Probability and Moment Inequalities for Demimartingales and Associated Random Variables

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Contents

Abstract

In recent years, concepts of dependence, including positive and negative association introduced by Esary et al. (1967) and Joag-Dev and Proschan (1983) respectively, have been the focus of substantial research activity. Among the various results presented in the literature are extensions and generalizations. In particular, Newman and Wright (1982) introduced the concept of a demimartingale and a demisubmartingale as a generalization of the notion of martingales and submartingales. The definition is rather a technical one and serves, among other things, the purpose of studying in a more general way the behavior of the partial sum of mean zero associated random variables. The class of N-demimartingales introduced later, generalizes in a natural way the concept of negative association and includes as special case the class of martingales equipped with the natural choice of σ-algebras. The aim of this work is to provide maximal and moment inequalities for the classes of demimartingales and N-demimartingales. The results presented in this thesis in many cases improve and generalize known results for martingales, and for positively and negatively associated random variables. The inequalities provided for these two new classes of random variables are instrumental in obtaining asymptotic results. The asymptotic results derived for demimartingales can also be applied to the case of partial sums of positively associated random variables while the asymptotic results concerning N-demimartingales can be applied to partial sums of negatively associated random variables and other statistical functions involving negatively associated random variables. As a natural generalization of demimartingales and demisubmartingales we introduce multidimen-

sionally indexed demi(sub)martingales. For this new class of random variables we prove a maximal inequality which becomes the source result for obtaining several inequalities for multidimensionally indexed associated random variables. These inequalities, when reduced to the case of single index, are in some cases sharper than the bounds already known in the literature.

Περίληψη

Τα τελευταία χρόνια έχουν μελετηθεί εκτενώς οι διάφορες έννοιες εξάρτησης τυχαίων μεταβλητών. Ιδιαίτερο ενδιαφέρον παρουσιάζουν οι έννοιες της θετικής και αρνητικής συσχέτισης οι οποίες έχουν εισαχθεί από τους Esary et al. (1967) και Joag-Dev και Proschan (1983) αντίστοιχα. Έκτοτε στη βιβλιογραφία γίνονται αναφορές και σε άλλες έννοιες εξάρτησης οι οποίες αποτελούν γενικεύσεις και επεκτάσεις των εννοιών της θετικής και αρνητικής εξάρτησης. Συγκεκριμένα οι Newman και Wright (1982) έχουν ορίσει τις έννοιες των demimartingales και demisubmartingales σε μια προσπάθεια γενίκευσης των martingales και submartingales. Ο ορισμός της ακολουθίας demimartingale εξυπηρετεί τη μελέτη της συμπεριφοράς των μερικών αθροισμάτων θετικά συσχετισμένων τυχαίων μεταβλητών με μέσο μηδέν. Η κλάση των N-demimartingales ορίστηκε με τρόπο ανάλογο των demimartingales και γενικεύει την έννοια των αρνητικά συσχετισμένων τυχαίων μεταβλητών και συμπεριλαμβάνει, όπως και οι demimartingales, ως ειδική περίπτωση την κλάση των martingales. Ο στόχος της παρούσας διατριβής είναι να παρουσιάσει μεγιστικές ανισότητες και ανισότητες ροπών για τις δύο αυτές κλάσεις τυχαίων μεταβλητών. Τα αποτελέσματα που παρουσιάζονται σε αυτή την εργασία γενικεύουν ή και βελτιώνουν ήδη γνωστά αποτελέσματα που αφορούν τις martingales και τις συσχετισμένες τυχαίες μεταβλητές. Οι ανισότητες που προκύπτουν είναι τα βασικά εργαλεία για την απόδειξη ασυμπτωτικών αποτελεσμάτων. Επιπλέον τα ασυμπτωτικά αποτελέσματα που αφορούν τις demimartingales μπορούν να εφαρμοστούν στην περίπτωση των μερικών αθροισμάτων θετικά συσχετισμένων τυχαίων μεταβλητών ενώ τα ασυμπτωτικά αποτελέσματα που αφορούν τις N-demimartingales εφαρμόζονται

στην παρούσα εργασία τόσο στην περίπτωση των μερικών αθροισμάτων όσο και στην περίπτωση άλλων στατιστικών συναρτήσεων που κατασκευάζονται με βάση αρνητικά συσχετισμένες τυχαίες μεταβλητές. Ως γενίκευση των demimartingales ορίζεται η αντίστοιχη κλάση με πολυδιάστατους δείκτες. Για αυτή την κλάση των τυχαίων μεταβλητών αποδεικνύεται μια μεγιστική ανισότητα από την οποία απορρέουν αποτελέσματα που αφορούν θετικά συσχετισμένες τυχαίες μεταβλητές με πολυδιάστατο δείκτη, τα οποία στη περίπτωση του μονοδιάστατου δείκτη βελτιώνουν τα αντίστοιχα γνωστά αποτελέσματα.

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Chapter 1

Introduction

1.1 Dependence in probability and statistics

The first encounter with dependent random variables that one faces is the drawing of balls without replacement from an urn: Suppose that an urn contains balls of two colors A and B. Drawing balls repeatedly from the urn, and putting $I_k = 1$ if the k ball has color A and 0 otherwise, produces a sequence of random variables. The sum of the first n indicators describes the number of balls with color A obtained after n draws. If we draw with replacement the indicators are independent; however if we draw without replacement then they are dependent.

There exist many notions of dependence. A fundamental notion is Markov dependence where vaguely speaking the future depends on the past only through the present. Another important dependence concept is martingale dependence.

There exist various concepts which are defined via some kind of decay, i.e., the further the two elements are apart in time or index, the weaker the dependence is. The simplest such concept is called m-dependence.

Definition 1.1.1 *The random variables $X_1, X_2, \ldots,$ are m-dependent if X_i and X_j are independent whenever $|i - j| > m$.*

Remark 1.1.2 *Independence is the same as 0-dependence.*

Since m-dependence is a natural generalization of independent random variables, classical results for sequences of independent random variables can also be established for m-dependent random variables. For example Diananda (1955) and Orey (1958) provide central limit theorems for a collection of random variables with m-dependence structure, Chen (1997) established the law of the iterated logarithm, while Romano and Wolf (2000) and Berk (1973) developed central limit theorems for m-dependent random variables with unbounded m. Furthermore, in the analysis of time series certain natural statistics have this m-dependence structure even if the underlying processes are independent such as long-memory processes (Beran (1994)) and the moving blocks bootstrap (Künsch (1989)).

Examples

1. We flip a coin repeatedly and let the events

$$B_n = \{\text{the } (n-1)\text{th and the } n\text{th toss both yield heads}\},\ n \geq 2.$$

It is obvious that we don't have independent events. However, the events with even indices are independent, and so are those with odd indices, i.e., B_i and B_j are independent if $|i-j| > 1$. Attaching indicators to the B-events, such that $I_n = 1$ whenever B_n occurs and 0 otherwise, we obtain a 1-dependent sequence of random variables.

2. Let $X_1, X_2, \ldots$ be independent random variables and let $g : \mathbb{R}^{m+1} \to \mathbb{R}$. Then the so called $(m+1)$-block factors sequence which is defined as

$$Y_n = g(X_n, X_{n+1}, \ldots, X_{n+m-1}, X_{n+m}),\ n \geq 1$$

is m-dependent.

In the m-dependence case the dependence stops abruptly. As a generalization of this case the concept of mixing is introduced which allows dependence to drop gradually. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $\mathcal{H}$ and $\mathcal{G}$ be sub-σ-algebras of $\mathcal{F}$. Then

$$\alpha(\mathcal{H}, \mathcal{G}) = \sup_{F \in \mathcal{H}, G \in \mathcal{G}} |P(F \cap G) - P(F)P(G)|,$$

$$\phi(\mathcal{H},\mathcal{G}) = \sup_{F\in\mathcal{H},G\in\mathcal{G}} |P(G|F) - P(G)|,\ P(F) > 0,$$

$$\psi(\mathcal{H},\mathcal{G}) = \sup_{F\in\mathcal{H},G\in\mathcal{G}} \frac{|P(F\bigcap G) - P(F)P(G)|}{P(F)P(G)}$$

$$\rho(\mathcal{H},\mathcal{G}) = \sup_{X\in L^2(\mathcal{H}),Y\in L^2(\mathcal{G})} |\rho_{X,Y}|$$

where $\rho_{X,Y}$ is the correlation coefficient between X and Y, are some measures of dependence. It can be shown that (see for example Berkes and Phillip (1978), Herrndorff (1983), Peligrad (1986,1990) etc.)

$$\alpha(\mathcal{H},\mathcal{G}) \leq \frac{1}{4},\ \psi(\mathcal{H},\mathcal{G}) \leq 1,\ \rho(\mathcal{H},\mathcal{G}) \leq 1,$$

$$4\alpha(\mathcal{H},\mathcal{G}) \leq 2\phi(\mathcal{H},\mathcal{G}) \leq \psi(\mathcal{H},\mathcal{G})$$

$$4\alpha(\mathcal{H},\mathcal{G}) \leq \rho(\mathcal{H},\mathcal{G}) \leq \psi(\mathcal{H},\mathcal{G})$$

$$\rho(\mathcal{H},\mathcal{G}) \leq 2\sqrt{\phi(\mathcal{H},\mathcal{G})}.$$

Corresponding to these measures we have the following mixing coefficients:

$$\alpha(n) = \sup_{k\in\mathbb{Z}^+} \alpha(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty),$$

$$\phi(n) = \sup_{k\in\mathbb{Z}^+} \phi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty),$$

$$\psi(n) = \sup_{k\in\mathbb{Z}^+} \psi(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty),$$

$$\rho(n) = \sup_{k\in\mathbb{Z}^+} \rho(\mathcal{F}_1^k, \mathcal{F}_{k+n}^\infty),$$

where $\mathcal{F}_i^j = \sigma\{X_k, i \leq k \leq j\}$.

These coefficients measure the dependence of those portions of the sequence $\{X_k, k \geq 1\}$ that are located n "time units" apart. From the inequalities above we can see that some measures of dependence are stronger than others. In the case of independent random variables all the above coefficients are equal to 0 and if the coefficients converge to 0 as $n \to \infty$ we may interpret this as asymptotic independence. Strong mixing conditions are used to establish strong laws of large numbers for non-independent random variables. A famous problem is the so called Ibragimov conjecture (1962) which states that a strictly stationary, centered ϕ-mixing sequence $X_1, X_2, \ldots$ such that $EX_1^2 < \infty$ and $Var\left(\sum_{k=1}^n X_k\right) \to \infty$ as $n \to \infty$ satisfies the central limit theorem. An extensive study of strong mixing conditions is presented by Bradley (2005).

One of the fundamental problems of dependence has been to obtain conditions on a multivariate vector $\mathbf{X} = (X_1, \ldots, X_n)$ such that the condition

$$P(X_1 > x_1, \ldots, X_n > x_n) \geq \prod_{i=1}^n P(X_i > x_i) \tag{1.1}$$

holds for all real x_i. Problems involving dependent pairs of variables (X, Y) have been studied intensively in the case of bivariate normal distributions. Studies involving the general case are focused on the definition and estimation of measures of association.

If for a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ the above condition (1.1) holds true then $\mathbf{X}$ is said to be positively upper orthant dependent (PUOD)and if the random vector satisfies

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \geq \prod_{i=1}^n P(X_i \leq x_i) \tag{1.2}$$

it is said to be positively lower orthant dependent (PLOD).

Remark 1.1.3 *A random vector* $\mathbf{X}$ *satisfying such a condition is called negatively upper orthant dependent (NUOD) (with the reverse inequalities inside parentheses we define negatively lower orthant dependent (NLOD)). An infinite sequence is said to be negatively orthant dependent (NOD) if it is both NUOD and NLOD.*

The pair (X, Y) is said to be positively quadrant dependent if

$$P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y) \tag{1.3}$$

and it is said to be negatively quadrant dependent if the inequality sign is reversed. By rewriting (1.3) as

$$P(Y \leq y | X \leq x) \geq P(Y \leq y) \tag{1.4}$$

one can say that the knowledge that X is small increases the probability of Y being small. Tukey (1958) and Lehmann (1959) discussed the stronger condition

$$P(Y \leq y | X = x) \text{ is nonincreasing in } x. \tag{1.5}$$

If (1.5) holds then Y is said to be positively regression dependent on X.

Association was introduced by Esary, Proschan and Walkup (1967) and it is a concept that is also considered and used in actuarial mathematics and mathematical physics. In actuarial science it was first considered by Norberg (1989) who used it in order to investigate some alternatives to the independence assumption for multilife statuses in life insurance, as well as to quantify the consequences of a possible dependence on the amounts of premium relating to multilife insurance contracts. In mathematical physics the concept of association is due to the so called FKG inequalities (the inequalities are named after Fortuin, Kastelyn and Ginibre) i.e., the positive lattice condition which holds for many natural families of events, implies positive association. Newman (1980) proved that in a translation invariant pure phase of a ferromagnet, finite susceptibility and the FKG inequalities together imply convergence of the block spin scaling limit to the infinite temperature Gaussian fixed point. Newman (1983) provides a central limit theorem which is applicable to (not necessarily monotonic) functions of random variables satisfying the FKG inequalities. Furthermore, Preston (1974) provides a generalization of Holley's inequality which is itself a generalization of the FKG inequalities while Holley (1974) generalizes the FKG inequalities to two probability distributions.

Definition 1.1.4 *A finite collection of random variables $X_1, \ldots, X_n$ is said to be (positively)associated if*

$$Cov(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0,$$

for any componentwise nondecreasing functions f, g on $\mathbb{R}^n$ such that the covariance is defined. An infinite collection is associated if every finite subcollection is associated.

It is known that if $\mathbf{X}$ is associated then $\mathbf{X}$ is PUOD and PLOD.

A weaker concept of association is obtained by assuming that, for every pair of disjoint subsets A_1, A_2 of $\{1, \ldots, n\}$,

$$Cov(f(X_i, i \in A_1), g(X_i, i \in A_2)) \geq 0,$$

for every pair of coordinatewise nondecreasing functions f, g of $\{x_i, i \in A_1\}$ and $\{x_i, i \in A_2\}$, respectively. In this case the sequence $X_1, \ldots, X_n$ is called weakly associated. One can easily verify that association implies weak association.

Association has the following properties (Esary et al. (1967)):

$\mathbf{P_1}$ Any subset of associated random variables is associated;

$\mathbf{P_2}$ If two sets of associated random variables are independent of one another then their union is a set of associated random variables;

$\mathbf{P_3}$ The set consisting of a single random variable is associated;

$\mathbf{P_4}$ Nondecreasing functions of associated random variables are associated;

$\mathbf{P_5}$ If $\mathbf{X}^{(k)}$ are associated, for each k, and $\mathbf{X}^{(k)} \to \mathbf{X}$ in distribution then $\mathbf{X}$ is associated.

Examples

1. Let $X_1, \ldots, X_n$ be independent random variables and let $S_n = \sum_{i=1}^{n} X_i$. Then $\{S_1, \ldots, S_n\}$ are associated.

2. The order statistics $X_{(1)}, \ldots, X_{(n)}$ of a sample $X_1, \ldots, X_n$ are associated.

In many cases, for distributions with nonpositive corellations, it is important to have checkable conditions, which imply the inequality

$$P(X_1 > x_1, \ldots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i)$$

for $x_i \in \mathbb{R}$, $i = 1, \ldots, n$.

Definition 1.1.5 *A finite collection of random variables $X_1, \ldots, X_n$ is said to be negatively associated if*

$$Cov(f(X_i, i \in A), g(X_j, j \in B)) \leq 0,$$

for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and for any two componentwise nondecreasing functions f, g on $\mathbb{R}^{|A|}$ and $\mathbb{R}^{|B|}$ respectively, where $|A| = card(A)$, provided that the covariance is defined. An infinite collection is negatively associated if every finite subcollection is negatively associated.

The concepts of negative orthant dependence and negative association were introduced by Joag-Dev and Proschan (1983). They pointed out that negative orthant dependence is weaker than negative association since negative association implies negative orthant dependence but neither negative upper orthant dependence nor negative lower orthant dependence implies negative association. In their work they presented an example in which $\mathbf{X}$ possesses the NOD property but it is not negatively associated. Negative association possesses the following properties (Joag-Dev and Proschan (1983)):
$\mathbf{P_1}$ A pair (X, Y) of random variables is negatively associated if and only if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y).$$

$\mathbf{P_2}$ For disjoint sets $A_1, \ldots, A_m$ of $\{1, \ldots, n\}$, and nondecreasing positive functions $f_1, \ldots, f_m$, $\mathbf{X}$ is negatively associated implies

$$E \prod_{i=1}^{m} f_i(\mathbf{X}_{A_i}) \leq \prod_{i=1}^{m} E f_i(\mathbf{X}_{A_i})$$

where $\mathbf{X}_{A_i} = (X_j, j \in A_i)$.

$\mathbf{P_3}$ If $\mathbf{X}$ is negatively associated then it is negatively orthant dependent.

$\mathbf{P_4}$ A subset of negatively associated random variables is negatively associated.

$\mathbf{P_5}$ If $\mathbf{X}$ has independent components then it is negatively associated.

$\mathbf{P_6}$ Increasing functions defined on disjoint subsets of a set of negatively associated random variables are negatively associated.

$\mathbf{P_7}$ If $\mathbf{X}$ is negatively associated and $\mathbf{Y}$ is negatively associated, and $\mathbf{X}$ is independent of $\mathbf{Y}$, then $(\mathbf{X}, \mathbf{Y})$ is negatively associated.

Examples

1. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a set of real numbers. A permutation distribution is the joint distribution of the vector $\mathbf{X}$, which takes as values all permutations of $\mathbf{x}$ with equal probabilities $1/n!$. Such a distribution is negatively associated (Joag-Dev and Proschan (1983)).

2. (Multivariate Hypergeometric) An urn contains M balls of different colors. Suppose that a random sample of N balls is chosen without replacement and Y_i indicates the presence of ball of the ith color in the sample. Then $\mathbf{Y}$ has a permutation distribution, and hence it is negatively associated. More generally, M_i balls are of the ith color, $i = 1, \ldots, n$, with $\sum_{i=1}^{n} M_i = M$, and let X_i be the number of balls of the ith color in the sample. Then X_i can be viewed as the sum of M_i indicators in the sample model above. Since X_i are sums over nonoverlapping sets of random variables, the negative association is transmitted.

Joag-Dev and Proschan (1983) pointed out that a number of well-known multivariate distributions possess the negative association such as, multinomial distribution, negatively correlated normal distribution and joint distribution of ranks. Negatively associated random variables have been studied extensively because of their wide applications in multivariate statistical analysis and reliability theory.

Throughout the years the need of deep generalizations and new constructions arises, and the concept of dependence is extended to multidimensionally indexed random variables. Classical results of probability theory, such as central limit theorems and strong laws of large numbers and other important tools, are established for this new class of random variables. Roussas (1993) studied the empirical distribution function of a random field of associated identically distributed random variables with distribution function F and probability density function f. He proves that under some additional conditions the empirical distribution function converges almost surely and uniformly to F. Moreover, Roussas (1994) under the assumption of positive or negative association establishes the asymptotic normality of partial sums of random variables. Kim and Seok (1998) in their work derive maximal inequalities of linearly quadrant dependent random variables and they also obtain weak convergence for 2-parameter arrays of linearly quadrant dependent random variables. An extensive study of the theory of associated random fields is presented by Bulinski and Shashkin (2007).

1.2 Outline

Demimartingales and N-demimartingales are collections of random variables which generalize in a natural way the concepts of association and negative association respectively. The main objective of this thesis is to provide useful inequalities for the general classes of demimartingales and N-demimartingales and further advance their theory. We establish maximal and moment inequalities which, in many cases improve and generalize known results for other classes of random variables such as martingales and positively and negatively associated random variables. Strong laws of large numbers are also established since these inequalities are very useful in obtaining asymptotic results.

Chapter 2 is dedicated to the concept of demimartingales. First, we present a literature review on demimartingales, which includes important findings as well as key results that are used in this thesis. Next, we prove a maximal inequality which becomes the source for obtaining various other results. By proving an inequality for nonnegative real numbers we derive moment inequalities for demimartingales. In the last section of the chapter we introduce the concept of conditional demimartingales. For this class of random elements the review of known results is followed by several inequalities and a strong law of large numbers.

Various results for the class of N-demimartingales are presented in Chapter 3. Initially we present maximal inequalities and in particular a maximal inequality using the concept of complete downcrossings of an interval by a sequence of N-demimartingales. Then we provide an extension of Azuma's inequality for martingales to the case of N-demimartingales. The Azuma-type inequality for N-demimartingales is the key result for proving several exponential inequalities and asymptotic results not only for N-demimartingales but for mean zero negatively associated random variables as well. Furthermore, the Marcinkiewicz-Zygmund and Blackwell-Ross type inequalities are extended to the case of N-demimartingales.

In Chapter 4 the concept of demimartingales is extended to the case of collections of random variables which are multidimensionally indexed. We prove a Chow-type inequality for multidimensionally indexed demimartingales which becomes the source result for obtaining a Doob-type inequality and a Hájek-Rényi inequality for mean zero multidimensionally indexed associated random variables.

In Chapter 5 we discuss the future work which can be initiated based on the results presented in this thesis. The class of multidimensionally indexed N-demi(super)martingales can be defined and for this new class of random variables we intend to provide several inequalities and asymptotic results. We also emphasize the fact that one can define a stochastic process with a demimartingale and N-demimartingale structure and extend results to the case of random elements with continuous time index. Finally, we define the concept of domination of a strong N-demisupermartingale by an N-demisupermartingale, following the idea of Prakasa Rao (2007). The three concepts provide seed elements for further research on probability and moment inequalities.

Chapter 2

Demimartingales

2.1 Introduction

The concept of demimartingales was introduced by Newman and Wright (1982) in order to study the relation between sums of associated random variables and martingales. It is known that the partial sum of mean zero independent random variables with the natural choice of σ-algebras is a martingale. The motivation for the definition of demimartingales was based on the following proposition which refers to mean zero positively associated random variables.

Throughout this thesis all random variables, unless otherwise stated, are defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Also L^k, will denote the class of random variables with finite moments of up to order k.

Proposition 2.1.1 *Suppose $\{X_n, n \in \mathbb{N}\}$ are L^1, mean zero, associated random variables and $S_n = \sum_{i=1}^{n} X_i$. Then*

$$E\left[(S_{j+1} - S_j) f(S_1, \ldots, S_n)\right] \geq 0, \;\; j = 1, 2, \ldots$$

for all coordinatewise nondecreasing functions f.

The definition of demimartingales which follows is motivated by the previous proposition.

Definition 2.1.2 *A sequence of L^1 random variables $\{S_n, n \in \mathbb{N}\}$ is called a demimartingale if for all $j = 1, 2, \ldots$*

$$E\left[(S_{j+1} - S_j) f(S_1, \ldots, S_j)\right] \geq 0,$$

for all componentwise nondecreasing functions f whenever the expectation is defined. Moreover, if f is assumed to be nonnegative, the sequence $\{S_n, n \in \mathbb{N}\}$ is called a demisubmartingale.

It is clear by Proposition 2.1.1 that the partial sum of mean zero associated random variables is a demimartingale.

A martingale with the natural choice of σ-algebras is a demimartingale. This can easily be proven since

$$\begin{aligned} E[(S_{n+1} - S_n) f(S_1, \ldots, S_n)] &= E\{E[(S_{n+1} - S_n) f(S_1, \ldots, S_n) | \mathcal{F}_n]\} \\ &= E\{f(S_1, \ldots, S_n) E[(S_{n+1} - S_n) | \mathcal{F}_n]\} \\ &= 0 \end{aligned}$$

where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

Following the same steps, it can be verified that a submartingale is a demisubmartingale. The converse statement is false as we can see by the following Example 2.1.3.

Example 2.1.3 *We define the random variables $\{X_1, X_2\}$ such that*

$$P(X_1 = -1, X_2 = -2) = p, \;\; P(X_1 = 1, X_2 = 2) = 1 - p$$

where $0 \leq p \leq \frac{1}{2}$. Then $\{X_1, X_2\}$ is a demisubmartingale since for every nonnegative nondecreasing function f

$$E\left[(X_2 - X_1) f(X_1)\right] = -p f(-1) + (1 - p) f(1) \geq p\left(f(1) - f(-1)\right) \geq 0.$$

Observe that $\{X_1, X_2\}$ *is not a submartingale since*

$$E\left[X_2|X_1=-1\right]=\sum_{x_2=-2,2} x_2 P(X_2=x_2|X_1=-1)=-2<-1.$$

As pointed out above, the partial sum of mean zero associated random variables is a demimartingale. The converse statement is false, as we can see in Example 2.1.4, where we construct a demimartingale so that the demimartingale differences do not possess the association property.

Example 2.1.4 *We define the random variables* X_1 *and* X_2 *such that*

$$P(X_1=5, X_2=7)=\frac{3}{8},\ P(X_1=-3, X_2=7)=\frac{1}{8},\ P(X_1=-3, X_2=-7)=\frac{4}{8}.$$

Let f *be a nondecreasing function. Then* $\{X_1, X_2\}$ *is a demimartingale since*

$$E\left[(X_2-X_1)f(X_1)\right]=\frac{6}{8}\left[f(5)-f(-3)\right]\geq 0.$$

Let g *be a nondecreasing function such that*

$$g(x)=0 \text{ for } x<0,\ g(2)=2,\ g(5)=5,\ g(10)=20.$$

By simple algebra we can verify that

$$E[g(X_1)]=\frac{3}{8}g(5)+\frac{5}{8}g(-3)=\frac{15}{8},$$

$$E[g(X_2-X_1)]=\frac{3}{8}g(2)+\frac{1}{8}g(10)+\frac{4}{8}g(-4)=\frac{26}{8},$$

$$E[g(X_1)g(X_2-X_1)]=\frac{3}{8}g(5)g(2)+\frac{1}{8}g(-3)g(10)+\frac{4}{8}g(-3)g(-4)=\frac{30}{8}.$$

The random variables X_1 *and* X_2-X_1 *are not associated since*

$$Cov\left(g(X_1), g(X_2-X_1)\right)=-\frac{75}{32}<0.$$

The partial sum of mean zero associated random variables is not the only special case of a demimartingale. Below we provide two more examples.

