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SYMMETRY METHODS FOR HIGHER-ORDER EVOLUTION EQUATIONS

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Abstract

An important tool for the solution of differential equations is the application of symmetry methods which reflect invariance under infinitesimal transformations. The Norwegian mathematician Marious Sophus Lie (1842–1899) was the developer of “Lie theory” [38,39]. Lie came to the study of the symmetries of differential equations [40] through his extensive work on continuous groups [44] of geometrical transformations [41–43] and, later, contact transformations [45].

Lie group methods are perhaps the most powerful tool currently available in finding exact solutions of nonlinear partial differential equations (PDEs). Probably the most useful method is the application of Lie point transformations which are those that form a continuous Lie group of transformations, leaving the PDE invariant. Symmetries of this PDE are then revealed, perhaps enabling new solutions to be found directly or via similarity reductions.

The idea of group classification of nonlinear equations introduced by Ovsiannikov [49] who studied the Lie symmetries of the well known nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u}{\partial x}\right).$$

In recent years, Lie group methods were extended. For example, Bluman and coauthors presented the nonclassical reduction method and also introduced the idea of potential symmetries.

The nonclassical method, introduced by Bluman in 1967 [3], generalises and includes Lie's classical method for obtaining solutions of PDEs. In this case we require the invariance of the PDE, in conjunction with the invariant surface condition, under the infinitesimal transformations.

Bluman and coauthors [4,5] introduced a method for finding a new class of symmetries for a system of PDEs $\Delta(t, x, u)$, in the case that the system possesses at least one con-

servation law. If we introduce potential variables v as further unknown functions using conservation laws of the system, we obtain a new system $Z(t, x, u, v)$. Any Lie symmetry of $Z(t, x, u, v)$ induces a symmetry for the system $\Delta(t, x, u)$. If at least one of the generators which correspond to the variables t, x and u depends explicitly on the potential variables v, then the local symmetry of $Z(t, x, u, v)$ induces a nonlocal symmetry of $\Delta(t, x, u)$, otherwise the symmetry of $Z(t, x, u, v)$ induces a local symmetry of $\Delta(t, x, u)$. These nonlocal symmetries are known as potential symmetries.

If we combine the method for finding potential symmetries and the nonclassical method we derive the so-called, nonclassical potential symmetries. In this method we search for nonclassical symmetries for the potential system or the potential equation. However, it turns out that it is easier to search for nonclassical symmetries for the potential equation which is the equation that arises if we eliminate the variable u from the potential system.

Equivalence transformations play an important role in the theory of Lie group classification. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group [39, 50].

The idea of Lie symmetries and their generalisations as described above will be applied to certain evolution equations. Chapters 2–3 come from the literature and Chapters 4–9 is the new contribution of this work. This thesis is organised as follows. In Chapter 2 we present the basic ideas and definitions that are needed in later chapters. In Chapter 3 we show the known results for a second-order nonlinear diffusion equation. In Chapters 4, 5 and 6 we prove symmetry properties for a chain of a third-, fourth-, fifth- and sixth-order equations. In Chapter 4 we exhibit symmetry properties for a fourth-order equation written in conservation form. It was introduced in the literature as a generalisation of the fourth-order thin-film equation. We derive equivalence transformations, Lie symmetries, nonclassical symmetries, potential symmetries and nonclassical potential symmetries. In Chapter 5 we give the analysis for a third-order equation and in Chapter 6 the symmetry properties for the sixth- and fifth-order equations of the chain. In Chapter 7 we show the Lie symmetries and the potential symmetries for a fourth- and third-order generalised evolution equations, respectively. In Chapter 8 we derive an enhanced Lie group classification for a class of dispersive equations. The complete list of form-preserving point transformations is presented. We show the nonclassical reductions, potential symmetries and nonclassical potential symmetries. In Chapter 9 we exhibit the Lie symmetry classification for a third-order generalised equation with variable coefficients. Finally, in Chapter 10, we give certain conclusions and we suggest certain problems that can be considered in the near future.

Περίληψη

Ένα πολύ χρήσιμο εργάλειο για την επίλυση διαφορικών εξισώσεων είναι η εφαρμογή των μεθόδων συμμετριών όπου τις αφήνουν αναλλοίωτες κάτω από την επίδραση απειροστών μετασχηματισμών. Ο Νορβηγός μαθηματικός Marious Sophus Lie (1842–1899) ήταν ο εμπνευστής της "θεωρίας Lie" [38, 39]. Ο Lie μελέτησε τις συμμετρίες των διαφορικών εξισώσεων [40] διά μέσου της εκτεταμένης μελέτης των συνεχών ομάδων [44] των γεωμετρικών μετασχηματισμών [41–43] και, αργότερα, των μετασχηματισμών επαφής [45].

Οι μέθοδοι συμμετριών Lie ισώς να αποτελούν το πιο δυνατό διαθέσιμο εργαλείο στην επίλυση μη-γραμμικών μερικών διαφορικών εξισώσεων. Πιθανόν η πιο χρήσιμη μέθοδος είναι η εφαρμογή των σημειακών μετασχηματισμών Lie όπου σχηματίζουν μια συνεχή ομάδα μετασχηματισμών Lie και αφήνουν τη μερική διαφορική εξίσωση αναλλοίωτη. Σχηματίζονται έτσι οι συμμετρίες της μερικής διαφορικής εξίσωσης από τις οποίες πιθανόν να προκύψουν νέες λύσεις είτε απευθείας είτε με μετασχηματισμούς ομοιότητας.

Η ιδέα της ταξινόμησης των ομάδων μιας μη-γραμμικής εξίσωσης πρωτοεμφανίστηκε από τον Ovsiannikov [49] ο οποίος μελέτησε τις συμμετρίες Lie της καλά γνωστής μη-γραμμικής εξίσωσης της διάχυσης

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u}{\partial x}\right).$$

Τα τελευταία χρόνια, οι μέθοδοι ομάδων επεκτάθηκαν. Για παράδειγμα, ο Bluman και οι συνεργάτες του παρουσίασαν την ιδέα των μη-κλασικών συμμετριών και των δυναμικών συμμετριών.

Η μη-κλασική μέθοδος, η οποία παρουσιάστηκε από τον Bluman το 1967 [3], γενικεύει και περιέχει την κλασική μέθοδο του Lie για τον προσδιορισμό λύσεων μερικών διαφορικών εξισώσεων. Σε αυτή την περίπτωση απαιτούμε να είναι αναλλοίωτη η μερική διαφορική εξίσωση σε συνύπαρξη με την συνθήκη αναλλοίωτης επιφάνειας, κάτω υπό την επίδραση των απειροστών μετασχηματισμών.

Ο Bluman και οι συνεργάτες του [4, 5] παρουσίασαν μια μέθοδο για τον υπολογισμό μιας νέας τάξης συμμετριών για ένα σύστημα μερικών διαφορικών εξισώσεων $\Delta(t, x, u)$, στην περίπτωση που το σύστημα επιδέχεται τουλάχιστον ένα νόμο διατήρησης. Εισάγουμε μεταβλητές δυναμικού v και χρησιμοποιώντας το νόμο διατήρησης παίρνουμε ένα νέο σύστημα $Z(t, x, u, v)$. Οποιαδήποτε συμμετρία Lie του συστήματος $Z(t, x, u, v)$ δίνει μια αντίστοιχη συμμετρία για το σύστημα $\Delta(t, x, u)$. Αν τουλάχιστον ένας από τους γεννήτορες ο οποίος αντιστοιχεί στις μεταβλητές t, x και u εξαρτάται από την μεταβλητή ομοιότητας v, τότε η τοπική συμμετρία του $Z(t, x, u, v)$ παράγει μια μη-τοπική συμμετρία για το σύστημα $\Delta(t, x, u)$, αλλιώς η συμμετρία του $Z(t, x, u, v)$ θα δώσει τοπική συμμετρία για το $\Delta(t, x, u)$. Αυτές οι μη-τοπικές συμμετρίες είναι γνωστές ως δυναμικές συμμετρίες.

Συνδυάζοντας τη μέθοδο υπολογισμού δυναμικών συμμετριών και τη μη-κλασική μέθοδο προκύπτει η ούτως καλούμενη, μέθοδος υπολογισμού μη-κλασικών δυναμικών συμμετριών. Με αυτή τη μέθοδο ψάχνουμε για μη-κλασικές συμμετρίες για το σύστημα δυναμικού ή την εξίσωση δυναμικού. Εν τούτοις, φαίνεται ότι είναι ευκολότερο να ψάχνουμε για μη-κλασικές συμμετρίες για την εξίσωση δυναμικού η οποία είναι η εξίσωση που προκύπτει απαλείφοντας τη μεταβλητή u από το δυναμικό σύστημα.

Οι ισοδύναμοι μετασχηματισμοί έχουν σημαντικό ρόλο στη θεωρία της ταξινόμησης ομάδων Lie. Το σύνολο όλων των ισοδύναμων μετασχηματισμών μιας δοσμένης οικογένειας διαφορικών εξισώσεων σχηματίζουν μια ομάδα η οποία ονομάζεται ισοδύναμη ομάδα [39, 50].

Η ιδέα των συμμετριών Lie και των γενικεύσεων τους όπως περιγράφηκαν πιο πάνω θα εφαρμοστούν σε συγκεκριμένες εξισώσεις εξέλιξης σε αύτη τη διατριβή η οποία έχει οργανωθεί ως εξής. Στο Κεφάλαιο 2 δίνουμε τις βασικές ιδέες και ορισμούς που χρειάζονται για τα μετέπειτα κεφάλαια και στο Κεφάλαιο 3 παρουσιάζουμε τα γνωστά αποτελέσματα της δεύτερης τάξης μη-γραμμικής εξίσωσης της διάχυσης. Στα Κεφάλαια 4, 5 και 6 παρουσιάζουμε τις ιδιότητες συμμετριών για μια αλυσίδα εξισώσεων τρίτης, τέταρτης, πέμπτης και έκτης ταξης. Στο Κεφάλαιο 4 παρουσιάζουμε τις ιδιότητες συμμετριών για μια εξίσωση τέταρτης τάξης η οποία είναι γραμμένη σε μορφή διατήρησης. Στη βιβλιογραφία εισήχθηκε ως μια γενίκευση της τέταρτης τάξης εξίσωσης thin-film. Εξάγουμε ισοδύναμους μετασχηματισμούς, συμμετρίες Lie, μη-κλασικές συμμετρίες, δυναμικές συμμετρίες και μη-κλασικές δυναμικές συμμετρίες. Στο Κεφάλαιο 5 παρουσιάζουμε την ανάλυση για την εξίσωση τρίτης τάξης και στο Κεφάλαιο 6 τις ιδιότητες συμμετριών για τις εξισώσεις έκτης και πέμπτης τάξης της αλυσίδας. Στο Κεφάλαιο 7 παρουσιάζουμε τις συμμετρίες Lie και τις δυναμικές συμμετρίες για τις γενικευμένες εξισώσεις εξέλιξης τέταρτης και τρίτης τάξης, αντιστοίχως. Στο Κεφάλαιο 8 παρουσιάζουμε εκτενή ταξινόμηση ομάδων Lie για μια κατηγορία εξισώσεων διασποράς. Παρουσιάζονται όλες οι μορφές των σημειακών μετασχηματισμών διατήρησης μορφής. Παρουσιάζουμε τους μη-κλασικούς μετασχηματισμούς, δυναμικές συμμετριές και μη-κλασικές δυναμικές συμμετρίες. Στο Κεφάλαιο 9 παρουσιάζουμε την ταξινόμηση συμμετριών Lie για μια γενικευμένη εξίσωση τρίτης τάξης με μεταβλητούς συντελεστές. Τελειώνοντας, στο Κεφάλαιο 10 παρουσιάζουμε συγκεκριμένα συμπεράσματα και προτείνουμε συγκεκριμένα προβλήματα τα οποία θα μπορούσαν να μελετηθούν μελλοντικά.

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Chapter 1

Introduction

"I am certain, absolutely certain that...these theories will be recognized as fundamental at some point in the future."

Marious Sophus Lie (1842–1899), a Norwegian mathematician, said these words more than one hundred years ago and he was absolutely right since Lie's theories are powerful tools for understanding the physical laws of Nature. Lie, was the establisher of group analysis of differential equations [38, 39].

Following the works of Lie, Ovsianikov in the late 1950's and 1960's and Bluman in the late 1960's and 1970's developed a major revival of interest in symmetry methods for differential equations. With the publication of the texts of Ovsiannikov [50], Bluman and Kumei [5] and Olver [47], there are now several comprehensive accounts of the basic theory as well as more recent applications and generalisations.

Nowadays, transformation methods are one of the most powerful tool currently available in the area of nonlinear PDEs. While there is no existing general theory for solving such equations, many special cases have yielded to appropriate changes of variables. Point transformations are the ones which are mostly used. These are transformations in the space of the dependent and the independent variables of a PDE. Probably the most useful point transformations of PDEs are those which form a continuous Lie group of transformations, which leave the equation invariant. Symmetries of this PDE are then revealed, perhaps enabling new solutions to be found directly or via similarity reductions. The classical method of finding Lie symmetries is first to find infinitesimal transformations, with the benefit of linearization, and then to extend these to groups of finite transformations.

The investigation of nonlinear diffusion equations by means of symmetry methods began in 1959 with Ovsiannikov's work [49] in which the author performed the group classification of the class of equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u}{\partial x}\right).$$

If $f(u) = u^n$ then we have the nonlinear diffusion equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right).$$

In Chapter 3 we recall the known results of the above second-order nonlinear diffusion equation. Namely, we present the equivalence transformations, Lie symmetries, nonclassical symmetries, potential symmetries and nonclassical potential symmetries.

Motivated by these results, in Chapters 4, 5 and 6, we present the symmetry properties for a chain of nonlinear diffusion equations. One of the generalisations of the fourth-order evolution PDE that were considered by King in [31] is

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n\frac{\partial^3 u}{\partial x^3} + au^{n-1}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + bu^{n-2}\left(\frac{\partial u}{\partial x}\right)^3\right). \tag{1.1}$$

Symmetry properties of this equation are presented in Chapter 4. Furthermore King, in the same article, introduced the sixth-order nonlinear thin-film equation

$$\begin{aligned}\frac{\partial u}{\partial t} \;&=\; -\frac{\partial}{\partial x}\left[u^n\frac{\partial^5 u}{\partial x^5} + a_1u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4} + a_2u^{n-1}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + a_3u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial x^3}\right.\\ &\qquad\left. + a_4u^{n-2}\frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + a_5u^{n-3}\left(\frac{\partial u}{\partial x}\right)^3\frac{\partial^2 u}{\partial x^2} + a_6u^{n-4}\left(\frac{\partial u}{\partial x}\right)^5\right],\end{aligned} \tag{1.2}$$

for which symmetry properties are presented in Chapter 6. However, there exist two missing pieces of the chain which are the third-order nonlinear equation,

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n\frac{\partial^2 u}{\partial x^2} + au^{n-1}\left(\frac{\partial u}{\partial x}\right)^2\right), \tag{1.3}$$

for which symmetry properties are presented in the Chapter 5 and the fifth-order nonlinear equation

$$\begin{aligned}\frac{\partial u}{\partial t} \;&=\; -\frac{\partial}{\partial x}\left[u^n\frac{\partial^4 u}{\partial x^4} + a_1u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + a_2u^{n-1}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + a_3u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2}\right.\\ &\qquad\left. + a_4u^{n-3}\left(\frac{\partial u}{\partial x}\right)^4\right],\end{aligned} \tag{1.4}$$

for which symmetry properties are presented in Chapter 6.

In Chapter 7 we consider the classes of a fourth- and third-order generalised evolution equations, respectively,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial^3 u}{\partial x^3} + g(u)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + h(u)\left(\frac{\partial u}{\partial x}\right)^3\right) \tag{1.5}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial^2 u}{\partial x^2} + g(u)\left(\frac{\partial u}{\partial x}\right)^2\right). \tag{1.6}$$

We note that equation (1.5) is the generalisation of the fourth-order equation (1.1) for which the complete group analysis is presented in Chapter 4. Equation (1.6) is the generalisation of the third-order equation (1.3) for which symmetry properties are presented in Chapter 5. In Chapter 7 we show the Lie symmetries and the potential symmetries for these equations.

In Chapter 8, we present an enhanced Lie group analysis for the class of dispersive equations of the form

$$u_t + \epsilon\left(u^m\right)_x + \frac{1}{b}\left[u^a\left(u^b\right)_{xx}\right]_x = 0, \tag{1.7}$$

which generalises the $K(m,n)$ equation

$$u_t + \epsilon\left(u^m\right)_x + \left(u^n\right)_{xxx} = 0.$$

Finally, in Chapter 9 we present the Lie symmetry classification for the generalised $K(m,n)$ equations with variable coefficients of the form

$$u_t + \epsilon(u^m)_x + f(t)\left(u^n\right)_{xxx} = 0. \tag{1.8}$$

Since the differential equations which we examine depend upon arbitrary elements, for certain values of these parameters we obtain useful symmetry properties. These values need not be those which arise in a physical situation. The existence of an additional symmetry motivates us to search for patterns of values of the parameters for which the exceptional symmetry occurs, specifically in Chapters 4, 5 and 6 in which we examine a chain of equations.

All the calculations involved in this thesis have been facilitated by the computer algebraic package “REDUCE” [21].

Chapter 2

Basic Definitions

2.1 Introduction

This Chapter introduces the basic ideas that are needed in the chapters that follow. We define Lie groups of transformations and infinitesimal transformations, we examine when a PDE is invariant under the action of an infinitesimal transformation and how we construct the so-called optimal system and the invariant solutions that arise from transformations which yield invariants. We also give the definitions of nonclassical symmetries, potential symmetries, nonclassical potential symmetries and finally equivalence transformations.

2.2 Lie Groups of Transformations

Sophus Lie developed a theory of transformations, currently known as Lie groups of transformations. These transformations map a given differential equation to itself. In other words, we can say that the differential equation remains invariant under some continuous group of transformations usually known as *symmetries* of the differential equation. In this Section we provide the definitions of the groups, the groups of transformations and finally, more specifically, the one-parameter Lie groups of transformations.

2.2.1 Groups

Definition 2.1. A *group* is a set G together with a law of composition ϕ such that for any two elements g and h of G the product $\phi(g, h)$ is again an element of G. The group operation is required to satisfy the following axioms:

(i) Associativity: If g, h and k are elements of G, then

$$\phi(g, \phi(h, k)) = \phi(\phi(g, h), k).$$

(ii) Identity Element: There is a distinguished element e of G, called the identity element, which has the property that

$$\phi(e, g) = g = \phi(g, e)$$

for all g in G.

(iii) Inverses: For each g in G there is an inverse, denoted by g^{-1}, with the property that

$$\phi(g, g^{-1}) = e = \phi(g^{-1}, g).$$

Definition 2.2. A group G is *abelian* if $\phi(g, h) = \phi(h, g)$ for $g, h \in G$.

Definition 2.3. A *subgroup* of G is a group formed by a subset of elements of G with the same group operation.

2.2.2 Examples of Groups

Example 2.1. G is the set of all integers with $\phi(g, h) = g + h$. Here $e = 0$ and $g^{-1} = -g$.

Example 2.2. G is the set of all positive reals with $\phi(g, h) = gh$. Here $e = 1$ and $g^{-1} = \frac{1}{g}$.

2.2.3 Groups of Transformations

Definition 2.4. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ lie in a region $D \subset \mathbb{R}^n$. The set of transformations

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}, \epsilon),$$

defined for each $\mathbf{x}$ in D, depending upon parameter ϵ lying in set $S \subset \mathbb{R}$, with $\phi(\epsilon, \delta)$ defining a law of composition of parameters, ϵ and δ, in S, forms a *one-parameter group of transformations* on D if:

(i) For each parameter ϵ in S the transformations are one-to-one onto D, in particular $\mathbf{x}^*$ lies in D.

(ii) S with the law of composition ϕ forms a group G.

(iii) $\mathbf{x}^* = \mathbf{x}$, $\forall\, \mathbf{x} \in D$ when $\epsilon = e$, i.e.,

$$\mathbf{X}(\mathbf{x}, e) = \mathbf{x}.$$

(iv) If $\mathbf{x}^* = \mathbf{X}(\mathbf{x}, \epsilon)$ and $\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}^*, \delta)$, then

$$\mathbf{x}^{**} = \mathbf{X}\left(\mathbf{x}, \phi(\epsilon, \delta)\right).$$

2.2.4 One-parameter Lie Group of Transformations

Definition 2.5. A group of transformations defines a *one-parameter Lie group of transformations* if in addition to satisfying axioms (i)-(iv):

(v) ϵ is a continuous parameter, i.e., S is an interval in $\mathbb{R}$. Without loss of generality $\epsilon = 0$ corresponds to the identity element e.

(vi) $\mathbf{X}$ is infinitely differentiable with respect to $\mathbf{x}$ in D and an analytic function of ϵ in S.

(vii) $\phi(\epsilon, \delta)$ is an analytic function of ϵ and δ for $\epsilon, \delta \in S$.

2.2.5 Examples of One-parameter Lie Groups of Transformations

Example 2.3. Group of Scaling in the Plane:

$$x^* = \alpha x, \quad y^* = \alpha^2 y, \quad 0 < \alpha < \infty.$$

Here $\phi(\alpha, \beta) = \alpha\beta$ and the identity element $\alpha = 1$. This group of transformations can also be reparametrized in terms of $\epsilon = \alpha - 1$:

$$x^* = (1+\epsilon)x, \quad y^* = (1+\epsilon)^2 y, \quad -1 < \epsilon < \infty,$$

so that the identity element is $\epsilon = 0$ with the law of composition of parameters given by $\phi(\epsilon, \delta) = \epsilon + \delta + \epsilon\delta$.

Example 2.4. Group of Rotations in the Plane:

$$x^* = x\cos\epsilon - y\sin\epsilon, \quad y^* = x\sin\epsilon + y\cos\epsilon,$$

where $\phi(\epsilon, \delta) = \epsilon + \delta$ and the identity element is $\epsilon = 0$.

Example 2.5. Affine Transformations :

$$x^* = x, \quad y^* = \alpha y.$$

In this case $\phi(\alpha, \beta) = \alpha\beta$ and the identity element is $\alpha = 1$.

Example 2.6. Perspective Transformations :

$$x^* = \alpha x, \quad y^* = \alpha y.$$

Here $\phi(\alpha, \beta) = \alpha\beta$ and the identity element is $\alpha = 1$.

2.3 Infinitesimal Transformations

An important tool for the solution of differential equations is the application of symmetry methods which reflect invariance under infinitesimal transformation.

Consider a one-parameter (ϵ) Lie group of transformations

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}, \epsilon) \tag{2.1}$$

with identity $\epsilon = 0$ and law of composition ϕ. Expanding (2.1) about $\epsilon = 0$, we get (for some neighborhood of $\epsilon = 0$)

$$\mathbf{x}^* = \mathbf{x} + \epsilon \left(\left. \frac{\partial \mathbf{X}(\mathbf{x}, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0} \right) + \frac{\epsilon^2}{2!} \left(\left. \frac{\partial^2 \mathbf{X}(\mathbf{x}, \epsilon)}{\partial \epsilon^2} \right|_{\epsilon=0} \right) + \cdots = \mathbf{x} + \epsilon \left(\left. \frac{\partial \mathbf{X}(\mathbf{x}, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0} \right) + O\left(\epsilon^2\right).$$

Let

$$\boldsymbol{\xi}(\mathbf{x}) = \left. \frac{\partial \mathbf{X}(\mathbf{x}, \epsilon)}{\partial \epsilon} \right|_{\epsilon=0}.$$

The transformation $\mathbf{x} + \epsilon \boldsymbol{\xi}(\mathbf{x})$ is called the *infinitesimal transformation* of the Lie group of transformations (2.1) and the components of $\boldsymbol{\xi}(\mathbf{x})$ are called the *infinitesimal functions* of (2.1).

2.3.1 First Fundamental Theorem of Lie

Theorem 2.1. *There exists a parameterization $\tau(\epsilon)$ such that the Lie group of transformations* (2.1) *is equivalent to the solution of the initial value problem of the system of first-order differential equations*

$$\frac{d\mathbf{x}^*}{d\tau} = \boldsymbol{\xi}(\mathbf{x}^*),$$

with

$$\mathbf{x}^* = \mathbf{x} \quad when \quad \tau = 0.$$

In particular

$$\tau(\epsilon) = \int_0^\epsilon \Theta(\epsilon')d\epsilon',$$

where

$$\Theta(\epsilon) = \left.\frac{\partial\phi(\alpha,\beta)}{\partial\beta}\right|_{(\alpha,\beta)=(\epsilon^{-1},\epsilon)}$$

and

$$\Theta(0) = 1.$$

(ϵ^{-1} denotes the inverse element to ϵ.)

2.3.2 Infinitesimal Generators

In view of Lie's First Fundamental Theorem, from now on, without loss of generality, we assume that a one-parameter (ϵ) Lie group of transformations is parameterized such that the law of composition is $\phi(\alpha,\beta) = \alpha + \beta$ so that $\epsilon^{-1} = -\epsilon$ and $\Theta(\epsilon) \equiv 1$. Thus in terms of its infinitesimal functions, $\boldsymbol{\xi}(\mathbf{x})$, the one-parameter Lie group of transformations (2.1) becomes

$$\frac{d\mathbf{x}^*}{d\epsilon} = \boldsymbol{\xi}(\mathbf{x}^*) \tag{2.2}$$

with

$$\mathbf{x}^* = \mathbf{x} \ \text{ at } \ \epsilon = 0.$$

Definition 2.6. The *infinitesimal generator* of the one-parameter Lie group of transformations (2.1) is the operator

$$\Gamma = \Gamma(\mathbf{x}) = \boldsymbol{\xi}(\mathbf{x}) \cdot \nabla = \sum_{i=1}^{n} \xi_i(\mathbf{x})\frac{\partial}{\partial x_i},$$

where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right).$$

For any differentiable function, $F(\mathbf{x}) = F(x_1, x_2, \dots, x_n)$, we have

$$\Gamma F(\mathbf{x}) = \boldsymbol{\xi}(\mathbf{x}) \cdot \nabla F(\mathbf{x}) = \sum_{i=1}^{n} \xi_i(\mathbf{x}) \frac{\partial F(\mathbf{x})}{\partial x_i}.$$

Example 2.7. We consider the group of projective transformations on the real plane

$$x^* = \frac{x}{1-\epsilon x}, \quad y^* = \frac{y}{1-\epsilon x}.$$

The *infinitesimal functions* for the projective transformations are

$$\left.\frac{dx^*}{d\epsilon}\right|_{\epsilon=0} = \left.\frac{x^2}{(1-\epsilon x)^2}\right|_{\epsilon=0} = x^2, \quad \left.\frac{dy^*}{d\epsilon}\right|_{\epsilon=0} = \left.\frac{xy}{(1-\epsilon x)^2}\right|_{\epsilon=0} = xy$$

and the *infinitesimal generator* is

$$\Gamma = x^2\partial x + xy\partial y.$$

Hence the system (2.2) has the following form

$$\frac{dx^*}{d\epsilon} = x^{*2}, \quad \frac{dy^*}{d\epsilon} = x^* y^*$$

with initial conditions

$$x^* = x, \; y^* = y \quad \text{when } \epsilon = 0.$$

2.3.3 Invariant Functions

Definition 2.7. An infinitely differentiable function $F(\mathbf{x})$ is called an *invariant function* of the Lie group of transformations (2.1) if $F(\mathbf{x}^*) = F(\mathbf{x})$ identically in $\mathbf{x}$ and ϵ in a neighborhood of $\epsilon = 0$.

Remark 2.1. Given an invariant function $F(\mathbf{x})$, any function $\Phi(F(\mathbf{x}))$ is also invariant.

Theorem 2.2. *A function $F(\mathbf{x})$ is absolute invariant of the Lie group of transformations* (2.1) *with the generator Γ if and only if it solves the homogeneous PDE*

$$\Gamma F(\mathbf{x}) \equiv \boldsymbol{\xi}(\mathbf{x}) \cdot \nabla F(\mathbf{x}) = 0.$$

Example 2.8. If we take the group of projective transformations, a function $F(x,y)$ is invariant if and only if

$$\Gamma F(x,y) \equiv x^2 \frac{\partial F}{\partial x} + xy \frac{\partial F}{\partial y} = 0.$$

When we use the method of characteristics, we can solve the above first-order linear PDE, that is,

$$\frac{dx}{x^2} = \frac{dy}{xy} = \frac{dF}{0},$$

from which we deduce that

$$F = \Psi\left(\frac{x}{y}\right).$$

Hence any function of the form $\Psi\left(\frac{x}{y}\right)$ remains invariant under the group of projective transformations.

2.4 Invariance of a PDE

In this Section we apply infinitesimal transformations to the construction of solutions of PDEs. We show that the infinitesimal criterion for invariance of PDEs leads directly to an algorithm to determine infinitesimal generators Γ admitted by given PDEs. Invariant surfaces of the corresponding Lie group of point transformations lead to *invariant solutions (similarity solutions)*. These solutions are obtained by solving PDEs with less independent variables than the given PDEs.

Theorem 2.3. *(Infinitesimal Criterion for Invariance of PDEs). Firstly we consider a nth-order PDE in two independent variables of the form*

$$E(t, x, u, u_t, u_x, \ldots, u_{ij}) = 0. \tag{2.3}$$

Here t and x denote the two independent variables, u denotes the coordinate corresponding to the dependent variable and u_{ij} denotes the nth-order partial derivatives of $\frac{\partial^{i+j}u}{\partial t^i \partial x^j}$ with respect to $t,\ x$ for $2 \le i+j \le n$.

Let

$$\Gamma = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u \tag{2.4}$$

be the generator of the one-parameter Lie group of infinitesimal transformations

$$\tau^* = T(t, x, u, \epsilon),\ \ \xi^* = X(t, x, u, \epsilon),\ \ u^* = U(t, x, u, \epsilon) \tag{2.5}$$

and

$$\Gamma^{(n)} = \Gamma + \eta^t(t,x,u,u_t,u_x)\partial u_t + \eta^x(t,x,u,u_t,u_x)\partial u_x + \ldots + \eta^{(n)}_{ij}(t,x,u,u_t,u_x,\ldots,u_{ij})\partial u_{ij}$$

be the nth-prolongation infinitesimal generator of (2.4)*, where* η^t *and* η^x *are given by*

$$\eta^t = D_t\eta - (D_t\tau)u_t - (D_t\xi)u_x \text{ and } \eta^x = D_x\eta - (D_x\tau)u_t - (D_x\xi)u_x$$

in terms of $\tau(t,x,u)$*,* $\xi(t,x,u)$ *and* $\eta(t,x,u)$*. For* $\eta^{(n)}_{ij}$*,we give some examples of expansions:*

$$\eta^{xx} = D_x\eta^x - (D_x\tau)u_{xt} - (D_x\xi)u_{xx},$$

$$\eta^{xxt} = D_t\eta^{xx} - (D_t\tau)u_{xxt} - (D_t\xi)u_{xxx},$$

$$\eta^{tttx} = D_x\eta^{ttt} - (D_x\tau)u_{tttt} - (D_x\xi)u_{xttt}.$$

(D_t *and* D_x *are the total derivative operators with respect to* t *and* x*, respectively.) Similarly, we can construct the other expansions of* $\eta^{(n)}_{ij}$*. See for example, [5].*

We say that the one-parameter Lie group of point transformations (2.5) *is admitted by PDE* (2.3)*, i.e., is a point symmetry of PDE* (2.3)*, if and only if*

$$\Gamma^{(n)}E(t,x,u,u_t,u_x\ldots,u_{ij}) = 0 \text{ when } E(t,x,u,u_t,u_x,\ldots,u_{ij}) = 0.$$

2.4.1 Optimal System

The following two definitions are useful:

Definition 2.8. One of the most important operations on vector fields is their commutator (Lie bracket). If Γ_i and Γ_j are vector fields on M (infinitesimal generators), then their commutator $[\Gamma_i, \Gamma_j]$ is the unique vector field satisfying

$$[\Gamma_i, \Gamma_j](f) = \Gamma_i(\Gamma_j(f)) - \Gamma_j(\Gamma_i(f))$$

for all smooth functions $f : M \to \mathbb{R}$.

The Lie bracket has the following properties:

(i) Bilinearity

$$[a\Gamma_i + b\Gamma_j, \Gamma_k] = a[\Gamma_i, \Gamma_k] + b[\Gamma_j, \Gamma_k],$$

$$[\Gamma_i, a\Gamma_j + b\Gamma_k] = a[\Gamma_i, \Gamma_j] + b[\Gamma_i, \Gamma_k],$$

where $a,\ b$ are constants.

(ii) Skew-Symmetry

$$[\Gamma_i, \Gamma_j] = -[\Gamma_j, \Gamma_i].$$

(iii) Jacobi Identity

$$[\Gamma_i, [\Gamma_j, \Gamma_k]] + [\Gamma_j, [\Gamma_k, \Gamma_i]] + [\Gamma_k, [\Gamma_i, \Gamma_j]] = 0.$$

Definition 2.9. The formula of the adjoint representation Ad (using Lie series), which is given by

$$\mathrm{Ad}\,[\exp(\epsilon\Gamma_i)]\,\Gamma_j = \Gamma_j - \epsilon[\Gamma_i, \Gamma_j] + \frac{\epsilon^2}{2!}[\Gamma_i, [\Gamma_i, \Gamma_j]] + \dots,$$

denotes the separate adjoint actions for each element Γ_i acting on all the other elements.

