DEPARTMENT OF MATHEMATICS AND STATISTICS

APPLICATIONS OF ROOT SYSTEMS IN GEOMETRY AND PHYSICS

CHARALAMPOS A. EVRIPIDOU

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Doctoral Candidate: Charalampos A. Evripidou

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Examination Committee:

Research supervisor: ______________________________

Pantelis Damianou,
Professor, University of Cyprus

Committee Member: ______________________________

Nikos Stylianopoulos,
Professor, University of Cyprus

Committee Member: ______________________________

Costas Daskaloyannis,
Associate Professor, Aristotle University of Thessaloniki, Thessaloniki

Committee Member: ______________________________

Christos Pallikaros,
Associate Professor, University of Cyprus

Committee Member: ______________________________

Mark Roberts,
Professor, University of Surrey, United Kingdom

DECLARATION OF DOCTORAL CANDIDATE

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Περίληψη

> Η περίληψη είναι εκείνο το μέρος ενός βιβλίου που γράφεται τελευταίο, τοποθετείται πρώτο και διαβάζεται λιγότερο. — Alfred Lotka.

Σε αυτή τη διατριβή μελετούμε περιοχές των μαθηματικών που μπορεί να φαίνονται ασυσχέτιστες αλλά έχουν κοινή προέλευση τη θεωρία των συστημάτων ριζών. Τα συστήματα ριζών χρησιμοποιούνται για την ταξινόμηση των αλγεβρών Lie αλλά εμφανίζονται και σε άλλες ταξινομήσεις, για παράδειγμα στην ταξινόμηση των πεπερασμένων ομάδων Coxeter. Επίσης χρησιμοποιούνται και στα ολοκληρώσιμα συστήματα στην κλασική και κβαντική μηχανική. Να αναφέρουμε τα συστήματα Toda, τα συστήματα Calogero-Moser και τα γενικευμένα συστήματα Volterra του Bogoyavlensky. Για ευκολία αυτή η διατριβή χωρίζεται σε δύο μέρη.

Στο πρώτο μέρος ασχολούμαστε με τα πολυώνυμα Coxeter των αφινικών αλγεβρών Lie και επίσης με τα πολυώνυμα Coxeter μιας οικογένειας ομάδων Coxeter οι οποίες ορίζονται μέσω γραφημάτων. Υπολογίζουμε τα πολυώνυμα Coxeter, τους αριθμούς Coxeter και τους εκθέτες για κάθε μία από τις αφινικές άλγεβρες Lie χρησιμοποιώντας στοιχειώδεις ιδιότητες των πολυωνύμων Chebyshev. Γενικεύουμε δύο μεθόδους των Steinberg και Berman, Lee και Moody για τον υπολογισμό των αφινικών αριθμών Coxeter και των αφινικών εκθετών στην περίπτωση της αφινικής άλγεβρας Lie τύπου $A_n^{(1)}$. Χρησιμοποιούμε αυτές τις μεθόδους για τον υπολογισμό των αφινικών αριθμών Coxeter και των εκθετών για κάθε μία από τις αφινικές άλγεβρες Lie. Επίσης υπολογίζουμε τα πολυώνυμα Coxeter μίας οικογένειας ομάδων Coxeter τις οποίες ορίζουμε μέσω γραφημάτων. Σε αυτά τα γραφήματα περιλαμβάνονται πολλά γνωστά γραφήματα, για παράδειγμα τα διαγράμματα Dynkin τύπου D_n, τα αφινικά διαγράμματα Dynkin τύπου $D_n^{(1)}$, τα διαγράμματα E_n και πολλά άλλα γνωστά γραφήματα. Υπολογίζουμε το όριο της φασματικής ακτίνας των μετασχηματισμών Coxeter αυτών των ομάδων Coxeter όταν ο αριθμός των κορυφών τους τείνει στο άπειρο.

Στο δεύτερο μέρος αυτής της διατριβής ασχολούμαστε με ολοκληρώσιμα συστήματα τα οποία είναι του τύπου Lotka-Volterra. Χρησιμοποιούμε μία καινούρια μέθοδο για να παράγουμε Χαμιλτονιανά συστήματα κατασκευάζοντας τα αντίστοιχα ζεύγη Lax. Αυτό το επιτυγχάνουμε χρησιμοποιώντας υποσύνολα των θετικών ριζών, συστημάτων ριζών απλών αλγεβρών Lie, τα οποία περιέχουν τις απλές ρίζες. Σε αρκετές περιπτώσεις η μέθοδος αυτή δίνει γνωστά Χαμιλτονιανά συστήματα τα οποία είναι τύπου Lotka-Volterra. Ονομάζουμε τα συστήματα τα οποία παίρνουμε από αυτή τη μέθοδο γενικευμένα συστήματα τύπου Lotka-Volterra. Ταξινομούμε όλα τα υποσύνολα των θετικών ριζών, που περιέχουν τις απλές ρίζες του συστήματος ριζών τύπου A_n, τα οποία μετά από μία απλή αλλαγή μεταβλητών δίνουν συστήματα τύπου Lotka-Volterra. Επίσης αποδεικνύουμε ότι η μέθοδος μας δουλεύει και για αρκετά άλλα υποσύνολα των θετικών ριζών.

Abstract

> The abstract is that part of a book which is written last, placed first and read least. — Alfred Lotka.

In this thesis we investigate some areas of mathematics which may be unrelated but nevertheless they have as common theme the theory of abstract root systems. Root systems of course are used in the classification of simple Lie algebras. They also appear in other classifications, for example the classification of finite Coxeter groups. They are also used in the theory of integrable systems in classical and quantum Mechanics. A number of mechanical systems are defined to correspond to simple or affine Lie algebras. We mention the various Toda lattices, Calogero-Moser systems and the generalized Volterra lattices of Bogoyavlensky. For convenience we divide this thesis in two parts.

The first part is concerned with the Coxeter polynomials of finite and affine Lie algebras and also with the Coxeter polynomials of a family of Coxeter groups arising from graphs. We define the Coxeter number and exponents with respect to each conjugacy class of the Coxeter elements of the simple and affine Lie algebras. In the case of the affine Lie algebra of type $A_n^{(1)}$ we have $\lfloor \frac{n+1}{2} \rfloor$ different conjugacy classes of Coxeter elements while for all the other cases we have only one. We compute the Coxeter polynomial, the Coxeter number and exponents of each one of the simple and affine Lie algebras using properties of Chebyshev polynomials. We generalize two methods of Steinberg and Berman, Lee and Moody for the computation of affine Coxeter number and affine exponents in the case of the affine Lie algebra of type $A_n^{(1)}$. We use these methods for the computation of the affine Coxeter number and affine exponents of each one of the affine Lie algebras. We also compute the Coxeter polynomials of a family of Coxeter groups defined by their Coxeter graphs. This family of graphs includes several well known graphs, e.g. the D_n Dynkin diagrams, the $D_n^{(1)}$ affine Dynkin diagrams, the E_n diagrams and many other diagrams. We find the limit of the spectral radius of the Coxeter elements of these graphs as the number of the vertices of their arms tends to infinity.

The second part of this thesis is concerned with the theory of integrable systems and more specifically with Lotka-Volterra systems. We device a new method for producing integrable systems by constructing the corresponding Lax pairs. This is achieved by considering a larger subset of the positive roots than the simple roots of a simple complex Lie algebra. In several cases these subsets of the positive roots recover well known Hamiltonian systems which are of Lotka-Volterra type. Therefore we call the systems produced by this method generalized Lotka-Volterra systems. We find all subsets of the positive roots of the simple Lie algebra of type A_n which produce after a suitable change of variables Lotka-Volterra systems. Furthermore we show that our method works for several other cases.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

> The greatest challenge to any thinker is stating the problem in a way that will allow a solution — Bertrand Russell

This thesis is divided in two parts. In the first part we investigate the connection between Chebyshev polynomials and root systems of affine Lie algebras. We generate the Coxeter polynomials and the characteristic polynomials of Cartan and adjacency matrices of the affine Lie algebras using properties of Chebyshev polynomials. We also generalize two results of Steinberg and Berman, Lie and Moody to the case of the affine Lie algebra of type $A_n^{(1)}$. We calculate the Coxeter polynomials of a family of Coxeter groups and we also find a family of Pisot numbers as limits of sequences of Salem numbers.

In his seminal paper [**?**], Coxeter gave the definition of Coxeter groups and classified the finite Coxeter groups. He also defined the Coxeter element of the Coxeter groups and observed that the eigenvalues of these elements have remarkable properties. These and several other properties were later developed by various mathematicians (see [**?**, **?**, **?**, **?**]). The order h, of a Coxeter element is the Coxeter number of the Coxeter group W and the eigenvalues of the Coxeter element are of the form $e^{\frac{\pi i m_j}{h}}$, for some integers $m_j \in \{1, 2, \ldots, h-1\}$. The integers m_j are called the exponents of the Coxeter group. Some of the properties of the exponents and the Coxeter number are

- The cardinality of W is $\prod(m_j + 1)$.
- The rank r of the corresponding Lie algebra is $\frac{|R|}{h}$, where $|R|$ is the cardinality of the root system.
- A word for the longest element is $\sigma^{\frac{h}{2}}$ (σ is a Coxeter element).
- The length of the longest element in W is $\sum m_j = \frac{r \cdot h}{2}$.
- The height of the highest root is $h-1$.

- When the Coxeter group is the Weyl group of a Lie algebra, the dimension of the Lie algebra is $r(h+1)$.

- Assume that $m_1 \leq m_2 \leq \ldots \leq m_r$ and that $n_1 \leq \ldots \leq n_{h-1}$ is the partition of $\sum m_j$ conjugate to that of m_i's. Then n_i is the number of reflections with trace i.

- The spectrum of the Coxeter graph is $\left\{2\cos\left(\frac{m_j\pi}{h}\right)\right\}$.

- The spectrum of the Cartan matrix is $\left\{4\cos^2\left(\frac{m_j\pi}{2h}\right)\right\}$.

- Let k_m denotes the number of roots of height m. Then $k_m - k_{m+1}$ is the number of times m occurs as an exponent of W.

- The Poincare polynomial of the Coxeter group W is defined as $W(x) = \sum_{w\in W} x^{\ell(w)}$ where $\ell(w)$is the length of w. It factors as $\prod(1 + x + \ldots + x^{m_j})$.

- The Poincare polynomial of a compact Lie group with W its Weyl group factors as $\prod(1 + x^{2m_j+1})$.

- The determinant of the Cartan matrix is

$$\det(C) = 2^{2r}\prod_{i=1}^{r}\sin^2\frac{m_i\pi}{2h}.$$

After the work of Coxeter, several authors (see e.g. [?, ?]) started investigating the Coxeter element of the affine Coxeter groups. The eigenvalues of these elements, satisfy similar properties, as the eigenvalues of the Coxeter elemens of the finite Coxeter groups. If σ is a Coxeter element of an affine Coxeter group then it satisfies a relation of the form $(\sigma - 1)(\sigma^h - 1) = 0$. The number h is the affine Coxeter number and the eigenvalues of σ are, again, of the form $e^{\frac{\pi i m_j}{h}}$, for some integers $m_j \in \{0, 1, \ldots, h\}$. The integers m_j are the affine exponents. In the first part of this thesis we are mainly concerned with the Coxeter polynomials associated with the affine Lie algebras. For completeness we include the corresponding results for the complex simple finite dimensional Lie algebras (section 3.2 and part of section 3.4) which are taken from [?] and [?].

For a Dynkin diagram Γ of a root system of type X, in addition to the Cartan matrix C_X, we associate the Coxeter adjacency matrix which is the matrix $A_X = 2I - C_X$. The characteristic polynomial of Γ is that of A_X and the spectral radius of Γ is

$$\rho(\Gamma) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A_X\}.$$

We use the following notation. The subscript n in all cases is equal to the degree of the polynomial except that $Q_n(x)$ is of degree $2n$.

- $p_n(x)$ will denote the characteristic polynomial of the Cartan matrix,
- $q_n(x) = \det(2xI + A_X)$,
- $a_n(x) = q_n\left(\frac{x}{2}\right)$ will denote the characteristic polynomial of $-A_X$ and finally,
- $Q_n(x) = x^n a_n\left(x + \frac{1}{x}\right)$.

Note the relation between the polynomials a_n, q_n and p_n

$$p_n(x) = a_n(x-2) = q_n\left(\frac{x}{2} - 1\right).$$

We prove the following result

Theorem 1. *Let C be the $n \times n$ affine Cartan matrix of an affine Lie algebra of type X. Then q_n is a polynomial related to Chebyshev polynomials as follows*

$$\begin{aligned}
\text{for } X = A_{n-1}^{(1)}, &\quad q_n(x) = 2\left(T_n(x) + (-1)^{n-1}\right), \\
\text{for } X = B_{n-1}^{(1)}, &\quad q_n(x) = 2\left(T_n(x) - T_{n-4}(x)\right), \\
\text{for } X = C_{n-1}^{(1)}, &\quad q_n(x) = 2\left(T_n(x) - T_{n-2}(x)\right) \text{ and} \\
\text{for } X = D_{n-1}^{(1)}, &\quad q_n(x) = 8x^2\left(T_{n-2}(x) - T_{n-4}(x)\right),
\end{aligned}$$

where $T_n(x)$ is the n^{th} Chebyshev polynomial of first kind.

Using the fact that for bipartite Dynkin diagrams the spectrum of the Coxeter adjacency matrix A is the same as the spectrum of $-A$ it follows that the eigenvalues of the Cartan matrix occur in pairs, λ and $4-\lambda$ (see e.g. [?, ?, ?]). In our case this happens in all cases except for $A_{n-1}^{(1)}$, n odd. In the bipartite cases, $a_n(x)$ is the characteristic polynomial of the Coxeter adjacency matrix.

Let L be a complex finite dimensional simple Lie algebra with Cartan matrix C of rank n and simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. The Killing form on L induces an inner product on the real vector space V with basis Π. The Weyl group W of L is a subgroup of $\operatorname{Aut} V$ which is generated by reflections on V. Namely, for each fixed root α_i consider the reflection σ_i through the hyperplane perpendicular to α_i

$$\sigma_i : V \longrightarrow V, \quad \alpha \mapsto \alpha - 2\frac{(\alpha, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i.$$

Then the Weyl group of L is $W = \langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle$. The Cartan matrix C satisfies

$$C_{i,j} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)},$$

and therefore $\sigma_i(\alpha_j) = \alpha_j - C_{j,i}\alpha_i$. The Weyl group of an affine Lie algebra of rank n and Cartan matrix C is $W = \langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle$, where

$$\sigma_i(\alpha_j) = \alpha_j - C_{j,i}\alpha_i$$

is a "reflection" in the real vector space with basis $\{\alpha_1, \ldots, \alpha_n\}$. If $z = (z_1, \ldots, z_n)$ is a left zero eigenvector of C (z can be taken to be in $\mathbb{Z}^n$, see [?]) and $\alpha = \sum_{k=1}^{n} z_k \alpha_k$ then $\sigma_i(\alpha) = \alpha$, $\forall i = 1, \ldots, n$. Thus the Weyl group W, acts on $\{k\alpha : k \in \mathbb{Z}\}$ as the identity.

A Coxeter polynomial f_n is the characteristic polynomial of $\sigma_{\pi(1)}\sigma_{\pi(2)} \ldots \sigma_{\pi(n)} \in gl(V)$ for some $\pi \in S_n$. When the Dynkin diagram does not contain cycles the Coxeter polynomial is uniquely defined and for bipartite Dynkin diagrams is closely related to the polynomial $Q_n(x)$; the polynomial $Q_n(x)$ turns out to be $Q_n(x) = f_n(x^2)$. For the case of $A_{n-1}^{(1)}$, Coleman showed in [?] that, there are $\left\lfloor \frac{n}{2} \right\rfloor$ different Coxeter polynomials. For n even, $Q_n\left(\sqrt{x}\right)$ is one of the Coxeter polynomials, the one corresponding to the largest conjugacy class of the Coxeter transformations. According to [?] the largest conjugacy class contains the Coxeter transformations with the property that the set

$$\left\{ i : \pi^{-1}(i) > \pi^{-1}(i+1 \pmod n), i = 1, 2, \ldots, n \right\}$$

has the largest cardinality, i.e. contains $\frac{n}{2}$ elements. For example one may choose the Coxeter transformation $\sigma_1\sigma_3 \ldots \sigma_{n-1}\sigma_2\sigma_4 \ldots \sigma_n$, which is the one considered in [?].

The roots of Q_n are in the unit disk and therefore by a theorem of Kronecker (theorem 8), $Q_n(x)$ is a product of cyclotomic polynomials. We determine the factorization of Q_n as a product of cyclotomic polynomials. This factorization in turn determines the factorization of f_n. The irreducible factors of Q_n are in one-to-one correspondence with the irreducible factors of $a_n(x)$.

The roots of a Coxeter polynomial f_n, of a Lie algebra of affine type, are of the form $e^{\frac{2m_j\pi i}{h}}$ for some integers $m_j \in \{0, 1, \ldots, h\}$, where the numbers m_j are the affine exponents and h the affine Coxeter number associated with the Coxeter transformation σ. These numbers are normally defined only for the bipartite case. For $A_n^{(1)}$, n odd one defines them with respect to the Coxeter polynomial corresponding to the largest conjugacy class of the Coxeter element. We examine in detail the case of $A_n^{(1)}$ for n both even and odd and we calculate the affine exponents and affine Coxeter number for each conjugacy class.

These numbers are related to the Cartan matrix and give a universal formula for the spectrum of the Dynkin diagram Γ and the eigenvalues of the Cartan matrix. For the bipartite case the spectrum is

$$\left\{ 2\cos\frac{m_j\pi}{h} : j = 1, \ldots n \right\}$$

and the eigenvalues of the Cartan matrix are $\left\{4\cos^2\frac{m_j\pi}{2h} : j = 1, \ldots n\right\}$.

The affine exponents, affine Coxeter number of $X_n^{(1)}$ and the roots of the corresponding simple Lie algebra X_n are related in an inquisitive way (see [?, ?]). Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the simple roots of X_n, $V = \mathbb{R}\text{-span}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and β the branch root of X_n. Let $w_{\beta^\vee} \in V^*$ be the weight corresponding to the co-root $\beta^\vee$. Then for some $c \in \mathbb{N}$ and a proper enumeration of m_j we have

$$c \cdot w_{\beta^\vee} = \sum_{j=1}^{n} m_j \alpha_j^\vee \ , \tag{1.1}$$

where c is the smallest integer such that $c \cdot w_{\beta^\vee}$ belongs to the co-root lattice. The coefficient of $\beta^\vee$ is the affine Coxeter number. Here we have identified V with V^* using the inner product induced by the Killing form. We generalize that result and show that the relation (1.1) is valid for all conjugacy classes in $A_n^{(1)}$.

Steinberg [?], in his explanation of the MacKay correspondence, shows a mysterious relation between affine Coxeter polynomials (for the simply laced Dynkin diagrams and later Stekolshchik in [?] for the multiple laced) and Coxeter polynomials of type A_n. Each affine Coxeter polynomial is a product of Coxeter polynomials of type A_n. From X_n remove the branch root. If $g(x)$ is the Coxeter polynomial of the reduced system then the Coxeter polynomial of $X_n^{(1)}$ is $(x-1)^2 g(x)$. The affine exponents and affine Coxeter number of an affine Lie algebra are easily computed using Steinberg's theorem. We include the table 2.1 listing the affine exponents and affine Coxeter number for affine Lie algebras. Furthermore, we demonstrate that the method of Steinberg works also in the case of $A_n^{(1)}$.

In [?], Lakatos proves a result about the spectral radius of Coxeter transformations of noncyclotomic starlike trees, which she called wild stars. Let $S^{(0)}_{p_1,\ldots,p_k}$ denote the wild star consisting of k paths of length $p_1, \ldots, p_k$ and one branching point. Lakatos proved that the limit of the spectral radius of the Coxeter transformations of $S^{(0)}_{p_1,\ldots,p_k}$ as $p_1, \ldots, p_k \to \infty$ is $k-1$. We define $S^{(i)}_{p_1,\ldots,p_k}$ to be the join of i Dynkin diagrams of type $D_{p_1}, \ldots, D_{p_i}$ and $k-i$ Dynkin diagrams of type $A_{p_{i+1}}, \ldots, A_{p_k}$. We generalize the sesult of Lakatos and show that the limit of the spectral radius of the Coxeter transformations of $S^{(i)}_{p_1,\ldots,p_k}$ as $p_1, \ldots, p_k \to \infty$ is $k-1$. We also calculate the Coxeter polynomials of the Coxeter graphs $S^{(i)}_{p_1,p_2,p_3}$ and find the limit of the spectral radius of the Coxeter transformations of $S^{(i)}_{p_1,p_2,p_3}$ as $p_j \to \infty$ and $p_j, p_l \to \infty$. These limits are Pisot numbers.

The second part of this thesis is concerned with the theory of integrable Hamiltonian systems. Some parts of chapter 6 and chapter 7 were done in collaboration with the postdoctoral fellow Stelios Charalambides.

Jurgen K. Moser made important contributions to the theory of completely integrable Hamiltonian systems. To quote Moser from [?]

> In the last twenty years one of the most fascinating developments in the theory of Hamiltonian systems is connected with the discovery of the new integrable systems, like the Toda lattice and various other systems. This subject grew very rapidly. Although it originated from applied problems, it has in the meantime spread to a variety of other more abstract fields such as Lie algebras.

In this part we investigate a new class of Hamiltonian systems which are connected with subsets of the positive roots of a root system of a complex simple Lie algebra.

The Volterra system (also known as the KM system) is a well-known integrable system defined by

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}) \qquad i = 1, 2, \ldots, n, \tag{1.2}$$

where $x_0 = x_{n+1} = 0$. It was studied by Lotka in [?] to model oscillating chemical reactions and by Volterra in [?] to describe population evolution in a hierarchical system of competing species. It was first solved by Kac and van-Moerbeke in [?], using a discrete version of inverse scattering due to Flaschka [?]. In [?] Moser gave a solution of the system using the method of continued fractions and in the process he constructed action-angle coordinates. Equations (1.2) can be considered as a finite-dimensional approximation of the Korteweg-de Vries (KdV) equation. The Poisson bracket for this system can be thought as a lattice generalization of the Virasoro algebra [?]. The Volterra system is associated with a simple Lie algebra of type A_n in the sense that it can be written in Lax pair form $\dot{L} = [B, L]$, where

$$L = \sum_{i=1}^{n} a_i \left(X_{\alpha_i} + X_{-\alpha_i} \right)$$

and

$$B = \sum_{i=1}^{n-1} a_i a_{i+1} \left(X_{\alpha_i + \alpha_{i+1}} - X_{-\alpha_i - \alpha_{i+1}} \right),$$

with $\{\alpha_1, \ldots, \alpha_n\}$ being the simple roots of the root system of the Lie algebra of type A_n and X_{α_i} the corresponding root vectors. This Lax pair is due to Moser [?]; it gives a polynomial (in fact cubic) system of differential equations. The change of variables $x_i = 2a_i^2$, produces equations (1.2). We generalize this Lax pair and produce a larger class of Hamiltonian systems which we call generalized Volterra systems.

We devise a new method for producing Hamiltonian systems by constructing the corresponding Lax pairs. This is achieved by considering a larger subset of the positive roots, than the simple roots, of a simple Lie algebra. In several cases these subsets of the positive roots recover well known Hamiltonian systems which are of Lotka-Volterra type. For example using the simple roots of the root system of type A_n we recover the KM system while using the simple roots and the highest root we recover the periodic KM system. We find and classify all subsets of the positive roots of the Lie algebra of type A_n which give rise to Lotka-Volterra systems and we also show that in several other cases our algorithm works. In higher dimensions we are able, using our method, to derive new completely integrable Hamiltonian systems.

Bogoyavlensky in [?, ?] and [?] produces Lotka-Volterra systems which are generalizations of the periodic KM system. Our method produces Lotka-Volterra systems which are different from these of Bogoyavlensky. The systems produce in [?] and [?] are described explicitly in [?] and they are different from the Lotka-Volterra systems produced by our algorithm (see section 5.5). Also the construction of Bogoyavlensky in [?] covers a wide variety of generalizations. However as it can be seen by the diagrams of interactions of the systems in [?], by restricting these systems one cannot obtain our systems in an obvious way.

This thesis is structured as follows.

In chapter 2 we give the basic definitions needed for the remaining chapters. We review the basic results about root systems, Lie algebras, Coxeter groups and the Mahler measure of integer polynomials. The rest of this thesis is divided in two parts.

The first part (made up of the chapters 2 to 4) is about is about the Coxeter polynomials of simple and affine Lie algebras and also about the Coxeter transformations of a family of Coxeter groups, defined by their Coxeter graphs. In chapter 3 we calculate the spectrum of Cartan matrices and the Coxeter polynomials of simple and affine Lie algebras, using properties of Chebyshev polynomials. We also compute the associated polynomials a_n, q_n and Q_n of the simple and affine Lie algebras and we use them for the explicit calculation of their Coxeter polynomials, Coxeter number and exponents. The Coxeter graphs of the simple and affine Lie algebras are all trees, except the one of the affine Lie algebra with root system of type $A_n^{(1)}$. For this Lie algebra the Coxeter polynomial is not uniquely defined. We calculate all of their Coxeter polynomials in section 3.5. In that section we generalize the methods of Steinberg and Berman, Lee and Moody, for the case of the affine Lie algebra with root system of type $A_n^{(1)}$. We show that these methods can be modified and applied to the case of the affine Lie algebra with root system of type $A_n^{(1)}$, and give all of their Coxeter polynomials. In chapter 4 we generalize a result of Piroska Lakatos about the spectral radius of the Coxeter transformations of the Coxeter graphs $S_{p_1,\ldots,p_k}^{(i)}$.

The new results of the first part can be summarized as follows.

1. Let $a_n(x)$ be the characteristic polynomial of the Coxeter adjacency matrix. The spectrum of this polynomial is called the spectrum of the Dynkin graph. Using the knowledge of the roots of $U_n(x)$, the Chebyshev polynomial of the second kind, we are able to compute the roots of $a_n(x)$ and in the bipartite case they turn out to be

$$2\cos\frac{m_j\pi}{h}\ .$$

Let C be the generalized Cartan matrix associated with the affine Lie algebra. The eigenvalues of C in the bipartite case are

$$4\cos^2\frac{m_j\pi}{2h}\ .$$

Let $f(x)$ be the affine Coxeter polynomial (in the case of $A_n^{(1)}$ with n odd we use the Coxeter polynomial corresponding to the largest conjugacy class). Then the roots of f in terms of the exponents and Coxeter number are

$$e^{\frac{2m_j\pi i}{h}}\ .$$

2. Let $\Pi=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ be the simple roots of the associated simple Lie algebra, $V=\mathbb{R}\text{-span}(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and β the branch root. Let $w_{\beta^\vee}\in V^*$ be the weight corresponding to the co-root $\beta^\vee$. Then for some $c\in\mathbb{N}$ and a proper enumeration of m_j we have

$$c\cdot w_{\beta^\vee}=\sum_{j=1}^{n}m_j\alpha_j^\vee\ ,$$

where $c\in\mathbb{N}$ is the smallest integer such that $c\cdot w_{\beta^\vee}$ belongs to the co-root lattice. The coefficient of $\beta^\vee$ is the affine Coxeter number. This method is extended for each conjugacy class in the Coxeter group of $A_n^{(1)}$.

3. One may use a procedure of Steinberg which relates affine Coxeter polynomials with the corresponding Coxeter polynomial of the reduced system obtained by removing a branch root. Each affine Coxeter polynomial is a product of Coxeter polynomials of type A_n. This method is also extended to the case of $A_n^{(1)}$.

4. We generalize a result of Piroska Lakatos about the Coxeter polynomials of the Coxeter graphs, $S^{(i)}_{p_1,\ldots,p_k}$. We show that for $k=3$ the limits

$$\lim_{p_j\to\infty}\rho\left(S^{(i)}_{p_1,p_2,p_3}\right)$$

and

$$\lim_{p_j,p_m\to\infty}\rho\left(S^{(i)}_{p_1,p_2,p_3}\right)$$

are Pisot numbers for all $i\in\{0,1,2,3\}$ and $j,m=1,2,3$. We also show that for all $k\in\mathbb{N}$ and $0\le i\le k$

$$\lim_{p_1,\ldots,p_k\to\infty}\rho\left(S^{(i)}_{p_1,\ldots,p_k}\right)=k-1.$$

The second part is about Lotka-Volterra systems and is made up of the chapters 5 to 7. In chapter 5 we give the basic definitions about Hamiltonian systems and Poisson brackets. In chapter 6 we explain the new algorithm for constructing Hamiltonian systems by producing the corresponding Lax pairs (section 6.2). In section 6.3 we present all systems produced by our algorithm for the cases of root systems of type A_3 and A_4 and we classify, in section 6.4, all subsets of the positive roots of the root system of type A_n which give, under the transformation $x_i = 2a_i^2$, systems of Lotka-Volterra type. We describe the corresponding systems and discuss their integrability. In section 6.5 we present two interesting methods for finding first integrals for Hamiltonian system emerging from Lax pairs and we use these methods in section 6.6, to find additional first integrals for systems corresponding to certain subsets of the positive roots of the root system of type A_n. Finally in chapter 7 we present a variation of our algorithm where we use complex coefficients.

The new results of the second part can be summarized as follows.

1. We device an algorithm for constructing Hamiltonian systems corresponding to subsets of the positive roots of root systems of simple Lie algebras. This algorithm produces the corresponding Lax pairs.

2. We explicitly present all the Lotka-Volterra systems produced by this algorithm from subsets of the positive roots of the root systems of type A_3 and A_4. In several cases we recover well known integrable systems but we also produce some new systems of Lotka-Volterra type which we show that they are integrable.

3. We classify all subsets of the positive roots of the root system of type A_n which produce, under the change of variables $x_i = 2a_i^2$, Lotka-Volterra systems. We explicitly describe the corresponding Lotka-Volterra systems. We also show that our algorithm produces consistent Lax pairs for certain families of subsets of the positive roots of the root system of type A_n, and we describe the corresponding Hamiltonian systems.

4. We present a variation of the algorithm which uses complex coefficients. We show that this method produces different Hamiltonian systems and more Lotka-Volterra systems, than the previous one.

Chapter 2

Background

> He turned the handle and the door opened. Beyond it was another door. He turned the handle and the other door stood wide. He opened doors, a hundred and twenty-four. Then he grew tired, and he collapsed. Beyond the hundred and twenty-fifth door, there is a garden where the roses have just opened, he though, drowsily dying. Beyond that door was another door.—Antanas Skema

2.1 Root systems

In the following V will be a finite dimensional real Euclidean space (with inner product written $(\,,\,)$). We write $\langle x, y\rangle := \frac{2(x,y)}{(y,y)}$.

Given a nonzero vector α orthogonal to the hyperplane H, the reflection in the hyperplane H, denoted s_α, is given by

$$s_\alpha(x) = x - \langle x, \alpha\rangle\alpha,\ \forall x \in V.$$

It is characterized by the properties, $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(x) = 0$ for all x orthogonal to α. We easily verify that $(s_\alpha(x), s_\alpha(y)) = (x, y)$ for all $x, y \in V$. Also, if w is an automorphism of V which preserves the inner product, then

$$ws_\alpha w^{-1}(\beta) = s_{w\alpha}.$$

Indeed,

$$ws_\alpha w^{-1}(w\alpha) = ws_\alpha(\alpha) = -w\alpha$$

and if $(w\alpha, \beta) = 0$, it follows that $(\alpha, w^{-1}\beta) = 0$. Therefore

$$ws_\alpha w^{-1}(\beta) = ws_\alpha(w^{-1}\beta) = 0.$$

Definition 1.

- A root system of V is a finite subset R of V which satisfies

  1. The set R spans V,
  2. If $\alpha \in R$ then $k\alpha \in R$ if and only if $k = \pm 1$,
  3. If $\alpha \in R$ then $s_\alpha(R) = R$

- A root system R of V is called crystallographic root system if $\langle x, y \rangle \in \mathbb{Z}$ for all $x, y \in R$.

Remark 1. Crystallographic root systems are the root systems corresponding to the semi-simple complex finite dimensional Lie algebras (see section 2.2).

The elements of R are called roots and the group generated by $\{s_\alpha : \alpha \in R\}$ will be denoted by W.

Lemma 1. *The group W is finite.*

Proof. If $\alpha \in R$ then s_α is a permutation of R. The restriction of W to R is faithfull because if $w_1 = w_2$ on R, then since $\operatorname{span}(R) = V$, it follows that $w_1 = w_2$ on V. Therefore W is a subgroup of the symmetric group on the elements of R and hence is finite. □

Definition 2.

- A subset Π of a root system R is called a base for the root system if it is a basis for the vector space V and each element of R is written as a linear combination of elements of Π where all coefficients are either non negative or non positive.

- The elements of R which are written as a non negative linear combination of the elements of Π are called positive roots of R with respect to the base Π. An element α of R is called negative with respect to the base Π if $-\alpha$ is positive with respect to the base Π.

- The subset of R containing the positive roots will be denoted by R^+ and the the subset containing the negative roots by R^-.

Theorem 2. *Every root system has a base.*

The proof of the previous theorem can be found in standard textbooks of Lie algebras (see for instance [**?**, **?**]).

Lemma 2. *If Π is a base for a root system R then*

1. $(\alpha, \beta) < 0$ *for all* $\alpha, \beta \in \Pi$ *with* $\alpha \neq \beta$.

2. *If* $p \in R^+$ *then there exists an* $\alpha \in \Pi$ *such that* $(p, \alpha) > 0$.

Proof. 1. If $\alpha, \beta \in \Pi$ and $(\alpha, \beta) > 0$ then $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in R$. This is a contradiction since $\langle \beta, \alpha \rangle < 0$.

2. Let $p = \sum_{\alpha \in \Pi} k_\alpha \alpha$ where all k_α are non negative. Then

$$0 < (p, p) = \sum_{\alpha \in \Pi} k_\alpha (p, \alpha)$$

and therefore $(p, \alpha) > 0$ for some $\alpha \in \Pi$.

□

Lemma 3. *If* Π *is a base of the root system* R *and* $\alpha \in \Pi$ *then* $s_\alpha (R^+ \setminus \{\alpha\}) = R^+ \setminus \{\alpha\}$.

Proof. Let $p \in R^+ \setminus \{\alpha\}$ and

$$p = \sum_{\beta \in \Pi} k_\beta \beta$$

with $k_\beta \geq 0$. Then

$$s_\alpha(p) = \sum_{\beta \in \Pi} k_\beta s_\alpha(\beta) = \sum_{\beta \in \Pi \setminus \{\alpha\}} k_\beta \beta + k'_\alpha \alpha.$$

But there is a $\beta_0 \in R \setminus \{\alpha\}$ such that $k_{\beta_0} > 0$. Therefore the coefficient of β_0 when we write $s_\alpha(p)$ as a linear combination of the elements of B is positive. Hence $s_\alpha(p)$ is positive and of course different from α. □

Definition 3. If R is a root system with a base Π and $p \in R$ with $p = \sum_{\alpha \in \Pi} k_\alpha \alpha$ then the number $\sum_{\alpha \in \Pi} k_\alpha$ is the height of p and is denoted $\mathrm{ht}(p)$.

For example $\mathrm{ht}(\alpha) = 1$ for all $\alpha \in \Pi$.

Example 1. In the vector space $\mathbb{R}^2$ with the usual euclidean inner product, let R be the subset of $\mathbb{R}^2$ with

$$R = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\},$$

where $\alpha = (1, 0)$, $\beta = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ as shown in fig. 2.1.

It is straightforward to verify that R is a crystallographic root system and that the following subsets are bases for R.

$$\{\alpha, \beta\}, \{\alpha, -\alpha - \beta\}, \{\alpha + \beta, -\beta\}, \{\alpha + \beta, -\alpha\}, \{\beta, -\alpha - \beta\}, \{-\alpha, -\beta\}.$$

This root system is said to be of type A_2. With respect to the base $\Pi = \{\alpha, \beta\}$ we have $\mathrm{ht}(\alpha + \beta) = 2$, while with respect to the base $\Pi = \{\beta, -\alpha - \beta\}$ we have $\mathrm{ht}(\alpha + \beta) = -1$.

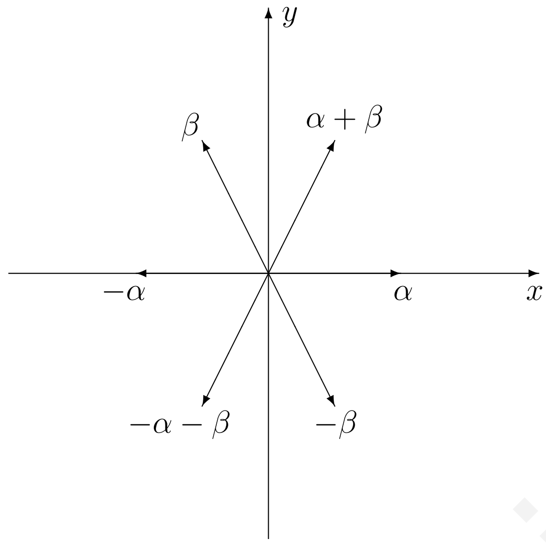


Figure 2.1: Root system of type A_2

The group W is generated by the reflections $s_\alpha, s_\beta, s_{\alpha+\beta}$. Note that

$$s_{\alpha+\beta} = s_\alpha s_\beta,$$

and therefore W is generated by s_α, s_β, i.e. $W = \langle s_\alpha, s_\beta \rangle$. The homomorphism $\phi : W \to S_3$ generated by $\phi(s_\alpha) = (1\,2)$ and $\phi(s_\beta) = (2\,3)$ is an isomorphism of the group W and the symmetric group on 3 elements. There is another way to describe this root system as shown in §2.1.

Lemma 4. *Let R be a root system with base Π and W_0 the group generated by $\{\alpha : \alpha \in \Pi\}$. If $w \in W_0,\ p' \in R^+$ and $p = w(p') \in R^+$ then* $\mathrm{ht}(p) \geq 1$.

Proof. Let $p \in W_0(R^+) \cap R^+$ be such that $\mathrm{ht}(p)$ is minimum. We will show that $\mathrm{ht}(p) = 1$. Assume, for contradiction that $\mathrm{ht}(p) < 1$. Then $p \notin \Pi$ and therefore there exists an $\alpha \in \Pi$ such that $(p, \alpha) > 0$. Hence $\mathrm{ht}(s_\alpha(p)) = \mathrm{ht}(p - \langle p, \alpha \rangle \alpha) = \mathrm{ht}(p) - \langle p, \alpha \rangle < \mathrm{ht}(p)$ and $s_\alpha(p) \in W_0(R^+) \cap R^+$, contradiction. Therefore $\mathrm{ht}(p) = 1$. □

We see, from the proof of the previous lemma, that if $\mathrm{ht}(p) = 1$ then p is a simple root. Therefore we have the following corollary.

Corollary 1. *If $p \in R^+ \setminus \Pi$ then* $\mathrm{ht}(p) > 1$.

Proof. Let $\alpha \in \Pi$ be such that $(p, \alpha) > 0$. Then $\mathrm{ht}(s_\alpha(p)) = \mathrm{ht}(p) - \langle p, \alpha \rangle < \mathrm{ht}(p)$ and therefore $\mathrm{ht}(p) > 1$ (from the previous lemma it couldn't be $\mathrm{ht}(p) = 1$). □

If R is a root system with base Π and $\alpha, \beta \in \Pi$ we denote the order of $s_\alpha s_\beta$ by $n(\alpha, \beta)$. For example for all $\alpha \in \Pi,\ n(\alpha, \alpha) = 1$ and $n(\alpha, \beta) > 1$ for all $\alpha, \beta \in \Pi$ with $\alpha \neq \beta$.

The next theorem shows, how the group W can be described through generators and relations. A proof of this theorem can be found in [**?**, **?**] or [**?**].

Theorem 3. *The group W' generated by the elements $\{\sigma_\alpha : \alpha \in \Pi\}$ and the relations $\{(\sigma_\alpha, \sigma_\beta)^{n(\alpha,\beta)} = 1 : \alpha, \beta \in \Pi\}$ is isomorphic to the group W.*

The groups with presentation as the one described in the previous theorem are known as Coxeter groups. This is the subject of section 2.5.

Next we present the root systems associated with the four classical simple Lie algebras (see section 2.2), as subsets of the euclidean spaces $\mathbb{R}^n$ with the usual inner product. These are the root systems of type A_n, B_n, C_n and D_n.

Root system of type A_n

Let V be the hyperplane of $\mathbb{R}^{n+1}$ for which the coordinates sum to 0 (i.e. vectors orthogonal to $(1, 1, \ldots, 1)$). Let R be the set of vectors in V of length $\sqrt{2}$ with integer coordinates. There are $\binom{n}{2}$ such vectors in all. We use the standard inner product in $\mathbb{R}^{n+1}$ and the standard orthonormal basis $\{e_1, e_2, \ldots, e_n\}$. Then, it is easy to see that

$$R = \{e_i - e_j \mid i, j \in \{1, 2, \ldots, n+1\},\ i \neq j\}.$$

The set R is a root system known as root system of type A_n. The set

$$\Pi = \{\alpha_i = e_i - e_{i+1} \mid i = 1, 2, \ldots, n\}$$

is a base of this root system in the sense that each vector in R is a linear combination of these n vectors with integer coefficients, either all nonnegative or all nonpositive. For example, the positive roots are written as $e_i - e_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$ for $1 \leq i < j \leq n+1$. Therefore $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and the set of positive roots R^+ is given by

$$R^+ = \{e_i - e_{j+1} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j \,|\, 1 \leq i \leq j \leq n\}.$$

The highest root for the root system A_n is the root

$$e_1 - e_n = \alpha_1 + \alpha_2 + \ldots + \alpha_n,$$

of height n. Note that the homomorphism $\phi : W \to S_{n+1}$ generated by

$$\phi(s_{\alpha_i}) = (i \quad i+1),\ i = 1, 2, \ldots, n$$

is an isomorphism between the group W and the symmetric group S_{n+1}.

Root system of type B_n

Let V be the the Euclidean space $\mathbb{R}^n$ with the usual inner product and let R be the subset of V which consist of the vectors with integer coefficient whose length is 1 or $\sqrt{2}$. The subset R consist of the vectors

$$R = \{\pm e_i, \pm e_j \pm e_k \,|\, i = 1, 2, \dots, n,\ 1 \leq j < k \leq n\},$$

where as usual $\{e_1, e_2, \dots, e_n\}$ is the orthonormal basis of $\mathbb{R}^n$. In total there are $2n^2$ such vectors. The set R is a root system known as root system of type B_n. A base for the root system B_n is

$$\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\},$$

where $\alpha_i = e_i - e_{i+1}$, $i = 1, 2, \dots, n-1$ and $\alpha_n = e_n$.
The set of positive roots of B_n is given by

$$R^+ = R_1 \cup R_2 \cup R_3,$$

where

$$R_1 = \{e_i = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n \,|\, i = 1, 2, \dots, n\},$$
$$R_2 = \{e_i - e_{j+1} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j \,|\, 1 \leq i \leq j \leq n-1\} \text{ and}$$
$$R_3 = \{e_i + e_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_n \,|\, 1 \leq i < j \leq n\}.$$

The highest root for the root system of type B_n is the root

$$e_1 + e_n = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_n,$$

of height $2n - 1$

Root system of type C_n

Let V be the the Euclidean space $\mathbb{R}^n$ with the usual inner product and let R be the subset of V which consist of the vectors

$$R = \{\pm 2e_i, \pm e_j \pm e_k \,|\, i = 1, 2, \dots, n,\ 1 \leq j < k \leq n\},$$

where as usual $\{e_1, e_2, \dots, e_n\}$ is the orthonormal basis of $\mathbb{R}^n$. In total there are $2n^2$ such vectors. The set R is a root system known as root system of type C_n. A base for the root system C_n is

$$\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\},$$

where $\alpha_i = e_i - e_{i+1}$, $i = 1, 2, \ldots, n-1$ and $\alpha_n = 2e_n$.
The set of positive roots of C_n is given by

$$R^+ = R_1 \cup R_2 \cup R_3,$$

where

$$R_1 = \{2e_i = 2\alpha_i + 2\alpha_{i+1} + \ldots + 2\alpha_{n-1} + \alpha_n \,|\, i = 1, 2, \ldots, n\},$$
$$R_2 = \{e_i - e_{j+1} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j \,|\, 1 \leq i \leq j \leq n-1\} \text{ and}$$
$$R_3 = \{e_i + e_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{n-1} + \alpha_n \,|\, 1 \leq i < j \leq n\}.$$

The highest root for the root system of type C_n is

$$2e_1 = 2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n,$$

of height $2n - 1$.

Root system of type D_n

Let V be the the Euclidean space $\mathbb{R}^n$ with the usual inner product and let R be the subset of V which consist of all vectors with integer coefficients and length $\sqrt{2}$. The set R is given by

$$R = \{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq n\},$$

where as usual $\{e_1, e_2, \ldots, e_n\}$ is the canonical orthonormal basis of $\mathbb{R}^n$. In total there are $2n^2 - 2n$ such vectors. The set R is a root system known as root system of type D_n. A base for the root system D_n is

$$\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\},$$

where $\alpha_i = e_i - e_{i+1}$, $i = 1, 2, \ldots, n-1$ and $\alpha_n = e_{n-1} + e_n$.
The set of positive roots of D_n is given by

$$R^+ = R_1 \cup R_2 \cup R_3,$$

where

$$R_1 = \{e_i - e_{j+1} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j \,|\, 1 \leq i \leq j \leq n-1\},$$
$$R_2 = \{e_i + e_j = \alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \,|\, 1 \leq i < j \leq n-1\},$$
$$R_3 = \{e_i + e_n = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{n-2} + \alpha_n \,|\, 1 \leq i \leq n-2\} \cup \{e_{n-1} + e_n = \alpha_n\}.$$

The highest root for the root system of type D_n is

$$e_1 + e_2 = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n,$$

of height $2n - 3$.

2.2 Lie algebras

Definition 4. A Lie algebra L over the field $\mathbb{F}$ is an $\mathbb{F}$-vector space with a bilinear map $[\,,]$, called the Lie bracket, which satisfies

- (skew-symmetry) $[x, x] = 0$ for all $x \in L$,
- (Jacobi identity) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in L$.

We give some basic examples of Lie algebras.

Example 2.

1. Write $\mathfrak{gl}_n(\mathbb{C})$ for the $\mathbb{C}$-vector space of all $n \times n$ matrices. The vector space $\mathfrak{gl}_n(\mathbb{C})$ becomes a Lie algebra, known as the general linear algebra, if we define the Lie bracket
$$[x, y] = xy - yx, \quad \text{for } x, y \in \mathfrak{gl}_n(\mathbb{C}).$$
It is straightforward to verify that $[\,,]$ is indeed a Lie bracket.

2. The vector space $\mathbb{R}^3$ becomes a Lie algebra with Lie bracket the cross product.

3. Write $\mathfrak{sl}_n(\mathbb{C})$ for the special linear algebra, the $\mathbb{C}$-vector subspace of $\mathfrak{gl}_n(\mathbb{C})$ of all $n \times n$ traceless matrices. Note that from the property $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ it follows that if $x, y, \in \mathfrak{sl}_n(\mathbb{C})$ then $[x, y] \in \mathfrak{sl}_n(\mathbb{C})$. Therefore the vector space $\mathfrak{sl}_n(\mathbb{C})$ becomes a Lie algebra with Lie bracket
$$[x, y] = xy - yx, \quad \text{for } x, y \in \mathfrak{gl}_n(\mathbb{C}).$$
For $n = 2$ the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, as a vector space, has a basis consisting of the matrices
$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
More generally, if we denote by $e_{i,j}$ the $n \times n$ matrix which has 1 in the i, j position and all other entries zero, then the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, as a vector space, has a basis consisting of the matrices $e_{i,j}$ for $i \neq j$ and $e_{i,i} - e_{i+1,i+1}$ for $i = 1, 2, \ldots, n - 1$.

Definition 5. Let L, L_1, L_2 be Lie algebras with Lie brackets $[\,,]\,,\ [\,,]_1\,,\ [\,,]_2$ respectively.

- A Lie subalgebra L' of L is a vector subspace of L which is a Lie algebra with Lie bracket the Lie bracket of L.
- An ideal L' of L is a Lie subalgebra of L with the property $[x, y] \in L'$ for all $x \in L, y \in L'$.
- A linear map $\phi : L_1 \to L_2$ is an homomorphism of Lie algebras if

  $$\phi\left([x, y]_1\right) = [\phi(x), \phi(y)]_2 \,.$$

  The map ϕ is called monomorphism or epimorphism of Lie algebras if ϕ is injective or surjective respectively. The map ϕ is called isomorphism of Lie algebras if it is both monomorphism and epimorphism. In that case the Lie algebras L_1, L_2 are called isomorphic Lie algebras.