Example 2.1.5 *Let $X_1, \ldots, X_n$ be associated random variables and let $h(x_1, \ldots, x_m)$ be a "kernel" mapping $\mathbb{R}^m$ to $\mathbb{R}$ for an integer $1 \leq m \leq n$. Without loss of generality assume that h is symmetric in its arguments, i.e., invariant under permutations of arguments. We construct the U-statistic*

$$U_n = \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \cdots < i_m \leq n} h(X_{i_1}, \ldots, X_{i_m})$$

where $\sum_{1 \leq i_1 < \cdots < i_m \leq n}$ denotes summation over the $\binom{n}{m}$ combinations of m distinct elements $\{i_1, \ldots, i_m\}$ from $\{1, \ldots, n\}$.
The following proposition is due to Christofides (2004).

Proposition 2.1.6 *Let U_n be a U-statistic based on associated random variables and on the kernel h. Assume that h is componentwise nondecreasing and without loss of generality $E(h) = 0$. Then $S_n = \binom{n}{m} U_n, n \geq m$ is a demimartingale.*

Example 2.1.7 *Let $X_1, X_2, \ldots$ be associated and identically distributed random variables with density (or probability mass) function $f(., \theta)$.*
Define the following "likelihood ratio" type statistical function

$$L_n = \prod_{k=1}^{n} \frac{f(X_k, \theta_1)}{f(X_k, \theta_0)}.$$

Observe that the above statistic in case of independent observations tests the hypotheses

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta = \theta_1.$$

Assume that the function

$$h(x) = \frac{f(x, \theta_1)}{f(x, \theta_0)}$$

is nondecreasing in x.
Then under H_0, $\{L_n, n \in \mathbb{N}\}$ is a demisubmartingale. Since

$$L_{n+1} - L_n = \left(\frac{f(X_{n+1}, \theta_1)}{f(X_{n+1}, \theta_0)} - 1 \right) L_n,$$

then for g a nonnegative componentwise nondecreasing function

$$\begin{aligned} E[(L_{n+1}-L_n)g(L_1,\ldots,L_n)] &= E\left[\left(\frac{f(X_{n+1},\theta_1)}{f(X_{n+1},\theta_0)}-1\right)L_n g(L_1,\ldots,L_n)\right] \\ &\geq E\left(\frac{f(X_{n+1},\theta_1)}{f(X_{n+1},\theta_0)}-1\right)E\left(L_n g(L_1,\ldots,L_n)\right) \\ &= 0, \end{aligned}$$

since under H_0

$$E\left(\frac{f(X_{n+1},\theta_1)}{f(X_{n+1},\theta_0)}\right)=1.$$

In particular, one can verify that, if $f(.,\theta)$ is the density of $N(\theta,\sigma^2)$ for $\theta_1 \geq \theta_0$ and under H_0 the process $\{L_n, n \in \mathbb{N}\}$ is a demimartingale and if $f(.,\theta)$ is the density of the exponential distribution with parameter θ, then under H_0 and provided that $\theta_0 \geq \theta_1$ $\{L_n, n \in \mathbb{N}\}$ is a demimartingale.

2.2 An overview of inequalities

Wang et al. (2010) and Wang and Hu (2009) provide maximal inequalities for demimartingales as well as Doob type maximal inequality and strong law of large numbers. Fakoor and Azarnoosh (2005) establish a maximal inequality which improves the maximal inequality provided by Christofides (2000). Wood (1984) presents Doob's maximal inequality and upcrossing inequality for demimartingales.

Newman and Wright (1982) established the following inequality using the concept of complete upcrossings of an interval by a demimartingale. The number of complete upcrossings of the interval $[a,b]$ is defined as the number of times a sequence of random variables passes from below a to above b. Their result will be used later in order to prove a maximal inequality for demimartingales.

Theorem 2.2.1 *Let $\{S_n, n \in \mathbb{N}\}$ be a demisubmartingale and let $U_{a,b}$ be the number of complete upcrossings of the interval $[a,b]$ by $S_1,\ldots,S_n$.*
Then

$$E(U_{a,b}) \leq \frac{1}{b-a}E\left[(S_n-a)^+-(S_1-a)^+\right].$$

The next lemma, which is a consequence of the properties of convex functions, establishes the fact that a nondecreasing convex function of a demi(sub)martingale is a demisubmartingale (Christofides (2000)).

Lemma 2.2.2 *Let $\{S_n, n \in \mathbb{N}\}$ be a demimartingale or a demisubmartingale and g a nondecreasing convex function. Then $\{g(S_n), n \in \mathbb{N}\}$ is a demisubmartingale.*

In particular, as special cases, Lemma 2.2.2 establishes the demisubmartingale property for two important sequences of random variables as it is shown in the following corollary.

Corollary 2.2.3 *If $\{S_n, n \in \mathbb{N}\}$ is a demimartingale, then $\{S_n^+, n \in \mathbb{N}\}$ and $\{S_n^-, n \in \mathbb{N}\}$ are demisubmartingales where $X^+ = \max\{0, X\}$ and $X^- = \max\{0, -X\}$.*

Corollary 2.2.3 was essential for the proof of a Chow-type maximal inequality for demisubmartingales provided by Christofides (2000). The result is presented in the following theorem and it is a generalization of the corresponding inequality for martingales.

Theorem 2.2.4 *Let $\{S_n, n \in \mathbb{N}\}$ be a demisubmartingale with $S_0 \equiv 0$. Let $\{c_k, k \geq 1\}$ be a nonincreasing sequence of positive numbers. Then for every $\varepsilon > 0$,*

$$\varepsilon P\left\{\max_{1 \leq k \leq n} c_k S_k \geq \varepsilon\right\} \leq \sum_{j=1}^{n} c_j E(S_j^+ - S_{j-1}^+).$$

The Chow-type inequality for demimartingales is the source result for the following Hájek-Rényi inequality for associated random variables (Christofides (2000)).

Corollary 2.2.5 *Let $X_1, \ldots, X_n$ be mean zero associated random variables and $\{c_j, j \geq 1\}$ a nonincreasing sequence of positive numbers. Let $S_n = \sum_{i=1}^{n} X_i$ with $S_0 \equiv 0$. Then for every $\varepsilon > 0$,*

$$P\left\{\max_{1 \leq k \leq n} c_k |S_k| \geq \varepsilon\right\} \leq 2\varepsilon^{-2}\left\{2\sum_{j=1}^{n} c_j^2 Cov(X_j, S_{j-1}) + \sum_{j=1}^{n} c_j^2 E(X_j^2)\right\}.$$

A Hájek-Rényi inequality for associated sequences is also presented in Prakasa Rao (2002a) and in Sung (2008).

The following Doob type inequality follows directly from Theorem 3 of Newman and Wright (1982).

Corollary 2.2.6 *Let* $\{S_n, n \geq 1\}$ *be a demisubmartingale. Then for* $\varepsilon > 0$,

$$\varepsilon P(\max_{1\leq k\leq n} S_k \geq \varepsilon) \leq E[S_n I\{\max_{1\leq k\leq n} S_k \geq \varepsilon\}]$$

Remark 2.2.7 *In case of a nonnegative demisubmartingale the above corollary immediately leads to the following moment inequalities by applying Lemma 9.1 of Gut (2005).*

$$E\left(\max_{1\leq k\leq n} S_k\right)^p \leq \left(\frac{p}{p-1}\right)^p E(S_n^p),\ p > 1$$

and

$$E\left(\max_{1\leq k\leq n} S_k\right)^p \leq \frac{e}{e-1}(1 + E(S_n \log^+ S_n)),\ p = 1$$

where $log^+ x = \max\{1, \log x\}$.

Wang et al (2010) have generalized the above two inequalities for nonnegative convex functions of demimartingales.

Wang (2004) establishes a maximal inequality for random variables which generalizes and improves the Chow-type inequality provided by Christofides (2000). His result is presented in the following theorem.

Theorem 2.2.8 *Let* $S_1, \ldots, S_n$ *be a demimartingale. Let* g *be a nonnegative convex function on* $\mathbb{R}$ *with* $g(0) = 0$ *and* $\{c_n, n \geq 1\}$ *be a nonincreasing sequence of positive numbers. Define* $S_n^* = \max\{c_1 g(S_1), \ldots, c_n g(S_n)\}$ *with* $S_0^* \equiv 0$. *Then for every* $\varepsilon > 0$,

$$\varepsilon P(S_n^* \geq \varepsilon) \leq \sum_{i=1}^{n} c_i E[(g(S_i) - g(S_{i-1}))I\{S_n^* \geq \varepsilon\}].$$

By using the result of Theorem 2.2.8, Wang (2004) proves the following Hájek-Rényi inequality for mean zero associated random fields.

Theorem 2.2.9 *Let $\{X_{(m,n)}, m \geq 1, n \geq 1\}$ be a mean zero double sequence of associated random variables. Let $\{b_{(m,n)}, m \geq 1, n \geq 1\}$ be an array of positive constants such that $b_{(m,n)} = 0$ if $m = 0$ or $n = 0$ and*

$$\Delta b_{(m,n)} = b_{(m,n)} - b_{(m-1,n)} - b_{(m,n-1)} + b_{(m-1,n-1)} \geq 0 \tag{2.1}$$

for all $m \geq 1, n \geq 1$. Then for all $\varepsilon > 0$

$$P(\max_{1\leq k\leq m} \max_{1\leq j\leq n} |S_{(k,j)}|/b_{(k,j)} \geq \varepsilon) \leq \frac{64}{\varepsilon^2} E\left(\sum_{k=1}^{m}\sum_{j=1}^{n} \frac{X_{(k,j)}}{b_{(k,j)}}\right)^2.$$

The following Whittle-type inequality for demisubmartingales was established by Prakasa Rao (2002c) and generalizes the Hájek-Rényi inequality obtained by Christofides (2000).

Theorem 2.2.10 *Let the sequence of random variables $\{S_n, n \in \mathbb{N}\}$ be a demisubmartingale and $\phi(.)$ be a nonnegative nondecreasing convex function such that $\phi(S_0) = 0$. Let $\psi(u)$ be a nonnegative nondecreasing function for $u > 0$. Let A_n be the event that $\phi(S_k) \leq \psi(u_k), 1 \leq k \leq n$, where $0 = u_0 < u_1 \leq \ldots \leq u_n$. Then*

$$P(A_n) \geq 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}.$$

If, in addition, there exist nonnegative real numbers Δ_k, $1 \leq k \leq n$, such that

$$\begin{aligned} 0 &\leq E[(\phi(S_k) - \phi(S_{k-1}))f(\phi(S_1), \ldots, \phi(S_{k-1}))] \\ &\leq \Delta_k E[f(\phi(S_1), \ldots, \phi(S_{k-1}))],\ 1 \leq k \leq n, \end{aligned}$$

for all componentwise nonnegative nondecreasing functions f such that the expectation is defined and

$$\psi(u_k) \geq \psi(u_{k-1}) + \Delta_k,\ 1 \leq k \leq n$$

then

$$P(A_n) \geq \prod_{k=1}^{n}\left(1 - \frac{\Delta_k}{\psi(u_k)}\right).$$

2.3 A maximal inequality for demimartingales

For a martingale sequence $\{(X_n, \mathcal{F}_n), n \in \mathbb{N}\}$ with $X_1 = 0$, Brown (1971) showed that for any $t > 0$,

$$P\left[\max_{1\le k\le n} |X_k| > 2t\right] \le P[|X_n| > t] + t^{-1}E[(|X_n| - 2t)I\{|X_n| > 2t\}].$$

Bhattacharya (2005) proved an extension of Brown's inequality for nonnegative submartingales.

Theorem 2.3.1 *(Bhatthacharya 2005) Suppose that $\{(X_k, \mathcal{F}_k), k \ge 1\}$ is a nonnegative submartingale. Then for any $t > 0$,*

$$P\left[\max_{1\le k\le n} X_k > 2t\right] \le P[X_1 > t] + t^{-1}E[X_n I\{X_n > t\}].$$

Brown's and Bhattacharya's results was the motivation for proving the following maximal inequality for demimartingales. The "key" theorem for obtaining the desired inequality is the upcrossing inequality of Theorem 2.2.1.

Theorem 2.3.2 *Let $\{S_n, n \in \mathbb{N}\}$ be a demimartingale. Then for $c > 0$*

$$P\left(\max_{k\le n} |S_k| > 2c\right) \le P\left(|S_n| > c\right) + \frac{1}{c}\int_{\{|S_n|>2c\}} (|S_n| - 2c)dP. \tag{2.2}$$

Proof. We define the sets $A_n = \{\min_{k\le n} S_k < -2c\}$ and let b_1 be the number of upcrossings of $[-2c, -c]$ by $S_0 \equiv 0, S_1, \ldots, S_n$. Then

$$\begin{aligned} P(A_n) &= P\left(A_n, S_n \ge -c\right) + P\left(A_n, S_n < -c\right) \\ &\le P\left(\min_{k\le n} S_k < -2c, S_n \ge -c\right) + P\left(S_n < -c\right) \\ &\le P\left(b_1 > 0\right) + P\left(S_n < -c\right) \\ &\le Eb_1 + P\left(S_n < -c\right). \end{aligned} \tag{2.3}$$

Let $B_n = \{\max_{k\le n} S_k > 2c\}$ and let b_2 be the number of upcrossings of $[-2c, -c]$ by $S_0, -S_1, \ldots, -S_n$. Then,

$$\begin{aligned}
P(B_n) &= P\left(B_n, S_n \le c\right) + P\left(B_n, S_n > c\right) \\
&\le P\left(\max_{k\le n} S_k > 2c, S_n \le c\right) + P\left(S_n > c\right) \\
&= P\left(-\max_{k\le n} S_k < -2c, -S_n \ge -c\right) + P\left(S_n > c\right) \\
&= P\left(\min_{k\le n}(-S_k) < -2c, -S_n \ge -c\right) + P\left(S_n > c\right) \\
&\le P\left(b_2 > 0\right) + P\left(S_n > c\right) \\
&\le Eb_2 + P\left(S_n > c\right). \qquad (2.4)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
P(A_n \cup B_n) &= P\left(\{\min_{k\le n} S_k < -2c\} \cup \{\max_{k\le n} S_k > 2c\}\right) \\
&= P\left(\{-\min_{k\le n} S_k > 2c\} \cup \{\max_{k\le n} S_k > 2c\}\right) \\
&= P\left(\{\max_{k\le n}(-S_k) > 2c\} \cup \{max_{k\le n} S_k > 2c\}\right) \\
&= P\left(\max_{k\le n} |S_k| > 2c\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
P\left(\max_{k\le n} |S_k| > 2c\right) &= P(A_n \cup B_n) \\
&\le P(A_n) + P(B_n) \\
&\le Eb_1 + Eb_2 + P\left(S_n < -c\right) + P\left(S_n > c\right) \qquad (2.5) \\
&= Eb_1 + Eb_2 + P\left(|S_n| > c\right) \qquad (2.6)
\end{aligned}$$

where inequality (2.5) follows by (2.3) and (2.4).

Applying Theorem 2.2.1 we have

$$\begin{aligned}
Eb_1 + Eb_2 &\le \frac{1}{c}\left\{E\left[S_n + 2c\right]^+ - E[S_0 + 2c]^+ + E\left[-S_n + 2c\right]^+ - E\left[-S_0 + 2c\right]^+\right\} \\
&= \frac{1}{c}\left\{\int_{\{S_n \ge -2c\}} (S_n + 2c)dP + \int_{\{S_n \le 2c\}} (-S_n + 2c)dP - 4c\right\} \\
&= \frac{1}{c}\left\{\int_{\{S_n \ge -2c\}} S_n dP + \int_{\{S_n \le 2c\}} (-S_n)dP - \int_{\{|S_n| > 2c\}} 2cdP\right\}
\end{aligned}$$

$$= \frac{1}{c}\left\{\int_{\{S_n>2c\}} S_n dP + \int_{\{S_n<-2c\}} -S_n dP - \int_{\{|S_n|>2c\}} 2cdP\right\}$$

$$= \frac{1}{c}\int_{\{|S_n|>2c\}} (|S_n| - 2c)dP. \tag{2.7}$$

Then (2.2) follows by (2.6) and (2.7). ∎

The next corollary gives a more convenient bound for the quantity $P\left(\max_{k\leq n}|S_k| > 2c\right)$.

Corollary 2.3.3 *Let $S_n, n \in \mathbb{N}$ be a demimartingale.*

Then for $c > 0$

$$P\left(\max_{k\leq n}|S_k| > 2c\right) \leq \frac{1}{c}E\left[|S_n| I\{|S_n| > c\}\right].$$

Proof. Follows easily from Theorem 2.3.2 since

$$(|S_n| - 2c)\, I\{|S_n| > 2c\} \leq (|S_n| - c)\, I\{|S_n| > c\}.$$

∎

As an application of Corollary 2.3.3 we immediately have the following maximal inequality.

Corollary 2.3.4 *Let $\{S_n, n \in \mathbb{N}\}$ be a demimartingale.*

Then for $c > 0$

$$P\left(\max_{k\leq n}|S_k| > 2c\right) \leq \frac{1}{c}\sqrt{ES_n^2}\sqrt{P(|S_n| > c)}.$$

Proof. Follows from Corollary 2.3.3 by using the Cauchy-Schwartz inequality. ∎

2.4 Moment inequalities for demimartingales

The result that follows provides a useful (deterministic) inequality for nonnegative real numbers. The inequality is used in the theorem that follows in which we obtain moment inequalities for demimartingales.

Lemma 2.4.1 *Let $x, y \geq 0$ and $p \geq 2$. Then,*

$$y^p \geq x^p + px^{p-1}(y - x) + (y - x)^p.$$

Proof. The lemma is trivially true if $x = 0$ and/or $y = 0$. In addition the result holds as an equality for $p = 2$. Therefore we assume that $y > 0$ and $p > 2$.

We can write y^p as:

$$y^p = x^p + px^{p-1}(y - x) + py(y^{p-1} - x^{p-1}) - (p-1)(y^p - x^p).$$

It needs to be shown that:

$$py(y^{p-1} - x^{p-1}) - (p-1)(y^p - x^p) \geq (y - x)^p. \tag{2.8}$$

We divide both sides of (2.8) by y^p and by defining $r = x/y$ it is sufficient to show that the function

$$g(r) = p(1 - r^{p-1}) - (p-1)(1 - r^p) - (1 - r)^p$$

is nonnegative for $r \geq 0$. For the first derivative of the function $g(r)$ we have that:

$$g'(r) = p(1 - r)\left[(1 - r)^{p-2} - (p-1)r^{p-2}\right].$$

The solutions of the equation $g'(r) = 0$ are $r = 1$ and $r = \frac{1}{a+1}$ where $a = (p-1)^{\frac{1}{p-2}}$.

Observe that $r = \frac{1}{a+1} \in (0, \frac{1}{2})$.

The second derivative of the function $g(r)$ is given by:

$$\begin{aligned} g''(r) &= -p(p-1)(p-2)r^{p-3} + p(p-1)^2 r^{p-2} - p(p-1)(1-r)^{p-2} \\ &= -p(p-1)r^{p-3}\left[(p-2) - (p-1)r\right] - p(p-1)(1-r)^{p-2} \\ &= -p(p-1)r^{p-3}\left[p(1-r) + (r-2)\right] - p(p-1)(1-r)^{p-2}. \end{aligned}$$

Since $g''(1) > 0$, the point $(1, 0)$ is a local minimum.

Let

$$f(r) = r^{p-3}\left[p(1-r) + (r-2)\right] + (1-r)^{p-2}.$$

Then for $0 < r < \frac{1}{2}$

$$\begin{aligned} f(r) &\geq r^{p-3}(2 - 2r + r - 2) + (1-r)^{p-2} \\ &= -r^{p-2} + (1-r)^{p-2} \\ &> 0. \end{aligned}$$

Since $g''\left(\frac{1}{a+1}\right) < 0$, the function $g(r)$ has a local maximum for $r = \frac{1}{a+1}$.

We need to prove that $g\left(\frac{1}{a+1}\right) > 0$.

It can be shown that:

$$g\left(\frac{1}{a+1}\right) = \left(\frac{1}{a+1}\right)^p \left[(a+1)^p - ap - a^p - 1\right].$$

We define :

$$h(a) = (a+1)^p - ap - a^p - 1$$

for $a \geq 0$.

The function h is nonnegative since $h(0) = 0$ and

$$\begin{aligned} h'(a) &= p\left[(a+1)^{p-1} - a^{p-1} - 1\right] \\ &= p\left[(a+1)^{p-1} - (a^{p-1} + 1)\right] \\ &> \left[(a+1)^{p-1} - (a+1)^{p-1}\right] \\ &= 0. \end{aligned}$$

Summarizing, we have that g (with $g(0) = 0$), has two points of inflection, $\frac{1}{a+1} \in (0, \frac{1}{2})$ and 1, the first being a maximum and the second a minimum.

Therefore we conclude that the function $g(r)$ is nonnegative. ■

Next, with the use of Lemma 2.4.1 we prove moment inequalities for demimartingales.

Theorem 2.4.2 *Let $\{S_n, n \in \mathbb{N}\}$ be a nonnegative demisubmartingale with $S_0 \equiv 0$. Then for $p \geq 2$,*

$$ES_n^p \geq \sum_{j=1}^{n} Ed_j^p$$

where $d_j = S_j - S_{j-1}$, $j = 1, \ldots, n$.

Proof. By Lemma 2.4.1 we have that:

$$\begin{aligned} ES_{j+1}^p &\geq ES_j^p + pE\left[S_j^{p-1}\left(S_{j+1} - S_j\right)\right] + Ed_{j+1}^p \\ &\geq ES_j^p + Ed_{j+1}^p \end{aligned}$$

where the last inequality follows by the demisubmartingale property. Using induction we finally have the desired result. ■

For the special case of p being a positive even number, the previous result can be extended for demimartingales.

Theorem 2.4.3 *Let $\{S_n, n \in \mathbb{N}\}$ be a demimartingale with $S_0 \equiv 0$ and let p be a positive even integer. Then,*

$$E|S_n|^p \geq \frac{1}{2^{p-1}} \sum_{j=1}^{n} E|d_j|^p$$

where $d_j = S_j - S_{j-1}$, $j = 1, \ldots, n$.

Proof. It is known that for every $a, b \in \mathbb{R}$ and $p \geq 2$

$$|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p). \tag{2.9}$$

Applying (2.9) for p positive even integer we have that

$$(x^+ - y^+)^p + (x^- - y^-)^p \geq \frac{1}{2^{p-1}}(x-y)^p. \tag{2.10}$$

Therefore, since by Lemma 2.2.3 $\{S_n^+, n \in \mathbb{N}\}$ and $\{S_n^-, n \in \mathbb{N}\}$ are nonnegative demisubmartingales then

$$\begin{aligned} E|S_n|^p &= E(S_n^+)^p + E(S_n^-)^p \\ &\geq \sum_{j=1}^{n} E(S_j^+ - S_{j-1}^+)^p + \sum_{j=1}^{n} E(S_j^- - S_{j-1}^-)^p \qquad (2.11) \\ &\geq \frac{1}{2^{p-1}} \sum_{j=1}^{n} E(S_j - S_{j-1})^p \qquad (2.12) \\ &= \frac{1}{2^{p-1}} \sum_{j=1}^{n} E|d_j|^p \end{aligned}$$

where inequality (2.11) follows by applying Theorem 2.4.2 to the sequences $\{S_n^+, n \in \mathbb{N}\}$ and $\{S_n^-, n \in \mathbb{N}\}$ and inequality (2.12) follows from (2.10). ■

An immediate application of Theorem 2.4.3 for mean zero associated random variables gives the following:

Corollary 2.4.4 *Let* $\{X_n, n \in \mathbb{N}\}$ *be mean zero positively associated random variables. Then for p positive even integer,*

$$E|S_n|^p \geq \frac{1}{2^{p-1}} \sum_{j=1}^{n} E|X_j|^p$$

where $S_n = \sum_{j=1}^{n} X_j$.

Proof. The result follows by Corollary 2.4.3 since S_n is a demimartingale. ■

The result of Corollary 2.4.4 is comparable to Theorem 3 of Christofides and Vaggelatou (2004). In the case where $\sum_{k=1}^{n} E|X_k|^p > \frac{1}{2} \max\left\{ (\sum_{k=1}^{n} E(X_k^2))^{\frac{p}{2}}, \sum_{k=1}^{n} E|X_k|^p \right\}$ and p is a positive even number, the result of Corollary 2.4.4 gives a sharper bound.

2.5 Conditional demimartingales

Chow and Teicher (1978), Majerak et al. (2005), Roussas (2008) and Prakasa Rao (2009) studied the concept of conditionally independent random variables as well as conditional association and provided several results such as conditional versions of generalized Borel-Cantelli lemma, generalized Kolmogorov's inequality, generalized Hájek-Rényi inequalities and further related applications.

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space and let $\mathcal{F}$ be a sub-algebra of $\mathcal{A}$. Let us recall the definition of $\mathcal{F}$ independent random events.

Definition 2.5.1 *The set of events* $A_1, \ldots, A_n$ *are said to be conditionally independent given* $\mathcal{F}$ *or* $\mathcal{F}$*-independent if*

$$E\left(\prod_{j=1}^{k} I_{A_{i_j}} | \mathcal{F}\right) = \prod_{j=1}^{k} E[I_{A_{i_j}} | \mathcal{F}] \ a.s.$$

for all $1 \leq i_1 < i_2 < \ldots < i_k \leq n,\ 2 \leq k \leq n$.