For the construction of the optimal system of subalgebras, firstly we make the commutator table for the Lie algebra of the Γ_i and then using the Lie series we construct a table showing the separate adjoint actions for each element Γ_i acting on all the other elements. This table enables us to derive the optimal system that provides all possible invariant solutions.

Ovsiannikov [50] proved that the optimal system of solutions consists of solutions invariant with respect to all proper inequivalent subalgebras of the symmetry algebra. More detail about the construction of optimal sets of subalgebras can be found in [47,50].

2.4.2 Similarity Transformations

The invariant solutions that arise from transformations which yield invariants allow one to obtain solutions through reducing the number of independent variables of a PDE by at least one. For example a PDE with two independent variables can be reduced to an ordinary differential equation (ODE) or even to algebraic equation. The similarity transformations are constructed from the solution of the *invariant surface condition*

$$\tau(t,x,u)u_t + \xi(t,x,u)u_x = \eta(t,x,u). \tag{2.6}$$

Now, if $\frac{\xi(t,x,u)}{\tau(t,x,u)}$ is independent of u, then the solution of (2.6) has the form

$$\eta(t,x) = \text{constant},$$

$$u(t,x) = F(t,x,\eta,f(\eta)), \tag{2.7}$$

where F is a known function. Equation (2.7) is the *invariant solution* and the function $\eta(t,x)$ is called the *similarity variable* that constitutes the independent variable of the ODE that we get from the transformation. The function $f(\eta)$ is the unknown function of the ODE.

2.5 Nonclassical Symmetries

The nonclassical method, introduced in Bluman [3], generalises and includes Lie's classical method for *obtaining solutions of PDEs*. In this case we require the invariance of the PDE,

$$E(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx},\ldots) = 0, \tag{2.8}$$

in conjunction with the invariant surface condition, (2.6), under the infinitesimal transformations generated by

$$\Gamma = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \tag{2.9}$$

which results in an overdetermined nonlinear system of PDEs for the determination of the coefficients $\tau(t,x,u)$, $\xi(t,x,u)$ and $\eta(t,x,u)$.

A "nonclassical symmetry" is not a symmetry of a given PDE (2.8) unless the infinitesimal coefficients yielding an infinitesimal generator (2.9) yield a point symmetry of (2.8). Otherwise a mapping resulting from such an infinitesimal generator maps *no* solution of (2.8) into a different solution of it. In other words the nonclassical method is not a "symmetry" method but an extension of Lie's symmetry method ("classical method") for the purpose of finding specific solutions of PDEs.

From the nature of the constraint invariant surface condition equation (2.6), without loss of generality, in using the nonclassical method, two simplifying cases need only be considered when solving the determining equations for finding the form of the infinitesimal coefficients, namely $\tau \neq 0$ and $\tau = 0$. In the case $\tau(t,x,u) \neq 0$ we can assume that $\tau = 1$, without loss of generality. Also, when $\tau = 0$, we can take $\xi = 1$, without loss of generality. In this latter case the invariant conditions result to a single nonlinear PDE in $\eta(t, x, u)$. Here we only consider the case where $\tau = 1$. For recent applications of this method see [35] and references therein.

2.6 Potential Symmetries

Bluman and coauthors [4, 5] introduced a method for finding a new class of symmetries, nonlocal symmetries, for a system of PDEs $\Delta(t, x, u)$, with independent variables t, x and dependent variables u in the case when at least one of the PDEs can be written in conserved form.

If we introduce new potential variables v, which are potentials for the PDEs written in conserved forms as further unknown functions, we obtain a new system $Z(t, x, u, v)$. By construction any solution $u(t, x)$, $v(t, x)$ of $Z(t, x, u, v)$ defines a solution $u(x, t)$ of $\Delta(t, x, u)$. The given system $\Delta(t, x, u)$ is the said to be embedded in the auxiliary system $Z(t, x, u, v)$ so that any Lie group of transformation for $Z(t, x, u, v)$ induces a symmetry for $\Delta(t, x, u)$. If at least one of the generators which correspond to the variables t, x and u depends explicitly on the potential variables v, then the local symmetry of $Z(t, x, u, v)$ induces a nonlocal symmetry of $\Delta(t, x, u)$. Otherwise the symmetry of $Z(t, x, u, v)$ induces a local symmetry of $\Delta(t, x, u)$. These nonlocal symmetries are known as *potential symmetries*. More details about potential symmetries and their uses can be found in [4,5,28].

2.7 Nonclassical Potential Symmetries

As we have seen for a given scalar PDE, a potential variable can be introduced through a conservation law. Such a conservation law yields an equivalent system (potential system) of PDEs with the given dependent variable and the potential variable as its dependent variables. Also the nonclassical method for obtaining solutions of PDEs is a generalisation of the classical method for obtaining invariant solutions from point symmetries admitted by a given PDE. Here we combine the two approaches to derive the so-called, *nonclassical potential symmetries*. In other words we search for nonclassical symmetries for the potential system or the potential equation. However, it turns out that it is easier to search for nonclassical symmetries for the potential equation which is the equation that arises if we eliminate the variable u from the potential system. More details about nonclassical potential symmetries can be found in [54].

2.8 Equivalence Transformations

Equivalence transformations play an important role in the theory of Lie group classification. An equivalence transformation of a class of PDEs is an invertible transformation of the independent and dependent variables of the form that maps every equation of the class into an equations of the same form. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group [39,50]. There exist two methods for calculation of equivalence transformations, the direct which was used first by Lie [39] and the Lie infinitesimal method which was introduced by Ovsyannikov [50]. Although the direct method involves considerable computational difficulties, it has the benefit of finding the most general equivalence group. For recent applications of the direct method one can refer, for example, to references [34, 52, 71, 72]. More detailed description and examples of both methods can be found in [27]. Here we use the direct method to derive the desired equivalence transformations.

Chapter 3

Group Analysis of a Second-Order Nonlinear Diffusion Equation

3.1 Introduction

The investigation of nonlinear heat (or diffusion if u represents mass concentration) equations by means of symmetry methods began in 1959 with Ovsiannikov's work [49] in which the author performed the group classification of the class of equations of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial u}{\partial x}\right). \tag{3.1}$$

Equation (3.1) describes the stationary motion of a boundary layer of fluid over a flat plate and a vortex of incompressible fluid in a porus medium with polytropic relation between gas density and pressure.

If we consider the case that the diffusion term is $f(u) = u^n$, then (3.1) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right). \tag{3.2}$$

Equation (3.2) is called a fast diffusion equation for $-2 < n < 0$ and a slow diffusion equation for $n > 0$. In the first case the spread of mass is much faster than in the linear case $n = 0$ and in the second case it is slower.

In this Chapter we present the known results for equation (3.2). We give the equivalence transformations, the group classification of point symmetries, the optimal system of one-dimensional subalgebras and all possible types of invariant solutions [49]. Also we give the nonclassical symmetries [1,20], the potential symmetries [5] and the nonclassical potential symmetries [54]. Motivated by the results of this Chapter, in next Chapters we show a complete group analysis for a chain of nonlinear evolution equations. Namely, we derive the symmetry properties for the third-order nonlinear equation (1.3), the fourth-order thin-film equation (1.1), the fifth-order nonlinear equation (1.4) and the sixth-order thin-film equation (1.2). Also, we present symmetry properties for the fourth- and third-order generalised equations (1.5) and (1.6), respectively. Finally, we give symmetry properties for the third-order dispersive equation (1.7) and the third-order generalised equation with variable coefficients (1.8).

3.2 Equivalence Transformations

We find that equation (3.2) admits the equivalence transformations

$$t' = c_1 t + c_2, \quad x' = c_3 x + c_4, \quad u' = c_1^{-1/n} c_3^{2/n} u, \quad n' = n,$$

where $c_1 c_3 \neq 0$. Furthermore in the case for which $n = -\frac{4}{3}$ we have the additional equivalence transformations

$$t' = c_1 t + c_2, \quad x' = \frac{c_3 x + c_4}{c_5 x + c_6}, \quad u' = c_1^{-1/n} (c_5 x + c_6)^{-4/n} u, \quad n' = n,$$

where $c_1 \neq 0$ and $c_3 c_6 - c_4 c_5 = \pm 1$.

3.3 Lie Symmetries

From the definition, a second-order PDE admits Lie point symmetries if and only if

$$\Gamma^{(2)} E\big|_{E=0} = 0,$$

where $\Gamma^{(2)}$ is the second prolongation of the generator

$$\Gamma = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

which is given by the relation

$$\begin{aligned}\Gamma^{(2)} &= \Gamma + [D_t\eta - (D_t\tau)u_t - (D_t\xi)u_x]\partial u_t + [D_x\eta - (D_x\tau)u_t - (D_x\xi)u_x]\partial u_x \\ &\quad + [D_x\eta^x - (D_x\tau)u_{xt} - (D_x\xi)u_{xx}]\partial u_{xx}.\end{aligned}$$

Here D_t and D_x are the total derivatives with respect to t and x respectively and η^x is the coefficient function of ∂u_x.

In this case we have that

$$E = u_t - u^n u_{xx} - nu^{n-1}u_x^2 = 0$$

and equation (3.2) admits Lie point symmetries if and only if

$$\Gamma^{(2)}[u_t - u^n u_{xx} - nu^{n-1}u_x^2] = 0 \tag{3.3}$$

for $u_t = u^n u_{xx} + nu^{n-1}u_x^2$.

After the elimination of u_t due to the above expression equation (3.3) becomes an identity in the variables u_x, u_{tx} and u_{xx}. The coefficients of different powers of these variables must be zero and these give the determining equations on the coefficients τ, ξ and η. Using the general results on point transformations between evolution equations [34] that $\tau = \tau(t)$ and $\xi = \xi(t, x)$, we take the following determining equations from the coefficients of u_{xx}, u_x^2, u_x and the term independent of derivatives of (3.3), respectively,

$$(\tau_t - 2\xi_x)u + n\eta = 0, \tag{3.4}$$

$$\eta_{uu}u^2 + n(\tau_t - 2\xi_x + \eta_u)u + n(n-1)\eta = 0, \tag{3.5}$$

$$(\xi_{xx} - 2\eta_{xu})u^{n+1} - 2n\eta_x u^n - \xi_t u = 0, \tag{3.6}$$

$$\eta_{xx}u^n - \eta_t = 0. \tag{3.7}$$

When we solve these equations (3.4)–(3.7), we observe that for the case where n is arbitrary, the symmetry Lie algebra is four-dimensional and is spanned by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 2t\partial_t + x\partial_x, \quad \Gamma_4 = \frac{nx}{2}\partial_x + u\partial_u.$$

An additional Lie symmetry exists for the specific value of the parameter $n = -\frac{4}{3}$. In particular, equation (3.2) admits a fifth symmetry

$$\Gamma_5 = x^2\partial_x + \frac{4xu}{n}\partial_u.$$

3.3.1 Invariant Solutions

The primary use of Lie symmetries is to obtain a reduction of variables. Similarity variables appear as constants of integration in the solution of the characteristic equations

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}u}{\eta}.$$

Reductions could be obtained from any symmetry which is an arbitrary linear combination, i.e.

$$a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 + a_5\Gamma_5.$$

To ensure that a minimal complete set of reductions is obtained from the Lie symmetries of equation (3.2), we construct the so-called optimal system of subalgebras. In the case for which n is arbitrary the optimal system and the corresponding similarity reductions that transform (3.2) into an ODE are given by the operators

$$\begin{aligned}
\langle\Gamma_2\rangle \quad &: \quad u = \phi(\omega), \quad \omega = t,\\
\langle\Gamma_1 + c\Gamma_2\rangle \quad &: \quad u = \phi(\omega), \quad \omega = x - ct,\\
\langle\Gamma_3 + c\Gamma_4\rangle \quad &: \quad u = t^{\frac{c}{2}}\phi(\omega),\ \omega = \begin{cases} x & \text{if } nc+2=0,\\ t^{-\frac{1}{2}}x^{\frac{2}{nc+2}} & \text{if } nc+2\neq 0,\end{cases}\\
\langle\Gamma_4 + c\Gamma_1\rangle \quad &: \quad u = \begin{cases} x^{\frac{2}{n}}\phi(\omega),\ \omega = e^t x^{-\frac{2c}{n}} & \text{if } n\neq 0,\\ e^{\frac{t}{c}}\phi(\omega),\ \omega = x & \text{if } n = 0,\end{cases}\\
\langle\Gamma_4 + c\Gamma_2 - \tfrac{n}{2}\Gamma_3\rangle \quad &: \quad u = \begin{cases} t^{-\frac{1}{n}}\phi(\omega),\ \omega = x + \frac{c}{n}\ln t & \text{if } n\neq 0,\\ e^{\frac{x}{c}}\phi(\omega),\ \omega = t & \text{if } n = 0.\end{cases}
\end{aligned}$$

In the special case $n = -\frac{4}{3}$, for which a fifth symmetry exists, we obtain the following additional reductions that correspond to the additional subalgebras:

$$\langle\Gamma_5 + c\Gamma_2 + 2k\Gamma_3\rangle : \ u = \begin{cases} ((x+k)^2+1)^{\frac{2}{n}}\exp\left[-\frac{4k}{n}\tan^{-1}(x+k)\right]\phi(\omega),\\ \omega = t\exp\left[-4k\tan^{-1}(x+k)\right] \ \text{if } c-k^2 = 1,\\ \\ ((x+k)^2-1)^{\frac{2}{n}}\exp\left[\frac{4k}{n}\tanh^{-1}(x+k)\right]\phi(\omega),\\ \omega = t\exp\left[4k\tanh^{-1}(x+k)\right] \ \text{if } c-k^2 = -1,\\ \\ (x+k)^{\frac{4}{n}}\exp\left[\frac{4k}{n(x+k)}\right]\phi(\omega),\\ \omega = t\exp\left[\frac{4k}{x+k}\right] \ \text{if } c-k^2 = 0,\end{cases}$$

where ω is the independent, ϕ the dependent variable of the reduced ODE, $c = 0, \pm 1$ and $k \in \mathbb{R}$.

The results of Sections 3.2 and 3.3 can be found in [49].

3.4 Nonclassical Symmetries

From the corresponding definition in Chapter 2, in this case we require the invariance of the system of PDEs,

$$u_t = (u^n u_x)_x,$$

$$\tau(t,x,u)u_t + \xi(t,x,u)u_x = \eta(t,x,u),$$

under the class of infinitesimal transformations generated by

$$\Gamma = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u.$$

This results in an overdetermined nonlinear system of PDEs for the determination of the coefficients $\tau(t,x,u)$, $\xi(t,x,u)$ and $\eta(t,x,u)$. After we choose $\tau = 1$, the nonclassical method applied to equation (3.2) gives rise to the four nonlinear determining equations

$$\xi_{uu}u - n\xi_u = 0, \tag{3.8}$$

$$(2\xi_{xu} - \eta_{uu})u^{n+2} - n\eta_u u^{n+1} + n\eta u^n - 2\xi\xi_u u^2 = 0, \tag{3.9}$$

$$(\xi_{xx} - 2\eta_{xu})u^{n+1} - 2n\eta_x u^n - (\xi_t + 2\xi\xi_x - 2\xi_u\eta)u + n\xi\eta = 0, \tag{3.10}$$

$$\eta_{xx}u^{n+1} - (2\xi_x\eta + \eta_t)u + n\eta^2 = 0. \tag{3.11}$$

After we have solved the determining system (3.8)–(3.11), we can assure that equation (3.2) admits a proper nonclassical symmetry only for $n = -\frac{1}{2}$. For $n \neq -\frac{1}{2}$ we only recover the classical symmetries. For $n = -\frac{1}{2}$ we obtain the nonclassical symmetry,

$$\Gamma = \partial_t + \phi(x)\sqrt{u}\partial_u,$$

where $\phi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \frac{1}{2}\phi^2 = 0.$$

A particular solution of this equation is $\phi = 12x^{-2}$. The nonclassical operator above produces the nonclassiclal reduction

$$u = \left[\tfrac{1}{2}\phi(x)t + F(x)\right]^2$$

which reduces (3.2) to the ODE

$$\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} - \frac{1}{2}\phi F = 0.$$

If $\phi = 12x^{-2}$, this ODE becomes an equation of Euler type with the form

$$x^2 \frac{\mathrm{d}^2 F}{\mathrm{d}x^2} - 6F = 0$$

and solution

$$F(x) = c_1 x^{-2} + c_2 x^3$$

which yields the explicit solution for (3.2)

$$u(t,x) = \left(6tx^{-2} + c_1 x^{-2} + c_2 x^3\right)^2.$$

The results which are presented in this Section can be found in [1, 20].

3.5 Potential Symmetries

If we introduce the potential variable v, we can write equation (3.2) as a system of two PDEs

$$\begin{aligned} v_x &= u, \\ v_t &= u^n u_x. \end{aligned} \tag{3.12}$$

Suppose (3.12) admits an infinitesimal generator of the form

$$\Gamma = \tau(t,x,u,v)\partial_t + \xi(t,x,u,v)\partial_x + \eta(t,x,u,v)\partial_u + \zeta(t,x,u,v)\partial_v. \tag{3.13}$$

We search for Lie point symmetries for the system (3.12) with the optimal goal of finding potential symmetries for equation (3.2). Lie symmetries of (3.12) induce potential symmetries for (3.2) if the following condition holds,

$${\tau_v}^2 + {\xi_v}^2 + {\eta_v}^2 \neq 0.$$

The system (3.12) admits Lie symmetries if and only if

$$\Gamma^{(1)}[v_x - u] = 0, \tag{3.14}$$

$$\Gamma^{(1)}[v_t - u^n u_x] = 0,$$

where system (3.12) holds. Here $\Gamma^{(1)}$ is the first extension of the generator (3.13) and is given by the relation

$$\begin{aligned}\Gamma^{(1)} &= \Gamma + [D_t\eta - (D_t\tau)u_t - (D_t\xi)u_x]\partial u_t + [D_x\eta - (D_x\tau)u_t - (D_x\xi)u_x]\partial u_x \\ &\quad + [D_t\zeta - (D_t\tau)v_t - (D_t\xi)v_x]\partial v_t + [D_x\zeta - (D_x\tau)v_t - (D_x\xi)v_x]\partial v_x.\end{aligned}$$

(Here D_t and D_x are the total derivatives with respect to t and x, respectively.)

Eliminating $v_t,\ v_x$ through substitution of (3.12) into (3.14), we obtain seven determining equations for $\tau,\ \xi,\ \eta$ and ζ which simplify to:

$$\tau_u = 0, \tag{3.15}$$

$$\tau_v u + \tau_x = 0, \tag{3.16}$$

$$\xi_u u - \zeta_u = 0, \tag{3.17}$$

$$\tau_v u^n - \xi_u = 0, \tag{3.18}$$

$$\eta_v u^{n+1} + \eta_x u^n + \xi_t u - \zeta_t = 0, \tag{3.19}$$

$$(\tau_t - \xi_x + \eta_u - \zeta_v)u + n\eta = 0, \tag{3.20}$$

$$\xi_v u^2 + (\xi_x - \zeta_v)u - \zeta_x + \eta = 0. \tag{3.21}$$

Solution of the determining equations (3.15)–(3.21) can be summarised as follows:

For the case that $n \neq -2$, the system (3.12) admits a five-parameter group with infinitesimal generators

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 2t\partial_t + x\partial_x + v\partial_v, \quad \Gamma_4 = \partial_v, \tag{3.22}$$

$$\Gamma_5 = x\partial_x + \frac{2u}{n}\partial_u + \left(1 + \frac{2}{n}\right) v\partial_v. \tag{3.23}$$

When $n = -2$, the system (3.12) admits an infinite-parameter group with infinitesimal generators (3.22), (3.23) and

$$\Gamma_6 = -xv\partial_x + u(xu + v)\partial_u + 2t\partial_v,$$

$$\Gamma_7 = 4t^2\partial_t + -x(2t+v^2)\partial_x + u(6t+2xuv+v^2)\partial_u + 4tv\partial_v,$$

$$\Gamma_\infty = f\partial_x - u^2 f_v \partial_u,$$

where $f = f(t, v)$ is a solution of the linear heat equation $f_t = f_{vv}$.

The Lie symmetries (3.22) and (3.23) project into local symmetries of (3.2) and the Lie symmetries Γ_6, Γ_7 and Γ_∞ induce potential symmetries for the corresponding equation (3.2).

The results of this Section can be found in [5].

3.6 Nonclassical Potential Symmetries

In this case we search for nonclassical symmetries for a potential system or potential equation. It is easier to search for nonclassical symmetries for the potential equation, which is obtained from the associated auxiliary system of (3.2) given by

$$v_x = u,$$

$$v_t = u^n u_x.$$

To archieve this we eliminate u from the above system to get

$$v_t = v_x^n v_{xx},$$

which is the potential form of equation (3.2). Now we take into consideration the case $n = -1$ for which the potential equation becomes

$$v_t = v_x^{-1} v_{xx}. \tag{3.24}$$

Here the invariance surface condition has the form

$$\tau(t, x, v)v_t + \xi(t, x, v)v_x = \zeta(t, x, v)$$

and the reduction operators have the general form

$$\Gamma = \tau(t, x, v)\partial_t + \xi(t, x, v)\partial_x + \zeta(t, x, v)\partial_v.$$

We assume that $\tau = 1$ without loss of generality. The four determining equations for the coefficients ξ and ζ have the form

$$\xi_{vv} - \xi\xi_v = 0, \tag{3.25}$$

$$\zeta_{xx} - \zeta\zeta_x = 0, \tag{3.26}$$

$$\xi_t - 2\xi_{xv} + \zeta_{vv} + \xi\zeta_v - \xi_v\zeta + \xi\xi_x = 0, \tag{3.27}$$

$$\zeta_t - 2\zeta_{xv} + \xi_{xx} + \xi_x\zeta - \xi\zeta_x + \zeta\zeta_v = 0. \tag{3.28}$$

The nonclassical symmetries of the potential fast diffusion equation (3.24) that result from the solution of the determining equations (3.25)–(3.28) are

$$\Gamma_1 = \partial_t + \varepsilon\partial_x + f(\omega)\partial_v, \quad \text{where } \omega = x + \varepsilon t,$$

$$\Gamma_2 = \partial_t + f(\omega)\left(\partial_x + \partial_v\right), \quad \text{where } \omega = x + v,$$

$$\Gamma_3 = \partial_t + \xi\partial_x + \left(\varphi_t + \varphi_x\xi\right)\partial_v, \quad \text{where } \xi = \frac{-2}{v+\varphi} \quad \text{and } \varphi\epsilon\left\{t + e^x, tf(x)\right\},$$

$$\Gamma_4 = \partial_t + \xi\partial_x - \frac{g_t + g_x\xi}{1+g^2}\partial_v, \quad \text{where } \xi = -2\frac{1 + g\tan v}{\tan v - g} \text{ and}$$
$$g\epsilon\left\{\tan(2t)\tanh x, \coth(2t)\cot x\right\},$$

$$\Gamma_5 = \partial_t + \xi\partial_x - \frac{g_t + g_x\xi}{1-g^2}\partial_v, \quad \text{where } \xi = -2\frac{1 - g\tanh v}{\tanh v - g} \text{ and}$$
$$g\epsilon\left\{\tanh(2t)\tanh x, \tanh(2t)\coth x, \coth(2t)\coth x, \frac{e^{2x}\tanh(2t)+1}{e^{2x}-\tanh(2t)}, \frac{2-e^{2x}-e^{4t}}{2+e^{2x}+e^{4t}}\right\}.$$

Here $\varepsilon \,\epsilon\, \{0, 1\}$ and f is an arbitrary nonconstant solution of the ODE $f_{\omega\omega} = ff_\omega$ with the solution being in parametric form

$$\omega = \int \frac{\mathrm{d}f}{\frac{f^2}{2} + c} + c_1$$

which leads to the particular solution

$$f = \begin{cases} \sqrt{2c}\tan\left[\sqrt{\frac{c}{2}}(\omega - c_1)\right] & \text{if } c > 0, \\ \sqrt{2|c|}\left[\frac{1+\exp\left(\sqrt{2|c|}(\omega-c_1)\right)}{1-\exp\left(\sqrt{2|c|}(\omega-c_1)\right)}\right] & \text{if } c < 0, \\ \frac{2}{c_1-\omega} & \text{if } c = 0. \end{cases}$$

The results of this Section can be found in [54].

Chapter 4

Group Analysis of a Fourth-Order Nonlinear Thin-Film Equation

4.1 Introduction

Thin-films arise in a variety of contexts. They can be between two solid surfaces in the form of a lubricant or with one solid surface and the other free as in water or oil on a road. When surface tension is a dominant physical effect driving the motion, the governing evolution PDE is nonlinear and of the fourth-order in the spatial derivatives. Myers [46] presented a review of research into thin-films of fluid for which the surface tension is a driving mechanism. The introduction of surface tension into standard lubrication theory leads to the nonlinear evolution PDE

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u^3 u_{xxx} + f\left(u,\, u_x,\, u_{xx}\right)\right) = 0,$$

where $u(t,x)$ is the height of the film of fluid. For suitable choices of the function f this equation has been applied to models for coating, drainage of foams and the movement of contact lenses.

King [31] provided some generalisations of the fourth-order evolution PDE,

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n \frac{\partial^3 u}{\partial x^3}\right),$$

which has been the subject of extensive investigations [2, 13, 22, 32, 46, 66]. For example, the case $n = 3$ corresponds to the study of capillary-driven flow and the case $n = 1$

it describes the thickness of a thin bridge between two masses of fluid in a Hele–Shaw cell [13]. One of the forms considered by King is

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n \frac{\partial^3 u}{\partial x^3} + a u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + b u^{n-2}\left(\frac{\partial u}{\partial x}\right)^3\right). \tag{4.1}$$

In [31] it is shown that for appropriate choices of $n,\ a$ and b in terms of new parameters $k,\ l$ and m, equation (4.1) can be written in the interestingly compact form

$$u_t = -\frac{1}{m}\left(u^k\left(u^l\left(u^m\right)_{xx}\right)_x\right)_x,$$

which immediately invites generalisation to an equation of higher order of the form

$$u_t = -\frac{1}{m_1}\left(u^{m_n}\left(u^{m_{n-1}}\left(\cdots\left(u^{m_2}\left(u^{m_1}\right)_{xx}\right)_x\right)\cdots\right)_x\right)_x.$$

Symmetry properties for this equation have been explored recently in [69].

If we introduce a potential variable v, equation (4.1) can be written as a system of two PDEs

$$\begin{aligned} v_x &= u, \\ v_t &= -(u^n u_{xxx} + a u^{n-1} u_x u_{xx} + b u^{n-2} u_x^3). \end{aligned} \tag{4.2}$$

Eliminating u from system (4.2) we have the, so-called, potential form of (4.1),

$$v_t = -(v_x^n v_{xxxx} + a v_x^{n-1} v_{xx} v_{xxx} + b v_x^{n-2} v_{xx}^3). \tag{4.3}$$

Equations (4.1)–(4.3) are equivalent in the sense that if $\{u(t,x), v(t,x)\}$ satisfy (4.2), then $u(t,x)$ solves (4.1) and $v(t,x)$ solves (4.3).

Our goal in this Chapter is to perform a complete group analysis of equation (4.1). We derive the equivalence group of (4.1) and Lie symmetries are presented. Also we present the nonclassical symmetries and give the special forms of (4.1) that admit potential symmetries. Finally we derive nonclassical potential symmetries. These results have already appeared in [10].

4.2 Equivalence Transformations

We call an equivalence transformation of a class of PDEs, $E(t,x,u) = 0$, an invertible transformation of the variables t, x and u of the form

$$t' = Q(t,x,u), \quad x' = P(t,x,u), \quad u' = R(t,x,u) \tag{4.4}$$

that maps every equation of the class into an equation of the same form, $E(t',x',u')=0$. For example, in the case of (4.1), an equivalence transformation maps (4.1) into

$$\frac{\partial u'}{\partial t'}=-\frac{\partial}{\partial x'}\left(u'^{n'}\frac{\partial^3 u'}{\partial x'^3}+a'u'^{n'-1}\frac{\partial u'}{\partial x'}\frac{\partial^2 u'}{\partial x'^2}+b'u'^{n'-2}\left(\frac{\partial u'}{\partial x'}\right)^3\right).$$

For invertibility we require that $\frac{\partial(Q,P,R)}{\partial(t,x,u)}\neq 0$.

Since evolution equation (4.1) is a multipolynomial in the derivatives of u with respect to x, equivalence transformations are restricted to the class [34]

$$t'=Q(t),\quad x'=P(t,x),\ u'=R(t,x,u),$$

where $Q_tP_xR_u\neq 0$.

Details of the method can be found in [34, 52, 71, 72]. We find that equation (4.1) admits the equivalence transformations

$$t'=c_1t+c_2,\quad x'=c_3x+c_4,\quad u'=c_1^{-1/n}c_3^{4/n}u,\quad (n',a',b')=(n,a,b), \tag{4.5}$$

where $c_1c_3\neq 0$. Furthermore in the case for which

$$(n,a,b)\in\left\{(-4,-6,6),\left(-\frac{8}{3},-\frac{16}{3},\frac{44}{9}\right),\left(-\frac{8}{5},-\frac{24}{5},\frac{104}{25}\right)\right\}, \tag{4.6}$$

we have the additional equivalence transformations

$$t'=c_1t+c_2,\quad x'=\frac{c_3x+c_4}{c_5x+c_6},\quad u'=c_1^{-1/n}(c_5x+c_6)^{-8/n}u,\quad (n',a',b')=(n,a,b), \tag{4.7}$$

where $c_1\neq 0$ and $c_3c_6-c_4c_5=\pm 1$. We note that a special case of these additional equivalence transformations is the cyclic group of order two,

$$t'=t,\ x'=\frac{1}{x},\ u'=x^nu.$$

We have similar definitions for the equivalence transformations of the system (4.2). Here the results separate into two branches:

$$t'=c_3^{-n}c_1^{n+4}t+c_5,\ \ x'=c_1x+c_2,\ \ u'=c_1^{-1}c_3u,\ \ v'=c_3v+c_4,\ \ (n',a',b')=(n,a,b) \tag{4.8}$$

and

$$t'=c_1^{-n}c_3^{n+4}t+c_5,\quad x'=c_1v+c_2,\quad u'=c_1^{-1}c_3u,\quad v'=c_3x+c_4,$$

$$(n', a', b') = (-n-4, -a-10, 3a+b+15), \tag{4.9}$$

where $c_1 c_3 \neq 0$.

Similarly we observe that the equivalence transformations for the potential form (4.3) are either

$$t' = c_3^{-n} c_1^{n+4} t + c_5, \quad x' = c_1 x + c_2, \quad v' = c_3 v + c_4, \quad (n', a', b') = (n, a, b) \tag{4.10}$$

or

$$t' = c_1^{-n} c_3^{n+4} t + c_5, \; x' = c_1 v + c_2, \; v' = c_3 x + c_4, \; (n', a', b') = (-n-4, -a-10, 3a+b+15). \tag{4.11}$$

Transformations such as those in (4.11) are known as potential equivalence transformations. More examples of such transformations can be found in [53, 67].

We note that a special case of the equivalence transformation (4.11) is the pure hodograph transformation

$$t' = t, \quad x' = v, \quad v' = x$$

which leaves

$$v_t = -(v_x^{-2} v_{xxxx} - 5 v_x^{-3} v_{xx} v_{xxx} + b v_x^{-4} v_{xx}^3)$$

invariant and also maps

$$v_t = -(v_x^{-4} v_{xxxx} - 10 v_x^{-5} v_{xx} v_{xxx} + 15 v_x^{-6} v_{xx}^3)$$

into the linear equation $v_t + v_{xxxx} = 0$.

4.3 Lie Symmetries

We look for vector fields of the form,

$$\Gamma = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u,$$

which generate one-parameter groups of point symmetry transformations of equation (4.1). These vector fields form the maximal Lie invariance algebra of this equation. Any such vector field, Γ, satisfies the criterion of infinitesimal invariance, i.e., the action of the fourth prolongation $\Gamma^{(4)}$ of Γ on equation (4.1) results in the conditions being an identity for all solutions of this equation, namely, we require that

$$\Gamma^{(4)}[u_t + u^n u_{xxxx} + (n+a)u^{n-1}u_x u_{xxx} + au^{n-1}u_{xx}^2 + [a(n-1)+3b]u^{n-2}u_x^2 u_{xx}$$

$$+b(n-2)u^{n-3}u_x^4] = 0 \tag{4.12}$$

identically, modulo equation (4.1).