If L is an algebra, that is a vector space with a bilinear form $(x, y) \mapsto xy$, then a derivation on L is a linear map $f : L \to L$ with the property

$$f(xy) = xf(y) + f(x)y, \text{ for all } x, y \in L.$$

For example, any Lie algebra is an algebra with bilinear form the Lie bracket. If L is a Lie algebra, then for $x \in L$, the adjoint map $\mathrm{ad}_x : L \to L$ defined by $\mathrm{ad}_x(y) = [x, y]$ is a derivation with respect to the Lie bracket, i.e. for all $x, y, z \in L$

$$\mathrm{ad}_x([y, z]) = [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]\,.$$

The last equality is exactly the Jacobi identity.

All Lie algebras considered from now on are over the field of complex numbers $\mathbb{C}$.

Example 3. If L is an algebra of dimension n, then the subspace $\mathrm{Der}(L)$ of $\mathfrak{gl}_n(\mathbb{C})$, containing the $n \times n$ matrices corresponding to the derivations of L is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. Indeed if $f, g \in \mathrm{Der}(L)$ then for all $x, y \in L$,

$$\begin{gathered}
[f, g]\,(xy) = \\
fg(xy) - gf(xy) = f(g(x)y + xg(y)) - g(f(x)y + xf(y)) = \\
f(g(x))y + g(x)f(y) + f(x)g(y)) + xf(g(y)) - \\
g(f(x))y - f(x)g(y) - g(x)f(y) - xg(f(y)) = \\
[f, g]\,(x)y + x\,[f, g]\,(y).
\end{gathered}$$

Therefore $[f, g] \in \mathrm{Der}(L)$ and $\mathrm{Der}(L)$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$

If $S \subseteq L$, then the Lie subalgebra spanned by S is the minimal Lie subalgebra of L containing the set $\{x : x \in S\}$. For $S_1, S_2 \subseteq L$ we denote by $[S_1, S_2]$ the Lie subalgebra of L spanned by $\{[x, y] : x \in S_1, x \in S_2\}$. Therefore if L' is a vector subspace of L then L' is a Lie subalgebra of L if $[L', L'] \subseteq L'$, while it is an ideal of L if $[L, L'] \subseteq L'$. If L_1, L_2 are ideals of L then, from the Jacobi identity, it follows that $[L_1, L_2]$ is also an ideal of L. The ideal $[L, L]$ of L is called the derived algebra of L and the Lie algebra L is called abelian if its derived algebra vanishes. The derived series of L is the decreasing series of ideals

$$L^{(0)} = L, L^{(i+1)} = \left[L^{(i)}, L^{(i)}\right] \ i \geq 0.$$

The Lie algebra L is called solvable if $L^{(k)} = 0$ for some $k \in \mathbb{N}$ while it is called simple if it is not abelian and has no non trivial ideals (i.e. no ideals other than 0 and L). The radical, $\mathrm{rad}(L)$, of L is the maximal solvable ideal of L.

The next lemma is an immediate consequence of the definitions.

Lemma 5. *Let L be a Lie algebra. Then the following are equivalent*

i) *The radical of L is trivial,* $\mathrm{rad}(L) = 0$.

ii) *The Lie algebra L has no nontrivial abelian ideals.*

Proof. If I is an abelian ideal of L then I is solvable and hence $\mathrm{rad}(L) \supseteq I$. Therefore *ii)* $\Rightarrow$ *i)*. Conversely if $I = \mathrm{rad}(L) \neq 0$ and $I^{(k)} = 0$ with k minimum then $I^{(k-1)}$ is a nontrivial abelian ideal of L. $\square$

Example 4. Let L be the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ with basis e, f, h as in example 2. Then we have the relations $[e, f] = h,\ [e, h] = -2e,\ [f, h] = 2f$. If I is an ideal of L and $e \in I$ then $[e, f] = h \in I$ and $\left[\frac{1}{2}f, h\right] = f \in I$. Similarly if $f \in I$ or $h \in I$ we deduce that $e, f, h \in I$ and therefore $I = L$. If $x = \lambda_1 e + \lambda_2 f + \lambda_3 h \in I$ then $\left[[e, x], \frac{1}{2}e\right] = \lambda_2 e$ and $\left[\frac{1}{2}f, [x, f]\right] = \lambda_2 f$. Therefore $I = L$ and we conclude that L is simple, $\mathrm{rad}(L) = 0$ and that $L^{(k)} = L$ for all $k \in \mathbb{N}$.

Definition 6. The Killing form $k : L \times L \to \mathbb{C}$ of the Lie algebra L is the bilinear map defined by

$$k(x, y) = \mathrm{tr}(\mathrm{ad}_x \, \mathrm{ad}_y).$$

Proposition 1. *For a Lie algebra L the following are equivalent.*

1. *The radical of L is trivial,* $\mathrm{rad}(L) = 0$.

2. *The Lie algebra L is the direct sum of finitely many simple Lie algebras.*

3. *The Killing form on L is nondegenerate.*

The equivalence of 1. and 2. is a theorem of Weyl while the equivalence of 1. and 3. is a theorem of Cartan known as Cartan's second criterion. For a proof of this proposition see [**?**, **?**, **?**].

Definition 7. A Lie algebra satisfying one of the properties of proposition 1 is called semisimple .

From now on all Lie algebras are assumed semisimple. An element $x \in L$ is called semisimple if the linear map ad_x is diagonalizable.

Definition 8. A Lie subalgebra H of a Lie algebra L is said to be a Cartan subalgebra if it is abelian, every element $x \in H$ is semisimple and it is maximal with these properties.

In the following we give some basic properties of semisimple Lie subalgebras

All Cartan subalgebras of L have the same dimension known as the rank of L. Let H be a Cartan subalgebra of L. A root of the Lie algebra L is a nonzero function $\alpha \in H^*$ such that for all $h \in H,\ \alpha(h)$ is an eigenvalue of ad_h corresponding to a common eigenvector $x \in L$. We denote by L_α the weight space $L_\alpha = \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}$. For all roots $\alpha \in H^*,\ \dim(L_\alpha) = 1$.

The Cartan subalgebra is a maximal abelian subalgebra, therefore if $\alpha = 0,\ L_\alpha = H$. The set of the roots of L is denoted by R. The Lie algebra L is decomposed as

$$L = H \oplus \bigoplus_{\alpha \in R} L_\alpha. \tag{2.1}$$

The above decomposition is known as the root space decomposition.

Let $X_\alpha \in L_\alpha$ be nonzero. Then $L_\alpha = \mathrm{span}(X_\alpha)$ and $[h, X_\alpha] = \alpha(h)X_\alpha$ for all $h \in H$ and $\alpha \in R$. If $\alpha, \beta \in R$ are such that $\alpha + \beta \in R$ then

$$[h, [X_\alpha, X_\beta]] = [X_\alpha, [h, X_\beta]] + [[h, X_\alpha], X_\beta] =$$
$$(\alpha(h) + \beta(h)) [X_\alpha, X_\beta] \in L_{\alpha+\beta},$$

and therefore $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

The roots R generate H^* as a vector space over $\mathbb{C}$. The Killing form on H is nondegenerate and defines an inner product, denoted $(\,,)$, on the vector space H^* as follows. For $\alpha \in H^*$ define h_α to be the element of H defined by the relation

$$k(h, h_\alpha) = \alpha(h),\ \forall h \in H.$$

If $\alpha, \beta \in H^*$ we define $(\alpha, \beta) = k(h_\alpha, h_\beta)$.

The set R, of the roots of L satisfies the following properties.

1. It spans H^* as a vector space over $\mathbb{R}$.

2. If $\alpha \in R$ then $k\alpha \in R$ if and only if $k = \pm 1$.

3. For all $\alpha, \beta \in H$, $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.

4. If $\alpha \in H$ and s_α is the reflection defined by the relation

$$s_\alpha(x) = x - \langle x, \alpha \rangle \alpha,$$

then $s_\alpha(R) = R$.

Therefore R is a root system in the sense of section 2.1. It can be proved that if R is a crystallographic root system then it is the root system of some finite dimensional complex semisimple Lie algebra (see [**?**, **?**, **?**, **?**]).

The root space decomposition (2.1) can be rewritten as

$$L = H \oplus \bigoplus_{\alpha \in R} L_\alpha = H \oplus \bigoplus_{\alpha \in R^+} (L_\alpha \oplus L_{-\alpha}) = L^- \oplus H \oplus L^+, \tag{2.2}$$

where $L^- = \bigoplus_{\alpha \in R^+} L_{-\alpha}$ and $L^+ = \bigoplus_{\alpha \in R^+} L_\alpha$.

The decomposition (2.2) is known as triangular decomposition of the Lie algebra L because L^+ can be represented by upper triangular matrices and L^+ by lower triangular matrices.

Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of simple roots of the Lie algebra L (i.e. a set of simple roots of the root system R of the Lie algebra L, see section 2.1) and let W be the group generated by Π. The group W is called Weyl group of the Lie algebra L.

Definition 9.

1. If $\alpha, \beta \in R$ and $\alpha + \beta \in R$ define $N_{\alpha,\beta}$ by the relation

$$[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}.$$

If $\alpha + \beta \notin R$ we define $N_{\alpha,\beta} = 0$

2. The α-string of roots through β is the sequence of roots

$$\beta - r\alpha, \beta - (r-1)\alpha, \ldots, \beta, \ldots, \beta + q\alpha$$

where $\beta + i\alpha \in R$ for all $i = -r, -r+1, \ldots, q$ and $\beta - (r+1)\alpha, \beta + (q+1)\alpha \notin R$.

For a proof of the following two propositions see [**?**, **?**, **?**]

Proposition 2. *The elements X_α can be chosen so that*

i) $[X_\alpha, X_{-\alpha}] = h_\alpha$.

ii) *If α, β are roots with $\beta \neq \pm\alpha$ and $\beta - r\alpha, \ldots, \beta + q\alpha$ is the α-string through β then*

$$N_{\alpha,\beta}^2 = q(r+1)\frac{|\alpha+\beta|^2}{|\beta|^2}$$

Proposition 3. *If $\alpha, \beta, \alpha+\beta \in R$ then*

$$q|\alpha+\beta|^2 = (r+1)|\beta|^2.$$

Therefore we have the following theorem.

Theorem 4 (C. Chevalley). *Let X_α be chosen as in proposition 2. Then the basis*

$$\{h_{\alpha_i} : i = 1, 2, \ldots, n\} \cup \{X_\alpha : \alpha \in R\}$$

satisfies

1. $\left[h_{\alpha_i}, h_{\alpha_j}\right] = 0$ *for* $i \neq j$.
2. $[h_{\alpha_i}, X_\alpha] = \langle \alpha, \alpha_i \rangle X_\alpha$.
3. $[X_\alpha, X_{-\alpha}] = h_\alpha$.
4. *If* $\alpha + \beta \in R$, $[X_\alpha, X_\beta] = \pm(r+1)X_{\alpha+\beta}$.
5. *If* $\alpha + \beta \notin R$ *and* $\alpha + \beta \neq 0$ *then* $[X_\alpha, X_\beta] = 0$.

Definition 10. A basis satisfying the properties of theorem 4 is said to be a Chevalley basis.

There is a matrix and a diagram associated to each complex semisimple Lie algebra known as the Cartan matrix and the Dynkin diagram respectively. These are of particular interest since they are used for the classification of the simple Lie algerbas. We can define the notion of isomorphic root systems and then one shows that two root systems are isomorphic if and only if they have the same Dynkin diagram (or equivalently the same Cartan matrix). Also if two Lie algebras have isomorphic root systems then they are isomorphic. Therefore for the classification of simple Lie algebras is sufficient to classify the connected Dynkin diagrams (it can be shown that a Lie algebra is simple if and only if its Dynkin diagram is connected), or equivalently the indecomposable Cartan matrices. It turns out that the Dynkin diagrams associated with the simple Lie algebras are four infinite families (where the associated Lie algebras are known as the classical simple Lie algebras) A_n, B_n, C_n and D_n and five exceptional cases (where the associated Lie algebras are known as the exceptional simple Lie algebras) E_6, E_7, E_8, F_4 and G_2.

Definition 11. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of simple roots for the root system of a semisimple complex Lie algebra L.

- The integers $C_{i,j} = \langle a_i, a_j \rangle = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ are the Cartan integers of the Lie algebra L.
- The matrix $C = (C_{i,j})$ is the Cartan matrix of the Lie algebra L.
- The graph Γ with n vertices $\mathcal{V} = \{v_1, \ldots, v_n\}$ and $C_{i,j}C_{j,i}$ edges between the vertices i, j is the Coxeter-Dynkin diagram of the Lie algebra L.
- The Dynkin diagram of the Lie algebra L is the Coxeter-Dynkin diagram of L where whenever we have multiple edges between two vertices v_i, v_j, we put an arrow pointing to the vertex v_i if the root α_i is shorter than the root α_j, or we put an arrow pointing to the vertex v_j if the root α_j is longer than the root α_j.

Now we give a linear representation of the four classical Lie algebras whose root systems are of type A_n, B_n, C_n and D_n (see [**?**, **?**], see also section 2.1). We also present the Cartan matrices and the Dynkin diagrams associated with these Lie algebras. By a linear representation we mean that we view our Lie algebra as a Lie subalgebra of the general linear algebra $\mathfrak{gl}_n(C)$ of $n \times n$ complex matrices with the usual Lie bracket $[x, y] = xy - yx$. All Lie algebras will be subalgebras of the special linear Lie algebra (the one with root system of type A_n). There will be of the form $\mathfrak{gl}_S(n, \mathbb{C})$ for a suitable matrix S, where $\mathfrak{gl}_S(n, \mathbb{C})$ is described in the next lemma.

Lemma 6. *Let S be an invertible $n \times n$ complex matrix. We define*

$$\mathfrak{gl}_S(n, \mathbb{C}) = \{M \in \mathfrak{gl}_n(\mathbb{C}) : M + S^{-1}M^T S = 0\}.$$

Then the vector space $\mathfrak{gl}_S(n, \mathbb{C})$ is a Lie subalgebra of the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of the traceless $n \times n$ complex matrices.

Proof. First, if $M \in \mathfrak{gl}_S(n, \mathbb{C})$ it follows that

$$\operatorname{tr}(-M) = \operatorname{tr}(S^{-1}M^T S) = \operatorname{tr}(M) \Rightarrow \operatorname{tr}(M) = 0$$

and therefore $M \in \mathfrak{sl}_n(\mathbb{C})$. Since for all $M, N \in \mathfrak{gl}_S(n, \mathbb{C})$, we have

$$S^{-1}[M, N]^T S = S^{-1}(N^T M^T - M^T N^T)S =$$
$$S^{-1}N^T S S^{-1} M^T S - S^{-1} M^T S S^{-1} N^T S = -[M, N]$$

it follows that $\mathfrak{gl}_S(n, \mathbb{C})$ is indeed a Lie subalgebra of $\mathfrak{sl}_n(\mathbb{C})$. $\square$

In the next lemma we show that if the matrices S_1, S_2 are congruent (i.e. $P^t S_1 P = S_2$ for some invertible matrix P) then the Lie algebras $\mathfrak{gl}_{S_1}(n, \mathbb{C})$ and $\mathfrak{gl}_{S_2}(n, \mathbb{C})$ are isomorphic.

Lemma 7. *Let S_1, S_2 be two invertible congruent $n \times n$ complex matrices; i.e. $P^t S_1 P = S_2$ for some invertible matrix P. Then the homomorphism*

$$\phi : \mathfrak{gl}_{S_1}(n, \mathbb{C}) \to \mathfrak{gl}_{S_2}(n, \mathbb{C}),\ M \mapsto P^{-1} M P$$

is an isomorphism of Lie algebras.

We denote by $E_{i,j}$ the $n \times n$ matrix with zeros everywhere except in the position i, j where it has 1. We denote by Δ_n the $n \times n$ matrix $\Delta_n = \sum_{i=1}^{n} E_{i,n+1-i}$ with ones on the antidiagonal and zeros elsewhere. Let $A = (a_{i,j})$ be an $n \times n$ matrix. The per transpose of the matrix A is the matrix $\Delta_n A^T \Delta_n$ and is denoted by A^{PT}. The i, j entry of the matrix A^{PT} is $a_{n+1-j,n+1-i}$. We say that the matrix A is per symmetric if $A^{PT} = A$ and we say that the matrix A is per skew-symmetric if $A^{PT} = -A$. Equivalently the matrix A is per symmetric if it is symmetric with respect to its anti diagonal (i.e. $a_{i,j} = a_{n+1-j,n+1-i}$ for all $1 \leq i, j \leq n$) and it is per skew-symmetric if it is skew-symmetric with respect to its anti diagonal (i.e. $a_{i,j} = -a_{n+1-j,n+1-i}$ for all $1 \leq i, j \leq n$).

Simple Lie algebra of type A_n

The simple Lie algebra with root system A_n is the special linear Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$, the vector space of the traceless $n+1 \times n+1$ matrices. The root system A_n with base $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ contains the roots $\pm(\alpha_i + \alpha_{i+1} + \ldots + \alpha_j)$ for all $1 \leq i \leq j \leq n$. The Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$, as a vector space has a basis containing the matrices $E_{i,j}$ for $1 \leq i \neq j \leq n$ and also the matrices $E_{i,i} - E_{i+1,i+1}$, $i = 1, 2, \ldots, n$. A Cartan subalgebra H of the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$ is the subalgebra spanned by the diagonal matrices in $\mathfrak{sl}_{n+1}(\mathbb{C})$, $E_{i,i} - E_{i+1,i+1}$, $i = 1, 2, \ldots, n$, i.e.

$$H = \left\{ \sum_{i=1}^{n+1} \lambda_i E_{i,i} : \sum_{i=1}^{n+1} \lambda_i = 0 \right\}.$$

The rank of the Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$ (i.e. the dimension of H) is n. If $h = \sum_{i=1}^{n+1} \lambda_i E_{i,i} \in H$ then $[h, E_{i,j}] = (\lambda_i - \lambda_j) E_{i,j}$ and therefore $E_{i,j}$ is a root vector corresponding to the root $\alpha \in H^*$, $h \mapsto \lambda_i - \lambda_j$. The root system R of type A_n contains the roots

$$R = \left\{ \alpha_{i,j} \in H^* : \alpha_{i,j}\left(\sum_{i=1}^{n+1} \lambda_i E_{i,i}\right) = \lambda_i - \lambda_j \right\}.$$

A base of the root system A_n consist of the roots

$$\Pi = \{\alpha_i := \alpha_{i,i+1} : i = 1, 2, \dots, n\}.$$

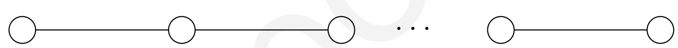
From §2.1 it follows that the Cartan integers for the root system A_n are

$$C_{i,j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = (\alpha_i, \alpha_j) = \begin{cases} 2, \text{ if } i = j, \\ -1, \text{ if } |i - j| = 1, \\ 0, \text{ if } |i - j| > 1. \end{cases}$$

Therefore its Cartan matrix is

$$C_{A_n} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \tag{2.3}$$

and its Dynkin diagram is



Simple Lie algebra of type B_n

Let S be the $2n + 1 \times 2n + 1$ matrix of the form

$$S = \begin{pmatrix} 0 & 0 & \Delta_n \\ 0 & -2 & 0 \\ \Delta_n & 0 & 0 \end{pmatrix}.$$

The simple Lie algebra with root system of type B_n is the orthogonal Lie algebra $\mathfrak{o}_{2n+1}(\mathbb{C})$ of the square $2n + 1 \times 2n + 1$ matrices M which satisfy $M + S^{-1}M^T S = 0$ or equivalently $\mathfrak{o}_{2n+1}(\mathbb{C}) = \mathfrak{gl}_S(n, \mathbb{C})$. The Lie algebra $L = \mathfrak{o}_{2n+1}(\mathbb{C})$ contains the matrices M of the form

$$M = \begin{pmatrix} A & 2\Delta_n w & B \\ v^T & 0 & w^T \\ C & 2\Delta_n v & D \end{pmatrix},$$

where $D = -A^{PT}$, the matrices B and C are per skew-symmetric matrices and v, w are column vectors of size n. The dimension of L is $n(2n + 1)$ and a basis of L contains the matrices

$$
\begin{gathered}
H_{\alpha_i} = E_{i,i} - E_{2n+2-i,2n+2-1},\ i = 1, 2, \ldots, n, \\
X_{\alpha_i+\ldots+\alpha_n} = 2E_{i,n+1} + E_{n+1,2n+2-i},\ i = 1, 2, \ldots, n, \\
X_{-\alpha_i-\ldots-\alpha_n} = -2E_{2n+2-i,n+1} - E_{n+1,i},\ i = 1, 2, \ldots, n, \\
X_{\alpha_i+\ldots+\alpha_j} = E_{i,j} - E_{2n+2-j,2n+2-i},\ 1 \le i \le j < n, \\
X_{-\alpha_i-\ldots-\alpha_j} = -E_{j,i} + E_{2n+2-i,2n+2-j},\ 1 \le i \le j < n, \\
X_{\alpha_i+\ldots+\alpha_{j-1}+2\alpha_j+\ldots+2\alpha_n} = E_{i,2n+2-j} - E_{j,2n+2-i},\ 1 \le i < j \le n, \\
X_{-\alpha_i-\ldots-\alpha_{j-1}-2\alpha_j-\ldots-2\alpha_n} = -E_{2n+2-j,i} + E_{2n+2-i,j},\ 1 \le i < j \le n.
\end{gathered}
$$

A Cartan subalgebra H of the Lie algebra $\mathfrak{o}_{2n+1}(\mathbb{C})$ is the subalgebra spanned by the diagonal matrices in $\mathfrak{o}_{2n+1}(\mathbb{C})$, H_{α_i}, $i = 1, 2, \ldots, n$. The rank of the Lie algebra $\mathfrak{o}_{2n+1}(\mathbb{C})$ is n.

If $h = \sum_{i=1}^{n+1} \lambda_i H_{\alpha_i} \in H$ then $[h, X_\alpha] = (\alpha(h))X_\alpha$ where $\alpha \in H^*$ is the corresponding root and $\alpha(h)$ is determined by the relations ($\delta_{i,j}$ is the Kronecker delta)

$$\alpha_i(H_{\alpha_j}) = \delta_{i,j} - \delta_{i+1,j},\ i = 1, 2, \ldots, n-1 \text{ and } \alpha_n(H_{\alpha_j}) = \delta_{j,n}.$$

Therefore X_α is a root vector corresponding to the root $\alpha \in H^*$. A base of the root system B_n consist of the roots

$$\Pi = \{\alpha_i : i = 1, 2, \ldots, n\}.$$

From §2.1 it follows that the Cartan integers for the root system B_n are

$$C_{i,j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = (\alpha_i, \alpha_j) = \begin{cases} 2, \text{ if } i = j, \\ -1, \text{ if } |i-j| = 1, \\ 0, \text{ if } |i-j| > 1 \end{cases} \quad \text{for all } 1 \le i, j \le n-1$$

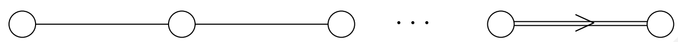
and

$$
\begin{gathered}
C_{i,n} = C_{n,i} = 0, \text{ for } 1 \le i \le n-2, \\
C_{n,n} = 2, \\
C_{n-1,n} = 2C_{n,n-1} = -2.
\end{gathered}
$$

Its Cartan matrix is

$$C_{B_n} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}. \tag{2.4}$$

and its Dynkin diagram is



Simple Lie algebra of type C_n

The simple Lie algebra with root system C_n is the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C}) = \mathfrak{gl}_S(2n, \mathbb{C})$ where S is the matrix

$$S = \begin{pmatrix} 0 & \Delta_n \\ -\Delta_n & 0 \end{pmatrix}.$$

We call it the symplectic Lie algebra. Note that $\mathfrak{sp}_{2n}(\mathbb{C})$ contains the $2n \times 2n$ matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $D = -A^{PT}$ and the matrices B and C are per symmetric. The dimension of this Lie algebra is $\frac{n(2n+1)}{2}$ and a basis contains the matrices

$$\begin{aligned} H_{\alpha_i} &= E_{i,i} - E_{2n+1-i,2n+1-1},\ i = 1, 2, \ldots, n, \\ X_{2\alpha_i+\ldots+2\alpha_{n-1}+\alpha_n} &= E_{i,2n+1-i},\ i = 1, 2, \ldots, n, \\ X_{-2\alpha_i-\ldots-2\alpha_{n-1}-\alpha_n} &= -E_{2n+1-i,i},\ i = 1, 2, \ldots, n \\ X_{\alpha_i+\ldots+\alpha_j} &= E_{i,j+1} - E_{2n-j,2n+1-i},\ 1 \leq i \leq j < n, \\ X_{-\alpha_i-\ldots-\alpha_j} &= -E_{j+1,i} + E_{2n+1-i,2n-j},\ 1 \leq i \leq j < n, \\ X_{\alpha_i+\ldots+\alpha_{j-1}+2\alpha_j+\ldots+2\alpha_{n-1}+\alpha_n} &= E_{i,2n+1-j} + E_{j,2n+1-i},\ 1 \leq i < j \leq n, \\ X_{-\alpha_i-\ldots-\alpha_{j-1}-2\alpha_j-\ldots-2\alpha_{n-1}-\alpha_n} &= -E_{2n+1-i,j} - E_{2n+1-j,i},\ 1 \leq i < j \leq n. \end{aligned}$$

A Cartan subalgebra H of the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is the subalgebra spanned by the diagonal matrices in $\mathfrak{sp}_{2n}(\mathbb{C})$, H_{α_i}, $i = 1, 2, \ldots, n$. The rank of the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is n.

If $h = \sum_{i=1}^{n+1} \lambda_i H_{\alpha_i} \in H$ then $[h, X_\alpha] = (\alpha(h))X_\alpha$ where $\alpha \in H^*$ is the corresponding root and $\alpha(h)$ is determined by the relations ($\delta_{i,j}$ is the Kronecker delta)

$$\alpha_i(H_{\alpha_j}) = \delta_{i,j} - \delta_{i+1,j},\ i = 1, 2, \dots, n-1 \text{ and } \alpha_n(H_{\alpha_j}) = 2\delta_{j,n}.$$

Therefore X_α is a root vector corresponding to the root $\alpha \in H^*$. A base of the root system C_n consist of the roots

$$\Pi = \{\alpha_i : i = 1, 2, \dots, n\}.$$

From §2.1 it follows that the Cartan integers for the root system C_n are

$$C_{i,j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = (\alpha_i, \alpha_j) = \begin{cases} 2, \text{ if } i = j, \\ -1, \text{ if } |i-j| = 1, \\ 0, \text{ if } |i-j| > 1 \end{cases} \quad \text{for all } 1 \le i, j \le n-1$$

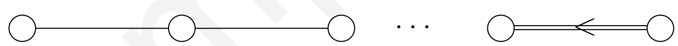
and

$$C_{i,n} = C_{n,i} = 0,\ \forall 1 \le i \le n-2,$$
$$C_{n-1,n-1} = C_{n,n} = 2, C_{n,n-1} = 2C_{n-1,n} = -2.$$

Therefore its Cartan matrix is the transpose of the Cartan matrix of the root system of type B_n

$$C_{C_n} = C_{B_n}^T$$

and its Dynkin diagram is



Simple Lie algebra of type D_n

The simple Lie algebra with root system D_n is the orthogonal Lie algebra $\mathfrak{o}_{2n}(\mathbb{C}) = \mathfrak{gl}_S(2n, \mathbb{C})$ where S is the matrix

$$S = \begin{pmatrix} 0 & \Delta_n \\ \Delta_n & 0 \end{pmatrix}.$$

The Lie algebra $\mathfrak{o}_{2n}(\mathbb{C})$ consists the $2n \times 2n$ matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $D = -A^{PT}$ and the matrices B and C are per skew-symmetric. Its dimension is $\frac{n(2n-1)}{2}$ and a basis contains the matrices

$$H_{\alpha_i} = E_{i,i} - E_{2n+2-i,2n+2-1},\ i = 1, 2, \ldots, n,$$
$$X_{\alpha_i+\ldots+\alpha_{n-2}+\alpha_n} = E_{i,n+1} - E_{n,2n+1-i},\ i = 1, 2, \ldots, n-2,$$
$$X_{-\alpha_i-\ldots-\alpha_{n-2}-\alpha_n} = -E_{n+1,i} + E_{2n+1-i,n},\ i = 1, 2, \ldots, n-2,$$
$$X_{\alpha_n} = E_{n-1,n+1} - E_{n,n+2},$$
$$X_{-\alpha_n} = -E_{n+1,n-1} + E_{n+2,n},$$
$$X_{\alpha_i+\ldots+\alpha_j} = E_{i,j+1} - E_{2n-j,2n+1-i},\ 1 \leq i \leq j < n,$$
$$X_{-\alpha_i-\ldots-\alpha_j} = -E_{j+1,i} + E_{2n+1-i,2n-j},\ 1 \leq i \leq j < n,$$
$$X_{\alpha_i+\ldots+\alpha_{j-1}+2\alpha_j+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n} = E_{i,2n+1-j} - E_{j,2n+1-i},\ 1 \leq i < j \leq n,$$
$$X_{-\alpha_i-\ldots-\alpha_{j-1}-2\alpha_j-\ldots-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n} = -E_{2n+1-j,i} + E_{2n+1-i,j},\ 1 \leq i < j \leq n.$$

A Cartan subalgebra H of the Lie algebra $\mathfrak{o}_{2n+1}(\mathbb{C})$ is the subalgebra spanned by the diagonal matrices in $\mathfrak{o}_{2n+1}(\mathbb{C})$, H_{α_i}, $i = 1, 2, \ldots, n$. The rank of the Lie algebra $\mathfrak{o}_{2n+1}(\mathbb{C})$ is n.

If $h = \sum_{i=1}^{n+1} \lambda_i H_{\alpha_i} \in H$ then $[h, X_\alpha] = (\alpha(h))X_\alpha$ where $\alpha \in H^*$ is the corresponding root and $\alpha(h)$ is determined by the relations ($\delta_{i,j}$ is the Kronecker delta)

$$\alpha_i(H_{\alpha_j}) = \delta_{i,j} - \delta_{i+1,j},\ i = 1, 2, \ldots, n-1 \text{ and } \alpha_n(H_{\alpha_j}) = \delta_{j,n-1} + \delta_{j,n}.$$

Therefore X_α is a root vector corresponding to the root $\alpha \in H^*$. A base of the root system D_n consist of the roots

$$\Pi = \{\alpha_i : i = 1, 2, \ldots, n\}.$$

From §2.1 it follows that the Cartan integers for the root system D_n are

$$C_{i,j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = (\alpha_i, \alpha_j) = \begin{cases} 2, \text{ if } i = j, \\ -1, \text{ if } |i-j| = 1, \\ 0, \text{ if } |i-j| > 1 \end{cases} \quad \text{for all } 1 \leq i, j \leq n-1$$

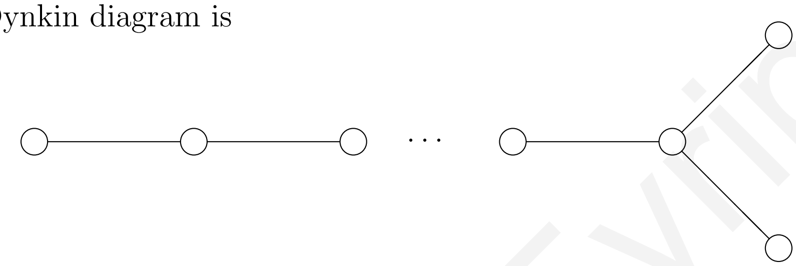
and

$$C_{i,n} = C_{n,i} = 0,\ \forall 1 \leq i \leq n-3,$$
$$C_{n-1,n-1} = C_{n,n} = 2, C_{n-2,n} = 2C_{n,n-2} = -1, C_{n,n-1} = C_{n-1,n} = 0.$$

Therefore its Cartan matrix is

$$C_{D_n} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix} \tag{2.5}$$

and its Dynkin diagram is



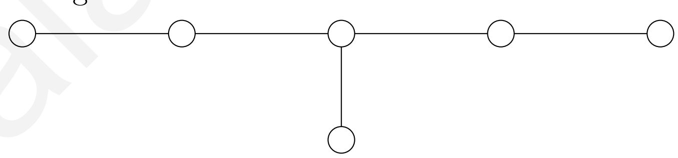
The Cartan matrices and Dynkin diagrams for the exceptional Lie algebras are

Simple Lie algebra of type E_6

Its Cartan matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{2.6}$$

and its Dynkin diagram is

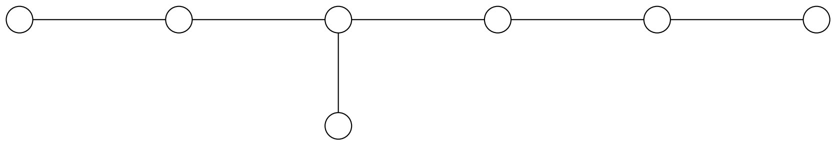


Simple Lie algebra of type E_7

Its Cartan matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{2.7}$$

and its Dynkin diagram is

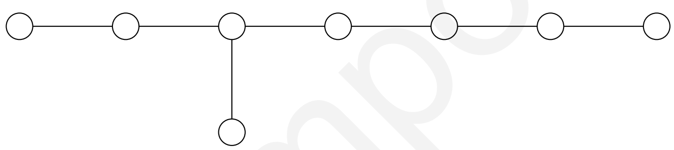


Simple Lie algebra of type E_8

Its Cartan matrix is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{2.8}$$

and its Dynkin diagram is

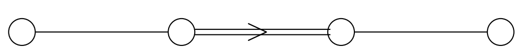


Simple Lie algebra of type F_4

Its Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{2.9}$$

and its Dynkin diagram is

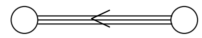


Simple Lie algebra of type G_2

Its Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \tag{2.10}$$

and its Dynkin diagram is



Many authors prefer to choose different matrices S than the ones we give here and get isomorphic representations of the classical Lie algebras (see lemma 7). We prefer the representations given in [**?**, **?**]. However the most common representations are given (see for instance [**?**, **?**]), for the case of the Lie algebra of type B_n, by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix} = P^T \begin{pmatrix} 0 & 0 & \Delta_n \\ 0 & -2 & 0 \\ \Delta_n & 0 & 0 \end{pmatrix} P,$$

where P is the invertible matrix

$$P = \begin{pmatrix} 0 & \Delta_n & 0 \\ \frac{\mathrm{i}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}.$$

For the Lie algebra of type C_n by the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = P^T \begin{pmatrix} 0 & \Delta_n \\ -\Delta_n & 0 \end{pmatrix} P,$$

where P is the invertible matrix

$$P = \begin{pmatrix} \Delta_n & 0 \\ 0 & I_n \end{pmatrix}.$$

For the Lie algebra of type D_n by the matrix

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} = P^T \begin{pmatrix} 0 & \Delta_n \\ \Delta_n & 0 \end{pmatrix} P,$$

where P is the invertible matrix

$$P = \begin{pmatrix} \Delta_n & 0 \\ 0 & I_n \end{pmatrix}.$$

The finite dimensional simple Lie algebras are classified via their root system. All data of the root system is encoded in the Cartan matrix, or in the Dynkin diagram.

Cartan matrices can be defined abstractly.

Definition 12. A Cartan matrix is an $n \times n$-integer matrix C which obeys

- $C_{i,i} = 2$,
- $C_{i,j} = 0 \Rightarrow C_{j,i} = 0, \ \forall i, j$
- $C_{i,j} \leq 0, \ \forall i \neq j$,
- $\det C > 0$.

Given a semisimple Lie algebra L with Cartan matrix C, then there is a set of generators of L,

$$\{x_i^{\pm}, h_i : 1 \leq i \leq n\},$$

where $x_i^+ \in L_{\alpha_i}, x_i^- \in L_{-\alpha_i}$, which are subjected to the Chevalley-Serre relations

- $[h_i, h_j] = 0$,
- $[h_i, x_j^{\pm}] = \pm C_{i,j} x_j^{\pm}$,
- $[x_i^+, x_j^-] = \delta_{i,j} h_i$,
- $(ad_{x_i^{\pm}})^{1-C_{j,i}}(x_j^{\pm})) = 0$.

The converse of the previous statement, is also true and is a theorem of Serre (see [**?**, **?**]).

Theorem 5 (J.-P. Serre). *Let C be a Cartan matrix and L the Lie algebra generated by the generators $\{x_i^{\pm}, h_i : 1 \leq i \leq n\}$, which are subjected to the Chevalley-Serre relations. Then L is a finite dimensional semisimple Lie algebra and its Cartan subalgebra is generated by h_i, $i = 1, 2, \ldots, n$.*

Relaxing the last condition on Cartan matrices we obtain the so called generalized Cartan matrices. Generalized Cartan matrices are classified into three disjoint categories, finite, affine and indefinite (see [**?**] Chapter 4). Finite are the usual Cartan matrices associated with complex semi-simple finite dimensional Lie algebras, while affine and indefinite, give rise to infinite dimensional Lie algebras.

An affine Cartan matrix is one for which $\det C = 0$ and each proper principal minor of C is positive. Thus each $(n-1) \times (n-1)$ submatrix of C obtained by removing an i^{th} row and i^{th} column is a Cartan matrix. Chevalley-Serre relations on affine Cartan matrices give rise to affine Lie algebras. This important subclass of generalized Cartan matrices is characterized by the property that they are symmetrizable and the corresponding symmetric matrices DC are positive semidefinite.

With each affine Cartan matrix C we associate a graph, which we also call the Dynkin diagram. It is a connected graph with n vertices $\{v_1, \ldots, v_n\}$, and $C_{i,j}C_{j,i}$ edges between the vertices v_i, v_j for $i \neq j$. In case $C_{i,j} < C_{j,i}$ we put an arrow in edge (v_i, v_j) pointing to

v_j. For an affine Cartan matrix it is customary to enumerate the vertices as $\{v_0, v_1, \ldots, v_n\}$ so that the corresponding Dynkin diagram has $n+1$ vertices.

Similar to root systems there are the so called affine root systems. We will not give the general definitions but instead we will describe how from an irreducible root system of type X we obtain an affine root system of type $X^{(1)}$. From this root system we define an affine Cartan matrix C and an affine Dynkin diagram and therefore we obtain an affine Lie algebra. These are the untwisted affine Lie algebras.

Let R be an irreducible root system on the Euclidean space V of type X. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a base of R and α_0 its highest root. In the Euclidean space $V \times \mathbb{R}$ (with inner product $((v_1, r_1), (v_2, r_2)) = (v_1, v_2) + r_1 r_2$) consider the set

$$\tilde{R} = \{(\alpha, i) : i \in \mathbb{Z}, (\alpha, i) \neq (0, 0)\}.$$

This set is called the affine root system of type $X^{(1)}$. As in the case of root systems we can show that the set

$$\tilde{\Pi} = \{\tilde{\alpha_0} = (-\alpha_0, 1), \tilde{\alpha_1} = (\alpha_1, 0), \tilde{\alpha_2} = (\alpha_2, 0), \ldots, \tilde{\alpha_n} = (\alpha_n, 0)\},$$

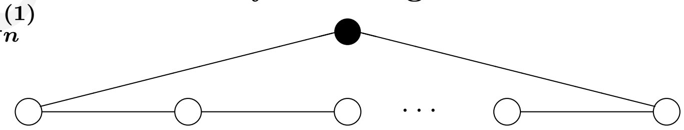
is a base of $\tilde{R}$, in the sense that every element of $\tilde{R}$ is a linear combination of elements of $\tilde{\Pi}$ where all coefficients are either non negative or non positive. This follows from the fact that for every root $\alpha \in R$, $\alpha_0 - \alpha \in R^+$.

The affine Cartan matrix associated to the root system $\tilde{R}$ (of type $X^{(1)}$) is the $n+1 \times n+1$ matrix C defined by $C_{i,j} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$, $i, j = 0, 1, \ldots, n$. We can easily verify that this matrix is indeed an affine Cartan matrix in the sense of definition 12. The affine Dynkin diagram of type $X^{(1)}$ is the graph with $n+1$ vertices $\{v_0, v_1, \ldots, v_n\}$ and $C_{i,j}C_{j,i}$ edges between the vertices v_i, v_j. If $C_{i,j}C_{j,i} > 1$ we put an arrow pointing to v_i if $(\tilde{\alpha}_i, \tilde{\alpha}_i) < (\tilde{\alpha}_j, \tilde{\alpha}_j)$ and to v_j if $(\tilde{\alpha}_i, \tilde{\alpha}_i) > (\tilde{\alpha}_j, \tilde{\alpha}_j)$. The Lie algebra defined by the affine Cartan matrix C is called untwisted affine Lie algebra of type $X^{(1)}$.

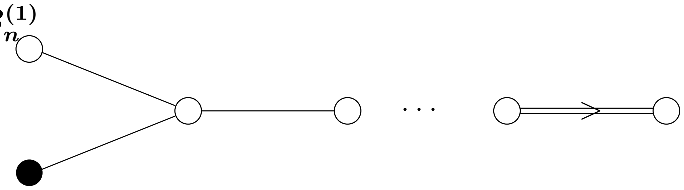
Next we display all the affine Dynkin diagrams. The corresponding affine Cartan matrices are presented in chapter 3. The black nodes correspond to the root $\tilde{\alpha}_0$.

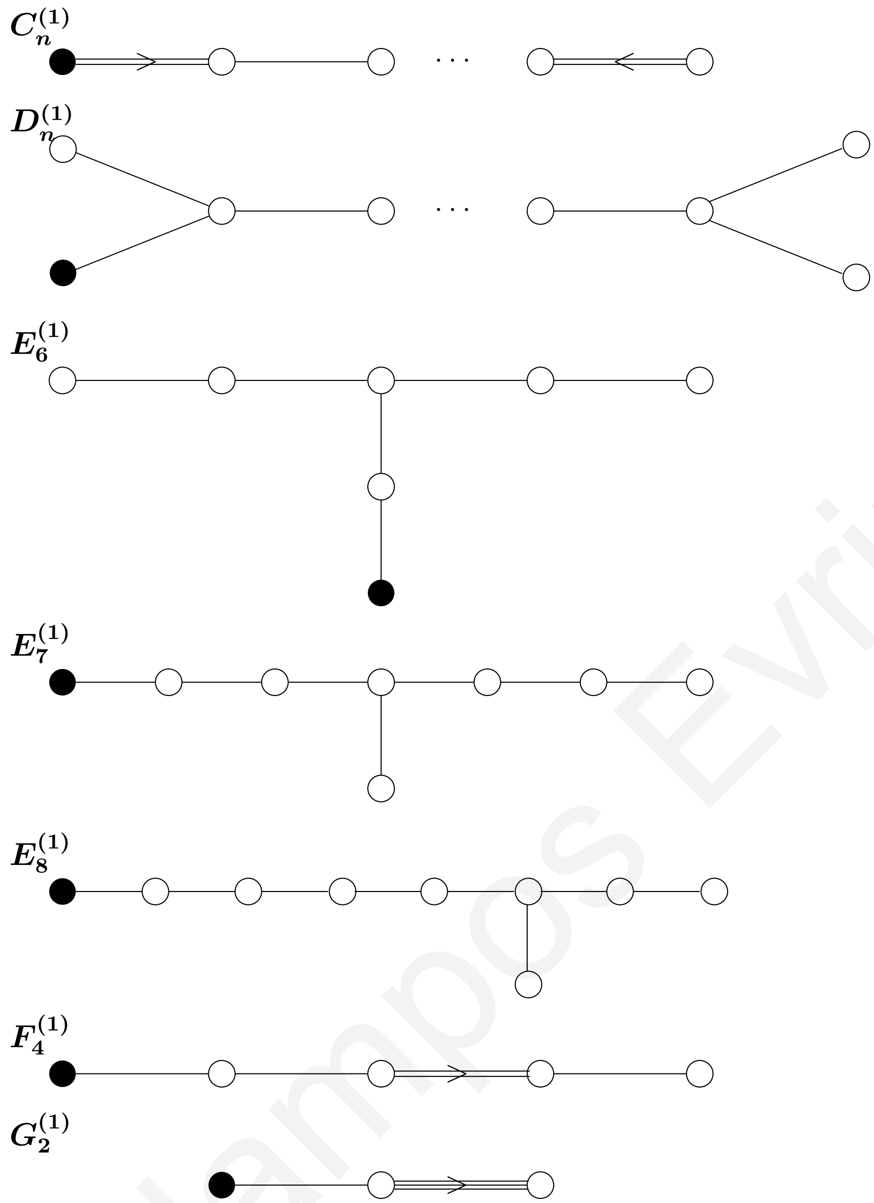
Dynkin Diagrams for affine Lie algebras

$\boldsymbol{A_n^{(1)}}$



$\boldsymbol{B_n^{(1)}}$





2.3 Chebyshev Polynomials

There are several kinds of Chebyshev polynomials which all play important role in modern developments (polynomial approximation, orthogonal polynomials, numerical approximation). We will show that these polynomials can also be used in the theory of Lie algebras for the explicit calculation of the Coxeter polynomials of the simple Lie algebras over $\mathbb{C}$. The Chebyshev polynomials of first and second kind (denoted respectively T_n and U_n) are defined by

$$T_n(x) = \frac{1}{2} \cdot \det \begin{pmatrix} 2x & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2x & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2x \end{pmatrix}$$

and

$$U_n(x) = \det \begin{pmatrix} 2x & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2x & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2x & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2x \end{pmatrix}.$$

The first few polynomials are

$$\begin{array}{rclrcl} T_0(x) & = & 1 & U_0(x) & = & 1 \\ T_1(x) & = & x & U_1(x) & = & 2x \\ T_2(x) & = & 2x^2-1 & U_2(x) & = & 4x^2-1 \\ T_3(x) & = & 4x^3-3x & U_3(x) & = & 8x^3-4x \\ T_4(x) & = & 8x^4-8x^2+1 & U_4(x) & = & 16x^4-12x^2+1 \\ T_5(x) & = & 16x^5-20x^3+5x & U_5(x) & = & 32x^5-32x^3+6x \\ T_6(x) & = & 32x^6-48x^4+18x^2-1 & U_6(x) & = & 64x^6-80x^4+24x^2-1. \end{array}$$

Expanding the determinants with respect to the first row we obtain the recurrence

$$F_{n+1} = 2xF_n - F_{n-1}.$$

For the initial values $F_0 = 1,\ F_1 = x$ and $F_0 = 1,\ F_1 = 2x$ we get the Chebyshev polynomials of first and second kind respectively.

For $x = \cos\theta$, the trigonometric identities

$$2x \cos n\theta = \cos(n+1)\theta + \cos(n-1)\theta$$

and

$$2x \sin(n+1)\theta = \sin(n+2)\theta + \cos n\theta,$$

give

$$T_n(x) = \cos n\theta,\ U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{2.11}$$

Proposition 4. *The Chebyshev polynomials of first kind $T_n(x)$, satisfy*

$$\begin{aligned} T_n(-x) &= (-1)^n T_n(x) \\ T_n(1) &= 1 \\ T_{2n}(0) &= (-1)^n \\ T_{2n-1}(0) &= 0, \end{aligned}$$

while the Chebyshev polynomials of second kind $U_n(x)$, satisfy

$$\begin{aligned} U_n(-x) &= (-1)^n U_n(x) \\ U_n(1) &= n+1 \\ U_{2n}(0) &= (-1)^n \\ U_{2n-1}(0) &= 0. \end{aligned}$$

Proof. All properties are easily verified for $n = 0, 1$. So let $n \in \mathbb{N}$ with $n \geq 2$. If $T_k(-x) = (-1)^k T_k(x)$ for $k = n-2, n-1$ then

$$T_n(-x) = -2xT_{n-1}(-x) - T_{n-2}(-x) =$$
$$(-1)^n \left(2xT_{n-1}(x) - T_{n-2}(x)\right) = (-1)^n T_n(x).$$

If $T_k(1) = T_{k-1}(1) = 1$ for $k = n-1$ then $T_n(1) = 2T_k(1) - T_{k-1}(1) = 1$.
If $T_{2k}(0) = (-1)^k$ and $T_{2k-1}(0) = 0$ then $T_{2k+1}(0) = 0 \cdot T_{2k}(1) - T_{2k-1}(1) = 0$, while if $T_{2k}(0) = (-1)^k$ and $T_{2k+1}(0) = 0$ then $T_{2k+2}(0) = 0 \cdot T_{2k+1}(1) - T_{2k}(1) = (-1)^{k+1}$.

The proof of the properties for the U_n polynomials is similar. □

In addition, (see [**?**]), using the formulas (2.11) we can easily find the roots of the Chebyshev polynomials.

$$T_n(x) = 2^{n-1} \prod_{j=1}^{n} \left(x - \cos\left(\frac{(2j-1)\pi}{2n}\right)\right)$$

and

$$U_n(x) = 2^n \prod_{j=1}^{n} \left(x - \cos\left(\frac{j\pi}{n+1}\right)\right).$$

In several cases it is useful and convenient to express the Chebyshev polynomials in terms of the monomials x^n. In our cases it is more convenient to express them in terms of the polynomials $(1-x)^n$ (see propositions 12 to 18). In the next proposition we write the Chebyshev polynomials explicitly in terms of powers of $(1-x)$.