Majerak et al. (2005) provided counterexamples showing that the independence of events does not imply conditional independence and that conditional independence of events does not imply their independence. Next we give the definition of conditionally independent random variables.

Definition 2.5.2 *A sequence of random variables $\{X_n, n \in \mathbb{N}\}$ defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ is said to be conditionally independent given a sub-algebra $\mathcal{F}$ or $\mathcal{F}$-independent if the sequence of events $\zeta_n = \sigma(X_n), n \geq 1$ are conditionally independent given $\mathcal{F}$. Equivalently, the sequence of random variables $\{X_n, n \in \mathbb{N}\}$ is said to be $\mathcal{F}$-independent if and only if for $(x_1, \ldots, x_n) \in \mathbb{R}^n$*

$$E\left(\prod_{i=1}^{n} I_{\{X_i \leq x_i\}} | \mathcal{F}\right) = \prod_{i=1}^{n} E[I_{\{X_i \leq x_i\}} | \mathcal{F}] \ a.s., \ for \ n = 2, 3 \ldots.$$

Prakasa Rao (2009) provides counterexamples where independent random variables lose their independence under conditioning and dependent random variables become independent under conditioning.

Conditional association is defined in analogy of (unconditional) association. Following Prakasa Rao (2009) for notational simplicity we will use the notation $E^{\mathcal{F}}(X_n)$ to denote $E[X_n|\mathcal{F}]$.

Definition 2.5.3 *A finite collection of random variables $X_1, \ldots, X_n$ is said to be $\mathcal{F}$-associated if*

$$Cov^{\mathcal{F}}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0,$$

for any componentwise nondecreasing function f, g on $\mathbb{R}^n$ where

$$Cov^{\mathcal{F}}(Y_1, Y_2) = E^{\mathcal{F}}(Y_1 Y_2) - E^{\mathcal{F}}(Y_1) E^{\mathcal{F}}(Y_2)$$

such that the covariance is defined. An infinite collection is $\mathcal{F}$-associated if every finite subcollection is $\mathcal{F}$-associated.

Independence, conditioned upon a given σ-field, is a useful concept, which occurs in many situations described by stochastic processes, such as Markov processes. Furthermore, conditioning is an effective tool in such classical cases as Rao-Blackwellization of an estimator, in establishing Wald's identity for sequential sampling, and in many cases as a tool for simplifying proofs. Conditioning is also crucial in studying efficiency and dealing with certain parametric models.

Since conditioning has an important role in statistics and motivated by the fact that the partial sum of mean zero associated random variables is a demimartingale we introduce the concept of $\mathcal{F}$-demi(sub)martingales. For this new class of random variables we provide maximal inequalities and related asymptotic results.

Definition 2.5.4 *Let $\{S_n, n \geq 1\}$ be a collection of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$. The sequence $\{S_n, n \geq 1\}$ is called an $\mathcal{F}$-demimartingale if for every componentwise nondecreasing function f*

$$E\left[(S_j - S_i)\, f(S_1, \ldots, S_i)|\mathcal{F}\right] \geq 0, \quad j > i$$

where $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{A}$. If moreover f is nonnegative then $\{S_n, n \geq 1\}$ is called an $\mathcal{F}$-demisubmartingale.

It is clear that a sequence of random variables which is an $\mathcal{F}$-demimartingale is always a demimartingale and if moreover f is nonnegative, then an $\mathcal{F}$-demisubmartingale is always a demisubmartingale. The converse cannot always be true as it can be seen by the following example.

Example 2.5.5 *We define the random variables X_1 and X_2 such that*

$$P(X_1 = 5, X_2 = 7) = \frac{3}{8}, P(X_1 = 5, X_2 = -7) = 0,$$

$$P(X_1 = -3, X_2 = 7) = \frac{1}{8}, P(X_1 = -3, X_2 = -7) = \frac{4}{8}.$$

As it has already been shown in Example 2.1.4 $\{X_1, X_2\}$ is a demimartingale. Moreover, we assume that f is a nonnegative function. Notice that, given the event $\{| X_1X_2 |= 21\}$, $\{X_1, X_2\}$ is not a demisubmartingale since

$$E\left[(X_2 - X_1)f(X_1)| \mid X_1X_2 \mid= 21\right] = -\frac{6}{8}f(-3) < 0. \tag{2.13}$$

Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of $\mathcal{F}$-associated random variables such that $E^{\mathcal{F}}X_k = 0$ for all $k \geq 1$. It is trivial to prove that $S_n = \sum_{k=1}^{n} X_k$ is an $\mathcal{F}$-demimartingale.

Lemma 2.5.6 *Let $S_1, S_2, \ldots$ be an $\mathcal{F}$-demimartingale or an $\mathcal{F}$-demisubmartingale and let g be a nondecreasing convex function. Then $\{g(S_n), n \in \mathbb{N}\}$ is an $\mathcal{F}$-demisubmartingale.*

Proof. Since g is a nondecreasing convex function

$$g(S_{n+1}) \geq g(S_n) + (S_{n+1} - S_n)h(S_n)$$

where h is the left derivative of g and by the convexity and monotonicity of g, the function h is a nonnegative nondecreasing function. Then for every f nonnegative componentwise nondecreasing function

$$E[(g(S_{n+1}) - g(S_n))f(g(S_1), \ldots, g(S_n))|\mathcal{F}] \geq E[(S_{n+1} - S_n)f_1(S_1, \ldots, S_n)|\mathcal{F}] \geq 0$$

since $\{S_n, n \in \mathbb{N}\}$ is an $\mathcal{F}$-demi(sub)martingale and $f_1(S_1, \ldots, S_n) = h(S_n)f(g(S_1), \ldots, g(S_n))$ is nonnegative componentwise nondecreasing function. ■

The following result provides a Chow type inequality for the sequence $\{g(S_n), n \in \mathbb{N}\}$ where $\{S_n, n \in \mathbb{N}\}$ is an $\mathcal{F}$-demimartingale and g is a nonnegative nondecreasing convex function.

Theorem 2.5.7 *Let $\{S_n, n \in \mathbb{N}\}$ be an $\mathcal{F}$-demimartingale, with $S_0 \equiv 0$ and let g be a nonnegative nondecreasing convex function such that $g(0) = 0$.*
Let

$$A = \{\max_{1 \leq i \leq n} c_i g(S_i) \geq \varepsilon\}$$

where $\{c_n, n \in \mathbb{N}\}$ is a nonincreasing sequence of positive $\mathcal{F}$-measurable random variables and ε an $\mathcal{F}$-measurable random variable such that $\varepsilon > 0$ a.s.
Then

$$\begin{aligned}\varepsilon P(A|\mathcal{F}) &\leq \sum_{i=1}^{n} c_i E^{\mathcal{F}}[g(S_i) - g(S_{i-1})] - c_n E^{\mathcal{F}}[g(S_n)I_{A^c}] \\ &\leq \sum_{i=1}^{n} c_i E^{\mathcal{F}}[g(S_i) - g(S_{i-1})] \ \text{a.s.}\end{aligned}$$

Proof. We define the sets:

$$A_i = \{c_k g(S_k) < \varepsilon, 1 \leq k < i, c_i g(S_i) \geq \varepsilon\}, \ i = 1, \ldots, n$$

Then

$$\begin{aligned}
\varepsilon P(A|\mathcal{F}) &= \varepsilon P(\bigcup_{i=1}^{n} A_i|\mathcal{F}) \\
&= \varepsilon \sum_{i=1}^{n} P(A_i|\mathcal{F}) \\
&= \sum_{i=1}^{n} \varepsilon E^{\mathcal{F}}[I_{A_i}] \\
&\leq \sum_{i=1}^{n} E^{\mathcal{F}}[c_i g(S_i) I_{A_i}] \\
&= c_1 E^{\mathcal{F}}[g(S_1) I_{A_1}] + \sum_{i=2}^{n} c_i E^{\mathcal{F}}[g(S_i) I_{A_i}] \\
&= c_1 E^{\mathcal{F}}[g(S_1)] - c_1 E^{\mathcal{F}}[g(S_1) I_{A_1^c}] + c_2 E^{\mathcal{F}}[g(S_2) I_{A_2}] + \sum_{i=3}^{n} c_i E^{\mathcal{F}}[g(S_i) I_{A_i}] \\
&\leq c_1 E^{\mathcal{F}}[g(S_1)] - c_2 E^{\mathcal{F}}[g(S_1) I_{A_1^c}] + c_2 E^{\mathcal{F}}[g(S_2) I_{A_1^c}] \\
&- c_2 E^{vF}[g(S_2) I_{A_1^c \bigcap A_2^c}] + \sum_{i=3}^{n} c_i E^{\mathcal{F}}[g(S_i) I_{A_i}] \\
&= c_1 E^{\mathcal{F}}[g(S_1)] + c_2 E^{\mathcal{F}}[(g(S_2) - g(S_1)) I_{A_1^c}] - c_2 E^{\mathcal{F}}[g(S_2) I_{A_1^c \bigcap A_2^c}] \\
&+ \sum_{i=3}^{n} c_i E^{\mathcal{F}}[g(S_i) I_{A_i}] \\
&= c_1 E^{\mathcal{F}}[g(S_1) - g(S_0)] + c_2 E^{\mathcal{F}}[g(S_2) - g(S_1)] - c_2 E^{\mathcal{F}}[(g(S_2) - g(S_1)) I_{A_1}] \\
&- c_2 E^{\mathcal{F}}[g(S_2) I_{A_1^c \bigcap A_2^c}] + \sum_{i=3}^{n} c_i E^{\mathcal{F}}[g(S_i) I_{A_i}].
\end{aligned}$$

By the convexity of the function g we have:

$$g(S_2) - g(S_1) \geq (S_2 - S_1) h(S_1)$$

where h is the left derivative of the function g which is a nonnegative nondecreasing function. Then:

$$E^{\mathcal{F}}[(g(S_2) - g(S_1)) I_{A_1}] \geq E^{\mathcal{F}}[(S_2 - S_1) h(S_1) I_{A_1}] \geq 0$$

where the last inequality follows from the $\mathcal{F}$-demimartingale property.

Therefore:

$$\varepsilon P(A|\mathcal{F}) \leq c_1 E^{\mathcal{F}}[g(S_1) - g(S_0)] + c_2 E^{\mathcal{F}}[g(S_2) - g(S_1)] - c_2 E^{\mathcal{F}}[g(S_2) I_{A_1^c \bigcap A_2^c}]$$

$$+ \sum_{i=3}^{n} c_i E^{\mathcal{F}}[g(S_i) I_{A_i}].$$

Working in the same way we have the desired result. ■

Theorem 2.5.7 was proved under the assumption that g is a nonnegative nondecreasing convex function. The following Theorem 2.5.8 shows that the assumption of monotonicity can be dropped.

Theorem 2.5.8 *Let $\{S_n, n \in \mathbb{N}\}$ be an $\mathcal{F}$-demimartingale, with $S_0 = 0$ and let g be a nonnegative convex function such that $g(0) = 0$.*
Let

$$A = \{\max_{1 \le i \le n} c_i g(S_i) \ge \varepsilon\}$$

where $\{c_n, n \in \mathbb{N}\}$ a nonincreasing sequence of positive $\mathcal{F}$-measurable random variables and ε an $\mathcal{F}$-measurable random variable such that $\varepsilon > 0$ a.s.
Then:

$$\begin{aligned} \varepsilon P(A|\mathcal{F}) &\le \sum_{i=1}^{n} c_i E^{\mathcal{F}}[g(S_i) - g(S_{i-1})] - c_n E^{\mathcal{F}}[g(S_n) I_{A^c}] \\ &\le \sum_{i=1}^{n} c_i E^{\mathcal{F}}[g(S_i) - g(S_{i-1})] \ a.s. \end{aligned}$$

Proof. We define the functions:

$$u(x) = g(x) I_{\{x \ge 0\}} \quad \text{and} \quad v(x) = g(x) I_{\{x < 0\}}.$$

Observe that the function $u(x)$ is nonnegative nondecreasing convex and the function $v(x)$ is nonnegative nonincreasing convex. From the definition of the functions u and v we have

$$g(x) = u(x) + v(x) = \max\{u(x), v(x)\}.$$

Then

$$\begin{aligned} P(A|\mathcal{F}) &= P(\max_{1 \le i \le n} c_i max\{u(S_i), v(S_i)\} \ge \varepsilon|\mathcal{F}) \\ &\le P(\max_{1 \le i \le n} c_i u(S_i) \ge \varepsilon|\mathcal{F}) + P(\max_{1 \le i \le n} c_i v(S_i) \ge \varepsilon|\mathcal{F}). \end{aligned} \tag{2.14}$$

By Theorem 2.5.7 we have

$$\varepsilon P(\max_{1\leq i\leq n} c_i u(S_i) \geq \varepsilon|\mathcal{F}) \leq \sum_{i=1}^{n} c_i E^{\mathcal{F}}[u(S_i) - u(S_{i-1})]. \tag{2.15}$$

We define the sets:

$$B_i = \{c_k v(S_k) < \varepsilon, 1 \leq k < i, c_i v(S_i) \geq \varepsilon\},\ i = 1, \ldots, n.$$

Following the steps of the proof of Theorem 2.5.7, we have

$$\begin{aligned}\varepsilon P(\max_{1\leq i\leq n} c_i v(S_i) \geq \varepsilon|\mathcal{F}) \quad &\leq \quad \sum_{i=3}^{n} c_i E^{\mathcal{F}}[v(S_i) I_{B_i}] + c_1 E^{\mathcal{F}}[v(S_1)] + c_2 E^{\mathcal{F}}[v(S_2) - v(S_1)] \\ &- \quad c_2 E^{\mathcal{F}}[v(S_2) I_{B_1^c \bigcap B_2^c}] - c_2 E^{\mathcal{F}}[(v(S_2) - v(S_1)) I_{B_1}].\end{aligned}$$

Let $h(x)$ be the left derivative of the function v. Then h is a nonpositive nondecreasing function. Then, since I_{B_1} is a nonincreasing function of S_1, $h(S_1)I_{B_1}$ is a nondecreasing function of S_1. Then by the $\mathcal{F}$-demimartingale property :

$$E^{\mathcal{F}}[(v(S_2) - v(S_1)) I_{B_1}] \geq 0.$$

Using the same arguments we can show that:

$$\varepsilon P(\max_{1\leq i\leq n} c_i v(S_i) \geq \varepsilon|\mathcal{F}) \leq \sum_{i=1}^{n} c_i E^{\mathcal{F}}[v(S_i) - v(S_{i-1})]. \tag{2.16}$$

By inequalities (2.14), (2.15) and (2.16) we have the desired result. ■

As an application of Theorem 2.5.8 we derive a Hájek-Rényi inequality for mean zero $\mathcal{F}$-associated random variables.

Corollary 2.5.9 *Let $\{X_n, n \geq 1\}$ be $\mathcal{F}$-associated random variables, with $E^{\mathcal{F}}(X_k) = 0$ for all k, $\{c_n, n \geq 1\}$ be a nonincreasing sequence of positive $\mathcal{F}$-measurable random variables and let ε be an $\mathcal{F}$-measurable random variable such that $\varepsilon > 0$ a.s.*
Then:

$$P(\max_{1\leq i\leq n} c_i|S_i| \geq \varepsilon|\mathcal{F}) \leq \varepsilon^{-2}\left\{\sum_{i=1}^{n} c_i^2 E^{\mathcal{F}}(X_i^2) + 2\sum_{i=1}^{n} c_i^2 Cov^{\mathcal{F}}(X_i, S_{i-1})\right\} \ a.s. \tag{2.17}$$

where $S_k = X_1 + \cdots + X_k$.

Proof. Since $g(x) = |x|^2$ is a nonnegative convex function by the Chow-type inequality we have:

$$\begin{aligned} P(\max_{1\le i\le n} c_i|S_i| \ge \varepsilon|\mathcal{F}) &= P(\max_{1\le i\le n} c_i^2|S_i|^2 \ge \varepsilon^2|\mathcal{F}) \\ &\le \varepsilon^{-2}\sum_{i=1}^{n} c_i^2 E^{\mathcal{F}}\left[(S_i)^2 - (S_{i-1})^2\right] \\ &= \varepsilon^{-2}\sum_{i=1}^{n} c_i^2 E^{\mathcal{F}}\left[X_i(X_i + 2S_{i-1})\right] \\ &= \varepsilon^{-2}\left\{\sum_{i=1}^{n} c_i^2 E^{\mathcal{F}}(X_i^2) + 2\sum_{i=1}^{n} c_i^2 Cov^{\mathcal{F}}(X_i, S_{i-1})\right\}. \end{aligned}$$

■

The next result, which is a Kronecker's type lemma will provide the link for obtaining a strong law of large numbers.

Lemma 2.5.10 *Let $S_0 \equiv 0, S_1, S_2, \ldots$ be a sequence of random variables and let $\{c_n, n \in \mathbb{N}\}$ be a nonincreasing sequence of positive $\mathcal{F}$-measurable random variables such that for $n \to \infty$*

$$c_n \to 0 \ a.s.$$

Further assume that

$$\sum_{k=1}^{\infty} c_k E^{\mathcal{F}}[g(S_k) - g(S_{k-1})] < \infty \ a.s.$$

where g is a real function such that $g(0) = 0$. Then for $n \to \infty$

$$c_n E^{\mathcal{F}}[g(S_n)] \to 0 \ a.s.$$

Proof. Let

$$A = \{\omega : \sum_{k=1}^{\infty} c_k(\omega) E^{\mathcal{F}}[g(S_k(\omega)) - g(S_{k-1}(\omega))] < \infty\}.$$

By assumption $P(A) = 1$. We define

$$B = \{\omega : c_n(\omega) E^{\mathcal{F}}[g(S_n(\omega))] \to 0\}.$$

Let $\omega_0 \in A$. Then

$$\sum_{k=1}^{\infty} c_k(\omega_0) E^{\mathcal{F}}[g(S_k(\omega_0)) - g(S_{k-1}(\omega_0))] < \infty$$

and by Kronecker's lemma

$$\sum_{k=1}^{n} c_n(\omega_0) E^{\mathcal{F}}[g(S_k(\omega_0)) - g(S_{k-1}(\omega_0))] \to 0 \ a.s.$$

which is equivalent to $c_n(\omega_0)E^{\mathcal{F}}[g(S_n(\omega_0))] \to 0 \ a.s.$

Therefore $\omega_0 \in B$ and finally $P(B) = 1$. ■

The next result provides a strong law of large numbers for $\mathcal{F}$-demimartingales.

Corollary 2.5.11 *Let $\{S_n, n \geq 1\}$ such that $S_0 \equiv 0$, be an $\mathcal{F}$-demimartingale, $\{c_n, n \geq 1\}$ a nonincreasing sequence of positive $\mathcal{F}$-measurable random variables such that for $n \to \infty$*

$$c_n \to 0 \ a.s.$$

and let g be a nonnegative convex function such that $g(0) = 0$. We further assume that:

$$\sum_{k=1}^{\infty} c_k E^{\mathcal{F}}[g(S_k) - g(S_{k-1})] < \infty \ a.s. \tag{2.18}$$

and

$$E^{\mathcal{F}}[g(S_n)] < \infty \ a.s.$$

Then conditionally on $\mathcal{F}$ for $n \to \infty$

$$c_n g(S_n) \to 0 \ a.s.$$

Proof. By the Chow type inequality and if ε is an $\mathcal{F}$-measurable random variable such that $\varepsilon > 0$ a.s.

$$\begin{aligned} \varepsilon P(\sup_{k \geq n} c_k g(S_k) \geq \varepsilon | \mathcal{F}) &\leq \sum_{k=n}^{\infty} c_k E^{\mathcal{F}}[g(S_k) - g(S_{k-1})] \\ &\leq c_n E^{\mathcal{F}}[g(S_n)] + \sum_{k=n+1}^{\infty} c_k E^{\mathcal{F}}[g(S_k) - g(S_{k-1})]. \end{aligned} \tag{2.19}$$

By (2.18) we have that:

$$\sum_{k=n+1}^{\infty} c_k E^{\mathcal{F}}[g(S_k) - g(S_{k-1})] \to 0 \text{ a.s.} \tag{2.20}$$

and by (2.18) and Lemma 2.5.10 we have that a.s.

$$c_n E^{\mathcal{F}}[g(S_n)] \to 0. \tag{2.21}$$

Inequalities (2.19), (2.20) and (2.21) give the desired result. ■

Chapter 3

N-demimartingales

3.1 Introduction

Motivated by the definition of a demimartingale, the idea of a similar generalization for negatively associated random variables leads to the concept of the so called N-demimartingales and N-demisupermartingales.

Definition 3.1.1 *A sequence of L^1 random variables $\{S_n, n \in \mathbb{N}\}$ is called an N-demimartingale if for all $j = 1, 2, \ldots$*

$$E\left[(S_{j+1} - S_j) f(S_1, \ldots, S_j)\right] \leq 0,$$

for all componentwise nondecreasing functions f provided the expectation is defined. Moreover, if f is assumed to be nonnegative, the sequence $\{S_n, n \in \mathbb{N}\}$ is called an N-demisupermartingale.

Various results and examples of N-demimartingales and N-demisupermartingales can be found in Christofides (2003) and Prakasa Rao (2004, 2007).

It is trivial to verify that the partial sum of mean zero negatively associated random variables is an N-demimartingale. The converse statement is false and a counterexample of an N-demimartingale so that N-demimartingale differences do not possess the negative association property is given in the following Example 3.1.2.

Example 3.1.2 *We define the random variables X_1, X_2 such that*

$$P(X_1 = 5, X_2 = 5) = \frac{1}{8},\ P(X_1 = -3, X_2 = 5) = \frac{2}{8},$$

$$P(X_1 = 5, X_2 = -3) = \frac{2}{8},\ P(X_1 = -3, X_2 = -3) = \frac{3}{8}$$

and let f be a nondecreasing function. Then:

$$E\left[(X_2 - X_1)f(X_1)\right] = 2[f(-3) - f(5)] \leq 0. \tag{3.1}$$

We define the random variable X_3 such that:

$$P(X_1 = 5, X_2 = 5, X_3 = -2) = P(X_1 = 5, X_2 = -3, X_3 = -2) = \frac{1}{12},$$

$$P(X_1 = -3, X_2 = 5, X_3 = -2) = P(X_1 = -3, X_2 = -3, X_3 = -2) = \frac{1}{12},$$

$$P(X_1 = 5, X_2 = 5, X_3 = 1) = \frac{1}{24},\ P(X_1 = 5, X_2 = -3, X_3 = 1) = \frac{4}{24},$$

$$P(X_1 = -3, X_2 = 5, X_3 = 1) = \frac{4}{24},\ P(X_1 = -3, X_2 = -3, X_3 = 1) = \frac{7}{24}.$$

Notice that $EX_1 = EX_2 = EX_3 = 0$.

Let g be a nondecreasing function. Then

$$E\left[(X_3 - X_2)g(X_1, X_2)\right] = \frac{18}{24}\left[g(5, -3) - g(5, 5)\right] + \frac{30}{24}\left[g(-3, -3) - g(-3, 5)\right] \leq 0. \tag{3.2}$$

By inequalities (3.1) and (3.2) we have that $\{X_1, X_2, X_3\}$ is an N-demimartingale.

Let h be a nondecreasing function. Then we have that:

$$E[h(X_1)] = \frac{3}{8}h(5) + \frac{5}{8}h(-3),$$

$$E[h(X_3 - X_2)] = \frac{1}{6}h(-7) + \frac{1}{6}h(1) + \frac{5}{24}h(-4) + \frac{11}{24}h(4),$$

$$\begin{aligned} E[h(X_1)h(X_3 - X_2)] &= h(5)\left[\frac{1}{12}h(-7) + \frac{1}{12}h(1) + \frac{1}{24}h(-4) + \frac{4}{24}h(4)\right] \\ &+ h(-3)\left[\frac{1}{12}h(-7) + \frac{1}{12}h(1) + \frac{4}{24}h(-4) + \frac{7}{24}h(4)\right]. \end{aligned}$$

We now take h *such that:* $h(-7) = h(-4) = h(-3) = 0, h(1) = 4, h(4) = 8$ *and* $h(5) = 16$.

The random variables X_1, $X_3 - X_2$ *are not negatively associated since*

$$Cov\left(h(X_1), h(X_3 - X_2)\right) = \frac{2}{3} > 0.$$

It can easily be shown that a martingale with the natural choice of σ-algebras is also an N-demimartingale. Furthermore, it can be verified that a supermartingale is an N-demisupermartingale. The converse statement is false as we can see in the following example.

Example 3.1.3 *We define the random variables* $\{X_1, X_2\}$ *such that*

$$P(X_1 = 1, X_2 = 0) = p, \;\; P(X_1 = 0, X_2 = 1) = 1 - p$$

where $\frac{1}{2} \leq p \leq 1$. *Then* $\{X_1, X_2\}$ *is an N-demisupermartingale since for every* f *nonnegative nondecreasing function*

$$E\left[(X_2 - X_1) f(X_1)\right] = (1-p) f(0) - p f(1) \leq p\left(f(0) - f(1)\right) \leq 0.$$

Observe that $\{X_1, X_2\}$ *is not a supermartingale since*

$$E\left[X_2 | X_1 = 0\right] = \sum_{x_2 = 0,1} x_2 P(X_2 = x_2 | X_1 = 0) = \frac{P(X_2 = 1, X_1 = 0)}{P(X_1 = 0)} = 1 > 0.$$

The partial sum of mean zero associated random variables is not the only special case of an N-demimartingale. One more special case is presented in the following proposition.

