad \left. + 7\left(B^2+5B+6\right)y^5 - \left(B^2+3B\right)y^7\right](1-x) \\
& - \frac{\alpha^2 B^{*4}}{648B^2}\left[\left(B^2+7B+4\right)y - 2\left(B^2+5B+2\right)y^3 + \left(B^2+3B\right)y^5\right] \\
& + \frac{\alpha^2 Re^2 B^{*7}}{943034400B^5}\left[3\left(2193B^5+35088B^4+221641B^3+731346B^2+1409100B+1358280\right)y\right. \\
& \qquad - 4\left(2839B^5+39746B^4+239316B^3+803880B^2+1576575B+1455300\right)y^3 \\
& \qquad + 1386\left(B^5+12B^4+111B^3+602B^2+1470B+1260\right)y^5 \\
& \qquad + 396\left(13B^5+130B^4+448B^3+560B^2+105B\right)y^7 \\
& \qquad \left.\left. -1925\left(B^5+8B^4+21B^3+18B^2\right)y^9 + 168\left(B^5+6B^4+9B^3\right)y^{11}\right]\right] + O\left(\varepsilon^3\right).
\end{aligned}
$$

3.3 Results and discussion

Let us first discuss the effect of the slip number on the two velocity components. In creeping flow (*Re*=0), the transverse velocity component, u_y, is zero at second order. The effect of the slip number *B* on the transverse velocity is shown in Fig. 3.2. The transverse velocity is reduced as the slip number is reduced from infinity (no slip) to zero (full-slip). As slip becomes stronger the velocity in the flow direction tends to become more uniform and thus the flow tends to become one-dimensional. Given that the transverse velocity component is always positive (Eq. (3.54)), the streamlines of the flow under study are either horizontal or have a slight positive slope which reaches its maximum value roughly in the middle of the *y*-interval [0,1]. The effect of slip on the transverse velocity component is more clearly illustrated in Fig. 3, where the reduced mean value,

$$
\frac{\bar{u}_y}{\alpha Re\,\varepsilon^2} \equiv \frac{1}{\alpha Re\,\varepsilon^2}\int_0^1 u_y(y)\,dy = \frac{B^3}{1120(B+3)^5}\left(19B^2+209B+504\right) + O\left(\varepsilon^3\right) \quad (3.59)
$$

is plotted versus the slip number *B*. Appreciable slip occurs in the range $1<B<100$ and slip may be considered as strong for $B<1$. In conclusion, the unidirectionality assumption is valid when the flow is creeping and/or slip is strong.

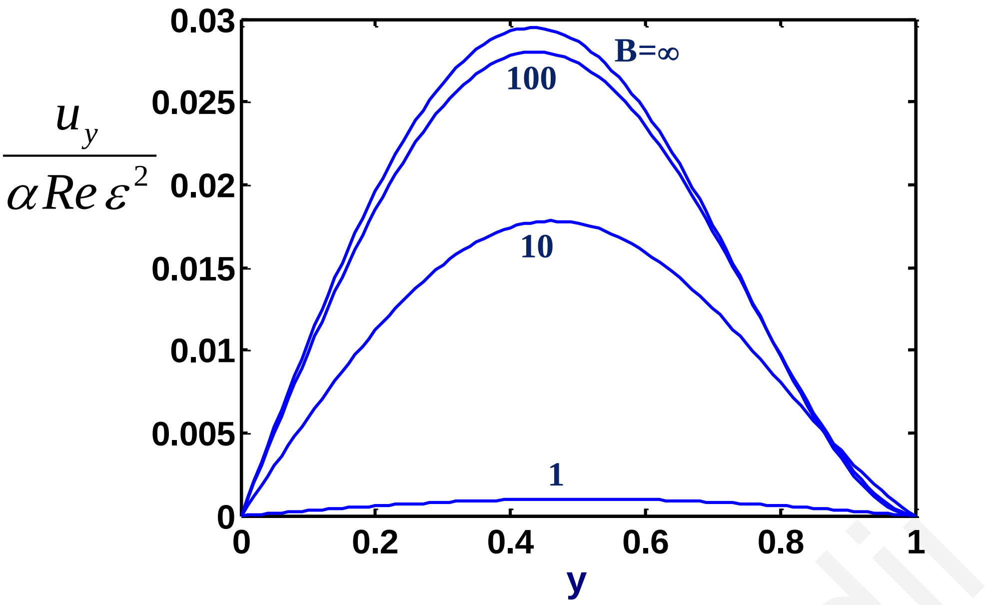


Figure 3.2: *Effect of the slip number on the transverse velocity component.*

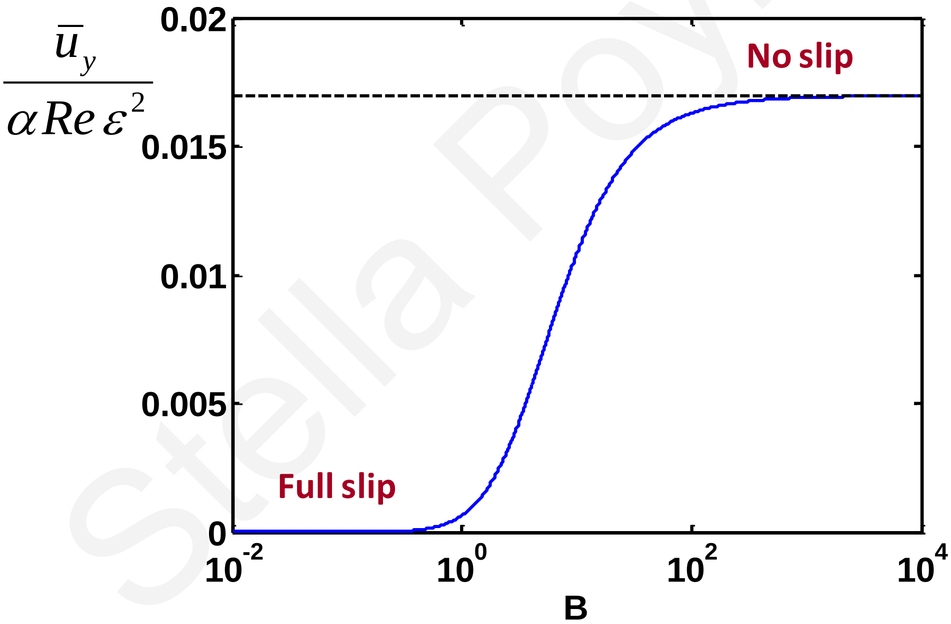


Figure 3.3: *The mean transverse velocity as a function of the slip number.*

In Fig. 3.4, the contours of the velocity in the flow direction for B=∞ (no slip) and 1 (strong slip) with Re=0, ε=0.1, and α=0.01 are compared. Even though the contour patterns are similar, the main difference is that the range of the velocity values, which in the case of no-slip is the interval [0, 1.5], shrinks with slip (Fig. 3.5); in the extreme case of full slip, u_x is uniform and equal to 1 at the channel exit.

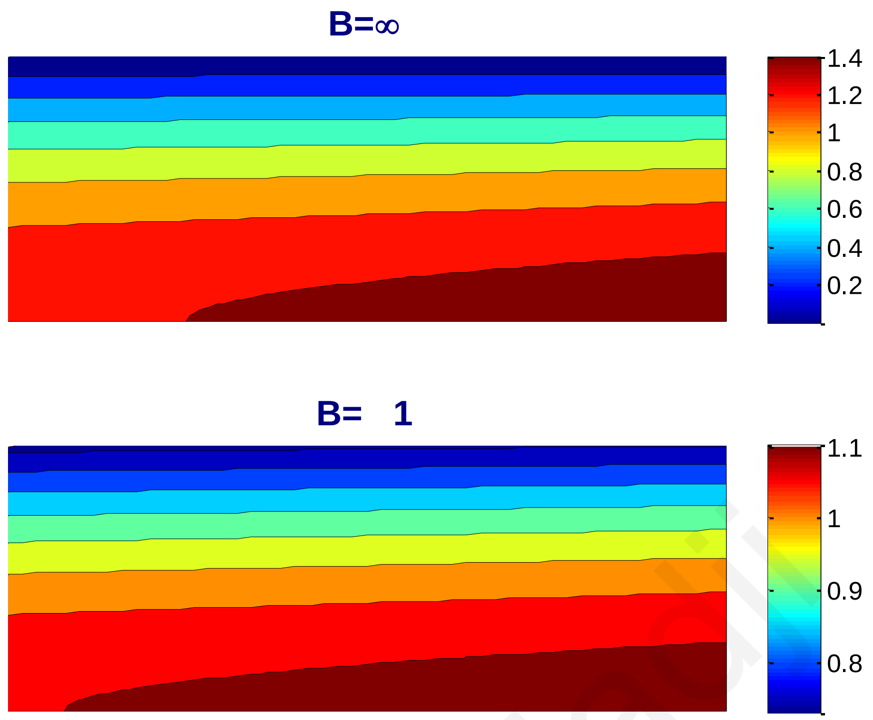


Figure 3.4: *Contours of u_x for $B=\infty$ (no slip) and 1 (strong slip); $Re=0$, $\varepsilon=0.1$, and $\alpha=0.01$*

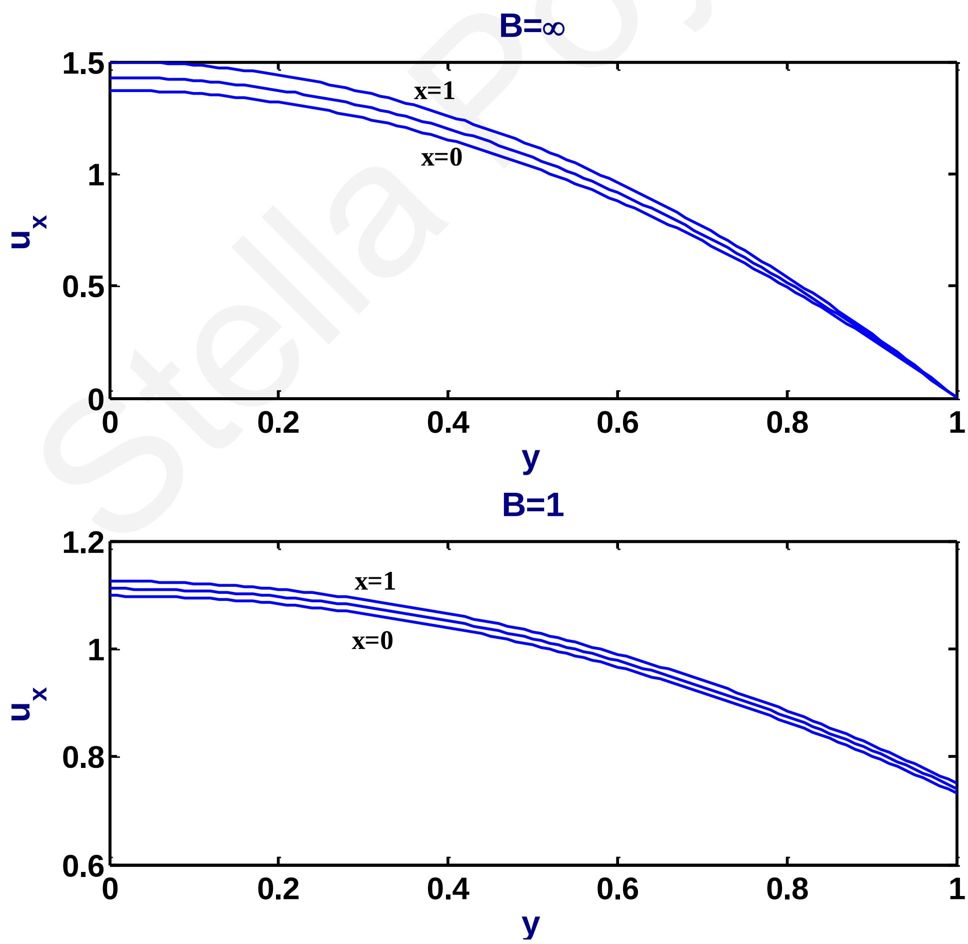


Figure 3.5: *Profiles of the velocity in the flow direction at $x=0$, 0.5 and 1 for $B=\infty$ (no slip) and 1; $\varepsilon=0.1$, $Re=0$, and $\alpha=0.01$.*

The effect of the compressibility number ε on the contours of u_x for Re=0, α=0.01 and B=1 (strong slip) is illustrated in Fig. 3.6. In Fig. 3.7, the velocity contours obtained with Re=0 and 100 and B=1, ε=0.1, and α=0.01 are shown. The results are essentially the same, since higher-order contributions contain the product αRe which is small. To magnify the effect of αRe, the velocity contours for a shorter channel with aspect ratio α=0.1 are plotted in Fig. 3.8. It is observed that the effect of Reynolds number becomes significant. Note that, the Mach number corresponding to Re=100, ε=0.1, and α=0.1 is equal to 0.6 (γ is of unity order) and the flow can no longer be considered weakly compressible. However, the asymptotic expansions are still valid since the compressibility number is still small. Note that, since $Re = 3(\gamma / a\varepsilon)Ma^2$, when the compressibility number ε and the Mach number are small (<0.3), solutions are admissible only below a critical value of the Reynolds number. (For example, the critical value for Re is 270 for the data in Fig. 3.7 and is reduced to 27 in Fig. 3.8 where α is increased from 0.01 to 0.1.) Generally, as the channel becomes shorter (α increasing) the admissible Reynolds numbers get smaller-the flow tends to creeping flow.

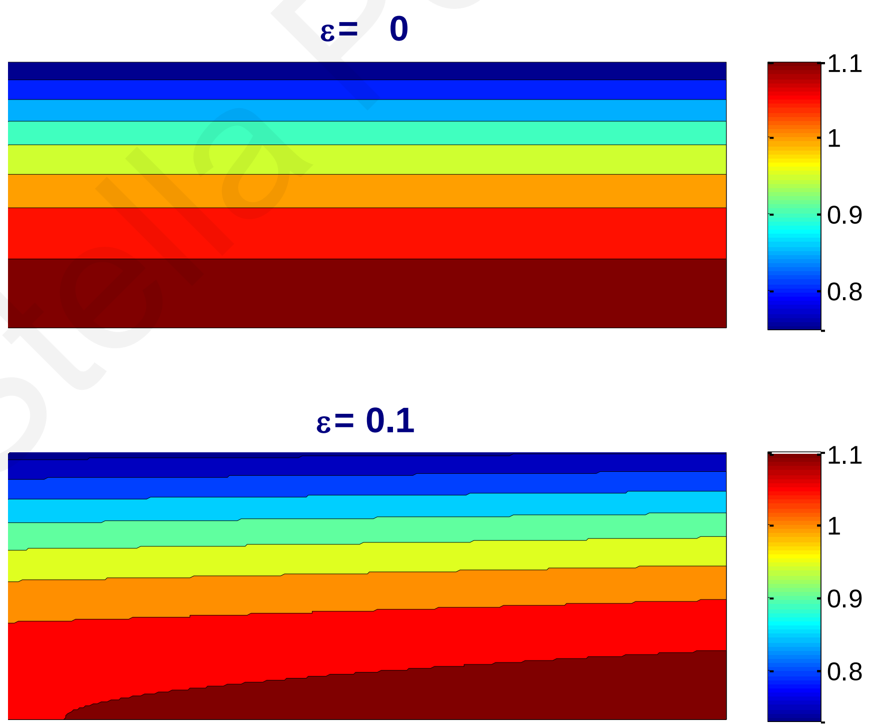


Figure 3.6: *Contours of u_x for ε=0 (incompressible flow) and 0.1; Re=0, B=1 and α=0.01.*

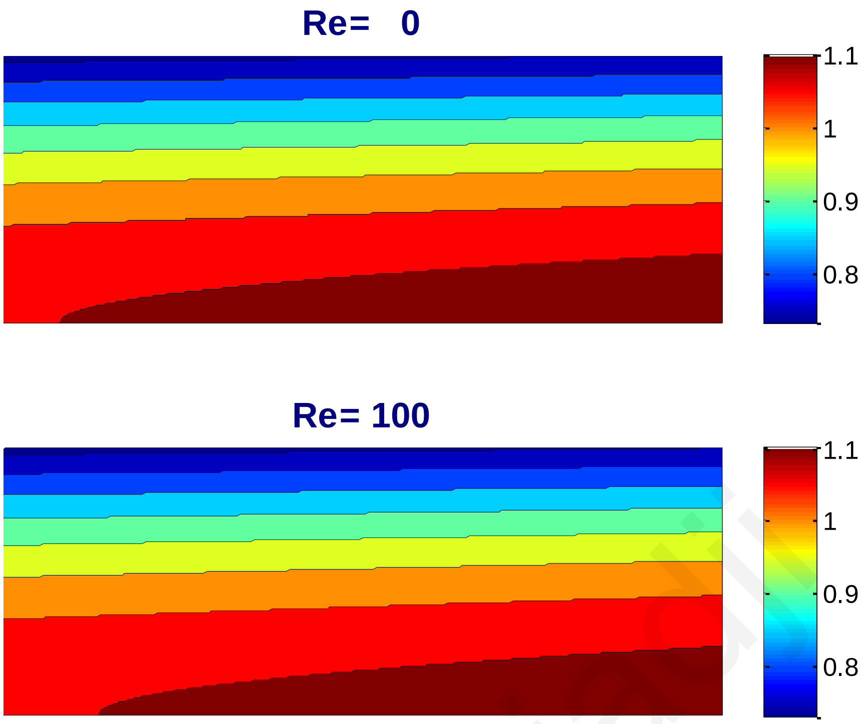


Figure 3.7: *Contours of* u_x *for Re=0 and 100;* α*=0.01 (long channel), B=1, and* ε*=0.1.*

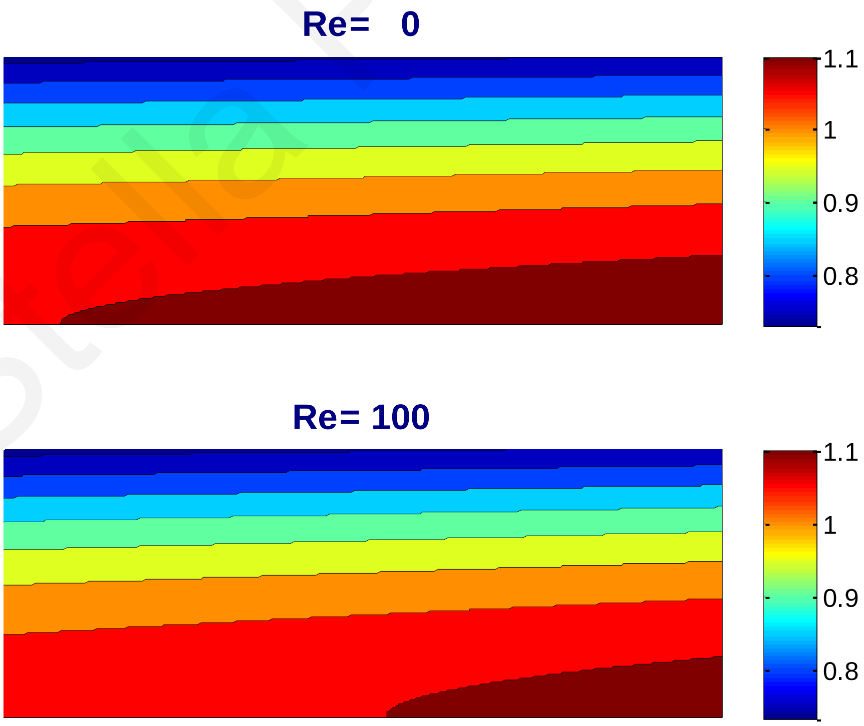


Figure 3.8: *Contours of* u_x *for Re=0 and 100;* α*=0.1 (shorter channel), B=1, and* ε*=0.1.*

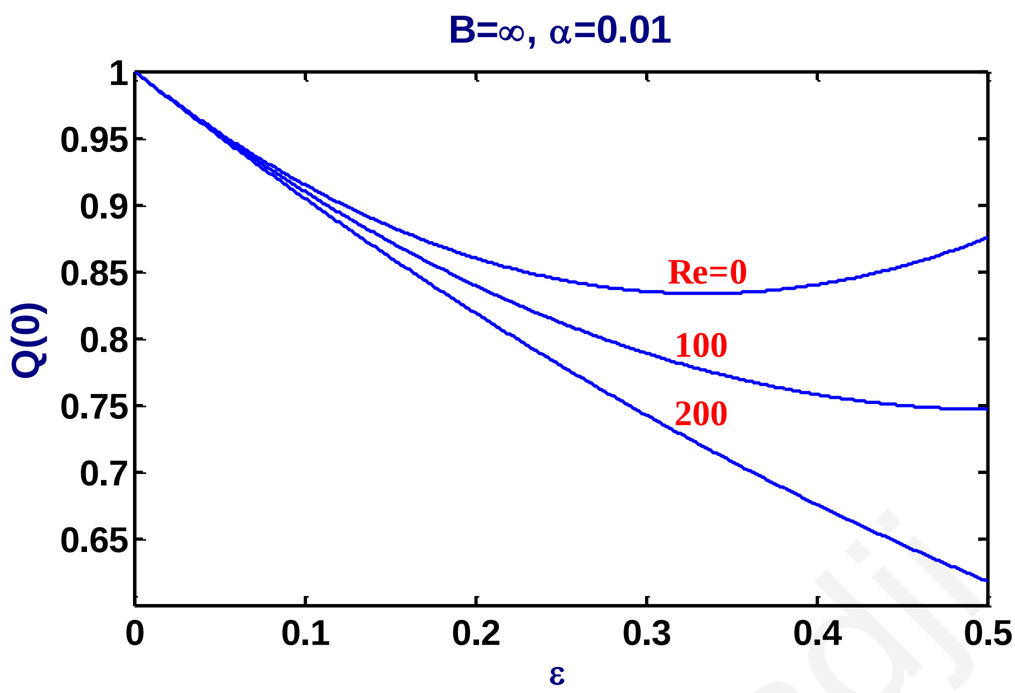


Figure 3.9: *The volumetric flow rate at the inlet plane for different Reynolds numbers with α=0.01 and no slip at the wall (B=∞).*

Another way to investigate the validity of our solution arises from looking into the volumetric flow rate given by Eq. (3.58). Since the solution is up to second order, Q is a parabolic function of ε for any value of x. At the exit plane,

$$Q(1)=1-\frac{\alpha^2 B^2 (2B+5)}{15(B+3)^3}\varepsilon^2+O(\varepsilon^3). \tag{3.60}$$

Obviously, $Q(1)$ is slightly below unity, given that $\alpha\varepsilon$ is small. Since the flow is compressible, the volumetric flow rate is reduced as we move upstream. A solution is assumed to be admissible if the volumetric flow rate $Q(0)$ at the inlet is a decreasing function of ε and positive. In Fig. 3.9, $Q(0)$ is plotted versus ε for various Reynolds numbers, with $B=\infty$ (no slip) and $\alpha=0.01$. In creeping flow ($Re=0$), solutions are admissible for $\varepsilon<1/3$. As the Reynolds number is increased, $Q(0)$ decreases faster with ε and may become negative for even smaller compressibility numbers. In other words, given the compressibility number, the aspect ratio, and the Mach number, solutions are admissible only below a critical value of the Reynolds number, which has also be noted above.

As shown in Fig. 3.10, slip weakens the compressibility effects and reduces the reduction of the volumetric flow rate upstream. As a result, slip extends the range of admissible solutions by shifting the minimum of $Q(0)$ to the right (Fig. 3.11).

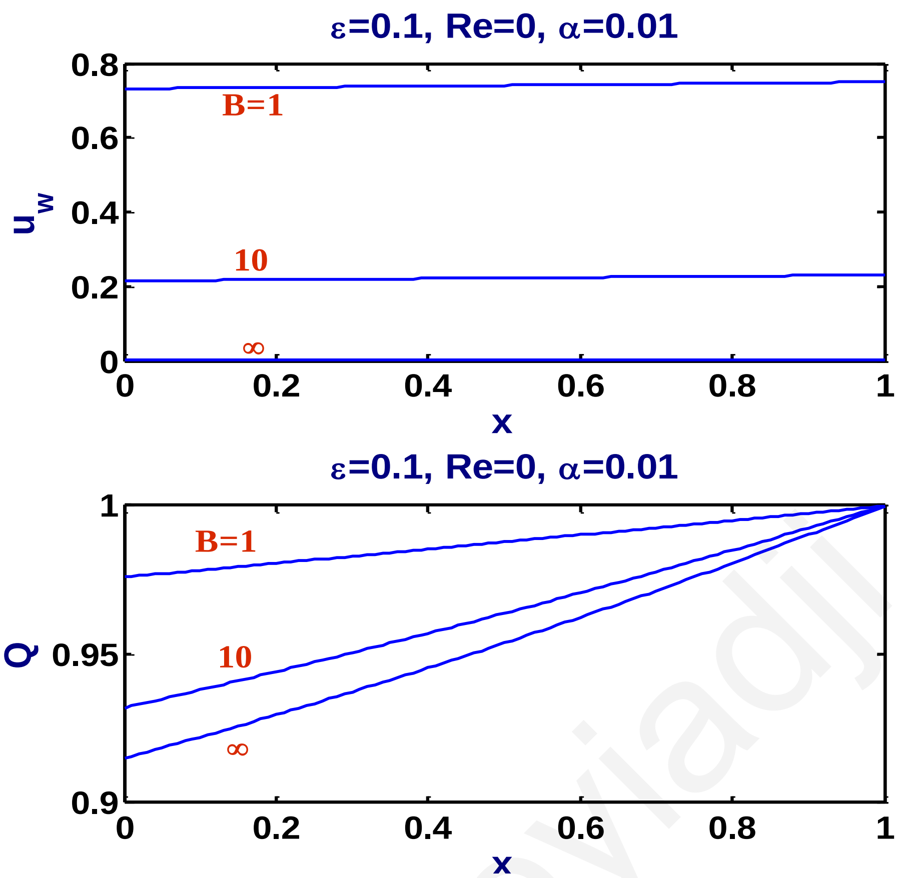


Figure 3.10: *Variations of the slip velocity and the volumetric flow in the channel for different slip numbers, ε=0.1, Re=0, and α=0.01.*

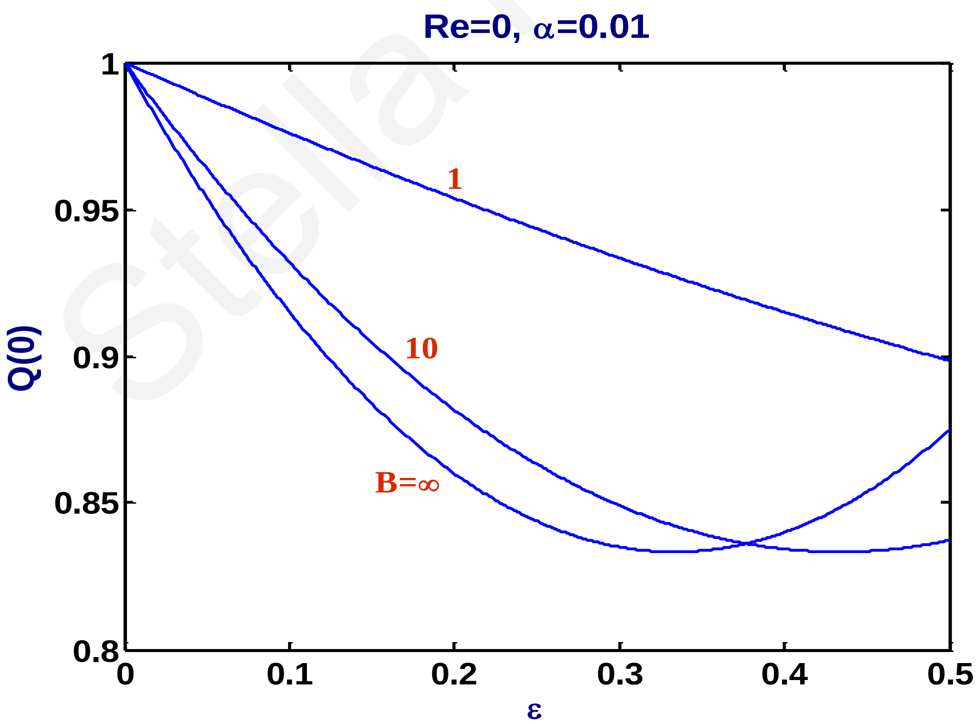


Figure 3.11: *Effect of slip number on the volumetric flow rate at the entrance plane; Re=0, α=0.01.*

From Eq. (3.56), we see that the density ρ at the exit plane is 1 at leading order. At the inlet plane, where the density obviously is maximized, we have

$$\rho(x=0)=1+\frac{B}{B+3}\varepsilon+O\left(\varepsilon^2\right). \tag{3.61}$$

The maximum value for ρ, obtained in the case of no slip (B=∞), is given by

$$\rho_{\max}=1+\varepsilon+O\left(\varepsilon^2\right), \tag{3.62}$$

and is independent of Re and α. In creeping flow, $\varepsilon<1/3$ and thus the maximum admissible value of the density for any α is $\rho_{max}=4/3$, which restricts the range of validity of the solution. However, more compression, which is expected for very small values of α (for very long channels), can be obtained only if higher values of the compressibility number are admissible, i.e. for lower values of the Reynolds number. In other words, moderately compressible flow is associated with finite, moderate Reynolds numbers. Recalling that for weakly compressible flow we have $\alpha\varepsilon Re<0.27$, such a combination of ε and Re is allowed only for smaller values of the aspect ratio α.

Generally, slip reduces the pressure in the channel and the required pressure drop. In Fig. 3.12, we show the distribution of the pressure along the centreline for different slip numbers, ε=0.1, Re=0 and α=0.01. As the slip number tends to zero (full slip) the pressure tends to become zero everywhere.

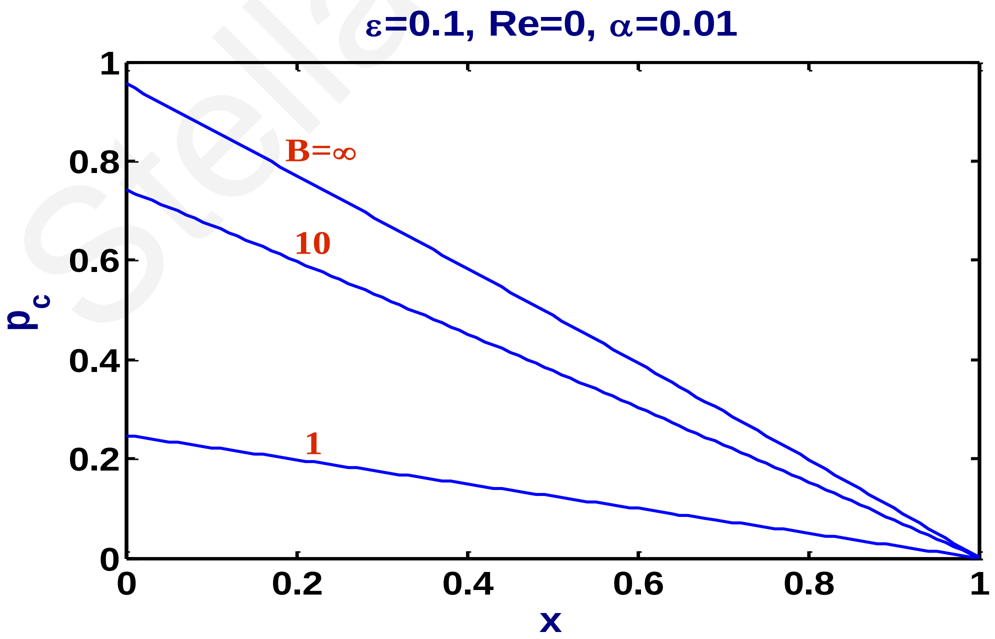


Figure 3.12: *Variation of the pressure along the centreline for various slip numbers; ε=0.1, Re=0, α=0.01*

Following Venerus and Bugajsky (2010) we calculate the mean pressure drop as follows

$$\overline{\Delta p} \equiv \bar{p}(0) - \bar{p}(1) \equiv \int_0^1 \left[p(0,y) - p(1,y) \right] dy \, , \tag{3.63}$$

which gives

$$\begin{aligned} \overline{\Delta p} = \frac{B^*}{3} - \varepsilon \left[\frac{B^{*2}}{18} - \frac{\alpha Re B^{*4}}{315 B^3} \left(2B^3 + 14B^2 + 35B + 35 \right) \right] \\ + \varepsilon^2 \left[\frac{B^{*3}}{54} - \frac{2\alpha Re B^{*5}}{945 B^3} \left(2B^3 + 14B^2 + 35B + 35 \right) - \frac{\alpha^2 B^{*4}}{243 B^2} \left(5B^2 + 25B + 36 \right) \right. \\ + \frac{\alpha^2 Re^2 B^{*7}}{29469825 B^6} \left(3044 B^6 + 42616 B^5 + 267036 B^4 + 951720 B^3 \right. \\ \left. \left. + 1964655 B^2 + 2182950 B + 1091475 \right) \right] + O\left(\varepsilon^3 \right). \end{aligned} \tag{3.64}$$

Equation (3.62) gives the pressure drop for channel flow of a compressible Newtonian fluid with slip at the wall. This is a generalization of the result provided by Venerus and Bugajsky (2010) for the no-slip case ($B \to \infty$):

$$\overline{\Delta p} = 1 - \left(\frac{1}{2} - \frac{18}{35} \alpha Re \right) \varepsilon + \left(\frac{1}{2} - \frac{5}{3} \alpha^2 - \frac{36}{35} \alpha Re + \frac{3044}{13475} \alpha^2 Re^2 \right) \varepsilon^2 + O\left(\varepsilon^3 \right). \tag{3.65}$$

(It should be noted that the Reynolds number in Venerus and Bugajsky (2010) is twice the present Reynolds number.) It is clear that the required pressure drop decreases with compressibility and increases with inertia, as illustrated in Fig. 3.13. The effect of slip is illustrated in Fig. 3.14 where the pressure drops for various slip numbers are plotted. Slip leads to the reduction of the pressure difference required to drive the flow and consequently alleviates compressibility effects. This is, of course, expected and also noted in previous works. For example, Zhang et al. (2009), in their analysis of slip flow characteristics of compressible gases in microchannels, reported that “slip effect makes the flow less compressible”. For the set of values used to construct Figs. 3.13 and 3.14, the wall and centerline pressures are essentially constant, i.e. the pressure is essentially a function of *x*. Hence, the pressure contours are practically straight lines, parallel to the inlet and exit planes (Fig. 3.15). This is not the case for short channels, e.g. when α=1, since the contributions of the higher-order terms become more important; this effect is illustrated in Fig. 3.16.