Proposition 5. *The Chebyshev polynomials of first and second kind are expressed in terms of powers of* $(1-x)$ *as*

$$T_n(x) = n\sum_{j=0}^{n}(-2)^j \frac{(n+j-1)!}{(n-j)!(2j)!}(1-x)^j \quad (n>0), \tag{2.12}$$

and

$$U_n(x) = \sum_{j=0}^{n}(-2)^j \binom{n+j+1}{2j+1}(1-x)^j. \tag{2.13}$$

Proof. The proof of these formulas is by induction using the three term recurrence relation of the Chebyshev polynomials. We prove only the formula for the T_n polynomials since the proof for the U_n polynomials is similar.

The polynomials T_n satisfy the recurrence relation

$$T_n(1-x) = 2(1-x)T_{n-1}(1-x) - T_{n-2}(1-x).$$

Therefore we only need to prove that

$$2(n-1)\frac{(n+k-2)!}{(n-k-1)!(2k)!} + (n-1)\frac{(n+k-3)!}{(n-k)!(2k-2)!} - (n-2)\frac{(n+k-3)!}{(n-k-2)!(2k)!} = n\frac{(n+k-1)!}{(n-k)!(2k)!},$$

which is a straightforward verification. □

2.4 Minimal polynomials of $2\cos\frac{2k\pi}{n}$

A complex number ω of order n is called a primitive n^{th} root of unity, e.g. $e^{\frac{2\pi i}{n}}$ is a primitive n^{th} root of unity. If ω is a primitive n^{th} root of unity then ω^k is a primitive n^{th} root of unity if and only if $\gcd(n,k)=1$. Since the root $e^{\frac{2\pi i}{n}}$ produces all n^{th} roots of unity (i.e. $e^{\frac{2k\pi i}{n}}$, $k=0,1,2,\ldots,n-1$ are all the n^{th} roots of unity) it follows that there are exactly $\phi(n)$ primitive n^{th} roots of unity where ϕ is Euler's totient function. Primitive n^{th} roots of unity are conjugate algebraic integers (i.e. their minimal polynomial over $\mathbb{Z}$ is the same). This polynomial is what we call n^{th} cyclotomic polynomial.

Let ω be a primitive n^{th} root of unity and $\Phi_n(x)$ its minimal polynomial. Then

$$\Phi_n(x) = (x-\omega^{k_1})(x-\omega^{k_2})\cdots(x-\omega^{k_{\phi(n)}}),$$

where $1 \leq k_1, k_2, \ldots, k_{\phi(n)} < n$ are the integers relatively prime to n. The polynomial $\Phi_n(x) \in \mathbb{Z}[x]$ is the n^{th} cyclotomic polynomial. From $\omega^{k\cdot n} - 1 = 0$ we conclude that the polynomial $\Phi_n(x)$ divides the polynomial $x^{k \cdot n} - 1, \ \forall k \in \mathbb{N}$. In fact

$$x^n - 1 = \prod_{d|n} \Phi_d(x). \tag{2.14}$$

Following Lehmer [?], using cyclotomic polynomials we can derive the minimal polynomials Ψ_n of the algebraic integers $2\cos\frac{2k\pi}{n}$, where $\gcd(k, n) = 1$ (for $n \geq 2$). The polynomial Φ_n, being reciprocal (i.e. $\Phi_n(x) = x^{\phi(n)}\Phi_n\left(\frac{1}{x}\right)$), it can be written in the form

$$\Phi_n(x) = x^{\frac{\phi(n)}{2}}\Psi_n\left(x + \frac{1}{x}\right), \tag{2.15}$$

for some monic irreducible polynomial Ψ_n with integer coefficients and degree half of that of Φ_n. The irreducibility of Ψ_n is equivalent to the irreducibility of Φ_n.

For $x = e^{\frac{2k\pi i}{n}}$, a primitive n^{th} root of unity, we have $x + \frac{1}{x} = 2\cos\frac{2k\pi}{n}$. From equation (2.15) we conclude that $2\cos\frac{2k\pi}{n}$ is a root of the irreducible polynomial Ψ_n and therefore the polynomial Ψ_n is the minimal polynomial of $2\cos\frac{2k\pi}{n}$. Equation (2.15) can also be used for the calculation of the polynomials Ψ_n. The first fifteen polynomials Ψ_n are

$$\begin{aligned}
\Psi_1(x) &= x - 2\\
\Psi_2(x) &= x + 2\\
\Psi_3(x) &= x + 1\\
\Psi_4(x) &= x\\
\Psi_5(x) &= x^2 + x - 1\\
\Psi_6(x) &= x - 1\\
\Psi_7(x) &= x^3 + x^2 - 2x - 1\\
\Psi_8(x) &= x^2 - 2\\
\Psi_9(x) &= x^3 - 3x + 1\\
\Psi_{10}(x) &= x^2 - x - 1\\
\Psi_{11}(x) &= x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1\\
\Psi_{12}(x) &= x^2 - 3\\
\Psi_{13}(x) &= x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1\\
\Psi_{14}(x) &= x^3 - x^2 - 2x + 1\\
\Psi_{15}(x) &= x^4 - x^3 - 4x^2 + 4x + 1.
\end{aligned}$$

Proposition 6. *The roots of the polynomials Ψ_n are*

$$2\cos\frac{2k\pi}{n}, \quad \textit{where } \gcd(k, n) = 1.$$

2.5 Coxeter groups

Weyl groups of Lie algebras belong to a larger class of groups known as Coxeter groups.

Definition 13.

- A Coxeter matrix with n vertices is an $n \times n$ symmetric matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ which satisfies $m_{i,j} \in \mathbb{N} \cup \{+\infty\}$ and $m_{i,j} = 1$ if and only if $i = j$.
- A Coxeter graph Γ, is a simple graph (i.e. a graph without multiple edges or loops) with n vertices $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ such that each edge (v_i, v_j) is labeled with an integer $m_{i,j} \geq 3$. If $m_{i,j} = 3$ we usually don't label the edge (v_i, v_j).

There is a one to one correspondence between Coxeter matrices and Coxeter graphs. If $M = (m_{i,j})_{1 \leq i,j \leq n}$ is a Coxeter matrix, the corresponding Coxeter graph is the graph with n vertices $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ such that there is an edge (v_i, v_j) between the vertices v_i and v_j if and only if $m_{i,j} \geq 3$. The edge (v_i, v_j) is labeled with $m_{i,j}$ if and only if $m_{i,j} \geq 4$.

Definition 14. The Coxeter group of a Coxeter matrix M (or of the corresponding Coxeter graph) is the group W with presentation

$$W = \langle s_1, s_2, \ldots, s_n : (s_i s_j)^{m_{i,j}} = 1 \rangle$$

Weyl groups of Lie algebras are important examples of Coxeter groups. If W is the Weyl group of a Lie algebra with Cartan matrix $C = (c_{i,j})_{1 \leq i,j \leq n}$ then W is a Coxeter group with Coxeter matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ where $m_{i,i} = 1$ for all $i = 1, 2, \ldots, n$ and the integers $m_{i,j}$ for $i \neq j$ are defined by

$m_{i,j}$	$c_{i,j}c_{j,i}$
2	0
3	1
4	2
6	3

Let W be a Coxeter group with Coxeter matrix M. We define a bilinear form B on the real vector space V with basis $\{e_1, e_2 \ldots, e_n\}$, as

$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{i,j}}\right).$$

Let $\sigma_i : V \to V$ be the involution defined by

$$\sigma_i(v) = v - 2B(v, e_i)e_i, \ v \in V. \tag{2.16}$$

The group generated by the involutions $\sigma_i, i = 1, 2, \ldots, n$ is isomorphic to the Coxeter group W.

It is known that the Coxeter group W is finite if and only if the bilinear form B is positive definite (see [**?**, **?**]). The affine Coxeter groups are the infinite Coxeter groups for which the bilinear form B is positive semidefinite.

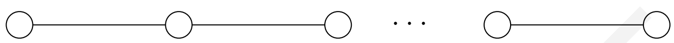
The finite Coxeter groups are classified as follows (the subscripts denote the number of the elements of the set S)

$$A_n, BC_n, D_n, E_6, E_7, E_8, F_4, G_2, H_3, H_4, I_2^{(m)}.$$

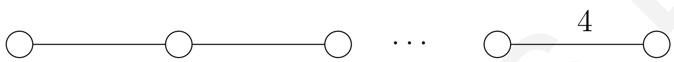
The corresponding Coxeter graphs are the following.

Coxeter graphs of finite Coxeter groups

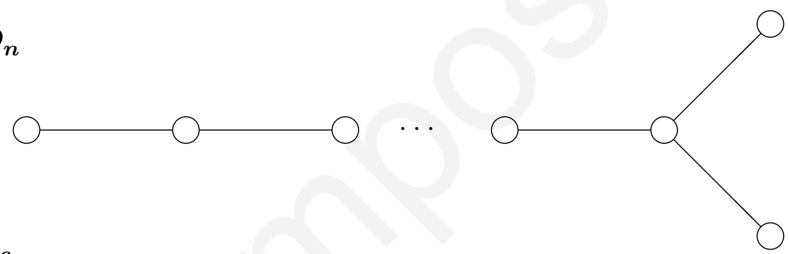
$\boldsymbol{A_n}$



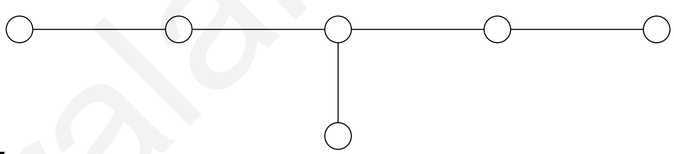
$\boldsymbol{BC_n}$



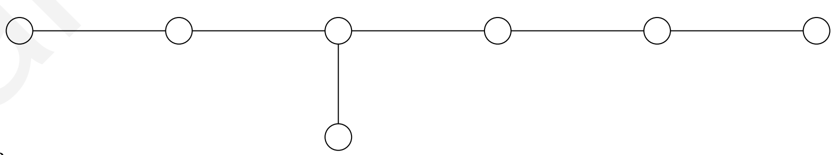
$\boldsymbol{D_n}$



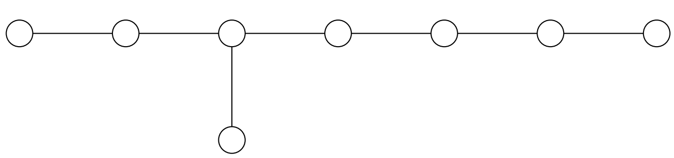
$\boldsymbol{E_6}$



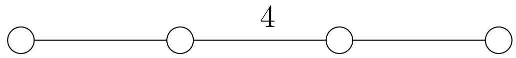
$\boldsymbol{E_7}$



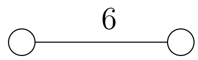
$\boldsymbol{E_8}$



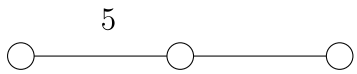
F_4



G_2

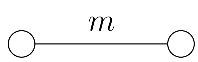


H_3



H_4

$I_2^{(m)}$



The Coxeter groups $A_n, BC_n, D_n, E_6, E_7, E_8, F_4, G_2$ are the Weyl groups of the finite dimensional complex simple Lie algebras (see section 2.2).

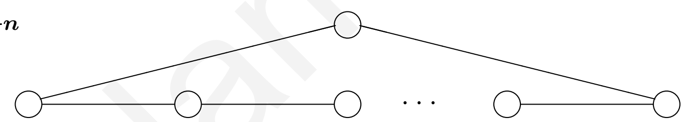
The affine Coxeter groups are

$$\tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2.$$

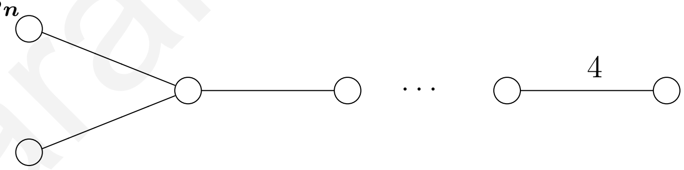
The corresponding Coxeter graphs are the following.

Coxeter graphs of affine Coxeter groups

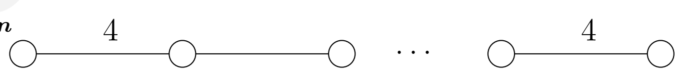
$\tilde{A}_n$



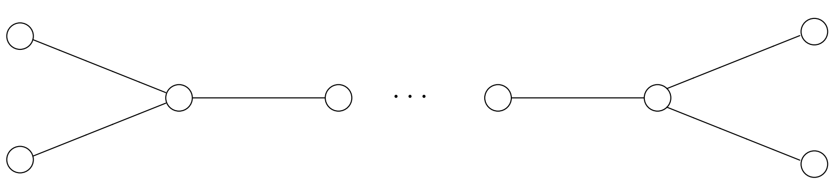
$\tilde{B}_n$



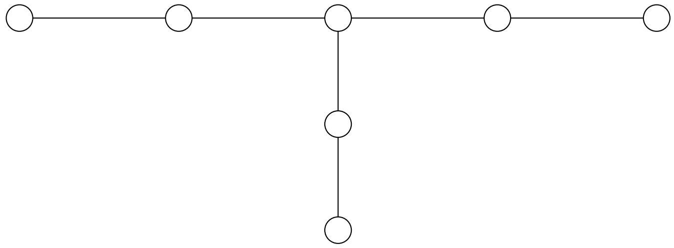
$\tilde{C}_n$



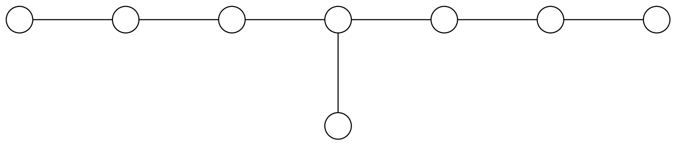
$\tilde{D}_n$



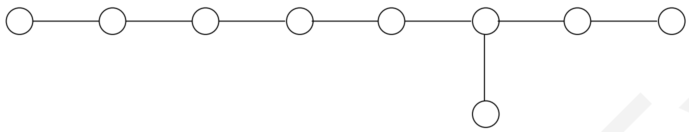
$\tilde{E}_6$



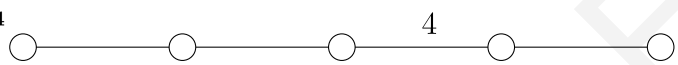
$\tilde{E}_7$



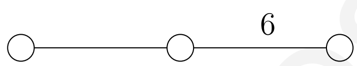
$\tilde{E}_8$



$\tilde{F}_4$



$\tilde{G}_2$



$\tilde{A}_1$

All these, except $\tilde{A}_1$ are the affine Weyl groups of the affine Lie algebras.

Let W be a Coxeter group, V a vector space over $\mathbb{R}$ and $\sigma_i : V \to V$, the involutions defined by (2.16).

Definition 15. An element $s \in W$ of the form

$$\sigma = \sigma_{\pi_1}\sigma_{\pi_2}\ldots\sigma_{\pi_n}$$

where $\pi \in S_n$ is a permutation of n elements is known as a Coxeter element (or Coxeter transformation) of the Coxeter group. The characteristic polynomial of the element σ is called Coxeter polynomial of the Coxeter group.

The next proposition shows that when the Coxeter graph of the Coxeter group is a tree then any two of the Coxeter elements are conjugate in W (see[**?**])

Proposition 7 (H. S. M. Coxeter). *Assume that W is a Coxeter group with Coxeter graph Γ. If Γ is a tree then any two Coxeter elements of the Coxeter system are conjugate in W.*

For the proof we need the following lemma.

Lemma 8. *Suppose that* Γ *is a simple graph with* n *vertices* $\mathcal{V} = \{v_1, v_2 \ldots, v_n\}$ *and edges* $\mathcal{E} = \{(v_i, v_j) : \text{there is an edge between the vertices } v_i \text{ and } v_j\}$*. Suppose also that we have an alphabet* $\{s_1, s_2, \ldots, s_n\}$*. In the set*

$$A = \{s_{\pi(1)}s_{\pi(2)} \cdots s_{\pi(n)} : \pi \in S_n\}$$

we define the equivalence relation generated by

1. $s_{\pi(1)}s_{\pi(2)} \cdots s_{\pi(n)} \sim s_{\pi(2)}s_{\pi(3)} \cdots s_{\pi(1)}$
2. $s_{\pi(1)} \cdots s_{\pi(i)}s_{\pi(i+1)} \cdots s_{\pi(n)} \sim s_{\pi(1)} \cdots s_{\pi(i+1)}s_{\pi(i)} \cdots s_{\pi(n)}$ *for all* $\pi \in S_n$ *such that* $(v_{\pi(i)}, v_{\pi_{i+1}}) \notin \mathcal{E}$*.*

Then $s \sim t$ *for all* $s, t \in A$*.*

Proof. We assume without loss of generality that the vertices of Γ are enumerated so that v_i is a leaf in the subgraph of Γ with vertex set $\{v_1, v_2, \ldots, v_i\}$ (i.e. there is one and only one vertex v_j which is connected with v_i with an edge). Let $s = s_{\pi(1)}s_{\pi(2)} \cdots s_{\pi(n)} \in A$. We may assume that $\pi(n) = n$ and therefore $s = s_{\pi(1)}s_{\pi(2)} \cdots s_{\pi(n-1)}s_n$.
If $(v_{\pi(1)}, v_n) \in \mathcal{E}$ then $(v_{\pi(i)}, v_n) \notin \mathcal{E}$ for all $i = 1, 2, \ldots, n$ and therefore

$$s \sim s_{\pi(2)} \ldots s_{\pi(n-1)}s_n s_{\pi(1)} \sim s_{\pi(2)} \ldots s_n s_{\pi(n-1)}s_{\pi(1)} \sim \cdots \sim$$
$$s_n s_{\pi(2)} \ldots s_{\pi(n-1)}s_{\pi(1)} \sim s_{\pi(2)} \ldots s_{\pi(n-1)}s_{\pi(1)}s_n$$

It follows that if $B = \{s_1, s_2, \ldots, s_{n-1}\}$ and $\sim'$ is the equivalence relation defined by the subgraph Γ' of Γ with vertex set $v_1, v_2, \ldots, v_{n-1}$, then if s', t' are words with the letters $s_1, s_2, \ldots, s_{n-1}$ and $s \sim' t$, then $s's_n \sim t's_n$. Using induction on n the lemma is proved. $\square$

As a corollary we obtain proposition 7.

Definition 16. Let W be a finite Coxeter group. The Coxeter transformation of W has finite order h. The integer h is known as the Coxeter number of the Coxeter group.

Lemma 9. *The Coxeter polynomial of the finite Coxeter group* W *is of the form*

$$f(x) = (x - \zeta^{m_1})(x - \zeta^{m_2}) \cdots (x - \zeta^{m_n}),$$

where $0 \leq m_i \leq h$ *and* ζ *is an* h^{th} *root of unity.*

Proof. Since the Coxeter transformation σ of the Coxeter group W has finite order h it follows that $\sigma^h = 1$ and therefore $\sigma^h - 1 = 0$. We conclude that the characteristic polynomial of σ divides the polynomial $x^h - 1$ and the result follows. $\square$

Definition 17. The integers m_i in lemma 9 are known as the exponents of the Coxeter group.

Coxeter was the first one (see [?]) who studied the Coxeter transformations of the finite Coxeter groups and observed that their eigenvalues have remarkable properties. For example $hn = |R|$ where R is the root system associated with the Coxeter group W. In the case of the Weyl groups the order of W is equal to

$$(m_1 + 1)(m_2 + 1) \cdots (m_n + 1).$$

If $\beta = k_1\alpha_1 + \cdots + k_n\alpha_n$ is the highest root in R, then $h = k_1 + k_2 + \cdots + k_n + 1$. All these and many other relations were proved later by B.Kostant in [?], by C. Chevalley in [?], by R. Steinberg in [?] and other authors (see also [?]).

For an affine Lie algebra with affine Cartan matrix C of rank n, the roots of the Coxeter polynomial $f(x)$ are in the unit disk. Thus, from Kronecker's theorem (theorem 8), the polynomial $f(x)$ is a product of cyclotomic polynomials.

Let $V = \mathbb{R} - \text{span}\,\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ and D a diagonal matrix with positive entries such that CD is symmetric. The matrix CD defines a semi-positive bilinear form $(\,,)$ on V. For $\alpha = \sum_{i=0}^{n} z_i\alpha_i$, $(\alpha, .) = 0$ if and only if $(z_0, z_1, \ldots, z_n)\, C = 0$. Thus if $(z_0, z_1, \ldots, z_n)$ is a left zero eigenvector of C and $\alpha = \sum_{i=0}^{n} z_i\alpha_i$, the induced bilinear form on $\tilde{V} = V/\langle\alpha\rangle$ is positive definite and corresponds to an $n \times n$ submatrix of C. Therefore if $\sigma : V \longrightarrow V$ is a Coxeter transformation of the affine Lie algebra, the induced transformation on $\tilde{V}$ has finite order h. It follows that $(\sigma - 1)(\sigma^h - 1) = 0$ and therefore the roots of f are h^{th} roots of unity and we have the following proposition.

Proposition 8. *The roots of $f(x)$ are of the form*

$$\left\{ e^{\frac{2m_j\pi i}{h}} : m_j \in \{0, 1, \ldots, h\} \right\}. \tag{2.17}$$

Definition 18. The integers m_j of the previous proposition are the affine exponents and h is the affine Coxeter number associated with the Coxeter transformation σ.

These numbers are uniquely defined for each affine Lie algebra except in the case of $A_n^{(1)}$ where we define them for each conjugacy class. Using the fact that the corank of the Cartan matrix $C_{X^{(1)}}$ is 1 and the relation between the polynomials $p(x)$ and $f(x)$ (for the bipartite case) it follows that $(x-1)^2 \mid f(x)$ and $(x-1)^3 \nmid f(x)$. For the factor $(x-1)^2$ we define the associated affine exponents to be 0 and h.

Definition 19. For a Dynkin diagram Γ with corresponding Cartan matrix C we define a weight function $b : V(\Gamma) \to \mathbb{N}$ on the vertices of Γ. If the vertex r_i has only one neighbor we define $b(r_i) = 1 - \sum_{j \neq i} C_{i,j}$ while if it has more than one neighbors we define

$b(r_i) = -\sum_{j\neq i} C_{i,j}$. For a Dynkin diagram Γ not of type A_n, we define the branch vertex r_i to be the one which maximize b. For the case of A_n, n odd, we define the branch vertex to be the middle one.

Example 5. For the case of the Dynkin diagrams D_n, E_6, E_7, E_8 the branch vertex is the one which is the common endpoint of three edges. For the Dynkin diagram of type C_n the branch vertex is the one which corresponds to the highest root.

Steinberg's theorem [**?**, p.591], provides a relation between the affine exponents and affine Coxeter number of an affine Lie algebra of type $X_n^{(1)} \neq A_n^{(1)}$ and the exponents and Coxeter number of the root system A_n.

Theorem 6 (**R. Steinberg**). *Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of the simple roots of the Lie algebra of type X_n. Define the branch root β to be that root which corresponds to the branch vertex of the corresponding Dynkin diagram. If we delete the branch root the reduced system is a product of root systems A_n. The Coxeter polynomial of $X_n^{(1)}$ is $f(x) = (x-1)^2 g(x)$ where $g(x)$ is the Coxeter polynomial of the reduced system.*

Therefore from Steinberg's theorem we conclude that the affine Coxeter number h is the Coxeter number of the reduced system and the affine exponents are obtained using the following procedure:
From the factor $(x-1)^2 \mid f(x)$ it follows that 0 and h are affine exponents. If Y appears in the reduced system and $m_j^{'}$ is an exponent, $h^{'}$ the Coxeter number of Y then $\frac{h}{h^{'}} m_j^{'}$ is an affine exponent of $X^{(1)}$.

.

Root system	Affine Exponents	Affine Coxeter number
$A_n^{(1)}$	$0, k_j, 2k_j, \ldots, jk_j,$ $n_j, 2n_j, \ldots, (n-j)n_j$	jk_j
$B_{2n+1}^{(1)}$	$0, 1, 2, 3, \ldots, 2n, n$	$2n$
$B_{2n}^{(1)}$	$0, 2, 4, \ldots, 2(2n-1), 2n-1$	$2(2n-1)$
$C_n^{(1)}$	$0, 1, 2, \ldots, n$	n
$D_{2n+1}^{(1)}$	$0, 2, 4, \ldots, 2(2n-1), 2n-1, 2n-1$	$2(2n-1)$
$D_{2n}^{(1)}$	$0, 1, 2, 3, \ldots, 2n-2, n-1, n-1$	$2n-2$
$E_6^{(1)}$	$0, 2, 2, 3, 4, 4, 6$	6
$E_7^{(1)}$	$0, 3, 4, 6, 6, 8, 9, 12$	12
$E_8^{(1)}$	$0, 6, 10, 12, 15, 18, 20, 24, 30$	30
$F_4^{(1)}$	$0, 2, 3, 4, 6$	6
$G_2^{(1)}$	$0, 1, 2$	2

Table 2.1: Affine Exponents and affine Coxeter number for affine root systems

Example 6. For the root system $E_8^{(1)}$ the reduced system is $A_1 \times A_2 \times A_4$. The exponents of A_n are $1, 2, \ldots, n$ and the Coxeter number $n+1$ (see [**?**]). We conclude that the affine Coxeter number is $lcm(2, 3, 5) = 30$ and the affine exponents are $0, 6, 10, 12, 15, 18, 20, 24, 30$.

In table 2.1 we list the affine exponents and the affine Coxeter number for the affine Lie algebras. In the case of $A_n^{(1)}$, for $j = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor$ we have denoted $k_j = \frac{n+1-j}{d_j}$ and $n_j = \frac{j}{d_j}$, where $d_j = \gcd(n+1, j)$. The affine exponents and affine Coxeter number of $A_n^{(1)}$ given in table 2.1, are those associated with the Coxeter polynomial $(x^j - 1)(x^{n+1-j} - 1)$. For n odd and $j = \frac{n+1}{2}$ we have $n_j = k_j = 1, d_j = j$ and we obtain the case considered in [**?**]. Note the duality in the set of affine exponents:

$$m_i + m_{n-i} = h, \ i = 0, 1, \ldots, n, \tag{2.18}$$

where h is the affine Coxeter number. This is a consequence of proposition 8.

The affine exponents, affine Coxeter number of $X_n^{(1)}$ and the roots of X_n are related in a mysterious way given by a theorem of Berman, Lee and Moody (see [**?**, **?**, **?**]).

Theorem 7 (**S. Berman, Y. S. Lee, R. V. Moody**)**.** *Let* $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *be the simple roots of the Lie algebra of type* $X_n \neq A_n$, $V = \mathbb{R}\text{-span}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *and* β *be the branch root of* X_n. *Denote* $\alpha_i^\vee = 2\frac{\alpha_i}{(\alpha_i, \alpha_i)}$ *the coroots and* $w_{\alpha_i^\vee} \in V^*$ *the corresponding weights. Write* $w_{\beta^\vee} = (v, \cdot)$, $v \in V$ *and let* $c \in \mathbb{N}$ *be the smallest integer such that* $c \cdot v \in \mathbb{Z}\text{-span}(\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee)$. *Then*

$$c \cdot v = \sum_{j=1}^{n} m_j \alpha_j^\vee$$

where m_j *are the nonzero affine exponents of* $X_n^{(1)}$ *and the coefficient of* $\beta^\vee$ *is the affine Coxeter number.*

Example 7. Removing the branch vertex of B_4 we obtain the root system $A_2 \times A_1$ with Coxeter polynomial $g(x) = (x^2 + x + 1)(x + 1)$; the Coxeter polynomial of $B_4^{(1)}$ is

$$f(x) = (x - 1)^2(x^2 + x + 1)(x + 1).$$

The Coxeter number of $A_2 \times A_1$ is the affine Coxeter number of $B_4^{(1)}$, that is $3 \cdot 2 = 6$.

The roots of the Coxeter polynomial are $1, 1, -1, \omega, \omega^2$, where ω is a primitive third root of unity. If $\zeta = e^{\frac{2\pi i}{6}}$ then $1 = \zeta^0, \omega = \zeta^2, -1 = \zeta^3, \omega^2 = \zeta^4, 1 = \zeta^6$. The numbers $0, 2, 3, 4, 6$ are the affine exponents of $B_4^{(1)}$.

From the representation of the root system B_4 given given in section 2.1 we conclude that the corresponding co-roots are $\alpha_i^\vee = e_i - e_{i+1} \in \mathbb{R}^4, \ i = 1, 2, 3$ and $\alpha_4^\vee = 2e_4$ (which is the root system of type C_4). The branch root is the root α_3 and the corresponding co-weight is $v = w_{\alpha_3^\vee} = (1,1,1,0) \in (\mathbb{R}^4)^*$. Now v does not belong to the co-root lattice but $2v = 2\alpha_1^\vee + 4\alpha_2^\vee + 6\alpha_3^\vee + 3\alpha_4^\vee$ does. Therefore for $B_4^{(1)}$, $c = 2$, the non zero affine exponents are $2, 3, 4, 6$ and the affine Coxeter number is 6.

Example 8. For the case of D_6 with root system $\{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ and branch root α_4, it can be easily verified that $v = w_{\alpha_4} = (1,1,1,1,0,0) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 2\alpha_6$. Therefore for $D_6^{(1)}$, $c = 1$, the nonzero affine exponents are $1, 2, 2, 2, 3, 4$ and the affine Coxeter number is 4.

The following proposition can be found in [**?**] and shows that the Coxeter polynomial of a Coxeter tree is reciprocal.

Proposition 9 (**S. Berman, Y. S. Lee, R. V. Moody**). *Let Γ be a Coxeter tree. The characteristic polynomial $\chi_\Gamma(x)$ of the graph Γ and the Coxeter polynomial $\Gamma(x)$ are related in the following way*

$$\Gamma(x^2) = x^n \chi_\Gamma\left(x + \frac{1}{x}\right),$$

where n is the degree of $\chi_\Gamma(x)$.

Proof. Let $\mathcal{V}(\Gamma) = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_{k+m}\}$ be the vertices of Γ enumerated such that if $i, j \leq k$ or $i, j > k$ then $(v_i, v_j) \notin \mathcal{E}(\Gamma)$. Let σ_i be the Coxeter reflections associated to v_i, i.e. if $\widehat{e} = \{e_1, e_2, \ldots, e_{k+m}\}$ is a basis of the vector space V then $\sigma_i(e_j) = e_j - (2\delta_{i,j} - A_{j,i})e_i$. Then with respect to the basis $\widehat{e}$ the Coxeter reflection σ_i is given by the matrix where its i^{th} row is the i^{th} row of the matrix $A - I$ and its j^{th} row is the j^{th} row of the identity matrix I. We see at once that $\sigma_i^2 = I$ for all i and that for $i, j \leq k$ or $i, j > k \Rightarrow \sigma_i\sigma_j = \sigma_i + \sigma_j - I$. Therefore we obtain the following relations

$$\sigma_1\sigma_2 \ldots \sigma_k = \sigma_1 + \sigma_2 + \ldots + \sigma_k - (k-1)I,$$
$$\sigma_{k+1}\sigma_{k+2} \ldots \sigma_{k+m} = \sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_{k+m} - (m-1)I.$$

Let's denote by C_1 the transformation $\sigma_1\sigma_2 \ldots \sigma_k$ and by C_2 the transformation defined by $\sigma_{k+1}\sigma_{k+2} \ldots \sigma_{k+m}$. It follows that $C_1^2 = I$, $C_2^2 = I$ and $C_1 + C_2 = A$. We thus get

$$2I + C_1C_2 + C_2C_1 = (C_1 + C_2)^2 = A^2.$$

If $e^{z_1}, e^{z_2}, \ldots$ are the roots of the Coxeter polynomial of the graph Γ then $2 + e^{z_1} + e^{-z_1}, 2 + e^{z_2} + e^{-z_2}, \ldots$ are the roots of $\chi_{A^2}(x)$. Since Γ is bipartite the polynomial $\chi_\Gamma(x)$ is of the form

$$\chi_\Gamma(x) = (x - r_1)(x + r_1)(x - r_2)(x + r_2) \ldots = (x^2 - r_1^2)(x^2 - r_2^2) \ldots$$

where $r_i^2 = 2 + e^{z_i} + e^{-z_i}$. It follows that

$$x^{\frac{n}{2}}\chi_\Gamma\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) = \left(x^2+\left(2-r_1^2\right)x+1\right)\left(x^2+\left(2-r_2^2\right)x+1\right)\ldots =$$
$$\left(x^2-\left(e^{z_1}+e^{-z_1}\right)x+1\right)\left(x^2-\left(e^{z_2}+e^{-z_2}\right)x+1\right)\ldots =$$
$$\left(x-e^{z_1}\right)\left(x-e^{-z_1}\right)\left(x-e^{z_2}\right)\left(x-e^{-z_2}\right)\ldots = \Gamma(x).$$

□

We immediately get the following corollary.

Corollary 2. *The Coxeter polynomial of a tree Γ is reciprocal.*

The next two propositions shows how the Coxeter polynomial of two trees Γ_1, Γ_2 and the Coxeter polynomial of the tree Γ' which is the join of Γ_1, Γ_2 are related.

Proposition 10 (**A. Boldt**). *Suppose that Γ is a tree which is the Coxeter graph of a Coxeter group W. Assume that v_0 is a vertex on Γ, Γ_1 is the tree obtained from Γ by adding an edge (v_0, v_1) and Γ_2 the tree obtained by adding an edge (v_1, v_2) on Γ_2. Then the Coxeter polynomials $\Gamma(x), \Gamma_1(x), \Gamma_2(x)$ of the Coxeter graphs $\Gamma, \Gamma_1, \Gamma_2$ are related in the following way*

$$\Gamma_2(x) = (x+1)\Gamma_1(x) - \Gamma(x).$$

The previous proposition (due to [**?**]) is a special case of the next one which is due to to Subbotin-Sumin (see [**?**]).

Proposition 11 (**V. F. Subbotin, M. V. Sumin**). *Let $e = (v_1, v_2) \in \mathcal{E}(\Gamma)$ be a splitting edge of the tree Γ that splits it to the simple graphs Γ_1 and Γ_2. Assume that $v_1 \in \mathcal{V}(\Gamma_1)$ and $v_2 \in \mathcal{V}(\Gamma_2)$. Then*

$$\Gamma(x) = \Gamma_1(x)\Gamma_2(x) - x\tilde{\Gamma}_1(x)\tilde{\Gamma}_2(x)$$

where $\tilde{\Gamma}_i$ denotes the subgraph of Γ_i with vertex set $\mathcal{V}(\Gamma_i) \setminus \{v_i\}$.

Proof. We enumerate the vertices of Γ as $\mathcal{V}(\Gamma_1) = \{u_1, u_2, \ldots, u_k\}$ the vertices of Γ_1 and $\mathcal{V}(\Gamma_2) = \{u_{k+1}, u_{k+2}, \ldots, u_{k+m}\}$ the vertices of Γ_2, where $v_1 = u_k$ and $v_2 = u_{k+1}$. Let $\widehat{e} = \widehat{e}_1 \cup \widehat{e}_2$ be a basis for the vector space V, where $\widehat{e}_1 = \{e_1, e_2, \ldots, e_k\}$ is a basis of V_1 and $\widehat{e}_2 = \{e_{k+1}, e_{k+2}, \ldots, e_{k+m}\}$ is a basis of V_2. Also let σ_i be the Coxeter reflections corresponding to u_i. Therefore $R_1 = \sigma_1\sigma_2\ldots\sigma_k$ is a Coxeter transformation of Γ_1, $R_2 = \sigma_{k+1}\sigma_{k+2}\ldots\sigma_{k+m}$ is a Coxeter transformation of Γ_2 and R_1R_2 is a Coxeter transformation of Γ. If we represent R_1, R_2 and R as matrices with respect to the basis $\widehat{e}$ we have

$$R = R_1R_2 = \begin{pmatrix} Q_1 & E_{k,1} \\ 0_{m,k} & I_m \end{pmatrix}\begin{pmatrix} I_k & 0_{k,m} \\ E_{1,k} & Q_2 \end{pmatrix},$$

where Q_i are the transformations R_i restricted to V_i, $E_{i,j}$ is the matrix with all entries zero except the i,j entry which is 1 and $0_{i,j}$ is the $i \times j$ zero matrix. The Coxeter polynomial of Γ is then given by

$$\Gamma(x) = \det(R - xI_{k+m}) = \det \begin{pmatrix} Q_1 + E_{k,k} - xI_k & E_{k,1}Q_2 \\ E_{1,k} & Q_2 - xI_m \end{pmatrix}.$$

Subtracting the $k+1^{\text{th}}$ row from the k^{th} row we obtain

$$\Gamma(x) = \det \begin{pmatrix} Q_1 - xI_k & xE_{k,1} \\ E_{1,k} & Q_2 - xI_m \end{pmatrix}.$$

Expanding the determinant with respect to the k^{th} row we deduce that

$$\Gamma(x) = \Gamma_1(x)\Gamma_2(x) - x\tilde{\Gamma}_1(x)\tilde{\Gamma}_2(x).$$

□

2.6 Mahler measure

This subsection is about the Mahler measure of integer monic polynomials and Lehmer's problem. For an excellent survey on this subject see [**?**].

Lehmer in [**?**], in order to construct large prime numbers, considers irreducible integer monic polynomials. Let

$$f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1x + a_0 = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_d)$$

be an irreducible monic polynomial with integer coefficients. Lehmer defines the numbers

$$\Delta_k(f) = \prod_{i=1}^{d}(\alpha_i^k - 1)$$

and

$$Q_k(f) = \prod_{i=1}^{d}(\alpha_i^k + 1).$$

Since $\Delta_k(f)$ and $Q_k(f)$ are symmetric polynomials on the roots $\alpha_1, \alpha_2, \ldots, \alpha_d$, it follows that they are polynomials on the coefficients $a_0, a_1, \ldots, a_d$ of f. Therefore the numbers $\Delta_k(f)$ and $Q_k(f)$ are integers. These integers were introduced and studied by Pierce in 1916 (see [**?**]).

Lehmer was able to describe the prime factors of these integers and therefore to produce large prime numbers. Note that for $f(x) = x - 2$ the integers $\Delta_k(f)$ are the Mersenne numbers which give rise to the Mersenne primes. In order to handle easily the integers $\Delta_k(f)$, Lehmer was looking for polynomials such that $\Delta_k(f)$ increase as slow as possible.

For polynomials whose roots are not on the unit circle, the limit

$$\lim_{n\to\infty}\left|\frac{\Delta_{n+1}}{\Delta_n}\right|$$

is a measure of the rate of growth of the sequence $(\Delta_n)_{n\in\mathbb{N}}$. It can be easily seen that this limit equals

$$\lim_{n\to\infty}\left|\frac{\Delta_{n+1}}{\Delta_n}\right| = \prod_{i=1}^{d}\max\{1,|\alpha_i|\}.$$

For lack of something better, Lehmer used the quantity $\prod_{i=1}^{d}\max\{1,|\alpha_i|\}$ to measure the rate of growth of the sequence $(\Delta_n)_{n\in\mathbb{N}}$, even when some of the roots of the polynomial f were on the unit circle.

Definition 20. Given an integer polynomial

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0 = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_d),$$

its Mahler measure is

$$M(f) = |a_d|\prod_{i=1}^{d}\max\{1,|\alpha_i|\}.$$

Mahler in [**?**] generalized the definition of the "Mahler measure" to polynomials f of several variables and called it the measure of f. Later Waldschmidt, Boyd and Durand coined the term Mahler measure for this quantity.

It can be easily seen that the functions $\Delta_n(f), Q_n(f)$ and the Mahler measure $M(f)$, are multiplicative, i.e. $\Delta_n(fg) = \Delta_n(f)\Delta_n(g)$, $Q_n(fg) = Q_n(f)Q_n(g)$ and $M(fg) = M(f)M(g)$. Therefore it is reasonable to consider only irreducible polynomials. From now on, unless otherwise said, all polynomials considered will be monic, irreducible with integer coefficients.

Plainly, for every polynomial f we have $M(f) \geq 1$ while if f is cyclotomic $M(f) = 1$. A classical theorem of Kronecker says that the converse is true (see [**?**, **?**]).

Theorem 8 (**L. Kronecker**)**.** *If f is a monic irreducible integer polynomial with integer coefficients then its Mahler measure equals* 1 *if and only if f is cyclotomic.*

For a cyclotomic polynomial f, the sequences $(\Delta_n(f))_{n\in\mathbb{N}}$ and $(Q_n(f))_{n\in\mathbb{N}}$ are finite. In that particular case, Lehmer, explicitly describes the sequences $(\Delta_n(f))_{n\in\mathbb{N}}$ and $(Q_n(f))_{n\in\mathbb{N}}$, and shows that they are of no importance for his purposes. Therefore he concentrates on the non-cyclotomic polynomials, or in other words, on the polynomials with

Mahler measure greater than one. The following question, known as Lehmer's problem, arises.

Question. If ϵ is a positive number, can we find a monic integer polynomial such that its Mahler measure lies between 1 and $1+\epsilon$?

In other words Lehmer's problem asks if Kronecker's theorem can be strengthened. The smallest known Mahler measure greater than 1 is

$$M(L) = 1.17628\ldots.$$

This number is the Mahler measure of Lehmer's polynomial

$$L(x) = x^{10} + x^9 + x^7 + x^6 + x^5 + x^4 + x^3 + x + 1$$

and is known as Lehmer's number. Lehmer conjectured in [**?**] that Lehmer's number is the smallest Mahler measure of the non-cyclotomic polynomials. Both Lehmer's problem and conjecture are still unanswered.

Definition 21. Let $f(x) \in \mathbb{R}[x]$ be a polynomial with real coefficients.

- The reciprocal of the polynomial f is the polynomial $f^*(x) = x^n f\left(\frac{1}{x}\right)$.
- The polynomial f is called reciprocal if $f = f^*$.

An equivalent definition is that the polynomial $f(x) = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0$ is reciprocal if $a_i = a_{d-i}$ for all $i = 0, 1, \ldots, d$. If α is a nonzero root of the polynomial f then $\frac{1}{\alpha}$ is a root of the polynomial f^*. Therefore the Mahler measure of the polynomials f and f^* is the same.

Definition 22. Let

$$f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_1 x + a_0 = (x - \alpha_1)\ldots(x - \alpha_d)$$

be a polynomial and suppose that only one of its roots, let us say α_d, lies outside the unit circle.

- The algebraic integer α_d is called a Salem number if $|\alpha_i| \leq 1$ for all $i = 1, 2, \ldots, d-1$ and at least one $|\alpha_i| = 1$. In that case the polynomial f is called Salem polynomial.

- The algebraic integer α_d is called a Pisot number if $|\alpha_i| < 1$ for all $i = 1, 2, \ldots, d-1$. In that case the polynomial f is called Pisot polynomial.

Remark 2. If α is a root of f then the complex conjugate $\bar{\alpha}$ of α is also a root of the polynomial f and therefore both Salem and Pisot numbers are real.

Lemma 10. *Let f be a polynomial of degree greater than 2 and suppose that only one of its roots, let us say α_d, lies outside the unit circle. Then f is a Salem polynomial if and only if f is reciprocal.*

Proof. If f is a Salem polynomial then there is a root α of f with $|\alpha| = 1$. Since $\bar{\alpha} = \frac{1}{\alpha}$ is also a root of f and f is irreducible it follows that f is the minimal polynomial of the algebraic integer $\frac{1}{\alpha}$. It can be easily seen that the minimal polynomial of $\frac{1}{\alpha}$ is the polynomial f^* and therefore $f = f^*$.

Conversely suppose that $f = f^*$. Then if α is a root of f, $\frac{1}{\alpha}$ is also a root of f and therefore only one root of f can lie inside the unit circle (since there is only one outside). Hence, f is a Salem polynomial. $\square$

There are polynomials which are neither Salem nor Pisot. For example the polynomial

$$x^{10} - x^8 + x^7 - x^5 + x^3 - x^2 + 1$$

is not a Salem polynomial. The polynomial

$$x^4 - x + 1$$

is not a Pisot polynomial. Both these polynomials have 2 roots outside the unit circle.

Siegel in [**?**] proved that the smallest Pisot number (that is the smallest Mahler measure among the Pisot polynomials) is $\theta_0 = 1.3247\ldots$, root of the Pisot polynomial $x^3 - x - 1$. Later Smith in [**?**] proves that θ_0 is the smallest Mahler measure among the non reciprocal polynomials and therefore solves Lehmer's problem and Lehmer's conjecture for this particular class of polynomials. Thus, to solve Lehmer's problem it suffices to look at reciprocal polynomials.

Salem in 1945 shows that every Pisot number is a Limit point of a sequence of Salem numbers (see [**?**]). Therefore, from Siegel's theorem, we conclude that for any $M > \theta_0$, there are infinite Salem numbers in the interval $(1, M)$. In chapter 4 we define new Pisot numbers and find sequences of Salem polynomials converging to them.

Chapter 3

Cartan matrices and Coxeter polynomials of Lie algebras

He (Wilhelm Killing) exhibited the characteristic equation of the Weyl group when Weyl was 3 years old and listed the orders of the Coxeter transformation 19 years before Coxeter was born.—A.J.Coleman

3.1 Introduction

In this chapter we compute the characteristic and Coxeter polynomials of affine Lie algebras. This allows us to compute the affine Coxeter number and the affine exponents of the affine Lie algebras. For completeness we include the analog results for simple Lie algebras following [?, ?]. We generalize the definition of the branch vertex of Dynkin diagrams (definition 19) to the case of the Dynkin diagram of type A_n. With the generalization, the two theorems, of Steinberg (theorem 6) and of Berman, Lee and Moody (theorem 7) are applicable, to the case of the affine Lie algebra of type $A_n^{(1)}$.

3.2 Cartan matrices of the classical finite Lie algebras

3.2.1 Cartan matrix of type A_n

Toeplitz matrices have constant entries on each diagonal parallel to the main diagonal. Tridiagonal Toeplitz matrices are commonly the result of discretizing differential equations.

The eigenvalues of the Toeplitz matrix

$$\begin{pmatrix} b & a & & & & \\ c & b & a & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & b & a \\ & & & & c & b \end{pmatrix}$$

are given by

$$\lambda_j = b + 2a\sqrt{\frac{c}{a}}\cos\frac{j\pi}{n+1} \quad j = 1, 2\ldots, n\ , \tag{3.1}$$

see e.g. [**?**, p. 59].

The Cartan matrix, C_{A_n}, of type A_n is a tri-diagonal matrix of the form (2.3). It appears in the classification theory of simple Lie algebras over $\mathbb{C}$.

Taking $a = c = -1$, $b = 2$ in (3.1) we deduce that the eigenvalues of A_n are given by

$$\lambda_j = 2 - 2\cos\frac{j\pi}{n+1} = 4\sin^2\frac{j\pi}{2(n+1)} \quad j = 1, 2, \ldots, n\ .$$

Let d_n be the determinant of C_{A_n}. One can compute it using expansion on the first row and induction. We obtain $d_n = 2d_{n-1} - d_{n-2}$, $d_1 = 2$, $d_2 = 3$. This is a simple linear recurrence with solution $d_n = n + 1$.

We conclude that

$$\prod_{j=1}^{n} 4\sin^2\frac{j\pi}{2(n+1)} = n + 1,$$

or equivalently that

$$2^{2n}\prod_{j=1}^{n}\sin^2\frac{j\pi}{2(n+1)} = n + 1.$$

Lemma 11. *The characteristic polynomial $p_n(x)$ of C_{A_n} satisfies*

$$p_n(x) = U_n\left(\frac{x}{2} - 1\right)\ ,$$

where U_n is the Chebyshev polynomial of the second kind.

Proof. We write the eigenvalue equation in the form $\det(xI_n - C_{A_n}) = 0$ where I_n is the

$n \times n$ identity matrix. Explicitly,

$$\det\left(xI_n - C_{A_n}\right) = \det \begin{pmatrix} x-2 & 1 & & & & \\ 1 & x-2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & x-2 & 1 \\ & & & & 1 & x-2 \end{pmatrix} =$$

$$\det \begin{pmatrix} 2\left(\frac{x-2}{2}\right) & 1 & & & & \\ 1 & 2\left(\frac{x-2}{2}\right) & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2\left(\frac{x-2}{2}\right) & 1 \\ & & & & 1 & 2\left(\frac{x-2}{2}\right) \end{pmatrix} = U_n\left(\frac{x}{2} - 1\right).$$

$\square$

Remark 3. Note that

$$p_n(0) = U_n(-1) = (-1)^n U_n(1) = (-1)^n (n+1) \ ,$$

which agrees (up to a sign) with the formula for the determinant of A_n.

We list the formula for the characteristic polynomial of the matrix C_{A_n} for small values of n.

$$\begin{aligned} p_1(x) &= x - 2 \\ p_2(x) &= x^2 - 4x + 3 = (x-1)(x-3) \\ p_3(x) &= x^3 - 6x^2 + 10x - 4 = (x-2)(x^2 - 4x + 2) \\ p_4(x) &= x^4 - 8x^3 + 21x^2 - 20x + 5 = (x^2 - 5x + 5)(x^2 - 3x + 1) \\ p_5(x) &= x^5 - 10x^4 + 36x^3 - 56x^2 + 35x - 6 = (x-1)(x-2)(x-3)(x^2 - 4x + 1) \\ p_6(x) &= x^6 - 12x^5 + 55x^4 - 120x^3 + 126x^2 - 56x + 7 \\ p_7(x) &= x^7 - 14x^6 + 78x^5 - 220x^4 + 330x^3 - 252x^2 + 84x - 8 \ . \end{aligned}$$

Using the formula (2.13) we prove the following.

