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PROBLEMS ON HIGHER AND LOWER DIMENSIONAL DYNAMICAL SYSTEMS

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The present doctoral dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy of the University of Cyprus. It is a product of original work of my own, unless otherwise mentioned through references, notes, or any other statements.

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Dedicated to my parents Rolandos and Androula
and to my brothers Dimitris and Marinos.
For their love, support and encouragement.

Περίληψη

Σε αυτή τη διδακτορική διατριβή μελετούμε προβλήματα από την περιοχή των δυναμικών συστημάτων. Πιο συγκεκριμένα μελετούμε δυναμικά συστημάτα υψηλοτέρων και χαμηλότερων διαστάσεων. Για να το κάνουμε αυτό χωρίζουμε τη διδακτορική διατριβή σε δύο μέρη. Γενικά, στο πρώτο μέρος, μελετούμε προβλήματα τα οποία είναι αντικείμενο των μερικών διαφορικών εξισώσεων, και τα οποία ανήκουν στην περιοχή των δυναμικών συστημάτων υψηλοτέρων διαστάσεων. Στην περίπτωση μας, τέτοια προβλήματα περιλαμβάνουν ειδικές περιπτώσεις απειροδιάστατων δυναμικών συστημάτων με διάχυση. Στο δεύτερο μέρος, γενικά, μελετούμε ένα προβλήμα το οποίο είναι αντικείμενο των συνήθων διαφορικών εξισώσεων, και το οποίο ανήκει στην περιοχή των δυναμικών συστημάτων χαμηλότερων διαστάσεων. Αυτή τη φορά περιοριζόμαστε σε ειδικές περιπτώσεις του δευτέρου μέρους του 16ου προβλήματος του Hilbert.

Για το πρώτο μέρος της διδακτορικής διατριβής συγκεκριμένοι ερευνητικοί σκοποί είναι η αναλυτική εξέταση της αναλυτικότητας των λύσεων για:

- συστήματα διάχυσης-διασποράς,
- μία οικογένεια μη γραμμικών εξελικτικών ψευδοδιαφορικών εξισώσεων στη μία χωρική διάσταση, και
- μία οικογένεια μη γραμμικών εξελικτικών ψευδοδιαφορικών εξισώσεων στις δύο χωρικές διάστασεις.

Για το πρώτο πρόβλημα όπως παραπάνω χρησημοποιούμε μία μέθοδο ημιομάδων, σε αντίθεση με τα δύο τελευταία προβλήματα στα οποία χρησιμοποιούμε μία φασματική μέθοδο. Αναπτύσσουμε όλα αυτά τα προβλήματα στα Κεφάλαια 3, 4 και 5.

Στο Κεφάλαιο 3 μελετούμε τις αναλυτικές ιδιότητες των λύσεων εξισώσεων τύπου Kuramoto–Sivashinsky και άλλων σχετικών συστημάτων, με περιοδικές αρχικές συνθήκες. Για να το κάνουμε αυτό, εξετάζουμε κατά πόσο η μέθοδος ημιομάδων, η οποία αναπτύχθηκαι από τους Collet *et al.* [11] μπορεί να εφαρμοστεί και σε συστήματα διάχυσης-διασποράς. Διαπιστώνουμε ότι η μέθοδος αυτή δουλεύει και για τέτοια συστήματα, και αποδεικνύουμαι ότι οι λύσεις τους είναι αναλυτικές ως προς τη χωρική μεταβλητή σε μία λωρίδα γύρω από την ευθεία των πραγματικών αριθμών. Επιπλέον, δίνεται ένα κάτω φράγμα για το πλάτος της λωρίδας αναλυτικότητας για καθένα από τα συστήματα που μελετούμε.

Στο Κεφάλαιο 4 μελετούμε τις αναλυτικές ιδιότητες των λύσεων για μία οικογένεια μη γραμμικών εξελικτικών ψευδοδιαφορικών εξισώσεων στη μία χωρική διάσταση, που έχουν ολικούς ελκυστές. Για να το κάνουμε αυτό, χρησιμοποιούμε ένα κριτήριο αναλυτικότητας για περιοδικές συναρτήσεις ως προς τη χωρική μεταβλητή, το οποίο περιλαμβάνει τον ρυθμό αύξησης κατάλληλης νόρμας των $n^{ης}$ τάξεως παραγώγους της λύσης, ως προς τη χωρική μεταβλητή, καθώς το n τείνει στο άπειρο. Χρησιμοποιώντας αυτό το κριτήριο και τη φασματική μέθοδο που αναπτύχθηκαι από τους Akrivis *et al.* [3] βελτειώνουμε αποτελέσματα που εμφανίζονται στο [3].

Στο Κεφάλαιο 5 μελετούμε τις αναλυτικές ιδιότητες των λύσεων για μία οικογένεια μη γραμμικών εξελικτικών ψευδοδιαφορικών εξισώσεων στις δύο χωρικές διάστασεις. Για να το κάνουμε αυτό, εξετάζουμε κατά πόσο η φασματική μέθοδος, η οποία αναπτύχθηκαι στο [3] μπορεί να εφαρμοστεί και σε συστήματα με δύο χωρικές μεταβλητές. Εισάγουμε ένα κριτήριο, το οποίο παρέχει μία ικανή συνθήκη για την αναλυτικότητα περιοδικών συναρτήσεων $u \in C^\infty$, το οποίο περιλαμβάνει τον ρυθμό αύξησης κατάλληλης νόρμας του $\nabla^n u$, καθώς το n τείνει στο άπειρο. Αυτό το κριτήριο μας επιτρέπει να αποδείξουμε αναλυτικότητα των λύσεων ως προς τις χωρικές μεταβλητές για διάφορα συστήματα, που περιλαμβάνουν τις εξισώσεις Topper–Kawahara, Frenkel–Indireshkumar και Coward–Hall καθώς και τις τροποποιημένες εκδοχές τους με διασπορά, υποθέτωντας ότι αυτά τα συστήματα έχουν ολικούς ελκυστές.

Στο δεύτερο μέρος της διδακτορικής διατριβής συγκεκριμένοι ερευνητικοί σκοποί είναι ο προσδιορισμός του άνω φράγματος του αριθμού των οριακών κύκλων σε πολυωνυμικά διανυσματικά πεδία στο επίπεδο, λαμβάνοντας υπόψη μόνο των βαθμών των πολυωνύμων και η εξέταση της σχέσης των θέσεων τους σε δύο συστήματα τα οποία αποτελούν γενικεύσεις της εξίσωσης Van der Pol. Η μεθόδος που χρησιμοποιούμε και για τις δύο γενικεύσεις της εξίσωσης Van der Pol είναι να δημιουργούμε οριακούς κύκλους διαταράσσοντας ένα σύστημα (στην περίπτωση μας, τον γραμμικό αρμονικό ταλαντωτή) το οποίο έχει κέντρο, κατά τετοίο τρόπο ώστε οριακοί κύκλοι να διακλαδώνοται στο διαταραγμένο σύστημα από τις περιοδικές λύσεις του μη διαταραγμένου συστήματος. Αναπτύσσουμε όλα αυτά τα προβλήματα στα Κεφάλαια 8 και 9.

Στο Κεφάλαιο 8 μελετούμε τη διακλάδωση των οριακών κύκλων από τον γραμμικό αρμονικό ταλαντωτή $\dot{x} = y$, $\dot{y} = -x$ στο σύστημα

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon y^{p+1}\bigl(1 - x^{2q}\bigr),$$

όπου το ε είναι μία μικρή θετική παράμετρος που τείνει στο 0, το $p \in \mathbb{N}_0$ είναι άρτιος και το $q \in \mathbb{N}$. Αποδεικνύουμε ότι το παραπάνω σύστημα, όπου το $p \in \mathbb{N}_0$ είναι άρτιος και το $q \in \mathbb{N}$ έχει μοναδικό οριακό κύκλο στο επίπεδο. Επίσης εξετάζουμε και μερικές άλλες ιδιότητες αυτού του μοναδικού οριακού κύκλου για μερικές ειδικές περιπτώσεις αυτού του συστήματος.

Στο Κεφάλαιο 9 μελετούμε τη διακλάδωση των οριακών κύκλων από τον γραμμικό αρμονικό ταλαντωτή $\dot{x} = y$, $\dot{y} = -x$ στο σύστημα

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon f(y)\bigl(1 - x^2\bigr),$$

όπου το ε είναι μία μικρή θετική παράμετρος που τείνει στο 0 και το f είναι ένα περιττό πολυώνυμο βαθμού $2n + 1$, με το n ένα αυθαίρετο αλλά φιξαρισμένο φυσικό αριθμό. Αποδεικνύουμε ότι το παραπάνω σύστημα, για κατάλληλα επιλεγμένα περιττά πολυώνυμα f βαθμού $2n + 1$ έχει ακριβώς $n + 1$ οριακούς κύκλους και ότι αυτός ο αριθμός είναι άνω φράγμα του αριθμού των οριακών κύκλων σε κάθε περίπτωση ενός αυθαίρετου περιττού πολυωνύμου f βαθμού $2n+1$. Επίσης εξετάζουμε τη σχέση των θέσεων των οριακών κύκλων αυτού του συστήματος. Συγκεκριμένα, κατασκευάζουμε συστήματα διαφορικών εξισώσεων με n οριακούς κύκλους σε καθορισμένες θέσεις και έναν οριακό κύκλο του οποίου η θέση εξαρτάται από τη θέση των προηγούμενων n οριακών κύκλων. Τελικά, δίνουμε μερικά παραδείγματα για να εξηγήσουμε τη γενική θεωρία που παρουσιάζουμε σε αυτό το κεφάλαιο.

Abstract

In this doctoral thesis, we study some problems from the area of dynamical systems; more precisely, on higher and lower dimensional dynamical systems. In order to do this we separate the current thesis in two parts. In general, in Part I, we study problems which are a subject of partial differential equations (PDEs), and which belong to the area of higher dimensional dynamical systems. In our case, such problems include special cases of infinite-dimensional dissipative dynamical systems. In Part II, in general, we study a problem which is a subject of ordinary differential equations (ODEs), and which belongs to the area of lower dimensional dynamical systems. At this time we restrict ourselves to special cases of the second part of Hilbert's 16th problem.

Specific research goals for the Part I of the doctoral thesis is the analytical investigation of the analyticity of the solutions for:

- dissipative-dispersive systems, such as the dispersively modified Kuramoto–Sivashinsky equation, a nonlocal Kuramoto–Sivashinsky equation and the dispersively modified Otto's model,
- a class of non-linear evolutionary pseudo-differential equations in one spatial dimension, and
- a class of non-linear evolutionary pseudo-differential equations in two spatial dimensions.

For the first problem as above we use semigroup methods, instead of the rest two problems in which the methods is spectral. We cover all these problems in Chapters 3, 4 and 5.

In Chapter 3 we study the analyticity properties of solutions of dissipative-dispersive systems, with periodic initial data. In order to do this, we explore the applicability of the semigroup method, which was developed in Collet *et al.* [11], and which was introduced in order to establish the analyticity of the Kuramoto–Sivashinsky equation. So, we prove that the solutions of a variety of dissipative-dispersive systems, which possess a global attractor, are analytic with respect to the spatial variable in a strip around the real axis. Furthermore, a lower bound for the width of the strip of analyticity is obtained in each case.

In Chapter 4 we study the analyticity properties of solutions for a class of non-linear evolutionary pseudo-differential equations possessing global attractors. In order to do this, we utilize an analyticity criterion for spatially periodic functions, which involves the rate of growth of a suitable norm of the n^{th} derivative of the solution, with respect to the spatial variable, as n tends to infinity. This criterion is applied to a general class of non-linear evolutionary pseudo-differential equations, under certain conditions, provided they possess global attractors. Using this criterion and the spectral method developed in Akrivis *et al.* [3] we have improved previous results which appear in [3].

In Chapter 5 we study the analyticity properties of solutions of Kuramoto–Sivashinsky type equations in two spatial dimensions, with periodic initial data. In order to do this we explore the applicability of the spectral method developed in [3], in three-dimensional models. We introduce a criterion, which provides a sufficient condition for analyticity of a periodic function $u \in C^\infty$, involving the rate of growth of $\nabla^n u$, in suitable norms, as n tends to infinity. This criterion allows us to establish spatial analyticity for the solutions of a variety of systems, including Topper–Kawahara, Frenkel–Indireshkumar and Coward–Hall equations and their dispersively modified versions, once we assume that these systems possess global attractors.

In Part II of the doctoral thesis specific research goals is the determination of the upper bound for the number of limit cycles in polynomial vector fields, depending only on the degree and an investigation of their relative positions inside two classes which constitute generalizations of the Van der Pol equation. The method used on both of the generalized Van der Pol equations is to produce limit cycles by perturbing a system (in our case, the linear harmonic oscillator) which has a center, in such a way that limit cycles bifurcate in the perturbed system from the periodic orbits of the period annulus of the center of the unperturbed system. We cover all these problems in Chapters 8 and 9.

In Chapter 8 we study the bifurcation of limit cycles from the linear harmonic oscillator $\dot{x} = y$, $\dot{y} = -x$ in the class

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon y^{p+1}\big(1 - x^{2q}\big),$$

where ε is a small positive parameter tending to 0, $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. We prove that the above differential system in the global plane, where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$ has a unique limit cycle. We also investigate and some other properties of this unique limit cycle for some special cases of this differential system.

In Chapter 9 we study the bifurcation of limit cycles from the linear harmonic oscillator $\dot{x} = y$, $\dot{y} = -x$ in the class

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon f(y)\big(1 - x^2\big),$$

where ε is a small positive parameter tending to 0 and f is an odd polynomial of degree $2n + 1$, with n a fixed but arbitrary natural number. We prove that, the above differential system, in the global plane, for particularly chosen odd polynomials f of degree $2n + 1$ has exactly $n + 1$ limit cycles and that this number is an upper bound for the number of limit cycles for every case of an arbitrary odd polynomial f of degree $2n + 1$. We also investigate the possible relative positions of the limit cycles for this differential system. In particular, we construct differential systems with n given limit cycles and one limit cycle whose position depends on the position of the previous n limit cycles. Finally, we give some examples in order to illustrate the general theory presented in this chapter.

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Xenakis Ioakim

Contents

Part I: Infinite-dimensional dissipative dynamical systems

Chapter 1

Higher dimensional dynamical systems

1.1 The Kuramoto–Sivashinsky equation

The Kuramoto–Sivashinsky (KS) equation

$$u_t + uu_x + u_{xx} + u_{xxxx} = 0, \tag{1.1}$$

is one of the simplest nonlinear PDEs exhibiting complex spatio-temporal dynamics. For example, it has been derived in the context of plasma ion mode instabilities by LaQuey *et al.* [34] (see also Cohen *et al.* [9]), reaction-diffusion systems by Kuramoto and Tsuzuki [33] (see also Kuramoto [32]), laminar flame fronts by Sivashinsky [46], viscous liquid flows on an inclined plane by Sivashinsky and Michelson [47], viscous film flow by Shlang and Sivashinsky [45] and delay-diffusion population models by Lin and Kahn [36].

The L-periodic in space solutions of (1.1) have received considerable attention both analytically and computationally. If u is L-periodic in space and satisfies (1.1), then its spatial average, i.e.,

$$\frac{1}{L}\int_0^L u(x,t)\,dx,$$

is independent of t. Also, (1.1) is invariant under the Galilean transformation

$$t = \tilde{t}, \quad x = \tilde{x} + st, \quad u = \tilde{u} + s,$$

and thus we may restrict our attention to zero average solutions of (1.1). If we express a solution u of (1.1) as a Fourier series

$$u(x,t) = \sum_{\mu \in q\mathbb{Z}} \hat{u}(\mu, t) e^{i\mu x},$$

where $q = 2\pi/L$, then its Fourier coefficients satisfy the infinite dimensional dynamical system

$$\frac{d}{dt}\hat{u}(\mu,t) \;=\; (\mu^2-\mu^4)\,\hat{u}(\mu,t) - \frac{i\mu}{2}\sum_{\mu'\in q\mathbb{Z}} \hat{u}(\mu',t)\,\hat{u}(\mu-\mu',t), \quad \mu\in q\mathbb{Z}. \tag{1.2}$$

Equations (1.2) reveal that high frequencies ($|\mu|>1$) are linearly stable, while the low frequencies ($0<|\mu|<1$) are linearly unstable. The nonlinear term in (1.2) causes transfer of energy from low to high frequencies and keeps the solution of (1.1) bounded in the L^2-norm.

A considerable corpus of analytical results exist for the KS equation and we review some of the most salient ones needed for our purposes here. It is shown by Constantin *et al.* [12] that the long-time dynamics of KS equation is governed by a finite dimensional dynamical system of size at least as large as the number of linearly unstable modes which are equal to $[L/(2\pi)]$ (the largest integer less than or equal to $L/(2\pi)$) for odd L-periodic solutions (i.e., for solutions of the form $u(x,t)=\sum_{n=1}^{\infty} a_n(t)\sin(qnx)$, where $q=2\pi/L$). For general L-periodic initial data we have boundedness of solutions as shown independently by Il'yashenko [23], Goodman [22] and Collet *et al.* [10]. The result was given by Collet *et al.* in [10] is the following theorem:

Let the initial data $u_0=u(\cdot,0)$ of the KS equation be L-periodic, and of zero mean. Then, there is a positive constant c_0, independent of L and u_0, such that

$$\limsup_{t\to\infty}\Big(\int_0^L |u(x,t)|^2\,dx\Big)^{1/2} \;\le\; R_L \;=\; c_0\,L^{8/5}. \tag{1.3}$$

Typically, the boundedness of the solution u of (1.1) is obtained by proving that $\|u-\varphi\|^2$ is a Lyapunov function, where $\|\cdot\|$ is the L^2-norm of L-periodic functions, for a suitable background flow $\varphi=\varphi(x)$, a smooth L-periodic function, when $\|u\|$ is sufficiently large; this (background flow) argument was first used in [12]. With this auxiliary function φ we derive the following energy estimate:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u-\varphi\|^2 &= -\frac{1}{2}\int_0^L \varphi_x u^2\,dx + \int_0^L (u-\varphi)_x u_x\,dx - \int_0^L (u-\varphi)_{xx} u_{xx}\,dx\\ &\le -\frac{1}{2}\Big(\int_0^L u_{xx}^2\,dx - 3\int_0^L u_x^2\,dx + \int_0^L \varphi_x u^2\,dx\Big) + \frac{1}{2}\Big(\int_0^L \varphi_{xx}^2\,dx + \int_0^L \varphi_x^2\,dx\Big).\end{aligned}$$

In order to obtain boundedness of the solution u we need to construct a function φ for which the term $\int_0^L u_{xx}^2\,dx - 3\int_0^L u_x^2\,dx + \int_0^L \varphi_x u^2\,dx$ above controls the L^2-norm of u, i.e.,

$$\|v\|^2 \;\le\; c\Big(\int_0^L v_{xx}^2\,dx - 3\int_0^L v_x^2\,dx + \int_0^L \varphi_x v^2\,dx\Big),$$

for some $c>0$ and every $v\in \mathrm{H}^2_{\mathrm{per}}(0,L)$, with zero average. Exploring this argument further one may derive a bound, for the solution of the KS equation, of the form

$$\limsup_{t\to\infty} \|u(\cdot,t)\| \,=\, \mathcal{O}(L^{\alpha}). \tag{1.4}$$

This exponent α is the one for which

$$\int_0^L \left(\varphi^2+\varphi_x^2+\varphi_{xx}^2\right)dx \,=\, \mathcal{O}(L^{\alpha}).$$

Nicolaenko *et al.* [39] obtained the first such bound, for odd solutions of the KS equation, with $\alpha = 5/2$. Bronski and Gambill [7] constructed a background flow φ which allowed them to obtain such a bound, for general (of zero average) solutions of the KS equation, with $\alpha = 1$. Using an alternative approach, Giacomelli and Otto [20] have provided the following improvement

$$\limsup_{t\to\infty} \|u(\cdot,t)\| \,=\, o(L). \tag{1.5}$$

A further improvement of (1.5) can be found in Otto [40] who finds

$$\limsup_{t\to\infty} \|u(\cdot,t)\| \,=\, \mathcal{O}\big(L^{1/2}(\log L)^{5/3}\big).$$

In [12] it is also shown that bounds in the L^2-norm of the solution of KS equation imply that these solutions are attracted by a set of finite dimension, the global attractor, and bounds in the dimension of the global attractor are provided. (See also Foias *et al.* [16].) Note that (1.4–1.5) can be used in turn to prove boundedness of the solution in any Sobolev norm.

In a companion paper Collet *et al.* [11] establish the analyticity of solutions of KS equation. In particular, they proved the following theorem:

For sufficiently large times, the solution of KS equation extends as a holomorphic function of x in a strip (in $\mathbb{C}$) of width

$$\beta_L \geq d_1\, L^{-16/25},$$

around the real axis, where d_1 is a positive constant independent of L.

This provides the following estimate for the spectral density at high wavenumbers,

$$\limsup_{t\to\infty} |\hat{u}(j,t)| \,=\, \mathcal{O}(e^{-cL^{-16/25}q|j|}),$$

where $\hat{u}(j,t)$ is the jth Fourier coefficient of $u(\cdot,t)$ and $q = 2\pi/L$.

In this thesis we investigate physically relevant extensions of the KS equation which arise in two- and three-dimensional hydrodynamics. These extensions are derived by including additional physical effects in the models (such as dispersive effects, more spatial dimensions and variations in the surface tension due to surfactants). Besides their physical relevance such models provide richer dynamics and pose significant mathematical challenges. The proposed directions of investigation are at the forefront of research in nonlinear dissipative dynamical systems and are original in at least two ways including:

- Three-dimensional interfacial flows
- Inclusion of dispersive effects yielding active-dissipative-dispersive systems.

1.2 The dispersively modified Kuramoto–Sivashinsky equation

The dispersively modified KS equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathcal{D}u \;=\; 0, \tag{1.6}$$

defined on 2π-periodic domains, with ν a positive constant and $\mathcal{D}$ a linear antisymmetric pseudo-differential operator defined by its symbol in Fourier space, that is,

$$(\widehat{\mathcal{D}w})_k \;=\; id_k\hat{w}_k, \quad d_{-k} \;=\; -d_k \in \mathbb{R}, \tag{1.7}$$

i.e., $\mathcal{D}$ is dispersive, has been derived in the context of interfacial hydrodynamics. In (1.7), $\hat{w}_k$ are the Fourier coefficients of w, whenever

$$w(x) \;=\; \sum_{k\in\mathbb{Z}} \hat{w}_k\, e^{ikx}.$$

Papageorgiou *et al.* [42] and Kas-Danouche *et al.* [28] derived an equation of the form of (1.6) to describe the stability of core-annular flows with applications to oil transport (lubricated pipe-lining). In particular, in this case the Fourier transform of the operator $\mathcal{D}$ can be expressed in terms of modified Bessel functions:

$$(\widehat{\mathcal{D}w})_k \;=\; \frac{ik^2 I_1(k)}{kI_1^2(k) - kI_0^2(k) + 2I_0(k)I_1(k)}\hat{w}_k, \tag{1.8}$$

where $I_\nu(k)$, with $\nu = 0$ and 1 denotes the modified Bessel function of the first kind of order ν. The well-posedness of (1.6) for periodic initial data can be derived from the work of Tadmor [48] since it constitutes a special case of the central theorem proved there. In particular, it can be shown that the corresponding initial value problem possesses a global (space periodic) solution which grows at most exponentially in time. (See also the relevant work of Biagioni *et al.* [5].)

For (one-dimensional) falling film flows a particular case of (1.6) where $\mathcal{D}u = \delta u_{xxx}$ and δ is a constant was originally derived by Topper and Kawahara [49] (see also

Kawahara and Toh [31], and Frenkel and Indireshkumar [19]). The resulting equation is the following KS/KdV equation (also known as Kawahara equation)

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + \nu u_{xxxx} \;=\; 0, \tag{1.9}$$

and note that the dispersive term is of lower order than the stabilizing term u_{xxxx}. Kawahara and Toh [31] were among the first to establish numerically the regularizing effect of dispersion on the dynamics with traveling wave pulses emerging at large times.

It is noteworthy that equation (1.9) with the inclusion of a fifth order dispersion term takes the form

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + \nu u_{xxxx} + \varepsilon u_{xxxxx} \;=\; 0, \tag{1.10}$$

known as the Benney–Lin equation, which has been derived in the context of the one-dimensional evolution of sufficiently small amplitude long waves in various problems in fluid dynamics. (See, for example, Benney [4] and Lin [35].) Global well-posedness of the periodic initial value problem for (1.10) with initial data in $\mathrm{H}^s_{\mathrm{per}}(\mathbb{R})$, $s \geq 0$, has been established by Biagioni and Linares [6]. (See also Chen and Li [8].) Here, $\mathrm{H}^s_{\mathrm{per}}(\mathbb{R})$ denotes the Sobolev space consisting of the 2π-periodic functions with finite norm $\|w\|_{\mathrm{H}^s}$, where

$$\|w\|_{\mathrm{H}^s} = \left(\sum_{k\in\mathbb{Z}} (1+k^2)^s |\hat{w}_k|^2 \right)^{1/2}.$$

Specific research goals for this part of the thesis is the rigorous analytical investigation of the analyticity of the solutions for the above equations.

1.3 A nonlocal Kuramoto–Sivashinsky equation

The nonlocal KS equation

$$u_t + uu_x \pm u_{xx} + \nu u_{xxxx} + \mu \mathcal{H}[u]_{xxx} = 0, \tag{1.11}$$

on a 2π-periodic interval, where ν a positive constant, μ a non negative constant and $\mathcal{H}$ the Hilbert transform operator defined by

$$\mathcal{H}[f](x) = \frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(\xi)}{x-\xi}\,d\xi, \tag{1.12}$$

where the integral is understood in the sense of a Cauchy principal value, exhibits a complex behavior including chaotic oscillations as in the case of the usual KS equation. Here, the operator $\mathcal{H}$ is defined by its symbol in Fourier space, that is,

$$(\widehat{\mathcal{H}[w]})_k \;=\; -i\,\mathrm{sgn}(\mathrm{Re}\,k)\hat{w}_k,$$

whenever $w(x) = \sum_{k\in\mathbb{Z}} \hat{w}_k\, e^{ikx}$. This equation was first derived by Gonzales and Castellanos [21] and also by Tseluiko and Papageorgiou [51], using formal asymptotics. A plus sign in front of the u_{xx} term corresponds to the linearly unstable hydrodynamic regime (the modified Kuramoto–Sivashinsky (MKS) equation) and a minus sign to the stable one (the modified damped Kuramoto–Sivashinsky (MDKS) equation). A weakly nonlinear analysis of the Navier–Stokes equations, the electrostatics equations and associated free surface conditions, leads to a MKS, or a MDKS equation which have an additional nonlocal term due to the effect of the electric field. Analytical results of global existence, uniqueness and uniform boundedness of solutions of the MKS equation were obtained by Duan and Ervin [15], who also obtain a bound for the radius of the absorbing ball in L^2. In general, global existence and uniqueness results for (1.11), as and an estimation of the radius of the absorbing ball in L^2 can be derived from the work of Tseluiko and Papageorgiou [50], who studied a generalized class of nonlocal evolution equations which includes as special case the equation (1.11). This generalized class, defined on 2π-periodic intervals, has the form

$$u_t + uu_x \pm u_{xx} + \nu u_{xxxx} - \mu(\mathcal{H}\circ\partial_x)^p[u] = 0, \tag{1.13}$$

where ν a positive constant and μ a non negative constant. Here $p \in [3,4)$ (for $p = 3$ equations (1.11) and (1.13) are identical), and the operator $(\mathcal{H}\circ\partial_x)^p$ is defined by its symbol in Fourier space. Now, specifically, for the radius $R_{\nu,\mu}$ of the absorbing ball of the equation (1.11) Tseluiko and Papageorgiou [50] established the following estimate:

$$R_{\nu,\mu} = c_1 \nu^{-31/10}\mu^{41/10}, \tag{1.14}$$

where c_1 is a positive constant.

Specific research goals for this part of the thesis is the rigorous analytical investigation of the analyticity of the solutions for the above equations.

1.4 The Burgers–Sivashinsky equation

The Burgers–Sivashinsky (BS) equation ([22])

$$u_t + uu_x - u - u_{xx} = 0,$$

superficially seems to have much in common with the KS equation (1.1). It too has low wave number instability, high wave number damping, and nonlinear stabilization via energy transfer. Despite the similarity between KS and BS, when L is large, where L is the period of the system, their solutions have different qualitative behavior. KS solutions are observed to have high dimensional chaos (see [37]) while BS solutions just approach time independent steady states as $t \to \infty$.

1.5 The α, β–model

Of interest is the model

$$u_t + uu_x - |\partial_x|^\alpha u + |\partial_x|^\beta u \;=\; 0,$$

with L-periodic initial data, where $\beta > \alpha \geq 0$ and the operator $|\partial_x|^\sigma$ defined by

$$|\partial_x|^\sigma \left(\sum_{k\in\mathbb{Z}} \hat{w}_k \, e^{iqkx} \right) = \sum_{k\in\mathbb{Z}} q^\sigma |k|^\sigma \hat{w}_k \, e^{iqkx}, \quad q = \frac{2\pi}{L},$$

which is due to Otto [41]. We call this equation as the α, β–model (or Otto's model). Note that the α, β–model reduces to the KS equation for $\alpha = 2$ and $\beta = 4$. Its dispersively modified version, which is the most general case, has the form

$$u_t + uu_x - |\partial_x|^\alpha u + |\partial_x|^\beta u + \mathcal{D}u \;=\; 0. \tag{1.15}$$

It is noteworthy that (1.15) with periodic initial data possesses a global attractor for $\alpha \geq 2$ (see [17, 18]).

Specific research goals for this part of the thesis is the rigorous analytical investigation of the analyticity of the solutions for the above equations.

1.6 Kuramoto–Sivashinsky type equations in 1D

The PDE

$$u_t + uu_x + \mathcal{P}u \;=\; 0, \tag{1.16}$$

defined on 2π-periodic domains, extends the dispersively modified KS equation (1.6). Here, $\mathcal{P}$ is a linear pseudo-differential operator defined by its symbol in Fourier space, that is,

$$(\widehat{\mathcal{P}w})_k \;=\; \lambda_k \, \hat{w}_k, \quad k \in \mathbb{Z},$$

whenever $w(x) = \sum_{k\in\mathbb{Z}} \hat{w}_k \, e^{ikx}$, and with λ_k satisfying

$$\operatorname{Re} \lambda_k \;\geq\; c_1 |k|^\gamma \quad \text{for all } \; |k| \geq k_0,$$

for some positive constants c_1, γ and k_0 a sufficiently large positive integer. Global existence of solutions of (1.16) has been established for $\gamma > 3/2$ (see [48]); when $\gamma \geq 2$, it can be deduced from [18] that equation (1.16) possesses a global attractor compact in every Sobolev norm. Analyticity of solutions of (1.16) is established when $\gamma > 5/2$, in [3].

Specific research goals for this part of the thesis is the rigorous analytical investigation of the analyticity of the solutions for the equation (1.16) when $\gamma > 2$.

1.7 Kuramoto–Sivashinsky type equations in 2D

All the equations discussed in the previous sections represent one-dimensional waves with the dependent variable $u(x,t)$ representing the scaled interfacial shape. In the case of falling films and core annular flows, x is in the direction of the flow (the former flows are driven by gravity due to the inclination of the substrate whereas the latter ones are typically driven by axial pressure gradients). When the flow becomes three-dimensional the interfacial shape is a function of two spatial variables and time – a surface embedded in three-dimensional space. The form of the model evolution equations depend on the application as we describe next.

In the case of falling film flows Topper and Kawahara [49] derived a rather general evolution equation for the liquid interface which takes the form:

$$u_t + uu_x + \alpha u_{xx} + \beta\Delta u + \gamma\Delta^2 u + \delta\Delta u_x \;=\; 0,$$

where $\gamma > 0,\ \Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, x is in the direction of the flow while y is the transverse coordinate. A particular case of this (and also a certain limit of a film flowing on the outside surface of a vertical circular cylinder) has been rederived by Frenkel and Indireshkumar [19] and takes the form

$$u_t + uu_x + u_{xx} + \Delta^2 u + \delta\Delta u_x \;=\; 0. \tag{1.17}$$

There are three parameters in the problem, the length of the two spatial periods (periodic boundary conditions are considered) and the dispersion parameter δ. This evolution equation extends the KS/KdV equation (1.9) to three-dimensional fluctuations and is particularly important since it can naturally capture transverse flow instabilities.

In the case of interfacial instability of rotating core-annular flow a model also retaining dispersive effects has been derived by Coward and Hall [13] and takes the form

$$u_t + uu_x + \alpha\Delta u + \Delta^2 u + \delta\mathcal{D}u \;=\; 0, \tag{1.18}$$

where Δ is as above, while x and y denote the axial and azimuthal coordinates in a cylindrical polar coordinate system. (Note that for core annular flows y is restricted in the interval $[0, 2\pi]$ due to the geometry of the problem as opposed to the falling film case in (1.17).) Due to this fact, there are three parameters in the problem, the spatial period in the axial direction, the parameter α and the dispersion parameter δ. The constant α is positive if the density of the annular fluid is smaller than that of the core fluid, and negative in the converse arrangement – linearly the former density ratio provides a destabilising mechanism; the dispersion parameter δ can be positive or negative if the viscosity ratio of annular to core fluid is smaller or larger than unity, respectively. The two-dimensional pseudo-differential operator $\mathcal{D}$ is best represented in terms of its symbol in Fourier space given by

$$\widehat{(\mathcal{D}w)}_{\xi,\eta} = \mathcal{N}_{\xi,\eta}\hat{w}_{\xi,\eta},$$

where

$$\begin{aligned}\mathcal{N}_{\xi,\eta} =& \frac{2i\eta^2 I_\eta(\xi I_\eta^2 - 2\eta I_{\eta+1}I_\eta - \xi I_\eta I_{\eta+1})}{2\xi I_{\eta+1}^2 I_{\eta-1} - \xi I_\eta^2 I_{\eta-1} - \xi I_\eta^2 I_{\eta+1} + 2(2+\eta)I_\eta I_{\eta+1} I_{\eta-1}} \\ &+ \frac{i\xi^2 I_{\eta+1}(\xi I_\eta I_{\eta-1} - 2(\eta-2)I_{\eta-1}I_{\eta+1} - \xi I_\eta I_{\eta+1})}{2\xi I_{\eta+1}^2 I_{\eta-1} - \xi I_\eta^2 I_{\eta-1} - \xi I_\eta^2 I_{\eta+1} + 2(2+\eta)I_\eta I_{\eta+1} I_{\eta-1}} + i\xi\eta,\end{aligned}$$

where ξ, η denote the wave numbers in the Fourier transforms in the x and y directions, respectively, and $I_\eta = I_\eta(\xi)$ denotes the modified Bessel function of the first kind of order η with $\eta \in \mathbb{Z}$. The axisymmetric case discussed above in equations (1.6) along with (1.8), derives by setting the wave number $\eta = 0$, as expected.

The systems (1.17) and (1.18) constitute model equations of fundamental interest since they are higher dimensional PDEs which exhibit complex dynamics of physical relevance. So far these problems have received very little attention (with the exception of limited computations by Indireshkumar and Frenkel [24] and Coward and Hall [13]) and thus a complete picture of their inevitably rich dynamics remains to be explored mathematically.

In the current thesis we intent to study these systems as initial value problems with initial data periodic in both x and y which is a natural setting for studying such problems. Specific research goals for this part of the thesis is the rigorous analytical investigation of the analyticity of the solutions for the above equations.

Chapter 2

Analyticity of the Kuramoto–Sivashinsky equation

In this chapter we present the details of the main result of [11] because this result can be directly extended to the dispersively modified KS equation. The approach in Section 2.2 belongs to Akrivis *et al.* [2].

2.1 Introduction

Let $m \in \mathbb{N}_0$ and $\mathrm{H}^m_{\mathrm{per}}[0, L]$ denotes the Sobolev space of the L-periodic functions with vanishing mean value, $\int_0^L v(x)\, dx = 0$.

We consider the Fourier series expansion of functions $v \in \mathrm{L}^2_{\mathrm{per}}[0, L] := \mathrm{H}^0_{\mathrm{per}}[0, L]$,

$$v(x) = \sum_{n\in\mathbb{Z}} \hat{v}_n\, e^{iqnx}, \quad q := \frac{2\pi}{L}, \tag{2.1}$$

with the Fourier coefficients $\hat{v}_n$ given by

$$\hat{v}_n := \frac{1}{L} \int_0^L v(x) e^{-iqnx}\, dx, \quad n \in \mathbb{Z}.$$

Obviously, $\hat{v}_0 = 0$, since the mean value of v vanishes.

We denote by $(\cdot, \cdot)$ the inner product in $\mathrm{L}^2_{\mathrm{per}}[0, L]$,

$$(v, w) := \int_0^L v(x)\overline{w}(x)\, dx, \quad v, w \in \mathrm{L}^2_{\mathrm{per}}[0, L].$$

Also, $\|u\| = (u, u)^{1/2}$. We next consider the differential operator

$$-\partial_x^2 : \mathrm{H}^2_{\mathrm{per}}[0, L] \to \mathrm{L}^2_{\mathrm{per}}[0, L], \quad -\partial_x^2 v := -v''(= -v_{xx}).$$

This operator is obviously self-adjoint, since, by periodicity,

$$(-\partial_x^2 v, w) = (\partial_x v, \partial_x w) = (v, -\partial_x^2 w) \quad \forall\, v, w \in \mathrm{H}^2_{\mathrm{per}}[0, L]. \tag{2.2}$$

Also, $-\partial_x^2$ is non-negative definite, since, in view of (2.2),

$$(-\partial_x^2 v, v) = \|\partial_x v\|^2 \quad \forall\, v \in \mathrm{H}^2_{\mathrm{per}}[0, L]. \tag{2.3}$$

The right-hand side of (2.3) is positive for all nonvanishing $v \in \mathrm{H}^2_{\mathrm{per}}[0, L]$ (since the only constant element of $\mathrm{H}^2_{\mathrm{per}}[0, L]$ is the zero function, due to the property of vanishing mean value of the elements of $\mathrm{H}^2_{\mathrm{per}}[0, L]$).

Therefore, the square root A of $-\partial_x^2$ is well defined. To give a representation of A, we first notice that the eigenvalues λ_n and corresponding eigenfunctions φ_n of $-\partial_x^2$ are well known,

$$\lambda_n = q^2 n^2, \quad \varphi_n(x) = e^{iqnx}, \quad n \in \mathbb{Z};$$

notice that

$$-\partial_x^2 e^{iqnx} = q^2 n^2 e^{iqnx}, \quad n \in \mathbb{Z}. \tag{2.4}$$

From (2.1) and (2.4) we obtain the desired representation of A, namely

$$(Av)(x) = \sum_{n \in \mathbb{Z}} q|n| \hat{v}_n e^{iqnx} \quad \forall\, v \in \mathrm{H}^1_{\mathrm{per}}[0, L]. \tag{2.5}$$

(Notice that $((qn)^2)^{1/2} = q|n|$.)