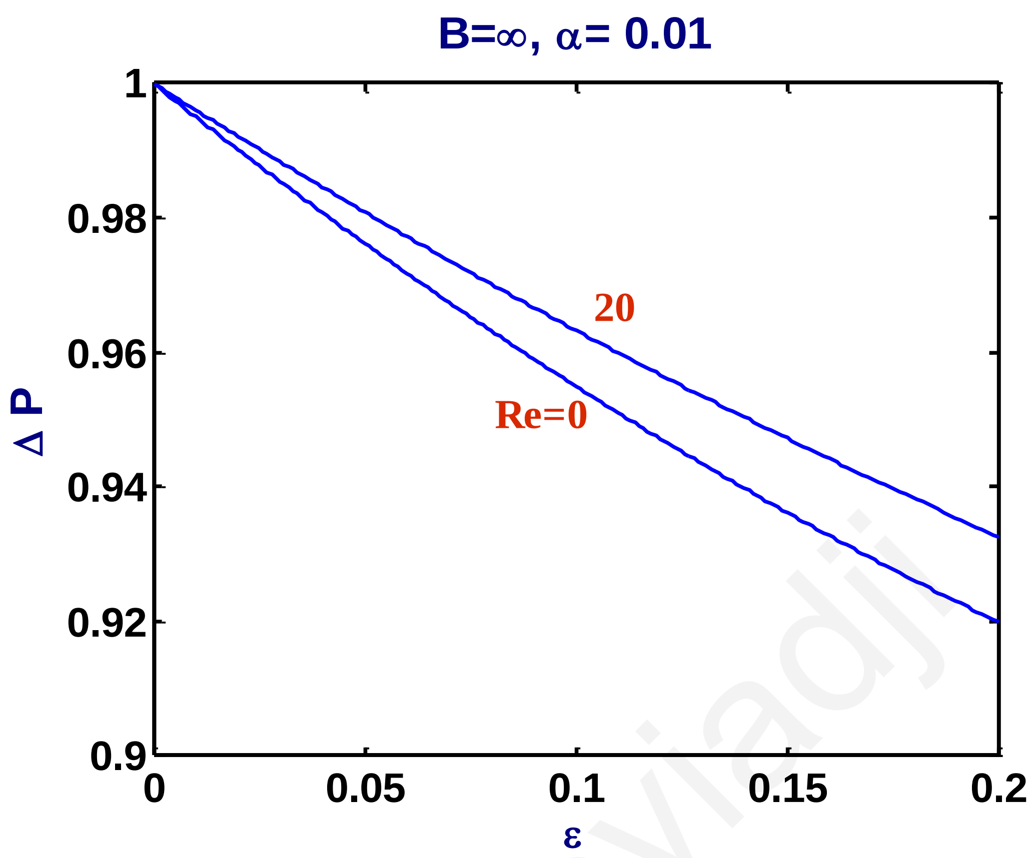


Figure 3.13: *Effect of the Reynolds number on the mean pressure drop; no slip, α=0.01*

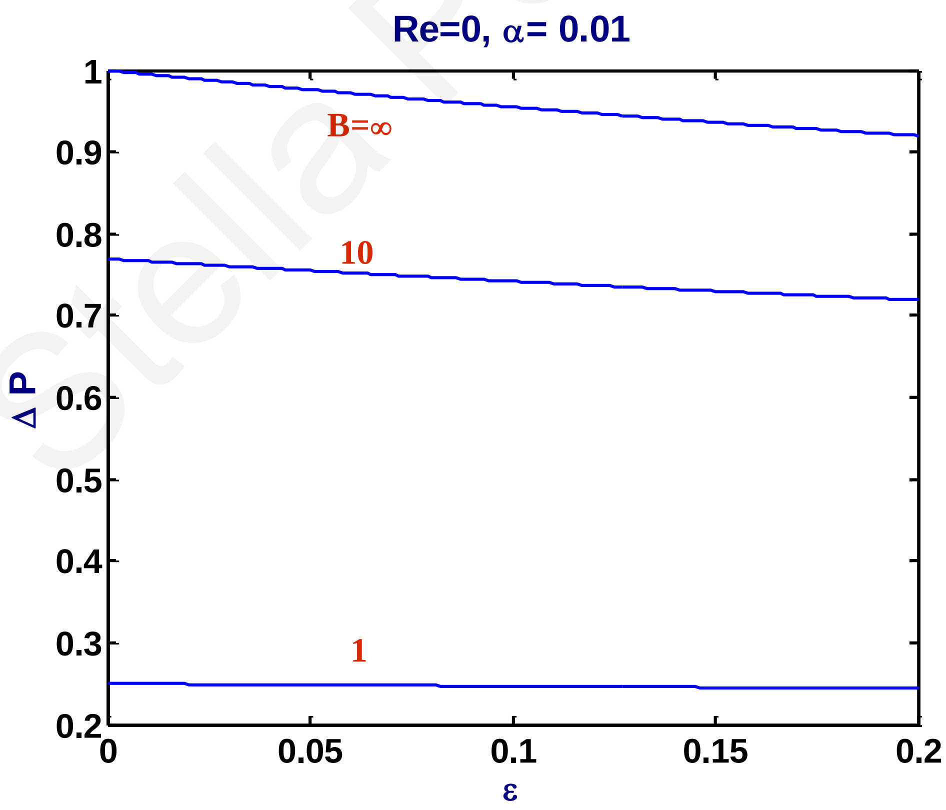


Figure 3.14: *Effect of the slip number on the mean pressure drop; Re=0, α=0.01.*

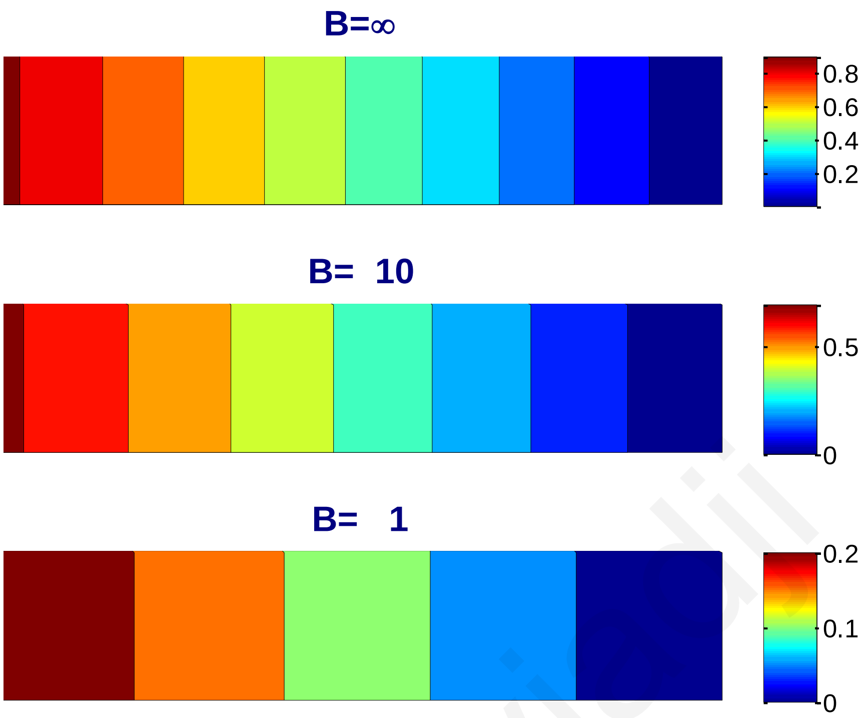


Figure 3.15: *Pressure contours for different compressibility numbers, ε=0.2 Re=0, and α=0.01 (long channel).*

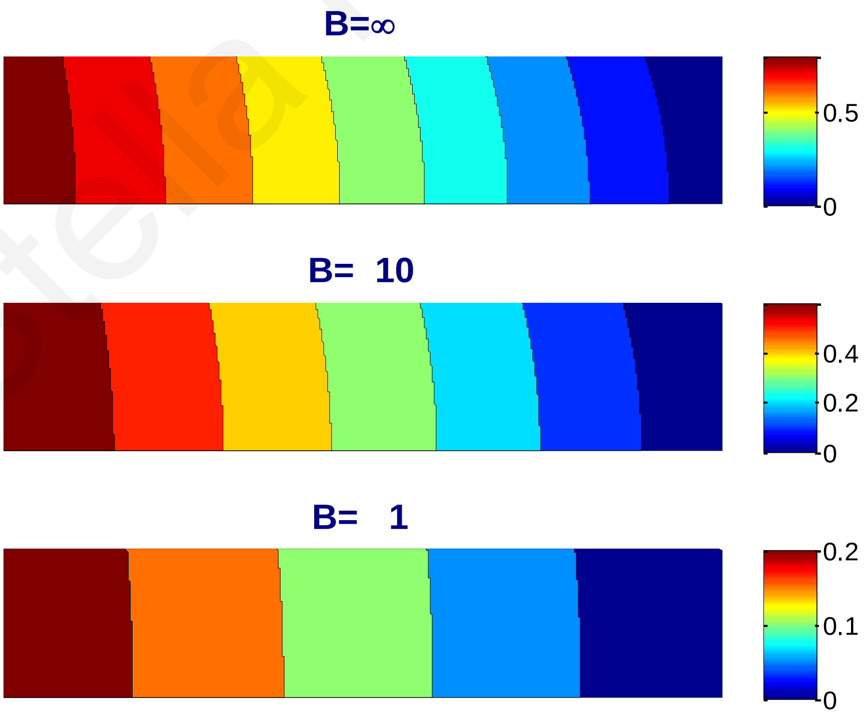


Figure 3.16: *Pressure contours for different compressibility numbers, ε=0.2, Re=0, and α=1 (short channel).*

The mean pressure drop for axisymmetric Poiseuille flow of a compressible Newtonian fluid with slip at the wall, defined by

$$\overline{\Delta p} \equiv \bar{p}(0) - \bar{p}(1) \equiv 2\int_0^1 \left[p(0,z) - p(1,z) \right] r dr \tag{3.66}$$

is:

$$\begin{aligned}\overline{\Delta p} = \frac{B^*}{8} - \left[\frac{B^{*2}}{128} - \frac{\alpha Re B^{*3}}{2048 B^2}\left(B^2+4B+8\right) \right]\varepsilon \\ + \left[\frac{B^{*3}}{1024} - \frac{\alpha Re B^{*4}}{8192 B^2}\left(B^2+4B+8\right) - \frac{\alpha^2 B^{*4}}{294912 B^2}\left(49B^2+300B+576\right) \right. \\ \left. + \frac{\alpha^2 Re^2 B^{*6}}{14155776 B^5}\left(2B^5+24B^4+171B^3+648B^2+1080B+864\right) \right]\varepsilon^2 + O\left(\varepsilon^3\right).\end{aligned} \tag{3.67}$$

The above equation generalizes the result in Venerus (2006) for the no-slip case:

$$\overline{\Delta p} = 1 - \left(\frac{1}{2} - \frac{\alpha Re}{4} \right)\varepsilon + \left(\frac{1}{2} - \frac{\alpha Re}{2} - \frac{49\alpha^2}{72} + \frac{\alpha^2 Re^2}{27} \right)\varepsilon^2 + O\left(\varepsilon^3\right). \tag{3.68}$$

We have derived a solution for equations (3.17)-(3.19) and (3.22)-(3.25) which is valid for all values of the channel aspect ratio α. It is, moreover, obvious from equation (3.19) that we recover the lubrication approximation (α^2<<1) with the transverse pressure gradient being zero when αRe<<1 if all terms of order α^2 or higher are neglected. (The aspect ratio α cannot be identically zero, since, in this limiting case, the pressure scale, i.e. the pressure required to drive the flow in a channel of infinite length with no slip at the wall, is infinite.) Therefore our solution gives the lubrication-theory solution in the presence of slip if we neglect the terms of order α^2 or higher and assume that αRe<<1. The transverse velocity component vanishes, the pressure and the density are functions of x only, and the pressure drop is given by

$$\Delta p = \frac{B^*}{3} - \frac{B^{*2}}{18}\varepsilon + \frac{B^{*3}}{54}\varepsilon^2 + O\left(\varepsilon^3\right). \tag{3.69}$$

The velocity in the flow direction is simplified to:

$$u_x = \frac{B^*}{2B}\left(B+2-By^2\right)\left[1 - \frac{B^*}{3}(1-x)\varepsilon + \frac{B^{*2}}{6}(1-x)^2\varepsilon^2 \right] + O\left(\varepsilon^3\right). \tag{3.70}$$

As already discussed, such a solution is admissible if $Q(0)$ is a decreasing function of the compressibility number ε. This condition is satisfied when

$$\varepsilon < \frac{B+3}{3B} = \frac{1}{B^*} . \tag{3.71}$$

If a more refined solution is desired, one could construct perturbation expansions using α as the perturbation parameter (for any compressible flow) or double asymptotic expansions where both ε and α are perturbation parameters.

In the case of the axisymmetric Poiseuille flow, the average Darcy friction factor, defined by

$$\overline{f} \equiv -\frac{8}{Re}\int_0^1 \frac{\partial u_z}{\partial r}(1,z)\,dz , \tag{3.72}$$

is of interest. Integrating the above equation yields

$$\begin{aligned}\frac{Re\,\overline{f}}{32} = \frac{B}{B+4}\Bigg\{1-\frac{B}{B+4}\left[\frac{1}{2}-\frac{B}{12(B+4)}\alpha Re\right]\varepsilon+\frac{B^2}{(B+4)^2}\Bigg[\frac{1}{2}-\frac{13B+12}{72(B+4)}\alpha^2 \\ -\frac{B^2+2B+4}{4B(B+4)}\alpha Re+\frac{17B^3+78B^2+360B+1440}{2160(B+4)^3}\alpha^2 Re^2\Bigg]\varepsilon^2\Bigg\}+O\left(\varepsilon^3\right).\end{aligned} \tag{3.73}$$

In the no-slip limit ($B \to \infty$), one finds that

$$\frac{Re\,\overline{f}}{32} = 1-\left(\frac{1}{2}-\frac{1}{12}\alpha Re\right)\varepsilon+\left(\frac{1}{2}-\frac{13}{72}\alpha^2-\frac{1}{4}\alpha Re+\frac{17}{2160}\alpha^2 Re^2\right)\varepsilon^2+O\left(\varepsilon^3\right). \tag{3.74}$$

Venerus (2006) compared the pressure drop and the friction factor for the no-slip case, defined respectively by Eqs. (3.66) and (3.72), and noted that the effect of inertia on pressure drop is significantly larger than on drag force. He also pointed out that the one-dimensional models for the no-slip case overpredict the friction factor by roughly 10%. Similarly to the pressure drop, the average Darcy friction factor is reduced dramatically with slip, as shown in Fig. 3.17. For a given slip number, it is essentially constant for a wide range of the parameter αRe corresponding to the weak compressibility regime and then increases rapidly. In Fig. 3.18, the average Darcy friction factor for Re=0 and 50, ε=0.2 and α=0.1 is plotted versus the slip number B. It can be seen that the friction factor is reduced with slip following a sigmoidal curve and also that the Reynolds number effect becomes weaker by slip.

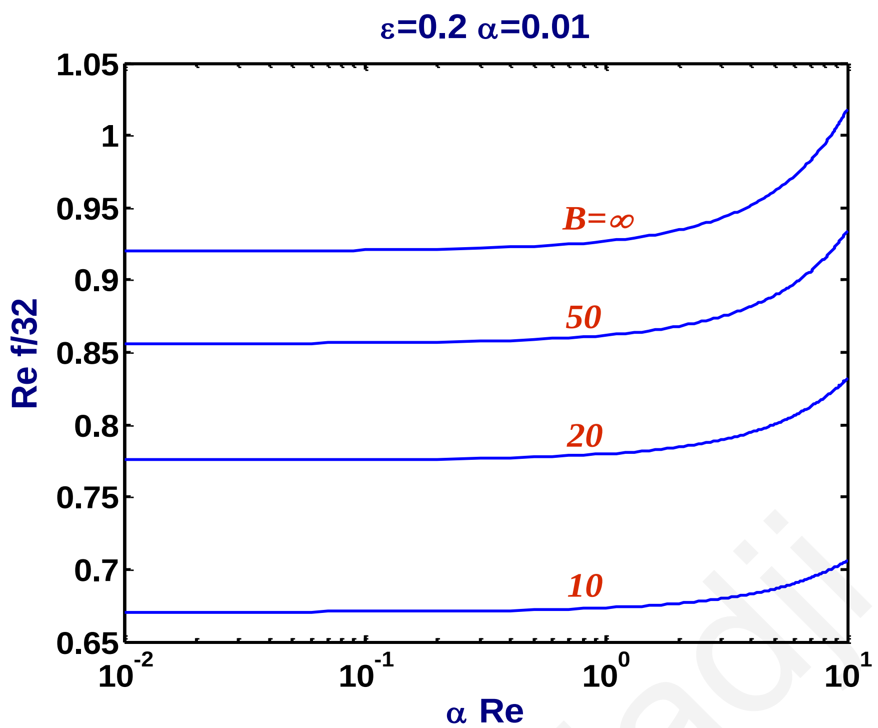


Figure 3.17: *The average Darcy friction factor for the axisymmetric Poiseuille flow versus αRe for various slip numbers; ε=0.2 and α=0.01.*

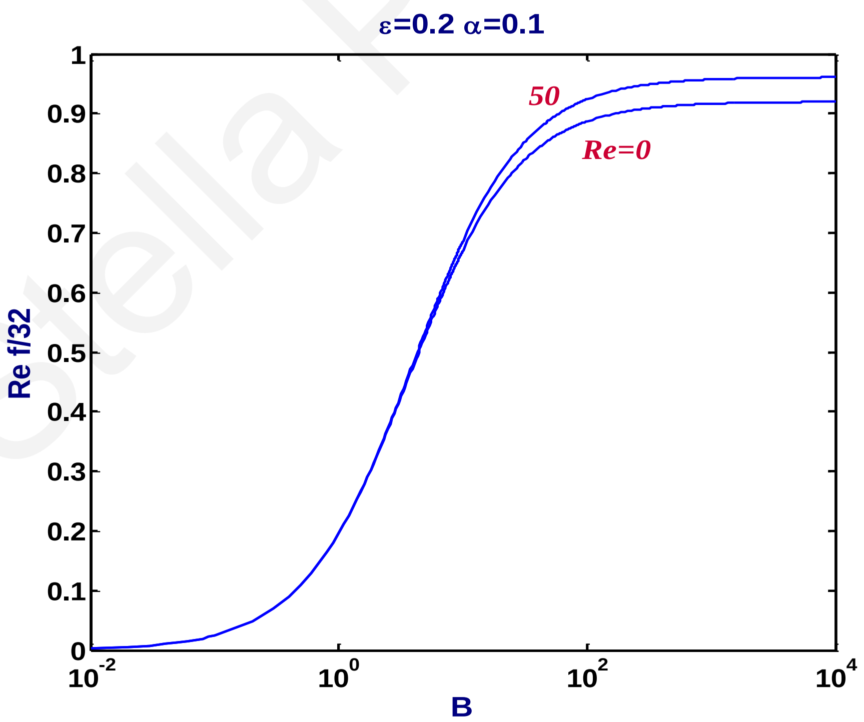


Figure 3.18: *Average Darcy friction factor for the axisymmetric Poiseuille flow versus the slip number for Re=0 and 50; ε=0.2 and α=0.1.*

3.4 Conclusions

We have derived perturbation solutions of the weakly compressible plane and axisymmetric Poiseuille flows with Navier's slip at the wall thus generalizing previous results by Taliadorou et al. (2009) and Venerus and Bugasjsky (2010). The density is assumed to be a linear function of pressure and the associated isothermal compressibility number is used as the perturbation parameter. In the proposed derivation, the primary flow variables, i.e. the two velocity components, the pressure, and the density, are perturbed. Solutions have been obtained up to second order. The corresponding expressions of the volumetric flow rate and the pressure drop are also provided and discussed. As expected, slip weakens the y-dependence of the solution. The unidirectionality assumption is valid if the Reynolds number is very small and/or slip along the wall is strong.

Chapter 4

Weakly compressible Poiseuille flows with pressure-dependent viscosity

4.1 Introduction

We consider the steady, laminar, plane, Poiseuille flow of a weakly compressible Newtonian fluid with a viscosity that is weakly dependent on the pressure, assuming that both the density and the viscosity vary linearly with pressure. A perturbation analysis is performed on all primary variables using the dimensionless isothermal compressibility and the dimensionless viscosity-pressure coefficient as the perturbation parameters. This double asymptotic expansion allows us to derive analytical perturbation solutions up to the second order. These generalise the solutions obtained in Taliadorou et al. (2009) for the constant-viscosity case and those in Chapter 2 for the incompressible case, which correspond to the two limiting cases, and allow the study of the combined effects of compressibility and the viscosity pressure-dependence.

The chapter is organized as follows: In Section 4.2 the governing equations of the plane Poiseuille flow are presented along with the appropriate boundary conditions, and then they are dedimensionalized. In Section 4.3 (subsections 4.3.1-4.3.6) the perturbation method is applied on the primary variables of the flow in terms of two perturbation parameters and approximate, analytical, perturbation solutions are obtained. In Section 4.4 the solutions are discussed in terms of the various parameters that appear in the solutions. The conclusions of the chapter are summarized in Section 4.5.

4.2 Governing equations

We consider the steady, laminar, plane Poiseuille flow of a Newtonian fluid in a slit of length *L* and width *2H* in Cartesian coordinates (x, y) as in Fig. 4.1

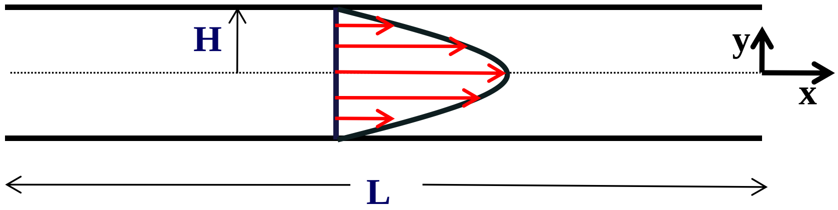


Figure 4.1: *Geometry of steady, laminar, plane Poiseuille flow.*

The fluid is assumed to be compressible with zero bulk viscosity, therefore the viscous stress tensor is given by

$$\boldsymbol{\tau} = \eta\left(2\mathbf{D} - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{u}\right), \tag{4.1}$$

where η is the pressure-dependent viscosity,

$$\eta = \eta(p), \tag{4.2}$$

I is the unit second-order tensor, and **D** is the rate of deformation tensor defined by

$$\mathbf{D} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]. \tag{4.3}$$

is the rate-of-deformation tensor and **u** is the velocity vector.

Under the further assumption of zero gravity, the momentum equation becomes

$$\rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \eta(p)\nabla^2\mathbf{u} + 2\eta'(p)\nabla p\cdot\mathbf{D} + \frac{1}{3}\eta(p)\nabla(\nabla\cdot\mathbf{u}) - \frac{2}{3}\eta'(p)\nabla p\nabla\cdot\mathbf{u}. \tag{4.4}$$

Since we assume bidirectional flow, $u_z = 0$, we consider only the *x*- and *y* momentum equations:

x-momentum

$$\rho\left(u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) + \frac{\eta}{3}\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x\partial y}\right)$$
$$+2\frac{\partial \eta}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial \eta}{\partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \frac{2}{3}\frac{\partial \eta}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) \tag{4.5}$$

y-momentum

$$\rho\left(u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\eta\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)+\frac{\eta}{3}\left(\frac{\partial^2 u_x}{\partial x\partial y}+\frac{\partial^2 u_y}{\partial y^2}\right)$$
$$+\frac{\partial\eta}{\partial x}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)+2\frac{\partial\eta}{\partial y}\frac{\partial u_y}{\partial y}-\frac{2}{3}\frac{\partial\eta}{\partial y}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\right). \tag{4.6}$$

The continuity equation for a steady compressible flow is given by

$$\nabla\cdot\left(\rho\mathbf{u}\right)=0 \tag{4.7}$$

or

$$\frac{\partial\left(\rho u_x\right)}{\partial x}+\frac{\partial\left(\rho u_y\right)}{\partial y}=0 \tag{4.8}$$

for the two dimensional flow.

As for the boundary conditions, we apply again the symmetry conditions along the symmetry plane ($y=0$) while u_x and u_y vanish along the wall (no slip and no penetration). The pressure at the upper right corner of the flow domain is taken to be equal to zero. Therefore the boundary conditions are:

$$\frac{\partial u_x}{\partial y}(x,0)=u_y(x,0)=0,\ \ x\in\left[0,L\right] \tag{4.9}$$

$$u_x(x,H)=u_y(x,H)=0,\ \ x\in\left[0,L\right] \tag{4.10}$$

$$p(L,H)=0\,. \tag{4.11}$$

We employ a linear equation of state for the density,

$$\rho=\rho_0\left[1+\kappa(p-p_0)\right], \tag{4.12}$$

where κ is the constant isothermal compressibility. The constant κ is a measure of the ability of the material to change its volume under applied pressure at constant temperature, and it is defined by

$$\kappa\equiv-\frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_{p_0,T_0}, \tag{4.13}$$

where V is the specific volume, ρ_0 and V_0 are respectively the density and the specific volume at the reference pressure, p_0, and temperature, T_0.

We also employ a linear equation for the viscosity

$$\eta = \eta_0 \left[1 + \lambda \left(p - p_0 \right) \right], \tag{4.14}$$

where η_0 is the viscosity at atmospheric pressure and λ is the viscosity-pressure coefficient (which depends on the temperature).

To nondimensionalize the governing equations and the boundary conditions, we scale *x* by the length of the channel *L*, *y* by the channel half-width *H*, the density ρ by the reference density ρ_0 and the viscosity η by the reference viscosity η_0. Furthermore, we nondimensionalize the horizontal velocity, u_x, by the mean velocity at the channel exit *U* which is defined by

$$U \equiv \frac{\dot{M}}{\rho_0 H W},$$

with $\dot{M}$ being the mass flow rate and *W* the unit length in the *z*-direction. The transverse velocity u_y is nondimensionalized by UH/L and the pressure by $3\eta_0 LU/H^2$.

The dimensionless forms of the equation of state (4.12) and of the viscosity equation (4.14) are

$$\rho = 1 + \varepsilon p \tag{4.15}$$

and

$$\eta = 1 + \delta p, \tag{4.16}$$

where

$$\varepsilon \equiv \frac{3\kappa \eta_0 LU}{H^2} \tag{4.17}$$

and

$$\delta \equiv \frac{3\lambda \eta_0 LU}{H^2} \tag{4.18}$$

are the dimensionless compressibility number and viscosity-pressure coefficient, respectively.

The dimensionless forms of the continuity and of the *x*-momentum and the *y*-momentum equations in the case of compressible Poiseuille flow under the assumptions of zero bulk velocity and zero gravity are as follows:

$$\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} = 0 \tag{4.19}$$

$$\alpha Re\rho\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=-3\frac{\partial p}{\partial x}+\eta\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y}{\partial x\partial y}\right)$$
$$+\frac{2\alpha^2}{3}\frac{\partial\eta}{\partial x}\left(2\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}\right)+\frac{\partial\eta}{\partial y}\left(\frac{\partial u_x}{\partial y}+\alpha^2\frac{\partial u_y}{\partial x}\right) \quad (4.20)$$

$$\alpha^3 Re\rho\left(u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}\right)=-3\frac{\partial p}{\partial y}+\alpha^2\eta\left(\alpha^2\frac{\partial^2 u_y}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x}{\partial x\partial y}\right)$$
$$+\alpha^2\frac{\partial\eta}{\partial x}\left(\alpha^2\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial\eta}{\partial y}\left(2\frac{\partial u_y}{\partial y}-\frac{\partial u_x}{\partial x}\right) \quad (4.21)$$

where

$$Re\equiv\frac{\rho_0 HU}{\eta_0} \quad (4.22)$$

is the Reynolds number, and

$$\alpha\equiv\frac{H}{L} \quad (4.23)$$

is the aspect ratio of the channel.

The dimensionless conditions that complete the system of the governing equations are:

$$\frac{\partial u_x}{\partial y}(x,0)=u_y(x,0)=0,\ \ x\in[0,1] \quad (4.24)$$

$$u_x(x,1)=u_y(x,1)=0,\ \ x\in[0,1] \quad (4.25)$$

$$p(1,1)=0\,. \quad (4.26)$$

Even though the density ρ and the viscosity η can be eliminated using Eqs.(4.15) and (4.16), they are kept in order to facilitate the derivation of the perturbation solution.

4.3 Perturbation solution

In Chapter 3, we had considered problems whose approximate analytical solution was presented as an asymptotic expansion in a single perturbation parameter which was the small compressibility number ε. In Chapter 2, the dimensionless viscosity-pressure coefficient δ appears in the exact analytical solution (in Chapter 2, δ is denoted by ε). In the current problem we assume that both the density and the viscosity depend weakly on the pressure, so we have two small numbers in the governing equations (4.19)-(4.21); the compressibility number ε and the dimensionless viscosity-pressure coefficient δ. Provided that both ε and δ are small, $\varepsilon \ll 1$, $\delta \ll 1$, we assume that the solution, in terms of the primary variables u_x, u_y, p, ρ and η, is represented as a double asymptotic expansion in ε and δ, as shown below:

$$\begin{aligned}
u_x &= u_x^{(00)} + \varepsilon u_x^{(10)} + \delta u_x^{(01)} + \varepsilon^2 u_x^{(20)} + \delta^2 u_x^{(02)} + \varepsilon\delta u_x^{(11)} + h.o.t. \\
u_y &= u_y^{(00)} + \varepsilon u_y^{(10)} + \delta u_y^{(01)} + \varepsilon^2 u_y^{(20)} + \delta^2 u_y^{(02)} + \varepsilon\delta u_y^{(11)} + h.o.t. \\
p &= p^{(00)} + \varepsilon p^{(10)} + \delta p^{(01)} + \varepsilon^2 p^{(20)} + \delta^2 p^{(02)} + \varepsilon\delta p^{(11)} + h.o.t. \\
\rho &= \rho^{(00)} + \varepsilon \rho^{(10)} + \delta \rho^{(01)} + \varepsilon^2 \rho^{(20)} + \delta^2 \rho^{(02)} + \varepsilon\delta \rho^{(11)} + h.o.t. \\
\eta &= \eta^{(00)} + \varepsilon \eta^{(10)} + \delta \eta^{(01)} + \varepsilon^2 \eta^{(20)} + \delta^2 \eta^{(02)} + \varepsilon\delta \eta^{(11)} + h.o.t.,
\end{aligned} \tag{4.27}$$

where *h.o.t.* stands for *higher order terms,* which in this case are terms of $O\left(\varepsilon^3, \delta^3, \varepsilon^2\delta, \varepsilon\delta^2\right)$ and higher. We substitute the expansions of Eq (4.27) into the governing equations (4.15), (4.16), (4.19)-(4.21) and into the boundary conditions (4.24)-(4.26) and collect the terms of the same order in ε and δ. Thus, we derive perturbation equations and boundary conditions for the zero-order as well as for the orders ε, δ, ε^2, $\delta^2$and $\varepsilon\delta$. The systems are solved analytically for all primary variables u_x, u_y, p, ρ and η. By retaining all orders above, we allow enough generality in our solution to be able to investigate the three possible cases: $\varepsilon \sim \delta$, $\varepsilon \ll \delta$ and $\varepsilon \gg \delta$.

The systems of orders 1, ε, δ, ε^2, δ^2 and $\varepsilon\delta$, that are formed from the above equations and boundary conditions are presented in Tables (4.1)-(4.6).

Table 4.1: *Zero-order equations and boundary conditions*

Continuity equation

$$\frac{\partial}{\partial x}\left(\rho^{(00)}u_x^{(00)}\right)+\frac{\partial}{\partial y}\left(\rho^{(00)}u_y^{(00)}\right)=0$$

x-component of the momentum equation

$$\alpha\, Re\, \rho^{(00)}\left(u_x^{(00)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(00)}}{\partial y}\right)=-3\frac{\partial p^{(00)}}{\partial x}+\eta^{(00)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(00)}}{\partial x^2}+\frac{\partial^2 u_x^{(00)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(00)}}{\partial x\partial y}\right)$$
$$+\frac{2\alpha^2}{3}\frac{\partial \eta^{(00)}}{\partial x}\left(2\frac{\partial u_x^{(00)}}{\partial x}-\frac{\partial u_y^{(00)}}{\partial y}\right)+\frac{\partial \eta^{(00)}}{\partial y}\left(\frac{\partial u_x^{(00)}}{\partial y}+\alpha^2\frac{\partial u_y^{(00)}}{\partial x}\right)$$

y-component of the momentum equation

$$a^3\, Re\, \rho^{(00)}\left(u_x^{(00)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(00)}}{\partial y}\right)=-3\frac{\partial p^{(00)}}{\partial y}+a^2\eta^{(00)}\left(\alpha^2\frac{\partial^2 u_y^{(00)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(00)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(00)}}{\partial x\partial y}\right)$$
$$+\alpha^2\frac{\partial \eta^{(00)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(00)}}{\partial x}+\frac{\partial u_x^{(00)}}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial \eta^{(00)}}{\partial y}\left(2\frac{\partial u_y^{(00)}}{\partial y}-\frac{\partial u_x^{(00)}}{\partial x}\right)$$

State and viscosity equations

$$\rho^{(00)}=1 \text{ and } \eta^{(00)}=1$$

Boundary conditions

$$\frac{\partial u_x^{(00)}}{\partial y}(x,0)=u_y^{(00)}(x,0)=0,\ \ x\in[0,1]$$

$$u_x^{(00)}(x,1)=u_y^{(00)}(x,1)=0,\ \ x\in[0,1]$$

$$p^{(00)}(1,1)=0$$

$$\int_0^1\rho^{(00)}u_x^{(00)}dy=1$$

Table 4.2: *Equations and boundary conditions of order ε*

Continuity equation

$$\frac{\partial}{\partial x}\left(\rho^{(00)}u_x^{(10)}+\rho^{(10)}u_x^{(00)}\right)+\frac{\partial}{\partial y}\left(\rho^{(00)}u_y^{(10)}+\rho^{(10)}u_y^{(00)}\right)=0$$

x-component of the momentum equation

$$\alpha\, Re\, \rho^{(00)}\left(u_x^{(0)}\frac{\partial u_x^{(10)}}{\partial x}+u_x^{(10)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(10)}}{\partial y}+u_y^{(10)}\frac{\partial u_x^{(00)}}{\partial y}\right)+\alpha\, Re\, \rho^{(10)}\left(u_x^{(00)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$=-3\frac{\partial p^{(10)}}{\partial x}+\eta^{(00)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(10)}}{\partial x^2}+\frac{\partial^2 u_x^{(10)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(10)}}{\partial x\partial y}\right)+\eta^{(10)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(00)}}{\partial x^2}+\frac{\partial^2 u_x^{(00)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(00)}}{\partial x\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial \eta^{(00)}}{\partial x}\left(2\frac{\partial u_x^{(10)}}{\partial x}-\frac{\partial u_y^{(10)}}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial \eta^{(10)}}{\partial x}\left(2\frac{\partial u_x^{(00)}}{\partial x}-\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\frac{\partial \eta^{(00)}}{\partial y}\left(\frac{\partial u_x^{(10)}}{\partial y}+\alpha^2\frac{\partial u_y^{(10)}}{\partial x}\right)+\frac{\partial \eta^{(10)}}{\partial y}\left(\frac{\partial u_x^{(00)}}{\partial y}+\alpha^2\frac{\partial u_y^{(00)}}{\partial x}\right)$$

y-component of the momentum equation

$$\alpha^3\, Re\, \rho^{(00)}\left(u_x^{(00)}\frac{\partial u_y^{(10)}}{\partial x}+u_x^{(10)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(10)}}{\partial y}+u_y^{(10)}\frac{\partial u_y^{(00)}}{\partial y}\right)+\alpha^3\, Re\, \rho^{(10)}\left(u_x^{(00)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$=-3\frac{\partial p^{(10)}}{\partial y}+a^2\eta^{(00)}\left(\alpha^2\frac{\partial^2 u_y^{(10)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(10)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(10)}}{\partial x\partial y}\right)+a^2\eta^{(10)}\left(\alpha^2\frac{\partial^2 u_y^{(00)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(00)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(00)}}{\partial x\partial y}\right)$$

$$+\alpha^2\frac{\partial \eta^{(00)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(10)}}{\partial x}+\frac{\partial u_x^{(10)}}{\partial y}\right)+\alpha^2\frac{\partial \eta^{(10)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(00)}}{\partial x}+\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial \eta^{(00)}}{\partial y}\left(2\frac{\partial u_y^{(10)}}{\partial y}-\frac{\partial u_x^{(10)}}{\partial x}\right)+\frac{2\alpha^2}{3}\frac{\partial \eta^{(10)}}{\partial y}\left(2\frac{\partial u_y^{(00)}}{\partial y}-\frac{\partial u_x^{(00)}}{\partial x}\right)$$

State and viscosity equations

$$\rho^{(10)}=p^{(01)}\ \text{ and }\ \eta^{(10)}=0$$

Boundary conditions

$$\frac{\partial u_x^{(10)}}{\partial y}(x,0)=u_y^{(10)}(x,0)=0,\ \ x\in[0,1]$$

$$u_x^{(10)}(x,1)=u_y^{(10)}(x,1)=0,\ \ x\in[0,1]$$

$$p^{(10)}(1,1)=0$$

$$\int_0^1\left(\rho^{(00)}u_x^{(10)}+\rho^{(10)}u_x^{(00)}\right)dy=0$$

Table 4.3: *Equations and boundary conditions of order δ*

Continuity equation

$$\frac{\partial}{\partial x}\left(\rho^{(00)}u_x^{(01)}+\rho^{(01)}u_x^{(00)}\right)+\frac{\partial}{\partial y}\left(\rho^{(00)}u_y^{(01)}+\rho^{(01)}u_y^{(00)}\right)=0$$

x-component of the momentum equation

$$\alpha\,Re\,\rho^{(00)}\left(u_x^{(00)}\frac{\partial u_x^{(01)}}{\partial x}+u_x^{(01)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(01)}}{\partial y}+u_y^{(01)}\frac{\partial u_x^{(00)}}{\partial y}\right)+\alpha\,Re\,\rho^{(01)}\left(u_x^{(00)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$=-3\frac{\partial p^{(01)}}{\partial x}+\eta^{(00)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(01)}}{\partial x^2}+\frac{\partial^2 u_x^{(01)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(01)}}{\partial x\partial y}\right)+\eta^{(01)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(00)}}{\partial x^2}+\frac{\partial^2 u_x^{(00)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(00)}}{\partial x\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial\eta^{(00)}}{\partial x}\left(2\frac{\partial u_x^{(01)}}{\partial x}-\frac{\partial u_y^{(01)}}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial\eta^{(01)}}{\partial x}\left(2\frac{\partial u_x^{(00)}}{\partial x}-\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\frac{\partial\eta^{(00)}}{\partial y}\left(\frac{\partial u_x^{(01)}}{\partial y}+\alpha^2\frac{\partial u_y^{(01)}}{\partial x}\right)+\frac{\partial\eta^{(01)}}{\partial y}\left(\frac{\partial u_x^{(00)}}{\partial y}+\alpha^2\frac{\partial u_y^{(00)}}{\partial x}\right)$$

y-component of the momentum equation

$$\alpha^3\,Re\,\rho^{(00)}\left(u_x^{(00)}\frac{\partial u_y^{(01)}}{\partial x}+u_x^{(01)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(01)}}{\partial y}+u_y^{(01)}\frac{\partial u_y^{(00)}}{\partial y}\right)+\alpha^3\,Re\,\rho^{(01)}\left(u_x^{(00)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$=-3\frac{\partial p^{(01)}}{\partial y}+a^2\eta^{(00)}\left(\alpha^2\frac{\partial^2 u_y^{(01)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(01)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(01)}}{\partial x\partial y}\right)+a^2\eta^{(01)}\left(\alpha^2\frac{\partial^2 u_y^{(00)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(00)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(00)}}{\partial x\partial y}\right)$$

$$+\alpha^2\frac{\partial\eta^{(00)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(01)}}{\partial x}+\frac{\partial u_x^{(01)}}{\partial y}\right)+\alpha^2\frac{\partial\eta^{(01)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(00)}}{\partial x}+\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial\eta^{(00)}}{\partial y}\left(2\frac{\partial u_y^{(01)}}{\partial y}-\frac{\partial u_x^{(01)}}{\partial x}\right)+\frac{2\alpha^2}{3}\frac{\partial\eta^{(01)}}{\partial y}\left(2\frac{\partial u_y^{(00)}}{\partial y}-\frac{\partial u_x^{(00)}}{\partial x}\right)$$