Proposition 12. *Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix C_{A_n}. Then*

$$p_n(x) = \sum_{j=0}^{n} (-1)^{n+j} \binom{n+j+1}{2j+1} x^j.$$

Proof. From the properties of Chebyshev polynomials (see section 2.3) it follows that $U_n\left(\frac{x}{2}-1\right)=(-1)^nU_n\left(1-\frac{x}{2}\right)$. Therefore from the formula (2.13) we deduce that

$$p_n(x)=(-1)^nU_n\left(1-\frac{x}{2}\right)=\sum_{j=0}^{n}(-1)^{n+j}\binom{n+j+1}{2j+1}x^j.$$

$\square$

3.2.2 Cartan matrix of type B_n and C_n

The Cartan matrix C_{B_n}, of type B_n, is a tri-diagonal matrix of the form (2.4). Since the Cartan matrix of type C_n is the transpose of this matrix we consider only the Cartan matrix of type B_n. Using expansion on the first row it is easy to prove that $\det(C_{B_n})=2$.

We list the formula for the characteristic polynomial of the matrix B_n for small values of n.

$$\begin{aligned}
p_2(x) &= x^2-4x+2\\
p_3(x) &= x^3-6x^2+9x-2=(x-2)(x^2-4x+1)\\
p_4(x) &= x^4-8x^3+20x^2-16x+2\\
p_5(x) &= x^5-10x^4+35x^3-50x^2+25x-2\\
&= (x-2)(x^4-8x^3+19x^2-12x+1)\\
p_6(x) &= x^6-12x^5+53x^4-104x^3+85x^2-20x+1\\
&= (x^2-4x+1)(x^4-8x^3+20x^2-16x+1)\\
p_7(x) &= x^7-14x^6+77x^5-210x^4+294x^3-196x^2+49x-2\\
&= (x-2)(x^6-12x^5+53x^4-104x^3+86x^2-24x+1)\ .
\end{aligned}$$

By expanding the determinant of the matrix $2xI+A_{B_n}$ with respect to the first row, we obtain the recurrence

$$q_1(x)=2x,\quad q_2(x)=4x^2-2,\quad q_{n+1}(x)=2xq_n(x)-q_{n-1}(x).$$

One may define $q_0(x)=2$. The recurrence implies that $q_n(x)=2T_n(x)$ where T_n is the n^{th} Chebyshev polynomial of the first kind.

Using the formula (2.12) we obtain the following result.

Proposition 13. *Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix (2.4). Then*

$$p_n(x)=\sum_{j=0}^{n-1}(-1)^{n+j}\frac{2n(n+j-1)!}{(n-j)!(2j)!}x^j\ .$$

3.2.3 Cartan matrix of type D_n

The Cartan matrix C_{D_n}, of type D_n, is a matrix of the form (2.5). Note that the matrix is no longer tri-diagonal. Using expansion on the first row and induction it is easy to prove that $\det(C_{D_n}) = 4$.

We list some formulas for the characteristic polynomial of the matrix C_{D_n} for small values of n.

$$\begin{aligned}
p_2(x) &= x^2 - 4x + 2 = (x-2)^2 \\
p_3(x) &= x^3 - 6x^2 + 10x - 4 = (x-2)(x^2 - 4x + 2) \\
p_4(x) &= x^4 - 8x^3 + 21x^2 - 20x + 4 = (x-2)^2(x^2 - 4x + 1) \\
p_5(x) &= x^5 - 10x^4 + 36x^3 - 56x^2 + 34x - 4 \\
&= (x-2)(x^4 - 8x^3 + 20x^2 - 16x + 2) \\
p_6(x) &= x^6 - 12x^5 + 55x^4 - 120x^3 + 125x^2 - 52x + 4 \\
&= (x-2)^2(x^4 - 8x^3 + 19x^2 - 12x + 1) \ .
\end{aligned}$$

By expanding the determinant of the matrix $2xI + A_{D_n}$ with respect to the first row, we deduce the recurrence

$$q_2(x) = 4x^2, \qquad q_3(x) = 8x^3 - 4x, \qquad q_{n+1}(x) = 2xq_n(x) - q_{n-1} \ .$$

We may define $q_1(x) = 4x$. It is clear that $q_n(x) = 4xT_{n-1}(x)$ where T_n is the n^{th} Chebyshev polynomial of the first kind.

Proposition 14. *Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix (2.5). Then*

$$p_n(x) = (2-x)\sum_{j=0}^{n-1}(-1)^{n+j}\frac{(2n-2)(n+j-2)!}{(n-j-1)!(2j)!}x^j \ .$$

Proof. From the formula $p_n(x) = q_n\left(\frac{x}{2} - 1\right)$ and (2.12) it follows that

$$p_n(x) = 2(x-2)(-1)^{n-1}T_{n-1}\left(1 - \frac{x}{2}\right) =$$
$$(2-x)\sum_{j=0}^{n-1}(-1)^{n+j}\frac{(2n-2)(n+j-2)!}{(n-j-1)!(2j)!}x^j \ .$$

□

3.3 Cartan matrices of the classical affine Lie algebras

3.3.1 Cartan matrix of type $A_n^{(1)}$

The Cartan matrix of type $A_n^{(1)}$ is the matrix

$$C_{A_n^{(1)}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{3.2}$$

We list some formulas for the characteristic polynomial of the matrix for small values of n.

$$\begin{aligned} p_3(x) &= x^3 - 6x^2 + 9x = x(x-3)^2 \\ p_4(x) &= x^4 - 8x^3 + 20x^2 - 16x = x(x-4)(x-2)^2 \\ p_5(x) &= x^5 - 10x^4 + 35x^3 - 50x^2 + 25x = x(x^2-5x+5)^2 \\ p_6(x) &= x^6 - 12x^5 + 54x^4 - 112x^3 + 105x^2 - 36x = x(x-4)(x-1)^2(x-3)^2 \\ p_7(x) &= x^7 - 14x^6 + 77x^5 - 210x^4 + 294x^3 - 196x^2 + 49x = \\ & \quad x(x^3 - 7x^2 + 14x - 7)^2 \\ p_8(x) &= x^8 - 16x^7 + 104x^6 - 352x^5 + 660x^4 - 672x^3 + 336x^2 - 64x = \\ & \quad x(x-4)(x-2)^2(x^2-4x+2)^2 . \end{aligned}$$

We define the sequence of polynomials $q_n(x)$ in the following way

$$q_n(x) = \det \begin{pmatrix} 2x & 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 2x & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2x & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2x & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2x & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 2x \end{pmatrix}.$$

By expanding the determinant we obtain the following formula for q_n

$$q_n(x) = 2xU_{n-1}(x) - 2U_{n-2}(x) + 2(-1)^{n-1} = U_n(x) - U_{n-2}(x) + 2(-1)^{n-1} = 2T_n(x) + 2(-1)^{n-1}.$$

It is easy then to compute the first few polynomials.

$$\begin{aligned}
q_3(x) &= 8x^3 - 6x + 2 = 2(x+1)(2x-1)^2 \\
q_4(x) &= 16x^4 - 16x^2 = 16x^2(x-1)(x+1) \\
q_5(x) &= 32x^5 - 40x^3 + 10x + 2 = 2(x+1)(4x^2-2x-1)^2 \\
q_6(x) &= 64x^6 - 96x^4 + 36x^2 - 4 = 4(x+1)(x-1)(2x-1)^2(2x+1)^2 \\
q_7(x) &= 128x^7 - 224x^5 + 112x^3 - 14x + 2 = 2(x+1)(8x^3 - 4x^2 - 4x + 1)^2.
\end{aligned}$$

For n even the polynomial q_n is divisible by $x-1$. Indeed

$$q_n(1) = U_n(1) - U_{n-2}(1) + 2(-1)^{n-1} = (n+1) - (n-1) - 2 = 0.$$

Remark 4. Note that

$$\begin{gathered}
p_n(0) = q_n(-1) = U_n(-1) - U_{n-2}(-1) + 2(-1)^{n-1} = \\
(-1)^n U_n(1) - (-1)^{n-2} U_{n-2}(1) + 2(-1)^{n-1} = 0.
\end{gathered}$$

Therefore the determinant of $A_n^{(1)}$ is zero and p_n is divisible by x.

Proposition 15. *Let p_n be the characteristic polynomial of the Cartan matrix (3.2). Then*

$$p_n(x) = \sum_{j=1}^{n} (-1)^{n+j} \frac{2n(n+j-1)!}{(n-j)!(2j)!} x^j.$$

Proof. Using the properties of Chebyshev polynomials it follows that

$$\frac{1}{2} p_n(x) + (-1)^n = \frac{1}{2} q_n\left(\frac{x}{2} - 1\right) + (-1)^n = T_n\left(\frac{x}{2} - 1\right) = (-1)^n T_n\left(1 - \frac{x}{2}\right).$$

Using the formula (2.12) we have

$$\begin{gathered}
T_n\left(1 - \frac{x}{2}\right) = n \sum_{j=0}^{n} (-2)^j \frac{(n+j-1)!}{(n-j)!(2j)!} \left(1 - \left(1 - \frac{x}{2}\right)\right)^j = \\
n \sum_{j=0}^{n} (-1)^j \frac{(n+j-1)!}{(n-j)!(2j)!} x^j.
\end{gathered}$$

Therefore

$$p_n(x) = \sum_{j=0}^{n} (-1)^{n+j} \frac{2n(n+j-1)!}{(n-j)!(2j)!} x^j + 2(-1)^{n-1}$$

and the result follows. □

3.3.2 Cartan matrix of type $B_n^{(1)}$

The Cartan matrix of type $B_n^{(1)}$ is the matrix

$$C_{B_n^{(1)}} = \begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{3.3}$$

Using expansion on the first row it is easy to show that $\det\left(C_{B_n^{(1)}}\right) = 0$.

We list some formulas for the characteristic polynomial of the matrix for small values of n.

$$\begin{aligned} p_4(x) &= x^4 - 8x^3 + 20x^2 - 16x = x(x-4)(x-2)^2 \\ p_5(x) &= x^5 - 10x^4 + 35x^3 - 50x^2 + 24x = x(x-1)(x-2)(x-3)(x-4) \\ p_6(x) &= x^6 - 12x^5 + 54x^4 - 112x^3 + 104x^2 - 32x = \\ & x(x-4)(x-2)^2(x^2-4x+2) \\ p_7(x) &= x^7 - 14x^6 + 77x^5 - 210x^4 + 293x^3 - 190x^2 + 40x = \\ & x(x-2)(x-4)(x^2-5x+5)(x^2-3x+1)\ . \end{aligned}$$

Define $q_n(x) = \det\left(2xI + A_{B_n^{(1)}}\right)$. By expanding the determinant we obtain the following formula for q_n

$$q_n(x) = 4x\left(T_{n-1}(x) - T_{n-3}(x)\right) = 8x\left(x^2-1\right)U_{n-3}(x).$$

Equivalently

$$q_n(x) = 2\left(T_n(x) - T_{n-4}(x)\right).$$

The first few polynomials are:

$$\begin{aligned} q_4(x) &= 16x^4 - 16x^2 = 16x^2(x-1)(x+1) \\ q_5(x) &= 32x^5 - 40x^3 + 8x = 8x(x-1)(2x+1)(2x-1)(x+1) \\ q_6(x) &= 64x^6 - 96x^4 + 32x^2 = 32x^2(x-1)(x+1)(2x^2-1) \\ q_7(x) &= 128x^7 - 224x^5 + 104x^3 - 8x = \\ & 8x(x-1)(x+1)(4x^2-2x-1)(4x^2+2x-1). \end{aligned}$$

As in the case of the Cartan matrix $C_{A_n^{(1)}}$, we can easily compute the explicit form of the p_n polynomial.

Proposition 16. *Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix (3.3). Then*

$$p_n(x) = x(x-2)(x-4)\sum_{j=0}^{n-3}(-1)^{n+j+1}\binom{n+j-2}{2j+1}x^j.$$

Proof. From $p_n(x) = q_n\left(\frac{x}{2}-1\right)$ and $q_n(x) = 8x\left(x^2-1\right)U_{n-3}(x)$ we only need to show that

$$U_{n-3}\left(\frac{x}{2}-1\right) = \sum_{j=0}^{n-3}(-1)^{n+j+1}\binom{n+j-2}{2j+1}x^j,$$

which is formula (2.13) combined with $U_n(-x) = (-1)^n U_n(x)$. □

3.3.3 Cartan matrix of type $C_n^{(1)}$

The Cartan matrix of type $C_n^{(1)}$ is the tri-diagonal matrix

$$C_{C_n^{(1)}} = \begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}. \tag{3.4}$$

Using expansion on the first row it is easy to show that $\det\left(C_{C_n^{(1)}}\right) = 0$.

We list the formula for the characteristic polynomial of the matrix (3.4) for small values of n.

$$\begin{aligned}
p_3(x) &= x^3 - 6x^2 + 8x = x(x-2)(x-4) \\
p_4(x) &= x^4 - 8x^3 + 19x^2 - 12x = x(x-1)(x-3)(x-4) \\
p_5(x) &= x^5 - 10x^4 + 34x^3 - 44x^2 + 16x = x(x-2)(x-4)(x^2-4x+2) \\
p_6(x) &= x^6 - 12x^5 + 53x^4 - 104x^3 + 85x^2 - 20x = \\
& \quad x(x-4)(x^2-5x+5)(x^2-3x+1) \\
p_7(x) &= x^7 - 14x^6 + 76x^5 - 200x^4 + 259x^3 - 146x^2 + 24x = \\
& \quad x(x-1)(x-2)(x-3)(x-4)(x^2-4x+1).
\end{aligned}$$

Define $q_n(x) = \det(2xI + A)$. By expanding the determinant with respect to the first row we obtain

$$q_n(x) = 4\left(xT_{n-1}(x) - T_{n-2}(x)\right),$$

where $T_n(x)$ is the n^{th} Chebyshev polynomial of the first kind. Equivalently,

$$q_n(x) = 2\left(T_n(x) - T_{n-2}(x)\right) = 4\left(x^2 - 1\right)U_{n-2}(x).$$

Proposition 17. *Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix (3.4). Then*

$$p_n(x) = x(x-4)\sum_{j=0}^{n-2}(-1)^{n+j}\binom{n+j-1}{2j+1}x^j.$$

3.3.4 Cartan matrix of type $D_n^{(1)}$

The Cartan matrix of type $D_n^{(1)}$ is the matrix

$$C_{D_n^{(1)}} = \begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 2 \end{pmatrix}. \tag{3.5}$$

We list the formula for the characteristic polynomial of the matrix $C_{D_n^{(1)}}$ for small values of n.

$$\begin{aligned} p_5(x) &= x(x-4)(x-2)^3 \\ p_6(x) &= x(x-1)(x-2)^2(x-3)(x-4) \\ p_7(x) &= x(x-2)^3(x-4)(x^2-4x+2) \\ p_8(x) &= x(x-2)^2(x-4)(x^2-3x+1)(x^2-5x+5) \\ p_9(x) &= x(x-2)^3(x-4)(x-1)(x-3)(x^2-4x+1). \end{aligned}$$

By expanding the determinant $q_n(x) = \det(2xI + A)$, with respect to the first row we get

$$q_n(x) = 2x\hat{q}_{n-1}(x) - 2x\hat{q}_{n-3}(x),$$

where $\hat{q}_n$ is the q_n polynomial of the matrix C_{D_n}. Therefore

$$q_n(x) = 8x^2(T_{n-2}(x) - T_{n-4}(x)) = 16x^2(x^2-1)U_{n-4}(x).$$

Proposition 18. *Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix (3.5). Then*

$$p_n(x) = x(x-2)^2(x-4)\sum_{j=0}^{n-4}(-1)^{n+j}\binom{n+j-3}{2j+1}x^j.$$

3.4 Coxeter polynomials

3.4.1 Coxeter polynomials for the classical finite Lie algebras

Associated polynomials for A_n

We present the factorization of the polynomial $Q_n(x)$ for small values of n, as a product of cyclotomic polynomials. The polynomial $Q_n(x)$ is an even polynomial. For, if n is even, then $U_n(x)$ is an even polynomial and $a_n(x) = U_n(\frac{x}{2})$ is also even. If n is odd then $U_n(x)$ and $a_n(x)$ are both odd functions. This implies that $Q_n(x)$ is even. The factorization of Q_n for small values of n, as product of cyclotomic polynomials is given in table 3.1.

Root system	Cyclotomic Factors
A_2	$\Phi_3\Phi_6$
A_3	$\Phi_4\Phi_8$
A_4	$\Phi_5\Phi_{10}$
A_5	$\Phi_3\Phi_4\Phi_6\Phi_{12}$
A_6	$\Phi_7\Phi_{14}$
A_7	$\Phi_4\Phi_8\Phi_{16}$
A_8	$\Phi_3\Phi_6\Phi_9\Phi_{18}$
A_9	$\Phi_4\Phi_5\Phi_{10}\Phi_{20}$
A_{10}	$\Phi_{11}\Phi_{22}$
A_{11}	$\Phi_3\Phi_4\Phi_6\Phi_8\Phi_{12}\Phi_{24}$

Table 3.1: Factorization of the polynomials $Q_n(x)$ for the root system A_n

It is not difficult to guess the factorization of $Q_n(x)$. The characteristic polynomial of the Coxeter transformation has roots ζ^k where ζ is a primitive h root of unity and k runs over the exponents of a root system of type A_n. Therefore

$$f_n(x) = (x-\zeta)(x-\zeta^2)\dots(x-\zeta^n)\ .$$
$$\Rightarrow (x-1)f_n(x) = x^{n+1}-1 \Rightarrow f_n(x) = \frac{x^{n+1}-1}{x-1}\ .$$

Using the formula (2.14) we obtain

$$f_n(x) = \prod_{\substack{j|n+1 \\ j\neq 1}} \Phi_d \ .$$

The next proposition is from [?].

Proposition 19. *The factorization of the polynomial Q_n for the root system A_n is given by*

$$Q_n(x) = \prod_{\substack{j|2n+2 \\ j\neq 1,2}} \Phi_j(x) \ . \tag{3.6}$$

Proof. Since

$$Q_n(x) = f_n(x^2) = \prod_{\substack{j|n+1 \\ j\neq 1}} \Phi_j(x^2)$$

we should know what is $\Phi_j(x^2)$.

It is well-known, see [?], that

$$\Phi_j(x^2) = \begin{cases} \Phi_{2j}(x), & if \quad j \quad \text{is} \quad even \\ \Phi_j(x)\Phi_{2j}(x), & if \quad j \quad \text{is} \quad odd \end{cases}$$

To complete the proof we must show that each divisor of $2n+2$ bigger than 2 appears in the product (3.6). Let d be a divisor of $2n+2$ bigger than 2. We consider two cases:
i) If d is odd then since $d|2(n+1)$ we have that $d|n+1$. Since Φ_d is a factor of $f_n(x)$, then $f_d(x^2) = \Phi_d(x)\Phi_{2d}(x)$, and therefore Φ_d appears.
ii) If d is even, then $d = 2s$ for some integer s bigger than 1. Since $2s|2(n+1)$ we have that $s|n+1$. Therefore Φ_s appears in the factorization of $f_n(x)$. If s is odd then $\Phi_s(x^2) = \Phi_s(x)\Phi_{2s}(x)$ and if s is even $\Phi_s(x^2) = \Phi_{2s}(x)$. In either case $\Phi_{2s} = \Phi_d$ appears. $\square$

An alternative way to derive the formula for f_n is the following. Note that the Coxeter adjacency matrix A_{A_n} is related to the Cartan matrix with $A_{A_n} = 2I - C_{A_n}$ and $a_n(x) = p_n(x+2) = q_n(\frac{x+2}{2} - 1) = q_n(\frac{x}{2})$. Therefore we have

$$a_n(x) = U_n\left(\frac{x}{2}\right)$$

and

$$Q_n(x) = x^n U_n\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right) \ .$$

Set $x = e^{i\theta}$ to obtain

$$Q_n(x) = e^{in\theta} U_n\left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right) = e^{in\theta} U_n(\cos\theta) =$$

$$e^{in\theta}\frac{\sin(n+1)\theta}{\sin\theta} = e^{in\theta}\frac{e^{i(n+1)\theta} + e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{x^{2(n+1)} - 1}{x^2 - 1} \ .$$

Setting $u = x^2$ we deduce that

$$f_n(u) = \frac{u^{n+1} - 1}{u - 1} = u^n + u^{n-1} + \cdots + u + 1 \ .$$

Therefore $Q_n(x) = x^{2n} + x^{2(n-1)} + \cdots + x^2 + 1$ for all $x \in \mathbb{C}$.

We present the characteristic polynomial of the adjacency matrix and the Coxeter polynomial for small values of n.

Root system	The polynomial $a_n(x)$	Coxeter polynomial
A_2	$a_2 = x^2 - 1$	$f_2 = \Phi_3$
A_3	$a_3 = x^3 - 2x$	$f_3 = \Phi_2\Phi_4$
A_4	$a_4 = x^4 - 3x^2 + 1$	$f_4 = \Phi_5$
A_5	$a_5 = x^5 - 4x^3 + 3x$	$f_5 = \Phi_2\Phi_3\Phi_6$
A_6	$a_6 = x^6 - 5x^4 + 6x^2 - 1$	$f_6 = \Phi_7$
A_7	$a_7 = x^7 - 6x^5 + 10x^3 - 4x$	$f_7 = \Phi_2\Phi_4\Phi_8$
A_8	$a_8 = x^8 - 7x^6 + 15x^4 - 10x^2 + 1$	$f_8 = \Phi_3\Phi_9$
A_9	$a_9 = x^9 - 8x^7 + 21x^5 - 20x^3 + 5x$	$f_9 = \Phi_2\Phi_5\Phi_{10}$
A_{10}	$a_{10} = x^{10} - 9x^8 + 26x^6 - 35x^4 + 15x^2 - 1$	$f_{10} = \Phi_{11}$

Table 3.2: Characteristic and Coxeter polynomials for the root system A_n

Note that $a_n(x)$ is explicitly given by the formula

$$a_n(x) = U_n\left(\frac{x}{2}\right) = \sum_{j=0}^{[\frac{n}{2}]}(-1)^j\binom{n-j}{j}(x)^{n-2j}$$

due to formula (2.13).

Associated Polynomials for B_n and C_n

In the case of B_n we have

$$a_n(x) = 2T_n\left(\frac{x}{2}\right)$$

and therefore

$$Q_n(x) = 2x^nT_n\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right) \ .$$

Set $x = e^{i\theta}$ to obtain

$$Q_n(x) = 2e^{in\theta}T_n\left(\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\right) = 2e^{in\theta}T_n(\cos\theta) =$$
$$2e^{in\theta}\cos n\theta = 2e^{in\theta}\frac{1}{2}\left(e^{in\theta}+e^{-in\theta}\right) = e^{2in\theta}+1 = x^{2n}+1.$$

Therefore $Q_n(x) = x^{2n}+1$ for all $x \in \mathbb{C}$. As a result the Coxeter polynomial is $f_n(x) = x^n + 1$. We present the factorization of $f_n(x)$ for small values of n.

Root system	The polynomial $a_n(x)$	Coxeter polynomial
B_2	$a_2 = x^2 - 2$	$f_2 = \Phi_4$
B_3	$a_3 = x^3 - 3x$	$f_2 = \Phi_2\Phi_6$
B_4	$a_4 = x^4 - 4x^2 + 2$	$f_4 = \Phi_8$
B_5	$a_5 = x^5 - 5x^3 + 5x$	$f_5 = \Phi_2\Phi_{10}$
B_6	$a_6 = x^6 - 6x^4 + 9x^2 - 2$	$f_6 = \Phi_4\Phi_{12}$
B_7	$a_7 = x^7 - 7x^5 + 14x^3 - 7x$	$f_7 = \Phi_2\Phi_{14}$
B_8	$a_8 = x^8 - 8x^6 + 20x^4 - 16x^2 + 2$	$f_8 = \Phi_{16}$
B_9	$a_9 = x^9 - 9x^7 + 27x^5 - 30x^3 + 9x$	$f_9 = \Phi_2\Phi_6\Phi_{18}$
B_{10}	$a_{10} = x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2 - 2$	$f_{10} = \Phi_4\Phi_{20}$

Table 3.3: Characteristic and Coxeter polynomials for the root system B_n

Write $n = 2^\alpha N$ where N is odd. As we already mentioned

$$f_n(x) = x^n + 1 = \prod_{d|N}\Phi_{2md}(x)\ ,$$

where $m = 2^\alpha$. Therefore

$$f_n(x) = x^n + 1 = \prod_{\substack{d|n\\ d\ \text{odd}}}\Phi_{2^{\alpha+1}d}(x) = \prod_{d|N}\Phi_{2^{\alpha+1}d}(x)\ .$$

Proposition 20. *Let $r = 2^{\alpha+2}$. Then*

$$Q_n(x) = \prod_{\substack{d|n\\ d\ \text{odd}}}\Phi_{rd}(x)\ .$$

Proof. It follows from the formula $\Phi_k(x^2) = \Phi_{2k}(x)$ when k is even. □

Note that the $a_n(x)$ polynomial is explicitly given in this case by the formula

$$a_n(x) = 2T_n\left(\frac{x}{2}\right) = \sum_{j=0}^{[\frac{n}{2}]}(-1)^j\frac{n(n-j-1)!}{j!(n-2j)!}(x)^{n-2j},$$

due to formula (2.12). Since $a_n(x) = 2T_n\left(\frac{x}{2}\right)$ these polynomials satisfy the recursion

$$a_{n+1} = xa_n(x) - a_{n-1}(x)$$

with $a_0(x) = 2$ and $a_1(x) = x$.

We mention a useful application of these polynomials. One can use them to express $x^n + x^{-n}$ as a function of $\zeta = x + \frac{1}{x}$. For $x = e^{i\theta}$ it is just the expression of $2\cos n\theta$ as a polynomial in $2\cos\theta$. This polynomial is clearly $a_n(x)$, the adjacency polynomial of B_n.

Example 9. Since

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \ .$$

it follows that

$$x^2 + \frac{1}{x^2} = \zeta^2 - 2 = a_2(\zeta) \ .$$

Similarly

$$x^3 + \frac{1}{x^3} = \zeta^3 - 3\zeta = a_3(\zeta) \ ,$$

and

$$x^4 + \frac{1}{x^4} = \zeta^4 - 4\zeta^2 + 2 = a_4(\zeta) \ .$$

Associated Polynomials for D_n

In the case of D_n we have

$$q_n(x) = 4xT_{n-1}(x).$$

Therefore,

$$a_n(x) = 2xT_{n-1}\left(\frac{x}{2}\right),$$

and

$$Q_n(x) = 2x^n\left(x + \frac{1}{x}\right)T_{n-1}\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right).$$

Using the methods of the previous section we obtain $Q_n(x) = x^{2n} + x^{2(n-1)} + x^2 + 1$. We conclude that $f_n(x) = x^n + x^{n-1} + x + 1$. We present the formula for $a_n(x)$ and the factorization of $f_n(x)$ for small values of n.

Proposition 21. *Write $n - 1 = 2^\alpha N$ where N is odd. Then*

$$f_n(x) = (x+1)(x^{n-1}+1) = \Phi_2(x)\prod_{d|N}\Phi_{2^{\alpha+1}d}(x) \ .$$

and

$$Q_n(x) = \Phi_4(x)\prod_{\substack{d|n-1 \\ d \text{ odd}}}\Phi_{2^{\alpha+2}d}(x) \ . \tag{3.7}$$

Root system	The polynomial $a_n(x)$	Coxeter polynomial
D_4	$a_4 = x^4 - 3x^2$	$f_4(x) = \Phi_2^2\Phi_6$
D_5	$a_5 = x^5 - 4x^3 + 2x$	$f_5 = \Phi_2\Phi_8$
D_6	$a_6 = x^6 - 5x^4 + 5x^2$	$f_6 = \Phi_2^2\Phi_{10}$
D_7	$a_7 = x^7 - 6x^5 + 9x^3 - 2x$	$f_7 = \Phi_2\Phi_4\Phi_{12}$
D_8	$a_8 = x^8 - 7x^6 + 14x^4 - 7x^2$	$f_8 = \Phi_2^2\Phi_{14}$
D_9	$a_9 = x^9 - 8x^7 + 20x^5 - 16x^3 + 2x$	$f_9 = \Phi_2\Phi_{16}$
D_{10}	$a_{10} = x^{10} - 9x^8 + 27x^6 - 30x^4 + 9x^2$	$f_{10} = \Phi_2^2\Phi_6\Phi_{18}$

Table 3.4: Characteristic and Coxeter polynomials for the root system D_n

3.4.2 Coxeter polynomials for the exceptional finite Lie algebras

Lie algebra of type G_2

The Cartan matrix for G_2 is (2.10), with characteristic polynomial

$$p_2(x) = x^2 - 4x + 1 \ ,$$

since

$$q_2(x) = 2T_2(x) - U_0(x) = 4x^2 - 3$$

and $p_2(x) = q_2\left(\frac{x}{2} - 1\right)$. The roots of $a_2(x) = x^2 - 3$ are

$$2\cos\frac{m_i\pi}{h}$$

where $m_1 = 1$ and $m_2 = 5$ are the exponents of the root system of type G_2. The Coxeter number h is 6. Finally,

$$Q_2(x) = x^4 - x^2 + 1 = \Phi_{12}(x) \ ,$$

and

$$f_2(x) = x^2 - x + 1 = \Phi_6(x) \ .$$

Lie algebra of type F_4

The Cartan matrix for F_4 is (2.9) with characteristic polynomial

$$p_4(x) = x^4 - 8x^3 + 20x^2 - 16x + 1 \ ,$$

and

$$a_4(x) = x^4 - 4x^2 + 1 = \psi_{24}(x) \ .$$

The roots of $a_4(x)$ are

$$\frac{1}{2}(\pm\sqrt{6}\pm\sqrt{2})$$

i.e.

$$2\cos\frac{m_i\ \pi}{12}$$

where $m_i \in \{1,5,7,11\}$. These are the exponents for F_4 and being the numbers less than 12 and prime to 12 imply

$$f_4(x) = x^4 - x^2 + 1 = \Phi_{12}(x)\ .$$

Lie algebras of type E_n

- $n = 6$

  The Cartan matrix for E_6 is (2.6). The associated polynomials are

$$q_6(x) = 64x^6 - 80x^4 + 20x^2 - 1 = (2x+1)(2x-1)(16x^4 - 16x^2 + 1),$$

$$p_6(x) = (x-1)(x-3)(x^4 - 8x^3 + 20x^2 - 16x + 1),$$

$$a_6(x) = x^6 - 5x^4 + 5x^2 - 1 = (x+1)(x-1)(x^4 - 4x^2 + 1) = \psi_3(x)\psi_6(x)\psi_{24}(x)$$

  and

$$Q_6(x) = (x^2+x+1)(x^2-x+1)(x^8 - x^4 + 1) = \Phi_3(x)\Phi_6(x)\Phi_{24}(x).$$

  The exponents of E_6 are $\{1,4,5,7,8,11\}$ and the Coxeter number is 12. The subset $\{1,5,7,11\}$ produces Φ_{12} and $\{4,8\}$ produces Φ_3. Therefore

$$f_6(x) = \Phi_3(x)\Phi_{12}(x)\ .$$

  The roots of $a_6(x)$ are

$$\pm 1,\quad \frac{1}{2}(\pm\sqrt{6}\pm\sqrt{2})$$

  i.e.

$$2\cos\frac{m_i\ \pi}{12}$$

  where $m_i \in \{1,4,5,7,8,11\}$. These are the exponents for E_6 and the Coxeter number is 12.

- $n = 7$

The Cartan matrix for E_7 is (2.7) The associated polynomials are

$$q_7(x) = 128x^7 - 192x^5 + 72x^3 - 6x = 2x(64x^6 - 96x^4 + 36x^2 - 3),$$

$$p_7(x) = (x-2)(x^6 - 12x^5 + 54x^4 - 112x^3 + 105x^2 + 1),$$

$$a_7(x) = x^7 - 6x^5 + 9x^3 - 3x = x(x^6 - 6x^4 + 9x^2 - 3) = \psi_4(x)\psi_{36}(x)$$

and

$$Q_7(x) = (x^2+1)(x^{12} - x^6 + 1) = \Phi_4(x)\Phi_{36}(x).$$

The exponents of E_7 are $\{1,5,7,9,11,13,17\}$ and the Coxeter number is 18. The subset $\{1,5,7,11,13,17\}$ produces Φ_{18} and $\{9\}$ produces Φ_2. Therefore the Coxeter polynomial factors out as

$$f_7(x) = \Phi_2(x)\Phi_{18}(x).$$

- $n = 8$

The Cartan matrix for E_8 is (2.8) The associated polynomials are

$$q_8(x) = 256x^8 - 448x^6 + 224x^4 - 32x^2 + 1,$$

$$p_8(x) = x^8 - 16x^7 + 105x^6 + 364x^5 + 714x^4 - 784x^3 + 440x^2 - 96x + 1,$$

$$a_8(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1 = \psi_{60}(x)$$

and

$$Q_8(x) = x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1 = \Phi_{60}(x).$$

The exponents of E_8 are $\{1,7,11,13,17,19,23,29\}$ which are the positive integers less than 30 and prime to 30. Therefore the Coxeter polynomial of E_8 is

$$f_8(x) = \Phi_{30}(x).$$

3.4.3 Coxeter polynomials for the classical affine Lie Algebras

The Coxeter polynomials for the affine Lie algebras are well-known, see e.g. [**?**]. We display their formulas in table 3.5 and then we derive the same formulas using Chebyshev polynomials.

Associated Polynomials for $A_n^{(1)}$

In the case of $A_{n-1}^{(1)}$ we have

Dynkin Diagram	Coxeter polynomial	Cyclotomic Factors
$A_n^{(1)}$	$(x^i-1)\cdot(x^{n+1-i}-1)$, $i=1,2,\ldots,\lfloor\frac{n+1}{2}\rfloor$	$\prod_{d\mid i}\Phi_d\prod_{d\mid n+1-i}\Phi_d$, $i=1,2,\ldots,\lfloor\frac{n+1}{2}\rfloor$
$B_n^{(1)}$	$(x^{n-1}-1)(x^2-1)$	$\Phi_1\Phi_2\prod_{d\mid n-1}\Phi_d$
$C_n^{(1)}$	$(x^n-1)(x-1)$	$\Phi_1\prod_{d\mid n}\Phi_d$
$D_n^{(1)}$	$(x^{n-2}-1)(x-1)(x+1)^2$	$\Phi_1\Phi_2^2\prod_{d\mid n-2}\Phi_d$
$E_6^{(1)}$	$x^7+x^6-2x^4-2x^3+x+1$	$\Phi_1^2\Phi_2\Phi_3^2$
$E_7^{(1)}$	$x^8+x^7-x^5-2x^4-x^3+x+1$	$\Phi_1^2\Phi_2^2\Phi_3\Phi_4^2$
$E_8^{(1)}$	$x^9+x^8-x^6-x^5-x^4-x^3+x+1$	$\Phi_1^2\Phi_2\Phi_3\Phi_5$
$F_4^{(1)}$	$x^5-x^3-x^2+1$	$\Phi_1^2\Phi_2\Phi_3$
$G_2^{(1)}$	x^3-x^2-x+1	$\Phi_1^2\Phi_2$

Table 3.5: Coxeter polynomials for Affine Graphs

$q_n(x)=2\left(T_n(x)+(-1)^{n-1}\right)$, so $a_n(x)=2\left(T_n\left(\frac{x}{2}\right)+(-1)^{n-1}\right)$ and $Q_n(x)=2x^n\left(T_n\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right)+(-1)^{n-1}\right)$.

If we set $x=e^{i\theta}$ we have

$$2x^nT_n\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right)=2x^nT_n(\cos\theta)=2x^n\cos(n\theta)$$
$$=2x^n\frac{1}{2}\left(e^{in\theta}+e^{-in\theta}\right)=x^n\left(x^n+\frac{1}{x^n}\right)=x^{2n}+1.$$

Therefore

$$Q_n(x)=x^{2n}+(-1)^{n-1}2x^n+1=\left(x^n+(-1)^{n-1}\right)^2.$$

and the factorization of Q_n is given by

$$Q_n(x)=\begin{cases}g_2 & n\ \text{even}\\ \frac{g_1}{g_2} & n\ \text{odd}\end{cases},$$

where $g_1=\prod_{d\mid 2n}\Phi_d^2,\ \ g_2=\prod_{d\mid n}\Phi_d^2.$

In the case of $A_n^{(1)}$ (since the graph $A_n^{(1)}$ is not a tree) the Coxeter polynomial is not uniquely defined. There are $\lfloor\frac{n+1}{2}\rfloor$ non conjugate Coxeter elements each one producing a different Coxeter polynomial. These polynomials are given by the formula (see [?])

$$(x^j - 1) \cdot (x^{n+1-j} - 1), \quad j = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor .$$

The factorization of these polynomials is given by

$$\prod_{d|j} \Phi_d(x) \prod_{d|n+1-j} \Phi_d(x), \quad j = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor$$

and for the first values of n we obtain table 3.6.

n	$f_{n+1}(x)$
3	$j=1:\ x^4 - x^3 - x + 1 = (x-1)(x^2-1) = \Phi_1^2\Phi_2$
	$j=2:\ x^4 - 2x^2 + 1 = (x^2-1)(x^2-1) = \Phi_1^2\Phi_2^2$
4	$j=1:\ x^5 - x^4 - x + 1 = (x-1)(x^4-1) = \Phi_1^2\Phi_2\Phi_4$
	$j=2:\ x^5 - x^3 - x^2 + 1 = (x^2-1)(x^3-1) = \Phi_1^2\Phi_2\Phi_3$
5	$j=1:\ x^6 - x^5 - x + 1 = (x-1)(x^5-1) = \Phi_1^2\Phi_5$
	$j=2:\ x^6 - x^4 - x^2 + 1 = (x^2-1)(x^4-1) = \Phi_1^2\Phi_2^2\Phi_4$
	$j=3:\ x^6 - 2x^3 + 1 = (x^3-1)(x^3-1) = \Phi_1^2\Phi_3^2$
6	$j=1:\ x^7 - x^6 - x + 1 = (x-1)(x^6-1) = \Phi_1^2\Phi_2\Phi_3\Phi_6$
	$j=2:\ x^7 - x^5 - x^2 + 1 = (x^2-1)(x^5-1) = \Phi_1^2\Phi_2\Phi_5$
	$j=3:\ x^7 - x^4 - x^3 + 1 = (x^3-1)(x^4-1) = \Phi_1^2\Phi_2\Phi_3\Phi_4$
7	$j=1:\ x^8 - x^7 - x + 1 = (x-1)(x^7-1) = \Phi_1^2\Phi_7$
	$j=2:\ x^8 - x^6 - x^2 + 1 = (x^2-1)(x^6-1) = \Phi_1^2\Phi_2^2\Phi_3\Phi_6$
	$j=3:\ x^8 - x^5 - x^3 + 1 = (x^3-1)(x^5-1) = \Phi_1^2\Phi_3\Phi_5$
	$j=4:\ x^8 - 2x^4 + 1 = (x^4-1)(x^4-1) = \Phi_1^2\Phi_2^2\Phi_4^2$
8	$j=1:\ x^9 - x^8 - x + 1 = (x-1)(x^8-1) = \Phi_1^2\Phi_2\Phi_4\Phi_8$
	$j=2:\ x^9 - x^7 - x^2 + 1 = (x^2-1)(x^7-1) = \Phi_1^2\Phi_2\Phi_7$
	$j=3:\ x^9 - x^6 - x^3 + 1 = (x^3-1)(x^6-1) = \Phi_1^2\Phi_2\Phi_3^2\Phi_6$
	$j=4:\ x^9 - x^5 - x^4 + 1 = (x^4-1)(x^5-1) = \Phi_1^2\Phi_2\Phi_4\Phi_5$

Table 3.6: Coxeter polynomials for $A_n^{(1)}$

Note that when n is even the polynomial $Q_n(x)$ can be written in the form $Q_n(x) = f_n(x^2)$ with $f_n(x)$ the Coxeter polynomial corresponding to the largest conjugacy class of Coxeter elements. In fact, for n even,

$$f_n(x) = \left(x^{\frac{n}{2}} - 1\right)^2 .$$

Using the formula we have found for the polynomial $Q_n(x)$ we can calculate the roots of the polynomial $a_n(x)$. Since $Q_n(x) = \left(x^n + (-1)^{n-1}\right)^2$, the roots of Q_n are given by

$$\begin{aligned} e^{\frac{2k\pi \mathrm{i}}{n}}, &\quad k = 0, 1, \ldots, n-1\ , \quad \text{for n even} \\ e^{\frac{(2k+1)\pi \mathrm{i}}{n}}, &\quad k = 0, 1, \ldots, n-1\ , \quad \text{for n odd} \end{aligned}$$

each one being a double root. Now if r is a root of a_n, it follows that $x-r$ is a factor of a_n so $x\left(x + \dfrac{1}{x} - r\right) = x^2 - rx + 1$ is a factor of $Q_n(x)$, meaning that $x^2 - rx + 1 = (x-c)(x-\bar{c})$, with c being one of the roots of Q_n and $r = 2\,\mathrm{Re}(c)$. We conclude that the roots of a_n are given by

$$\begin{aligned} 2\cos\frac{2k\pi}{n}, &\quad k = 0, 1, \ldots, n-1 \ \text{ for } n \text{ even}, \\ 2\cos\frac{(2k+1)\pi}{n}, &\quad k = 0, 1, \ldots, n-1 \ \text{ for } n \text{ odd}. \end{aligned}$$

From the identity $\cos(-x) = \cos x$ it follows that the roots of $a_{2n+2}(x)$ are given by

$$2\cos\frac{k\pi}{n+1}, \quad k = 0, 1, 1, 2, 2, \ldots, n, n, n+1,$$

where $k = 0, 1, 1, 2, 2, \ldots, n, n, n+1$ are the affine exponents and $h = n+1$ is the affine Coxeter number associated with the Coxeter polynomial $(x^{n+1} - 1)^2$.

Example 10. In the case of $A_5^{(1)}$ we have

$$a_6(x) = \left(x^2 - 1\right)^2 \left(x^2 - 4\right).$$

Therefore the roots of $a_6(x)$ are

$$1, 1, -1, -1, 2, -2,$$

and they have the form

$$2\cos\frac{m_i \pi}{h},$$

where m_i are the affine exponents and h is the affine Coxeter number associated with the Coxeter polynomial $(x^3 - 1)^2$.

Associated Polynomials for $B_n^{(1)}$

In the case of $B_{n-1}^{(1)}$ we have

$$q_n\left(x\right) = 8x\left(x^2 - 1\right) U_{n-3}\left(x\right).$$

Therefore,

$$a_n(x) = q_n\left(\frac{x}{2}\right) = x\left(x^2-4\right)U_{n-3}\left(\frac{x}{2}\right),$$

and

$$Q_n(x) = x^n\left(x^3 - x - \frac{1}{x} + \frac{1}{x^3}\right)U_{n-3}\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right).$$

Set $x = e^{i\theta}$ to obtain

$$\begin{aligned}
Q_n(x) &= x^{n-3}\left(x^6 - x^4 - x^2 + 1\right)U_{n-3}(\cos\theta)\\
&= x^{n-3}\left(x^4-1\right)\left(x^2-1\right)\frac{\sin(n-2)\theta}{\sin\theta}\\
&= x^{n-3}\left(x^4-1\right)\left(x^2-1\right)\frac{\left(e^{i(n-2)\theta} - e^{-i(n-2)\theta}\right)}{e^{i\theta}-e^{-i\theta}}\\
&= x^{n-3}\left(x^4-1\right)\left(x^2-1\right)\frac{x}{x^{n-2}}\frac{x^{2(n-2)}-1}{x^2-1} = x^{2n} - x^{2(n-2)} - x^4 + 1.
\end{aligned}$$

Therefore

$$Q_n(x) = x^{2n} - x^{2(n-2)} - x^4 + 1 = \left(x^4-1\right)\left(x^{2(n-2)}-1\right) = \Phi_1\Phi_2\Phi_4\prod_{d\mid 2(n-2)}\Phi_d,$$

for all $x \in \mathbb{C}$. The Coxeter polynomial for $B_n^{(1)}$ is then

$$f_{n+1}(x) = x^{n+1} - x^{n-1} - x^2 + 1 = \left(x^{n-1}-1\right)\left(x^2-1\right) = \Phi_1\Phi_2\prod_{d\mid n-1}\Phi_d$$

and the factorization of a_n is given by

$$a_n(x) = \Psi_4\prod_{j\mid 2(n-2)}\Psi_j(x).$$

We present the factorization of $f_n(x)$, in table 3.7, for small values of n.

In general we have two cases:

1) For the case of $B_{2n+1}^{(1)}$

$$a_{2n+2}(x) = x\left(x^2-4\right)U_{2n-1}\left(\frac{x}{2}\right).$$

Since the roots of $U_n(x)$ are

$$\cos\left(\frac{k\pi}{n+1}\right), \qquad k = 1, 2, \ldots, n,$$

Root system	The polynomial $a_n(x)$	Coxeter polynomial
$B_3^{(1)}$	$a_4 = x^4 - 4x^2$	$f_4(x) = \Phi_1^2\Phi_2^2$
$B_4^{(1)}$	$a_5 = x^5 - 5x^3 + 4x$	$f_5 = \Phi_1^2\Phi_2\Phi_3$
$B_5^{(1)}$	$a_6 = x^6 - 6x^4 + 8x^2$	$f_6 = \Phi_1^2\Phi_2^2\Phi_4$
$B_6^{(1)}$	$a_7 = x^7 - 7x^5 + 13x^3 - 4x$	$f_7 = \Phi_1^2\Phi_2\Phi_5$
$B_7^{(1)}$	$a_8 = x^8 - 8x^6 + 19x^4 - 12x^2$	$f_8 = \Phi_1^2\Phi_2^2\Phi_3\Phi_6$
$B_8^{(1)}$	$a_9 = x^9 - 9x^7 + 26x^5 - 25x^3 + 4x$	$f_9 = \Phi_1^2\Phi_2\Phi_7$
$B_9^{(1)}$	$a_{10} = x^{10} - 10x^8 + 34x^6 - 44x^4 + 16x^2$	$f_{10} = \Phi_1^2\Phi_2^2\Phi_4\Phi_8$

Table 3.7: Characteristic and Coxeter polynomials for the root system $B_n^{(1)}$

the roots of a_{2n+2} are $0, \pm 2$ and

$$2\cos\frac{k\pi}{2n}, \qquad k = 1, 2, \ldots, 2n-1.$$

Therefore the affine exponents are $0, 1, 2, \ldots, n-1, n, n, n+1, \ldots, 2n-1, 2n$ and the affine Coxeter number is $h = 2n$.

2) For the case of $B_{2n}^{(1)}$

$$a_{2n+1}(x) = x\left(x^2 - 4\right) U_{2n-2}\left(\frac{x}{2}\right).$$

Since the roots of $U_n(x)$ are

$$\cos\left(\frac{k\pi}{n+1}\right), \qquad k = 1, 2, \ldots, n,$$

the roots of a_{2n+1} are $0, \pm 2$ and

$$\cos\frac{2k\pi}{2(2n-1)}, \qquad k = 1, 2, \ldots, 2n-2.$$

It follows that the affine exponents are $0, 2, \ldots, 2n-2, 2n-1, 2n, \ldots, 2(2n-1)$ and the affine Coxeter number is $h = 2(2n-1)$.

Associated Polynomials for $C_n^{(1)}$

For $C_{n-1}^{(1)}$ we have

$$q_n(x) = 4\left(x^2 - 1\right) U_{n-2}(x).$$

Therefore,

$$a_n(x) = q_n\left(\frac{x}{2}\right) = \left(x^2 - 4\right) U_{n-2}\left(\frac{x}{2}\right),$$

and

$$Q_n(x) = x^n \left(x^2 - 2 + \frac{1}{x^2}\right) U_{n-2}\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right).$$

Using the same method as in the previous case we obtain

$$Q_n(x) = x^{2n} - x^{2(n-1)} - x^2 + 1 = \left(x^{2(n-1)} - 1\right)\left(x^2 - 1\right) = \Phi_1\Phi_2 \prod_{d|2(n-1)} \Phi_d,$$

for all $x \in \mathbb{C}$. The Coxeter polynomial for $C_n^{(1)}$ is then

$$f_{n+1}(x) = x^{n+1} - x^n - x + 1 = (x^n - 1)(x - 1) = \Phi_1 \prod_{d|n} \Phi_d.$$

We present the factorization of $f_n(x)$ for small values of n.