Similarly, the operator $e^{\alpha t A}$, with $\alpha, t \in \mathbb{R}$, is defined by

$$(e^{\alpha t A} v)(x) = \sum_{n \in \mathbb{Z}} e^{\alpha t q|n|} \hat{v}_n e^{iqnx}.$$

2.2 Analyticity of solutions

Let now $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a smooth, L-periodic in the spatial variable, function, such that $u(\cdot, t)$ has vanishing mean value, for all $t \geq 0$, and satisfies the KS equation

$$u_t + u u_x + u_{xx} + u_{xxxx} \; = \; 0, \quad x \in \mathbb{R},\ t \geq 0. \tag{2.6}$$

For a positive constant α and the operator A introduced in (2.5), we define the function v by

$$v(x, t) := (e^{\alpha t A} u)(x, t).$$

Then, (2.6) takes the form

$$(e^{-\alpha t A} v)_t + (e^{-\alpha t A})(v_{xx} + v_{xxxx}) + u u_x = 0,$$

i.e.,

$$(e^{-\alpha tA})v_t - \alpha e^{-\alpha tA} Av + e^{-\alpha tA}(v_{xx} + v_{xxxx}) + uu_x = 0. \tag{2.7}$$

Taking in (2.7) the L^2 inner product with $e^{\alpha tA}v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - \alpha(Av, v) - \|v_x\|^2 + \|v_{xx}\|^2 + (uu_x, e^{\alpha tA}v) = 0. \tag{2.8}$$

Next, we focus on the last term on the left-hand side of (2.8) that is due to the nonlinearity of the KS equation. With the trilinear form b,

$$b(v_1, v_2, v_3) := \int_0^L v_1(x)(\partial_x v_2)(x) v_3(x)\, dx,$$

we obviously have

$$b(u, u, e^{\alpha tA}v) = (uu_x, e^{\alpha tA}v).$$

Lemma 2.2.1. *There exists a constant C such that*

$$|b(u, u, e^{\alpha tA}v)| \leq C\sqrt{\alpha t}\, \|v\|\, \|Av\|^2. \tag{2.9}$$

Proof. First, notice that

$$b(v, v, w) = -\frac{1}{2} b(v, w, v). \tag{2.10}$$

Indeed,

$$b(v, v, w) = \int_0^L v v_x w\, dx = \frac{1}{2}\int_0^L (v^2)_x w\, dx = -\frac{1}{2}\int_0^L v^2 w_x\, dx = -\frac{1}{2}\int_0^L v w_x v\, dx$$

and (2.10) follows. In particular, (2.10) yields

$$b(v, v, v) = 0. \tag{2.11}$$

Now, using (2.10) and (2.11), we have

$$\begin{aligned} b(u, u, e^{\alpha tA}v) &= -\frac{1}{2} b(u, e^{\alpha tA}v, u) = -\frac{1}{2} b(e^{-\alpha tA}v, e^{\alpha tA}v, e^{-\alpha tA}v) \\ &= -\frac{1}{2}\big[b(e^{-\alpha tA}v, e^{\alpha tA}v, e^{-\alpha tA}v) - b(v, v, v)\big], \end{aligned}$$

i.e.,

$$b(u, u, e^{\alpha tA}v) = -\frac{1}{2}\Big[\int_0^L e^{-\alpha tA}v (e^{\alpha tA}v)_x e^{-\alpha tA}v\, dx - \int_0^L v v_x v\, dx\Big]. \tag{2.12}$$

Now,

$$
\begin{aligned}
&\int_0^L e^{-\alpha t A} v (e^{\alpha t A} v)_x e^{-\alpha t A} v \, dx \\
&= \int_0^L \Big(\sum_{m\in\mathbb{Z}} e^{-\alpha t q|m|} \hat{v}_m e^{iqmx} \Big) iq \Big(\sum_{n\in\mathbb{Z}} e^{\alpha t q|n|} n \hat{v}_n e^{iqnx} \Big) \Big(\sum_{k\in\mathbb{Z}} e^{-\alpha t q|k|} \hat{v}_k e^{iqkx} \Big) \, dx \\
&= iq \int_0^L \sum_{m,n,k\in\mathbb{Z}} n \hat{v}_n \hat{v}_m \hat{v}_k e^{\alpha t q(|n|-|m|-|k|)} e^{iq(n+m+k)x} \, dx \\
&= iqL \sum_{n\in\mathbb{Z}} \sum_{m+k=-n} n \hat{v}_n \hat{v}_m \hat{v}_k e^{\alpha t q(|n|-|m|-|k|)} \\
&= iqL \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} (-n) \hat{v}_{-n} \hat{v}_{n-m} \hat{v}_m e^{\alpha t q(|n|-|n-m|-|m|)} \\
&= -iqL \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} n \hat{v}_{-n} \hat{v}_{n-m} \hat{v}_m e^{\alpha t q(|n|-|m|-|n-m|)}.
\end{aligned}
$$

Similarly,

$$
\int_0^L v v_x v \, dx = -iqL \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} n \hat{v}_{-n} \hat{v}_{n-m} \hat{v}_m .
$$

Therefore, (2.12) takes the form

$$
b(u,u,e^{\alpha t A} v) = \frac{i}{2} qL \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} n \hat{v}_{-n} \hat{v}_{n-m} \hat{v}_m \big(e^{\alpha t q(|n|-|m|-|n-m|)} - 1 \big). \tag{2.13}
$$

Now, note that

$$
|n| - |m| - |n-m| = \begin{cases} 0, & \text{if } \; n \geq m > 0 \; \text{ or } \; n \leq m < 0, \\ -2|m|, & \text{if } \; n > 0 > m \; \text{ or } \; n < 0 < m, \\ -2|n-m|, & \text{if } \; m > n > 0 \; \text{ or } \; m < n < 0, \end{cases}
$$

i.e.,

$$
|n| - |m| - |n-m| = \begin{cases} 0, & \text{if } mn > 0, \; |n| \geq |m|, \\ -2|m|, & \text{if } mn < 0, \\ -2|n-m|, & \text{if } mn > 0, \; |m| > |n|. \end{cases} \tag{2.14}
$$

Using (2.14), we easily obtain from (2.13)

$$
\begin{aligned}
b(u,u,e^{\alpha t A} v) = \frac{i}{2} qL \Big[&\sum_{mn<0} n \hat{v}_{-n} \hat{v}_{n-m} \hat{v}_m \big(e^{-2\alpha t q|m|} - 1 \big) \\
&+ \sum_{mn>0,\, |m|>|n|} n \hat{v}_{-n} \hat{v}_{n-m} \hat{v}_m \big(e^{-2\alpha t q|n-m|} - 1 \big) \Big].
\end{aligned} \tag{2.15}
$$

We have that

$$
\frac{i}{2} qL \sum_{mn<0} n \hat{v}_{-n} \hat{v}_{n-m} \hat{v}_m \big(e^{-2\alpha t q|m|} - 1 \big) = \frac{i}{2} qL \sum_{n>0>m} n \hat{v}_{-n} \hat{v}_{n-m} \hat{v}_m \big(e^{-2\alpha t q|m|} - 1 \big)
$$

$$+\frac{i}{2}qL\sum_{m>0>n} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|m|}-1\big).$$

Setting $m' = -m$ and $n' = -n$, the second sum becomes

$$\begin{aligned}\frac{i}{2}qL\sum_{m>0>n} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|m|}-1\big) &= \frac{i}{2}qL\sum_{n'>0>m'}(-n')\hat{v}_{n'}\hat{v}_{m'-n'}\hat{v}_{-m'}\big(e^{-2\alpha tq|m'|}-1\big)\\ &= \frac{i}{2}qL\sum_{n'>0>m'}(-n')\bar{\hat{v}}_{-n'}\bar{\hat{v}}_{n'-m'}\bar{\hat{v}}_{m'}\big(e^{-2\alpha tq|m'|}-1\big)\\ &= \frac{i}{2}qL\sum_{n>0>m}(-n)\bar{\hat{v}}_{-n}\bar{\hat{v}}_{n-m}\bar{\hat{v}}_{m}\big(e^{-2\alpha tq|m|}-1\big),\end{aligned}$$

which shows that this second sum is minus the complex conjugate of the first sum. Thus,

$$\begin{aligned}\frac{i}{2}qL\sum_{mn<0} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|m|}-1\big) &= \frac{i}{2}qL\cdot 2i\,\mathrm{Im}\sum_{n>0>m} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|m|}-1\big)\\ &= -\,qL\,\mathrm{Im}\sum_{n>0>m} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|m|}-1\big)\\ &= -\,qL\,\mathrm{Im}\sum_{k,n>0} n\hat{v}_{-n}\hat{v}_{n+k}\hat{v}_{-k}\big(e^{-2\alpha tqk}-1\big).\end{aligned} \tag{2.16}$$

Considering the second sum on the right-hand side of (2.15), we have

$$\begin{aligned}&\frac{i}{2}qL\sum_{mn>0,\,|m|>|n|} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|n-m|}-1\big)\\ &= \frac{i}{2}qL\sum_{m>n>0} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|n-m|}-1\big)+\frac{i}{2}qL\sum_{m<n<0} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|n-m|}-1\big).\end{aligned}$$

Setting $m' = -m$ and $n' = -n$, the second sum becomes

$$\begin{aligned}&\frac{i}{2}qL\sum_{m<n<0} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|n-m|}-1\big)\\ &= \frac{i}{2}qL\sum_{m'>n'>0}(-n')\hat{v}_{n'}\hat{v}_{m'-n'}\hat{v}_{-m'}\big(e^{-2\alpha tq|n'-m'|}-1\big)\\ &= \frac{i}{2}qL\sum_{m>n>0}(-n)\bar{\hat{v}}_{-n}\bar{\hat{v}}_{n-m}\bar{\hat{v}}_{m}\big(e^{-2\alpha tq|n-m|}-1\big),\end{aligned}$$

which shows that this second sum is minus the complex conjugate of the first sum. Thus,

$$\frac{i}{2}qL\sum_{mn>0,\,|m|>|n|} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|n-m|}-1\big)$$

$$= -qL \,\mathrm{Im} \sum_{m>n>0} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|n-m|} - 1\big).$$

A further change of indices $k = m - n$ yields

$$-qL \,\mathrm{Im} \sum_{m>n>0} n\hat{v}_{-n}\hat{v}_{n-m}\hat{v}_m\big(e^{-2\alpha tq|n-m|} - 1\big) = -qL \,\mathrm{Im} \sum_{k,n>0} n\hat{v}_{-n}\hat{v}_{-k}\hat{v}_{k+n}\big(e^{-2\alpha tqk} - 1\big). \tag{2.17}$$

Combining (2.16) and (2.17) we finally obtain that

$$b\big(u, u, e^{\alpha tA}v\big) \;=\; -2qL \,\mathrm{Im} \sum_{k,n>0} n\hat{v}_{-n}\hat{v}_{-k}\hat{v}_{k+n}\big(e^{-2\alpha tqk} - 1\big). \tag{2.18}$$

Note that (2.18) is identical to the formula

$$b(u, u, e^{\alpha tA}v) = 2qL \,\mathrm{Im} \sum_{k>m>0} (k - m)\hat{v}_{-m}\hat{v}_k\hat{v}_{-(k-m)}\big(1 - e^{-2\alpha tq|m|}\big), \tag{2.19}$$

which appears in [11]. To see this set $n = k - m$ in (2.19). Cauchy–Schwarz now provides

$$\begin{aligned}
|b(u, u, e^{atA}v)| &\leq 2L \sum_{m,n>0} qm\, qn\, |\hat{v}_n|\, |\hat{v}_m|\, |\hat{v}_{m+n}| \Big(\frac{1 - e^{-2atqm}}{qm}\Big) \\
&\leq 2L\Big(\sum_{m,n>0} (qn)^2\, |\hat{v}_n|^2\, |\hat{v}_m|^2\Big)^{1/2} \Big(\sum_{m,n>0} (qm)^2|\hat{v}_{m+n}|^2\Big(\frac{1 - e^{-2atqm}}{qm}\Big)^2\Big)^{1/2}.
\end{aligned} \tag{2.20}$$

Clearly,

$$\Big(\sum_{m,n>0} (qn)^2\, |\hat{v}_n|^2\, |\hat{v}_m|^2\Big)^{1/2} \;=\; \Big(\sum_{m>0} |\hat{v}_m|^2\Big)^{1/2} \Big(\sum_{n>0} (qn)^2\, |\hat{v}_n|^2\Big)^{1/2}, \tag{2.21}$$

while

$$\begin{aligned}
\Big(\sum_{m,n>0}(qm)^2|\hat{v}_{m+n}|^2\Big(\frac{1-e^{-2atqm}}{qm}\Big)^2\Big)^{1/2} &\leq \Big(\sum_{m,n>0} \big(q(m+n)\big)^2|\hat{v}_{m+n}|^2\Big(\frac{1-e^{-2atqm}}{qm}\Big)^2\Big)^{1/2} \\
&= \Big(\sum_{k>m>0} (qk)^2|\hat{v}_k|^2\Big(\frac{1-e^{-2atqm}}{qm}\Big)^2\Big)^{1/2} \\
&\leq \Big(\sum_{k,m>0} (qk)^2|\hat{v}_k|^2\Big(\frac{1-e^{-2atqm}}{qm}\Big)^2\Big)^{1/2} \\
&= \Big(\sum_{k>0} (qk)^2\, |\hat{v}_k|^2\Big)^{1/2} \Big(\sum_{m>0} \Big(\frac{1-e^{-2atqm}}{qm}\Big)^2\Big)^{1/2}.
\end{aligned} \tag{2.22}$$

Since the function $f(x) = \Big(\dfrac{1-e^{-2atqx}}{qx}\Big)^2$ is monotonically decreasing, we have that

$$\sum_{m>0}\Big(\frac{1-e^{-2atqm}}{qm}\Big)^2 \;\le\; \int_0^\infty f(x)\,dx \;=\; \frac{2at}{q}\int_0^\infty \Big(\frac{1-e^{-x}}{x}\Big)^2 dx. \tag{2.23}$$

Combining (2.20–2.23) we obtain that

$$\begin{aligned}
|b(u,u,e^{atA}v)| \;&\le 2(2\pi)^{1/2}cL\Big(\frac{2at}{q}\Big)^{1/2}\Big(\sum_{m>0}|\hat{v}_m|^2\Big)^{1/2}\sum_{n>0}(qn)^2\,|\hat{v}_n|^2\\
&= 2^{3/2}cL^{3/2}(at)^{1/2}\Big(\sum_{m>0}|\hat{v}_m|^2\Big)^{1/2}\sum_{n>0}(qn)^2\,|\hat{v}_n|^2\\
&= 2^{3/2}cL^{3/2}(at)^{1/2}\|v\|\|Av\|^2,
\end{aligned}$$

where

$$c \;=\; \Big(\frac{1}{2\pi}\int_0^\infty \Big(\frac{1-e^{-x}}{x}\Big)^2 dx\Big)^{1/2}$$

and so Lemma 2.2.1 follows with $C \;=\; 2^{3/2}cL^{3/2}$. □

From now on we assume (2.9). Combining (2.8) with (2.9), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le \alpha\|A^{\frac{1}{2}}v\|^2 + \|Av\|^2 - \|A^2v\|^2 + C\sqrt{\alpha t}\,\|v\|\,\|Av\|^2. \tag{2.24}$$

Now,

$$\|Av\|^2 = (Av,Av) = (v,A^2v),$$

whence

$$\|Av\|^2 \le \|v\|\,\|A^2v\|. \tag{2.25}$$

Furthermore,

$$\|A^{\frac{1}{2}}v\|^2 = (A^{\frac{1}{2}}v, A^{\frac{1}{2}}v) = (v,Av) \le \|v\|\,\|Av\|,$$

whence, in view of (2.25),

$$\|A^{\frac{1}{2}}v\|^2 \le \|v\|^{\frac{3}{2}}\,\|A^2v\|^{\frac{1}{2}}. \tag{2.26}$$

Combination of (2.24), (2.25) and (2.26) provides that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le \alpha\|v\|^{\frac{3}{2}}\,\|A^2v\|^{\frac{1}{2}} + \|v\|\,\|A^2v\| - \|A^2v\|^2 + C\sqrt{\alpha t}\,\|v\|^2\,\|A^2v\|.$$

Using here Young's inequality, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2 \le \frac{\big(\alpha\|v\|^{\frac{3}{2}}\big)^{\frac{4}{3}}}{\frac{4}{3}} + \frac{\big(\|A^2v\|^{\frac{1}{2}}\big)^4}{4} + \|v\|^2 + \frac{1}{4}\|A^2v\|^2 - \|A^2v\|^2\\
+\frac{C^2}{2}\alpha t\,\|v\|^4 + \frac{1}{2}\|A^2v\|^2,
\end{aligned}$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq \frac{3}{4}\alpha^{\frac{4}{3}}\|v\|^2 + \|v\|^2 + \frac{C^2}{2}\alpha t\,\|v\|^4,$$

whence

$$\frac{d}{dt}\|v\|^2 \leq \Big(2 + \frac{3}{2}\alpha^{\frac{4}{3}}\Big)\|v\|^2 + C^2\alpha t\,\|v\|^4. \tag{2.27}$$

Setting $\Phi(t) := \|v(\cdot,t)\|^2$, we write (2.27) in the form

$$\Phi'(t) \leq \big(C_1 + C_2\alpha^{\frac{4}{3}}\big)\Phi(t) + C_3\alpha t\big(\Phi(t)\big)^2. \tag{2.28}$$

Assume now that $\Phi(0) \leq R_L^2$, with $R_L^2 \geq \limsup_{t\to\infty}\int_0^L |u(x,t)|^2\,dx$. As long as $\Phi(t) \leq 4R_L^2$ holds, relation (2.28) implies

$$\Phi'(t) \leq \big(C_1 + C_2\alpha^{\frac{4}{3}} + 4C_3R_L^2\alpha t\big)\Phi(t),$$

whence

$$\Phi(t) \leq \Phi(0)\exp\big[(C_1 + C_2\alpha^{\frac{4}{3}})t + 2C_3R_L^2\alpha t^2\big].$$

As long as

$$(C_1 + C_2\alpha^{\frac{4}{3}})t + 2C_3R_L^2\alpha t^2 \leq \log 4,$$

we obviously have $\Phi(t) \leq 4R_L^2$.

This holds for $t \leq t_L$ which is the positive root of the quadratic

$$2C_3R_L^2\alpha t^2 + (C_1 + C_2\alpha^{\frac{4}{3}})t - \log 4 \;=\; 0,$$

which is

$$t_L \;=\; \frac{-(C_1 + C_2\alpha^{\frac{4}{3}}) + \big((C_1 + C_2\alpha^{\frac{4}{3}})^2 + 8\log 4\, C_3R_L^2\alpha\big)^{1/2}}{4C_3R_L^2\alpha}.$$

Note that t_L depends on α and L. The objective is to maximize the product αt_L, for large L (equivalently, large R_L), in order to optimize the width of the band of analyticity. Writing $\alpha = \alpha_0 R_L^{\gamma}$, it is easy to show that $\alpha_L t_L$ is optimal when $\gamma = 6/5$, which leads to

$$\alpha_L t_L \;\geq\; kR_L^{-2/5}, \tag{2.29}$$

for large R_L, where k is a suitable positive constant. □

Therefore, the following has been proved.

Theorem 2.2.1. *For sufficiently large t, the solution of the KS equation extends as a holomorphic function of x in a strip (in $\mathbb{C}$) of width*

$$\beta_L \geq kR_L^{-2/5},$$

around the real axis, where k is a positive constant independent of L.

Remark 2.2.1. An alternative way to see how (2.29) arises, is to use Lagrange multipliers as we describe next. We want to maximize

$$H(\alpha, t) = \alpha t,$$

subject to

$$F(\alpha, t) = 2C_3 R_L^2 \alpha t^2 + C_1 t + C_2 \alpha^{\frac{4}{3}} t - \log 4 \leq 0. \tag{2.30}$$

Proof. First, note that, it suffices to maximize H, subject to $F(\alpha, t) = 0$, instead of (2.30). We consider the Lagrange function

$$G(\alpha, t, \lambda) = \alpha t + 2C_3 R_L^2 \alpha t^2 \lambda + C_1 t\lambda + C_2 \alpha^{\frac{4}{3}} t\lambda - (\log 4)\lambda.$$

For the partial derivatives of G we have that

$$\begin{aligned} G_\alpha(\alpha, t, \lambda) &= t + 2C_3 R_L^2 t^2 \lambda + \frac{4}{3} C_2 \alpha^{\frac{1}{3}} t\lambda, \\ G_t(\alpha, t, \lambda) &= \alpha + 4C_3 R_L^2 \alpha t\lambda + C_1 \lambda + C_2 \alpha^{\frac{4}{3}} \lambda, \\ G_\lambda(\alpha, t, \lambda) &= 2C_3 R_L^2 \alpha t^2 + C_1 t + C_2 \alpha^{\frac{4}{3}} t - \log 4. \end{aligned}$$

Now, we have to solve the system $G_\alpha = G_t = G_\lambda = 0$. More precisely we have the system

$$t + 2C_3 R_L^2 t^2 \lambda + \frac{4}{3} C_2 \alpha^{\frac{1}{3}} t\lambda = 0, \tag{2.31}$$

$$\alpha + 4C_3 R_L^2 \alpha t\lambda + C_1 \lambda + C_2 \alpha^{\frac{4}{3}} \lambda = 0, \tag{2.32}$$

$$2C_3 R_L^2 \alpha t^2 + C_1 t + C_2 \alpha^{\frac{4}{3}} t - \log 4 = 0. \tag{2.33}$$

Multiplying (2.31) with α and (2.32) with $-t$, and then summing the two equations we get

$$-2C_3 R_L^2 \alpha t^2 \lambda - C_1 t\lambda + \frac{1}{3} C_2 \alpha^{\frac{4}{3}} t\lambda = 0,$$

i.e.,

$$-2C_3 R_L^2 \alpha t^2 - C_1 t + \frac{1}{3} C_2 \alpha^{\frac{4}{3}} t = 0. \tag{2.34}$$

Summing (2.33) and (2.34), we get

$$\frac{4}{3} C_2 \alpha^{\frac{4}{3}} t - \log 4 = 0,$$

which gives us that

$$t = \frac{3 \log 4}{4C_2} \alpha^{-\frac{4}{3}},$$

i.e.,

$$t = C_5 \alpha^{-\frac{4}{3}}. \tag{2.35}$$

Combining (2.33) and (2.35), we obtain that

$$C_6 R_L^2 \alpha^{-\frac{5}{3}} + C_7 \alpha^{-\frac{4}{3}} + C_2 C_5 - \log 4 = 0. \tag{2.36}$$

Now, note that

$$C_2 C_5 - \log 4 = C_2 \frac{3 \log 4}{4C_2} - \log 4 = -\frac{\log 4}{4} < 0.$$

Let $\xi = \alpha^{-\frac{1}{3}}$ and (2.36) becomes

$$\xi^5 + \frac{C_8}{R_L^2}\xi^4 - \frac{C_9}{R_L^2} = 0.$$

Consider now the function

$$g(\xi) = \xi^5 + \frac{C_8}{R_L^2}\xi^4 - \frac{C_9}{R_L^2}, \quad \text{where } \xi \in [0, \infty).$$

We have that

$$g(0) = -\frac{C_9}{R_L^2} < 0 \quad \text{and} \quad \lim_{\xi \to \infty} g(\xi) = \infty.$$

For the derivative of g we have that

$$g'(\xi) = 5\xi^4 + \frac{4C_8}{R_L^2}\xi^3,$$

hence $g' > 0$ in $(0, \infty)$ and so the function g is strictly increasing in $(0, \infty)$. Since g is a continuous function we finally have that intersects the axis of Ox at exactly one point, say at $(\xi_*, 0)$. So

$$\xi_*^5 \leq \frac{C_9}{R_L^2}, \text{ i.e., } \xi_* \leq C_9^{\frac{1}{5}} R_L^{-\frac{2}{5}}.$$

Furthermore, since the function g is strictly increasing, we get

$$\xi_*^5 + \frac{C_8}{R_L^2}\left(C_9^{\frac{1}{5}} R_L^{-\frac{2}{5}}\right)^4 - \frac{C_9}{R_L^2} \geq 0,$$

i.e.,

$$\xi_*^5 \geq \frac{C_9}{R_L^2} - \frac{C_8 C_9^{\frac{4}{5}}}{R_L^{\frac{18}{5}}},$$

which gives that (notice that $\frac{18}{5} > 2$),

$$\xi_*^5 \geq \frac{1}{2}\frac{C_9}{R_L^2}, \quad \text{for sufficiently large } R_L.$$

So we have that

$$\xi_* \geq C_{10} R_L^{-\frac{2}{5}}, \quad \text{for sufficiently large } R_L,$$

which implies that

$$\alpha_*^{-\frac{1}{3}} \geq C_{10} R_L^{-\frac{2}{5}}, \quad \text{for sufficiently large } R_L,$$

i.e.,

$$\alpha_* \geq C_{11} R_L^{\frac{6}{5}}, \quad \text{for sufficiently large } R_L. \tag{2.37}$$

From (2.35), $t_* = C_5 \alpha_*^{-\frac{4}{3}}$, whence, in view of (2.37),

$$t_* \geq C_5 C_{11}^{-\frac{4}{3}} R_L^{-\frac{8}{5}}, \quad \text{for sufficiently large } R_L. \tag{2.38}$$

Finally, combining (2.37) and (2.38) we have that

$$\alpha_* t_* \geq C_{11} R_L^{\frac{6}{5}} C_5 C_{11}^{-\frac{4}{3}} R_L^{-\frac{8}{5}} = C_{12} R_L^{-\frac{2}{5}}, \quad \text{for sufficiently large } R_L,$$

where C_{12} is a suitable positive constant. □

Chapter 3

Analyticity of dissipative-dispersive systems

In this chapter, we study the analyticity properties of solutions of dissipative-dispersive systems, with periodic initial data, by exploring the applicability of the semigroup method, which was developed in Collet *et al.* [11]. We establish the analyticity, with respect to the spatial variable in a strip around the real axis, for a variety of such systems, including the dispersively modified KS equation (1.6), the nonlocal KS equation (1.11) with the plus sign in front of the u_{xx} term and the dispersively modified Otto's model (1.15), and which possess global attractors. We also provide lower bounds for the width of the strip of analyticity. This chapter follows the paper [25].

3.1 Introduction

It is noteworthy that in the case of vanishing dispersion, i.e., $\mathcal{D} \equiv 0$, equation (1.6) reduces to the well-known KS equation (1.1) defined on L-periodic intervals. Now, note that equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} \, = \, 0, \tag{3.1}$$

defined on 2π-periodic intervals, is obtained, from equation (1.1) given on L-periodic intervals by the following rescaling (dropping the bars):

$$\bar{t} \, = \, \nu t, \quad \bar{x} \, = \, \nu^{1/2} x, \quad \bar{u} \, = \, \nu^{-1/2} u, \tag{3.2}$$

where $\nu = (2\pi/L)^2$. Let us explain how this rescaling works. First, by setting $\bar{x} = (2\pi/L)x$ we see that the interval $[0, L]$ transformed to $[0, 2\pi]$. Also by using

$$\frac{\partial}{\partial x} \mapsto \frac{2\pi}{L}\frac{\partial}{\partial \bar{x}},$$

in (1.1), we get

$$u_t + \frac{2\pi}{L} u u_{\bar{x}} + \left(\frac{2\pi}{L}\right)^2 u_{\bar{x}\bar{x}} + \left(\frac{2\pi}{L}\right)^4 u_{\bar{x}\bar{x}\bar{x}\bar{x}} \, = \, 0, \tag{3.3}$$

and then by using

$$\frac{\partial}{\partial t} \mapsto \zeta \frac{\partial}{\partial \bar{t}}, \quad \text{with } \zeta \in \mathbb{R},$$

in (3.3), we get

$$\zeta u_{\bar{t}} + \frac{2\pi}{L} u u_{\bar{x}} + \left(\frac{2\pi}{L}\right)^2 u_{\bar{x}\bar{x}} + \left(\frac{2\pi}{L}\right)^4 u_{\bar{x}\bar{x}\bar{x}\bar{x}} = 0. \tag{3.4}$$

Finally, if we set $u = \eta\, \bar{u}$, with $\eta \in \mathbb{R}$, we write (3.4) in the form

$$\zeta \bar{u}_{\bar{t}} + \frac{2\pi}{L} \eta\, \bar{u} \bar{u}_{\bar{x}} + \left(\frac{2\pi}{L}\right)^2 \bar{u}_{\bar{x}\bar{x}} + \left(\frac{2\pi}{L}\right)^4 \bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}} = 0, \tag{3.5}$$

and by choosing in (3.5), $\zeta = (2\pi/L)^2$ and $\eta = 2\pi/L$ we take

$$\bar{u}_{\bar{t}} + \bar{u} \bar{u}_{\bar{x}} + \bar{u}_{\bar{x}\bar{x}} + \left(\frac{2\pi}{L}\right)^2 \bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}} = 0. \tag{3.6}$$

So, in (3.6) if we drop the bars and set $\nu = (2\pi/L)^2$ we get equation (3.1).

Equation (1.6) is obtained, from the following equation given on L-periodic interval

$$u_t + u u_x + u_{xx} + u_{xxxx} + \mathcal{D} u = 0,$$

by the rescaling given in (3.2) (dropping the bars) where $\nu = (2\pi/L)^2$, in such a way as we have seen before for equations (1.1) and (3.1).

Equation (1.11) with the plus sign in front of the u_{xx} term is obtained, from the following equation given on L-periodic interval

$$u_t + u u_x + u_{xx} + u_{xxxx} + \gamma \mathcal{H}[u]_{xxx} = 0,$$

where $\gamma \geq 0$, by the rescaling given in (3.2) (dropping the bars) where $\nu = (2\pi/L)^2$ and $\mu = (2\pi/L)\gamma$, in such a way as we have seen before for equations (1.1) and (3.1).

3.2 Analyticity of solutions

3.2.1 The dispersively modified Kuramoto–Sivashinsky equation

Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a smooth, 2π-periodic in the spatial variable, function, such that $u(\cdot, t)$ has vanishing mean value, for all $t \geq 0$, and satisfies the dispersively modified KS equation

$$u_t + u u_x + u_{xx} + \nu u_{xxxx} + \mathcal{D} u = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{3.7}$$

where ν is a positive constant and $\mathcal{D}$ is a linear antisymmetric pseudo-differential operator.

For a positive constant α and the operator A introduced in (2.5), we define the function v by

$$v(x,t) := (e^{\alpha t A}u)(x,t). \tag{3.8}$$

Then, (3.7) takes the form

$$(e^{-\alpha t A}v)_t + (e^{-\alpha t A})(v_{xx} + \nu v_{xxxx}) + uu_x + \mathcal{D}u = 0,$$

i.e.,

$$(e^{-\alpha t A})v_t - \alpha e^{-\alpha t A}Av + e^{-\alpha t A}(v_{xx} + \nu v_{xxxx}) + uu_x + \mathcal{D}u = 0. \tag{3.9}$$

Taking in (3.9) the L^2 inner product with $e^{\alpha t A}v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - \alpha(Av,v) - \|v_x\|^2 + \nu\|v_{xx}\|^2 + (\mathcal{D}u, e^{\alpha t A}v) + (uu_x, e^{\alpha t A}v) = 0. \tag{3.10}$$

For $(\mathcal{D}u, e^{\alpha t A}v)$, we have

$$\begin{aligned}
(\mathcal{D}u, e^{\alpha t A}v) &= \int_0^{2\pi} (\mathcal{D}u)(e^{\alpha t A}\bar{v})\,dx = \int_0^{2\pi} \Big(\mathcal{D}\sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx}\Big)\Big(\sum_{\ell\in\mathbb{Z}} e^{\alpha t|\ell|}\bar{\hat{v}}_\ell e^{-i\ell x}\Big)dx \\
&= \int_0^{2\pi} \Big(\sum_{k\in\mathbb{Z}} i d_k \hat{u}_k e^{ikx}\Big)\Big(\sum_{\ell\in\mathbb{Z}} e^{\alpha t|\ell|}\bar{\hat{v}}_\ell e^{-i\ell x}\Big)dx \\
&= i\int_0^{2\pi} \sum_{k,\ell\in\mathbb{Z}} d_k \hat{u}_k \bar{\hat{v}}_\ell e^{\alpha t|\ell|} e^{i(k-\ell)x}dx = 2\pi i \sum_{\substack{\ell=k \\ \text{and } k\in\mathbb{Z}}} d_k \hat{u}_k \bar{\hat{v}}_\ell e^{\alpha t|k|} \\
&= 2\pi i \sum_{k\in\mathbb{Z}} d_k \hat{u}_k \bar{\hat{v}}_k e^{\alpha t|k|} = 0,
\end{aligned} \tag{3.11}$$

using in the last equality from (1.7) that $d_{-k} = -d_k \in \mathbb{R}$, and so (3.10), in view of (3.11), becomes

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - \alpha(Av,v) - \|v_x\|^2 + \nu\|v_{xx}\|^2 + (uu_x, e^{\alpha t A}v) = 0. \tag{3.12}$$

Now, combining (3.12) with (2.9), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le \alpha\|A^{\frac{1}{2}}v\|^2 + \|Av\|^2 - \nu\|A^2 v\|^2 + C\sqrt{\alpha t}\,\|v\|\,\|Av\|^2. \tag{3.13}$$

Combination of (3.13), (2.25) and (2.26) provides that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le \alpha\|v\|^{\frac{3}{2}}\,\|A^2 v\|^{\frac{1}{2}} + \|v\|\,\|A^2 v\| - \nu\|A^2 v\|^2 + C\sqrt{\alpha t}\,\|v\|^2\,\|A^2 v\|.$$

Using here Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \le \frac{\big(\frac{\alpha}{\varepsilon_1}\|v\|^{\frac{3}{2}}\big)^{\frac{4}{3}}}{\frac{4}{3}} + \frac{\big(\varepsilon_1\|A^2 v\|^{\frac{1}{2}}\big)^4}{4} + \varepsilon_2\|v\|^2 + \frac{1}{4\varepsilon_2}\|A^2 v\|^2 - \nu\|A^2 v\|^2$$

$$+\frac{C^2}{2\varepsilon_3^2}\alpha t\,\|v\|^4+\frac{\varepsilon_3^2}{2}\|A^2v\|^2,$$

i.e.,

$$\frac{1}{2}\frac{d}{dt}\|v\|^2\le\frac{3\alpha^{\frac{4}{3}}}{4\varepsilon_1^{\frac{4}{3}}}\|v\|^2+\frac{\varepsilon_1^4}{4}\|A^2v\|^2+\varepsilon_2\|v\|^2+\frac{1}{4\varepsilon_2}\|A^2v\|^2-\nu\|A^2v\|^2+\frac{C^2}{2\varepsilon_3^2}\alpha t\,\|v\|^4$$
$$+\frac{\varepsilon_3^2}{2}\|A^2v\|^2,$$

whence, choosing $\varepsilon_1=\nu^{1/4}$, $\varepsilon_2=1/\nu$, $\varepsilon_3=\nu^{1/2}$ we get

$$\frac{d}{dt}\|v\|^2\le\left(\frac{2}{\nu}+\frac{3\alpha^{\frac{4}{3}}}{2\nu^{\frac{1}{3}}}\right)\|v\|^2+\frac{C^2}{\nu}\alpha t\,\|v\|^4. \tag{3.14}$$

In the next, all C_n where $n=1,2,\dots,13$ are positive constants. Setting $\Phi(t)=\|v(\cdot,t)\|^2$, we write (3.14) in the form

$$\Phi'(t)\le\Big(\frac{C_1}{\nu}+\frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}\Big)\Phi(t)+\frac{C_3}{\nu}\alpha t\big(\Phi(t)\big)^2. \tag{3.15}$$

Assume now that $\Phi(0)\le R_\nu^2$. As long as $\Phi(t)\le 4R_\nu^2$ holds, relation (3.15) implies

$$\Phi'(t)\le\Big(\frac{C_1}{\nu}+\frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}+4\frac{C_3}{\nu}R_\nu^2\alpha t\Big)\Phi(t),$$

whence

$$\Phi(t)\le\Phi(0)\exp\left[\Big(\frac{C_1}{\nu}+\frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}\Big)t+2\frac{C_3}{\nu}R_\nu^2\alpha t^2\right].$$

As long as

$$\Big(\frac{C_1}{\nu}+\frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}\Big)t+2\frac{C_3}{\nu}R_\nu^2\alpha t^2\le\log 4,$$

we obviously have $\Phi(t)\le 4R_\nu^2$. This holds for $t\le t_\nu$ which is the positive root of the quadratic

$$2\frac{C_3}{\nu}R_\nu^2\alpha t^2+\Big(\frac{C_1}{\nu}+\frac{C_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}}\Big)t-\log 4\;=\;0. \tag{3.16}$$

Now, note that

$$R_\nu=c_2\nu^{-21/20}, \tag{3.17}$$

where c_2 is a positive constant. To see this, notice that from (1.3) we have

$$\limsup_{t\to\infty}\Big(\int_0^L|u(x,t)|^2\,dx\Big)^{1/2}\;\le\;c_0\,L^{8/5},\quad\text{i.e.,}$$
$$\limsup_{t\to\infty}\Big(\nu^{1/2}\int_0^{2\pi}|\bar u(\bar x,\bar t\,)|^2\,d\bar x\Big)^{1/2}\;\le\;c_2\,\nu^{-4/5},\quad\text{i.e.,}$$
$$\limsup_{t\to\infty}\Big(\int_0^{2\pi}|\bar u(\bar x,\bar t\,)|^2\,d\bar x\Big)^{1/2}\;\le\;c_2\,\nu^{-21/20}\;=\;R_\nu,$$

using in the second inequality (3.2) and the fact that $\nu = (2\pi/L)^2$.

Substituting (3.17) into (3.16), we obtain that

$$2C_4\nu^{-\frac{31}{10}}\alpha t^2 + C_1\nu^{-1}t + C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t - \log 4 = 0.$$

The objective is to maximize the product αt, for small ν, in order to optimize the width of the strip of analyticity.