State and viscosity equations

$$\rho^{(01)}=0 \text{ and } \eta^{(01)}=p^{(10)}$$

Boundary conditions

$$\frac{\partial u_x^{(01)}}{\partial y}(x,0)=u_y^{(01)}(x,0)=0,\ \ x\in[0,1]$$

$$u_x^{(01)}(x,1)=u_y^{(01)}(x,1)=0,\ \ x\in[0,1]$$

$$p^{(01)}(1,1)=0$$

$$\int_0^1\left(\rho^{(00)}u_x^{(01)}+\rho^{(01)}u_x^{(00)}\right)dy=0$$

Table 4.4: *Equations and boundary conditions of order ε^2*

Continuity equation

$$\frac{\partial}{\partial x}\left(\rho^{(00)}u_x^{(20)}+\rho^{(10)}u_x^{(10)}+\rho^{(10)}u_x^{(00)}\right)+\frac{\partial}{\partial y}\left(\rho^{(00)}u_y^{(20)}+\rho^{(10)}u_y^{(10)}+\rho^{(20)}u_y^{(00)}\right)=0$$

x-component of the momentum equation

$$\alpha\,Re\,\rho^{(00)}\left(u_x^{(00)}\frac{\partial u_x^{(20)}}{\partial x}+u_x^{(10)}\frac{\partial u_x^{(10)}}{\partial x}+u_x^{(20)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(20)}}{\partial y}+u_y^{(10)}\frac{\partial u_x^{(10)}}{\partial y}+u_y^{(20)}\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$+\alpha\,Re\,\rho^{(10)}\left(u_x^{(00)}\frac{\partial u_x^{(10)}}{\partial x}+u_x^{(10)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(10)}}{\partial y}+u_y^{(10)}\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$+\alpha\,Re\,\rho^{(20)}\left(u_x^{(00)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(00)}}{\partial y}\right)=-3\frac{\partial p^{(20)}}{\partial x}+\eta^{(00)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(20)}}{\partial x^2}+\frac{\partial^2 u_x^{(20)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(20)}}{\partial x\partial y}\right)$$

$$+\eta^{(10)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(10)}}{\partial x^2}+\frac{\partial^2 u_x^{(10)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(10)}}{\partial x\partial y}\right)+\eta^{(20)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(00)}}{\partial x^2}+\frac{\partial^2 u_x^{(00)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(00)}}{\partial x\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial \eta^{(00)}}{\partial x}\left(2\frac{\partial u_x^{(20)}}{\partial x}-\frac{\partial u_y^{(20)}}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial \eta^{(10)}}{\partial x}\left(2\frac{\partial u_x^{(10)}}{\partial x}-\frac{\partial u_y^{(10)}}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial \eta^{(20)}}{\partial x}\left(2\frac{\partial u_x^{(00)}}{\partial x}-\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\frac{\partial \eta^{(00)}}{\partial y}\left(\frac{\partial u_x^{(20)}}{\partial y}+\alpha^2\frac{\partial u_y^{(20)}}{\partial x}\right)+\frac{\partial \eta^{(10)}}{\partial y}\left(\frac{\partial u_x^{(10)}}{\partial y}+\alpha^2\frac{\partial u_y^{(10)}}{\partial x}\right)+\frac{\partial \eta^{(20)}}{\partial y}\left(\frac{\partial u_x^{(00)}}{\partial y}+\alpha^2\frac{\partial u_y^{(00)}}{\partial x}\right)$$

y-component of the momentum equation

$$\alpha^3\,Re\,\rho^{(00)}\left(u_x^{(00)}\frac{\partial u_y^{(20)}}{\partial x}+u_x^{(10)}\frac{\partial u_y^{(10)}}{\partial x}+u_x^{(20)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(20)}}{\partial y}+u_y^{(10)}\frac{\partial u_y^{(10)}}{\partial y}+u_y^{(20)}\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\alpha^3\,Re\,\rho^{(10)}\left(u_x^{(00)}\frac{\partial u_y^{(10)}}{\partial x}+u_x^{(10)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(10)}}{\partial y}+u_y^{(10)}\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\alpha^3\,Re\,\rho^{(20)}\left(u_x^{(00)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(00)}}{\partial y}\right)=-3\frac{\partial p^{(20)}}{\partial y}$$

$$+a^2\eta^{(00)}\left(\alpha^2\frac{\partial^2 u_y^{(20)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(20)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(20)}}{\partial x\partial y}\right)+a^2\eta^{(10)}\left(\alpha^2\frac{\partial^2 u_y^{(10)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(10)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(10)}}{\partial x\partial y}\right)$$

Table 4.4: *(continued)*

$$+a^2\eta^{(20)}\left(\alpha^2\frac{\partial^2 u_y^{(00)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(00)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(00)}}{\partial x\partial y}\right)+\alpha^2\frac{\partial\eta^{(00)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(20)}}{\partial x}+\frac{\partial u_x^{(20)}}{\partial y}\right)$$

$$+\alpha^2\frac{\partial\eta^{(10)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(10)}}{\partial x}+\frac{\partial u_x^{(10)}}{\partial y}\right)+\alpha^2\frac{\partial\eta^{(20)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(00)}}{\partial x}+\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial\eta^{(00)}}{\partial y}\left(2\frac{\partial u_y^{(20)}}{\partial y}-\frac{\partial u_x^{(20)}}{\partial x}\right)+\frac{2\alpha^2}{3}\frac{\partial\eta^{(10)}}{\partial y}\left(2\frac{\partial u_y^{(10)}}{\partial y}-\frac{\partial u_x^{(10)}}{\partial x}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial\eta^{(20)}}{\partial y}\left(2\frac{\partial u_y^{(00)}}{\partial y}-\frac{\partial u_x^{(00)}}{\partial x}\right)$$

State and viscosity equations

$$\rho^{(20)}=p^{(10)}\ \text{ and }\ \eta^{(20)}=0$$

Boundary conditions

$$\frac{\partial u_x^{(20)}}{\partial y}(x,0)=u_y^{(20)}(x,0)=0,\ \ x\in[0,1]$$

$$u_x^{(20)}(x,1)=u_y^{(20)}(x,1)=0,\ \ x\in[0,1]$$

$$p^{(20)}(1,1)=0$$

$$\int_0^1\left(\rho^{(00)}u_x^{(20)}+\rho^{(10)}u_x^{(10)}+\rho^{(20)}u_x^{(00)}\right)dy=0$$

Table 4.5: *Equations and boundary conditions of order δ^2*

Continuity equation

$$\frac{\partial}{\partial x}\left(\rho^{(00)}u_x^{(02)}+\rho^{(01)}u_x^{(01)}+\rho^{(02)}u_x^{(00)}\right)+\frac{\partial}{\partial y}\left(\rho^{(00)}u_y^{(02)}+\rho^{(01)}u_y^{(01)}+\rho^{(02)}u_y^{(00)}\right)=0$$

x-component of the momentum equation

$$\alpha\,Re\,\rho^{(00)}\left(u_x^{(00)}\frac{\partial u_x^{(02)}}{\partial x}+u_x^{(01)}\frac{\partial u_x^{(01)}}{\partial x}+u_x^{(02)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(02)}}{\partial y}+u_y^{(01)}\frac{\partial u_x^{(01)}}{\partial y}+u_y^{(02)}\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$+\alpha\,Re\,\rho^{(01)}\left(u_x^{(00)}\frac{\partial u_x^{(01)}}{\partial x}+u_x^{(01)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(01)}}{\partial y}+u_y^{(01)}\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$+\alpha\,Re\,\rho^{(02)}\left(u_x^{(00)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(00)}}{\partial y}\right)=-3\frac{\partial p^{(02)}}{\partial x}$$

$$+\eta^{(00)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(02)}}{\partial x^2}+\frac{\partial^2 u_x^{(02)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(02)}}{\partial x\partial y}\right)+\eta^{(01)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(01)}}{\partial x^2}+\frac{\partial^2 u_x^{(01)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(01)}}{\partial x\partial y}\right)$$

$$+\eta^{(02)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(00)}}{\partial x^2}+\frac{\partial^2 u_x^{(00)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(00)}}{\partial x\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial\eta^{(00)}}{\partial x}\left(2\frac{\partial u_x^{(02)}}{\partial x}-\frac{\partial u_y^{(02)}}{\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial\eta^{(01)}}{\partial x}\left(2\frac{\partial u_x^{(01)}}{\partial x}-\frac{\partial u_y^{(01)}}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial\eta^{(02)}}{\partial x}\left(2\frac{\partial u_x^{(00)}}{\partial x}-\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\frac{\partial\eta^{(00)}}{\partial y}\left(\frac{\partial u_x^{(02)}}{\partial y}+\alpha^2\frac{\partial u_y^{(02)}}{\partial x}\right)+\frac{\partial\eta^{(01)}}{\partial y}\left(\frac{\partial u_x^{(01)}}{\partial y}+\alpha^2\frac{\partial u_y^{(01)}}{\partial x}\right)+\frac{\partial\eta^{(02)}}{\partial y}\left(\frac{\partial u_x^{(00)}}{\partial y}+\alpha^2\frac{\partial u_y^{(00)}}{\partial x}\right)$$

y-component of the momentum equation

$$\alpha^3\,Re\,\rho^{(00)}\left(u_x^{(00)}\frac{\partial u_y^{(02)}}{\partial x}+u_x^{(01)}\frac{\partial u_y^{(01)}}{\partial x}+u_x^{(02)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(02)}}{\partial y}+u_y^{(01)}\frac{\partial u_y^{(01)}}{\partial y}+u_y^{(02)}\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\alpha^3\,Re\,\rho^{(01)}\left(u_x^{(00)}\frac{\partial u_y^{(01)}}{\partial x}+u_x^{(01)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(01)}}{\partial y}+u_y^{(01)}\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\alpha^3\,Re\,\rho^{(02)}\left(u_x^{(00)}\frac{\partial u_y^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_y^{(00)}}{\partial y}\right)=-3\frac{\partial p^{(02)}}{\partial y}$$

$$+a^2\eta^{(00)}\left(\alpha^2\frac{\partial^2 u_y^{(02)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(02)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(02)}}{\partial x\partial y}\right)$$

Table 4.5: *(continued)*

$$+a^2\eta^{(01)}\left(\alpha^2\frac{\partial^2 u_y^{(01)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(01)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(01)}}{\partial x\partial y}\right)$$

$$+a^2\eta^{(02)}\left(\alpha^2\frac{\partial^2 u_y^{(00)}}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y^{(00)}}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x^{(00)}}{\partial x\partial y}\right)$$

$$+\alpha^2\frac{\partial\eta^{(00)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(02)}}{\partial x}+\frac{\partial u_x^{(02)}}{\partial y}\right)+\alpha^2\frac{\partial\eta^{(01)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(01)}}{\partial x}+\frac{\partial u_x^{(01)}}{\partial y}\right)$$

$$+\alpha^2\frac{\partial\eta^{(02)}}{\partial x}\left(\alpha^2\frac{\partial u_y^{(00)}}{\partial x}+\frac{\partial u_x^{(00)}}{\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial\eta^{(00)}}{\partial y}\left(2\frac{\partial u_y^{(02)}}{\partial y}-\frac{\partial u_x^{(02)}}{\partial x}\right)+\frac{2\alpha^2}{3}\frac{\partial\eta^{(01)}}{\partial y}\left(2\frac{\partial u_y^{(01)}}{\partial y}-\frac{\partial u_x^{(01)}}{\partial x}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial\eta^{(02)}}{\partial y}\left(2\frac{\partial u_y^{(00)}}{\partial y}-\frac{\partial u_x^{(00)}}{\partial x}\right)$$

State and viscosity equations

$$\rho^{(02)}=0 \text{ and } \eta^{(02)}=p^{(01)}$$

Boundary conditions

$$\frac{\partial u_x^{(02)}}{\partial y}(x,0)=u_y^{(02)}(x,0)=0,\ \ x\in[0,1]$$

$$u_x^{(02)}(x,1)=u_y^{(02)}(x,1)=0,\ \ x\in[0,1]$$

$$p^{(02)}(1,1)=0$$

$$\int_0^1\left(\rho^{(00)}u_x^{(02)}+\rho^{(01)}u_x^{(01)}+\rho^{(02)}u_x^{(00)}\right)dy=0$$

Table 4.6: *Equations and boundary conditions of order εδ*

Continuity equation

$$\frac{\partial}{\partial x}\left(\rho^{(00)}u_x^{(11)}+\rho^{(10)}u_x^{(01)}+\rho^{(01)}u_x^{(10)}+\rho^{(11)}u_x^{(00)}\right)+\frac{\partial}{\partial y}\left(\rho^{(00)}u_y^{(11)}+\rho^{(10)}u_y^{(01)}+\rho^{(01)}u_y^{(10)}+\rho^{(11)}u_y^{(00)}\right)=0$$

x-component of the momentum equation

$$\alpha\, Re\, \rho^{(00)}\left(u_x^{(00)}\frac{\partial u_x^{(11)}}{\partial x}+u_x^{(10)}\frac{\partial u_x^{(01)}}{\partial x}+u_x^{(01)}\frac{\partial u_x^{(10)}}{\partial x}+u_x^{(11)}\frac{\partial u_x^{(00)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(11)}}{\partial y}\right.$$

$$\left.+u_y^{(10)}\frac{\partial u_x^{(01)}}{\partial y}+u_y^{(01)}\frac{\partial u_x^{(10)}}{\partial y}+u_y^{(11)}\frac{\partial u_x^{(00)}}{\partial y}\right)+\alpha\, Re\, \rho^{(10)}\left(u_x^{(00)}\frac{\partial u_x^{(01)}}{\partial x}+u_y^{(00)}\frac{\partial u_x^{(01)}}{\partial y}\right)$$

$$=-3\frac{\partial p^{(11)}}{\partial x}+\eta^{(00)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(11)}}{\partial x^2}+\frac{\partial^2 u_x^{(11)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(11)}}{\partial x\partial y}\right)$$

$$+\eta^{(10)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(01)}}{\partial x^2}+\frac{\partial^2 u_x^{(01)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(01)}}{\partial x\partial y}\right)$$

$$+\eta^{(01)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(10)}}{\partial x^2}+\frac{\partial^2 u_x^{(10)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(10)}}{\partial x\partial y}\right)$$

$$+\eta^{(11)}\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x^{(00)}}{\partial x^2}+\frac{\partial^2 u_x^{(00)}}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y^{(00)}}{\partial x\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial \eta^{(00)}}{\partial x}\left(2\frac{\partial u_x^{(11)}}{\partial x}-\frac{\partial u_y^{(11)}}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial \eta^{(10)}}{\partial x}\left(2\frac{\partial u_x^{(01)}}{\partial x}-\frac{\partial u_y^{(01)}}{\partial y}\right)$$

$$+\frac{2\alpha^2}{3}\frac{\partial \eta^{(01)}}{\partial x}\left(2\frac{\partial u_x^{(10)}}{\partial x}-\frac{\partial u_y^{(10)}}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial \eta^{(11)}}{\partial x}\left(2\frac{\partial u_x^{(00)}}{\partial x}-\frac{\partial u_y^{(00)}}{\partial y}\right)$$

$$+\frac{\partial \eta^{(00)}}{\partial y}\left(\frac{\partial u_x^{(11)}}{\partial y}+\alpha^2\frac{\partial u_y^{(11)}}{\partial x}\right)+\frac{\partial \eta^{(10)}}{\partial y}\left(\frac{\partial u_x^{(01)}}{\partial y}+\alpha^2\frac{\partial u_y^{(01)}}{\partial x}\right)$$

$$+\frac{\partial \eta^{(01)}}{\partial y}\left(\frac{\partial u_x^{(10)}}{\partial y}+\alpha^2\frac{\partial u_y^{(10)}}{\partial x}\right)+\frac{\partial \eta^{(11)}}{\partial y}\left(\frac{\partial u_x^{(00)}}{\partial y}+\alpha^2\frac{\partial u_y^{(00)}}{\partial x}\right)$$

Table 4.6: *(continued)*

y-component of the momentum equation

$$
\begin{aligned}
&\alpha^{3}\, Re\, \rho^{(00)} \left(u_x^{(00)} \frac{\partial u_y^{(11)}}{\partial x} + u_x^{(10)} \frac{\partial u_y^{(01)}}{\partial x} + u_x^{(01)} \frac{\partial u_y^{(10)}}{\partial x} + u_x^{(11)} \frac{\partial u_y^{(00)}}{\partial x} \right. \\
&\left. + u_y^{(00)} \frac{\partial u_y^{(11)}}{\partial y} + u_y^{(10)} \frac{\partial u_y^{(01)}}{\partial y} + u_y^{(01)} \frac{\partial u_y^{(10)}}{\partial y} + u_y^{(11)} \frac{\partial u_y^{(00)}}{\partial y} \right) \\
&+\alpha^{3}\, Re\, \rho^{(10)} \left(u_x^{(00)} \frac{\partial u_y^{(01)}}{\partial x} + u_x^{(01)} \frac{\partial u_y^{(00)}}{\partial x} + u_y^{(00)} \frac{\partial u_y^{(01)}}{\partial y} + u_y^{(01)} \frac{\partial u_y^{(00)}}{\partial y} \right) \\
&+\alpha^{3}\, Re\, \rho^{(01)} \left(u_x^{(00)} \frac{\partial u_y^{(10)}}{\partial x} + u_x^{(01)} \frac{\partial u_y^{(00)}}{\partial x} + u_y^{(00)} \frac{\partial u_y^{(10)}}{\partial y} + u_y^{(10)} \frac{\partial u_y^{(00)}}{\partial y} \right) \\
&+\alpha^{3}\, Re\, \rho^{(11)} \left(u_x^{(00)} \frac{\partial u_y^{(00)}}{\partial x} + u_y^{(00)} \frac{\partial u_y^{(00)}}{\partial y} \right) = -3 \frac{\partial p^{(11)}}{\partial y} \\
&+ a^{2}\eta^{(00)} \left(\alpha^{2} \frac{\partial^{2} u_y^{(11)}}{\partial x^{2}} + \frac{4}{3} \frac{\partial^{2} u_y^{(11)}}{\partial y^{2}} + \frac{1}{3} \frac{\partial^{2} u_x^{(11)}}{\partial x \partial y} \right) + a^{2}\eta^{(10)} \left(\alpha^{2} \frac{\partial^{2} u_y^{(01)}}{\partial x^{2}} + \frac{4}{3} \frac{\partial^{2} u_y^{(01)}}{\partial y^{2}} + \frac{1}{3} \frac{\partial^{2} u_x^{(01)}}{\partial x \partial y} \right) \\
&+ a^{2}\eta^{(01)} \left(\alpha^{2} \frac{\partial^{2} u_y^{(10)}}{\partial x^{2}} + \frac{4}{3} \frac{\partial^{2} u_y^{(10)}}{\partial y^{2}} + \frac{1}{3} \frac{\partial^{2} u_x^{(10)}}{\partial x \partial y} \right) + a^{2}\eta^{(11)} \left(\alpha^{2} \frac{\partial^{2} u_y^{(00)}}{\partial x^{2}} + \frac{4}{3} \frac{\partial^{2} u_y^{(00)}}{\partial y^{2}} + \frac{1}{3} \frac{\partial^{2} u_x^{(00)}}{\partial x \partial y} \right) \\
&+\alpha^{2} \frac{\partial \eta^{(00)}}{\partial x} \left(\alpha^{2} \frac{\partial u_y^{(11)}}{\partial x} + \frac{\partial u_x^{(11)}}{\partial y} \right) + \alpha^{2} \frac{\partial \eta^{(10)}}{\partial x} \left(\alpha^{2} \frac{\partial u_y^{(01)}}{\partial x} + \frac{\partial u_x^{(01)}}{\partial y} \right) \\
&+\alpha^{2} \frac{\partial \eta^{(01)}}{\partial x} \left(\alpha^{2} \frac{\partial u_y^{(10)}}{\partial x} + \frac{\partial u_x^{(10)}}{\partial y} \right) + \alpha^{2} \frac{\partial \eta^{(11)}}{\partial x} \left(\alpha^{2} \frac{\partial u_y^{(00)}}{\partial x} + \frac{\partial u_x^{(00)}}{\partial y} \right) \\
&+ \frac{2\alpha^{2}}{3} \frac{\partial \eta^{(00)}}{\partial y} \left(2 \frac{\partial u_y^{(11)}}{\partial y} - \frac{\partial u_x^{(11)}}{\partial x} \right) + \frac{2\alpha^{2}}{3} \frac{\partial \eta^{(10)}}{\partial y} \left(2 \frac{\partial u_y^{(01)}}{\partial y} - \frac{\partial u_x^{(01)}}{\partial x} \right) \\
&+ \frac{2\alpha^{2}}{3} \frac{\partial \eta^{(01)}}{\partial y} \left(2 \frac{\partial u_y^{(10)}}{\partial y} - \frac{\partial u_x^{(10)}}{\partial x} \right) + \frac{2\alpha^{2}}{3} \frac{\partial \eta^{(11)}}{\partial y} \left(2 \frac{\partial u_y^{(00)}}{\partial y} - \frac{\partial u_x^{(00)}}{\partial x} \right)
\end{aligned}
$$

State and viscosity equations

$$\rho^{(11)} = p^{(01)} \text{ and } \eta^{(11)} = p^{(10)}$$

Table 4.6: *(continued)*

Boundary conditions
$$\frac{\partial u_x^{(11)}}{\partial y}(x,0)=u_y^{(11)}(x,0)=0,\ \ x\in[0,1]$$
$$u_x^{(11)}(x,1)=u_y^{(11)}(x,1)=0,\ \ x\in[0,1]$$
$$p^{(11)}(1,1)=0$$
$$\int_0^1\left(\rho^{(00)}u_x^{(11)}+\rho^{(10)}u_x^{(01)}+\rho^{(01)}u_x^{(10)}+\rho^{(11)}u_x^{(00)}\right)dy=0$$

In the following subsections, we outline the methodology that was followed to obtain analytical solutions for the zero-order and the orders ε, δ, ε^2, δ^2 and $\varepsilon\delta$.

4.3.1 Zero-order solution

Assuming that $u_y^{(00)}$ is equal to zero and using $\rho^{(00)}=1$ and $\eta^{(00)}=1$ we have from the continuity equation that

$$\frac{\partial u_x^{(00)}}{\partial x}=0\ \Rightarrow\ u_x^{(00)}=u_x^{(00)}(y). \tag{4.28}$$

From the y-momentum equation we get

$$\frac{\partial p^{(00)}}{\partial y}=0\ \Rightarrow\ p^{(00)}=p^{(00)}(x). \tag{4.29}$$

The x-momentum equation gives us the differential equations

$$3\frac{\partial p^{(00)}}{\partial x}=\frac{\partial^2 u_x^{(00)}}{\partial y^2}=A, \tag{4.30}$$

where A is an unknown constant. Solving the two differential equations in (4.30) we find that

$$u_x^{(00)}=\frac{A}{2}y^2+c_1y+c_2\ \text{ and }\ p^{(00)}=\frac{A}{3}x+c_3, \tag{4.31}$$

where c_1, c_2, and c_3 are unknown constants. Applying the boundary conditions

$$\frac{\partial u_x^{(00)}}{\partial y}(x,0)=0,\ u_x^{(00)}(x,1)=0 \text{ and } p^{(00)}(1,1)=0$$

we calculate that

$$A=-3,\ c_1=0,\ c_2=\frac{3}{2},\ c_3=1.$$

Therefore, the zero-order solution is:

$$\begin{aligned} u_x^{(00)} &= \frac{3}{2}\left(1-y^2\right) \\ u_y^{(00)} &= 0 \\ p^{(00)} &= 1-x \\ \rho^{(00)} &= 1 \\ \eta^{(00)} &= 1 \end{aligned} \qquad (4.32)$$

As expected, the solution at zero order is that of a steady, laminar incompressible plane, flow with constant viscosity.

4.3.2 Solution of order ε

The equation of state $\rho^{(10)}=p^{(01)}$ and the viscosity equation $\eta^{(10)}=0$ lead to

$$\rho^{(10)}=p^{(00)}=1-x \text{ and } \eta^{(10)}=0. \qquad (4.33)$$

Assuming that $u_y^{(10)}=0$, the continuity equation becomes

$$u_x^{(00)}\frac{\partial \rho^{(10)}}{\partial x}+\frac{\partial u_x^{(10)}}{\partial x}=0. \qquad (4.34)$$

By integrating Eq. (4.34) with respect to *x* we get

$$u_x^{(10)}=-\frac{3}{2}\left(1-y^2\right)(1-x)+F(y), \qquad (4.35)$$

where $F(y)$ is an unknown function to be determined.

The *y*-momentum equation is simplified to

$$-3\frac{\partial p^{(10)}}{\partial y}+\frac{a^2}{3}\frac{\partial^2 u_x^{(10)}}{\partial x \partial y}=0.$$

Integrating the above with respect to *y* we get

$$p^{(10)} = \frac{a^2}{6}\left(1-y^2\right)+G(x), \tag{4.36}$$

where $G(x)$ is an unknown function to be determined.

The *x*-momentum equation becomes

$$\alpha\, Re\, u_x^{(00)} \frac{\partial u_x^{(10)}}{\partial x} = -3\frac{\partial p^{(10)}}{\partial x} + \frac{\partial^2 u_x^{(10)}}{\partial y^2}, \tag{4.37}$$

which if we substitute $u_x^{(00)}$ from Eq. (4.32) and use Eqs. (4.35) and (4.36) leads to

$$\frac{9\alpha\, Re}{4}\left(1-y^2\right)^2 - F''(y) = -3G'(x)+3(1-x) = A, \tag{4.38}$$

where *A* is an unknown constant.

The solutions of the above ODEs for *F* and *G* respectively, are:

$$F(y) = \frac{9\alpha\, Re}{4}\left(\frac{y^2}{2}-\frac{y^4}{6}+\frac{y^6}{30}\right)+\frac{Ay^2}{2}+c_1 y + c_2 \tag{4.39}$$

and

$$G(x) = -\frac{(1-x)^2}{2} - \frac{A}{3}(1-x)+c_3, \tag{4.40}$$

where c_1, c_2, and c_3 are unknown constants.

Applying the conditions

$$\frac{\partial u_x^{(10)}}{\partial y}(x,0) = u_x^{(10)}(x,0) = 0 \quad \text{and} \quad p^{(10)}(1,1)=0,$$

we find that

$$F(1) = F'(0) = 0 \ \text{ and } \ G(1) = 0.$$

Using now the order ε mass-flow condition

$$\int_0^1 \left(\rho^{(00)} u_x^{(10)} + \rho^{(10)} u_x^{(00)}\right) dy = 0,$$

we easily see that

$$A = -\frac{54\alpha Re}{35}, \ c_1 = c_3 = 0, \ \text{and} \ c_2 = -\frac{3\alpha Re}{56}. \tag{4.41}$$

Therefore the solution of order ε is:

$$
\begin{aligned}
&u_x^{(10)} = -\frac{3}{2}\left(1-y^2\right)\left(1-x\right)+\frac{3\alpha Re}{280}\left(-5+33y^2-35y^4+7y^6\right)\\
&u_y^{(10)} = 0\\
&p^{(10)} = -\frac{\left(1-x\right)^2}{2}+\frac{18\alpha Re}{35}\left(1-x\right)+\frac{\alpha^2}{6}\left(1-y^2\right)\\
&\rho^{(10)} = 1-x\\
&\eta^{(10)} = 0
\end{aligned}
\tag{4.42}
$$

The solution represents the order ε effect of the compressibility of the fluid and as expected, the solution agrees with that in Chapter 3 for the limiting case $B \to \infty$ (no slip limit) and in Taliadorou et al. (2009) for order ε that there is no effect on the flow due to the pressure-dependence of viscosity at this order since $\eta^{(10)} = 0$.

4.3.3 Solution of order δ

Once more we assume that $u_y^{(01)} = 0$ and from the state equation $\rho^{(01)} = 0$ and the viscosity equation $\eta^{(01)} = p^{(10)}$, we have that

$$
\rho^{(01)} = 0 \text{ and } \eta^{(01)} = p^{(00)} = 1-x.
$$

The order δ continuity equation leads to

$$
u_x^{(01)} = u_x^{(01)}\left(y\right) = F\left(y\right),
$$

where $F\left(y\right)$ is an unknown function. Substituting all the known quantities into the y-momentum equation anf integrating with respect to y we find

$$
p^{(01)} = \frac{a^2y^2}{2}+G\left(x\right), \tag{4.43}
$$

where $G\left(x\right)$ is an unknown function. Substituting all the known quantities into the x-momentum equation and separating variables we find

$$
F''\left(y\right) = 3G'\left(x\right)+3\left(1-x\right) = A, \tag{4.44}
$$

where A is an unknown constant. We can easily solve the two ODEs of Eq. (4.44) for F and G:

$$
F\left(y\right) = \frac{Ay^2}{2}+c_1y+c_2 \tag{4.45}
$$

and

$$G(x)=\frac{(1-x)^2}{2}-\frac{A}{3}(1-x)+c_3\,, \tag{4.46}$$

where c_1, c_2, and c_3 are unknown constants. Applying the boundary conditions

$$\frac{\partial u_x^{(01)}}{\partial y}(x,0)=u_x^{(01)}(x,1)=0\,,$$

the condition for the mass flow rate and the condition $p(1,1)=0$, the constants c_1, c_2, and c_3 are easily calculated:

$$A=c_1=c_2=0,\; c_3=-\frac{\alpha^2}{2}\,. \tag{4.47}$$

Therefore the solution of order δ is:

$$\begin{aligned}
&u_x^{(01)}=0\\
&u_y^{(01)}=0\\
&p^{(01)}=\frac{(1-x)^2}{2}-\frac{\alpha^2}{2}\left(1-y^2\right)\\
&\rho^{(01)}=0\\
&\eta^{(01)}=1-x
\end{aligned} \tag{4.48}$$

The solution represents the order δ effect due to the dependence of the viscosity on the pressure (there are no compressibility effects as indicated by $\rho^{(01)}=0$). Furthermore, we note that it agrees with $O(\delta)$ term in the expansion of the exact solution in Chapter 2, upon fixing some constants appropriately.

4.3.4 Solution of order ε^2

To obtain the solution of order ε^2 we assume again that $u_y^{(20)}=u_y^{(20)}(y)$, and from the equations of state $\rho^{(20)}=p^{(10)}$ and the viscosity equation $\eta^{(20)}=0$ we obtain

$$\rho^{(20)}=p^{(10)}=-\frac{1}{2}(1-x)^2+\frac{18\alpha Re}{35}(1-x)+\frac{\alpha^2}{6}\left(1-y^2\right) \tag{4.49}$$

and

$$\eta^{(20)}=0\,. \tag{4.50}$$

Substituting all the known quantities into the order ε^2 continuity equation and integrating with respect to x gives

$$u_x^{(20)} = \frac{9}{4}\left(1-y^2\right)\left(1-x\right)^2 - \frac{3\alpha Re}{280}\left(67-39y^2-35y^4+7y^6\right)\left(1-x\right) + \frac{\partial u_y^{(20)}}{\partial y}\left(1-x\right) - \frac{\alpha^2}{4}\left(1-y^2\right)^2 + F\left(y\right) \tag{4.51}$$

where $F\left(y\right)$ is an unknown function.

Substituting all the known quantities into the *y*-component of the order ε^2 momentum equation and integrating with respect to *y* gives

$$p^{(20)} = -\frac{\alpha^2}{2}\left(1-y^2\right)\left(1-x\right) + \frac{\alpha^3 Re}{840}\left(67-39y^2-35y^4+7y^6\right) + \frac{\alpha^2}{3}\frac{\partial u_y^{(20)}}{\partial y} + G\left(x\right), \tag{4.52}$$

where $G\left(x\right)$ is an unknown function.

Substituting all the known quantities in the second-order *x*-momentum equation and after some calculations, we get:

$$\begin{aligned} &-\frac{3\alpha\, Re}{2}\left(1-y^2\right)\frac{\partial u_y^{(20)}}{\partial y} - 3\alpha\, Re\, y u_y^{(20)} - \frac{\alpha^2}{2}\left(11-15y^2\right) \\ &+\frac{9\alpha^2\, Re^2}{280}\left(31-34y^2-32y^4+42y^6-7y^8\right) - F''\left(y\right) = \\ &-3G'\left(x\right) - \frac{9}{2}\left(1-x\right)^2 + \frac{\partial^3 u_y^{(20)}}{\partial y^3}\left(1-x\right) + \frac{9\alpha\, Re}{70}\left(59-70y^2+35y^4\right)\left(1-x\right). \end{aligned} \tag{4.53}$$

The equations and the boundary conditions we need for this problem are the same as those used in Taliadorou et al. (2009). Hence, we skip all further calculations and the solution of order ε^2 is immediately given by:

$$\begin{aligned} u_x^{(20)} &= \frac{9}{4}\left(1-y^2\right)\left(1-x\right)^2 - \frac{3\alpha Re}{280}\left(1-y^2\right)\left(57+84y^2-21y^4\right)\left(1-x\right) - \frac{\alpha^2}{8}\left(1-y^2\right)\left(1+3y^2\right) \\ &\quad - \frac{3\alpha^2 Re^2}{431200}\left(1-y^2\right)\left(2193-9163y^2-6853y^4+5159y^6-6161y^8\right) \\ u_y^{(20)} &= \frac{3\alpha Re}{140}\, y\left(1-y^2\right)^2\left(5-y^2\right) \\ p^{(20)} &= \frac{1}{2}\left(1-x\right)^3 - \frac{36\alpha Re}{35}\left(1-x\right)^2 - \frac{\alpha^2}{6}\left(11-3y^2\right)\left(1-x\right) + \frac{3044\alpha^2 Re^2}{13475}\left(1-x\right) \\ &\quad + \frac{\alpha^3 Re}{840}\left(1-y^2\right)\left(97-140y^2+35y^4\right) \\ \rho^{(20)} &= -\frac{1}{2}\left(1-x\right)^2 + \frac{18\alpha Re}{35}\left(1-x\right) + \frac{\alpha^2}{6}\left(1-y^2\right) \\ \eta^{(20)} &= 0 \end{aligned} \tag{4.54}$$

The pressure-dependence of the viscosity, as expected, does not have an effect at order ε^2, as indicated by. $\eta^{(20)} = 0$.

4.3.5 Solution of order δ^2

Once more we assume that $u_y^{(02)} = u_y^{(02)}(y)$. From the equation of state and the equation of viscosity we have that

$$\rho^{(02)} = 0 \tag{4.55}$$

and

$$\eta^{(02)} = p^{(01)} = \frac{(1-x)^2}{2} - \frac{\alpha^2}{2}\left(1-y^2\right). \tag{4.56}$$

Substituting all the known quantities into the continuity equation at order ε^2 and integrating with respect to *x* we find that

$$u_x^{(02)} = \frac{\partial u_y^{(02)}}{\partial y}(1-x) + F(y), \tag{4.57}$$

where $F(y)$ is an unknown function. From the *y*-momentum we get

$$p^{(02)} = \frac{\alpha^2}{3}\frac{\partial u_y^{(02)}}{\partial y} - \frac{\alpha^2}{2}\left(1-y^2\right)(1-x) + G(x), \tag{4.58}$$

where $G(x)$ is also an unknown function. From the *x*-momentum we have

$$-\frac{3\alpha^2}{2}\left(1-y^2\right)\frac{\partial u_y^{(02)}}{\partial y} - 3\alpha\, Re\, y u_y^{(02)} + 3\alpha^2 y^2 - F''(y) = -3G'(x) - \frac{3}{2}(1-x)^2 + \frac{\partial^3 u_y^{(02)}}{\partial y^3}(1-x). \tag{4.59}$$

In order to be able to separate variables we demand that the last term of Eq. (4.59) is a scalar multiple of $(1-x)$; therefore, we set

$$\frac{\partial^3 u_y^{(02)}}{\partial y^3} = \gamma, \tag{4.60}$$

with γ being a constant to be determined. Integrating (4.60) we get

$$u_y^{(02)} = \frac{\gamma}{6}y^3 + \frac{c_1}{2}y^2 + c_2 y + c_3,$$

where c_1, c_2, and c_3 are unknown constants.