Root system	The polynomial $a_n(x)$	Coxeter polynomial
$C_3^{(1)}$	$a_4 = x^4 - 5x^2 + 4$	$f_4(x) = \Phi_1^2\Phi_3$
$C_4^{(1)}$	$a_5 = x^5 - 6x^3 + 8x$	$f_5 = \Phi_1^2\Phi_2\Phi_4$
$C_5^{(1)}$	$a_6 = x^6 - 7x^4 + 13x^2 - 4$	$f_6 = \Phi_1^2\Phi_5$
$C_6^{(1)}$	$a_7 = x^7 - 8x^5 + 19x^3 - 12x$	$f_7 = \Phi_1^2\Phi_2\Phi_3\Phi_6$
$C_7^{(1)}$	$a_8 = x^8 - 9x^6 + 26x^4 - 25x^2 + 4$	$f_8 = \Phi_1^2\Phi_7$
$C_8^{(1)}$	$a_9 = x^9 - 10x^7 + 34x^5 - 44x^3 + 16x$	$f_9 = \Phi_1^2\Phi_2\Phi_4\Phi_8$
$C_9^{(1)}$	$a_{10} = x^{10} - 11x^8 + 43x^6 - 70x^4 + 41x^2 - 4$	$f_{10} = \Phi_1^2\Phi_3\Phi_9$

Table 3.8: Characteristic and Coxeter polynomials for the root system $C_n^{(1)}$

The factorization of a_n is given by

$$a_n(x) = \prod_{j|2(n-1)} \Psi_j(x).$$

In general we have

$$a_{n+1}(x) = \left(x^2 - 4\right) U_{n-1}\left(\frac{x}{2}\right)$$

and therefore the roots of a_{n+1} are ± 2 and

$$2\cos\frac{k\pi}{n} \qquad k = 1, 2, \ldots, n-1.$$

The affine exponents are $0, 1, \ldots, n-1, n$ and the affine Coxeter number is $h = n$.

Associated Polynomials for $D_n^{(1)}$

In the case of $D_{n-1}^{(1)}$ we have

$$q_n(x) = 16x^2\left(x^2-1\right)U_{n-4}(x).$$

Therefore,

$$a_n(x) = q_n\left(\frac{x}{2}\right) = x^2\left(x^2-4\right)U_{n-4}\left(\frac{x}{2}\right)$$

and

$$Q_n(x) = x^n\left(x^4-2+\frac{1}{x^4}\right)U_{n-4}\left(\frac{1}{2}\left(x+\frac{1}{x}\right)\right).$$

It follows that

$$Q_n(x) = \left(x^4-1\right)\left(x^{2(n-2)}+x^{2(n-3)}-x^2-1\right) = \left(x^4-1\right)\left(x^2+1\right)\left(x^{2(n-3)}-1\right)$$

and the Coxeter polynomial for $D_n^{(1)}$ is

$$f_{n+1}(x) = \left(x^{n-2}-1\right)\left(x^2-1\right)(x+1).$$

We present the factorization of $f_n(x)$ for small values of n.

Root system	The polynomial $a_n(x)$	Coxeter polynomial
$D_4^{(1)}$	$a_5 = x^5-4x^3$	$f_5 = \Phi_1^2\Phi_2^3$
$D_5^{(1)}$	$a_6 = x^6-5x^4+4x^2$	$f_6 = \Phi_1^2\Phi_2^2\Phi_3$
$D_6^{(1)}$	$a_7 = x^7-6x^5+8x^3$	$f_7 = \Phi_1^2\Phi_2^3\Phi_4$
$D_7^{(1)}$	$a_8 = x^8-7x^6+13x^4-4x^2$	$f_8 = \Phi_1^2\Phi_2^2\Phi_5$
$D_8^{(1)}$	$a_9 = x^9-8x^7+19x^5-12x^3$	$f_9 = \Phi_1^2\Phi_2^3\Phi_3\Phi_6$
$D_9^{(1)}$	$a_{10} = x^{10}-9x^8+26x^6-25x^4+4x^2$	$f_{10} = \Phi_1^2\Phi_2^2\Phi_7$

Table 3.9: Characteristic and Coxeter polynomials for the root system $D_n^{(1)}$

Note that the factorization of $Q_n(x)$ is

$$Q_n(x) = \Phi_1\Phi_2\Phi_4^2\prod_{j|2(n-3)}\Phi_j(x),$$

and the factorization of $f_n(x)$ is

$$f_n(x) = \Phi_1\Phi_2^2\prod_{j|(n-3)}\Phi_j(x).$$

The corresponding factorization of $a_n(x)$ is

$$a_n(x) = \Psi_4^2 \prod_{j|2(n-3)} \Psi_j(x).$$

In general we have two cases:

1) For the case of $D_{2n+1}^{(1)}$ we have

$$a_{2n+2}(x) = x^2(x^2-4)U_{2n-2}\left(\frac{x}{2}\right).$$

The roots of a_{2n+2} are $0, 0, \pm 2$ and

$$2\cos\frac{2k\pi}{2(2n-1)}, \qquad k = 1, 2, \ldots, 2n-2.$$

The affine exponents are $0, 2, \ldots, 2n-2, 2n-1, 2n-1, 2n, \ldots, 2(2n-1)$ and the affine Coxeter number $h = 2(2n-1)$.

2) For the case of $D_{2n}^{(1)}$

$$a_{2n+1}(x) = x^2(x^2-4)U_{2n-3}\left(\frac{x}{2}\right)$$

and the roots of a_{2n+1} are $0, 0, \pm 2$ and

$$2\cos\frac{k\pi}{2(n-1)} \qquad k = 1, 2, \ldots, 2n-3.$$

Therefore the affine exponents are $0, 1, \ldots, n-2, n-1, n-1, n, 2n-3, 2n-2$ and the affine Coxeter number is $h = 2n-2$.

3.4.4 Coxeter polynomials for the exceptional affine Lie algebras

Affine Lie algebra of type $E_6^{(1)}$

The Cartan matrix of $E_6^{(1)}$ is

$$C_{E_6^{(1)}} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

The polynomial $p_7(x)$ is

$$p_7(x) = x^7 - 14x^6 + 78x^5 - 220x^4 + 329x^3 - 246x^2 + 72x$$

and the polynomial $a_7(x)$

$$a_7(x) = x^7 - 6x^5 + 9x^3 - 4x.$$

The roots of $a_7(x)$ are

$$0, 1, -1, 1, -1, 2, -2$$

i.e.

$$2\cos\frac{m_i\ \pi}{6},$$

where $m_i \in \{0, 2, 2, 3, 4, 4, 6\}$. These are the affine exponents for $E_6^{(1)}$ and the affine Coxeter number is $h = 6$.

The Coxeter polynomial is

$$f_7(x) = x^7 + x^6 - 2x^4 - 2x^3 + x + 1 = \prod_{m_i}\left(x - e^{\frac{2m_i\pi}{6}}\right) = \Phi_1^2\Phi_2\Phi_3^2.$$

Affine Lie algebra of type $E_7^{(1)}$

The Cartan matrix of $E_7^{(1)}$ is

$$C_{E_7^{(1)}} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The polynomial $p_8(x)$ is

$$p_8(x) = x^8 - 16x^7 + 105x^6 - 364x^5 + 714x^4 - 784x^3 + 440x^2 - 96x$$

and the polynomial $a_8(x)$

$$a_8(x) = x^8 - 7x^6 + 14x^4 - 8x^2.$$

The roots of $a_8(x)$ are

$$0, 0, 1, -1, \sqrt{2}, -\sqrt{2}, 2, -2$$

i.e.

$$2\cos\frac{m_i\,\pi}{12},$$

where $m_i \in \{0, 3, 4, 6, 6, 8, 9, 12\}$. These are the affine exponents for $E_7^{(1)}$ and $h = 12$ is the affine Coxeter number.

The Coxeter polynomial is

$$f_8(x) = x^8 + x^7 - x^5 - 2x^4 - x^3 + x + 1 = \prod_{m_i}\left(x - e^{\frac{2m_i\pi}{12}}\right) = \Phi_1^2\Phi_2^2\Phi_3\Phi_4^2.$$

Affine Lie algebra of type $E_8^{(1)}$

The Cartan matrix for $E_8^{(1)}$ is

$$\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}$$

and

$$p_9(x) = x^9 - 18x^8 + 136x^7 - 560x^6 + 1364x^5 - 1992x^4 + 1679x^3 - 730x^2 + 120x.$$

The polynomial $a_9(x)$ is given by

$$a_9(x) = x^9 - 8x^7 + 20x^5 - 17x^3 + 4x,$$

with roots

$$0, 1, -1, 2, -2, \frac{\sqrt{5}-1}{2}, \frac{-\sqrt{5}+1}{2}, \frac{\sqrt{5}+1}{2}, \frac{-\sqrt{5}-1}{2}$$

i.e.

$$2\cos\frac{m_i\,\pi}{30}$$

where $m_i = 0, 6, 10, 12, 15, 18, 20, 24, 30$ are the affine exponents and $h = 30$ is the affine Coxeter number.

The Coxeter polynomial is

$$f_9(x) = x^9 + x^8 - x^6 - x^5 - x^4 - x^3 + x + 1 = \prod_{m_i} \left(x - e^{\frac{2m_i \pi}{30}} \right) = \Phi_1^2 \Phi_2 \Phi_3 \Phi_5.$$

Affine Lie algebra of type $F_4^{(1)}$

The Cartan matrix for $F_4^{(1)}$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

with

$$p_5(x) = x^5 - 10x^4 + 33x^3 - 38x^2 + 2x + 12$$

and

$$a_5(x) = x^5 - 5x^3 + 4x.$$

The roots of the polynomial $a_5(x)$ are

$$0, 1, -1, 2, -2$$

i.e.

$$2 \cos \frac{m_i \ \pi}{6},$$

where $m_i = 0, 2, 3, 4, 6$ are the affine exponents and $h = 6$ is the affine Coxeter number.

The Coxeter polynomial is

$$f_5(x) = x^5 - x^3 - x^2 + 1 = \prod_{m_i} \left(x - e^{\frac{2m_i \pi}{6}} \right) = \Phi_1^2 \Phi_2 \Phi_3.$$

Affine Lie algebra of type $G_2^{(1)}$

The Cartan matrix for $G_2^{(1)}$ is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

and the polynomial $p_3(x)$ is given by

$$p_3(x) = x^3 - 6x^2 + 8x.$$

The polynomial $a_3(x)$ is

$$a_3(x) = x^3 - 4x,$$

with roots

$$0, 2, -2$$

i.e.

$$2\cos\frac{m_i\ \pi}{2},$$

where $m_i \in \{0, 1, 2\}$. These are the affine exponents for $G_2^{(1)}$ and the affine Coxeter number is $h = 2$.

The Coxeter polynomial is

$$f_3(x) = x^3 - x^2 - x + 1 = \prod_{m_i}(x - e^{m_i \pi}) = \Phi_1^2 \Phi_2.$$

3.5 The $A_n^{(1)}$ case

The aim of this section is to show that the formulas of Berman, Lee, Moody (see [?]) and Steinberg (see [?, ?]), for the Coxeter polynomials can be modified and applied to the case of $A_n^{(1)}$. In particular we show in propositions 22 and 23 that these formulas can be used for the explicit calculation of all the Coxeter polynomials for any affine Lie algebra. We compute and list in table 2.1 the affine exponents and affine Coxeter number associated with each Coxeter polynomial of $A_n^{(1)}$.

First we fix some notation. Let $X_n^{(1)}$ be an affine Lie algebra with Cartan matrix C, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ a set of simple roots of the root system of type X_n and α_0 minus the highest root of X_n. Let $V = \mathbb{R} - \mathrm{span}\{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n\}$, $z = (z_0, z_1, \ldots, z_n) \in \mathbb{Z}^{n+1}$ the left zero eigenvector of C, $\alpha = \sum_{i=0}^n z_i \alpha_i$ and $\tilde{V} = V/\langle\alpha\rangle$ as in section 2.5. Let $\sigma = \sigma_{\pi(0)}\sigma_{\pi(1)}\sigma_{\pi(2)}\ldots\sigma_{\pi(n)} \in gl(V)$ and $\pi \in S_{n+1}$ be a Coxeter transformation of $X_n^{(1)}$. From the definition of the simple reflections as $\sigma_j(\alpha_i) = \alpha_i - C_{i,j}\alpha_j$ it follows that σ leaves α invariant. Therefore σ is defined on $\tilde{V}$ and it has finite order. Its order is the affine Coxeter number h, associated with σ.

3.5.1 Berman, Lee, Moody's method

The defect map of the Coxeter transformation σ is the map $\partial : V \to \mathbb{R}\alpha$, defined by $\partial = Id_V - \sigma^h : V \to V$. In [?] Berman, Lee and Moody consider all cases of affine Lie algebras where the Dynkin diagram is bipartite and they show that for all i, $\partial(\alpha_i) \in c\mathbb{Z}\alpha$ for some $c \in \mathbb{N}$ (for the case $A_n^{(1)}, n$ odd, they consider the defect map of the Coxeter transformation corresponding to the largest conjugacy class). Further, they prove that if β is the branch root of X, then c is the least positive integer such that $cw_{\beta^\vee}$ belongs to the co-root lattice and $cw_{\beta^\vee} = \sum_{i=0}^{n} m_i\alpha_i^\vee$, where m_i are the affine exponents. We generalize this result to include all Coxeter transformations corresponding to $A_n^{(1)}$, for n both even and odd.

First we generalize the notion of a branch vertex of the Dynkin diagram of a simple Lie algebra to the case of A_n.

Definition 23. Let Γ be a finite Dynkin diagram of type X_n and $b : V(\Gamma) \to \mathbb{N}$ the weight function of definition 19. A vertex r_i is said to be a branch vertex of Γ if b attains its maximum value on r_i.

Therefore, for the case of the Dynkin diagram of type A_n, all vertices are branch vertices with $b(r_i) = 2$ for all $i = 1, 2, \ldots, n$. Now we can extent Berman, Lee and Moody's result to the case of $A_n^{(1)}$.

Proposition 22. *Let Γ be the Dynkin diagram of the simple finite dimensional Lie algebra of type X_n and let $\beta = \alpha_{i_0}$ be a root corresponding to a branch vertex of Γ. If c is the least positive integer such that $cw_{\beta^\vee}$ belongs to the co-root lattice of X_n, then*

$$cw_{\beta^\vee} = \sum_{i=0}^{n} m_i\alpha_i^\vee,$$

where m_i are the affine exponents and m_{i_0} is the affine Coxeter number of $X_n^{(1)}$ associated with a Coxeter polynomial of $X_n^{(1)}$.

Proof. We consider only the case $X_n = A_n$ since the other cases follow from theorem 7.

We realize (see section 2.2) the root system A_n as the set of vectors in $\mathbb{R}^{n+1}$ with length $\sqrt{2}$ and whose coordinates are integers and sum to zero. The inner product is the usual inner product in $\mathbb{R}^{n+1}$. One choice of a base of the root system of type A_n is

$$\alpha_i = (\underbrace{0, 0, \ldots, 0, 1}_{i \text{ terms}}, -1, 0, \ldots, 0, 0).$$

The co-roots are $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)} = \alpha_i$ and the corresponding weights are

$$w_i = \frac{1}{n+1}(\underbrace{n+1-i, n+1-i, \ldots, n+1-i}_{i \text{ terms}}, -i, -i, \ldots, -i).$$

Let $d_i = \gcd(n+1-i, i) = \gcd(n+1, i)$. If we choose as branch root the root $\beta = \alpha_{i_0}$ (or $\beta = \alpha_{n+1-i_0}$), $i_0 \in \left\{1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\right\}$ then the smallest positive integer c for which cw_β belongs to the co-root lattice is $c = \dfrac{n+1}{d_{i_0}}$ (in the case where n is odd and $i = \frac{n+1}{2}$ we have $c = 2$; for the other cases $c > 2$). For that c we have

$$\begin{aligned} cw_\beta &= \left(\frac{n+1-i_0}{d_{i_0}}, \frac{n+1-i_0}{d_{i_0}}, \ldots, \frac{n+1-i_0}{d_{i_0}}, -\frac{i_0}{d_{i_0}}, -\frac{i_0}{d_{i_0}}, \ldots, -\frac{i_0}{d_{i_0}}\right) \\ &= \frac{n+1-i_0}{d_{i_0}}\alpha_1 + 2\frac{n+1-i_0}{d_{i_0}}\alpha_2 + \ldots + i_0\frac{n+1-i_0}{d_{i_0}}\alpha_{i_0} + \\ &\quad i_0\frac{n-i_0}{d_{i_0}}\alpha_{i_0+1} + i_0\frac{n-1-i_0}{d_{i_0}}\alpha_{i_0+2} + \ldots + \frac{i_0}{d_{i_0}}\alpha_n. \end{aligned}$$

The coefficients

$$0, \frac{n+1-i_0}{d_{i_0}}, 2\frac{n+1-i_0}{d_{i_0}}, \ldots, i_0\frac{n+1-i_0}{d_{i_0}}, i_0\frac{n-i_0}{d_{i_0}}, i_0\frac{n-1-i_0}{d_{i_0}}, \ldots, \frac{i_0}{d_{i_0}}$$

are precisely the affine exponents and $i_0\frac{n+1-i_0}{d_{i_0}}$ is the affine Coxeter number corresponding to the Coxeter polynomial

$$(x^{i_0} - 1)(x^{n+1-i_0} - 1).$$

□

We illustrate with two examples for the cases $A_4^{(1)}$ and $A_5^{(1)}$.

Example 11. In the case of the root system of type A_4

the simple roots are

$$\alpha_1 = (1, -1, 0, 0, 0), \alpha_2 = (0, 1, -1, 0, 0), \alpha_3 = (0, 0, 1, -1, 0), \alpha_4 = (0, 0, 0, 1, -1).$$

If we choose α_1 (or α_4) as the branch root then

$$w_1 = \frac{1}{5}(4, -1, -1, -1, -1)$$

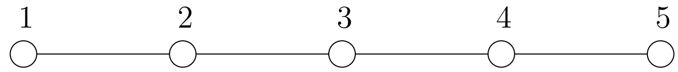
and $5w_1 = 4\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$. The affine Coxeter number is 4 and the affine exponents $0, 1, 2, 3, 4$ which give rise to the Coxeter polynomial $(x-1)(x^4-1)$.

If we choose α_2 (or α_3) as the branch root then

$$w_2 = \frac{1}{5}(3, 3, -2, -2, -2)$$

and $5w_1 = 3\alpha_1 + 6\alpha_2 + 4\alpha_3 + 2\alpha_4$. The affine Coxeter number is 6 and the affine exponents $0, 2, 3, 4, 6$ which give rise to the Coxeter polynomial $(x^2 - 1)(x^3 - 1)$.

Example 12. Dynkin diagram of A_5:



In this case the simple roots for A_5 are

$$\alpha_1 = (1, -1, 0, 0, 0, 0), \alpha_2 = (0, 1, -1, 0, 0, 0), \alpha_3 = (0, 0, 1, -1, 0, 0),$$
$$\alpha_4 = (0, 0, 0, 1, -1, 0), \alpha_5 = (0, 0, 0, 0, 1, -1).$$

The affine Coxeter number corresponding to the Coxeter polynomial $(x - 1)(x^5 - 1)$ is 5 and the affine exponents are $0, 1, 2, 3, 4, 5$. They correspond to the branch root α_1 (or α_5) for which the co-weight is

$$w_1 = \frac{1}{6}(5, -1, -1, -1, -1, -1)$$

and $6w_1 = 5\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$.

The branch root α_2 (or α_4) corresponds to the Coxeter polynomial $(x^2 - 1)(x^4 - 1)$ which give rise to the affine Coxeter number 4 and the affine exponents $0, 1, 2, 2, 3, 4$.

If we choose the middle root α_3, as the branch root then

$$w_3 = \frac{1}{2}(1, 1, 1, -1, -1, -1)$$

and $2w_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$. The affine Coxeter number is 3 and the affine exponents are $0, 1, 2, 3, 2, 1$ which give rise to the Coxeter polynomial $(x^3 - 1)^2$.

3.5.2 Steinberg's method

Steinberg in [**?**] (see also [**?**]) shows that for the affine root systems considered in definition 19, their Coxeter polynomial is a product of Coxeter polynomials of type A_i. Removing the branch root, if $g(x)$ is the Coxeter polynomial of the resulting root system then $(x - 1)^2 g(x)$ is the Coxeter polynomial of $X_n^{(1)}$.

We generalize Steinberg result to the case of root systems of type $A_n^{(1)}$.

Proposition 23. *Let β be a branch root, as defined in definition 23 of an affine root system of type $X_n^{(1)}$. If $g(x)$ is the Coxeter polynomial of the root system obtained by removing β from the root system of type X_n, then $(x - 1)^2 g(x)$ is a Coxeter polynomial of $X_n^{(1)}$.*

Proof. For $X_n \neq A_n$ we have Steinberg's theorem. For $X_n = A_n$ if we take as branch root $\beta = \alpha_{i_0},\ i_0 \in \left\{1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\right\}$, then the root system obtained by removing β is $A_{i_0-1} \times A_{n-i_0}$ with Coxeter polynomial

$$g(x) = \left(x^{i_0-1} + x^{i_0-2} + \ldots + x + 1\right)\left(x^{n-i_0} + x^{n-i_0-1} + \ldots + x + 1\right).$$

Then $(x-1)^2 g(x) = (x^{i_0} - 1)\left(x^{n-i_0} - 1\right)$ is one of the Coxeter polynomials of $A_n^{(1)}$ $\square$

Example 13. In the case of the root system of type $A_4^{(1)}$, if we choose α_1 (or α_4) as the branch root then the root system obtained by removing α_1 is A_3 with Coxeter polynomial x^3+x^2+x+1. We obtain the Coxeter polynomial $(x-1)^2(x^3+x^2+x+1) = (x-1)(x^4-1)$.

If we choose α_2 (or α_3) as the branch root then the root system obtained by removing the branch root is $A_1 \times A_2$ with Coxeter polynomial $(x+1)(x^2+x+1)$. The corresponding Coxeter polynomial is $(x-1)^2(x+1)(x^2+x+1) = (x^2-1)(x^3-1)$.

Example 14. In the case of the root system of type $A_5^{(1)}$ if we choose α_1 (or α_5) as the branch root then the root system obtained by removing the branch root is A_4 with Coxeter polynomial $x_4 + x^3 + x^2 + x + 1$. We obtain the Coxeter polynomial $(x-1)^2(x^4 + x^3 + x^2 + x + 1) = (x-1)(x^5-1)$.

If we choose α_2 (or α_4) as the branch root then the root system obtained by removing the branch root is $A_1 \times A_3$ with Coxeter polynomial $(x+1)(x^3+x^2+x+1)$. We obtain the Coxeter polynomial $(x-1)^2(x+1)(x^3+x^2+x+1) = (x^2-1)(x^4-1)$.

If we choose α_3 as the branch root then the root system obtained by removing α_3 is $A_2 \times A_2$ with Coxeter polynomial $(x^2+x+1)^2$. We obtain the Coxeter polynomial $(x-1)^2(x^2+x+1)^2 = (x^3-1)^2$.

Chapter 4

Coxeter polynomials of Salem trees

If only I had the theorems! Then I should find the proofs easily enough. — Bernhard Riemann

4.1 Introduction

In this chapter we will be concerned only with simple graphs that are trees. This class of graphs Γ has the property that all their Coxeter elements are conjugate in the corresponding Coxeter group W_Γ (see proposition 7 and also [?, ?, ?]) and therefore we can speak about the Coxeter polynomial of Γ which is the characteristic polynomial of a Coxeter element. We denote this polynomial by $\Gamma(x)$. Another important property of trees is that they are bipartite, i.e. the set of their vertices, $\mathcal{V}(\Gamma)$, can be partitioned into two sets $\mathcal{V}_1, \mathcal{V}_2$ with the property $v_1, v_2 \in \mathcal{V}_1$ or $v_1, v_2 \in \mathcal{V}_2$ implies $(v_1, v_2) \notin \mathcal{E}(\Gamma)$. The adjacency matrix of Γ is the $n \times n$ symmetric matrix $A \in \mathbb{M}_n(\mathbb{Z})$, with $A_{i,j} = 1$ if $(v_i, v_j) \in \mathcal{E}(V)$ and $A_{i,j} = 0$ if $(v_i, v_j) \notin \mathcal{E}(V)$. The characteristic polynomial of Γ is the polynomial $\chi_\Gamma(x) = \det(xI_n - A)$.

For a monic polynomial $p(x) \in \mathbb{Z}[x]$ the set of its zeros $\{z \in \mathbb{C} : p(z) = 0\}$ will be denoted by $Z(p)$ and the maximum value of the set $\{|z| : z \in Z(p)\}$ by $\rho(p)$. With the polynomial $p(x)$ we associate the polynomial $f(x) = x^n p\left(x + \frac{1}{x}\right)$, where $n = \deg(p)$, which is reciprocal (see section 2.6). The sets $Z(p)$ and $Z(f)$ are related in the following way. If $r \in Z(p)$ is real and $z = \frac{r \pm \sqrt{r^2-4}}{2}$ then $z \in Z(f)$. Furthermore, if $|r| \leq 2$ then $|z| = 1$ while if $|r| > 2$ then $z \in \mathbb{R}$ and $|z| > 1$.

A Coxeter graph Γ is called cyclotomic graph if all the roots of its Coxeter polynomial are on the unit disk or equivalently if its Coxeter polynomial is a product of cyclotomic polynomials. It is called a Salem graph if its Coxeter polynomial has only one root outside the unit circle or equivalently its Coxeter polynomial is a product of a Salem and cyclotomic polynomials.

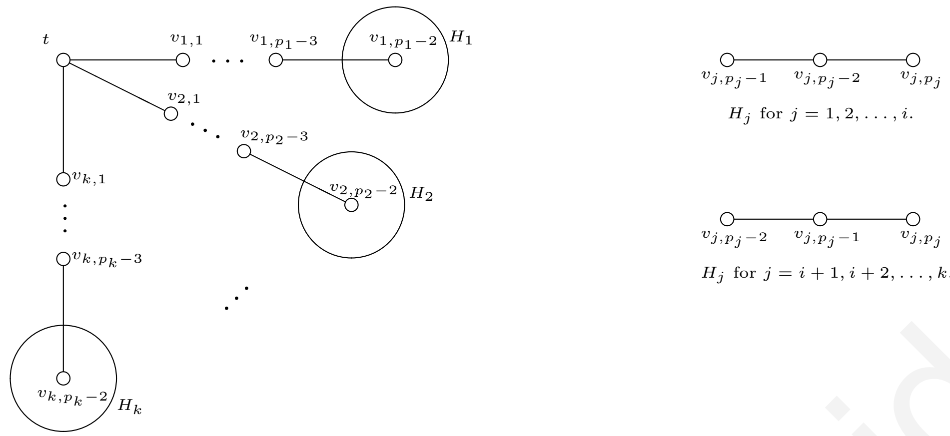


Figure 4.1: The graphs $S^{(i)}_{p_1,\ldots,p_k}$

The join of the Coxeter graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ on the vertices $v_i \in \mathcal{V}(\Gamma_i)$ is the graph obtained by adding a new vertex and joining that to v_i for all $i = 1, 2, \ldots, k$. In [?] it was shown that if a noncyclotomic tree is the join of cyclotomic trees then it is a Salem tree. For $k \in \mathbb{N}$, $p_1, \ldots, p_k \in \mathbb{N}$ and $i \in \{0, 1, 2, \ldots, k\}$ consider the graph $S^{(i)}_{p_1,\ldots,p_k}$ which is the join of the Dynkin diagrams $D_{p_1}, \ldots, D_{p_i}$ and $A_{p_{i+1}}, \ldots, A_{p_k}$, as shown in fig. 4.1. For particular values of i and p_j the graphs $S^{(i)}_{p_1,\cdots,p_k}$ give rise to well known graphs. For $k = 2, i = 0$ we obtain the Dynkin diagrams $A_{p_1+p_2+1}$, for $k = 3, i = 0, p_1 = 2, p_2 = 1$ we obtain the graphs E_{p_3+4}, for $k = 3, i = 0, p_j = 2$ the affine Dynkin diagram $E_6^{(1)}$, for $k = 3, i = 1, p_2 = p_3 = 1$ the affine Dynkin diagrams $D^{(1)}_{p_1+2}$ and many others. The polynomial $S^{(0)}_{1,2,6}(x)$ is Lehmer's polynomial which is conjectured to have the smallest Mahler measure among the monic integer polynomials (see section 2.6 and [?]). We prove three theorems about the Coxeter polynomials $S^{(i)}_{p,q,r}(x)$ for $i = 0, 1, 2, 3$. In theorem 9 we explicitly calculate the Coxeter polynomials $S^{(i)}_{p,q,r}(x)$ for $i = 0, 1, 2, 3$ and in theorem 10 we show that the limits $\lim_{p\to\infty} \rho\left(S^{(i)}_{p,q,r}\right)$, $\lim_{q\to\infty} \rho\left(S^{(i)}_{p,q,r}\right)$ and $\lim_{r\to\infty} \rho\left(S^{(i)}_{p,q,r}\right)$ are Pisot numbers. We also prove that $\lim_{p,q,r\to\infty} \rho\left(S^{(i)}_{p,q,r}\right) = 2$ for all $i = 0, 1, 2, 3$. In [?] Lakatos showed that $\lim_{p_1,\ldots,p_k\to\infty} \rho\left(S^{(0)}_{p_1,\ldots,p_k}\right) = k - 1$. In theorem 11 we generalize that result by showing that $\lim_{p_1,\ldots,p_k\to\infty} \rho\left(S^{(i)}_{p_1,\ldots,p_k}\right) = k - 1$ for all $i \in \{0, 1, \ldots, k\}$.

Remark 5. James McKee and Chris Smyth in [?] gave a characterization of all Salem trees. It follows from their characterization that the cyclotomic cases of $S^{(i)}_{p_1,\ldots,p_k}$ are those for $k = i = 2$ or $k = 3, i = 0, p_1 = p_2 = p_3 = 2$ or $k = 3, i = 0, p_1 = 1, p_2 = p_3 = 2$ or $k = 3, i = 0, p_1 = 1, p_2 = 2, p_3 = 5$ and subgraphs of these. For all the other cases, $S^{(i)}_{p_1,\ldots,p_k}$ are Salem trees.

4.2 Coxeter polynomials of the $S^{(i)}_{p_1,\ldots,p_k}$ graphs

Theorem 9. *For $i \le 2$ the Coxeter polynomial $S^{(i)}_{p,q,r}(x)$, of the Coxeter graph $S^{(i)}_{p,q,r}$ is given by the formula*

$$\frac{x-1}{(x+1)^i} S^{(i)}_{p,q,r}(x) = x^{r+2} F^{(i)}_{p,q}(x) - \left(F^{(i)}_{p,q}\right)^* (x).$$

where the polynomials $F^{(i)}_{p,q}$ are

$$\begin{aligned} F^{(0)}_{p,q}(x) &= x^{p+q} - A_{p-1}(x)A_{q-1}(x), \\ F^{(1)}_{p,q}(x) &= x^{p+q-2}(x-1) - \left(x^{p-2}+1\right) A_{q-1}(x) \text{ and} \\ F^{(2)}_{p,q}(x) &= x^{p+q-4}(x-1)^2 - \left(x^{p-2}+1\right)\left(x^{q-2}+1\right). \end{aligned}$$

The Coxeter polynomial $S^{(3)}_{p,q,r}(x)$ is given by the formula

$$\frac{1}{(x+1)^3} S^{(3)}_{p,q,r}(x) = x^r F^{(3)}_{p,q}(x) + \left(F^{(3)}_{p,q}\right)^* (x)$$

with $F^{(3)}_{p,q}(x) = F^{(2)}_{p,q}(x)$.

Proof. For simplicity of notation, we will write u_j, v_j, w_j instead of $v_{1,j}, v_{2,j}, v_{3,j}$ respectively. Applying proposition 11 to the splitting edge (t, u_1) of the graph $S^{(0)}_{p,q,r}$ we see that

$$S^{(0)}_{p,q,r}(x) = A_p(x)A_{q+r+1}(x) - xA_{p-1}(x)A_q(x)A_r(x).$$

The Coxeter polynomial $A_m(x)$ can be easily calculated using proposition 11 (see also section 3.3). It satisfies the recurrence

$$A_m(x) = A_{m-1}(x) + x\left(A_{m-1}(x) - A_{m-2}(x)\right)$$

and is given by the formula $A_m(x) = x^m + x^{m-1} + \ldots + x + 1$. Therefore

$$\begin{aligned} (x-1)^3 S^{(0)}_{p,q,r}(x) &= x^{p+q+r+4} - 2x^{p+q+r+3} + x^{p+r+2} + x^{q+r+2} - x^{r+2} \\ &\quad + x^{p+q+2} - x^{p+2} - x^{q+2} + 2x - 1 \Rightarrow \\ (x-1)^2 S^{(0)}_{p,q,r}(x) &= x^{p+q+r+2}(x-1) - x^{r+2}(x^q-1)A_{p-1}(x) \\ &\quad + x^2(x^q-1)A_{p-1}(x) - x + 1 \Rightarrow \\ (x-1) S^{(0)}_{p,q,r}(x) &= x^{p+q+r+2} - x^{r+2}A_{p-1}(x)A_{q-1}(x) + x^2 A_{p-1}(x)A_{q-1}(x) - 1 \\ &= x^{r+2}F^{(0)}_{p,q}(x) - \left(F^{(0)}_{p,q}\right)^* (x). \end{aligned}$$

The cases $i = 1, 2$ are proved similarly by applying proposition 11 to the splitting edges (u_{p-2}, u_p) and using the formula for $S^{(i-1)}_{p,q,r}(x)$.
For the Coxeter polynomial $S^{(3)}_{p,q,r}$ we apply proposition 11 to the splitting edge (w_{r-2}, w_r) to obtain

$$S^{(3)}_{p,q,r}(x) = (x+1)S^{(2)}_{p,q,r-1} - x(x+1)S^{(2)}_{p,q,r-3}.$$

Therefore

$$\begin{gathered}\frac{x-1}{(x+1)^3}S^{(3)}_{p,q,r}(x) = \frac{x-1}{(x+1)^2}S^{(2)}_{p,q,r-1} - x\frac{x-1}{(x+1)^2}S^{(2)}_{p,q,r-3} = \\ x^{r+1}F^{(2)}_{p,q}(x) - \left(F^{(2)}_{p,q}\right)^*(x) - x^r F^{(2)}_{p,q}(x) + x\left(F^{(2)}_{p,q}\right)^*(x) \Rightarrow \\ \frac{1}{(x+1)^3}S^{(3)}_{p,q,r}(x) = x^r F^{(2)}_{p,q}(x) + \left(F^{(2)}_{p,q}\right)^*(x)\end{gathered}$$

□

Remark 6. For the case $i = 1$ we could have applied proposition 11 to the splitting edge (u_{p-2}, u_p) to obtain

$$\frac{1}{(x+1)}S^{(1)}_{p,q,r}(x) = x^p F^{(0)}_{q,r}(x) + \left(F^{(0)}_{q,r}\right)^*(x).$$

Similarly by noting that q, r are interchangeable in $S^{(1)}_{p,q,r}$ and p, q are interchangeable in $S^{(2)}_{p,q,r}$, then applying proposition 11 to the splitting edge (v_{q-2}, v_q) we obtain

$$\frac{1}{(x+1)^2}S^{(2)}_{p,q,r}(x) = x^p F^{(1)}_{q,r}(x) + \left(F^{(1)}_{q,r}\right)^*(x).$$

For the next theorem we need two lemmas first. The first lemma is due to Hoffman and Smith (see [**?**]).

Lemma 12 (A. J. Hoffman, J. H. Smith)**.** *If $k, p_1, \ldots, p_k \in \mathbb{N}$, $0 \le i \le k$ and $p'_j > p_j$ for some $1 \le j \le k$ then*

1. $\rho\left(S^{(i)}_{p_1,\ldots,p_j,\ldots,p_k}\right) \le \rho\left(S^{(i)}_{p_1,\ldots,p'_j,\ldots,p_k}\right)$ *if $j > i$ and*
2. $\rho\left(S^{(i)}_{p_1,\ldots,p_j,\ldots,p_k}\right) \ge \rho\left(S^{(i)}_{p_1,\ldots,p'_j,\ldots,p_k}\right)$ *if $j \le i$.*

Equality can happen if and only if the graph $S^{(i)}_{p_1,\ldots,p'_j,\ldots,p_k}$ is cyclotomic.

We also need the following lemma.

Lemma 13. *Suppose that $f_n(x) = x^n g(x) + h(x)$ is a sequence of functions such that g, h are continuous, for all $n \in \mathbb{N}$ $f_n(z_n) = 0$ and that $\lim_{n\to\infty} z_n = z_0$. If $|z_0| > 1$ then $g(z_0) = 0$ while if $|z_0| < 1$ then $h(z_0) = 0$.*

Proof. Suppose that $|z_0| > 1$. Since h is continuous and $|g(z_n)| = \frac{|h(z_n)|}{|z_n^n|}$ it follows that $\lim_{n\to\infty} |g(z_n)| = 0$. Using $|g(z_0)| - |g(z_n)| \leq |g(z_0) - g(z_n)| \xrightarrow[n\to\infty]{} 0$ we conclude that $g(z_0) = 0$. The proof for the case $|z_0| < 1$ is similar. $\square$

Theorem 10. *The spectral radius, $\rho\left(S^{(i)}_{p,q,r}\right)$ of the Coxeter transformation of $S^{(i)}_{p,q,r}$ satisfies*

i) $\lim\limits_{r\to\infty} \rho\left(S^{(i)}_{p,q,r}\right) = \rho\left(F^{(i)}_{p,q}\right)$ *and* $\rho\left(F^{(i)}_{p,q}\right)$ *is a Pisot number for* $i = 0, 1, 2$,

ii) $\lim\limits_{p\to\infty} \rho\left(S^{(i)}_{p,q,r}\right) = \rho\left(F^{(i-1)}_{q,r}\right)$ *for* $i = 1, 2, 3$,

iii) $\lim\limits_{p,q\to\infty} \rho\left(S^{(i)}_{p,q,r}\right) = \rho\left(x^{r+2} - 2x^{r+1} + 1\right)$ *for* $i = 0, 1, 2$,

iv) $\lim\limits_{q,r\to\infty} \rho\left(S^{(i)}_{p,q,r}\right) = \rho\left(x^{p} - 2x^{p-1} - 1\right)$ *for* $i = 1, 2, 3$ *and*

v) $\lim\limits_{p,q,r\to\infty} \rho\left(S^{(i)}_{p,q,r}\right) = 2$ *for all* $i = 0, 1, 2, 3$.

Proof. From theorem 9 and lemma 13 we conclude that in order to prove (i) is enough to show that the sequence $\left(\rho(S^{(i)}_{p,q,r})\right)_{r\in\mathbb{N}}$ is convergent. It follows from lemma 12 that for $i = 0, 1, 2$ the sequence $\left(\rho(S^{(i)}_{p,q,r})\right)_{r\in\mathbb{N}}$ is increasing. Since the polynomials $S^{(i)}_{p,q,r}(x)$ are of the form $S^{(i)}_{p,q,r}(x) = x^{r+2}F(x) + G(x)$ where $F(x), G(x)$ are monic polynomials, the sequence $\left(\rho(S^{(i)}_{p,q,r})\right)_{r\in\mathbb{N}}$ is also bounded. For, if M is large enough such that $F(x), G(x) > 0$ for all $x \geq M$, then $z < M$ for all $z \in Z\left(S^{(i)}_{p,q,r}\right)$. We now prove that $\rho\left(F^{(i)}_{p,q}\right)$ is a Pisot number (cf. lemma 4.3 in [?]). Let $\epsilon > 0$ be small enough and r be large enough such that $\rho\left(S^{(i)}_{p,q,r}\right) > 1 + \epsilon$ and $\left|x^{r+2}F^{(i)}_{p,q}(x)\right| > \left|\left(F^{(i)}_{p,q}\right)^*(x)\right|$ for every $|x| = 1 + \epsilon$. From Rouche's theorem it follows that the polynomial $F^{(i)}_{q,r}(x)$ has only one root, let's say z_0, outside the unit circle. If z_0 was a Salem number then we would have $F^*(z_0) = 0$ and therefore $S^{(i)}_{p,q,r}(z_0) = 0$ for all large r, contrary to lemma 12. Therefore $z_0 = \rho\left(F^{(i)}_{p,q}(x)\right)$ and it is a Pisot number.

The proof of (ii) is similar to that of (i) and it follows from lemma 13 using the alternative form of $S^{(i)}_{p,q,r}, i = 1, 2, 3$ given in remark 6. If we set $\ell_{q,r} = \lim_{p\to\infty} S^{(0)}_{p,q,r}$ then it follows from lemma 12 that $\ell_{q,r}$ is increasing on q. Since $\ell_{q,r} = \rho\left(F^{(0)}_{q,r}\right) > 1$ and $(x-1)^2F^{(0)}_{q,r} = x^q\left(x^{r+2} - 2x^{r+1} + 1\right) + x^r - 1$ we deduce from lemma 13 that $\lim_{p,q\to\infty} \rho\left(S^{(0)}_{p,q,r}(x)\right) = \rho\left(x^{r+2} - 2x^{r+1} + 1\right)$. The other cases of (iii) and (iv) are done similarly.

It remains to prove (v). Let $\ell_p = \lim_{q,r\to\infty} \rho\left(S^{(0)}_{p,q,r}(x)\right)$. The polynomial $F(x) = x^{p+2} - 2x^{p+1} + 1$ is decreasing in $\left(1, \frac{2p+2}{p+2}\right)$, increasing in $\left(\frac{2p+2}{p+2}, 2\right)$, $F(1) = 0$ and $F(2) = 1$. Therefore the only root of H outside the unit circle is ℓ_p and satisfies $\frac{2p+2}{p+2} < \ell_p < 2$. We conclude that $\lim_{p,q,r\to\infty} \rho\left(S^{(0)}_{p,q,r}(x)\right) = 2$. The cases $i = 1, 2, 3$ are similar. $\square$

Theorem 11. *For $k, p_1, \ldots, p_k \in \mathbb{N}$ and all $i \in \{0, 1, \ldots, k\}$*

$$\lim_{p_1,\ldots,p_k\to\infty} \rho\left(S^{(i)}_{p_1,\ldots,p_k}\right) = k-1$$

Proof. The proposition 11 applied to the splitting edge $(t, v_{k,1})$ yields $(x-1)S^{(i)}_{p_1,\ldots,p_k}(x) = x^{p_k}G(x) - G^*(x)$ where

$$G(x) = S^{(i)}_{p_1,\cdots,p_{k-1}}(x) - D_{p_1}(x)\ldots D_{p_i}(x)A_{p_{i+1}}(x)A_{p_{k-1}}(x),$$

for any $i \in \{0, 1, \ldots, k\}$. Therefore $\lim_{p_k\to\infty} \rho\left(S^{(i)}_{p_1,\ldots,p_k}(x)\right) = \rho(G(x))$. Inductively we show that $\lim_{p_2,\ldots,p_k\to\infty} \rho\left(S^{(i)}_{p_1,\ldots,p_k}(x)\right) = \rho(H(x))$ where the polynomial $H(x)$ is given by

$$H(x) = \begin{cases} x^{p_1} - (k-1)x^{p_1-1} - k + 2, & \text{if} i \neq 0, \\ x^{p_1+1} - (k-1)x^{p_1} + k - 2, & \text{if} i = 0. \end{cases}$$

Hence $\lim_{p_1,p_2,\ldots,p_k\to\infty} \rho\left(S^{(i)}_{p_1,\ldots,p_k}(x)\right) = k-1$. □

Example 15. For the case of the Dynkin diagrams D_n, theorem 9 gives

$$D_n(x) = S^{(0)}_{1,1,n-3}(x) = \frac{1}{x-1}\left(x^{n-1}(x^2-1) + x^2 - 1\right) = x^n + x^{n+1} + x + 1.$$

For the affine Dynkin diagrams $D^{(1)}_n$, theorem 9 gives

$$D^{(1)}_n(x) = S^{(1)}_{n-2,1,1}(x) = \frac{x+1}{x-1}\left(x^3(x^{n-2} - x^{n-3} - x^{n-4} - 1) + x^{n-2} + x^2 + x - 1\right) = \left(x^{n-2}-1\right)(x-1)(x+1)^2.$$

For the E_n diagrams it gives

$$E_n(x) = S^{(0)}_{1,2,n-4}(x) = \frac{1}{x-1}\left(x^{n-2}(x^3 - x - 1) + x^3 + x^2 - 1\right).$$

All these agree with the known formulas of Coxeter polynomials of the Dynkin and E_n diagrams (see section 3.4 and also [**?**] and [**?**]).

We also prove the following theorem concerning joins of Coxeter graphs.

Theorem 12. *Let Γ be the join of the simple graphs Γ_i, $i = 1, 2, \ldots, n$. Suppose that z is a zero of $\Gamma_i(x)$ with multiplicity m_i. Then z is a zero of the Coxeter polynomial $\Gamma(x)$ with multiplicity at least*

$$\min\{m - m_i : i = 1, 2, \ldots, n\}$$

where $m = m_1 + m_2 + \ldots + m_n$.

Proof. Let us denote by $\Gamma_{(k)}$ the join of the graphs Γ_i at the vertices $v_i \in \mathcal{V}(\Gamma_i),\ i = 1, 2, \ldots, k$. The graph $\Gamma_{(n)}$ looks like the one in fig. 4.2.

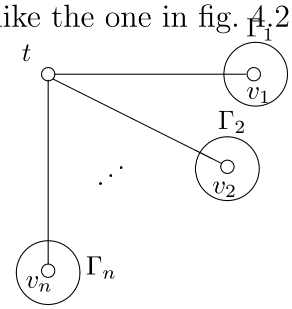


Figure 4.2: The join of the graphs Γ_i

Applying proposition 11 to the splitting edge (t, v_n) we obtain

$$\Gamma_{(n)}(x) = \Gamma_{(n-1)}(x)\Gamma_n(x) - x\tilde{\Gamma}_n(x)\Gamma_1(x)\Gamma_2(x)\ldots\Gamma_{n-1}(x),$$

where $\tilde{\Gamma}_n$ is the subgraph of Γ_n on the vertices $\mathcal{V}(\Gamma_n) \setminus \{v_n\}$. Since the polynomial $\Gamma_{(2)}$ satisfies the relation

$$\Gamma_{(2)}(x) = (x-z)^{m_1+m_2} f(x) - (x-z)^{m_1} g(x) - (x-z)^{m_2} h(x)$$

for some polynomials f, g, h we can now proceed by induction and see that the theorem is true. $\square$

Theorem 12 generalize a theorem of V.F.Kolmykov (see [**?**]) which says that if Γ is the join of the Coxeter graphs Γ_1, Γ_2 and z is a root of the Coxeter polynomials $\Gamma_1(x)$ and $\Gamma_2(x)$ then $\Gamma(z) = 0$. According to [**?**] if $z \neq \pm 1$ then $m_i \in \{0, 1\}$ and therefore in that case, theorem 12 can be found in [**?**] where the authors have proved that z is a zero of $\Gamma(x)$ with multiplicity at least $m-1$. For $z = \pm 1$ however z can be a zero of $\Gamma(x)$ with multiplicity less than $m-1$. For example consider the join Γ of the affine Dynkin diagrams $D_4^{(1)}$ as shown in fig. 4.3. The polynomials $\Gamma(x)$ and $D_4^{(1)}(x)$ both have 1 as a zero with multiplicity 2.

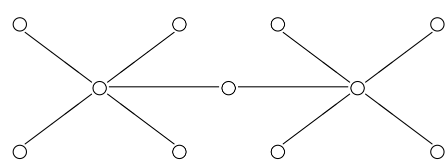


Figure 4.3: The join of two $D_4^{(1)}$

Chapter 5

Hamiltonian systems

> Integrable systems, what are they? It's not easy to answer precisely. The question can occupy a whole book, or be dismissed as Louis Armstrong is reputed to have done once when asked what jazz was—'If you gotta ask, you'll never know!'—Nigel Hitchin

5.1 Hamiltonian systems

One of the most interesting type of nonlinear systems of differential equations are the Hamiltonian systems. They arise in many physical problems. The Hamiltonian equations for such a system are obtained by a single function, the Hamiltonian.

Definition 24. Let $H \in C^{\infty}(\mathbb{R}^{2n})$ where $H = H(q_1, q_2, \ldots, q_n, p_1, \ldots, p_n) = H(q, p)$. A dynamical system of the form

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \ i = 1, 2, \ldots, n, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \ i = 1, 2, \ldots, n \end{aligned} \tag{5.1}$$

is called Hamiltonian system. Equations (5.1) are known as Hamilton's equations. The coordinates q, p denote respectively, the positions and the momenta and the Hamiltonian as the total energy of the system.