We want to maximize

$$H(\alpha, t) = \alpha t,$$

subject to

$$F(\alpha, t) = 2C_4\nu^{-\frac{31}{10}}\alpha t^2 + C_1\nu^{-1}t + C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t - \log 4 \leq 0. \tag{3.18}$$

First, note that, it suffices to maximize H subject to $F(\alpha, t) = 0$, instead of (3.18). We consider the Lagrange function

$$G(\alpha, t, \lambda) = \alpha t + 2C_4\nu^{-\frac{31}{10}}\alpha t^2\lambda + C_1\nu^{-1}t\lambda + C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t\lambda - (\log 4)\lambda.$$

For the partial derivatives of G we have that

$$\begin{aligned}
G_\alpha(\alpha, t, \lambda) &= t + 2C_4\nu^{-\frac{31}{10}}t^2\lambda + \frac{4}{3}C_2\nu^{-\frac{1}{3}}\alpha^{\frac{1}{3}}t\lambda, \\
G_t(\alpha, t, \lambda) &= \alpha + 4C_4\nu^{-\frac{31}{10}}\alpha t\lambda + C_1\nu^{-1}\lambda + C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}\lambda, \\
G_\lambda(\alpha, t, \lambda) &= 2C_4\nu^{-\frac{31}{10}}\alpha t^2 + C_1\nu^{-1}t + C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t - \log 4.
\end{aligned}$$

Now, we have to solve the system $G_\alpha = G_t = G_\lambda = 0$. More precisely we have the system

$$t + 2C_4\nu^{-\frac{31}{10}}t^2\lambda + \frac{4}{3}C_2\nu^{-\frac{1}{3}}\alpha^{\frac{1}{3}}t\lambda = 0, \tag{3.19}$$

$$\alpha + 4C_4\nu^{-\frac{31}{10}}\alpha t\lambda + C_1\nu^{-1}\lambda + C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}\lambda = 0, \tag{3.20}$$

$$2C_4\nu^{-\frac{31}{10}}\alpha t^2 + C_1\nu^{-1}t + C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t - \log 4 = 0. \tag{3.21}$$

Multiplying (3.19) with α and (3.20) with $-t$, and then summing the two equations we get

$$-2C_4\nu^{-\frac{31}{10}}\alpha t^2\lambda - C_1\nu^{-1}t\lambda + \frac{1}{3}C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t\lambda = 0,$$

i.e.,

$$-2C_4\nu^{-\frac{31}{10}}\alpha t^2 - C_1\nu^{-1}t + \frac{1}{3}C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t = 0. \tag{3.22}$$

Summing (3.21) and (3.22), we get

$$C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t + \frac{1}{3}C_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t - \log 4 = 0,$$

which gives us that

$$t = \frac{3\log 4}{4C_2}\nu^{\frac{1}{3}}\alpha^{-\frac{4}{3}},$$

i.e.,

$$t = C_5\nu^{\frac{1}{3}}\alpha^{-\frac{4}{3}}. \tag{3.23}$$

Combining (3.21) and (3.23), we obtain that

$$C_6\nu^{-\frac{73}{30}}\alpha^{-\frac{5}{3}} + C_7\nu^{-\frac{2}{3}}\alpha^{-\frac{4}{3}} + C_2C_5 - \log 4 = 0. \tag{3.24}$$

Now, note that

$$C_2C_5 - \log 4 = C_2\frac{3\log 4}{4C_2} - \log 4 = -\frac{\log 4}{4} < 0.$$

Let $\xi = \alpha^{-\frac{1}{3}}$ and (3.24) becomes

$$\xi^5 + C_8\nu^{\frac{53}{30}}\xi^4 - C_9\nu^{\frac{73}{30}} = 0.$$

Consider now the function

$$g(\xi) = \xi^5 + C_8\nu^{\frac{53}{30}}\xi^4 - C_9\nu^{\frac{73}{30}}, \quad \text{where } \xi \in [0,\infty).$$

We have that

$$g(0) = -C_9\nu^{\frac{73}{30}} < 0 \quad \text{and} \quad \lim_{\xi\to\infty} g(\xi) = \infty.$$

For the derivative of g we have that

$$g'(\xi) = 5\xi^4 + 4C_8\nu^{\frac{53}{30}}\xi^3,$$

hence $g' > 0$ in $(0,\infty)$ and so the function g is strictly increasing in $(0,\infty)$. Since g is a continuous function we finally have that intersects the axis of Ox at exactly one point, say at $(\xi_*, 0)$. So

$$\xi_*^5 \le C_9\nu^{\frac{73}{30}}, \text{ i.e., } \xi_* \le C_9^{\frac{1}{5}}\nu^{\frac{73}{150}}.$$

Furthermore, since the function g is strictly increasing, we get

$$\xi_*^5 + C_8\nu^{\frac{53}{30}}(C_9^{\frac{1}{5}}\nu^{\frac{73}{150}})^4 - C_9\nu^{\frac{73}{30}} \ge 0,$$

i.e.,

$$\xi_*^5 \ge C_9\nu^{\frac{73}{30}} - C_8C_9^{\frac{4}{5}}\nu^{\frac{557}{150}},$$

which gives that (notice that $\frac{557}{150} > \frac{73}{30}$),

$$\xi_*^5 \geq \frac{1}{2} C_9 \nu^{\frac{73}{30}}, \quad \text{for sufficiently small } \nu.$$

So we have that

$$\xi_* \geq C_{10} \nu^{\frac{73}{150}}, \quad \text{for sufficiently small } \nu,$$

which implies that

$$\alpha_*^{-\frac{1}{3}} \geq C_{10} \nu^{\frac{73}{150}}, \quad \text{for sufficiently small } \nu,$$

i.e.,

$$\alpha_* \geq C_{11} \nu^{-\frac{73}{50}}, \quad \text{for sufficiently small } \nu. \tag{3.25}$$

From (3.23), $t_* = C_5 \nu^{\frac{1}{3}} \alpha_*^{-\frac{4}{3}}$, whence, in view of (3.25),

$$t_* \geq C_{12} \nu^{\frac{57}{25}}, \quad \text{for sufficiently small } \nu. \tag{3.26}$$

Finally, combining (3.25) and (3.26) we have that

$$\alpha_* t_* \geq C_{11} \nu^{-\frac{73}{50}} C_{12} \nu^{\frac{57}{25}} = C_{13} \nu^{\frac{41}{50}}, \quad \text{for sufficiently small } \nu,$$

where C_{13} is a suitable positive constant. □

Therefore, the following has been proved.

Theorem 3.2.1. *For sufficiently large t, the solution $u(x,t)$ of the equation* (1.6) *extends as a holomorphic function of x in a strip* (*in* $\mathbb{C}$) *of width*

$$\beta_\nu \geq b\, \nu^{41/50},$$

around the real axis, where b is a positive constant.

3.2.2 A nonlocal Kuramoto–Sivashinsky equation

Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a smooth, 2π-periodic in the spatial variable, function, such that $u(\cdot, t)$ has vanishing mean value, for all $t \geq 0$, and satisfies the nonlocal KS equation

$$u_t + u u_x + u_{xx} + \nu u_{xxxx} + \mu \mathcal{H}[u]_{xxx} = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{3.27}$$

where ν is a positive constant, μ is a non negative constant and $\mathcal{H}$ is the Hilbert transform operator defined in (1.12).

Using the definition of the function v, which given by (3.8), equation (3.27) takes the form

$$(e^{-\alpha t A} v)_t + (e^{-\alpha t A})(v_{xx} + \nu v_{xxxx} + \mu \mathcal{H}[v]_{xxx}) + u u_x = 0,$$

i.e.,

$$(e^{-\alpha t A})v_t - \alpha e^{-\alpha t A}Av + e^{-\alpha t A}(v_{xx} + \nu v_{xxxx} + \mu \mathcal{H}[v]_{xxx}) + uu_x = 0. \tag{3.28}$$

Taking in (3.28) the L^2 inner product with $e^{\alpha t A}v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - \alpha(Av, v) - \|v_x\|^2 + \nu\|v_{xx}\|^2 + \mu(\mathcal{H}[v]_{xxx}, v) + (uu_x, e^{\alpha t A}v) = 0. \tag{3.29}$$

Now, combining (3.29) with (2.9), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq \alpha\|A^{\frac{1}{2}}v\|^2 + \|Av\|^2 - \nu\|A^2 v\|^2 + \mu\|A^{\frac{3}{2}}v\|^2 + C\sqrt{\alpha t}\,\|v\|\,\|Av\|^2. \tag{3.30}$$

In arriving at the result above, we have used the fact

$$(\mathcal{H}[v]_{xxx}, v) = \|A^{\frac{3}{2}}v\|^2.$$

Combining (2.25), (2.26) and

$$\|A^{\frac{3}{2}}v\|^2 \leq \|v\|^{\frac{1}{2}}\,\|A^2 v\|^{\frac{3}{2}},$$

(3.30) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq\; &\alpha\|v\|^{\frac{3}{2}}\,\|A^2 v\|^{\frac{1}{2}} + \|v\|\,\|A^2 v\| - \nu\|A^2 v\|^2 + \mu\|v\|^{\frac{1}{2}}\,\|A^2 v\|^{\frac{3}{2}}\\ &+ C\sqrt{\alpha t}\,\|v\|^2\,\|A^2 v\|.\end{aligned}$$

Using here Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq &\frac{\left(\frac{\alpha}{\varepsilon_1}\|v\|^{\frac{3}{2}}\right)^{\frac{4}{3}}}{\frac{4}{3}} + \frac{\left(\varepsilon_1\|A^2 v\|^{\frac{1}{2}}\right)^4}{4} + \varepsilon_2\|v\|^2 + \frac{1}{4\varepsilon_2}\|A^2 v\|^2 - \nu\|A^2 v\|^2\\ &+ \frac{\left(\frac{\mu}{\varepsilon_3}\|v\|^{\frac{1}{2}}\right)^4}{4} + \frac{\left(\varepsilon_3\|A^2 v\|^{\frac{3}{2}}\right)^{\frac{4}{3}}}{\frac{4}{3}} + \frac{C^2}{2\varepsilon_4^2}\alpha t\,\|v\|^4 + \frac{\varepsilon_4^2}{2}\|A^2 v\|^2,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq &\frac{3\alpha^{\frac{4}{3}}}{4\varepsilon_1^{\frac{4}{3}}}\|v\|^2 + \frac{\varepsilon_1^4}{4}\|A^2 v\|^2 + \varepsilon_2\|v\|^2 + \frac{1}{4\varepsilon_2}\|A^2 v\|^2 - \nu\|A^2 v\|^2\\ &+ \frac{\mu^4}{4\varepsilon_3^4}\|v\|^2 + \frac{3\varepsilon_3^{\frac{4}{3}}}{4}\|A^2 v\|^2 + \frac{C^2}{2\varepsilon_4^2}\alpha t\,\|v\|^4 + \frac{\varepsilon_4^2}{2}\|A^2 v\|^2,\end{aligned}$$

whence, choosing $\varepsilon_1 = \nu^{1/4}/\sqrt{2}$, $\varepsilon_2 = 2/\nu$, $\varepsilon_3 = \nu^{3/4}$, $\varepsilon_4 = \nu^{1/2}/\sqrt{8}$ we get

$$\frac{d}{dt}\|v\|^2 \leq \left(\frac{4}{\nu} + \frac{3\alpha^{\frac{4}{3}}}{2^{\frac{1}{3}}\nu^{\frac{1}{3}}} + \frac{\mu^4}{2\nu^3}\right)\|v\|^2 + 8\frac{C^2}{\nu}\alpha t\,\|v\|^4. \tag{3.31}$$

In the next, all D_n where $n = 1, 2, \dots, 16$ are positive constants. Setting $\Phi(t) = \|v(\cdot,t)\|^2$, we write (3.31) in the form

$$\Phi'(t) \leq \Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3}\Big)\Phi(t) + \frac{D_4}{\nu}\alpha t\big(\Phi(t)\big)^2. \tag{3.32}$$

Assume now that $\Phi(0) \leq R^2_{\nu,\mu}$. As long as $\Phi(t) \leq 4R^2_{\nu,\mu}$ holds, relation (3.32) implies

$$\Phi'(t) \leq \Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3} + 4\frac{D_4}{\nu}R^2_{\nu,\mu}\alpha t\Big)\Phi(t),$$

whence

$$\Phi(t) \leq \Phi(0)\exp\left[\Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3}\Big)t + 2\frac{D_4}{\nu}R^2_{\nu,\mu}\alpha t^2\right].$$

As long as

$$\Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3}\Big)t + 2\frac{D_4}{\nu}R^2_{\nu,\mu}\alpha t^2 \leq \log 4,$$

we obviously have $\Phi(t) \leq 4R^2_{\nu,\mu}$. This holds for $t \leq t_{\nu,\mu}$ which is the positive root of the quadratic

$$2\frac{D_4}{\nu}R^2_{\nu,\mu}\alpha t^2 + \Big(\frac{D_1}{\nu} + \frac{D_2}{\nu^{\frac{1}{3}}}\alpha^{\frac{4}{3}} + \frac{D_3\mu^4}{\nu^3}\Big)t - \log 4 \;=\; 0. \tag{3.33}$$

Substituting (1.14) into (3.33), we obtain that

$$D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t^2 + D_1\nu^{-1}t + D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t + D_3\nu^{-3}\mu^4 t - \log 4 = 0.$$

The objective is to maximize the product αt, for small ν and large μ, in order to optimize the width of the strip of analyticity.

We want to maximize

$$Q(\alpha, t) = \alpha t,$$

subject to

$$Y(\alpha,t) = D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t^2 + D_1\nu^{-1}t + D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t + D_3\nu^{-3}\mu^4 t - \log 4 \leq 0. \tag{3.34}$$

First, note that, it suffices to maximize Q subject to $Y(\alpha,t) = 0$, instead of (3.34). We consider the Lagrange function

$$V(\alpha,t,\lambda) = \alpha t + D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t^2\lambda + D_1\nu^{-1}t\lambda + D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t\lambda + D_3\nu^{-3}\mu^4 t\lambda - (\log 4)\lambda.$$

For the partial derivatives of V we have that

$$V_\alpha(\alpha,t,\lambda) = t + D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}t^2\lambda + \frac{4}{3}D_2\nu^{-\frac{1}{3}}\alpha^{\frac{1}{3}}t\lambda,$$
$$V_t(\alpha,t,\lambda) = \alpha + 2D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t\lambda + D_1\nu^{-1}\lambda + D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}\lambda + D_3\nu^{-3}\mu^4\lambda,$$

$$V_\lambda(\alpha,t,\lambda) = D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t^2 + D_1\nu^{-1}t + D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t + D_3\nu^{-3}\mu^4 t - \log 4.$$

Now, we have to solve the system $V_\alpha = V_t = V_\lambda = 0$. More precisely we have the system

$$t + D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}t^2\lambda + \frac{4}{3}D_2\nu^{-\frac{1}{3}}\alpha^{\frac{1}{3}}t\lambda = 0, \tag{3.35}$$

$$\alpha + 2D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t\lambda + D_1\nu^{-1}\lambda + D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}\lambda + D_3\nu^{-3}\mu^4\lambda = 0, \tag{3.36}$$

$$D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t^2 + D_1\nu^{-1}t + D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t + D_3\nu^{-3}\mu^4 t - \log 4 = 0. \tag{3.37}$$

Multiplying (3.35) with α and (3.36) with $-t$, and then summing the two equations we get

$$-D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t^2\lambda - D_1\nu^{-1}t\lambda + \frac{1}{3}D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t\lambda - D_3\nu^{-3}\mu^4 t\lambda = 0,$$

i.e.,

$$-D_5\nu^{-\frac{36}{5}}\mu^{\frac{41}{5}}\alpha t^2 - D_1\nu^{-1}t + \frac{1}{3}D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t - D_3\nu^{-3}\mu^4 t = 0. \tag{3.38}$$

Summing (3.37) and (3.38), we get

$$D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t + \frac{1}{3}D_2\nu^{-\frac{1}{3}}\alpha^{\frac{4}{3}}t - \log 4 = 0,$$

which gives us that

$$t = \frac{3\log 4}{4D_2}\nu^{\frac{1}{3}}\alpha^{-\frac{4}{3}},$$

i.e.,

$$t = D_6\nu^{\frac{1}{3}}\alpha^{-\frac{4}{3}}. \tag{3.39}$$

Combining (3.37) and (3.39), we obtain that

$$D_7\nu^{-\frac{98}{15}}\mu^{\frac{41}{5}}\alpha^{-\frac{5}{3}} + (D_8\nu^{-\frac{2}{3}} + D_9\nu^{-\frac{8}{3}}\mu^4)\alpha^{-\frac{4}{3}} + D_2D_6 - \log 4 = 0. \tag{3.40}$$

Now, note that

$$D_2D_6 - \log 4 = D_2\frac{3\log 4}{4D_2} - \log 4 = -\frac{\log 4}{4} < 0.$$

Let $\psi = \alpha^{-\frac{1}{3}}$ and (3.40) becomes

$$\psi^5 + (D_{10}\nu^{\frac{88}{15}}\mu^{-\frac{41}{5}} + D_{11}\nu^{\frac{58}{15}}\mu^{-\frac{21}{5}})\psi^4 - D_{12}\nu^{\frac{98}{15}}\mu^{-\frac{41}{5}} = 0.$$

Consider now the function

$$p(\psi) = \psi^5 + (D_{10}\nu^{\frac{88}{15}}\mu^{-\frac{41}{5}} + D_{11}\nu^{\frac{58}{15}}\mu^{-\frac{21}{5}})\psi^4 - D_{12}\nu^{\frac{98}{15}}\mu^{-\frac{41}{5}}, \quad \text{where } \psi \in [0,\infty).$$

We have that

$$p(0) = -D_{12}\nu^{\frac{98}{15}}\mu^{-\frac{41}{5}} < 0 \quad \text{and} \quad \lim_{\psi\to\infty} p(\psi) = \infty.$$

For the derivative of p we have that

$$p'(\psi) = 5\psi^4 + 4(D_{10}\nu^{\frac{88}{15}}\mu^{-\frac{41}{5}} + D_{11}\nu^{\frac{58}{15}}\mu^{-\frac{21}{5}})\psi^3,$$

hence $p' > 0$ in $(0,\infty)$ and so the function p is strictly increasing in $(0,\infty)$. Since p is a continuous function we finally have that intersects the axis of Ox at exactly one point, say at $(\psi_*, 0)$. So

$$\psi_*^5 \leq D_{12}\nu^{\frac{98}{15}}\mu^{-\frac{41}{5}}, \text{ i.e., } \psi_* \leq D_{12}^{\frac{1}{5}}\nu^{\frac{98}{75}}\mu^{-\frac{41}{25}}.$$

Furthermore, since the function p is strictly increasing, we get

$$\psi_*^5 + (D_{10}\nu^{\frac{88}{15}}\mu^{-\frac{41}{5}} + D_{11}\nu^{\frac{58}{15}}\mu^{-\frac{21}{5}})(D_{12}^{\frac{1}{5}}\nu^{\frac{98}{75}}\mu^{-\frac{41}{25}})^4 - D_{12}\nu^{\frac{98}{15}}\mu^{-\frac{41}{5}} \geq 0,$$

i.e.,

$$\psi_*^5 \geq D_{12}\nu^{\frac{98}{15}}\mu^{-\frac{41}{5}} - D_{10}D_{12}^{\frac{4}{5}}\nu^{\frac{832}{75}}\mu^{-\frac{369}{25}} - D_{11}D_{12}^{\frac{4}{5}}\nu^{\frac{682}{75}}\mu^{-\frac{269}{25}},$$

which gives that (notice that $\frac{832}{75} > \frac{682}{75} > \frac{98}{15}$ and $\frac{369}{25} > \frac{269}{25} > \frac{41}{5}$),

$$\psi_*^5 \geq \frac{1}{2}D_{12}\nu^{\frac{98}{15}}\mu^{-\frac{41}{5}}, \quad \text{for sufficiently small } \nu \text{ and sufficiently large } \mu.$$

So we have that

$$\psi_* \geq D_{13}\nu^{\frac{98}{75}}\mu^{-\frac{41}{25}}, \quad \text{for sufficiently small } \nu \text{ and sufficiently large } \mu,$$

which implies that

$$\alpha_*^{-\frac{1}{3}} \geq D_{13}\nu^{\frac{98}{75}}\mu^{-\frac{41}{25}}, \quad \text{for sufficiently small } \nu \text{ and sufficiently large } \mu,$$

i.e.,

$$\alpha_* \geq D_{14}\nu^{-\frac{98}{25}}\mu^{\frac{123}{25}}, \quad \text{for sufficiently small } \nu \text{ and sufficiently large } \mu. \tag{3.41}$$

From (3.39), $t_* = D_6\nu^{\frac{1}{3}}\alpha_*^{-\frac{4}{3}}$, whence, in view of (3.41),

$$t_* \geq D_{15}\nu^{\frac{417}{75}}\mu^{-\frac{164}{25}}, \quad \text{for sufficiently small } \nu \text{ and sufficiently large } \mu. \tag{3.42}$$

Finally, combining (3.41) and (3.42) we have that

$$\alpha_* t_* \geq D_{14}\nu^{-\frac{98}{25}}\mu^{\frac{123}{25}}D_{15}\nu^{\frac{417}{75}}\mu^{-\frac{164}{25}} = D_{16}\left(\frac{\nu}{\mu}\right)^{\frac{41}{25}},$$

for sufficiently small ν and sufficiently large μ, where D_{16} is a suitable positive constant. $\square$

Therefore, the following has been proved.

Theorem 3.2.2. *For sufficiently large t, the solution $u(x,t)$ of the equation* (1.11) *with the plus sign in front of the u_{xx} term extends as a holomorphic function of x in a strip (in $\mathbb{C}$) of width*

$$\delta_{\nu,\mu} \geq d\Big(\frac{\nu}{\mu}\Big)^{41/25},$$

around the real axis, where d is a positive constant.

3.2.3 The dispersively Otto's model

Let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a smooth, L-periodic in the spatial variable, function, such that $u(\cdot, t)$ has vanishing mean value, for all $t \geq 0$, and satisfies the dispersively Otto's model

$$u_t + uu_x - |\partial_x|^\alpha u + |\partial_x|^\beta u + \mathcal{D}u = 0, \quad x \in \mathbb{R},\ t \geq 0, \tag{3.43}$$

where $\beta > \alpha \geq 0$, $\beta > 2$ and $\mathcal{D}$ is a linear antisymmetric pseudo-differential operator.

For a positive constant ϑ and the operator A introduced in (2.5), we define the function v by

$$v(x,t) = (e^{\vartheta t A}u)(x,t).$$

Then, (3.43) takes the form

$$(e^{-\vartheta t A}v)_t + (e^{-\vartheta t A})(-|\partial_x|^\alpha v + |\partial_x|^\beta v) + uu_x + \mathcal{D}u = 0,$$

i.e.,

$$(e^{-\vartheta t A})v_t - \vartheta e^{-\vartheta t A}Av + e^{-\vartheta t A}(-|\partial_x|^\alpha v + |\partial_x|^\beta v) + uu_x + \mathcal{D}u = 0. \tag{3.44}$$

Taking in (3.44) the L^2 inner product with $e^{\vartheta t A}v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - \vartheta(Av, v) - \||\partial_x|^{\frac{\alpha}{2}}v\|^2 + \||\partial_x|^{\frac{\beta}{2}}v\|^2 + (\mathcal{D}u, e^{\vartheta t A}v) + (uu_x, e^{\vartheta t A}v) = 0. \tag{3.45}$$

Now, (3.45), in view of (3.11), becomes

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 - \vartheta(Av, v) - \||\partial_x|^{\frac{\alpha}{2}}v\|^2 + \||\partial_x|^{\frac{\beta}{2}}v\|^2 + (uu_x, e^{\vartheta t A}v) = 0. \tag{3.46}$$

Combining (3.46) with (2.9), we get

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq \vartheta\|A^{\frac{1}{2}}v\|^2 + \|A^{\frac{\alpha}{2}}v\|^2 - \|A^{\frac{\beta}{2}}v\|^2 + C\sqrt{\vartheta t}\,\|v\|\,\|Av\|^2. \tag{3.47}$$

Lemma 3.2.1. *For every $\beta > \alpha > 0$ and v sufficiently smooth L-periodic function*

$$\|A^{\frac{\alpha}{2}} v\|^2 \leq \|v\|^{2-\frac{2\alpha}{\beta}} \, \|A^{\frac{\beta}{2}} v\|^{\frac{2\alpha}{\beta}}. \tag{3.48}$$

Proof. Using Hölder's inequality, we obtain

$$\begin{aligned}
\|A^{\frac{\alpha}{2}} v\|^2 &= \sum_{k=1}^{\infty} |k|^{\alpha} |\hat{v}_k|^2 = \sum_{k=1}^{\infty} |k|^{\alpha} |\hat{v}_k|^{\frac{2\alpha}{\beta}} |\hat{v}_k|^{\frac{2\beta-2\alpha}{\beta}} \\
&\leq \left(\sum_{k=1}^{\infty} \left(|\hat{v}_k|^{\frac{2(\beta-\alpha)}{\beta}} \right)^{\frac{\beta}{\beta-\alpha}} \right)^{\frac{\beta-\alpha}{\beta}} \left(\sum_{k=1}^{\infty} \left(|k|^{\alpha} |\hat{v}_k|^{\frac{2\alpha}{\beta}} \right)^{\frac{\beta}{\alpha}} \right)^{\frac{\alpha}{\beta}} \\
&= \left(\sum_{k=1}^{\infty} |\hat{v}_k|^2 \right)^{\frac{\beta-\alpha}{\beta}} \left(\sum_{k=1}^{\infty} |k|^{\beta} |\hat{v}_k|^2 \right)^{\frac{\alpha}{\beta}} = \|v\|^{2-\frac{2\alpha}{\beta}} \, \|A^{\frac{\beta}{2}} v\|^{\frac{2\alpha}{\beta}},
\end{aligned}$$

whenever $0 < \alpha < \beta < \infty$. □

Combining, (3.47) and (3.48), we obtain that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq \vartheta \|v\|^{2-\frac{2}{\beta}} \, \|A^{\frac{\beta}{2}} v\|^{\frac{2}{\beta}} + \|v\|^{2-\frac{2\alpha}{\beta}} \, \|A^{\frac{\beta}{2}} v\|^{\frac{2\alpha}{\beta}} - \|A^{\frac{\beta}{2}} v\|^2 + C\sqrt{\vartheta t} \, \|v\|^{3-\frac{4}{\beta}} \, \|A^{\frac{\beta}{2}} v\|^{\frac{4}{\beta}}.$$

Using here Young's inequality, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq{}& \frac{\left(\frac{\vartheta}{\varepsilon_1}\|v\|^{2-\frac{2}{\beta}}\right)^{\frac{\beta}{\beta-1}}}{\frac{\beta}{\beta-1}} + \frac{\left(\varepsilon_1 \|A^{\frac{\beta}{2}} v\|^{\frac{2}{\beta}}\right)^{\beta}}{\beta} + \frac{\left(\frac{1}{\varepsilon_2}\|v\|^{2-\frac{2\alpha}{\beta}}\right)^{\frac{\beta}{\beta-\alpha}}}{\frac{\beta}{\beta-\alpha}} + \frac{\left(\varepsilon_2 \|A^{\frac{\beta}{2}} v\|^{\frac{2\alpha}{\beta}}\right)^{\frac{\beta}{\alpha}}}{\frac{\beta}{\alpha}} \\
&- \|A^{\frac{\beta}{2}} v\|^2 + \frac{\left(\frac{1}{\varepsilon_3} C\sqrt{\vartheta t}\|v\|^{3-\frac{4}{\beta}}\right)^{\frac{\beta}{\beta-2}}}{\frac{\beta}{\beta-2}} + \frac{\left(\varepsilon_3 \|A^{\frac{\beta}{2}} v\|^{\frac{4}{\beta}}\right)^{\frac{\beta}{2}}}{\frac{\beta}{2}},
\end{aligned}$$

i.e.,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v\|^2 \leq{}& \frac{(\beta-1)\vartheta^{\frac{\beta}{\beta-1}}}{\beta \varepsilon_1^{\frac{\beta}{\beta-1}}}\|v\|^2 + \frac{\varepsilon_1^{\beta}\|A^{\frac{\beta}{2}} v\|^2}{\beta} + \frac{\beta-\alpha}{\beta \varepsilon_2^{\frac{\beta}{\beta-\alpha}}}\|v\|^2 + \frac{\alpha \varepsilon_2^{\frac{\beta}{\alpha}} \|A^{\frac{\beta}{2}} v\|^2}{\beta} - \|A^{\frac{\beta}{2}} v\|^2 \\
&+ \frac{(\beta-2)C^{\frac{\beta}{\beta-2}}(\vartheta t)^{\frac{\beta}{2(\beta-2)}}}{\beta \varepsilon_3^{\frac{\beta}{\beta-2}}}\|v\|^{\frac{3\beta-4}{\beta-2}} + \frac{2\varepsilon_3^{\frac{\beta}{2}} \|A^{\frac{\beta}{2}} v\|^2}{\beta},
\end{aligned}$$

whence, choosing $\varepsilon_1 = (\beta-2)^{1/\beta}$, $\varepsilon_2 = (1/\alpha)^{\alpha/\beta}$, $\varepsilon_3 = (1/2)^{2/\beta}$ we get

$$\frac{d}{dt}\|v\|^2 \leq \left(\frac{2(\beta-1)\vartheta^{\frac{\beta}{\beta-1}}}{\beta(\beta-2)^{\frac{1}{\beta-1}}} + \frac{2(\beta-\alpha)}{\beta\left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\beta-\alpha}}} \right)\|v\|^2 + \frac{2(\beta-2)C^{\frac{\beta}{\beta-2}}(\vartheta t)^{\frac{\beta}{2(\beta-2)}}}{\beta\left(\frac{1}{2}\right)^{\frac{2}{\beta-2}}}\|v\|^{\frac{3\beta-4}{\beta-2}}. \tag{3.49}$$

In the next, all E_n where $n = 1, 2, \ldots, 12$ are positive constants. Setting $\Phi(t) = \|v(\cdot,t)\|^2$, we write (3.49) in the form

$$\Phi'(t) \leq \left(E_1 \vartheta^{\frac{\beta}{\beta-1}} + E_2\right)\Phi(t) + E_3 (\vartheta t)^{\frac{\beta}{2(\beta-2)}} \left(\Phi(t)\right)^{\frac{3\beta-4}{2(\beta-2)}}. \tag{3.50}$$

Assume now that $\varPhi(0) \leq R_L^2$. As long as $\varPhi(t) \leq 4R_L^2$ holds, relation (3.50) implies

$$\varPhi'(t) \leq \big(E_1\vartheta^{\frac{\beta}{\beta-1}} + E_2 + E_3(2R_L)^{\frac{\beta}{\beta-2}}(\vartheta t)^{\frac{\beta}{2(\beta-2)}}\big)\varPhi(t),$$

whence

$$\varPhi(t) \leq \varPhi(0)\exp\Big[\big(E_1\vartheta^{\frac{\beta}{\beta-1}} + E_2\big)t + \frac{2(\beta-2)E_3(2R_L)^{\frac{\beta}{\beta-2}}\vartheta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}t^{\frac{3\beta-4}{2(\beta-2)}}\Big].$$

As long as

$$\big(E_1\vartheta^{\frac{\beta}{\beta-1}} + E_2\big)t + \frac{2(\beta-2)E_3(2R_L)^{\frac{\beta}{\beta-2}}\vartheta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}t^{\frac{3\beta-4}{2(\beta-2)}} \leq \log 4,$$

we obviously have $\varPhi(t) \leq 4R_L^2$. This holds for $t \leq t_L$ which is the positive root of the equation

$$\frac{2(\beta-2)E_3(2R_L)^{\frac{\beta}{\beta-2}}\vartheta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}t^{\frac{3\beta-4}{2(\beta-2)}} + \big(E_1\vartheta^{\frac{\beta}{\beta-1}} + E_2\big)t - \log 4 = 0.$$

Let us explain why

$$\frac{2(\beta-2)E_3(2R_L)^{\frac{\beta}{\beta-2}}\vartheta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4}t^{\frac{3\beta-4}{2(\beta-2)}} + \big(E_1\vartheta^{\frac{\beta}{\beta-1}} + E_2\big)t - \log 4 \leq 0, \text{ for } t \in [0, t_L]. \quad (3.51)$$

First, note that

$$\kappa = \frac{3\beta-4}{2(\beta-2)} \in (3/2, \infty) \quad \text{when } \beta > 2.$$

Also we have that both

$$E_4 = \frac{2(\beta-2)E_3(2R_L)^{\frac{\beta}{\beta-2}}\vartheta^{\frac{\beta}{2(\beta-2)}}}{3\beta-4} \quad \text{and} \quad \mu = E_1\vartheta^{\frac{\beta}{\beta-1}} + E_2 \quad \text{are positive.}$$

Consider now the function

$$f(t) = E_4 t^{\kappa} + \mu t - \log 4, \quad \text{where } t \in [0, \infty),$$

with $\kappa \in (3/2, \infty)$, $E_4 > 0$ and $\mu > 0$. We have that

$$f(0) = -\log 4 < 0 \quad \text{and} \quad \lim_{t\to\infty} f(t) = \infty.$$

For the derivative of f we have that

$$f'(t) = \kappa E_4 t^{\kappa-1} + \mu,$$

hence $f' > 0$ in $(0, \infty)$ and so the function f is strictly increasing in $(0, \infty)$. Since f

is a continuous function we finally have that intersects the axis of Ox at exactly one point, say at $(t_L, 0)$; and (3.51) is clear. The objective is to maximize the product ϑt, for large R_L, in order to optimize the width of the strip of analyticity.

We want to maximize

$$H(\vartheta, t) = \vartheta t,$$

subject to

$$F(\vartheta, t) = E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} + E_1 \vartheta^{\frac{\beta}{\beta-1}} t + E_2 t - \log 4 \leq 0. \tag{3.52}$$

First, note that, it suffices to maximize H, subject to $F(\vartheta, t) = 0$, instead of (3.52). We consider the Lagrange function

$$G(\vartheta, t, \lambda) = \vartheta t + E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} \lambda + E_1 \vartheta^{\frac{\beta}{\beta-1}} t\lambda + E_2 t\lambda - (\log 4)\lambda.$$

For the partial derivatives of G we have that

$$\begin{aligned} G_\vartheta(\vartheta, t, \lambda) &= t + \frac{\beta}{2(\beta-2)} E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{4-\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} \lambda + \frac{\beta}{\beta-1} E_1 \vartheta^{\frac{1}{\beta-1}} t\lambda, \\ G_t(\vartheta, t, \lambda) &= \vartheta + \frac{3\beta-4}{2(\beta-2)} E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{\beta}{2(\beta-2)}} t^{\frac{\beta}{2(\beta-2)}} \lambda + E_1 \vartheta^{\frac{\beta}{\beta-1}} \lambda + E_2 \lambda, \\ G_\lambda(\vartheta, t, \lambda) &= E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} + E_1 \vartheta^{\frac{\beta}{\beta-1}} t + E_2 t - \log 4. \end{aligned}$$

Now, we have to solve the system $G_\vartheta = G_t = G_\lambda = 0$. More precisely we have the system

$$t + \frac{\beta}{2(\beta-2)} E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{4-\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} \lambda + \frac{\beta}{\beta-1} E_1 \vartheta^{\frac{1}{\beta-1}} t\lambda = 0, \tag{3.53}$$

$$\vartheta + \frac{3\beta-4}{2(\beta-2)} E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{\beta}{2(\beta-2)}} t^{\frac{\beta}{2(\beta-2)}} \lambda + E_1 \vartheta^{\frac{\beta}{\beta-1}} \lambda + E_2 \lambda = 0, \tag{3.54}$$

$$E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} + E_1 \vartheta^{\frac{\beta}{\beta-1}} t + E_2 t - \log 4 = 0. \tag{3.55}$$

Multiplying (3.53) with ϑ and (3.54) with $-t$, and then summing the two equations we get

$$-E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} \lambda + \frac{1}{\beta-1} E_1 \vartheta^{\frac{\beta}{\beta-1}} t\lambda - E_2 t\lambda = 0,$$

i.e.,

$$-E_5 R_L^{\frac{\beta}{\beta-2}} \vartheta^{\frac{\beta}{2(\beta-2)}} t^{\frac{3\beta-4}{2(\beta-2)}} + \frac{1}{\beta-1} E_1 \vartheta^{\frac{\beta}{\beta-1}} t - E_2 t = 0. \tag{3.56}$$

Summing (3.55) and (3.56), we get

$$E_1 \vartheta^{\frac{\beta}{\beta-1}} t + \frac{1}{\beta-1} E_1 \vartheta^{\frac{\beta}{\beta-1}} t - \log 4 = 0,$$

which gives us that

$$t = \frac{(\beta-1)\log 4}{\beta E_1}\vartheta^{-\frac{\beta}{\beta-1}},$$

i.e.,

$$t = E_6\vartheta^{-\frac{\beta}{\beta-1}}. \tag{3.57}$$

Combining (3.55) and (3.57), we obtain that

$$E_7 R_L^{\frac{\beta}{\beta-2}}\vartheta^{-\frac{\beta(2\beta-3)}{2(\beta-1)(\beta-2)}} + E_2E_6\vartheta^{-\frac{\beta}{\beta-1}} + E_1E_6 - \log 4 = 0. \tag{3.58}$$

Now, note that

$$E_1E_6 - \log 4 = E_1\frac{(\beta-1)\log 4}{\beta E_1} - \log 4 = -\frac{\log 4}{\beta} < 0.$$

Let $\xi = \vartheta^{-\frac{\beta}{2(\beta-1)(\beta-2)}}$ and (3.58) becomes

$$\xi^{2\beta-3} + \frac{E_8}{R_L^{\frac{\beta}{\beta-2}}}\xi^{2(\beta-2)} - \frac{E_9}{R_L^{\frac{\beta}{\beta-2}}} = 0.$$

Consider now the function

$$g(\xi) = \xi^{2\beta-3} + \frac{E_8}{R_L^{\frac{\beta}{\beta-2}}}\xi^{2(\beta-2)} - \frac{E_9}{R_L^{\frac{\beta}{\beta-2}}}, \quad \text{where } \xi \in [0,\infty).$$

We have that

$$g(0) = -\frac{E_9}{R_L^{\frac{\beta}{\beta-2}}} < 0 \quad \text{and} \quad \lim_{\xi\to\infty} g(\xi) = \infty.$$

For the derivative of g we have that

$$g'(\xi) = (2\beta-3)\xi^{2\beta-4} + \frac{2(\beta-2)E_8}{R_L^{\frac{\beta}{\beta-2}}}\xi^{2\beta-5},$$

hence $g' > 0$ in $(0,\infty)$ and so the function g is strictly increasing in $(0,\infty)$. Since g is a continuous function we finally have that intersects the axis of Ox at exactly one point, say at $(\xi_*, 0)$. So

$$\xi_*^{2\beta-3} \le \frac{E_9}{R_L^{\frac{\beta}{\beta-2}}}, \text{ i.e., } \xi_* \le E_9^{\frac{1}{2\beta-3}} R_L^{-\frac{\beta}{(\beta-2)(2\beta-3)}}.$$

Furthermore, since the function g is strictly increasing, we get

$$\xi_*^{2\beta-3} + \frac{E_8}{R_L^{\frac{\beta}{\beta-2}}}\left(E_9^{\frac{1}{2\beta-3}} R_L^{-\frac{\beta}{(\beta-2)(2\beta-3)}}\right)^{2(\beta-2)} - \frac{E_9}{R_L^{\frac{\beta}{\beta-2}}} \ge 0,$$

i.e.,

$$\xi_*^{2\beta-3} \geq \frac{E_9}{R_L^{\frac{\beta}{\beta-2}}} - \frac{E_8 E_9^{\frac{2(\beta-2)}{2\beta-3}}}{R_L^{\frac{\beta(4\beta-7)}{(\beta-2)(2\beta-3)}}},$$

which gives that (notice that $(4\beta-7)/(2\beta-3) > 1$ for $\beta > 2$),

$$\xi_*^{2\beta-3} \geq \frac{1}{2}\frac{E_9}{R_L^{\frac{\beta}{\beta-2}}}, \quad \text{for sufficiently large } R_L.$$

So we have that

$$\xi_* \geq E_{10} R_L^{-\frac{\beta}{(\beta-2)(2\beta-3)}}, \quad \text{for sufficiently large } R_L,$$

which implies that

$$\vartheta_*^{-\frac{\beta}{2(\beta-1)(\beta-2)}} \geq E_{10} R_L^{-\frac{\beta}{(\beta-2)(2\beta-3)}}, \quad \text{for sufficiently large } R_L,$$

i.e.,

$$\vartheta_* \geq E_{11} R_L^{\frac{2(\beta-1)}{2\beta-3}}, \quad \text{for sufficiently large } R_L. \tag{3.59}$$

From (3.57), $t_* = E_6 \vartheta_*^{-\frac{\beta}{\beta-1}}$, whence, in view of (3.59),

$$t_* \geq E_6 E_{11}^{-\frac{\beta}{\beta-1}} R_L^{-\frac{2\beta}{2\beta-3}}, \quad \text{for sufficiently large } R_L. \tag{3.60}$$

Finally, combining (3.59) and (3.60) we have that

$$\vartheta_* t_* \geq E_{11} R_L^{\frac{2(\beta-1)}{2\beta-3}} E_6 E_{11}^{-\frac{\beta}{\beta-1}} R_L^{-\frac{2\beta}{2\beta-3}} = E_{12} R_L^{-\frac{2}{2\beta-3}}, \quad \text{for sufficiently large } R_L,$$

where E_{12} is a suitable positive constant. □

Therefore, the following has been proved.

Theorem 3.2.3. *For sufficiently large t, the solution $u(x,t)$ of the equation* (1.15) *extends as a holomorphic function of x in a strip* (*in* $\mathbb{C}$) *of width*

$$\gamma_L \geq c\, R_L^{2/(3-2\beta)},$$

around the real axis, where c is a positive constant.