Proposition 3.1.4 *Let* $\{X_n, n \in \mathbb{N}\}$ *be negatively associated random variables with* $E(X_k) \leq 0, \ \forall\, k$ *and let* $\{Y_n, n \in \mathbb{N}\}$ *be a sequence of nonnegative, independent random variables and independent of the* X_i*'s. Let*

$$T_n = \sum_{i=1}^{n} X_i Y_i.$$

Then T_n *is an N-demisupermartingale. In case* $E(X_k) = 0 \ \forall\, k$, T_n *is an N-demimartingale.*

Proof. Let f be a nonnegative componentwise nondecreasing function. Then for $y_1, \ldots, y_n$ real numbers

$$\begin{aligned}
E & \quad [(T_{n+1} - T_n) f(T_1, \ldots, T_n)] \\
&= E\left[X_{n+1} Y_{n+1} f(T_1, \ldots, T_n)\right] \\
&= E(Y_{n+1}) E\left[X_{n+1} f(T_1, \ldots, T_n)\right] \\
&= E(Y_{n+1}) E\left\{E\left[X_{n+1} f(T_1, \ldots, T_n) | Y_1, \ldots, Y_n\right]\right\} \\
&= E(Y_{n+1}) \int E\left[X_{n+1} f\left(X_1 y_1, \ldots, \sum_{i=1}^{n} X_i y_i\right)\right] dF_{Y_1}(y_1) \ldots dF_{Y_n}(y_n) \\
&\leq E(Y_{n+1}) \int E(X_{n+1}) E\left[f\left(X_1 y_1, \ldots, \sum_{i=1}^{n} X_i y_i\right)\right] dF_{Y_1}(y_1) \ldots dF_{Y_n}(y_n) \quad (3.3) \\
&\leq 0 \quad (3.4)
\end{aligned}$$

where (3.3) follows from the negative association property and (3.4) follows from the fact that $E(Y_{n+1}) \geq 0$, $E(X_{n+1}) \leq 0$ and $E(f(T_1, \ldots, T_n)) \geq 0$.

If $E(X_k) = 0 \ \forall \ k$ then f need not be nonnegative and (3.4) follows immediately implying that $\{T_n, n \in \mathbb{N}\}$ is an N-demimartingale. ■

Remark 3.1.5 *Following the same steps, for the sequences of the previous proposition and considering the random variables*

$$V_n = \sum_{1 \leq i < j \leq n} c_{ij} \min\{X_i, X_j\} \tag{3.5}$$

and

$$U_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} X_i Y_j \tag{3.6}$$

where c_{ij} are nonnegative real numbers, one can show that $\{V_n, n \in \mathbb{N}\}$ is an N-demisupermartingale and for each m $\{U_{n,m}, n \in \mathbb{N}\}$ is an N-demisupermartingale or an N-demimartingale depending on whether $E(X_k) \leq 0 \ \forall \ k$ or $E(X_k) = 0 \ \forall \ k$.
This can easily be proven since for the sequence of $\{V_n, n \in \mathbb{N}\}$ we have that

$$E[(V_{n+1} - V_n) f(V_1, \ldots, V_n) \quad = \quad E\left[\sum_{i=1}^{n} c_{in+1} \min\{X_i, X_{n+1}\} f(V_1, \ldots, V_n)\right]$$

$$
\begin{aligned}
&\leq E\left[X_{n+1}\left(\sum_{i=1}^{n} c_{in+1}\right) f(V_1\ldots,V_n)\right] \\
&\leq E(X_{n+1})E\left[\left(\sum_{i=1}^{n} c_{in+1}\right) f(V_1\ldots,V_n)\right] \\
&\leq 0.
\end{aligned}
$$

For the sequence of $\{U_{n,m}, n \in \mathbb{N}\}$ the proof follows by using similar arguments to those we used for the proof of the Proposition 3.1.4.

The random variable in (3.5) can be considered as a special case of an one sample U-statistic while the random variable in (3.6) can be considered as a special case of a generalized U-statistic.

As it has already been mentioned in Chapter 2 the sequence $\{g(S_n), n \in \mathbb{N}\}$, where $\{S_n, n \in \mathbb{N}\}$ is a demimartingale and g a nondecreasing convex function, is always a demisubmartingale. The same question arises for the case of N-demimartingales, i.e., for what functions g the N-demisupermartingale property is not violated for the sequence $\{g(S_n), n \in \mathbb{N}\}$ if $\{S_n, n \in \mathbb{N}\}$ is an N-demimartingale.

Christofides (2003) (Remark 1.4) states that if $\{S_n, n \in \mathbb{N}\}$ is an N-demimartingale and $Y_n = aS_n + b$ with $a, b \in \mathbb{R}$ then $\{Y_n, n \in \mathbb{N}\}$ is also an N-demimartingale. This can be seen in the next lemma. However, the problem of finding a general class of functions with this property remains open.

Lemma 3.1.6 *Let $\{S_n, n \in \mathbb{N}\}$ be an N-demimartingale and let $Y_n = aS_n + b$ with $a, b \in \mathbb{R}$. Then $\{Y_n, n \in \mathbb{N}\}$ is an N-demimartingale. If $\{S_n, n \in \mathbb{N}\}$ is an N-demisupermartingale then $Y_n = aS_n + b$ is also an N-demisupermartingale provided $a > 0$.*

Proof. Let f be a componentwise nondecreasing function. Then

$$E[(Y_{n+1} - Y_n)f(Y_1, \ldots, Y_n)] = E[a(S_{n+1} - S_n)f(aS_1 + b, \ldots, aS_n + b)] \leq 0.$$

since $f_1(S_1, \ldots, S_n) = af(aS_1 + b, \ldots, aS_n + b)$ is a componentwise nondecreasing function of $S_1, \ldots, S_n$ for $a \in \mathbb{R}$.

In the case where $\{S_n, n \in \mathbb{N}\}$ is an N-demisupermartingale and if f is a nonnegative componentwise nondecreasing function then

$$E[(Y_{n+1} - Y_n)f(Y_1, \ldots, Y_n)] \leq 0$$

since the function $f_1(S_1, \ldots, S_n) = af(aS_1 + b, \ldots, aS_n + b)$ is a nonnegative componentwise nondecreasing function for $a > 0$. ■

Although no general class of functions can be found so that the N-demimartingale or the N-demisupermartingale property is maintained, we can establish that some functions of N-demimartingales also form an N-demi(super)martingale by direct verification. We provide two such examples.

Example 3.1.7 *Let $X_1, X_2, \ldots$ be negatively associated and identically distributed random variables. Let $\psi(t) = M_{X_1}(t) = E[e^{tX_1}]$ for $t \in \mathbb{R}$ be the moment generating function of X_1. Let $S_n = \sum_{k=1}^{n} X_k$ and*

$$Y_n(t) = \frac{e^{tS_n}}{[\psi(t)]^n} = \prod_{k=1}^{n} \frac{e^{tX_k}}{\psi(t)}$$

for $n \geq 1$ and $t \in \mathbb{R}$. Then $\{Y_n(t), n \in \mathbb{N}\}$ for $t \geq 0$ is an N-demisupermartingale. This can easily be proven since for f nonnegative componentwise nondecreasing function

$$E[(Y_{n+1} - Y_n)f(Y_1, \ldots, Y_n)] = E\left[\left(\frac{e^{tX_{n+1}}}{\psi(t)} - 1\right) f_1(Y_1, \ldots, Y_n)\right]. \qquad (3.7)$$

For $t \geq 0$ the function $g(X) = e^{tX}$ is a nondecreasing function of X, therefore $f_1(Y_1, \ldots, Y_n)$ is a nonnegative componentwise nondecreasing function of $X_1, \ldots, X_n$. Then by equality 3.7

$$E[(Y_{n+1} - Y_n)f(Y_1, \ldots, Y_n)] \leq E\left(\frac{e^{tX_{n+1}}}{\psi(t)} - 1\right) E\left[f_1(Y_1, \ldots, Y_n)\right] = 0$$

where the first inequality follows by the negative association property and the last equality by the fact that $E\left[\frac{e^{tX_{n+1}}}{\psi(t)}\right] = 1$.

Example 3.1.8 *Let $\{S_n, n \in \mathbb{N}\}$ be an N-demimartingale such that*

$$-a \leq S_j - S_{j-1} \leq b \text{ for all } j$$

where $a, b > 0$. *Then*

$$Z_n = \exp\{S_n - a - bn\}$$

is an N-demisupermartingale. Let

$$L(x) = \frac{a}{a+b}e^b + \frac{b}{a+b}e^{-a} + \frac{e^b - e^{-a}}{a+b}x$$

be the line through the points $(-a, e^{-a}), (b, e^b)$. *By the convexity of the exponential function we have that* $e^x \le L(x)$ *for* $x \in [-a, b]$. *Let* f *be a nonnegative componentwise nondecreasing function. In what follows* f^* *denotes a nonnegative, componentwise nondecreasing function which is allowed to change from line to line.*

$$\begin{aligned}
E\left[(Z_{n+1} - Z_n)f(Z_1, \ldots, Z_n)\right] &= E\left[(e^{(S_{n+1}-S_n)-b} - 1)f^*(S_1, \ldots, S_n)\right] \\
&\le E\left[(e^{-b}L(S_{n+1} - S_n) - 1)f^*(S_1, \ldots, S_n)\right] \\
&= c_1 E f^*(S_1, \ldots, S_n) + c_2 e^{-b} E\left[(S_{n+1} - S_n)f^*(S_1, \ldots, S_n)\right] \\
&\le 0
\end{aligned}$$

since $c_1 = \left(\frac{a}{a+b}e^b + \frac{b}{a+b}e^{-a}\right)e^{-b} - 1 \le 0$, $c_2 = \frac{e^b - e^{-a}}{a+b} \ge 0$ *and* $\{S_n, n \in \mathbb{N}\}$ *is an N-demimartingale.*

Several maximal inequalities can be found in the literature for sequences of N-demimartingales. In particular Prakasa Rao (2004) provides the following inequality for N-demimartingales for a special case of real functions.

Theorem 3.1.9 *Let* $\{S_n, n \in \mathbb{N}\}$ *be an N-demimartingale. Let* $m(.)$ *be a nonnegative nondecreasing function on* $\mathbb{R}$ *with* $m(0) = 0$. *Let* $g(.)$ *be a function such that* $g(0) = 0$ *and suppose that*

$$g(x) - g(y) \ge (y - x)h(y)$$

for all x, y *where* $h(.)$ *is a nonnegative nondecreasing function. Further, suppose that* $\{c_k, 1 \le k \le n\}$ *is a sequence of positive numbers such that* $(c_k - c_{k-1})g(S_k) \ge 0, 1 \le k \le n - 1$. *Define*

$$Y_k = \max\{c_1 g(S_1), \ldots, c_k g(S_k)\}, k \ge 1, Y_0 \equiv 0.$$

Then

$$E\left(\int_0^{Y_n} udm(u)\right) \leq \sum_{k=1}^{n} c_k E[(g(S_k) - g(S_{k-1}))m(Y_n)].$$

For the special case of $m(t) = I\{t \geq \varepsilon\}$ Prakasa Rao's inequality has the following useful form

$$\varepsilon P(Y_n \geq \varepsilon) \leq \sum_{k=1}^{n} c_k E[(g(S_k) - g(S_{k-1}))I\{Y_n \geq \varepsilon\}]. \tag{3.8}$$

3.2 Maximal inequalities for N-demimartingales

Theorem 3.1.9 can serve as a source to obtain several useful probability and moment inequalities for the maximum of an N-demimartingale.

Theorem 3.2.1 *Let $\{S_n, n \in \mathbb{N}\}$ be an N-demimartingale. Then for $\varepsilon > 0$*

$$\varepsilon P(\max_{1\leq k\leq n} S_k \geq \varepsilon) \leq E\left(S_n I\{\max_{1\leq k\leq n} S_k \geq \varepsilon\}\right).$$

Proof. Since $\{S_n, n \in \mathbb{N}\}$ is an N-demimartingale then by Lemma 3.1.6 $\{-S_n, n \in \mathbb{N}\}$ is also an N-demimartingale. Then by (3.8) for $g(x) = -x$ and $c_k = 1 \ \forall \ k$, we have

$$\begin{aligned} \varepsilon P(\max_{1\leq k\leq n} S_k \geq \varepsilon) &\leq \sum_{k=1}^{n} E\left[(S_k - S_{k-1})\, I\{\max_{1\leq k\leq n} S_k \geq \varepsilon\}\right] \\ &= E\left(S_n I\{\max_{1\leq k\leq n} S_k \geq \varepsilon\}\right). \end{aligned} \tag{3.9}$$

■

From the previous theorem, we immediately have the following.

Corollary 3.2.2 *If $\{S_n, n \in \mathbb{N}\}$ is a nonnegative N-demimartingale then for $\varepsilon > 0$,*

$$\varepsilon P(\max_{1\leq k\leq n} S_k \geq \varepsilon) \leq E(S_n).$$

Remark 3.2.3 *Following the same steps as above, one can show that for a nonnegative N-demimartingale $\{S_n, n \in \mathbb{N}\}$ and $\varepsilon > 0$ we have that*

$$\varepsilon P(\max_{n\leq k\leq L} S_k \geq \varepsilon) \leq E(S_n)$$

and since the right hand side does not depend on L, we can immediately infer that

$$\varepsilon P(\sup_{k\geq n} S_k \geq \varepsilon) \leq E(S_n).$$

Corollary 3.2.4 *Let $\{S_n, n \in \mathbb{N}\}$ be a nonnegative N-demimartingale. Then*

$$E\left(\max_{1\leq k\leq n} S_k\right)^p \leq \left(\frac{p}{p-1}\right)^p E(S_n^p),\ p>1$$

and

$$E\left(\max_{1\leq k\leq n} S_k\right)^p \leq \left(\frac{e}{e-1}\right)\left(1+E(S_n log^+ S_n)\right),\ p=1.$$

Proof. Follows by combining (3.9) and Lemma 9.1 of Gut (2005). ∎

The next maximal inequality for N-demimartingales that is provided by the following theorem is proved through the concept of complete downcrossings of an interval by a sequence of random variables. The numbers of complete downcrossings of the interval $[a, b]$ is defined as the number of times a sequence of random variables passes from above b to below a. The next lemma is due to Prakasa Rao (2002b).

Lemma 3.2.5 *(Prakasa Rao (2002)) Let $\{S_n, n \in \mathbb{N}\}$ be an N-demimartingale. Then for any real numbers a, b such that $a < b$*

$$E(D_{a,b}) \leq \frac{1}{b-a}\left[E(b-S_n)^+ - E(b-S_1)^+\right]$$

where $D_{a,b}$ is the number of complete down crossings of the interval $[a, b]$ by $S_1, \ldots, S_n$.

Theorem 3.2.6 *Let $\{S_n, n \in \mathbb{N}\}$ be an N-demimartingale.*
Then for $c > 0$

$$\begin{aligned} P\left(\max_{k\leq n}|S_k| > 2c\right) &\leq P(|\ S_n\ |> c) + \frac{1}{c}\left[\int_{\{|S_n|>2c\}}(|S_n|-2c)dP \right.\\ &\left. - \int_{\{|S_1|>2c\}}(|S_1|-2c)dP\right]. \end{aligned} \quad (3.10)$$

Proof. Following similar steps to those in the proof of Theorem 2.3.2, we define the sets $A_n = \{\max_{k\leq n} S_k > 2c\}$ and let d_1 be the number of downcrossings of $[c, 2c]$ by $S_1, \ldots, S_n$. Then,

$$\begin{aligned} P(A_n) &= P\left(A_n, S_n \leq c\right) + P\left(A_n, S_n > c\right) \\ &\leq P\left(\max_{k\leq n} S_k > 2c, S_n \leq c\right) + P\left(S_n > c\right) \\ &\leq P\left(d_1 > 0\right) + P\left(S_n > c\right) \\ &\leq Ed_1 + P\left(S_n > c\right). \end{aligned} \tag{3.11}$$

Let $B_n = \{\min_{k\leq n} S_k < -2c\}$ and let d_2 be the number of downcrossings of $[c, 2c]$ by $-S_1, \ldots, -S_n$. Then,

$$\begin{aligned} P(B_n) &= P\left(B_n, S_n \geq -c\right) + P\left(B_n, S_n < -c\right) \\ &\leq P\left(\min_{k\leq n} S_k < -2c, S_n \geq -c\right) + P\left(S_n < -c\right) \\ &= P\left(-\min_{k\leq n} S_k > 2c, -S_n \leq c\right) + P\left(S_n < -c\right) \\ &= P\left(\max_{k\leq n}(-S_k) > 2c, -S_n \leq c\right) + P\left(S_n < -c\right) \\ &\leq P\left(d_2 > 0\right) + P\left(S_n < -c\right) \\ &\leq Ed_2 + P\left(S_n < -c\right). \end{aligned} \tag{3.12}$$

Furthermore,

$$\begin{aligned} P(A_n \cup B_n) &= P\left(\{\min_{k\leq n} S_k < -2c\} \cup \{\max_{k\leq n} S_k > 2c\}\right) \\ &= P\left(\{-\min_{k\leq n} S_k > 2c\} \cup \{\max_{k\leq n} S_k > 2c\}\right) \\ &= P\left(\{\max_{k\leq n}(-S_k) > 2c\} \cup \{\max_{k\leq n} S_k > 2c\}\right) \\ &= P\left(\max_{k\leq n} |S_k| > 2c\right). \end{aligned}$$

Therefore,

$$P\left(\max_{k\leq n} |S_k| > 2c\right) = P(A_n \cup B_n)$$

$$
\begin{aligned}
&\leq P(A_n) + P(B_n) \\
&\leq Ed_1 + Ed_2 + P\left(S_n < -c\right) + P\left(S_n > c\right) \qquad (3.13) \\
&= Ed_1 + Ed_2 + P\left(|S_n| > c\right) \qquad (3.14)
\end{aligned}
$$

where inequality (3.13) follows by (3.11) and (3.12).

Applying Lemma 3.2.5 we have

$$
\begin{aligned}
Ed_1 + Ed_2 &\leq \frac{1}{c}\left\{E\left[2c - S_n\right]^+ - E\left[2c - S_1\right]^+ + E\left[2c + S_n\right]^+ - E\left[2c + S_1\right]^+\right\} \\
&= \frac{1}{c}\left\{\int_{\{|S_n|>2c\}} (|S_n| - 2c)dP - \int_{\{|S_1|>2c\}} (|S_1| - 2c)dP\right\}. \qquad (3.15)
\end{aligned}
$$

Then (3.10) follows by (3.14) and (3.15). ■

By applying Theorem 3.2.6 for nonnegative N-demimartingales we have the following corollary.

Corollary 3.2.7 *Let $\{S_n, n \in \mathbb{N}\}$ be a nonnegative N-demimartingale. Then for all $c > 0$*

$$
P(\max_{k\leq n} S_k > 2c) \leq P(S_n > c) + \frac{1}{c}\int_A (S_n - 2c)dP
$$

where $A = \{S_n > 2c, S_1 \leq 2c\}$.

Proof. Since $\{S_n, n \in \mathbb{N}\}$ is a nonnegative N-demimartingale by Theorem 3.2.6 we have

$$
\begin{aligned}
P(\max_{k\leq n} S_k > 2c) \leq\; & P(S_n > c) + \frac{1}{c}\{E[(S_n - 2c)I\{S_n > 2c\}] \\
& -E[(S_1 - 2c)I\{S_1 > 2c\}]\} \qquad (3.16)
\end{aligned}
$$

We need to find an upper bound for the quantity

$$
B \equiv E[(S_n - 2c)I\{S_n > 2c\}] - E[(S_1 - 2c)I\{S_1 > 2c\}].
$$

Therefore,

$$
B = E[(S_n - 2c)(I\{S_1 > 2c\} + I\{S_n > 2c\} - I\{S_1 > 2c\})]
$$

$$
\begin{aligned}
& -E[(S_1 - 2c)I\{S_1 > 2c\}] \\
= \; & E[(S_n - S_1)I\{S_1 > 2c\}] + E[(S_n - 2c)(I\{S_n > 2c\} - I\{S_1 > 2c\})] \\
\leq \; & E[(S_n - 2c)(I\{S_n > 2c\} - I\{S_1 > 2c\})] \\
= \; & E[(S_n - 2c)I\{S_n > 2c, S_1 \leq 2c\}, \qquad (3.17)
\end{aligned}
$$

where the last inequality follows by the N-demimartingale property. Inequalities (3.16) and (3.17) together give the desired result. ∎

3.3 Azuma's inequality

Hoeffding (1963) obtained the following inequality for independent random variables.

Theorem 3.3.1 *Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of independent random variables such that $a_i \leq X_i \leq b_i$ for $i = 1, 2, \ldots$. Then for $t > 0$*

$$P\left(\frac{S_n}{n} - \mu \geq t\right) \leq \exp\left\{\frac{-2n^2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\}$$

where $S_n = \sum_{i=1}^n X_i$ and $\mu = \frac{1}{n}ES_n$.

Hoeffding's result was extended to the case of bounded martingale differences. The following result for martingale differences was given by Azuma (1967).

Theorem 3.3.2 *Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of martingale differences such that $|X_i| < \alpha < \infty$ for all $i = 1, 2, \ldots$ and let $S_n = X_1 + \cdots + X_n$. Then for every $\varepsilon > 0$,*

$$P(S_n \geq n\varepsilon) \leq \exp\left\{\frac{-n\varepsilon^2}{2\alpha^2}\right\}.$$

Given that a martingale with the natural choice of σ-algebras is an N-demimartingale, it is of interest to see whether an analog of the above inequality holds true for N-demimartingales. The answer is given by the following result.

Theorem 3.3.3 *Let $\{S_n, n \in \mathbb{N}\}$ (with $S_0 \equiv 0$) be an N-demimartingale and assume that*

$$|S_i - S_{i-1}| \leq c_i < \infty \quad i = 1, 2, \ldots,$$

where $c_1, c_2, \ldots$ are positive real numbers. Then for every $\varepsilon > 0$,

$$P(S_n - E(S_n) \geq n\varepsilon) \leq \exp\left\{\frac{-n^2\varepsilon^2}{2\sum_{i=1}^n c_i^2}\right\} \tag{3.18}$$

and

$$P(|S_n - E(S_n)| \geq n\varepsilon) \leq 2\exp\left\{\frac{-n^2\varepsilon^2}{2\sum_{i=1}^n c_i^2}\right\}. \tag{3.19}$$

Proof. Without loss of generality we assume that $E(S_1) = 0$. Let $t \in \mathbb{R}$ and $x \in [-c_i, c_i]$. We can write

$$tx = \frac{1}{2}\left(1 + \frac{x}{c_i}\right)(c_i t) + \frac{1}{2}\left(1 - \frac{x}{c_i}\right)(-c_i t).$$

By the convexity of the exponential function we have:

$$e^{tx} \leq \cosh(c_i t) + \frac{x}{c_i}\sinh(c_i t).$$

Then

$$E\left[e^{tS_n}\right] = E\left[\prod_{i=1}^n e^{t(S_i - S_{i-1})}\right] \leq E\left[\prod_{i=1}^n \left[\cosh(c_i t) + (S_i - S_{i-1})\frac{\sinh(c_i t)}{c_i}\right]\right].$$

By induction we will prove that

$$E\left[e^{tS_n}\right] \leq \prod_{i=1}^n \cosh(c_i t) \quad \forall\, t > 0. \tag{3.20}$$

First observe that,

$$\begin{aligned} E\left[e^{tS_2}\right] &\leq E\left[\left(\cosh(c_1 t) + S_1\frac{\sinh(c_1 t)}{c_1}\right)\left(\cosh(c_2 t) + (S_2 - S_1)\frac{\sinh(c_2 t)}{c_2}\right)\right] \\ &= \cosh(c_1 t)\cosh(c_2 t) + \frac{\sinh(c_1 t)\sinh(c_2 t)}{c_1 c_2}E[S_1(S_2 - S_1)] \\ &\quad + \frac{\cosh(c_1 t)\sinh(c_2 t)}{c_2}E(S_2 - S_1) + \frac{\cosh(c_2 t)\sinh(c_1 t)}{c_1}E(S_1) \\ &= \cosh(c_1 t)\cosh(c_2 t) + \frac{\sinh(c_1 t)\sinh(c_2 t)}{c_1 c_2}E[S_1(S_2 - S_1)] \\ &\leq \cosh(c_1 t)\cosh(c_2 t) \end{aligned}$$

where the first inequality follows by the fact that $ES_1 = 0$ and $ES_2 = ES_1$ (implied by the N-demimartingale property) and the second inequality follows from the

N-demimartingale property. Thus (3.20) is true for $n = 2$. Assume now that the statement is true for $n = k$. We will show that it is true for $n = k + 1$.

$$\begin{aligned} E\left[e^{tS_{k+1}}\right] &= E\left[e^{t(S_{k+1}-S_k)}.e^{tS_k}\right] \\ &\leq E\left[\left(\cosh(c_{k+1}t) + (S_{k+1}-S_k)\frac{\sinh(c_{k+1}t)}{c_{k+1}}\right)e^{tS_k}\right] \\ &= \cosh(c_{k+1}t)E\left[e^{tS_k}\right] + \frac{\sinh(c_{k+1}t)}{c_{k+1}}E\left[(S_{k+1}-S_k)e^{tS_k}\right] \\ &\leq \prod_{i=1}^{k+1}\cosh(c_i t) \end{aligned}$$

where the last inequality follows from the N-demimartingale property and the induction hypothesis. Thus (3.20) is established.

Since $\cosh(c_i t) \leq e^{\frac{c_i^2 t^2}{2}}$, by inequality (3.20) we have:

$$E\left[e^{tS_n}\right] \leq \exp\left\{\frac{t^2\sum_{i=1}^n c_i^2}{2}\right\}.$$

For $\varepsilon, t > 0$

$$\begin{aligned} P(S_n \geq n\varepsilon) &= P(tS_n \geq tn\varepsilon) \\ &= P(e^{tS_n} \geq e^{tn\varepsilon}) \\ &\leq e^{-tn\varepsilon}E\left[e^{tS_n}\right] \\ &\leq \exp\left\{-tn\varepsilon + \frac{t^2\sum_{i=1}^n c_i^2}{2}\right\}. \end{aligned}$$

The above upper bound is minimized with respect to t by choosing $t = n\varepsilon/\sum_{i=1}^n c_i^2$ and (3.18) is established.