Conditions $u_x^{(02)}(x,1) = 0$ and $\frac{\partial u_x^{(02)}}{\partial y}(x,0) = 0$ lead to

$$\left.\frac{\partial u_y^{(02)}}{\partial y}\right|_{y=1} = \left.\frac{\partial^2 u_y^{(02)}}{\partial y^2}\right|_{y=0} = 0 \tag{4.61}$$

and

$$F(1) = F'(0) = 0 \ . \tag{4.62}$$

Applying the conditions $u_y^{(02)}(x,0) = u_y^{(02)}(x,1) = 0$ and the conditions in Eq. (4.61) we find that γ, c_1, c_2, and c_3 are equal to zero. Hence,

$$u_y^{(02)} = 0 \, . \tag{4.63}$$

By means of Eq. (4.63) the x-momentum equation (4.59) takes the simpler form

$$3\alpha^2 y^2 - F''(y) = -3G'(x) - \frac{3}{2}(1-x)^2 = A \, , \tag{4.64}$$

where A is an unknown constant. The two ODEs of separate variables are easily solved under the conditions

$$F(1) = F'(0) = 0 \ \text{ and } \ p^{(02)}(1,1) = 0$$

to get

$$u_x^{(02)} = F(y) = \frac{\alpha^2}{20}\left(1-y^2\right)\left(1-5y^2\right) \tag{4.65}$$

and

$$G(x) = \frac{1}{6}(1-x)^3 - \frac{\alpha^2}{5}(1-x) \, . \tag{4.66}$$

Finally, we find that the solution of order δ^2 is:

$$\boxed{\begin{aligned} u_x^{(02)} &= \frac{\alpha^2}{20}\left(1-y^2\right)\left(1-5y^2\right) \\ u_y^{(02)} &= 0 \\ p^{(02)} &= \frac{1}{6}(1-x)^3 - \frac{\alpha^2}{10}\left(3-5y^2\right)(1-x) \\ \rho^{(02)} &= 0 \\ \eta^{(02)} &= \frac{(1-x)^2}{2} - \frac{\alpha^2}{2}\left(1-y^2\right) \end{aligned}} \tag{4.67}$$

This solution represents the order δ^2 perturbation due to the dependence of the viscosity on the pressure. We notice that there is no contribution from the compressibility at order δ^2 as indicated by $\rho^{(02)} = 0$). Furthermore, we note that it is essentially the same as the $O(\delta^2)$ term in the expansion of the exact solution in Chapter 2 (assuming that δ is small enough in order to assure the validity of the expansion), upon fixing some constants appropriately.

4.3.6 Solution of order εδ

To obtain the solution at order *εδ* we assume that $u_y^{(11)} = u_y^{(11)}(y)$. From the equation of state of the density and from the viscosity equation

$$\rho^{(11)} = p^{(01)} \text{ and } \eta^{(11)} = p^{(10)},$$

we get

$$\rho^{(11)} = \frac{(1-x)^2}{2} - \frac{\alpha^2}{2}(1-y^2) \tag{4.68}$$

and

$$\eta^{(11)} = -\frac{(1-x)^2}{2} + \frac{18\alpha Re}{35}(1-x) + \frac{\alpha^2}{6}(1-y^2), \tag{4.69}$$

respectively.

The continuity equation is simplified to

$$u_x^{(00)} \frac{\partial \rho^{(11)}}{\partial x} + \frac{\partial u_x^{(11)}}{\partial x} + \frac{\partial u_y^{(11)}}{\partial y} = 0. \tag{4.70}$$

Substituting all the known quantities into Eq. (4.70) and integrating with respect to *x* we get

$$u_x^{(11)} = -\frac{3}{4}(1-y^2)(1-x)^2 - \frac{3\alpha^2}{4}(1-y^2)^2 + (1-x)\frac{\partial u_y^{(11)}}{\partial y} + F(y), \tag{4.71}$$

where $F(y)$ is an unknown function.

The *y*-momentum equation is simplified to

$$-3\frac{\partial p^{(11)}}{\partial y} + a^2\eta^{(00)}\left(\frac{4}{3}\frac{\partial^2 u_y^{(11)}}{\partial y^2} + \frac{1}{3}\frac{\partial^2 u_x^{(11)}}{\partial x \partial y}\right) + \frac{a^2}{3}\eta^{(01)}\frac{\partial^2 u_x^{(10)}}{\partial x \partial y} + \alpha^2\frac{\partial \eta^{(10)}}{\partial x}\frac{\partial u_x^{(01)}}{\partial y} + \alpha^2\frac{\partial \eta^{(11)}}{\partial x}\frac{\partial u_x^{(00)}}{\partial y} = 0. \tag{4.72}$$

Substituting all the known quantities into Eq. (4.72) and integrating with respect to *y*, we find that the pressure is given by

$$p^{(11)} = \frac{\alpha^2}{3}\frac{\partial u_y^{(11)}}{\partial y} + \frac{4\alpha^2}{3}\left(1-y^2\right)\left(1-x\right) - \frac{\alpha^3 Re}{280}\left(67-39y^2-35y^4+7y^6\right)+G\left(x\right), \quad (4.73)$$

where $G\left(x\right)$ is an unknown function.

Substituting all the known quantities in the *x*-momentum we end up with

$$\begin{aligned} -\frac{3}{2}\alpha\, Re\left(1-y^2\right)\frac{\partial u_y^{(11)}}{\partial y} - 3\alpha\, Re\, y u_y^{(11)} + \frac{7\alpha^2}{2}\left(1-3y^2\right) - F''\left(y\right) = \\ -3G'\left(x\right)+6\left(1-x\right)^2+\left(\frac{\partial^3 u_y^{(11)}}{\partial y^3}-\frac{108\alpha\, Re}{35}\right)\left(1-x\right) \end{aligned} \quad (4.74)$$

In order to separate variables we demand that the last term of Eq. (4.74) is a scalar multiple of $\left(1-x\right)$, therefore we set

$$\frac{\partial^3 u_y^{(11)}}{\partial y^3}-\frac{108\alpha\, Re}{35} = \alpha\, Re\,\gamma\,, \quad (4.75)$$

with γ being a constant to be determined.

Integrating Eq. (4.75) three times we find that

$$u_y^{(11)} = \frac{\alpha Re}{210}\left(108+35\gamma\right)y^3 + \frac{c_1}{2}y^2 + c_2 y + c_3\,,$$

where c_1, c_2, and c_3 are unknown constants.

The boundary conditions $u_x^{(11)}\left(x,1\right)=0$ and $\frac{\partial u_x^{(11)}}{\partial y}\left(x,0\right)=0$ give respectively

$$\left(1-x\right)\left.\frac{\partial u_y^{(11)}}{\partial y}\right|_{y=1} + F\left(1\right)=0 \quad \text{and} \quad \left(1-x\right)\left.\frac{\partial^2 u_y^{(11)}}{\partial y^2}\right|_{y=0} + F'\left(0\right)=0$$

and in order for these to apply for every *x* in our domain we must have

$$\left.\frac{\partial u_y^{(11)}}{\partial y}\right|_{y=1} = \left.\frac{\partial^2 u_y^{(11)}}{\partial y^2}\right|_{y=0} = 0 \text{ and } F\left(1\right)=F'\left(0\right)=0.$$

Applying the conditions

$$\left.\frac{\partial u_y^{(11)}}{\partial y}\right|_{y=1} = \left.\frac{\partial^2 u_y^{(11)}}{\partial y^2}\right|_{y=0} = 0 \quad \text{and} \quad u_y^{(11)}\left(x,0\right)=u_y^{(11)}\left(x,1\right)=0$$

we find that

$$c_1 = c_2 = c_3 = 0 \text{ and } \gamma = -\frac{108}{35},$$

and therefore $u_y^{(11)}$ is zero.

Equation (4.74) is simplified to

$$\frac{7\alpha^2}{2}\left(1-3y^2\right) - F''(y) = -3G'(x) + 6(1-x)^2 - \frac{108\alpha\, Re}{35}(1-x) = A = const. \quad (4.76)$$

The first ODE in Eq. (4.76) gives

$$F(y) = \frac{7\alpha^2}{2}\left(\frac{1}{2}y^2 - \frac{1}{4}y^4\right) - \frac{A}{2}y^2 + c_4 y + c_5, \quad (4.77)$$

where c_4 and c_5 are unknown constants.

Applying the conditions

$$\int_0^1 \left(\rho^{(00)}u_x^{(11)} + \rho^{(10)}u_x^{(01)} + \rho^{(01)}u_x^{(10)} + \rho^{(11)}u_x^{(00)}\right)dy = 0$$

and

$$F(1) = F'(0) = 0,$$

we find, after some calculations, that

$$A = \frac{7\alpha^2}{5}, \; c_4 = 0 \text{ and } c_5 = -\frac{7\alpha^2}{40}. \quad (4.78)$$

Therefore,

$$F(y) = -\frac{7\alpha^2}{2}\left(1-y^2\right)\left(1-5y^2\right). \quad (4.79)$$

The second ODE in Eq. (4.76) gives

$$G(x) = -\frac{2}{3}(1-x)^3 + \frac{18\alpha\, Re}{35}(1-x)^2 + \frac{A}{3}(1-x) + c_6, \quad (4.80)$$

where c_6 is an unknown constant and from the condition for the pressure $p^{(11)}(1,1) = 0$ we find that $G(1) = 0$ which leads to $c_6 = 0$.

Finally, the solution of order $\varepsilon\delta$ is:

$$
\begin{aligned}
u_x^{(11)} &= -\frac{3}{4}\left(1-y^2\right)\left(1-x\right)^2 + \frac{\alpha^2}{40}\left(1-y^2\right)\left(23+5y^2\right) \\
u_y^{(11)} &= 0 \\
p^{(11)} &= -\frac{2}{3}\left(1-x\right)^3 + \frac{18\alpha \,\mathrm{Re}}{35}\left(1-x\right)^2 + \frac{\alpha^2}{15}\left(27-20y^2\right)\left(1-x\right) \\
&\quad - \frac{\alpha^3 Re}{280}\left(1-y^2\right)\left(67+28y^2-7y^4\right) \\
\rho^{(11)} &= \frac{1}{2}\left(1-x\right)^2 - \frac{\alpha^2}{2}\left(1-y^2\right) \\
\eta^{(11)} &= -\frac{1}{2}\left(1-x\right)^2 + \frac{18\alpha Re}{35}\left(1-x\right) + \frac{\alpha^2}{6}\left(1-y^2\right)
\end{aligned}
\tag{4.81}
$$

At this order the solution represents the combined effect of the compressibility and of the pressure-dependence of the viscosity as both $\rho^{(11)}$ and $\eta^{(11)}$ are not zero. To understand this solution better, we consider the limiting case where the aspect ratio of the channel α, is equal to zero (lubrication approximation). We see then that for any fixed y, $\rho^{(11)}$ decreases monotonically and $\eta^{(11)}$ increases monotonically with x as we move from the left-end of the channel, $x=0$, to the right-end of the channel, $x=1$.

4.3.7 The solution up to order εδ

Combining the solutions of zero-order and of orders ε, δ, ε^2, δ^2 and $\varepsilon\delta$, we find that the approximate perturbation, analytical solution is given by:

$$
\begin{aligned}
u_x = {} & \frac{3}{2}\left(1-y^2\right)+\varepsilon\left[-\frac{3}{2}\left(1-y^2\right)\left(1-x\right)+\frac{3\alpha Re}{280}\left(-5+33y^2-35y^4+7y^6\right)\right] \\
& +\varepsilon^2\left[\frac{9}{4}\left(1-y^2\right)\left(1-x\right)^2-\frac{3\alpha Re}{280}\left(1-y^2\right)\left(57+84y^2-21y^4\right)\left(1-x\right)-\frac{\alpha^2}{8}\left(1-y^2\right)\left(1+3y^2\right)\right. \\
& \left.-\frac{3\alpha^2 Re^2}{431200}\left(1-y^2\right)\left(2193-9163y^2-6853y^4+5159y^6-6161y^8\right)\right] \\
& +\delta^2\frac{\alpha^2}{20}\left(1-y^2\right)\left(1-5y^2\right)+\varepsilon\delta\left[-\frac{3}{4}\left(1-y^2\right)\left(1-x\right)^2+\frac{\alpha^2}{40}\left(1-y^2\right)\left(23+5y^2\right)\right]+h.ot.
\end{aligned}
\tag{4.82}
$$

$$
u_y = \varepsilon^2\frac{3\alpha Re}{140}\,y\left(1-y^2\right)^2\left(5-y^2\right)+h.o.t. \tag{4.83}
$$

$$
\begin{aligned}
p = {} & 1-x+\varepsilon\left[-\frac{\left(1-x\right)^2}{2}+\frac{18\alpha Re}{35}\left(1-x\right)+\frac{\alpha^2}{6}\left(1-y^2\right)\right]+\delta\left[\frac{\left(1-x\right)^2}{2}-\frac{\alpha^2}{2}\left(1-y^2\right)\right] \\
& +\varepsilon^2\left[\frac{1}{2}\left(1-x\right)^3-\frac{36\alpha Re}{35}\left(1-x\right)^2-\frac{\alpha^2}{6}\left(11-3y^2\right)\left(1-x\right)+\frac{3044\alpha^2 Re^2}{13475}\left(1-x\right)\right. \\
& \left.+\frac{\alpha^3 Re}{840}\left(1-y^2\right)\left(97-140y^2+35y^4\right)\right]+\delta^2\left[\frac{1}{6}\left(1-x\right)^3-\frac{\alpha^2}{10}\left(3-5y^2\right)\left(1-x\right)\right] \\
& +\varepsilon\delta\left[-\frac{2}{3}\left(1-x\right)^3+\frac{18\alpha\,\mathrm{Re}}{35}\left(1-x\right)^2+\frac{\alpha^2}{15}\left(27-20y^2\right)\left(1-x\right)\right. \\
& \left.-\frac{\alpha^3 Re}{280}\left(1-y^2\right)\left(67+28y^2-7y^4\right)\right]+h.o.t
\end{aligned}
\tag{4.84}
$$

$$
\begin{aligned}
\rho = {} & 1+\varepsilon\left(1-x\right)+\varepsilon^2\left[-\frac{1}{2}\left(1-x\right)^2+\frac{18\alpha Re}{35}\left(1-x\right)+\frac{\alpha^2}{6}\left(1-y^2\right)\right] \\
& +\varepsilon\delta\left[\frac{\left(1-x\right)^2}{2}-\frac{\alpha^2}{2}\left(1-y^2\right)\right]+h.o.t.
\end{aligned}
\tag{4.85}
$$

$$
\begin{aligned}
\eta = {} & 1+\delta\left(1-x\right)+\delta^2\left[\frac{\left(1-x\right)^2}{2}-\frac{\alpha^2}{2}\left(1-y^2\right)\right] \\
& +\varepsilon\delta\left[-\frac{\left(1-x\right)^2}{2}+\frac{18\alpha Re}{35}\left(1-x\right)+\frac{\alpha^2}{6}\left(1-y^2\right)\right]+h.o.t.
\end{aligned}
\tag{4.86}
$$

4.4 Results and discussion

The basic features of the velocity and the pressure fields are as follows:

- The zero-order solution is just the solution of incompressible flow with constant viscosity.
- The terms of orders ε and ε^2 are the terms obtained in Taliadorou et al. (2009) and Venerus and Bugajsky (2010), for the compressible flow with constant viscosity
- The terms of orders δ and δ^2 represent the effects of the pressure dependence of the viscosity.

These terms agree with the expansion of the exact solution derived in Chapter 2 (upon fixing some constants appropriately). (As already mentioned, in Chapter 2 the viscosity-pressure coefficient is denoted by ε.) Assuming that δ is small enough in order to ensure the validity of an infinite expansion in terns of δ, we find that the expansion of the solution of the plane flow in Section 2.5 is given by

$$\begin{aligned} u_x &= \frac{A}{2}\left(1-y^2\right) - \frac{A^3\alpha^2}{108}\left(1-y^4\right)\delta^2 + O\left(\delta^4\right) \\ u_y &= 0 \\ p &= -\frac{A}{3}x + \frac{A^2}{18}\left(x^2+\alpha^2 y^2\right)\delta - \frac{A^3}{54}\left(x^3+3\alpha^2 xy^2\right)\delta^2 + O\left(\delta^3\right) \\ \eta &= 1 - \frac{A}{3}x\delta + \frac{A^2}{18}\left(x^2+\alpha^2 y^2\right)\delta^2 + O\left(\delta^3\right) \end{aligned} \tag{4.87}$$

Giving an appropriate value to the constant A and fixing the Reynolds number to zero in the solution we obtained here, (4.87) is essentially the same as solution (4.82)-(4.84), but shifted to the right as the origin of the coordinates is located at the exit of the plane in Section 2.5.

Let us examine now the expressions (4.80)-(4.84) for the primary variables in more detail.

Transverse velocity

The transverse velocity u_y is zero at first order in ε (by assumption). It is also zero for all other orders of ε and δ except at the second order in ε. u_y is always positive, it only depends on the y coordinate and it varies linearly with the aspect ratio and the Reynolds number.

Velocity in the flow direction

The horizontal velocity u_x deviates from the parabolic incompressible solution at first order in *ε* due to fluid inertia and due to geometric effects (exhibited by the terms involving the aspect ratio α .

The deviation of first order in *ε* depends on both the x and y and it may be positive or negative depending on the value of α and *Re*. Since $u_x^{(10)}$ is linear in x we can easily find that

$$u_x^{(10)} \geq 0 \ \text{ if } \quad x \geq 1 - \frac{\alpha Re}{140}\left(-5 + 28y^2 - 7y^4\right)$$

where $-5 + 28y^2 - 7y^4 < 0$ for $0 < y < y^*$ where $y^* > \sqrt{\frac{1}{7}\left(14 - \sqrt{161}\right)} = 0.43$. Therefore for $0 < y \leq y^*$, $u_x^{(10)} \leq 0$ for all $0 \leq x \leq 1$. For $y^* < y < 1$ then $u_x^{(10)} \geq 0$ for $x \geq 1 - \frac{\alpha Re}{140}\left(-5 + 28y^2 - 7y^4\right)$.

When $\alpha \to 0$ we have $u_x^{(10)} \leq 0$ for the whole interval $0 \leq x \leq 1$ and therefore we have a decrease of order *ε* in the horizontal velocity. The same holds as $Re \to 0$. At second order in *ε* there is a reduction of the horizontal velocity that is independent of inertia and which does not alter its parabolicity.

At first order in *δ*, $u_x^{(01)}$ is zero so the pressure-dependence of the viscosity does not affect the flow at this order. At second order in *δ*, $u_x^{(02)}$ depends only on the y coordinate, and it is always positive for $0 \leq y \leq 1$. It also increases with the square of the aspect ratio *α*. For long channels where $\alpha << 1$, $u_x^{(02)}$ is therefore very small.

The order *εδ* is a *coupling* term representing the interaction of the compressibility and the pressure dependence of viscosity. This term can be either positive or negative depending on the value of the aspect ratio *α*. Specifically, since $u_x^{(11)}$ is a quadratic in x we can easily show that

$$u_x^{(11)} \geq 0 \ \text{ for } \ x \geq 1 - \alpha\sqrt{\frac{23}{30}\left(1 + \frac{5}{23}y^2\right)}.$$

Therefore, since $0.87 \le \sqrt{\frac{23}{30}\left(1+\frac{5}{23}y^2\right)} \le 0.97$ when α is approximately 1 (short channel) $u_x^{(11)}$ is positive for almost all values of x and y. When the channel is long **(** $a \ll 1$), since $0.87 \le \sqrt{\frac{23}{30}\left(1+\frac{5}{23}y^2\right)} \le 0.97$**,** we find that $u_x^{(11)}$ is negative for almost all values of x and y and the pressure-dependence of the viscosity decreases the horizontal velocity at this order.

Contour plots of the horizontal velocity field are presented in Figs. 4.2-4.6. In Fig. 4.2, we have the case of incompressible flow with constant viscosity ($\varepsilon = \delta = 0$). The contours are horizontal lines. In Fig. 4.3 we fix the pressure dependence of viscosity at $\delta = 0.01$ and we vary the compressibility effect with ε=0.01, 0.1, and 0.2. In Fig. 4.3a ($\varepsilon = \delta = 0.01$) the contours are almost horizontal lines (as in the case of the incompressible flow with constant viscosity), since there is almost no x-dependence. In Fig. 4.3b we have $\varepsilon = 0.1$ and the x-dependence kicks in, making the contours bend. The larger deviation here is the term of $O(\varepsilon)$, $u_x^{(10)}$ which increases linearly with x as mentioned above. This explains the strong bend of the contours close to the symmetry plane y=0. Furthermore, the overall value of u_x decreases. In Fig. 4.4 we fix the pressure dependence of viscosity at $\delta = 0.1$ and we vary the compressibility effect with ε=0.01, 0.1, and 0.2. In Fig. 4.4a, even though $\delta = 0.1$, since $u_x^{01} = 0$ the contours are still almost horizontal lines. Again as ε increases the x-dependence becomes stronger and u_x decreases (note the x-y region of high values of u_x at the bottom right corner of the contour plot decreasing as ε increases). In Fig. 4.5 we fix the pressure dependence of viscosity at $\delta = 0.2$ and we vary ε=0.01, 0.1, and 0.2. In Fig. 4.6 we fix $\varepsilon = 0.1$ and we vary δ=0.01, 0.1, and 0.2.

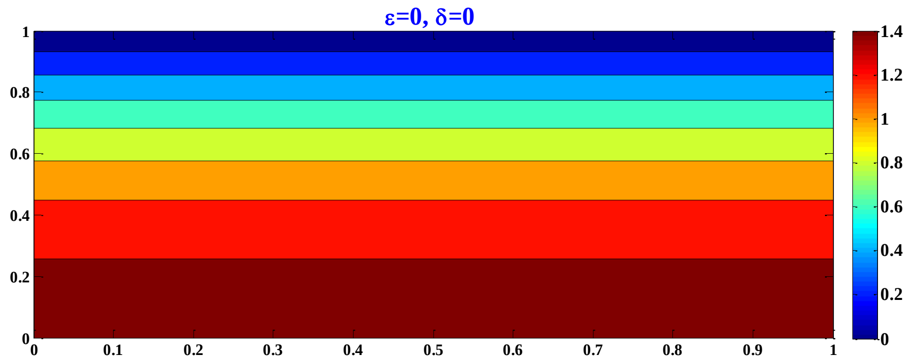
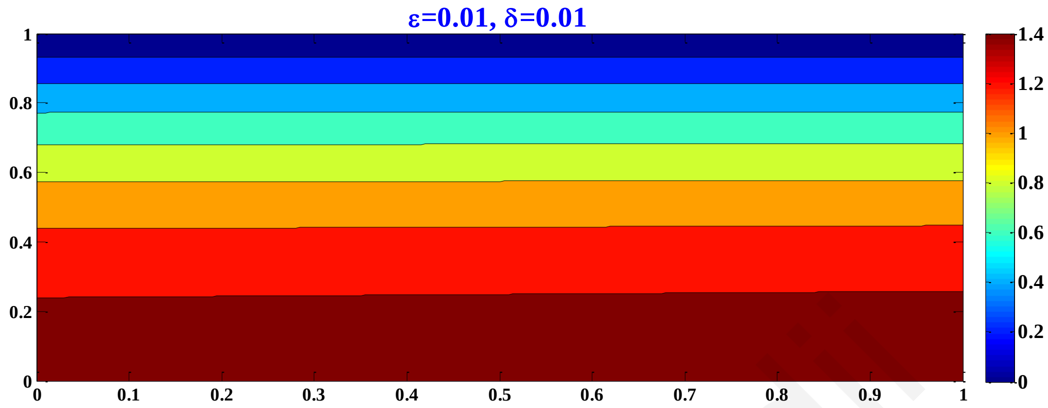
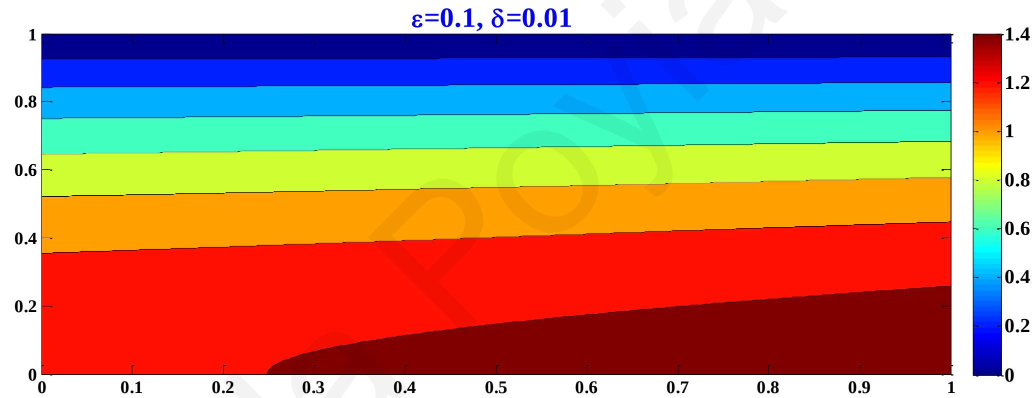


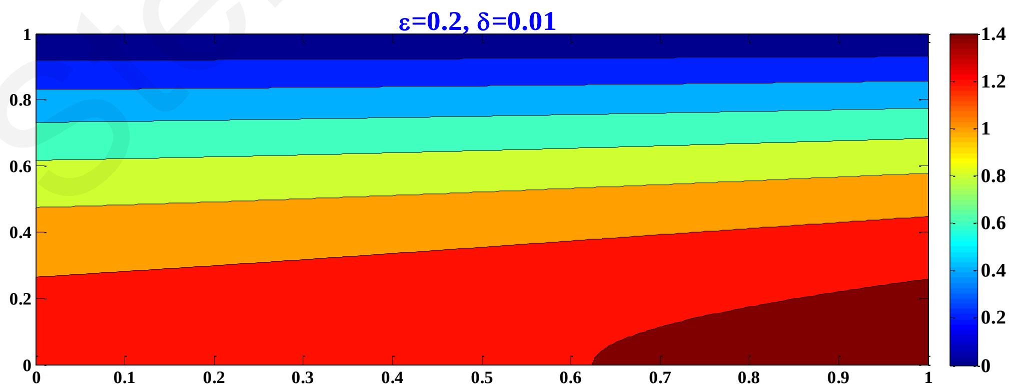
Figure 4.2: *Contours of* u_x *for* ε*=0,* $\delta = 0$*; Re=0, α=0.1.*



(a)

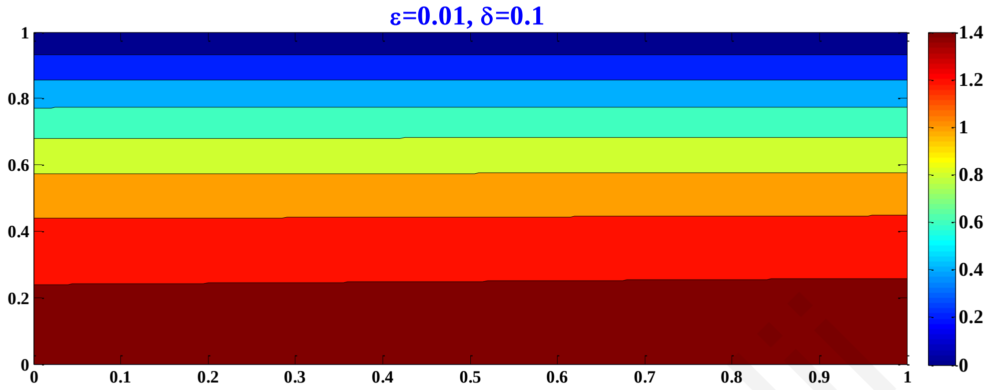


(b)

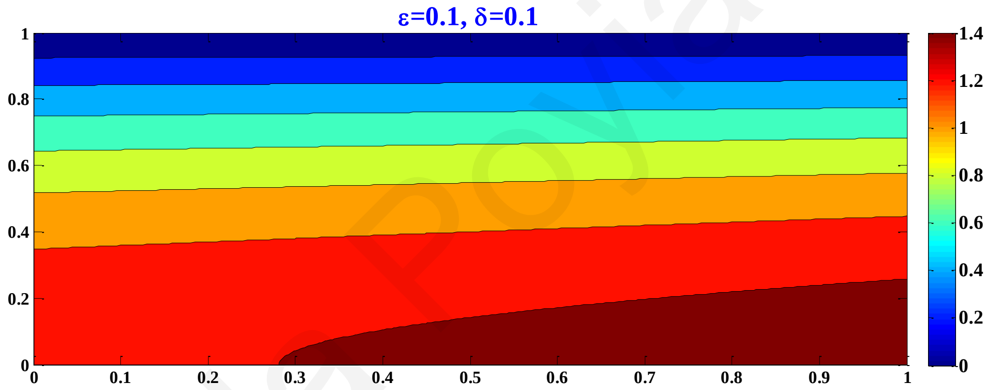


(c)

Figure 4.3: *Contours of u_x for δ=0.01 and ε=0.01, 0.1 and 0.2; Re=0, α=0.1.*



(a)



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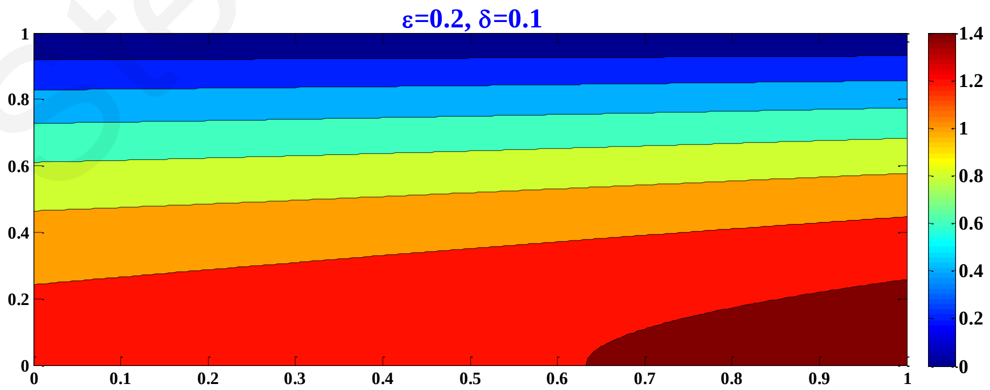
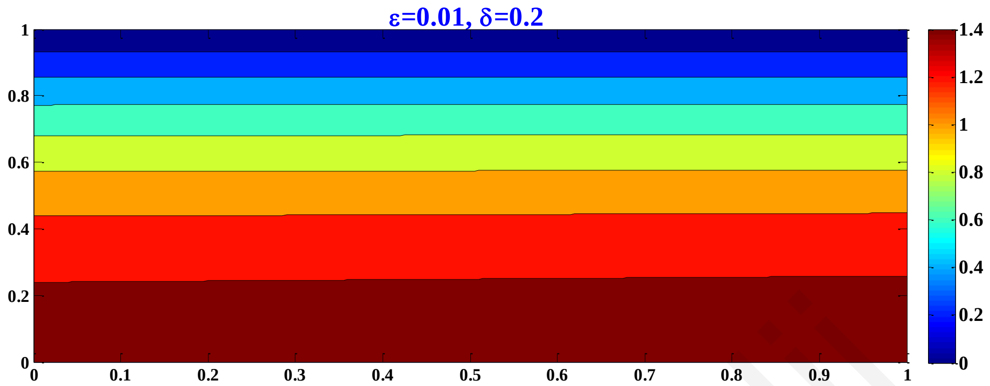
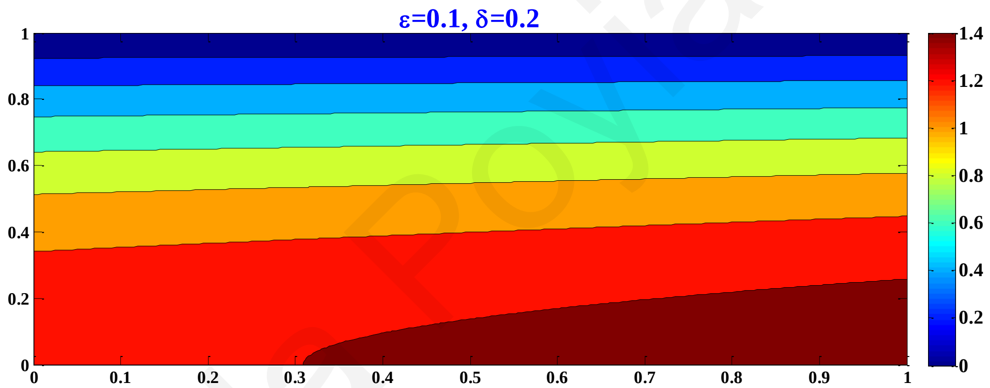


(c)

Figure 4.4: *Contours of* u_x *for* δ*=0.1 and* ε*=0.01, 0.1 and 0.2; Re=0,* α*=0.1.*



(a)



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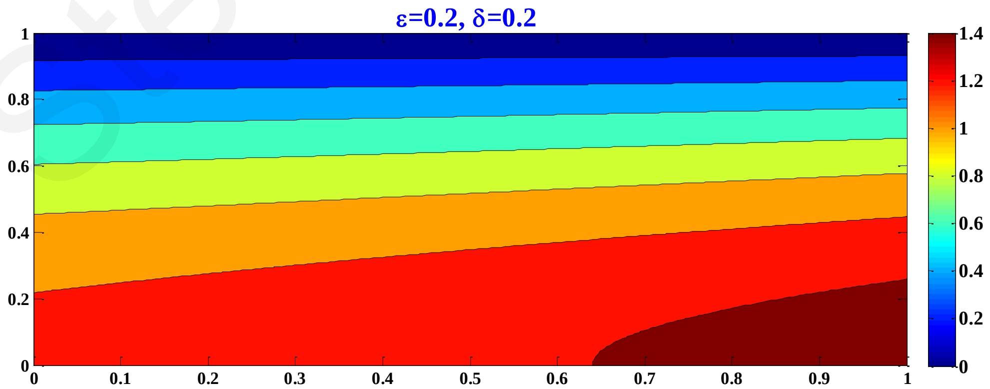
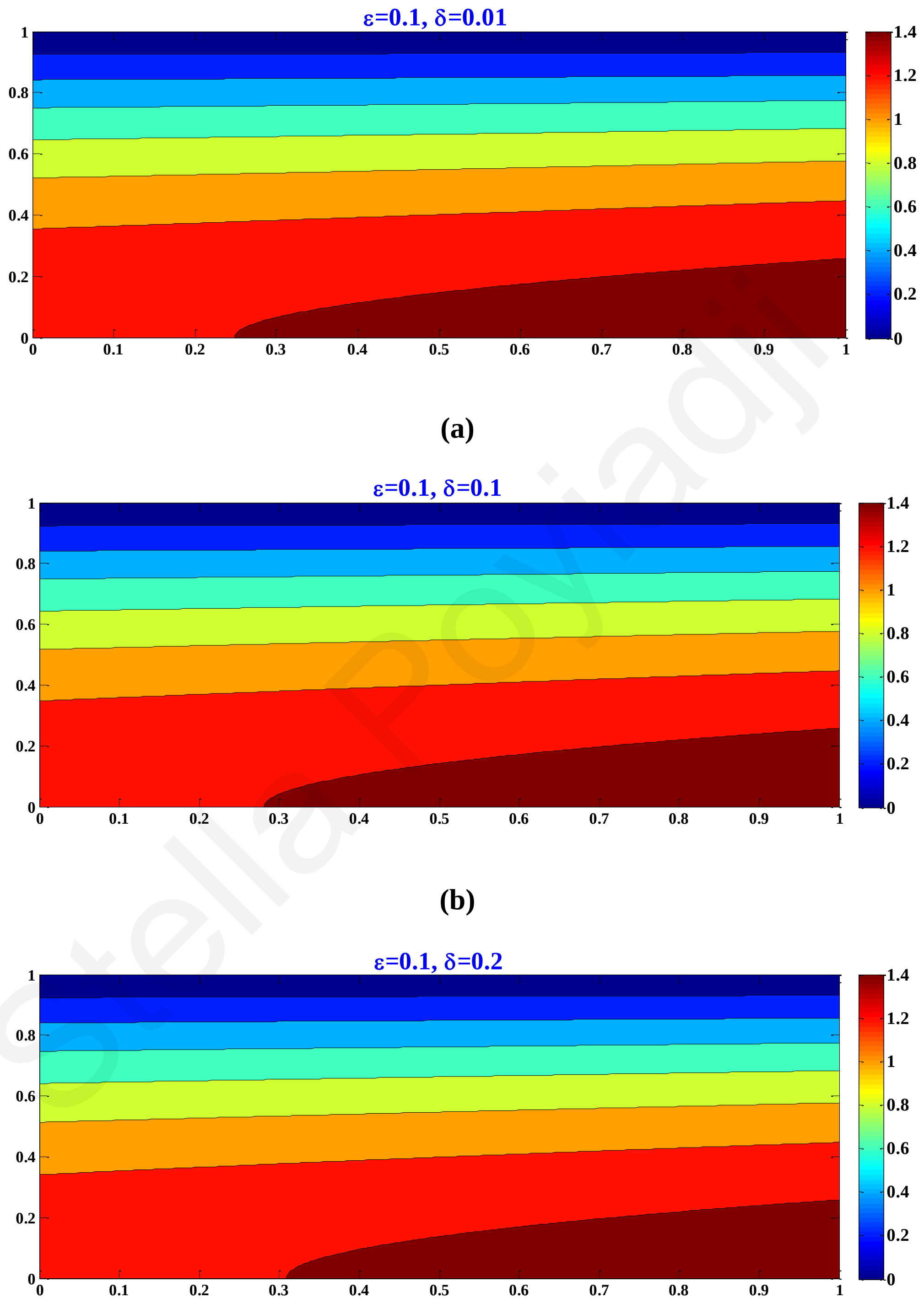


(c)

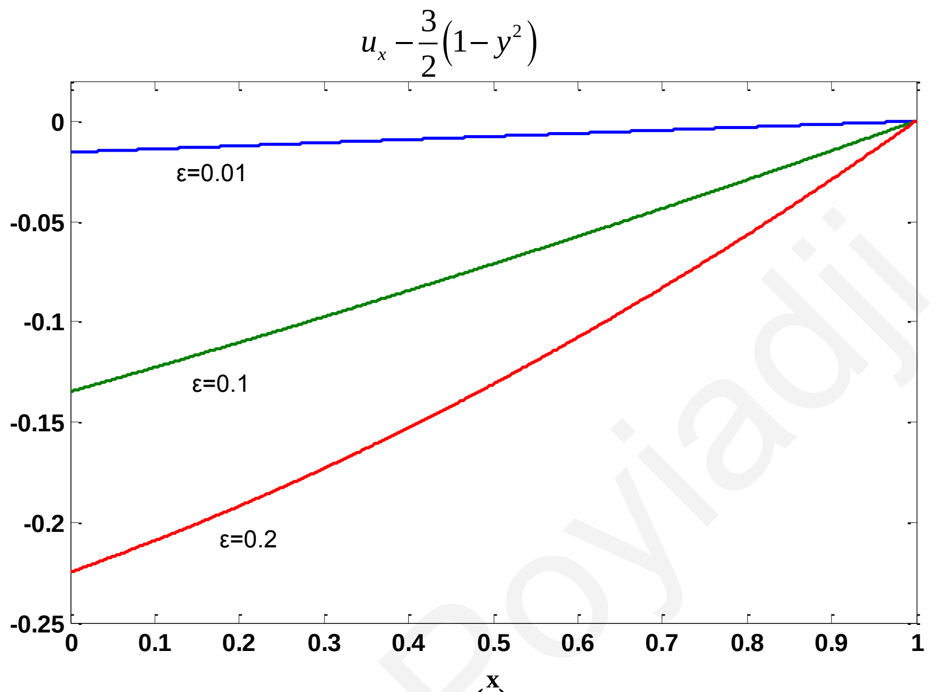
Figure 4.5: *Contours of* u_x *for* δ*=0.2 and* ε*=0.01, 0.1 and 0.2; Re=0,* α*=0.1.*



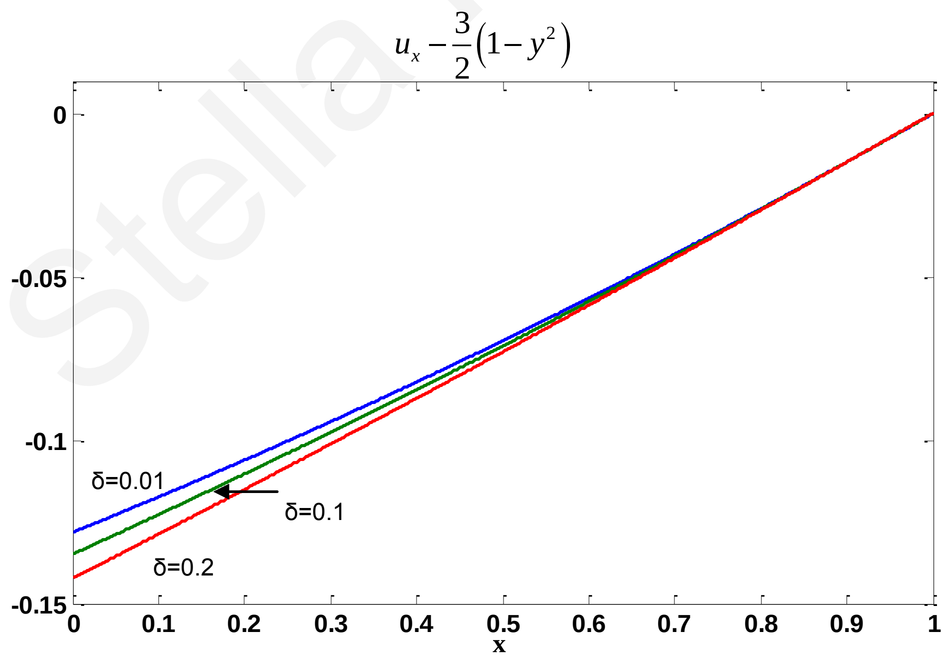
(c)

Figure 4.6: *Contours of u_x for ε=0.1 and δ=0.01, 0.1 and, 0.2; Re=0, α=0.1.*

Profiles of the deviation of u_x from $u_x^{(00)} = (3/2)(1-y^2)$ at $y=0$ are presented in Figs. 4.7. In Fig. 4.7a we fix $\delta = 0.1$ and we vary the compressibility number ε=0.01, 0.1 and 0.2, (*Re*=0, *α*=0.1). As expected, the deviation is negative along the whole axis of the channel. In Fig. 4.7b we fix $\varepsilon = 0.1$ and we vary δ=0.01, 0.1 and 0.2, (*Re*=0, *α*=0.1). As expected, the deviation is negative along the axis of the channel.