After the elimination of u_t due to (4.1) equation (4.12) becomes a multivariable polynomial in the variables u_x, u_{xx}, u_{tx}, u_{xxx}, u_{txx}, u_{xxxx} and u_{txxx}. The coefficients of different powers of these variables must be zero, giving the determining equations on the coefficients τ, ξ and η. If we use the general results on point transformations between evolution equations [34], then the remaining determining equations produce the functional forms of $\tau(t)$, $\xi(t,x)$ and $\eta(t,x,u)$. Also from the coefficient of u_{xxxx} we deduce that

$$\eta = \frac{(4\xi_x - \tau_t)u}{n}. \tag{4.13}$$

After we use the fact that τ is a function of t, ξ is a function of t and x and the expression (4.13) for η, from the coefficients of u_{xxx}, u_{xx}, $u_x u_{xx}$, u_x^3, u_x^2, u_x and the term independent of derivatives of (4.12) we have the following determining equations, respectively,

$$(2a-n+8)\xi_{xx} = 0, \tag{4.14}$$

$$(2a-n+6)\xi_{xxx} = 0, \tag{4.15}$$

$$(3an+20a+24b-3n^2+12n)\xi_{xx} = 0, \tag{4.16}$$

$$(9an-8a-8b-an^2+13bn)\xi_{xx} = 0, \tag{4.17}$$

$$(3an+8a+12b-n^2+12n)\xi_{xxx} = 0, \tag{4.18}$$

$$(4a+3n+16)\xi_{xxxx}u^n - n\xi_t = 0, \tag{4.19}$$

$$4\xi_{xxxxx}u^n + 4\xi_{tx} - \tau_{tt} = 0. \tag{4.20}$$

When we solve the determining equations (4.14)–(4.20), we conclude that, if n, a and b are arbitrary, the symmetry Lie algebra is four-dimensional and is spanned by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x, \quad \Gamma_4 = \frac{nx}{4}\partial_x + u\partial_u.$$

An additional Lie symmetry exists for specific values of the parameters n, a and b. In particular equation (4.1) admits a fifth symmetry

$$\Gamma_5 = x^2\partial_x + \frac{8xu}{n}\partial_u$$

if $(n,a,b) \in \left\{(-4,-6,6), \left(-\frac{8}{3}, -\frac{16}{3}, \frac{44}{9}\right), \left(-\frac{8}{5}, -\frac{24}{5}, \frac{104}{25}\right)\right\}$. The three cases of (4.1) correspond to the following equations:

$$\begin{aligned}
u_t &= -\left[u^{-4}u_{xxx} - 6u^{-5}u_x u_{xx} + 6u^{-6}u_x^3\right]_x, \\
u_t &= -\left[u^{-8/3}u_{xx} - \frac{4}{3}u^{-11/3}u_x^2\right]_{xx}, \\
u_t &= -\left[u^{-8/5}u_x\right]_{xxx}.
\end{aligned}$$

We note that transformations (4.5) and (4.7) do not change arbitrary elements and hence their projections onto the space of the variables of equation (4.1) form point symmetry groups. In fact these symmetry groups are the finite forms of the infinitesimal groups derived in the present Section. We recall that finite forms arise from infinitesimal forms using Lie first fundamental theorem.

4.3.1 Invariant Solutions

The next step is the construction of the optimal system of subalgebras. Firstly we make the commutator table for the Lie algebra of the Γ_i and then using the Lie series we construct a table showing the separate adjoint actions for each element Γ_i acting on all the other elements. This table enables us to derive the optimal system of subalgebras that provides all possible invariant solutions (Lie ansatzes).

In the following tables we present the commutator relations and the adjoint actions for the Lie algebra of the Γ_i.

Table 4.1: Commutation relations for the Lie algebra of equation (4.1)

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	0	0	$4\Gamma_1$	0	0
Γ_2	0	0	Γ_2	$\frac{n}{4}\Gamma_2$	$\frac{8}{n}\Gamma_4$
Γ_3	$-4\Gamma_1$	$-\Gamma_2$	0	0	Γ_5
Γ_4	0	$-\frac{n}{4}\Gamma_2$	0	0	$\frac{n}{4}\Gamma_5$
Γ_5	0	$-\frac{8}{n}\Gamma_4$	$-\Gamma_5$	$-\frac{n}{4}\Gamma_5$	0

Table 4.2: Adjoint actions for the Lie algebra of equation (4.1)

Ad	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	Γ_1	Γ_2	$\Gamma_3 - 4\epsilon\Gamma_1$	Γ_4	Γ_5
Γ_2	Γ_1	Γ_2	$\Gamma_3 - \epsilon\Gamma_2$	$\Gamma_4 - \frac{n\epsilon}{4}\Gamma_2$	$\Gamma_5 - \frac{8\epsilon}{n}\Gamma_4 + \epsilon^2\Gamma_2$
Γ_3	$e^{4\epsilon}\Gamma_1$	$e^{\epsilon}\Gamma_2$	Γ_3	Γ_4	$e^{-\epsilon}\Gamma_5$
Γ_4	Γ_1	$e^{\frac{n\epsilon}{4}}\Gamma_2$	Γ_3	Γ_4	$e^{-\frac{n\epsilon}{4}}\Gamma_5$
Γ_5	Γ_1	$\Gamma_2 + \frac{8\epsilon}{n}\Gamma_4 + \epsilon^2\Gamma_5$	$\Gamma_3 + \epsilon\Gamma_5$	$\Gamma_4 + \frac{n\epsilon}{4}\Gamma_5$	Γ_5

In the case for which n, a, b are arbitrary the optimal system consists of the following inequivalent subalgebras

$$\langle\Gamma_2\rangle,\ \langle\Gamma_1 + c\Gamma_2\rangle,\ \langle\Gamma_3 + c\Gamma_4\rangle,\ \langle\Gamma_4 + c\Gamma_1\rangle,\ \langle\Gamma_4 + c\Gamma_2 - \tfrac{n}{4}\Gamma_3\rangle.$$

In the case for which (n, a, b) are given by equation (4.6), in addition to the above list, we have a reduction that corresponds to the subalgebra

$$\langle\Gamma_5 + c\Gamma_2 + 2k\Gamma_3\rangle.$$

For each component of the optimal system we construct the corresponding similarity reduction that transforms (4.1) into an ODE. We obtain the following results:

$$\langle\Gamma_2\rangle \quad : \quad u=\phi(\omega), \quad \omega=t,$$

$$\langle\Gamma_1+c\Gamma_2\rangle \quad : \quad u=\phi(\omega), \quad \omega=x-ct,$$

$$\langle\Gamma_3+c\Gamma_4\rangle \quad : \quad u=t^{\frac{c}{4}}\phi(\omega),\ \omega=\begin{cases} x & \text{if } nc+4=0,\\ t^{-\frac{1}{4}}x^{\frac{4}{nc+4}} & \text{if } nc+4\neq 0,\end{cases}$$

$$\langle\Gamma_4+c\Gamma_1\rangle \quad : \quad u=\begin{cases} x^{\frac{4}{n}}\phi(\omega),\ \omega=\mathrm{e}^t x^{-\frac{4c}{n}} & \text{if } n\neq 0,\\ \mathrm{e}^{\frac{t}{c}}\phi(\omega),\ \omega=x & \text{if } n=0,\end{cases}$$

$$\langle\Gamma_4+c\Gamma_2-\tfrac{n}{4}\Gamma_3\rangle \quad : \quad u=\begin{cases} t^{-\frac{1}{n}}\phi(\omega),\ \omega=x+\frac{c}{n}\ln t & \text{if } n\neq 0,\\ \mathrm{e}^{\frac{x}{c}}\phi(\omega),\ \omega=t & \text{if } n=0.\end{cases}$$

In the case for which we have five symmetries we obtain the following additional reduction

$$\langle\Gamma_5+c\Gamma_2+2k\Gamma_3\rangle : \ u=\begin{cases} ((x+k)^2+1)^{\frac{4}{n}}\exp\left[-\frac{8k}{n}\tan^{-1}(x+k)\right]\phi(\omega),\\ \omega=t\exp\left[-8k\tan^{-1}(x+k)\right]\ \text{if } c-k^2=1,\\ \\ ((x+k)^2-1)^{\frac{4}{n}}\exp\left[\frac{8k}{n}\tanh^{-1}(x+k)\right]\phi(\omega),\\ \omega=t\exp\left[8k\tanh^{-1}(x+k)\right]\ \text{if } c-k^2=-1,\\ \\ (x+k)^{\frac{8}{n}}\exp\left[\frac{8k}{n(x+k)}\right]\phi(\omega),\\ \omega=t\exp\left[\frac{8k}{x+k}\right]\ \text{if } c-k^2=0,\end{cases}$$

where ω is the independent, ϕ is the dependent variable of the reduced ODE, $c=0,\pm1$ and $k\in\mathbb{R}$.

We give some examples of reduced ODEs. The reduction that corresponds to the subalgebra $\langle\Gamma_1+c\Gamma_2\rangle$ leads to the equation

$$c\phi_\omega-[\phi^{n-2}(\phi^2\phi_{\omega\omega\omega}+a\phi\phi_\omega\phi_{\omega\omega}+b\phi_\omega^3)]_\omega=0$$

which provides traveling-wave solutions for the thin-film equation (4.1). We integrate this equation by parts and the integral has the form

$$\phi^{n-2}(\phi^2\phi_{\omega\omega\omega}+a\phi\phi_\omega\phi_{\omega\omega}+b\phi_\omega^3)=c\phi+c_1, \tag{4.21}$$

where c_1 is the constant of integration.

For the special case that $(n, a, b) = (2, 3, 0)$ and $c_1 = 0$ this has the form

$$\phi\phi_{\omega\omega\omega} + 3\phi_\omega\phi_{\omega\omega} = c,$$

with solution [51]

$$\phi^2 = A_2\omega^2 + A_1\omega + A_0 + \frac{c(\omega - x_0)^3}{3}, \text{ where } x_0 \text{ is an arbitrary number.}$$

Consequently, the form of u is

$$u = \pm\sqrt{A_2(x-ct)^2 + A_1(x-ct) + A_0 + \frac{c(x-ct-x_0)^3}{3}}.$$

For $c_1 = 0$, $c = -k_1 \neq 0$ and $n = 0$, equation (4.21) admits the symmetry generator $\phi\partial_\phi$. By means of the substitution $y(\omega) = \frac{\phi_\omega}{\phi}$, (4.21) can be reduced to

$$y_{\omega\omega} + (a+3)yy_\omega + (a+b+1)y^3 + k_1 = 0. \tag{4.22}$$

For the case that $(a, b) = (-2, 1)$, equation (4.22) has the form

$$y_{\omega\omega} + yy_\omega + k_1 = 0.$$

This equation can be integrated and we obtain a Riccati equation

$$y_\omega = -\frac{y^2}{2} - k_1\omega - k_0,$$

where k_0 is the constant of integration. If we use the substitution $\omega = s - \frac{k_0}{k_1}$ we obtain the ODE

$$y_s = -\frac{y^2}{2} - k_1 s.$$

The solutions to this equation can be given in term of Bessel functions [57] as

$$y(s) = \frac{\sqrt{2k_1}\sqrt{s}\left(c_0 J_{\frac{2}{3}}(\chi) - J_{-\frac{2}{3}}(\chi)\right)}{J_{\frac{1}{3}}(\chi) + c_0 J_{-\frac{1}{3}}(\chi)}, \tag{4.23}$$

with $\chi(s) = -\frac{1}{3}\sqrt{2k_1}s^{3/2}$ and c_0 arbitrary constant. Since $y = \frac{\phi_\omega}{\phi}$, from (4.23) we can obtain solutions for the equation (4.21) and consequently we take the corresponding traveling-wave solutions of equation (4.1).

We note that for the case that $(n, a, b) = (0, -2, 1)$ equation (4.1) has the form

$$u_t = -\left(u_{xxx} - 2u^{-1}u_x u_{xx} + u^{-2}u_x^3\right)_x \equiv -\left[u\left(\ln u\right)_{xx}\right]_{xx}$$

which is the Derriba–Lebowitz–Speer–Spohn equation [16].

In the case $nc+4=0$ the subalgebra $\langle \Gamma_3 + c\Gamma_4 \rangle$ leads to the reduced equation

$$\phi - n[\phi^{n-2}(\phi^2\phi_{xxx} + a\phi\phi_x\phi_{xx} + b\phi_x^3)]_x = 0.$$

From subalgebra $\langle \Gamma_4 + c\Gamma_1 \rangle$ in the case $n=0$ we obtain

$$\phi + c[\phi^{-2}(\phi^2\phi_{xxx} + a\phi\phi_x\phi_{xx} + b\phi_x^3)]_x = 0.$$

The subalgebra $\langle \Gamma_4 + c\Gamma_2 - \frac{n}{4}\Gamma_3 \rangle$ leads to the reduced ODE

$$c\phi_\omega - \phi + n[\phi^{n-2}(\phi^2\phi_{\omega\omega\omega} + a\phi\phi_\omega\phi_{\omega\omega} + b\phi_\omega^3)]_\omega = 0,$$

where $n \neq 0$. In the case $n=0$, we obtain the solution

$$u = c_1 \exp\left[\frac{1}{c^4}\left(c^3x - (a+b+1)t\right)\right].$$

In the case for which we have five symmetries and $k=0$ we obtain the following solutions: For $c=\pm 1$ and $(n,a,b) = \left(-\frac{8}{5}, -\frac{24}{5}, \frac{104}{25}\right)$ we have

$$u = (x^2+c)^{-5/2}(24t+c_1)^{5/8}$$

and for $(n,a,b) = \left(-\frac{8}{3}, -\frac{16}{3}, \frac{44}{9}\right)$ we have

$$u = (x^2+c)^{-3/2}(8t+c_1)^{3/8}.$$

4.4 Nonclassical Symmetries

From the definition, in the case of nonclassical symmetries we require invariance of equation (4.1) in conjunction with its invariant surface condition,

$$\tau(t,x,u)u_t + \xi(t,x,u)u_x = \eta(t,x,u)$$

under the infinitesimal transformations generated by

$$\Gamma = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u.$$

We can assume that $\tau = 1$ without loss of generality. After the application of the method to equation (4.1), from the coefficient of $u_{xx}u_{xxx}$ we can conclude that $\xi = \xi(t,x)$. Using this fact, the system of PDEs that results in the determination of the coefficients ξ and η comprises of the equations

$$2(3\xi_{xx} - 2\eta_{xu})u - (a+n)\eta_x = 0, \tag{4.24}$$

$$4\eta_{uu}u^2 + (a+n)\eta_u u - (a+n)\eta = 0, \tag{4.25}$$

$$3\eta_{uu}u^2 + a\eta_u u - a\eta = 0, \tag{4.26}$$

$$6\eta_{uuu}u^3 + (5a+3n)\eta_{uu}u^2 + 2(an-a+3b)\eta_u u - 2(an-a+3b)\eta = 0, \tag{4.27}$$

$$12\eta_{xuu}u^2 - [(5a+3n)\xi_{xx} - (7a+3n)\eta_{xu}]u + 2(an-a+3b)\eta_x = 0, \tag{4.28}$$

$$(2\xi_{xxx} - 3\eta_{xxu})u - a\eta_{xx} = 0, \tag{4.29}$$

$$\begin{aligned}&\eta_{uuuu}u^4 + (a+n)\eta_{uuu}u^3 + (an-a+3b)\eta_{uu}u^2 + 3b(n-2)\eta_u u\\ &-3b(n-2)\eta = 0,\end{aligned} \tag{4.30}$$

$$4\eta_{xuuu}u^3 + 3(a+n)\eta_{xuu}u^2 - (an-a+3b)(\xi_{xx} - 2\eta_{xu})u + 4b(n-2)\eta_x = 0, \tag{4.31}$$

$$6\eta_{xxuu}u^2 - (a+n)(\xi_{xxx} - 3\eta_{xxu})u + (an-a+3b)\eta_{xx} = 0, \tag{4.32}$$

$$(\xi_{xxxx} - 4\eta_{xxxu})u^{n+1} - (a+n)\eta_{xxx}u^n + (\xi_t + 4\xi\xi_x)u - n\xi\eta = 0, \tag{4.33}$$

$$\eta_{xxxx}u^{n+1} + (4\xi_x\eta + \eta_t)u - n\eta^2 = 0. \tag{4.34}$$

After we have solved the equations (4.24)–(4.34), it turns out that equation (4.1) admits a nonclassical symmetry only in the case where $(n, a, b) = (-\frac{1}{2}, -\frac{3}{2}, \frac{3}{4})$. This symmetry has the form

$$\Gamma = \partial_t + \phi(x)\sqrt{u}\partial_u,$$

where $\phi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4} + \frac{1}{2}\phi^2 = 0.$$

A particular solution of this equation is $\phi = -1680x^{-4}$ [51]. The nonclassical operator above produces the ansatz

$$u = \left[\tfrac{1}{2}\phi(x)t + F(x)\right]^2 \tag{4.35}$$

which reduces (4.1) to the ODE

$$\frac{\mathrm{d}^4F}{\mathrm{d}x^4} + \frac{1}{2}\phi F = 0. \tag{4.36}$$

For the particular solution $\phi = -1680x^{-4}$, equation (4.36) becomes an equation of Euler type of the form

$$x^4 \frac{\mathrm{d}^4 F}{\mathrm{d}x^4} - 840F = 0$$

and solution

$$F(x) = c_1 x^{-4} + c_2 x^7 + x^{3/2}\left[c_3 \cos\left(\frac{\sqrt{111}\ln|x|}{2}\right) + c_4 \sin\left(\frac{\sqrt{111}\ln|x|}{2}\right)\right].$$

Consequently we have that the solution of (4.1) is

$$u(t,x) = \left\{-840tx^{-4} + c_1 x^{-4} + c_2 x^7 + x^{3/2}\left[c_3 \cos\left(\frac{\sqrt{111}\ln|x|}{2}\right) + c_4 \sin\left(\frac{\sqrt{111}\ln|x|}{2}\right)\right]\right\}^2.$$

Equation (4.1) with $(n,a,b) = (-\frac{1}{2}, -\frac{3}{2}, \frac{3}{4})$ can be written in the form

$$\frac{\partial u}{\partial t} = -2\frac{\partial^4}{\partial x^4}\sqrt{u}. \tag{4.37}$$

If we interpret (4.35) as an ansatz with the two new unknown functions ϕ and F of the single variable x, then this ansatz reduces equation (4.37) to a system of the two ODEs above with respect to the functions ϕ and F,

$$\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4} + \frac{1}{2}\phi^2 = 0 \quad \text{and} \quad \frac{\mathrm{d}^4 F}{\mathrm{d}x^4} + \frac{1}{2}\phi F = 0.$$

This means that the generalised vector field $(\sqrt{u})_{tt}\partial_u$ associated with the ansatz (4.35) is a generalised conditional symmetry of the equation (4.37).

The ansatz (4.35) first appeared in [48] for the $u^{-1/2}$-diffusion equation. Γ is an extension of a nonclassical symmetry operator presented, e.g. in [1] (p. 26), for the $u^{-1/2}$-diffusion equation to equations of the form (4.1). The same ansatz, the same generalised conditional symmetry and similar nonclassical symmetries are possessed by any equation of the form

$$\frac{\partial u}{\partial t} = \sum_{i=0}^{N} c^i(x)\frac{\partial^i}{\partial x^i}\sqrt{u}.$$

4.5 Potential Symmetries

As we have said, in this case we search for Lie symmetries for the system (4.2) with the optimal goal of finding potential symmetries for equation (4.1).

The system (4.2) admits Lie symmetries if and only if

$$\Gamma^{(1)}[v_x - u] = 0, \tag{4.38}$$

$$\Gamma^{(3)}[v_t + (u^n u_{xxx} + au^{n-1}u_x u_{xx} + bu^{n-2}u_x^3)] = 0 \tag{4.39}$$

for $v_x = u$ and $v_t = -(u^n u_{xxx} + au^{n-1}u_x u_{xx} + bu^{n-2}u_x^3)$. Here $\Gamma^{(1)}$ and $\Gamma^{(3)}$ are the first and third extensions, respectively, of the generator

$$\Gamma = \tau(t,x,u,v)\partial_t + \xi(t,x,u,v)\partial_x + \eta(t,x,u,v)\partial_u + \zeta(t,x,u,v)\partial_v.$$

From the coefficients of $u_x u_{xxx}$, u_{xxx} and u_x of (4.38) and from the coefficient u_{xx}^2 of (4.39) we have the determining equations, respectively,

$$\tau_u = 0,$$

$$\tau_v u + \tau_x = 0,$$

$$\xi_u u - \zeta_u = 0,$$

$$\xi_u = 0,$$

from which we can deduce that the coefficients τ is a function of t and ξ and ζ are functions of t, x and v. After we have used these results, equation (4.38) gives that

$$\eta = -\xi_v u^2 - (\xi_x - \zeta_v)u + \zeta_x. \tag{4.40}$$

We substitute the form of η (4.40) into (4.39) and we have the simplifying determining system for the determination of the coefficients $\tau,\ \xi,\ \eta$ and ζ which has the form

$$(n+4)\xi_v u^2 - [\tau_t - (n+4)\xi_x + n\zeta_v]u - n\zeta_x = 0, \tag{4.41}$$

$$(an+5a+10)\xi_v u^2 - a[\tau_t + n\zeta_v - (n+4)\xi_x]u - a(n-1)\zeta_x = 0, \tag{4.42}$$

$$(a+10)\xi_{vv}u^3 + [2(a+8)\xi_{xv} - (a+4)\zeta_{vv}]u^2 + [(a+6)\xi_{xx} - 2(a+2)\zeta_{xv}]u$$
$$-a\zeta_{xx} = 0, \tag{4.43}$$

$$(3a+bn+6b)\xi_v u^2 - b[\tau_t + n\zeta_v - (n+4)\xi_x]u - b(n-2)\zeta_x = 0, \tag{4.44}$$

$$(2a+b+5)\xi_{vv}u^3 + [(3a+2b+4)\xi_{xv} - (a+b+1)\zeta_{vv}]u^2$$
$$+[(a+b)\xi_{xx} - (a+2b)\zeta_{xv}]u - b\zeta_{xx} = 0, \tag{4.45}$$

$$(a+10)\xi_{vvv}u^4 + [3(a+8)\xi_{xvv} - (a+6)\zeta_{vvv}]u^3 + [3(a+6)\xi_{xxv} - 3(a+4)\zeta_{xvv}]u^2$$

$$+[(a+4)\xi_{xxx} - 3(a+2)\zeta_{xxv}]u - a\zeta_{xxx} = 0, \tag{4.46}$$

$$\xi_{vvvv}u^{n+5} + (4\xi_{xvvv} - \zeta_{vvvv})u^{n+4} + 2(3\xi_{xxvv} - 2\zeta_{xvvv})u^{n+3}$$

$$+2(2\xi_{xxxv} - 3\zeta_{xxvv})u^{n+2} + (\xi_{xxxx} - 4\zeta_{xxxv})u^{n+1} - \zeta_{xxxx}u^n + \xi_t u - \zeta_t = 0. \tag{4.47}$$

Equation (4.47) breaks up into more equations depending upon the values of the parameter n.

When we have solved the determining system (4.41)–(4.47), we deduce that potential symmetries exist in two cases. The Lie symmetries of (4.2) with the forms

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x - u\partial_u, \quad \Gamma_4 = \partial_v, \quad \Gamma_5 = u\partial_u + v\partial_v$$

project into local symmetries of (4.1). We obtain that the following Lie symmetries of the system (4.2) induce potential symmetries for the corresponding equation (4.1):

(1) $(n,a,b) = (0,-4,3)$: $\Gamma_6 = 2uv\partial_u + v^2\partial_v$.

(2) $(n,a,b) = (-4,-10,15)$: $\Gamma_7 = \psi(t,v)\partial_x - u^2\psi_v\partial_u$, where $\psi_t + \psi_{vvvv} = 0$.

Since the system

$$v_x = u,$$

$$v_t = -(u_{xxx} - 4u^{-1}u_x u_{xx} + 3u^{-2}u_x^3),$$

admits six linearly independent Lie symmetries (Γ_6 and five that project into Lie symmetries of the original equation), we can construct the optimal system. In the next set of tables we present the commutator table and the adjoint table for the Lie algebra of the Γ_i, $i = 1, \ldots, 6$, for $(n,a,b) = (0,-4,3)$, from which we are going to construct the optimal system that provides all possible invariant solutions.

Table 4.3: Commutation relations for the Lie algebra of the system (4.2)

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	$4\Gamma_1$	0	0	0
Γ_2	0	0	Γ_2	0	0	0
Γ_3	$-4\Gamma_1$	$-\Gamma_2$	0	0	0	0
Γ_4	0	0	0	0	Γ_4	$2\Gamma_5$
Γ_5	0	0	0	$-\Gamma_4$	0	Γ_6
Γ_6	0	0	0	$-2\Gamma_5$	$-\Gamma_6$	0

Table 4.4: Adjoint actions for the Lie algebra of the system (4.2)

Ad	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	Γ_1	Γ_2	$\Gamma_3 - 4\epsilon\Gamma_1$	Γ_4	Γ_5	Γ_6
Γ_2	Γ_1	Γ_2	$\Gamma_3 - \epsilon\Gamma_2$	Γ_4	Γ_5	Γ_6
Γ_3	$e^{4\epsilon}\Gamma_1$	$e^{\epsilon}\Gamma_2$	Γ_3	Γ_4	Γ_5	Γ_6
Γ_4	Γ_1	Γ_2	Γ_3	Γ_4	$\Gamma_5 - \epsilon\Gamma_4$	$\Gamma_6 - 2\epsilon\Gamma_5 + \epsilon^2\Gamma_4$
Γ_5	Γ_1	Γ_2	Γ_3	$e^{\epsilon}\Gamma_4$	Γ_5	$e^{-\epsilon}\Gamma_6$
Γ_6	Γ_1	Γ_2	Γ_3	$\Gamma_4 + 2\epsilon\Gamma_5 + \epsilon^2\Gamma_6$	$\Gamma_5 + \epsilon\Gamma_6$	Γ_6

We find that the only linear combination of Lie symmetries that is not equivalent to any component of the optimal system derived in Subsection 4.3.1, is the following:

$$\Delta = 4\varepsilon t\partial_t + \varepsilon x\partial_x + (2uv - \varepsilon u)\partial_u + (v^2 + \alpha)\partial_v,$$

where $\varepsilon = \pm 1$ and $\alpha \in \{-1, 0, 1\}$. Depending upon the value of the constant α, symmetry Δ leads to the following reductions:

$$v = \tan[\varepsilon \ln x + f(\omega)], \quad u = g(\omega)(v^2 + 1)\exp\left(-\varepsilon \tan^{-1} v\right), \quad \alpha = 1,$$

$$v = -\tanh[\varepsilon \ln x + f(\omega)], \quad u = g(\omega)(v^2-1)\exp\left(\varepsilon \tanh^{-1} v\right), \quad \alpha = -1,$$
$$v = -\frac{\varepsilon}{\ln x + f(\omega)}, \quad u = g(\omega) v^2 \mathrm{e}^{\varepsilon/v}, \quad \alpha = 0,$$
where $\omega = x^4/t$.

4.5.1 Further Potential Symmetries

For a complete investigation of potential symmetries the first step is to calculate the conservation laws. For PDEs in two independent variables, t and x, the general form of (local) conservation laws is

$$D_t F + D_x G = 0,$$

where D_t and D_x are the total derivatives with respect to t and x. The above equality is assumed to be satisfied for any solution of the corresponding system of equations. The components F and G of the conserved vector (F, G) are functions of t, x and derivatives of u and are called the density and the flux of the conservation law, respectively. Basic definitions and statements on conservation laws can be found in [47].

Here we have derived the conservation laws using the direct method [71, 72] in the special case where

$$F = F(t, x, u), \quad G = G(t, x, u, u_x, u_{xx}, u_{xxx}).$$

In fact if the density F does not depend upon the derivatives of u, the above is the most general form of conservation laws for evolution equation (4.1) [55]. In the case for which the parameters a and b are arbitrary we have the conservation law

$$F = -u, \quad G = u^{n-2}\left(u^2 u_{xxx} + a u u_x u_{xx} + b u_x^3\right).$$

If the parameters satisfy the relation $2b - (n-1)(a-n) = 0$, then equation (4.1) admits a second conservation law

$$\begin{aligned} F = -xu, \quad G \;&=\; \tfrac{1}{2}u^{n-2}\left[2xu^2 u_{xxx} + 2axuu_x u_{xx} - 2u^2 u_{xx}\right. \\ &\qquad \left. + (n-1)(a-n)xu_x^3 + (n-a)uu_x^2\right]. \end{aligned}$$

Finally, if $a = 3n$ and $b = n(n-1)$, we have two additional conservation laws

$$\begin{aligned} F = -x^2 u, \quad G \;&=\; u^{n-2}\left[x^2u^2u_{xxx} + 3nx^2uu_xu_{xx} - 2xu^2u_{xx}\right. \\ &\qquad \left. + n(n-1)x^2u_x^3 - 2nxuu_x^2 + 2u^2u_x\right] \end{aligned}$$

and

$$F = -x^3 u, \quad G \;=\; u^{n-2}\left[x^3 u^2 u_{xxx} + 3nx^3 u u_x u_{xx} - 3x^2 u^2 u_{xx} \right.$$
$$\left. + n(n-1)x^3 u_x^3 - 3nx^2 u u_x^2 + 6xu^2 u_x - \tfrac{6}{n+1}u^3\right].$$

System (4.2) which corresponds to the first conservation law, was employed in the present Section to derive potential symmetries for the thin-film equation (4.1). On the other hand Lie symmetries of the auxiliary systems that correspond to the second and third conservation laws project into Lie symmetries of equation (4.1), that is, we obtain no further potential symmetries. Examples of potential symmetries for diffusion type equations that are obtained form different potential systems can be found in [28, 68].

The exhaustive analysis of potential symmetries of (4.1) can only be achieved when all conservation laws are derived, that is, all potential systems are known. We also point out that it is possible to construct potential systems using linear combinations of basis conservation laws. For example, the conservation law with $F = -(x^2 + c)u$ leads to a potential system which is not equivalent to the above.

4.5.2 Linearising Mappings

In [5] it is shown that an invertible mapping which transforms a nonlinear PDE to a linear one does not exist if the nonlinear PDE does not admit an infinite-parameter Lie group of contact transformations. Also such mappings which transform a nonlinear system of PDEs to a linear one do not exist if a system does not admit an infinite-parameter Lie group of transformations. If such infinite-parameter groups exist then the nonlinear PDE (or system of nonlinear PDEs) can be transformed into a linear PDE (or into a system of linear PDEs) provided that these groups satisfy certain criteria [5].

Here we have seen that the nonlinear system (4.2) for $(n, a, b) = (-4, -10, 15)$ with the form

$$v_x = u,$$
$$v_t = -(u^{-4}u_{xxx} - 10u^{-5}u_x u_{xx} + 15u^{-6}u_x^3),$$

admits the infinite-dimensional Lie symmetry Γ_7. This Lie symmetry leads to the point transformation

$$t' = t, \quad x' = v, \quad u' = \frac{1}{u}, \quad v' = x$$

that connects the above nonlinear system and the linear system

$$v'_{x'} = u',$$

$$v'_{t'} = -u'_{x'x'x'}.$$

In turn this mapping produces the one-to-one contact transformation

$$\mathrm{d}t' = \mathrm{d}t, \quad \mathrm{d}x' = u\mathrm{d}x - (u^{-4}u_{xx} - 3u^{-5}u_x^2)_x\mathrm{d}t, \quad u' = \frac{1}{u}$$

which transforms the linear equation $u'_{t'} + u'_{x'x'x'x'} = 0$ into the nonlinear PDE

$$u_t + \left(u^{-4}u_{xx} - 3u^{-5}u_x^2\right)_{xx} = 0.$$

4.6 Nonclassical Potential Symmetries

Following the idea of Section 4.4, we search for nonclassical symmetries for the potential equation (4.3). If we take $\tau = 1$, the invariance surface condition has the form

$$v_t = \zeta(t, x, v) - \xi(t, x, v)v_x$$

and the reduction operator has the form

$$\Gamma = \partial_t + \xi(t, x, v)\partial_x + \zeta(t, x, v)\partial_v.$$

The determining system for the coefficients ξ and ζ has the form

$$(a + 10)\xi_{vv} = 0, \tag{4.48}$$

$$2(a + 8)\xi_{xv} - (a + 4)\zeta_{vv} = 0, \tag{4.49}$$

$$(a + 6)\xi_{xx} - 2(a + 2)\zeta_{xv} = 0, \tag{4.50}$$

$$a\zeta_{xx} = 0, \tag{4.51}$$

$$a\zeta_x = 0, \tag{4.52}$$

$$(a + 10)\xi_v = 0, \tag{4.53}$$

$$(3a + 2b)\xi_v = 0, \tag{4.54}$$

$$b\zeta_x = 0, \tag{4.55}$$

$$(2a+b+5)\xi_{vv}=0, \tag{4.56}$$

$$(3a+2b+4)\xi_{xv}-(a+b+1)\zeta_{vv}=0, \tag{4.57}$$

$$(a+b)\xi_{xx}-(a+2b)\zeta_{xv}=0, \tag{4.58}$$

$$b\zeta_{xx}=0, \tag{4.59}$$

$$(a+10)\xi_{vvv}=0, \tag{4.60}$$

$$3(a+8)\xi_{xvv}-(a+6)\zeta_{vvv}=0, \tag{4.61}$$

$$(a+6)\xi_{xxv}-(a+4)\zeta_{xvv}=0, \tag{4.62}$$

$$(a+4)\xi_{xxx}-3(a+2)\zeta_{xxv}=0, \tag{4.63}$$

$$a\zeta_{xxx}=0, \tag{4.64}$$

$$\begin{aligned}&\xi_{vvvv}v_x^{n+6}+(4\xi_{xvvv}-\zeta_{vvvv})v_x^{n+5}+2(3\xi_{xxvv}-2\zeta_{xvvv})v_x^{n+4}+2(2\xi_{xxxv}-3\zeta_{xxvv})v_x^{n+3}\\ &+(\xi_{xxxx}-4\zeta_{xxxv})v_x^{n+2}-\zeta_{xxxx}v_x^{n+1}+(n+4)\xi_v\xi v_x^3+[\xi_t+(n+4)(\xi_x\xi-\xi_v\zeta)-n\xi\zeta_v]v_x^2\\ &-[(n+4)\xi_x\zeta+\zeta_t+n(\xi\zeta_x-\zeta_v\zeta)]v_x+n\zeta\zeta_x=0.\end{aligned} \tag{4.65}$$

Depending upon the values of the parameter n, equation (4.65) can be split into more equations.