Remark 3.2.1. In the special case of KS equation without dispersion, the solution extends as a holomorphic function of x, for sufficiently large t, in a strip (in $\mathbb{C}$) of width

$$\gamma_L \geq c\, R_L^{-2/5},$$

around the real axis (see statement in Subsection 1.1 and [11]). This is also a consequence of Theorem 3.2.3 with $\beta = 4$, since the case where $\beta = 4$ corresponds to the

KS equation.

Chapter 4

Analyticity for pseudo-differential equations in 1D

In this chapter, we study the analyticity properties of solutions for a class of non-linear evolutionary pseudo-differential equations possessing global attractors. In order to do this, we utilize an analyticity criterion for spatially periodic functions, which involves the rate of growth of a suitable norm of the n^{th} derivative of the solution, with respect to the spatial variable, as n tends to infinity. This criterion is applied to a general class of non-linear evolutionary pseudo-differential equations, under certain conditions, provided they possess global attractors. Using this criterion and the spectral method developed in Akrivis *et al.* [3] we have improved previous results which appear in [3]. This chapter follows the paper [26].

4.1 Introduction

We present analyticity properties of zero mean, spatially 2π-periodic solutions of PDEs of the form

$$u_t + uu_x + \mathcal{P}u \;=\; 0, \tag{4.1}$$

possessing a global attractor. Here, $\mathcal{P}$ is a linear pseudo-differential operator defined by its symbol in Fourier space, that is,

$$\widehat{(\mathcal{P}w)}_k \;=\; \lambda_k\,\hat{w}_k, \quad k \in \mathbb{Z}, \tag{4.2}$$

whenever $w(x) = \sum_{k\in\mathbb{Z}} \hat{w}_k\, e^{ikx}$, and with λ_k satisfying

$$\operatorname{Re}\lambda_k \;\geq\; c_1|k|^\gamma \quad \text{for all } \; |k| \geq k_0, \tag{4.3}$$

for some positive constants c_1, γ and k_0 a sufficiently large positive integer. Global existence of solutions of (4.1) has been established for $\gamma > 3/2$ (see [48]); when $\gamma \geq 2$, it can be deduced from [18] that equation (4.1) possesses a global attractor compact in every Sobolev norm. Analyticity of solutions of (4.1) is established when $\gamma > 5/2$, in [3]. Here, we shall establish that the solutions of (4.1) are analytic even when $\gamma > 2$.

A special case of equation (4.1) is the dispersively modified KS equation

$$u_t + uu_x + u_{xx} + \nu u_{xxxx} + \mathcal{D}u \,=\, 0, \tag{4.4}$$

with ν a positive constant and $\mathcal{D}$ a linear antisymmetric pseudo-differential operator; in Fourier space

$$(\widehat{\mathcal{D}w})_k \,=\, id_k\hat{w}_k, \quad d_{-k} \,=\, -d_k \in \mathbb{R},$$

that is, $\mathcal{D}$ is dispersive. When $d_k = -k^3$, we obtain the Kawahara equation [29, 30]; another application that emerges from the dynamics of two-phase core-annular flows yields d_k in terms of modified Bessel functions of the first kind [42]. Note that such spatially extended systems are typically defined on L-periodic domains and equations (4.1) and (4.4) have been scaled to have 2π periodicity. This rescaling provides a canonical equation with a "viscosity" parameter $\nu = (2\pi/L)^2$ in front of the highest derivative. (On this rescaling see Subsection 3.1.) It can be deduced from [18] that the 2π-periodic solutions of (4.1) possess a global attractor, bounded in every Sobolev norm; in fact, such proofs are possible for $\gamma \geq 2$ in (4.3). This Sobolev norm boundedness is used in our analyticity estimates to obtain a lower bound on the band of analyticity.

The approach in this chapter is distinct from that in [11] which uses semigroup methods on the L-periodic KS equation (a special case of (4.4) with $\mathcal{D} \equiv 0$),

$$u_t + uu_x + u_{xx} + u_{xxxx} \,=\, 0.$$

Given the bound (see, for example, [10, 22, 20, 40])

$$\limsup_{t\to\infty} \int_0^L |u(x,t)|^2 dx \,\leq\, R_L^2,$$

the idea is to obtain a lower bound for αt so that the L^2-norm of $v := e^{\alpha t \boldsymbol{A}}u$ stays bounded. Here, $\boldsymbol{A}$ is the pseudo-differential operator, which is defined in the Fourier space as

$$(\widehat{\boldsymbol{A}u})_k = |k|\hat{u}_k,$$

and thus if $u = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikqx}$, where $q = 2\pi/L$, then

$$v = e^{at\boldsymbol{A}} = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikqx+at|k|}.$$

4.2 An analyticity criterion

A real analytic and periodic function $f : \mathbb{R} \to \mathbb{C}$ extends holomorphically in a neighborhood

$$\Omega_\beta = \{x + iy : x, y \in \mathbb{R} \text{ and } |y| < \beta\}$$

for some $\beta > 0$. The maximum such $\beta \in (0, \infty]$ is called the band of analyticity of f. For completeness, we say that the band of analyticity of f is zero if and only if f is not real analytic. Next, we state an analyticity criterion for periodic functions which involves the rate of growth of suitable norms of f.

Theorem 4.2.1 (Analyticity criterion). *Let $u : \mathbb{R} \to \mathbb{C}$ be an L-periodic C^∞ function, $p \in [1, \infty]$ and*

$$\mu := \limsup_{s\to\infty} \frac{\|u\|_{p,s}^{1/s}}{s},$$

where

$$\|u\|_{p,s} = \Big(\sum_{k\in\mathbb{Z}} |k|^{ps} |\hat{u}_k|^p\Big)^{1/p},$$

with $\hat{u}_k = \frac{1}{L}\int_0^L u(x)e^{-ikqx}dx$ and $q = 2\pi/L$. Then the band of analyticity β of u is given by

$$\beta = \begin{cases} \infty & \text{if } \mu = 0, \\ \dfrac{1}{e\mu} & \text{if } \mu \in (0, \infty), \\ 0 & \text{if } \mu = \infty. \end{cases}$$

Proof. Clearly, if $1 \le p \le \infty$, then there exist positive constants C_1 and C_2, such that

$$C_1\|u\|_{p,n} \le \|u^{(n)}\|_\infty \le C_2\|u\|_{p,n+1}, \tag{4.5}$$

for every $n \ge 1$ and $u \in C^\infty(\mathbb{R})$, which is L-periodic. It is readily seen that (4.5) implies

$$\limsup_{n\to\infty} \frac{\|u\|_{p,n}^{1/n}}{n} = \limsup_{n\to\infty} \frac{\|u^{(n)}\|_\infty^{1/n}}{n}. \tag{4.6}$$

Formula (4.6) implies that it suffices to show the theorem for the $\|u\|_{p,\infty}$-norm, instead of the $\|u\|_{p,s}$-norm. Due to Stirling's formula we have that

$$\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = e,$$

which in combination with (4.6), yields that

$$\tilde{\mu} = \limsup_{n\to\infty} \Big(\frac{\|u\|_{p,\infty}}{n!}\Big)^{1/n} = \limsup_{n\to\infty} \frac{n}{(n!)^{1/n}} \cdot \frac{\|u\|_{p,\infty}^{1/n}}{n} = e\mu.$$

Therefore, in order to prove our analyticity criterion it suffices to establish the following two claims.

Claim I. *If* $\tilde{\mu} < \infty$ *and* $\gamma := \begin{cases} \infty & \text{if } \tilde{\mu} = 0, \\ \dfrac{1}{\tilde{\mu}} & \text{if } \tilde{\mu} > 0, \end{cases}$ *then* u *extends holomorphically in* Ω_γ.

Claim II. *If* $\gamma \in (0, \infty)$ *and* u *extends holomorphically in* Ω_γ*, then* $\tilde{\mu} \leq 1/\gamma$.

Proof of Claim I. It can be readily seen that the function

$$U(x+iy) \;=\; \sum_{n=0}^{\infty} \frac{u^{(n)}(x)}{n!}(iy)^n$$

is well defined (cf. n^{th}-root test for series) and differentiable (in fact C^∞), with respect to both x and y, for every $(x, y) \in \mathbb{R} \times (-\gamma, \gamma)$, and satisfies the Cauchy–Riemann equations, i.e., $U_y = iU_x$. Therefore, U is holomorphic in Ω_γ, and since $U(x) = u(x)$, for $x \in \mathbb{R}$, then u extends holomorphically in Ω_γ.

Proof of Claim II. Let U be holomorphic in Ω_γ and agrees with u in $\mathbb{R}$, and $\varepsilon \in (0, \gamma)$. Set

$$M_\varepsilon = \max\big\{|U(x+iy)| : x \in [0, L] \text{ and } |y| \leq \gamma - \varepsilon\big\}.$$

We have

$$M_\varepsilon \;=\; \sup_{z \in \overline{\Omega}_{\gamma-\varepsilon}} |U(z)|,$$

since U is also L-periodic. Also, for every $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$u^{(n)}(x) \;=\; U^{(n)}(x) \;=\; \frac{n!}{2\pi i}\int_{|z-x|=\gamma-\varepsilon} \frac{U(z)}{(z-x)^{n+1}}\,dz,$$

whence

$$|u^{(n)}(x)| \;\leq\; \frac{n! M_\varepsilon}{(\gamma-\varepsilon)^n},$$

and thus

$$\tilde{\mu} \;=\; \limsup_{n\to\infty} \left(\frac{\|u^{(n)}\|_\infty}{n!}\right)^{1/n} \;\leq\; \frac{1}{\gamma-\varepsilon}$$

for every $\varepsilon \in (0, \gamma)$. Consequently, $\tilde{\mu} \leq 1/\gamma$. □

4.3 Analyticity of solutions

We shall apply our analyticity criterion to 2π-periodic solutions (with zero spatial mean) of (4.1), where $\mathcal{P}$ is a linear pseudo-differential operator with a symbol in Fourier space given by (4.2). Well-posedness and global existence (in time) of solutions of (4.1) is established in [48]. Existence of a global attractor $\mathcal{X}$ can be derived from the results in [18]. In fact, when $t > 0$, every solution of (4.1) becomes C^∞ with respect to x. In particular, for every $n \in \mathbb{N}$, there exists an R_n, depending on $\mathcal{P}$, but independent of

u_0, such that

$$\limsup_{t\to\infty} \|\partial_x^n u(\cdot,t)\| \;\le\; R_n.$$

We follow now the approach of Akrivis *et al.* from [3]. Expressing $u(x,t) = \sum_{k\in\mathbb{Z}} \hat{u}_k(t)\, e^{ikx}$, equation (4.1) is transformed into the following infinite dimensional dynamical system

$$\frac{d}{dt}\hat{u}_k = -\lambda_k \hat{u}_k - ik\hat{\varphi}_k, \quad k\in\mathbb{Z}, \tag{4.7}$$

with

$$\hat{\varphi}_k(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}u^2(x,t)\, e^{-ikx}\, dx = \frac{1}{2}\sum_{j=1}^{k-1} \hat{u}_j(t)\hat{u}_{k-j}(t) + \sum_{j=1}^{\infty} \hat{u}_{-j}(t)\hat{u}_{k+j}(t). \tag{4.8}$$

Clearly, (4.7) implies that

$$\hat{u}_k(t) \;=\; e^{-\lambda_k t}\hat{u}_k(0) - ik\int_0^t e^{-\lambda_k(t-s)}\hat{\varphi}_k(s)\, ds, \tag{4.9}$$

and consequently

$$\limsup_{t\to\infty} |\hat{u}_k(t)| \;\le\; \frac{|k|}{\operatorname{Re}\lambda_k}\limsup_{t\to\infty} |\hat{\varphi}_k(t)|, \tag{4.10}$$

whenever $\operatorname{Re}\lambda_k > 0$.

Remark 4.3.1. Let us explain how (4.10) arises from (4.9).

Proof. Let $M := \limsup_{t\to\infty} |\hat{\varphi}_k(t)|$. Also, let $\varepsilon > 0$. Then from the definition of M there exists a $T := T(\varepsilon)$ such that $|\hat{\varphi}_k(t)| \le M + \varepsilon \;\forall\; t \ge T$. Now, we can write (4.9) as follows

$$\hat{u}_k(t) \;=\; e^{-\lambda_k t}\hat{u}_k(0) - ik\int_0^T e^{-\lambda_k(t-s)}\hat{\varphi}_k(s)ds - ik\int_T^t e^{-\lambda_k(t-s)}\hat{\varphi}_k(s)ds. \tag{4.11}$$

Taking absolute values in both sides of (4.11) we take that

$$|\hat{u}_k(t)| \;\le\; e^{-\operatorname{Re}\lambda_k t}|\hat{u}_k(0)| + |k|\|\hat{\varphi}_k\|_\infty\int_0^T e^{-\operatorname{Re}\lambda_k(t-s)}ds + |k|\int_T^t e^{-\operatorname{Re}\lambda_k(t-s)}|\hat{\varphi}_k(s)|ds. \tag{4.12}$$

Now, from (4.12), we have for $t \ge T$

$$|\hat{u}_k(t)| \le e^{-\operatorname{Re}\lambda_k t}|\hat{u}_k(0)| + |k|\|\hat{\varphi}_k\|_\infty \frac{e^{-\operatorname{Re}\lambda_k t}\left(e^{\operatorname{Re}\lambda_k T} - 1\right)}{\operatorname{Re}\lambda_k} + |k|(M+\varepsilon)\frac{1 - e^{-\operatorname{Re}\lambda_k(t-T)}}{\operatorname{Re}\lambda_k},$$

and consequently

$$|\hat{u}_k(t)| \longrightarrow \frac{|k|(M+\varepsilon)}{\operatorname{Re}\lambda_k} \quad \text{for every } \varepsilon > 0.$$

Thus, $\limsup_{t\to\infty} |\hat{u}_k(t)| \le \frac{|k|}{\operatorname{Re}\lambda_k} M$. □

We next define for $p > 2$

$$h_p(s) \;=\; \limsup_{t\to\infty} \left(\sum_{k=1}^{\infty} k^{ps} |\hat{u}_k(t)|^p \right)^{1/p}, \quad s \in \mathbb{R}.$$

Note that, if $n \in \mathbb{N}$ and $n \le s$, then

$$\begin{aligned} 2^{1/p} h_p(s) = \limsup_{t\to\infty} \left(\sum_{k\in\mathbb{Z}} |k|^{ps} |\hat{u}_k(t)|^p \right)^{1/p} &\ge \limsup_{t\to\infty} \left(\sum_{k\in\mathbb{Z}} |k|^{pn} |\hat{u}_k(t)|^p \right)^{1/p} \\ = \limsup_{t\to\infty} \|u(\cdot,t)\|_{p,n}. \end{aligned}$$

Also,

$$\limsup_{t\to\infty} |\hat{u}_m(t)| \;\le\; \frac{h_p(s)}{|m|^s} \quad \text{for all } m \in \mathbb{Z} \setminus \{0\}\,. \tag{4.13}$$

Our target is to show the following claim.

Claim I. *There exist positive constants M and a, such that, for every $s \ge 0$,*

$$h_p(s) \;\le\; M(as)^s. \tag{4.14}$$

This result in turn implies that

$$\limsup_{s\to\infty} \left(\frac{1}{s} \limsup_{t\to\infty} \|u(\cdot,t)\|_{p,s}^{1/s} \right) = \limsup_{s\to\infty} \frac{2^{1/(ps)} h_p^{1/s}(s)}{s} \le \limsup_{s\to\infty} \frac{2^{1/(ps)} M^{1/s} as}{s} \;\le\; a.$$

By using our analyticity criterion, we shall consequently obtain a lower bound for the band of analyticity β of solutions u in the attractor, namely $\beta \ge 1/(ea)$.

The claim will be proved by the following inductive method.

First, we pick $M, a > 0$, so that

$$h_p(s) \le M(as)^s, \quad \text{for every } s \in [0,2].$$

Suitable values are, for example,

$$M \ge 2^{1/2} R_2 \ge 2^{1/2} \limsup_{t\to\infty} \|u_{xx}(\cdot,t)\| \quad \text{and} \quad a \;\ge\; 1.$$

Indeed, noting that

$$(as)^s \ge e^{-1/(ea)} > \frac{1}{2}, \quad \text{for all } a \ge 1 \text{ and } s \ge 0,$$

we obtain

$$M(as)^s > \frac{M}{2} \ge \frac{1}{\sqrt{2}} \limsup_{t\to\infty} \|u_{xx}(\cdot,t)\| = \frac{1}{\sqrt{2}} \limsup_{t\to\infty} \left(\sum_{k\in\mathbb{Z}} k^4 |\hat{u}_k(t)|^2 \right)^{1/2}$$

$$= \limsup_{t\to\infty}\left(\frac{1}{2}\sum_{k\in\mathbb{Z}} k^4|\hat{u}_k(t)|^2\right)^{1/2} = \limsup_{t\to\infty}\left(\sum_{k=1}^{\infty} k^4|\hat{u}_k(t)|^2\right)^{1/2}$$

$$= \limsup_{t\to\infty}\left(\sum_{k=1}^{\infty}\left(k^2|\hat{u}_k(t)|\right)^2\right)^{1/2} \geq \limsup_{t\to\infty}\left(\sum_{k=1}^{\infty}\left(k^2|\hat{u}_k(t)|\right)^p\right)^{1/p} = h_p(2) \geq h_p(s)$$

for all $s\in[0,2]$, since $p>2$. Next we shall prove (by selecting a possibly larger a) that (4.14) holds for every $s\in[\sigma,\sigma+1]$, provided that the same inequality holds for every $s\in[0,\sigma]$ and $\sigma\geq 2$. This in turn establishes that (4.14) holds for every $s\geq 0$. It suffices to show the following claim.

Claim II. *If* (4.14) *holds for every* $s\in[0,\sigma]$ *and* $\sigma\geq 1$, *then it also holds and for* $s=\sigma+\sigma_1$, *where* $\sigma_1\in(0,\gamma-\frac{2p+1}{p})$.

Proof of Claim II. For every $j=1,\ldots,k-1$, we have, by virtue of (4.13),

$$\limsup_{t\to\infty}|\hat{u}_j(t)| \leq \frac{h_p\left(\frac{\sigma j}{k}\right)}{j^{\frac{\sigma j}{k}}} \leq \frac{M\left(a\frac{\sigma j}{k}\right)^{\frac{\sigma j}{k}}}{j^{\frac{\sigma j}{k}}},$$

and thus, the first sum on the right-hand side of (4.8) is estimated as follows:

$$\begin{aligned}\limsup_{t\to\infty}\sum_{j=1}^{k-1}|\hat{u}_j(t)|\,|\hat{u}_{k-j}(t)| &\leq \sum_{j=1}^{k-1}\frac{h_p\left(\frac{\sigma j}{k}\right)}{j^{\frac{\sigma j}{k}}}\cdot\frac{h_p\left(\frac{\sigma(k-j)}{k}\right)}{(k-j)^{\frac{\sigma(k-j)}{k}}}\\ &\leq \sum_{j=1}^{k-1}\frac{M\left(a\frac{\sigma j}{k}\right)^{\frac{\sigma j}{k}}}{j^{\frac{\sigma j}{k}}}\cdot\frac{M\left(a\frac{\sigma(k-j)}{k}\right)^{\frac{\sigma(k-j)}{k}}}{(k-j)^{\frac{\sigma(k-j)}{k}}}\\ &= \frac{(k-1)M^2(a\sigma)^\sigma}{k^\sigma} \leq \frac{M^2(a\sigma)^\sigma}{k^{\sigma-1}}. \qquad (4.15)\end{aligned}$$

For the second sum in the right-hand side of (4.8), using inequality (4.13) and the fact that $|\hat{u}_{-j}(t)|=|\hat{u}_j(t)|$, we obtain that

$$\begin{aligned}\limsup_{t\to\infty}\sum_{j=1}^{\infty}|\hat{u}_j(t)|\,|\hat{u}_{k+j}(t)| &\leq \limsup_{t\to\infty}\left(\sum_{j=1}^{\infty}|\hat{u}_j(t)|^p\right)^{1/p}\limsup_{t\to\infty}\left(\sum_{j=1}^{\infty}|\hat{u}_{k+j}(t)|^q\right)^{1/q}\\ &\leq h_p(0)\left(\sum_{j=1}^{\infty}\frac{h_p^q(\sigma)}{(k+j)^{q\sigma}}\right)^{1/q} \leq M\,h_p(\sigma)\left(\int_0^\infty\frac{dx}{(x+k)^{q\sigma}}\right)^{1/q}\\ &\leq M^2(a\sigma)^\sigma\left(\frac{1}{q\sigma-1}\cdot\frac{1}{k^{q\sigma-1}}\right)^{1/q} = \frac{M^2(a\sigma)^\sigma}{(q\sigma-1)^{1/q}k^{\sigma-(1/q)}}\\ &\leq \frac{M^2(a\sigma)^\sigma}{(q-1)^{1/q}k^{\sigma-1}}, \qquad (4.16)\end{aligned}$$

assuming that $p,q\in(1,\infty)$ with $\dfrac{1}{p}+\dfrac{1}{q}=1$. In arriving at the result above, we have used the fact

$$\limsup_{t\to\infty}\sum_{j=1}^{\infty}|\hat{u}_{k+j}(t)|^q \le \sum_{j=1}^{\infty}\limsup_{t\to\infty}|\hat{u}_{k+j}(t)|^q \le \sum_{j=1}^{\infty}\frac{h_p^q(\sigma)}{(k+j)^{q\sigma}},$$

along with (4.13). Also, we note that

$$\begin{aligned}\int_0^{\infty}\frac{dx}{(x+k)^{q\sigma}} &= \lim_{t\to\infty}\int_0^t \frac{dx}{(x+k)^{q\sigma}} = \lim_{t\to\infty}\int_0^t (x+k)^{-q\sigma}dx = \lim_{t\to\infty}\frac{(x+k)^{-q\sigma+1}}{-q\sigma+1}\Big]_{x=0}^{x=t}\\ &= \lim_{t\to\infty}\left[\frac{1}{(-q\sigma+1)(t+k)^{q\sigma-1}} - \frac{1}{(-q\sigma+1)k^{q\sigma-1}}\right] = \frac{1}{(q\sigma-1)k^{q\sigma-1}},\end{aligned}$$

since $q\sigma \ge q > 1$. Finally, notice that

$$q\sigma - 1 \ge q-1 \iff (q\sigma-1)^{1/q} \ge (q-1)^{1/q} \quad \text{and} \quad \sigma - \frac{1}{q} > \sigma - 1 \iff k^{\sigma-(1/q)} > k^{\sigma-1}.$$

Now, from (4.3), we have

$$\operatorname{Re}\lambda_k \ge c_1 k^{\gamma} \quad \text{for } k \ge k_0. \tag{4.17}$$

Combination of (4.10), (4.15), (4.16) and (4.17) provides that

$$\limsup_{t\to\infty}|\hat{u}_k(t)| \le \frac{\big(2+(q-1)^{1/q}\big)M^2(a\sigma)^{\sigma}}{2c_1(q-1)^{1/q}\,k^{\sigma+\gamma-2}} \quad \text{for } k \ge k_0.$$

Thus,

$$\begin{aligned}&\limsup_{t\to\infty}\sum_{k=1}^{\infty}k^{p\sigma+p\sigma_1}|\hat{u}_k(t)|^p\\ &\le \sum_{k=k_0}^{\infty}\frac{k^{p\sigma+p\sigma_1}\big(2+(q-1)^{1/q}\big)^pM^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p(q-1)^{p/q}\,k^{p\sigma+p\gamma-2p}}\\ &\quad + \limsup_{t\to\infty}\sum_{k=1}^{k_0-1}k^{p\sigma+p\sigma_1}|\hat{u}_k(t)|^p\\ &\le \frac{\big(2+(q-1)^{1/q}\big)^pM^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p(q-1)^{p/q}}\sum_{k=k_0}^{\infty}\frac{1}{k^{p(\gamma-2-\sigma_1)}}\\ &\quad + (k_0-1)^{p\sigma+p\sigma_1-2p}\limsup_{t\to\infty}\sum_{k=1}^{k_0-1}k^{2p}|\hat{u}_k(t)|^p\\ &= \frac{\big(2+(q-1)^{1/q}\big)^pM^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p(q-1)^{p/q}}\sum_{k=k_0}^{\infty}\frac{1}{k^{p(\gamma-2-\sigma_1)}}\\ &\quad + (k_0-1)^{p\sigma+p\sigma_1-2p}\limsup_{t\to\infty}\sum_{k=1}^{k_0-1}\big(k^4|\hat{u}_k(t)|^2\big)^{p/2}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\big(2+(q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p(q-1)^{p/q}} \sum_{k=k_0}^{\infty} \frac{1}{k^{p(\gamma-2-\sigma_1)}} \\
&\quad + (k_0-1)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \left(\sum_{k=1}^{k_0-1} k^4 |\hat{u}_k(t)|^2 \right)^{p/2} \\
&\le \frac{\big(2+(q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p(q-1)^{p/q}} \int_{k_0-1}^{\infty} \frac{dx}{x^{p(\gamma-2-\sigma_1)}} \\
&\quad + (k_0-1)^{p\sigma+p\sigma_1-2p} R_2^p \\
&= \frac{\big(2+(q-1)^{1/q}\big)^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p(q-1)^{p/q}} \cdot \frac{1}{(p(\gamma-2-\sigma_1)-1)(k_0-1)^{p(\gamma-2-\sigma_1)-1}} \\
&\quad + (k_0-1)^{p\sigma+p\sigma_1-2p} R_2^p,
\end{aligned}
$$

because of the fact that

$$
p(\gamma-2-\sigma_1) \;>\; 1 \iff \sigma_1 \;<\; \gamma - \frac{2p+1}{p}.
$$

Since

$$
h_p^p(\sigma+\sigma_1) = \limsup_{t\to\infty} \sum_{k=1}^{\infty} k^{p\sigma+p\sigma_1} |\hat{u}_k(t)|^p,
$$

we have

$$
h_p(\sigma+\sigma_1) \;\le\; CM^2(a\sigma)^\sigma + (k_0-1)^{\sigma+\sigma_1-2} M,
$$

where

$$
C \;=\; \frac{2+(q-1)^{1/q}}{2c_1(q-1)^{1/q}\Big(\big(p(\gamma-2-\sigma_1)-1\big)(k_0-1)^{p(\gamma-2-\sigma_1)-1}\Big)^{1/p}}.
$$

In arriving at the result above, we have used the fact

$$
(\vartheta+\varphi)^{1/p} \;\le\; \vartheta^{1/p} + \varphi^{1/p} \quad \text{for all } \vartheta, \varphi > 0 \text{ and } p \ge 1.
$$

This inductive step is complete if we can find positive constants M and a satisfying

$$
CM^2(a\sigma)^\sigma + (k_0-1)^{\sigma+\sigma_1-2} M \;\le\; M\big(a(\sigma+1)\big)^{\sigma+1} \quad \text{for every } \sigma \ge 1. \tag{4.18}
$$

Clearly, for every $M > 0$, there exists an $a_0 > 0$, such that (4.18) holds for every $a \ge a_0$. □

Therefore, the following has been proved.

Theorem 4.3.1. *Let $\mathcal{X}$ be the global attractor of the equation*

$$
u_t + uu_x + \mathcal{P}u = 0,
$$

with 2π-periodic initial data in L^2, where $\mathcal{P}$ is a linear pseudo-differential operator *defined by its symbol in Fourier space, that is,*

$$\left(\widehat{\mathcal{P}w}\right)_k = \lambda_k\, \hat{w}_k, \quad k \in \mathbb{Z},$$

whenever $w(x) = \sum_{k\in\mathbb{Z}} \hat{w}_k\, e^{ikx}$, *and with the eigenvalues* λ_k *satisfying the condition*

$$\operatorname{Re}\lambda_k \;\geq\; c_1|k|^{\gamma} \quad \textit{for all } \ |k| \geq k_0,$$

for some positive constants c_1, $\gamma > 2$ *and* k_0 *a sufficiently large positive integer. Then, every* $w \in \mathcal{X}$ *extends to a holomorphic function in* Ω_β, *for a suitable* $\beta > 0$.

Chapter 5

Analyticity for pseudo-differential equations in 2D

In this chapter, we investigate the analyticity properties of solutions of KS type equations in two spatial dimensions, with periodic initial data. In order to do this we explore the applicability of the spectral method developed in [3], in three-dimensional models. We introduce a criterion, which provides a sufficient condition for analyticity of a periodic function $u \in C^\infty$, involving the rate of growth of $\nabla^n u$, in suitable norms, as n tends to infinity. This criterion allows us to establish spatial analyticity for the solutions of a variety of systems, including Topper–Kawahara, Frenkel–Indireshkumar and Coward–Hall equations and their dispersively modified versions, once we assume that these systems possess global attractors. This chapter follows the paper [27].

5.1 Introduction

We present analyticity properties of spatially 2π-periodic solutions in both x and y of equations of the form

$$u_t + uu_x + \mathcal{P}u = 0, \tag{5.1}$$

assuming that possess a global attractor. The conservative nature of (5.1) allows us to assume that u is of zero mean. Here, $\mathcal{P}$ is a linear pseudo-differential operator in the spatial variables defined by its symbol in Fourier space, that is,

$$\big(\widehat{\mathcal{P}w}\big)_{k,\ell} = \lambda_{k,\ell}\, \hat{w}_{k,\ell}, \quad (k,\, \ell) \in \mathbb{Z}^2, \tag{5.2}$$

whenever $w(x,y) = \sum_{(k,\ell)\in\mathbb{Z}^2} \hat{w}_{k,\ell}\, e^{i(kx+\ell y)}$. The operator $\mathcal{P}$ is assumed to contain a dissipative component of sufficiently high order, i.e., with eigenvalues $\lambda_{k,\ell}$ satisfying the condition

$$\operatorname{Re}\lambda_{k,\ell} \;\geq\; c_1(|k|+|\ell|)^\gamma \quad \text{for all } \; |k|+|\ell| \geq n_0, \tag{5.3}$$

for some positive constants c_1, γ and n_0 a sufficiently large positive integer. Here, we shall establish analyticity of solutions to (5.1) when $\gamma > 3$.

We now continue with analytical results in variations of the KS equation in space dimension two. Such equations are very challenging and many questions about these are still open. We mention several interesting works here. In [44] the existence of a bounded local absorbing set and an attractor is shown in thin 2D domain with restricted initial data, for the equation

$$\boldsymbol{u}_t + \frac{1}{2}\nabla(\boldsymbol{u}\cdot\boldsymbol{u}) + \Delta\boldsymbol{u} + \Delta^2\boldsymbol{u} \;=\; 0, \quad \nabla\times\boldsymbol{u} \;=\; 0,$$

where $\boldsymbol{u} = (u_1, u_2)$. In [38] this result is improved by showing that

$$\begin{aligned} &\limsup_{t\to\infty} \|u_1(\cdot,\cdot,t)\|_{\mathrm{L}^2} \;\le\; C L_x^{8/5} L_y^{1/2}, \\ &\lim_{t\to\infty} \|u_2(\cdot,\cdot,t)\|_{\mathrm{L}^2} \;=\; 0, \end{aligned}$$

on the bounded domain $(0, L_x) \times (0, L_y)$ with the assumption $L_y \le C_1 L_x^{-67/35}$.

In [43] the following variation of the KS equation in 2D is studied:

$$\begin{cases} u_t + uu_x + u_{xx} + \Delta^2 u \;=\; 0, \\ u(\cdot,\cdot,0) \;=\; u_0, \\ u(x,y;t) \;=\; u(x+L,y;t) \;=\; u(x,y+L;t) \quad \forall\,(x,y)\in\mathbb{R}^2,\; t\ge 0, \end{cases} \tag{5.4}$$

and under some assumptions it is shown that the solutions of this equation are bounded for all positive time in L^2 and in other spaces H^m. Then it is proved that the equation (5.4) possesses a global attractor and that the attractor has finite dimension.

In [14] the following variation of the KS equation in 2D is studied:

$$\begin{cases} u_t + uu_x + uu_y + \Delta u + \Delta^2 u - g(x) \;=\; 0, \\ u(\cdot,\cdot,0) \;=\; u_0, \\ u(x,y;t) \;=\; u(x+2L,y;t) \;=\; u(x,y+2L;t) \quad \forall\,(x,y)\in\mathbb{R}^2,\; t\ge 0, \end{cases}$$

and under some assumptions the globally well-posed and the existence of a global attractor in the periodic case is proved.

Sharp numerical estimates for the size of the attractor of (5.4) are presented in [1]. In particular, these numerical estimates suggest that

$$\limsup_{t\to\infty} \|u(\cdot,\cdot,t)\| \;=\; \mathcal{O}(L^2), \quad \text{where } L_x = L_y = L.$$

5.2 An analytic extensibility criterion

A C^∞-function $u : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{R}^k$ open, is said to be real analytic at $\boldsymbol{x}^0 \in \Omega$, if there exists $\beta > 0$ such that

$$u(\boldsymbol{x}^0 + \boldsymbol{h}) = \sum_{a \in \mathbb{N}^k} \frac{1}{a!} D^a u(\boldsymbol{x}^0) \boldsymbol{h}^a, \tag{5.5}$$

for every $\boldsymbol{h} = (h_1, \ldots, h_k) \in \mathbb{R}^k$, with $\|\boldsymbol{h}\| = (h_1^2 + \cdots + h_k^2)^{1/2} < \beta$. (Note that, with $\mathbb{N}$ we mean the set of all the non negative integers, i.e., $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.) In such case the function u extends holomorphically in an open ball in $\mathbb{C}^k$ of radius β, centered at $\boldsymbol{x}^0$. In particular, if $u : \mathbb{R}^k \to \mathbb{C}$ is C^∞, 2π-periodic in every argument and analytic for all $\boldsymbol{x}^0 \in \mathbb{R}^k$, then there exists $\beta > 0$ such that (5.5) converges for all $\boldsymbol{x}^0 \in \mathbb{R}^k$ and $\|\boldsymbol{h}\| < \beta$, and thus u extends holomorphically in the open domain

$$\Omega_\beta = \big\{ (x_1 + iy_1, \ldots, x_k + iy_k) : x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{R} \text{ and } |y_1|, \ldots, |y_k| < \beta \big\} \subset \mathbb{C}^k.$$

Next we provide a condition which is sufficient for a C^∞ and periodic function $u : \mathbb{R}^k \to \mathbb{C}$ to extends holomorphically in Ω_β.

Lemma 5.2.1 (***Analytic extensibility criterion***). *Let $u : \mathbb{R}^k \to \mathbb{C}$ be a C^∞-function which is 2π-periodic in every argument, i.e.,*

$$u(x_1 + 2\mu_1\pi, \ldots, x_k + 2\mu_k\pi) = u(x_1, \ldots, x_k) \quad \textit{for all} \quad x_1, \ldots, x_k \in \mathbb{R},\ \mu_1, \ldots, \mu_k \in \mathbb{Z},$$

and hence u is expressed as

$$u(\boldsymbol{x}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^k} e^{i\boldsymbol{m}\cdot\boldsymbol{x}} \hat{u}_{\boldsymbol{m}} = \sum_{(m_1, \ldots, m_k) \in \mathbb{Z}^k} e^{i(m_1x_1 + \cdots + m_kx_k)} \hat{u}_{\boldsymbol{m}}. \tag{5.6}$$

If

$$\mu := \limsup_{s \to \infty} \frac{\|u\|_{\mathrm{H}^s}^{1/s}}{s} < \infty, \tag{5.7}$$

where

$$\|u\|_{\mathrm{H}^s} = \Big(\sum_{\boldsymbol{m} \in \mathbb{Z}^k} \|\boldsymbol{m}\|^{2s} |\hat{u}_{\boldsymbol{m}}|^2 \Big)^{1/2} = \Big(\sum_{(m_1, \ldots, m_k) \in \mathbb{Z}^k} \big(m_1^2 + \cdots + m_k^2\big)^s |\hat{u}_{m_1, \ldots, m_k}|^2 \Big)^{1/2}$$
$$= \|(-\Delta)^{s/2} u\|_{\mathrm{L}^2},$$

then u extends holomorphically in Ω_β, where

$$\beta = \begin{cases} \infty & \textit{if} \quad \mu = 0, \\ \dfrac{1}{e\mu} & \textit{if} \quad \mu \in (0, \infty). \end{cases}$$

Analyticity for u can be obtained even in the case where $\|u\|_{\mathrm{H}^s}$ *is replaced in* (5.7) *by*

$$\|u\|_{p,s} = \Bigg(\sum_{(m_1,\ldots,m_k)\in\mathbb{Z}^k} \big(|m_1| + \cdots + |m_k|\big)^{ps} |\hat{u}_{m_1,\ldots,m_k}|^p \Bigg)^{1/p}, \quad s \in \mathbb{R}, \tag{5.8}$$

for $p > 2$.

Proof. We set the function $U : \mathbb{C}^k \to \mathbb{C}$ as

$$U(z_1,\ldots,z_k) = U(x_1+iy_1,\ldots,x_k+iy_k) = \sum_{a\in\mathbb{N}^k} \frac{1}{a!} i^{|a|} \boldsymbol{y}^a D^a u(\boldsymbol{x}). \tag{5.9}$$

We can write (5.9) in the form

$$U(x_1+iy_1,\ldots,x_k+iy_k) = \sum_{n=0}^{\infty} \frac{1}{n!} \Bigg(\sum_{|a|=n} \frac{n!}{a!} i^{|a|} \boldsymbol{y}^a D^a u(\boldsymbol{x}) \Bigg),$$

and hence, for $|y_i| < \beta$ with $i = 1,\ldots,k$ we get

$$|U(x_1+iy_1,\ldots,x_k+iy_k)| \leq \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \Bigg(\sum_{|a|=n} \frac{n!}{a!} |D^a u(\boldsymbol{x})| \Bigg).$$

Our target is to show that

$$n! \sum_{|a|=n} \frac{1}{a!} |D^a u(\boldsymbol{x})| \leq \text{positive constant},$$

and in such a case (5.9) converges for all $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^k \times (-\beta,\beta)^k$. From (5.6) we take

$$D^a u = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} i^{|a|} \boldsymbol{m}^a e^{i\boldsymbol{m}\cdot\boldsymbol{x}} \hat{u}_{\boldsymbol{m}}$$

and so we have that

$$\sum_{|a|=n} \frac{1}{a!} |D^a u(\boldsymbol{x})| = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} \Bigg(\sum_{|a|=n} \frac{\|\boldsymbol{m}\|^a}{a!} \Bigg) |\hat{u}_{\boldsymbol{m}}| = \frac{1}{n!} \sum_{\boldsymbol{m}\in\mathbb{Z}^k} \|\boldsymbol{m}\|^n |\hat{u}_{\boldsymbol{m}}|, \tag{5.10}$$

where $\boldsymbol{m} = (m_1,\ldots,m_k)$ and $\|\boldsymbol{m}\| = (m_1^2+\cdots+m_k^2)^{1/2}$. Now, we have that

$$\begin{aligned} \Bigg(\sum_{\boldsymbol{m}\in\mathbb{Z}^k\setminus\{(0,\ldots,0)\}} \|\boldsymbol{m}\|^n |\hat{u}_{\boldsymbol{m}}| \Bigg)^2 &= \Bigg(\sum_{\boldsymbol{m}\in\mathbb{Z}^k\setminus\{(0,\ldots,0)\}} \big(m_1^2+\cdots+m_k^2\big)^{n/2} |\hat{u}_{\boldsymbol{m}}| \Bigg)^2 \\ &\leq c_{k,d} \sum_{\boldsymbol{m}\in\mathbb{Z}^k\setminus\{(0,\ldots,0)\}} \big(m_1^2+\cdots+m_k^2\big)^{\frac{n}{2}+d} |\hat{u}_{\boldsymbol{m}}|^2 \\ &\leq c_{k,d} \sum_{\boldsymbol{m}\in\mathbb{Z}^k\setminus\{(0,\ldots,0)\}} \big(m_1^2+\cdots+m_k^2\big)^{n+d} |\hat{u}_{\boldsymbol{m}}|^2, \end{aligned} \tag{5.11}$$

where $c_{k,d}$ is a constant depending on k but not on n. For the first inequality in (5.11), we have used the corresponding result of

$$\left(\sum_{m=1}^{\infty} m a_m\right)^2 \leq \lambda \sum_{m=1}^{\infty} m^d a_m^2,$$

for suitable constant λ, in k-variables.