(a)



(b)

Figure 4.7: *Deviation of u_x from the incompressible flow along y=0 for: (a) δ=0.1 and ε=0.01, 0.1, and 0.2. (b) ε=0.1 and δ=0.01, 0.1, and 0.2; Re=0, α=0.1.*

Pressure

The pressure for planar, steady, incompressible, laminar flow with constant viscosity is linearly decreasing with x along the channel and is independent of y. The compressibility and the pressure-dependent viscosity both introduce dependence on the y coordinate. The deviation at first order in ε is positive for $0 \le x \le x^*$ where

$$x^* = 1 - \frac{18\alpha Re}{35}\left(1 + \sqrt{1 + \left(\frac{1}{3}\right)\left(\frac{35}{18Re}\right)^2 \left(1 - y^2\right)}\right).$$

For long channels ($a << 1$), x^* is $1 - O(\alpha)$ and the deviation at first order in ε is *negative* for almost all values of x and y. (Note that the Reynolds number is assumed to be $O(1)$ - bounds on the values of the Reynolds number *Re* which ensure that the assumptions of the asymptotic expansions are respected are provided in Section 4.4.4.) Therefore, compressibility causes a reduction to the horizontal velocity at this order.

At first order in δ, $p^{(01)}$ is parabolic in both x and y and the y-dependence becomes stronger as α increases. $p^{(01)}$ decreases proportionally to the square of α; it is positive for $0 \le x \le x^*$ where $x^* = 1 - \alpha\sqrt{1 - y^2}$. For long channels ($a << 1$), x^* is approximately equal to 1 and the deviation at first order in ε is therefore *positive* for almost all values of x and y.

Comparing the perturbations $\varepsilon p^{(10)}$ and $\delta p^{(01)}$ when $a << 1$ (and $\varepsilon \sim \delta$) we conclude that the effects of the pressure-dependence of viscosity and of compressibility compete with each other in that case. At second order in ε, the dependence of $p^{(20)}$ on y is not only due to geometric effects but also due to the fluid's inertia (as exhibited by terms involving the Reynolds number). At the order $\varepsilon\delta$, $p^{(11)}$ depends on x and y. It is a cubic in x and a cubic in y^2 and in the aspect ratio α. For long channels ($\alpha << 1$) we neglect the terms that involve α and we find

$$p^{(11)} \approx -\frac{2}{3}(1 - x)^3,$$

which is independent of y and negative throughout the channel.

Contour plots of the pressure field are presented in Figs. 4.8-4.11 for various values of δ and ε. We choose *Re*=0 but now we choose $\alpha = 1$ since in the expression (4.86), for $\alpha << 1$, the *y*-dependence is negligible and thus no bending in the contours would be observed. For $\varepsilon << 1$ and $\delta = 0$ the pressure contours are almost vertical since $p^{(00)}$ depends only on x. The contours bend for larger values of ε and δ. The competing effects of compressibility and the pressure-dependence of viscosity discussed above are clearly visible in Fig. 4.11 where we have $\varepsilon = 0.2$, $\delta = 0.01, 0.1, 0.2$ (*Re*=0, *α*=1). In Fig. 4.11a ($\varepsilon = 0.2$ $\delta = 0.01$) the contours bend to the left since the positive term $\varepsilon \frac{a^2}{6}\left(1-y^2\right)$ in $\varepsilon p^{(10)}$ is larger than the term $-\delta \frac{a^2}{2}\left(1-y^2\right)$ in $\delta p^{(01)}$. In Fig. 4.11b ($\varepsilon = 0.2$, $\delta = 0.1$) the positive term $\varepsilon \frac{a^2}{6}\left(1-y^2\right)$ in $\varepsilon p^{(10)}$ is somewhat smaller than the term $-\delta \frac{a^2}{2}\left(1-y^2\right)$ in $\delta p^{(01)}$ and the contours are almost vertical. In Fig. 4.11c ($\varepsilon = 0.2$ $\delta = 0.2$) the positive term $\varepsilon \frac{a^2}{6}\left(1-y^2\right)$ in $\varepsilon p^{(10)}$ is smaller than the term $-\delta \frac{a^2}{2}\left(1-y^2\right)$ in $\delta p^{(01)}$ and the contours bend to the right.

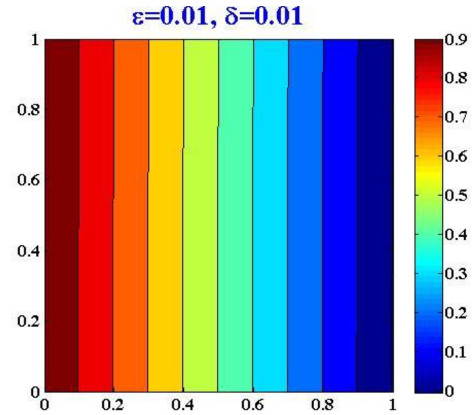
Density

The density is constant at zero-order. The deviations of order ε and ε^2 are both decreasing functions of x and y which is expected since the fluid is decompressed as it moves downstream (for constant viscosity). The deviations of order δ and order δ^2 are zero. The deviation of order *εδ* is, thus, the first order where the effect of the pressure-dependence of the viscosity appear and it is an increasing function of x and y. This means that the fluid is compressed due to the pressure-dependence of the viscosity. We therefore conclude that, as the fluid moves downstream, the compression due to the pressure-dependence of the viscosity slightly reduces the decompression caused by compressibility.

As an interesting remark, when $\delta \sim \varepsilon$, at the exit of the channel ($x = 1$)

$$\rho = 1 + \Delta\rho \quad \text{where} \quad \Delta\rho = \frac{1}{2}\varepsilon^2\alpha^2\left(1-y^2\right)\left(\frac{\varepsilon}{3}-\delta\right)$$

and since $\left(\frac{\varepsilon}{3}-\delta\right) < 0$ we have $\Delta\rho < 0$ which is different from the case with constant viscosity where the corresponding value at the exit was $\Delta\rho = \frac{1}{2}\varepsilon^2\alpha^2\left(1-y^2\right) > 0$.



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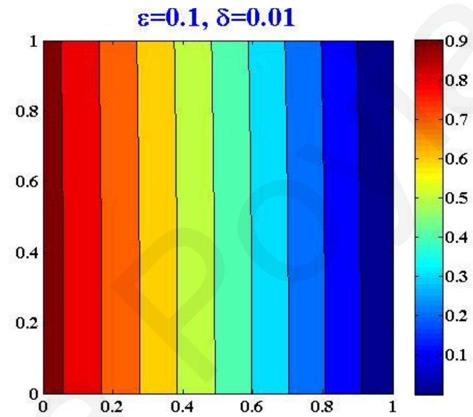

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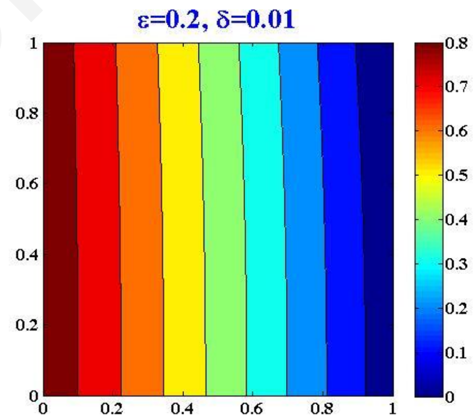
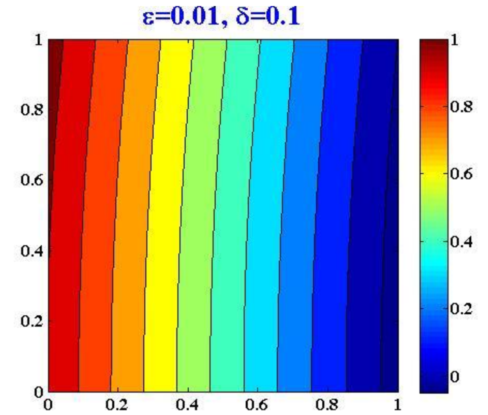
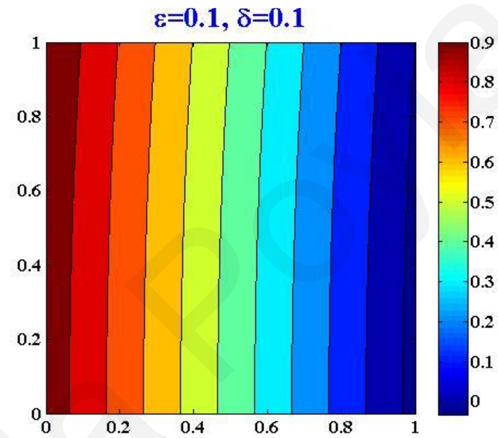


(c)

Figure 4.8: *Contours of pressure for δ=0.01 and ε=0.01, 0.1 and 0.2; Re=0, α=1.*



(a)



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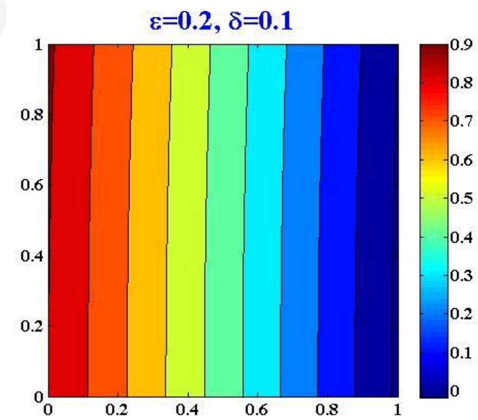
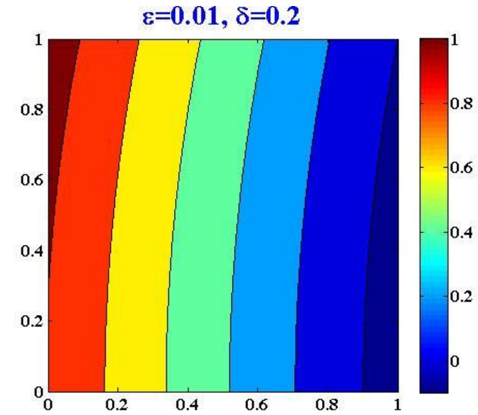


(c)

Figure 4.9: *Contours of pressure for δ=0.1 and ε=0.01, 0.1 and 0.2; Re=0, α=1.*



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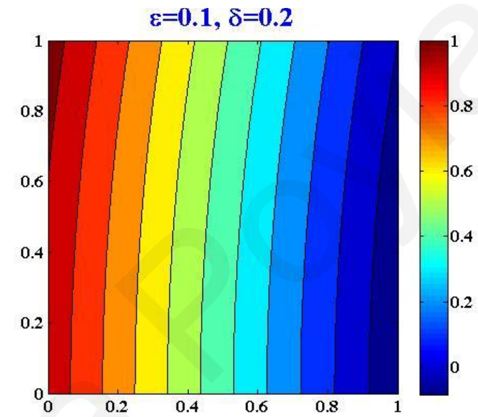

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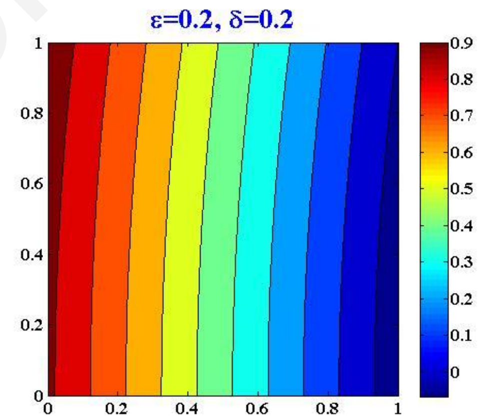
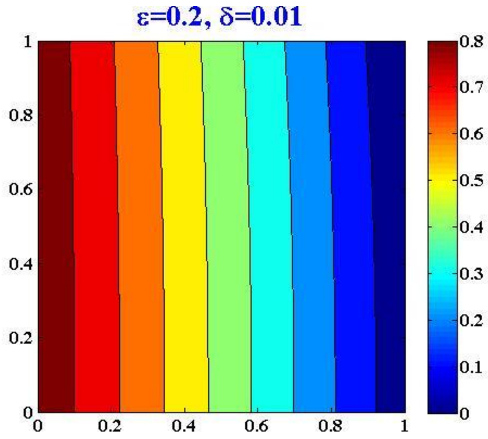


(c)

Figure 4.10: *Contours of pressure for* δ=0.*2 and* ε*=0.01, 0.1 and 0.2; Re=0,* α*=1.*



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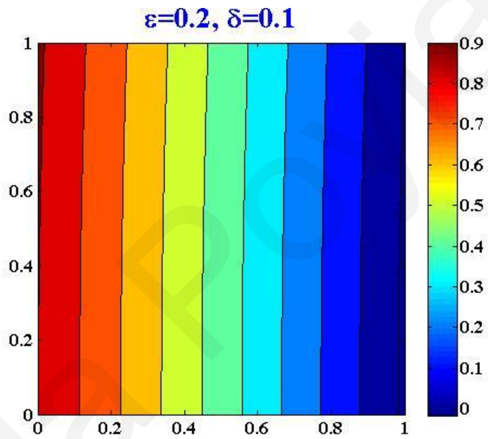

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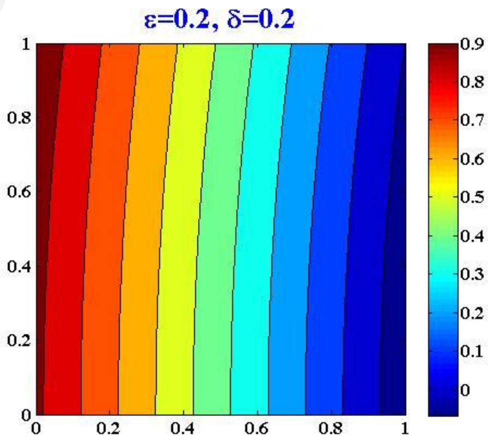


(c)

Figure 4.11: *Contours of pressure for ε=0.2 and δ=0.01, 0.1 and 0.2; Re=0, α=1.*

4.4.1 Volumetric flow rate and streamfunction

The volumetric flow rate,

$$Q(x) \equiv \int_0^1 u_x(x,y)\,dy \tag{4.88}$$

is given by

$$Q(x) = 1 - \varepsilon(1-x) + \varepsilon^2\left[\frac{3}{2}(1-x)^2 - \frac{18\alpha Re}{35}(1-x) - \frac{2\alpha^2}{35}\right] + \varepsilon\delta\left[-\frac{1}{2}(1-x)^2 + \frac{2\alpha^2}{5}\right]. \tag{4.89}$$

The streamfunction $\psi(x,y)$, defined by

$$\frac{\partial \psi}{\partial x} \equiv \rho u_y, \quad \frac{\partial \psi}{\partial y} \equiv -\rho u_x,$$

is found to be as follows:

$$\begin{aligned}\psi = & -\frac{1}{2}y\left(3-y^2\right) - \varepsilon\frac{3\alpha\,\mathrm{Re}}{280}\left(1-y^2\right)^2\left(-5+y^2\right) - \varepsilon^2\left[\frac{3\alpha\,\mathrm{Re}}{140}y\left(1-y^2\right)\left(5-y^2\right)(1-x)\right. \\ & \left. + \frac{\alpha^2}{8}y\left(1-y^2\right)^2 - \frac{\alpha^2\,\mathrm{Re}^2}{431200}y\left(1-y^2\right)^2\left(6579+1802y^2-1589y^4+168y^6\right)\right] - \delta^2\frac{\alpha^2}{20}y\left(1-y^2\right)^2 \\ & -\varepsilon\delta\left[-\frac{1}{2}y\left(3-y^2\right)(1-x)^2 + \frac{9\alpha\,\mathrm{Re}}{35}\left(3-y^2\right)(1-x) + \frac{\alpha^2}{120}y\left(9-y^2\right)\left(11-3y^2\right)\right] + h.o.t.\end{aligned} \tag{4.90}$$

4.4.2 Mean Pressure Drop

The mean pressure drop is a very useful quantity defined by

$$\overline{\Delta p} \equiv \overline{p}(0) - \overline{p}(1) \equiv \int_0^1\left[p(0,y) - p(1,y)\right]dy \tag{4.91}$$

which gives

$$\begin{aligned}\overline{\Delta p} = & 1 + \varepsilon\left(-\frac{1}{2} + \frac{18}{35}\alpha Re\right) + \frac{\delta}{2} + \varepsilon^2\left(\frac{1}{2} - \frac{5}{3}\alpha^2 - \frac{36}{35}\alpha\, Re + \frac{3044}{13475}\alpha^2\, Re^2\right) \\ & + \delta^2\left(\frac{1}{6} - \frac{2}{15}\alpha^2\right) + \varepsilon\delta\left(-\frac{2}{3} + \frac{61}{45}\alpha^2 + \frac{18}{35}\alpha\, Re\right)\end{aligned} \tag{4.92}$$

When $\varepsilon << \delta$:

$$\overline{\Delta p} = 1 + \frac{\delta}{2} + \delta^2\left(\frac{1}{6} - \frac{2}{15}a^2\right) + \varepsilon\left(-\frac{1}{2} + \frac{18}{35}\alpha\, Re\right)$$

When $\delta << \varepsilon$:

$$\overline{\Delta p} = 1 + \varepsilon\left(-\frac{1}{2} + \frac{18}{35}\alpha\, Re\right) + \varepsilon^2\left(\frac{1}{2} - \frac{5}{3}a^2 - \frac{36}{35}a\, Re + \frac{3044}{13475}a^2\, Re^2\right) + \frac{\delta}{2}.$$

Therefore, compared to the case of weakly compressible flow (see Taliadorou et al. (2009)) with constant viscosity the mean pressure drop increases by $\delta/2$. In Fig. 4.12(a) we plot $\overline{\Delta p}$ as a function of δ for fixed $\varepsilon = 0.1$ (Re=0, α=1). We see that $\overline{\Delta p}$ increases monotonically as the dependence of viscosity on pressure increases, while the compressibility has a constant value. (The discontinuity at $\delta = 0.045$ is due to the fact that as the relative size of ε and δ changes we have to use different asymptotic expansions for $\overline{\Delta p}$ as explained above.) In Fig. 4.12(b) we plot $\overline{\Delta p}$ as a function of ε for fixed $\delta = 0.1$ (Re=0, α=1). We see that $\overline{\Delta p}$ decreases monotonically as the compressibility increases, while the dependence of viscosity on the pressure has a constant value. (The discontinuity of the curve at $\varepsilon = 0.045$ is due to the fact that as the relative size of ε and δ changes we have to use different asymptotic expansions as explained above.)

By examining expression (4.92) we can again conclude that the effect of compressibility – $\varepsilon/2$ acts opposite to the effect due to the pressure-dependence of viscosity, $\delta/2$. When $\varepsilon \sim \delta$ the effects are two effects are almost cancelled out, when $\varepsilon >> \delta$ the mean pressure drop decreases and when $\varepsilon << \delta$ the mean pressure drop increases as compared to the mean pressure drop for an incompressible, constant-viscosity flow.

4.4.3. Validity of the asymptotic expansion

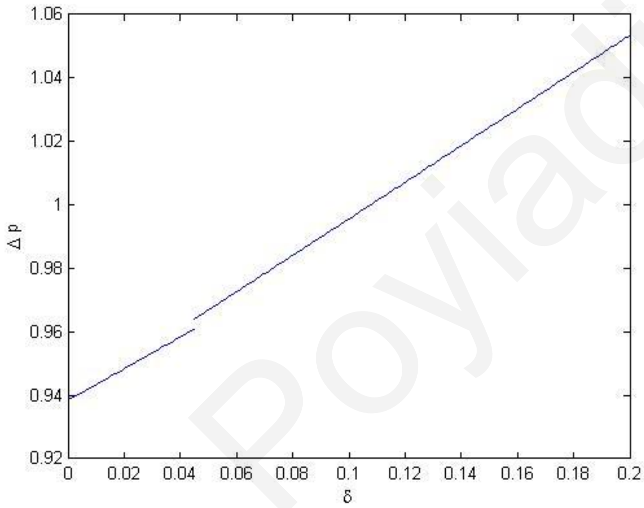
We always need to ensure that the parameter values we use in our plots do not violate the assumptions of the asymptotic expansions of the solution. We therefore need to ensure that the coefficients of all powers of ε and δ are of order 1. The values of α we consider are at most 1 since we are primarily interested in long channels. Therefore, we examine only the terms that involve the Reynolds number Re and obtain constraints on the value of Re so that the asymptotic expansions are valid.

Considering expression (4.92) we need to ensure that:

- $\frac{18}{35}Re = O(1)$ so, approximately $\frac{18}{35}Re \le 5$ so approximately $Re \le 10$.
- $\frac{36}{35}\alpha Re = O(1)$ so, approximately $\frac{36}{35}\alpha Re \le 5$ and therefore, approximately $Re \le \frac{5}{\alpha}$.

- $\frac{3044}{13475}\alpha^2 Re^2 = O(1)$ so, approximately $\frac{3044}{13475}\alpha^2 Re^2 \le 5$ and therefore, approximately $Re \le \frac{2}{\alpha}$.

Therefore the second constraint on the Reynolds number is less stringent than the third constraint, and, thus, not needed. Combining the first and the third constraint we conclude that when $\alpha \ge \frac{1}{5}$ we need to ensure $Re \le \frac{2}{\alpha}$ and when $\alpha \le \frac{1}{5}$ we need to ensure that $Re \le 10$. (These constraints are respected in all the plots in Section 4.4.5.)



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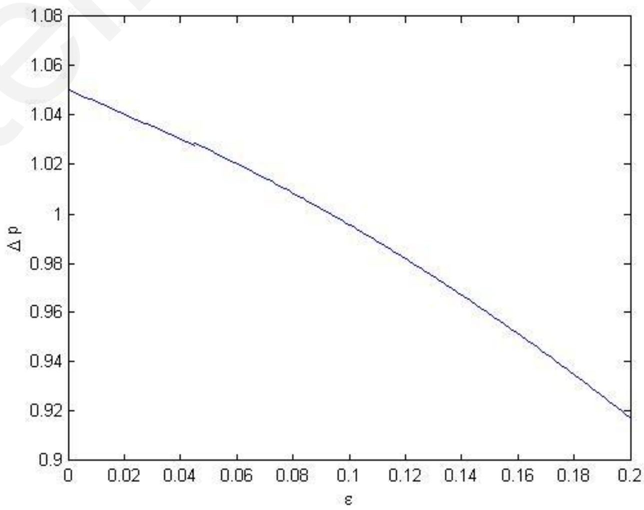

(b)

Figure 4.12: *The average pressure drop* $\overline{\Delta p}$ *for: (a) ε=0.1 and varying δ and (b) δ=0.1 and varying ε; Re=0, α=1*

4.5 Conclusions

In the present work we derive second-order perturbation solutions for the planar isothermal Poiseuille flows of weakly compressible Newtonian liquids where the viscosity also weakly depends on the pressure. A linear equation of state is employed and the isothermal compressibility and the viscosity pressure coefficient are taken as the perturbation parameters. The no slip boundary condition is assumed along the wall. (The shear and bulk viscosities are assumed to be zero.) The primary unknown variables are perturbed in the present work and explicit analytical solutions for pressure, density and velocity are obtained up to the second order. The derivation of the solution of the axisymmetric flow is provided in the Appendix. Our results extend previous work on the weakly compressible flow with constant viscosity and also on the incompressible flow with pressure-dependent viscosity. The effects of compressibility and the pressure dependence of viscosity, aspect ratio and Reynolds number on the velocity and pressure fields are analysed and discussed. When the compressibility number and the viscosity-pressure coefficient are of the same order, the viscosity-pressure coefficient perturbs the velocity component in the flow direction at the second-order only. The transverse velocity is not affected by the viscosity's pressure dependence at any order; it is only affected by compressibility. The pressure field is affected by compressibility and the viscosity's pressure-dependence at both the first order and the second order, and these two effects compete with each other.

Chapter 5

Summary and recommendations for future work

In this thesis, we have solved three different steady, laminar Poiseuille flows of Newtonian fluids: (a) the incompressible flow with viscosity varying with pressure; (b) the weakly compressible flow with Navier slip at the wall assuming a linear equation of state; and, (c) the weakly compressible flow with viscosity with pressure dependent viscosity.

In Chapter 2, we considered the unidirectional Poiseuille flow of an incompressible Newtonian fluid with viscosity that increases linearly with pressure. Under these assumptions we obtained semi-analytical solutions for the velocity and the pressure for the plane, the axisymmetric and the annular flows. The velocity and the pressure vary with $\alpha\varepsilon$ where α is the aspect ratio of the channel and ε is the dimensionless pressure-dependence coefficient. We observe that as $\alpha\varepsilon$ increases and approaches a critical value, the velocity which is independent of the axial coordinate, tends from a parabolic profile to a symmetric triangular one. The pressure, which depends on both the axial and the radial coordinate, increases exponentially as we move upstream. Thus, the pressure required to drive the flow increases rapidly with the length of the channel. At the inlet plane the pressure depends weakly on the radial coordinate for all values of $\alpha\varepsilon$ but the dependence becomes stronger towards the outlet plane. This effect is more noticeable for higher values of $\alpha\varepsilon$.

In Chapter 3, we obtained perturbation solutions for the plane and axisymmetric Poiseuille flows of a weakly compressible Newtonian fluid with wall slip. We assumed that slip obeys the linear Navier's slip equation and the density obeys a linear equation of state and used a regular perturbation method with the dimensionless isothermal compressibility number ε as the perturbation parameter and the primary flow fields as the dependent variables. Solutions have been obtained up to second order. Our results reveal that slip weakens the y-dependence of the solution. The transverse velocity decreases with slip and takes it maximum value near the middle-plane of the wall and the axis of symmetry. The horizontal velocity in a compressible flow increases faster when slip appears, in a smaller range of values. In addition, we noted that when the flow becomes compressible, a dependence on the horizontal coordinate appears which comes in contrast with the incompressible flow. Examining the effects of inertia we saw that when the Reynolds number obtains larger values for a flow in a short channel, the velocity increases slower compared to a small Reynolds number. This was not the case in a long channel. As for the pressure, we saw that it increases slower upstream when slip appears. As the flow becomes more compressible the required pressure drop decreases but this effect is weakened by the appearance of slip and by large values of the Reynolds number. Finally, we observed that the volumetric flow rate decreases faster with compressibility and larger values of the Reynolds number and this is more intense with slip.

In Chapter 4, we obtained perturbation solutions for the plane and axisymmetric steady, laminar Poiseuille flows of a weakly compressible Newtonian fluid with viscosity that is also weakly dependent on the pressure. As before, the density and the viscosity vary linearly with the pressure. Due to the nature of the flow it was reasonable to perform a perturbation analysis in terms of two small numbers: the isothermal compressibility number ε and the viscosity-pressure coefficient δ. This choice gave a double expansion for the solution which allowed ε and δ to be decoupled. We analytically derived the terms of the expansion up to the second order in terms of the two parameters. We then studied the combined effects of the compressibility and the viscosity. Our results extend previous work on the weakly compressible flow with constant viscosity and also on the incompressible flow with pressure-dependent viscosity. The effects of compressibility and the pressure dependence of viscosity, aspect ratio and Reynolds number on the velocity and pressure fields are analysed and discussed. When the compressibility number and the viscosity-pressure coefficient are of the same order, they perturb the velocity component in the flow direction at first- and second-order, respectively. The transverse velocity is not affected by the viscosity's pressure dependence at any order; it is only affected by compressibility. The pressure field is affected by compressibility and the viscosity's pressure-dependence at both the first order and the second order, and these two effects compete each other.

The results of this thesis, gave rise to questions and ideas for future work: Possible directions are the following:

(a) it would be interesting to study the flows of generalized Newtonian fluids with pressure-dependent material parameters, such as power law fluid with pressure-dependent consistency index and/or a Bingham fluid with pressure-dependent plastic viscosity and yield stress.

(b) It would be nice if the regular perturbation method developed in Chapter 3 be extended to Bingham fluids with or without slip at the wall. This flow is of great importance to waxy crude oil transport.

(c) In addition to the density and the viscosity, the slip coefficient may also be taken as a function of pressure.

Appendix A

Poiseuille flows with pressure-dependent viscosity and wall slip

In this Appendix we provide the solutions for the steady, two-dimensional, plane isothermal Poiseuille flow of an incompressible Newtonian fluid with pressure-dependent viscosity under zero gravity and zero bulk viscosity presented in chapter 2, replacing the no-slip boundary condition with by a linear slip condition:

$$\tau_w = \beta u_w, \tag{A.1}$$

where τ_w is the wall shear stress, β is the constant slip coefficient, and u_w is the slip velocity. The limiting case $\beta \to \infty$ corresponds to the no-slip boundary.

The governing equations are:

Continuity equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{A.2}$$

x-momentum

$$\rho\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) + 2\eta'(p)\frac{\partial p}{\partial x}\frac{\partial u_x}{\partial x} + \eta'(p)\frac{\partial p}{\partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{A.3}$$

y-momentum

$$\rho\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + 2\eta'(p)\frac{\partial p}{\partial y}\frac{\partial u_y}{\partial y} + \eta'(p)\frac{\partial p}{\partial x}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \tag{A.4}$$

We assume that the viscosity dependence obeys the linear equation of state

$$\eta = \eta_0\left(1 + \lambda p\right).$$

In order to nondimensionalize all the above equations, we scale x by the length of the channel L, y by the channel half-width H, η by the reference viscosity η_0 and the horizontal velocity u_x by the mean velocity U at the exit, defined by

$$U \equiv \frac{\dot{M}}{\rho H W},$$

where $\dot{M}$ is the mass flow rate and W is the unit length in the x-direction. The transversal velocity u_y is scaled by UH/L and finally the pressure p by $3\eta_0 LU/H^2$. The last scale is taken so that the dimensionless pressure gradient of the incompressible flow is equal to 1 Using the above scales the dimensionless equations turn out to be:

Continuity equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{A.5}$$

x-momentum

$$\begin{aligned} \alpha Re\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) &= -3\frac{\partial p}{\partial x} + \eta(p)\left(\alpha^2 \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \\ &+ \eta'(p)\left[2\alpha^2 \frac{\partial p}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial p}{\partial y}\left(\frac{\partial u_x}{\partial y} + \alpha^2 \frac{\partial u_y}{\partial x} \right) \right] \end{aligned} \tag{A.6}$$

y-momentum

$$\begin{aligned} \alpha^3 \mathrm{Re}\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \right) &= -3\frac{\partial p}{\partial y} + \alpha^2 \eta(p)\left(\alpha^2 \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) \\ &+ \alpha^2 \eta'(p)\left[\frac{\partial p}{\partial x}\left(\alpha^2 \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) + 2\frac{\partial p}{\partial y}\frac{\partial u_y}{\partial y} \right] \end{aligned} \tag{A.7}$$

Viscosity equation

$$\eta = 1 + \varepsilon p, \tag{A.8}$$

where the dimensionless numbers

$$a \equiv \frac{H}{L} \qquad Re \equiv \frac{\rho_0 H U}{\eta_0} \qquad \varepsilon \equiv \frac{\lambda \eta_0 B^* L U}{H^2}$$

are the aspect ratio of the channel, the Reynolds number and the isothermal compressibility, respectively.