To prove inequality (3.19) we write:

$$P(|S_n| \geq n\varepsilon) = P(S_n \geq n\varepsilon) + P(-S_n \geq n\varepsilon).$$

Since by Lemma 3.1.6 the collection $\{-S_n, n \in \mathbb{N}\}$ is also an N-demimartingale, inequality (3.19) follows by applying inequality (3.18) twice. ■

Remark 3.3.4 *In case the N-demimartingale differences are uniformly bounded, i.e., when $|S_i - S_{i-1}| \le c < \infty$ for $i = 1, 2, \ldots$, then (3.18) and (3.19) have the simple form*

$$P(S_n - E(S_n) \ge n\varepsilon) \le \exp\left\{\frac{-n\varepsilon^2}{2c^2}\right\},$$

and

$$P(|S_n - E(S_n)| \ge n\varepsilon) \le 2\exp\left\{\frac{-n\varepsilon^2}{2c^2}\right\}$$

respectively.

Given that the partial sum of mean zero negatively associated random variables is an N-demimartingale, we immediately have the following result.

Corollary 3.3.5 *Let $\{X_n, n \in \mathbb{N}\}$ be mean zero negatively associated random variables such that $|X_k| \le c_k,\ k = 1, 2, \ldots$. Let $S_n = X_1 + \cdots + X_n$. Then for every $\varepsilon > 0$,*

$$P(S_n \ge n\varepsilon) \le \exp\left\{\frac{-n^2\varepsilon^2}{2\sum_{i=1}^n c_i^2}\right\},$$

and

$$P(|S_n| \ge n\varepsilon) \le 2\exp\left\{\frac{-n^2\varepsilon^2}{2\sum_{i=1}^n c_i^2}\right\}.$$

Remark 3.3.6 *Exponential inequalities for mean zero negatively associated random variables $\{X_i, i \ge 1\}$ have been obtained by Han (2007) following a different approach, and in particular a Hoeffding inequality under the assumption that $a_i \le X_i \le b_i$ for all i. It is worth mentioning that for $b_i = c_i$ and $a_i = -c_i$ for $i = 1, 2\ldots$, Corollary 3.3.5 provides the same bound.*

As an application of Corollary 3.3.5 we give the following result which was first proven by Matula (1997).

Corollary 3.3.7 *Let $\{X_n, n \in \mathbb{N}\}$ be a sequence of negatively associated random variables with common distribution function F. Then for every $\varepsilon > 0$ and for $x \in \mathbb{R}$,*

$$P(|F_n(x) - F(x)| > \varepsilon) \le 2\exp\left\{\frac{-n\varepsilon^2}{2}\right\}$$

where F_n denotes the empirical distribution function.

Corollary 3.3.7, follows immediately from Corollary (3.3.5), given that for fixed x, $\{I_{\{X_n \le x\}} - F(x), n \in \mathbb{N}\}$ is a sequence of mean zero, negatively associated random variables satisfying

$$|I_{\{X_n \le x\}} - F(x)| \le 1, \quad n = 1, 2, \ldots,$$

where $I_{\{X_n \le x\}}$ is the indicator function of the set $\{X_n \le x\}$.

The following exponential inequality for mean zero negatively associated random variables is due to Matula (1997).

Theorem 3.3.8 *Let $\{X_n, n \in \mathbb{N}\}$ be mean zero negatively associated random variables such that $|X_k| \le \alpha \ \ \forall\ k$. Let $S_n = X_1 + \cdots + X_n$. Then for every $\varepsilon > 0$,*

$$P(S_n \ge n\varepsilon) \le \exp\left\{\frac{-n\varepsilon}{2\alpha} \sinh^{-1}\left(\frac{n\varepsilon\alpha}{2b_n}\right)\right\}.$$

where $b_n = \sum_{i=1}^n EX_i^2$.

The bounds of Corollary 3.3.5 (for the special case where $c_k = \alpha$ for $k = 1, 2, \ldots$) and Theorem 3.3.8 are not directly comparable given that the bound in Theorem 3.3.8 is expressed in terms of the variances of the random variables. Let $y = \varepsilon/\alpha$. To compare the two bounds it is sufficient to check under what conditions the function

$$f(y) = y - \sinh^{-1}\left(\frac{ny\alpha^2}{2b_n}\right), \ y > 0$$

is nonnegative. Simple calculations show that if $n\alpha^2 \le 2b_n$ or $y \ge 1$ then f is nondecreasing and since $f(0) = 0$, the function is nonnegative. This means that if $n\alpha^2 \le 2b_n$ or $\varepsilon \ge \alpha$ the bound of Corollary 3.3.5 is sharper than the bound of Theorem 3.3.8. However, if $n\alpha^2 > 2b_n$ and $y < \sqrt{\alpha^4 - 4b_n^2/n^2}$, i.e., if $n\alpha^2 > 2b_n$ and $\varepsilon < \alpha^{-1}\sqrt{\alpha^4 - 4b_n^2/n^2}$, the function is nonincreasing implying that Matula's bound is sharper than the bound of Corollary 3.3.5.

Remark 3.3.9 *By applying Theorem 3.3.3, one can obtain large deviation inequalities for other statistical functions involving negatively associated random variables such as the ones described in (3.5) and (3.6).*

Remark 3.3.10 *The problem of providing exponential bounds for the tail probabilities $P(\mid S_n \mid \geq n\varepsilon)$ is of great importance in probability and statistics. From a statistical view-point such inequalities can be used for obtaining rates of convergence for estimates of various quantities especially in a nonparametric setting. Qin and Li (2010) discuss the construction of confidence intervals for the regression vector β in a linear model under negatively associated errors. Lemma 2 of their paper provides an exponential inequality for the tail probability of the sum of the negatively associated errors which is the "key" result for obtaining a central limit theorem leading to the desired result.*

3.4 Marcinkiewicz-Zygmund inequality

The previous section provides exponential inequalities for the tail probability $P(S_n \geq n\varepsilon)$ under the assumption of bounded N-demimartingale differences. It is of interest to study the case where this assumption is replaced by

$$\|S_{n+1} - S_n\|_p < M < \infty$$

for $p > 1$, where $\|X\|_p = [E(|X|^p)]^{\frac{1}{p}}$. The answer is given through the so called Marcinkiewicz-Zygmund inequality for N-demimartingales.

The Marcinkiewicz-Zygmund inequalities, named after Jozef Marcinkiewicz and Antoni Zygmund, give relations between moments of sums and moments of summands. In the following theorem we present two of those inequalities.

Theorem 3.4.1 *Let $p \geq 1$. Suppose that $\{X_n, n \in \mathbb{N}\}$ are mean zero independent random variables such that $E|X_k|^p < \infty$ for all k, and let $\{S_n, n \in \mathbb{N}\}$ denote the partial sums. Then there exist constants A_p and B_p depending only on p such that*

$$A_p E\left(\sum_{k=1}^{n} X_k^2\right)^{\frac{p}{2}} \leq E|S_n|^p \leq B_p E\left(\sum_{k=1}^{n} X_k^2\right)^{\frac{p}{2}}$$

or equivalently

$$(A_p)^{\frac{1}{p}} \|Q_n(X)\|_p \leq \|S_n\|_p \leq (B_p)^{\frac{1}{p}} \|Q_n(X)\|_p$$

where

$$Q_n(X) = \left(\sum_{k=1}^{n} X_k^2\right)^{\frac{1}{2}}$$

is the quadratic variation of the summands and $\|S_n\|_p = [E(|S_n|^p)]^{\frac{1}{p}}$.

Burkholder (1966) proved the following Marcinkiewicz-Zygmund inequality for martingales.

Theorem 3.4.2 *Let* $p > 1$. *Suppose that* $\{(X_n, \mathcal{F}_n), n \in \mathbb{N}\}$ *is a martingale with martingale differences* $\{Y_k, k \geq 0\}$ *and let* $S_n(X) = (\sum_{k=0}^{n} Y_k^2)^{\frac{1}{2}}$ *be the square function. Then there exist constants* A_p *and* B_p *depending only on* p *such that*

$$A_p \|S_n(X)\|_p \leq \|X_n\|_p \leq B_p \|S_n(X)\|_p.$$

Remark 3.4.3 *If the increments are independent mean zero, Theorem 3.4.2 reduces to Theorem 3.4.1.*

It is worth mentioning that a lot of attention has been given to the best constants or the growth rates of the constants appearing in these moment inequalities.

We now present a known, useful inequality for real numbers. The proof follows by standard arguments and it is therefore omitted.

Lemma 3.4.4 *Let* a, b *be real numbers and let* $p \in (1, 2]$.
Then

$$|a + b|^p \leq |a|^p + p|a|^{p-1} sign(a) b + 2^{2-p}|b|^p. \tag{3.21}$$

The next result provides a Marcinkiewicz-Zygmund inequality for nonnegative N-demimartingales. Although important by itself, the result is used in the theorem that follows to establish an upper bound for the tail probability $P(S_n \geq n\varepsilon)$.

Lemma 3.4.5 *Let $\{S_n, n \in \mathbb{N}\}$ (with $S_0 \equiv 0$) be a nonnegative N-demimartingale. Then*

(i) For $p \in (1, 2]$,

$$\begin{aligned} \|S_n\|_p^p &\leq \|d_1\|_p^p + 2^{2-p} \sum_{j=1}^{n-1} \|d_{j+1}\|_p^p \\ &\leq 2^{2-p} \sum_{j=1}^{n} \|d_j\|_p^p. \end{aligned}$$

(ii) For p positive even integer,

$$\begin{aligned} \|S_n\|_p^2 &\leq \|d_1\|_p^2 + (p-1) \sum_{j=1}^{n-1} \|d_{j+1}\|_p^2 \\ &\leq (p-1) \sum_{j=1}^{n} \|d_j\|_p^2, \end{aligned}$$

where $d_j = S_j - S_{j-1},\ j = 1, 2, \ldots, n$.

Proof. Let $p \in (1, 2]$. By applying Lemma 3.4.4 for $a = S_j,\ b = S_{j+1} - S_j$ we have that:

$$\begin{aligned} ES_{j+1}^p &\leq ES_j^p + pE[S_j^{p-1}(S_{j+1} - S_j)] + 2^{2-p} E|d_{j+1}|^p \\ &\leq ES_j^p + 2^{2-p} E|d_{j+1}|^p \end{aligned}$$

where the last inequality follows by the N-demimartingale property.

By induction we have that,

$$\begin{aligned} ES_n^p &\leq Ed_1^p + 2^{2-p} \sum_{j=2}^{n} E|d_j|^p \\ &= \|d_1\|_p^p + 2^{2-p} \sum_{j=2}^{n} \|d_j\|_p^p \\ &\leq 2^{2-p} \sum_{j=1}^{n} \|d_j\|_p^p. \end{aligned}$$

Let p be positive even integer.

We assume that $\|S_j\|_p > 0$, for all j, otherwise the result is trivially true. Without loss of generality we assume that $\|S_j\|_p = 1$.

Following Rio (2009), we define the function φ on $[0, \infty)$ by

$$\varphi(t) = \|S_j + t(S_{j+1} - S_j)\|_p^p.$$

We need to prove that,

$$\varphi(t) \leq \left[1 + (p-1)\|S_{j+1} - S_j\|_p^2 t^2\right]^{\frac{p}{2}}. \tag{3.22}$$

By Taylor's integral formula for $\varphi_1(t) = |S_j + t(S_{j+1} - S_j)|^p$,

$$\begin{aligned} |S_j + t(S_{j+1} - S_j)|^p &= S_j^p + pt(S_{j+1} - S_j)S_j^{p-1} \\ &\quad + p(p-1)\int_0^t (t-s)(S_{j+1} - S_j)^2 |S_j + s(S_{j+1} - S_j)|^{p-2} ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi(t) &= ES_j^p + ptE[(S_{j+1} - S_j)S_j^{p-1}] \\ &\quad + p(p-1)\int_0^t (t-s)E\left[(S_{j+1} - S_j)^2 |S_j + s(S_{j+1} - S_j)|^{p-2}\right] ds. \end{aligned}$$

By Hölder's inequality and the N-demimartingale property we have that,

$$\varphi(t) \leq 1 + p(p-1)\|d_{j+1}\|_p^2 \int_0^t (t-s)[\varphi(s)]^{1-\frac{2}{p}} ds := \psi(t).$$

For the first two derivatives of the function $\psi(t)$ we may write,

$$\psi'(t) = p(p-1)\|d_{j+1}\|_p^2 \int_0^t [\varphi(s)]^{1-\frac{2}{p}} ds$$

and

$$\psi''(t) = p(p-1)\|d_{j+1}\|_p^2 [\varphi(t)]^{1-\frac{2}{p}} \leq p(p-1)\|d_{j+1}\|_p^2 [\psi(t)]^{1-\frac{2}{p}}.$$

By multiplying the above inequality by $2\psi'$ and integrating between 0 and x:

$$\int_0^x 2\psi'(t)\psi''(t)dt \leq 2p(p-1)\|d_{j+1}\|_p^2 \int_0^x \psi'(t)[\psi(t)]^{1-\frac{2}{p}} dt$$

$$\psi'(x) \leq p\|d_{j+1}\|_p \sqrt{[\psi(x)]^{2-\frac{2}{p}} - 1}.$$

By Lemma 2.1 of Rio (2009) we have that:

$$\psi'(t) \leq p\sqrt{p-1}\|d_{j+1}\|_p[\psi(t)]^{1-\frac{2}{p}}\sqrt{[\psi(t)]^{\frac{2}{p}}-1}.$$

By defining $z = \psi(t)^{\frac{2}{p}}$ and solving the differential inequality we finally obtain,

$$\psi(t) \leq [1 + (p-1)\|d_{j+1}\|_p^2 t^2]^{\frac{p}{2}}.$$

Since $\varphi \leq \psi$ (3.22) is established.

For $t = 1$ and by induction we finally have the desired result. ■

Let us consider the case of an N-demimartingale difference sequence with finite moments of order p.

Theorem 3.4.6 *Let $\{S_n, n \in \mathbb{N}\}$ be a nonnegative N-demimartingale such that for $p > 1$*

$$\|S_{j+1} - S_j\|_p < M_{j+1} < \infty, \text{ for } j = 1, 2, \ldots.$$

Then for every $\varepsilon > 0$,

(i)

$$P(S_n > n\varepsilon) \leq \frac{4}{(2n\varepsilon)^p}\sum_{j=1}^{n} M_j^p, \ 1 < p \leq 2$$

(ii)

$$P(S_n > n\varepsilon) \leq \frac{(p-1)^{\frac{p}{2}}}{n^p\varepsilon^p}\left(\sum_{j=1}^{n} M_j^2\right)^{\frac{p}{2}}, \text{ p positive even integer.}$$

Proof. Let $p \in (1, 2]$. By Lemma 3.4.5 and since $\|S_{j+1} - S_j\|_p < M_{j+1}$ we have that

$$ES_n^p \leq 2^{2-p}\sum_{j=1}^{n}\|d_j\|_p^p \leq 2^{2-p}\sum_{j=1}^{n} M_j^p.$$

Therefore for every $\varepsilon > 0$,

$$P(S_n > n\varepsilon) \leq \frac{ES_n^p}{n^p\varepsilon^p} \leq \frac{4}{(2n\varepsilon)^p}\sum_{j=1}^{n} M_j^p.$$

Let p be a positive even integer. Using Lemma 3.4.5 we can write

$$ES_n^p \leq (p-1)^{\frac{p}{2}}\left(\sum_{j=1}^{n} M_j^2\right)^{\frac{p}{2}}.$$

Then for every $\varepsilon > 0$,

$$P(S_n > n\varepsilon) \leq \frac{ES_n^p}{n^p\varepsilon^p} \leq \frac{(p-1)^{\frac{p}{2}}}{n^p\varepsilon^p}\left(\sum_{j=1}^{n} M_j^2\right)^{\frac{p}{2}}.$$

■

For the special case of uniformly bounded N-demimartingale difference sequences of order p the result of Theorem 3.4.6 has the following form.

Corollary 3.4.7 *Let $\{S_n, n \in \mathbb{N}\}$ be a nonnegative N-demimartingale such that*

$$\|S_{j+1} - S_j\|_p < M < \infty$$

for $p > 1$.

Then for every $\varepsilon > 0$,

(i)

$$P(S_n > n\varepsilon) \leq \frac{4}{n^{p-1}}\left(\frac{M}{2\varepsilon}\right)^p, \; 1 < p \leq 2$$

(ii)

$$P(S_n > n\varepsilon) \leq \left(\frac{p-1}{n}\right)^{\frac{p}{2}}\left(\frac{M}{\varepsilon}\right)^p, \; p \text{ positive even integer.}$$

3.5 Asymptotic results

Large deviation inequalities serve, among other things, the purpose of studying the asymptotic behavior of statistical functions, and in particular those of estimators.
The following asymptotic results concern the complete convergence of bounded N-demimartingales. First, let us recall the definition of complete convergence.

Definition 3.5.1 *Let $X_1, X_2, \ldots$ be random variables. X_n converges completely to the random variable X as $n \to \infty$, if and only if*

$$\sum_{n=1}^{\infty} P\left(| X_n - X |> \varepsilon\right) < \infty, \quad \text{for all } \; \varepsilon > 0.$$

Theorem 3.5.2 *Let $\{S_n, n \in \mathbb{N}\}$ be an N-demimartingale such that $\mid S_i - S_{i-1} \mid \leq \alpha$ for all $i = 1, 2, \ldots$. Then for $r > \frac{1}{2}$,*

$$n^{-r} S_n \longrightarrow 0 \quad \textit{completely.}$$

Proof. Assume without loss of generality that $ES_1 = 0$. Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(\mid S_n \mid \geq n^r \varepsilon) &\leq 2 \sum_{n=1}^{\infty} \exp\left\{ \frac{-n^{2r-1}\varepsilon^2}{2\alpha^2} \right\} \\
&= 2 \sum_{n=1}^{\infty} \exp\left\{-n^{2r-1} d\right\} \\
&= 2 \sum_{n=1}^{\infty} \left(\exp(-d)\right)^{n^{2r-1}} < \infty
\end{aligned}$$

where $d = \varepsilon^2/2\alpha^2$ and the first inequality follows from Azuma's inequality for N-demimartingales given in Remark 3.3.4. ■

As an application of Theorem 3.5.2 we immediately have the following asymptotic result for mean zero negatively associated random variables.

Corollary 3.5.3 *Let $\{X_n, n \in \mathbb{N}\}$ be mean zero negatively associated random variables such that $|X_k| \leq \alpha$ for $k = 1, 2, \ldots$. Let $S_n = X_1 + \cdots + X_n$. Then for $r > \frac{1}{2}$,*

$$n^{-r} S_n \longrightarrow 0 \quad \textit{completely.}$$

The next result generalizes Theorem 3.5.2 since the assumption of uniformly bounded N-demimartingale differences is dropped.

Theorem 3.5.4 *Let $\{S_n, n \in \mathbb{N}\}$ be an N-demimartingale such that $\mid S_i - S_{i-1} \mid \leq c_i$, for all $i = 1, 2, \ldots$. Let the following two conditions :*

(i) $\sum_{i=1}^{\infty} c_i^2 < \infty$ and let r be a positive number,

(ii) $\sum_{i=1}^{n} c_i^2 = O(n^\rho)$ where $\rho > 0$ and let $r > \frac{\rho}{2}$.

If (i) or (ii) is true, then

$$n^{-r} S_n \longrightarrow 0 \quad \textit{completely.}$$

Proof. Assume without loss of generality that $ES_1 = 0$. Furthermore, assume that (i) is true. Then

$$
\begin{aligned}
\sum_{n=1}^{\infty} P(| S_n |\geq n^r \varepsilon) &\leq 2\sum_{n=1}^{\infty} \exp\left\{\frac{-n^2 n^{2r-2}\varepsilon^2}{2\sum_{i=1}^{n} c_i^2}\right\} \\
&= 2\sum_{n=1}^{\infty} \exp\left\{\frac{-n^{2r}\varepsilon^2}{2\sum_{i=1}^{n} c_i^2}\right\} \\
&\leq 2\sum_{n=1}^{\infty} \exp\left\{\frac{-n^{2r}\varepsilon^2}{2\sum_{i=1}^{\infty} c_i^2}\right\} \\
&= 2\sum_{n=1}^{\infty} (\exp(-d))^{n^{2r}} < \infty
\end{aligned}
$$

where $d = \varepsilon^2/2\sum_{i=1}^{\infty} c_i^2$ and the first inequality follows from inequality (3.19). Assume now that (ii) is valid. By using Theorem 3.3.3 we can write

$$
\begin{aligned}
P(S_n \geq n^r \varepsilon) &\leq \exp\left\{-\frac{n^{2r}\varepsilon^2}{2\sum_{i=1}^{n} c_i^2}\right\} \\
&= \exp\left\{-\frac{\varepsilon^2}{2} O(n^{2r-\rho})\right\}.
\end{aligned}
$$

Therefore we obtain

$$
\sum_{n=1}^{\infty} P(S_n \geq n^r \varepsilon) \leq \sum_{n=1}^{\infty} \exp\left\{-\frac{\varepsilon^2}{2} O(n^{2r-\rho})\right\} < \infty
$$

for $r > \frac{\rho}{2}$. ■

Remark 3.5.5 *Observe that for $\rho = 1$ the second part of Theorem 3.5.4 reduces to the result of Theorem 3.5.2.*

By applying Theorem 3.5.4 for mean zero negatively associated random variables we have the following result.

Corollary 3.5.6 *Let $\{X_n, n \in \mathbb{N}\}$ be mean zero negatively associated random variables such that $|X_i| \leq c_i$ for all $i = 1, 2, \ldots$ and let $S_n = X_1 + \cdots + X_n$. Assume that $\sum_{i=1}^{\infty} c_i^2 < \infty$. Then for $r > 0$,*

$$
n^{-r} S_n \longrightarrow 0 \quad \textit{completely.}
$$

Complete convergence results for nonnegative N-demimartingales can also be established by using the large deviation inequality of Theorem 3.4.6.

Theorem 3.5.7 *Let $\{S_n, n \in \mathbb{N}\}$ be a nonnegative N-demimartingale such that for $p > 1$*

$$\|S_{j+1} - S_j\|_p < M_{j+1} < \infty.$$

Let us consider the following conditions for $p \in (1, 2]$:
(i) $\sum_{j=1}^{\infty} M_j^p < \infty$ and let r be a positive number such that $pr > 1$,
(ii) $\sum_{j=1}^{n} M_j^p = O(n^\alpha)$ and let r be a positive number such that $pr - 1 > \alpha$,
(iii) $\sum_{j=2}^{\infty} \frac{M_j^p}{(j-1)^{pr-1}} < \infty$ and let r be a positive number such that $pr > 1$.
For p positive even integer let the following conditions:
(iv) $\sum_{j=1}^{\infty} M_j^2 < \infty$ and let r be a positive number such that $pr > 1$,
(v) $\sum_{j=1}^{n} M_j^2 = O(n^\alpha)$, and let r be a positive number such that $pr - \frac{\alpha p}{2} > 1$.
If anyone of the above conditions is true, then

$$n^{-r} S_n \to 0, \ \textit{completely}.$$

Proof. Let $p \in (1, 2]$. Assume that (i) is true. Then by using condition (i) and by applying Theorem 3.4.6 we have that

$$\begin{aligned} \sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) &\leq \frac{4}{(2\varepsilon)^p} \sum_{n=1}^{\infty} \frac{\left(\sum_{j=1}^{n} M_j^p\right)}{n^{pr}} \\ &\leq \frac{4}{(2\varepsilon)^p} \sum_{n=1}^{\infty} \frac{\left(\sum_{j=1}^{\infty} M_j^p\right)}{n^{pr}} < \infty. \end{aligned}$$

Assume now that condition (ii) is valid. By applying Theorem 3.4.6 we can write

$$P(S_n > n^r \varepsilon) \leq \frac{4}{(2\varepsilon)^p} \frac{1}{n^{pr}} O(n^\alpha) = \frac{4}{(2\varepsilon)^p} O(n^{\alpha - pr}).$$

Therefore

$$\sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) \leq \frac{4}{(2\varepsilon)^p} \sum_{n=1}^{\infty} O(n^{\alpha - pr}) < \infty$$

for $pr - 1 > \alpha$.

Finally for the case $p \in (1,2]$ assume that condition (iii) holds. By applying Theorem 3.4.6 we arrive at the following inequality

$$\begin{aligned}
\sum_{n=1}^{\infty} P(S_n \geq n^r \varepsilon) &\leq \frac{4}{(2\varepsilon)^p} \sum_{n=1}^{\infty} \frac{1}{n^{pr}} \sum_{j=1}^{n} M_j^p \\
&= \frac{4}{(2\varepsilon)^p} \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \frac{M_j^p}{n^{pr}} \\
&= \frac{4}{(2\varepsilon)^p} \left\{ \sum_{n=1}^{\infty} \frac{M_1^p}{n^{pr}} + \sum_{j=2}^{\infty} \sum_{n=j}^{\infty} \frac{M_j^p}{n^{pr}} \right\}. \qquad (3.23)
\end{aligned}$$

It can easily be verified that

$$\sum_{n=j}^{\infty} \frac{1}{n^{pr}} \leq \int_{j-1}^{\infty} x^{-pr} dx = \frac{1}{pr-1} (j-1)^{1-pr}. \qquad (3.24)$$

We can write

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{M_1^p}{n^{pr}} &= M_1^p + M_1^p \sum_{n=2}^{\infty} \frac{1}{n^{pr}} \\
&\leq \frac{pr M_1^p}{pr-1}
\end{aligned}$$

where the last inequality follows from (3.24).