Solving the system (4.48)–(4.65), we deduce that equation (4.3) admits such symmetries in two cases, for which

$$(n,a,b)\in\{(-1,0,0),(-3,-10,15)\}.$$

In particular, we find that for $(n,a,b)=(-1,0,0)$ equation (4.3) has the form

$$v_t=-{v_x}^{-1}v_{xxxx} \tag{4.66}$$

and it admits the reduction operator

$$\Gamma_1=\partial_t+c\partial_x+\phi(x+ct)\partial_v,$$

where $\phi(\omega)$ is a solution of the ODE

$$\frac{\mathrm{d}^4\phi}{\mathrm{d}\omega^4}+\phi\frac{\mathrm{d}\phi}{\mathrm{d}\omega}=0. \tag{4.67}$$

Now the pure hodograph transformation connects equation (4.66) and

$$v_t = -(v_x^{-3} v_{xxxx} - 10 v_x^{-4} v_{xx} v_{xxx} + 15 v_x^{-5} v_{xx}^3), \tag{4.68}$$

which is the form of equation (4.3) for $(n, a, b) = (-3, -10, 15)$. Hence equation (4.68) admits the reduction operators

$$\Gamma_2 = \partial_t + \phi(v + ct)\partial_x + c\partial_v,$$

where $\phi(\omega)$ is a solution of the ODE (4.67).

Chapter 5

Group Analysis of a Third-Order Nonlinear Evolution Equation

5.1 Introduction

In this Chapter we show the symmetry properties of third-order nonlinear evolution equations of the class

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n \frac{\partial^2 u}{\partial x^2} + a u^{n-1}\left(\frac{\partial u}{\partial x}\right)^2\right). \tag{5.1}$$

Examples of this class of equations are the well-known Harry–Dym equation, which is one of the most interesting integrable models for physicists and mathematicians,

$$u_t = 2\left(u^{-1/2}\right)_{xxx}.$$

It arises, e.g., in the analysis of the Saffman–Taylor problem with surface tension [30]. Other examples are the integrable equations [9],

$$u_t = 2\left(u^{-2}\right)_{xxx}$$

and

$$u_t = -\left(u^{-6} u_{xx} - 3u^{-7} u_x^2\right)_x.$$

If we introduce a potential variable v, equation (5.1) can be written as a system of two PDEs

$$v_x = u, \tag{5.2}$$

$$v_t = -\left(u^n u_{xx} + a u^{n-1} u_x^2\right)$$

and elimination of u from this system gives the potential form of equation (5.1),

$$v_t = -\left(v_x^n v_{xxx} + a v_x^{n-1} v_{xx}^2\right). \tag{5.3}$$

In the subsequent analysis we present equivalence transformations, Lie symmetries, nonclassical symmetries, potential symmetries and potential nonclassical symmetries. Furthermore we find those forms of (5.1) that can be linearised and we provide the corresponding linearising mappings [10].

5.2 Equivalence Transformations

Unlike equations (4.1) which admit equivalence transformations in the case in which the primed variables are equal to the corresponding unprimed variables, here we have additional equivalence transformations when these variables are not equal. In the first case, the equivalence group of the class (5.1) consists of the transformations

$$t' = c_1 t + c_2, \quad x' = c_3 x + c_4, \quad u' = c_1^{-1/n} c_3^{3/n} u, \quad (n', a') = (n, a),$$

where $u'(t', x')$ satisfies (5.1) with the variables and parameters being primed and $c_1 c_3 \neq 0$. In the case where

$$(n, a) \in \left\{ \left(-3, -\frac{3}{2}\right), \left(-\frac{3}{2}, -\frac{3}{2}\right) \right\}$$

we have the additional equivalence transformation

$$t' = c_1 t + c_2, \quad x' = \frac{c_3 x + c_4}{c_5 x + c_6}, \quad u' = c_1^{-1/n} (c_5 x + c_6)^{-6/n} u, \quad (n', a') = (n, a)$$

in which $c_1 \neq 0$ and $c_3 c_6 - c_4 c_5 = \pm 1$.

In the second case we have the additional equivalence transformations

$$t' = c_1 t + c_2, \quad x' = c_3 x + c_4, \quad u' = c_1^{-1/n'} c_3^{3/n'} u^{n/n'}, \tag{5.4}$$

where

$$a = \frac{nn' - 2n' + n}{2n'}, \; a' = \frac{nn' + n' - 2n}{2n}$$

and $n,\ n'$ are arbitrary. Furthermore in the case for which $a = a' = -\frac{3}{2}$ and $n = -3,\ n' = -\frac{3}{2}$ we have, additionally,

$$t' = c_1 t + c_2, \quad x' = \frac{c_3 x + c_4}{c_5 x + c_6}, \quad u' = c_1^{2/3} c_5^{-2} (c_5 x + c_6)^4 u^2.$$

Note that in the above two cases we assumed that $nn' \neq 0$.

The equivalence group of the system (5.2) is generated by the transformations

$$t' = c_3^{-n} c_1^{n+3} t + c_5, \quad x' = c_1 x + c_2, \quad u' = c_1^{-1} c_3 u, \quad v' = c_3 v + c_4$$

if $(n', a') = (n, a)$ and by the transformations

$$t' = c_1^{-n} c_3^{n+3} t + c_5, \quad x' = c_1 v + c_2, \quad u' = c_1^{-1} c_3 u, \quad v' = c_3 x + c_4$$

if $(n', a') = (-n-3, -a-3)$, where $c_1 c_3 \neq 0$.

5.3 Lie Symmetries

Equation (5.1) admits Lie point symmetries if and only if

$$\Gamma^{(3)}[u_t + u^n u_{xxx} + (2a+n)u^{n-1} u_x u_{xx} + a(n-1)u^{n-2} u_x^3] = 0 \tag{5.5}$$

for $u_t = -\left[u^n u_{xxx} + (2a+n)u^{n-1} u_x u_{xx} + a(n-1)u^{n-2} u_x^3\right]$.

If we use the above expression, we can eliminate u_t and equation (5.5) becomes an identity in the variables $u_x,\ u_{xx},\ u_{tx},\ u_{xxx}$ and u_{txx}. From coefficients of different powers of these variables, which must separately be equal to zero, we derive the determining equations for the coefficients $\tau,\ \xi$ and η. We again use the general results on point transformations between evolution equations [34], that $\tau = \tau(t)$ and $\xi = \xi(t, x)$. From the coefficient of u_{xxx} we have that

$$\eta = \frac{(3\xi_x - \tau_t)u}{n}.$$

If we use the above facts, from the coefficients of u_{xx}, u_x^2, u_x and the term independent of derivatives of (5.5) we have the following determining equations, respectively,

$$(2a+3)\xi_{xx} = 0, \tag{5.6}$$

$$(7an + 3a - n^2 + 6n)\xi_{xx} = 0, \tag{5.7}$$

$$(6a + 2n + 9)\xi_{xxx}u^n - n\xi_t = 0, \tag{5.8}$$

$$3\xi_{xxxx}u^n + 3\xi_{tx} - \tau_{tt} = 0. \tag{5.9}$$

The solution of the determining equations (5.6)–(5.9) gives that for n and a arbitrary the symmetry Lie algebra is four-dimensional and is spanned by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 3t\partial_t + x\partial_x, \quad \Gamma_4 = \frac{nx}{3}\partial_x + u\partial_u.$$

If $(n, a) \in \left\{\left(-3, -\frac{3}{2}\right), \left(-\frac{3}{2}, -\frac{3}{2}\right)\right\}$, then equation (5.1) also admits a fifth Lie symmetry

$$\Gamma_5 = x^2\partial_x + \frac{6xu}{n}\partial_u$$

and the two cases of (5.1) correspond to the equations

$$\begin{aligned} u_t &= -\left[u^{-3}u_{xx} - \frac{3}{2}u^{-4}u_x^2\right]_x \text{ and} \\ u_t &= -\left[u^{-3/2}u_x\right]_{xx}. \end{aligned}$$

Furthermore, for the case that $n = 0$, from the coefficient of u_{xxx} we have that

$$3\xi_x - \tau_t = 0$$

from which we can conclude that

$$\xi = \frac{1}{3}\tau_t x + L(t).$$

If we use this expression, from the coefficient of $u_x u_{xx}$ we have that

$$3\eta_{uu}u^2 + 2a\eta_u u - 2a\eta = 0 \tag{5.10}$$

which is an equation of Euler type with solution

$$\eta = g(t, x)u + \phi(t, x)u^{-2a/3}. \tag{5.11}$$

After we have substituted the expression η (5.11) into (5.5), from the coefficients of u_{xx}, u_x^3, u_x^2, u_x and the term independent of derivatives of (5.5), we have the following determining equations, respectively,

$$(2a + 3)g_x = 0, \tag{5.12}$$

$$a\phi(8a^2 + 18a + 9) = 0, \tag{5.13}$$

$$a\left[3g_x u^{\frac{2a+3}{3}} - (4a+3)\phi_x\right] = 0, \tag{5.14}$$

$$3(2a+3)g_{xx} - \tau_{tt}x - 3L_t = 0, \tag{5.15}$$

$$(g_t + g_{xxx})u^{\frac{2a+3}{3}} + \phi_t + \phi_{xxx} = 0. \tag{5.16}$$

In this case, after we have solved the determining equations (5.12)–(5.16), we conclude that, if $(n, a) = (0, -\frac{3}{4})$, in addition to the four Lie symmetries, equation (5.1) admits the infinite-dimensional Lie symmetry

$$\Gamma_\infty = \phi(t, x)\sqrt{u}\partial_u,$$

where $\phi(t, x)$ is a solution of the linear equation $\phi_t + \phi_{xxx} = 0$.

5.3.1 Linearising Mappings

The existence of infinite-dimensional Lie symmetries suggests linearisation of the equation being studied. For the criteria of the existence of such mappings and how to construct them, one can read [5].

The equation

$$u_t = -\left(u_{xx} - \frac{3}{4}u^{-1}u_x^2\right)_x$$

that admits the infinite-dimensional Lie symmetry Γ_∞ can be mapped into the linear PDE

$$u'_{t'} + u'_{x'x'x'} = 0 \tag{5.17}$$

by the transformation

$$t' = t, \quad x' = x, \quad u' = \sqrt{u}.$$

In the case of the class of fourth-order equations (4.1) we have seen that no subclass exists that can be linearised by a local mapping. It appears that only equations of odd-order from the chain can be linearised by a local mapping.

5.3.2 Invariant Solutions

In the following tables we present the commutator table and the adjoint table for the Lie algebra of the Γ_i which, as we know, are needed for the construction of the optimal system of subalgebras.

Table 5.1: Commutation relations for the Lie algebra of equation (5.1)

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	0	0	$3\Gamma_1$	0	0
Γ_2	0	0	Γ_2	$\frac{n}{3}\Gamma_2$	$\frac{6}{n}\Gamma_4$
Γ_3	$-3\Gamma_1$	$-\Gamma_2$	0	0	Γ_5
Γ_4	0	$-\frac{n}{3}\Gamma_2$	0	0	$\frac{n}{3}\Gamma_5$
Γ_5	0	$-\frac{6}{n}\Gamma_4$	$-\Gamma_5$	$-\frac{n}{3}\Gamma_5$	0

Table 5.2: Adjoint actions for the Lie algebra of equation (5.1)

Ad	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	Γ_1	Γ_2	$\Gamma_3 - 3\epsilon\Gamma_1$	Γ_4	Γ_5
Γ_2	Γ_1	Γ_2	$\Gamma_3 - \epsilon\Gamma_2$	$\Gamma_4 - \frac{n\epsilon}{3}\Gamma_2$	$\Gamma_5 - \frac{6\epsilon}{n}\Gamma_4 + \epsilon^2\Gamma_2$
Γ_3	$e^{3\epsilon}\Gamma_1$	$e^{\epsilon}\Gamma_2$	Γ_3	Γ_4	$e^{-\epsilon}\Gamma_5$
Γ_4	Γ_1	$e^{\frac{n\epsilon}{3}}\Gamma_2$	Γ_3	Γ_4	$e^{-\frac{n\epsilon}{3}}\Gamma_5$
Γ_5	Γ_1	$\Gamma_2 + \frac{6\epsilon}{n}\Gamma_4 + \epsilon^2\Gamma_5$	$\Gamma_3 + \epsilon\Gamma_5$	$\Gamma_4 + \frac{n\epsilon}{3}\Gamma_5$	Γ_5

In the case in which n and a are arbitrary the optimal system consists of the list of inequivalent subalgebras

$$\langle\Gamma_2\rangle,\ \langle\Gamma_1 + c\Gamma_2\rangle,\ \langle\Gamma_3 + c\Gamma_4\rangle,\ \langle\Gamma_4 + c\Gamma_1\rangle,\ \langle\Gamma_4 + c\Gamma_2 - \tfrac{n}{3}\Gamma_3\rangle.$$

If $(n,a) \in \left\{\left(-3,-\frac{3}{2}\right),\left(-\frac{3}{2},-\frac{3}{2}\right)\right\}$, in addition to the above list, we have a reduction that corresponds to the subalgebra

$$\langle \Gamma_5 + c\Gamma_2 + 2k\Gamma_3 \rangle.$$

The similarity reductions that result from the components of the optimal system that transform (5.1) into an ODE are:

$$\begin{aligned}
\langle \Gamma_2 \rangle \quad &: \quad u = \phi(\omega), \quad \omega = t, \\
\langle \Gamma_1 + c\Gamma_2 \rangle \quad &: \quad u = \phi(\omega), \quad \omega = x - ct, \\
\langle \Gamma_3 + c\Gamma_4 \rangle \quad &: \quad u = t^{\frac{c}{3}}\phi(\omega),\ \omega = \begin{cases} x & \text{if } nc+3=0, \\ t^{-\frac{1}{3}}x^{\frac{3}{nc+3}} & \text{if } nc+3\neq 0, \end{cases} \\
\langle \Gamma_4 + c\Gamma_1 \rangle \quad &: \quad u = \begin{cases} x^{\frac{3}{n}}\phi(\omega),\ \omega = \mathrm{e}^t x^{-\frac{3c}{n}} & \text{if } n \neq 0, \\ \mathrm{e}^{\frac{t}{c}}\phi(\omega),\ \omega = x & \text{if } n = 0, \end{cases} \\
\langle \Gamma_4 + c\Gamma_2 - \tfrac{n}{3}\Gamma_3 \rangle \quad &: \quad u = \begin{cases} t^{-\frac{1}{n}}\phi(\omega),\ \omega = x + \frac{c}{n}\ln t & \text{if } n \neq 0, \\ \mathrm{e}^{\frac{x}{c}}\phi(\omega),\ \omega = t & \text{if } n = 0. \end{cases}
\end{aligned}$$

In the case for which we have five symmetries, we obtain the following additional reduction

$$\langle \Gamma_5 + c\Gamma_2 + 2k\Gamma_3 \rangle : \ u = \begin{cases} ((x+k)^2+1)^{\frac{3}{n}} \exp\left[-\frac{6k}{n}\tan^{-1}(x+k)\right]\phi(\omega), \\ \ \omega = t\exp\left[-6k\tan^{-1}(x+k)\right] \text{ if } c-k^2=1, \\ \\ ((x+k)^2-1)^{\frac{3}{n}} \exp\left[\frac{6k}{n}\tanh^{-1}(x+k)\right]\phi(\omega), \\ \omega = t\exp\left[6k\tanh^{-1}(x+k)\right] \text{ if } c-k^2=-1, \\ \\ (x+k)^{\frac{6}{n}}\exp\left[\frac{6k}{n(x+k)}\right]\phi(\omega), \\ \ \omega = t\exp\left[\frac{6k}{x+k}\right] \text{ if } c-k^2=0, \end{cases}$$

where ω is the independent and ϕ the dependent variable of the reduced ODE, $c = 0, \pm 1$ and $k \in \mathbb{R}$.

We give some examples of reduced ODEs. The reduction that corresponds to the subalgebra $\langle \Gamma_1 + c\Gamma_2 \rangle$ leads to the equation

$$c\phi_\omega - [\phi^{n-1}(\phi\phi_{\omega\omega} + a\phi_\omega^2)]_\omega = 0$$

which provides traveling-wave solutions for the equation (5.1). If we integrate the above equation by parts, the integral has the form

$$\phi^{n-1}(\phi\phi_{\omega\omega} + a\phi_\omega^2) = c\phi + c_1, \tag{5.18}$$

where c_1 is the constant of integration. If $(n, a) = (1, -1)$, this has the form

$$\phi\phi_{\omega\omega} - \phi_\omega^2 = c\phi + c_1$$

with solutions [51]:

(1) $\phi = A_1 \sinh(A_3\omega) + A_2 \cosh(A_3\omega) + cA_3^{-2}$, where the constants A_1, A_2 and A_3 are related by the constraint $(A_1^2 - A_2^2)A_3^2 + c_1 + c^2 A_3^{-2} = 0$. In this case the form of u is

$$u = A_1 \sinh(A_3(x - ct)) + A_2 \cosh(A_3(x - ct)) + cA_3^{-2}.$$

(2) $\phi = A_1 \sin(A_3\omega) + A_2 \cos(A_3\omega) - cA_3^{-2}$, where the constants A_1, A_2 and A_3 are related by the constraint $(A_1^2 + A_2^2)A_3^2 + c_1 - c^2 A_3^{-2} = 0$. Here the form of u is

$$u = A_1 \sin(A_3(x - ct)) + A_2 \cos(A_3(x - ct)) - cA_3^{-2}.$$

For $c_1 = 0$, $c \neq 0$ and $n = 0$, equation (5.18) admits the symmetry generator $\phi\partial_\phi$. By means of the substitution $y(\omega) = \frac{\phi_\omega}{\phi}$, (5.18) can be reduced to

$$y_\omega + (a + 1)y^2 - c = 0. \tag{5.19}$$

For the case that $a = -1$, the solution of (5.19) is $y = c\omega + k_1$ and since $y(\omega) = \frac{\phi_\omega}{\phi}$ we obtain that

$$u = \phi(\omega) = \exp\left[\frac{1}{2}c(x - ct)^2 + k_1(x - ct) + k_2\right].$$

Otherwise, when $a \neq -1$ and $a + 1 > 0$, the solution for (5.19) is

$$y = \begin{cases} \sqrt{\frac{c}{a+1}} \left[\frac{1+\exp\left(2\sqrt{c(a+1)}(k_1-\omega)\right)}{1-\exp\left(2\sqrt{c(a+1)}(k_1-\omega)\right)}\right] & \text{if } c > 0, \\ \sqrt{\frac{|c|}{a+1}} \tan\left(\sqrt{|c|(a+1)}(k_1 - \omega)\right) & \text{if } c < 0. \end{cases}$$

The corresponding travelling wave solutions to the equation (5.1) can be written as

$$u = \begin{cases} k_2 \exp\left[-\sqrt{\frac{c}{(a+1)}}\omega\right] \left[\left(\exp\left[2\sqrt{c(a+1)}\omega\right] - \exp\left[2k_1\sqrt{c(a+1)}\right]\right)\right]^{1/(a+1)} & \text{if } c > 0, \\ k_2 \left[\tan^2\left(\sqrt{|c|(a+1)}(k_1 - \omega)\right) + 1\right]^{-1/2(a+1)} & \text{if } c < 0. \end{cases}$$

Here, k_1 and k_2 are arbitrary constants and $\omega = x - ct$. Similarly we can find the form of u for the case that $a + 1 < 0$.

In the case $nc + 3 \neq 0$ the subalgebra $\langle \Gamma_3 + c\Gamma_4 \rangle$ leads to the reduced equation

$$81\phi^{n+2}\phi_{\omega\omega\omega}\omega^{-nc} + 81(2a + n)\phi^{n+1}\phi_\omega\phi_{\omega\omega}\omega^{-nc} - 81nc\phi^{n+2}\phi_{\omega\omega}\omega^{-nc-1}$$

$$+81(n-1)a\phi^n\phi_\omega^3\omega^{-nc} - 27(2a+n)nc\phi^{n+1}\phi_\omega^2\omega^{-nc-1} + 9(2nc+3)nc\phi^{n+2}\phi_\omega\omega^{-nc-2}$$

$$-(n^3c^3 + 9n^2c^2 + 27nc + 27)\phi^2\phi_\omega\omega + (n^3c^3 + 9n^2c^2 + 27nc + 27)c\phi^3 = 0$$

and for $nc + 3 = 0$ we have the ODE

$$\phi - n[\phi^{n-1}(\phi\phi_{xx} + a\phi_x^2)]_x = 0.$$

From subalgebra $\langle \Gamma_4 + c\Gamma_1 \rangle$ in the case $n \neq 0$, we obtain

$$27c^3\phi^{n+2}\phi_{\omega\omega\omega}\omega^3 + 27(2a+n)c^3\phi^{n+1}\phi_\omega\phi_{\omega\omega}\omega^3 - 27(2a-3c+3)c^2\phi^{n+2}\phi_{\omega\omega}\omega^2$$

$$+27(n-1)ac^3\phi^n\phi_\omega^3\omega^3 + 9c^2(6ac - 7an - 3a + 3nc + n^2 - 6n)\phi^{n+1}\phi_\omega^2\omega^2$$

$$-3(18ac - 15an - 27a - 9c^2 + 27c + 4n^2 - 9n - 27)c\phi^{n+2}\phi_\omega\omega$$

$$-n^3\phi^2\phi_\omega\omega - 3(3an + 9a - n^2 + 9)\phi^{n+3} = 0$$

and for the case that $n = 0$, we have that

$$\phi + c[\phi^{-1}(\phi\phi_{xx} + a\phi_x^2)]_x = 0.$$

The final example is the subalgebra $\langle \Gamma_4 + c\Gamma_2 - \frac{n}{3}\Gamma_3 \rangle$ which leads to the reduced ODE

$$c\phi_\omega - \phi + n[\phi^{n-1}(\phi\phi_{\omega\omega} + a\phi_\omega^2)]_\omega = 0,$$

where $n \neq 0$. In the case $n = 0$, we obtain the solution $u = c_1 \exp\left[\frac{1}{c^3}\left(c^2x - (a+1)t\right)\right]$.

5.4 Nonclassical Symmetries

Here we apply the nonclassical method to equation (5.1). From the coefficient of u_{xx}^2 we conclude that $\xi = \xi(t,x)$. We use this fact and the form of the determining system for the coefficients ξ and η becomes

$$3\eta_{uu}u^2 + (2a+n)\eta_u u - (2a+n)\eta = 0, \tag{5.20}$$

$$3(\xi_{xx} - \eta_{xu})u - (2a+n)\eta_x = 0, \tag{5.21}$$

$$\eta_{uuu}u^3 + (2a+n)\eta_{uu}u^2 + 2a(n-1)\eta_u u - 2a(n-1)\eta = 0, \tag{5.22}$$

$$3\eta_{xuu}u^2 - (2a+n)(\xi_{xx} - 2\eta_{xu})u + 3a(n-1)\eta_x = 0, \tag{5.23}$$

$$(\xi_{xxx} - 3\eta_{xxu})u^{n+1} - (2a+n)\eta_{xx}u^n + (\xi_t + 3\xi\xi_x)u - n\xi\eta = 0, \tag{5.24}$$

$$\eta_{xxx}u^{n+1} + (3\xi_x\eta + \eta_t)u - n\eta^2 = 0. \tag{5.25}$$

We solve the equations (5.20)–(5.25) to deduce that equation (5.1) admits nonclassical symmetries in two cases. If $(n, a) = (-\frac{1}{2}, -\frac{1}{2})$ it admits the reduction operator

$$\Gamma_1 = \partial_t + \phi(x)\sqrt{u}\partial_u,$$

where $\phi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3} + \frac{1}{2}\phi^2 = 0.$$

A special solution of this latter equation is $\phi = 120x^{-3}$. If $(n, a) = (-\frac{1}{3}, -\frac{5}{6})$, then (5.1) admits the reduction operator

$$\Gamma_2 = \partial_t + \psi(x)u^{2/3}\partial_u,$$

where $\psi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^3\psi}{\mathrm{d}x^3} + \frac{1}{3}\psi^2 = 0.$$

A special solution is $\psi = 180x^{-3}$. We point out that these two equations, which admit nonclassical symmetries, are connected by the mapping

$$u \mapsto u^{3/2}$$

which is a special case of the equivalence transformation (5.4).

In the case for which $(n, a) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$, Γ_1 leads to the ansatz (4.35)

$$u = \left[\tfrac{1}{2}\phi(x)t + F(x)\right]^2$$

which reduces (5.1) to the ODE

$$\frac{\mathrm{d}^3F}{\mathrm{d}x^3} + \frac{1}{2}\phi F = 0. \tag{5.26}$$

For the solution $\phi = 120x^{-3}$, equation (5.26) becomes an equation of Euler type of the form

$$x^3\frac{\mathrm{d}^3F}{\mathrm{d}x^3} + 60F = 0 \tag{5.27}$$

which has the solution

$$F(x) = c_1 x^{-3} + x^3 \left[c_2 \cos\left(\sqrt{11}\ln|x|\right) + c_3 \sin\left(\sqrt{11}\ln|x|\right)\right], \tag{5.28}$$

which produces the solution of (5.1),

$$u(t,x) = \left[60tx^{-3} + c_1 x^{-3} + x^3 \left(c_2 \cos\left(\sqrt{11}\ln|x|\right) + c_3 \sin\left(\sqrt{11}\ln|x|\right)\right)\right]^2.$$

The corresponding solution for the case $(n,a) = \left(-\frac{1}{3}, -\frac{5}{6}\right)$ can be determined either by using the above mapping or the ansatz that can be obtained from Γ_2. Namely, we have the ansatz

$$u = \left[\tfrac{1}{3}\psi(x)t + F(x)\right]^3 \tag{5.29}$$

with the form of the reduced ODE (5.1)

$$\frac{\mathrm{d}^3 F}{\mathrm{d}x^3} + \frac{1}{3}\psi F = 0. \tag{5.30}$$

For $\psi = 180x^{-3}$ equation (5.30) becomes an equation of Euler type with the form of (5.27)

$$x^3 \frac{\mathrm{d}^3 F}{\mathrm{d}x^3} + 60F = 0$$

and solution the same as the solution of $F(x)$ in (5.28). Consequently in this case the solution of (5.1) is

$$u(t,x) = \left[60tx^{-3} + c_1 x^{-3} + x^3 \left(c_2 \cos\left(\sqrt{11}\ln|x|\right) + c_3 \sin\left(\sqrt{11}\ln|x|\right)\right)\right]^3.$$

5.5 Potential Symmetries

In this case the system (5.2) admits Lie symmetries if and only if

$$\Gamma^{(1)}\left[v_x - u\right] = 0, \tag{5.31}$$

$$\Gamma^{(2)}\left[v_t + \left(u^n u_{xx} + a u^{n-1} u_x^2\right)\right] = 0 \tag{5.32}$$

for $v_x = u$ and $v_t = -\left(u^n u_{xx} + a u^{n-1} u_x^2\right)$. $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are the first and second extensions, respectively, of the generator

$$\Gamma = \tau(t,x,u,v)\partial_t + \xi(t,x,u,v)\partial_x + \eta(t,x,u,v)\partial_u + \zeta(t,x,u,v)\partial_v.$$

From the coefficients of $u_x u_{xx}$, u_{xx} and u_x of (5.31) and from the coefficient $u_x u_{xx}$ of (5.32) we have the determining equations, respectively,

$$\tau_u = 0,$$

$$\tau_v u + \tau_x = 0,$$

$$\xi_u u - \zeta_u = 0,$$

$$\xi_u = 0.$$

After we have solved these equations we can deduce that the coefficient τ is a function of t and the coefficients ξ and ζ are functions of t, x and v. We use these results and from equation (5.31) we have that

$$\eta = -\xi_v u^2 - (\xi_x - \zeta_v)u + \zeta_x. \tag{5.33}$$

We substitute the form of η (5.33) into (5.32) and the simplified determining system, that arises from the coefficients of u_{xx}, u_x^2, u_x and the term independent of derivatives, for the determination of the coefficients τ, ξ η and ζ has the following form, respectively,

$$(n+3)\xi_v u^2 - [\tau_t - (n+3)\xi_x + n\zeta_v]u - n\zeta_x = 0, \tag{5.34}$$

$$(an+4a+3)\xi_v u^2 - a[\tau_t + n\zeta_v - (n+3)\xi_x]u - a(n-1)\zeta_x = 0, \tag{5.35}$$

$$\begin{aligned}&2(a+3)\xi_{vv}u^3 + [(4a+9)\xi_{xv} - (2a+3)\zeta_{vv}]u^2 + [(2a+3)\xi_{xx} - (4a+3)\zeta_{xv}]u\\ &-2a\zeta_{xx} = 0,\end{aligned} \tag{5.36}$$

$$\begin{aligned}&\xi_{vvv}u^{n+4} + (3\xi_{xvv} - \zeta_{vvv})u^{n+3} + 3(\xi_{xxv} - \zeta_{xvv})u^{n+2} + (\xi_{xxx} - 3\zeta_{xxv})u^{n+1}\\ &-\zeta_{xxx}u^n + \xi_t u - \zeta_t = 0.\end{aligned} \tag{5.37}$$

Equation (5.37) can be broken into more equations in proportion the values of the parameter n.

When we have solved the determining system (5.34)–(5.37), we obtain that the system (5.2) admits Lie symmetries which induce potential symmetries of (5.1) in two cases.

In particular, if $(n,a) = (-3,-3)$ equation (5.1) admits the potential symmetry

$$\Gamma = \psi(t,v)\partial_x - u^2\psi_v\partial_u,$$

where $\psi(t,v)$ is a solution of the linear equation $\psi_t + \psi_{vvv} = 0$.

If $(n,a) = \left(0, -\frac{3}{2}\right)$, it admits the potential symmetry

$$\Gamma = 2uv\partial_u + v^2\partial_v.$$

5.5.1 Further Potential Symmetries

Equation (5.1) can be also written in the conserved form

$$\left[u^{2a-n+2}\right]_t + (2a-n+2)\left[u^{2a+1}u_{xx} + \tfrac{1}{2}(n-1)u^{2a}u_x^2\right]_x = 0,$$

where $a \neq \frac{n}{2} - 1$. If we introduce a potential variable v, we obtain the auxiliary system

$$\begin{aligned} v_x &= -\frac{u^{2a-n+2}}{2a-n+2}, \\ v_t &= u^{2a+1}u_{xx} + \tfrac{1}{2}(n-1)u^{2a}u_x^2. \end{aligned} \tag{5.38}$$

In the case for which $a = \frac{n}{2} - 1$, the corresponding auxiliary system takes the form

$$\begin{aligned} v_x &= -\ln u, \\ v_t &= u^{n-2}\left(uu_{xx} + \tfrac{1}{2}(n-1)u_x^2\right). \end{aligned} \tag{5.39}$$

Lie symmetries of the system (5.38) induce potential symmetries of (5.1) in two cases, while Lie symmetries of the system (5.39) lead only to Lie symmetries of (5.1). The first case is when $(n,a) = (3,0)$ system (5.38) admits the following Lie symmetry which is a potential symmetry of (5.1)

$$\Gamma = \psi(t,v)\partial_x + \psi_v\partial_u,$$

where $\psi(t,v)$ is a solution of the linear equation $\psi_t + \psi_{vvv} = 0$. We point out that the potential equation (5.3) that corresponds to $(n,a) = (-3,-3)$ is the same as the potential equation that is obtained by eliminating u in (5.38) when $(n,a) = (3,0)$.

The second case that the system (5.38) produces potential symmetry is when $(n,a) = \left(0, -\frac{3}{2}\right)$ and the symmetry has the form

$$\Gamma = -2uv\partial_u + v^2\partial_v.$$

Here, when $(n,a) = \left(0,-\frac{3}{2}\right)$ the potential equation (5.3) is the same as the potential equation obtained from system (5.38).

Equation (5.1) can be written in other conserved forms when the parameters n and a satisfy certain relations. For example, if $a = n$, then equation (5.1) can be written as a system of two equations

$$v_x = -xu,$$

$$v_t = u^{n-1}\left(xuu_{xx} + nxu_x^2 - uu_x\right)$$

or

$$v_x = -x^2 u,$$

$$v_t = u^{n-1}\left(x^2uu_{xx} + nx^2u_x^2 - 2xuu_x + \frac{2}{n+1}u^2\right).$$

However Lie symmetries, of the above two systems do not induce potential symmetries for equation (5.1).