The boundary conditions that complete the above equations are in their dimensionless form

$$\frac{\partial u_x}{\partial y}(x,0)=0,\ \ x\in[0,1] \tag{A.9}$$

$$\frac{\partial u_x}{\partial y}(x,1)=-Bu_x(x,1),\ \ x\in[0,1] \tag{A.10}$$

$$p(0,0)=0 \tag{A.11}$$

$$\int_0^1 u_x dy=1, \tag{A.12}$$

where we use for simplicity

$$B\equiv\frac{\beta H}{\eta_0}.$$

Under the assumption of one-dimensional flow, $u_y=0$, the continuity equation is simplified to

$$\frac{\partial u_x}{\partial x}=0. \tag{A.13}$$

The x-component of the momentum equation becomes

$$\alpha Reu_x\frac{\partial u_x}{\partial x}=-3\frac{\partial p}{\partial x}+\eta(p)\left(\alpha^2\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}\right)+\eta'(p)\left(2\alpha^2\frac{\partial p}{\partial x}\frac{\partial u_x}{\partial x}+\frac{\partial p}{\partial y}\frac{\partial u_x}{\partial y}\right) \tag{A.14}$$

and the y-component

$$-3\frac{\partial p}{\partial y}+\eta'(p)\alpha^2\frac{\partial p}{\partial x}\frac{\partial u_x}{\partial y}=0. \tag{A.15}$$

From the continuity equation (A.13) we get that $u_x=u_x(y)$. Considering this, as well as the state equation (A.8), the x-momentum and the y-momentum are simplified to

$$-3\frac{\partial p}{\partial x}+(1+\varepsilon p)\frac{d^2u_x}{dy^2}+\varepsilon\frac{\partial p}{\partial y}\frac{du_x}{dy}=0 \tag{A.16}$$

and

$$-3\frac{dp}{dy}+\varepsilon\alpha^2\frac{\partial p}{\partial x}\frac{du_x}{dy}=0, \tag{A.17}$$

respectively.

From Eq. (A.17) we get that

$$\frac{\partial p}{\partial y} = \frac{\varepsilon \alpha^2}{3} \frac{\partial p}{\partial x} \frac{du_x}{dy}.$$

When we substitute the above equation into Eq. (A.16) we obtain

$$\frac{3}{1+\varepsilon p} \frac{\partial p}{\partial x} = \frac{\dfrac{d^2 u_x}{dy^2}}{1 - \dfrac{\varepsilon^2 \alpha^2}{9} \left(\dfrac{du_x}{dy} \right)^2} = -A = const., \tag{A.18}$$

where A is a constant that we need to determine.To solve the above equations we set for simplicity $\partial u_x / \partial y \equiv f(y)$ and $E \equiv \varepsilon\alpha / 3$, and by using these, the second differential equation becomes

$$\frac{f'(y)}{1 - E^2 f^2(y)} = -A. \tag{A.19}$$

The solution of the above equation is

$$\frac{1}{E} \tanh^{-1}\left(Ef(y) \right) = -Ay + c,$$

which gives

$$f(y) = \frac{1}{E} \tanh\left(-AEy + c_1 \right). \tag{A.20}$$

Using the symmetry condition $\partial u_x / \partial y(x,0) = 0$, we get $f(0) = 0$ which gives that $c_1 = 0$. Therefore, Eq. (A.20) becomes

$$\frac{\partial u_x}{\partial y} = -\frac{1}{E} \tanh\left(AEy \right). \tag{A.21}$$

Integrating the above equation we get

$$u_x = -\frac{1}{AE^2} \ln\left[cosh\left(AEy \right) \right] + c_2.$$

By applying the slip condition (A.10) we get that

$$c_2 = \frac{1}{AE^2} \ln\left[\cosh\left(AE \right) \right] + \frac{1}{BE} \tanh\left(AE \right)$$

and the horizontal velocity component turns out to be

$$u_x(y) = -\frac{9}{A\varepsilon^2\alpha^2}\ln\left[\frac{\cosh\left(\frac{A\varepsilon\alpha}{3}y\right)}{\cosh\left(\frac{A\varepsilon\alpha}{3}\right)}\right] + \frac{3}{B\varepsilon\alpha}\tanh\left(\frac{A\varepsilon\alpha}{3}\right). \tag{A.22}$$

To find the pressure we solve the first differential equation of (A.18) and we obtain

$$p(x,y) = \frac{1}{\varepsilon}\left[C(y)e^{\frac{-A\varepsilon x}{3}} - 1\right]. \tag{A.23}$$

Differentiating *p* with respect to *x* and *y* we get that

$$\frac{\partial p}{\partial x} = -\frac{A}{3}C(y)e^{\frac{-A\varepsilon x}{3}} \tag{A.24}$$

and

$$\frac{\partial p}{\partial y} = \frac{1}{\varepsilon}C'(y)e^{\frac{-A\varepsilon x}{3}}, \tag{A.25}$$

respectively. When we substitute the above derivatives into Eq. (A.17) and using (A.21) we get the following ODE:

$$\frac{C'(y)}{C(y)} = AE\tanh(AEy).$$

The solution of this is

$$C(y) = c_3\cosh(AEy).$$

Condition (A.11) gives $C(0) = 0$ which leads to $c_3 = 1$ and the pressure turns out to be

$$p(x,y) = \frac{1}{\varepsilon}\left[\cosh\left(\frac{A\varepsilon\alpha}{3}y\right)e^{\frac{-A\varepsilon x}{3}} - 1\right].$$

The constant *A* is determined by demanding that the volumetric flow rate is equal to unity. It turns out that *A* is the root of

$$\int_0^1 \ln\left[\cosh\left(\frac{A\varepsilon\alpha}{3}y\right)\right]dy - \ln\left[\cosh\left(\frac{A\varepsilon\alpha}{3}\right)\right] - \frac{A\varepsilon\alpha}{3B}\tanh\left(\frac{A\varepsilon\alpha}{3}\right) + \frac{A\varepsilon^2\alpha^2}{9} = 0.$$

Therefore the analytical solution is given by:

$$u_x(y) = -\frac{9}{A\varepsilon^2\alpha^2}\ln\left[\frac{\cosh\left(\frac{A\varepsilon\alpha}{3}y\right)}{\cosh\left(\frac{A\varepsilon\alpha}{3}\right)}\right] + \frac{3}{B\varepsilon\alpha}\tanh\left(\frac{A\varepsilon\alpha}{3}\right)$$

$$p(x,y) = \frac{1}{\varepsilon}\left[\cosh\left(\frac{A\varepsilon\alpha}{3}y\right)e^{\frac{-A\varepsilon x}{3}} - 1\right],$$

where A is the solution of

$$\int_0^1 \ln\left[\cosh\left(\frac{A\varepsilon\alpha}{3}y\right)\right]dy - \ln\left[\cosh\left(\frac{A\varepsilon\alpha}{3}\right)\right] - \frac{A\varepsilon\alpha}{3B}\tanh\left(\frac{A\varepsilon\alpha}{3}\right) + \frac{A\varepsilon^2\alpha^2}{9} = 0.$$

Appendix B

Perturbation solutions of axisymmetric Poiseuille flow with wall slip

In this Appendix, we obtain the perturbation solutions up to the second order for the axisymmetric, steady, laminar, two-dimensional, isothermal Poiseuille flow of a weakly compressible Newtonian fluid with constant (pressure-independent) viscosity. We employ the isothermal compressibility as the perturbation parameter and derive analytically the perturbation solutions up to the second order.

B.1 Governing Equations and Boundary conditions

In this section, we derive the second order perturbation solution for the steady, two-dimensional, axisymmetric isothermal Poiseuille flow of a compressible Newtonian fluid with constant viscosity. The governing equations under zero bulk viscosity and zero gravity are:

the continuity equation,

$$\frac{\partial(\rho u_z)}{\partial z}+\frac{1}{r}\frac{\partial(r\rho u_r)}{\partial r}=0 \tag{B.1}$$

r-momentum

$$\rho\left(u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)+\frac{\partial^2 u_r}{\partial z^2}\right]+\frac{\eta}{3}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_z}{\partial z}\right] \tag{B.2}$$

z-momentum

$$\rho\left(u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{\partial^2 u_z}{\partial z^2}\right]+\frac{\eta}{3}\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_z}{\partial z}\right] \tag{B.3}$$

where p is the pressure, u_z is the axial velocity component, u_r is the radial velocity component, ρ is the density and η is the constant viscosity. We assume that the fluid obeys a linear equation of state

$$\rho = \rho_0 \left(1 + \kappa p\right), \tag{B.4}$$

where κ is the constant isothermal compressibility,. We also assume that it follows a liner slip equation along the wall

$$\tau_w = -\tau_{rz}\big|_{r=R} = \beta u_w ,$$

where R is the channel radius, β is the slip parameter which is a constant and u_w is the slip velocity.

We solve the problem under the following boundary conditions

$$u_r\left(z,0\right) = 0 \text{ and } \tau_{rz}\left(z,0\right) = \eta \frac{\partial u_z}{\partial r}\left(z,0\right) = 0 \Rightarrow \frac{\partial u_z}{\partial r}\left(z,0\right) = 0, \; z \in \left[0, L\right] \tag{B.5}$$

$$u_r\left(z,R\right) = 0 \text{ and } \tau_w = -\tau_{rz}\big|_{r=R} = \beta u_w, \; z \in \left[0, L\right] \tag{B.6}$$

$$p\left(L,R\right) = 0 \tag{B.7}$$

$$2\int_0^R \rho u_z r dr = \frac{\dot{M}}{\pi R}. \tag{B.8}$$

For the nondimensionalisation of Eqs. (B.1)-(B.4), we scale z by the length L of the channel, r by R, ρ by the reference density ρ_0 and the axial velocity u_z by the mean velocity at the exit U, defined as:

$$U = \frac{\dot{M}}{\rho_0 \pi R^2},$$

where $\dot{M}$ is the mass flow rate. The radial velocity u_r is scaled by UR/L and finally the pressure p by $8\eta LU/R^2$ so that the dimensionless pressure gradient of the incompressible flow with no-slip at the wall is equal to 1.

Imposing the scales on the governing equations (B.1)-(B.4) and on the boundary conditions (B.5)-(B.8) we find that the flow is governed by the following dimensionless equations and boundary conditions:

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho u_r\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)=0 \tag{B.9}$$

r-momentum

$$\alpha^3 \operatorname{Re}\rho\left(u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}\right)=-8\frac{\partial p}{\partial r}+\frac{4\alpha^2}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)\right)+\alpha^4\frac{\partial^2 u_r}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_z}{\partial r\partial z} \tag{B.10}$$

z-momentum

$$\alpha \operatorname{Re}\rho\left(u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)=-8\frac{\partial p}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)\right) \tag{B.11}$$

Equation of state

$$\rho=1+\varepsilon p \tag{B.12}$$

Boundary conditions

$$u_r\left(z,0\right)=0 \text{ and } \frac{\partial u_z}{\partial r}\left(z,0\right)=0 \tag{B.13}$$

$$u_r\left(z,1\right)=0 \text{ and } -\frac{\partial u_z}{\partial r}\left(z,1\right)=Bu_z\left(z,1\right) \tag{B.14}$$

$$p\left(1,1\right)=0 \tag{B.15}$$

$$2\int_0^1 \rho u_z r dr=1 \tag{B.16}$$

Dimensionless numbers

$$a\equiv\frac{R}{L},\qquad Re\equiv\frac{\rho_0 RU}{\eta},\qquad \varepsilon\equiv\frac{8\kappa\eta LU}{R^2},\qquad B\equiv\frac{\beta R}{\eta} \tag{B.17}$$

B.2 Perturbation method

We consider the compressibility number ε as the perturbation parameter so the expansions for all primary variables are:

$$\begin{aligned}
u_z &= u_z^{(0)}+\varepsilon u_z^{(0)}+\varepsilon^2 u_z^{(2)}+O\left(\varepsilon^3\right)\\
u_r &= u_r^{(0)}+\varepsilon u_r^{(1)}+\varepsilon^2 u_r^{(2)}+O\left(\varepsilon^3\right)\\
p &= p^{(0)}+\varepsilon p^{(1)}+\varepsilon^2 p^{(2)}+O\left(\varepsilon^3\right)\\
\rho &= \rho^{(0)}+\varepsilon\rho^{(1)}+\varepsilon^2\rho^{(2)}+O\left(\varepsilon^3\right).
\end{aligned} \tag{B.18}$$

As in the plane flow, we substitute equations (B.18) into the governing equations (B.9)-(B.12) and also into the boundary conditions (B.13)-(B.16) and we collect the terms of the same order in ε up to the second order.

The equations up to the second order along with the respective boundary conditions are tabulated in Tables B.1-B3.

Table B.1: *Zero-order equations and boundary conditions*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho^{(0)}u_r^{(0)}\right)+\frac{\partial}{\partial z}\left(\rho^{(0)}u_z^{(0)}\right)=0 \tag{B.19}$$

r-component of the momentum equation

$$\alpha^3 \operatorname{Re}\rho^{(0)}\left(u_r^{(0)}\frac{\partial u_r^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_r^{(0)}}{\partial z}\right)=-8\frac{\partial p^{(0)}}{\partial r}+\alpha^4\frac{\partial^2 u_r^{(0)}}{\partial z^2}+\frac{4\alpha^2}{3}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(0)}\right)\right]+\frac{\alpha^2}{3}\frac{\partial^2 u_z^{(0)}}{\partial r\partial z} \tag{B.20}$$

z-component of the momentum equation

$$\alpha \operatorname{Re}\rho^{(0)}\left(u_r^{(0)}\frac{\partial u_z^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_z^{(0)}}{\partial z}\right)=-8\frac{\partial \rho^{(0)}}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(0)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(0)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(0)}\right)\right] \tag{B.21}$$

Equation of state

$$\rho^{(0)}=1 \tag{B.22}$$

Boundary conditions

$$u_r^{(0)}\left(z,0\right)=0 \ \text{ and } \ \frac{\partial u_z^{(0)}}{\partial r}\left(z,0\right)=0, \ \ z\in\left[0,1\right] \tag{B.23}$$

$$u_r^{(0)}\left(z,1\right)=0 \ \text{ and } \ -\frac{\partial u_z^{(0)}}{\partial r}\left(z,1\right)=Bu_z^{(0)}\left(z,1\right), \ \ z\in\left[0,1\right] \tag{B.24}$$

$$p^{(0)}\left(1,1\right)=0 \tag{B.25}$$

$$2\int_0^1 \rho^{(0)}u_z^{(0)}rdr=1 \tag{B.26}$$

Table B.2: *First-order equations and boundary conditions*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\rho^{(0)}u_r^{(1)}+\rho^{(1)}u_r^{(0)}\right)\right]+\frac{\partial}{\partial z}\left(\rho^{(0)}u_z^{(1)}+\rho^{(1)}u_z^{(0)}\right)=0 \qquad \text{(B.27)}$$

r -component of the momentum equation

$$\alpha^3\,\mathrm{Re}\,\rho^{(1)}\left(u_r^{(0)}\frac{\partial u_r^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_r^{(0)}}{\partial z}\right)+\alpha^3\,\mathrm{Re}\,\rho^{(0)}\left(u_r^{(0)}\frac{\partial u_r^{(1)}}{\partial r}+u_r^{(1)}\frac{\partial u_r^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_r^{(1)}}{\partial z}+u_z^{(1)}\frac{\partial u_r^{(0)}}{\partial z}\right)$$
$$=-8\frac{\partial p^{(1)}}{\partial r}+\alpha^4\frac{\partial^2 u_r^{(1)}}{\partial z^2}+\frac{4\alpha^2}{3}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(1)}\right)\right]+\frac{a^2}{3}\frac{\partial^2 u_z^{(1)}}{\partial r\partial z}$$

(B.28)

z -component of the momentum equation

$$\alpha\,\mathrm{Re}\,\rho^{(1)}\left(u_r^{(0)}\frac{\partial u_z^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_z^{(0)}}{\partial z}\right)+\alpha\,\mathrm{Re}\,\rho^{(0)}\left(u_r^{(0)}\frac{\partial u_z^{(1)}}{\partial r}+u_r^{(1)}\frac{\partial u_z^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_z^{(1)}}{\partial z}+u_z^{(1)}\frac{\partial u_z^{(0)}}{\partial z}\right)$$
$$=-8\frac{\partial p^{(1)}}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(1)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(1)}}{\partial z^2}+\frac{a^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(1)}\right)\right)$$

(B.29)

Equation of state

$$\rho^{(1)}=p^{(0)} \qquad \text{(B.30)}$$

Boundary conditions

$$u_r^{(1)}(z,0)=0 \text{ and } \frac{\partial u_z^{(1)}}{\partial r}(z,0)=0,\ z\in[0,1] \qquad \text{(B.31)}$$

$$u_r^{(1)}(z,1)=0 \text{ and } -\frac{\partial u_z^{(1)}}{\partial r}(z,1)=Bu_z^{(1)}(z,1),\ z\in[0,1] \qquad \text{(B.32)}$$

$$p^{(1)}(1,1)=0 \qquad \text{(B.33)}$$

$$2\int_0^1\left(\rho^{(0)}u_z^{(1)}+\rho^{(1)}u_z^{(0)}\right)rdr=0 \qquad \text{(B.34)}$$

Table B.3: *Second-order equations and boundary conditions*

Continuity equation

$$\frac{1}{r}\frac{\partial}{dr}\left[r\left(\rho^{(0)}u_r^{(2)}+\rho^{1}u_r^{(1)}+\rho^{(2)}u_r^{(0)}\right)\right]+\frac{\partial}{\partial z}\left(\rho^{(0)}u_z^{(2)}+\rho^{(1)}u_z^{(1)}+\rho^{(2)}u_z^{(0)}\right)=0 \quad \text{(B.35)}$$

r-component of the momentum equation

$$a^3\,\mathrm{Re}\,\rho^{(2)}u_r^{(0)}\frac{\partial u_r^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_r^{(0)}}{\partial z}+a^3\,\mathrm{Re}\,\rho^{(1)}\left(u_r^{(0)}\frac{\partial u_r^{(1)}}{\partial r}+u_r^{(1)}\frac{\partial u_r^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_r^{(1)}}{\partial z}+u_z^{(1)}\frac{\partial u_r^{(0)}}{\partial z}\right)$$
$$+\rho^{(0)}u_r^{(0)}\frac{\partial u_r^{(2)}}{\partial r}+u_r^{(1)}\frac{\partial u_r^{(1)}}{\partial r}+u_r^{(2)}\frac{\partial u_r^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_r^{(0)}}{\partial z}+u_z^{(1)}\frac{\partial u_r^{(1)}}{\partial z}+u_z^{(2)}\frac{\partial u_r^{(0)}}{\partial z}$$
$$=-8\frac{\partial p^{(2)}}{\partial r}+\alpha^4\frac{\partial^2 u_r^{(2)}}{\partial z^2}+\frac{4\alpha^2}{3}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(2)}\right)\right]+\frac{a^2}{3}\frac{\partial^2 u_z^{(2)}}{\partial r\partial z}$$

(B.36)

z-component of the momentum equation

$$a\,\mathrm{Re}\,\rho^{(2)}\left(u_r^{(0)}\frac{\partial u_z^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_z^{(0)}}{\partial z}\right)+\rho^{(1)}\left(u_r^{(0)}\frac{\partial u_z^{(1)}}{\partial r}+u_r^{(1)}\frac{\partial u_z^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_z^{(1)}}{\partial z}+u_z^{(1)}\frac{\partial u_z^{(0)}}{\partial z}\right)$$
$$+\rho^{(0)}\left(u_r^{(0)}\frac{\partial u_z^{(2)}}{\partial r}+u_r^{(1)}\frac{\partial u_z^{(1)}}{\partial r}+u_r^{(2)}\frac{\partial u_z^{(0)}}{\partial r}+u_z^{(0)}\frac{\partial u_z^{(2)}}{\partial z}+u_z^{(1)}\frac{\partial u_r^{(1)}}{\partial z}+u_z^{(2)}\frac{\partial u_z^{(0)}}{\partial z}\right)$$
$$=-8\frac{\partial p^{(2)}}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(2)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(2)}}{\partial z^2}+\frac{a^2}{3}\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(2)}\right)\right]$$

(B.37)

Equation of state

$$\rho^{(2)}=p^{(1)} \quad \text{(B.38)}$$

Boundary conditions

$$u_r^{(2)}(z,0)=0 \text{ and } \frac{\partial u_z^{(2)}}{\partial r}(z,0)=0,\ z\in[0,1] \quad \text{(B.39)}$$

$$u_r^{(2)}(z,1)=0 \text{ and } -\frac{\partial u_z^{(2)}}{\partial r}(z,1)=Bu_z^{(2)}(z,1),\ z\in[0,1] \quad \text{(B.40)}$$

$$p^{(2)}(1,1)=0 \quad \text{(B.41)}$$

$$2\int_0^1\left(\rho^{(0)}u_z^{(2)}+\rho^{(1)}u_z^{(1)}+\rho^{(2)}u_z^{(0)}\right)rdr=0 \quad \text{(B.42)}$$

B.3 The solution up to the second order

The solution of the axisymmetric flow up to the second order is:

$$
\begin{aligned}
u_z(r,z) = {} & \frac{B^*}{4B}\left(B+2-Br^2\right) + \varepsilon\left[-\frac{B^{*2}}{32B}\left(B+2-Br^2\right)(1-z)\right. \\
& \left. + \frac{\alpha Re B^{*4}}{73728B^2}\left[-2\left(B^2+10B+24\right)+9\left(B^2+8B+16\right)r^2-9\left(B^2+6B+8\right)r^4+2\left(B^2+4B\right)r^6\right]\right] \\
& + \varepsilon^2\left[\frac{3B^{*3}}{512B}\left(B+2-Br^2\right)(1-z)^2 - \frac{\alpha Re B^{*5}}{196608B^4}\left[B^4+10B^3+72B^2+240B+192\right.\right. \\
& \qquad \left. +6\left(B^4+8B^3+12B^2-16B\right)r^2 - 9(B^4+6B^3+8B^2)r^4 + 2(B^4+4B^3)r^6\right](1-z) \\
& + \frac{\alpha^2 B^{*4}}{294912B^2}\left[B^2+32B-48-4\left(7B^2+48B\right)r^2+27\left(B^4+4B^3\right)r^4\right] \\
& + \frac{\alpha^2 Re^2 B^{*7}}{4529848300B^5}\left[43B^5+774B^4+1328B^3-42720B^2-268800B-460800\right. \\
& \qquad -200\left(5B^5+80B^4+360B^3-96B^2-4032B-6912\right)r^2 \\
& \qquad +100\left(33B^5+462B^4+2112B^3+2736B^2-3456B-6912\right)r^4 \\
& \qquad -1200\left(3B^5+36B^4+148B^3+224B^2+64B\right)r^6 \\
& \qquad \left.\left. +1425\left(B^5+10B^4+32B^3+32B^2\right)r^8 - 168\left(B^5+8B^4+16B^3\right)r^{10}\right]\right] + O\left(\varepsilon^3\right)
\end{aligned}
$$

$$
u_r(r) = \varepsilon^2\,\frac{\alpha Re B^{*5}}{1179648B^2}\,r\left(1-r^2\right)\left[4\left(B^2+10B+24\right)-\left(5B^2+32B+48\right)r^2+\left(B^2+4B\right)r^4\right]
$$

$$
\begin{aligned}
p(r,z) = {} & \frac{B^*}{8}(1-z) + \varepsilon\left[-\frac{B^{*2}}{128}(1-z)^2 + \frac{\alpha Re B^{*4}}{16384B^3}\left(B^3+8B^2+24B+32\right)(1-z)\right. \\
& \left. + \frac{\alpha^2 B^{*2}}{768}\left(1-r^2\right)\right] + \varepsilon^2\left[-\frac{B^{*3}}{1024}(1-z)^3 - \frac{\alpha Re B^{*5}}{65536B^3}\left(B^3+8B^2+24B+32\right)(1-z)^2\right. \\
& - \frac{\alpha^2 B^{*4}}{147456B^2}\left[29B^2+168B+288-9(B^2+4B)r^2\right](1-z) \\
& + \frac{\alpha^2 Re^2 B^{*7}}{113246208B^6}\left(2B^6+32B^5+267B^4+1332B^3+3672B^2+5184B+3456\right)(1-z) \\
& + \frac{\alpha^3 Re B^{*5}}{14155776B^3}\left[19B^3+202B^2+576B+288-18\left(3B^3+24B^2+52B+16\right)r^2\right. \\
& \qquad \left.\left. +45\left(B^3+6B^2+8B\right)r^4-10\left(B^3+4B^2\right)r^6\right]\right] + O\left(\varepsilon^3\right)
\end{aligned}
$$

$$
\rho(r,z) = 1+\varepsilon\frac{B^*}{8}(1-z)+\varepsilon^2\left[-\frac{B^{*2}}{128}(1-z)^2+\frac{\alpha Re B^{*4}}{16384B^3}\left(B^3+8B^2+24B+32\right)(1-z)+\frac{\alpha^2 B^{*2}}{768}\left(1-r^2\right)\right]+O\left(\varepsilon^3\right)
$$

Taking into consideration the limiting case $B \to \infty$ (no-slip case) we see that the solution agrees with the one found in Taliadorou et al. (2009) and Venerus and Bugajsky (2010).

The volumetric flow rate,

$$Q(z) \equiv 2\int_0^1 u_z(r,z)\,r\,dr \tag{B.43}$$

is given by

$$Q(z) = 1 - \varepsilon\frac{B^*}{8}(1-z) + \varepsilon^2\left[\frac{3B^{*2}}{128}(1-z)^2 - \frac{\alpha Re B^{*3}}{2048B^2}\left(B^2+4B+8\right)(1-z) - \frac{\alpha^2 B^{*3}}{9216B}(B+3)\right] + O\left(\varepsilon^3\right). \tag{B.44}$$

The mean pressure drop is obtained by

$$\overline{\Delta p} \equiv \bar{p}(0) - \bar{p}(1) \equiv 2\int_0^1 \left[p(0,r) - p(1,r)\right] r\,dr\,, \tag{B.45}$$

which gives

$$\begin{aligned}\overline{\Delta p} = \frac{B^*}{8} - \varepsilon\left[\frac{B^{*2}}{128} - \frac{\alpha Re B^{*3}}{2048B^2}\left(B^2+4B+8\right)\right] \\ + \varepsilon^2\left[\frac{B^{*3}}{1024} - \frac{\alpha Re B^{*4}}{8192B^2}\left(B^2+4B+8\right) - \frac{\alpha^2 B^{*4}}{294912B^2}\left(49B^2+300B+576\right)\right. \\ \left. + \frac{\alpha^2 Re^2 B^{*6}}{14155776B^5}\left(2B^5+24B^4+171B^3+648B^2+1080B+864\right)\right] + O\left(\varepsilon^3\right).\end{aligned} \tag{B.46}$$

The stream function $\psi(x,y)$ is defined by

$$\frac{\partial\psi}{\partial r} = \rho r u_z \quad \text{and} \quad \frac{\partial\psi}{\partial z} = -r\rho u_r\,. \tag{B.47}$$

Solving Eqs. (B.47) , we find that the stream function is given by

$$\begin{aligned}\psi(x,y) = \frac{B^*}{16B}r^2\left[2(B+2) - Br^2\right] + \varepsilon\frac{\alpha Re B^{*3}}{36864B}r^2\left(1-r^2\right)\left[-4(B+6) + (5B+12)r^2 - Br^4\right] \\ + \varepsilon^2\left[\frac{\alpha \mathrm{Re} B^{*4}}{147456B}r^2\left(1-r^2\right)\left[4(B+6) - (5B+12)r^2 + Br^4\right]\right. \\ + \frac{\alpha^2 B^{*4}}{589824B^2}r^2\left[13B^2+104B+48 - 2\left(13B^2+78B+24\right)r^2 + 13\left(B^2+3B\right)r^4\right] \\ + \frac{\alpha^2 Re^2 B^{*6}}{11324620800B^4}r^2\left[43B^4+602B^3-1080B^2-38400B-115200\right. \\ -100\left(5B^4+60B^3+120B^2-576B-1728\right)r^2 \\ +100\left(11B^4+110B^3+264B^2-144B-576\right)r^4 - 300\left(3B^4+24B^3+52B^2+16B\right)r^6 \\ \left.\left. +285\left(B^4+6B^3+8B^2\right)r^8 - 28\left(B^4+4B^3\right)r^{10}\right]\right] + O\left(\varepsilon^3\right)\end{aligned} \tag{B.48}$$

Appendix C
Weakly compressible axisymmetric Poiseuille flow with pressure-dependent viscosity

In this appendix, we consider the axisymmetric, steady, laminar, Poiseuille flow of a weakly compressible Newtonian fluid with a pressure-dependent viscosity employing linear equations of state for the density and for the viscosity. We perturb all primary variables using the dimensionless isothermal compressibility ε and the dimensionless viscosity-pressure coefficient δ as the perturbation parameters and thus obtaining a double expansion of the solution in terms of ε and δ.

C.1 Governing equations and boundary conditions

We assume as in chapter 4 that the flow is bidirectional with $u_\theta = 0$ with no bulk viscosity and with zero gravity. The governing equations and boundary conditions that describe this problem are listed below:

Density state equation

$$\rho = \rho_0 \left[1 + \kappa (p - p_0)\right], \tag{C.1}$$

where κ is the constant isothermal compressibility which is a measure of the ability of the material to change its volume under applied pressure at constant temperature. This is defined by

$$\kappa \equiv -\frac{1}{V_0}\left(\frac{\partial V}{\partial p}\right)_{p_0, T_0},$$

where V is the specific volume, ρ_0 and V_0 are respectively the density and the specific volume at the reference pressure, p_0, and temperature, T_0.

Viscosity equation

$$\eta = \eta_0 \left[1 + \lambda \left(p - p_0 \right) \right], \tag{C.2}$$

where η_0 is the viscosity at atmospheric pressure and λ is the temperature dependent pressure-viscosity coefficient.

Continuity equation

$$\frac{1}{r}\frac{\partial \left(r\rho u_r \right)}{\partial r} + \frac{\partial \left(\rho u_z \right)}{\partial z} = 0 \tag{C.3}$$

r-momentum

$$\begin{aligned} \rho\left(u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} \right) &= -\frac{\partial p}{\partial r} + \eta\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r \right) \right) + \frac{1}{3}\frac{\partial^2 u_z}{\partial r \partial z} + \frac{\partial^2 u_r}{\partial z^2} \right] \\ &\quad + 2\frac{\partial \eta}{\partial r}\left[\frac{\partial u_r}{\partial r} - \frac{1}{3}\frac{1}{r}\frac{\partial \left(r u_r \right)}{\partial r} - \frac{1}{3}\frac{\partial u_z}{\partial z} \right] + \frac{\partial \eta}{\partial z}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \end{aligned} \tag{C.4}$$

z-momentum

$$\begin{aligned} \rho\left(u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) &= -\frac{\partial p}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r} \right) + \frac{4}{3}\frac{\partial^2 u_z}{\partial z^2} + \frac{1}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r \right) \right) \right] \\ &\quad + 2\frac{\partial \eta}{\partial z}\left[\frac{2}{3}\frac{\partial u_z}{\partial z} - \frac{1}{3r}\frac{\partial}{\partial r}\left(r u_r \right) \right] + \frac{\partial \eta}{\partial r}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \end{aligned} \tag{C.5}$$

Boundary conditions

$$\frac{\partial u_z}{\partial r}\left(z,0 \right) = u_r\left(z,0 \right) = 0, \;\; z \in \left[0,L \right] \tag{C.6}$$

$$u_z\left(z,R \right) = u_r\left(z,R \right) = 0, \;\; z \in \left[0,L \right] \tag{C.7}$$

$$p\left(L,R \right) = 0 \tag{C.8}$$

$$2\int_0^R \rho u_z r dr = \frac{\dot{M}}{\pi R}. \tag{C.9}$$

For the nondimensionalisation of Eqs. (C.1)-(C.9), we scale z by the length L of the channel, r by the radius of the channel R, ρ by the reference density ρ_0, the axial velocity u_z by the mean velocity at the exit U,

$$U = \frac{\dot{M}}{\rho_0 \pi R^2},$$

where $\dot{M}$ is the mass flow rate and the radial velocity u_r is scaled by UR/L. The pressure p by $8\eta_0 LU/R^2$ which again is taken so that the dimensionless pressure gradient of the incompressible flow is equal to 1.

As before the asterisks in the following equations denote the dimensionless quantities.

Applying the scales mentioned above on the state equations (C.1) and (C.2) we find that

$$\rho = \rho_0(1+\kappa p) \Rightarrow \rho^*\rho_0 = \rho_0\left(1+\kappa\frac{8\eta_0 LU}{R^2}p^*\right) \Rightarrow \rho^* = 1+\varepsilon p^*$$

and

$$\eta = \eta_0(1+\lambda p) \Rightarrow \eta^*\eta_0 = \eta_0\left(1+\lambda\frac{8\eta_0 LU}{R^2}p^*\right) \Rightarrow \eta^* = 1+\delta p^*,$$

where

$$\varepsilon \equiv \frac{3\kappa\eta_0 LU}{R^2} \text{ and } \delta \equiv \frac{3\lambda\eta_0 LU}{R^2}$$

are the dimensionless compressibility number and the dimensionless viscosity-pressure coefficient respectively. The continuity equation (C.3) nondimensionalizes as follows

$$\frac{1}{R^2 r^*}\frac{\partial}{\partial r^*}\left(Rr^*\frac{UR}{L}\rho_0\rho^* U u_r^*\right)+\frac{\partial}{L\partial z^*}\left(\rho_0\rho^*\frac{U}{L}u_z^*\right)=0 \Rightarrow$$
$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\rho^* u_r^*\right)+\frac{\partial}{\partial z^*}\left(\rho^* u_z^*\right)=0.$$

The *r*-momentum equation (C.4) becomes

$$\rho_0\rho^*\left(\frac{U^2R^2}{L^2R}u_r^*\frac{\partial u_r^*}{\partial r^*}+\frac{U^2R}{L^2}u_z^*\frac{\partial u_r^*}{\partial z^*}\right)=$$
$$-\frac{8\eta_0 LU}{R^3}\frac{1}{R}\frac{\partial p^*}{\partial r^*}+\eta_0\eta^*\left[\frac{4}{3}\frac{UR^2}{R^3L}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*u_r^*\right)\right)+\frac{1}{3}\frac{U}{RL}\frac{\partial^2\partial u_z^*}{\partial r^*\partial z^*}+\frac{UR}{L^3}\frac{\partial^2 u_r^*}{\partial z^{*2}}\right]$$
$$+2\frac{\eta_0}{R}\frac{\partial\eta^*}{\partial r^*}\left[\frac{UR}{LR}\frac{\partial u_r^*}{\partial r^*}-\frac{1}{3}\frac{UR^2}{R^2L}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*u_r^*\right)-\frac{1}{3}\frac{U}{L}\frac{\partial u_z^*}{\partial z^*}\right]+\frac{\eta_0}{L}\frac{\partial\eta^*}{\partial z^*}\left(\frac{UR}{L^2}\frac{\partial u_r^*}{\partial z^*}+\frac{U}{R}\frac{\partial u_z^*}{\partial r^*}\right) \Rightarrow$$

$$\alpha^3 Re\rho^*\left(u_r^*\frac{\partial u_r^*}{\partial r^*}+u_z^*\frac{\partial u_r^*}{\partial z^*}\right)=-8\frac{\partial p^*}{\partial r^*}+\alpha^2\eta^*\left[\frac{4}{3}\frac{\partial}{\partial r^*}\left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*u_r^*\right)\right)+\frac{1}{3}\frac{\partial^2\partial u_z^*}{\partial r^*\partial z^*}+\alpha^2\frac{\partial^2 u_r^*}{\partial z^{*2}}\right]$$
$$+2\alpha^2\frac{\partial\eta^*}{\partial r^*}\left[\frac{\partial u_r^*}{\partial r^*}-\frac{1}{3r^*}\frac{\partial\left(r^*u_r^*\right)}{\partial r^*}-\frac{1}{3}\frac{\partial u_z^*}{\partial z^*}\right]+\alpha^2\frac{\partial\eta^*}{\partial z^*}\left(\alpha^2\frac{\partial u_r^*}{\partial z^*}+\frac{\partial u_z^*}{\partial r^*}\right),$$

where

$$\alpha \equiv \frac{R}{L} \text{ and } Re \equiv \frac{\rho_0 RU}{\eta_0}$$

are the aspect ratio of the channel and the Reynolds number respectively.