Now, the definition of μ given in (5.7) implies that, for every $\varepsilon > 0$, there exists an s_0, such that

$$\|u\|_{\mathrm{H}^s} \leq \big(s(\mu+\varepsilon)\big)^s, \quad \text{for all } \; s > s_0. \tag{5.12}$$

Combining (5.10), (5.11) and (5.12) we get

$$\sum_{|a|=n} \frac{1}{a!}|D^a u(\boldsymbol{x})| \leq \frac{1}{n!}\Big(c_{k,d}\|u\|_{\mathrm{H}^{n+d}}\Big)^{1/2} \leq \frac{\vartheta_0}{n!}\,\big((n+d)(\mu+\varepsilon)\big)^{(n+d)/2},$$

where $\vartheta_0 = (c_{k,d})^{1/2}$, and so

$$\Big(\sum_{|a|=n} \frac{1}{a!}|D^a u(\boldsymbol{x})|\Big)^{1/n} \leq \Big(\frac{\vartheta_0}{n!}\,\big((n+d)(\mu+\varepsilon)\big)^{(n+d)/2}\Big)^{1/n} \longrightarrow e\,(\mu+\varepsilon) \quad \text{for every } \varepsilon > 0.$$

Therefore,

$$\limsup_{n\to\infty} \Big(\sum_{|a|=n} \frac{1}{a!}|D^a u(\boldsymbol{x})|\Big)^{1/n} \leq e\,(\mu+\varepsilon) \quad \text{for every } \varepsilon > 0$$

and thus

$$\limsup_{n\to\infty} \Big(\sum_{|a|=n} \frac{1}{a!}|D^a u(\boldsymbol{x})|\Big)^{1/n} \leq e\,\mu.$$

It can be readily seen that the function (5.9) is differentiable (in fact C^∞), with respect to both $\boldsymbol{x}$ and $\boldsymbol{y}$, for every $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^k \times (-\beta,\beta)^k$, and satisfies the Cauchy-Riemann equations with respect to $z_1,\ldots,z_k$, i.e., $U_{y_j} = iU_{x_j}$, where $j = 1,\ldots,k$ or equivalently $\bar{\partial}_{z_j} U = 0$ for $j = 1,\ldots,k$. Therefore, U is holomorphic in Ω_β, and since $U(x) = u(x)$, for $x \in \mathbb{R}$, then u extends holomorphically in Ω_β.

Now, in order to show that in (5.7) the $\|u\|_{\mathrm{H}^s}$ can be replaced by the norm (5.8) it suffices to show

$$c\|(-\Delta)^{(s-\vartheta)/2}u\|_{\mathrm{L}^2} \leq \|u\|_{p,s}, \tag{5.13}$$

with $u : \mathbb{R}^k \to \mathbb{C}$ a C^∞-function which is 2π-periodic in every argument, and for a suitable positive constants ϑ and c, not depending on u, to be defined later. Using Hölder's inequality, we obtain

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^k} \|\boldsymbol{m}\|^{2(s-\vartheta)}|\hat{u}_{\boldsymbol{m}}|^2 = \sum_{\boldsymbol{m}\in\mathbb{Z}^k} \|\boldsymbol{m}\|^{-2\vartheta}\|\boldsymbol{m}\|^{2s}|\hat{u}_{\boldsymbol{m}}|^2$$

$$
\begin{aligned}
&\leq \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^k} \left(\|\boldsymbol{m}\|^{-2\vartheta}\right)^{p/(p-2)} \right)^{(p-2)/p} \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^k} \left(\|\boldsymbol{m}\|^{2s}|\hat{u}_{\boldsymbol{m}}|^2\right)^{p/2} \right)^{2/p} \\
&= \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^k} \|\boldsymbol{m}\|^{-(2\vartheta p)/(p-2)} \right)^{(p-2)/p} \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^k} \|\boldsymbol{m}\|^{ps}|\hat{u}_{\boldsymbol{m}}|^p \right)^{2/p}.
\end{aligned}
$$

The series

$$
\sum_{\boldsymbol{m}\in\mathbb{Z}^k\smallsetminus\{(0,\dots,0)\}} \|\boldsymbol{m}\|^{-(2\vartheta p)/(p-2)} = \sum_{\boldsymbol{m}\in\mathbb{Z}^k\smallsetminus\{(0,\dots,0)\}} \frac{1}{(m_1^2+\cdots+m_k^2)^{(\vartheta p)/(p-2)}}
$$

converges if and only if

$$
\frac{2\vartheta p}{p-2} > k,
$$

and in particular for

$$
\vartheta = \frac{(k+1)(p-2)}{2p} =: \vartheta_{k,p}.
$$

So,

$$
\begin{aligned}
\|(-\Delta)^{(s-\vartheta)/2}u\|_{\mathrm{L}^2} &= \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^k} \|\boldsymbol{m}\|^{2(s-\vartheta)}|\hat{u}_{\boldsymbol{m}}|^2 \right)^{1/2} \leq C_{k,p}\left(\sum_{\boldsymbol{m}\in\mathbb{Z}^k} \|\boldsymbol{m}\|^{ps}|\hat{u}_{\boldsymbol{m}}|^p \right)^{1/p} \\
&\leq C_{k,p}\|u\|_{p,s},
\end{aligned}
$$

where $C_{k,p} = \left(\sum_{\boldsymbol{m}\in\mathbb{Z}^k} \|\boldsymbol{m}\|^{-(2\vartheta p)/(p-2)}\right)^{(p-2)/(2p)}$, and the inequality in (5.13) holds for $c = 1/C_{k,p}$. □

5.3 Analyticity of solutions

We shall apply our analytic extensibility criterion to 2π-periodic solutions in both x and y, with zero spatial mean, of (5.1), where $\mathcal{P}$ is a linear pseudo-differential operator with a symbol in Fourier space given by (5.2) and satisfying (5.3) with $\gamma > 3$. In our analysis it is necessary to assume that (5.1) possesses a global attractor $\mathcal{V}$. In particular, we assume that there exist real numbers R_s, independent of the initial data u_0, such that

$$
\limsup_{t\to\infty} \|u(\cdot,\cdot,t)\|_{\mathrm{H}^s} = R_s, \quad \text{for all } s \leq 4.
$$

We follow now the approach of Akrivis *et al.* from [3]. We define for $p > 2$

$$
h_p(s) = \limsup_{t\to\infty} \left(\sum_{(k,\ell)\in\mathbb{Z}^2} (|k|+|\ell|)^{ps}|\hat{u}_{k,\ell}(t)|^p \right)^{1/p}, \quad s\in\mathbb{R}.
$$

Note that

$$
h_p(s) \geq \limsup_{t\to\infty} (|\zeta|+|\theta|)^s|\hat{u}_{\zeta,\theta}(t)|,
$$

and consequently

$$\limsup_{t\to\infty} |\hat{u}_{\zeta,\theta}(t)| \le \frac{h_p(s)}{(|\zeta|+|\theta|)^s} \quad \text{for all } (\zeta,\theta)\in\mathbb{Z}\times\mathbb{Z}\smallsetminus\{(0,0)\}. \tag{5.14}$$

Our target is to show the following claim.

Claim I. *There exist positive constants* M *and* a*, such that, for every* $s \ge 0$*,*

$$h_p(s) \;\le\; M(as)^s. \tag{5.15}$$

Expressing

$$u(x,y,t) = \sum_{(k,\ell)\in\mathbb{Z}^2} \hat{u}_{k,\ell}(t)\, e^{i(kx+\ell y)},$$

equation (5.1) is transformed into the following infinite dimensional dynamical system

$$\frac{d}{dt}\hat{u}_{k,\ell} = -\lambda_{k,\ell}\hat{u}_{k,\ell} - ik\hat{\varphi}_{k,\ell}, \quad (k,\ell)\in\mathbb{Z}^2, \tag{5.16}$$

with

$$\begin{aligned}
&\hat{\varphi}_{k,\ell}(t) = \\
&= \frac{1}{(2\pi)^2}\int_0^{2\pi}\!\!\int_0^{2\pi} \frac{1}{2}u^2(x,y,t)\, e^{-i(kx+\ell y)}\, dxdy \\
&= \frac{1}{2(2\pi)^2}\int_0^{2\pi}\!\!\int_0^{2\pi} \Big[\sum_{(m,n)\in\mathbb{Z}^2} \hat{u}_{m,n}(t)\, e^{i(mx+ny)}\Big]\Big[\sum_{(\sigma,\tau)\in\mathbb{Z}^2} \hat{u}_{\sigma,\tau}(t)\, e^{i(\sigma x+\tau y)}\Big]\, e^{-i(kx+\ell y)}\, dxdy \\
&= \frac{1}{2(2\pi)^2}\int_0^{2\pi}\!\!\int_0^{2\pi} \sum_{(m,n,\sigma,\tau)\in\mathbb{Z}^4} \hat{u}_{m,n}(t)\hat{u}_{\sigma,\tau}(t)\, e^{i(m+\sigma-k)x}\, e^{i(n+\tau-\ell)y}\, dxdy \\
&= \frac{1}{2(2\pi)^2}(2\pi)^2 \sum_{m+\sigma=k}\sum_{n+\tau=\ell} \hat{u}_{m,n}(t)\hat{u}_{\sigma,\tau}(t) \\
&= \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2} \hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t), \quad (k,\ell)\in\mathbb{Z}^2.
\end{aligned} \tag{5.17}$$

Now, without loss of generality assuming in (5.17) that $(k,\ell)\in\mathbb{N}^2$, we get

$$\begin{aligned}
\hat{\varphi}_{k,\ell}(t) = &\frac{1}{2}\sum_{\substack{1\le m\le k\\ 1\le n\le \ell}} \hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) + \sum_{\substack{1\le m\le k\\ n\ge 1}} \hat{u}_{k-m,\ell+n}(t)\hat{u}_{m,-n}(t) \\
&+ \sum_{\substack{m\ge 1\\ 1\le n\le \ell}} \hat{u}_{k+m,\ell-n}(t)\hat{u}_{-m,n}(t) + \sum_{\substack{m\ge 1\\ n\ge 1}} \hat{u}_{k+m,\ell+n}(t)\hat{u}_{-m,-n}(t) \\
&+ \sum_{\substack{m\ge k+1\\ n\ge 1}} \hat{u}_{k-m,\ell+n}(t)\hat{u}_{m,-n}(t) + \frac{1}{2}\sum_{m\ge 1} \hat{u}_{k-m,\ell}(t)\hat{u}_{m,0}(t)
\end{aligned}$$

$$
+\frac{1}{2}\sum_{m\geq 1}\hat{u}_{k+m,\ell}(t)\hat{u}_{-m,0}(t)+\frac{1}{2}\sum_{n\geq 1}\hat{u}_{k,\ell-n}(t)\hat{u}_{0,n}(t)+\frac{1}{2}\sum_{n\geq 1}\hat{u}_{k,\ell+n}(t)\hat{u}_{0,-n}(t). \tag{5.18}
$$

In order to see how we take (5.18) from (5.17), first notice that

$$
\begin{aligned}
\sum_{(m,n)\in\mathbb{Z}^2}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) = &\sum_{\substack{1\leq m\leq k\\ 1\leq n\leq \ell}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)+\sum_{\substack{1\leq m\leq k\\ n\leq -1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)\\
&+\sum_{\substack{m\leq -1\\ 1\leq n\leq \ell}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)+\sum_{\substack{1\leq m\leq k\\ n\geq \ell+1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)\\
&+\sum_{\substack{m\geq k+1\\ 1\leq n\leq \ell}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)+\sum_{\substack{m\leq -1\\ n\leq -1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)\\
&+\sum_{\substack{m\geq k+1\\ n\leq -1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)+\sum_{\substack{m\leq -1\\ n\geq \ell+1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)\\
&+\sum_{\substack{m\geq k+1\\ n\geq \ell+1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t)+\sum_{m\in\mathbb{Z}}\hat{u}_{k-m,\ell}(t)\hat{u}_{m,0}(t)\\
&+\sum_{n\in\mathbb{Z}}\hat{u}_{k,\ell-n}(t)\hat{u}_{0,n}(t),
\end{aligned}
$$

and then

$$
\begin{aligned}
\sum_{\substack{1\leq m\leq k\\ n\leq -1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) &= \sum_{\substack{1\leq m\leq k\\ n\geq \ell+1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) = \sum_{\substack{1\leq m\leq k\\ n\geq 1}}\hat{u}_{k-m,\ell+n}(t)\hat{u}_{m,-n}(t),\\
\sum_{\substack{m\leq -1\\ 1\leq n\leq \ell}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) &= \sum_{\substack{m\geq k+1\\ 1\leq n\leq \ell}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) = \sum_{\substack{m\geq 1\\ 1\leq n\leq \ell}}\hat{u}_{k+m,\ell-n}(t)\hat{u}_{-m,n}(t),\\
\sum_{\substack{m\leq -1\\ n\leq -1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) &= \sum_{\substack{m\geq k+1\\ n\geq \ell+1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) = \sum_{\substack{m\geq 1\\ n\geq 1}}\hat{u}_{k+m,\ell+n}(t)\hat{u}_{-m,-n}(t),\\
\sum_{\substack{m\geq k+1\\ n\leq -1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) &= \sum_{\substack{m\leq -1\\ n\geq \ell+1}}\hat{u}_{k-m,\ell-n}(t)\hat{u}_{m,n}(t) = \sum_{\substack{m\geq k+1\\ n\geq 1}}\hat{u}_{k-m,\ell+n}(t)\hat{u}_{m,-n}(t).
\end{aligned}
$$

Clearly, (5.16) implies that

$$
\hat{u}_{k,\ell}(t) \;=\; e^{-\lambda_{k,\ell}t}\hat{u}_{k,\ell}(0)-ik\int_0^t e^{-\lambda_{k,\ell}(t-s)}\hat{\varphi}_{k,\ell}(s)\,ds,
$$

and consequently

$$
\limsup_{t\to\infty}|\hat{u}_{k,\ell}(t)| \;\leq\; \frac{|k|}{\operatorname{Re}\lambda_{k,\ell}}\limsup_{t\to\infty}|\hat{\varphi}_{k,\ell}(t)|, \tag{5.19}
$$

whenever $\operatorname{Re}\lambda_{k,\ell}>0$.

The claim will be proved by the following inductive method. First, we pick $M,a>$

0, so that

$$h_p(s) \leq M(as)^s, \quad \text{for every } s \in [0,4].$$

Suitable values are, for example,

$$M \geq 32R_4 \geq 32 \limsup_{t\to\infty} \|u(\cdot,\cdot,t)\|_{\mathrm{H}^4} \quad \text{and} \quad a \geq 1.$$

Indeed, noting that

$$(as)^s \geq e^{-1/(ea)} > \frac{1}{2}, \quad \text{for all } a \geq 1 \text{ and } s \geq 0,$$

we obtain

$$\begin{aligned}
M(as)^s > \frac{M}{2} &\geq 16 \limsup_{t\to\infty} \|u(\cdot,\cdot,t)\|_{\mathrm{H}^4} \\
&= 16 \limsup_{t\to\infty} \Bigg(\sum_{(k,\ell)\in\mathbb{Z}^2} (k^2+\ell^2)^4 |\hat{u}_{k,\ell}(t)|^2 \Bigg)^{1/2} \\
&\geq 16 \limsup_{t\to\infty} \Bigg(\sum_{(k,\ell)\in\mathbb{Z}^2} \Big(\frac{1}{2}(|k|+|\ell|)^2\Big)^4 |\hat{u}_{k,\ell}(t)|^2 \Bigg)^{1/2} \\
&= \limsup_{t\to\infty} \Bigg(\sum_{(k,\ell)\in\mathbb{Z}^2} (|k|+|\ell|)^8 |\hat{u}_{k,\ell}(t)|^2 \Bigg)^{1/2} \\
&= \limsup_{t\to\infty} \Bigg(\sum_{(k,\ell)\in\mathbb{Z}^2} \big((|k|+|\ell|)^4 |\hat{u}_{k,\ell}(t)|\big)^2 \Bigg)^{1/2} \\
&\geq \limsup_{t\to\infty} \Bigg(\sum_{(k,\ell)\in\mathbb{Z}^2} \big((|k|+|\ell|)^4 |\hat{u}_{k,\ell}(t)|\big)^p \Bigg)^{1/p} = h_p(4) \geq h_p(s),
\end{aligned}$$

for all $s \in [0,4]$, since $p > 2$. Next we shall prove (by selecting a possibly larger a) that (5.15) holds for every $s \in [\sigma, \sigma+1]$, provided that the same inequality holds for every $s \in [0,\sigma]$ and $\sigma \geq 4$. This in turn establishes that (5.15) holds for every $s \geq 0$. It suffices to show the following claim.

Claim II. *If* (5.15) *holds for every* $s \in [0,\sigma]$ *and* $\sigma \geq 3$*, then it also holds for* $s = \sigma + \sigma_1$*, where* $\sigma_1 \in (0, \gamma - \frac{3p+2}{p})$.

Before we prove this, we first need the following estimate:

Lemma 5.3.1. *For every* $(k,\ell) \in \mathbb{N}^2$ *it holds that*

$$\limsup_{t\to\infty} |\hat{\varphi}_{k,\ell}(t)| \leq \frac{13M^2(a\sigma)^\sigma}{2(k+\ell)^{\sigma-2}}. \tag{5.20}$$

Proof. For every $m = 1, \ldots, k$ and $n = 1, \ldots, \ell$, we have, by virtue of (5.14),

$$\limsup_{t\to\infty} |\hat{u}_{m,n}(t)| \le \frac{h_p\left(\frac{\sigma(m+n)}{k+\ell}\right)}{(m+n)^{\frac{\sigma(m+n)}{k+\ell}}} \le \frac{M\left(a\frac{\sigma(m+n)}{k+\ell}\right)^{\frac{\sigma(m+n)}{k+\ell}}}{(m+n)^{\frac{\sigma(m+n)}{k+\ell}}},$$

and thus, the first sum on the right-hand side of (5.18) is estimated as follows:

$$\begin{aligned}
&\limsup_{t\to\infty} \sum_{\substack{m=1 \\ (m,n)\neq}}^{k} \sum_{\substack{n=1 \\ (k,\ell)}}^{\ell} |\hat{u}_{k-m,\ell-n}(t)|\,|\hat{u}_{m,n}(t)| \\
&\le \sum_{\substack{m=1 \\ (m,n)\neq}}^{k} \sum_{\substack{n=1 \\ (k,\ell)}}^{\ell} \frac{h_p\left(\frac{\sigma(k+\ell-(m+n))}{k+\ell}\right)}{(k+\ell-(m+n))^{\frac{\sigma(k+\ell-(m+n))}{k+\ell}}} \cdot \frac{h_p\left(\frac{\sigma(m+n)}{k+\ell}\right)}{(m+n)^{\frac{\sigma(m+n)}{k+\ell}}} \\
&\le \sum_{\substack{m=1 \\ (m,n)\neq}}^{k} \sum_{\substack{n=1 \\ (k,\ell)}}^{\ell} \frac{M\left(a\frac{\sigma(k+\ell-(m+n))}{k+\ell}\right)^{\frac{\sigma(k+\ell-(m+n))}{k+\ell}}}{(k+\ell-(m+n))^{\frac{\sigma(k+\ell-(m+n))}{k+\ell}}} \cdot \frac{M\left(a\frac{\sigma(m+n)}{k+\ell}\right)^{\frac{\sigma(m+n)}{k+\ell}}}{(m+n)^{\frac{\sigma(m+n)}{k+\ell}}} \\
&= \sum_{\substack{m=1 \\ (m,n)\neq}}^{k} \sum_{\substack{n=1 \\ (k,\ell)}}^{\ell} \frac{M^2(a\sigma)^\sigma}{(k+\ell)^\sigma} = \frac{(k\ell-1)M^2(a\sigma)^\sigma}{(k+\ell)^\sigma} \le \frac{k\ell M^2(a\sigma)^\sigma}{(k+\ell)^\sigma} \\
&\le \frac{(k+\ell)^2 M^2(a\sigma)^\sigma}{(k+\ell)^\sigma} = \frac{M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-2}}.
\end{aligned} \tag{5.21}$$

For the second sum in the right-hand side of (5.18), using inequality (5.14), we obtain that

$$\begin{aligned}
&\limsup_{t\to\infty} \sum_{m=1}^{k} \sum_{n=1}^{\infty} |\hat{u}_{k-m,\ell+n}(t)|\,|\hat{u}_{m,-n}(t)| \\
&\le \sum_{m=1}^{k} \sum_{n=1}^{\infty} \frac{h_p\left(\frac{\sigma(k+\ell-(m-n))}{k+\ell+2n}\right)}{(k+\ell-(m-n))^{\frac{\sigma(k+\ell-(m-n))}{k+\ell+2n}}} \cdot \frac{h_p\left(\frac{\sigma(m+n)}{k+\ell+2n}\right)}{(m+n)^{\frac{\sigma(m+n)}{k+\ell+2n}}} \\
&\le \sum_{m=1}^{k} \sum_{n=1}^{\infty} \frac{M\left(a\frac{\sigma(k+\ell-(m-n))}{k+\ell+2n}\right)^{\frac{\sigma(k+\ell-(m-n))}{k+\ell+2n}}}{(k+\ell-(m-n))^{\frac{\sigma(k+\ell-(m-n))}{k+\ell+2n}}} \cdot \frac{M\left(a\frac{\sigma(m+n)}{k+\ell+2n}\right)^{\frac{\sigma(m+n)}{k+\ell+2n}}}{(m+n)^{\frac{\sigma(m+n)}{k+\ell+2n}}} \\
&= \sum_{m=1}^{k} \sum_{n=1}^{\infty} \frac{M^2(a\sigma)^\sigma}{(k+\ell+2n)^\sigma} = kM^2(a\sigma)^\sigma \sum_{n=1}^{\infty} \frac{1}{(k+\ell+2n)^\sigma} \\
&\le kM^2(a\sigma)^\sigma \int_0^\infty \frac{dx}{(2x+k+\ell)^\sigma} \le \frac{(k+\ell)M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-1}} = \frac{M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-2}}.
\end{aligned} \tag{5.22}$$

In arriving at the result of (5.22), we have used the fact

$$\int_0^\infty \frac{dx}{(2x+k+\ell)^\sigma} = \lim_{\xi\to\infty}\int_0^\xi (2x+k+\ell)^{-\sigma}dx \;=\; \frac{1}{2(\sigma-1)(k+\ell)^{\sigma-1}} \;\le\; \frac{1}{(k+\ell)^{\sigma-1}}, \tag{5.23}$$

since $\sigma \geq 3$.

For the third sum in the right-hand side of (5.18), using inequality (5.14), we obtain that

$$\begin{aligned}
&\limsup_{t\to\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\ell}|\hat{u}_{k+m,\ell-n}(t)|\,|\hat{u}_{-m,n}(t)| \;\le \\
&\le \sum_{m=1}^{\infty}\sum_{n=1}^{\ell}\frac{h_p\left(\frac{\sigma(k+\ell+(m-n))}{k+\ell+2m}\right)}{(k+\ell+(m-n))^{\frac{\sigma(k+\ell+(m-n))}{k+\ell+2m}}}\cdot\frac{h_p\left(\frac{\sigma(m+n)}{k+\ell+2m}\right)}{(m+n)^{\frac{\sigma(m+n)}{k+\ell+2m}}} \\
&\le \sum_{m=1}^{\infty}\sum_{n=1}^{\ell}\frac{M\left(a\frac{\sigma(k+\ell+(m-n))}{k+\ell+2m}\right)^{\frac{\sigma(k+\ell+(m-n))}{k+\ell+2m}}}{(k+\ell+(m-n))^{\frac{\sigma(k+\ell+(m-n))}{k+\ell+2m}}}\cdot\frac{M\left(a\frac{\sigma(m+n)}{k+\ell+2m}\right)^{\frac{\sigma(m+n)}{k+\ell+2m}}}{(m+n)^{\frac{\sigma(m+n)}{k+\ell+2m}}} \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\ell}\frac{M^2(a\sigma)^\sigma}{(k+\ell+2m)^\sigma} \;\le\; \ell M^2(a\sigma)^\sigma\sum_{m=1}^{\infty}\frac{1}{(k+\ell+2m)^\sigma} \\
&\le \ell M^2(a\sigma)^\sigma\int_0^\infty\frac{dx}{(2x+k+\ell)^\sigma} \le \frac{(k+\ell)M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-1}} = \frac{M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-2}}.
\end{aligned}\tag{5.24}$$

In arriving at the result of (5.24), we have used (5.23).

For the fourth sum in the right-hand side of (5.18), using inequality (5.14), we obtain that

$$\begin{aligned}
&\limsup_{t\to\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|\hat{u}_{k+m,\ell+n}(t)|\,|\hat{u}_{-m,-n}(t)| \\
&\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{h_p\left(\frac{\sigma(k+\ell+m+n)}{k+\ell+2m+2n}\right)}{(k+\ell+m+n)^{\frac{\sigma(k+\ell+m+n)}{k+\ell+2m+2n}}}\cdot\frac{h_p\left(\frac{\sigma(m+n)}{k+\ell+2m+2n}\right)}{(m+n)^{\frac{\sigma(m+n)}{k+\ell+2m+2n}}} \\
&\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{M\left(a\frac{\sigma(k+\ell+m+n)}{k+\ell+2m+2n}\right)^{\frac{\sigma(k+\ell+m+n)}{k+\ell+2m+2n}}}{(k+\ell+m+n)^{\frac{\sigma(k+\ell+m+n)}{k+\ell+2m+2n}}}\cdot\frac{M\left(a\frac{\sigma(m+n)}{k+\ell+2m+2n}\right)^{\frac{\sigma(m+n)}{k+\ell+2m+2n}}}{(m+n)^{\frac{\sigma(m+n)}{k+\ell+2m+2n}}} \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{M^2(a\sigma)^\sigma}{(k+\ell+2m+2n)^\sigma} \;\le\; M^2(a\sigma)^\sigma\int_0^\infty\int_0^\infty\frac{dydx}{(2x+2y+k+\ell)^\sigma} \le \frac{M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-2}}.
\end{aligned}\tag{5.25}$$

In arriving at the result of (5.25), we have used the fact

$$\int_0^\infty\int_0^\infty\frac{dydx}{(2x+2y+k+\ell)^\sigma} = \lim_{\xi\to\infty}\lim_{\psi\to\infty}\int_0^\xi\int_0^\psi\frac{dydx}{(2x+2y+k+\ell)^\sigma},$$

from which it is easy to see that

$$\lim_{\psi\to\infty}\int_0^{\psi}(2y+2x+k+\ell)^{-\sigma}dy \;=\; \frac{1}{2}\lim_{\psi\to\infty}\left.\frac{(2y+2x+k+\ell)^{-\sigma+1}}{-\sigma+1}\right]_{y=0}^{y=\psi}$$
$$= \frac{1}{2(\sigma-1)(2x+k+\ell)^{\sigma-1}}$$

and finally, notice that

$$\lim_{\xi\to\infty}\int_0^{\xi}\frac{(2x+k+\ell)^{-\sigma+1}}{2(\sigma-1)}dx = \lim_{\xi\to\infty}\left.\frac{(2x+k+\ell)^{-\sigma+2}}{4(\sigma-1)(-\sigma+2)}\right]_{x=0}^{x=\xi}$$
$$= \frac{1}{4(\sigma-1)(\sigma-2)(k+\ell)^{\sigma-2}} \;\leq\; \frac{1}{(k+\ell)^{\sigma-2}},$$

since $\sigma \geq 3$.

For the fifth sum in the right-hand side of (5.18), using inequality (5.14), we obtain that

$$\limsup_{t\to\infty}\sum_{m=k+1}^{\infty}\sum_{n=1}^{\infty}|\hat{u}_{k-m,\ell+n}(t)|\,|\hat{u}_{m,-n}(t)|$$
$$\leq \sum_{m=k+1}^{\infty}\sum_{n=1}^{\infty}\frac{h_p\left(\frac{\sigma(-k+\ell+m+n)}{-k+\ell+2m+2n}\right)}{(-k+\ell+m+n)^{\frac{\sigma(-k+\ell+m+n)}{-k+\ell+2m+2n}}}\cdot\frac{h_p\left(\frac{\sigma(m+n)}{-k+\ell+2m+2n}\right)}{(m+n)^{\frac{\sigma(m+n)}{-k+\ell+2m+2n}}}$$
$$\leq \sum_{m=k+1}^{\infty}\sum_{n=1}^{\infty}\frac{M\left(a\frac{\sigma(-k+\ell+m+n)}{-k+\ell+2m+2n}\right)^{\frac{\sigma(-k+\ell+m+n)}{-k+\ell+2m+2n}}}{(-k+\ell+m+n)^{\frac{\sigma(-k+\ell+m+n)}{-k+\ell+2m+2n}}}\cdot\frac{M\left(a\frac{\sigma(m+n)}{-k+\ell+2m+2n}\right)^{\frac{\sigma(m+n)}{-k+\ell+2m+2n}}}{(m+n)^{\frac{\sigma(m+n)}{-k+\ell+2m+2n}}}$$
$$= \sum_{m=k+1}^{\infty}\sum_{n=1}^{\infty}\frac{M^2(a\sigma)^{\sigma}}{(-k+\ell+2m+2n)^{\sigma}} \;\leq\; M^2(a\sigma)^{\sigma}\int_k^{\infty}\int_0^{\infty}\frac{dydx}{(2x+2y-k+\ell)^{\sigma}}$$
$$\leq \frac{M^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma-2}}. \tag{5.26}$$

In arriving at the result of (5.26), we have used the fact

$$\int_k^{\infty}\int_0^{\infty}\frac{dydx}{(2x+2y-k+\ell)^{\sigma}} = \lim_{\xi\to\infty}\lim_{\psi\to\infty}\int_k^{\xi}\int_0^{\psi}\frac{dydx}{(2x+2y-k+\ell)^{\sigma}},$$

from which it is easy to see that

$$\lim_{\psi\to\infty}\int_0^{\psi}(2y+2x-k+\ell)^{-\sigma}dy = \frac{1}{2}\lim_{\psi\to\infty}\left.\frac{(2y+2x-k+\ell)^{-\sigma+1}}{-\sigma+1}\right]_{y=0}^{y=\psi}$$
$$= \frac{1}{2(\sigma-1)(2x-k+\ell)^{\sigma-1}}$$

and finally, notice that

$$\lim_{\xi\to\infty}\int_k^{\xi}\frac{(2x-k+\ell)^{-\sigma+1}}{2(\sigma-1)}dx = \lim_{\xi\to\infty}\frac{(2x-k+\ell)^{-\sigma+2}}{4(\sigma-1)(-\sigma+2)}\Bigg]_{x=k}^{x=\xi}$$
$$= \frac{1}{4(\sigma-1)(\sigma-2)(k+\ell)^{\sigma-2}} \le \frac{1}{(k+\ell)^{\sigma-2}},$$

since $\sigma \ge 3$.

For the sixth sum in the right-hand side of (5.18) we have that

$$\frac{1}{2}\sum_{m=1}^{\infty}\hat{u}_{k-m,\ell}(t)\,\hat{u}_{m,0}(t) = \frac{1}{2}\left[\sum_{m=1}^{k}\hat{u}_{k-m,\ell}(t)\,\hat{u}_{m,0}(t) + \sum_{m=k+1}^{\infty}\hat{u}_{k-m,\ell}(t)\,\hat{u}_{m,0}(t)\right] \tag{5.27}$$

For the first sum in the right-hand side of (5.27), using inequality (5.14), we obtain that

$$\limsup_{t\to\infty}\sum_{m=1}^{k}|\hat{u}_{k-m,\ell}(t)|\,|\hat{u}_{m,0}(t)| \le \sum_{m=1}^{k}\frac{h_p\left(\frac{\sigma(k+\ell-m)}{k+\ell}\right)}{(k+\ell-m)^{\frac{\sigma(k+\ell-m)}{k+\ell}}}\cdot\frac{h_p\left(\frac{\sigma m}{k+\ell}\right)}{m^{\frac{\sigma m}{k+\ell}}}$$
$$\le \sum_{m=1}^{k}\frac{M\left(a\frac{\sigma(k+\ell-m)}{k+\ell}\right)^{\frac{\sigma(k+\ell-m)}{k+\ell}}}{(k+\ell-m)^{\frac{\sigma(k+\ell-m)}{k+\ell}}}\cdot\frac{M\left(a\frac{\sigma m}{k+\ell}\right)^{\frac{\sigma m}{k+\ell}}}{m^{\frac{\sigma m}{k+\ell}}} = \sum_{m=1}^{k}\frac{M^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma}} = \frac{kM^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma}}$$
$$\le \frac{(k+\ell)M^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma}} \le \frac{(k+\ell)^2M^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma}} = \frac{M^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma-2}}. \tag{5.28}$$

For the second sum in the right-hand side of (5.27), using inequality (5.14), we obtain that

$$\limsup_{t\to\infty}\sum_{m=k+1}^{\infty}|\hat{u}_{k-m,\ell}(t)|\,|\hat{u}_{m,0}(t)| \le \sum_{m=k+1}^{\infty}\frac{h_p\left(\frac{\sigma(-k+\ell+m)}{-k+\ell+2m}\right)}{(-k+\ell+m)^{\frac{\sigma(-k+\ell+m)}{-k+\ell+2m}}}\cdot\frac{h_p\left(\frac{\sigma m}{-k+\ell+2m}\right)}{m^{\frac{\sigma m}{-k+\ell+2m}}}$$
$$\le \sum_{m=k+1}^{\infty}\frac{M\left(a\frac{\sigma(-k+\ell+m)}{-k+\ell+2m}\right)^{\frac{\sigma(-k+\ell+m)}{-k+\ell+2m}}}{(-k+\ell+m)^{\frac{\sigma(-k+\ell+m)}{-k+\ell+2m}}}\cdot\frac{M\left(a\frac{\sigma m}{-k+\ell+2m}\right)^{\frac{\sigma m}{-k+\ell+2m}}}{m^{\frac{\sigma m}{-k+\ell+2m}}} = \sum_{m=k+1}^{\infty}\frac{M^2(a\sigma)^{\sigma}}{(-k+\ell+2m)^{\sigma}}$$
$$\le M^2(a\sigma)^{\sigma}\int_k^{\infty}\frac{dx}{(-k+\ell+2x)^{\sigma}} \le \frac{M^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma-2}}. \tag{5.29}$$

In arriving at the result of (5.29), we have used the fact

$$\int_k^{\infty}\frac{dx}{(2x-k+\ell)^{\sigma}} = \lim_{\xi\to\infty}\int_k^{\xi}\frac{dx}{(2x-k+\ell)^{\sigma}} = \lim_{\xi\to\infty}\frac{(2x-k+\ell)^{-\sigma+1}}{2(-\sigma+1)}\Bigg]_{x=k}^{x=\xi}$$
$$= \frac{1}{2(\sigma-1)(k+\ell)^{\sigma-1}} \le \frac{k+\ell}{(k+\ell)^{\sigma-1}} = \frac{1}{(k+\ell)^{\sigma-2}},$$

since $\sigma \geq 3$. Combination of (5.27), (5.28) and (5.29) provides that

$$\limsup_{t\to\infty} \sum_{m=1}^{\infty} |\hat{u}_{k-m,\ell}(t)|\,|\hat{u}_{m,0}(t)| \;\leq\; \frac{M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-2}}. \tag{5.30}$$

For the seventh sum in the right-hand side of (5.18), using inequality (5.14), we obtain that

$$\begin{aligned}
\limsup_{t\to\infty} \sum_{m=1}^{\infty} |\hat{u}_{k+m,\ell}(t)|\,|\hat{u}_{-m,0}(t)| \;&\leq\; \sum_{m=1}^{\infty} \frac{h_p\Big(\frac{\sigma(k+\ell+m)}{k+\ell+2m}\Big)}{(k+\ell+m)^{\frac{\sigma(k+\ell+m)}{k+\ell+2m}}} \cdot \frac{h_p\Big(\frac{\sigma m}{k+\ell+2m}\Big)}{m^{\frac{\sigma m}{k+\ell+2m}}} \\
\leq \sum_{m=1}^{\infty} \frac{M\Big(a\frac{\sigma(k+\ell+m)}{k+\ell+2m}\Big)^{\frac{\sigma(k+\ell+m)}{k+\ell+2m}}}{(k+\ell+m)^{\frac{\sigma(k+\ell+m)}{k+\ell+2m}}} \cdot \frac{M\Big(a\frac{\sigma m}{k+\ell+2m}\Big)^{\frac{\sigma m}{k+\ell+2m}}}{m^{\frac{\sigma m}{k+\ell+2m}}} \;&=\; \sum_{m=1}^{\infty} \frac{M^2(a\sigma)^\sigma}{(k+\ell+2m)^\sigma} \\
\leq M^2(a\sigma)^\sigma \int_0^\infty \frac{dx}{(k+\ell+2x)^\sigma} \;\leq\; \frac{M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-2}}.
\end{aligned} \tag{5.31}$$

In arriving at the result of (5.31), we have used (5.23).