5.5.2 Linearising Mappings

The infinite-dimensional Lie symmetry admitted by the system (5.2) for $(n, a) = (-3, -3)$ with the form

$$v_x = u,$$

$$v_t = -\left(u^{-3}u_x\right)_x$$

leads to the mapping

$$t' = t, \quad x' = v, \quad u' = \frac{1}{u}, \quad v' = x$$

that transform any solution $(u(t, x), v(t, x))$ of this system into a solution $(u'(t', x'), v'(t', x'))$ of the linear system

$$v'_{x'} = u', \tag{5.40}$$

$$v'_{t'} = -u'_{x'x'}.$$

In turn this mapping produces the one-to-one contact transformation

$$\mathrm{d}t' = \mathrm{d}t, \quad \mathrm{d}x' = u\mathrm{d}x - (u^{-3}u_x)_x\mathrm{d}t, \quad u' = \frac{1}{u}$$

which transforms the linear equation (5.17) into the nonlinear PDE

$$u_t + \left(u^{-3}u_x\right)_{xx} = 0.$$

We have seen also that the system (5.38) for $(n, a) = (3, 0)$ with the form

$$v_x = \frac{1}{u},$$

$$v_t = (uu_x)_x$$

admits an infinite-dimensional Lie symmetry. This symmetry leads to the mapping

$$t' = t, \quad x' = v, \quad u' = u, \quad v' = x$$

that connects this nonlinear system and the linear system (5.40). In turn this mapping produces the one-to-one contact transformation

$$\mathrm{d}t' = \mathrm{d}t, \quad \mathrm{d}x' = u\mathrm{d}x + (uu_x)_x\mathrm{d}t, \quad u' = u$$

which transforms the linear equation (5.17) into the nonlinear PDE

$$u_t + u^2 \left(uu_x\right)_{xx} = 0.$$

5.6 Nonclassical Potential Symmetries

Here we list the nonclassical symmetries of the potential equation (5.3) that induce potential nonclassical symmetries of equation (5.1). Firstly, we present the determining system for the determination of the coefficients ξ and ζ.

$$(a+3)\xi_v = 0, \tag{5.41}$$

$$a\zeta_x = 0, \tag{5.42}$$

$$(a+3)\xi_{vv} = 0, \tag{5.43}$$

$$(4a+9)\xi_{xv} - (2a+3)\zeta_{vv} = 0, \tag{5.44}$$

$$(2a+3)\xi_{xx} - (4a+3)\zeta_{xv} = 0, \tag{5.45}$$

$$a\zeta_{xx} = 0, \tag{5.46}$$

$$\begin{aligned}
&\xi_{vvv}v_x^{n+5} + (3\xi_{xvv} - \zeta_{vvv})v_x^{n+4} + 3(\xi_{xxv} - \zeta_{xvv})v_x^{n+3} + (\xi_{xxx} - 3\zeta_{xxv})v_x^{n+2} \\
&-\zeta_{xxx}v_x^{n+1} + (n+3)\xi_v\xi v_x^3 + [\xi_t + (n+3)(\xi_x\xi - \xi_v\zeta) - n\xi\zeta_v]v_x^2 \\
&-[(n+3)\xi_x\zeta + \zeta_t + n(\xi\zeta_x - \zeta_v\zeta)]v_x + n\zeta_x\zeta = 0.
\end{aligned} \tag{5.47}$$

Depending upon the values of the parameter n, equation (5.47) is able to be broken into more equations.

We solve the system of equations (5.41)–(5.47) and we conclude that we have nonclassical potential symmetries in two cases.

If $(n, a) = (-1, 0)$, then equation (5.3) admits the reduction operator

$$\Gamma_1 = \partial_t + c\partial_x + \phi(x + ct)\partial_v$$

and if, $(n, a) = (-2, -3)$, it admits the reduction operator

$$\Gamma_2 = \partial_t + \phi(v + ct)\partial_x + c\partial_v.$$

In both operators $\phi(\omega)$ is a solution of the ODE

$$\frac{\mathrm{d}^3\phi}{\mathrm{d}\omega^3} + \phi\frac{\mathrm{d}\phi}{\mathrm{d}\omega} = 0.$$

This latter equation can be integrated to give

$$\int \frac{\mathrm{d}\phi}{\sqrt{-\frac{1}{3}\phi^3 + c_1\phi + c_2}} = \omega + c_3.$$

The corresponding potential equations to these two cases are connected via the pure hodograph transformation.

Chapter 6

Group Analysis of a Sixth-Order and a Fifth-Order Nonlinear Evolution Equations

6.1 Introduction

Equation (3.1) is one of the most known and well studied nonlinear PDEs. King [31] introduced and studied the fourth-order nonlinear thin-film equation (4.1) which was examined in the Chapter 4 from the point of view of Lie group analysis. Furthermore King [31] introduced the sixth-order nonlinear thin-film equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}\left[u^n\frac{\partial^5 u}{\partial x^5}+a_1u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4}+a_2u^{n-1}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3}+a_3u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial x^3}\right.\\ &\quad\left.+a_4u^{n-2}\frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)^2+a_5u^{n-3}\left(\frac{\partial u}{\partial x}\right)^3\frac{\partial^2 u}{\partial x^2}+a_6u^{n-4}\left(\frac{\partial u}{\partial x}\right)^5\right]. \qquad (6.1)\end{aligned}$$

A study of an elementary form of (6.1), namely

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^3\frac{\partial^5 u}{\partial x^5}\right),$$

was made from the viewpoints of numerical solution and asymptotic solution by Smith et al. [65]. Another example is the sixth-order equation of the form

$$u_t=\left[u\left(\frac{1}{u}\left(u\left(\ln u\right)_{xx}\right)_{xx}+\frac{1}{2}\left(\left(\ln u\right)_{xx}\right)^2\right)_x\right]_x,$$

which serves as an extension of the Derriba–Lebowitz–Speer–Spohn equation of the fourth-order [16],

$$u_t = -\left[u\left(\ln u\right)_{xx}\right]_{xx}.$$

This equation originates from the generalised quantum drift-diffusion model for semiconductors of Degond et al. [14] in the $O(\hbar^6)$ approximation, with $\hbar$ denoting the reduced Planck constant. The element $u(t,x)$ represents the particle density. After we expand the derivatives, the above equation can be written as

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left[\frac{\partial^5 u}{\partial x^5} - 3u^{-1}\frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4} - 5u^{-1}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + 8u^{-2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial x^3}\right.\\ &\qquad \left. + 11u^{-2}\frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - 18u^{-3}\left(\frac{\partial u}{\partial x}\right)^3\frac{\partial^2 u}{\partial x^2} + 6u^{-4}\left(\frac{\partial u}{\partial x}\right)^5\right].\end{aligned}$$

One can observe that the second-order equation (3.2), the fourth-order equation (4.1) and the sixth-order equation (6.1) consist of a chain of nonlinear evolution equations. However, there exist two missing pieces of the chain which are the third-order nonlinear equation (5.1),

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n\frac{\partial^2 u}{\partial x^2} + au^{n-1}\left(\frac{\partial u}{\partial x}\right)^2\right),$$

for which symmetry properties presented in the Chapter 5 and the fifth-order nonlinear equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}\left[u^n\frac{\partial^4 u}{\partial x^4} + a_1u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + a_2u^{n-1}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + a_3u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2}\right.\\ &\qquad \left. + a_4u^{n-3}\left(\frac{\partial u}{\partial x}\right)^4\right]. \end{aligned} \tag{6.2}$$

In this Chapter we present symmetry properties for the equations (6.1) and (6.2). We show the Lie point symmetries of the equations, the nonclassical symmetries and we give the special forms of the equations that admit potential symmetries. Finally, we give the nonclassical potential symmetries of the equations. Part of the results of this Chapter have already appeared in [11].

6.2 Symmetry Properties for a Sixth-Order Evolution Equation

In this Section we summarise the symmetry properties of sixth-order nonlinear evolution equations of the class (6.1)

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[u^n\frac{\partial^5 u}{\partial x^5} + a_1 u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4} + a_2 u^{n-1}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + a_3 u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^3 u}{\partial x^3}\right.$$
$$\left. + a_4 u^{n-2}\frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + a_5 u^{n-3}\left(\frac{\partial u}{\partial x}\right)^3\frac{\partial^2 u}{\partial x^2} + a_6 u^{n-4}\left(\frac{\partial u}{\partial x}\right)^5\right].$$

If we introduce a potential variable v, we can write equation (6.1) as a system of two PDEs with the form

$$v_x = u, \tag{6.3}$$

$$v_t = -\left(u^n u_{xxxxx} + a_1 u^{n-1}u_x u_{xxxx} + a_2 u^{n-1}u_{xx}u_{xxx} + a_3 u^{n-2}u_x^2 u_{xxx}\right.$$
$$\left. + a_4 u^{n-2}u_x u_{xx}^2 + a_5 u^{n-3}u_x^3 u_{xx} + a_6 u^{n-4}u_x^5\right)$$

and, if we eliminate u from this system, we derive the potential form of equation (6.1),

$$v_t = -\left(v_x^n v_{xxxxxx} + a_1 v_x^{n-1}v_{xx}v_{xxxxx} + a_2 v_x^{n-1}v_{xxx}v_{xxxx} + a_3 v_x^{n-2}v_{xx}^2 v_{xxxx}\right.$$
$$\left. + a_4 v_x^{n-2}v_{xx}v_{xxx}^2 + a_5 v_x^{n-3}v_{xx}^3 v_{xxx} + a_6 v_x^{n-4}v_{xx}^5\right). \tag{6.4}$$

6.2.1 Lie Symmetries

The equation for the sixth-order model, (6.1), has the Lie point symmetries

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 6t\partial_t + x\partial_x, \quad \Gamma_4 = \frac{nx}{6}\partial_x + u\partial_u.$$

In the cases that the parameters $(n, a_1, a_2, a_3, a_4, a_5, a_6)$ take the particular values

$$\begin{gathered}\left(-6, -15, a_2, \frac{3}{2}(-a_2+50), -6a_2, 15(a_2-10), 9(-a_2+10)\right),\\ \left(-4, -12, a_2, \frac{4}{3}(-a_2+37), \frac{(-31a_2-8)}{6}, \frac{104}{9}(a_2-7), \frac{56}{9}(-a_2+7)\right),\\ \left(-3, -\frac{21}{2}, -\frac{33}{2}, \frac{237}{4}, \frac{153}{2}, -\frac{885}{4}, \frac{225}{2}\right),\\ \left(-\frac{12}{5}, -\frac{48}{5}, -\frac{84}{5}, \frac{1326}{25}, \frac{1836}{25}, -\frac{24684}{125}, \frac{60588}{625}\right) \quad \text{or}\\ \left(-\frac{12}{7}, -\frac{60}{7}, -\frac{120}{7}, \frac{2280}{49}, \frac{3420}{49}, -\frac{59280}{343}, \frac{195624}{2401}\right)\end{gathered} \tag{6.5}$$

equation (6.1) admits a fifth Lie point symmetry, namely

$$\Gamma_5 = x^2\partial_x + \frac{12xu}{n}\partial_u.$$

The following equations correspond to the above five cases:

$$\begin{aligned}
u_t &= -\Big[u^{-6}u_{xxxxx} - 15u^{-7}u_x u_{xxxx} + a_2 u^{-7}u_{xx}u_{xxx} + \frac{3}{2}(-a_2+50)u^{-8}u_x^2 u_{xxx} \\
&\quad - 6a_2 u^{-8}u_x u_{xx}^2 + 15(a_2-10)u^{-9}u_x^3 u_{xx} + 9(-a_2+10)u^{-10}u_x^5\Big]_x, \\
u_t &= -\Big[u^{-4}u_{xxxx} - 8u^{-5}u_x u_{xxx} + \frac{1}{2}(a_2+8)u^{-5}u_{xx}^2 \\
&\quad + \frac{4}{3}(-a_2+7)u^{-6}u_x^2 u_{xx} + \frac{8}{9}(a_2-7)u^{-7}u_x^4\Big]_{xx}, \\
u_t &= -\left[u^{-3}u_{xxx} - \frac{9}{2}u^{-4}u_x u_{xx} + \frac{15}{4}u^{-5}u_x^3\right]_{xxx}, \\
u_t &= -\left[u^{-12/5}u_{xx} - \frac{6}{5}u^{-17/5}u_x^2\right]_{xxxx}, \\
u_t &= -\left[u^{-12/7}u_x\right]_{xxxxx}.
\end{aligned}$$

When n, a_1, a_2, a_3, a_4, a_5 and a_6 are arbitrary, the optimal system consists of the list of inequivalent subalgebras

$$\langle\Gamma_2\rangle,\ \langle\Gamma_1 + c\Gamma_2\rangle,\ \langle\Gamma_3 + c\Gamma_4\rangle,\ \langle\Gamma_4 + c\Gamma_1\rangle,\ \langle\Gamma_4 + c\Gamma_2 - \tfrac{n}{6}\Gamma_3\rangle$$

and when $(n, a_1, a_2, a_3, a_4, a_5, a_6)$ are given by equation (6.5), in addition to the above list, we have a reduction that corresponds to the subalgebra

$$\langle\Gamma_5 + c\Gamma_2 + 2k\Gamma_3\rangle.$$

We present some solutions for u. For the element $\langle\Gamma_4 + c\Gamma_2 - \frac{n}{6}\Gamma_3\rangle$, in the case that $n = 0$, the similarity reduction has the form $u = \exp\left(\frac{x}{c}\right)\phi(t)$ and it leads to the solution

$$u = c_1 \exp\left[\frac{1}{c^6}\left(c^5 x - (a_1+a_2+a_3+a_4+a_5+a_6+1)t\right)\right].$$

In the case for which we have five symmetries, the additional reduction has the form

$$\langle \Gamma_5 + c\Gamma_2 + 2k\Gamma_3 \rangle : \ u = \begin{cases} ((x+k)^2+1)^{\frac{6}{n}} \exp\left[-\frac{12k}{n}\tan^{-1}(x+k)\right]\phi(\omega), \\ \ \omega = t\exp\left[-12k\tan^{-1}(x+k)\right] \ \text{if } c-k^2=1, \\ \\ ((x+k)^2-1)^{\frac{6}{n}} \exp\left[\frac{12k}{n}\tanh^{-1}(x+k)\right]\phi(\omega), \\ \omega = t\exp\left[12k\tanh^{-1}(x+k)\right] \ \text{if } c-k^2=-1, \\ \\ (x+k)^{\frac{12}{n}} \exp\left[\frac{12k}{n(x+k)}\right]\phi(\omega), \\ \ \omega = t\exp\left[\frac{12k}{x+k}\right] \ \text{if } c-k^2=0, \end{cases}$$

where ω is the independent, ϕ the dependent variable of the reduced ODE, $c = 0, \pm 1$ and $k \in \mathbb{R}$. If $c = \pm 1$ and $k = 0$ the reduction has the form $u = (x^2+c)^{\frac{6}{n}}\phi(t)$ and we obtain the solutions:

For $(n, a_1, a_2, a_3, a_4, a_5, a_6) = \left(-\frac{12}{7}, -\frac{60}{7}, -\frac{120}{7}, \frac{2280}{49}, \frac{3420}{49}, -\frac{59280}{343}, \frac{195624}{2401}\right)$ we have the solution

$$u = (x^2+c)^{-7/2}(540ct + c_1)^{7/12},$$

for $(n, a_1, a_2, a_3, a_4, a_5, a_6) = \left(-\frac{12}{5}, -\frac{48}{5}, -\frac{84}{5}, \frac{1326}{25}, \frac{1836}{25}, -\frac{24684}{125}, \frac{60588}{625}\right)$ we have

$$u = (x^2+c)^{-5/2}(108ct + c_1)^{5/12}$$

and finally, when

$(n, a_1, a_2, a_3, a_4, a_5, a_6) = \left(-4, -12, a_2, \frac{4}{3}(-a_2+37), \frac{(-31a_2-8)}{6}, \frac{104}{9}(a_2-7), \frac{56}{9}(-a_2+7)\right)$, we have

$$u = (x^2+c)^{-3/2}[(-18a_2 - 324)ct + c_1]^{1/4}.$$

6.2.2 Nonclassical Symmetries

We find that equation (6.1) admits nonclassical symmetries in one case. If $(n, a_1, a_2, a_3, a_4, a_5, a_6) = (-\frac{1}{2}, -\frac{5}{2}, -5, \frac{15}{2}, \frac{45}{4}, -\frac{75}{4}, \frac{105}{16})$, (6.1) admits the reduction operator

$$\Gamma = \partial_t + \phi(x)\sqrt{u}\partial_u,$$

where $\phi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^6\phi}{\mathrm{d}x^6} + \frac{1}{2}\phi^2 = 0.$$

A special solution of this equation is $\phi = -665280x^{-6}$. The nonclassical operator above produces the ansatz (4.35)

$$u = \left[\tfrac{1}{2}\phi(x)t + F(x)\right]^2$$

which reduces (6.1) to the ODE

$$\frac{\mathrm{d}^6 F}{\mathrm{d}x^6} + \frac{1}{2}\phi F = 0. \tag{6.6}$$

For the particular solution $\phi = -665280x^{-6}$, equation (6.6) becomes an equation of Euler type with the form

$$x^6\frac{\mathrm{d}^6 F}{\mathrm{d}x^6} - 332640F = 0.$$

6.2.3 Potential Symmetries

We obtain that the system (6.3) admits Lie symmetries which induce potential symmetries of (6.1) in two cases.

Equation (6.1) admits the potential symmetry

$$\Gamma = \psi(t,v)\partial_x - u^2\psi_v\partial_u,$$

where $\psi(t,v)$ is a solution of the linear equation $\psi_t + \psi_{vvvvvv} = 0$ in the case $(n, a_1, a_2, a_3, a_4, a_5, a_6) = (-6, -21, -35, 210, 280, -1260, 945)$ and the potential symmetry

$$\Gamma = 2uv\partial_u + v^2\partial_v$$

in the case $(n, a_1, a_2, a_3, a_4, a_5, a_6) = (0, -6, \frac{(-2a_6-45)}{9}, \frac{(a_6+45)}{3}, \frac{2}{9}(4a_6+45), (-2a_6-15), a_6)$.

The existence of the infinite-dimensional Lie symmetry

$$\Gamma = \psi(t,v)\partial_x - u^2\psi_v\partial_u,$$

indicates possible linearisation. In fact the mapping

$$t \mapsto t, \quad x \mapsto v, \quad u \mapsto \frac{1}{u}, \quad v \mapsto x$$

connects system (6.3) in the case where
$(n, a_1, a_2, a_3, a_4, a_5, a_6) = (-6, -21, -35, 210, 280, -1260, 945)$ with the linear system

$$v_x = u,$$

$$v_t = -u_{xxxxx}.$$

6.2.4 Nonclassical Potential Symmetries

Here we present the nonclassical symmetries of the potential equation (6.4) that induce potential nonclassical symmetries of equation (6.1). We have nonclassical potential symmetries in two cases. Firstly, for $(n, a_1, a_2, a_3, a_4, a_5, a_6) = (-1, 0, 0, 0, 0, 0, 0)$, equation (6.4) admits the reduction operator

$$\Gamma_1 = \partial_t + c\partial_x + \phi(x + ct)\partial_v$$

and secondly, when $(n, a_1, a_2, a_3, a_4, a_5, a_6) = (-5, -21, -35, 210, 280, -1260, 945)$, it admits the reduction operator with the form

$$\Gamma_2 = \partial_t + \phi(v + ct)\partial_x + c\partial_v.$$

In both operators $\phi(\omega)$ is a solution of the ODE

$$\frac{\mathrm{d}^6\phi}{\mathrm{d}\omega^6} + \phi\frac{\mathrm{d}\phi}{\mathrm{d}\omega} = 0.$$

We recall that the potential equations corresponding to these two cases are connected via the pure hodograph transformation.

6.3 Symmetry Properties for a Fifth-Order Evolution Equation

Here we summarise the symmetry properties of fifth-order nonlinear evolution equations of the class (6.2)

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}\left[u^n\frac{\partial^4 u}{\partial x^4} + a_1u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + a_2u^{n-1}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + a_3u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2}\right.\\ &\qquad\left. + a_4u^{n-3}\left(\frac{\partial u}{\partial x}\right)^4\right].\end{aligned}$$

If we introduce a potential variable v, equation (6.2) can be written as a system of two PDEs

$$v_x = u, \tag{6.7}$$

$$v_t = -\left(u^n u_{xxxx} + a_1 u^{n-1} u_x u_{xxx} + a_2 u^{n-1} u_{xx}^2 + a_3 u^{n-2} u_x^2 u_{xx} + a_4 u^{n-3} u_x^4\right)$$

and elimination of u from this system gives the potential form of equation (6.2), namely,

$$v_t = -\left(v_x^n v_{xxxxx} + a_1 v_x^{n-1} v_{xx} v_{xxxx} + a_2 v_x^{n-1} v_{xxx}^2 + a_3 v_x^{n-2} v_{xx}^2 v_{xxx} + a_4 v_x^{n-3} v_{xx}^4\right). \tag{6.8}$$

6.3.1 Lie Symmetries

The fifth-order equation, (6.2), possesses four Lie point symmetries if n, a_1, a_2, a_3 and a_4 are arbitrary. They are

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 5t\partial_t + x\partial_x, \quad \Gamma_4 = \frac{nx}{5}\partial_x + u\partial_u.$$

If the parameters (n, a_1, a_2, a_3, a_4) take the particular values

$$\begin{array}{l} \left(-5, -10, a_2, 3(-a_2+10), \dfrac{9}{4}(a_2-10)\right), \\ \left(-\dfrac{10}{3}, -\dfrac{25}{3}, -5, 35, -\dfrac{640}{27}\right), \\ \left(-\dfrac{5}{2}, -\dfrac{15}{2}, -5, \dfrac{245}{8}, -\dfrac{315}{16}\right) \quad \text{or} \\ \left(-\dfrac{5}{3}, -\dfrac{20}{3}, -5, \dfrac{80}{3}, -\dfrac{440}{27}\right), \end{array} \tag{6.9}$$

where a_2 is a free parameter, there is a fifth Lie point symmetry given by

$$\Gamma_5 = x^2\partial_x + \frac{10xu}{n}\partial_u.$$

The four cases of (6.9) correspond to the following equations:

$$\begin{aligned} u_t &= -\left[u^{-5}u_{xxxx} - 10u^{-6}u_x u_{xxx} + a_2 u^{-6}u_{xx}^2 + 3(-a_2+10)u^{-7}u_x^2 u_{xx} + \frac{9}{4}(a_2-10)u^{-8}u_x^4\right]_x, \\ u_t &= -\left[u^{-10/3}u_{xxx} - 5u^{-13/3}u_x u_{xx} + \frac{40}{9}u^{-16/3}u_x^3\right]_{xx}, \\ u_t &= -\left[u^{-5/2}u_{xx} - \frac{5}{4}u^{-7/2}u_x^2\right]_{xxx}, \\ u_t &= -\left[u^{-5/3}u_x\right]_{xxxx}. \end{aligned}$$

Furthermore, if $(n, a_1, a_2, a_3, a_4) = (0, -\frac{5}{2}, -\frac{5}{4}, 5, -\frac{35}{16})$, equation (6.2) admits the infinite-dimensional Lie symmetry

$$\Gamma_\infty = \phi(t,x)\sqrt{u}\partial_u,$$

where $\phi(t,x)$ is a solution of the linear equation $\phi_t + \phi_{xxxxx} = 0$, in addition to the generic four symmetries.

In the case for which n, a_1, a_2, a_3 and a_4 are arbitrary the optimal system comprises the list of inequivalent subalgebras

$$\langle\Gamma_2\rangle,\ \langle\Gamma_1 + c\Gamma_2\rangle,\ \langle\Gamma_3 + c\Gamma_4\rangle,\ \langle\Gamma_4 + c\Gamma_1\rangle,\ \langle\Gamma_4 + c\Gamma_2 - \tfrac{n}{5}\Gamma_3\rangle$$

and in the case for which (n, a_1, a_2, a_3, a_4) are given by equation (6.9), in addition to the above list, we have a reduction that corresponds to the subalgebra

$$\langle\Gamma_5 + c\Gamma_2 + 2k\Gamma_3\rangle.$$

We note that $c = 0, \pm 1$ and $k \in \mathbb{R}$.

For the subalgebra $\langle\Gamma_4 + c\Gamma_2 - \frac{n}{5}\Gamma_3\rangle$ in the case $n = 0$, the similarity reduction has the form $u = \exp\left(\frac{x}{c}\right)\phi(t)$. For this case we obtain the solution

$$u = c_1 \exp\left[\frac{1}{c^5}\left(c^4 x - (a_1 + a_2 + a_3 + a_4 + 1)t\right)\right].$$

The existence of infinite-dimensional Lie symmetry indicates linearisation. The equation

$$u_t = -\left(u_{xxxx} - \frac{5}{2}u^{-1}u_x u_{xxx} - \frac{5}{4}u^{-1}u_{xx}^2 + 5u^{-2}u_x^2 u_{xx} - \frac{35}{16}u^{-3}u_x^4\right)_x$$

that admits the infinite-dimensional Lie symmetry Γ_∞ can be mapped into the linear equation

$$u_t + u_{xxxxx} = 0$$

by the mapping

$$u \mapsto \sqrt{u}.$$

In the case of the class of fourth-order equations (4.1) and sixth-order equations (6.1) we have seen there exists no member that can be a linearised by local mapping. From the other hand the equation of the third-order (5.1) was found to possess an infinite-dimensional Lie symmetry. Whilst such examples do not constitute a proof of the property, it appears that only equations of odd-order from the chain can be linearised by a local mapping.

6.3.2 Nonclassical Symmetries

Equation (6.2) admits nonclassical symmetries in two cases.

For the case that $(n, a_1, a_2, a_3, a_4) = (-\frac{1}{2}, -2, -\frac{3}{2}, \frac{9}{2}, -\frac{15}{8})$ it admits the reduction operator

$$\Gamma_1 = \partial_t + \phi(x)\sqrt{u}\partial_u,$$

where $\phi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^5\phi}{\mathrm{d}x^5} + \frac{1}{2}\phi^2 = 0$$

and a special solution is $\phi = 30240x^{-5}$. In this case Γ_1 leads to the ansatz (4.35)

$$u = \left[\tfrac{1}{2}\phi(x)t + F(x)\right]^2$$

which reduces (6.2) to the ODE

$$\frac{\mathrm{d}^5F}{\mathrm{d}x^5} + \frac{1}{2}\phi F = 0. \tag{6.10}$$

For the solution $\phi = 30240x^{-5}$ equation (6.10) becomes an equation of Euler type with the form

$$x^5\frac{\mathrm{d}^5F}{\mathrm{d}x^5} + 15120F = 0. \tag{6.11}$$

If $(n, a_1, a_2, a_3, a_4) = (-\frac{1}{3}, -3, -\frac{11}{6}, \frac{64}{9}, -\frac{88}{27})$ then (6.2) admits the reduction operator

$$\Gamma_2 = \partial_t + \psi(x)u^{2/3}\partial_u,$$

where $\psi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^5\psi}{\mathrm{d}x^5} + \frac{1}{3}\psi^2 = 0$$

with particualar solution $\psi = 45360x^{-5}$. From Γ_2 we have the nonclassical reduction (5.29)

$$u = \left[\tfrac{1}{3}\psi(x)t + F(x)\right]^3.$$

Equation (6.2) is reduced to the ODE

$$\frac{d^5F}{dx^5} + \frac{1}{3}\psi F = 0$$

which for $\psi = 45360x^{-5}$ becomes an equation of Euler type just as (6.11).

We point out that these two equations that admit nonclassical symmetries are connected by the mapping

$$u \mapsto u^{3/2}.$$

6.3.3 Potential Symmetries

System (6.7) admits Lie symmetries which induce potential symmetries of (6.2) in two cases. In particular, if $(n, a_1, a_2, a_3, a_4) = (-5, -15, -10, 105, -105)$, equation (6.2) admits the potential symmetry

$$\Gamma = \psi(t, v)\partial_x - u^2\psi_v\partial_u,$$

where $\psi(t, v)$ is a solution of the linear equation $\psi_t + \psi_{vvvvv} = 0$. As we know, the existence of infinite-dimensional Lie symmetry indicates linearisation. Namely, the mapping

$$t \mapsto t, \quad x \mapsto v, \quad u \mapsto \frac{1}{u}, \quad v \mapsto x$$

connects system (6.7) in the case where $(n, a_1, a_2, a_3, a_4) = (-5, -15, -10, 105, -105)$ with the linear system

$$v_x = u,$$

$$v_t = -u_{xxxx}.$$

If $(n, a_1, a_2, a_3, a_4) = (0, -5, \frac{4a_4}{9}, \frac{(-4a_4+15)}{3}, a_4)$, (6.2) admits the potential symmetry

$$\Gamma = 2uv\partial_u + v^2\partial_v.$$

6.3.4 Nonclassical Potential Symmetries

Here we list the nonclassical symmetries of the potential equation (6.8). It appears that we have only two cases that nonclassical symmetries of the potential equation (6.8) induce potential nonclassical symmetries of equation (6.2). In the case $(n, a_1, a_2, a_3, a_4) = (-1, 0, 0, 0, 0)$ equation (6.8) admits the reduction operator

$$\Gamma_1 = \partial_t + c\partial_x + \phi(x + ct)\partial_v$$

and in the case that $(n, a_1, a_2, a_3, a_4) = (-4, -15, -10, 105, -105)$, (6.8) admits the reduction operator

$$\Gamma_2 = \partial_t + \phi(v + ct)\partial_x + c\partial_v.$$

We state that in both operators, $\phi(\omega)$ is a solution of the ODE

$$\frac{\mathrm{d}^5\omega}{\mathrm{d}\omega^5} + \phi\frac{\mathrm{d}\phi}{\mathrm{d}\omega} = 0$$

and also that the potential equations in these two cases are connected via the pure hodograph transformation.

Chapter 7

Lie and Potential Symmetries of a Fourth- and Third-Order Generalised Evolution Equations

7.1 Introduction

In this Chapter we consider the class of a fourth- and third-order generalised evolution equations, respectively,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial^3 u}{\partial x^3} + g(u)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + h(u)\left(\frac{\partial u}{\partial x}\right)^3\right) \tag{7.1}$$

and

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(f(u)\frac{\partial^2 u}{\partial x^2} + g(u)\left(\frac{\partial u}{\partial x}\right)^2\right). \tag{7.2}$$

Here $f = f(u)$, $g = g(u)$ and $h = h(u)$ are arbitrary smooth functions. In both equations $f(u) \neq 0$. We note that equation (7.1) is the generalisation of equation (4.1) for which we presented the complete group analysis in Chapter 4 and equation (7.2) is the generalisation of equation (5.1) for which symmetry properties are presented in Chapter 5. We know that, if we introduce a potential variable, v, equations (7.1) and (7.2) can be written as a system of two PDEs, respectively,

$$v_x = u, \quad v_t = \left(f(u)u_{xxx} + g(u)u_x u_{xx} + h(u)u_x^3\right) \tag{7.3}$$

and

$$v_x = u, \quad v_t = \left(f(u)u_{xx} + g(u)u_x^2\right). \tag{7.4}$$

In this Chapter we present the Lie symmetries of equations (7.1) and (7.2) and the Lie symmetries of the systems (7.3) and (7.4) that induce potential symmetries for the corresponding equations.

7.2 Equation (7.1)

7.2.1 Lie Symmetries

Equation (7.1) admits Lie point symmetries if and only if

$$\Gamma^{(4)}[u_t - fu_{xxxx} - (f_u + g)u_x u_{xxx} - gu_{xx}^2 - (g_u + 3h)u_x^2 u_{xx} - h_u u_x^4] = 0 \tag{7.5}$$

identically, modulo equation (7.1).

We eliminate u_t due to (7.1) and equation (7.5) becomes an identity in the variables u_x, u_{xx}, u_{tx}, u_{xxx}, u_{txx}, u_{xxxx} and u_{txxx}. From the coefficients of different powers of these variables, which must be zero, we derive the determining equations on the coefficients τ, ξ and η. We use the general results on point transformations between evolution equations [34] and the remaining determining equations produce the functional forms of $\tau(t)$, $\xi(t,x)$ and $\eta(t,x,u)$. From the coefficient of u_{xxxx} we deduce that

$$\eta = \frac{f}{f_u}(4\xi_x - \tau_t) \text{ where } f \neq \text{constant}. \tag{7.6}$$

For the special case that $f(u)$ is an arbitrary function, from (7.6) we have that $\eta = 0$ and $\xi = \frac{1}{4}\tau_t x + L(t)$. Substituting the forms of η and ξ into (7.5), from the coefficient of u_x, the only nonzero coefficient, we get that $\tau_{tt}x + 4L_t = 0$. We deduce that $\tau = c_1 t + c_2$ and $L =$ constant. We mention that the forms of $g(u)$ and $h(u)$ also must be arbitrary functions of u. In this case the Lie algebra is three-dimensional and is given by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x.$$

If we assume that $f(u)$ is not an arbitrary function and after we have used the fact that τ is a function of t, ξ is a function of t and x and the expression (7.6) for η, from the coefficient of u_{xx}^2 we have the equation

$$(4\xi_x - \tau_t)\left(3ff_uf_{uuu} - 6ff_{uu}^2 + 3f_u^2f_{uu} + f_uf_{uu}g - f_u^2g_u\right) = 0, \tag{7.7}$$

where $4\xi_x - \tau_t \neq 0$, because otherwise we get the results of case where f is arbitrary. Equation (7.7) can be written in the form

$$g_u - \left(\frac{f_{uu}}{f_u}\right)g = -3f_u\left(\frac{f}{f_u}\right)_{uu}$$

which is a first-order linear ODE with solution

$$g = -3f_u\left(\frac{f}{f_u}\right)_u + kf_u, \text{ where } k = \text{constant}. \tag{7.8}$$

We substitute this form of g from (7.8) into (7.5) and from the coefficient of u_xu_{xxx} we take the equation

$$ff_uf_{uuu} - 2ff_{uu}^2 + f_u^2f_{uu} = 0$$

which can be written in the form

$$-f_u^3\left(\frac{f}{f_u}\right)_{uu} = 0.$$

This equation has solution $f(u) = u^n$ (with limiting case $f(u) = e^u$), where n is an arbitrary constant. We conclude that we have to examine the following forms of $f(u)$:

(1) $f(u) = u^n$,

(2) $f(u) = e^u$ and

(3) $f =$ constant.