Similarly from the z-momentum Eq. (C.5) we have that

$$\rho_0 \rho^* \left(\frac{U^2 R}{LR} u_r^* \frac{\partial u_z^*}{\partial r} + \frac{U^2}{L} u_z^* \frac{\partial u_z^*}{\partial z^*} \right) = -\frac{8\eta_0 LU}{R^2 L} \frac{\partial p^*}{\partial z^*}$$
$$+ \eta_0 \eta^* \left[\frac{4}{3} \frac{RU}{R^3} \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_z^*}{\partial r^*} \right) + \frac{4}{3} \frac{U}{L^2} \frac{\partial^2 \partial u_z^*}{\partial z^{*2}} + \frac{1}{3} \frac{R^2 U}{L^2 R^2} \frac{\partial}{\partial z^*} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* u_r^* \right) \right) \right]$$
$$+ 2 \frac{\eta_0}{L} \frac{\partial \eta^*}{\partial z^*} \left[\frac{2}{3} \frac{U}{L} \frac{\partial u_z^*}{\partial z^*} - \frac{1}{3} \frac{R^2 U}{R^2 L} \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* u_r^* \right) \right] + \frac{\eta_0}{R} \frac{\partial \eta^*}{\partial r^*} \left(\frac{U}{R} \frac{\partial u_z^*}{\partial r^*} + \frac{UR}{L} \frac{\partial u_r^*}{\partial z^*} \right),$$

so we find the nondimensionalised *z*-momentum equation:

$$\alpha Re \rho^* \left(u_r^* \frac{\partial u_z^*}{\partial r} + u_z^* \frac{\partial u_z^*}{\partial z^*} \right) = -8 \frac{\partial p^*}{\partial z^*} + \eta^* \left[\frac{4}{3} \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_z^*}{\partial r^*} \right) + \frac{4\alpha^2}{3} \frac{\partial^2 \partial u_z^*}{\partial z^{*2}} + \frac{\alpha^2}{3} \frac{\partial}{\partial z^*} \left(\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* u_r^* \right) \right) \right]$$
$$+ 2\alpha^2 \frac{\partial \eta^*}{\partial z^*} \left[\frac{2}{3} \frac{\partial u_z^*}{\partial z^*} - \frac{1}{3r^*} \frac{\partial}{\partial r^*} \left(r^* u_r^* \right) \right] + \frac{\partial \eta^*}{\partial r^*} \left(\frac{\partial u_z^*}{\partial r^*} + \alpha^2 \frac{\partial u_r^*}{\partial z^*} \right).$$

Nondimensionalising the boundary conditions we get:

- *at the centreline*

$$\frac{UR}{L} u_r^* \left(z^*, 0 \right) = 0 \;\Rightarrow\; u_r^* \left(z^*, 0 \right) = 0$$

and

$$\frac{U}{R} \frac{\partial u_z^*}{\partial r^*} \left(z^*, 0 \right) = 0 \quad \Rightarrow \quad \frac{\partial u_z^*}{\partial r^*} \left(z^*, 0 \right) = 0$$

- *at the boundary*

$$u_r^* \left(z^*, 1 \right) = u_z^* \left(z^*, 1 \right) = 0$$

- *the pressure condition*

$$8 \frac{\eta_0 LU}{R^2} p^* \left(1, 1 \right) = 0 \quad \Rightarrow \quad p^* \left(1, 1 \right) = 0$$

- *the mass flow rate*

$$2 \int_0^1 \rho_0 \rho^* U u_z^* R r dr^* = \frac{\dot{M}}{\pi R} \;\; \Rightarrow 2 \int_0^1 \rho^* u_z^* r dr^* = 1.$$

In summary, the flow is governed by the following dimensionless equations and boundary conditions where the stars have been dropped:

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho u_r\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)=0 \qquad \text{(C.10)}$$

r-momentum

$$\alpha^3 Re\rho\left(u_r\frac{\partial u_r}{\partial r}+u_z\frac{\partial u_r}{\partial z}\right)=-8\frac{\partial p}{\partial r}+\alpha^2\eta\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)\right)+\frac{1}{3}\frac{\partial^2\partial u_z}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r}{\partial z^2}\right]$$
$$+2\alpha^2\frac{\partial\eta}{\partial r}\left[\frac{\partial u_r}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r\right)-\frac{1}{3}\frac{\partial u_z}{\partial z}\right]+\alpha^2\frac{\partial\eta}{\partial z}\left(\alpha^2\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right)$$

(C.11)

z-momentum

$$\alpha Re\rho\left(u_r\frac{\partial u_z}{\partial r}+u_z\frac{\partial u_z}{\partial z}\right)=-8\frac{\partial p}{\partial z}+\eta\left[\frac{4}{3}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2\partial u_z}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)\right)\right]$$
$$+2\alpha^2\frac{\partial\eta}{\partial z}\left[\frac{2}{3}\frac{\partial u_z}{\partial z}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r\right)\right]+\frac{\partial\eta}{\partial r}\left(\frac{\partial u_z}{\partial r}+\alpha^2\frac{\partial u_r}{\partial z}\right)$$

(C.12)

Equation of state and viscosity equation

$$\rho=1+\varepsilon p \quad \text{and} \quad \eta=1+\delta p \qquad \text{(C.13)}$$

Boundary conditions

$$\frac{\partial u_z}{\partial r}\left(z,0\right)=u_r\left(z,0\right)=0,\ \ z\in\left[0,1\right] \qquad \text{(C.14)}$$

$$u_z\left(z,1\right)=u_r\left(z,1\right)=0,\ \ z\in\left[0,1\right] \qquad \text{(C.15)}$$

$$p\left(1,1\right)=0 \qquad \text{(C.16)}$$

$$2\int_0^1\rho u_z r dr=1 \qquad \text{(C.17)}$$

Dimensionless numbers

$$\varepsilon\equiv\frac{3\kappa\eta_0 LU}{R^2},\qquad \delta\equiv\frac{3\lambda\eta_0 LU}{R^2},\qquad \alpha\equiv\frac{R}{L},\qquad Re\equiv\frac{\rho_0 RU}{\eta_0} \qquad \text{(C.18)}$$

C.2 Perturbation method

We consider the dimensionless numbers ε and δ as the perturbation parameters and we perform perturbation on all primary variables u_z, u_r, p, ρ and η:

$$\begin{aligned}
u_z &= u_z^{(00)} + \varepsilon u_z^{(10)} + \delta u_z^{(01)} + \varepsilon^2 u_z^{(20)} + \delta^2 u_z^{(02)} + \varepsilon\delta u_z^{(11)} + h.o.t. \\
u_r &= u_r^{(00)} + \varepsilon u_r^{(10)} + \delta u_r^{(01)} + \varepsilon^2 u_r^{(20)} + \delta^2 u_r^{(02)} + \varepsilon\delta u_r^{(11)} + h.o.t. \\
p &= p^{(00)} + \varepsilon p^{(10)} + \delta p^{(01)} + \varepsilon^2 p^{(20)} + \delta^2 p^{(02)} + \varepsilon\delta p^{(11)} + h.o.t. \\
\rho &= \rho^{(00)} + \varepsilon \rho^{(10)} + \delta \rho^{(01)} + \varepsilon^2 \rho^{(20)} + \delta^2 \rho^{(02)} + \varepsilon\delta \rho^{(11)} + h.o.t. \\
\eta &= \eta^{(00)} + \varepsilon \eta^{(10)} + \delta \eta^{(01)} + \varepsilon^2 \eta^{(20)} + \delta^2 \eta^{(02)} + \varepsilon\delta \eta^{(11)} + h.o.t.,
\end{aligned} \tag{C.19}$$

where *h.o.t.* stands for *higher order terms* which in this case are terms of $O\left(\varepsilon^3, \delta^3, \varepsilon^2\delta, \varepsilon\delta^2\right)$ and higher.

We substitute the above expansions into the governing equations and the boundary conditions (C.10)-(C.17) and collect the terms of the same order in ε and δ, thus obtaining differential systems for which analytical solutions have been of derived for all primary variables u_z, u_r, p, ρ and η, for the zero-order and for the orders ε, δ, ε^2, $\delta^2$and $\varepsilon\delta$.

The systems of orders 1, ε, δ, ε^2, δ^2 and $\varepsilon\delta$, that are formed from the above equations and boundary conditions are presented in Tables (C.1)-(C.6).

Table C.1 Zero-*order equations and boundary conditions*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho^{(00)}u_r^{(00)}\right)+\frac{\partial}{\partial z}\left(\rho^{(00)}u_z^{(00)}\right)=0 \qquad \text{(C.20)}$$

r-component of the momentum equation

$$\alpha^3 Re\rho^{(00)}\left(u_r^{(00)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(00)}}{\partial z}\right)=$$
$$-8\frac{\partial p^{(00)}}{\partial r}+\alpha^2\eta^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right)+\frac{1}{3}\frac{\partial^2\partial u_z^{(00)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(00)}}{\partial z^2}\right]$$
$$+2\alpha^2\frac{\partial\eta^{(00)}}{\partial r}\left[\frac{\partial u_r^{(00)}}{\partial r}-\frac{1}{3r}\frac{\partial\left(ru_r^{(00)}\right)}{\partial r}-\frac{1}{3}\frac{\partial u_z^{(00)}}{\partial z}\right]+\alpha^2\frac{\partial\eta^{(00)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(00)}}{\partial z}+\frac{\partial u_z^{(00)}}{\partial r}\right)$$

(C.21)

z-component of the momentum equation

$$\alpha Re\rho^{(00)}\left(u_r^{(00)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(00)}}{\partial z}\right)=$$
$$-8\frac{\partial p^{(00)}}{\partial z}+\eta\left[\frac{4}{3}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2\partial u_z^{(00)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right)\right] \qquad \text{(C.22)}$$
$$+2\alpha^2\frac{\partial\eta^{(00)}}{\partial z}\left[\frac{2}{3}\frac{\partial u_z^{(00)}}{\partial z}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right]+\frac{\partial\eta^{(00)}}{\partial r}\left(\frac{\partial u_z^{(00)}}{\partial r}+\alpha^2\frac{\partial u_r^{(00)}}{\partial z}\right)$$

State and viscosity equations

$$\rho^{(00)}=1 \text{ and } \eta^{(00)}=1 \qquad \text{(C.23)}$$

Boundary conditions

$$\frac{\partial u_z^{(00)}}{\partial r}(z,0)=u_r^{(00)}(z,0)=0,\ z\in[0,1] \qquad \text{(C.24)}$$

$$u_z^{(00)}(z,1)=u_r^{(00)}(z,1)=0,\ z\in[0,1] \qquad \text{(C.25)}$$

$$p^{(00)}(1,1)=0 \qquad \text{(C.26)}$$

$$2\int_0^1\rho^{(00)}u_z^{(00)}rdr=1 \qquad \text{(C.27)}$$

Table C.2 *Equations and boundary conditions of order ε*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\rho^{(00)}u_r^{(10)}+\rho^{(10)}u_r^{(00)}\right)\right]+\frac{\partial}{\partial z}\left(\rho^{(00)}u_z^{(10)}+\rho^{(10)}u_z^{(00)}\right)=0 \qquad \text{(C.28)}$$

r-component of the momentum equation

$$\begin{aligned}
&\alpha^3 Re\rho^{(00)}\left(u_r^{(00)}\frac{\partial u_r^{(10)}}{\partial r}+u_r^{(10)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(10)}}{\partial z}+u_z^{(10)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&+\alpha^3 Re\rho^{(10)}\left(u_r^{(00)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(10)}}{\partial r}\\
&+\alpha^2 n^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(10)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(10)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(10)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(00)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(00)}}{\partial z^2}\right]\\
&+2\alpha^2\frac{\partial n^{(00)}}{\partial r}\left[\frac{\partial u_r^{(10)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)-\frac{1}{3}\frac{\partial u_z^{(10)}}{\partial z}\right]+2\alpha^2\frac{\partial n^{(10)}}{\partial r}\left[\frac{\partial u_r^{(00)}}{\partial r}-\frac{1}{3r}\left(ru_r^{(00)}\right)-\frac{1}{3}\frac{\partial u_z^{(00)}}{\partial z}\right]\\
&+\alpha^2\frac{\partial n^{(00)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(10)}}{\partial z}+\frac{\partial u_z^{(10)}}{\partial r}\right)+\alpha^2\frac{\partial n^{(10)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(00)}}{\partial z}+\frac{\partial u_z^{(00)}}{\partial r}\right)
\end{aligned} \qquad \text{(C.29)}$$

z-component of the momentum equation

$$\begin{aligned}
&\alpha Re\rho^{(00)}\left(u_r^{(00)}\frac{\partial u_z^{(10)}}{\partial r}+u_r^{(10)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(10)}}{\partial z}+u_z^{(10)}\frac{\partial u_z^{(00)}}{\partial z}\right)\\
&+\alpha Re\rho^{(10)}\left(u_r^{(00)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(10)}}{\partial z}\\
&+n^{(00)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(10)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(10)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)\right)\right]\\
&+n^{(10)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(00)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_z^{(00)}\right)\right)\right]\\
&+\frac{2\alpha^2}{3}\frac{\partial n^{(00)}}{\partial z}\left[2\frac{\partial u_z^{(10)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)\right]+\frac{2\alpha^2}{3}\frac{\partial n^{(10)}}{\partial z}\left[2\frac{\partial u_z^{(00)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right]\\
&+\frac{\partial n^{(00)}}{\partial r}\left(\frac{\partial u_z^{(10)}}{\partial r}+\alpha^2\frac{\partial u_r^{(10)}}{\partial z}\right)+\frac{\partial n^{(10)}}{\partial r}\left(\frac{\partial u_z^{(00)}}{\partial r}+\alpha^2\frac{\partial u_r^{(00)}}{\partial z}\right)
\end{aligned} \qquad \text{(C.30)}$$

Table C.2 *(continued)*

State and viscosity equations

$$\rho^{(10)} = p^{(01)} \text{ and } \eta^{(10)} = 0 \tag{C.31}$$

Boundary conditions

$$\frac{\partial u_z^{(10)}}{\partial r}(z,0) = u_r^{(10)}(z,0) = 0, \;\; z \in [0,1] \tag{C.32}$$

$$u_z^{(10)}(z,1) = u_r^{(10)}(z,1) = 0, \;\; z \in [0,1] \tag{C.33}$$

$$p^{(10)}(1,1) = 0 \tag{C.34}$$

$$\int_0^1 \left(\rho^{(00)} u_z^{(10)} + \rho^{(10)} u_z^{(00)} \right) r dr = 0 \tag{C.35}$$

Table C.3 *Equations and boundary conditions of order δ*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\rho^{(00)}u_r^{(01)}+\rho^{(01)}u_r^{(00)}\right)\right]+\frac{\partial}{\partial z}\left(\rho^{(00)}u_z^{(01)}+\rho^{(01)}u_z^{(00)}\right)=0 \qquad \text{(C.36)}$$

r-component of the momentum equation

$$\begin{aligned}
&\alpha^3 Rep^{(00)}\left(u_r^{(00)}\frac{\partial u_r^{(01)}}{\partial r}+u_r^{(01)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(01)}}{\partial z}+u_z^{(01)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&+\alpha^3 Rep^{(01)}\left(u_r^{(00)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(01)}}{\partial r}\\
&+\alpha^2 n^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(01)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(01)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(01)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(00)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(00)}}{\partial z^2}\right]\\
&+2\alpha^2\frac{\partial n^{(00)}}{\partial r}\left[\frac{\partial u_r^{(01)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)-\frac{1}{3}\frac{\partial u_z^{(01)}}{\partial z}\right]+2\alpha^2\frac{\partial n^{(01)}}{\partial r}\left[\frac{\partial u_r^{(00)}}{\partial r}-\frac{1}{3r}\left(ru_r^{(00)}\right)-\frac{1}{3}\frac{\partial u_z^{(00)}}{\partial z}\right]\\
&+\alpha^2\frac{\partial n^{(00)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(01)}}{\partial z}+\frac{\partial u_z^{(01)}}{\partial r}\right)+\alpha^2\frac{\partial n^{(01)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(00)}}{\partial z}+\frac{\partial u_z^{(00)}}{\partial r}\right)
\end{aligned} \qquad \text{(C.37)}$$

z-component of the momentum equation

$$\begin{aligned}
&\alpha Re\rho^{(00)}\left(u_r^{(00)}\frac{\partial u_z^{(01)}}{\partial r}+u_r^{(01)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(01)}}{\partial z}+u_z^{(01)}\frac{\partial u_z^{(00)}}{\partial z}\right)\\
&+\alpha Re\rho^{(01)}\left(u_r^{(00)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(01)}}{\partial z}\\
&+n^{(00)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(01)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(01)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)\right)\right]\\
&+n^{(01)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(00)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_z^{(00)}\right)\right)\right]\\
&+\frac{2\alpha^2}{3}\frac{\partial n^{(00)}}{\partial z}\left[2\frac{\partial u_z^{(01)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)\right]+\frac{2\alpha^2}{3}\frac{\partial n^{(01)}}{\partial z}\left[2\frac{\partial u_z^{(00)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right]\\
&+\frac{\partial n^{(00)}}{\partial r}\left(\frac{\partial u_z^{(01)}}{\partial r}+\alpha^2\frac{\partial u_r^{(01)}}{\partial z}\right)+\frac{\partial n^{(01)}}{\partial r}\left(\frac{\partial u_z^{(00)}}{\partial r}+\alpha^2\frac{\partial u_r^{(00)}}{\partial z}\right)
\end{aligned} \qquad \text{(C.38)}$$

Table C.3 *(continued)*

State and viscosity equations

$$\rho^{(01)} = 0 \text{ and } \eta^{(01)} = p^{(10)} \tag{C.39}$$

Boundary conditions

$$\frac{\partial u_z^{(01)}}{\partial r}(z,0) = u_r^{(01)}(z,0) = 0, \ z \in [0,1] \tag{C.40}$$

$$u_z^{(01)}(z,1) = u_r^{(01)}(z,1) = 0, \ z \in [0,1] \tag{C.41}$$

$$p^{(01)}(1,1) = 0 \tag{C.42}$$

$$\int_0^1 \left(\rho^{(00)} u_z^{(01)} + \rho^{(01)} u_z^{(00)} \right) r dr = 0 \tag{C.43}$$

Table C.4 *Equations and boundary conditions of order ε^2*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\rho^{(00)}u_r^{(20)}+\rho^{(10)}u_r^{(10)}+\rho^{(20)}u_r^{(00)}\right)\right]+\frac{\partial}{\partial z}\left(\rho^{(00)}u_z^{(20)}+\rho^{(10)}u_r^{(10)}+\rho^{(20)}u_z^{(00)}\right)=0 \quad \text{(C.44)}$$

r-component of the momentum equation

$$\begin{aligned}
&\alpha^3 Rep^{(00)}\left(u_r^{(00)}\frac{\partial u_r^{(20)}}{\partial r}+u_r^{(10)}\frac{\partial u_r^{(10)}}{\partial r}+u_r^{(20)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(20)}}{\partial z}+u_z^{(10)}\frac{\partial u_r^{(10)}}{\partial z}+u_z^{(20)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&\alpha^3 Rep^{(10)}\left(u_r^{(00)}\frac{\partial u_r^{(10)}}{\partial r}+u_r^{(10)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(10)}}{\partial z}+u_z^{(10)}\frac{\partial u_r^{(00)}}{\partial z}+u_z^{(20)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&+\alpha^3 Rep^{(20)}\left(u_r^{(00)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(20)}}{\partial r}\\
&+\alpha^2 n^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(20)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(20)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(10)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(10)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(10)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(20)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(00)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(00)}}{\partial z^2}\right]\\
&+2\alpha^2\frac{\partial n^{(00)}}{\partial r}\left[\frac{\partial u_r^{(20)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)-\frac{1}{3}\frac{\partial u_z^{(20)}}{\partial z}\right]\\
&+2\alpha^2\frac{\partial n^{(10)}}{\partial r}\left[\frac{\partial u_r^{(10)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)-\frac{1}{3}\frac{\partial u_z^{(10)}}{\partial z}\right]\\
&+2\alpha^2\frac{\partial n^{(20)}}{\partial r}\left[\frac{\partial u_r^{(00)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)-\frac{1}{3}\frac{\partial u_z^{(00)}}{\partial z}\right]\\
&+\alpha^2\frac{\partial n^{(00)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(20)}}{\partial z}+\frac{\partial u_z^{(20)}}{\partial r}\right)+\alpha^2\frac{\partial n^{(10)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(10)}}{\partial z}+\frac{\partial u_z^{(10)}}{\partial r}\right)\\
&+\alpha^2\frac{\partial n^{(20)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(00)}}{\partial z}+\frac{\partial u_z^{(00)}}{\partial r}\right)
\end{aligned}$$

(C.45)

Table C.4 *(continued)*

z-component of the momentum equation

$$
\alpha Re\rho^{(00)}\left(u_r^{(00)}\frac{\partial u_z^{(20)}}{\partial r}+u_r^{(10)}\frac{\partial u_z^{(10)}}{\partial r}+u_r^{(20)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(20)}}{\partial z}+u_z^{(10)}\frac{\partial u_z^{(10)}}{\partial z}+u_z^{(20)}\frac{\partial u_z^{(00)}}{\partial z}\right)
$$

$$
+\alpha Re\rho^{(10)}\left(u_r^{(00)}\frac{\partial u_z^{(10)}}{\partial r}+u_r^{(10)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(10)}\frac{\partial u_z^{(00)}}{\partial z}+u_z^{(00)}\frac{\partial u_z^{(10)}}{\partial z}\right)
$$

$$
+\alpha Re\rho^{(20)}\left(u_r^{(00)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(20)}}{\partial z}
$$

$$
+n^{(00)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(20)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(20)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)\right)\right]
$$

$$
+n^{(10)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(10)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(10)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_z^{(10)}\right)\right)\right]
$$

$$
+n^{(20)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(00)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_z^{(00)}\right)\right)\right]
$$

$$
+\frac{2\alpha^2}{3}\frac{\partial n^{(00)}}{\partial z}\left[2\frac{\partial u_z^{(20)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)\right]+\frac{2\alpha^2}{3}\frac{\partial n^{(10)}}{\partial z}\left[2\frac{\partial u_z^{(10)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)\right]
$$

$$
+\frac{2\alpha^2}{3}\frac{\partial n^{(20)}}{\partial z}\left[2\frac{\partial u_z^{(00)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right]
$$

$$
+\frac{\partial n^{(00)}}{\partial r}\left(\frac{\partial u_z^{(20)}}{\partial r}+\alpha^2\frac{\partial u_r^{(20)}}{\partial z}\right)+\frac{\partial n^{(10)}}{\partial r}\left(\frac{\partial u_z^{(10)}}{\partial r}+\alpha^2\frac{\partial u_r^{(10)}}{\partial z}\right)+\frac{\partial n^{(20)}}{\partial r}\left(\frac{\partial u_z^{(00)}}{\partial r}+\alpha^2\frac{\partial u_r^{(00)}}{\partial z}\right)
\tag{C.46}
$$

State and viscosity equations

$$
\rho^{(20)}=p^{(10)}\ \text{ and }\ \eta^{(20)}=0 \tag{C.47}
$$

Boundary conditions

$$
\frac{\partial u_z^{(20)}}{\partial r}(z,0)=u_r^{(20)}(z,0)=0,\ \ z\in[0,1] \tag{C.48}
$$

$$
u_z^{(20)}(z,1)=u_r^{(20)}(z,1)=0,\ \ z\in[0,1] \tag{C.49}
$$

$$
p^{(20)}(1,1)=0 \tag{C.50}
$$

$$
\int_0^1\left(\rho^{(00)}u_z^{(20)}+\rho^{(10)}u_z^{(10)}+\rho^{(20)}u_z^{(00)}\right)rdr=0 \tag{C.51}
$$

Table C.5 *Equations and boundary conditions of order δ^2*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\rho^{(00)}u_r^{(02)}+\rho^{(01)}u_r^{(01)}+\rho^{(02)}u_r^{(00)}\right)\right]+\frac{\partial}{\partial z}\left(\rho^{(00)}u_z^{(02)}+\rho^{(01)}u_r^{(01)}+\rho^{(02)}u_z^{(00)}\right)=0 \quad \text{(C.52)}$$

r-component of the momentum equation

$$\begin{aligned}
&\alpha^3 Rep^{(00)}\left(u_r^{(00)}\frac{\partial u_r^{(02)}}{\partial r}+u_r^{(01)}\frac{\partial u_r^{(01)}}{\partial r}+u_r^{(02)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(02)}}{\partial z}+u_z^{(01)}\frac{\partial u_r^{(01)}}{\partial z}+u_z^{(01)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&\alpha^3 Rep^{(01)}\left(u_r^{(00)}\frac{\partial u_r^{(01)}}{\partial r}+u_r^{(01)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(01)}}{\partial z}+u_z^{(01)}\frac{\partial u_r^{(00)}}{\partial z}+u_z^{(02)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&+\alpha^3 Rep^{(02)}\left(u_r^{(00)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(02)}}{\partial r}\\
&+\alpha^2 n^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(02)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(02)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(01)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(01)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(01)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(02)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(00)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(00)}}{\partial z^2}\right]\\
&+2\alpha^2\frac{\partial n^{(00)}}{\partial r}\left[\frac{\partial u_r^{(02)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)-\frac{1}{3}\frac{\partial u_z^{(02)}}{\partial z}\right]\\
&+2\alpha^2\frac{\partial n^{(01)}}{\partial r}\left[\frac{\partial u_r^{(01)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)-\frac{1}{3}\frac{\partial u_z^{(01)}}{\partial z}\right]\\
&+2\alpha^2\frac{\partial n^{(02)}}{\partial r}\left[\frac{\partial u_r^{(00)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)-\frac{1}{3}\frac{\partial u_z^{(00)}}{\partial z}\right]\\
&+\alpha^2\frac{\partial n^{(00)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(02)}}{\partial z}+\frac{\partial u_z^{(02)}}{\partial r}\right)+\alpha^2\frac{\partial n^{(01)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(01)}}{\partial z}+\frac{\partial u_z^{(01)}}{\partial r}\right)\\
&+\alpha^2\frac{\partial n^{(02)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(00)}}{\partial z}+\frac{\partial u_z^{(00)}}{\partial r}\right)
\end{aligned}$$

(C.53)

Table C.5 *(continued)*

z-component of the momentum equation

$$\alpha Re\rho^{(00)}\left(u_r^{(00)}\frac{\partial u_z^{(02)}}{\partial r}+u_r^{(01)}\frac{\partial u_z^{(01)}}{\partial r}+u_r^{(02)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(02)}}{\partial z}+u_z^{(01)}\frac{\partial u_z^{(01)}}{\partial z}+u_z^{(02)}\frac{\partial u_z^{(00)}}{\partial z}\right)$$
$$+\alpha Re\rho^{(01)}\left(u_r^{(00)}\frac{\partial u_z^{(01)}}{\partial r}+u_r^{(01)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(01)}\frac{\partial u_z^{(00)}}{\partial z}+u_z^{(00)}\frac{\partial u_z^{(01)}}{\partial z}\right)$$
$$+\alpha Re\rho^{(02)}\left(u_r^{(00)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(02)}}{\partial z}$$
$$+n^{(00)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(02)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(02)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right)\right]$$
$$+n^{(01)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(01)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(01)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_z^{(01)}\right)\right)\right]$$
$$+n^{(02)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(00)}}{\partial z^2}+\frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_z^{(00)}\right)\right)\right]$$
$$+\frac{2\alpha^2}{3}\frac{\partial n^{(00)}}{\partial z}\left[2\frac{\partial u_z^{(02)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right]+\frac{2\alpha^2}{3}\frac{\partial n^{(01)}}{\partial z}\left[2\frac{\partial u_z^{(01)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)\right]$$
$$+\frac{2\alpha^2}{3}\frac{\partial n^{(02)}}{\partial z}\left[2\frac{\partial u_z^{(00)}}{\partial z}-\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right]$$
$$+\frac{\partial n^{(00)}}{\partial r}\left(\frac{\partial u_z^{(20)}}{\partial r}+\alpha^2\frac{\partial u_r^{(20)}}{\partial z}\right)+\frac{\partial n^{(01)}}{\partial r}\left(\frac{\partial u_z^{(01)}}{\partial r}+\alpha^2\frac{\partial u_r^{(01)}}{\partial z}\right)+\frac{\partial n^{(02)}}{\partial r}\left(\frac{\partial u_z^{(00)}}{\partial r}+\alpha^2\frac{\partial u_r^{(00)}}{\partial z}\right)$$

(C.54)

State and viscosity equations

$$\rho^{(02)}=0 \text{ and } \eta^{(02)}=p^{(01)} \quad \text{(C.55)}$$

Boundary conditions

$$\frac{\partial u_z^{(02)}}{\partial r}(z,0)=u_r^{(02)}(z,0)=0,\ \ z\in[0,1] \quad \text{(C.56)}$$

$$u_z^{(02)}(z,1)=u_r^{(02)}(z,1)=0,\ \ z\in[0,1] \quad \text{(C.57)}$$

$$p^{(02)}(1,1)=0 \quad \text{(C.58)}$$

$$\int_0^1\left(\rho^{(00)}u_z^{(02)}+\rho^{(01)}u_z^{(01)}+\rho^{(02)}u_z^{(00)}\right)rdr=0 \quad \text{(C.59)}$$

Table C.6 *Equations and boundary conditions of order εδ*

Continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(p^{(00)}u_r^{(11)}+p^{(01)}u_r^{(10)}+p^{(10)}u_r^{(01)}+p^{(11)}u_r^{(00)}\right)\right]+\frac{\partial}{\partial z}\left(p^{(00)}u_z^{(11)}+p^{(01)}u_z^{(10)}+p^{(10)}u_z^{(01)}+p^{(11)}u_z^{(00)}\right)=0 \tag{C.60}$$

r-component of the momentum equation

$$\begin{aligned}
&\alpha^3 Rep^{(00)}\left(u_r^{(00)}\frac{\partial u_r^{(11)}}{\partial r}+u_r^{(01)}\frac{\partial u_r^{(10)}}{\partial r}+u_r^{(10)}\frac{\partial u_r^{(01)}}{\partial r}+u_r^{(11)}\frac{\partial u_r^{(00)}}{\partial r}\right.\\
&\qquad\left.+u_z^{(00)}\frac{\partial u_r^{(11)}}{\partial z}+u_z^{(01)}\frac{\partial u_r^{(10)}}{\partial z}+u_z^{(10)}\frac{\partial u_r^{(01)}}{\partial z}+u_z^{(11)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&+\alpha^3 Rep^{(01)}\left(u_r^{(00)}\frac{\partial u_r^{(10)}}{\partial r}+u_r^{(10)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(10)}}{\partial z}+u_z^{(10)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&+\alpha^3 Rep^{(10)}\left(u_r^{(00)}\frac{\partial u_r^{(01)}}{\partial r}+u_r^{(01)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(01)}}{\partial z}+u_z^{(01)}\frac{\partial u_r^{(00)}}{\partial z}\right)\\
&+\alpha^3 Rep^{(11)}\left(u_r^{(00)}\frac{\partial u_r^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_r^{(00)}}{\partial z}\right)=-8\frac{\partial p^{(11)}}{\partial r}+\alpha^2 n^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(11)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(11)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(01)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(10)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(10)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(10)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(01)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(01)}}{\partial z^2}\right]\\
&+\alpha^2 n^{(11)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(00)}}{\partial r\partial z}+\alpha^2\frac{\partial^2 u_r^{(00)}}{\partial z^2}\right]\\
&+2\alpha^2\frac{\partial n^{(00)}}{\partial r}\left[\frac{\partial u_r^{(11)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right)-\frac{1}{3}\frac{\partial u_z^{(11)}}{\partial z}\right]+2\alpha^2\frac{\partial n^{(01)}}{\partial r}\left[\frac{\partial u_r^{(10)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(10)}\right)-\frac{1}{3}\frac{\partial u_z^{(10)}}{\partial z}\right]\\
&2\alpha^2\frac{\partial n^{(10)}}{\partial r}\left[\frac{\partial u_r^{(01)}}{\partial r}-\frac{1}{3r}\frac{\partial}{\partial r}\left(ru_r^{(01)}\right)-\frac{1}{3}\frac{\partial u_z^{(01)}}{\partial z}\right]+2\alpha^2\frac{\partial n^{(11)}}{\partial r}\left[\frac{\partial u_r^{(00)}}{\partial r}-\frac{1}{3r}\left(ru_r^{(00)}\right)-\frac{1}{3}\frac{\partial u_z^{(00)}}{\partial z}\right]\\
&+\alpha^2\frac{\partial n^{(00)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(11)}}{\partial z}+\frac{\partial u_z^{(11)}}{\partial r}\right)+\alpha^2\frac{\partial n^{(01)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(10)}}{\partial z}+\frac{\partial u_z^{(10)}}{\partial r}\right)\\
&+\alpha^2\frac{\partial n^{(10)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(01)}}{\partial z}+\frac{\partial u_z^{(01)}}{\partial r}\right)+\alpha^2\frac{\partial n^{(11)}}{\partial z}\left(\alpha^2\frac{\partial u_r^{(00)}}{\partial z}+\frac{\partial u_z^{(00)}}{\partial r}\right)
\end{aligned} \tag{C.61}$$

Table C.6 *(continued)*

z-component of the momentum equation

$$
\begin{aligned}
&\alpha Re\rho^{(00)}\left(u_r^{(00)}\frac{\partial u_z^{(11)}}{\partial r} + u_r^{(01)}\frac{\partial u_z^{(10)}}{\partial r} + u_r^{(10)}\frac{\partial u_z^{(01)}}{\partial r} + u_r^{(11)}\frac{\partial u_z^{(00)}}{\partial r} \right.\\
&\left. + u_z^{(00)}\frac{\partial u_z^{(11)}}{\partial z} + u_z^{(01)}\frac{\partial u_z^{(10)}}{\partial z} + u_z^{(10)}\frac{\partial u_z^{(01)}}{\partial z} + u_z^{(11)}\frac{\partial u_z^{(00)}}{\partial z} \right)\\
&+\alpha Re\rho^{(01)}\left(u_r^{(00)}\frac{\partial u_z^{(10)}}{\partial r} + u_r^{(10)}\frac{\partial u_z^{(00)}}{\partial r} + u_z^{(10)}\frac{\partial u_z^{(00)}}{\partial z} + u_z^{(00)}\frac{\partial u_z^{(10)}}{\partial z} \right)\\
&+\alpha Re\rho^{(10)}\left(u_r^{(00)}\frac{\partial u_z^{(01)}}{\partial r} + u_r^{(01)}\frac{\partial u_z^{(00)}}{\partial r} + u_z^{(00)}\frac{\partial u_z^{(01)}}{\partial z} + u_z^{(01)}\frac{\partial u_z^{(00)}}{\partial z} \right)\\
&+\alpha Re\rho^{(11)}\left(u_r^{(00)}\frac{\partial u_z^{(00)}}{\partial r} + u_z^{(00)}\frac{\partial u_z^{(00)}}{\partial z} \right) = -8\frac{\partial p^{(11)}}{\partial z}\\
&+n^{(00)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(12)}}{\partial r} \right) + \frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(11)}}{\partial z^2} + \frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)} \right) \right) \right]\\
&+n^{(01)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(10)}}{\partial r} \right) + \frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(10)}}{\partial z^2} + \frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_z^{(10)} \right) \right) \right]\\
&+n^{(10)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(01)}}{\partial r} \right) + \frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(01)}}{\partial z^2} + \frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(01)} \right) \right) \right]\\
&+n^{(11)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r} \right) + \frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(00)}}{\partial z^2} + \frac{\alpha^2}{3}\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_z^{(00)} \right) \right) \right]\\
&+\frac{2\alpha^2}{3}\frac{\partial n^{(00)}}{\partial z}\left[2\frac{\partial u_z^{(11)}}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)} \right) \right] + \frac{2\alpha^2}{3}\frac{\partial n^{(01)}}{\partial z}\left[2\frac{\partial u_z^{(10)}}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(10)} \right) \right]\\
&+\frac{2\alpha^2}{3}\frac{\partial n^{(10)}}{\partial z}\left[2\frac{\partial u_z^{(01)}}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(01)} \right) \right] + \frac{2\alpha^2}{3}\frac{\partial n^{(11)}}{\partial z}\left[2\frac{\partial u_z^{(00)}}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(00)} \right) \right]\\
&+\frac{\partial n^{(00)}}{\partial r}\left(\frac{\partial u_z^{(11)}}{\partial r} + \alpha^2\frac{\partial u_r^{(11)}}{\partial z} \right) + \frac{\partial n^{(01)}}{\partial r}\left(\frac{\partial u_z^{(10)}}{\partial r} + \alpha^2\frac{\partial u_r^{(10)}}{\partial z} \right)\\
&+\frac{\partial n^{(10)}}{\partial r}\left(\frac{\partial u_z^{(01)}}{\partial r} + \alpha^2\frac{\partial u_r^{(01)}}{\partial z} \right) + \frac{\partial n^{(11)}}{\partial r}\left(\frac{\partial u_z^{(00)}}{\partial r} + \alpha^2\frac{\partial u_r^{(00)}}{\partial z} \right)
\end{aligned}
\tag{C.62}
$$

State and viscosity equations

$$\rho^{(11)} = p^{(01)} \text{ and } \eta^{(11)} = p^{(10)} \tag{C.63}$$

Table C.6 *(continued)*

Boundary conditions

$$\frac{\partial u_z^{(11)}}{\partial r}(z,0) = u_r^{(11)}(z,0) = 0, \;\; z \in [0,1] \tag{C.64}$$

$$u_z^{(11)}(z,1) = u_r^{(11)}(z,1) = 0, \;\; z \in [0,1] \tag{C.65}$$

$$p^{(11)}(1,1) = 0 \tag{C.66}$$

$$\int_0^1 \left(\rho^{(00)} u_z^{(11)} + \rho^{(01)} u_z^{(10)} + \rho^{(10)} u_z^{(01)} + \rho^{(11)} u_z^{(00)} \right) r dr = 0 \tag{C.67}$$

C.2.1 Zero-order solution

From Eqs. (C.23) we have that

$$\rho^{(00)} = 1 \text{ and } \eta^{(00)} = 1 . \tag{C.68}$$

Assuming that $u_r^{(00)} = 0$, the continuity equation and the *r*- momentum equations, when integrated with respect to *z* and *r*, give respectively:

$$\frac{\partial u_z^{(00)}}{\partial x} = 0 \;\Rightarrow\; u_z^{(00)} = u_z^{(00)}(r),$$

$$-8\frac{\partial p^{(00)}}{\partial r} = 0 \;\Rightarrow\; p^{(00)} = p^{(00)}(z).$$

The *z*- momentum equation simplifies to

$$8\frac{\partial p^{(00)}}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r} \right) = A, \tag{C.69}$$

where *A* is an unknown constant. Integrating the first ODE of Eq. (C.69) twice, we get

$$u_z^{(00)} = \frac{A}{4} r^2 + c_1 \, log \, r + c_2 , \tag{C.70}$$

and integrating the first ODE of Eq. (C.69) we have that

$$p^{(00)} = -\frac{A}{8}(1-z) + c_3 , \tag{C.71}$$

where A, c_1, c_2, c_3 are constants to be determined.