For the eighth sum in the right-hand side of (5.18) we have that

$$\frac{1}{2}\sum_{n=1}^{\infty} \hat{u}_{k,\ell-n}(t)\hat{u}_{0,n}(t) \;=\; \frac{1}{2}\left[\sum_{n=1}^{\ell} \hat{u}_{k,\ell-n}(t)\hat{u}_{0,n}(t) + \sum_{n=\ell+1}^{\infty} \hat{u}_{k,\ell-n}(t)\hat{u}_{0,n}(t)\right] \tag{5.32}$$

For the first sum in the right-hand side of (5.32), using inequality (5.14), we obtain that

$$\begin{aligned}
\limsup_{t\to\infty} \sum_{n=1}^{\ell} |\hat{u}_{k,\ell-n}(t)|\,|\hat{u}_{0,n}(t)| \;&\leq\; \sum_{n=1}^{\ell} \frac{h_p\Big(\frac{\sigma(k+\ell-n)}{k+\ell}\Big)}{(k+\ell-n)^{\frac{\sigma(k+\ell-n)}{k+\ell}}} \cdot \frac{h_p\Big(\frac{\sigma n}{k+\ell}\Big)}{n^{\frac{\sigma n}{k+\ell}}} \\
\leq \sum_{n=1}^{\ell} \frac{M\Big(a\frac{\sigma(k+\ell-n)}{k+\ell}\Big)^{\frac{\sigma(k+\ell-n)}{k+\ell}}}{(k+\ell-n)^{\frac{\sigma(k+\ell-n)}{k+\ell}}} \cdot \frac{M\Big(a\frac{\sigma n}{k+\ell}\Big)^{\frac{\sigma n}{k+\ell}}}{n^{\frac{\sigma n}{k+\ell}}} \;&=\; \sum_{n=1}^{\ell} \frac{M^2(a\sigma)^\sigma}{(k+\ell)^\sigma} \;=\; \frac{\ell M^2(a\sigma)^\sigma}{(k+\ell)^\sigma} \\
\leq \frac{(k+\ell)M^2(a\sigma)^\sigma}{(k+\ell)^\sigma} \;\leq\; \frac{(k+\ell)^2M^2(a\sigma)^\sigma}{(k+\ell)^\sigma} \;&=\; \frac{M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-2}}.
\end{aligned} \tag{5.33}$$

For the second sum in the right-hand side of (5.32), using inequality (5.14), we obtain that

$$\begin{aligned}
\limsup_{t\to\infty} \sum_{n=\ell+1}^{\infty} |\hat{u}_{k,\ell-n}(t)|\,|\hat{u}_{0,n}(t)| \;&\leq\; \sum_{n=\ell+1}^{\infty} \frac{h_p\Big(\frac{\sigma(k-\ell+n)}{k-\ell+2n}\Big)}{(k-\ell+n)^{\frac{\sigma(k-\ell+n)}{k-\ell+2n}}} \cdot \frac{h_p\Big(\frac{\sigma n}{k-\ell+2n}\Big)}{n^{\frac{\sigma n}{k-\ell+2n}}} \\
\leq \sum_{n=\ell+1}^{\infty} \frac{M\Big(a\frac{\sigma(k-\ell+n)}{k-\ell+2n}\Big)^{\frac{\sigma(k-\ell+n)}{k-\ell+2n}}}{(k-\ell+n)^{\frac{\sigma(k-\ell+n)}{k-\ell+2n}}} \cdot \frac{M\Big(a\frac{\sigma n}{k-\ell+2n}\Big)^{\frac{\sigma n}{k-\ell+2n}}}{n^{\frac{\sigma n}{k-\ell+2n}}} \;&=\; \sum_{n=\ell+1}^{\infty} \frac{M^2(a\sigma)^\sigma}{(k-\ell+2n)^\sigma} \\
\leq M^2(a\sigma)^\sigma \int_\ell^\infty \frac{dx}{(k-\ell+2x)^\sigma} \;\leq\; \frac{M^2(a\sigma)^\sigma}{(k+\ell)^{\sigma-2}}.
\end{aligned} \tag{5.34}$$

In arriving at the result of (5.34), we have used the fact

$$\int_{\ell}^{\infty} \frac{dx}{(2x+k-\ell)^{\sigma}} = \lim_{\xi\to\infty} \int_{\ell}^{\xi} \frac{dx}{(2x+k-\ell)^{\sigma}} = \lim_{\xi\to\infty} \frac{(2x+k-\ell)^{-\sigma+1}}{2(-\sigma+1)}\Bigg]_{x=\ell}^{x=\xi}$$

$$= \frac{1}{2(\sigma-1)(k+\ell)^{\sigma-1}} \le \frac{k+\ell}{(k+\ell)^{\sigma-1}} = \frac{1}{(k+\ell)^{\sigma-2}},$$

since $\sigma \ge 3$. Combination of (5.32), (5.33) and (5.34) provides that

$$\limsup_{t\to\infty} \sum_{n=1}^{\infty} |\hat{u}_{k,\ell-n}(t)|\,|\hat{u}_{0,n}(t)| \le \frac{M^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma-2}}. \tag{5.35}$$

For the ninth sum in the right-hand side of (5.18), using inequality (5.14), we obtain that

$$\limsup_{t\to\infty} \sum_{n=1}^{\infty} |\hat{u}_{k,\ell+n}(t)|\,|\hat{u}_{0,-n}(t)| \le \sum_{n=1}^{\infty} \frac{h_p\left(\frac{\sigma(k+\ell+n)}{k+\ell+2n}\right)}{(k+\ell+n)^{\frac{\sigma(k+\ell+n)}{k+\ell+2n}}} \cdot \frac{h_p\left(\frac{\sigma n}{k+\ell+2n}\right)}{n^{\frac{\sigma n}{k+\ell+2n}}}$$

$$\le \sum_{n=1}^{\infty} \frac{M\left(a\frac{\sigma(k+\ell+n)}{k+\ell+2n}\right)^{\frac{\sigma(k+\ell+n)}{k+\ell+2n}}}{(k+\ell+n)^{\frac{\sigma(k+\ell+n)}{k+\ell+2n}}} \cdot \frac{M\left(a\frac{\sigma n}{k+\ell+2n}\right)^{\frac{\sigma n}{k+\ell+2n}}}{n^{\frac{\sigma n}{k+\ell+2n}}} = \sum_{n=1}^{\infty} \frac{M^2(a\sigma)^{\sigma}}{(k+\ell+2n)^{\sigma}}$$

$$\le M^2(a\sigma)^{\sigma} \int_0^{\infty} \frac{dx}{(k+\ell+2x)^{\sigma}} \le \frac{M^2(a\sigma)^{\sigma}}{(k+\ell)^{\sigma-2}}. \tag{5.36}$$

In arriving at the result of (5.36), we have used (5.23).

Finally, combination of (5.18), (5.21), (5.22), (5.24), (5.25), (5.26), (5.30), (5.31), (5.35) and (5.36) provides (5.20). □

We now continue the proof of Claim II.

Proof of Claim II. For the $h_p^p(\sigma+\sigma_1)$ we have that

$$h_p^p(\sigma+\sigma_1) = \limsup_{t\to\infty} \sum_{(k,\ell)\in\mathbb{Z}^2} (|k|+|\ell|)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p. \tag{5.37}$$

Now, we split (5.37) in the following eight sums and we get

$$h_p^p(\sigma+\sigma_1) =$$

$$= \limsup_{t\to\infty} \Bigg[\sum_{(k,\ell)\in\mathbb{N}^2} (k+\ell)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p + \sum_{k\in-\mathbb{N},\ell\in\mathbb{N}} (-k+\ell)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p$$

$$+ \sum_{(k,\ell)\in-\mathbb{N}^2} (-k-\ell)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p + \sum_{k\in\mathbb{N},\ell\in-\mathbb{N}} (k-\ell)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p$$

$$+ \sum_{k\in\mathbb{N}} k^{p\sigma+p\sigma_1} |\hat{u}_{k,0}(t)|^p + \sum_{k\in-\mathbb{N}} (-k)^{p\sigma+p\sigma_1} |\hat{u}_{k,0}(t)|^p$$

$$+ \sum_{\ell\in\mathbb{N}} \ell^{p\sigma+p\sigma_1}|\hat{u}_{0,\ell}(t)|^p + \sum_{\ell\in-\mathbb{N}} (-\ell)^{p\sigma+p\sigma_1}|\hat{u}_{0,\ell}(t)|^p \Bigg]$$

$$= \limsup_{t\to\infty} \Bigg[\sum_{(k,\ell)\in\mathbb{N}^2} (k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p + \sum_{(k,\ell)\in\mathbb{N}^2} (k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{-k,\ell}(t)|^p$$

$$+ \sum_{(k,\ell)\in\mathbb{N}^2} (k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{-k,-\ell}(t)|^p + \sum_{(k,\ell)\in\mathbb{N}^2} (k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,-\ell}(t)|^p$$

$$+ \sum_{k\in\mathbb{N}} k^{p\sigma+p\sigma_1}|\hat{u}_{k,0}(t)|^p + \sum_{k\in\mathbb{N}} k^{p\sigma+p\sigma_1}|\hat{u}_{-k,0}(t)|^p$$

$$+ \sum_{\ell\in\mathbb{N}} \ell^{p\sigma+p\sigma_1}|\hat{u}_{0,\ell}(t)|^p + \sum_{\ell\in\mathbb{N}} \ell^{p\sigma+p\sigma_1}|\hat{u}_{0,-\ell}(t)|^p \Bigg]$$

$$= \limsup_{t\to\infty} \Bigg[2\sum_{(k,\ell)\in\mathbb{N}^2} (k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p + 2\sum_{(k,\ell)\in\mathbb{N}^2} (k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{-k,\ell}(t)|^p$$

$$+ 2\sum_{k\in\mathbb{N}} k^{p\sigma+p\sigma_1}|\hat{u}_{k,0}(t)|^p + 2\sum_{\ell\in\mathbb{N}} \ell^{p\sigma+p\sigma_1}|\hat{u}_{0,\ell}(t)|^p \Bigg], \tag{5.38}$$

since $|\hat{u}_{k,\ell}(t)| = |\hat{u}_{-k,-\ell}(t)|$, $|\hat{u}_{-k,\ell}(t)| = |\hat{u}_{k,-\ell}(t)|$, $|\hat{u}_{k,0}(t)| = |\hat{u}_{-k,0}(t)|$ and $|\hat{u}_{0,\ell}(t)| = |\hat{u}_{0,-\ell}(t)|$.

For the first sum in (5.38) we have that

$$\limsup_{t\to\infty} \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p$$

$$= \limsup_{t\to\infty} \Bigg[\sum_{k=1}^{\infty}\sum_{\ell=1}^{n_0-1}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p + \sum_{k=1}^{\infty}\sum_{\ell=n_0}^{\infty}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p \Bigg]$$

$$= \limsup_{t\to\infty} \Bigg[\sum_{k=1}^{n_0-1}\sum_{\ell=1}^{n_0-1}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p + \sum_{k=n_0}^{\infty}\sum_{\ell=1}^{n_0-1}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p$$

$$+ \sum_{k=1}^{n_0-1}\sum_{\ell=n_0}^{\infty}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p + \sum_{k=n_0}^{\infty}\sum_{\ell=n_0}^{\infty}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p \Bigg]. \tag{5.39}$$

For the first sum in (5.39) we have that

$$\limsup_{t\to\infty} \sum_{k=1}^{n_0-1}\sum_{\ell=1}^{n_0-1}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p$$

$$= \limsup_{t\to\infty} \sum_{k=1}^{n_0-1}\sum_{\ell=1}^{n_0-1}(k+\ell)^{p\sigma+p\sigma_1-2p}(k+\ell)^{2p}|\hat{u}_{k,\ell}(t)|^p$$

$$\leq (2n_0-2)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \sum_{k=1}^{n_0-1}\sum_{\ell=1}^{n_0-1}(k+\ell)^{2p}|\hat{u}_{k,\ell}(t)|^p$$

$$= (2n_0-2)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \sum_{k=1}^{n_0-1}\sum_{\ell=1}^{n_0-1}\Big((k+\ell)^4|\hat{u}_{k,\ell}(t)|^2\Big)^{p/2}$$

$$
\begin{aligned}
&\leq (2n_0-2)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \left(\sum_{k=1}^{n_0-1} \sum_{\ell=1}^{n_0-1} (k+\ell)^4 |\hat{u}_{k,\ell}(t)|^2 \right)^{p/2} \\
&\leq (2n_0-2)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \left(\sum_{k=1}^{n_0-1} \sum_{\ell=1}^{n_0-1} (k^2+\ell^2)^4 |\hat{u}_{k,\ell}(t)|^2 \right)^{p/2} \\
&\leq (2n_0-2)^{p\sigma+p\sigma_1-2p} R_4^p.
\end{aligned} \tag{5.40}
$$

For the second sum in (5.39) we have that

$$
\begin{aligned}
&\limsup_{t\to\infty} \sum_{k=n_0}^{\infty} \sum_{\ell=1}^{n_0-1} (k+\ell)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p \\
&= \limsup_{t\to\infty} \sum_{k=n_0}^{\infty} \sum_{\ell=n_0}^{2n_0-2} (k+\ell-n_0+1)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p \\
&\leq \limsup_{t\to\infty} \sum_{k=n_0}^{\infty} \sum_{\ell=n_0}^{\infty} (k+\ell-n_0+1)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p \\
&\leq \limsup_{t\to\infty} \sum_{k=n_0}^{\infty} \sum_{\ell=n_0}^{\infty} (k+\ell)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p.
\end{aligned} \tag{5.41}
$$

For the third sum in (5.39) we have that

$$
\begin{aligned}
&\limsup_{t\to\infty} \sum_{k=1}^{n_0-1} \sum_{\ell=n_0}^{\infty} (k+\ell)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p \\
&= \limsup_{t\to\infty} \sum_{k=n_0}^{2n_0-2} \sum_{\ell=n_0}^{\infty} (k+\ell-n_0+1)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p \\
&\leq \limsup_{t\to\infty} \sum_{k=n_0}^{\infty} \sum_{\ell=n_0}^{\infty} (k+\ell-n_0+1)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p \\
&\leq \limsup_{t\to\infty} \sum_{k=n_0}^{\infty} \sum_{\ell=n_0}^{\infty} (k+\ell)^{p\sigma+p\sigma_1} |\hat{u}_{k,\ell}(t)|^p.
\end{aligned} \tag{5.42}
$$

Now, from (5.3), we have

$$
\operatorname{Re} \lambda_{k,\ell} \;\geq\; c_1(k+\ell)^{\gamma} \quad \text{for } k+\ell \geq n_0,
$$

which in particular implies that

$$
\operatorname{Re} \lambda_{k,\ell} \;\geq\; c_1(k+\ell)^{\gamma} \quad \text{for } k,\ell \geq n_0. \tag{5.43}
$$

Combination of (5.19), (5.20) and (5.43) provides that

$$
\limsup_{t\to\infty} |\hat{u}_{k,\ell}(t)| \;\leq\; \frac{k+\ell}{\operatorname{Re} \lambda_{k,\ell}} \limsup_{t\to\infty} |\hat{\varphi}_{k,\ell}(t)| \;\leq\; \frac{13M^2(a\sigma)^{\sigma}}{2c_1(k+\ell)^{\sigma+\gamma-3}} \quad \text{for } k,\ell \geq n_0.
$$

Thus, for the fourth sum in (5.39) we have that

$$\begin{aligned}
&\limsup_{t\to\infty}\sum_{k=n_0}^{\infty}\sum_{\ell=n_0}^{\infty}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p\\
&\leq \sum_{k=n_0}^{\infty}\sum_{\ell=n_0}^{\infty}\frac{(k+\ell)^{p\sigma+p\sigma_1}13^pM^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p(k+\ell)^{p\sigma+p\gamma-3p}}\\
&= \left(\frac{13M^2}{2c_1}\right)^p(a\sigma)^{p\sigma}\sum_{k=n_0}^{\infty}\sum_{\ell=n_0}^{\infty}\frac{1}{(k+\ell)^{p(\gamma-3-\sigma_1)}}\\
&\leq \left(\frac{13M^2}{2c_1}\right)^p(a\sigma)^{p\sigma}\int_{n_0-1}^{\infty}\int_{n_0-1}^{\infty}\frac{dydx}{(x+y)^{p(\gamma-3-\sigma_1)}}\\
&= \frac{(2n_0-2)^{-p(\gamma-3-\sigma_1)+2}}{\big(p(\gamma-3-\sigma_1)-1\big)\big(p(\gamma-3-\sigma_1)-2\big)}\left(\frac{13M^2}{2c_1}\right)^p(a\sigma)^{p\sigma}. \qquad (5.44)
\end{aligned}$$

In arriving at the result of (5.44), we have used the fact

$$\int_{n_0-1}^{\infty}\int_{n_0-1}^{\infty}\frac{dydx}{(x+y)^{p(\gamma-3-\sigma_1)}}=\lim_{\xi\to\infty}\lim_{\psi\to\infty}\int_{n_0-1}^{\xi}\int_{n_0-1}^{\psi}\frac{dydx}{(x+y)^{p(\gamma-3-\sigma_1)}},$$

from which it is easy to see that

$$\begin{aligned}
\lim_{\psi\to\infty}\int_{n_0-1}^{\psi}(x+y)^{-p(\gamma-3-\sigma_1)}dy &= \lim_{\psi\to\infty}\frac{(x+y)^{-p(\gamma-3-\sigma_1)+1}}{-p(\gamma-3-\sigma_1)+1}\Bigg]_{y=n_0-1}^{y=\psi}\\
&= \frac{(x+n_0-1)^{-p(\gamma-3-\sigma_1)+1}}{p(\gamma-3-\sigma_1)-1}
\end{aligned}$$

and finally, notice that

$$\begin{aligned}
&\lim_{\xi\to\infty}\int_{n_0-1}^{\xi}\frac{(x+n_0-1)^{-p(\gamma-3-\sigma_1)+1}}{p(\gamma-3-\sigma_1)-1}dx\\
&=\frac{1}{p(\gamma-3-\sigma_1)-1}\lim_{\xi\to\infty}\frac{(x+n_0-1)^{-p(\gamma-3-\sigma_1)+2}}{-p(\gamma-3-\sigma_1)+2}\Bigg]_{x=n_0-1}^{x=\xi}\\
&=\frac{(2n_0-2)^{-p(\gamma-3-\sigma_1)+2}}{\big(p(\gamma-3-\sigma_1)-1\big)\big(p(\gamma-3-\sigma_1)-2\big)},
\end{aligned}$$

because of the fact that

$$p(\gamma-3-\sigma_1)-1>1\Longleftrightarrow\sigma_1<\gamma-\frac{3p+2}{p}.$$

Notice that the sums in (5.41) and (5.42) are estimated as the fourth sum as above and so for these two sums (5.44) holds. Finally, combination of (5.39), (5.40), (5.41), (5.42) and (5.44) provides that

$$\limsup_{t\to\infty}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{k,\ell}(t)|^p$$
$$\leq (2n_0-2)^{p\sigma+p\sigma_1-2p}R_4^p+\frac{3(2n_0-2)^{-p(\gamma-3-\sigma_1)+2}}{\big(p(\gamma-3-\sigma_1)-1\big)\big(p(\gamma-3-\sigma_1)-2\big)}\left(\frac{13M^2}{2c_1}\right)^p(a\sigma)^{p\sigma}. \tag{5.45}$$

For the second sum in (5.38) we have by splitting it in such a way like the first sum in (5.38), and by following the same steps as above for the first sum in (5.38) that (5.45) again holds, namely,

$$\limsup_{t\to\infty}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}(k+\ell)^{p\sigma+p\sigma_1}|\hat{u}_{-k,\ell}(t)|^p$$
$$\leq (2n_0-2)^{p\sigma+p\sigma_1-2p}R_4^p+\frac{3(2n_0-2)^{-p(\gamma-3-\sigma_1)+2}}{\big(p(\gamma-3-\sigma_1)-1\big)\big(p(\gamma-3-\sigma_1)-2\big)}\left(\frac{13M^2}{2c_1}\right)^p(a\sigma)^{p\sigma}. \tag{5.46}$$

The only noteworthy modification for the estimation of the second sum in (5.38) is that (5.20) will be replaced by:

Remark 5.3.1. For every $(k,\ell)\in\mathbb{N}^2$ it holds that

$$\limsup_{t\to\infty}|\hat{\varphi}_{-k,\ell}(t)| \leq \frac{13M^2(a\sigma)^\sigma}{2(k+\ell)^{\sigma-2}}. \tag{5.47}$$

It is easy to see that the proof of (5.47) is along the lines of the proof of Lemma 5.3.1.

Also, for the estimation of the second sum in (5.38), we once again need (5.3), to get that

$$\operatorname{Re}\lambda_{-k,\ell} \geq c_1(|-k|+\ell)^\gamma = c_1(k+\ell)^\gamma \quad \text{for } |-k|+\ell\geq n_0, \text{ i.e., for } k+\ell\geq n_0,$$

which in particular implies that

$$\operatorname{Re}\lambda_{-k,\ell} \geq c_1(k+\ell)^\gamma \quad \text{for } k,\ell\geq n_0. \tag{5.48}$$

Finally, for the estimation of the second sum in (5.38), combination of (5.19), (5.47) and (5.48) provides that

$$\limsup_{t\to\infty}|\hat{u}_{-k,\ell}(t)| \leq \frac{|-k|}{\operatorname{Re}\lambda_{-k,\ell}}\limsup_{t\to\infty}|\hat{\varphi}_{-k,\ell}(t)| \leq \frac{k+\ell}{\operatorname{Re}\lambda_{-k,\ell}}\limsup_{t\to\infty}|\hat{\varphi}_{-k,\ell}(t)|$$
$$\leq \frac{13M^2(a\sigma)^\sigma}{2c_1(k+\ell)^{\sigma+\gamma-3}} \quad \text{for } k,\ell\geq n_0.$$

Now, from (5.3) with $\ell = 0$, we have

$$\operatorname{Re}\lambda_{k,0} \;\geq\; c_1 k^{\gamma} \quad \text{for } k \geq n_0. \tag{5.49}$$

Combination of (5.19) and (5.20) with $\ell = 0$ and (5.49) provides that

$$\limsup_{t\to\infty} |\hat{u}_{k,0}(t)| \;\leq\; \frac{k}{\operatorname{Re}\lambda_{k,0}} \limsup_{t\to\infty} |\hat{\varphi}_{k,0}(t)| \;\leq\; \frac{13M^2(a\sigma)^{\sigma}}{2c_1 k^{\sigma+\gamma-3}} \quad \text{for } k \geq n_0.$$

Thus, for the third sum in (5.38) we have that

$$\begin{aligned}
&\limsup_{t\to\infty} \sum_{k=1}^{\infty} k^{p\sigma+p\sigma_1} |\hat{u}_{k,0}(t)|^p \\
&\leq \sum_{k=n_0}^{\infty} \frac{k^{p\sigma+p\sigma_1} 13^p M^{2p} (a\sigma)^{p\sigma}}{(2c_1)^p k^{p\sigma+p\gamma-3p}} + \limsup_{t\to\infty} \sum_{k=1}^{n_0-1} k^{p\sigma+p\sigma_1} |\hat{u}_{k,0}(t)|^p \\
&\leq \left(\frac{13M^2}{2c_1}\right)^p (a\sigma)^{p\sigma} \sum_{k=n_0}^{\infty} \frac{1}{k^{p(\gamma-3-\sigma_1)}} + (n_0-1)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \sum_{k=1}^{n_0-1} k^{2p} |\hat{u}_{k,0}(t)|^p \\
&\leq \left(\frac{13M^2}{2c_1}\right)^p (a\sigma)^{p\sigma} \int_{n_0-1}^{\infty} \frac{dx}{x^{p(\gamma-3-\sigma_1)}} + (n_0-1)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \sum_{k=1}^{n_0-1} \big(k^4 |\hat{u}_{k,0}(t)|^2\big)^{p/2} \\
&\leq \frac{(n_0-1)^{-p(\gamma-3-\sigma_1)+1}}{p(\gamma-3-\sigma_1)-1} \left(\frac{13M^2}{2c_1}\right)^p (a\sigma)^{p\sigma} \\
&\quad + (n_0-1)^{p\sigma+p\sigma_1-2p} \limsup_{t\to\infty} \left(\sum_{k=1}^{n_0-1} k^4 |\hat{u}_{k,0}(t)|^2\right)^{p/2} \\
&\leq \frac{1}{\big(p(\gamma-3-\sigma_1)-1\big)(n_0-1)^{p(\gamma-3-\sigma_1)-1}} \left(\frac{13M^2}{2c_1}\right)^p (a\sigma)^{p\sigma} + (n_0-1)^{p\sigma+p\sigma_1-2p} R_2^p,
\end{aligned} \tag{5.50}$$

because of the fact that

$$p(\gamma-3-\sigma_1) > 1 \iff \sigma_1 < \gamma - \frac{3p+1}{p}.$$

Now, from (5.3) with $k = 0$, we have

$$\operatorname{Re}\lambda_{0,\ell} \;\geq\; c_1 \ell^{\gamma} \quad \text{for } \ell \geq n_0. \tag{5.51}$$

Combination of (5.19) and (5.20) with $k = 0$ and (5.51) provides that

$$\limsup_{t\to\infty} |\hat{u}_{0,\ell}(t)| \;\leq\; \frac{k}{\operatorname{Re}\lambda_{0,\ell}} \limsup_{t\to\infty} |\hat{\varphi}_{0,\ell}(t)| \;\leq\; \frac{13kM^2(a\sigma)^{\sigma}}{2c_1 \ell^{\sigma+\gamma-2}} \quad \text{for } \ell \geq n_0.$$

Thus, for the fourth sum in (5.38) we have that

$$
\begin{aligned}
&\limsup_{t\to\infty}\sum_{\ell=1}^{\infty}\ell^{p\sigma+p\sigma_1}|\hat{u}_{0,\ell}(t)|^p\\
&\le \sum_{\ell=n_0}^{\infty}\frac{\ell^{p\sigma+p\sigma_1}13^p k^p M^{2p}(a\sigma)^{p\sigma}}{(2c_1)^p\ell^{p\sigma+p\gamma-2p}}+\limsup_{t\to\infty}\sum_{\ell=1}^{n_0-1}\ell^{p\sigma+p\sigma_1}|\hat{u}_{0,\ell}(t)|^p\\
&\le \left(\frac{13kM^2}{2c_1}\right)^p(a\sigma)^{p\sigma}\sum_{\ell=n_0}^{\infty}\frac{1}{\ell^{p(\gamma-2-\sigma_1)}}+(n_0-1)^{p\sigma+p\sigma_1-2p}\limsup_{t\to\infty}\sum_{\ell=1}^{n_0-1}\ell^{2p}|\hat{u}_{0,\ell}(t)|^p\\
&\le \left(\frac{13kM^2}{2c_1}\right)^p(a\sigma)^{p\sigma}\int_{n_0-1}^{\infty}\frac{dx}{x^{p(\gamma-2-\sigma_1)}}+(n_0-1)^{p\sigma+p\sigma_1-2p}\limsup_{t\to\infty}\sum_{\ell=1}^{n_0-1}\left(\ell^4|\hat{u}_{0,\ell}(t)|^2\right)^{p/2}\\
&\le \frac{(n_0-1)^{-p(\gamma-2-\sigma_1)+1}}{p(\gamma-2-\sigma_1)-1}\left(\frac{13kM^2}{2c_1}\right)^p(a\sigma)^{p\sigma}\\
&\quad+(n_0-1)^{p\sigma+p\sigma_1-2p}\limsup_{t\to\infty}\left(\sum_{\ell=1}^{n_0-1}\ell^4|\hat{u}_{0,\ell}(t)|^2\right)^{p/2}\\
&\le \frac{1}{\big(p(\gamma-2-\sigma_1)-1\big)(n_0-1)^{p(\gamma-2-\sigma_1)-1}}\left(\frac{13kM^2}{2c_1}\right)^p(a\sigma)^{p\sigma}+(n_0-1)^{p\sigma+p\sigma_1-2p}R_2^p,
\end{aligned}
\tag{5.52}
$$

because of the fact that

$$p(\gamma-2-\sigma_1)>1 \iff \sigma_1<\gamma-\frac{2p+1}{p}.$$

Combination of (5.38), (5.45), (5.46), (5.50) and (5.52) provides that

$$h_p(\sigma+\sigma_1)\ \le\ (C_1+C_2+C_3)M^2(a\sigma)^{\sigma}+4^{1/p}\left((2n_0-2)^{\sigma+\sigma_1-2}R_4+(n_0-1)^{\sigma+\sigma_1-2}R_2\right),$$

where

$$C_1=\frac{13\cdot 3^{1/p}}{2^{1-(2/p)}c_1\Big[\big(p(\gamma-3-\sigma_1)-1\big)\big(p(\gamma-3-\sigma_1)-2\big)(2n_0-2)^{p(\gamma-3-\sigma_1)-2}\Big]^{1/p}},$$

$$C_2=\frac{13}{2^{1-(1/p)}c_1\Big[\big(p(\gamma-3-\sigma_1)-1\big)(n_0-1)^{p(\gamma-3-\sigma_1)-1}\Big]^{1/p}}$$

and

$$C_3=\frac{13k}{2^{1-(1/p)}c_1\Big[\big(p(\gamma-2-\sigma_1)-1\big)(n_0-1)^{p(\gamma-2-\sigma_1)-1}\Big]^{1/p}}.$$

This inductive step is complete if we can find positive constants M and a satisfying

$$C_4M^2(a\sigma)^{\sigma}+4^{1/p}\big((2n_0-2)^{\sigma+\sigma_1-2}R_4+(n_0-1)^{\sigma+\sigma_1-2}R_2\big)\ \le\ M\big(a(\sigma+3)\big)^{\sigma+3}$$

for every $\quad \sigma \geq 3, \qquad (5.53)$

where $C_4 = C_1 + C_2 + C_3$. Clearly, for every $M > 0$, there exists an $a_0 > 0$, such that (5.53) holds for every $a \geq a_0$. □

Therefore, the following has been proved.

Theorem 5.3.1. *Assuming the existence of a global attractor $\mathcal{V}$ for the equation*

$$u_t + uu_x + \mathcal{P}u \;=\; 0,$$

where $u = u(x, y, t)$, with 2π-periodic initial data in L^2, where $\mathcal{P}$ is a linear pseudo-differential operator defined by its symbol in Fourier space, that is,

$$\big(\widehat{\mathcal{P}w}\big)_{k,\ell} = \lambda_{k,\ell}\, \hat{w}_{k,\ell}, \quad (k, \ell) \in \mathbb{Z}^2,$$

whenever $w(x, y) = \sum_{(k,\ell)\in\mathbb{Z}^2} \hat{w}_{k,\ell}\, e^{i(kx+\ell y)}$, and with the eigenvalues $\lambda_{k,\ell}$ satisfying the condition

$$\mathrm{Re}\,\lambda_{k,\ell} \;\geq\; c_1(|k| + |\ell|)^{\gamma} \quad \textit{for all } |k| + |\ell| \geq n_0,$$

for some positive constants c_1, $\gamma > 3$ and n_0 a sufficiently large positive integer, we get that, every $w \in \mathcal{V}$ extends to a holomorphic function with respect to the spatial variables.

As a consequence of the Theorem 5.3.1 we have the following result.

Corollary 5.3.1. *Let $\mathcal{M}$ be the global attractor of the equation* (5.4). *Then, every solution of* (5.4) *which is in $\mathcal{M}$ extends to a holomorphic function with respect to the spatial variables.*

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Part II: Qualitative theory of polynomial vector fields

Chapter 6

Lower dimensional dynamical systems

6.1 Historical references on Hilbert's 16th problem

In the qualitative theory of differential equations, research on limit cycles is an interesting and difficult topic. Limit cycles of planar vector fields were defined in the famous paper "Mémoire sur les courbes définies par une équation différentielle" (see [23], [24]). At the end of the 1920s Van der Pol [29], Liénard [16] and Andronov [1] proved that a periodic orbit of a self-sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After this observation, the existence and nonexistence, uniqueness, and other properties of limit cycles have been studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, and economists. However, one of the main problems in the qualitative theory of planar differential equations in the 20th century was the second part of Hilbert's 16th problem [9].

Hilbert's 16th problem was posed by David Hilbert at the Paris conference of the International Congress of Mathematicians in 1900, together with the other 22 problems. The original problem was posed as the Problem of the topology of algebraic curves and surfaces. Actually the problem consists of two similar problems in different branches of mathematics:

- An investigation of the relative positions of the branches of real algebraic curves of degree n (and similarly for algebraic surfaces).
- The determination of the upper bound for the number of limit cycles in polynomial vector fields of degree n and an investigation of their relative positions.

The first part of Hilbert's 16th problem.

In 1876 Harnack investigated algebraic curves and found that curves of degree n could have no more than $(n^2 - 3n + 4)/2$ separate components in the real plane. Furthermore, he showed how to construct curves that attained that upper bound and thus that it was the best possible bound. Curves with that number of components are called M-curves. Hilbert had investigated the M-curves of degree 6 and found that the 11 components always were grouped in a certain way. His challenge to the mathematical community now was to completely investigate the possible configurations of the components of the M-curves. Furthermore, he requested a generalization of Harnack's theorem to algebraic surfaces and a similar investigation of the surfaces with the maximum number of components.

The second part of Hilbert's 16th problem.

Here we are going to consider polynomial vector fields on the real plane, that is a system of differential equations of the form:

$$\frac{dx}{dt} = p(x, y) \quad \text{and} \quad \frac{dy}{dt} = q(x, y),$$

where both p and q are real polynomials of degree n. The second part of Hilbert's 16th problem is to decide an upper bound for the number of limit cycles in polynomial vector fields of degree n and, similar to the first part, investigate their relative positions.

The original formulation of the problems.

In his speech, Hilbert presented the problems as:

"The upper bound of closed and separate branches of an algebraic curve of degree n was decided by Harnack; from this arises the further question as of the relative positions of the branches on the plane. As of the curves of degree 6, I have - admittedly in a rather elaborate way - convinced myself that the 11 branches, that they can have according to Harnack, never all can be separate, rather there must exist one branch, which have another branch running in its interior and nine branches running in its exterior, or opposite. It seems to me that a thorough investigation of the relative positions of the upper bound for separate branches is of great interest, and similarly the corresponding investigation of the number, shape and position of the sheets of an algebraic surface in space - it is not yet even known, how many sheets a surface of degree 4 in three-dimensional space can maximally have."

Hilbert continues:

"Following this purely algebraic problem I would like to raise a question that, it seems to me, can be attacked by the same method of continuous coefficient changing, and whose answer is of similar importance to the topology of the families of curves defined by differential equations - that is the question of the upper bound and position of the Poincaré boundary cycles (cycles limites) for a differential equation of first order in the form:

$$\frac{dy}{dx} = \frac{Y}{X},$$

where X, Y are integer, rational functions of n-th degree in resp. x, y."

Since the statement of the second part of Hilbert's 16th problem, this remains open even for quadratic polynomial vector fields. The contributions of Bamon [3], Golitsina

[8] and Kotova [15] for the particular case of quadratic vector fields, and mainly of Écalle [6] and Il'yashenko [11] in proving that any polynomial vector field has but finitely many limit cycles have been the best results in this area. But until now it has not been proved that there exists an uniform upper bound depending only on the degree.

6.2 Generalized Van der Pol and Liénard equations

The Van der Pol equation

$$\ddot{x} - \varepsilon\dot{x}\big(1 - x^2\big) + x = 0,$$

could perhaps be regarded as the fundamental example of a nonlinear ordinary differential equation. It plays an important role in the theory of nonlinear electrical circuits and in fact was first considered by Van der Pol when he studying vacuum tube oscillators. Van der Pol equation possesses a limit cycle; no linear equation can have this property. This limit cycle is one of the most frequently studied limit cycles. However, note that, it was unknown until 1995 that the limit cycle of the Van der Pol equation is not algebraic [22].

In this thesis, we study the second part of Hilbert's 16th problem for two generalized Van der Pol equations. More specifically, we consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon y^{p+1}\big(1 - x^{2q}\big), \end{cases} \tag{6.1}$$

where $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$, and the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon f(y)\big(1 - x^2\big), \end{cases} \tag{6.2}$$

where f is an odd polynomial of degree $2n + 1$, with n a fixed but arbitrary natural number and $0 < \varepsilon \ll 1$. Systems (6.1) and (6.2) reduce to the Van der Pol equation for $p = 0$, $q = 1$, and $f(y) = y$, respectively. Our purpose here is to find an upper bound for the number of limit cycles for systems (6.1) and (6.2), depending only on the degree of their polynomials and investigate their relative positions.

System (6.1) is the generalized Van der Pol equation of the form

$$\ddot{x} - \varepsilon(\dot{x})^{p+1}\big(1 - x^{2q}\big) + x = 0, \tag{6.3}$$

where $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$. We search to find an upper bound for the number of limit cycles for equation (6.3), depending only on p and q. We prove that the generalized Van der Pol equation (6.3) has a unique limit cycle, and it is simple and stable. We also examine the manner in which the position and size of the limit cycle depend on p and q.

Several other generalizations of the Van der Pol equation have been considered in the literature. Minorsky [19] has considered a generalized Van der Pol equation of the form

$$\ddot{x} - \varepsilon\dot{x}\big(1 - x^{2q}\big) + x = 0, \tag{6.4}$$

where $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$. For $q = 1$, equation (6.4) reduces to the Van der Pol equation. For $p = 0$ equations (6.3) and (6.4) are identical. By applying a perturbation method, he showed for (6.4) that the stationary amplitude A_0, to first order in ε, is

$$A_0 = \left(\frac{\int_0^{2\pi} \sin^2 t\, dt}{\int_0^{2\pi} \sin^2 t \cos^{2q} t\, dt}\right)^{1/(2q)}. \tag{6.5}$$

For $q = 1$, 2 and 3, Minorsky found from (6.5) that $A_0 = 2$, 1.68 and 1.53, respectively.

The solution of the generalized Rayleigh equation

$$\ddot{y} - \varepsilon\dot{y}\Big(1 - \frac{1}{2q+1}(\dot{y})^{2q}\Big) + y = 0, \tag{6.6}$$

where $q \in \mathbb{N}$, is closely related to the solution of (6.4). For, if we differentiate (6.6) with respect to t and let $\dot{y} = x$, then x satisfies (6.4). Hence, results for (6.6) can be derived from the corresponding results for (6.4).

Holmes and Rand [10] have examined the qualitative behaviour of the non-linear oscillations governed by a differential equation of the form

$$\ddot{x} + \dot{x}\big(\alpha + \gamma x^2\big) + \beta x + \delta x^3 = 0,$$

where α, β, γ and δ are constants; $\alpha = -1$, $\beta = 1$, $\gamma = 1$ and $\delta = 0$ corresponds to the Van der Pol equation. They investigated the presence of local and global bifurcations and considered their physical significance.

A more general class of equations, containing (6.3) as a special case, has the form

$$\ddot{x} + \dot{x}\phi(x, \dot{x}) + x = 0, \tag{6.7}$$

and was studied in [26] and [27]. They obtained conditions about the existence and uniqueness of limit cycles of (6.7). In general, we observe that the existence and uniqueness theorem for limit cycles of (6.7) proved there does not apply for equation (6.3).

System (6.2) is the generalized Van der Pol equation of the form

$$\ddot{x} - \varepsilon f(\dot{x})\big(1 - x^2\big) + x = 0, \tag{6.8}$$

where f is an odd polynomial of degree $2n + 1$, with n a fixed but arbitrary natural number and $0 < \varepsilon \ll 1$. The problem is again to find an upper bound for the number of limit cycles for equation (6.8), depending only on the degree $2n+1$ of the odd polynomial f and investigate their relative positions. We prove that the generalized Van der Pol equation (6.8) has exactly $n+1$ limit cycles for particularly chosen odd polynomials f of degree $2n+1$ and that this number is an upper bound for the number of limit cycles for every case of an arbitrary odd polynomial f of degree $2n+1$. Furthermore, we show how to construct these polynomials of equation (6.8) which attain that upper bound. On the possible relative positions of the $n+1$ limit cycles we show that there exists a limit cycle whose position depends on the position of the rest n limit cycles (actually, this limit cycle is close to the circle with the dependent radius (see Definition 9.2.4)).

The Liénard equation

$$\ddot{x} + g(x)\dot{x} + x = 0, \tag{6.9}$$

where g is a polynomial, is another generalization of the Van der Pol equation. Equation (6.9) can be studied in a phase plane as a system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - g(x)y, \end{cases} \tag{6.10}$$

or in the so-called Liénard plane as

$$\begin{cases} \dot{x} = y - G(x), \\ \dot{y} = -x, \end{cases} \tag{6.11}$$

where $G(x) = \int_0^x g(s)\,ds$. The systems (6.10) and (6.11) are analytically conjugate. We observe that system (6.2) is not of the form of Liénard's equation (6.10), except when $f(y) = y$. Obviously, for $f(y) \neq y$, (6.10) can not reduce to (6.2). So, in general, (6.2) is not a special case of (6.10) and (6.10) is not a special case of (6.2).

Liénard [16] proved that, if G is a continuous odd function, which has a unique positive root at $x = a$ and is monotone increasing for $x \geq a$, then (6.11) has a unique limit cycle. Rychkov [25] proved that, if G is an odd polynomial of degree 5, then (6.11) has at most two limit cycles.

Lins, de Melo and Pugh [17] have studied the Liénard equation (6.11), where G is a polynomial of degree d. They proved that, if $G(x) = a_3x^3 + a_2x^2 + a_1x$, then (6.11) has at most one limit cycle. In fact, they gave a complete classification of the phase space of the cubic Liénard's equation, in terms of some explicit algebraic conditions on the coefficients of G. Also, using a method due to Poincaré they proved that, if $d = 2n+1$ or $2n+2$, then for any $k \in \mathbb{N}_0$ with $0 \leq k \leq n$ there exists a polynomial $G(x) = a_dx^d + \cdots + a_1x$ such that the system (6.11) has exactly k closed orbits. Motivated by this, they conjectured that the maximum number of limit cycles for system (6.11), where G is a polynomial of degree n would be equal to $\left[\frac{n-1}{2}\right]$ (the largest integer less than or equal to $\frac{n-1}{2}$).