A more compact form of Hamilton's equations is

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial q}, \end{aligned}$$

where $\dot{q} = (\dot{q_1}, \ldots, \dot{q_n}), \dot{p} = (\dot{p_1}, \ldots, \dot{p_n})$ and $\frac{\partial H}{\partial q} = \left(\frac{\partial H}{\partial q_1} \ldots \frac{\partial H}{\partial q_n}\right), \frac{\partial H}{\partial p} = \left(\frac{\partial H}{\partial p_1} \ldots \frac{\partial H}{\partial p_n}\right)$. Introducing $x = (q, p)$ and the symplectic matrix J defined by

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix, then Hamilton's equations (5.1) can be written as

$$\dot{x} = J \nabla H,$$

where ∇H is the column vector $\left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \ldots, \frac{\partial H}{\partial x_{2n}}\right)^T$.

Example 16. Consider the Hamiltonian

$$H = \sum_{i=1}^{n} e^{p_i + \frac{1}{2}(q_{i+1} - q_{i-1})} \in C^{\infty}\left(\mathbb{R}^{2n}\right),$$

where $q_0 = q_{n+1} = 0$. Then H gives rise to the system

$$\begin{aligned} \dot{q_i} &= e^{p_i + \frac{1}{2}(q_{i+1} - q_{i-1})}, \ i = 1, 2, \ldots, n \\ \dot{p_1} &= \frac{1}{2} e^{p_2 + \frac{1}{2}(q_3 - q_1)}, \\ \dot{p_i} &= \frac{1}{2}\left(e^{p_{i+1} + \frac{1}{2}(q_{i+2} - q_i)} - e^{p_{i-1} + \frac{1}{2}(q_i - q_{i-2})}\right), i = 2, \ldots, n-1, \\ \dot{p_n} &= -\frac{1}{2} e^{p_{n-1} + \frac{1}{2}(q_n - q_{n-2})}, \end{aligned}$$

The transformation $u_i = e^{p_i + \frac{1}{2}(q_{i+1} - q_{i-1})}, \ i = 1, 2, \ldots, n$ transforms the system to the system

$$\begin{aligned} \dot{u_1} &= u_1 u_2, \\ \dot{u_i} &= u_i \left(u_{i+1} - u_{i-1}\right), \ i = 2, 3, \ldots, n-1, \\ \dot{u_n} &= -u_n u_{n-1}. \end{aligned}$$

This system is the well known KM system. It was studied first by Volterra in [?] and it was first solved by Kac and van-Moerbecke in [?].

Lemma 14. *For a Hamiltonian system (5.1) the total energy (i.e. its Hamiltonian) remains constant along the solutions of (5.1).*

Proof. If $x = (q, p)$ is a solution of (5.1) then

$$\dot{H} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \dot{q_i} + \frac{\partial H}{\partial p_i} \dot{p_i}\right) = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i}\right) = 0,$$

and therefore H is constant. $\square$

Example 17. The Newtonian system

$$\ddot{x} = f(x)$$

is a Hamiltonian system. It can be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x). \end{aligned}$$

A special type of Hamiltonian that often occurs in physical systems is

$$H = \frac{1}{2}y^2 + U(x),$$

where $\frac{1}{2}y^2$ is the kinetic energy and $U(x) = -\int_{x_0}^{x} f(t)dt$ is the potential energy.

5.2 Poisson brackets

The symplectic matrix J can be used to define an operation, called the canonical Poisson bracket, between two smooth functions on $\mathbb{R}^{2n}$. Let f, g be two smooth functions on $\mathbb{R}^{2n}$, on the variables q, p. The (canonical) Poisson bracket of f and g is defined by

$$\{f, g\} = \nabla f^T J \nabla g = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right).$$

With respect to the Poisson bracket, Hamilton's equations can be written as

$$\dot{x_i} = \{x_i, H\}, \ i = 1, 2, \ldots, 2n.$$

In general for any smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^{2n})$, if x is a solution of (5.1) then

$$\dot{f} = \nabla f^T J \nabla H = \{f, H\}.$$

The canonical Poisson bracket satisfies the following properties.

1. (skew-symmetry) $\{f, g\} = -\{g, f\}$,
2. (Bilinearity) $\{\lambda f + g, h\} = \lambda \{f, h\} + \{g, h\}$,
3. (Jacobi identity) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$,

4. (Leibniz rule) $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

A smooth function f is called a first integral (or a constant of motion) of the system (5.1), if $\dot{f}(x) = 0$ for any solution x of (5.1). Equivalently, if $\{f, H\} = 0$.

For example, due to property 1 of Poisson brackets we derive lemma 14, i.e. that the Hamiltonian of the system (5.1) is constant of motion (in Hamiltonian systems the total energy is conserved).

Example 18. (The Holt Hamiltonian, (see [?, ?]))

A straightforward computation shows that the Hamiltonian system defined by the Hamiltonian

$$H_1 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + e^{\left(q_2 - \sqrt{3}q_1\right)} + e^{\left(q_2 + \sqrt{3}q_1\right)} + e^{-2q_2},$$

has the function

$$I_1 = p_1^3 - 3p_1p_2^2 + 3\left(e^{\left(q_2-\sqrt{3}q_1\right)} + e^{\left(q_2+\sqrt{3}q_1\right)} - 2e^{-2q_2}\right)p_1 - 3\sqrt{3}\left(e^{\left(q_2+\sqrt{3}q_1\right)} - e^{\left(q_2-\sqrt{3}q_1\right)}\right)p_2$$

as a constant of motion.

The Hamiltonian system defined by the Hamiltonian function

$$H_2 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{3}{4}q_1^{\frac{4}{3}} + \left(q_2^2 + \lambda q_1\right)^{-\frac{2}{3}},$$

where λ is a constant, has the following function as a constant of motion

$$I_2 = p_2^3 + \frac{3}{2}p_2p_1^2 + \left(-\frac{9}{2}q_1^{\frac{4}{3}} + 3q_2^2q_1^{-\frac{2}{3}} + 3\lambda q_1^{-\frac{2}{3}}\right)p_2 + 9p_1q_2q_1^{\frac{1}{3}}.$$

Example 19 (The Toda lattice). The Hamiltonian system defined by the Hamiltonian function

$$H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) = \sum_{i=1}^{n}\frac{1}{2}p_1^2 + \sum_{i=1}^{n-1}e^{q_i - q_{i+1}},$$

is the well known classical, non-periodic Toda lattice. It was first studied by Morikazu Toda in [?]. Hamilton's equations become

$$\dot{q_i} = p_i, \; i = 1, 2, \ldots, n,$$
$$\dot{p_i} = e^{q_{i-1}-q_i} - e^{q_i - q_{i+1}}, \; i = 1, 2, \ldots, n,$$

where we set $q_0 = q_{n+1} = 0$. Flaschka's transformation (see [?, ?, ?])

$$a_i = \frac{1}{2}e^{\frac{1}{2}(q_i - q_{i+1})}, b_i = -\frac{1}{2}p_i,$$

transforms the system to

$$\dot{a_i} = a_i(b_{i+1} - b_i), \ i = 1, 2, \ldots, n-1$$
$$\dot{b_i} = 2(a_i^2 - a_{i-1}^2), \ i = 1, 2, \ldots, n.$$

Flaschka's transformation can be used to find plenty constants of motion for this system. In fact, if L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & & \cdots & a_{n-1} & b_n \end{pmatrix}$$

then it can be proved that the functions

$$H_i = \frac{1}{i} \operatorname{tr}\left(L^i\right),$$

are all constants of motion for the Toda lattice (see proposition 26). Note that

$$H_1 = \sum_{i=1}^{n} b_i = -\frac{1}{2}(p_1 + p_2 + \ldots + p_n)$$

corresponds to the total momentum and

$$H_2 = \frac{1}{2} \sum_{i=1}^{n} b_i^2 + \sum_{i=1}^{n} a_i^2$$

is the Hamiltonian of the system.

Poisson who introduced, together with his teacher Joseph-Louis Lagrange, the Poisson bracket, proved the following proposition.

Proposition 24 (S. D. Poisson). *If I, J are first integrals of a Hamiltonian system, then the function $\{I, J\}$ is also a first integral of the system.*

Jacobi gave a simple proof of Poisson's proposition, which rely on the Jacobi identity of the Poisson bracket.

Proof. (Jacobi) If I, J are first integrals of the Hamiltonian system with Hamiltonian function H, then $\{I, H\} = \{J, H\} = 0$. From the Jacobi identity of the Poisson bracket it follows that

$$\{\{I,J\},H\} = \{I,\{J,H\}\} + \{\{I,H\},J\} = 0$$

and therefore $\{I,J\}$ is indeed a first integral. □

Definition 25. A first integral I is polynomial if $I \in \mathbb{R}[x]$ while it is rational if $I \in \mathbb{R}(x)$.

There is a more general definition of Poisson brackets, than the canonical Poisson bracket, defined on associative algebras; the canonical Poisson bracket is defined on the algebra of smooth function on $\mathbb{R}^{2n}$.

Definition 26.
- If $\mathcal{A}$ is a commutative associative algebra with unity (with respect to an operation "$\cdot$"), then a Poisson bracket, $\{,\}$, is a Lie bracket which further, it satisfies the Leibniz rule

$$\{fg,h\} = f\{g,h\} + g\{f,h\}, \ \forall f,g,h \in \mathcal{A},$$

  i.e. the linear map $\mathrm{ad}_f(g) = \{f,g\}$ is a derivation with respect to the operation "$\cdot$".
- A smooth Poisson structure (or simply a Poisson structure) on a smooth finite dimensional manifold $\mathcal{M}$ is a Poisson bracket defined on the algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$.

Hamiltonian systems can be defined in a more general framework, in the sense that the corresponding Poisson structure is not canonical.

Let $\{\,,\}$, be a Poisson bracket on the manifold M, $(x_1, x_2, \ldots, x_n)$ local coordinates on M and $H \in \mathcal{C}^\infty(M)$. A dynamical system of the form

$$\dot{x_i} = \{x_i, H\}, \ i = 1,2,\ldots,n, \tag{5.2}$$

is called a Hamiltonian system.

For simplicity we give the definitions in the case $\mathcal{M} = \mathbb{R}^n$ with coordinates $(x_1,\ldots,x_n)$.

The Poisson structure can be encoded in a matrix known as the Poisson matrix of the Poisson structure. This matrix is defined by the functions $\{x_i, x_j\}$; i.e. is the $n \times n$ matrix $\pi = (x_{i,j})_{1\le i,j\le n}$ with $x_{i,j} = \{x_i, x_j\}$.

Example 20. The Poisson matrix of the canonical Poisson structure is the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The Poisson bracket of two functions f, g on $\mathbb{R}^n$ can be written in terms of the Poisson matrix π, as

$$\{f,g\} = \nabla f^T \pi \nabla g.$$

In terms of the Poisson matrix, the Hamiltonian system 5.2, can be written as

$$\dot{x} = \pi \nabla H.$$

Note that due to the skew-symmetry of the Poisson bracket it follows that the Poisson matrix is skew-symmetric. However, given a skew-symmetric matrix π, then, the bracket $\{f, g\} = \nabla f^T \pi \nabla g$ satisfies all but the Jacobi identity, the properties of the Poisson bracket. The Leibniz rule rests upon the Leibniz rule for the derivation of the product of two functions. From the Leibniz rule and the fact that the second order partial derivatives of a smooth function commute, we obtain the following formula.
If f, g, h are smooth functions on $\mathbb{R}^n$, then

$$\begin{gathered}
\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = \\
\sum_{1 \le i,j,k,l \le n} x_{l,k} \frac{\partial x_{i,j}}{\partial x_l} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_k} + \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k} \frac{\partial h}{\partial x_i} + \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial x_i} \frac{\partial h}{\partial x_j} \right) = \\
\sum_{1 \le i,j,k \le n} \sum_{l=1}^{n} \left(x_{l,k} \frac{\partial x_{i,j}}{\partial x_l} + x_{l,i} \frac{\partial x_{j,k}}{\partial x_l} + x_{l,j} \frac{\partial x_{k,i}}{\partial x_l} \right) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_k}.
\end{gathered}$$

Therefore we have the following proposition

Proposition 25. *If $\pi = (x_{i,j})$ is a skew-symmetric $n \times n$ matrix of smooth functions on $\mathbb{R}^n = (x_1, x_2, \ldots, x_n)$, then the bracket, $\{,\}$, defined by the formula*

$$\{f, g\} = \nabla f^T \pi \nabla g, \; f, g \in \mathbb{R}^n$$

is a Poisson bracket if and only if for all $1 \le i < j < k \le n$,

$$\{\{x_i, x_j\}, x_k\} + \{\{x_j, x_k\}, x_i\} + \{\{x_k, x_i\}, x_j\} = 0.$$

Definition 27. The rank of the Poisson matrix of the Poisson structure, $\{,\}$, on $\mathbb{R}^n$ at the point $x_0 \in \mathbb{R}^n$ is the rank of the Poisson matrix evaluated at the point x_0, and is denoted $\mathrm{Rank}_{x_0}(\pi)$. The maximum $\max_{x_0 \in \mathbb{R}^n} \mathrm{Rank}_{x_0}(\pi)$ is the rank of the Poisson matrix and is denoted $\mathrm{Rank}\,(\pi)$.

Remark 7. Note that since the Poisson matrix is skew-symmetric, its rank is always even.

Definition 28. A Casimir function is a smooth function $C \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\{C, f\} = 0, \; \forall f \in C^\infty(\mathbb{R}^n).$$

If the rank of the Poisson structure is k, then there are $n - k$ Casimirs.

Example 21. For the Toda lattice (19) the function $H_1 = b_1 + b_2 + \ldots + b_n$ is a Casimir.

5.3 Liouville integrability

Definition 29. The set of elementary functions $f(x)$ in $\mathbb{R}^n$ is the set $\mathcal{E} \supseteq \mathbb{R}(x)$ with the property

1. If $f_1, f_2 \in \mathcal{E}$ then $f_1 + f_2, f_1 - f_2, f_1 \cdot f_2, \frac{f_1}{f_2} \in \mathcal{E}$.

2. If $f_0, f_1, \ldots, f_n \in \mathcal{E},\ f$ a smooth function in $\mathbb{R}^n$ and $f_0 + f_1 f + f_2 f^2 + \ldots + f_k f^k$ then $f \in \mathcal{E}$.

3. If $f \in \mathcal{E}$ then $\frac{\partial f}{\partial x_i} \in \mathcal{E}$.

4. If $f \in E$ then e^f and $\log f \in \mathcal{E}$.

For example the function $F(x) = x\sqrt[3]{e^{x^2+y}} - e^{x+y^2\log^2(\sqrt{x}+2y)}$ is elementary. The function $\int_0^x te^{t^2}dt$ is elementary. There are functions which are not elementary, but to prove that a function is not elementary is far from being obvious. For example it was proved by Liouville himself (see [?]) that the function $\int_0^x e^{-t^2}dt$ is not elementary.

Definition 30.

- The set Λ containing the elementary functions and also the functions satisfying the property

  5. If $f \in \Lambda$ then $\int_0^x f dx_i \in \Lambda$

  is called the set of Liouvillian functions. If $f \in \Lambda$ then we say that f can be represented by quadratures.

- A set of functions $\{f_1, f_2, \ldots, f_n\}$ is said to be involutive (or that the functions $f_1, f_2, \ldots, f_n$ are in involution) if $\{f_i, f_j\} = 0$ for all $i, j = 1, 2, \ldots, n$.

- The Hamiltonian system (5.1) is said to be (Liouville) integrable if it admits n independent first integrals in involution.

- In general the Hamiltonian system (5.2) on $\mathbb{R}^n$ defined by a Poisson bracket of rank $2r$ is said to be (Liouville) integrable if it admits $n - r$ independent first integrals in involution.

Remark 8. The terminology integrable system is in the sense that if a Hamiltonian system is integrable then it can be solved by quadratures; the solutions of the system are Liouvillian functions (see [?]).

5.4 Lax pairs

Peter Lax in [?] noted that for $L = -6\frac{d^2}{dx^2} - u$ and $B = -4\frac{d^3}{dx^3} - u\frac{d}{dx} - \frac{1}{2}u_x$, the equation

$$\frac{dL}{dt} = [B, L]\,,$$

is equivalent to the KdV equation

$$u_t + uu_x + u_{xxx} = 0.$$

Definition 31.
- A Lax pair is a pair of matrices $L, B \in M_k(C^\infty(\mathbb{R}^n))$ such that $\dot{L} = [B, L]$.
- A Lax pair L, B for which the equation $\dot{L} = [B, L]$ is equivalent to the Hamiltonian system (5.2) is called a Lax pair for (5.2).

The definition will be better understood with the following examples

Example 22. The system

$$\begin{aligned} \dot{x_1} &= x_1(x_2 - x_3),\\ \dot{x_2} &= x_2(x_3 - x_1),\\ \dot{x_3} &= x_3(x_1 - x_2) \end{aligned}$$

is equivalent to the Lax pair L, B with

$$L = \begin{pmatrix} 0 & 1 & x_1 \\ x_2 & 0 & 1 \\ 1 & x_3 & 0 \end{pmatrix}, \; B = \begin{pmatrix} x_1 + x_2 & 0 & 1 \\ 1 & x_2 + x_3 & 0 \\ 0 & 1 & x_1 + x_3 \end{pmatrix}.$$

It is straightforward to verify that

$$L = \begin{pmatrix} 0 & 0 & \dot{x_1} \\ \dot{x_2} & 0 & 0 \\ 0 & \dot{x_3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x_1(x_2 - x_3) \\ x_2(x_3 - x_1) & 0 & 0 \\ 0 & x_3(x_1 - x_2) & 0 \end{pmatrix} = [L, B]\,.$$

In the next example we see that a system can admit several Lax pairs.

Example 23. The KM system is defined by the equations

$$\begin{aligned} \dot{x_1} &= x_1 x_2,\\ \dot{x_i} &= x_i(x_{i+1} - x_{i-1}),\; i - 2, 3, \dots, n-1,\\ \dot{x_n} &= -x_n x_{n-1}. \end{aligned}$$

It admits several Lax pairs. The pair L, B with

$$L = \begin{pmatrix} x_1 & 0 & \sqrt{x_1x_2} & 0 & \cdots & & 0 \\ 0 & x_1+x_2 & 0 & \sqrt{x_2x_3} & & & \vdots \\ \sqrt{x_1x_2} & 0 & x_2+x_3 & & \ddots & & \\ 0 & \sqrt{x_2x_3} & & & & & \\ \vdots & & \ddots & & & & \sqrt{x_{n-1}x_n} \\ & & & & & x_{n-1}+x_n & 0 \\ 0 & & \cdots & & \sqrt{x_{n-1}x_n} & 0 & x_n \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{x_1x_2} & 0 & \cdots & & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{x_2x_3} & & & \vdots \\ -\frac{1}{2}\sqrt{x_1x_2} & 0 & 0 & & \ddots & & \\ 0 & -\frac{1}{2}\sqrt{x_2x_3} & & & & & \\ \vdots & & \ddots & & & & \frac{1}{2}\sqrt{x_{n-1}x_n} \\ & & & & & 0 & 0 \\ 0 & & \cdots & & -\frac{1}{2}\sqrt{x_{n-1}x_n} & 0 & 0 \end{pmatrix}$$

is a Lax pair of the KM system. So is the Lax pair

$$L = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ x_1 & 0 & 1 & \ddots & & \vdots \\ 0 & x_2 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & x_n & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & & \vdots \\ x_1x_2 & 0 & 0 & \ddots & & \vdots \\ \vdots & x_2x_3 & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & x_{n-1}x_n & 0 & 0 \end{pmatrix}.$$

Finally, there is a symmetric version due to Moser (see [**?**]) where

$$L = \begin{pmatrix} 0 & \sqrt{x_1} & 0 & \cdots & \cdots & 0 \\ \sqrt{x_1} & 0 & \sqrt{x_2} & \ddots & & \vdots \\ 0 & \sqrt{x_2} & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & \sqrt{x_n} \\ 0 & \cdots & \cdots & 0 & \sqrt{x_n} & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & \sqrt{x_1x_2} & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ -\sqrt{x_1x_2} & 0 & 0 & \ddots & \sqrt{x_2x_3} & \vdots \\ \vdots & -\sqrt{x_2x_3} & \ddots & \ddots & & \sqrt{x_{n-1}x_n} \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & -\sqrt{x_{n-1}x_n} & 0 & 0 \end{pmatrix}.$$

Example 24. The Lax pair

$$L = \begin{pmatrix} 0 & \sqrt{x_1} & 0 & \cdots & \cdots & -\sqrt{x_{n+1}} \\ \sqrt{x_1} & 0 & \sqrt{x_2} & \ddots & & \vdots \\ 0 & \sqrt{x_2} & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & \sqrt{x_n} \\ -\sqrt{x_{n+1}} & \cdots & \cdots & 0 & \sqrt{x_n} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \sqrt{x_1x_2} & \cdots & \sqrt{x_nx_{n+1}} & 0 \\ 0 & 0 & 0 & \ddots & & \sqrt{x_1x_{n+1}} \\ -\sqrt{x_1x_2} & 0 & 0 & \ddots & \sqrt{x_2x_3} & \vdots \\ \vdots & -\sqrt{x_2x_3} & \ddots & \ddots & & \sqrt{x_{n-1}x_n} \\ -\sqrt{x_nx_{n+1}} & & & \ddots & \ddots & 0 \\ 0 & -\sqrt{x_1x_{n+1}} & \cdots & -\sqrt{x_{n-1}x_n} & 0 & 0 \end{pmatrix}.$$

defines the periodic KM system,

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}),\ i - 2, 3, \ldots, n-1,$$

where $x_0 = x_n$.

Example 25. The Lax pair (see [**?**])

$$L = \begin{pmatrix} b_1 & 1 & 0 & \cdots & \cdots & a_n \\ a_1 & b_2 & 1 & \ddots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ -1 & \cdots & \cdots & 0 & a_{n-1} & b_n \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & a_n \\ a_1 & 0 & 0 & \ddots & & \vdots \\ 0 & a_2 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & a_{n-1} & 0 \end{pmatrix}.$$

gives rise to the periodic Toda lattice which is defined by the equations

$$\dot{b_i} = a_i - a_{i-1}, i = 1, 2, \ldots, n,$$
$$\dot{a_i} = a_i(b_{i+1} - b_i),\ i = 1, 2, 3, \ldots, n-1,$$

where $a_0 = a_n$.

Lax pairs are useful for finding first integrals for the system of differential equations they define. The next proposition shows that for the system defined by the Lax pair L, B, the traces of L^k are all constants of motion.

Proposition 26. *If L, B is a Lax pair for the system (5.1) then the functions $\operatorname{tr}(L^k)$ are constants of motion for the system (5.1).*

Proof. We compute the derivative of the functions $\operatorname{tr}(L^k)$.

$$\frac{1}{k}\operatorname{tr}\left(L^k\right) = \operatorname{tr}(L^{k-1}\dot{L}) = \operatorname{tr}(L^kB - L^{k-1}BL) =$$
$$\operatorname{tr}(L^kB) - \operatorname{tr}(L^KB) = 0.$$

We have used the property of the trace

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Therefore $\operatorname{tr} L^k$ is indeed a constant of motion. □

We immediately get the following result.

Corollary 3. *If L, B is a Lax pair and*

$$\chi_L(\lambda) = \lambda^n + f_{n-1}\lambda^{n-1} + \ldots + f_1\lambda + f_0$$

is the characteristic polynomial of L, then the functions f_i are constants of motion.

Example 26. For the case of the KM system it can be proved that the traces of L give enough independed constants of motion, in involution, so that the KM system is integrable.

5.5 Lotka-Volterra systems

The Lotka-Volterra equations were discovered independently by Alfred Lotka and Vito Volterra around 1925. Volterra was trying to make sense of the fact that the predator fish increased in numbers after WWI. This question was posed to him by his son-in-law Umberto D'Ancona a marine biologist who collected data of fish catches in the Adriatic for the years during and after the war. Volterra proposed the following simple system to model the interaction between predator and prey fish.

$$\begin{aligned} \dot{x} &= x(a - by) \\ \dot{y} &= y(-c + dx) \end{aligned}$$

where a, b, c, $d > 0$. This system and its generalizations to n dimensions is one of the most basic models in population dynamics. The variable x denotes the density of prey fish while y is the density of predator fish. Note that if there are no predators ($y = 0$) then x grows at a constant rate $\dot{x} = ax$, the so called Malthusian law of population. Volterra made the assumption that the interaction between predator and prey fish depends on both x and y, hence the Malthusian law is modified by subtracting a term bxy. Note that he did not take into account a possible death of prey fish due to other causes. Similarly, the density of the predator fish increases at a rate proportional to both x and y, i.e. a factor dxy. Assuming that they die at the rate $\dot{y} = -cy$ we get the second equation. The same model was also derived by Lotka [?] in the context of chemical reaction theory.

Note that the vector field vanishes at the origin $(0, 0)$ and at the point $(\frac{c}{d}, \frac{a}{b})$. The origin is saddle point while the second point is a center, i.e. it corresponds to a periodic solution. It is not difficult to produce a constant of motion. We multiply the first equation by $\frac{c-dx}{x}$ and the second by $\frac{a-by}{y}$ and then we add the result. We obtain

$$\frac{\dot{x}}{x}(c - dx) + \frac{\dot{y}}{y}(a - by) = 0 \ .$$

This equation is equivalent to

$$\frac{d}{dt}\left(c \ln x - dx + a \ln y - by\right) = 0 \ .$$

Therefore the function

$$H(x, y) = c \ln x + a \ln y - dx - by$$

is a constant of motion. The function H is actually a Hamiltonian. By defining the Poisson bracket on $\mathbb{R}^2$ by $\{x, y\} = xy$ we produce the following Hamiltonian formulation

$$\begin{aligned} \dot{x} &= \{x, H\} = x(a - by) \\ \dot{y} &= \{y, H\} = y(-c + dx) \ . \end{aligned}$$

The Lotka-Volterra equations generalize from two to n species. The most general form of the equations is

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad i = 1, 2, \ldots, n \ , \tag{5.3}$$

where x_i denotes the density of the ith species, ε_i is its intrinsic growth (or decay) rate and the matrix $A = (a_{ij})$ is called the interaction matrix. We consider Lotka-Volterra equations without linear terms ($\varepsilon_i = 0$), i.e., the population of the ith species stays constant if there is no interraction with other species. We also assume that the matrix of interaction coefficients $A = (a_{ij})$ is skew-symmetric. This assumption places the problem in the context of the so called conservative Lotka-Volterra systems.

These systems can be written in Hamiltonian form using the Hamiltonian function

$$H = x_1 + x_2 + \cdots + x_n.$$

Hamilton's equations take the form $\dot{x}_i = \{x_i, H\} = \sum_{j=1}^{n} \pi_{ij}$ with quadratic functions

$$\pi_{ij} = \{x_i, x_j\} = a_{ij} x_i x_j, \quad i, j = 1, 2, \ldots, n. \tag{5.4}$$

From the skew symmetry of the matrix $A = (a_{ij})$ it follows that the Schouten-Nijenhuis bracket $[\pi, \pi]$ vanishes identically:

$$\begin{aligned} [\pi, \pi]_{ijk} &= 2\left(a_{ij}\{x_i x_j, x_k\} + a_{jk}\{x_j x_k, x_i\} + a_{ki}\{x_k x_i, x_j\}\right) \\ &= 2\left(a_{ij}(a_{jk} + a_{ik}) + a_{jk}(a_{ki} + a_{ji}) + a_{ki}(a_{ij} + a_{kj})\right) x_i x_j x_k = 0 \, . \end{aligned}$$

The bivector field π is an example of a *diagonal Poisson structure.*

The Poisson tensor (5.4) is Poisson isomorphic to the constant Poisson structure defined by the constant matrix A (see [**?**]).

Proposition 27. *If* $\mathbf{k} = (k_1, k_2 \cdots, k_n)$ *is a vector in the kernel of* A *then the function*

$$f = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

is a Casimir.

Proof. For an arbitrary function g the Poisson bracket $\{f, g\}$ is

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} k_i \right) x_j f \frac{\partial g}{\partial x_j} = 0\,.$$

□

If the matrix A has rank r then there are $n - r$ functionally independent Casimirs. This type of integral can be traced back to Volterra [**?**]; see also [**?**, **?**, **?**].

Chapter 6

Generalized Lotka-Volterra systems

> If you would be a real seeker after truth, it is necessary that at least once in your life you doubt, as far as possible, all things. — René Descartes

6.1 Introduction

Recall the system (1.2), also known as the Volterra system. It is associated with a simple Lie algebra of type A_n in the sense that it can be written in Lax pair form $\dot{L} = [B, L]$ where

$$L = \begin{pmatrix} 0 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & 0 & a_2 & \ddots & & \vdots \\ 0 & a_2 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & a_n \\ 0 & \cdots & \cdots & 0 & a_n & 0 \end{pmatrix}, \tag{6.1}$$

and

$$B = \begin{pmatrix} 0 & 0 & a_1 a_2 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \vdots \\ -a_1 a_2 & 0 & 0 & \ddots & a_2 a_3 & \vdots \\ \vdots & -a_2 a_3 & \ddots & \ddots & & a_{n-1} a_n \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & -a_{n-1} a_n & 0 & 0 \end{pmatrix}.$$

This Lax pair is due to Moser [**?**]; it gives a polynomial (in fact cubic) system of differential equations. The change of variables $x_i = 2a_i^2$ gives equations (1.2). It is evident from the form of L in the Lax pair, that the position of the variables a_i corresponds to the simple root vectors of a root system of type A_n. On the other hand a non-zero entry of the matrix B occurs at a position corresponding to the sum of two simple roots α_i and α_j.

There is a similar Lax pair which gives rise to the periodic KM system (see section 5.5). Bogoyavlensky generalized this system for each simple Lie algebra and showed that the corresponding systems are also integrable. In [?], Bogoyavlensky studies the systems defined by the equations

$$\dot{a}_i = a_i \left(\sum_{j=1}^{p} a_{i+j} - \sum_{j=1}^{p} a_{i-j} \right), i = 1, 2, \ldots, n$$

where $a_{n+i} = a_i$ for all i. These systems are generalizations of the periodic KM system; for $p = 1$ we obtain the periodic KM system. Bogoyavlensky finds Lax pairs for these systems and proves that for $n = 4$ and $n = 5$ they are integrable. In [?, ?], Bogoyavlensky constructed integrable systems connected with simple Lie algebras which generalize the periodic KM system. He constructs the systems by defining the corresponding Lax pairs as follows. He considers the matrices

$$L = \sum_{i=1}^{n} b_i X_{\alpha_i} + X_{-\alpha_0} + \sum_{1 \leq i < j \leq n} \left[X_{\alpha_i}, X_{\alpha_j} \right]$$

and

$$B = \sum_{i=1}^{n} \frac{k_i}{b_i} X_{-\alpha_i} + X_{\alpha_0},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the simple roots, α_0 is the highest positive root and X_{α_i} the corresponding root vectors of a root system of a simple Lie algebra. Also $\alpha_0 = \sum_{k=1}^{n} k_i \alpha_i$. The systems produce by these Lax pairs are integrable and are called Bogoyavlensky-Volterra systems. There is a complete description of these systems in [?]. See [?] and [?] for more details.

In this chapter we generalize the Lax pair of Moser and produce a larger class of Hamiltonian systems which we call generalized Volterra systems since in some cases by a simple change of variables we produce Lotka-Volterra systems. It is clear that the systems we obtain are not subsystems of the ones defined by Bogoyavlensky. The systems defined in [?, ?] (by a completely different approach) are of a different nature. For example the defining matrix is not even skew-symmetric. The systems obtained in [?] are also different as one easily notices by comparing the resulting equations. By restricting the systems in [?] it is impossible to obtain our type of systems. This can be seen by examining the associated graph of the Lotka-Volterra systems.

We generalize the Lax pair of Moser (6.1) as follows. Instead of considering the set of simple roots Π, we begin with a subset Φ of the positive roots R^+ of a root system of a complex simple Lie algebra, which contains Π, i.e. $\Pi \subset \Phi \subset R^+$. For each such choice of a set Φ we produce a Lax pair and thus a new Hamiltonian system. We restrict our attention to some examples in the A_n case, however this algorithm applies more generally, for each complex simple Lie algebra. In dimension 3, this procedure produces only two systems, the KM system and the periodic KM system. In dimensions 4 and 5 (i.e. the cases of A_3 and A_4) and by allowing the use of complex coefficients (see chapter 7) this method works in all possible cases and in fact we have verified using Maple that all the resulting systems are Liouville integrable.

6.2 The procedure

We recall the following procedure from [**?**]. Let L be any simple Lie algebra equipped with its Killing form $\langle \cdot \mid \cdot \rangle$. One chooses a Cartan subalgebra H of L, and a base Π of simple roots for the root system R of H in L. To each positive root α one can associate a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of vectors in L which generate a Lie subalgebra isomorphic to $sl_2(\mathbf{C})$. The set $(X_\alpha, X_{-\alpha})_{\alpha \in R^+} \cup (H_\alpha)_{\alpha \in \Pi}$ is a basis of L, called a root basis. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ and let $X_{\alpha_1}, \ldots, X_{\alpha_\ell}$ be the corresponding root vectors in L. Define

$$L = \sum_{\alpha_i \in \Pi} a_i (X_{\alpha_i} + X_{-\alpha_i}) \ .$$

To find the matrix B we use the following procedure. For each i, j form the vectors $\left[X_{\alpha_i}, X_{\alpha_j}\right]$. If $\alpha_i + \alpha_j$ is a root then include a term of the form $a_i a_j \left[X_{\alpha_i}, X_{\alpha_j}\right]$ in B. We make B skew-symmetric by including the corresponding negative root vectors $a_i a_j [X_{-\alpha_i}, X_{-\alpha_j}]$. Finally, we define the system using the Lax pair equation

$$\dot{L} = [L, B] \ .$$

For a root system of type A_n we obtain the KM system.

We generalize this algorithm as follows. Consider a subset Φ of R^+ such that

$$\Pi \subset \Phi \subset R^+ \ .$$

The Lax matrix is easy to construct

$$L = \sum_{\alpha_i \in \Phi} a_i (X_{\alpha_i} + X_{-\alpha_i}) \ .$$

Here we use the following enumeration of Φ which we assume to have m elements. The variables a_j correspond to the simple roots α_j for $j = 1, 2, \ldots, \ell$. We assign the variables a_j for $j = \ell+1, \ell+2, \ldots, m$ to the remaining roots in Φ. To construct the matrix B we use the following algorithm. Consider the set $\Phi \cup \Phi^-$ which consists of all the roots in Φ together with their negatives. Let

$$\Psi = \left\{ \alpha + \beta \mid \alpha, \beta \in \Phi \cup \Phi^-, \quad \alpha + \beta \in R^+ \right\},$$

and define

$$B = \sum c_{ij} a_i a_j \left(X_{\alpha_i + \alpha_j} - X_{-\alpha_i - \alpha_j} \right), \tag{6.2}$$

where $c_{ij} = \pm 1$ if $\alpha_i + \alpha_j \in \Psi$ with $\alpha_i, \alpha_j \in \Phi \cup \Phi^-$ and 0 otherwise. In all eight cases in A_3 we are able to make the proper choices of the sign of the c_{ij} so that we can produce a Lax pair. This method produces a Lax pair in all but five out of sixty four cases in A_4. However, when we allow the c_{ij} to take the complex values $\pm i$ we are able to produce a Lax pair in all 64 cases. By using Maple we were able to check that all these examples in A_3 and A_4 are in fact Liouville integrable. We will not attempt to prove the integrability of these systems in general due to the complexity of their definition. We restrict our attention to some examples in the A_n case and we prove that for several subsets Φ of special form the algorithm works; i.e., there is a choice of the signs of $c_{i,j}$ which produce a consistent Lax pair.

This algorithm for certain subsets Φ recovers well known integrable systems. For example for $\Phi = \Pi$, the simple roots of the root system A_n, and $c_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$ we obtain the KM system while for $\Phi = \Pi \cup \{\alpha_{n+1}\}$, the simple roots and the highest root, the choice of the signs $c_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$ and $c_{1,n+1} = c_{n,n+1} = -1$ produces the periodic KM system.

We have to point out that in [**?**] there is a similar construction for the case of the Toda lattice where Hamiltonian systems were defined which interpolate between the classical Kostant-Toda lattice and the full Kostant-Toda lattice. In that case there is a simple criterion on the set Φ which ensures the construction of the Lax pair. In our case there is no such simple criterion. However, in the next proposition we present a sufficient (but not necessary) condition on the subset Φ which gives a consistent Lax pair.

Proposition 28. *Let $\Pi \subset \Phi \subset R^+$ be a subset of the positive roots with the property that whenever $\alpha, \beta, \gamma \in \Phi \cup \Phi^-$ then $\alpha + \beta + \gamma \neq 0$ and if $\alpha + \beta + \gamma \in R^+$ then $\alpha + \beta + \gamma \in \Phi$. Also let B be the matrix constructed using the algorithm described in (6.2). Then for any choice of the signs $c_{i,j}$ the pair L, B is a Lax pair.*

Proof. Let K be the following subset of the positive roots

$$K = \{\alpha + \beta + \gamma : \alpha, \beta, \gamma \in \Phi \cup \Phi^-, \ \alpha + \beta + \gamma \in R^+\}.$$

It is evident, from the construction of the matrix B, that for all possible choices of the signs $c_{i,j}$, the nonzero entries of the bracket $[L, B]$ appear in the positions corresponding to the root vectors $X_\alpha,\ \alpha \in K$. The condition $\alpha, \beta, \gamma \in \Phi \cup \Phi^- \Rightarrow \alpha + \beta + \gamma \neq 0$ implies that there are no variables in the diagonal of $[L, B]$ while the condition $\alpha, \beta, \gamma \in \Phi \cup \Phi^-$ and $\alpha + \beta + \gamma \in R^+ \Rightarrow \alpha + \beta + \gamma \in \Phi$ implies that $K \subset \Phi$. Since we also have $\Phi \subset K$ we deduce that $\Phi = K$ and therefore the pair L, B is a Lax pair. $\square$

This condition is of course not necessary. For example the KM and the periodic KM systems do not fall in this class. In theorem 13 and proposition 30 we find several other families of subsets Φ which give consistent Lax pairs.

A corollary of proposition 28 is the following.

Corollary 4. *Let Φ be the subset of the positive roots of the root system of any simple Lie algebra containing all the roots of odd height. If L and B are the matrices constructed as described in (6.2) then for all all possible choices of the signs $c_{i,j}$, L, B is a consistent Lax pair.*

We illustrate the previous corollary with examples from the roots systems of the classical simple Lie algebras.

Example 27. For the root system of type A_3 the subset of the positive roots of odd height is $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_1 + \alpha_2 + \alpha_3\}$. This choice gives rise to the matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_4 \\ a_1 & 0 & a_2 & 0 \\ 0 & a_2 & 0 & a_3 \\ a_4 & 0 & a_3 & 0 \end{pmatrix}.$$

The corresponding subset Ψ is

$$\Psi = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$$

and the roots in Ψ are formed as $\alpha_1 + \alpha_2 = \alpha_4 = \alpha_3$ and $\alpha_2 + \alpha_3 = \alpha_4 - \alpha_1$. Therefore the skew-symmetric matrix B constructed using (6.2) is

$$\begin{pmatrix} 0 & 0 & c_{1,2}a_1a_2 + c_{3,4}a_3a_4 & 0 \\ 0 & 0 & 0 & c_{1,4}a_1a_4 + c_{2,3}a_2a_3 \\ -c_{1,2}a_1a_2 - c_{3,4}a_3a_4 & 0 & 0 & 0 \\ 0 & -c_{1,4}a_1a_4 - c_{2,3}a_2a_3 & 0 & 0 \end{pmatrix}.$$

We can easily verify that all 16 possible choices of the signs $c_{i,j}$ give a consistent Lax pair.

The equation $\dot{L} = [L, B]$ is equivalent to the system

$$\begin{aligned}
\dot{a_1} &= -c_{1,2}a_1a_2^2 - c_{1,4}a_1a_4^2 - c_{2,3}a_2a_3a_4 - c_{3,4}a_2a_3a_4, \\
\dot{a_2} &= c_{1,2}a_2a_1^2 - c_{2,3}a_2a_3^2 + c_{3,4}a_1a_3a_4 - c_{1,4}a_1a_3a_4, \\
\dot{a_3} &= c_{2,3}a_3a_2^2 + c_{3,4}a_3a_4^2 + c_{1,2}a_1a_2a_4 + c_{1,4}a_1a_2a_4, \\
\dot{a_4} &= c_{1,4}a_1^2a_4 - c_{3,4}a_3^2a_4 + c_{2,3}a_1a_2a_3 - c_{1,2}a_1a_2a_3.
\end{aligned}$$

Of course only half of the choices of the signs give possibly non-isomorphic systems and only one of them gives a Lotka-Volterra system (see theorem 13 below), the well known periodic KM system.

Example 28. For the root system of type B_3 the subset of the positive roots of odd height is

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_1 + \alpha_2 + \alpha_3, \alpha_5 = \alpha_2 + 2\alpha_3, \alpha_6 = \alpha_1 + 2\alpha_2 + 2\alpha_3\}.$$

This choice of the positive roots gives rise to the matrix

$$L = \begin{pmatrix}
0 & a_1 & 0 & 2a_4 & 0 & 2a_6 & 0 \\
a_1 & 0 & a_2 & 0 & 2a_5 & 0 & -2a_6 \\
0 & a_2 & 0 & 2a_3 & 0 & -2a_5 & 0 \\
a_4 & 0 & a_3 & 0 & a_3 & 0 & a_4 \\
0 & 2a_5 & 0 & 2a_3 & 0 & -a_2 & 0 \\
2a_6 & 0 & -2a_5 & 0 & -a_2 & 0 & -a_1 \\
0 & -2a_6 & 0 & 2a_4 & 0 & -a_1 & 0
\end{pmatrix}$$

or equivalently

$$L = \sum_{i=1}^{6} a_i \left(X_{\alpha_i} + X_{-\alpha_i}\right).$$

The subset Ψ is

$$\Psi = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3\}$$

where the roots in Ψ are formed as $\alpha_1+\alpha_2 = \alpha_4-\alpha_3 = \alpha_6-\alpha_5$, $\alpha_2+\alpha_3 = \alpha_4-\alpha_1 = \alpha_6-\alpha_4$ and $\alpha_1 + \alpha_2 + 2\alpha_3 = \alpha_3 + \alpha_4 = \alpha_1 + \alpha_5 = \alpha_6 - \alpha_2$.

Therefore the matrix B constructed using (6.2) is given by

$$\begin{gathered}
\left(c_{1,2}a_1a_2 + c_{4,3}a_3a_4 + c_{5,6}a_5a_6\right)\left(X_{\alpha_1+\alpha_2} - X_{-\alpha_1-\alpha_2}\right) + \\
\left(c_{2,3}a_2a_3 + c_{1,4}a_1a_4 + c_{4,6}a_4a_6 + c_{3,5}a_3a_5\right)\left(X_{\alpha_2+\alpha_3} - X_{-\alpha_2-\alpha_3}\right) + \\
\left(c_{3,4}a_3a_4 + c_{1,5}a_1a_5 + c_{2,6}a_2a_6\right)\left(X_{\alpha_1+\alpha_2+\alpha_2+2\alpha_3} - X_{-\alpha_1-\alpha_2-2\alpha_3}\right).
\end{gathered}$$

In matrix form (see section 2.2) is the 7×7 matrix of the form

$$\begin{pmatrix} A_1 & 2\Delta w & A_2 \\ v^T & 0 & w^T \\ A_3 & 2\Delta v & A_4 \end{pmatrix}$$

where A_1, A_2, A_3, A_4 are 3×3 matrices, A_2 is per skew-symmetric, $A_3 = -A_2^T$ and A_4 is minus the per transpose matrix of A_1. The matrices A_1 and A_4 are skew-symmetric and v and w are 3×1 vectors. The matrix A_1 is

$$A_1 = \begin{pmatrix} 0 & 0 & c_{1,2}a_1a_2 + c_{4,3}a_3a_4 + c_{5,6}a_5a_6 \\ 0 & 0 & 0 \\ -c_{1,2}a_1a_2 - c_{4,3}a_3a_4 - c_{5,6}a_5a_6 & 0 & 0 \end{pmatrix},$$

the matrix A_2 is given by

$$A_2 = \begin{pmatrix} c_{3,4}a_3a_4 + c_{1,5}a_1a_5 + c_{2,6}a_2a_6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c_{3,4}a_3a_4 - c_{1,5}a_1a_5 - c_{2,6}a_2a_6 \end{pmatrix}.$$

The vectors v, w are $w = -v = (0, c_{2,3}a_2a_3 + c_{1,4}a_1a_4 + 2c_{4,6}a_4a_6 + 2c_{3,5}a_3a_5, 0, 0)$. Now it is straightforward to verify that the pair L, B is indeed a Lax pair. The equation $\dot{L} = [L, B]$ is equivalent to the equations

$$\begin{gathered}
\dot{a}_1 = -2c_{2,3}a_2a_3a_4 - 2c_{1,4}a_1a_4^2 - 4c_{4,6}a_4^2a_6 - 4c_{1,5}a_1a_5^2 - 4c_{3,5}a_3a_4a_5 - \\
4c_{3,4}a_3a_4a_5 - 4c_{2,6}a_2a_5a_6 - c_{1,2}a_1a_2^2 - 2c_{4,3}a_2a_3a_4 - 4c_{5,6}a_2a_5a_6, \\
\dot{a}_2 = c_{1,2}a_1^2a_2 + 2c_{4,3}a_1a_3a_4 + 4c_{5,6}a_1a_5a_6 - 4c_{1,5}a_1a_5a_6 - 4c_{3,5}a_3^2a_5 - \\
4c_{3,4}a_3a_4a_6 - 4c_{2,6}a_2a_6^2 - 2c_{2,3}a_2a_3^2 - 2c_{1,4}a_1a_3a_4 - 4c_{4,6}a_3a_4a_6, \\
\dot{a}_3 = c_{1,2}a_1a_2a_4 + 2c_{4,3}a_3a_4^2 + 4c_{5,6}a_4a_5a_6 + 2c_{1,5}a_1a_4a_5 + 2c_{3,4}a_3a_4^2 + 2c_{2,6}a_2a_4a_6 + \\
2c_{2,3}a_2a_3a_5 + 2c_{1,4}a_1a_4a_5 + 4c_{4,6}a_4a_5a_6 + c_{2,3}a_2^2a_3 + c_{1,4}a_1a_2a_4 + 2c_{4,6}a_2a_4a_6 + \\
2c_{3,5}a_2a_3a_5 + 4c_{3,5}a_3a_5^2, \\
\dot{a}_4 = -c_{1,2}a_1a_2a_3 - 2c_{4,3}a_3^2a_4 - 4c_{5,6}a_3a_5a_6 - 2c_{1,5}a_1a_3a_5 - 2c_{3,4}a_3^2a_4 - 2c_{2,6}a_2a_3a_6 - \\
2c_{2,3}a_2a_3a_6 - 2c_{1,4}a_1a_4a_6 - 4c_{4,6}a_4a_6^2 + c_{2,3}a_1a_2a_3 + c_{1,4}a_1^2a_4 + 2c_{4,6}a_1a_4a_6 + \\
2c_{3,5}a_1a_3a_5 - 4c_{3,5}a_3a_5a_6, \\
\dot{a}_5 = c_{1,5}a_1^2a_5 + c_{3,4}a_1a_3a_4 + c_{2,6}a_1a_2a_6 - c_{1,2}a_1a_2a_6 - 2c_{3,5}a_3^2a_5 \\
2c_{4,3}a_3a_4a_6 - 4c_{5,6}a_5a_6^2 - c_{2,3}a_2a_3^2 - c_{1,4}a_1a_3a_4 - 2c_{4,6}a_3a_4a_6, \\
\dot{a}_6 = c_{2,3}a_2a_3a_4 + c_{1,4}a_1a_4^2 + 2c_{4,6}a_4^2a_6 + c_{1,5}a_1a_2a_5 +
\end{gathered}$$

$$c_{3,4}a_2a_3a_4 + c_{2,6}a_2^2a_6 + c_{1,2}a_1a_2a_5 + 2c_{4,3}a_3a_4a_5 + 4c_{5,6}a_5^2a_6 + 2c_{3,5}a_3a_4a_5.$$

Example 29. For the root system of type C_3 the subset of the positive roots of odd height is

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_1 + \alpha_2 + \alpha_3, \alpha_5 = 2\alpha_2 + \alpha_3, \alpha_6 = 2\alpha_1 + 2\alpha_2 + \alpha_3\}.$$

This choice of the positive roots gives rise to the matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_4 & 0 & 2a_6 \\ a_1 & 0 & a_2 & 0 & 2a_5 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & a_4 \\ a_4 & 0 & a_3 & 0 & -a_2 & 0 \\ 0 & 2a_5 & 0 & -a_2 & 0 & -a_1 \\ 2a_6 & 0 & a_4 & 0 & -a_1 & 0 \end{pmatrix}$$

The matrix B constructed using (6.2) is the skew-symmetric 6×6 matrix of the form

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where A_1, A_2, A_3, A_4 are 3×3 matrices, A_2 is per symmetric and A_4 is minus the per transpose matrix of A_1. The matrix A_1 is

$$A_1 = \begin{pmatrix} 0 & 0 & c_{1,2}a_1a_2 + c_{3,4}a_3a_4 + 2c_{4,6}a_4a_6 \\ 0 & 0 & 0 \\ -c_{1,2}a_1a_2 - c_{3,4}a_3a_4 - 2c_{4,6}a_4a_6 & 0 & 0 \end{pmatrix}$$

and the per symmetric part of the matrix A_2 is

$$A_2 = \begin{pmatrix} 0 & 2c_{1,5}a_1a_5 + c_{2,4}a_2a_4 + 2c_{1,6}a_1a_6 & 0 \\ c_{1,4}a_1a_4 + c_{2,3}a_2a_3 + 2c_{2,5}a_2a_5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to verify that the pair L, B is indeed a Lax pair. The equation $\dot{L} = [L, B]$ is equivalent to the system

$$\begin{gathered} \dot{a_1} = -c_{1,4}a_1a_4^2 - 2c_{2,5}a_2a_4a_5 - c_{2,3}a_2a_3a_4 - 4c_{1,6}a_1a_6^2 - \\ 4c_{1,5}a_1a_5a_6 - 2c_{2,4}a_2a_4a_6 - c_{1,2}a_1a_2^2 - c_{3,4}a_2a_3a_4 - \\ 2c_{4,6}a_2a_4a_6 - 4c_{1,5}a_1a_5^2 - 4c_{1,6}a_1a_5a_6 - 2c_{2,4}a_2a_4a_5, \\ \dot{a_2} = c_{1,2}a_1^2a_2 + c_{3,4}a_1a_3a_4 + 2c_{4,6}a_1a_4a_6 - 2c_{1,4}a_1a_4a_5 - \end{gathered}$$

$$4c_{2,5}a_2a_5^2 - 2c_{2,3}a_2a_3a_5 - 2c_{1,5}a_1a_4a_5 - 2c_{1,6}a_1a_4a_6 -$$
$$c_{2,4}a_2a_4^2 - c_{1,4}a_1a_3a_4 - c_{2,3}a_2a_3^2 - 2c_{2,5}a_2a_3a_5,$$
$$\dot{a_3} = 2c_{1,4}a_1a_2a_4 + 2c_{2,3}a_2^2a_3 + 4c_{2,5}a_2^2a_5 +$$
$$2c_{1,2}a_1a_2a_4 + 2c_{3,4}a_3a_4^2 + 4c_{4,6}a_4^2a_6,$$
$$\dot{a_4} = 2c_{1,5}a_1a_2a_5 + 2c_{1,6}a_1a_2a_6 + c_{2,4}a_2^2a_4 -$$
$$c_{1,2}a_1a_2a_3 - c_{3,4}a_3^2a_4 - 2c_{4,6}a_3a_4a_6 + 2c_{1,2}a_1a_2a_6 +$$
$$2c_{3,4}a_3a_4a_6 + 4c_{4,6}a_4a_6^2 + c_{1,4}a_1^2a_4 + c_{2,3}a_1a_2a_3 + 2c_{2,5}a_1a_2a_5,$$
$$\dot{a_5} = 2c_{1,5}a_1^2a_5 + 2c_{1,6}a_1^2a_6 + c_{2,4}a_1a_2a_4 +$$
$$c_{1,4}a_1a_2a_4 + c_{2,3}a_2^2a_3 + 2c_{2,5}a_2^2a_5,$$
$$\dot{a_6} = 2c_{1,5}a_1^2a_5 + 2c_{1,6}a_1^2a_6 + c_{2,4}a_1a_2a_4 -$$
$$c_{1,2}a_1a_2a_4 - c_{3,4}a_3a_4^2 - 2c_{4,6}a_4^2a_6.$$

Example 30. The root system D_3 is exactly the root system A_3 and therefore the corresponding systems are the same as in the case of A_3. For the root system of type D_4 the subset of the positive roots of odd height is

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}.$$

This choice of the positive roots gives rise to the matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_5 & a_6 & 0 & a_8 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & a_7 & 0 & -a_8 \\ 0 & a_2 & 0 & a_3 & a_4 & 0 & -a_7 & 0 \\ a_5 & 0 & a_3 & 0 & 0 & -a_4 & 0 & -a_6 \\ a_6 & 0 & a_4 & 0 & 0 & -a_3 & 0 & -a_5 \\ 0 & a_7 & 0 & -a_4 & -a_3 & 0 & -a_2 & 0 \\ a_7 & 0 & -a_7 & 0 & 0 & -a_2 & 0 & -a_1 \\ 0 & -a_7 & 0 & -a_6 & -a_5 & 0 & -a_1 & 0 \end{pmatrix}.$$

The matrix B constructed using (6.2) is the skew-symmetric 8×8 matrix of the form

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where A_1, A_2, A_3, A_4 are 4×4 matrices, A_2 is per skew-symmetric and A_4 is minus the per transponse matrix of A_1. The matrix A_1 is

$$A_1 = \begin{pmatrix} 0 & 0 & c_{1,2}a_1a_2 + c_{3,5}a_3a_5 + c_{4,6}a_4a_6 + c_{7,8}a_7a_8 & 0 \\ 0 & 0 & 0 & c_{1,5}a_1a_5 + c_{2,3}a_2a_3 + c_{4,7}a_4a_7 + c_{6,8}a_6a_8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the per skew-symmetric part of the matrix A_2 is

$$A_2 = \begin{pmatrix} 0 & c_{1,7}a_1a_7 + c_{2,8}a_2a_8 + c_{3,6}a_3a_6 + c_{4,5}a_4a_5 & 0 & 0 \\ c_{1,6}a_1a_6 + c_{2,4}a_2a_4 + c_{3,7}a_3a_7 + c_{5,8}a_5a_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to verify that the pair L, B is indeed a Lax pair.