Case 1. $f(u) = u^n$.

From (7.8) we have that $g = (kn-3)u^{n-1}$. We can take $a = (kn-3)$, where a is an arbitrary constant so that $g = au^{n-1}$. We substitute the form of g into (7.5) and from the coefficient of $u_x^2u_{xx}$ we take the equation with the form

$$(4\xi_x - \tau_t)\left[h_uu - (n-2)h\right] = 0 \quad \text{with} \quad 4\xi_x - \tau_t \neq 0.$$

We conclude that from the solution of the first-order ODE, $h_uu - (n-2)h = 0$, the form of h is $h = bu^{n-2}$. Finally, from the forms of f, g and h we have that equation (7.1) has the same form as equation (4.1), the equation for which we presented complete group analysis in Chapter 4.

Case 2. $f(u) = e^u$.

In this case, from (7.8), $g = ke^u$. After we have substituted the form of g into (7.5) from the coefficient of $u_x^2 u_{xx}$ we can take the form of h. The coefficient of $u_x^2 u_{xx}$ has the form

$$(4\xi_x - \tau_t)\,(h_u - h) = 0 \ \text{ with } \ 4\xi_x - \tau_t \neq 0$$

and we conclude that $h = me^u$. We use this form of h and we take the simplifying determining system for the determination of the coefficients τ, ξ and η which has the form

$$(2k-1)\xi_{xx} = 0, \tag{7.9}$$

$$(2k-1)\xi_{xxx} = 0, \tag{7.10}$$

$$(k+8m-1)\xi_{xx} = 0, \tag{7.11}$$

$$(k-13m)\xi_{xx} = 0, \tag{7.12}$$

$$(3k+12m-1)\xi_{xxx} = 0, \tag{7.13}$$

$$(4k+3)\xi_{xxxx}e^u + \xi_t = 0, \tag{7.14}$$

$$4\xi_{xxxxx}e^u - 4\xi_{tx} + \tau_{tt} = 0. \tag{7.15}$$

We solve the determining equations (7.9)–(7.15) and we conclude that the symmetry Lie algebra is four-dimensional with infinitesimal generators

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x, \quad \Gamma_4 = x\partial_x + 4\partial_u.$$

Case 3. $f = \text{constant}$ $(f \neq 0)$.

In this case, without loss of generality, we can assume that $f = 1$. From the coefficient of u_{xxxx} we have that

$$4\xi_x - \tau_t = 0.$$

We deduce that the form of ξ is given by

$$\xi = \frac{1}{4}\tau_t x + L(t).$$

We use the above expression for ξ and we take the simplifying determining system for the determination of the coefficients τ, ξ and η. The system is

$$4\eta_{xu} + g\eta_x = 0, \tag{7.16}$$

$$4\eta_{uu} + g\eta_u + g_u\eta = 0, \tag{7.17}$$

$$3\eta_{uu} + g\eta_u + g_u\eta = 0, \tag{7.18}$$

$$3\eta_{xxu} + g\eta_{xx} = 0, \tag{7.19}$$

$$12\eta_{xuu} + 7g\eta_{xu} + 2(g_u + 3h)\eta_x = 0, \tag{7.20}$$

$$6\eta_{uuu} + 5g\eta_{uu} + 2(g_u + 3h)\eta_u + (g_{uu} + 3h_u)\eta = 0, \tag{7.21}$$

$$\eta_{uuuu} + g\eta_{uuu} + (g_u + 3h)\eta_{uu} + 3h_u\eta_u + h_{uu}\eta = 0, \tag{7.22}$$

$$4\eta_{xuuu} + 3g\eta_{xuu} + 2(g_u + 3h)\eta_{xu} + 4h_u\eta_x = 0, \tag{7.23}$$

$$6\eta_{xxuu} + 3g\eta_{xxu} + (g_u + 3h)\eta_{xx} = 0, \tag{7.24}$$

$$16\eta_{xxxu} + 4g\eta_{xxx} + \tau_{tt}x + 4L_t = 0, \tag{7.25}$$

$$\eta_{xxxx} - \eta_t = 0. \tag{7.26}$$

In order to solve the determining equations (7.16)–(7.26) we need to consider the following cases depending on the forms of $g(u)$ and $h(u)$:

(1) $g(u) = au^{-1}$ and $h = bu^{-2}$,

(2) $g =$ constant and $h(u)$ is arbitrary,

(3) $g =$ constant and $h =$ constant,

(4) $g = h = 0$.

Subcase 3.1: $g(u) = au^{-1}$ and $h = bu^{-2}$.

In this case we have the same results as the case of equation (4.1) with a and b to be arbitrary and $n = 0$. Namely, PDE (7.1) admits a four-parameter group with infinitesimal generators

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x, \quad \Gamma_4 = u\partial_u.$$

Subcase 3.2: $g =$ constant and $h(u)$ is arbitrary.

This case is a subcase for $f(u)$, $g(u)$ and $h(u)$ be arbitrary functions. We recall that the Lie algebra in this case is three-dimensional spanned by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x.$$

Subcase 3.3: $g =$ constant and $h =$ constant.

The Lie algebra in this case is four-dimensional and is spanned by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x, \quad \Gamma_4 = \partial_u.$$

Subcase 3.4: $g = h = 0$.

In this case equation (7.1) becomes a linear PDE and it admits an infinite-parameter group with infinitesimal generators

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x, \quad \Gamma_4 = u\partial_u \quad \text{and}$$

$$\Gamma_\infty = \phi(t,x)\partial_u,$$

where $\phi(t,x)$ is a solution of the linear equation $\phi_t = \phi_{xxxx}$.

In the following table we present briefly the Lie point symmetries for the different forms of $f(u)$, $g(u)$ and $h(u)$.

Table 7.1: Classification of equation (7.1)

Cases	$f(u)$	$g(u)$	$h(u)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	$\forall$	$A^{\ker} = \langle \partial_t,\, \partial_x,\, 4t\partial_t + x\partial_x \rangle$
General case of (n,a,b)				
2i	u^n	au^{n-1}	bu^{n-2}	$A^{\ker} \oplus \left\langle \frac{nx}{4}\partial_x + u\partial_u \right\rangle$
$(n,a,b) \in \left\{(-4,-6,6), (-\frac{8}{3}, -\frac{16}{3}, \frac{44}{9}), (-\frac{8}{5}, -\frac{24}{5}, \frac{104}{25})\right\}$				
2ii	u^n	au^{n-1}	bu^{n-2}	$A^{\ker} \oplus \left\langle \frac{nx}{4}\partial_x + u\partial_u,\, x^2\partial_x + \frac{8xu}{n}\partial_u \right\rangle$
3	e^u	ke^u	me^u	$A^{\ker} \oplus \langle x\partial_x + 4\partial_u \rangle$
4	1	constant	constant	$A^{\ker} \oplus \langle \partial_u \rangle$
5	1	0	0	$A^{\ker} \oplus \langle u\partial_u,\, \phi(t,x)\partial_u \rangle$

The function $\phi(t,x)$ is a solution of $\phi_t = \phi_{xxxx}$.

7.2.2 Potential Symmetries

Here we search for Lie symmetries for the system (7.3) with the optimal goal of finding potential symmetries for equations (7.1).

The system (7.3) admits Lie symmetries if and only if

$$\Gamma^{(1)}[v_x - u] = 0, \tag{7.27}$$

$$\Gamma^{(3)}\left[v_t - \left(f(u)u_{xxx} + g(u)u_x u_{xx} + h(u)u_x^3\right)\right] = 0 \tag{7.28}$$

for $v_x = u$ and $v_t = f(u)u_{xxx} + g(u)u_x u_{xx} + h(u)u_x^3$. From the determining system we have that the coefficient τ is a function of t and the coefficients ξ and ζ are functions of t, x and v. We use these results and from equation (7.27) we get that

$$\eta = -\xi_v u^2 - (\xi_x - \zeta_v)u + \zeta_x. \tag{7.29}$$

We substitute the form of η (7.29) and from equation (7.28) we have the simplifying determining system for the determination of the coefficients τ, ξ, η and ζ. From the coefficient of u_{xxx} we have that

$$\frac{f_u}{f} = \frac{-4\xi_v u + \tau_t - 4\xi_x}{\xi_v u^2 + (\xi_x - \zeta_v)u - \zeta_x}. \tag{7.30}$$

If we rewrite (7.30) as

$$\frac{f_u}{f} = \frac{-4\lambda_1 u + \lambda_2}{\lambda_1 u^2 + \lambda_3 u + \lambda_4} \text{ for } \lambda_1 = 1,$$

we derive the form of f which is

$$f = (u^2 + pu + q)^{-2} \exp\left[\int \frac{r\mathrm{d}u}{u^2 + pu + q}\right]$$

with p, q and r being arbitrary constants such that $4p^2 - 16q - r^2 \neq 0$. We point out that for the case $\lambda_1 = 0$ and $4p^2 - 16q - r^2 = 0$ we recover the results which we have presented in Chapter 4. We substitute the form of f and, after we have solved the determining system, we conclude that the forms of functions g and h are given by the forms

$$g = (-10u + k)(u^2 + pu + q)^{-3} \exp\left[\int \frac{r\mathrm{d}u}{u^2 + pu + q}\right],$$

$$h = (15u^2 - 3ku + c)(u^2 + pu + q)^{-4} \exp\left[\int \frac{r\mathrm{d}u}{u^2 + pu + q}\right].$$

We conclude that a potential symmetry exists in one case. The Lie symmetries of (7.3) with the forms

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 4t\partial_t + x\partial_x + v\partial_v \text{ and } \Gamma_4 = \partial_v$$

project into local symmetries of (7.1) and the Lie symmetry

$$\Gamma_5 = (r - 2p)t\partial_t + v\partial_x - (u^2 + pu + q)\partial_u - (qx + pv)\partial_v$$

induces a potential symmetry for the corresponding equation (7.1).

7.3 Equation (7.2)

7.3.1 Lie Symmetries

Equation (7.2) admits Lie point symmetries if and only if

$$\Gamma^{(3)}[u_t - fu_{xxx} - (f_u + 2g)u_x u_{xx} - g_u u_x^3] = 0 \tag{7.31}$$

for $u_t = fu_{xxx} + (f_u + 2g)u_x u_{xx} + g_u u_x^3$.

We use the above expression to eliminate u_t and the equation (7.31) becomes an identity in the variables u_x, u_{xx}, u_{tx}, u_{xxx} and u_{txx}. From coefficients of different powers of these variables, which must be equal to zero, we derive the determining equations for the coefficients τ, ξ and η. We use again the general results on point transformations between evolution equations [34], that $\tau = \tau(t)$ and $\xi = \xi(t, x)$. From the coefficient of u_{xxx} we have that

$$\eta = \frac{f}{f_u}(3\xi_x - \tau_t), \text{ where } f \neq \text{constant}.$$

If $f(u)$ is an arbitrary function, then the Lie algebra is three-dimensional given by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 3t\partial_t + x\partial_x.$$

If we use the above facts, from the coefficients of u_{xx}, $u_x u_{xx}$, u_x^3, u_x^2, u_x and the term independent of derivatives of (7.31) we have the following determining equations, respectively,

$$(3ff_{uu} - 3f_u^2 - 2f_u g)\xi_{xx} = 0, \tag{7.32}$$

$$3ff_uf_{uuu} - 6ff_{uu}^2 + 3f_u^2 f_{uu} + 2f_u f_{uu} g - 2f_u^2 g_u = 0, \tag{7.33}$$

$$\begin{aligned} &f^2 f_u^2 f_{uuuu} - 6f^2 f_u f_{uu} f_{uuu} + 3ff_u^3 f_{uuu} + 2ff_u^2 f_{uuu} g + 6f^2 f_{uu}^3 \\ &-5ff_u^2 f_{uu}^2 - 4ff_u f_{uu}^2 g + f_u^4 f_{uu} + 2f_u^3 f_{uu} g + 2ff_u^2 f_{uu} g_u \\ &-f_u^4 g_u - ff_u^3 g_{uu} = 0, \end{aligned} \tag{7.34}$$

$$\begin{aligned} &(9f^2 f_u f_{uuu} - 18f^2 f_{uu}^2 + 15ff_u^2 f_{uu} + 12ff_u f_{uu} g - 5f_u^4 - 10f_u^3 g \\ &-9ff_u^2 g_u)\xi_{xx} = 0, \end{aligned} \tag{7.35}$$

$$f_u^2 \xi_t - (9f^2 f_{uu} - 11ff_u^2 - 6ff_u g)\xi_{xxx} = 0, \tag{7.36}$$

$$\tau_{tt} - 3\xi_{tx} + 3f\xi_{xxxx} = 0. \tag{7.37}$$

From equation (7.32) we can distinguish two special cases. In summary we have the following special cases:

(1) $\xi_{xx} = 0$,

(2) $3ff_{uu} - 3f_u^2 - 2f_u g = 0$.

Case 1. $\xi_{xx} = 0$.

In this case we have that equations (7.32) and (7.35) are identically to zero, from (7.36) we have that ξ is not a function of t and also equation (7.37) gives that $\tau = c_1 t + c_2$. Equation (7.33) can be written in the form

$$g_u - \left(\frac{f_{uu}}{f_u}\right) g = -\frac{3}{2} f_u \left(\frac{f}{f_u}\right)_{uu}$$

which is a first-order linear ODE with solution

$$g = -\frac{3}{2} f_u \left(\frac{f}{f_u}\right)_u + kf_u, \text{ where } k = \text{constant}. \tag{7.38}$$

From (7.38) we have that

$$f_{uu} = \frac{2f_u}{3f}\left[g + \left(\frac{3}{2} - k\right) f_u\right]. \tag{7.39}$$

We substitute the form of f_{uu} in (7.39) into (7.31) and from (7.34) we have a relation between functions f and g

$$3(4k-3)ff_u g_u - 8(2k-3)f_u g^2 + 9f^2 g_{uu} - 36fgg_u + 16g^3 = 0. \tag{7.40}$$

Finally we conclude that the symmetry Lie algebra is four-dimensional and is spanned by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 3t\partial_t + x\partial_x \text{ and } \Gamma_4 = x\partial_x + \frac{3f}{f_u}\partial_u$$

when functions $f(u)$ and $g(u)$ satisfy equation (7.40).

Case 2. $3ff_{uu} - 3f_u^2 - 2f_u g = 0$.

From the above condition, we have that the form of g is

$$g = -\frac{3}{2}f_u\left(\frac{f}{f_u}\right)_u. \tag{7.41}$$

This form of g satisfies (7.33). From equation (7.36) we have that ξ equals to $\xi = \frac{1}{2}a_1x^2 + a_2x + a_3$ and from equation (7.37) we have that τ is a linear equation respect to t. After we have used these facts, from equation (7.35) we take that

$$9f^2f_uf_{uuu} - 27f^2f_{uu}^2 + 36ff_u^2f_{uu} - 20f_u^4 = 0 \tag{7.42}$$

which satisfies (7.34).

To conclude, we have that the symmetry Lie algebra is five-dimensional and is spanned by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 3t\partial_t + x\partial_x, \quad \Gamma_4 = x\partial_x + \frac{3f}{f_u}\partial_u \text{ and}$$

$$\Gamma_5 = x^2\partial_x + \frac{6xf}{f_u}\partial_u,$$

where the function $f(u)$ satisfies equation (7.42).

Case 3. $f = \text{constant}$ ($f \neq 0$).

Finally, we have to examine the special case for which $f = \text{constant}$. We assume that $f = 1$. From the coefficient of u_{xxx} we have that

$$3\xi_x - \tau_t = 0,$$

which means that $\xi = \frac{1}{3}\tau_t x + L(t)$. After we have used the expression for ξ, the simplifying determining system for the determination of the coefficients τ, ξ and η is

$$3\eta_{xu} + 2g\eta_x = 0, \tag{7.43}$$

$$3\eta_{uu} + 2g\eta_u + 2g_u\eta = 0, \tag{7.44}$$

$$\eta_{uuu} + 2g\eta_{uu} + 2g_u\eta_u + g_{uu}\eta = 0, \tag{7.45}$$

$$3\eta_{xuu} + 4g\eta_{xu} + 3g_u\eta_x = 0, \tag{7.46}$$

$$9\eta_{xxu} + 6g\eta_{xx} + \tau_{tt}x + 3L_t = 0, \tag{7.47}$$

$$\eta_{xxx} - \eta_t = 0. \tag{7.48}$$

If we write equation (7.44) as

$$\eta_{uu} = -\frac{2}{3}\left(g\eta\right)_u \tag{7.49}$$

and substitute it into equation (7.45), we derive the equation

$$\left(3g_u - 4g^2\right)\eta_u = -\frac{1}{2}\left(3g_u - 4g^2\right)_u \eta. \tag{7.50}$$

From (7.50) we distinguish three special subcases:

(1) $(3g_u - 4g^2) = 0$,

(2) $(3g_u - 4g^2) \neq 0$,

(3) $g = 0$.

Subcase 3.1: $(3g_u - 4g^2) = 0$.

We solve the ODE $(3g_u - 4g^2) = 0$ and we have that $g(u)$ has the form

$$g = \frac{3}{3\mu - 4u} \text{ where } \mu = \text{constant.}$$

We integrate (7.49) and using the form of $g(u)$ we obtain the first-order ODE with the form

$$\eta_u + \frac{2}{3\mu - 4u}\eta = k_1(t, x)$$

and solution

$$\eta = -\frac{(3\mu - 4u)}{2}k_1(t, x) + \sqrt{|3\mu - 4u|}\varphi(t, x). \tag{7.51}$$

Finally, after we have used the form of η (7.51), we infer that equation (7.2) admits an infinite-parameter group with infinitesimal generators

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 3t\partial_t + x\partial_x, \quad \Gamma_4 = (3\mu - 4u)\partial_u \text{ and}$$

$$\Gamma_\infty = \sqrt{|3\mu - 4u|}\varphi(t, x)\partial_u,$$

where $\varphi(t, x)$ satisfies the linear equation $\varphi_t = \varphi_{xxx}$.

Subcase 3.2: $(3g_u - 4g^2) \neq 0$.

In this case the solution of (7.50) gives

$$\eta = \frac{1}{\sqrt{|3g_u - 4g^2|}}\psi(t, x).$$

We substitute the form of η into (7.31) and we calculate that $\tau = c_1 t + c_2$ and functions L and ψ must be constants. From equation (7.44) we derive a relation with function $g(u)$ with the form

$$2(3g_u - 4g^2)g_{uuu} - 9g_{uu}^2 + 36gg_ug_{uu} + 16g^3g_{uu} - 24g_u^3 - 32g^2g_u^2 = 0$$

which satisfies equation (7.45). In this subcase the Lie algebra is four-dimensional and is spanned by

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 3t\partial_t + x\partial_x, \quad \Gamma_4 = \frac{1}{\sqrt{|3g_u - 4g^2|}}\partial_u.$$

Subcase 3.3: $g = 0$.

Equation (7.2) becomes a linear PDE with the form $u_t = u_{xxx}$ and this admits an infinite-parameter group with infinitesimal generators

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = 3t\partial_t + x\partial_x, \quad \Gamma_4 = u\partial_u \quad \text{and}$$

$$\Gamma_\infty = \varphi(t, x)\partial_u,$$

where $\varphi(t, x)$ is a solution of the linear equation $\varphi_t = \varphi_{xxx}$.

In Table 7.2 we summarize the Lie point symmetries for the different forms of $f(u)$ and $g(u)$.

Table 7.2: Classification of equation (7.2)

Cases	$f(u)$	$g(u)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	$A^{\ker} = \langle \partial_t,\, \partial_x,\, 3t\partial_t + x\partial_x \rangle$
2	$f(u)$	$-\frac{3}{2} f_u \left(\frac{f}{f_u}\right)_u + k f_u$	$A^{\ker} \oplus \left\langle x\partial_x + \frac{3f}{f_u}\partial_u \right\rangle$
3	$f(u)$	$-\frac{3}{2} f_u \left(\frac{f}{f_u}\right)_u$	$A^{\ker} \oplus \left\langle x\partial_x + \frac{3f}{f_u}\partial_u,\, x^2\partial_x + \frac{6xf}{f_u}\partial_u \right\rangle$
4	1	$\frac{3}{3\mu - 4u}$	$A^{\ker} \oplus \left\langle (3\mu - 4u)\partial_u,\, \sqrt{\lvert 3\mu - 4u \rvert}\varphi(t,x)\partial_u \right\rangle$
5	1	$\neq \frac{3}{3\mu - 4u}$	$A^{\ker} \oplus \left\langle \frac{1}{\sqrt{\lvert 3g_u - 4g^2 \rvert}}\partial_u \right\rangle$
6	1	0	$A^{\ker} \oplus \langle u\partial_u,\, \phi(t,x)\partial_u \rangle$

In Case 2 $k =$ constant and the functions $f(u)$ and $g(u)$ satisfy the relation

$$3(4k-3)ff_ug_u - 8(2k-3)f_ug^2 + 9f^2g_{uu} - 36fgg_u + 16g^3 = 0.$$

In Case 3 the function $f(u)$ satisfies

$$9f^2f_uf_{uuu} - 27f^2f_{uu}^2 + 36ff_u^2f_{uu} - 20f_u^4 = 0.$$

In Cases 4 and 6 the function $\varphi(t,x)$ is a solution of $\varphi_t = \varphi_{xxx}$.

In Cases 4 and 5 $\mu =$ constant.

In Case 5 the function $g(u)$ satisfies

$$2(3g_u - 4g^2)g_{uuu} - 9g_{uu}^2 + 36gg_ug_{uu} + 16g^3g_{uu} - 24g_u^3 - 32g^2g_u^2 = 0.$$

7.3.2 Potential Symmetries

The system (7.4) admits Lie symmetries if and only if

$$\Gamma^{(1)}[v_x - u] = 0 \text{ and} \tag{7.52}$$

$$\Gamma^{(2)}\left[v_t - \left(f(u)u_{xx} + g(u)u_x^2\right)\right] = 0 \tag{7.53}$$

for $v_x = u$ and $v_t = f(u)u_{xx} + g(u)u_x^2$. From the determining system we have that that the coefficient τ is a function of t and the coefficients ξ and ζ are functions of t, x and v. After we have used these results from equation (7.52) we have that

$$\eta = -\xi_v u^2 - (\xi_x - \zeta_v)u + \zeta_x. \tag{7.54}$$

We use the form of η, (7.54), and from equation (7.53) we have the simplifying determining system for the determination of the coefficients $\tau,\ \xi,\ \eta$ and ζ. The coefficient of u_{xx} gives that

$$\frac{f_u}{f} = \frac{-3\xi_v u + \tau_t - 3\xi_x}{\xi_v u^2 + (\xi_x - \zeta_v)u - \zeta_x},$$

with solution the form of f:

$$f = (u^2 + pu + q)^{-3/2} \exp\left[\int \frac{r\mathrm{d}u}{u^2 + pu + q}\right].$$

In this case p, q and r are arbitrary constants such that $9p^2 - 36q - 4r^2 \neq 0$. We substitute the form of f and after we have solved the determining system we conclude that the form of g is

$$g = (-3u + k)(u^2 + pu + q)^{-5/2} \exp\left[\int \frac{r\mathrm{d}u}{u^2 + pu + q}\right].$$

Here the system (7.4) admits a five-parameter group with infinitesimal generators

$$\Gamma_1 = \partial_t,\quad \Gamma_2 = \partial_x,\quad \Gamma_3 = 3t\partial_t + x\partial_x + v\partial_v,\quad \Gamma_4 = \partial_v \text{ and}$$

$$\Gamma_5 = \left(r - \frac{3}{2}p\right) t\partial_t + v\partial_x - (u^2 + pu + q)\partial_u - (qx + pv)\partial_v.$$

The Lie symmetries $\Gamma_1,\ \Gamma_2\ \Gamma_3$ and Γ_4 project into local symmetries of (7.2) and the Lie symmetry Γ_5 induces a potential symmetry for the corresponding equation (7.2).

Chapter 8

Group Analysis for a Class of Nonlinear Dispersive Equations

8.1 Introduction

We consider the class of nonlinear dispersive equations [63]

$$u_t + \epsilon \left(u^m\right)_x + \frac{1}{b}\left[u^a \left(u^b\right)_{xx}\right]_x = 0 \tag{8.1}$$

which is of interest in Mathematical Physics. Special cases of this class have been used to model successfully physical situations in a wide range of fields. For example, if $a = 0$ and $b = n$, we have the generalisation of the KdV equation [58, 59],

$$u_t + \epsilon \left(u^m\right)_x + \frac{1}{n}\left(u^n\right)_{xxx} = 0, \tag{8.2}$$

and the equation that corresponds to the values $m = 2,\ a = b = 1$ describes a motion of a diluted suspension [64]. Equations of the type (8.2) with values of the parameters m and n are denoted by $K(m, n)$. For example, the properties of equation $K(2, 2)$ were examined in [58]. Further applications of the class (8.1) can be found in [61–63] and references therein.

Our goal in this Chapter is to extend certain results of the recent work [8]. In particular we give an enhanced Lie group classification for the class (8.1). The complete list of form-preserving point transformations is presented. We show the nonclassical reductions, potential symmetries and nonclassical potential symmetries [12].

8.2 Equivalence Transformations

We recall that an equivalence transformation of a class of PDEs, $E(x,t,u)=0$, is an invertible transformation of the independent and dependent variables of the form

$$t'=Q(x,t,u),\quad x'=P(x,t,u),\quad u'=R(x,t,u) \tag{8.3}$$

that maps every equation of the class into an equations of the same form, $E(x',t',u')=0$. A complete classification of transformations of the class (8.3) that connect equations (8.1) and

$$u'_{t'}+\epsilon'\left(u'^{m'}\right)_{x'}+\frac{1}{b'}\left[u'^{a'}\left(u'^{b'}\right)_{x'x'}\right]_{x'}=0 \tag{8.4}$$

provides us the so-called form-preserving transformations [34] (or admissible transformations [56]) of equations (8.1). Equivalence transformations can be regarded as a subset of such transformations. The complete list of preserving transformations is presented in Section 8.5.

In order to derive the desired equivalence group of transformations we need to consider two cases:

(1) $a+b-1\neq 0$ and

(2) $a+b-1=0$.

Case 1. $a+b-1\neq 0$.
In this case we find that

$$t'=\beta t+\gamma,\quad x'=\alpha x+\delta,\quad u'=\alpha^{-\frac{1}{a+b-1}}\beta^{\frac{3}{a+b-1}}u \tag{8.5}$$

where

$$\epsilon'\beta^{\frac{a+b-m}{a+b-1}}=\epsilon\alpha^{\frac{a+b-3m+2}{a+b-1}}.$$

From the above relation we deduce that $\epsilon\epsilon'>0$ or $\epsilon=\epsilon'=0$ and $\alpha,\ \beta$ are nonnegative. Furthermore nonzero ϵ and ϵ' may be fixed. For example an equation with $\epsilon>0$ can be transformed into one with $\epsilon'=1$ and an equation with $\epsilon<0$ can be transformed into one with $\epsilon'=-1$. That is, we can take, without loss of generality, $\epsilon=\epsilon'=\pm1$. Hence, for this last possibility, equation (8.1) admits a three-parameter group of transformations

$$t'=\alpha^{\frac{a+b-3m+2}{a+b-m}}t+\gamma,\ \ x'=\alpha x+\delta,\ \ u'=\alpha^{\frac{2}{a+b-m}}u,\quad (a',b',m',\epsilon'=\pm1)=(a,b,m,\epsilon=\pm1). \tag{8.6}$$

While in the case $\epsilon = \epsilon' = 0$ equation (8.1) takes the form

$$u_t + \frac{1}{b}\left[u^a\left(u^b\right)_{xx}\right]_x = 0 \tag{8.7}$$

and it admits the four-parameter equivalence group (8.5).

Now if $m = m' = 1$ we obtain the 4-parameter equivalence group

$$t' = \beta t + \gamma, \quad x' = \alpha x + (\beta\epsilon' - \alpha\epsilon)t + \delta, \quad u' = \alpha^{-\frac{1}{a+b-1}}\beta^{\frac{3}{a+b-1}}u, \quad (a', b') = (a, b). \tag{8.8}$$

From this equivalence transformation we deduce that equations (8.1) and (8.4) are connected with ϵ or ϵ' being zero. In other words, in the case where $m = 1$, equation (8.1) can be mapped into equation (8.7).

Case 2. $a + b - 1 = 0$.

In this case, which also implies that $a' + b' = 1$, we have

$$t' = \alpha^3 t + \gamma, \quad x' = \alpha x + \delta, \quad u' = \beta u \tag{8.9}$$

where

$$\epsilon\beta^{1-m} = \alpha^2\epsilon'.$$

As in the previous case we have $\epsilon\epsilon' > 0$ or $\epsilon = \epsilon' = 0$. Hence, in the case $\epsilon = \epsilon' = \pm 1$ equation (8.1) admits a three-parameter equivalence group

$$t' = \alpha^3 t + \gamma, \quad x' = \alpha x + \delta, \quad u' = \alpha^{\frac{2}{(1-m)}}u \tag{8.10}$$

while in the case $\epsilon = \epsilon' = 0$, it admits the four-parameter equivalence group (8.9).

Finally, if $m = 1$ which also implies that $m' = 1$, equation (8.1) admits the four-parameter equivalence group

$$t' = \alpha^3 t + \gamma, \quad x' = \alpha x + (\alpha^3\epsilon' - \alpha\epsilon)t + \delta, \quad u' = \beta u.$$

Clearly, as in the previous case, if $m = 1$ equation (8.1) can be mapped into the simpler equation (8.7).

8.3 Lie Symmetries

Equation (8.1) admits Lie point symmetries if and only if

$$\Gamma^{(3)}[u_t + u^{a+b-1}u_{xxx} + (a+3b-3)u^{a+b-2}u_x u_{xx} + (b-1)(a+b-2)u^{a+b-3}u_x^3 + \epsilon m u^{m-1}u_x] = 0 \tag{8.11}$$

identically, modulo equation (8.1).

After elimination of u_t due to (8.1), equation (8.11) becomes a multivariable polynomial in the variables u_x, u_{xx}, u_{tx}, u_{xxx} and u_{txx}. The coefficients of the different powers of these variables must be zero, giving the determining equations on the coefficients τ, ξ and η. Since (8.1) has a specific form (it is a quasilinear evolution equation, the right-hand side of (8.1) is a polynomial in the pure derivatives of u with respect to x), the forms of the coefficients can be simplified. That is, $\tau = \tau(t)$ and $\xi = \xi(t, x)$ [34].

The coefficient of u_{xxx} in identity (8.11) gives

$$(a + b - 1)\eta + (\tau_t - 3\xi_x)u = 0$$

which implies that the analysis needs to be split into two cases:

(1) $a + b - 1 \neq 0$ and

(2) $a + b - 1 = 0$.

Case 1. $a + b - 1 \neq 0$.

If $a + b - 1 \neq 0$, we have

$$\eta = \frac{(3\xi_x - \tau_t)u}{a + b - 1}$$

and the coefficients of u_{xx}, u_x^2, u_x and the term independent of derivatives in (8.11) produce the following overdetermined system

$$(2b + 1)\xi_{xx} = 0, \tag{8.12}$$

$$(a^2 - 6b^2 - 5ab - a + 3b + 3)\xi_{xx} = 0, \tag{8.13}$$

$$\epsilon m \left[(a + b - m)\tau_t - (a + b - 3m + 2)\xi_x\right] u^m + (2a + 8b + 1)\xi_{xxx} u^{a+b}$$

$$-(a + b - 1)\xi_t u = 0, \tag{8.14}$$

$$3\epsilon m \xi_{xx} u^m + 3\xi_{xxxx} u^{a+b} - (\tau_{tt} - 3\xi_{tx})u = 0. \tag{8.15}$$

This system (8.12)–(8.15) provides the forms of $\tau(t)$ and $\xi(t, x)$ and consequently the desired Lie symmetries are obtained. They are tabulated as Cases 1–3 and 6–11 in the Table 8.1.

Case 2. $a + b - 1 = 0$.

In this case, we have

$$\xi = \frac{1}{3}\tau_t x + L(t)$$

and the coefficients of $u_x u_{xx},\ u_{xx},\ u_x^3,\ u_x^2,\ u_x$ and the term independent of derivatives in (8.11) produce the following overdetermined system

$$3\eta_{uu}u^2 - 2a\eta_u u + 2a\eta = 0, \tag{8.16}$$

$$3\eta_{xu}u - 2a\eta_x = 0, \tag{8.17}$$

$$\eta_{uuu}u^3 - 2a\eta_{uu}u^2 + 2a\eta_u u - 2a\eta = 0, \tag{8.18}$$

$$3\eta_{xuu}u^2 - 4a\eta_{xu}u + 3a\eta_x = 0, \tag{8.19}$$

$$2\epsilon m\tau_t u^{m+1} + 3\epsilon m(m-1)\eta u^m + (9\eta_{xxu} - \tau_{tt}x - 3L_t)u^2 - 6a\eta_{xx}u = 0, \tag{8.20}$$

$$\epsilon m\eta_x u^m + (\eta_t + \eta_{xxx})u = 0. \tag{8.21}$$

The solution of the above system (8.16)–(8.21) leads to the Lie symmetries which are tabulated as Cases 4, 5 and 12–15 in the Table 8.1.