Therefore by applying (3.24) to the second term of (3.23) we finally have

$$\sum_{n=1}^{\infty} P(S_n \geq n^r \varepsilon) \leq \frac{4}{(2\varepsilon)^p} \left\{ \frac{pr M_1^p}{pr-1} + \frac{1}{pr-1} \sum_{j=2}^{\infty} \frac{M_j^p}{(j-1)^{pr-1}} \right\} < \infty.$$

Let p be a positive even integer. Assume that condition (iv) holds. By following the same steps as above we can verify that

$$\sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) \leq \frac{(p-1)^{\frac{p}{2}}}{\varepsilon^p} \sum_{n=1}^{\infty} \frac{1}{n^{pr}} \left(\sum_{j=1}^{\infty} M_j^2 \right)^{\frac{p}{2}} < \infty.$$

Finally under the assumption that condition (v) is valid,

$$\sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) \leq \frac{(p-1)^{\frac{p}{2}}}{\varepsilon^p} \sum_{n=1}^{\infty} O\left(\frac{1}{n^{pr-\frac{\alpha p}{2}}} \right) < \infty.$$

for $pr - \frac{\alpha p}{2} > 1$. ■

Using Corollary 3.4.7 we establish complete convergence for nonnegative N-demimartingales in the case where the N-demimartingale differences are uniformly bounded.

Theorem 3.5.8 *Let* $\{S_n, n \in \mathbb{N}\}$ *be a nonnegative N-demimartingale such that for* $p > 1$

$$\|S_{j+1} - S_j\|_p < M < \infty.$$

Let the following two conditions:

(i) Let $p \in (1, 2]$ *and let* r *be a positive number such that* $pr > 2$.

(ii) Let p *be a positive even integer and let* r *be a positive number such that* $p(r - \frac{1}{2}) > 1$.

If (i) or (ii) is true, then

$$n^{-r} S_n \to 0, \text{ completely.}$$

Proof. Let $p \in (1, 2]$. By applying Theorem 3.4.7 we have that

$$\sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) \leq \sum_{n=1}^{\infty} 4 \left(\frac{M}{2\varepsilon}\right)^p \frac{1}{n^{pr-1}} < \infty.$$

Following the same steps we can verify that for $p > 2$

$$\sum_{n=1}^{\infty} P(S_n > n^r \varepsilon) \leq \sum_{n=1}^{\infty} \left(\frac{M}{\varepsilon}\right)^p (p-1)^{\frac{p}{2}} \frac{1}{n^{pr-\frac{p}{2}}} < \infty.$$

∎

3.6 Blackwell-Ross inequality

Blackwell (1954) presented the following gambling problem: Let us consider a game where the player's gain or loss does not exceed one unit and that the player's expectation doesn't exceed $-ug$, where g is his maximum loss or gain and u is a constant $0 < u < 1$. The player wants to maximize his probability of becoming at least t units ahead, where t is a positive constant. The system of play is a sequence $X_1, X_2 \ldots,$ of chance variables satisfying

$$|X_n| \leq 1 \tag{3.25}$$

and

$$E(X_n | X_1, \ldots, X_{n-1}) \leq -u(\max |X_n| | X_1, \ldots, X_{n-1}). \tag{3.26}$$

Blackwell proved that for any system $X_1, X_2, \ldots$ satisfying conditions (3.25) and (3.26) and any positive number t

$$P(S_n \geq t, \text{ for some } n) \leq \left(\frac{1-u}{1+u}\right)^t.$$

where $S_n = \sum_{i=1}^n X_i$. Blackwell's result proves that this is the best system in the sense of maximizing the probability of attaining at least t.

Blackwell (1997) extended his inequalities for the case of martingales.

Theorem 3.6.1 *Let* $\{S_n = \sum_{i=1}^n X_i, S_0 = 0, \mathcal{F}_n, n \geq 0\}$ *be a supermartingale such that* $|X_n| \leq 1$ *and* $E(X_n|\mathcal{F}_{n-1}) \leq -\gamma$ *for all* n, $0 < \gamma < 1$. *Then for any* $a > 0$,

$$P(S_n \geq a \text{ for some } n \geq 1) \leq \left(\frac{1-\gamma}{1+\gamma}\right)^a.$$

Theorem 3.6.2 *Let* $\{S_n = \sum_{i=1}^n X_i, \mathcal{F}_n\}$ *be a martingale where* $|X_i| \leq 1, \ i = 1, 2, \ldots$. *Then for any positive constants* a, b

$$P(S_n \geq a + bn \text{ for some } n \geq 1) \leq e^{-2ab}$$

and for $0 < b < 1$ *and suitable* r,

$$P(S_n \geq bn \text{ for some } n \geq m) \leq r^m \leq e^{\frac{-mb^2}{2}}.$$

Ross (1995) extended the previous inequalities for martingales of the form $S_n = \sum_{i=1}^n X_i$ when $-a \leq X_n \leq b, \ n = 1, 2, \ldots$ for nonnegative a, b.

Khan (2007) generalizes the Blackwell-Ross inequalities for martingales by assuming a suitable condition on the conditional moment generating function

$$\phi_n(\theta) = E[\exp(\theta X_n)|\mathcal{F}_{n-1}].$$

The main result of his paper is presented in the following theorem.

Theorem 3.6.3 *Let* $\{S_n = \sum_{1 \leq i \leq n} X_i, \mathcal{F}_n, n \geq 0\}$ *be a (super)martingale such that the conditional moment generating function* $\phi_n(\theta)$ *satisfies*

$$\phi_n(\theta) \leq f(\theta) \leq \exp(-\gamma\theta + \lambda\theta^2), \ \gamma \geq 0, \lambda > 0, \theta > 0$$

where f is a continuous positive function such that $f(0) = 1$. Then for positive a and b,

$$P(S_n \geq a + bn \text{ for some } n \geq m) \leq A^m \exp[-a(b+\gamma)/\lambda]$$

where $A = e^{-b\theta_0} f(\theta_0) \leq 1$ and $\theta_0 = (b+\gamma)/\lambda$. Moreover,

$$P(S_n \geq bn \text{ for some } n \geq m) \leq A_0^m \exp\left(-\frac{m(b+\gamma)^2}{4\lambda}\right),$$

where $A_0 = \exp(-(b-\gamma)\theta_0/2) f(\theta_0)$ and $\theta_0 = (b+\gamma)/2\lambda$.

Khan (2007) also provides as an extension of the above theorem inequalities of the form

$$P\left(\mid S_n \mid \geq b \sum_{i=1}^{n} v_i \text{ for some } n \geq m\right) \leq 2\exp\left\{-\frac{b}{2}\delta\left(\frac{b}{2}\right)\sum_{i=1}^{m} v_i\right\}$$

where $v_n = E(X_n^2 | \mathcal{F}_{n-1})$ and $\delta(b)$ is the unique solution of the equation $e^{\theta} - (1-b)\theta - 1 = 0$.

These inequalities are useful for certain convergence problems. Clearly $S_n / \sum_{i=1}^n v_i$ converges to zero a.s. provided $\sum_{i=1}^n v_i \to \infty$ as $n \to \infty$.

In the following theorem we prove a Blackwell-Ross inequality for N-demimartingales.

Theorem 3.6.4 *Let $\{S_n, n \in \mathbb{N}\}, (S_0 \equiv 0)$ be an N-demimartingale such that $E(S_1) = 0$ and*

$$-a \leq S_j - S_{j-1} \leq b \text{ for all } j$$

where $a, b > 0$.

Then for $m \in \mathbb{N}$

$$P(S_n \geq a + bn \text{ for some } n \geq m) \leq \frac{a+b}{b(1-e^{-(a+b)})} e^{-a} \left(\frac{a + be^{-(a+b)}}{a+b}\right)^m.$$

Proof. Using Markov's inequality we have that

$$\begin{aligned} P(S_n \geq a + bn \text{ for some } n \geq m) &\leq \sum_{n=m}^{\infty} P(\exp(S_n - a - bn) \geq 1) \\ &\leq \sum_{n=m}^{\infty} e^{-a} e^{-nb} E(e^{S_n}). \end{aligned}$$

We need to find an upper bound for the quantity $E(e^{S_n})$. Let

$$L(x) = \frac{a}{a+b}e^b + \frac{b}{a+b}e^{-a} + \frac{e^b - e^{-a}}{a+b}x$$

be the line through the points $(-a, e^{-a}), (b, e^b)$. By the convexity of the exponential function $e^x \leq L(x)$ and by using simple algebra we have that

$$\begin{aligned} E\left[e^{S_n}\right] &= E\left[e^{\sum_{i=1}^n (S_i - S_{i-1})}\right] \\ &= E\left[\prod_{i=1}^n e^{(S_i - S_{i-1})}\right] \\ &\leq E\left[\prod_{i=1}^n [A + B(S_i - S_{i-1})]\right] \end{aligned}$$

where $A = \frac{a}{a+b}e^b + \frac{b}{a+b}e^{-a}$ and $B = \frac{e^b - e^{-a}}{a+b}$. We will prove that $Ee^{S_n} \leq A^n$.
For $n = 2$ the statement holds true since

$$\begin{aligned} E\left[e^{S_2}\right] &\leq E\left[(A + BS_1)(A + B(S_2 - S_1))\right] \\ &= A^2 + ABE(S_2 - S_1) + ABE(S_1) + B^2 E[S_1(S_2 - S_1)] \\ &\leq A^2 \end{aligned}$$

where the last inequality follows by the N-demimartingale property.
We assume that the statement is true for $n = k$. For $n = k + 1$

$$\begin{aligned} E\left[e^{S_{k+1}}\right] &= E\left[e^{S_{k+1} - S_k} e^{S_k}\right] \\ &\leq E\left[(A + B(S_{k+1} - S_k))e^{S_k}\right] \qquad (3.27) \\ &= AE\left[e^{S_k}\right] + BE\left[(S_{k+1} - S_k)e^{S_k}\right] \\ &\leq A^{k+1} \end{aligned}$$

where (3.27) follows from the fact that $e^x \leq L(x)$ and the last inequality follows by applying the N-demimartingale property and the induction hypothesis.
Therefore

$$\begin{aligned} P(S_n \geq a + bn \ for \ some \ n \geq m) &\leq \sum_{n=m}^{\infty} e^{-a} \left(\frac{a + be^{-(a+b)}}{a+b}\right)^n \\ &= \frac{a+b}{b(1 - e^{-(a+b)})} e^{-a} \left(\frac{a + be^{-(a+b)}}{a+b}\right)^m . \end{aligned}$$

■

Remark 3.6.5 *The result of Theorem 3.6.4 can easily be applied to the case of mean zero negatively associated random variables. In particular let* $\{X_n, n \in \mathbb{N}\}$ *be a collection of mean zero negatively associated random variables such that* $-a \leq X_i \leq b$, $a, b > 0$, *and let* $S_n = \sum_{i=1}^{n} X_i$. *Then for* $m \in \mathbb{N}$,

$$P(S_n \geq a + bn \ for \ some \ n \geq m) \leq \frac{a+b}{b(1-e^{-(a+b)})} e^{-a} \left(\frac{a + be^{-(a+b)}}{a+b} \right)^m .$$

In the next corollary we establish a Blackwell-Ross inequality for N-demimartingales for the case of uniformly bounded N-demimartingale differences.

Corollary 3.6.6 *Let* $\{S_n, n \in \mathbb{N}\}, (S_0 \equiv 0)$ *be an N-demimartingale such that* $E(S_1) = 0$ *and*

$$|S_j - S_{j-1}| \leq a \ for \ all \ j,$$

where $a > 0$.

Then for $m \in \mathbb{N}$

$$P(S_n \geq a(1+n) \ for \ some \ n \geq m) \leq \frac{2}{1-e^{-2a}} e^{-a} \left(\frac{e^{-2a}+1}{2} \right)^m .$$

Proof. The result follows by applying Theorem 3.6.4 for $a = b$. ■

Remark 3.6.7 *The bound provided by Theorem 3.6.4 might not be useful for certain values of* a, b *and* m, *and in particular for values of* a *and* b *such that* $a + b$ *is very small.*

Remark 3.6.8 *Theorem 3.6.4 provides uniform exponential rates for the strong law of large numbers for N-demimartingales with bounded differences.*

Remark 3.6.9 *The result of Theorem 3.6.4 can be viewed as a generalization of the result provided by Theorem 3.3.3 since the event* $\{S_m \geq a + bm\}$ *is replaced by the larger event* $\{S_n \geq a + bn \ for \ some \ n \geq m\}$.

Chapter 4

Multidimensionally Indexed Extensions

4.1 Introduction

In this chapter we define the class of multidimensionally indexed demimartingales and demisubmartingales as a natural generalization of the notion of Newman and Wright (1982).

Let d be a positive integer. We denote by $\mathbb{N}^d$ the d-dimensional positive integer lattice. For $\mathbf{n}, \mathbf{m} \in \mathbb{N}^d$ with $\mathbf{n} = (n_1, \ldots, n_d)$ and $\mathbf{m} = (m_1, \ldots, m_d)$ the notation $\mathbf{n} \leq \mathbf{m}$ means that $n_i \leq m_i$ for all $i = 1, \ldots, d$, while the notation $\mathbf{n} < \mathbf{m}$ means that $n_i \leq m_i$ for all $i = 1, \ldots, d$ with at least one inequality strict. Finally the notation $|\mathbf{n}|$ stands for $\prod_{i=1}^{d} n_i$.

Definition 4.1.1 *A collection of multidimensionally indexed random variables* $\{X_{\mathbf{i}}, \mathbf{i} \leq \mathbf{n}\}$ *is said to be associated if for any two coordinatewise nondecreasing functions* $f, g : \mathbb{R}^{|\mathbf{n}|} \to \mathbb{R}$

$$Cov\,(f(X_{\mathbf{i}}, \mathbf{i} \leq \mathbf{n}), g(X_{\mathbf{i}}, \mathbf{i} \leq \mathbf{n})) \geq 0,$$

provided that the covariance is defined. An infinite collection is associated if every finite subcollection is associated.

The above definition is just the classical definition of association stated for the case of multidimensionally indexed random variables. The index of the variables in no way affects the qualitative property of association, i.e., that nondecreasing functions of all (or some) of the variables are nonnegatively correlated.

Definition 4.1.2 *An array of random variables $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is called a multidimensionally indexed demimartingale if:*

$$E\{(X_{\mathbf{j}} - X_{\mathbf{i}}) f(X_{\mathbf{k}}, \mathbf{k} \leq \mathbf{i})\} \geq 0, \quad \forall \;\; \mathbf{i}, \; \mathbf{j} \in \mathbb{N}^d \text{ with } \mathbf{i} \leq \mathbf{j},$$

and for all componentwise nondecreasing functions f. If in addition f is required to be nonnegative then $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to be a multidimensionally indexed demisubmartingale.

It is easy to verify that the partial sum of mean zero associated multidimensionally indexed random variables is a multidimensionally indexed demimartingale. Furthermore, a multidimensionally indexed martingale equipped with the natural choice of σ-algebras, is a multidimensionally indexed demimartingale.

In this chapter we present maximal inequalities and related asymptotic results for multidimensionally indexed demimartingales and demisubmartingales.

4.2 Chow-type maximal inequality

The following result will provide a Chow-type maximal inequality for the collection $\{g(Y_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$ where $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is a multidimensionally indexed demimartingale and g is a nondecreasing convex function. The monotonicity assumption of g will be relaxed later.

Lemma 4.2.1 *Let $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a multidimensionally indexed demimartingale with $Y_{\mathbf{k}} \equiv 0$ when $\prod_{i=1}^d k_i = 0$. Furthermore, let $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a nonincreasing array of* *positive numbers and let g be a nonnegative nondecreasing convex function on $\mathbb{R}$ with $g(0) = 0$. Then for every $\varepsilon > 0$*

$$\varepsilon P\left(\max_{\mathbf{k}\leq\mathbf{n}} c_{\mathbf{k}} g(Y_{\mathbf{k}}) \geq \varepsilon\right) \quad \leq \quad \min_{1\leq s\leq d}\left\{\sum_{\mathbf{k}\leq\mathbf{n}} c_{\mathbf{k}} E[g(Y_{\mathbf{k};s;i}) - g(Y_{\mathbf{k};s;i-1})]\right\},$$

where $Y_{\mathbf{k};s;i} = Y_{k_1\ldots k_{s-1} i k_{s+1}\ldots k_d}$, i.e., at the s^{th} position of the index $\mathbf{k}$ the component k_s is equal to i.

Proof. For simplicity the proof is presented for $d = 2$. The case $d > 2$ is similar. Define the sets

$$\begin{aligned}
A &= \left\{\max_{(k_1,k_2)\leq(n_1,n_2)} c_{k_1k_2} g(Y_{k_1k_2}) \geq \varepsilon\right\}, \\
B_{1k_2} &= \left\{c_{1k_2} g(Y_{1k_2}) \geq \varepsilon\right\}, \; 1 \leq k_2 \leq n_2, \\
B_{k_1k_2} &= \left\{c_{lk_2} g(Y_{lk_2}) < \varepsilon, \; 1 \leq l < k_1, \; c_{k_1k_2} g(Y_{k_1k_2}) \geq \varepsilon\right\}, \;\; 2 \leq k_1 \leq n_1, \; 1 \leq k_2 \leq n_2.
\end{aligned}$$

By the definitions of the sets A and $B_{k_1k_2}$ we have that $A = \bigcup_{k_1,k_2} B_{k_1k_2}$ and thus

$$\begin{aligned}
\varepsilon P(A) &= \varepsilon P\left(\bigcup_{(i,j)\leq(n_1,n_2)} B_{ij}\right) \\
&\leq \varepsilon \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} P\left(B_{ij}\right) \\
&= \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} E\left(\varepsilon I_{B_{ij}}\right) \\
&\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} E\left(c_{ij} g(Y_{ij}) I_{B_{ij}}\right) \\
&= \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j}) I_{B_{1j}}\right] + \sum_{j=1}^{n_2}\sum_{i=2}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\
&= \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] - \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j}) I_{B_{1j}^c}\right] + \sum_{j=1}^{n_2} E\left[c_{2j} g(Y_{2j}) I_{B_{2j}}\right] \\
&\quad + \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\
&\leq \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} c_{2j} E\left[g(Y_{2j}) I_{B_{2j}} - g(Y_{1j}) I_{B_{1j}^c}\right]
\end{aligned}$$

$$+\sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right]$$

where the last inequality follows from the monotonicity of the array $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$. Since $B_{2j} \subseteq B_{1j}^c$ we can write $I_{B_{2j}} = I_{B_{1j}^c} - I_{B_{1j}^c \bigcap B_{2j}^c}$. Then:

$$\begin{aligned}
\varepsilon P(A) \quad &\leq \quad \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j})) I_{B_{1j}^c}\right] \\
&\qquad - \sum_{j=1}^{n_2} c_{2j} E\left[g(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] + \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\
&= \quad \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] \\
&- \quad \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j})) I_{B_{1j}}\right] - \sum_{j=1}^{n_2} c_{2j} E\left[g(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] \\
&+ \quad \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right].
\end{aligned}$$

Since g is nondecreasing convex, we can write

$$g(y) - g(x) \geq (y - x) h(x)$$

where

$$h(y) = \lim_{x \to y^-} \frac{g(x) - g(y)}{x - y}$$

is the left derivative of g. Observe that $I_{B_{1j}} h(Y_{1j})$ is a nonnegative and nondecreasing function of Y_{1j} and by the demimartingale property of $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{\mathbf{2}}\}$ we have that

$$E\left[(g(Y_{2j}) - g(Y_{1j})) I_{B_{1j}}\right] \geq E\left[(Y_{2j} - Y_{1j}) h(Y_{1j}) I_{B_{1j}}\right] \geq 0, \quad \text{for } j = 1, 2, \ldots, n_2.$$

Then,

$$\begin{aligned}
\varepsilon P(A) \quad &\leq \quad \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] - \sum_{j=1}^{n_2} c_{2j} E\left[g(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] \\
&+ \quad \sum_{j=1}^{n_2} E\left[c_{3j} g(Y_{3j}) I_{B_{3j}}\right] + \sum_{j=1}^{n_2}\sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\
&\leq \quad \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] \\
&+ \quad \sum_{j=1}^{n_2} c_{3j} E\left[g(Y_{3j}) I_{B_{3j}} - g(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] + \sum_{j=1}^{n_2}\sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \qquad (4.1)
\end{aligned}$$

where (4.1) follows from the monotonicity of the array $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$.

Since $B_{3j} \subseteq B_{1j}^c \bigcap B_{2j}^c$ then $I_{B_{3j}} = I_{B_{1j}^c \bigcap B_{2j}^c} - I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}$ and we further have:

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] \\
&+ \sum_{j=1}^{n_2} c_{3j} E\left[(g(Y_{3j}) - g(Y_{2j})) I_{B_{1j}^c \bigcap B_{2j}^c}\right] - \sum_{j=1}^{n_2} c_{3j} E\left[g(Y_{3j}) I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}\right] \\
&+ \sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\
&= \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] + \sum_{j=1}^{n_2} c_{3j} E\left[(g(Y_{3j}) - g(Y_{2j}))\right] \\
&- \sum_{j=1}^{n_2} c_{3j} E\left[(g(Y_{3j}) - g(Y_{2j})) I_{B_{1j} \bigcup B_{2j}}\right] - \sum_{j=1}^{n_2} c_{3j} E\left[g(Y_{3j}) I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}\right] \\
&+ \sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right].
\end{aligned}$$

Using the same arguments as before regarding the demimartingale property of $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{\mathbf{2}}\}$ it can be shown that, since $I_{B_{1j} \bigcup B_{2j}}$ is a nonnegative nondecreasing function of Y_{1j} and Y_{2j},

$$E\left[(g(Y_{3j}) - g(Y_{2j}))\, I_{B_{1j} \bigcup B_{2j}}\right] \geq 0, \quad \text{for } j = 1, 2, \ldots, n_2.$$

Therefore

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] + \sum_{j=1}^{n_2} c_{3j} E\left[(g(Y_{3j}) - g(Y_{2j}))\right] \\
&- \sum_{j=1}^{n_2} c_{3j} E\left[g(Y_{3j}) I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}\right] + \sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right].
\end{aligned}$$

Continuing in the same manner and since by definition $Y_{0j} = 0$ we finally have:

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij} E\left[g(Y_{ij}) - g(Y_{i-1j})\right] - \sum_{j=1}^{n_2} c_{n_1 j} E\left[g(Y_{n_1 j}) I_{\bigcap_{i=1}^{n_1} B_{ij}^c}\right] && (4.2) \\
&\leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij} E\left[g(Y_{ij}) - g(Y_{i-1j})\right]. && (4.3)
\end{aligned}$$

Similarly it can be shown that:

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij} E\left[g(Y_{ij}) - g(Y_{ij-1})\right] - \sum_{i=1}^{n_1} c_{in_2} E\left[g(Y_{in_2}) I_{\bigcap_{j=1}^{n_2} B_{ij}^c}\right] \\
&\leq \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij} E\left[g(Y_{ij}) - g(Y_{ij-1})\right]. \qquad (4.4)
\end{aligned}$$

Inequalities (4.3) and (4.4) together give the desired result. ■

Remark 4.2.2 *Lemma 4.2.1 was proved under the assumption that g is nondecreasing. However,as the next result shows, the assumption can be dropped. The proof of Theorem 4.2.3 uses Lemma 4.2.1 as an auxiliary result.*

Theorem 4.2.3 *Let $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a multidimensionally indexed demimartingale such that $Y_{\mathbf{k}} \equiv 0$ when $\prod_{i=1}^d k_i = 0$. Let $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a nonincreasing array of positive numbers and let g be a nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$. Then for every $\varepsilon > 0$:*

$$\varepsilon P\left(\max_{\mathbf{k}\leq\mathbf{n}} c_{\mathbf{k}} g(Y_{\mathbf{k}}) \geq \varepsilon\right) \leq \min_{1\leq s\leq d}\left\{\sum_{\mathbf{k}\leq\mathbf{n}} c_{\mathbf{k}} E\left[g(Y_{\mathbf{k};s;i}) - g(Y_{\mathbf{k};s;i-1})\right]\right\}.$$

Proof. (For $d = 2$.)