The equation $\dot{L} = [L, B]$ is equivalent to the system

$$\begin{gathered}
\dot{a_1} = -c_{1,5}a_1a_5^2 - a_5a_2a_3c_{2,3} - a_5a_4a_7c_{4,7} - a_6c_{6,8}a_8a_5 - c_{1,6}a_6^2a_1 - a_6a_2c_{2,4}a_4 - \\
a_6a_7a_3c_{3,7} - a_5c_{5,8}a_8a_6 - c_{1,7}a_7^2a_1 - a_8a_7a_2c_{2,8} - a_6a_7a_3c_{3,6} - a_7a_4c_{4,5}a_5 - \\
c_{1,2}a_2^2a_1 - a_5a_3a_2c_{3,5} - a_2a_4c_{4,6}a_6 - a_8a_7a_2c_{7,8}, \\
\dot{a_2} = c_{1,2}a_2a_1^2 + a_1a_3c_{3,5}a_5 + a_6a_4a_1c_{4,6} + a_8a_7a_1c_{7,8} - a_8a_7a_1c_{1,7} - c_{2,8}a_8^2a_2 - \\
a_8a_6a_3c_{3,6} - a_4c_{4,5}a_5a_8 - a_6a_4a_1c_{1,6} - c_{2,4}a_4^2a_2 - a_7a_4a_3c_{3,7} - a_4a_5c_{5,8}a_8 - \\
a_5a_3a_1c_{1,5} - c_{2,3}a_3^2a_2 - a_7a_4a_3c_{4,7} - a_8a_6a_3c_{6,8}, \\
\dot{a_3} = a_2a_5a_1c_{1,5} + c_{2,3}a_3a_2^2 + a_7a_4a_2c_{4,7} + a_8a_6a_2c_{6,8} - a_6a_7a_1c_{1,6} - a_7a_4a_2c_{2,4} - \\
c_{3,7}a_7^2a_3 - a_5c_{5,8}a_8a_7 - a_6a_7a_1c_{1,7} - a_8a_6a_2c_{2,8} - a_3c_{3,6}a_6^2 - a_6a_4c_{4,5}a_5 + \\
a_2a_5a_1c_{1,2} + a_5^2a_3c_{3,5} + a_4c_{4,6}a_6a_5 + a_5a_7c_{7,8}a_8, \\
\dot{a_4} = a_2a_6a_1c_{1,6} + c_{2,4}a_4a_2^2 + a_7a_3a_2c_{3,7} + a_8a_5a_2c_{5,8} - a_5a_7a_1c_{1,5} - a_7a_3a_2c_{2,3} - \\
c_{4,7}a_7^2a_4 - a_6c_{6,8}a_8a_7 - a_5a_7a_1c_{1,7} - a_8a_5a_2c_{2,8} - a_6a_5a_3c_{3,6} - a_4c_{4,5}a_5^2 + \\
a_2a_6a_1c_{1,2} + a_6a_3c_{3,5}a_5 + a_4c_{4,6}a_6^2 + a_6a_7c_{7,8}a_8, \\
\dot{a_5} = c_{1,5}a_5a_1^2 + a_2a_3a_1c_{2,3} + a_7a_4a_1c_{4,7} + a_8a_6a_1c_{6,8} + a_8a_6a_1c_{1,6} + a_8a_4a_2c_{2,4} + \\
a_8a_7a_3c_{3,7} + a_5c_{5,8}a_8^2 + a_7a_4a_1c_{1,7} + a_8a_4a_2c_{2,8} + a_6a_4a_3c_{3,6} + a_4^2c_{4,5}a_5 - \\
a_2a_3a_1c_{1,2} - a_3^2c_{3,5}a_5 - a_4a_3c_{4,6}a_6 - a_8a_7a_3c_{7,8}, \\
\dot{a_6} = c_{1,6}a_6a_1^2 + a_1a_2c_{2,4}a_4 + a_7a_3a_1c_{3,7} + a_8a_5a_1c_{5,8} + a_8a_5a_1c_{1,5} + a_8a_3a_2c_{2,3} + \\
a_8a_7a_4c_{4,7} + a_6c_{6,8}a_8^2 + a_7a_3a_1c_{1,7} + a_8a_3a_2c_{2,8} + a_3^2c_{3,6}a_6 + a_5a_4a_3c_{4,5} - \\
a_4a_2a_1c_{1,2} - a_4a_3c_{3,5}a_5 - a_4^2c_{4,6}a_6 - a_8a_7a_4c_{7,8}, \\
\dot{a_7} = c_{1,7}a_7a_1^2 + a_8a_2a_1c_{2,8} + a_6a_3a_1c_{3,6} + a_5a_4a_1c_{4,5} - a_8a_2a_1c_{1,2} - a_8a_5a_3c_{3,5} - \\
a_8a_6a_4c_{4,6} - a_7c_{7,8}a_8^2 + a_6a_3a_1c_{1,6} + a_4a_3a_2c_{2,4} + c_{3,7}a_7a_3^2 + a_8a_5a_3c_{5,8} + \\
a_5a_4a_1c_{1,5} + a_4a_3a_2c_{2,3} + c_{4,7}a_7a_4^2 + a_8a_6a_4c_{6,8},
\end{gathered}$$

$$\dot{a_8} = -a_6a_5a_1c_{1,6} - a_5a_4a_2c_{2,4} - a_5a_7a_3c_{3,7} - a_5^2c_{5,8}a_8 - a_6a_5a_1c_{1,5} - a_6a_3a_2c_{2,3} - a_6a_7a_4c_{4,7} - a_6^2c_{6,8}a_8 + a_2a_7a_1c_{1,7} + c_{2,8}a_8a_2^2 + a_6a_3a_2c_{3,6} + a_5a_4a_2c_{4,5} + a_2a_7a_1c_{1,2} + a_5a_7a_3c_{3,5} + a_6a_7a_4c_{4,6} + a_7^2c_{7,8}a_8.$$

6.3 Examples in A_3 and A_4

Example 31. For the root system A_3 if we take $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$ then

$$\Phi \cup \Phi^- = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_1 - \alpha_2\}$$

and $\Psi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. In this example the variables a_i for $i = 1, 2, 3$ correspond to the three simple roots $\alpha_1, \alpha_2, \alpha_3$. We associate the variable a_4 to the root $\alpha_1 + \alpha_2$. We obtain the following Lax pair:

$$L = \begin{pmatrix} 0 & a_1 & a_4 & 0 \\ a_1 & 0 & a_2 & 0 \\ a_4 & a_2 & 0 & a_3 \\ 0 & 0 & a_3 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & -a_4a_2 & a_1a_2 & -a_4a_3 \\ a_4a_2 & 0 & -a_1a_4 & a_2a_3 \\ -a_1a_2 & a_1a_4 & 0 & 0 \\ a_4a_3 & -a_2a_3 & 0 & 0 \end{pmatrix}.$$

Using the substitution $x_i = a_i^2$ followed by scaling, the Lax pair is equivalent to the following Lotka-Volterra system.

$$\begin{aligned} \dot{x_1} &= x_1x_2 - x_1x_4, \\ \dot{x_2} &= -x_2x_1 + x_2x_3 + x_2x_4, \\ \dot{x_3} &= -x_3x_2 + x_3x_4, \\ \dot{x_4} &= x_4x_1 - x_4x_2 - x_4x_3\,. \end{aligned}$$

This system is integrable. There exist two functionally independent Casimir functions $F_1 = x_1x_3 = \det L$ and $F_2 = x_1x_2x_4$. The additional integral is the Hamiltonian $H = x_1 + x_2 + x_3 + x_4 = \operatorname{tr} L^2$.

The standard quadratic Poisson bracket (5.4) is given by

$$\pi = \begin{pmatrix} 0 & x_1x_2 & 0 & -x_1x_4 \\ -x_2x_1 & 0 & x_2x_3 & x_2x_4 \\ 0 & -x_3x_2 & 0 & x_3x_4 \\ x_4x_1 & -x_4x_2 & -x_4x_3 & 0 \end{pmatrix}.$$

One can find the Casimirs by computing the kernel of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}.$$

The two eigenvectors with eigenvalue 0 are $(1,0,1,0)$ and $(1,1,0,1)$. We obtain the two Casimirs $F_1 = x_1^1 x_2^0 x_3^1 x_4^0 = x_1 x_3$ and $F_2 = x_1^1 x_2^1 x_3^0 x_4^1 = x_1 x_2 x_4$.

There is a similar Lax pair defined by the subset $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3\}$, where the L matrix is

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ a_1 & 0 & a_2 & a_4 \\ 0 & a_2 & 0 & a_3 \\ 0 & a_4 & a_3 & 0 \end{pmatrix}.$$

The resulting system is isomorphic to the one of example 31.

Example 32. A Lax pair L, B corresponding to $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_4 \\ a_1 & 0 & a_2 & 0 \\ 0 & a_2 & 0 & a_3 \\ a_4 & 0 & a_3 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & a_1 a_2 - a_4 a_3 & 0 \\ 0 & 0 & 0 & -a_1 a_4 + a_2 a_3 \\ -a_1 a_2 + a_4 a_3 & 0 & 0 & 0 \\ 0 & a_1 a_4 - a_2 a_3 & 0 & 0 \end{pmatrix}.$$

Using the substitution $x_i = 2a_i^2$ we obtain the periodic KM-system

$$\begin{aligned} \dot{x_1} &= x_1 x_2 - x_1 x_4, \\ \dot{x_2} &= -x_2 x_1 + x_2 x_3, \\ \dot{x_3} &= x_3 x_4 - x_3 x_2, \\ \dot{x_4} &= x_4 x_1 - x_4 x_3\,. \end{aligned} \tag{6.3}$$

The Poisson matrix (which can be read from the right hand side of (6.3)) is

$$\pi = \begin{pmatrix} 0 & x_1x_2 & 0 & -x_1x_4 \\ -x_1x_2 & 0 & x_2x_3 & 0 \\ 0 & -x_2x_3 & 0 & x_3x_4 \\ x_1x_4 & 0 & -x_3x_4 & 0 \end{pmatrix}$$

of rank 2. In addition to the Hamiltonian

$$H = x_1 + x_2 + x_3 + x_4$$

it possesses two Casimirs $C_1 = x_1x_3$ and $C_2 = x_2x_4$.

Example 33. The Lax equation $\dot{L} = [B, L]$, corresponding to the subset of the positive roots of the root system A_3,

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$$

with

$$L = \begin{pmatrix} 0 & a_1 & a_4 & 0 \\ a_1 & 0 & a_2 & a_5 \\ a_4 & a_2 & 0 & a_3 \\ 0 & a_5 & a_3 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & -a_4a_2 & a_1a_2 & -a_1a_5 - a_4a_3 \\ a_4a_2 & 0 & -a_1a_4 - a_5a_3 & a_2a_3 \\ -a_1a_2 & a_1a_4 + a_5a_3 & 0 & -a_2a_5 \\ a_1a_5 + a_4a_3 & -a_2a_3 & a_2a_5 & 0 \end{pmatrix}$$

is equivalent to the following equations of motion

$$\begin{aligned} \dot{a_1} &= a_1a_2^2 - a_1a_5^2 - a_1a_4^2 - 2a_3a_4a_5, \\ \dot{a_2} &= a_2a_4^2 + a_2a_3^2 - a_2a_1^2 - a_2a_5^2, \\ \dot{a_3} &= a_3a_5^2 + a_3a_4^2 - a_3a_2^2 + 2a_1a_4a_5, \\ \dot{a_4} &= a_4a_1^2 - a_4a_2^2 - a_4a_3^2, \\ \dot{a_5} &= a_5a_1^2 - a_5a_3^2 + a_5a_2^2. \end{aligned}$$

Note that the system is not Lotka-Volterra. It is Hamiltonian with Hamiltonian function

$H=\frac{1}{2}\left(a_1^2+a_2^2+a_3^2+a_4^2+a_5^2\right)$. The system has Poisson matrix

$$\pi=\begin{pmatrix} 0 & a_1a_2 & -2\,a_4a_5 & -a_1a_4 & -a_1a_5 \\ -a_1a_2 & 0 & a_2a_3 & a_2a_4 & -a_2a_5 \\ 2\,a_4a_5 & -a_2a_3 & 0 & a_3a_4 & a_3a_5 \\ a_1a_4 & -a_2a_4 & -a_3a_4 & 0 & 0 \\ a_1a_5 & a_2a_5 & -a_3a_5 & 0 & 0 \end{pmatrix}$$

of rank 4 . The determinant $C=(a_1a_3-a_4a_5)^2$ of L is the Casimir of the system. The trace of L^3 gives the additional constant of motion

$$F=\frac{1}{6}\operatorname{tr}\left(L^3\right)=a_1a_2a_4+a_2a_3a_5\ .$$

Since the three constants of motion are evidently independent, the system is Liouville integrable.

Example 34. Let

$$L=\begin{pmatrix} 0 & a_1 & 0 & a_5 \\ a_1 & 0 & a_2 & a_4 \\ 0 & a_2 & 0 & a_3 \\ a_5 & a_4 & a_3 & 0 \end{pmatrix}$$

and

$$B=\begin{pmatrix} 0 & a_4a_5 & a_1a_2+a_5a_3 & -a_1a_4 \\ -a_4a_5 & 0 & -a_4a_3 & a_1a_5+a_2a_3 \\ -a_1a_2-a_5a_3 & a_4a_3 & 0 & -a_4a_2 \\ a_1a_4 & -a_1a_5-a_2a_3 & a_4a_2 & 0 \end{pmatrix}.$$

This Lax pair arises when we choose $\Phi=\{\alpha_1,\alpha_2,\alpha_3,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3\}$. The corresponding Poisson matrix has rank 4. In addition to the Hamiltonian H it possesses a Casimir $C=\det(L)=(a_1a_3-a_2a_5)^2$ and the integral $F=\frac{1}{6}\operatorname{tr}\left(L^3\right)=a_1a_4a_5+a_2a_3a_4$.

The Lax pair corresponding to the subset of the positive roots $\Phi=\{\alpha_1,\alpha_2,\alpha_3,\alpha_1+\alpha_2,\alpha_1+\alpha_2+\alpha_3\}$ gives rise to a system which is isomorphic to the one of the previous example.

Example 35. For the case of

$$\Phi=\{\alpha_1,\alpha_2,\alpha_3,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3\}$$

there are three different choices of the signs of $c_{i,j}$ which give consistent Lax pairs.

The Lax matrix is given by

$$L = \begin{pmatrix} 0 & a_1 & a_4 & a_6 \\ a_1 & 0 & a_2 & a_5 \\ a_4 & a_2 & 0 & a_3 \\ 0 & a_5 & a_3 & 0 \end{pmatrix}$$

In order to have a consistent Lax pair the upper triangular part of the skew-symmetric matrix B should be

$$\begin{pmatrix} 0 & c_{5,6}a_5a_6 - a_2a_4 & a_1a_2 - a_3a_6 & -c_{5,6}a_1a_5 + a_3a_4 \\ 0 & 0 & -c_{3,5}a_3a_5 - a_1a_4 & c_{5,6}a_1a_6 + c_{3,5}a_2a_3 \\ 0 & 0 & 0 & -c_{3,5}a_2a_5 - a_4a_6 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $c_{3,5} = c_{5,6} = 1$ we obtain the system

$$\begin{array}{rcl} \dot{a}_1 & = & -a_1a_2^2 + a_1a_4^2 + a_1a_5^2 - a_1a_6^2 \\ \dot{a}_2 & = & a_1^2a_2 - 2a_1a_3a_6 - a_2a_3^2 - a_2a_4^2 + a_2a_5^2 + 2a_4a_5a_6 \\ \dot{a}_3 & = & 2a_1a_2a_6 - 2a_1a_4a_5 + a_2^2a_3 + a_3a_4^2 - a_3a_5^2 - a_3a_6^2 \\ \dot{a}_4 & = & -a_1^2a_4 + a_2^2a_4 - a_3^2a_4 + a_4a_6^2 \\ \dot{a}_5 & = & -a_1^2a_5 + 2a_1a_3a_4 - a_2^2a_5 - 2a_2a_4a_6 + a_3^2a_5 + a_5a_6^2 \\ \dot{a}_6 & = & a_1^2a_6 + a_3^2a_6 - a_4^2a_6 - a_5^2a_6 \end{array},$$

for $c_{3,5} = -c_{5,6} = 1$ the system

$$\begin{array}{rcl} \dot{a}_1 & = & -a_1a_2^2 + a_1a_4^2 - a_1a_5^2 + a_1a_6^2 \\ \dot{a}_2 & = & a_1^2a_2 - a_2a_3^2 - a_2a_4^2 + a_2a_5^2 \\ \dot{a}_3 & = & a_2^2a_3 + a_3a_4^2 - a_3a_5^2 - a_3a_6^2 \\ \dot{a}_4 & = & -a_1^2a_4 - 2a_1a_3a_5 + a_2^2a_4 + 2a_2a_5a_6 - a_3^2a_4 + a_4a_6^2 \\ \dot{a}_5 & = & a_1^2a_5 + 2a_1a_3a_4 - a_2^2a_5 - 2a_2a_4a_6 + a_3^2a_5 - a_5a_6^2 \\ \dot{a}_6 & = & -a_1^2a_6 + a_3^2a_6 - a_4^2a_6 + a_5^2a_6 \end{array},$$

for $c_{3,5} = -c_{5,6} = -1$ the system

$$\begin{array}{rcl} \dot{a}_1 & = & -a_1a_2^2 + a_1a_4^2 + a_1a_5^2 - a_1a_6^2 + 2a_2a_3a_6 - 2a_3a_4a_5 \\ \dot{a}_2 & = & a_1^2a_2 - 2a_1a_3a_6 + a_3^2a_2 - a_2a_4^2 - a_2a_5^2 + 2a_4a_5a_6 \\ \dot{a}_3 & = & 2a_1a_2a_6 - 2a_1a_4a_5 - a_2^2a_3 + a_3a_4^2 + a_3a_5^2 - a_3a_6^2 \\ \dot{a}_4 & = & -a_1^2a_4 + 2a_1a_3a_5 + a_2^2a_4 - 2a_2a_5a_6 - a_3^2a_4 + a_4a_6^2 \\ \dot{a}_5 & = & -a_1^2a_5 + 2a_1a_3a_4 + a_2^2a_5 - 2a_2a_4a_6 - a_3^2a_5 + a_5a_6^2 \\ \dot{a}_6 & = & a_1^2a_6 - 2a_1a_2a_3 + 2a_2a_4a_5 + a_3^2a_6 - a_4^2a_6 - a_5^2a_6 \end{array}$$

and for $c_{3,5} = c_{5,6} = -1$ the system

$$\begin{array}{rcl}
\dot{a}_1 & = & -a_1a_2^2 + a_1a_4^2 - a_1a_5^2 + a_1a_6^2 + 2a_2a_3a_6 - 2a_3a_4a_5 \\
\dot{a}_2 & = & a_1^2a_2 + a_3^2a_2 - a_2a_4^2 - a_2a_5^2 \\
\dot{a}_3 & = & -a_2^2a_3 + a_3a_4^2 + a_3a_5^2 - a_3a_6^2 \\
\dot{a}_4 & = & -a_1^2a_4 + a_2^2a_4 - a_3^2a_4 + a_4a_6^2 \\
\dot{a}_5 & = & a_1^2a_5 + 2a_3a_4a_1 + a_2^2a_5 - 2a_4a_6a_2 - a_3^2a_5 - a_5a_6^2 \\
\dot{a}_6 & = & -a_1^2a_6 - 2a_2a_3a_1 + 2a_4a_5a_2 + a_3^2a_6 - a_4^2a_6 + a_5^2a_6
\end{array}$$

Example 36. For the root system of type A_4 the Lax pair corresponding to

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2 + \alpha_3\}$$

is given by the matrices

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & a_5 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & a_5 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & a_1a_2 & -a_1a_5 & 0 \\ 0 & 0 & -a_5a_3 & a_2a_3 & -a_5a_4 \\ -a_1a_2 & a_5a_3 & 0 & -a_2a_5 & a_3a_4 \\ a_1a_5 & -a_2a_3 & a_2a_5 & 0 & 0 \\ 0 & a_5a_4 & -a_3a_4 & 0 & 0 \end{pmatrix}.$$

Using the change of variables $x_i = 2a_i^2$ the corresponding Lotka-Volterra system becomes

$$\dot{x_1} = x_1x_2 - x_1x_5,$$

$$\dot{x_2} = -x_2x_5 + x_2x_3 - x_2x_1,$$

$$\dot{x_3} = x_3x_5 + x_3x_4 - x_3x_2,$$

$$\dot{x_4} = x_4x_5 - x_4x_3,$$

$$\dot{x_5} = -x_5x_4 - x_5x_3 + x_5x_1 + x_5x_2\,.$$

The associated Poisson matrix is of rank 4. The constants of motion are

$$\begin{aligned}
H &= x_1 + x_2 + x_3 + x_4 + x_5 \text{ (Hamiltonian)},\\
F &= x_1x_3 + x_1x_4 + x_2x_4,\\
C &= x_2x_3x_5 \text{ (Casimir)}\,.
\end{aligned}$$

Example 37. The Lax pair corresponding to the subset

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$$

of the positive roots of the root system of type A_4 is given by the matrices

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & a_5 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ a_5 & 0 & 0 & a_4 & 0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 0 & 0 & a_1a_2 & -a_4a_5 & 0 \\ 0 & 0 & 0 & a_2a_3 & -a_1a_5 \\ -a_1a_2 & 0 & 0 & 0 & a_3a_4 \\ a_4a_5 & -a_2a_3 & 0 & 0 & 0 \\ 0 & a_1a_5 & -a_3a_4 & 0 & 0 \end{pmatrix}.$$

Using the change of variables $x_i = 2a_i^2$ we obtain the periodic KM system

$$\begin{aligned}
\dot{x_1} &= x_1x_2 - x_1x_5,\\
\dot{x_2} &= x_2x_3 - x_1x_2,\\
\dot{x_3} &= x_3x_4 - x_2x_3,\\
\dot{x_4} &= x_4x_5 - x_3x_4,\\
\dot{x_5} &= x_1x_5 - x_4x_5\,.
\end{aligned}$$

The associated Poisson matrix is of rank 4. The traces of L^2 and L^4 together with the Casimir, $C = x_1x_2x_3x_4x_5$ ensure the integrability of the system.

Example 38. For the root system of type A_4 we obtain two isomorphic Lotka-Volterra systems corresponding to the subsets

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2 + \alpha_3\} \text{ and } \Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2 + \alpha_3 + \alpha_4\}.$$

The corresponding matrices are

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_5 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ a_5 & 0 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & a_1a_2 - a_3a_5 & 0 & -a_4a_5 \\ 0 & 0 & 0 & -a_1a_5 + a_2a_3 & 0 \\ -a_1a_2 + a_3a_5 & 0 & 0 & 0 & a_3a_4 \\ 0 & a_1a_5 - a_2a_3 & 0 & 0 & 0 \\ a_4a_5 & 0 & -a_3a_4 & 0 & 0 \end{pmatrix}$$

and

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & a_5 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ 0 & a_5 & 0 & a_4 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & a_1a_2 & 0 & -a_1a_5 \\ 0 & 0 & 0 & a_2a_3 - a_4a_5 & 0 \\ -a_1a_2 & 0 & 0 & 0 & -a_2a_5 + a_3a_4 \\ 0 & -a_2a_3 + a_4a_5 & 0 & 0 & 0 \\ a_1a_5 & 0 & a_2a_5 - a_3a_4 & 0 & 0 \end{pmatrix}.$$

We describe the corresponding systems and the isomorphism between them in the next section.

6.4 Subsets Φ corresponding to Lotka-Volterra systems

In the previous section we have presented several examples of cubic systems which (after a simple change of variables) are equivalent to Lotka-Volterra systems. In this section we classify all subsets Φ of the positive roots of A_n which produce, after a suitable change of variables, Lotka-Volterra systems. We prove the following theorem.

Theorem 13. *The only choices for the subset Φ of R^+ so that the corresponding gen-*

eralized Volterra system transforms into a Lotka-Volterra system, using the substitution $x_i = 2a_i^2$*, are the following five.*

1. $\Phi = \Pi$,
2. $\Phi = \Pi \cup \{\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}\}$,
3. $\Phi = \Pi \cup \{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\}$,
4. $\Phi = \Pi \cup \{\alpha_2 + \alpha_3 + \cdots + \alpha_n\}$,
5. $\Phi = \Pi \cup \{\alpha_1 + \alpha_2 + \cdots + \alpha_n\}$.

Case (1) gives rise to the KM system while case (5) gives rise to the periodic KM system.

Case (2) corresponds to the Lax equation $\dot{L} = [L, B]$ with L matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & & 0 & a_{n+1} & 0 \\ 0 & a_2 & 0 & a_3 & \ddots & & 0 & 0 \\ \vdots & 0 & a_3 & \ddots & \ddots & & & 0 \\ 0 & & \ddots & \ddots & 0 & a_{n-2} & 0 & \vdots \\ 0 & 0 & & & a_{n-2} & 0 & a_{n-1} & 0 \\ 0 & a_{n+1} & 0 & & 0 & a_{n-1} & 0 & a_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n & 0 \end{pmatrix}.$$

The skew-symmetric matrix B is defined using the method described in section (6.2) (see also the proof of proposition 29). Its upper triangular part is given by the formula

$$\sum_{i=1}^{n-1} a_i a_{i+1} X_{\alpha_i+\alpha_{i+1}} - a_{n-1}a_{n+1}X_{\alpha_{n+1}-\alpha_{n-1}} - a_2 a_{n+1} X_{\alpha_{n+1}-\alpha_2} -$$
$$a_1 a_{n+1} X_{\alpha_1+\alpha_{n+1}} - a_n a_{n+1} X_{\alpha_{n+1}+\alpha_n}.$$

After substituting $x_i = 2a_i^2$ for $i = 1, \ldots, n+1$, the Lax pair L, B becomes equivalent to the following equations of motion:

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 - x_{n+1}), \\ \dot{x}_2 &= x_2(x_3 - x_1 - x_{n+1}), \\ \dot{x}_i &= x_i(x_{i+1} - x_{i-1}), \qquad\qquad i = 3, 4, \ldots, n-2, n \\ \dot{x}_{n-1} &= x_{n-1}(x_n - x_{n-2} + x_{n+1}), \\ \dot{x}_{n+1} &= x_{n+1}(x_1 + x_2 - x_{n-1} - x_n). \end{aligned}$$

It is easily verified that for n even, the rank of the Poisson matrix is n and the function $f = x_2 x_3 \cdots x_{n-1} x_{n+1}$ is the Casimir of the system, while for n odd, the rank of the Poisson matrix is $n-1$ and the functions $f_1 = x_1 x_3 \cdots x_n = \sqrt{\det L}$ and $f_2 = x_2 x_3 \cdots x_{n-1} x_{n+1}$ are the Casimirs.

Case (3) corresponds to the Lax pair whose Lax matrix L is given by

$$
L = \begin{pmatrix}
0 & a_1 & 0 & \cdots & & 0 & a_{n+1} & 0 \\
a_1 & 0 & a_2 & 0 & & & 0 & 0 \\
0 & a_2 & 0 & a_3 & \ddots & & & 0 \\
\vdots & 0 & a_3 & \ddots & \ddots & & & \vdots \\
& & \ddots & \ddots & 0 & a_{n-2} & 0 & \\
0 & & & & a_{n-2} & 0 & a_{n-1} & 0 \\
a_{n+1} & 0 & & & 0 & a_{n-1} & 0 & a_n \\
0 & 0 & 0 & \cdots & & 0 & a_n & 0
\end{pmatrix}.
$$

The upper triangular part of the skew-symmetric matrix B is

$$
\sum_{i=1}^{n-1} a_i a_{i+1} X_{\alpha_i + \alpha_{i+1}} - a_{n-1} a_{n+1} X_{\alpha_{n+1} - \alpha_{n-1}} - a_1 a_{n+1} X_{\alpha_{n+1} - \alpha_1} - a_n a_{n+1} X_{\alpha_{n+1} + \alpha_n}.
$$

After substituting $x_i = 2a_i^2$ for $i = 1, \ldots, n+1$, we obtain the following equivalent equations of motion:

$$
\begin{aligned}
\dot{x}_1 &= x_1(x_2 - x_{n+1}) \\
\dot{x}_i &= x_i(x_{i+1} - x_{i-1}), \; i = 2, 3, 4, \ldots, n-2, n \\
\dot{x}_{n-1} &= x_{n-1}(x_n - x_{n-2} + x_{n+1}) \\
\dot{x}_{n+1} &= x_{n+1}(x_1 - x_n - x_{n-1}).
\end{aligned}
$$

For n even, the rank of the Poisson matrix is n and the function $f = x_1 x_2 \cdots x_{n-1} x_{n+1}$ is the Casimir, while for n odd, the rank of the Poisson matrix is $n-1$ and the functions $f_1 = x_1 x_3 x_5 \cdots x_n = \sqrt{\det L}$ and $f_2 = x_1 x_2 \cdots x_{n-1} x_{n+1}$ are Casimirs.

The system obtained in case (4) turns out to be isomorphic to the one in case (3). In fact, the change of variables $u_{n+1-i} = -x_i$ for $i = 1, 2, \ldots, n$ and $u_{n+1} = -x_{n+1}$ in case (3) gives the corresponding system of case (4).

Since subsystems of Lotka-Volterra systems are also Lotka- Volterra, in order to prove theorem 13 it is enough to consider the case where the subset Φ contains the simple roots and only one extra root. The following proposition shows that we have only four possible choices for the extra root in Φ which give rise to a Lotka-Volterra system. Therefore the proof of theorem 13 is a case by case verification of the 16 possible subsets Φ containing the simple roots and roots given in the following proposition.

Proposition 29. *Let $\Phi = \{\alpha_1, \dots, \alpha_{n+1}\}$ be the subset of the positive roots of the root system A_n containing the simple roots and the additional extra root α_{n+1}. Suppose that $\alpha_{n+1} = \alpha_k + \alpha_{k+1} + \dots + \alpha_m$ for some $1 \leq k < m \leq n$. Then the only possible choices of k, m that lead to a Lotka-Volterra system are*

$$(k,m) = (1,n), (1,n-1), (2,n) \text{ and } (2,n-1).$$

Proof. The matrix L is given by

$$L = \sum_{i=1}^{n} a_i \left(X_{\alpha_i} + X_{-\alpha_i}\right) + a_{n+1}\left(X_{\alpha_{n+1}} + X_{-\alpha_{n+1}}\right).$$

The matrix B is the skew-symmetric matrix constructed using the algorithm described in section 6.2, and its upper triangular part is

$$\begin{gathered}\sum_{1 \leq i \leq n-1} c_{i,i+1} a_i a_{i+1} \left[X_{\alpha_i}, X_{\alpha_{i+1}}\right] + \\ c_{k-1,n+1} a_{k-1} a_{n+1} \left[X_{\alpha_{k-1}}, X_{\alpha_{n+1}}\right] + c_{k,n+1} a_k a_{n+1} \left[X_{-\alpha_k}, X_{\alpha_{n+1}}\right] + \\ c_{m,n+1} a_m a_{n+1} \left[X_{-\alpha_m}, X_{\alpha_{n+1}}\right] + c_{m+1,n+1} a_{m+1} a_{n+1} \left[X_{\alpha_{m+1}}, X_{\alpha_{n+1}}\right].\end{gathered}$$

Note that this is the generic form of the matrix B. In some special cases B will be different; i.e. for $k = 1$ the term $c_{k-1,n+1} a_{k-1} a_{n+1} \left[X_{\alpha_{k-1}}, X_{\alpha_{n+1}}\right]$ will be missing and for $m = n$ the term $c_{m+1,n+1} a_{m+1} a_{n+1} \left[X_{\alpha_{m+1}}, X_{\alpha_{n+1}}\right]$ will be missing from B.

In order to determine the signs $c_{i,j}$ in a way such that the corresponding system, after the substitution $x_i = 2a_i^2$, is transformed into a Lotka-Volterra system we calculate the bracket $[L, B]$. Its upper triangular part is given by

$$\begin{gathered}\sum_{2 \leq i \leq n-1} c_{i,i+1} a_{i-1} a_i a_{i+1} \left[X_{\alpha_{i-1}}, \left[X_{\alpha_i}, X_{\alpha_{i+1}}\right]\right] + \\ \sum_{1 \leq i \leq n-2} c_{i,i+1} a_i a_{i+1} a_{i+2} \left[X_{\alpha_{i+2}}, \left[X_{\alpha_i}, X_{\alpha_{i+1}}\right]\right] + \\ \sum_{1 \leq i \leq n-1} c_{i,i+1} \left(a_i^2 a_{i+1} \left[X_{-\alpha_i}, \left[X_{\alpha_i}, X_{\alpha_{i+1}}\right]\right] + a_i a_{i+1}^2 \left[X_{-\alpha_{i+1}}, \left[X_{\alpha_i}, X_{\alpha_{i+1}}\right]\right]\right) + \\ c_{k-1,n+1} a_{k-1}^2 a_{n+1} \left[X_{-\alpha_{k-1}}, \left[X_{\alpha_{k-1}}, X_{\alpha_{n+1}}\right]\right] + \\ c_{k-1,n+1} a_{k-2} a_{k-1} a_{n+1} \left[X_{\alpha_{k-2}} \left[X_{\alpha_{k-1}}, X_{\alpha_{n+1}}\right]\right] + \\ c_{k-1,n+1} a_{k-1} a_m a_{n+1} \left[X_{-\alpha_m}, \left[X_{\alpha_{k-1}}, X_{\alpha_{n+1}}\right]\right] + \\ c_{k-1,n+1} a_{k-1} a_{m+1} a_{n+1} \left[X_{\alpha_{m+1}}, \left[X_{\alpha_{k-1}}, X_{\alpha_{n+1}}\right]\right] +\end{gathered}$$

$$
\begin{gathered}
c_{k-1,n+1}a_{k-1}a_{n+1}^2\left[X_{-\alpha_{n+1}},\left[X_{\alpha_{k-1}},X_{\alpha_{n+1}}\right]\right]+\\
c_{k,n+1}a_k^2a_{n+1}\left[X_{\alpha_k},\left[X_{-\alpha_k},X_{\alpha_{n+1}}\right]\right]+\\
c_{k,n+1}a_ka_{k+1}a_{n+1}\left[X_{-\alpha_{k+1}},\left[X_{-\alpha_k},X_{\alpha_{n+1}}\right]\right]+\\
c_{k,n+1}a_ka_ma_{n+1}\left[X_{-\alpha_m},\left[X_{-\alpha_k},X_{\alpha_{n+1}}\right]\right]+\\
c_{k,n+1}a_ka_{m+1}a_{n+1}\left[X_{\alpha_{m+1}},\left[X_{-\alpha_k},X_{\alpha_{n+1}}\right]\right]+\\
c_{k,n+1}a_ka_{n+1}^2\left[X_{\alpha_{n+1}},\left[X_{\alpha_k},X_{-\alpha_{n+1}}\right]\right]+\\
c_{m,n+1}a_{k-1}a_ma_{n+1}\left[X_{\alpha_{k-1}},\left[X_{-\alpha_m},X_{\alpha_{n+1}}\right]\right]+\\
c_{m,n+1}a_ka_ma_{n+1}\left[X_{-\alpha_k},\left[X_{-\alpha_m},X_{\alpha_{n+1}}\right]\right]+\\
c_{m,n+1}a_m^2a_{n+1}\left[X_{\alpha_m},\left[X_{-\alpha_m},X_{\alpha_{n+1}}\right]\right]+\\
c_{m,n+1}a_{m-1}a_ma_{n+1}\left[X_{-\alpha_{m-1}},\left[X_{-\alpha_m},X_{\alpha_{n+1}}\right]\right]+\\
c_{m,n+1}a_ma_{n+1}^2\left[X_{\alpha_{n+1}},\left[X_{\alpha_m},X_{-\alpha_{n+1}}\right]\right]+\\
c_{m+1,n+1}a_{k-1}a_{m+1}a_{n+1}\left[X_{\alpha_{k-1}},\left[X_{\alpha_{m+1}},X_{\alpha_{n+1}}\right]\right]+\\
c_{m+1,n+1}a_{m+1}a_{m+2}a_{n+1}\left[X_{\alpha_{m+2}},\left[X_{\alpha_{m+1}},X_{\alpha_{n+1}}\right]\right]+\\
c_{m+1,n+1}a_ka_{m+1}a_{n+1}\left[X_{-\alpha_k},\left[X_{\alpha_{m+1}},X_{\alpha_{n+1}}\right]\right]+\\
c_{m+1,n+1}a_{m+1}^2a_{n+1}\left[X_{-\alpha_{m+1}},\left[X_{\alpha_{m+1}},X_{\alpha_{n+1}}\right]\right]+\\
c_{m+1,n+1}a_{m+1}a_{n+1}^2\left[X_{-\alpha_{n+1}},\left[X_{\alpha_{m+1}},X_{\alpha_{n+1}}\right]\right]+\\
c_{k-2,k-1}a_{k-2}a_{k-1}a_{n+1}\left[X_{\alpha_{n+1}},\left[X_{\alpha_{k-2}},X_{\alpha_{k-1}}\right]\right]+\\
c_{k,k+1}a_ka_{k+1}a_{n+1}\left[X_{\alpha_{n+1}},\left[X_{-\alpha_k},X_{-\alpha_{k+1}}\right]\right]+\\
c_{m-1,m}a_{m-1}a_ma_{n+1}\left[X_{\alpha_{n+1}},\left[X_{-\alpha_{m-1}},X_{-\alpha_m}\right]\right]+\\
c_{m+1,m+2}a_{m+1}a_{m+2}a_{n+1}\left[X_{\alpha_{n+1}},\left[X_{\alpha_{m+1}},X_{\alpha_{m+2}}\right]\right].
\end{gathered}
$$

Note that as in the case of the matrix B, for some special cases of k and m, some extra terms in the bracket $[L,B]$ will be missing; e.g. for $k=2$ the terms corresponding to the positions $\left[X_{\alpha_{k-2}}\left[X_{\alpha_{k-1}},X_{\alpha_{n+1}}\right]\right]$ and $\left[X_{\alpha_{n+1}},\left[X_{\alpha_{k-2}},X_{\alpha_{k-1}}\right]\right]$ will be missing. In the general case the system will be transformed to a Lotka-Volterra system if the signs satisfy

$$
\begin{array}{lll}
c_{i,i+1}=c_{i+1,i+2}=c & i=1,2,\ldots,n-2, & \\
c_{m+1,n+1}=-c_{k-1,n+1}, & c_{k,n+1}=-c_{k,k+1}=-c, & c_{k-1,n+1}=-c_{m,n+1},\\
c_{m+1,n+1}=-c_{m+1,m+2}=-c, & c_{k,n+1}=-c_{m,n+1}, & c_{m,n+1}=c_{m-1,m}=c\\
c_{m+1,n+1}=-c_{k,n+1}, & c_{k-1,n+1}=c_{k-2,k-1}=c. &
\end{array}
$$

This linear system is solvable if and only if the equations $c_{m+1,n+1}=-c_{m+1,m+2}$ and $c_{k-1,n+1}=c_{k-2,k-1}$ are missing. Therefore the system is solvable if and only if $m\geq n-1$ and $k\leq 2$, and the result follows. $\square$

6.5 Two Lax pair techniques

In this section we present two techniques that we use to prove the integrability of the generalized Lotka-Volterra systems presented in the next section. The first one is due to Deift, Li, Nanda and Tomei (see [**?**]). It was used to establish the complete integrability of the full Kostant Toda lattice. The traces of powers of L were not enough to prove integrability, therefore the method of chopping was used to obtain additional integrals. First we describe the method: For $k = 0, \ldots, \lfloor \frac{n-1}{2} \rfloor$, denote by $(L - \lambda\, Id)_{(k)}$ the result of removing the first k rows and last k columns from $L - \lambda\, \mathrm{I}d$, and let

$$\det(L - \lambda\, \mathrm{Id})_{(k)} = E_{0k}\lambda^{n-2k} + \cdots + E_{n-2k,k}\ .$$

Set

$$\frac{\det(L - \lambda\, \mathrm{Id})_{(k)}}{E_{0k}} = \lambda^{n-2k} + I_{1k}\lambda^{n-2k-1} + \cdots + I_{n-2k,k}\ .$$

The functions I_{rk}, $r = 1, \ldots, n-2k$, are constants of motion for the full Kostant Toda lattice.

Example 39. We consider in detail the $gl(3, \mathbb{C})$ case of the full Toda. Let

$$L = \begin{pmatrix} f_1 & 1 & 0 \\ g_1 & f_2 & 1 \\ h_1 & g_2 & f_3 \end{pmatrix}\ ,$$

and take B to be the strictly lower part of L. The function $H_2 = \frac{1}{2}\operatorname{tr} L^2$ is the Hamiltonian, and using the standard Lie-Poisson bracket the equations

$$\dot{x} = \{H_2, x\}$$

are equivalent to

$$\begin{aligned}
\dot{f}_1 &= -g_1 \\
\dot{f}_2 &= g_1 - g_2 \\
\dot{f}_3 &= g_2 \\
\dot{g}_1 &= g_1(f_1 - f_2) - h_1 \\
\dot{g}_2 &= g_2(f_2 - f_3) + h_1 \\
\dot{h}_1 &= h_1(f_1 - f_3)\ .
\end{aligned}$$

Note that $H_1 = f_1 + f_2 + f_3$ while $H_2 = \frac{1}{2}(f_1^2 + f_2^2 + f_3^2) + g_1 + g_2$. The chopped matrix is given by

$$\begin{pmatrix} g_1 & f_2 - \lambda \\ h_1 & g_2 \end{pmatrix}\ .$$

The determinant of this matrix is $h_1\lambda + g_1g_2 - h_1f_2$ and one obtains the rational integral

$$I_{11} = \frac{g_1g_2 - h_1f_2}{h_1} \ .$$

Note that the phase space is six dimensional, we have two Casimirs H_1, I_{11} and the functions H_2, H_3 are enough to ensure integrability.

In the next example we use this technique to obtain the Casimir of a generalized Lotka-Volterra system.