Table 8.1: Classification of equation (8.1)

Cases	a	b	m	Conditions	Basis of $A^{\max}$
1	$\forall$	$\forall$	$\forall$		$A^{\ker} = \langle \partial_t,\, \partial_x,\, (a+b-3m+2)t\partial_t + (a+b-m)x\partial_x + 2u\partial_u \rangle$
2	$\forall$	$\forall$	0		$A^{\ker} \oplus \langle 3t\partial_t + x\partial_x \rangle$
3	$\forall$	$\forall$	1		$A^{\ker} \oplus \langle 3t\partial_t + (x+2\epsilon t)\partial_x \rangle$
4	0	1	2		$A^{\ker} \oplus \langle 2\epsilon t\partial_x + \partial_u \rangle$
5	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{2}$		$A^{\ker} \oplus \langle 3\epsilon t\partial_x + 4\sqrt{u}\partial_u \rangle$
6	0	$-\frac{1}{2}$	0		$A^{\ker} \oplus \left\langle 3t\partial_t + x\partial_x,\, x^2\partial_x - 4xu\partial_u \right\rangle$
7	$-\frac{3}{2}$	$-\frac{1}{2}$	0		$A^{\ker} \oplus \left\langle 3t\partial_t + x\partial_x,\, x^2\partial_x - 2xu\partial_u \right\rangle$
8	0	$-\frac{1}{2}$	1		$A^{\ker} \oplus \left\langle 3t\partial_t + (x+2\epsilon t)\partial_x,\, (x-\epsilon t)^2\partial_x - 4(x-\epsilon t)u\partial_u \right\rangle$
9	$-\frac{3}{2}$	$-\frac{1}{2}$	1		$A^{\ker} \oplus \left\langle 3t\partial_t + (x+2\epsilon t)\partial_x,\, (x-\epsilon t)^2\partial_x - 2(x-\epsilon t)u\partial_u \right\rangle$
10	0	$-\frac{1}{2}$	$-\frac{1}{2}$		$A^{\ker}\oplus$
				$\epsilon = 1$	$\left\langle \sqrt{2}e^{\frac{x}{\sqrt{2}}}\partial_x - 2ue^{\frac{x}{\sqrt{2}}}\partial_u,\, \sqrt{2}e^{-\frac{x}{\sqrt{2}}}\partial_x + 2ue^{-\frac{x}{\sqrt{2}}}\partial_u \right\rangle$
				$\epsilon = -1$	$\left\langle \sqrt{2}\sin\left(\frac{x}{\sqrt{2}}\right)\partial_x - 2u\cos\left(\frac{x}{\sqrt{2}}\right)\partial_u,\, \sqrt{2}\cos\left(\frac{x}{\sqrt{2}}\right)\partial_x + 2u\sin\left(\frac{x}{\sqrt{2}}\right)\partial_u \right\rangle$
11	$-\frac{3}{2}$	$-\frac{1}{2}$	-2		$A^{\ker}\oplus$
				$\epsilon = 1$	$\left\langle \sqrt{2}e^{\sqrt{2}x}\partial_x - 2ue^{\sqrt{2}x}\partial_u,\, \sqrt{2}e^{-\sqrt{2}x}\partial_x + 2ue^{-\sqrt{2}x}\partial_u \right\rangle$
				$\epsilon = -1$	$\langle \sqrt{2}\sin(\sqrt{2}x)\partial_x - 2u\cos(\sqrt{2}x)\partial_u,\, \sqrt{2}\cos(\sqrt{2}x)\partial_x + 2u\sin(\sqrt{2}x)\partial_u \rangle$
12	0	1	0		$A^{\ker} \oplus \langle 3t\partial_t + x\partial_x,\, \psi(t,x)\partial_u \rangle$
13	$\frac{3}{4}$	$\frac{1}{4}$	0		$A^{\ker} \oplus \langle 3t\partial_t + x\partial_x,\, \sqrt{u}\psi(t,x)\partial_u \rangle$
14	0	1	1		$A^{\ker} \oplus \langle 3t\partial_t + (x+2\epsilon t)\partial_x,\, \phi(t,x)\partial_u \rangle$
15	$\frac{3}{4}$	$\frac{1}{4}$	1		$A^{\ker} \oplus \langle 3t\partial_t + (x+2\epsilon t)\partial_x,\, \sqrt{u}\phi(t,x)\partial_u \rangle$

The function $\phi(t,x)$ is a solution of $\phi_t + \phi_{xxx} + \epsilon\phi_x = 0$ and the function $\psi(t,x)$ is a solution of $\psi_t + \psi_{xxx} = 0$.

Now as we have seen in the previous section, if $m = 1$, then equation (8.1) is mapped into (8.7). Hence in the above Table Cases 3, 8, 9, 14 and 15 are equivalent to Cases 2, 6, 7, 12 and 13, respectively. Furthermore we have additional equivalence transformations that connect the following cases:

$$
\begin{aligned}
\tilde{4} \mapsto 5 \quad &: \quad \tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = \frac{3\epsilon}{4\epsilon'}\sqrt{u} \\
\tilde{6} \mapsto 7 \quad &: \quad \tilde{t} = t, \quad \tilde{x} = \frac{1}{x}, \quad \tilde{u} = x^4 u \\
\tilde{6} \mapsto 10_{\epsilon=1} \quad &: \quad \tilde{t} = t, \quad \tilde{x} = \int \frac{\mathrm{dx}}{\alpha^2 e^{x/\sqrt{2}} - 2\sqrt{2}\alpha + 2e^{-x/\sqrt{2}}}, \quad \tilde{u} = \left(\alpha^2 e^{x/\sqrt{2}} - 2\sqrt{2}\alpha + 2e^{-x/\sqrt{2}}\right)^2 u \\
\tilde{6} \mapsto 10_{\epsilon=-1} \quad &: \quad \tilde{t} = t, \quad \tilde{x} = \int \frac{\mathrm{dx}}{\left(2\sqrt{2}\sin\frac{x}{2\sqrt{2}} + \alpha\cos\frac{x}{2\sqrt{2}}\right)^2}, \quad \tilde{u} = \left(2\sqrt{2}\sin\frac{x}{2\sqrt{2}} + \alpha\cos\frac{x}{2\sqrt{2}}\right)^4 u \\
\tilde{6} \mapsto 11_{\epsilon=1} \quad &: \quad \tilde{t} = t, \quad \tilde{x} = \int \frac{2\mathrm{dx}}{e^{-\sqrt{2}x} - 2\alpha + 2(\alpha^2+1)e^{\sqrt{2}x}}, \quad \tilde{u} = \frac{1}{4}\left(e^{-\sqrt{2}x} - 2\alpha + 2(\alpha^2+1)e^{\sqrt{2}x}\right)^2 u^2 \\
\tilde{6} \mapsto 11_{\epsilon=-1} \quad &: \quad \tilde{t} = t, \quad \tilde{x} = \int \frac{\mathrm{dx}}{\left(\sqrt{2}\sin\frac{x}{\sqrt{2}} + \alpha\cos\frac{x}{\sqrt{2}}\right)^2}, \quad \tilde{u} = \left(\sqrt{2}\sin\frac{x}{\sqrt{2}} + \alpha\cos\frac{x}{\sqrt{2}}\right)^4 u^2 \\
\tilde{12} \mapsto 13 \quad &: \quad \tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = \sqrt{u}
\end{aligned}
$$

Therefore we can summarize the results of the symmetry classification in the following theorem:

Theorem 8.1. *Equation* (8.1) *admits*

(1) a three-parameter Lie group if a, b *and* m *are arbitrary;*

(2) a four-parameter Lie group if (a) a, b *are arbitrary and* $m = 0$ *and if (b)* $a = 0$, $b = 1$ *and* $m = 2$;

(3) a five-parameter Lie group if $a = 0$, $b = -\frac{1}{2}$ *and* $m = 0$;

(4) an infinite-dimensional Lie group if $a = 0$, $b = 1$ *and* $m = 0$.

Any other member of the class (8.1) *is equivalent to the above five cases.*

In the case $m = 0$ which is equation (8.7), if we set $n = a + b - 1$ and $k = b - 1$, we obtain the class of equations (5.1)

$$u_t + \left[u^n u_{xx} + k u^{n-1} u_x^2\right]_x = 0$$

which was studied in Chapter 5. The results on Lie symmetries presented in Theorem 8.1 in the case $m = 0$ agree with those in Chapter 5.

8.3.1 Invariant Solutions

We give the optimal system which, consists a list of inequivalent subalgebras for the three cases of Theorem 8.1. and we give some examples of reduced ODEs.

(1) Here we have three Lie symmetries, $\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = (a+b-3m+2)t\partial_t + (a+b-m)x\partial_x + 2u\partial_u$, that produce an optimal system which depends upon the values of the parameters $a,\ b$ and m. We get four subcases.

(i) $a+b-3m+2 \neq 0, \quad a+b-m \neq 0$

$$\langle\Gamma_3\rangle, \quad \langle\Gamma_1 + c\Gamma_2\rangle, \quad \langle\Gamma_2\rangle,$$

where $c = 0, \pm 1$. For each component of the optimal system we construct the corresponding similarity reduction that transforms (8.1) into an ODE. We obtain the following results:

$$\begin{aligned}
\langle\Gamma_2\rangle \quad &: \quad u = \phi(\omega), \quad \omega = t, \\
\langle\Gamma_1 + c\Gamma_2\rangle \quad &: \quad u = \phi(\omega), \quad \omega = x - ct, \\
\langle\Gamma_3\rangle \quad &: \quad u = t^{\frac{2}{a+b-3m+2}}\phi(\omega), \quad \omega = xt^{-\frac{a+b-m}{a+b+2-3m}}.
\end{aligned}$$

The reduction that corresponds to the subalgebra $\langle\Gamma_1 + c\Gamma_2\rangle$ leads to the equation

$$c\phi_\omega - [\phi^{a+b-2}(\phi\phi_{\omega\omega} + (b-1)\phi_\omega^2) + \epsilon\phi^m]_\omega = 0$$

which provides traveling-wave solutions for equation (8.1).

(ii) $a+b-3m+2 \neq 0, \quad a+b-m = 0$

$$\langle\Gamma_3 + \alpha\Gamma_2\rangle, \quad \langle\Gamma_1 + c\Gamma_2\rangle, \quad \langle\Gamma_2\rangle,$$

where $c = 0, \pm 1$ and $\alpha \in \mathbb{R}$. The subalgebra $\langle\Gamma_3 + \alpha\Gamma_2\rangle$ produces the reduction

$$u = t^{1/(1-m)}\phi(\omega), \quad \omega = t^{\frac{\alpha}{2(m-1)}}e^x.$$

(iii) $a+b-3m+2 = 0, \quad a+b-m \neq 0$

$$\langle\Gamma_3 + \alpha\Gamma_1\rangle, \quad \langle\Gamma_1 + c\Gamma_2\rangle, \quad \langle\Gamma_2\rangle,$$

where $c = 0, \pm 1$ and $\alpha \in \mathbb{R}$. The subalgebra $\langle\Gamma_3 + \alpha\Gamma_1\rangle$ produces the reduction

$$u = x^{1/(m-1)}\phi(\omega), \quad \omega = x^{\frac{\alpha}{2(1-m)}}e^t.$$

(iv) $a+b-3m+2=0, \quad a+b-m=0 \quad \Rightarrow \quad m=1, \quad a+b=1,$

$$\langle \Gamma_3+\alpha\Gamma_2+\beta\Gamma_1\rangle, \quad \langle \Gamma_1+\gamma\Gamma_2\rangle, \quad \langle \Gamma_2\rangle,$$

where $\alpha,\beta,\gamma\in\mathbb{R}$. The subalgebra $\langle \Gamma_3+\alpha\Gamma_2+\beta\Gamma_1\rangle$ produces the reduction

$$u=e^{\frac{t}{\beta}}\phi(\omega), \quad \omega=x-\frac{\alpha}{\beta}t, \quad \text{if } \beta\neq 0,$$
$$u=e^{\frac{x}{\alpha}}\phi(\omega), \quad \omega=t, \quad \text{if } \beta=0.$$

In the case $\beta\neq 0$ the subalgebra $\langle \Gamma_3+\alpha\Gamma_2+\beta\Gamma_1\rangle$ leads to the equation

$$\alpha\phi_\omega-\phi-\beta[\phi_{\omega\omega}+(b-1)\phi^{-1}\phi_\omega^2+\epsilon\phi]_\omega=0$$

and in the case $\beta=0$, we obtain the solution

$$u=c_1\exp\left[\frac{1}{\alpha^3}\left(\alpha^2x-(\epsilon\alpha^2+b)t\right)\right].$$

(2a) Here we have four Lie symmetries, $\Gamma_1,\ \Gamma_2,\Gamma_3$ $(m=0)$ and $\Gamma_4=3t\partial_t+x\partial_x$, which in addition to the subcases 1(i)-1(iii) produce the reduction that corresponds to the subalgebra

$$\langle \Gamma_4+c\Gamma_3\rangle,$$

where $c=0,\pm1$. This subalgebra gives

$$u=t^{\frac{2c}{3+(a+b+2)c}}\phi(\omega), \quad \omega=xt^{-\frac{1+(a+b)c}{3+(a+b+2)c}}, \quad \text{if } 3+(a+b+2)c\neq 0,$$
$$u=x^{-c/(c+1)}\phi(\omega), \quad \omega=t, \quad \text{if } 3+(a+b+2)c=0.$$

In the case $3+(a+b+2)c=0$, we obtain the solution

$$u=x^{-c/(c+1)}\left[\frac{3(a+b+2)(1+2b-a)t}{(a+b-1)^2}+c_1\right]^{-1/(a+b-1)}.$$

(2b) Here we have four Lie symmetries, $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ $(a=0,\ b=1,\ m=2)$ and $\Gamma_4=2\epsilon t\partial_x+\partial_u$, which in addition to the subcase 1(i) produce the reduction that corresponds to the subalgebra

$$\langle \Gamma_4+c\Gamma_1\rangle,$$

where $c = 0, \pm 1$. We obtain

$$u = \frac{t}{c} + \phi(\omega), \quad \omega = x - \frac{\epsilon}{c}t^2, \quad \text{if} \;\; c \neq 0,$$
$$u = \frac{x}{2\epsilon t} + \phi(\omega), \quad \omega = t, \quad \text{if} \;\; c = 0.$$

In the case $c \neq 0$ we obtain

$$\phi_{\omega\omega\omega} + 2\epsilon\phi\phi_\omega + \frac{1}{c} = 0.$$

We integrate this equation and the integral has the form

$$\phi_{\omega\omega} + \epsilon\phi^2 + \frac{\omega}{c} = c_1, \text{ where } c_1 \;\text{ is the constant of integration.}$$

Subcase of the above equation, is the First Painlevé transcendent with the form

$$\phi_{\omega\omega} = 6\phi^2 + \omega.$$

Finally, in the case $c = 0$, we obtain the solution

$$u = \frac{x + 2\epsilon c_1}{2\epsilon t}.$$

(3) Here we have five Lie symmetries, $\Gamma_1,\ \Gamma_2,\ \Gamma_3$ $(a = 0,\ b = -\frac{1}{2},\ m = 0)$, $\Gamma_4 = 3t\partial_t + x\partial_x$ and $\Gamma_5 = x^2\partial_x - 4xu\partial_u$, which in addition to the subcases 1(i) and (2a) produce the reduction that corresponds to the subalgebra

$$\langle \Gamma_5 + \alpha\Gamma_2 + \beta\Gamma_4 \rangle,$$

where $\alpha, \beta \in \mathbb{R}$. The reductions in this subcase can be found in Chapter 5.

8.4 Nonclassical Symmetries

Here we require invariance of equation (8.1) in conjunction with its invariant surface condition,

$$u_t = \eta(t, x, u) - \xi(t, x, u)u_x$$

under the infinitesimal transformation generated by

$$\Gamma = \partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u.$$

The nonclassical conditions results in an overdetermined nonlinear system of PDEs for finding the forms of the coefficient functions ξ and η.

Coefficient of u_{xx}^2 gives $\xi_u = 0$. Coefficients of $u_x u_{xx}$, u_{xx}, u_x^3, u_x^2, u_x and the term independent of derivatives give, respectively,

$$3\eta_{uu}u^2 + (a+3b-3)\eta_u u - (a+3b-3)\eta = 0, \tag{8.22}$$

$$3(\xi_{xx} - \eta_{xu})u - (a+3b-3)\eta_x = 0, \tag{8.23}$$

$$\begin{aligned}&\eta_{uuu}u^3 + (a+3b-3)\eta_{uu}u^2 + 2(ab-a+b^2-3b+2)\eta_u u \\ &-2(ab-a+b^2-3b+2)\eta = 0,\end{aligned} \tag{8.24}$$

$$3\eta_{xuu}u^2 - (a+3b-3)(\xi_{xx} - 2\eta_{xu})u + 3(ab-a+b^2-3b+2)\eta_x = 0, \tag{8.25}$$

$$\begin{aligned}&2\epsilon m\xi_x u^{m+1} - \epsilon m(a+b-m)\eta u^m - (\xi_{xxx} - 3\eta_{xxu})u^{a+b+1} + (a+3b-3)\eta_{xx}u^{a+b} \\ &-(\xi_t + 3\xi\xi_x)u^2 + (a+b-1)\xi\eta u = 0,\end{aligned} \tag{8.26}$$

$$\epsilon m u^m \eta_x + u^{a+b}\eta_{xxx} + (3\xi_x\eta + \eta_t)u - (a+b-1)\eta^2 = 0. \tag{8.27}$$

This overdetermined nonlinear system (8.22)–(8.27) is solved to give the desired nonclassical reductions. It turns out that such reductions exist in two cases, $m = 1$ and $m \neq 1$. We have seen that, if $m = 1$, when we use the equivalence transformations equation (8.1) can be mapped into (8.7) which is equivalent with (5.1). Nonclassical reductions for equation (5.1) can be found in Chapter 5.

Solution of the above system, when $m \neq 1$, provides a nonclassical symmetry for two special cases of equation (8.1). In particular, if $(a, b, m) = (0, \frac{1}{2}, \frac{1}{2})$, it admits the reduction operator:

$$\Gamma_1 = \partial_t + \phi(x)\sqrt{u}\partial_u,$$

where $\phi(x)$ is a solution of the ODE

$$2\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3} + \epsilon\frac{\mathrm{d}\phi}{\mathrm{d}x} + \phi^2 = 0$$

and, if $(a, b, m) = (\frac{1}{2}, \frac{1}{6}, \frac{2}{3})$, it admits the reduction operator:

$$\Gamma_2 = \partial_t + \psi(x)u^{2/3}\partial_u,$$

where $\psi(x)$ is a solution of the ODE

$$3\frac{\mathrm{d}^3\psi}{\mathrm{d}x^3} + 2\epsilon\frac{\mathrm{d}\psi}{\mathrm{d}x} + \psi^2 = 0.$$

The operator Γ_1 leads to the ansatz (4.35)

$$u = \left[\tfrac{1}{2}\phi(x)t + F(x)\right]^2$$

which reduces (8.1) with $(a, b, m) = (0, \frac{1}{2}, \frac{1}{2})$ to the ODE

$$\frac{\mathrm{d}^3F}{\mathrm{d}x^3} + \frac{1}{2}\left(\phi F + \epsilon\frac{\mathrm{d}F}{\mathrm{d}x}\right) = 0.$$

Here we can interpret (4.35) as an ansatz with the two new unknown functions ϕ and F which reduces equation

$$u_t + \epsilon\left(\sqrt{u}\right)_x + 2\left(\sqrt{u}\right)_{xxx} = 0 \tag{8.28}$$

to a system of two ODEs:

$$2\frac{\mathrm{d}^3\phi}{\mathrm{d}x^3} + \epsilon\frac{\mathrm{d}\phi}{\mathrm{d}x} + \phi^2 = 0 \quad \text{and} \quad \frac{\mathrm{d}^3F}{\mathrm{d}x^3} + \frac{1}{2}\left(\phi F + \epsilon\frac{\mathrm{d}F}{\mathrm{d}x}\right) = 0.$$

Similarly, Γ_2 produces the ansatz (5.29)

$$u = \left[\tfrac{1}{3}\psi(x)t + F(x)\right]^3$$

which reduces (8.1) with $(a, b, m) = (\frac{1}{2}, \frac{1}{6}, \frac{2}{3})$ to the ODE

$$\frac{\mathrm{d}^3F}{\mathrm{d}x^3} + \frac{1}{3}\left(\psi F + 2\epsilon\frac{\mathrm{d}F}{\mathrm{d}x}\right) = 0$$

which can also be interpreted as a mapping that reduces the PDE into a system of two ODEs.

8.5 Form-preserving Transformations

Probably the most useful point transformations of PDEs are those which form a continuous (Lie) group of transformations, each member of which leaves an equation invariant. As we have seen in order to achieve this goal, we employ the classical method where we find these transformations in the infinitesimal form. Then, we can extend these to groups of finite transformations. However, this method may well overlook discrete symmetries such as simple reflection or hodograph transformations. Also infinitesimal transformations are not appropriate for directly linking a PDE with an equation of a different form. Therefore there is merit in studying point transformations directly in finite form with the ultimate dual goals of finding the complete set of point transformation symmetries of PDEs and discovering new links between different equations.

Here we present point transformations of the class (8.3) that connect equations (8.1) and (8.4). Since both equations (8.1) and (8.4) are polynomials in the derivatives in the spatial variable, it can be shown that the most general class of point transformations that connects them has the form [34]

$$t' = Q(t), \quad x' = P(x,t), \quad u' = R(x,t,u). \tag{8.29}$$

Such transformations are the equivalence transformations and the additional equivalence transformations derived in Section 8.3. In this Section we complete list of form-preserving transformations. Details of how such transformations are derived can be found in [34,71–73].

The analysis is split into two main cases:

(1) $a+b \neq 1$ and

(2) $a+b=1$.

Case 1. $a+b-1 \neq 0$.

Equation (8.1) is connected with (8.4) under the mapping

$$t' = \beta t + \gamma, \quad x' = \alpha x + \delta, \quad u' = \alpha^{\frac{3}{a+b-1}} \beta^{-\frac{1}{a+b-1}} u^{\frac{a+b-1}{a'+b'-1}} \tag{8.30}$$

where $\alpha\beta \neq 0$ and

$$a' = \frac{a-3b}{2(a-b-1)}, \quad b' = -\frac{a+b}{2(a-b-1)}. \tag{8.31}$$

Furthermore if $mm' \neq 0$, the following identities must hold

$$m'\epsilon'\beta^{\frac{a'+b'+m'}{a'+b'-1}} = m\epsilon\alpha^{\frac{a'+b'+3m'+2}{a'+b'-1}}, \quad (m-1)(a'+b'-1) = (m'-1)(a+b-1).$$

We also note that $a'+b' \neq a+b$, otherwise we obtain the results derived earlier. An example of such mapping is the following

$$t' = t, \quad x' = x, \quad u' = u^{3/2}$$

that connects equation (8.28) and equation

$$u'_{t'} + \frac{3}{4}\epsilon\left(u'^{2/3}\right)_{x'} + 6\left[u'^{1/2}\left(u'^{1/6}\right)_{x'x'}\right]_{x'} = 0.$$

As we have seen in the previous Section, both of these latter equations admit nonclassical symmetries. A second example is the mapping

$$t' = t, \quad x' = x, \quad u' = u^3$$

that connects equation

$$u_t + \left(u^2\right)_x + \frac{1}{2}\left(u^2\right)_{xxx} = 0$$

which was studied in [58] and equation

$$u'_{t'} + \frac{3}{2}\left(u'^{4/3}\right)_{x'} + 3\left[u'\left(u'^{1/3}\right)_{x'x'}\right]_{x'} = 0.$$

In the special case $m' = m = 1$, we obtain the transformation

$$t' = \beta t + \gamma, \quad x' = \alpha x + (\beta\epsilon' - \alpha\epsilon)t + \delta, \quad u' = \alpha^{\frac{3}{a+b-1}}\beta^{-\frac{1}{a+b-1}}u^{\frac{a+b-1}{a'+b'-1}},$$

where $\alpha\beta \neq 0$ and the relations (8.31) hold.

Case 2. $a + b - 1 = 0$.

This case, also implies that $a' + b' = 1$. The form preserving transformation

$$t' = \alpha^3 t + \gamma, \quad x' = \alpha x + \delta, \quad u' = \kappa u^{2b} \tag{8.32}$$

connects equations (8.1) and (8.4), where $\alpha\kappa \neq 0$ and

$$b' = \frac{1}{4b}.$$

Furthermore if $mm' \neq 0$, the following identities must hold

$$m = 2bm' + 1 - 2b, \quad \alpha^2\epsilon' m' - (2\epsilon m' b - 2\epsilon b + \epsilon)\kappa^{1-m'} = 0.$$

Finally, in the special case $m' = m = 1$, we obtain the transformation

$$t' = \alpha^3 t + \gamma, \quad x' = \alpha x + (\alpha^3\epsilon' - \alpha\epsilon)t + \delta, \quad u' = \kappa u^{2b},$$

where $\alpha\kappa \neq 0$ and the parameters b' and b are related as above.

8.6 Potential Symmetries

In this case we consider the potential system,

$$v_x = u, \tag{8.33}$$

$$v_t = -\epsilon u^m - \frac{1}{b}\left[u^a\left(u^b\right)_{xx}\right],$$

which it admits Lie symmetries if and only if

$$\Gamma^{(1)}\left[v_x - u\right] = 0, \tag{8.34}$$

$$\Gamma^{(2)}\left[v_t + \epsilon u^m + \frac{1}{b}\left[u^a\left(u^b\right)_{xx}\right]\right] = 0 \tag{8.35}$$

for $v_x = u$ and $v_t = -\left[\epsilon u^m + u^{a+b-1}u_{xx} + (b-1)u^{a+b-2}u_x^2\right]$. We recall that $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are the first and second extensions of the generator

$$\Gamma = \tau(t,x,u,v)\partial_t + \xi(t,x,u,v)\partial_x + \eta(t,x,u,v)\partial_u + \zeta(t,x,u,v)\partial_v.$$

From the coefficients of $u_x u_{xx}$, u_{xx} and u_x of (8.34) and from the coefficient $u_x u_{xx}$ of (8.35) we get that the coefficient τ is a function of t and the coefficients ξ and ζ are functions of t, x and v. From equation (8.34) we have that

$$\eta = -\xi_v u^2 - (\xi_x - \zeta_v)u + \zeta_x.$$

After we have used the above results, we have that functions τ, ξ, η and ζ are satisfying the following determining system

$$(a+b+2)\xi_v u^2 - \left[\tau_t - (a+b+2)\xi_x + (a+b-1)\zeta_v\right]u - (a+b-1)\zeta_x = 0, \tag{8.36}$$

$$(b^2+ab+2b-a)\xi_v u^2 - (b-1)\left[\tau_t - (a+b+2)\xi_x + (a+b-1)\zeta_v\right]u$$

$$-(b-1)(a+b-2)\zeta_x = 0, \tag{8.37}$$

$$2(b+2)\xi_{vv}u^3 + \left[(4b+5)\xi_{xv} - (2b+1)\zeta_{vv}\right]u^2 + \left[(2b+1)\xi_{xx} - (4b-1)\zeta_{xv}\right]u \tag{8.38}$$

$$-2(b-1)\zeta_{xx} = 0, \tag{8.39}$$

$$\epsilon(m-1)\xi_v u^{m+2} - \epsilon\left[\tau_t - m\xi_x + (m-1)\zeta_v\right]u^{m+1} - \epsilon m\zeta_x u^m$$

$$\xi_{vvv}u^{a+b+4} + (3\xi_{xvv} - \zeta_{vvv})u^{a+b+3} + 3(\xi_{xxv} - \zeta_{xvv})u^{a+b+2} + (\xi_{xxx} - 3\zeta_{xxv})u^{a+b+1}$$

$$-\zeta_{xxx}u^{a+b} + \xi_t u^2 - \zeta_t u = 0. \tag{8.40}$$

Equation (8.40) can break up into more equations in proportion the values of the parameters $a,\ b$ and m.

From the coefficient of u^2 in equation (8.36), we get two cases: $\xi_v = 0$ or $a+b+2=0$. After we have solved the determining system (8.36)–(8.40) we observe that for the case that $\xi_v = 0$ and $a+b+2=0$ we do not find potential symmetries. The system (8.33) admits Lie symmetries which induce potential symmetries for the corresponding equation (8.1) in two cases:

(1) $a+b+2=0$ and $\xi_v \neq 0$,

(2) $a+b+2 \neq 0$ and $\xi_v = 0$.

We present only the potential symmetries when $m \neq 1$ because, as we have seen in Section 8.2, if $m=1$ and we use the equivalence transformations, equation (8.1) can be mapped into (8.7) which is equivalent to (5.1). Potential symmetries for equation (5.1) can be found in Chapter 5. We obtain that the following Lie symmetries of the system (8.33) induce potential symmetries for the corresponding equation (8.1):

(1) $(a,b,m) = (0,-2,-1)$

$$\Gamma = v\partial_x - u^2\partial_u - 2\epsilon t\partial_v.$$

(2) $(a,b,m) = (\frac{3}{2}, -\frac{1}{2}, 3)$

(i) $\epsilon > 0$

$$\Gamma_1 = \sqrt{2\epsilon}u\cos(\sqrt{2\epsilon}v)\partial_u + \sin(\sqrt{2\epsilon}v)\partial_v,$$

$$\Gamma_2 = -\sqrt{2\epsilon}u\sin(\sqrt{2\epsilon}v)\partial_u + \cos(\sqrt{2\epsilon}v)\partial_v.$$

(ii) $\epsilon < 0$

$$\Gamma_1 = \sqrt{2|\epsilon|}ue^{\sqrt{2|\epsilon|}v}\partial_u + e^{\sqrt{2|\epsilon|}v}\partial_v,$$

$$\Gamma_2 = -\sqrt{2|\epsilon|}ue^{-\sqrt{2|\epsilon|}v}\partial_u + e^{-\sqrt{2|\epsilon|}v}\partial_v.$$

8.6.1 Further Potential Symmetries

Equation (8.1) can be written in other conserved forms when the parameters n, a and b satisfy certain relations. For example, if $a \neq b+1,\ a \neq b$ and $m \neq a-b$, the auxiliary system takes the form

$$
\begin{aligned}
&v_x = u^{b-a+1}, \\
&v_t = (a-b-1)\left[u^{2b-1}u_{xx} + \frac{1}{2}(a+b-2)u^{2b-2}u_x^2 - \frac{\epsilon m}{a-b-m}u^{b-a+m}\right]
\end{aligned}
\tag{8.41}
$$

for which Lie symmetries induce potential symmetries of (8.1) in two cases. The first case is when $(a,b,m) = (3,1,3)$ the system admits the following Lie symmetry which is a potential symmetry of (8.1)

$$\Gamma = v\partial_x + \partial_u + 6\epsilon t\partial_v.$$

The second case that the system (8.41) produces potential symmetries is when $(a,b,m) = (\frac{3}{2}, -\frac{1}{2}, -1)$ and the symmetries have the form

(i) $\epsilon > 0$

$$
\begin{aligned}
&\Gamma_1 = -\frac{2\epsilon u}{3}e^{\sqrt{\frac{2\epsilon}{3}}v}\partial_u + \sqrt{\frac{2\epsilon}{3}}e^{\sqrt{\frac{2\epsilon}{3}}v}\partial_v, \\
&\Gamma_2 = \frac{2\epsilon u}{3}e^{-\sqrt{\frac{2\epsilon}{3}}v}\partial_u + \sqrt{\frac{2\epsilon}{3}}e^{-\sqrt{\frac{2\epsilon}{3}}v}\partial_v.
\end{aligned}
$$

(ii) $\epsilon < 0$

$$
\begin{aligned}
&\Gamma_1 = -\frac{2|\epsilon|u}{3}\cos\left(\sqrt{\frac{2|\epsilon|}{3}}v\right)\partial_u + \sqrt{\frac{2|\epsilon|}{3}}\sin\left(\sqrt{\frac{2|\epsilon|}{3}}v\right)\partial_v, \\
&\Gamma_2 = \frac{2|\epsilon|u}{3}\sin\left(\sqrt{\frac{2|\epsilon|}{3}}v\right)\partial_u + \sqrt{\frac{2|\epsilon|}{3}}\cos\left(\sqrt{\frac{2|\epsilon|}{3}}v\right)\partial_v.
\end{aligned}
$$

In the case for which $m = a-b,\ a \neq b+1$ and $a \neq b$ equation (8.1) can be written as a system of two equations

$$
\begin{aligned}
&v_x = u^{b-a+1}, \\
&v_t = (a-b-1)\left[u^{2b-1}u_{xx} + \frac{1}{2}(a+b-2)u^{2b-2}u_x^2 + \epsilon(a-b)\ln u\right].
\end{aligned}
$$

If $a = b$ and $m \neq 0$, then equation (8.1) admits the conservation law

$$
\begin{aligned}
&v_x = u\ln u, \\
&v_t = -u^{2b-1}(1+\ln u)u_{xx} + \frac{1}{2}u^{2b-2}\left[2(1-b)\ln u + 3 - 2b\right]u_x^2 + \frac{\epsilon}{m}(1-m-m\ln u)u^m.
\end{aligned}
$$

Finally, if $a = b+1$ and $m \neq 1$ we have the system

$$
\begin{aligned}
&v_x = \ln u, \\
&v_t = -u^{2b-1}u_{xx} + \frac{1}{2}(1-2b)u^{2b-2}u_x^2 - \frac{\epsilon m}{m-1}u^{m-1}.
\end{aligned}
$$

Lie symmetries of the above three systems lead only to Lie symmetries of (8.1).