Following a standard argument (see for example, Wang (2004)) let $u(x) = g(x)I\{x \geq 0\}$ and $v(x) = g(x)I\{x < 0\}$. Clearly u is a nonnegative nondecreasing convex function while v a nonnegative nonincreasing convex function. From the definition of $u(x)$ and $v(x)$ we have:

$$g(x) = u(x) + v(x) = \max\{u(x), v(x)\}.$$

Then,

$$\begin{aligned}
\varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} g(Y_{ij}) \geq \varepsilon\right) &= \varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} \max\{u(Y_{ij}), v(Y_{ij})\} \geq \varepsilon\right) \\
&\leq \varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} u(Y_{ij}) \geq \varepsilon\right) \\
&+ \varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} v(Y_{ij}) \geq \varepsilon\right).
\end{aligned}$$

Since u is nonnegative nondecreasing convex, by Lemma 4.2.1 we have:

$$\varepsilon P\left(\max_{(i,j)\le(n_1,n_2)} c_{ij}u(Y_{ij})\ge\varepsilon\right) \le \min\left\{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}E\left[u(Y_{ij})-u(Y_{i-1j})\right], \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}E\left[u(Y_{ij})-u(Y_{ij-1})\right]\right\}.$$

We will show that

$$\varepsilon P\left(\max_{(i,j)\le(n_1,n_2)} c_{ij}v(Y_{ij})\ge\varepsilon\right) \le \min\left\{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}E\left[v(Y_{ij})-v(Y_{i-1j})\right], \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}E\left[v(Y_{ij})-v(Y_{ij-1})\right]\right\}. \quad (4.5)$$

Define the sets

$$\begin{aligned} A &= \left\{\max_{(i,j)\le(n_1,n_2)} c_{ij}v(Y_{ij})\ge\varepsilon\right\}, \\ B_{1j} &= \left\{c_{1j}v(Y_{1j})\ge\varepsilon\right\},\ 1\le j\le n_2, \\ B_{ij} &= \left\{c_{lj}v(Y_{ij})<\varepsilon,\ \ 1\le l<i,\ \ c_{ij}v(Y_{ij})\ge\varepsilon\right\},\ \ 2\le i\le n_1,\ \ 1\le j\le n_2. \end{aligned}$$

Then,

$$\begin{aligned} \varepsilon P(A) &= \varepsilon P\left(\bigcup_{(i,j)\le(n_1,n_2)} B_{ij}\right) \\ &\le \varepsilon\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} P(B_{ij}) \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} E\left(\varepsilon I_{B_{ij}}\right) \\ &\le \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} E\left[c_{ij}v(Y_{ij})I_{B_{ij}}\right] \\ &= \sum_{j=1}^{n_2} c_{1j}E\left[v(Y_{1j})I_{B_{1j}}\right] + \sum_{j=1}^{n_2}\sum_{i=2}^{n_1} E\left[c_{ij}v(Y_{ij})I_{B_{ij}}\right] \\ &= \sum_{j=1}^{n_2} c_{1j}E\left[v(Y_{1j})\right] - \sum_{j=1}^{n_2} c_{1j}E\left[v(Y_{1j})I_{B_{1j}^c}\right] \\ &\quad + \sum_{j=1}^{n_2} c_{2j}E\left[v(Y_{2j})I_{B_{2j}}\right] + \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij}v(Y_{ij})I_{B_{ij}}\right] \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{n_2} c_{1j} E\left[v(Y_{1j})\right] + \sum_{j=1}^{n_2} c_{2j} E\left[(v(Y_{2j}) - v(Y_{1j}))I_{B_{1j}^c}\right] \\
&\quad - \sum_{j=1}^{n_2} c_{2j} E\left[v(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] + \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right] \\
&= \sum_{j=1}^{n_2} c_{1j} E\left[v(Y_{1j})\right] + \sum_{j=1}^{n_2} c_{2j} E\left(v(Y_{2j}) - v(Y_{1j})\right) \\
&\quad - \sum_{j=1}^{n_2} c_{2j} E\left[(v(Y_{2j}) - v(Y_{1j}))I_{B_{1j}}\right] - \sum_{j=1}^{n_2} c_{2j} E\left[v(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] \\
&\quad + \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right].
\end{aligned}
$$

Since $v(x)$ is a nonnegative nonincreasing convex function, the function

$$
h(y) = \lim_{x \to y^-} \frac{v(x) - v(y)}{x - y}
$$

is a nonpositive nondecreasing function. By the convexity of the function v,

$$
v(Y_{2j}) - v(Y_{1j}) \geq (Y_{2j} - Y_{1j}) h(Y_{1j}).
$$

Since $h(Y_{1j})$ is a nonpositive nondecreasing function, the function $-h(Y_{1j})$ is nonnegative nonincreasing and $-h(Y_{1j})I_{B_{1j}}$ is a nonincreasing function of Y_{1j}, since by definition the indicator function $I_{B_{1j}}$ is a nonincreasing function of Y_{1j}. Then $h(Y_{1j})I_{B_{1j}}$ is a nondecreasing function of Y_{1j}. Furthermore, by the demimartingale property of $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{\mathbf{2}}\}$ we have:

$$
E\left[(v(Y_{2j}) - v(Y_{1j}))I_{B_{1j}}\right] \geq E\left[(Y_{2j} - Y_{1j}) I_{B_{1j}} h(Y_{1j})\right] \geq 0.
$$

Thus,

$$
\begin{aligned}
\varepsilon P(A) \quad &\leq \quad \sum_{j=1}^{n_2} c_{1j} E\left[v(Y_{1j})\right] + \sum_{j=1}^{n_2} c_{2j} E\left[(v(Y_{2j}) - v(Y_{1j}))\right] - \sum_{j=1}^{n_2} c_{2j} E\left[v(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] \\
&\quad + \quad \sum_{j=1}^{n_2} c_{3j} E\left[v(Y_{3j}) I_{B_{3j}}\right] + \sum_{j=1}^{n_2}\sum_{i=4}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right] \\
&\leq \quad \sum_{j=1}^{n_2}\sum_{i=1}^{2} c_{ij} E\left[(v(Y_{ij}) - v(Y_{i-1j}))\right] + \sum_{j=1}^{n_2} c_{3j} E\left[v(Y_{3j}) I_{B_{3j}} - v(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{j=1}^{n_2}\sum_{i=4}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right] \\
&= \sum_{j=1}^{n_2}\sum_{i=1}^{3} c_{ij} E\left[(v(Y_{ij}) - v(Y_{i-1j}))\right] - \sum_{j=1}^{n_2} c_{3j} E\left[(v(Y_{3j}) - v(Y_{2j})) I_{B_{1j}\bigcup B_{2j}}\right] \\
&- \sum_{j=1}^{n_2} c_{3j} E\left[v(Y_{3j}) I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}\right] + \sum_{j=1}^{n_2}\sum_{i=4}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right].
\end{aligned}
$$

The indicator $I_{B_{1j}\bigcup B_{2j}}$ is a nonincreasing function of Y_{1j}, Y_{2j}, so by using the same arguments as before we have:

$$
E[(v(Y_{3j}) - v(Y_{2j})) I_{B_{1j}\bigcup B_{2j}}] \geq 0.
$$

Continuing in the same way we finally have:

$$
\begin{aligned}
\varepsilon P(A) &\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{ij} E\left[v(Y_{ij}) - v(Y_{i-1j})\right] - \sum_{j=1}^{n_2} c_{n_1 j} E\left[v(Y_{n_1 j}) I_{\bigcap_{i=1}^{n_1} B_{ij}^c}\right] \\
&\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{ij} E\left[v(Y_{ij}) - v(Y_{i-1j})\right]. \qquad (4.6)
\end{aligned}
$$

By symmetry it can be shown that:

$$
\begin{aligned}
\varepsilon P(A) &\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{ij} E\left[v(Y_{ij}) - v(Y_{ij-1})\right] - \sum_{i=1}^{n_1} c_{in_2} E\left[v(Y_{in_2}) I_{\bigcap_{j=1}^{n_2} B_{ij}^c}\right] \\
&\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{ij} E\left[v(Y_{ij}) - v(Y_{ij-1})\right]. \qquad (4.7)
\end{aligned}
$$

Inequalities (4.6) and (4.7) together finally give

$$
\begin{aligned}
\varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} v(Y_{ij}) \geq \varepsilon\right) \leq \min\Bigg\{ &\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij} E\left[v(Y_{ij}) - v(Y_{i-1j})\right], \\
&\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij} E\left[v(Y_{ij}) - v(Y_{ij-1})\right] \Bigg\}
\end{aligned}
$$

and (4.5) is established. ∎

4.3 Doob-type inequality

The following result gives a Doob type inequality for multidimensionally indexed demimartingales.

Corollary 4.3.1 *Let* $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$*, be a multidimensionally indexed demimartingale with* $Y_{\mathbf{k}} \equiv 0$ *when* $\prod_{i=1}^d k_i = 0$ *and let* g *be a nonnegative, nondecreasing convex function on* $\mathbb{R}$ *with* $g(0) = 0$.

Let

$$A = \left\{ \max_{\mathbf{k} \leq \mathbf{n}} g(Y_{\mathbf{k}}) \geq \varepsilon \right\}, \quad C_{\mathbf{k}^{(s)}} = \left\{ \max_{1 \leq k_s \leq n_s} g(Y_{\mathbf{k}}) \geq \varepsilon \right\}$$

and

$$\mathbf{k}^{(s)} = (k_1, \ldots, k_{s-1}, k_{s+1}, \ldots, k_d).$$

Then, for every $\varepsilon > 0$*:*

$$\varepsilon P(A) \leq \min_{1 \leq s \leq d} \left\{ \sum_{\mathbf{k}^{(s)} \leq \mathbf{n}^{(s)}} \int_{C_{\mathbf{k}^{(s)}}} g(Y_{\mathbf{n}}) dP \right\}.$$

Proof. We give the proof for $d = 2$. The case for $d > 2$ is similar.

By (4.2) and for $c_{ij} = 1 \ \ \forall \ \ (i, j) \leq (n_1, n_2)$ we have:

$$\begin{aligned} \varepsilon P \left(\max_{(i,j) \leq (n_1,n_2)} g(Y_{ij}) \geq \varepsilon \right) &\leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} E[g(Y_{ij}) - g(Y_{i-1j})] - \sum_{j=1}^{n_2} E\left[g(Y_{n_1 j}) I_{\bigcap_{i=1}^{n_1} B_{ij}^c} \right] \\ &\leq \sum_{j=1}^{n_2} E\left[g(Y_{n_1 j}) I_{C_j} \right] \\ &= \sum_{j=1}^{n_2} \int_{C_j} g(Y_{n_1 j}) dP \\ &\leq \sum_{j=1}^{n_2} \int_{C_j} g(Y_{n_1 n_2}) dP, \end{aligned} \tag{4.8}$$

where the last inequality follows from the demimartingale property.

Similarly it can be shown that:

$$\varepsilon P \left(\max_{(i,j) \leq (n_1,n_2)} g(Y_{ij}) \geq \varepsilon \right) \leq \sum_{i=1}^{n_1} \int_{C_i} g(Y_{n_1 n_2}) dP. \tag{4.9}$$

Inequalities (4.8) and (4.9) together give the desired result. ■

4.4 Hájek-Rényi inequality

Using Theorem 4.2.3 we derive a Hájek-Rényi inequality for arrays of mean zero associated random variables.

Corollary 4.4.1 *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be mean zero multidimensionally indexed associated random variables and $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ a nonincreasing array of positive numbers. Let*

$$S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}, \quad \textit{with} \quad S_{\mathbf{k}} \equiv 0 \quad \textit{if} \quad \prod_{i=1}^{d} k_i = 0.$$

Then for every $\varepsilon > 0$,

$$P\left(\max_{\mathbf{k} \leq \mathbf{n}} c_{\mathbf{k}} \mid S_{\mathbf{k}} \mid \geq \varepsilon\right) \leq \min_{1 \leq s \leq d} \left\{\varepsilon^{-2} \sum_{\mathbf{k} \leq \mathbf{n}} c_{\mathbf{k}}^2 \left[2Cov(S_{\mathbf{k};s;i-1}, S_{\mathbf{k}}^{(s)}) + E(S_{\mathbf{k}}^{(s)})^2\right]\right\}$$

where

$$S_{\mathbf{k}}^{(s)} = \sum_{i_1=1}^{k_1} \cdots \sum_{i_{s-1}=1}^{k_{s-1}} \sum_{i_{s+1}=1}^{k_{s+1}} \cdots \sum_{i_d=1}^{k_d} X_{i_1 \ldots i_{s-1} k_s i_{s+1} \ldots i_d}$$

and

$$S_{\mathbf{k};s;k_s-1} = \sum_{l_1=1}^{k_1} \cdots \sum_{l_{s-1}=1}^{k_{s-1}} \sum_{l_s=1}^{k_s-1} \sum_{l_{s+1}=1}^{k_{s+1}} \cdots \sum_{l_d=1}^{k_d} X_{\mathbf{l}}.$$

Proof. (For $d = 2$)

It can be easily verified that the array $\{S_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$ is a 2-indexed demimartingale. Let $g(x) =\mid x \mid^2$. Then g is a nonnegative convex function.

$$\begin{aligned}
P\left(\max_{(i,j) \leq (n_1,n_2)} c_{ij} \mid S_{ij} \mid \geq \varepsilon\right) &= P\left(\max_{(i,j) \leq (n_1,n_2)} c_{ij}^2 \mid S_{ij} \mid^2 \geq \varepsilon^2\right) \\
&\leq \varepsilon^{-2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij}^2 E\left(\mid S_{ij} \mid^2 - \mid S_{i-1j} \mid^2\right) \\
&= \varepsilon^{-2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij}^2 E\left[(S_{ij} + S_{i-1j})(S_{ij} - S_{i-1j})\right] \\
&= \varepsilon^{-2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij}^2 E\left[\sum_{m=1}^{j} X_{im}\left(2S_{i-1j} + \sum_{m=1}^{j} X_{im}\right)\right] \\
&= \varepsilon^{-2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij}^2 \left[2Cov\left(S_{i-1j}, \sum_{m=1}^{j} X_{im}\right)\right.
\end{aligned}$$

$$+E\left(\sum_{m=1}^{j} X_{im}\right)^2\Bigg] \tag{4.10}$$

where the first inequality follows from Theorem 4.2.3. Similarly it can be shown that:

$$P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} \mid S_{ij} \mid\geq \varepsilon\right) \leq \varepsilon^{-2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}^2\left[2Cov\left(S_{ij-1}, \sum_{m=1}^{i} X_{mj}\right) + E\left(\sum_{m=1}^{i} X_{mj}\right)^2\right]. \tag{4.11}$$

The result now follows from (4.10) and (4.11). ■

Remark 4.4.2 *The bound derived for $d = 1$ is sharper than the bound provided by Corollary 2.2.5.*

Remark 4.4.3 *For $d = 2$ and $c_{ij} = 1$ for all $(i,j) \in \mathbb{N}^2$ the right hand side of Corollary 4.4.1 becomes*

$$\varepsilon^{-2}\min\Bigg\{E[\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_1}\sum_{l=1}^{n_2} X_{ij}X_{kl}] + \sum_{s=1}^{n_2-1}\sum_{j=1}^{n_2-s}\sum_{i_1=1}^{n_1-1}\sum_{i_2=1, i_2>i_1}^{n_1} E[X_{i_1 j} + X_{i_2 j}]^2,$$

$$E[\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_1}\sum_{l=1}^{n_2} X_{ij}X_{kl}] + \sum_{s=1}^{n_1-1}\sum_{i=1}^{n_1-s}\sum_{j_1=1}^{n_2-1}\sum_{j_2=1, j_2>j_1}^{n_2} E[X_{ij_1} + X_{ij_2}]^2\Bigg\}$$

and for small values of n_1 and n_2 it compares favorably with the Hájek-Rényi inequality in Theorem 2.2.9. Observe that the bound of Theorem 2.2.9 is

$$\frac{64}{\varepsilon^2}E[\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_1}\sum_{l=1}^{n_2} X_{ij}X_{kl}]$$

for $b_{n_1 n_2} = 1$ for all $n_1 \geq 1, n_2 \geq 1$. The same conclusion holds for general arrays $\{c_{ij}, (i,j) \in \mathbb{N}^2\}$ satisfying (2.1).

4.5 Further maximal inequalities

Using Theorem 4.2.3 as a source result, one can obtain various maximal probability and maximal moment inequalities, such as those provided by the next two corollaries.

Corollary 4.5.1 *Let* $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$*, be a multidimensionally indexed demimartingale with* $Y_{\mathbf{k}} = 0$ *when* $\prod_{i=1}^d k_i = 0$ *and let* g *be a nonnegative convex function on* $\mathbb{R}$ *with* $g(0) = 0$.

Let

$$A = \left\{\max_{\mathbf{k}\leq\mathbf{n}} g(Y_{\mathbf{k}}) \geq \varepsilon\right\}, \quad C_{\mathbf{k}^{(s)}} = \left\{\max_{1\leq k_s\leq n_s} g(Y_{\mathbf{k}}) \geq \varepsilon\right\}$$

and

$$\mathbf{k}^{(s)} = (k_1, \ldots, k_{s-1}, k_{s+1}, \ldots, k_d).$$

Then for every $\varepsilon > 0$*:*

$$\varepsilon P(A) \leq \min_{1\leq s\leq d}\left\{\sum_{\mathbf{k}^{(s)}\leq\mathbf{n}^{(s)}} \int_{C_{\mathbf{k}^{(s)}}} g(Y_{\mathbf{n}})dP\right\}.$$

Proof. (For $d = 2$)

Let $u(x)$ and $v(x)$ be the functions defined in Theorem 4.2.3. We further define:

$$A^{(u)} = \left\{\max_{(i,j)\leq(n_1,n_2)} u(Y_{ij}) \geq \varepsilon\right\}$$

and

$$B_{1k_2}^{(u)} = \{u(Y_{1k_2}) \geq \varepsilon\} \quad , \quad B_{k_1k_2}^{(u)} = \{u(Y_{lk_2}) < \varepsilon, \ 1 \leq l < k_1, \ u(Y_{k_1k_2}) \geq \varepsilon\}.$$

Since u is nondecreasing convex, by Doob's inequality in Corollary 4.3.1 we have,

$$\varepsilon P(A^{(u)}) \leq \sum_{j=1}^{n_2} \int_{\bigcup_{i=1}^{n_1} B_{ij}^{(u)}} u(Y_{n_1n_2})dP. \tag{4.12}$$

We need to prove a similar result for the quantity $P(A^{(v)})$:

$$\begin{aligned}
\varepsilon P(A^{(v)}) &\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} E\left[v(Y_{ij}) - v(Y_{i-1j})\right] - \sum_{j=1}^{n_2} c_{n_1j} E\left[v(Y_{n_1j}) I_{(\bigcup_{i=1}^{n_1} B_{ij}^{(v)})^c}\right] \\
&= \sum_{j=1}^{n_2} E\left[v(Y_{n_1j}) - v(Y_{n_1j}) I_{(\bigcup_{i=1}^{n_1} B_{ij}^{(v)})^c}\right] = \sum_{j=1}^{n_2} E\left[v(Y_{n_1j}) I_{\bigcup_{i=1}^{n_1} B_{ij}^{(v)}}\right] \\
&= \sum_{j=1}^{n_2} \int_{\bigcup_{i=1}^{n_1} B_{ij}^{(v)}} v(Y_{n_1j})dP \leq \sum_{j=1}^{n_2} \int_{\bigcup_{i=1}^{n_1} B_{ij}^{(v)}} v(Y_{n_1n_2})dP
\end{aligned} \tag{4.13}$$

where the first inequality follows from Theorem 4.2.3. By (4.12) and (4.13) we have that:

$$\begin{aligned}
\varepsilon P(A) &\leq \varepsilon P(A^{(u)}) + \varepsilon P(A^{(v)}) \\
&\leq \sum_{j=1}^{n_2} \left\{ \int_{\left\{\max_{1\leq i\leq n_1} u(Y_{ij})\geq\varepsilon\right\}} u(Y_{n_1n_2}) + \int_{\{\max_{1\leq i\leq n_1} v(Y_{ij})\geq\varepsilon\}} v(Y_{n_1n_2}) \right\} dP \\
&\leq \sum_{j=1}^{n_2} \int_{\{\max_{1\leq i\leq n_1} \max\{u(Y_{ij}),v(Y_{ij})\}\geq\varepsilon\}} (u(Y_{n_1n_2}) + v(Y_{n_1n_2}))dP \\
&= \sum_{j=1}^{n_2} \int_{C_j} g(Y_{n_1n_2})dP. \qquad (4.14)
\end{aligned}$$

Similarly,

$$\varepsilon P(A) \leq \sum_{i=1}^{n_1} \int_{C_i} g(Y_{n_1n_2})dP. \qquad (4.15)$$

Inequalities (4.14) and (4.15) together give the desired result. ■

Corollary 4.5.2 *Let $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$ be a 2-indexed demimartingale with $Y_{i0} = Y_{0j} = 0$ for all $(i,j) \leq (n_1, n_2)$. Let g be a nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$. Then for all $p > 1$:*

$$E\left(\max_{(i,j)\leq(n_1,n_2)} g(Y_{ij})\right)^p \leq C_{p,n_1,n_2} E\left(g(Y_{n_1n_2})\right)^p$$

with $C_{p,n_1,n_2} = \min\{n_1, n_2\} \left(\frac{p}{p-1}\right)^{2p-1}$.

Proof.

$$\begin{aligned}
E\left[\max_{(i,j)\leq(n_1,n_2)} g(Y_{ij})\right]^p &= p\int_0^\infty x^{p-1} P\left(\max_{(i,j)\leq(n_1,n_2)} g(Y_{ij}) \geq x\right) dx \\
&\leq p\int_0^\infty x^{p-2} \sum_{j=1}^{n_2} E\left[g(Y_{n_1n_2}) I\{\max_{1\leq i\leq n_1} g(Y_{ij}) \geq x\}\right] dx \qquad (4.16) \\
&= p\sum_{j=1}^{n_2} E\left[g(Y_{n_1n_2}) \int_0^{\max_{1\leq i\leq n_1}\{g(Y_{ij})\}} x^{p-2}dx\right] \\
&= \frac{p}{p-1} \sum_{j=1}^{n_2} E\left[g(Y_{n_1n_2}) \left(\max_{1\leq i\leq n_1} g(Y_{ij})\right)^{p-1}\right] \\
&\leq \frac{p}{p-1} \left[E(g(Y_{n_1n_2}))^p\right]^{\frac{1}{p}} \sum_{j=1}^{n_2} \left\{E\left(\max_{1\leq i\leq n_1} g(Y_{ij})\right)^p\right\}^{\frac{p-1}{p}} \qquad (4.17)
\end{aligned}$$

where inequality (4.16) follows from Corollary 4.5.1. Now observe that for $j = 1, \ldots, n_2$, $\{Y_{ij}, i \geq 1\}$ is a single index demimartingale. Therefore by Theorem 3.3 of Wang and Hu (2009)

$$\begin{aligned} E\left(\max_{1\leq i\leq n_1} g(Y_{ij})\right)^p &\leq \left(\frac{p}{p-1}\right)^p E\left(g(Y_{n_1 j})\right)^p \\ &\leq \left(\frac{p}{p-1}\right)^p E\left(\max_{1\leq j\leq n_2} g(Y_{n_1 j})\right)^p \\ &\leq \left(\frac{p}{p-1}\right)^{2p} E\left(g(Y_{n_1 n_2})\right)^p \end{aligned} \tag{4.18}$$

where (4.18) follows from Theorem 3.3 of Wang and Hu (2009) and the fact that $\{Y_{n_1 j}, j \geq 1\}$ is a single index demimartingale. Combining (4.17) with (4.18) we have that

$$E\left(\max_{(i,j)\leq(n_1,n_2)} g(Y_{ij})\right)^p \leq n_2 \left(\frac{p}{p-1}\right)^{2p-1} E\left(g(Y_{n_1 n_2})\right)^p. \tag{4.19}$$

Similarly,

$$E\left(\max_{(i,j)\leq(n_1,n_2)} g(Y_{ij})\right)^p \leq n_1 \left(\frac{p}{p-1}\right)^{2p-1} E\left(g(Y_{n_1 n_2})\right)^p. \tag{4.20}$$

The desired result follows by combining (4.19) and (4.20). ∎

Remark 4.5.3 *The upper bound of Corollary 4.5.2 unfortunately depends on n_1, n_2 and therefore this result might be useful only for small values of n_1, n_2 or for asymptotic results in case of n_1 or n_2 is fixed or bounded.*

Next, we present a strong law of large numbers for multidimensionally indexed demimartingales.

Corollary 4.5.4 *Assume that $\{Y_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$, $\{c_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ and the function g are as in Theorem 4.2.3. We also assume that there exists a number $p \geq 1$ such that $E(g(Y_{\mathbf{k}}))^p < \infty$ and for some $1 \leq s \leq d$*

$$\sum_{\mathbf{k}} c_{\mathbf{k}}^p E([g(Y_{\mathbf{k}})]^p - [g(Y_{\mathbf{k};s;k_s-1})]^p) < \infty \ \text{ and } \sum_{k_i, i\neq s} c_{\mathbf{k};s;N}^p E[g(Y_{\mathbf{k};s;N})]^p < \infty \ \text{ for each } N \in \mathbb{N}. \tag{4.21}$$

Then

$$c_{\mathbf{k}} g(Y_{\mathbf{k}}) \to 0 \ a.s \text{ as } \mathbf{k} \to \infty$$

where $\mathbf{k} \to \infty$ *means* $k_i \to \infty$ *for all* $i = 1, \ldots, d$.

Proof. We give the proof for $d = 2$. Without loss of generality we assume that $s = 2$. Then, conditions (4.21) can be written in the form

$$\sum_{(n_1,n_2)} c^p_{n_1 n_2} E([g(Y_{n_1 n_2})]^p - [g(Y_{n_1 n_2 - 1})]^p) < \infty \ , \ \sum_{n_1} c^p_{n_1 N} E[g(Y_{n_1 N})]^p < \infty \text{ for each } N \in \mathbb{N}. \tag{4.22}$$

Then by applying Theorem 4.2.3 for $\varepsilon > 0$,

$$\begin{aligned} \varepsilon^p P(\max_{(n_1,n_2)\geq(N,N)} c_{n_1 n_2} g(Y_{n_1 n_2}) \geq \varepsilon) &= \varepsilon^p P(\max_{(n_1,n_2)\geq(N,N)} c^p_{n_1 n_2}[g(Y_{n_1 n_2})]^p \geq \varepsilon^p) \\ &\leq \sum_{(n_1,n_2)\geq(N,N)} c^p_{n_1 n_2} E\left([g(Y_{n_1 n_2})]^p - [g(Y_{n_1 n_2 - 1})]^p\right) \\ &\leq \sum_{n_1=N}^{\infty} c^p_{n_1 N} E[g(Y_{n_1 N})]^p \\ &+ \sum_{(n_1,n_2)\geq(N,N+1)} c^p_{n_1 n_2} E\left([g(Y_{n_1 n_2})]^p - [g(Y_{n_1 n_2 - 1})]^p\right). \end{aligned}$$

By using conditions (4.22) it is straightforward to verify that

$$P(\max_{(n_1,n_2)\geq(N,N)} c_{n_1 n_2} g(Y_{n_1 n_2}) \geq \varepsilon) \to 0 \text{ as } N \to \infty.$$

■

Given that the partial sum of mean zero multidimensionally indexed associated random variables is a multidimensionally indexed demimartingale we immediately have the following result.