Example 40. Consider the generalized Lotka-Volterra system defined by the Lax matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_5 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ a_5 & 0 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix}$$

which corresponds to the subset $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2 + \alpha_3\}$. According to proposition 29 a suitable choice of signs for the entries of B gives rise to a Lotka-Volterra system. However, there is a second choice of sings which results in a different system. Define the matrix B to be

$$\begin{pmatrix} 0 & 0 & a_1a_2 + a_3a_5 & 0 & a_4a_5 \\ 0 & 0 & 0 & a_2a_3 + a_1a_5 & 0 \\ -a_1a_2 - a_3a_5 & 0 & 0 & 0 & a_3a_4 \\ 0 & -a_2a_3 - a_1a_5 & 0 & 0 & 0 \\ -a_4a_5 & 0 & -a_3a_4 & 0 & 0 \end{pmatrix} .$$

In this case the Lax equation $\dot{L} = [B, L]$ corresponds to the following system

$$\begin{aligned} \dot{a_1} &= a_1a_2^2 + a_1a_5^2 + 2a_2a_3a_5 \\ \dot{a_2} &= -a_2a_1^2 + a_2a_3^2 \\ \dot{a_3} &= -a_3a_2^2 + a_3a_4^2 - a_3a_5^2 - 2a_1a_2a_5 \\ \dot{a_4} &= -a_4a_3^2 - a_4a_5^2 \\ \dot{a_5} &= -a_5a_1^2 + a_5a_3^2 + a_5a_4^2 \ . \end{aligned}$$

The Hamiltonian of the system is $H = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2\right)$ and the Poisson matrix (of rank 4) is

$$\begin{pmatrix} 0 & a_1a_2 & 2a_2a_5 & 0 & a_1a_5 \\ -a_1a_2 & 0 & a_2a_3 & 0 & 0 \\ -2a_2a_5 & -a_2a_3 & 0 & a_3a_4 & -a_3a_5 \\ 0 & 0 & -a_3a_4 & 0 & -a_4a_5 \\ -a_1a_5 & 0 & a_3a_5 & a_4a_5 & 0 \end{pmatrix}.$$

The system is integrable with constants of motion $H = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2\right)$ and

$$F = \frac{1}{2}a_1^4 + a_1^2a_5^2 + \frac{1}{2}a_5^4 + a_1^2a_2^2 + 2a_1a_5a_2a_3 + a_3^2a_5^2 + a_4^2a_5^2 + \frac{1}{2}a_2^4 + a_2^2a_3^2 + \frac{1}{2}a_3^4 + a_4^2a_3^2 + \frac{1}{2}a_4^4.$$

In fact F is equal to $\operatorname{tr}\left(\frac{L^4}{4}\right)$. The Casimir of the system is $C = a_2^2 - \dfrac{a_1a_2a_3}{a_5}$ and may be obtained by the method of chopping as follows. We have

$$x \cdot I_5 - L = \begin{pmatrix} x & -a_1 & 0 & -a_5 & 0 \\ -a_1 & x & -a_2 & 0 & 0 \\ 0 & -a_2 & x & -a_3 & 0 \\ -a_5 & 0 & -a_3 & x & -a_4 \\ 0 & 0 & 0 & -a_4 & x \end{pmatrix}$$

and the one-chopped matrix is

$$\begin{pmatrix} -a_1 & x & -a_2 & 0 \\ 0 & -a_2 & x & -a_3 \\ -a_5 & 0 & -a_3 & x \\ 0 & 0 & 0 & -a_4 \end{pmatrix}$$

with determinant $a_4a_5x^2 + a_1a_2a_3a_4 - a_2^2a_4a_5$. Dividing the constant term of this polynomial by the leading term a_4a_5 we obtain the Casimir C.

The second method that we use is an old recipe of Moser. Moser in [?] describes a relation between the KM system and the non–periodic Toda lattice. The procedure is the following.

Form L^2 which is not anymore a tridiagonal matrix but is similar to one. Let $\hat{e} = \{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Also let $E_o = span\{e_{2i-1},\ i = 1, 2, \ldots\}$ and $E_e = span\{e_{2i},\ i = 1, 2, \ldots\}$. Then L^2 leaves E_o and E_e invariant and reduces in each of these spaces to a tridiagonal symmetric Jacobi matrix. For example, if we omit all even columns and all even rows we obtain a tridiagonal Jacobi matrix and the entries of this new matrix define the transformation from the KM system to the Toda lattice. We illustrate with a simple example where $n = 5$.

We use the symmetric version of the KM system Lax pair given by

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix} .$$

It is simple to calculate that L^2 is the matrix

$$\begin{pmatrix} a_1^2 & 0 & a_1 a_2 & 0 & 0 \\ 0 & a_1^2 + a_2^2 & 0 & a_2 a_3 & 0 \\ a_1 a_2 & 0 & a_2^2 + a_3^2 & 0 & a_3 a_4 \\ 0 & a_2 a_3 & 0 & a_3^2 + a_4^2 & 0 \\ 0 & 0 & a_3 a_4 & 0 & a_4^2 \end{pmatrix} .$$

Omitting even columns and even rows of L^2 we obtain the matrix

$$\begin{pmatrix} a_1^2 & a_1 a_2 & 0 \\ a_1 a_2 & a_2^2 + a_3^2 & a_3 a_4 \\ 0 & a_3 a_4 & a_4^2 \end{pmatrix} .$$

This is a tridiagonal Jacobi matrix. It is natural to define new variables $A_1 = a_1 a_2$, $A_2 = a_3 a_4$, $B_1 = a_1^2$, $B_2 = a_2^2 + a_3^2$, $B_3 = a_4^2$. The new variables A_1, A_2, B_1, B_2, B_3 satisfy the Toda lattice equations.

This procedure shows that the KM-system and the Toda lattice are closely related. The explicit transformation which is due to Hénon maps one system to the other. The mapping in the general case is given by

$$A_i = -\frac{1}{2}\sqrt{a_{2i} a_{2i-1}} \, , \qquad B_i = \frac{1}{2}\left(a_{2i-1} + a_{2i-2}\right) \, .$$

The equations satisfied by the new variables A_i, B_i are given by:

$$\begin{aligned} \dot{A}_i &= A_i \left(B_{i+1} - B_i\right) \\ \dot{B}_i &= 2\left(A_i^2 - A_{i-1}^2\right) . \end{aligned}$$

These are precisely the Toda equations in Flaschka's form.

This idea of Moser was applied with success to establish transformations from the generalized Volterra lattices of Bogoyavlensky [?, ?] to generalized Toda systems. The relation between the Volterra systems of type B_n and C_n and the corresponding Toda systems is in [?]. The similar construction of the Volterra lattice of type D_n and the generalized Toda lattice of type D_n is in [?]. We use this method in the next section to obtain a missing integral for some generalized Lotka-Volterra systems.

6.6 2-diagonal systems

In this section we define an infinite family of systems with a cubic Hamiltonian vector field. We present each such system in Lax pair form $\dot{L} = [B, L]$ which allows us to obtain a large family of first integrals, $H_i = \operatorname{tr}(L^i)$. Additional integrals are obtained by the method of Moser described in the previous section. In the examples we present, these integrals are enough to ensure the Liouville integrability of the systems. We believe that all these systems are Liouville integrable.

We begin with the definition of the matrices L and B. For convenience we let d_i denote the i^{th} diagonal starting from the upper right corner and moving towards the main diagonal. We take L to be an $n \times n$ symmetric matrix with the only non-zero entries on two diagonals d_m and d_{n-1} where $n \geqslant 2m$ and $m \geqslant 2$. Note that for $m = 1$ we obtain the periodic KM system.

The matrix L is given by

$$L = \begin{pmatrix}
0 & a_1 & 0 & \cdots & 0 & a_n & 0 & \cdots & 0 \\
a_1 & 0 & a_2 & 0 & & 0 & a_{n+1} & \ddots & \vdots \\
0 & a_2 & 0 & a_3 & \ddots & & \ddots & \ddots & 0 \\
\vdots & 0 & a_3 & \ddots & \ddots & & & 0 & a_{n+m-1} \\
0 & & & \ddots & & & & & 0 \\
a_n & 0 & & & & & a_{n-2} & 0 & \vdots \\
0 & a_{n+1} & \ddots & & & a_{n-2} & 0 & a_{n-2} & 0 \\
\vdots & \ddots & \ddots & 0 & & 0 & a_{n-2} & 0 & a_{n-1} \\
0 & \cdots & 0 & a_{n+m-1} & 0 & \cdots & 0 & a_{n-1} & 0
\end{pmatrix}.$$

That is, L is a symmetric $n \times n$ matrix whose non-zero upper diagonals are:

$$\begin{aligned}
d_{n-1} &= (a_1, a_2, \ldots, a_{n-1}) \\
d_m &= (a_n, a_{n+1}, \ldots, a_{n+m-1})
\end{aligned}$$

To put it in the terminology of section (6.2) this matrix has variables in the positions corresponding to the simple roots and also at the positive roots of height $n-m$, i.e.

$$L=\sum_{\alpha_i\in\Phi} a_i(X_{\alpha_i}+X_{-\alpha_i}),$$

where

$$\Phi=\{\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\alpha_1+\alpha_2+\ldots+\alpha_{n-m},\ldots,\alpha_m+\alpha_{m+1}+\ldots+\alpha_{n-1}\}.$$

By considering the set $\Psi=\{\alpha+\beta \mid \alpha,\beta\in\Phi\cup\Phi^-,\alpha+\beta\in R^+\}$ we define B to be the matrix

$$B=\sum c_{ij}a_ia_j\left(X_{\alpha_i+\alpha_j}+X_{-\alpha_i-\alpha_j}\right), \tag{6.4}$$

where the non-zero terms are taken over all $\alpha_i+\alpha_j\in\Psi$ with $\alpha_i,\alpha_j\in\Phi\cup\Phi^-$ and $c_{ij}=\pm1$. The following proposition shows that there is a choice of the signs $c_{i,j}$ that leads to a consistent Lax pair.

Proposition 30. *Let Φ be the subset of the positive roots of a root system of type A_{n-1} containing the simple roots and the roots of height $n-m$ where $n\geq 2m$ and $m\geq 2$. The skew-symmetric matrix B constructed using the algorithm of section 6.2 has nonzero variables in the positions corresponding to the root vectors $X_{\pm\alpha}$ where α runs through the positive roots of height 2, $n-m-1$ and $n-m+1$. The following choice of the signs $c_{i,j}$ gives a consistent Lax pair.*

$$\begin{aligned}
c_{i,i+1}&=1,\ i=1,2,\ldots,n-1,\\
c_{n-m+i,n+i}=c_{i,n+i-1}&=-1,\ i=1,2,\ldots,m,\\
c_{n-m+i,n+i-1}=c_{i,n+i}&=1,\ i=1,2,\ldots,m-1.
\end{aligned}$$

Proof. We form the subset K of the positive roots as in the proof of proposition 28

$$K=\{\alpha+\beta+\gamma:\alpha,\beta,\gamma\in\Phi\cup\Phi^-,\ \alpha+\beta+\gamma\in R^+\}.$$

In order to have a consistent Lax pair we must choose the signs $c_{i,j}$ in a way such that the variables in $[L,B]$ corresponding to the roots in $K\setminus\Phi$ vanish. The set $K\setminus\Phi$ contains the roots of height 3, $n-m-2$ and $n-m+2$. Each corresponding root vector is formed only in two ways as $[X_\alpha,[X_\beta,X_\gamma]]$ and $[X_\gamma,[X_\alpha,X_\beta]]$. From the Jacobi identity we conclude that, in order for the corresponding variables to vanish, the coefficients of these two root vectors must be the same. For example the root vectors corresponding to roots of height 3 are formed as $\left[X_{\alpha_i},\left[X_{\alpha_{i+1}},X_{\alpha_{i+2}}\right]\right]$ and $\left[X_{\alpha_{i+2}},\left[X_{\alpha_i},X_{\alpha_{i+1}}\right]\right]$ for $i=1,2,\ldots,n-3$ and we obtain the conditions $c_{i,i+1}=1$ for all $i=1,2,\ldots,n-2$. The root vectors corresponding to roots of height $n-m-2$ are formed as $\left[X_{-\alpha_{i+1}},\left[X_{-\alpha_i},X_{\alpha_{n+i-1}}\right]\right]$ and

$\left[X_{\alpha_{n+i-1}}, \left[X_{-\alpha_i}, X_{-\alpha_{i+1}}\right]\right]$ for $i = 1, 2, \ldots, m$. Note that here the root $\alpha_{n+i-1} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{n-m+i-1}$. We obtain the conditions $-c_{i,i+1} = c_{i,n+i-1}$ for all $i = 1, 2, \ldots, m$. Similar formulas for the roots of height $n-m-2$ and $n-m+2$ give our result. $\square$

Therefore the matrix B is the $n \times n$ skew-symmetric matrix with non-zero upper diagonals:

$$\begin{aligned}
d_{n-2} &= (a_1a_2, a_2a_3, \ldots, a_{n-2}a_{n-1}),\\
d_{m+1} &= (-a_{n-m}a_n, -a_{n-m+1}a_{n+1} - a_1a_n, \ldots, -a_{n-1}a_{n+m-1} - a_{m-1}a_{n+m-2}, -a_ma_{n+m-1}),\\
d_{m-1} &= (a_{n-m+1}a_n + a_1a_{n+1}, a_{n-m+2}a_{n+1} + a_2a_{n+2}, \ldots, a_{n-1}a_{n+m-2} + a_{m-1}a_{n+m-1}).
\end{aligned} \tag{6.5}$$

The Poisson bracket $\{\,,\}$ is determined by the $N \times N$ Poisson matrix $\pi = q - q^t$, where $N = n + m - 1$, and the non-zero entries of q are given by:

$$\begin{array}{lcll}
q_{i,i+n} & = & a_ia_{i+n} & \text{for } 1 \leqslant i \leqslant m-1,\\
q_{i,i+n-1} & = & -a_ia_{i+n-1} & \text{for } 1 \leqslant i \leqslant m,\\
q_{i+n-m-1,i+n-1} & = & a_{i+n-1}a_{i+n-m-1} & \text{for } 1 \leqslant i \leqslant m,\\
q_{i+n-m,i+n-1} & = & -a_{i+n-1}a_{i+n-m} & \text{for } 1 \leqslant i \leqslant m-1,\\
q_{i,i+1} & = & a_ia_{i+1} & \text{for } 1 \leqslant i \leqslant n-2,\\
q_{i+n-1,i+n} & = & 2a_ia_{i+n-m} & \text{for } 1 \leqslant i \leqslant m-1\,.
\end{array} \tag{6.6}$$

6.6.1 Special case with two diagonals, $m = 2$

In this subsection we consider the case where $m = 2$. The matrix L is defined by

$$L = \begin{pmatrix}
0 & a_1 & 0 & \cdots & 0 & a_n & 0\\
a_1 & 0 & a_2 & \ddots & & 0 & a_{n+1}\\
0 & a_2 & 0 & \ddots & & & 0\\
\vdots & \ddots & \ddots & \ddots & & 0 & \vdots\\
0 & & & & & a_{n-2} & 0\\
a_n & 0 & & 0 & a_{n-2} & 0 & a_{n-1}\\
0 & a_{n+1} & 0 & \cdots & 0 & a_{n-1} & 0
\end{pmatrix}$$

and corresponds to the subset Φ of the positive roots containing the simple roots and the roots of length $n-2$. The matrix B is defined by equation (6.4) and its upper triangular part is

$$\begin{pmatrix}
0 & 0 & a_1a_2 & 0 & \cdots & 0 & -a_{n-2}a_n & 0 & a_1a_{n+1}+a_{n-1}a_n \\
0 & 0 & 0 & a_2a_3 & & & 0 & -a_1a_n-a_{n-1}a_{n+1} & 0 \\
\vdots & \ddots & 0 & 0 & \ddots & & & 0 & -a_2a_{n+1} \\
 & & & \ddots & \ddots & & & & 0 \\
 & & & & & & \ddots & 0 & \vdots \\
 & & & & & & 0 & a_{n-3}a_{n-2} & 0 \\
 & & & & & & 0 & 0 & a_{n-2}a_{n-1} \\
\vdots & & & & & & \ddots & 0 & 0 \\
0 & \cdots & & & & & \cdots & 0 & 0
\end{pmatrix},$$

The Lax equation $\dot{L} = [B, L]$ is equivalent to the following system:

$$\begin{aligned}
\dot{a}_1 &= a_1a_2^2 + a_1a_{n+1}^2 - a_1a_n^2, \\
\dot{a}_2 &= a_2a_3^2 - a_1^2a_2 - a_2a_{n+1}^2, \\
\vdots & \qquad\qquad \vdots \\
\dot{a}_i &= a_ia_{i+1}^2 - a_{i-1}^2a_i, \qquad\qquad i = 3, 4, \ldots, n-3 \\
\vdots & \qquad\qquad \vdots \\
\dot{a}_{n-2} &= a_{n-2}a_n^2 - a_{n-3}^2a_{n-2} + a_{n-2}a_{n-1}^2, \\
\dot{a}_{n-1} &= a_{n-1}a_{n+1}^2 - a_{n-2}^2a_{n-1} - a_{n-1}a_n^2, \\
\dot{a}_n &= a_1^2a_n + a_{n-1}^2a_n - a_{n-2}^2a_n + 2a_1a_{n-1}a_{n+1}, \\
\dot{a}_{n+1} &= a_2^2a_{n+1} - a_1^2a_{n+1} - a_{n+1}a_{n-1}^2 - 2a_1a_{n-1}a_n\,.
\end{aligned}$$

The Poisson matrix π is defined by equations (6.6) and its upper triangular part is

$$\begin{pmatrix}
0 & a_1a_2 & 0 & \cdots & & & 0 & -a_1a_n & a_1a_{n+1} \\
\vdots & 0 & a_2a_3 & 0 & & & & 0 & -a_2a_{n+1} \\
 & & 0 & a_3a_4 & & & & & 0 \\
 & & & \ddots & & \ddots & 0 & & \\
 & & & & \ddots & \ddots & 0 & 0 & \vdots \\
 & & & & & 0 & a_{n-2}a_{n-1} & a_{n-2}a_n & 0 \\
 & & & & & & 0 & -a_{n-1}a_n & a_{n-1}a_{n+1} \\
\vdots & & & & & & & 0 & 2a_1a_{n-1} \\
0 & \cdots & & & & & & \cdots & 0
\end{pmatrix}.$$

For n even, the system has $n+1$ variables and the Poisson matrix has rank n and thus the Poisson structure has one Casimir. The traces of L give $\frac{n}{2}$ functionally independent first integrals in involution. Hence the system is integrable in the sense of Liouville. The Casimir is

$$C = \det L = (a_3a_5 \dots a_{n-5}a_{n-3}a_na_{n+1} - a_1a_3 \dots a_{n-3}a_{n-1})^2.$$

For n odd, the system has $n+1$ variables and the Poisson matrix has rank $n+1$. Therefore the Poisson structure is non-degenerate with no Casimirs. The traces $\operatorname{tr}(L^i)$ give only $\frac{n+1}{2} - 1$ functionally independent first integrals in involution. For the integrability of the system we need one more constant of motion which we obtain using the procedure of Moser described in section (6.5).

We give two examples for $n = 7, n = 9$.

Example 41. Consider the following matrices

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & a_7 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & a_8 \\ 0 & a_2 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & a_5 & 0 \\ a_7 & 0 & 0 & 0 & a_5 & 0 & a_6 \\ 0 & a_8 & 0 & 0 & 0 & a_6 & 0 \end{pmatrix}$$

and

$$\Lambda_o(L^2) = \begin{pmatrix} a_1^2 + a_2^2 + a_8^2 & a_2a_3 & a_1a_7 + a_6a_8 \\ a_2a_3 & a_3^2 + a_4^2 & a_4a_5 \\ a_1a_7 + a_6a_8 & a_4a_5 & a_5^2 + a_6^2 + a_7^2 \end{pmatrix} ,$$

where $\Lambda_o(L^2)$ denotes the matrix obtained from L^2 by omitting all odd rows and columns. We define a new set of variables $A_1 = a_2a_3, A_2 = a_4a_5, A_3 = a_1a_7 + a_6a_8, B_1 = a_1^2 + a_2^2 + a_8^2, B_2 = a_3^2 + a_4^2$ and $B_3 = a_5^2 + a_6^2 + a_7^2$. These variables satisfy the periodic Toda equations which are equivalent to the Lax equation $\dot{\Lambda}_o(L^2) = [C, \Lambda_o(L^2)]$ with

$$\Lambda_o(L^2) = \begin{pmatrix} B_1 & A_1 & A_3 \\ A_1 & B_2 & A_2 \\ A_3 & A_2 & B_3 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & A_1 & -A_3 \\ -A_1 & 0 & A_2 \\ A_3 & -A_2 & 0 \end{pmatrix}.$$

This system has two Casimirs $B_1 + B_2 + B_3$ and $A_1A_2A_3$. The Casimir $B_1 + B_2 + B_3$ expressed as a function of the original variables gives the Hamiltonian while the Casimir

$A_1A_2A_3$ gives the extra integral

$$A_1A_2A_3 = a_2a_3a_4a_5\,(a_1a_7 + a_6a_8)\,.$$

We could also obtain this integral from the system $\dot{\Lambda}_e(L^2) = [C, \Lambda_e(L^2)]$ where $\Lambda_e\,(L^2)$ denotes the matrix obtained from L^2 by omitting all even rows and columns.

$$\Lambda_e(L^2) = \begin{pmatrix} a_1^2 + a_7^2 & a_1a_2 & a_5a_7 & a_1a_8 + a_6a_7 \\ a_1a_2 & a_2^2 + a_3^2 & a_3a_4 & a_2a_8 \\ a_5a_7 & a_3a_4 & a_4^2 + a_5^2 & a_5a_6 \\ a_1a_8 + a_7a_6 & a_2a_8 & a_5a_6 & a_6^2 + a_8^2 \end{pmatrix} = \begin{pmatrix} B_1 & A_1 & A_4 & A_6 \\ A_1 & B_2 & A_2 & A_5 \\ A_4 & A_2 & B_3 & A_3 \\ A_6 & A_5 & A_3 & B_4 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & A_1 & -A_4 & A_6 \\ -A_1 & 0 & A_2 & -A_5 \\ A_4 & -A_2 & 0 & A_3 \\ -A_6 & A_5 & -A_3 & 0 \end{pmatrix}.$$

This system is not the full symmetric Toda lattice of Deift, Li, Nanda and Tomei [**?**]. Although the L matrix is the same, the C matrix is different. This system has two polynomial Casimirs, $B_1 + B_2 + B_3 + B_4$ and $A_1A_2A_4 + A_2A_3A_5$, with

$$A_1A_2A_4 + A_2A_3A_5 = a_2a_3a_4a_5\,(a_1a_7 + a_6a_8)\,.$$

Example 42. We take L to be

$$\begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & a_{10} \\ 0 & a_2 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5 & 0 & a_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & 0 & a_7 & 0 \\ a_9 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 & a_8 \\ 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_8 & 0 \end{pmatrix}.$$

The matrix

$$\Lambda_o(L^2) = \begin{pmatrix} a_1^2+a_2^2+a_{10}^2 & a_2a_3 & 0 & a_1a_9+a_8a_{10} \\ a_2a_3 & a_3^2+a_4^2 & a_4a_5 & 0 \\ 0 & a_4a_5 & a_5^2+a_6^2 & a_6a_7 \\ a_1a_9+a_8a_{10} & 0 & a_6a_7 & a_7^2+a_8^2+a_9^2 \end{pmatrix} =$$
$$\begin{pmatrix} B_1 & A_1 & 0 & A_4 \\ A_1 & B_2 & A_2 & 0 \\ 0 & A_2 & B_3 & A_3 \\ A_4 & 0 & A_3 & B_4 \end{pmatrix}$$

produces the periodic-Toda lattice which can be written in Lax pair form $\dot{\Lambda}_o(L^2) = [C, \Lambda_o(L^2)]$ with

$$C = \begin{pmatrix} 0 & A_1 & 0 & -A_4 \\ -A_1 & 0 & A_2 & 0 \\ 0 & -A_2 & 0 & A_3 \\ A_4 & 0 & -A_3 & 0 \end{pmatrix}.$$

This system also has two polynomial Casimirs $B_1+B_2+B_3+B_4$ and $A_1A_2A_3A_4$. By writing the latter one in the original variables we obtain the extra integral, namely

$$A_1A_2A_3A_4 = a_2a_3a_4a_5a_6a_7\left(a_1a_9+a_{10}a_8\right).$$

The intermediate Toda system $\dot{\Lambda}_e(L^2) = [C, \Lambda_e(L^2)]$ with

$$\Lambda_e(L^2) = \begin{pmatrix} B_1 & A_1 & 0 & A_5 & A_7 \\ A_1 & B_2 & A_2 & 0 & A_6 \\ 0 & A_2 & B_3 & A_3 & 0 \\ A_5 & 0 & A_3 & B_4 & A_4 \\ A_7 & A_6 & 0 & A_4 & B_5 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0 & A_1 & 0 & -A_5 & A_7 \\ -A_1 & 0 & A_2 & 0 & -A_6 \\ 0 & -A_2 & 0 & A_3 & 0 \\ A_5 & 0 & -A_3 & 0 & A_4 \\ -A_7 & A_6 & 0 & -A_4 & 0 \end{pmatrix}.$$

has two Casimirs $B_1+B_2+B_3+B_4+B_5$ and $A_1A_2A_3A_5+A_2A_3A_4A_6$. The second Casimir gives the extra constant of motion.

$$A_1A_2A_3A_5+A_2A_3A_4A_6 = a_2a_3a_4a_5a_6a_7\left(a_1a_9+a_8a_{10}\right).$$

Note that this intermediate Toda system is not of the type considered in [?].

6.6.2 Special case with two diagonals, $m = 3$

In this subsection we consider the case where $m = 3$.

The matrix L is given by

$$L = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & a_n & 0 & 0 \\ a_1 & 0 & a_2 & \ddots & & 0 & a_{n+1} & 0 \\ 0 & a_2 & 0 & \ddots & & & 0 & a_{n+2} \\ \vdots & \ddots & \ddots & \ddots & & & & 0 \\ 0 & \ddots & & & & \ddots & \ddots & \vdots \\ a_n & 0 & & & \ddots & \ddots & a_{n-2} & 0 \\ 0 & a_{n+1} & 0 & & \ddots & a_{n-2} & 0 & a_{n-1} \\ 0 & 0 & a_{n+2} & 0 & \cdots & 0 & a_{n-1} & 0 \end{pmatrix}$$

That is L is a symmetric $n \times n$ matrix whose non-zero upper diagonals are:

$$\begin{aligned} d_{n-1} &= (a_1, a_2, a_3, a_4, \dots, a_{n-1}), \\ d_3 &= (a_n, a_{n+1}, a_{n+2}). \end{aligned}$$

It corresponds to the subset Φ of the positive roots of the root system of type A_{n-1} containing the simple roots and the roots of length $n-3$. The matrix B constructed using the procedure described in section 6.2 is the $n \times n$ skew-symmetric matrix whose non-zero upper diagonals are:

$$\begin{aligned} d_{n-2} &= (a_1 a_2, a_2 a_3, a_3 a_4, \dots, a_{n-2} a_{n-1}), \\ d_4 &= (-a_{n-3} a_n, -a_{n-2} a_{n+1} - a_1 a_n, -a_{n-1} a_{n+2} - a_2 a_{n+1}, -a_3 a_{n+2}), \\ d_2 &= (a_{n-2} a_n + a_1 a_{n+1}, a_{n-1} a_{n+1} + a_2 a_{n+2}). \end{aligned}$$

These systems are Hamiltonian systems with a Poisson matrix determined by equations (6.6). For n even, the Poisson structure has two Casimirs and the traces of L^i together with an extra constant of motion obtained by Moser's technique give the integrability of the system. For n odd the system has one Casimir and the traces of the L^i give enough first integrals to ensure the integrability of the system. We illustrate this with two examples for $n = 7$ and $n = 8$.

Example 43. For $n = 7$ the matrix L is given by

$$\begin{pmatrix} 0 & a_1 & 0 & 0 & a_7 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & a_8 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & 0 & a_9 \\ 0 & 0 & a_3 & 0 & a_4 & 0 & 0 \\ a_7 & 0 & 0 & a_4 & 0 & a_5 & 0 \\ 0 & a_8 & 0 & 0 & a_5 & 0 & a_6 \\ 0 & 0 & a_9 & 0 & 0 & a_6 & 0 \end{pmatrix}.$$

The Casimir for the corresponding Poisson bracket is given by

$$\det L = -2a_1a_3a_4a_6(a_1a_5a_9 + a_2a_6a_7 - a_7a_8a_9).$$

Note that the constants of motion $H_i = \operatorname{tr} L^i$ for $i = 4, 5, 6$, together with the Hamiltonian $H_2 = \frac{1}{2}\left(a_1^2 + a_2^2 + \cdots + a_9^2\right)$ are functionally independent and in involution. Therefore the system is integrable.

Example 44. For n=8 the matrix L is given by

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & a_8 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & a_9 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & a_3 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & a_5 & 0 & 0 \\ a_8 & 0 & 0 & 0 & a_5 & 0 & a_6 & 0 \\ 0 & a_9 & 0 & 0 & 0 & a_6 & 0 & a_7 \\ 0 & 0 & a_{10} & 0 & 0 & 0 & a_7 & 0 \end{pmatrix}$$

and the matrix B is determined by the relations (6.5). It defines a Hamiltonian system with Poisson structure determined by the Poisson matrix

$$\begin{pmatrix} 0 & a_1a_2 & 0 & 0 & 0 & 0 & 0 & -a_1a_8 & a_1a_9 & 0 \\ -a_1a_2 & 0 & a_2a_3 & 0 & 0 & 0 & 0 & 0 & -a_2a_9 & a_2a_{10} \\ 0 & -a_2a_3 & 0 & a_3a_4 & 0 & 0 & 0 & 0 & 0 & -a_3a_{10} \\ 0 & 0 & -a_3a_4 & 0 & a_4a_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_4a_5 & 0 & a_5a_6 & 0 & a_5a_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_5a_6 & 0 & a_6a_7 & -a_6a_8 & a_6a_9 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_6a_7 & 0 & 0 & -a_7a_9 & a_7a_{10} \\ a_1a_8 & 0 & 0 & 0 & -a_5a_8 & a_6a_8 & 0 & 0 & 2a_1a_6 & 0 \\ -a_1a_9 & a_2a_9 & 0 & 0 & 0 & -a_6a_9 & a_7a_9 & -2a_1a_6 & 0 & 2a_2a_7 \\ 0 & -a_2a_{10} & a_3a_{10} & 0 & 0 & 0 & -a_7a_{10} & 0 & -2a_2a_7 & 0 \end{pmatrix},$$

which has rank 8. The Hamiltonian of the system is $H_2 = \frac{1}{2}\left(a_1^2 + a_2^2 + \cdots + a_{10}^2\right)$. A constant of motion is obtained using Moser's technique.
If we delete the odd numbered rows and columns of L^2 we obtain the matrix

$$\Lambda_o(L^2) = \begin{pmatrix} a_1^2 + a_2^2 + a_9^2 & a_2a_3 & a_1a_8 + a_6a_9 & a_7a_9 + a_2a_{10} \\ a_2a_3 & a_3^2 + a_4^2 & a_4a_5 & a_3a_{10} \\ a_1a_8 + a_6a_9 & a_4a_5 & a_5^2 + a_6^2 + a_8^2 & a_6a_7 \\ a_7a_9 + a_2a_{10} & a_3a_{10} & a_6a_7 & a_7^2 + a_{10}^2 \end{pmatrix} = \begin{pmatrix} B_1 & A_1 & A_4 & A_6 \\ A_1 & B_2 & A_2 & A_5 \\ A_4 & A_2 & B_3 & A_3 \\ A_6 & A_5 & A_3 & B_4 \end{pmatrix}.$$

We have

$$\begin{aligned} \dot{A}_1 &= (\dot{a_2a_3}) \\ &= \dot{a}_2a_3 + a_2\dot{a}_3 = (a_2a_3^2 + a_2a_{10}^2 - a_1^2a_2 - a_2a_9^2)a_3 + a_2(a_3a_4^2 - a_2^2a_3 - a_3a_{10}^2) \\ &= a_2a_3(a_3^2 + a_4^2 - a_1^2 - a_2^2 - a_9^2) \\ &= A_1(B_2 - B_1) \end{aligned}$$

and similarly the new variables B_i, A_i satisfy the system

$$\begin{aligned} \dot{B}_1 &= 2(A_1^2 + A_6^2 - A_4^2)\,, \dot{B}_2 = 2(A_2^2 - A_1^2 - A_5^2)\,, \\ \dot{B}_3 &= 2(A_3^2 + A_4^2 - A_2^2)\,, \dot{B}_4 = 2(A_5^2 - A_3^2 - A_6^2)\,, \\ \dot{A}_1 &= A_1(B_2 - B_1)\,, \dot{A}_2 = A_2(B_3 - B_2)\,, \\ \dot{A}_3 &= A_3(B_4 - B_3)\,, \dot{A}_4 = A_4(B_1 - B_3) + 2A_3A_6\,, \\ \dot{A}_5 &= A_5(B_2 - B_4) - 2A_1A_6\,, \dot{A}_6 = A_6(B_4 - B_1) - 2A_3A_4 + 2A_1A_5\,. \end{aligned} \tag{6.7}$$

This system can be written in Lax pair form $\Lambda_o(\dot{L^2}) = [C, \Lambda_o(L^2)]$ with

$$C = \begin{pmatrix} 0 & A_1 & -A_4 & A_6 \\ -A_1 & 0 & A_2 & -A_5 \\ A_4 & -A_2 & 0 & A_3 \\ -A_6 & A_5 & -A_3 & 0 \end{pmatrix}.$$

It is Hamiltonian with Hamiltonian function

$$H = \operatorname{tr}\left(\frac{\Lambda_o\left(L^2\right)^2}{2}\right) = \frac{1}{2}(B_1^2 + B_2^2 + B_3^2 + B_4^2) + A_1^2 + A_2^2 + A_3^2 + A_4^2 + A_5^2 + A_6^2$$

and Poisson matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 & A_1 & 0 & 0 & -A_4 & 0 & A_6 \\
0 & 0 & 0 & 0 & -A_1 & A_2 & 0 & 0 & -A_5 & 0 \\
0 & 0 & 0 & 0 & 0 & -A_2 & A_3 & A_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -A_3 & 0 & A_5 & -A_6 \\
-A_1 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -A_2 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -A_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_4 & 0 & -A_4 & 0 & 0 & 0 & 0 & 0 & 0 & A_3 \\
0 & A_5 & 0 & -A_5 & 0 & 0 & 0 & 0 & 0 & -A_1 \\
-A_6 & 0 & 0 & A_6 & 0 & 0 & 0 & -A_3 & A_1 & 0
\end{pmatrix}.$$

It has 2 Casimir functions $B_1 + B_2 + B_3 + B_4$ and $A_1A_2A_4 + A_2A_3A_5$. The function

$$F = A_1A_2A_4 + A_2A_3A_5 = a_2a_3a_4a_5\,(a_1a_8 + a_6a_9) + a_3a_4a_5a_6a_7a_{10} =$$
$$a_1a_2a_3a_4a_5a_8 + a_2a_3a_4a_5a_6a_9 + a_3a_4a_5a_6a_7a_{10}$$

is a constant of motion for the original system. The integrals H_2, H_4, H_6, F together with the two Casimirs given by

$$\begin{aligned}
C_1 &= a_1a_3a_5a_7, \\
C_2 &= \sqrt{\det L} - C_1 = a_1a_4a_6a_{10} + a_2a_4a_7a_8 - a_4a_8a_9a_{10}
\end{aligned}$$

ensure the integrability of the original system.

In general for n even Moser's technique gives the following additional constant of motion.

n	$F = a_2a_3\ldots a_{n-3}(a_1a_n + a_{n-2}a_{n+1}) + a_3a_4\ldots a_{n-1}a_{n+2}$
6	$a_1a_2a_3a_6 + a_2a_3a_4a_7 + a_3a_4a_5a_8$
8	$a_1a_2a_3a_4a_5a_8 + a_2a_3a_4a_5a_6a_9 + a_3a_4a_5a_6a_7a_{10}$
10	$a_1a_2a_3a_4a_5a_6a_7a_{10} + a_2a_3a_4a_5a_6a_7a_8a_{11} + a_3a_4a_5a_6a_7a_8a_9a_{12}$

Table 6.1: Additional constant of motion obtained using Moser's technique

The following two tables contain the Casimirs of the Poisson structure for $m = 3$.

n	$C = -\frac{1}{2}\det L$
7	$a_1a_3a_4a_6(a_1a_5a_9 + a_2a_6a_7 - a_7a_8a_9)$
9	$a_1a_3a_4a_5a_6a_8(a_1a_7a_{11} + a_2a_8a_9 - a_9a_{10}a_{11})$
11	$a_1a_3a_4a_5a_6a_7a_8a_{10}(a_1a_9a_{13} + a_2a_{10}a_{11} - a_{11}a_{12}a_{13})$
13	$a_1a_3a_4a_5a_6a_7a_8a_9a_{10}a_{12}(a_1a_{11}a_{15} + a_2a_{12}a_{13} - a_{13}a_{14}a_{15})$
n	$a_1a_3a_4\cdots a_{n-3}a_{n-1}(a_1a_{n-2}a_{n+2} + a_2a_{n-1}a_n - a_na_{n+1}a_{n+2})$

Table 6.2: Casimirs for $m = 3$ and n odd

n	C_1	$C_2 = \sqrt{\lvert\det L\rvert} - C_1$
6	$a_1a_3a_5$	$-(a_1a_4a_8 + a_2a_5a_6 - a_6a_7a_8)$
8	$a_1a_3a_5a_7$	$a_4(a_1a_6a_{10} + a_2a_7a_8 - a_8a_9a_{10})$
10	$a_1a_3a_5a_7a_9$	$-a_4a_6(a_1a_8a_{12} + a_2a_9a_{10} - a_{10}a_{11}a_{12})$
12	$a_1a_3a_5a_7a_9a_{11}$	$a_4a_6a_8(a_1a_{10}a_{14} + a_2a_{11}a_{12} - a_{12}a_{13}a_{14})$
14	$a_1a_3a_5a_7a_9a_{11}a_{13}$	$-a_4a_6a_8a_{10}(a_1a_{12}a_{16} + a_2a_{13}a_{14} - a_{14}a_{15}a_{16})$
n	$a_1a_3\cdots a_{n-3}a_{n-1}$	$a_4a_6\cdots a_{n-6}a_{n-4}(a_1a_{n-2}a_{n+2} + a_2a_{n-1}a_n - a_na_{n+1}a_{n+2})$

Table 6.3: Casimirs for $m = 3$ and n even

Chapter 7

Using complex coefficients

Now I will have less distraction.— Leonhard Euler, upon losing his eyesight.

As we noted in the introduction of the previous chapter, we may produce more Lotka-Volterra systems by changing the matrix L from symmetric to Hermitian. The aim of this chapter is to describe this idea of using complex coefficients and give some examples of new Lotka-Volterra systems produced by this new method. Let us begin with an example

Example 45. Consider the case of a root system of type A_4 and the subset of positive roots $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 = \alpha_1 + \alpha_2\}$. It turns out that if

$$L = \sum_{i=1}^{4} a_i \left(X_{\alpha_i} + X_{-\alpha_i}\right) + a_5 \left(X_{\alpha_5} + X_{-\alpha_5}\right)$$

then the corresponding linear system of signs (described in the proof of proposition 29) does not have a solution, while if

$$L = \sum_{i=1}^{4} a_i \left(X_{\alpha_i} + X_{-\alpha_i}\right) + \mathrm{i}a_5 \left(X_{\alpha_5} - X_{-\alpha_5}\right)$$

then the corresponding system of signs does have a solution and gives rise to the system

$$\begin{aligned}
\dot{a}_1 &= a_1 a_2^2 + a_1 a_5^2 \\
\dot{a}_2 &= -a_1^2 a_2 + a_2 a_3^2 - a_2 a_5^2 \\
\dot{a}_3 &= -a_2^2 a_3 + a_3 a_4^2 - a_3 a_5^2 \\
\dot{a}_4 &= -a_3^2 a_4 \\
\dot{a}_5 &= -a_1^2 a_5 + a_2^2 a_5 + a_3^2 a_5.
\end{aligned} \tag{7.1}$$

This system can be easily transformed to a Lotka-Volterra system which is integrable with one rational Casimir $\frac{a_1 a_2}{a_5}$, and an extra constant of motion, $\mathrm{tr}(L^4)$.

In general, the idea is to make the matrix L Hermitian by replacing some terms of the form $a_i\left(X_{\alpha_i}+X_{-\alpha_i}\right)$ with $\mathrm{i}a_i\left(X_{\alpha_i}-X_{-\alpha_i}\right)$ where i is the imaginary unit. The construction of the matrix B is the same as in section 6.2 with the only difference of B being skew-Hermitian. Example 45 suggest that we may replace with $\mathrm{i}a_i\left(X_{\alpha_i}-X_{-\alpha_i}\right)$ all variables in L corresponding to the roots of height 2. But doing so we see that the only possible way to have a consistent Lax pair is to replace with $\mathrm{i}a_i\left(X_{\alpha_i}-X_{-\alpha_i}\right)$ all variables in L corresponding to roots of even height. Therefore we end up with the following alternative method of constructing Lax pairs.

We begin with a subset Φ of the positive roots containing the simple roots. We write $\Phi=\Phi_1\cup\Phi_2$ where Φ_1 are the roots in Φ of odd height and Φ_2 are the roots in Φ of even height. The Lax matrix is constructed as

$$L=\sum_{\alpha_i\in\Phi_1}a_i(X_{\alpha_i}+X_{-\alpha_i})+\sum_{\alpha_i\in\Phi_2}\mathrm{i}a_i(X_{\alpha_i}-X_{-\alpha_i})=\sum_{\alpha_i\in\Phi}b_i(X_{\alpha_i}\pm X_{-\alpha_i})\ , \tag{7.2}$$

where the variables b_i are defined as $b_i=a_i$ if $\alpha_i\in\Phi_1$ and $b_i=\mathrm{i}a_i$ if $\alpha_i\in\Phi_2$. Consider the set $\Phi\cup\Phi^-$ which consists of all the roots in Φ together with their negatives. Let

$$\Psi=\left\{\alpha+\beta\mid\alpha,\beta\in\Phi\cup\Phi^-,\alpha+\beta\in R^+\right\}\ .$$

We define the upper triangular part of the skew-Hermitian matrix B as

$$\sum c_{ij}b_ib_jX_{\alpha_i+\alpha_j}, \tag{7.3}$$

where $c_{ij}=\pm1$ if $\alpha_i+\alpha_j\in\Psi$ with $\alpha_i,\alpha_j\in\Phi\cup\Phi^-$ and 0 otherwise.

An easy consequence of the construction of the matrices L and B is the following lemma.

Lemma 15. *Let Φ be a subset of the positive roots containing the simple roots and L,B the matrices constructed in (7.2) and (7.3). Also let K be the subset of the positive roots defined by*

$$K=\{\alpha+\beta+\gamma:\alpha,\beta,\gamma\in\Phi\cup\Phi^-,\ \alpha+\beta+\gamma\in R^+\}.$$

Let's write $K=K_1\cup K_2$ where K_1 are the roots in K of odd height and K_2 are the roots in K of even height. Then the bracket $[L,B]$ is decomposed into $[L,B]=A_1+\mathrm{i}A_2$ for a symmetric matrix A_1 and a skew-symmetric matrix A_2 where the nonzero entries of A_1 and A_2 appear in the "correct" positions; i.e. those of A_1 in positions corresponding to root vectors $X_\alpha,\ \alpha\in K_1$ while those of A_2 in positions corresponding to root vectors $X_\alpha,\ \alpha\in K_2$.

Next we show that this method in general produces more Lax pairs than the one described in section 6.2. Since for this method we don't have to worry about the diagonal entries of the bracket $[L, B]$ (see lemma 15) we end up with the following proposition.

Proposition 31. *Let $\Pi \subset \Phi \subset R^+$ be a subset of the positive roots containing the simple roots with the property that whenever $\alpha, \beta, \gamma \in \Phi \cup \Phi^-$ and $\alpha+\beta+\gamma \in R^+$ then $\alpha+\beta+\gamma \in \Phi$. Also let L, B be the matrices constructed using the algorithms described in (7.2) and (7.3) respectively. Then for any choice of the signs $c_{i,j}$ the pair L, B is a Lax pair.*

Example 46. Let $k, n \in \mathbb{N}$ with $1 \leq k < n$. If Φ is the subset of the positive roots of the root system A_n containing the simple roots and all the roots of height larger than k then for all possible choices of the signs $c_{i,j}$ we have a consistent Lax pair.

Example 47. For the root system of type A_3 all Lax pairs corresponding to

$$\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

are given by the matrices

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_5 \\ a_1 & 0 & a_2 & \mathrm{i}a_4 \\ 0 & a_2 & 0 & a_3 \\ a_5 & -\mathrm{i}a_4 & a_3 & 0 \end{pmatrix}$$

and B, whose upper triangular part is

$$\begin{pmatrix} 0 & \mathrm{i}c_{4,5}a_4a_5 & c_{1,2}a_1a_2 - c_{3,5}a_3a_5 & -\mathrm{i}c_{3,4}a_1a_4 \\ 0 & 0 & -\mathrm{i}c_{3,4}a_3a_4 & c_{1,5}a_1a_5 + c_{2,3}a_2a_3 \\ 0 & 0 & 0 & \mathrm{i}c_{4,5}a_2a_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can verify that for all 32 choices of the signs $c_{i,j}$ no one of the corresponding systems is the same as the one produced by the method of section 6.2. Therefore this procedure produces systems which in general are different from the ones of section 6.2.

Example 48. Define the matrix L to be

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & a_9 \\ a_1 & 0 & a_2 & \mathrm{i}a_6 & 0 & \mathrm{i}a_8 \\ 0 & a_2 & 0 & a_3 & 0 & 0 \\ 0 & -\mathrm{i}a_6 & a_3 & 0 & a_4 & \mathrm{i}a_7 \\ 0 & 0 & 0 & a_4 & 0 & a_5 \\ a_9 & -\mathrm{i}a_8 & 0 & -\mathrm{i}a_7 & a_5 & 0 \end{pmatrix}$$

and the upper triangular part of the skew-Hermitian matrix B to be

$$\begin{pmatrix} 0 & \mathrm{i}a_8a_9 & -a_1a_2 & -\mathrm{i}a_1a_6-\mathrm{i}a_7a_9 & a_5a_9 & \mathrm{i}a_1a_8 \\ 0 & 0 & -\mathrm{i}a_3a_6 & -a_2a_3+a_7a_8 & -\mathrm{i}a_4a_6+\mathrm{i}a_5a_8 & a_1a_9+a_6a_7 \\ 0 & 0 & 0 & -\mathrm{i}a_2a_6 & -a_3a_4 & \mathrm{i}a_2a_8-\mathrm{i}a_3a_7 \\ 0 & 0 & 0 & 0 & -\mathrm{i}a_5a_7 & -a_4a_5+a_6a_8 \\ 0 & 0 & 0 & 0 & 0 & -\mathrm{i}a_4a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This Lax pair gives rise to a Lotka-Volterra system. The associated Poisson matrix is given by

$$\pi = \begin{pmatrix} 0 & -a_1a_2 & 0 & 0 & 0 & -a_1a_6 & 0 & a_1a_8 & a_1a_9 \\ a_1a_2 & 0 & -a_2a_3 & 0 & 0 & -a_2a_6 & 0 & a_2a_8 & 0 \\ 0 & a_2a_3 & 0 & -a_3a_4 & 0 & a_3a_6 & -a_3a_7 & 0 & 0 \\ 0 & 0 & a_3a_4 & 0 & -a_4a_5 & a_4a_6 & -a_4a_7 & 0 & 0 \\ 0 & 0 & 0 & a_4a_5 & 0 & 0 & a_5a_7 & -a_5a_8 & -a_5a_9 \\ a_1a_6 & a_2a_6 & -a_3a_6 & -a_4a_6 & 0 & 0 & -a_6a_7 & a_6a_8 & 0 \\ 0 & 0 & a_3a_7 & a_4a_7 & -a_5a_7 & a_6a_7 & 0 & -a_7a_8 & -a_7a_9 \\ -a_1a_8 & -a_2a_8 & 0 & 0 & a_5a_8 & -a_6a_8 & a_7a_8 & 0 & a_8a_9 \\ -a_1a_9 & 0 & 0 & 0 & a_5a_9 & 0 & a_7a_9 & -a_8a_9 & 0 \end{pmatrix},$$

which has the following Casimirs.

$$a_2a_4a_9, \frac{a_2a_4a_8}{a_1}, \frac{a_1a_3a_7}{a_4}, \frac{a_6}{a_2a_3}, a_1a_3a_5.$$

The additional integral is $H_4 = \mathrm{tr}\,(L^4)$.

We prove the following proposition which is the equivalent of proposition 29 and shows that this method gives more Lotka-Volterra systems than the one described in section 6.2.

Proposition 32. *Let $\Phi = \{\alpha_1, \ldots, \alpha_{n+1}\}$ be the subset of the positive roots of the root system A_n containing the simple roots and the additional extra root α_{n+1}. Suppose that $\alpha_{n+1} = \alpha_k + \alpha_{k+1} + \ldots + \alpha_m$ for some $1 \le k < m \le n$. Then the only possible choices of k, m that lead to a Lotka-Volterra system are*

$$(k, m) = (1, n), (1, n-1), (2, n), (2, n-1) \text{ and } (i, i+1) \text{ for } i = 1, 2, \ldots, n-1.$$

Proof. The proof for the case $m - k > 1$ is the same as the proof of proposition 29. When $m - k = 1$, since the matrix $[L, B]$ is Hermitian, from lemma 15 it follows that its diagonal entries are zero. The corresponding linear system of signs becomes

$$
\begin{aligned}
&c_{i,i+1} = c_{i+1,i+2} = c \qquad i = 1, 2, \dots, n-2, \\
&c_{m+1,n+1} = -c_{k-1,n+1}, \quad c_{k-1,n+1} = -c_{m,n+1} \quad c_{m+1,n+1} = -c_{m+1,m+2} = -c, \\
&c_{m+1,n+1} = -c_{k,n+1}, \quad c_{k-1,n+1} = c_{k-2,k-1} = c.
\end{aligned}
$$

which has a solution and therefore the Lax equation $\dot{L} = [L, B]$ is transformed to a Lotka-Volterra system. □

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Index