8.7 Nonclassical Potential Symmetries

In this Section we search for nonclassical symmetries for the potential form of (8.1) which is given by

$$v_t = -\epsilon v_x^m - \frac{1}{b}\left[v_x^a\left(v_x^b\right)_{xx}\right]. \tag{8.42}$$

The invariance surface condition has the form

$$v_t = \zeta(t,x,v) - \xi(t,x,v)v_x$$

and the reduction operators have the form

$$\Gamma = \partial_t + \xi(t,x,v)\partial_x + \zeta(t,x,v)\partial_v.$$

The determining system for the determination of the coefficients ξ and ζ is

$$(b+2)\xi_v = 0, \tag{8.43}$$

$$(b-1)\zeta_x = 0, \tag{8.44}$$

$$(b+2)\xi_{vv} = 0, \tag{8.45}$$

$$(4b+5)\xi_{xv} - (2b+1)\zeta_{vv} = 0, \tag{8.46}$$

$$(2b+1)\xi_{xx} - (4b-1)\zeta_{xv} = 0, \tag{8.47}$$

$$(b-1)\zeta_{xx} = 0, \tag{8.48}$$

$$\begin{aligned}
&\epsilon(a+b-m+3)\xi_v v_x^{m+2} + \epsilon[(a+b-m+2)\xi_x - (a+b-m)\zeta_v]v_x^{m+1}\\
&-\epsilon(a+b-m-1)\zeta_x v_x^m - \xi_{vvv}v_x^{a+b+4} - (3\xi_{xvv} - \zeta_{vvv})v_x^{a+b+3}\\
&-3(\xi_{xxv} - \zeta_{xvv})v_x^{a+b+2} - (\xi_{xxx} - 3\zeta_{xxv})v_x^{a+b+1} + \zeta_{xxx}v_x^{a+b} - (a+b+2)\xi_v\xi v_x^3\\
&-[\xi_t + (a+b+2)(\xi_x\xi - \xi_v\zeta) - (a+b-1)\xi\zeta_v]v_x^2\\
&+[(a+b+2)\xi_x\zeta + \zeta_t + (a+b-1)(\xi\zeta_x - \zeta_v\zeta)]v_x - (a+b-1)\zeta\zeta_x = 0.
\end{aligned} \tag{8.49}$$

Depending upon the values of the parameters a, b and m, equation (8.49) is able to break up into more equations.

After the solution of the system (8.43)–(8.49), we deduce that equation (8.42) admits such symmetries in two special cases for which

(1) $a = -1$ and $b = 1$,

(2) $a = 1$ and $b = -2$.

(We do not present the results for $m = 1$.)

Case 1. $a = -1$ and $b = 1$.

For the case that $(a, b, m) = (-1, 1, -1)$ equation (8.1) admits nonclassical potential symmetries. In particular we find that equation (8.42) admits the reduction operator

$$\Gamma_1 = \partial_t + c\partial_x + \phi(\omega)\partial_v,$$

respectively, where $\omega = x + ct$.

Case 2. $a = 1$ and $b = -2$.

In this case equation (8.1) admits nonclassical potential symmetries for $m = 2$. Equation (8.42) admits the reduction

$$\Gamma_2 = \partial_t + \phi(\omega)\partial_x + c\partial_v,$$

respectively, where $\omega = v + ct$.

In both reductions the function $\phi(\omega)$ is a solution of the ODE

$$\frac{\mathrm{d}^3\phi}{\mathrm{d}\omega^3} + \phi\frac{\mathrm{d}\phi}{\mathrm{d}\omega} = 0.$$

We note that the corresponding potential equations to these two cases are connected by the pure hodograph transformation

$$x \mapsto v, \quad v \mapsto x.$$

Chapter 9

Lie Symmetry Classification for a $K(m, n)$ Equation with Variable Coefficients

9.1 Introduction

In this Chapter we present the Lie symmetry classification for the following generalised equation with variable coefficients

$$u_t + \epsilon(u^m)_x + f(t)\,(u^n)_{xxx} = 0, \tag{9.1}$$

where $f(t)$ is a function of t and ϵ is an arbitrary constant. If f is constant is known as $K(m, n)$ equation. A study of the generalised $K(n, n)$ equation with variable coefficients with the form

$$u_t + a(t)(u^n)_x + b(t)\,(u^n)_{xxx} = 0, \quad n \neq 0, 1,$$

for which $a(t)$ and $b(t)$ are functions of t, was done in [74]. Equation (9.1) is a subcase of this equation.

9.2 Lie Symmetries

Equation (9.1) admits Lie point symmetries if and only if

$$\Gamma^{(3)}[u_t + f\left(nu^{n-1}u_{xxx} + 3n(n-1)u^{n-2}u_xu_{xx} + (n^3 - 3n^2 + 2n)u^{n-3}u_x^3\right)$$

$$+\epsilon m u^{m-1} u_x] = 0 \tag{9.2}$$

for $u_t = -\left[f\left(n u^{n-1} u_{xxx} + 3n(n-1) u^{n-2} u_x u_{xx} + (n^3 - 3n^2 + 2n) u^{n-3} u_x^3\right) + \epsilon m u^{m-1} u_x\right]$.

After we have used the above expression we can eliminate u_t and equation (9.2) becomes an identity in the variables $u_x,\ u_{xx},\ u_{tx},\ u_{xxx}$ and u_{txx}. From coefficients of different powers of these variables, which must be equal to zero, we derive the determining equations on the coefficients τ, ξ and η. We use the general results again on point transformations between evolution equations [34] and the forms of the coefficients can be simplified, that is, $\tau = \tau(t)$ and $\xi = \xi(t,x)$.

From the coefficient of u_{xxx} we have that

$$\left[f_t \tau + f(\tau_t - 3\xi_x)\right] u + (n-1) f \eta = 0.$$

We deduce that the analysis needs to be split in two cases:

(1) $n \neq 1$ and

(2) $n = 1$.

Case 1. $n \neq 1$.

In this case the form of η is

$$\eta = -\frac{[f_t \tau + f(\tau_t - 3\xi_x)] u}{(n-1) f}$$

and the coefficients of u_{xx}, u_x^2, u_x and the term independent of derivatives in (9.2) produce the following determining equations, respectively,

$$n(2n+1) f \xi_{xx} = 0 \text{ (The coefficient of } u_{xx} \text{ is the same as the coefficient of } u_x^2\text{)}, \tag{9.3}$$

$$\begin{aligned}&\left[\epsilon m(m-n) f \tau_t - \epsilon m(3m-n-2) f \xi_x + \epsilon m(m-1) f_t \tau\right] u^m \\ &-8n(n+1) f^2 \xi_{xxx} u^n + (n-1) f \xi_t u = 0,\end{aligned} \tag{9.4}$$

$$3\epsilon m f^2 \xi_{xx} u^m + 3n f^3 \xi_{xxxx} u^n - \left[f^2 \tau_{tt} + f f_t \tau_t + f f_{tt} \tau - f_t^2 \tau - 3\xi_{tx} f^2\right] u = 0. \tag{9.5}$$

After we have solved the determining equations (9.3)–(9.5), we take the forms of $\tau(t)$, $\xi(t,x)$ and the function $f(t)$. The Lie symmetries according the form of $f(t)$ are tabulated in the Table 9.1.

Table 9.1: Classification of equation (9.1) ($n \neq 1$)

Cases	n	m	$f(t)$	Conditions	Basis of $A^{\max}$
$n \neq 1$					
1	$\forall$	$\forall$	$\forall$		∂_x
2	$3m-n-2=0$		$\forall$		$\partial_x,\ x\partial_x + \frac{3u}{n-1}\partial_u$
3	$\forall$	0	$\forall$		$\frac{1}{f}\partial_t,\ \partial_x,\ x\partial_x + \frac{3u}{n-1}\partial_u,\ 3\frac{\int f\mathrm{d}t}{f}\partial_t + x\partial_x$
4	$-\frac{1}{2}$	0	$\forall$		$\frac{1}{f}\partial_t,\ \partial_x,\ x\partial_x - 2u\partial_u,\ 3\frac{\int f\mathrm{d}t}{f}\partial_t + x\partial_x,$ $x^2\partial_x - 4xu\partial_u$
5	$\forall$	$\forall$	constant		$\partial_t,\ \partial_x,\ (3m-n-2)t\partial_t + (m-n)x\partial_x - 2u\partial_u$
6	$-\frac{1}{2}$	$-\frac{1}{2}$	constant		$\partial_t,\ \partial_x,\ 3t\partial_t + 2u\partial_u,$
				$\frac{f}{\epsilon} > 0$	$\sqrt{\frac{f}{\epsilon}}\sin\left(\sqrt{\frac{\epsilon}{f}}x\right)\partial_x - 2u\cos\left(\sqrt{\frac{\epsilon}{f}}x\right)\partial_u,$ $\sqrt{\frac{f}{\epsilon}}\cos\left(\sqrt{\frac{\epsilon}{f}}x\right)\partial_x + 2u\sin\left(\sqrt{\frac{\epsilon}{f}}x\right)\partial_u$
				$\frac{f}{\epsilon} < 0$	$\sqrt{\lvert\frac{f}{\epsilon}\rvert}e^{\sqrt{\lvert\frac{\epsilon}{f}\rvert}x}\partial_x - 2ue^{\sqrt{\lvert\frac{\epsilon}{f}\rvert}x}\partial_u,$ $\sqrt{\lvert\frac{f}{\epsilon}\rvert}e^{-\sqrt{\lvert\frac{\epsilon}{f}\rvert}x}\partial_x + 2ue^{-\sqrt{\lvert\frac{\epsilon}{f}\rvert}x}\partial_u$
7	$\forall$	$\forall$	t^k		$\partial_x,\ (3m-n-2)t\partial_t + (km-k+m-n)x\partial_x$ $+(k-2)u\partial_u$
8	$3m-n-2=0$		t^2		$\partial_x,\ x\partial_x + \frac{3u}{n-1}\partial_u,\ t\partial_t + x\partial_x$
9	$\forall$	$\forall$	e^{kt}		$\partial_x,\ (3m-n-2)\partial_t + k(m-1)x\partial_x + ku\partial_u$

For the Cases 3 and 4, for which $m=0$, we can introduce a new time $T = \int f\mathrm{d}t$.

Case 2. $n = 1$.

In this case, from the coefficient of $u_x u_{xx}$ we have that $\eta_{uu} = 0$, so $\eta = a_1(t,x)u + a_2(t,x)$. We use the fact that $\tau = \tau(t)$, $\xi = \xi(t,x)$ and the form for η and from (9.2) we obtain the following determining equations

$$f_t\tau + f(\tau_t - 3\xi_x) = 0, \tag{9.6}$$

$$a_{1x} - \xi_{xx} = 0, \tag{9.7}$$

$$\epsilon m \left[\tau_t - \xi_x + (m-1)a_1\right] u^{m+1} + \epsilon m(m-1)a_2 u^m + (3fa_{1xx} - \xi_t - f\xi_{xxx})u^2 = 0, \tag{9.8}$$

$$\epsilon m a_{1x} u^{m+1} + \epsilon m a_{2x} u^m + (a_{1t} + f a_{1xxx})u^2 + (a_{2t} + f a_{2xxx})u = 0. \tag{9.9}$$

We solve the system (9.6)-(9.9) and in Table 9.2 we present the different forms for the Lie algebra according to the possible forms of the function $f(t)$.

Table 9.2: Classification of equation (9.1) ($n = 1$)

Cases	$f(t)$	Conditions	Basis of $A^{\max}$
		$n = 1$	
		$m \neq 2$	
1	$\forall$		∂_x
2	constant		∂_t, ∂_x, $3(m-1)t\partial_t + (m-1)x\partial_x - 2u\partial_u$
3	t^k		∂_x, $3(m-1)t\partial_t + (m-1)(k+1)x\partial_x + (k-2)u\partial_u$
4	e^{kt}		∂_x, $3(m-1)\partial_t + k(m-1)x\partial_x + ku\partial_u$
		$m = 2$	
5	$\forall$		∂_x, $2\epsilon t\partial_x + \partial_u$
6	constant		∂_t, ∂_x, $2\epsilon t\partial_x + \partial_u$, $3t\partial_t + x\partial_x - 2u\partial_u$
7	t^k	$k \neq 1$	∂_x, $2\epsilon t\partial_x + \partial_u$, $3t\partial_t + (k+1)x\partial_x + (k-2)u\partial_u$
8	t		∂_x, $2\epsilon t\partial_x + \partial_u$, $3t\partial_t + 2x\partial_x - u\partial_u$, $2\epsilon t^2\partial_t + 2\epsilon tx\partial_x + (x - 2\epsilon tu)\partial_u$
9	e^{kt}		∂_x, $2\epsilon t\partial_x + \partial_u$, $3\partial_t + kx\partial_x + ku\partial_u$
10	$f_1(t)$	$p^2 - 4q - 4r^2 \neq 0$	∂_x, $2\epsilon t\partial_x + \partial_u$, $6\epsilon(t^2 + pt + q)\partial_t + \epsilon(6t + 2r + 3p)x\partial_x$ $-(6\epsilon tu - 2\epsilon ru + 3\epsilon pu - 3x)\partial_u$

In Case 10, $f_1(t) = \sqrt{t^2 + pt + q}\exp\left(\int \frac{r\mathrm{d}t}{t^2+pt+q}\right)$ and p, q and r are arbitrary constants such that $p^2 - 4q - 4r^2 \neq 0$ because then we revert to Case 8.

Chapter 10

Conclusions

The main goal of this thesis was the investigation of symmetry properties for special classes of nonlinear evolution PDEs. Our motivation started from the known results of the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^n \frac{\partial u}{\partial x}\right).$$

In Chapters 4, 5 and 6 we gave the symmetry properties for a chain of nonlinear diffusion equations. Namely, we have seen the symmetry properties for the third-, fourth-, fifth- and sixth-order equations, respectively,

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n \frac{\partial^2 u}{\partial x^2} + a u^{n-1}\left(\frac{\partial u}{\partial x}\right)^2\right),$$

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(u^n \frac{\partial^3 u}{\partial x^3} + a u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + b u^{n-2}\left(\frac{\partial u}{\partial x}\right)^3\right),$$

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}\left[u^n \frac{\partial^4 u}{\partial x^4} + a_1 u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^3 u}{\partial x^3} + a_2 u^{n-1}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + a_3 u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2}\right. \\ &\qquad \left. + a_4 u^{n-3}\left(\frac{\partial u}{\partial x}\right)^4\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}\left[u^n \frac{\partial^5 u}{\partial x^5} + a_1 u^{n-1}\frac{\partial u}{\partial x}\frac{\partial^4 u}{\partial x^4} + a_2 u^{n-1}\frac{\partial^2 u}{\partial x^2}\frac{\partial^3 u}{\partial x^3} + a_3 u^{n-2}\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^3 u}{\partial x^3}\right. \\ &\qquad \left. + a_4 u^{n-2}\frac{\partial u}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + a_5 u^{n-3}\left(\frac{\partial u}{\partial x}\right)^3 \frac{\partial^2 u}{\partial x^2} + a_6 u^{n-4}\left(\frac{\partial u}{\partial x}\right)^5\right].\end{aligned}$$

In Chapter 7 we presented the Lie symmetries and the potential symmetries for the third- and fourth-order generalised evolution equations, respectively,

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(f(u)\frac{\partial^2 u}{\partial x^2}+g(u)\left(\frac{\partial u}{\partial x}\right)^2\right),$$

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(f(u)\frac{\partial^3 u}{\partial x^3}+g(u)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+h(u)\left(\frac{\partial u}{\partial x}\right)^3\right),$$

which are the generalisations of the the third-order equation for which symmetry properties presented in Chapter 5 and the fourth-order equation for which the complete group analysis presented in Chapter 4.

In Chapter 8 we performed an enhanced group analysis for the class of dispersive equations

$$u_t+\epsilon\left(u^m\right)_x+\frac{1}{b}\left[u^a\left(u^b\right)_{xx}\right]_x=0$$

and we saw the relation with the third-order equation (5.1) for which symmetry properties presented in Chapter 5.

Finally in Chapter 9 we presented the Lie symmetry classification for the generalised $K(m,n)$ equation with variable coefficients

$$u_t+\epsilon(u^m)_x+f(t)\left(u^n\right)_{xxx}=0.$$

We have studied differential equations which depend upon parameters and for certain values of these parameters we obtained useful symmetry properties. One of the main goals that we had was to find patterns between the values of the parameters for which exceptional symmetries occur. Such investigation occurs in Chapters 4, 5 and 6 in which we examine a chain of equations. Possible patterns between the values will be useful for the investigation of higher-order equations of the chain.

We present some interesting conclusions from the analysis that we had through the results of these three chapters.

Firstly, we have seen that the symmetry Lie algebra is four-dimensional and is spanned by

$$\Gamma_1=\partial_t,\quad \Gamma_2=\partial_x,\quad \Gamma_3=\lambda t\partial_t+x\partial_x,\quad \Gamma_4=\frac{nx}{\lambda}\partial_x+u\partial_u,$$

where λ is the order of the equation and $\lambda \in \mathbb{N} - \{1\}$. An additional Lie symmetry exists for specific values of the parameters. In particular the equations (including the second-order equation) admit a fifth symmetry

$$\Gamma_5 = x^2\partial_x + \frac{2\lambda x u}{n}\partial_u.$$

We have seen that the second-, third-, fourth-, fifth- and sixth-order equations admit a fifth symmetry when the parameters take the particular values, respectively,

$$\begin{aligned}
&n = -\frac{4}{3}, \\
&(n,a) = \left(-\frac{3}{2}, -\frac{3}{2}\right), \\
&(n,a,b) = \left(-\frac{8}{5}, -\frac{24}{5}, \frac{104}{25}\right), \\
&(n,a_1,a_2,a_3,a_4) = \left(-\frac{5}{3}, -\frac{20}{3}, -5, \frac{80}{3}, -\frac{440}{27}\right), \\
&(n,a_1,a_2,a_3,a_4,a_5,a_6) = \left(-\frac{12}{7}, -\frac{60}{7}, -\frac{120}{7}, \frac{2280}{49}, \frac{3420}{49}, -\frac{59280}{343}, \frac{195624}{2401}\right).
\end{aligned}$$

The above five cases, correspond to the following equations:

$$\begin{aligned}
u_t &= -\left[u^{-4/3}u_x\right]_x, \\
u_t &= -\left[u^{-6/4}u_x\right]_{xx}, \\
u_t &= -\left[u^{-8/5}u_x\right]_{xxx}, \\
u_t &= -\left[u^{-10/6}u_x\right]_{xxxx}, \\
u_t &= -\left[u^{-12/7}u_x\right]_{xxxxx}
\end{aligned}$$

and the generalisation of this class of equations is given by

$$u_t = -\frac{\partial^{\lambda-1}}{\partial x^{\lambda-1}}\left[u^{-2\lambda/(\lambda+1)}u_x\right], \text{ where } \lambda \in \mathbb{N} - \{1\}.$$

Also, we have seen that the third-, fourth-, fifth- and sixth-order equations admit a fifth symmetry when the parameters take the particular values, respectively,

$$\begin{aligned}
&(n,a) = \left(-3, -\frac{3}{2}\right), \\
&(n,a,b) = \left(-\frac{8}{3}, -\frac{16}{3}, \frac{44}{9}\right), \\
&(n,a_1,a_2,a_3,a_4) = \left(-\frac{5}{2}, -\frac{15}{2}, -5, \frac{245}{8}, -\frac{315}{16}\right), \\
&(n,a_1,a_2,a_3,a_4,a_5,a_6) = \left(-\frac{12}{5}, -\frac{48}{5}, -\frac{84}{5}, \frac{1326}{25}, \frac{1836}{25}, -\frac{24684}{125}, \frac{60588}{625}\right).
\end{aligned}$$

The above four cases, correspond to the following equations:

$$
\begin{aligned}
u_t &= -\left[u^{-6/2}u_{xx} - \frac{3}{2}u^{-8/2}u_x^2\right]_x, \\
u_t &= -\left[u^{-8/3}u_{xx} - \frac{4}{3}u^{-11/3}u_x^2\right]_{xx}, \\
u_t &= -\left[u^{-10/4}u_{xx} - \frac{5}{4}u^{-14/4}u_x^2\right]_{xxx}, \\
u_t &= -\left[u^{-12/5}u_{xx} - \frac{6}{5}u^{-17/5}u_x^2\right]_{xxxx}
\end{aligned}
$$

and the generalised form for the above equations is

$$
u_t = -\frac{\partial^{\lambda-2}}{\partial x^{\lambda-2}}\left[u^{-2\lambda/(\lambda-1)}u_{xx} - \frac{\lambda}{\lambda-1}u^{(-3\lambda+1)/(\lambda-1)}u_x^2\right], \text{ where } \lambda \in \mathbb{N} - \{1,2\}.
$$

Finally, the fourth-, fifth- and sixth-order equations admit a fifth symmetry for the specific values of the parameters, respectively,

$$
\begin{aligned}
&(n,a,b) = (-4,-6,6), \\
&(n,a_1,a_2,a_3,a_4) = \left(-\frac{10}{3}, -\frac{25}{3}, -5, 35, -\frac{640}{27}\right), \\
&(n,a_1,a_2,a_3,a_4,a_5,a_6) = \left(-3, -\frac{21}{2}, -\frac{33}{2}, \frac{237}{4}, \frac{153}{2}, -\frac{885}{4}, \frac{225}{2}\right).
\end{aligned}
$$

These three cases correspond to the following equations:

$$
\begin{aligned}
u_t &= -\left[u^{-8/2}u_{xxx} - \frac{12}{2}u^{-10/2}u_xu_{xx} + \frac{4\cdot 6}{2^2}u^{-12/2}u_x^3\right]_x, \\
u_t &= -\left[u^{-10/3}u_{xxx} - \frac{15}{3}u^{-13/3}u_xu_{xx} + \frac{5\cdot 8}{3^2}u^{-16/3}u_x^3\right]_{xx}, \\
u_t &= -\left[u^{-12/4}u_{xxx} - \frac{18}{4}u^{-16/4}u_xu_{xx} + \frac{6\cdot 10}{4^2}u^{-20/4}u_x^3\right]_{xxx}.
\end{aligned}
$$

In this case the generalisation of this class of equations is given by

$$
u_t = -\frac{\partial^{\lambda-3}}{\partial x^{\lambda-3}}\left[u^{-\frac{2\lambda}{(\lambda-2)}}u_{xxx} - \frac{3\lambda}{(\lambda-2)}u^{\frac{(-3\lambda+2)}{(\lambda-2)}}u_xu_{xx} + \frac{\lambda(2\lambda-2)}{(\lambda-2)^2}u^{-\frac{4(\lambda-1)}{(\lambda-2)}}u_x^3\right],
$$

where $\lambda \in \mathbb{N} - \{1,2,3\}$.

We have seen three generalisations from which we can derive the particular values of the parameters of higher-order equations for which they admit a fifth symmetry. When the free parameter a_2 involved in our results, for the fifth- and sixth-order equations, it was difficult to find such generalisations for the corresponding equations. More thorough examination of the higher-order equations of the chain could give us some answers.

In the case for which the parameters are arbitrary, the optimal system and the corresponding similarity reductions that transform the corresponding equation into an ODE are given by the operators

$$\begin{aligned}
\langle\Gamma_2\rangle &: \quad u=\phi(\omega), \quad \omega=t,\\
\langle\Gamma_1+c\Gamma_2\rangle &: \quad u=\phi(\omega), \quad \omega=x-ct,\\
\langle\Gamma_3+c\Gamma_4\rangle &: \quad u=t^{\frac{c}{\lambda}}\phi(\omega),\ \omega=\begin{cases} x & \text{if } nc+\lambda=0,\\ t^{-\frac{1}{\lambda}}x^{\frac{\lambda}{nc+\lambda}} & \text{if } nc+\lambda\neq 0,\end{cases}\\
\langle\Gamma_4+c\Gamma_1\rangle &: \quad u=\begin{cases} x^{\frac{\lambda}{n}}\phi(\omega),\ \omega=e^t x^{-\frac{\lambda c}{n}} & \text{if } n\neq 0,\\ e^{\frac{t}{c}}\phi(\omega),\ \omega=x & \text{if } n=0,\end{cases}\\
\langle\Gamma_4+c\Gamma_2-\tfrac{n}{\lambda}\Gamma_3\rangle &: \quad u=\begin{cases} t^{-\frac{1}{n}}\phi(\omega),\ \omega=x+\frac{c}{n}\ln t & \text{if } n\neq 0,\\ e^{\frac{x}{c}}\phi(\omega),\ \omega=t & \text{if } n=0.\end{cases}
\end{aligned}$$

In the case that a fifth symmetry exists, we obtain the following additional reductions, correspond to the additional subalgebras:

$$\langle\Gamma_5+c\Gamma_2+2k\Gamma_3\rangle:\ u=\begin{cases} ((x+k)^2+1)^{\frac{\lambda}{n}}\exp\left[-\frac{2\lambda k}{n}\tan^{-1}(x+k)\right]\phi(\omega),\\ \omega=t\exp\left[-2\lambda k\tan^{-1}(x+k)\right]\ \text{if } c-k^2=1,\\ \\ ((x+k)^2-1)^{\frac{\lambda}{n}}\exp\left[\frac{2\lambda k}{n}\tanh^{-1}(x+k)\right]\phi(\omega),\\ \omega=t\exp\left[2\lambda k\tanh^{-1}(x+k)\right]\ \text{if } c-k^2=-1,\\ \\ (x+k)^{\frac{2\lambda}{n}}\exp\left[\frac{2\lambda k}{n(x+k)}\right]\phi(\omega),\\ \omega=t\exp\left[\frac{2\lambda k}{x+k}\right]\ \text{if } c-k^2=0,\end{cases}$$

where ω is the independent, ϕ the dependent variable of the reduced ODE, $c=0,\pm1$, $k\in\mathbb{R}$ and $\lambda\in\mathbb{N}-\{1\}$, where λ denotes the order of the equation.

We noticed that for the subalgebra $\langle\Gamma_4+c\Gamma_2-\frac{n}{\lambda}\Gamma_3\rangle$ in the special case $n=0$, we obtain the generalised form of the solution

$$u=c_1\exp\left[\frac{1}{c^{\lambda}}\left(c^{\lambda-1}x-\left(\sum a_i+1\right)t\right)\right],\ \text{where } \lambda\in\mathbb{N}-\{1,2\}$$

and a_i the parameters involved in the corresponding equations.

Also, for the fourth-order equation (4.1), in the case for which we have five symmetries and $k=0$ we obtain the following solutions:

For $c = \pm 1$ and $(n, a, b) = \left(-\frac{8}{5}, -\frac{24}{5}, \frac{104}{25}\right)$ we have the solution

$$u = (x^2 + c)^{-5/2}(24t + c_1)^{5/8}$$

and for $(n, a, b) = \left(-\frac{8}{3}, -\frac{16}{3}, \frac{44}{9}\right)$ we have the solution

$$u = (x^2 + c)^{-3/2}(8t + c_1)^{3/8}.$$

Similarly, the sixth-order equation (6.1), admits the solutions:
For $(n, a_1, a_2, a_3, a_4, a_5, a_6) = \left(-\frac{12}{7}, -\frac{60}{7}, -\frac{120}{7}, \frac{2280}{49}, \frac{3420}{49}, -\frac{59280}{343}, \frac{195624}{2401}\right)$

$$u = (x^2 + c)^{-7/2}(540ct + c_1)^{7/12},$$

for $(n, a_1, a_2, a_3, a_4, a_5, a_6) = \left(-\frac{12}{5}, -\frac{48}{5}, -\frac{84}{5}, \frac{1326}{25}, \frac{1836}{25}, -\frac{24684}{125}, \frac{60588}{625}\right)$

$$u = (x^2 + c)^{-5/2}(108ct + c_1)^{5/12}$$

and finally, when
$(n, a_1, a_2, a_3, a_4, a_5, a_6) = \left(-4, -12, a_2, \frac{4}{3}(-a_2 + 37), \frac{(-31a_2 - 8)}{6}, \frac{104}{9}(a_2 - 7), \frac{56}{9}(-a_2 + 7)\right)$,
we have

$$u = (x^2 + c)^{-3/2}[(-18a_2 - 324)ct + c_1]^{1/4}.$$

The third- and fifth-order equations do not admit such forms of solutions. We suppose that the higher even-order equations will admit such solutions but the odd-order equations will not.

The final comment for the Lie group analysis of the chain is that, in the case of the class of fourth-order equations (4.1) and sixth-order equations (6.1) we have seen there exists no member that can be a linearised by local mapping. On the other hand the equation of the third-order (5.1) and fifth-order (6.2) was found to possess an infinite-dimensional Lie symmetry, for the special case that $n = 0$, with the form

$$\Gamma_\infty = \phi(t, x)\sqrt{u}\partial_u,$$

where $\phi(t, x)$ is a solution of the linear equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial^{2\lambda_1 - 1}}{\partial x^{2\lambda_1 - 1}}\phi = 0, \text{ for } \lambda_1 \in \mathbb{N} - \{1\},$$

where $2\lambda_1 - 1$ indicates the order of the equation. Equations that admit the infinite-dimensional Lie symmetry Γ_∞ can be mapped into the linear equation

$$\frac{\partial u}{\partial t} + \frac{\partial^{2\lambda_1 - 1}}{\partial x^{2\lambda_1 - 1}} u = 0, \text{ for } \lambda_1 \in \mathbb{N} - \{1\},$$

by the mapping

$$u \mapsto \sqrt{u}.$$

It appears that only equations of odd-order from the chain can be linearised by local mappings.

We have seen that the second-, third-, fourth-, fifth- and sixth-order equations admit proper nonclassical symmetry

$$\Gamma_1 = \partial_t + \phi(x)\sqrt{u}\partial_u,$$

where $\phi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^{\lambda_2+1}\phi}{\mathrm{d}x^{\lambda_2+1}} + \frac{1}{2}\phi^2 = 0, \text{ where } \lambda_2 \in \mathbb{N}.$$

Here, $\lambda_2 + 1$ denotes the order of the equation. A special solution for the obove ODE is

$$\phi = (-1)^{\lambda_2} \frac{2(2\lambda_2 + 1)!}{\lambda_2!} x^{-(\lambda_2+1)}.$$

In this case, Γ_1 leads to the ansatz (4.35)

$$u = \left[\tfrac{1}{2}\phi(x)t + F(x)\right]^2$$

which reduces the corresponding equation to the ODE

$$\frac{\mathrm{d}^{\lambda_2+1}F}{\mathrm{d}x^{\lambda_2+1}} + \frac{1}{2}\phi F = 0.$$

The third- and fifth-order equations, and we suppose all odd-order equations of the chain, admit also the reduction operator

$$\Gamma_2 = \partial_t + \psi(x)u^{2/3}\partial_u,$$

where $\psi(x)$ is a solution of the ODE

$$\frac{\mathrm{d}^{2\lambda_3+1}\psi}{\mathrm{d}x^{2\lambda_3+1}} + \frac{1}{3}\psi^2 = 0, \text{ where } \lambda_3 \in \mathbb{N}$$

and particualar solution

$$\psi = \frac{3(4\lambda_3 + 1)!}{(2\lambda_3)!} x^{-(2\lambda_3+1)}.$$

We have that $2\lambda_3 + 1$ indicates the order of the equation. In this case, Γ_2 leads to the nonclassical reduction (5.29)

$$u = \left[\tfrac{1}{3}\psi(x)t + F(x)\right]^3.$$

Equation is reduced to the ODE

$$\frac{d^{2\lambda_3+1}F}{dx^{2\lambda_3+1}} + \frac{1}{3}\psi F = 0.$$

The two special forms of the odd-order equations that admit nonclassical symmetries are connected by the mapping $u \mapsto u^{3/2}$.

The four equations admit proper potential symmetries with the forms:

$$\Gamma = \psi(t, v)\partial_x - u^2\psi_v\partial_u,$$

where $\psi(t, v)$ is a solution of the linear equation

$$\frac{\partial\psi}{\partial t} + \frac{\partial^\lambda}{\partial v^\lambda}\psi = 0, \text{ for } \lambda \in \mathbb{N} - \{1, 2\}$$

and

$$\Gamma = 2uv\partial_u + v^2\partial_v.$$

Finally, the four equations admit nonclassical potential symmetries. If $n = -1$ and the other parameters are equal to zero then the corresponding potential equation admits the reduction operator

$$\Gamma_1 = \partial_t + c\partial_x + \phi(x + ct)\partial_v.$$

Also, it admits a second form of reduction operator

$$\Gamma_2 = \partial_t + \phi(v + ct)\partial_x + c\partial_v.$$

In both operators $\phi(\omega)$ is a solution of the ODE

$$\frac{\mathrm{d}^\lambda\phi}{\mathrm{d}\omega^\lambda} + \phi\frac{\mathrm{d}\phi}{\mathrm{d}\omega} = 0, \text{ where } \lambda \in \mathbb{N} - \{1, 2\}.$$

For the above cases we recall that λ denotes the order of the equation. The corresponding potential equations to these two cases are connected via the pure hodograph transformation

$$t' = t, \quad x' = v, \quad v' = x.$$

Lengthier investigation of the chain could give us new interesting answers and new patterns for the construction of the different forms of symmetries. Also we can use Lie group analysis for the ODEs that arise through the analysis and construct exact solutions.

The French scholar Jean Dieudonne said:

"Lie theory is the process of becoming the most important part of modern mathematics. Little by little it became obvious that the most unexpected theories, from arithmetic to quantum physics, came to encircle the Lie field like a gigantic axis."

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