Conditions $\partial u_z^{(00)}(z,0)/\partial r = 0$ and $p^{(00)}(1,1) = 0$ give $c_1 = c_3 = 0$. The no-slip condition and the mass flow rate condition give respectively

$$\frac{A}{4} + c_2 = 0 \text{ and } \frac{A}{8} + c_2 = 1. \tag{C.72}$$

The solution of the system of equations (C.72) is $A = -8$ and $c_2 = 2$. We substitute constants A, c_1, c_2 and and c_3 into Eqs.(C.70) and (C.71) and we find that the zero-order solution is:

$$\begin{aligned} u_z^{(00)} &= 2\left(1 - r^2\right) \\ u_r^{(00)} &= 0 \\ p^{(00)} &= 1 - z \\ \rho^{(00)} &= 1 \\ \eta^{(00)} &= 1 \end{aligned} \tag{C.73}$$

C.2.2 Solution of order ε

From equations (C.31) we get that

$$\rho^{(10)} = p^{(00)} = 1 - z \text{ and } \eta^{(10)} = 0, \tag{C.74}$$

and when we substitute all the known quantities into the continuity equation we find that

$$\frac{\partial}{\partial z}\left(u_z^{(10)} + \rho^{(10)} u_z^{(00)}\right) = 0.$$

We assume that the radial velocity component is equal to zero: $u_r^{(10)} = 0$ and integrating the above with respect to *z* and using $\rho^{(10)} = 1 - z$ we find that the horizontal velocity is

$$u_z^{(10)} = -2\left(1 - r^2\right)(1 - z) + F(r), \tag{C.75}$$

where $F(r)$ is an unknown function.

Using all the necessary quantities from Eq. (C.73) and Eqs. (C.74) the *r* momentum equation in Eq. (C.29) becomes

$$-8\frac{\partial p^{(10)}}{\partial r} + \frac{a^2}{3}\frac{\partial^2 u_z^{(10)}}{\partial r \partial z} = 0$$

and by integration with respect to *r* it gives

$$p^{(10)} = \frac{a^2}{12}\frac{\partial u_z^{(10)}}{\partial z} + G(z),$$

where $G(z)$ is an unknown function to be determined. Substituting Eq. (C.75) we find

$$p^{(10)} = \frac{a^2}{12}\left(1-r^2\right) + G(z).$$

Substituting all the known quantities in the *z*-momentum equation from Eq. (C.30) we obtain

$$\alpha\, Re\, u_z^{(00)}\frac{\partial u_z^{(10)}}{\partial z} = -8\frac{\partial p^{(10)}}{\partial z} + \eta^{(00)}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(10)}}{\partial r}\right) \Rightarrow$$

$$4\alpha\, Re\left(1-r^2\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}\left(rF'(r)\right) = -8G'(z) + 8(1-z) = A, \tag{C.76}$$

with *A* being a constant that we will determine. From the first ODE of (C.76)

$$4\alpha\, Re\left(1-r^2\right)^2 - \frac{1}{r}\frac{\partial}{\partial r}\left(rF'(r)\right) = A,$$

we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rF'(r)\right) = 4\alpha\, Re\left(1-r^2\right)^2 - A. \tag{C.77}$$

Integrating Eq. (C.77) we find

$$F'(r) = 2\alpha\, Re\left(r - r^3 + \frac{r^5}{3}\right) - \frac{A}{2}r + \frac{c_1}{r}, \tag{C.78}$$

where c_1 is an unknown constant. From the boundary condition $\partial u_z^{(10)}(z,0)/\partial r = 0$ we get $F'(0) = 0$, and so $c_1 = 0$. Integrating Eq. (C.78) we find

$$F(r) = \frac{\alpha\, Re}{18}\left(18r^2 - 9r^4 + 2r^6\right) - \frac{A}{4}r^2 + c_2, \tag{C.79}$$

with c_2 being an unknown constant. The condition $u_z^{(10)}(z,1) = 0$ gives $F(1) = 0$ so we get

$$\frac{A}{4} - c_2 = \frac{11\alpha Re}{18}. \tag{C.80}$$

From Eq. (C.35) we get $\int_0^1 F(r)\, r dr$ and this leads to

$$\frac{A}{8} - c_2 = \frac{13\alpha Re}{36}. \tag{C.81}$$

The solution of the system of Eqs. (C.80) and (C.81) is

$$A = 2\alpha Re \text{ and } c_2 = -\frac{\alpha Re}{9}. \tag{C.82}$$

Integrating the second ODE of (C.76) gives

$$G(z) = -\frac{1}{2}(1-z)^2 + \frac{A}{8}(1-z) + c_3,$$

where c_3 is an unknown constant The condition for the pressure $p^{(10)}(1,1) = 0$ leads to $G(1) = 0$ and we find that $c_3 = 0$.

Therefore the solution of order ε is:

$$\begin{aligned}
u_x^{(10)} &= -2(1-r^2)(1-z) - \frac{\alpha Re}{18}(1-r^2)(2-7r^2+2r^4) \\
u_y^{(10)} &= 0 \\
p^{(10)} &= -\frac{1}{2}(1-z)^2 + \frac{\alpha Re}{4}(1-z) + \frac{\alpha^2}{12}(1-r^2) \\
\rho^{(10)} &= 1-z \\
\eta^{(10)} &= 0
\end{aligned} \tag{C.83}$$

C.2.3 Solution of order δ

Eqs. (C.39) gives

$$\rho^{(01)} = 0 \text{ and } \eta^{(01)} = p^{(00)} = 1 - z. \tag{C.84}$$

The continuity equation in Eq. (C.36) becomes $\partial u_z^{(01)} / \partial z = 0$ and under the assumption that $u_r^{(01)} = 0$ we integrate the continuity equation to find

$$u_z^{(01)} = u_z^{(01)}(r) = F(r),$$

where $F(r)$ is an unknown function to be determined.

The r-momentum in Eq. (C.37)becomes

$$-8\frac{\partial p^{(01)}}{\partial r}+\alpha^2\frac{\partial \eta^{(01)}}{\partial z}\frac{\partial u_z^{(00)}}{\partial r}=0,$$

and by integrating the above equation with respect to *r* we find

$$p^{(01)}=-\frac{\alpha^2}{4}\left(1-r^2\right)+G(z),$$

with $G(z)$ being an unknown function.

The *z*-momentum simplifies to

$$-3\frac{\partial p^{(01)}}{\partial z}+\eta^{(00)}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(01)}}{\partial r}\right)+\eta^{(01)}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r}\right)=0, \tag{C.85}$$

and from this we get the two following ODEs

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rF'(r)\right)=8G'(z)+8(1-z)=A, \tag{C.86}$$

where *A* is an unknown constant. Integrating twice we find that the solution of the first ODE of Eq. (C.86) is

$$F(r)=\frac{A}{4}r^2+c_1\,log\,r+c_2, \tag{C.87}$$

where c_1 and c_2 are unknown constants. Integrating the second ODE we find

$$G(z)=\frac{1}{2}(1-z)^2-\frac{A}{8}(1-z)+c_3. \tag{C.88}$$

Applying the boundary condition $\partial u_z^{(01)}(z,0)/\partial r=0$ we get $F'(0)=0$ and imposing this condition on Eq. (C.87) we find that $c_1=0$. The boundary condition $u_z^{(01)}(z,1)=0$ when applied to Eq. (C.87) gives $F(1)=0$. Applying the mass flow rate condition gives

$$\int_0^1 F(r)\,rdr=0.$$

These two conditions give respectively

$$\frac{A}{4}+c_2=0 \text{ and } \frac{A}{8}+c_2=0.$$

The solution of the above system is $A=c_2=0$, so we conclude from Eq. (C.87) that $F(r)$ and therefore the horizontal velocity component is equal to zero.

The condition for the pressure gives $G(1)=0$ and from this we find that $c_3=0$.

Therefore, the solution of order δ is

$$\boxed{\begin{aligned} u_z^{(01)} &= 0 \\ u_r^{(01)} &= 0 \\ p^{(01)} &= \frac{1}{2}(1-z)^2-\frac{\alpha^2}{4}\left(1-r^2\right) \\ \rho^{(01)} &= 0 \\ \eta^{(01)} &= 1-z \end{aligned}} \tag{C.89}$$

C.2.4 Solution of order ε^2

As we can see from equation (C.47) the viscosity in this order is equal to zero and therefore does affect the flow. As expected the solution of order ε^2 is the same as the solution of order ε^2 in Taliadorou et al. (2009), so only the basic steps of the solution are presented. Here we assume that the radial velocity component in a function of r instead of being equal to zero so we have

$$u_r^{(20)}=u_r^{(20)}(r). \tag{C.90}$$

From Eqs. (C.47) we obtain

$$\rho^{(20)}=p^{(10)}=-\frac{1}{2}(1-z)^2+\frac{\alpha Re}{4}(1-z)+\frac{\alpha^2}{12}\left(1-r^2\right) \tag{C.91}$$

and

$$\eta^{(20)}=0.$$

The continuity equation (C.44) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho^{(00)}u_r^{(20)}\right)+\frac{\partial}{\partial z}\left(\rho^{(00)}u_z^{(20)}+\rho^{(10)}u_r^{(10)}+\rho^{(20)}u_z^{(00)}\right)=0.$$

Integrating with respect to x we get

$$u_z^{(20)}=3\left(1-r^2\right)(1-z)^2+\frac{\alpha Re}{18}\left(-7+9r^4-2r^6\right)(1-z)+\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)(1-z)-\frac{\alpha^2}{6}\left(1-r^2\right)^2+F(r), \tag{C.92}$$

where $F(r)$ is an unknown function. Conditions

$$u_z^{(20)}(z,1)=0 \quad \text{and} \quad \frac{\partial u_z^{(20)}}{\partial r}(z,0)=0$$

give respectively

$$(1-z)\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)\bigg|_{r=1}+F(1)=0 \text{ and } (1-z)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)\right)\bigg|_{r=0}+F'(0)=0 \quad \text{(C.93)}$$

and in order for these to apply for every z in our domain we must have that

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)\right)\bigg|_{r=0}=\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)\bigg|_{r=1}=0 \text{ and } F'(0)=F(1)=0 \ . \quad \text{(C.94)}$$

The r-momentum equation (C.45) becomes

$$-8\frac{\partial p^{(20)}}{\partial r}+\alpha^2\eta^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)\right)+\frac{1}{3}\frac{\partial^2 u_z^{(20)}}{\partial r\partial z}\right]=0$$

and integrating the above with respect to r we find

$$p^{(20)}=-\frac{\alpha^2}{4}\left(1-r^2\right)(1-z)-\frac{\alpha^3 Re}{432}\left(-7+9r^4-2r^6\right)+\frac{\alpha^2}{8}\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(20)}\right)+G(z), \quad \text{(C.95)}$$

with $G(z)$ being an unknown function. The x-momentum in Eq. (C.46) becomes

$$\alpha\, Re\,\rho^{(00)}\left(u_r^{(20)}\frac{\partial u_z^{(00)}}{\partial r}+u_z^{(00)}\frac{\partial u_z^{(20)}}{\partial z}+u_z^{(10)}\frac{\partial u_z^{(10)}}{\partial z}\right)+\alpha\, Re\,\rho^{(10)}u_z^{(00)}\frac{\partial u_z^{(10)}}{\partial z}=$$
$$-8\frac{\partial p^{(20)}}{\partial z}+n^{(00)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(20)}}{\partial r}\right)+\frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(20)}}{\partial z^2}\right].$$

(C.96)

Eqs. (C.91)-(C.96) are the same as those found in Taliadorou et al. (2009) so we immediately have the solution of order ε^2:

$$u_z^{(20)}=2\left(1-r^2\right)\left[\frac{3}{2}(1-z)^2-\frac{\alpha Re}{12}\left(1+7r^2-2r^4\right)(1-z)+\frac{\alpha^2}{144}\left(1-27r^2\right)\right.$$
$$\left.+\frac{\alpha^2 Re^2}{43200}\left(43-957r^2+2343r^4-1257r^6+168r^8\right)\right]$$
$$u_r^{(20)}=\frac{\alpha Re}{36}r\left(1-r^2\right)^2\left(4-r^2\right)$$
$$p^{(20)}=\frac{1}{2}(1-z)^3-\frac{\alpha Re}{2}(1-z)^2+\left[-\frac{\alpha^2}{36}\left(29-9r^2\right)+\frac{\alpha^2 Re^2}{27}\right](1-z)+\frac{\alpha^3 Re}{432}\left(1-r^2\right)\left(19-35r^2+10r^4\right)$$
$$\rho^{(20)}=-\frac{1}{2}(1-z)^2+\frac{\alpha Re}{4}(1-z)+\frac{\alpha^2}{12}\left(1-r^2\right)$$
$$\eta^{(20)}=0$$

(C.97)

C.2.5 Solution of order δ^2

From the equations of state in Eq. (C.55) we find that

$$\rho^{(02)} = 0 \tag{C.98}$$

and

$$\eta^{(02)} = p^{(01)} = \frac{1}{2}(1-z)^2 - \frac{\alpha^2}{4}(1-r^2). \tag{C.99}$$

From the continuity equation (C.52) is simplified to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\rho^{(00)}u_r^{(02)}\right) + \frac{\partial}{\partial z}\left(\rho^{(00)}u_z^{(02)}\right) = 0. \tag{C.100}$$

We assume that the radial velocity is a function of *r* so we have

$$u_r^{(02)} = u_r^{(02)}(r) \tag{C.101}$$

and when we integrate Eq. (C.100) with respect to *r* the solution of Eq. (C.100) for the horizontal velocity component is

$$u_z^{(02)} = \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)(1-z) + F(r), \tag{C.102}$$

where $F(r)$ is an unknown function. Conditions $u_z^{(02)}(z,1) = 0$ and $\partial u_z^{(02)}(z,0)/\partial r = 0$ give respectively

$$(1-z)\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\bigg|_{r=1} + F(1) = 0 \text{ and } (1-z)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right)\bigg|_{r=0} + F'(0) = 0 \tag{C.103}$$

and in order for these to apply for every *z* in our domain we must have that

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right)\bigg|_{r=0} = \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\bigg|_{r=1} = 0 \text{ and } F'(0) = F(1) = 0\ . \tag{C.104}$$

The *r*-component of the momentum equation in Eq. (C.53) is simplified to

$$-8\frac{\partial p^{(02)}}{\partial r} + \alpha^2 n^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right) + \frac{1}{3}\frac{\partial^2 u_z^{(02)}}{\partial r\partial z}\right] + \alpha^2\frac{\partial n^{(02)}}{\partial z}\frac{\partial u_z^{(00)}}{\partial r} = 0.$$

We solve the above equation for $p^{(02)}$ by integrating with respect to *r* to find

$$p^{(02)} = \frac{\alpha^2}{6}\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right) + \frac{\alpha^2}{24}\frac{\partial u_z^{(02)}}{\partial z} + \frac{\alpha^2}{8}\frac{\partial n^{(02)}}{\partial z}u_z^{(00)} + G(z), \tag{C.105}$$

with $G(z)$ being an unknown function. By replacing all the known quantities into Eq. (C.105) we get

$$p^{(02)} = \frac{\alpha^2}{8}\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right) + \frac{\alpha^2}{4}\left(1-r^2\right)\left(1-z\right) + G\left(z\right). \tag{C.106}$$

By replacing all the known quantities, the *z*-component of the momentum equation is simplified to

$$\alpha \operatorname{Re} \rho^{(00)} \left(u_r^{(02)} \frac{\partial u_z^{(00)}}{\partial r} + u_z^{(00)} \frac{\partial u_z^{(02)}}{\partial z} \right) = -8 \frac{\partial p^{(02)}}{\partial z} + n^{(00)} \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z^{(02)}}{\partial r} \right) + n^{(02)} \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z^{(00)}}{\partial r} \right) + \frac{\partial n^{(02)}}{\partial r}\frac{\partial u_z^{(00)}}{\partial r} \tag{C.107}$$

By substituting all the known quantities into Eq. (C.107) results in

$$-4\alpha\, Re\, ru_r^{(02)} - 2\alpha\, Re\left(1-r^2\right)\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(rF'\left(r\right)\right) + 2\alpha^2 r^2 = -8G'\left(z\right) - 4\left(1-z\right)^2 + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right)\right)\left(1-z\right) \tag{C.108}$$

In order to be able to separate variables we demand that the last term of Eq. (C.108) is a scalar multiple of $\left(1-z\right)$ therefore we set

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right)\right) = \gamma, \tag{C.109}$$

with γ being an unknown constant. By integrating Eq. (C.109) once, we get

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right) = \frac{\gamma}{2}r^2 + c_1, \tag{C.110}$$

where c_1 is an unknown constant. Applying the condition

$$\left.\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right)\right|_{r=0} = 0,$$

we find $c_1 = 0$. By integrating Eq. (C.110) twice, we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right) = \frac{\gamma}{4}r^2 + c_2 \tag{C.111}$$

and

$$ru_r^{(02)} = \frac{\gamma}{16} r^4 + \frac{c_2}{2} r^2 + c_3, \quad \text{(C.112)}$$

where c_2 and c_3 are unknown constants. Applying the boundary condition $u_r^{(02)}(z,0) = 0$ on Eq. (C.112) we get $c_3 = 0$ and applying the conditions

$$u_r^{(02)}(z,1) = 0 \ \text{ and } \ \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\bigg|_{r=1} = 0,$$

we get

$$\frac{\gamma}{8} + c_2 = 0 \ \text{ and } \ \frac{\gamma}{4} + c_2 = 0, \quad \text{(C.113)}$$

respectively. Solving the system in Eq. (C.113) we obtain $\gamma = c_2 = 0$, thus the radial velocity component is equal to zero:

$$u_r^{(02)} = 0.$$

Since the radial velocity component is equal to zero, Eq. (C.108) is simplified to

$$2\alpha^2 r^2 - \frac{1}{r}\frac{\partial}{\partial r}\left(rF'(r)\right) = -8G'(z) - 4(1-z)^2 = A, \quad \text{(C.114)}$$

where A is an unknown constant. Integrating the first ODE of Eq. (C.114) twice with respect to r we find

$$rF'(r) = \frac{\alpha^2}{2} r^4 - \frac{A}{2} r^2 + c_4$$

and

$$F(r) = \frac{\alpha^2}{8} r^4 - \frac{A}{4} r^2 + c_5, \quad \text{(C.115)}$$

where c_4, and c_5 are unknown constants. Imposing the condition $F'(0) = 0$ we get that $c_4 = 0$ and imposing the condition $F(1) = 0$ we get

$$\frac{A}{2} - c_5 = \frac{\alpha^2}{4}. \quad \text{(C.116)}$$

The mass flow rate condition (C.59) yields the condition $\int_0^1 F(r)\,rdr = 0$ which, when imposed to Eq. (C.115) leads to

$$\frac{A}{8} - c_5 = \frac{\alpha^2}{24}. \qquad \text{(C.117)}$$

The solution of the system of Eqs. (C.116) and (C.117) is $A = \frac{2\alpha^2}{3}$ and $c_5 = \frac{\alpha^2}{24}$. So, we find that

$$u_z^{(02)} = F(r) = \frac{\alpha^2}{24}\left(1-r^2\right)\left(1-3r^2\right).$$

The solution of the second ODE of Eq. (C.114) is

$$G(z) = \frac{1}{6}(1-z)^3 - \frac{A}{8}(1-z) + c_6, \qquad \text{(C.118)}$$

where c_6 is an unknown constant. Applying the condition $p^{(02)}(1,1) = 0$ on Eq. (C.115) we get $G(1) = 0$. Applying this on Eq. (C.118) we find that $c_6 = 0$. Finally, from Eqs. (C.118)and (C.106) we find that

$$p^{(02)} = \frac{1}{6}(1-z)^3 - \frac{\alpha^2}{12}\left(2-3r^2\right)(1-z).$$

Hence the solution of order δ^2 is given by

$$\begin{aligned}
u_z^{(02)} &= \frac{\alpha^2}{24}\left(1-r^2\right)\left(1-3r^2\right)\\
u_r^{(02)} &= 0\\
p^{(02)} &= \frac{1}{6}(1-z)^3 - \frac{\alpha^2}{12}\left(2-3r^2\right)(1-z)\\
\rho^{(02)} &= 0\\
\eta^{(02)} &= \frac{1}{2}(1-z)^2 - \frac{\alpha^2}{4}\left(1-r^2\right)
\end{aligned}$$

(3.119)

C.2.6 Solution of order εδ

We assume that $u_r^{(11)} = u_r^{(11)}(r)$. From the equation of state and the equation of viscosity in Eq. (C.63) we find that

$$\rho^{(11)} = \frac{1}{2}(1-z)^2 - \frac{\alpha^2}{4}\left(1-r^2\right) \qquad \text{(C.120)}$$

and

$$\eta^{(11)} = -\frac{1}{2}(1-z)^2 + \frac{\alpha Re}{4}(1-z) + \frac{\alpha^2}{12}(1-r^2). \tag{C.121}$$

From the continuity equation we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right) + \frac{\partial}{\partial z}\left(u_z^{(11)} + \rho^{(11)} u_z^{(00)}\right) = 0 . \tag{C.122}$$

Integrating equation (C.122) with respect to z we obtain

$$u_z^{(11)} = -\rho^{(11)} u_z^{(00)} + \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right)(1-z) + F(r). \qquad \text{(C.123).}$$

Substituting $u_z^{(00)}$ and $\rho^{(11)}$ from Eq. (C.73) and Eq. (C.120) we find

$$u_z^{(11)} = -\left(1-r^2\right)(1-z)^2 + \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right)(1-z) + \frac{\alpha^2}{2}\left(1-r^2\right)^2 + F(r). \tag{C.124}$$

Conditions $u_z^{(11)}(z,1) = 0$ and $\frac{\partial u_z^{(11)}}{\partial r}(z,0) = 0$ give respectively

$$(1-z)\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right)\Bigg|_{r=1} + F(1) = 0 \;\text{ and }\; (1-z)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right)\right)\Bigg|_{r=0} + F'(0) = 0 \tag{C.125}$$

and in order for these to apply for every z in our domain we must have that

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right)\right)\Bigg|_{r=0} = \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right)\Bigg|_{r=1} = 0 \;\text{ and }\; F'(0) = F(1) = 0 . \tag{C.126}$$

The r-component of the momentum equation in Eq. (C.61) is simplified to

$$-8\frac{\partial p^{(11?)}}{\partial r} + \alpha^2 n^{(00)}\left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right)\right) + \frac{1}{3}\frac{\partial^2 u_z^{(11)}}{\partial r \partial z}\right] + \frac{\alpha^2}{3} n^{(01)} \frac{\partial^2 u_z^{(10)}}{\partial r \partial z}$$
$$+\alpha^2 \frac{\partial n^{(01)}}{\partial z}\frac{\partial u_z^{(10)}}{\partial r} + \alpha^2 \frac{\partial n^{(11)}}{\partial z}\frac{\partial u_z^{(00)}}{\partial r} = 0$$

We solve the above equation for $p^{(11)}$ by integrating with respect to r to find

$$p^{(11)} = \frac{\alpha^2}{6}\frac{1}{r}\frac{\partial}{\partial r}\left(r u_r^{(11)}\right) + \frac{\alpha^2}{24}\frac{\partial u_z^{(11)}}{\partial z} + \frac{\alpha^2}{24}\frac{\partial u_z^{(10)}}{\partial z} - \frac{\alpha^2}{8} u_z^{(10)} + \frac{\alpha^2}{8}\left[(1-z) - \frac{\alpha Re}{4}\right] u_z^{(00)} + G(z). \tag{C.127}$$

By replacing all the known quantities in Eq. (C.127) we get

$$p^{(11)} = \frac{2\alpha^2}{2}\left(1-r^2\right)\left(1-z\right) + \frac{\alpha^2}{8}\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right) + \frac{\alpha^3 \,\mathrm{Re}}{144}\left(-7+9r^4-2r^6\right) + G\left(z\right). \quad \text{(C.128)}$$

The z-component of the momentum equation it is simplified to

$$\alpha \,\mathrm{Re}\, \rho^{(00)}\left(u_r^{(00)}\frac{\partial u_z^{(11)}}{\partial r} + u_z^{(00)}\frac{\partial u_z^{(11)}}{\partial z}\right) = -8\frac{\partial p^{(11)}}{\partial z} + n^{(00)}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(12)}}{\partial r}\right) + \frac{4\alpha^2}{3}\frac{\partial^2 u_z^{(11)}}{\partial z^2}\right]$$
$$+ n^{(01)}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(10)}}{\partial r}\right) + n^{(11)}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z^{(00)}}{\partial r}\right) + \frac{4\alpha^2}{3}\frac{\partial n^{(01)}}{\partial z}\frac{\partial u_z^{(10)}}{\partial z} + \frac{\partial n^{(11)}}{\partial r}\frac{\partial u_z^{(00)}}{\partial r}$$

which by substituting all the known quantities results in

$$-4\alpha \,Re\, ru_r^{(11)} - 2\alpha \,Re\left(1-r^2\right)\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(rF'\left(r\right)\right) + \frac{2\alpha^2}{3}\left(7-13r^2\right) =$$
$$16\left(1-z\right)^2 - 8G'\left(z\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right)\right)\right)\left(1-z\right) - 4\alpha \,Re\left(1-z\right)$$
$$\text{(C.129)}$$

In order to be able to separate variables we demand that the terms involving both r and z are scalar multiples of $\left(1-z\right)$ therefore we set

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(02)}\right)\right)\right) - 4\alpha \,Re = \alpha \,Re\,\gamma\,, \quad \text{(C.130)}$$

with γ being an unknown constant.

By integrating Eq. (C.130) once we get

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right)\right) = \frac{\alpha \,Re}{2}\left(4+\gamma\right)r^2 + c_1\,, \quad \text{(C.131)}$$

where c_1 is an unknown constant. Applying the condition

$$\left.\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right)\right)\right|_{r=0} = 0\,,$$

we find $c_1 = 0$. By integrating Eq. (C.131) twice we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right) = \frac{\alpha \,Re}{4}\left(4+\gamma\right)r^2 + c_2 \quad \text{(C.132)}$$

and

$$ru_r^{(11)} = \frac{\alpha \,Re}{16}\left(4+\gamma\right)r^4 + \frac{c_2}{2}r^2 + c_3\,, \quad \text{(C.133)}$$

where c_2, and c_3 are unknown constants.

Applying the boundary condition $u_r^{(11)}(z,0)=0$ on Eq. (C.133) we get $c_3=0$ and applying the conditions

$$u_r^{(11)}(z,1)=0 \text{ and } \frac{1}{r}\frac{\partial}{\partial r}\left(ru_r^{(11)}\right)\Big|_{r=1}=0,$$

we get

$$\frac{\alpha Re}{8}(4+\gamma)+c_2=0 \text{ and } \frac{\alpha Re}{4}(4+\gamma)+c_2=0, \tag{C.134}$$

respectively. Solving the system in Eq. (C.113) we obtain $\gamma=-4$ and $c_2=0$, thus the radial velocity component is equal to zero:

$$u_r^{(11)}=0.$$

Since the radial velocity component is equal to zero, Eq.(C.108) is simplified to

$$\frac{2\alpha^2}{3}\left(7-13r^2\right)-\frac{1}{r}\frac{\partial}{\partial r}\left(rF'(r)\right)=16(1-z)^2-8G'(z)-4\alpha\,Re(1-z)=A, \tag{C.135}$$

where A is an unknown constant.

Integrating the first ODE of Eq. (C.114) twice with respect to r we find

$$rF'(r)=\frac{14\alpha^2}{3}\left(\frac{r^2}{2}-\frac{r^4}{2}\right)-\frac{A}{2}r^2+c_4$$

and

$$F(r)=\frac{7\alpha^2}{12}\left(2r^2-r^4\right)-\frac{A}{4}r^2+c_5, \tag{C.136}$$

where c_4, and c_5 are unknown constants.

Imposing the condition $F'(0)=0$ we get that $c_4=0$ and imposing the condition $F(1)=0$ we get

$$\frac{A}{4}-c_5=\frac{7\alpha^2}{12}. \tag{C.137}$$

The mass flow rate condition yields the condition $\int_0^1 F(r)rdr=0$ which, when imposed to Eq. (C.136) leads to

$$\frac{A}{8}-c_5=\frac{7\alpha^2}{18}. \tag{C.138}$$

The solution of the system of Eqs. (C.137) and (C.138) is $A=\frac{14\alpha^2}{9}$ and $c_5=-\frac{7\alpha^2}{36}$.

So, we find that

$$u_z^{(11)}=-\left(1-r^2\right)\left(1-z\right)^2+\frac{\alpha^2}{72}\left(1-r^2\right)\left(23+3r^2\right).$$

The solution of the second ODE of Eq. (C.135) is

$$G(z)=-\frac{2}{3}(1-z)^3+\frac{\alpha\, Re}{4}(1-z)^2+\frac{A}{8}(1-z)+c_6, \tag{C.139}$$

where c_6 is an unknown constant.

Applying the condition $p^{(11)}(1,1)=0$ on Eq. (C.128) we get $G(1)=0$ and from Eq. (C.139) we find that $c_6=0$.

Finally combining Eqs. (C.128) and (C.139) we find that

$$p^{(11)}=-\frac{2}{3}(1-z)^3+\frac{\alpha\,\mathrm{Re}}{4}(1-z)^2+\frac{2\alpha^2}{9}\left(4-3r^2\right)(1-z)-\frac{\alpha^3 Re}{144}\left(1-r^2\right)\left(7+6y^2-2y^4\right).$$

Finally summarizing we have that the solution of order $\varepsilon\delta$ is

$$\begin{aligned}
u_z^{(11)} &= -\left(1-r^2\right)\left(1-z\right)^2+\frac{\alpha^2}{72}\left(1-r^2\right)\left(23+3r^2\right)\\
u_r^{(11)} &= 0\\
p^{(11)} &= -\frac{2}{3}(1-z)^3+\frac{\alpha\,\mathrm{Re}}{4}(1-z)^2+\frac{2\alpha^2}{9}\left(4-3r^2\right)(1-z)\\
&\quad -\frac{\alpha^3 Re}{144}\left(1-r^2\right)\left(7+6y^2-2y^4\right)\\
\rho^{(11)} &= \frac{1}{2}(1-z)^2-\frac{\alpha^2}{4}\left(1-r^2\right)\\
\eta^{(11)} &= -\frac{1}{2}(1-z)^2+\frac{\alpha Re}{4}(1-z)+\frac{\alpha^2}{12}\left(1-r^2\right)
\end{aligned} \tag{C.140}$$

C.3 The solution up to the second order

Combining the solutions of zero-order and of orders ε, δ, ε^2, δ^2 and $\varepsilon\delta$, we find that the perturbation solution that includes these orders is given by:

the perturbation solution is as follows:

$$
\begin{aligned}
u_z &= 2\left(1-r^2\right)+\varepsilon\left(1-r^2\right)\left[-2\left(1-z\right)-\frac{\alpha Re}{18}\left(2-7r^2+2r^4\right)\right] \\
&\quad +\varepsilon^2 2\left(1-r^2\right)\left[\frac{3}{2}\left(1-z\right)^2-\frac{\alpha Re}{12}\left(1+7r^2-2r^4\right)\left(1-z\right)+\frac{\alpha^2}{144}\left(1-27r^2\right)\right. \\
&\quad \left. +\frac{\alpha^2 Re^2}{43200}\left(43-957r^2+2343r^4-1257r^6+168r^8\right)\right] \\
&\quad +\delta^2\ \frac{\alpha^2}{24}\left(1-r^2\right)\left(1-3r^2\right)+\varepsilon\delta\left(1-r^2\right)\left[-\left(1-z\right)^2+\frac{\alpha^2}{72}\left(23+3r^2\right)\right]+h.o.t.
\end{aligned}
$$

$$
u_r = \varepsilon^2\,\frac{\alpha Re}{36}\,r\left(1-r^2\right)^2\left(4-r^2\right)+h.o.t.
$$

$$
\begin{aligned}
p &= 1-z+\varepsilon\left[-\frac{1}{2}\left(1-z\right)^2+\frac{\alpha Re}{4}\left(1-z\right)+\frac{\alpha^2}{12}\left(1-r^2\right)\right]+\delta\left[\frac{1}{2}\left(1-z\right)^2-\frac{\alpha^2}{4}\left(1-r^2\right)\right] \\
&\quad +\varepsilon^2\left[\frac{1}{2}\left(1-z\right)^3-\frac{\alpha Re}{2}\left(1-z\right)^2-\frac{\alpha^2}{36}\left(29-9r^2\right)\left(1-z\right)+\frac{\alpha^2 Re^2}{27}\left(1-z\right)\right. \\
&\quad \left. +\frac{\alpha^3 Re}{432}\left(1-r^2\right)\left(19-35r^2+10r^4\right)\right]+\delta^2\left[\frac{1}{6}\left(1-z\right)^3-\frac{\alpha^2}{12}\left(2-3r^2\right)\left(1-z\right)\right] \\
&\quad +\varepsilon\delta\left[-\frac{2}{3}\left(1-z\right)^3+\frac{\alpha\,\mathrm{Re}}{4}\left(1-z\right)^2+\frac{2\alpha^2}{9}\left(4-3r^2\right)\left(1-z\right)-\frac{\alpha^3 Re}{144}\left(1-r^2\right)\left(7+6y^2-2y^4\right)\right]+h.o.t.
\end{aligned}
$$

$$
\rho = 1+\varepsilon\left(1-z\right)+\varepsilon^2\left[-\frac{1}{2}\left(1-z\right)^2+\frac{\alpha Re}{4}\left(1-z\right)+\frac{\alpha^2}{12}\left(1-r^2\right)\right]+\varepsilon\delta\left[\frac{1}{2}\left(1-z\right)^2-\frac{\alpha^2}{4}\left(1-r^2\right)\right]+h.o.t.
$$

$$
\eta = 1+\delta\left(1-z\right)+\delta^2\left[\frac{1}{2}\left(1-z\right)^2-\frac{\alpha^2}{4}\left(1-r^2\right)\right]+\varepsilon\delta\left[-\frac{1}{2}\left(1-z\right)^2+\frac{\alpha Re}{4}\left(1-z\right)+\frac{\alpha^2}{12}\left(1-r^2\right)\right]+h.o.t.
$$

The volumetric flow rate,

$$
Q(z)\equiv 2\int_0^1 u_z\left(r,z\right)rdr \tag{C.141}
$$

is given by

$$Q(z) = 1 - \varepsilon(1-z) + \varepsilon^2\left[\frac{3}{2}(1-z)^2 + \frac{5\alpha Re}{36}(1-z) - \frac{2\alpha^2}{27}\right] + \varepsilon\delta\left[-\frac{3}{8}(1-z)^2 + \frac{37\alpha^2}{120}\right] + h.o.t.. \tag{C.142}$$

The mean pressure drop for axisymmetric Poiseuille flow of a compressible Newtonian fluid, defined by

$$\overline{\Delta p} \equiv \bar{p}(0) - \bar{p}(1) \equiv 2\int_0^1 \left[p(0,z) - p(1,z) \right] r dr \tag{C.143}$$

is:

$$\begin{aligned} \overline{\Delta p} = 1 &- \varepsilon\left(\frac{1}{2} - \frac{1}{4}\alpha \, \mathrm{Re}\right) + \frac{\delta}{2} + \varepsilon^2\left(\frac{1}{2} - \frac{49}{72}\alpha^2 - \frac{\alpha \, \mathrm{Re}}{2} + \frac{\alpha^2 \, \mathrm{Re}^2}{27}\right) \\ &+ \delta^2\left(\frac{1}{6} - \frac{\alpha^2}{24}\right) + \varepsilon\delta\left(-\frac{2}{3} + \frac{17\alpha^2}{15} + \frac{18\alpha \, \mathrm{Re}}{35}\right) + h.o.t. \end{aligned} \tag{C.144}$$

For the axisymmetric Poiseuille flow, the average Darcy friction factor, defined by

$$\bar{f} \equiv -\frac{8}{Re}\int_0^1 \frac{\partial u_z}{\partial r}(1,z)\, dz , \tag{C.145}$$

gives

$$\frac{Re\,\bar{f}}{32} = 1 - \varepsilon\left(\frac{1}{2} - \frac{1}{12}\alpha Re\right) + \varepsilon^2\left(\frac{1}{2} - \frac{13}{72}\alpha^2 - \frac{1}{4}\alpha Re + \frac{17}{2160}\alpha^2 Re^2\right) - \delta^2\frac{\alpha^2}{24} + \varepsilon\delta\left(-\frac{1}{8} + \frac{7}{20}\alpha^2\right) + h.o.t. \tag{C.146}$$

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