Corollary 4.5.5 *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{\mathbf{d}}\}$ *be multidimensionally indexed mean zero associated random variables and let* $\{c_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ *be as in Theorem 4.2.3. Let* $S_{\mathbf{n}} = \sum_{\mathbf{i} \leq \mathbf{n}} X_{\mathbf{i}}$. *We also assume that there exists a number* $p \geq 1$ *such that* $E(|S_{\mathbf{k}}|^p) < \infty$ *and for some* $1 \leq s \leq d$

$$\sum_{\mathbf{k}} c^p_{\mathbf{k}} E(|S_{\mathbf{k}}|^p - |S_{\mathbf{k};s;k_s-1}|^p) < \infty \quad and \quad \sum_{k_i, i\neq s} c^p_{\mathbf{k};s;N} E|S_{\mathbf{k};s;N}|^p < \infty \quad for\ each \quad N \in \mathbb{N}.$$

Then

$$c_{\mathbf{k}} S_{\mathbf{k}} \to 0 \ a.s \ as \ \mathbf{k} \to \infty.$$

Proof. The result follows by Corollary 4.5.4 since the function $g(x) = |x|$ is nonnegative convex and $g(0) = 0$. ■

For the special case where $c_{\mathbf{k}} = \left\{\prod_{i=1}^{d} k_i\right\}^{-1}$ and $p = 2$, Corollary 4.5.5 gives a generalization of Kolmogorov's strong law of large numbers.

Corollary 4.5.6 *Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{\mathbf{d}}\}$ be multidimensionally indexed mean zero associated random variables and let $S_{\mathbf{n}} = \sum_{\mathbf{i} \leq \mathbf{n}} X_{\mathbf{i}}$. We assume that $E(|S_{\mathbf{k}}|^2) < \infty$ and for some $1 \leq s \leq d$*

$$\sum_{\mathbf{k}} a_{\mathbf{k}}^2 E(|S_{\mathbf{k}}|^2 - |S_{\mathbf{k};s;k_s-1}|^2) < \infty$$

and

$$\sum_{k_i, i \neq s} a_{\mathbf{k};s;N}^2 E|S_{\mathbf{k};s;N}|^2 < \infty \quad \text{for each} \quad N \in \mathbb{N}.$$

where $a_{\mathbf{k}} = \left(\prod_{i=1}^{d} k_i\right)^{-1}$ and $a_{\mathbf{k};s;N} = a_{k_1 \ldots k_{s-1} N k_{s+1} \ldots k_d}$.
Then

$$\frac{S_{\mathbf{n}}}{|\,\mathbf{n}\,|} \to 0 \ a.s. \ as \quad \mathbf{n} \to \infty.$$

Chapter 5

Future Work

5.1 Introduction

The results presented in this thesis in no way exhaust the research on the concepts of demimartingales and N-demimartingales. It would be naive to state that everything one wants to know is included in this manuscript. However the results can serve as a starting point for further advances on this important area of probability theory. In what follows we briefly describe plans for future work on three topics, namely on multidimensionally indexed N-demimartingales, on continuous time demimartingales (and N-demimartingales) and on strong N-demisupermartingales.

The class of multidimensionally indexed N-demimartingales is a natural generalization of the notion of N-demimartingales defined in Chapter 3. In a future work, maximal and moment inequalities can be provided for this new class of random variables as well as asymptotic results.

The class of continuous time demimartingales and N-demimartingales is an extension to the case of continuous time index and one might conjecture that in the future could be proven to be useful given the fact that continuous time martingales are applicable to important areas of human activity such as finance and economics.

The class of strong N-demisupermartingales, is mainly of theoretical interest and it is defined following Prakasa Rao (2007) who considers the case of strong demisubmartingales. This concept is closely related to the ideas of strong martingales and domination, studied in detail by Osekowski (2007).

5.2 Multidimensionally indexed N-demimartingales

Motivated by the definition of multidimensionally indexed demimartingales which are presented in Chapter 4, the class of multidimensionally indexed N-demimartingales can be defined in a similar way as a natural generalization of the notion of N-demimartingales discussed in Chapter 3. Closely related to multidimensionally indexed N-demimartingales is the notion of negatively associated multidimensionally indexed random variables.

Definition 5.2.1 *A collection of multidimensionally indexed random variables* $\{X_{\mathbf{i}}, \mathbf{i} \leq \mathbf{n}\}$ *is said to be negatively associated if*

$$Cov(f(X_{\mathbf{i}}, \mathbf{i} \in A), g(X_{\mathbf{j}}, \mathbf{j} \in B)) \leq 0$$

where A *and* B *are disjoint subsets of the set* $\{\mathbf{i}, \mathbf{i} \leq \mathbf{n}\}$ *and* f, g *are componentwise nondecreasing functions on* $\mathbb{R}^{|A|}$ *and* $\mathbb{R}^{|B|}$ *respectively. An infinite collection is negatively associated if every subcollection is associated.*

The above definition is the classical definition of negatively associated random variables stated for the case of multidimensionally indexed random variables. Let us now introduce the definitions of multidimensionally indexed N-demimartingales and N-demisupermartingales.

Definition 5.2.2 *An array of random variables* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *is called a multidimensionally indexed N-demimartingale if:*

$$E\left\{(X_{\mathbf{j}} - X_{\mathbf{i}}) f(X_{\mathbf{k}}, \mathbf{k} \leq \mathbf{i})\right\} \leq 0, \quad \forall \;\; \mathbf{i},\ \mathbf{j} \in \mathbb{N}^d \text{ with } \mathbf{i} \leq \mathbf{j},$$

and for all componentwise nondecreasing functions f*. If in addition* f *is required to be nonnegative, then* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *is said to be a multidimensionally indexed N-demisupermartingale.*

It is trivial to verify that the partial sum of mean zero negatively associated multidimensionally indexed random variables is a multidimensionally indexed N-demimartingale.

Clearly a multidimensionally indexed martingale with the natural choice of σ-algebras is a multidimensionally indexed N-demimartingale (and of course, a multidimensionally indexed demimartingale).

In a future work, our goal is to provide maximal and moment inequalities for this new class of random variables. These inequalities can lead us to asymptotic results which will be also valid for the case of negatively associated multidimensionally indexed random variables.

5.3 Continuous time demimartingales

A stochastic process is the mathematical generalization of an empirical process whose development is governed by probabilistic laws. Many applications of stochastic processes occur in physics, engineering, biology, medicine, psychology as well as other branches of mathematical analysis.

Let a filtered complete probability space, $(\Omega, \mathcal{F}, F, P)$ with filtration $F = (\mathcal{F}_t, t \in \mathcal{T})$ be given. A time-index set, $\mathcal{T}$, is considered to be continuous: $\mathcal{T} = \mathbb{R}_+ := [0, \infty)$ or discrete: $\mathcal{T} = \mathbb{N} := \{0, 1, 2, \ldots\}$.

In the case of a time-index set $\mathcal{T} = \mathbb{R}_+$, the filtration, F is an increasing and right continuous family of σ-algebras. A filtered probability space $(\Omega, \mathcal{F}, F, P)$ is also called a stochastic basis.

Let $(E, \mathcal{E})$ be a measurable space, usually Polish, i.e., a complete separable metric space, for example $\mathbb{R}^d$.

Definition 5.3.1 *An F-adapted stochastic process $X = \{X_t, t \in \mathcal{T}\}$ is given by a family of E-valued random variables that is, for each $t \in \mathcal{T}$, $X_t(\omega)$ is $\mathcal{F}_t$-measurable, for each $\omega \in \Omega$. $X(\omega)$ is an E-valued function on $\mathcal{T}$ and called a trajectory or path of X.*

For a given F-adapted stochastic process $X = \{X_t, t \in \mathcal{T}\}$, the natural filtration of X is:

$$\mathcal{F}_t^X := \sigma\{X_s, s \leq t\},\ t \in \mathcal{T},$$

generated by paths of X_s up to and including time t. By definition, X is always $\mathcal{F}_t^X$-adapted and $\mathcal{F}_t^X \subset \mathcal{F}_t$ for each $t \in \mathcal{T}$.

Next we introduce the concept of stopping times which turn out to have several attractive properties in the martingale theory. For example, many martingales can be suitably "stopped" in such a way that the martingale property is maintained.

Definition 5.3.2 *A random variable $\tau = \tau(\omega) : \Omega \to \bar{\mathcal{T}}$ is called a stopping time, if for all $t \in \mathcal{T}$*

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

where $\bar{\mathcal{T}} := \mathcal{T} \bigcup \{\infty\}$. The stopping time τ is finite if $P\{\tau < \infty\} = 1$ and is bounded if $P\{\tau \leq c\} = 1$ for some constant c.

The right continuity of the filtration $F = \{\mathcal{F}_t, t \in \mathbb{R}_+\}$ implies that $\{\omega : \tau(\omega) < t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.

The stopping time σ-algebra, $\mathcal{F}_\tau$, is defined to be:

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \bigcap \{\tau \leq t\} \in \mathcal{F}_t,\ \forall\ t \in \mathcal{T}\}.$$

The important feature is that stopping times are measurable with respect to "what has happened so far", and hence, do not depend on the future. Typical stopping times are first entrance times, such as the first time a random walk reaches a certain level, the first time a simple, symmetric random walk returns to 0, and so on. Such questions can be answered by looking at what has happened until "now". Typical random indices which are not stopping times are last exit times, for example, the last time a simple, symmetric random walk returned to 0. Such questions cannot be answered without knowledge of the future.

The following properties of stopping times can easily be verified (see for example Karlin and Taylor (1975), Szekli (1995), Borovskikh and Korolyuk (1997)) .

1. If τ_1 and τ_2 are two stopping times, then $\tau_1 + \tau_2$, $\tau_1 \wedge \tau_2 := \min\{\tau_1, \tau_2\}$ and $\tau_1 \vee \tau_2 := \max\{\tau_1, \tau_2\}$ are also stopping times and

$$\mathcal{F}_{\tau_1 \wedge \tau_2} = \mathcal{F}_{\tau_1} \bigcap \mathcal{F}_{\tau_2}, \qquad \mathcal{F}_{\tau_1 \vee \tau_2} = \mathcal{F}_{\tau_1} \bigcup \mathcal{F}_{\tau_2}$$

2. Let $\{\tau_n, n \geq 1\}$, be a sequence of stopping times. Then $\wedge_n \tau_n$ and $\vee_n \tau_n$ are also stopping times and

$$\mathcal{F}_{\wedge_n \tau_n} = \bigcap \mathcal{F}_{\tau_n}.$$

3. Let $\tau \leq \sigma$. Then $\mathcal{F}_\tau \subseteq \mathcal{F}_\sigma$.

4. Let $X = \{X_t,\ t \in \mathcal{T}\}$ be an $\mathcal{F}$-adapted stochastic process. Then X_τ is $\mathcal{F}_\tau$-measurable and the process stopped at τ. Also $X^\tau = \{X_{t\wedge\tau}, t \in \mathcal{T}\}$ is $\mathcal{F}_{t\wedge\tau}$-measurable where $X_{t\wedge\tau} = X_t I_{\{t<\tau\}} + X_\tau I_{\{t\geq\tau\}}$.

Let us now introduce the concept of a continuous time-indexed martingale. As in the case of discrete index, a martingale, in the context of games, is the mathematical expression of the fairness of the game, in the sense that the conditional expectation of a gain in the next game is its current value.

Definition 5.3.3 *A real valued F-adapted stochastic process $X = \{X_t, t \in \mathcal{T}\}$ is called a martingale (respectively submartingale, supermartingale) if for all $t \in \mathcal{T}$, X_t is integrable $E \mid X_t \mid < \infty$ and for all $s < t$*

$$E[X_t|\mathcal{F}_s] = X_s \ a.s.$$

(respectively $E[X_t|\mathcal{F}_s] \geq X_s$, $E[X_t|\mathcal{F}_s] \leq X_s$).

As in the case of discrete index, if $\{X_t, t \in \mathcal{T}\}$ has independent increments whose means are zero, then $\{X_t, t \in \mathcal{T}\}$ is a continuous time indexed martingale.

One question that is important to be answered is under what conditions a stopped martingale (submartingale) remains a martingale (submartingale). The answer is given by the following theorem which is known as Doob's optional sampling theorem.

Theorem 5.3.4 *Let $Z \in L^1$, and suppose that $\{(X_n, \mathcal{F}_n), n \geq 0\}$ is a martingale of the form $X_n = E(Z|\mathcal{F}_n)$, $n \geq 0$, and that τ is a stopping time. Then $\{(X_\tau, \mathcal{F}_\tau), (Z, \mathcal{F}_\infty)\}$ is a martingale, and in particular,*

$$EX_\tau = EZ.$$

A property of martingales is that they have constant expectation. As a corollary of the previous result we can derive the fact that martingales evaluated at bounded stopping times have constant expectation. The following theorem proves that this property characterizes martingales.

Theorem 5.3.5 *Suppose that $X_1, X_2, \ldots$ is an $\mathcal{F}_n$-adapted sequence. Then $\{(X_n, \mathcal{F}_n), n \geq 0\}$ is a martingale if and only if*

$$EX_\tau = constant \quad for\ all\ bounded\ stopping\ times \quad \tau.$$

Doob's optional sampling theorem can be extended in order to cover a sequence of non-decreasing stopping times.

Theorem 5.3.6 *Let $Z \in L^1$ and suppose that $\{(X_n, \mathcal{F}_n), n \geq 0\}$ is a martingale of the form $X_n = E(Z|\mathcal{F}_n)$, $n \geq 0$. If $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_k$ are stopping times, then $\{X_0, X_{\tau_1}, X_{\tau_2}, \ldots, X_{\tau_k}, Z\}$ is a martingale, and*

$$EX_0 = EX_{\tau_1} = \cdots = EZ.$$

For further study on the continuous time index martingales see Gut (2005) or Borovskikh and Korolyuk (1997).

As a natural generalization of the definition of demi(sub)martingales we can define continuous time-indexed demi(sub)martingales.

Let $\mathcal{M}(k)$ denote the class of real-valued coordinatewise nondecreasing functions on $\mathbb{R}^k$, $k \in \mathbb{N}$.

Definition 5.3.7 *The stochastic process $\{X_t, t \in \mathcal{T}\}$ is called a demimartingale if for all $t \in \mathcal{T}$ and for all $s < t$*

$$E[(X_t - X_s)f(X_{u_1}, \ldots, X_{u_k},\ u_i \leq s,\ i = 1, \ldots, k)] \geq 0,$$

for all $f \in \mathcal{M}(k)$ such that the expectation is defined. Moreover, if f is also assumed to be nonnegative, the sequence is called a demisubmartingale.

Remark 5.3.8 *In the case where the time-index set is of the form $\mathcal{T} = \{t_1, t_2, \ldots\}$ the sequence will be called a discrete time-index demi(sub)martingale while in the case of $\mathcal{T} = \mathbb{R}_+$ the sequence is called a continuous time-index demi(sub)martingale.*

Let us now introduce the concept of (positive) association for a stochastic process $\{X_t, t \in \mathcal{T}\}$.

Definition 5.3.9 *Let $\mathcal{T}$ be a time index. The stochastic process $\{X_t, t \in \mathcal{T}\}$ is said to be (positively) associated if*

$$Cov(f(X_t, t \in \mathcal{T}_1), g(X_t, t \in \mathcal{T}_2)) \geq 0$$

for all $\mathcal{T}_1$ and $\mathcal{T}_2$ finite subsets of $\mathcal{T}$ and $f \in \mathcal{M}(\mid T_1 \mid)$ and $g \in \mathcal{M}(\mid T_2 \mid)$ provided such that the covariance is defined.

Remark 5.3.10 *Let $\{X_t, t \in \mathcal{T}\}$ be an associated stochastic process such that $E(X_t) =$ 0 for all $t \in \mathcal{T} = \{t_1, t_2, \ldots\}$. We define*

$$S_t = \sum_{t_i \leq t} X_{t_i}.$$

Then $\{S_t, t \in \mathcal{T}\}$ is a demimartingale since, for $t > s$

$$\begin{aligned} E[(S_t - S_s)f(S_l, l \leq s)] &= E\left[\left(\sum_{t_i \leq t} X_{t_i} - \sum_{t_i \leq s} X_{t_i}\right) f(X_{t_i}, t_i \leq s)\right] \\ &= E\left[\left(\sum_{s < t_i \leq t} X_{t_i}\right) f(X_{t_i}, t_i \leq s)\right] \\ &\geq E\left(\sum_{s < t_i \leq t} X_{t_i}\right) E[f(X_{t_i}, t_i \leq s)] \\ &= 0, \end{aligned}$$

where $f \in \mathcal{M}(k)$ *with* k *being the cardinality of the set* $T_s = \{t \in \mathcal{T} : t \leq s\}$.

The class of continuous time-index N-demimartingales can be defined in a similar way.

Definition 5.3.11 *The stochastic process* $\{X_t, t \in \mathcal{T}\}$ *is called an N-demimartingale if for all* $t \in \mathcal{T}$ *and for all* $s < t$

$$E[(X_t - X_s)f(X_{u_1}, \ldots, X_{u_k},\ u_i \leq s,\ i = 1, \ldots, k)] \leq 0$$

for all $f \in \mathcal{M}(k)$ *such that the expectation is defined. Moreover, if* f *is also assumed to be nonnegative, the sequence is called an N-demisupermartingale.*

Remark 5.3.12 *In the case where the time-index set is of the form* $\mathcal{T} = \{t_1, t_2, \ldots\}$ *the sequence will be called a discrete time-index N-demi(super)martingale while in the case of* $\mathcal{T} = \mathbb{R}_+$ *the sequence is called a continuous time-index N-demi(super)martingale.*

Our goal is to explore the relation between continuous time index martingales and continuous time index demimartingales and between continuous time index martingales and continuous time index N-demimartingales. It is worth trying to see if the results of this thesis and other results are valid for the case of continuous index. Finally, we will investigate the role of stopping times in the theory of demimartingales. It would be interesting to see if stopping times have as crucial a role as the one they have in the theory of martingales.

5.4 Strong N-demisupermartingales

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space equipped with the discrete filtration (assume that $\mathcal{F}_0 = \{\Omega, \emptyset\}$). Suppose that $\{M_n, n \in \mathbb{N}\}$ and $\{N_n, n \in \mathbb{N}\}$ are Hilbert-space-valued martingales such that $M_0 = N_0 = 0$ a.s.. Define the difference sequences $\{d_n, n \in \mathbb{N}\}$ and $\{e_n, n \in \mathbb{N}\}$ where $d_n = M_n - M_{n-1}$ and $e_n = N_n - N_{n-1}$, $n = 1, 2, \ldots$. Burkholder

(1988, 1989) states that if $\{M_n, n \in \mathbb{N}\}$ is differentially subordinated by $\{N_n, n \in \mathbb{N}\}$, that is, with probability 1

$$|d_n| \leq |e_n|$$

then for all $t > 0$ and all n,

$$tP(|M_n| \geq t) \leq 2E(|N_n|)$$

and for any $1 < p < \infty$ and any n,

$$(E|M_n|^p)^{\frac{1}{p}} \leq (p^* - 1)(E|N_n|^p)^{\frac{1}{p}} \tag{5.1}$$

where $p^* = \max\{p, p/(p-1)\}$.

Burkholder's results have many extensions since the subordination condition can be replaced by other conditions called dominations. Ozekowski (2007) introduced the concept of dominated martingales.

Definition 5.4.1 *Let $\{M_n, n \in \mathbb{N}\}$ and $\{N_n, n \in \mathbb{N}\}$ be $\mathcal{F}_n$-adapted martingales and let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a nondecreasing convex function. Then M_n is dominated by the martingale N_n if for any $n \geq 1$*

$$E(\phi(|d_n|)|\mathcal{F}_{n-1}) \leq E(\phi(|e_n|)|\mathcal{F}_{n-1}) \ a.s.$$

The notation $M \prec_C N$ is used to denote domination of M_n by N_n.

Ozekowski (2007) provides the following weak-type inequality for dominated martingales.

Theorem 5.4.2 *For all martingales $(M_n), (N_n)$ taking values in the Hilbert space $\mathcal{H}$, such that $M \prec_C N$, and any $t > 0$, we have*

$$tP(|M_n| \geq t) \leq 6E|N_n|, \quad n = 0, 1, 2, \ldots.$$

Motivated by the work of Burkholder (1988, 1989) and Osekowski (2007) for inequalities for dominated martingales, Prakasa Rao (2007) defines strong demisubmartingales.

Definition 5.4.3 *Let $\{M_n, n \geq 0\}$ with $M_0 = 0$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that*

$$E\left[(M_{n+1} - M_n) f(M_0, \ldots, M_n) | \zeta_n\right] \geq 0$$

for any nonnegative coordinatewise nondecreasing function f given the filtration $\{\zeta_n, n \geq 0\}$ contained in $\mathcal{F}$. Then the sequence $\{M_n, n \geq 0\}$ is said to be a strong demisubmartingale with respect to the filtration $\{\zeta_n, n \geq 0\}$.

It is obvious that a strong demisubmartingale is always a demisubmartingale.

In the following definition, Prakasa Rao (2007) introduces the concept of domination of strong demisubmartingales by demisubmartingales.

Definition 5.4.4 *Suppose that $\{M_n, n \geq 0\}$ is a strong demisubmartingale with respect to the filtration generated by a demisubmartingale $\{N_n, n \geq 0\}$. Let $M_0 = N_0 = 0$. The strong demisubmartingale $\{M_n, n \geq 0\}$ is said to be weakly dominated by the demisubmartingale $\{N_n, n \geq 0\}$, if for every nondecreasing convex function $\phi : \mathbb{R}^+ \to \mathbb{R}$, and for any nonnegative coordinatewise nondecreasing function $f : \mathbb{R}^{2n} \to \mathbb{R}$,*

$$E\left[(\phi(|e_n|) - \phi(|d_n|)) f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1}) | N_0, \ldots, N_{n-1}\right] \geq 0, \ \text{a.s.},$$

for all $n \geq 1$, where $d_n = M_n - M_{n-1}$ and $e_n = N_n - N_{n-1}$. In such a case we use the notation $M \ll N$.

In analogy with the inequalities for dominated martingales developed in Osekowski (2007), Prakasa Rao (2007) establishes the following inequality regarding the domination of a strong demisubmartingale by a demisubmartingale.

Theorem 5.4.5 *Let $\{M_n, n \geq 0\}$ be a strong demisubmartingale with respect to the filtration generated by a demisubmartingale $\{N_n, n \geq 0\}$. Furthermore suppose that $M \ll N$. Then for any $\lambda > 0$,*

$$\lambda P(|M_n| \geq \lambda) \leq 6E|N_n|, \ n \geq 0. \tag{5.2}$$

Motivated by these definitions and results, we can define the class of strong N-demisupermartingales.

Definition 5.4.6 *Let* $\{Y_n, n \geq 0\}$ *with* $Y_0 = 0$ *be a sequence of random variables defined on a probability space* $(\Omega, \mathcal{F}, \mathcal{P})$. *Suppose that*

$$E\left[(Y_{n+1} - Y_n) f(Y_0, \ldots, Y_n) | \mathcal{F}_n\right] \leq 0$$

for any nonnegative coordinatewise nondecreasing function f, *given the filtration* $\{\mathcal{F}_n, n \geq 0\}$ *contained in* $\mathcal{F}$. *The sequence* $\{Y_n, n \geq 0\}$ *is said to be a strong N-demisupermartingale with respect to the filtration* $\{\mathcal{F}_n, n \geq 0\}$.

Remark 5.4.7 *It is straightforward to verify that a strong N-demisupermartingale is an N-demisupermartingale in the sense of N-demimartingales discussed in Chapter 3.*

Definition 5.4.8 *Let* $\{Y_n, n \geq 0\}$ *be a strong N-demisupermartingale with respect to the filtration generated by an N-demisupermartingale* $\{X_n, n \geq 0\}$ *with* $Y_0 = X_0 \equiv 0$. *The strong N-demisupermartingale* $\{Y_n, n \geq 0\}$ *is said to be weakly dominated by the N-demisupermartingale* $\{X_n, n \geq 0\}$ *if for every nondecreasing convex function* $\phi : \mathbb{R}_+ \to \mathbb{R}$, *and for any nonnegative coordinatewise nondecreasing function* $f : \mathbb{R}^{2n} \to \mathbb{R}$,

$$E[(\phi(|e_n|) - \phi(|d_n|)) f(Y_0, \ldots, Y_{n-1}; X_0, \ldots, X_{n-1}) \mid X_0, \ldots, X_{n-1}] \leq 0 \quad a.s.,$$

for all $n \geq 1$ *where* $d_n = Y_n - Y_{n-1}$ *and* $e_n = X_n - X_{n-1}$. *The notation* $Y \ll X$ *is used to denote domination of* $\{Y_n, n \geq 0\}$ *by* $\{X_n, n \geq 0\}$.

In analogy to the case of strong demisubmartingales studied by Prakasa Rao (2007), we intend to investigate the concept of domination of a strong N-demisupermartingale by an N-demisupermartingale. In particular, it would be of interest to see if results such as (5.2) can be obtained or norm inequalities similar to (5.1) can be established for the case of domination of a strong N-demisupermartingale by an N-demisupermartingale. In addition it is natural to see how these results are related to convergence concepts of the two sequences. Furthermore, the classes of strong demisubmartingales and strong N-demisubmartingales will be fully investigated. This investigation could include maximal and moment inequalities as well as related asymptotic results.

5.5 Best constants for the Marcinkiewicz-Zygmund inequalities

An interesting subject for future work is to determine the optimal constants for the Marcinkiewicz-Zygmund-type inequalities for nonnegative N-demimartingales provided in Section 3.4. In the present work we didn't focus on this task. However the various constants which appear in the literature for the corresponding inequalities for independent random variables and martingales raise the question whether the constant $c_p = p - 1$ which appears in our result can be improved.

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