However, in [5] it has been proven by Dumortier, Panazzolo and Roussarie the existence of classical Liénard equations (6.11) of degree 7 with at least 4 limit cycles. This easily implied the existence of classical Liénard equations of degree n, $n \geq 7$, with $\left[\frac{n-1}{2}\right] + 1$ limit cycles. The counterexamples were proven to occur in systems

$$\begin{cases} \dot{x} = y - \Big(x^7 + \displaystyle\sum_{i=2}^{6} c_i x^i\Big), \\ \dot{y} = \varepsilon(b - x), \end{cases}$$

for small $\varepsilon > 0$. Recently, in [18] it has been proven by De Maesschalck and Dumortier the existence of classical Liénard equations (6.11) of degree 6 having 4 limit cycles. It implies the existence of classical Liénard equations of degree n, $n \geq 6$, having at least $\left[\frac{n-1}{2}\right] + 2$ limit cycles.

Il'yashenko and Panov [12] proved that, if

$$G(x) = x^n + \sum_{i=1}^{n-1} a_i x^i, \quad |a_i| \leq C, \quad C \geq 4, \quad n \geq 5,$$

and suppose that n is odd, then the number $L(n, C)$ of limit cycles of (6.11) admits the upper estimate

$$L(n, C) \leq \exp(\exp C^{14n}).$$

Caubergh and Dumortier [4] proved that the maximal number of limit cycles for (6.11) of even degree is finite when restricting the coefficients to a compact, thus proving the existential part of Hilbert's 16th problem for Liénard equations when restricting the coefficients to a compact set.

Chapter 7

Small perturbation of a Hamiltonian system

In this chapter, we make some elementary remarks about small perturbation of a Hamiltonian system. This chapter follows partly the book of Arnol'd (see [2]).

7.1 An introduction

We consider the system

$$\begin{cases} \dot{x} = y + \varepsilon f_1(x, y), \\ \dot{y} = -x + \varepsilon f_2(x, y), \end{cases} \tag{7.1}$$

where $0 < \varepsilon \ll 1$ and f_1, f_2 are C^1 functions of x and y, which is a perturbation of the linear harmonic oscillator

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x, \end{cases}$$

which has all the solutions periodic with:

$$x^0(t) = A\cos(t - t_0) \quad \text{and} \quad y^0(t) = -A\sin(t - t_0).$$

In general, the phase curves of (7.1) are not closed and it is possible to have the form of a spiral with a small distance of order ε between neighboring turns. In order to decide if the phase curve approaches the origin or recedes from it, we consider the function (mechanic energy)

$$E(x, y) = \frac{1}{2}\big(x^2 + y^2\big).$$

It is easy to compute the derivative of the energy and it is proportional to ε:

$$\frac{d}{dt}E(x, y) = x\dot{x} + y\dot{y} = \varepsilon\big(xf_1(x, y) + yf_2(x, y)\big) =: \varepsilon\dot{E}(x, y). \tag{7.2}$$

We want information for the sign of the quantity

$$\int_0^{T(\varepsilon)} \varepsilon \dot{E}\big(x^\varepsilon(t), y^\varepsilon(t)\big)dt =: \Delta E, \tag{7.3}$$

which corresponds to the change of energy of $(x^\varepsilon(t), y^\varepsilon(t))$ in one complete turn:

$$y^\varepsilon(0) = y^\varepsilon(T(\varepsilon)) = 0.$$

Using the theorem of continuous dependence on parameters in ODEs, one can prove the following lemma:

Lemma 7.1.1. *For* (7.3) *we have*

$$\Delta E = \varepsilon \int_0^{2\pi} \dot{E}\big(A\cos(t - t_0), -A\sin(t - t_0)\big)dt + o(\varepsilon). \tag{7.4}$$

Let

$$F(A) := \int_0^{2\pi} \dot{E}\big(x^0(t), y^0(t)\big)dt, \tag{7.5}$$

and we write (7.4) as

$$\Delta E = \varepsilon\Big[F(A) + \frac{o(\varepsilon)}{\varepsilon}\Big].$$

Using the implicit function theorem, one can prove the following theorem, which is the Poincaré's method:

Theorem 7.1.1. *If the function F given by* (7.5), *has a positive simple root A_0, namely*

$$F(A_0) = 0 \quad \textit{and} \quad F'(A_0) \neq 0,$$

then (7.1) *has a periodic solution with amplitude $A_0 + O(\varepsilon)$ for $0 < \varepsilon \ll 1$.*

Proof. Let

$$Q(\varepsilon, A) := F(A) + \frac{o(\varepsilon)}{\varepsilon}.$$

Using the smooth dependence on ε in ODEs, we have that Q is a C^1 function such that

$$Q(0, A_0) = 0 \quad \text{and} \quad \frac{\partial Q}{\partial A}(0, A_0) \neq 0.$$

Then, by the implicit function theorem, there exists a unique function $A(\varepsilon)$, such that

$$A(0) = A_0 \quad \text{and} \quad Q(\varepsilon, A(\varepsilon)) = 0.$$

Therefore (7.1) has a solution which is a closed curve, so it is periodic. □

Chapter 8

A generalized Van der Pol equation

In this chapter, we study the bifurcation of limit cycles from the linear oscillator $\dot{x} = y$, $\dot{y} = -x$ in the class

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon y^{p+1}\big(1 - x^{2q}\big),$$

where ε is a small positive parameter tending to 0, $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. We prove that the above differential system, in the global plane where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$, has a unique limit cycle. More specifically, the existence of a limit cycle, which is the main result in this work, is obtained by using the Poincaré's method, and the uniqueness can be derived from the work of Sabatini and Villari [28]. We also investigate and some other properties of this unique limit cycle for some special cases of this differential system. Such special cases have been studied by Minorsky [19] and Moremedi et al. [20]. This chapter follows the paper [13].

8.1 Existence, uniqueness and other properties of a limit cycle

Our main result is given in the following theorem:

Theorem 8.1.1. *System* (6.1)*, where $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$ has the unique limit cycle*

$$x^2 + y^2 = \left[\frac{(p+2q+2)(p+2q)\ldots(2q+2)}{(p+2)p\ldots 4\cdot 2}\,\frac{2q(2q-2)\ldots 4\cdot 2}{(2q-1)(2q-3)\ldots 3\cdot 1}\right]^{1/q} + O(\varepsilon),$$

and it is simple and stable.

Proof. From (7.2) we have

$$\dot{E}(x,y) = y^{p+2}\big(1 - x^{2q}\big), \tag{8.1}$$

where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. Substituting (8.1) into (7.5), we obtain that

$$F(A) = \int_0^{2\pi} (y^0(t))^{p+2}\big(1 - (x^0(t))^{2q}\big)dt, \tag{8.2}$$

where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. Substituting $x^0(t) = A\cos(t - t_0)$ and $y^0(t) = -A\sin(t - t_0)$ into (8.2), and using the assumption that $p \in \mathbb{N}_0$ is even we get

$$F(A) = A^{p+2}\left[\int_0^{2\pi} \sin^{p+2}(t - t_0)dt - A^{2q}\int_0^{2\pi} \sin^{p+2}(t - t_0)\cos^{2q}(t - t_0)dt\right]. \tag{8.3}$$

Let

$$c_1 := \int_0^{2\pi} \sin^{p+2}(t - t_0)dt$$

and

$$c_2 := \int_0^{2\pi} \sin^{p+2}(t - t_0)\cos^{2q}(t - t_0)dt,$$

where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$. Using the fact that

$$c_1 = 4\int_0^{\pi/2} \sin^{p+2}(t - t_0)dt,$$

we get

$$c_1 = 2\frac{(p+1)(p-1)\dots 3\cdot 1}{(p+2)p\dots 4\cdot 2}\pi.$$

In arriving at the result above, we have used the fact that for each $n \in \mathbb{N}$

$$\int_0^{\pi/2} \sin^{2n} t\, dt = \frac{1\cdot 3\dots(2n-1)}{2\cdot 4\dots 2n}\frac{\pi}{2}.$$

Using the fact that

$$c_2 = 4\int_0^{\pi/2} \sin^{p+2}(t - t_0)\cos^{2q}(t - t_0)dt,$$

we get

$$c_2 = 2\frac{(p+1)(p-1)\dots 5\cdot 3\cdot 1}{(p+2q+2)(p+2q)\dots(2q+2)}\frac{(2q-1)(2q-3)\dots 3\cdot 1}{2q(2q-2)\dots 4\cdot 2}\pi.$$

In arriving at the result above, we have used the fact that for each m, $n \in \mathbb{N}$ and even

$$\int_0^{\pi/2} \sin^m t\cos^n t\, dt = \frac{(m-1)(m-3)\dots 5\cdot 3\cdot 1}{(m+n)(m+n-2)\dots(n+2)}\frac{(n-1)(n-3)\dots 3\cdot 1}{n(n-2)\dots 4\cdot 2}\frac{\pi}{2}. \tag{8.4}$$

Substituting c_1 and c_2 given as above into (8.3) it follows that

$$F(A) = 2\pi A^{p+2}\left[\frac{(p+1)(p-1)\dots 3\cdot 1}{(p+2)p\dots 4\cdot 2} - \frac{(p+1)(p-1)\dots 5\cdot 3\cdot 1}{(p+2q+2)(p+2q)\dots(2q+2)}\frac{(2q-1)\dots 3\cdot 1}{2q\dots 4\cdot 2}A^{2q}\right].$$

Now, for $A > 0$ the polynomial F has the root

$$A = \left[\frac{(p+2q+2)(p+2q)\dots(2q+2)}{(p+2)p\dots 4\cdot 2}\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\right]^{1/(2q)}.$$

Let

$$A_0 = A_0(p,q) := \left[\frac{(p+2q+2)(p+2q)\dots(2q+2)}{(p+2)p\dots 4\cdot 2}\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\right]^{1/(2q)}, \tag{8.5}$$

where $p \in \mathbb{N}_0$ is even and $q \in \mathbb{N}$.

For the derivative of F we have that

$$F'(A) = 2\pi A^{p+1}\left[\frac{(p+1)(p-1)\dots 3\cdot 1}{p(p-2)\dots 4\cdot 2} - \frac{(p+1)(p-1)\dots 3\cdot 1}{(p+2q)(p+2q-2)\dots(2q+2)}\frac{(2q-1)\dots 3\cdot 1}{2q\dots 4\cdot 2}A^{2q}\right].$$

We compute the derivative of F at A_0 and we get

$$F'(A_0) = -4\pi A_0^{p+1}\frac{(p+1)(p-1)\dots 3\cdot 1}{(p+2)p\dots 4\cdot 2}\cdot q \neq 0,$$

using the assumptions that $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $A_0 > 0$. So, from Theorem 7.1.1, it follows that (6.1) has a limit cycle close to the circle $x^2 + y^2 = A_0^2$. Moreover, since $F'(A_0) < 0$, this limit cycle is simple and stable.

Let now prove that the number of limit cycles for system (6.1), with ε small is exactly one. The proof of this can be derived from the work of Sabatini and Villari [28] using Corollary 1 proved there. We first note that the system (6.1) can be written and in the form

$$\begin{cases} \dot{x} = y - \varepsilon x^{p+1}\big(y^{2q} - 1\big), \\ \dot{y} = -x, \end{cases}$$

where $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$. As we already saw, Poincaré's method (see Theorem 7.1.1) ensures the existence of a limit cycle for (6.1). Since $a = -1, b = 1, G(x) = \frac{x^2}{2}$, one has $G(a) = G(b)$, so the hypotheses of Corollary 1 hold (see [28]), and the system (6.1) has exactly one limit cycle. This completes the proof that (6.1) has exactly one limit cycle.

So, we prove that (6.1) has a unique limit cycle, and it is simple and stable. □

Remark 8.1.1. The expression (6.5) obtained by Minorsky, is a special case of the expression (8.5) which we found. Indeed, for $p = 0$ it can be verified that (8.5) equals (6.5). This may be done by evaluating the integral in the denominator of (6.5), using (8.4).

Proposition 8.1.1. *System* (6.1)*, with* $p \in \mathbb{N}_0$ *is even,* $q \in \mathbb{N}$ *satisfying* $p + 2 = 2q$*, and* $0 < \varepsilon \ll 1$ *has the unique limit cycle* $x^2 + y^2 = 4 + O(\varepsilon)$*, and it is simple and stable.*

Proof. From Theorem 8.1.1 it follows that system (6.1), with $p \in \mathbb{N}_0$ is even, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$ has a unique limit cycle, and it is simple and stable. It remains to prove that

$$\left[\frac{(p+2q+2)(p+2q)\ldots(2q+2)}{(p+2)p\ldots 4\cdot 2}\,\frac{2q(2q-2)\ldots 4\cdot 2}{(2q-1)(2q-3)\ldots 3\cdot 1}\right]^{1/q} = 4, \tag{8.6}$$

when $p + 2 = 2q$.

By the assumption that $p + 2 = 2q$ the left-hand side of (8.6) gives

$$\left[\frac{2^q(2q)(2q-1)(2q-2)\ldots(q+2)(q+1)}{(2q-1)(2q-3)(2q-5)\ldots 5\cdot 3\cdot 1}\right]^{1/q} = 2\left[\frac{2q(2q-1)\ldots(q+2)(q+1)}{(2q-1)(2q-3)\ldots 5\cdot 3\cdot 1}\right]^{1/q}.$$

Hence it suffices to show that

$$\left[\frac{2q(2q-1)(2q-2)\ldots(q+2)(q+1)}{(2q-1)(2q-3)(2q-5)\ldots 5\cdot 3\cdot 1}\right]^{1/q} = 2.$$

CLAIM. *It is valid that*

$$\frac{2q(2q-1)(2q-2)\ldots(q+2)(q+1)}{(2q-1)(2q-3)(2q-5)\ldots 5\cdot 3\cdot 1} = 2^q, \quad \forall\, q \in \mathbb{N}.$$

Proof of Claim. The claim will be proved by induction on q. For $q = 1$, we have $\frac{2}{1} = 2^1$, therefore the claim is valid for $q = 1$. Supposing that the claim is valid for q, we will prove that it is true and for $q + 1$, namely

$$\frac{\left[2(q+1)\right](2q+1)(2q)(2q-1)\ldots(q+3)(q+2)}{(2q+1)(2q-1)(2q-3)(2q-5)\ldots 5\cdot 3\cdot 1} = 2^{q+1}. \tag{8.7}$$

The left-hand side of (8.7) is equal to

$$2(q+1)\frac{2q(2q-1)(2q-2)\ldots(q+2)}{(2q-1)(2q-3)\ldots 5\cdot 3\cdot 1} = 2\cdot 2^q = 2^{q+1},$$

which is the right-hand side of (8.7). Therefore, the claim is valid for every $q \in \mathbb{N}$. This completes the proof of the proposition. □

Remark 8.1.2. It is well known that the Van der Pol equation with $0 < \varepsilon \ll 1$ has the unique limit cycle $x^2 + y^2 = 4 + O(\varepsilon)$, and it is simple and stable. This arises and from Proposition 8.1.1 with $p = 0$ and $q = 1$.

In the next proposition, we give a different proof, much more elementary than the proof has been given by Moremedi *et al.* [20], concerning the decreases of the amplitude of the limit cycle of system (6.1) with $p = 0$ and $0 < \varepsilon \ll 1$, as q increases.

Proposition 8.1.2. *System* (6.1)*, with* $p = 0$*,* $q \in \mathbb{N}$ *and* $0 < \varepsilon \ll 1$ *has a unique limit cycle which is simple, stable and its amplitude decreases monotonically from* 2 *to* 1 *as* q *increases from* $q = 1$*. Therefore, the unique limit cycle of the system* (6.1)*, with* $p = 0$ *has the equation* $x^2 + y^2 = 1 + O(\varepsilon)$ *as* $q \to \infty$*.*

Proof. From Theorem 8.1.1 it follows that system (6.1), with $p = 0$, $q \in \mathbb{N}$ and $0 < \varepsilon \ll 1$ has a unique limit cycle, and it is simple and stable. From (8.5) when $p = 0$ it follows that

$$A_0 = \left[\frac{2q+2}{2}\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\right]^{1/(2q)}.$$

Let

$$A_0(q) := \left[\frac{2q+2}{2}\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\right]^{1/(2q)}, \quad q \in \mathbb{N}. \tag{8.8}$$

Clearly, $A_0(1) = 2$. In order to prove that the sequence $A_0(q)$, $q \in \mathbb{N}$ given by (8.8) is strictly decreasing we must show that $A_0(q+1) < A_0(q)$ for all $q \in \mathbb{N}$.

We have that

$$\begin{aligned} A_0(q+1) &= \left[\frac{2q+4}{2}\frac{(2q+2)(2q)\dots 4\cdot 2}{(2q+1)(2q-1)\dots 3\cdot 1}\right]^{\frac{1}{2(q+1)}} \\ &= \left[\frac{2q+4}{2q+1}\right]^{\frac{1}{2(q+1)}}\left[\frac{2q+2}{2}\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\right]^{\frac{1}{2q}-\frac{1}{2q(q+1)}} \\ &= \left[s(q)\right]^{\frac{1}{2(q+1)}} A_0(q), \end{aligned}$$

where

$$s(q) = \frac{2q+4}{2q+1}\left[\frac{1}{q+1}\frac{(2q-1)(2q-3)\dots 3\cdot 1}{2q(2q-2)\dots 4\cdot 2}\right]^{1/q}, \quad q \in \mathbb{N}.$$

Now, in order to show that $A_0(q+1) < A_0(q)$, it suffices to show that

$$s(q) < 1, \quad \forall\, q \in \mathbb{N}.$$

We have that

$$s(q) < \frac{2q+4}{2q+1}\frac{1}{(q+1)^{1/q}}.$$

CLAIM I. *It is valid that*

$$\frac{2q+4}{2q+1} \leq (q+1)^{1/q}, \quad \forall\, q \in \mathbb{N}. \tag{8.9}$$

Proof of Claim I. The inequality (8.9) is valid for $q = 1, \dots, 5$, as it can easily be checked. In order to prove (8.9) for $q \in \mathbb{N}$, $q \geq 6$ we will show that

$$1 + \frac{2}{q} < q^{1/q} \Longleftrightarrow \Big(1 + \frac{2}{q}\Big)^q < q, \quad \forall\, q \in \mathbb{N},\, q \geq 6. \tag{8.10}$$

One can easily check that the inequality (8.10) is valid for $q = 6$ and 7. Since

$$\lim_{q\to\infty} \Big(1 + \frac{2}{q}\Big)^q = \mathrm{e}^2,$$

in order to prove (8.10) for $q \in \mathbb{N}$, $q \geq 8$, it suffices to show that the sequence $\big(1 + \frac{2}{q}\big)^q$, $q \in \mathbb{N}$, is strictly increasing. Notice that

$$\Big(1 + \frac{2}{q}\Big)^q < \Big(1 + \frac{2}{q+1}\Big)^{q+1} \Longleftrightarrow \frac{q+1}{q+3} < \Big[\frac{q(q+3)}{(q+1)(q+2)}\Big]^q$$
$$\Longleftrightarrow 1 - \frac{2}{q+3} < \Big[1 - \frac{2}{(q+1)(q+2)}\Big]^q.$$

Now, using Bernoulli's inequality, we have for $q \in \mathbb{N}$ that

$$\Big[1 - \frac{2}{(q+1)(q+2)}\Big]^q \geq 1 - \frac{2q}{(q+1)(q+2)}.$$

Since is valid that

$$\frac{1}{q+3} > \frac{q}{(q+1)(q+2)},$$

the proof that the sequence $\big(1 + \frac{2}{q}\big)^q$, $q \in \mathbb{N}$ is strictly increasing is complete.

So, we have proved the inequality (8.9) for every $q \in \mathbb{N}$. Therefore,

$$s(q) < 1, \quad \forall\, q \in \mathbb{N},$$

which proves that the sequence $A_0(q)$, $q \in \mathbb{N}$ is strictly decreasing.

Now, note that (8.8) gives

$$A_0(q) = \big[(q+1)^{1/q}\big]^{1/2} \big[(2q+1)^{1/(2q)}\big]^{1/2} \left[\frac{1}{2q+1}\left(\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\right)^2\right]^{1/(4q)}. \tag{8.11}$$

CLAIM II. *It is valid that*

$$\lim_{q\to\infty} \left[\frac{1}{2q+1}\left(\frac{2q(2q-2)\dots 4\cdot 2}{(2q-1)(2q-3)\dots 3\cdot 1}\right)^2\right]^{1/(4q)} = 1. \tag{8.12}$$

Proof of Claim II. From the inequality $0 < \sin t < 1$, $t \in (0, \pi/2)$ (with induction) we have that $\sin^{2q+1} t < \sin^{2q} t < \sin^{2q-1} t$, for every $t \in (0, \pi/2)$ and $q \in \mathbb{N}$. So, we have that

$$\int_0^{\pi/2} \sin^{2q+1} t\, dt < \int_0^{\pi/2} \sin^{2q} t\, dt < \int_0^{\pi/2} \sin^{2q-1} t\, dt. \tag{8.13}$$

Using that, for each $n \in \mathbb{N}$

$$\int_0^{\pi/2} \sin^{2n-1} t\, dt = \frac{2\cdot 4 \dots (2n-2)}{1\cdot 3\dots (2n-1)} \quad \text{and}$$

$$\int_0^{\pi/2} \sin^{2n} t\, dt = \frac{1\cdot 3\dots (2n-1)}{2\cdot 4\dots 2n}\frac{\pi}{2},$$

(8.13) leads to

$$\frac{1\cdot 3\dots (2q-1)}{2\cdot 4\dots (2q-2)} < \frac{2\cdot 4\dots (2q-2)2q}{1\cdot 3\dots (2q-3)(2q-1)}\frac{2}{\pi} < \frac{1\cdot 3\dots (2q+1)}{2\cdot 4\dots 2q}. \tag{8.14}$$

Multiplying (8.14) by

$$\frac{2\cdot 4\dots (2q-2)2q}{1\cdot 3\dots (2q-1)(2q+1)}\frac{\pi}{2},$$

we get

$$\frac{2q}{2q+1}\frac{\pi}{2} < \frac{1}{2q+1}\left[\frac{2\cdot 4\dots (2q-2)2q}{1\cdot 3\dots (2q-3)(2q-1)}\right]^2 < \frac{\pi}{2}, \tag{8.15}$$

and then the inequality

$$\left(\frac{2q}{2q+1}\right)^{1/(4q)}\left(\frac{\pi}{2}\right)^{1/(4q)} < \left[\frac{1}{2q+1}\left(\frac{2\cdot 4\dots (2q-2)2q}{1\cdot 3\dots (2q-3)(2q-1)}\right)^2\right]^{1/(4q)} < \left(\frac{\pi}{2}\right)^{1/(4q)},$$

implies (8.12).

Using (8.12), from (8.11), we easily obtain that $\lim\limits_{q\to\infty} A_0(q) = 1$.

The proof of the proposition is complete. □

Remark 8.1.3. The uniqueness of the limit cycle for the system (6.1), with $p = 0$, $q \in \mathbb{N}$ studied in Proposition 8.1.2 follows and from the fact that the function $\varphi(x, y) = -\varepsilon(1 - x^{2q})$ is strictly star-shaped (see [26],[27]).

Remark 8.1.4. From (8.15) it follows that

$$\lim_{q\to\infty} \frac{1}{2q+1}\left[\frac{2\cdot 4\dots (2q-2)2q}{1\cdot 3\dots (2q-3)(2q-1)}\right]^2 = \frac{\pi}{2},$$

which is the Wallis's product. It is exciting and unexpected how this limit of Wallis appears in the proof of Proposition 8.1.2.

Proposition 8.1.3. *System* (6.1)*, with* $p \in \mathbb{N}_0$ *is even,* $q = 1$ *and* $0 < \varepsilon \ll 1$ *has a unique limit cycle which is simple, stable and its amplitude increases monotonically from* 2 *to infinity as* p *increases from* $p = 0$.

Proof. From Theorem 8.1.1 it follows that system (6.1), with $p \in \mathbb{N}_0$ is even, $q = 1$ and $0 < \varepsilon \ll 1$ has a unique limit cycle, and it is simple and stable. From (8.5) when

$q = 1$ it follows that

$$A_0 = \left[\frac{(p+4)(p+2)p \dots 6 \cdot 4}{(p+2)p \dots 4 \cdot 2} \cdot \frac{2}{1} \right]^{1/2} = (p+4)^{1/2}.$$

Let $A_0(p) := (p+4)^{1/2}$, $p \in \mathbb{N}_0$ is even. Clearly, $A_0(0) = 2$. Obviously $A_0(p) < A_0(p+1)$ for all $p \in \mathbb{N}_0$ is even and $A_0(p) \to \infty$ as $p \to \infty$ and so the proof is complete. □

Remark 8.1.5. We make now an observation on the type of the bifurcation phenomenon of limit cycles encountered in Proposition 8.1.3. Not the "large amplitude limit cycle" is encountered in Proposition 8.1.3 but the "medium amplitude limit cycle". For given p the limit cycle of (6.1), with $q = 1$, has a finite limiting radius and therefore is called "medium amplitude limit cycle". When increasing p also the radius of the limiting circle increases; in particular when $p \to \infty$ then the limiting radius also tends to ∞. The "large amplitude limit cycle" would disappear at ∞ when the bifurcation parameter ε tends to 0.

Chapter 9

Another generalized Van der Pol equation

In this chapter, we study the bifurcation of limit cycles from the harmonic oscillator $\dot{x} = y$, $\dot{y} = -x$ in the class

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon f(y)\big(1 - x^2\big),$$

where ε is a small positive parameter tending to 0 and f is an odd polynomial of degree $2n+1$, with n an arbitrary but fixed natural number. We prove that, the above differential system, in the global plane, for particularly chosen odd polynomials f of degree $2n+1$ has exactly $n+1$ limit cycles and that this number is an upper bound for the number of limit cycles for every case of an arbitrary odd polynomial f of degree $2n+1$. More specifically, the existence of the limit cycles, which is the first of the main results in this work, is obtained by using the Poincaré's method, and the upper bound for the number of limit cycles can be derived from the work of Iliev [11]. We also investigate the possible relative positions of the limit cycles for this differential system, which is the second main problem studying in this work. In particular, we construct differential systems with n given limit cycles and one limit cycle whose position depends on the position of the previous n limit cycles. Finally, we give some examples in order to illustrate the general theory presented in this work. This chapter follows the paper [14].

9.1 Existence and other properties of limit cycles

Now, we state the main results of this chapter, which are the following theorems. The proofs of these theorems will be given in Section 9.3. For the definitions appear in these theorems, like the sets V^n, V^n_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, V^1 and the dependent radius, see the next section. The first and second of our results, consider the system (6.2), with f an odd polynomial of degree $2n+1$, where $n \in \mathbb{N}$, $n \geq 2$.

Theorem 9.1.1. *Let* $(\lambda_1, \lambda_2, \dots, \lambda_n) \in V^n$ *be such that* $(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) \in V^n_{n+1}$, *where* λ_{n+1} *is the dependent radius given by* (9.4), *if* $n \in \mathbb{N}$, $n \geq 2$. *Then the system* (6.2), *with* $0 < \varepsilon \ll 1$ *and*

$$\begin{aligned} f(y) = {} & \tau y^{2n+1} + \cdots + \tau(2n-2k+3)\dots(2n+1) \\ & \times \Big[1 - \frac{1}{2(n+2)}\sum_{i_1=1}^{n+1}\lambda_{i_1} + \frac{1}{4(n+1)(n+2)}\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1}\lambda_{i_1}\lambda_{i_2} + \dots \\ & + \frac{1}{2^k(n-k+3)\dots(n+2)}(-1)^k \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\cdots<i_k}}^{n+1}\lambda_{i_1}\dots\lambda_{i_k}\Big] y^{2(n-k)+1} \\ & + \cdots + \tau\Big[\frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!}(-1)^n \prod_{i_1=1}^{n+1}\lambda_{i_1}\Big] y, \end{aligned} \tag{9.1}$$

where $\tau \in \mathbb{R} \setminus \{0\}$ *and* $1 \leq k \leq n-1$, *has exactly the following* $n+1$ *limit cycles:*

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_2 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon).$$

Furthermore, (assuming from now on an ordering such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1}$, *where now* λ_{n+1} *is not necessary the dependent radius) we have for the stability of the limit cycles that, if* $\tau > 0$ *(respectively* $\tau < 0$*),*

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_3 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon)$$

are stable (respectively unstable) and

$$x^2 + y^2 = \lambda_2 + O(\varepsilon), \quad x^2 + y^2 = \lambda_4 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_n + O(\varepsilon)$$

are unstable (respectively stable) for n *even; and*

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_3 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_n + O(\varepsilon)$$

are unstable (respectively stable) and

$$x^2 + y^2 = \lambda_2 + O(\varepsilon), \quad x^2 + y^2 = \lambda_4 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon)$$

are stable (respectively unstable) for n *odd.*

Theorem 9.1.2. *For system* (6.2), *where* ε *is small and* f *is an arbitrary odd polynomial of degree* $2n+1$ *we have that the number of* $n+1$ *limit cycles is an upper bound for the number of limit cycles. Moreover, from the set of all the odd polynomials, the polynomials* f *given by* (9.1), *are the only that attain that upper bound.*

Our third and fourth results, concern the system (6.2), with f an odd polynomial of degree 3.

Theorem 9.1.3. *Let $\lambda_1 \in V^1$. Then the system* (6.2)*, with $0 < \varepsilon \ll 1$ and*

$$f(y) = \tau y^3 - \tau \frac{1}{8} \lambda_1 \lambda_2 y, \tag{9.2}$$

where $\tau \in \mathbb{R} \setminus \{0\}$ and λ_2 is the dependent radius given by (9.5)*, has exactly the following* 2 *limit cycles:*

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_2 + O(\varepsilon).$$

Furthermore, (assuming from now on an ordering such that $\lambda_1 < \lambda_2$, where now λ_2 is not necessary the dependent radius) we have for the stability of the limit cycles that, if $\tau > 0$ (respectively $\tau < 0$) $x^2 + y^2 = \lambda_1 + O(\varepsilon)$ is unstable (respectively stable) and $x^2 + y^2 = \lambda_2 + O(\varepsilon)$ is stable (respectively unstable).

Theorem 9.1.4. *For system* (6.2)*, where ε is small and f is an arbitrary odd polynomial of degree* 3 *we have that the number of* 2 *limit cycles is an upper bound for the number of limit cycles. Moreover, from the set of all the odd polynomials, the polynomials f given by* (9.2)*, are the only that attain that upper bound.*

Remark 9.1.1. It is important to note that the above theorems don't inform us which limit cycles we have for a differential equation of the form (6.2). That we succeed through these theorems is to construct differential equations of the form (6.2) with n given limit cycles and one limit cycle which is close to the circle with the dependent radius, for particularly chosen odd polynomials f of degree $2n+1$. So, we show how to construct differential equations of the form (6.2) that attain the upper bound of $n+1$ limit cycles, when the odd polynomial f is of degree $2n+1$. Evenly important it is still and one negative result which can be obtained by these theorems, that we know a priori which limit cycles we can't have for system (6.2) with odd polynomials f of degree $2n+1$. Substantially, we construct the set of all the possible limiting radii of limit cycles for the system (6.2) with odd polynomials f of degree $2n+1$. This is the set V_{n+1}^n which contains the Λ-points (see Definition 9.2.7).

Remark 9.1.2. It is surprising the connection between the dependent radius for a circle (see Definition 9.2.4) and the existence of one branch which can not separate from the rest branches for an algebraic curve. More specifically, relatively to the existence of such branch we refer the following of Hilbert's speech about the first part of Hilbert's 16th problem "As of the curves of degree 6, I have -admittedly in a rather elaborate way- convinced myself that the 11 branches, that they can have according to Harnack, never all can be separate, rather there must exist one branch, which have another branch running in its interior and nine branches running in its exterior, or opposite". Here, we have for the relative positions of limit cycles that the limit cycle which is close to the circle with the dependent radius can not lie wherever, contrary the position of this limit cycle depends on the position of the rest limit cycles. In this sense, we can say that the first and second part of Hilbert's 16th problem come closer.

Remark 9.1.3. I would mention for system (6.2) that by forcing the coefficients of an arbitrary odd polynomial to be those given in the Theorem 9.1.1 when $n \in \mathbb{N}$, $n \geq 2$ (respectively in the Theorem 9.1.3 when $n = 1$), do not allow us to put $n + 1$ (respectively 2) limit cycles in arbitrary placements. The reason for this is the Theorem 9.1.2 (respectively the Theorem 9.1.4); in the statement of these theorems we see that the proposing polynomials f (given in Theorems 9.1.1 and 9.1.3) are the only that attain the upper bound of the $n + 1$ limit cycles. Now it is easy to see that in the coefficients of these polynomials (unless in the first monomial in each case) appears the dependent radius, and this observation in turn implies that one limit cycle do not lie in arbitrary placements.

In order to see this more clearly consider for the system (6.2) the case where $n = 1$. Once we chose λ_1 from V^1, the dependent radius λ_2 follows from (9.5) will be positive (see Proposition 9.2.1) and different from the associated λ_1 (see Remark 9.2.3), and then for the system (6.2) with $n = 1$, the polynomial f given by (9.2) is the only that realizing the maximal number of 2 limit cycles, and are asymptotic to the circles $x^2 + y^2 = \lambda_1$ and $x^2 + y^2 = \lambda_2$ (note that for this circle the placement is not arbitrary, it depends on λ_1) as $\varepsilon \to 0$ (see Theorems 9.1.3 and 9.1.4). (See and Example 9.4.6.)

9.2 Definitions

In this section, we introduce some new definitions. These definitions are obtained by using technical integral expressions (see Remark 9.2.1) and properties of symmetric functions of the roots of polynomials (Vieta's formulas). The first of these definitions has an important role in the construction of the sinusoidal-type sets and also the advantage played by this definition along with the Definition 9.2.2 is going to be understandable in the proof of Proposition 9.2.1.

Remark 9.2.1. I adopt the sinusoidal terminology for the next two definitions, due to the formula

$$\int_0^{2\pi} \sin^{2n} t \, dt = \frac{1 \cdot 3 \dots (2n-3)(2n-1)}{2^{n-1} n!} \pi, \quad \text{for} \quad n \in \mathbb{N}, \tag{9.3}$$

(see [21]) which gives the coefficients of the sums and products.

Definition 9.2.1 (sinusoidal-type numbers)**.** Let $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$ be distinct positive real numbers, where $n \in \mathbb{N}$, $n \geq 2$. We define for $n \in \mathbb{N}$, $n \geq 3$, the sinusoidal-type numbers of order n, associated to the $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$

$$\begin{aligned}
\bar{s}^n &:= 2(n+2) + \dots + \frac{(-1)^k}{2^{k-1}(n-k+3)\dots(n+1)} \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} \lambda_{i_1}\dots\lambda_{i_k} + \dots \\
&\quad + \frac{(-1)^n}{2^{n-2}(n+1)!} \prod_{i_1=1}^{n} \lambda_{i_1}, \\
\hat{s}^n &:= 2(n+1) + \dots + \frac{(-1)^k}{2^{k-1}(n-k+2)(n-k+3)\dots n} \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} \lambda_{i_1}\dots\lambda_{i_k} + \dots \\
&\quad + \frac{(-1)^n}{2^{n-1}n!} \prod_{i_1=1}^{n} \lambda_{i_1}, \\
\tilde{s}^n &:= \frac{1}{4n(n+1)} \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} + \dots \\
&\quad + \frac{(-1)^k(k-1)}{2^k(n-k+2)(n-k+3)\dots(n+1)} \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} \lambda_{i_1}\dots\lambda_{i_k} + \dots \\
&\quad + \frac{(-1)^n(n-1)}{2^n(n+1)!} \prod_{i_1=1}^{n} \lambda_{i_1},
\end{aligned}$$

where $2 \le k \le n-1$ for the $\bar{s}^n, \hat{s}^n$ and $3 \le k \le n-1$ for $\tilde{s}^n$. In the special case where $n = 2$, we define the sinusoidal-type numbers of second order, associated to the λ_1, λ_2

$$\begin{aligned}
\bar{s}^2 &:= 8 + \frac{1}{6}\lambda_1\lambda_2, \\
\hat{s}^2 &:= 6 + \frac{1}{4}\lambda_1\lambda_2, \\
\tilde{s}^2 &:= \frac{1}{24}\lambda_1\lambda_2.
\end{aligned}$$

For $n = 1$, we define the sinusoidal-type numbers of first order

$$\bar{s}^1 := 6, \quad \hat{s}^1 := 4.$$

For the sinusoidal-type numbers we have the following result.

Lemma 9.2.1. *For $n \in \mathbb{N}$, $n \ge 2$, we have that*

$$\begin{aligned}
\bar{s}^n = \hat{s}^n &\iff \tilde{s}^n = 1, \\
\bar{s}^n > \hat{s}^n &\iff \tilde{s}^n < 1, \\
\bar{s}^n < \hat{s}^n &\iff \tilde{s}^n > 1,
\end{aligned}$$

where $\bar{s}^n$, $\hat{s}^n$ and $\tilde{s}^n$ are the sinusoidal-type numbers of order n, with $n \in \mathbb{N}$, $n \ge 2$, of the Definition 9.2.1.

Proof. Clearly, for $n \in \mathbb{N}$, $n \geq 3$,

$$\begin{aligned}
&2(n+2) + \frac{1}{2(n+1)} \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \frac{1}{4n(n+1)} \sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{n} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} + \dots \\
&+ \frac{(-1)^k}{2^{k-1}(n-k+3)\dots(n+1)} \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} \lambda_{i_1}\dots\lambda_{i_k} + \dots + \frac{(-1)^n}{2^{n-2}(n+1)!} \prod_{i_1=1}^{n} \lambda_{i_1} \\
&= 2(n+1) + \frac{1}{2n} \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \frac{1}{4(n-1)n} \sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{n} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} + \dots \\
&\quad + \frac{(-1)^k}{2^{k-1}(n-k+2)(n-k+3)\dots n} \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} \lambda_{i_1}\dots\lambda_{i_k} + \dots + \frac{(-1)^n}{2^{n-1}n!} \prod_{i_1=1}^{n} \lambda_{i_1}
\end{aligned}$$

if and only if

$$\begin{aligned}
&\frac{1}{4n(n+1)} \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \frac{2}{8(n-1)n(n+1)} \sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{n} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} + \dots \\
&\quad + \frac{(-1)^k(k-1)}{2^k(n-k+2)(n-k+3)\dots(n+1)} \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} \lambda_{i_1}\dots\lambda_{i_k} + \dots \\
&\quad + \frac{(-1)^n(n-1)}{2^n(n+1)!} \prod_{i_1=1}^{n} \lambda_{i_1} = 1,
\end{aligned}$$

and the first equivalence has been proved. Similarly, one can prove and the rest two equivalences. For $n = 2$, (i.e. for the sinusoidal-type numbers of second order), it is easy to see that the above equivalences hold. The proof of the lemma is complete. □

We continue with another definition. The role played by this definition is that the square roots of the coordinates $\lambda_1, \lambda_2, \dots, \lambda_n$ of the points $(\lambda_1, \lambda_2, \dots, \lambda_n)$, where $n \in \mathbb{N}$, $n \geq 2$, are going to be the limiting radii of the limit cycles which are asymptotic to circles of radii $\sqrt{\lambda_i}$ for $i = 1, 2, \dots, n$ centered at the origin when the small positive parameter of our system tending to 0. This will be done by forcing $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}$ to be simple roots of the polynomial F defined in (7.5) (see Theorem 7.1.1). For this reason, we assume that the given $\lambda_1, \lambda_2, \dots, \lambda_n$ are all positive and with $\lambda_i \neq \lambda_j$ for all $i \neq j$ where $i, j = 1, 2, \dots, n$. In this way, we are going to construct n limit cycles for system (6.2). But Iliev in [11] proved that the maximal number of limit cycles due to polynomial perturbations of degree n of the harmonic oscillator is equal to $[\frac{n-1}{2}]$ (the largest integer less than or equal to $\frac{n-1}{2}$). Since in our case the polynomial perturbations are of degree $2n+3$ we can achieve $n+1$ limit cycles. Now, we see that we can have an additional limit cycle. About the position of this limit cycle we later give the definition of the dependent radius.

The same observation of all the above is valid and in the case where $n = 1$.

Definition 9.2.2 (sinusoidal-type sets)**.** For $n \in \mathbb{N}$, $n \geq 2$, we define the sinusoidal-type sets of order n as

$$S_1^n := \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0$$
$$\forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i < \bar{s}^n \text{ when } \tilde{s}^n > 1\Big\},$$
$$S_2^n := \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0$$
$$\forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i < \hat{s}^n \text{ when } \tilde{s}^n < 1\Big\},$$
$$S_3^n := \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0$$
$$\forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i < \bar{s}^n = \hat{s}^n \text{ when } \tilde{s}^n = 1\Big\},$$
$$S_4^n := \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0$$
$$\forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i > \bar{s}^n \text{ when } \tilde{s}^n < 1\Big\},$$
$$S_5^n := \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0$$
$$\forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i > \hat{s}^n \text{ when } \tilde{s}^n > 1\Big\},$$
$$S_6^n := \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0$$
$$\forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i > \bar{s}^n = \hat{s}^n \text{ when } \tilde{s}^n = 1\Big\},$$

where $\bar{s}^n$, $\hat{s}^n$ and $\tilde{s}^n$ are the sinusoidal-type numbers of order n, with $n \in \mathbb{N}$, $n \geq 2$, of the Definition 9.2.1. For $n = 1$, we define the sinusoidal-type sets of first order

$$S_1^1 := \big\{\lambda_1 : \lambda_1 \in (0, 4)\big\},$$
$$S_2^1 := \big\{\lambda_1 : \lambda_1 \in (6, +\infty)\big\}.$$

Definition 9.2.3. We define for $n \in \mathbb{N}$, $n \geq 2$, the set V^n as the set

$$V^n := \cup_{i=1}^{6} S_i^n.$$

For $n = 1$, we define the set V^1 as the set

$$V^1 := S_1^1 \cup S_2^1.$$

Now, we continue with the last statement of the observation that we made before the Definition 9.2.2. The positions of the n limit cycles have to satisfy an algebraic relation in order that there is an odd polynomial f, realizing the maximal number of limit cycles, and the position of the $(n+1)$-th limit cycle is estimated in terms of the positions of these n limit cycles. On this we have the following definition.

Definition 9.2.4 (dependent radius)**.** Let $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$. We call dependent radius representing with λ_{n+1} the quantity (when is defined) given by the formula

$$\lambda_{n+1} = \lambda_{n+1}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) := \frac{\Xi}{\Psi}, \tag{9.4}$$

where

$$\Xi = 2^{n+1}(n+2)! + \cdots + (-1)^k 2^{n-k+1}(n-k+2)! \sum_{\substack{i_1,\dots,i_k=1 \\ i_1<\cdots<i_k}}^{n} \lambda_{i_1} \dots \lambda_{i_k}$$
$$+ \cdots + 4(-1)^n \prod_{i_1=1}^{n} \lambda_{i_1}$$

and

$$\Psi = 2^n(n+1)! + \cdots + (-1)^k 2^{n-k}(n-k+1)! \sum_{\substack{i_1,\dots,i_k=1 \\ i_1<\cdots<i_k}}^{n} \lambda_{i_1} \dots \lambda_{i_k}$$
$$+ \cdots + (-1)^n \prod_{i_1=1}^{n} \lambda_{i_1},$$

where $1 \leq k \leq n-1$. So, the dependent radius is the $(n+1)$-th radius associated to the radii $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$.

For $n = 1$, let $\lambda_1 \in \mathbb{R}$, then we call dependent radius representing with λ_2 the quantity (when is defined) given by the formula

$$\lambda_2 = \lambda_2(\lambda_1) := \frac{24 - 4\lambda_1}{4 - \lambda_1}. \tag{9.5}$$

So, in this case the dependent radius is the second radius associated to the radius λ_1.

For the dependent radius we have the following result.

Proposition 9.2.1. *If $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) \in V^n$, $n \in \mathbb{N}$, $n \geq 2$, then the dependent radius $\lambda_{n+1} = \lambda_{n+1}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n)$, $n \in \mathbb{N}$, $n \geq 2$, is positive. If $n = 1$ and suppose that $\lambda_1 \in V^1$, then the dependent radius $\lambda_2 = \lambda_2(\lambda_1)$ is positive.*

On the other hand if $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$, $n \in \mathbb{N}$, $n \geq 2$, are distinct positive real numbers so that the dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated with the radii $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$ is positive, then $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) \in V^n$, $n \in \mathbb{N}$, $n \geq 2$. If λ_1 is a positive real number so that the dependent radius λ_2 associated to the radius λ_1 is positive, then $\lambda_1 \in V^1$.

Proof. To prove that the dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated to the radii $\lambda_1, \lambda_2, \dots, \lambda_n$ is positive when $(\lambda_1, \lambda_2, \dots, \lambda_n) \in V^n$, $n \in \mathbb{N}$, $n \geq 2$, it suffices to show that both numerator and denominator of (9.4) are of the same sign.

We will check the case where $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n) \in S_1^n$, $n \in \mathbb{N}$, $n \geq 2$. Similarly, one can prove and the other cases.

In the set S_1^n, $n \in \mathbb{N}$, $n \geq 2$, we have $\sum_{i_1=1}^{n} \lambda_{i_1} < \bar{s}^n$. Now, for $n \in \mathbb{N}$, $n \geq 3$, using the definition of $\bar{s}^n$ and multiplying the last inequality by $-2^n(n+1)!$ we have

$$2^{n+1}(n+2)! - 2^n(n+1)! \sum_{i_1=1}^{n} \lambda_{i_1} + 2^{n-1}n! \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \cdots + 4(-1)^n \prod_{i_1=1}^{n} \lambda_{i_1} > 0,$$

which shows that the numerator of (9.4) is positive.

Since $\sum_{i_1=1}^{n} \lambda_{i_1} < \bar{s}^n$ in the set S_1^n, we have that $-\sum_{i_1=1}^{n} \lambda_{i_1} > -\bar{s}^n$. Using this observation we obtain the first inequality for the denominator of (9.4)

$$\begin{aligned}
&2^n(n+1)! - 2^{n-1}n! \sum_{i_1=1}^{n} \lambda_{i_1} + 2^{n-2}(n-1)! \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \cdots + (-1)^n \prod_{i_1=1}^{n} \lambda_{i_1} \\
&> 2^n(n+1)! - 2^{n-1}n!\bar{s}^n + 2^{n-2}(n-1)! \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \cdots + (-1)^n \prod_{i_1=1}^{n} \lambda_{i_1} \\
&= 2^n(n+1)! - 2^n n!(n+2) - 2^{n-2}\frac{n!}{n+1} \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} + \cdots - \frac{2(-1)^n}{n+1} \prod_{i_1=1}^{n} \lambda_{i_1} \\
&\quad + 2^{n-2}(n-1)! \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \cdots + (-1)^n \prod_{i_1=1}^{n} \lambda_{i_1} \\
&= -2^n n! + 2^n n!\tilde{s}^n > -2^n n! + 2^n n! = 0.
\end{aligned}$$

Here, we have used in the first equality the definition of $\bar{s}^n$ for $n \in \mathbb{N}$, $n \geq 3$ and in the last inequality that $\tilde{s}^n > 1$ in the set S_1^n.

So, we proved that both numerator and denominator of (9.4) when $n \in \mathbb{N}$, $n \geq 3$, are positive, which show that the dependent radius λ_{n+1} is positive if $(\lambda_1, \ldots, \lambda_n) \in S_1^n$, $n \in \mathbb{N}$, $n \geq 3$.

In the case where $n = 2$, it is easy to show that the dependent radius λ_3 associated to the radii λ_1, λ_2 is positive if $(\lambda_1, \lambda_2) \in S_1^2$, since in that case both numerator and denominator of (9.4) with $n = 2$, are positive.

If $n = 1$, it is easy to show that the dependent radius λ_2 associated to the radius λ_1 is positive if $\lambda_1 \in V^1$, since in that case both numerator and denominator of (9.5) are of the same sign.

Let us now show the inverse. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$, $n \in \mathbb{N}$, $n \geq 2$, be distinct positive real numbers so that the dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated to the radii $\lambda_1, \lambda_2, \ldots, \lambda_n$ is positive.

First, we examine the case where both numerator and denominator of λ_{n+1} are positive.

Since we suppose that the numerator of λ_{n+1} is positive, we have that

$$2^{n+1}(n+2)! - 2^n(n+1)!\sum_{i_1=1}^{n}\lambda_{i_1} + 2^{n-1}n!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2} - \cdots + 4(-1)^n\prod_{i_1=1}^{n}\lambda_{i_1} > 0,$$

and dividing this inequality by $-2^n(n+1)!$ we have that $\sum_{i_1=1}^{n}\lambda_{i_1} < \bar{s}^n$.

Since we suppose that the denominator of λ_{n+1} is positive, we have that

$$2^n(n+1)! - 2^{n-1}n!\sum_{i_1=1}^{n}\lambda_{i_1} + 2^{n-2}(n-1)!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2} - \cdots + (-1)^n\prod_{i_1=1}^{n}\lambda_{i_1} > 0,$$

and dividing this inequality by $-2^{n-1}n!$ we have that $\sum_{i_1=1}^{n}\lambda_{i_1} < \hat{s}^n$.

Now, we have the following possibilities: $\bar{s}^n = \hat{s}^n$ or $\bar{s}^n > \hat{s}^n$ or $\bar{s}^n < \hat{s}^n$, where $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n = \hat{s}^n$ we know from Lemma 9.2.1 that $\tilde{s}^n = 1$ and hence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_3^n$, $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n > \hat{s}^n$ we know from Lemma 9.2.1 that $\tilde{s}^n < 1$ and hence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_2^n$, $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n < \hat{s}^n$ we know from Lemma 9.2.1 that $\tilde{s}^n > 1$ and hence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_1^n$, $n \in \mathbb{N}$, $n \geq 2$.

Let us now examine the case where both numerator and denominator of λ_{n+1} are negative.

Since we suppose that the numerator of λ_{n+1} is negative, we have that

$$2^{n+1}(n+2)! - 2^n(n+1)!\sum_{i_1=1}^{n}\lambda_{i_1} + 2^{n-1}n!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2} - \cdots + 4(-1)^n\prod_{i_1=1}^{n}\lambda_{i_1} < 0,$$

and dividing this inequality by $-2^n(n+1)!$ we have that $\sum_{i_1=1}^{n}\lambda_{i_1} > \bar{s}^n$.

Since we suppose that the denominator of λ_{n+1} is negative, we have that

$$2^n(n+1)! - 2^{n-1}n!\sum_{i_1=1}^{n}\lambda_{i_1} + 2^{n-2}(n-1)!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2} - \cdots + (-1)^n\prod_{i_1=1}^{n}\lambda_{i_1} < 0,$$

and dividing this inequality by $-2^{n-1}n!$ we have that $\sum_{i_1=1}^{n}\lambda_{i_1} > \hat{s}^n$.

Now, we have the following possibilities: $\bar{s}^n = \hat{s}^n$ or $\bar{s}^n > \hat{s}^n$ or $\bar{s}^n < \hat{s}^n$, where $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n = \hat{s}^n$ we know from Lemma 9.2.1 that $\tilde{s}^n = 1$ and hence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_6^n$, $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n > \hat{s}^n$ we know from Lemma 9.2.1 that $\tilde{s}^n < 1$ and hence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_4^n$, $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n < \hat{s}^n$ we know from Lemma 9.2.1 that $\tilde{s}^n > 1$ and hence $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in S_5^n$, $n \in \mathbb{N}$, $n \geq 2$.

We have thus proved the inverse of Proposition 9.2.1 for $n \in \mathbb{N}$, $n \geq 2$. In fact we proved a stronger result. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$, $n \in \mathbb{N}$, $n \geq 2$, be distinct positive real numbers. Supposing that both numerator and denominator of the positive dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated to the radii $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive, then $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \cup_{i=1}^3 S_i^n$, $n \in \mathbb{N}$, $n \geq 2$. If we suppose that both numerator and denominator of the positive dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated to the radii $\lambda_1, \lambda_2, \ldots, \lambda_n$ are negative, then $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \cup_{i=4}^6 S_i^n$, $n \in \mathbb{N}$, $n \geq 2$.

If $n = 1$, let λ_1 be a positive real number so that the dependent radius λ_2 associated to the radius λ_1 is positive.

First, we examine the case where both numerator and denominator of λ_2 are positive. In that case we have for the numerator that $24 - 4\lambda_1 > 0$ which implies that $\lambda_1 < 6$ and for the denominator that $4 - \lambda_1 > 0$ which implies that $\lambda_1 < 4$. Combining the last two results about λ_1, we have that $0 < \lambda_1 < 4$ and hence $\lambda_1 \in S_1^1$.

Let now examine the case where both numerator and denominator of λ_2 are negative. In that case we have for the numerator that $24 - 4\lambda_1 < 0$ which implies that $\lambda_1 > 6$ and for the denominator that $4 - \lambda_1 < 0$ which implies that $\lambda_1 > 4$. Combining the last two results about λ_1, we have that $6 < \lambda_1 < +\infty$ and hence $\lambda_1 \in S_2^1$.

So, we proved that, if λ_1 is a positive real number so that the dependent radius λ_2 associated to the radius λ_1 is positive, then $\lambda_1 \in V^1$. The proof of the proposition is complete. □

Remark 9.2.2. According to Proposition 9.2.1, the set V^n is the biggest set from which we can choose the points $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ so that the corresponding dependent radius λ_{n+1} given by (9.4), is positive if $n \in \mathbb{N}$, $n \geq 2$ and the set V^1 is the biggest set from which we can choose the numbers λ_1 so that the corresponding dependent radius λ_2 given by (9.5), is positive.

Now, is following the definition which has the central role. The advantage played by this definition is that the square roots of the coordinates $\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}$ (where λ_{n+1} is the dependent radius associated to the radii $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$) of the points $(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1})$, where $n \in \mathbb{N}$, $n \geq 2$, are going to be the limiting radii of the limit cycles which are asymptotic to circles of radii $\sqrt{\lambda_i}$ for $i = 1, 2, \ldots, n, n+1$ centered at the origin when the small positive parameter of our system tending to 0. This will be done by forcing $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ to be all the simple roots of the polynomial F defined in (7.5) (see Theorem 7.1.1). For this reason, we assume that the given points $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ belong to V^n and we want for the corresponding dependent radius $\lambda_{n+1} = \lambda_{n+1}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ (which from Proposition 9.2.1 is positive) to satisfy that $\lambda_{n+1} \neq \lambda_j$ for all $j = 1, 2, \ldots, n$. In this way, we construct $n + 1$ limit cycles for system (6.2), and so we achieve the maximal number of limit cycles due to polynomial perturbations of degree $2n + 3$ of the harmonic oscillator (see [11]).

The same observation of all the above is valid and in the case where $n = 1$.

Definition 9.2.5. We define now for $n \in \mathbb{N}$, $n \geq 2$, the sets

$$
\begin{aligned}
S^n_{1,n+1} &:= \big\{(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_1,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n), \\
&\qquad\quad \lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\}, \\
S^n_{2,n+1} &:= \big\{(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_2,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n), \\
&\qquad\quad \lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\}, \\
S^n_{3,n+1} &:= \big\{(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_3,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n), \\
&\qquad\quad \lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\}, \\
S^n_{4,n+1} &:= \big\{(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_4,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n), \\
&\qquad\quad \lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\}, \\
S^n_{5,n+1} &:= \big\{(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_5,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n), \\
&\qquad\quad \lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\}, \\
S^n_{6,n+1} &:= \big\{(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_6,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n), \\
&\qquad\quad \lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\},
\end{aligned}
$$

where $S^n_1, S^n_2, S^n_3, S^n_4, S^n_5$ and S^n_6 are the sinusoidal-type sets of order n, with $n \in \mathbb{N}$, $n \geq 2$, of the Definition 9.2.2 and λ_{n+1} is the dependent radius given by (9.4). For $n = 1$, we define the sets

$$
\begin{aligned}
S^1_{1,2} &:= \big\{(\lambda_1, \lambda_2) : \lambda_1 \in S^1_1,\ \lambda_2 = \lambda_2(\lambda_1),\ \lambda_2 \neq \lambda_1\big\}, \\
S^1_{2,2} &:= \big\{(\lambda_1, \lambda_2) : \lambda_1 \in S^1_2,\ \lambda_2 = \lambda_2(\lambda_1),\ \lambda_2 \neq \lambda_1\big\},
\end{aligned}
$$

where S^1_1 and S^1_2 are the sinusoidal-type sets of first order of the Definition 9.2.2 and λ_2 is the dependent radius given by (9.5).

Remark 9.2.3. Notice that, if $\lambda_1 \in (0, 4)$, then the dependent radius λ_2 given by (9.5), belongs to $(6, +\infty)$ and so we have that $(\lambda_1, \lambda_2) \in S^1_{1,2}$. If $\lambda_1 \in (6, +\infty)$, then the dependent radius λ_2 given by (9.5), belongs to $(0, 4)$ and so we have that $(\lambda_1, \lambda_2) \in S^1_{2,2}$.

Remark 9.2.4. According to Remark 9.2.3, the sets $S^1_{1,2}$ and $S^1_{2,2}$, which defined as above, take the more simple form

$$
\begin{aligned}
S^1_{1,2} &= \big\{(\lambda_1, \lambda_2) : \lambda_1 \in S^1_1,\ \lambda_2 = \lambda_2(\lambda_1)\big\}, \\
S^1_{2,2} &= \big\{(\lambda_1, \lambda_2) : \lambda_1 \in S^1_2,\ \lambda_2 = \lambda_2(\lambda_1)\big\},
\end{aligned}
$$

where S^1_1 and S^1_2 are the sinusoidal-type sets of first order of the Definition 9.2.2 and λ_2 is the dependent radius given by (9.5).

Definition 9.2.6. We define for $n \in \mathbb{N}$, $n \geq 2$, the set V_{n+1}^n as the set

$$V_{n+1}^n := \cup_{i=1}^6 S_{i,n+1}^n.$$

For $n = 1$, we define the set V_2^1 as the set

$$V_2^1 := S_{1,2}^1 \cup S_{2,2}^1.$$

Remark 9.2.5. In the set V_{n+1}^n, $n \in \mathbb{N}$, the positive dependent radius λ_{n+1}, $n \in \mathbb{N}$, is obviously different from $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$, $n \in \mathbb{N}$.

Remark 9.2.6. It is possible, for $n \in \mathbb{N}$, $n \geq 2$, the point $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in V^n$ but the point $(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}) \notin V_{n+1}^n$, where λ_{n+1} is the dependent radius given by (9.4), associated to the radii $\lambda_1, \lambda_2, \ldots, \lambda_n$. (For example, it is easy to see that the point $(4, 6) \in V^2$; in particular belongs to S_3^2. We calculate the dependent radius λ_3, associated to the $4, 6$, which from Proposition 9.2.1 is positive and we have that $\lambda_3 = 6$. Now, the point $(4, 6, 6) \notin V_3^2$.)

For $n = 1$, according to Remark 9.2.3, if $\lambda_1 \in V^1$, then $(\lambda_1, \lambda_2) \in V_2^1$, where λ_2 is the dependent radius given by (9.5), associated to the radius λ_1.

Definition 9.2.7 (Λ-points (lambda points))**.** We will call the points

$$(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}) \in V_{n+1}^n, \quad n \in \mathbb{N},$$

the Λ-points.

9.3 Proofs of existential theorems for limit cycles

Proof of Theorem 9.1.1. From (7.2) we have

$$\dot{E}(x, y) = y f(y)\big(1 - x^2\big), \tag{9.6}$$

where f is the polynomial introduced in (9.1). Substituting (9.6) into (7.5), we obtain that

$$F(A) = \int_0^{2\pi} y^0(t) f(y^0(t))\big(1 - (x^0(t))^2\big)dt, \tag{9.7}$$

where f is the polynomial introduced in (9.1). We insert the definition of f given by (9.1) in (9.7) to obtain

$$\begin{aligned} F(A) = \tau \int_0^{2\pi} \Big[(y^0(t))^{2(n+1)} + (2n+1)\Big(1 - \frac{1}{2(n+2)}\sum_{i_1=1}^{n+1} \lambda_{i_1}\Big)(y^0(t))^{2n} \\ + \cdots + \Big(\frac{1 \cdot 3 \ldots (2n+1)}{2^{n+1}(n+2)!}(-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1}\Big)(y^0(t))^2\Big]\big(1 - (x^0(t))^2\big)dt. \end{aligned} \tag{9.8}$$

Substituting $x^0(t) = A\cos(t - t_0)$ and $y^0(t) = -A\sin(t - t_0)$ into (9.8) we get

$$\begin{aligned} F(A) = \tau A^2 \int_0^{2\pi} &\Big[A^{2n} \sin^{2(n+1)}(t - t_0) \\ &+ (2n+1)\Big(1 - \frac{1}{2(n+2)} \sum_{i_1=1}^{n+1} \lambda_{i_1}\Big) A^{2(n-1)} \sin^{2n}(t - t_0) \\ &+ \dots + \Big(\frac{1 \cdot 3 \dots (2n+1)}{2^{n+1}(n+2)!}(-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1}\Big) \sin^2(t - t_0)\Big] \\ &\times \big[1 - A^2 + A^2 \sin^2(t - t_0)\big] dt, \end{aligned}$$

whence, after multiplying the terms in the two brackets we get

$$\begin{aligned} F(A) = \tau A^2 \int_0^{2\pi} &\Big[A^{2n} \sin^{2(n+1)}(t - t_0) - A^{2(n+1)} \sin^{2(n+1)}(t - t_0) \\ &+ A^{2(n+1)} \sin^{2(n+2)}(t - t_0) \\ &+ (2n+1)\Big(1 - \frac{1}{2(n+2)} \sum_{i_1=1}^{n+1} \lambda_{i_1}\Big) A^{2(n-1)} \sin^{2n}(t - t_0) \\ &- (2n+1)\Big(1 - \frac{1}{2(n+2)} \sum_{i_1=1}^{n+1} \lambda_{i_1}\Big) A^{2n} \sin^{2n}(t - t_0) \\ &+ (2n+1)\Big(1 - \frac{1}{2(n+2)} \sum_{i_1=1}^{n+1} \lambda_{i_1}\Big) A^{2n} \sin^{2(n+1)}(t - t_0) \\ &+ \dots + \Big(\frac{1 \cdot 3 \dots (2n+1)}{2^{n+1}(n+2)!}(-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1}\Big) \sin^2(t - t_0) \\ &- \Big(\frac{1 \cdot 3 \dots (2n+1)}{2^{n+1}(n+2)!}(-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1}\Big) A^2 \sin^2(t - t_0) \\ &+ \Big(\frac{1 \cdot 3 \dots (2n+1)}{2^{n+1}(n+2)!}(-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1}\Big) A^2 \sin^4(t - t_0)\Big] dt. \end{aligned}$$

Using now (9.3), we finally obtain

$$\begin{aligned} F(A) = \pi\tau A^2 \frac{1 \cdot 3 \dots (2n+1)}{2^{n+1}(n+2)!}\Big[&- A^{2(n+1)} + \sum_{i_1=1}^{n+1} \lambda_{i_1} A^{2n} - \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} A^{2(n-1)} \\ &+ \dots - (-1)^k \sum_{\substack{i_1,\dots,i_k=1 \\ i_1<\dots<i_k}}^{n+1} \lambda_{i_1} \dots \lambda_{i_k} A^{2(n-k+1)} - \dots \\ &- (-1)^n \sum_{\substack{i_1,\dots,i_n=1 \\ i_1<\dots<i_n}}^{n+1} \lambda_{i_1} \dots \lambda_{i_n} A^2 + (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1}\Big]. \end{aligned}$$

We show now that $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ are roots of the polynomial F. Let

$$W(A) := A^{2(n+1)} - \sum_{i_1=1}^{n+1} \lambda_{i_1} A^{2n} + \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} A^{2(n-1)} - \dots$$
$$+ (-1)^n \sum_{\substack{i_1,\dots,i_n=1 \\ i_1<\dots<i_n}}^{n+1} \lambda_{i_1} \dots \lambda_{i_n} A^2 - (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1},$$

namely, we write

$$F(A) = -\pi\tau A^2 \frac{1 \cdot 3 \dots (2n+1)}{2^{n+1}(n+2)!} \cdot W(A).$$

Now, it suffices to show that $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ are roots of the polynomial W. Without loss of generality we consider the quantity $\sqrt{\lambda_1}$. For $W\big(\sqrt{\lambda_1}\big)$ we have

$$\begin{aligned} W\big(\sqrt{\lambda_1}\big) &= \lambda_1^{n+1} - \lambda_1^n \sum_{i_1=1}^{n+1} \lambda_{i_1} + \lambda_1^{n-1} \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} - \dots - (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1} \\ &= \lambda_1^{n+1} - \lambda_1^{n+1} - \lambda_1^n \sum_{i_1=2}^{n+1} \lambda_{i_1} + \lambda_1^n \sum_{i_1=2}^{n+1} \lambda_{i_1} + \lambda_1^{n-1} \sum_{\substack{i_1,i_2=2 \\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} \\ &\quad - \lambda_1^{n-1} \sum_{\substack{i_1,i_2=2 \\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} - \lambda_1^{n-2} \sum_{\substack{i_1,i_2,i_3=2 \\ i_1<i_2<i_3}}^{n+1} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} \\ &\quad + \lambda_1^{n-2} \sum_{\substack{i_1,i_2,i_3=2 \\ i_1<i_2<i_3}}^{n+1} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} + \dots + (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1} - (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1} \\ &= 0. \end{aligned}$$

So, $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ are roots of the polynomial W and therefore and for the polynomial F.

Now, using Theorem 7.1.1, it suffices to show that $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ are not roots of the polynomial W'; therefore they are not roots and for polynomial F'. For the derivative of W we have that

$$\begin{aligned} W'(A) = 2A\Big[&(n+1)A^{2n} - n \sum_{i_1=1}^{n+1} \lambda_{i_1} A^{2(n-1)} + (n-1) \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} A^{2(n-2)} \\ &- \dots + (-1)^n \sum_{\substack{i_1,\dots,i_n=1 \\ i_1<\dots<i_n}}^{n+1} \lambda_{i_1} \dots \lambda_{i_n}\Big]. \end{aligned}$$

Now, we have that one root of W' is $A = 0$ and we also have another $2n$ roots. From those $2n$ roots, n are positive and the other n are negative (these roots are opposite numbers). Let

$$G(A) := (n+1)A^{2n} - n\sum_{i_1=1}^{n+1}\lambda_{i_1}A^{2(n-1)} + (n-1)\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1}\lambda_{i_1}\lambda_{i_2}A^{2(n-2)}$$
$$-\cdots + (-1)^n \sum_{\substack{i_1,\dots,i_n=1\\ i_1<\cdots<i_n}}^{n+1}\lambda_{i_1}\dots\lambda_{i_n},$$

namely, we write

$$W'(A) = 2A\cdot G(A).$$

Now, we check if the roots $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ of the polynomial W are possible to be roots and for the polynomial W', therefore and for the polynomial G. Without loss of generality we consider the root $\sqrt{\lambda_1}$ of W. For $G\big(\sqrt{\lambda_1}\big)$ we have

$$\begin{aligned}
G\big(\sqrt{\lambda_1}\big) &= (n+1)\lambda_1^n - n\lambda_1^{n-1}\sum_{i_1=1}^{n+1}\lambda_{i_1} + (n-1)\lambda_1^{n-2}\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1}\lambda_{i_1}\lambda_{i_2}\\
&\quad -\cdots + (-1)^n \sum_{\substack{i_1,\dots,i_n=1\\ i_1<\cdots<i_n}}^{n+1}\lambda_{i_1}\dots\lambda_{i_n}\\
&= \lambda_1^n - \lambda_1^{n-1}\sum_{i_1=2}^{n+1}\lambda_{i_1} + \lambda_1^{n-2}\sum_{\substack{i_1,i_2=2\\ i_1<i_2}}^{n+1}\lambda_{i_1}\lambda_{i_2} - \lambda_1^{n-3}\sum_{\substack{i_1,i_2,i_3=2\\ i_1<i_2<i_3}}^{n+1}\lambda_{i_1}\lambda_{i_2}\lambda_{i_3}\\
&\quad +\cdots - (-1)^n\lambda_1 \sum_{\substack{i_1,\dots,i_n=2\\ i_1<\cdots<i_n}}^{n+1}\lambda_{i_1}\dots\lambda_{i_n} + (-1)^n\lambda_2\lambda_3\dots\lambda_n\lambda_{n+1}\\
&= (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)(\lambda_1-\lambda_{n+1}).
\end{aligned}$$

Obviously, $W'\big(\sqrt{\lambda_1}\big)$ is not zero since in the set V_{n+1}^n we have that $\lambda_1 \neq \lambda_j$ for $j = 2, 3, \dots, n, n+1$. Similarly, none of the $\sqrt{\lambda_2}, \sqrt{\lambda_3}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ is a root of W'.

Therefore, we have that the roots $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ of W are not roots of W'. Finally, none of the roots $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ of F is a root of F'. That is essential so that the $n+1$ simple roots of F create $n+1$ limit cycles. Hence, from Poincaré's method (see Theorem 7.1.1) it follows that (6.2), with f be the polynomial introduced in (9.1), has at least $n+1$ limit cycles, and are asymptotic to circles of radius $\sqrt{\lambda_i}$ for $i = 1, 2, \dots, n+1$ centered at the origin as $\varepsilon \to 0$.

Let now prove that the number of limit cycles for system (6.2), with ε small and f be the polynomial introduced in (9.1), is exactly $n+1$. The proof of this can be derived from the work of Iliev [11] since it constitutes a special case of the Theorem 1 proved there. Actually, applying this theorem from [11] for the special case $k = 1$, since the degree of (6.2) is $2n+3$ we can obtain at most $n+1$ limit cycles. Finally, combining this result with the result that (6.2), with f be the polynomial introduced in (9.1), has at least $n+1$ limit cycles we get the desired result, namely that the number of limit cycles for system (6.2), with ε small and f be the polynomial introduced in (9.1) is exactly $n+1$.

Now, concerning the stability of the limit cycles we have the following.

From now on we will suppose for the coordinates $\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}$ of the points $(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) \in V^n_{n+1}$, $n \in \mathbb{N}$, $n \geq 2$, an ordering such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1}$. We can always achieve this since the positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}$ are distinct. Note that in this order with λ_{n+1} we do not necessary mean the dependent radius. Since

$$G\big(\sqrt{\lambda_1}\,\big) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\dots(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n+1}),$$

we have that $W'\big(\sqrt{\lambda_1}\big) < 0$ if n is odd and that $W'\big(\sqrt{\lambda_1}\big) > 0$ if n is even.

So using the fact that, if $F'\big(\sqrt{\lambda_i}\big) < 0$ the limit cycle $x^2 + y^2 = \lambda_i + O(\varepsilon)$ is stable and if $F'\big(\sqrt{\lambda_i}\big) > 0$ the limit cycle is unstable we have for the stability of the $n+1$ limit cycles that, if $\tau > 0$ (respectively $\tau < 0$)

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_3 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon)$$

are stable (respectively unstable) and

$$x^2 + y^2 = \lambda_2 + O(\varepsilon), \quad x^2 + y^2 = \lambda_4 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_n + O(\varepsilon)$$

are unstable (respectively stable) for n even; and

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_3 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_n + O(\varepsilon)$$

are unstable (respectively stable) and

$$x^2 + y^2 = \lambda_2 + O(\varepsilon), \quad x^2 + y^2 = \lambda_4 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon)$$

are stable (respectively unstable) for n odd. The proof is complete. □

Proof of Theorem 9.1.2. As we already saw, according to Theorem 1 from [11] the number of $n+1$ limit cycles is an upper bound for the number of limit cycles for system (6.2), where ε is small and f is an arbitrary odd polynomial of degree $2n+1$.

Now, it is easy to see that for system (6.2), where f is an arbitrary odd polynomial of degree $2n+1$, the associated F given by (7.5) is an even polynomial of degree $2n+4$, with 0 as a double root. Therefore, in general the polynomial F has at most $n+1$ simple positive roots. Furthermore, since V^n, $n \in \mathbb{N}$, $n \geq 2$ is the biggest set from which we can choose the points $(\lambda_1, \lambda_2, \dots, \lambda_n)$ so that the dependent radius λ_{n+1} given by (9.4), is positive if $n \in \mathbb{N}$, $n \geq 2$ (see Remark 9.2.2) and F as we showed has at most $n+1$ simple positive roots, we must choose the points $(\lambda_1, \lambda_2, \dots, \lambda_{n+1}) \in V^n_{n+1}$, in order the polynomial F has exactly $n+1$ simple positive roots, and thus, from the set of all the odd polynomials, the polynomials f given by (9.1) are the only such that the system (6.2) attains the upper bound of the $n+1$ limit cycles. The proof is complete. $\square$

Proof of Theorem 9.1.3. The proof is identical as in Theorem 9.1.1; the only modification is that the polynomial f given by (9.1) will be replaced by the polynomial f introduced in (9.2). $\square$

Proof of Theorem 9.1.4. The proof is identical as in Theorem 9.1.2; the only modification is that the case where $n \in \mathbb{N}$, $n \geq 2$ will be replaced by $n = 1$. $\square$

9.4 Examples

In this section we illustrate the general theory of this work by some examples.

Example 9.4.1. We consider $\lambda_1 = 4$, $\lambda_2 = 5$. These λ_1, λ_2 are distinct and positive. We have according to Definition 9.2.1 that the sinusoidal-type numbers of second order, associated to the $4, 5$ are

$$\begin{aligned} \bar{s}^2 &:= 8 + \frac{1}{6}\lambda_1\lambda_2 = 8 + \frac{1}{6}\cdot 4\cdot 5 = \frac{34}{3}, \\ \hat{s}^2 &:= 6 + \frac{1}{4}\lambda_1\lambda_2 = 6 + \frac{1}{4}\cdot 4\cdot 5 = 11, \\ \tilde{s}^2 &:= \frac{1}{24}\lambda_1\lambda_2 = \frac{1}{24}\cdot 4\cdot 5 = \frac{5}{6}. \end{aligned}$$

Since $\lambda_1 + \lambda_2 = 4 + 5 = 9$, we have that $(4, 5) \in S_2^2$ and therefore $(4, 5) \in V^2$. We calculate the dependent radius λ_3, associated to the $4, 5$, which from Proposition 9.2.1 is positive and we have that

$$\lambda_3 := \frac{192 - 24(\lambda_1 + \lambda_2) + 4\lambda_1\lambda_2}{24 - 4(\lambda_1 + \lambda_2) + \lambda_1\lambda_2} = \frac{192 - 216 + 80}{24 - 36 + 20} = 7.$$

So, we have the Λ-point $(4, 5, 7)$ which belongs to the set V_3^2.

Now, using Theorem 9.1.1, for $\tau = 16$, we have that the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\big(16y^5 - 80y^3 + 175y\big)\big(1 - x^2\big), \end{cases} \tag{9.9}$$

with $0 < \varepsilon \ll 1$ has exactly, the limit cycles $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 5 + O(\varepsilon)$ and $x^2 + y^2 = 7 + O(\varepsilon)$.

Since $4 < 5 < 7$, the limit cycles $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 7 + O(\varepsilon)$ are stable and the limit cycle $x^2 + y^2 = 5 + O(\varepsilon)$ is unstable.

From Theorem 9.1.1 we have for the system (9.9) that, if we change τ from 16 to -16 the unstable limit cycle $x^2 + y^2 = 5 + O(\varepsilon)$ becomes stable and the stable limit cycles $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 7 + O(\varepsilon)$ become unstable.

Example 9.4.2. We consider $\lambda_1 = 4, \lambda_2 = 16$. These λ_1, λ_2 are distinct and positive. We have according to Definition 9.2.1 that the sinusoidal-type numbers of second order, associated to the $4, 16$ are

$$\bar{s}^2 := 8 + \frac{1}{6}\lambda_1\lambda_2 = 8 + \frac{1}{6}\cdot 4\cdot 16 = \frac{56}{3},$$
$$\hat{s}^2 := 6 + \frac{1}{4}\lambda_1\lambda_2 = 6 + \frac{1}{4}\cdot 4\cdot 16 = 22,$$
$$\tilde{s}^2 := \frac{1}{24}\lambda_1\lambda_2 = \frac{1}{24}\cdot 4\cdot 16 = \frac{8}{3}.$$

Since $\lambda_1 + \lambda_2 = 4 + 16 = 20$, we have that $(4, 16) \notin S_1^2$, $(4, 16) \notin S_5^2$ and therefore $(4, 16) \notin V^2$. Therefore from Proposition 9.2.1 the dependent radius λ_3, associated to the $4, 16$ is not positive.

So, according to the Theorem 9.1.1 it does not exist a system of the form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\big(a_0 y^5 + a_1 y^3 + a_2 y\big)\big(1 - x^2\big), \end{cases}$$

where $0 < \varepsilon \ll 1$ and $a_0, a_1, a_2 \in \mathbb{R}$, which has exactly three limit cycles whereof the two of them have the equations $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 16 + O(\varepsilon)$.

Example 9.4.3. We consider $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. These $\lambda_1, \lambda_2, \lambda_3$ satisfy our assertions, since $\lambda_i \neq \lambda_j$ for all $i \neq j$ where $i, j = 1, 2, 3$ and are positive. We have according to Definition 9.2.1 that the sinusoidal-type numbers of third order, associated to the $1, 2, 3$ are

$$\bar{s}^3 := 10 + \frac{1}{8}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \frac{1}{48}\lambda_1\lambda_2\lambda_3 = 10 + \frac{11}{8} - \frac{1}{8} = \frac{45}{4},$$
$$\hat{s}^3 := 8 + \frac{1}{6}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \frac{1}{24}\lambda_1\lambda_2\lambda_3 = 8 + \frac{11}{6} - \frac{1}{4} = \frac{115}{12},$$
$$\tilde{s}^3 := \frac{1}{48}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \frac{1}{96}\lambda_1\lambda_2\lambda_3 = \frac{11}{48} - \frac{1}{16} = \frac{1}{6}.$$

Since $\lambda_1+\lambda_2+\lambda_3 = 1+2+3 = 6$, we have that $(1, 2, 3) \in S_2^3$ and therefore $(1, 2, 3) \in V^3$. We calculate the dependent radius λ_4, associated to the $1, 2, 3$, which from Proposition 9.2.1 is positive and we have that

$$\lambda_4 := \frac{1920 - 192(\lambda_1 + \lambda_2 + \lambda_3) + 24(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - 4\lambda_1\lambda_2\lambda_3}{192 - 24(\lambda_1 + \lambda_2 + \lambda_3) + 4(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3} = \frac{504}{43}.$$

So, we have the Λ-point $(1, 2, 3, 504/43)$ which belongs to the set V_4^3.

Now, using Theorem 9.1.1, for $\tau = 43/8$, we have that the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\Big(\frac{43}{8}y^7 - \frac{581}{20}y^5 + \frac{5887}{128}y^3 - \frac{1323}{64}y\Big)\big(1 - x^2\big), \end{cases} \tag{9.10}$$

with $0 < \varepsilon \ll 1$ has exactly, the limit cycles $x^2 + y^2 = 1 + O(\varepsilon)$, $x^2 + y^2 = 2 + O(\varepsilon)$, $x^2 + y^2 = 3 + O(\varepsilon)$ and $x^2 + y^2 = (504/43) + O(\varepsilon)$.

Since $1 < 2 < 3 < 504/43$, the limit cycles $x^2 + y^2 = 1 + O(\varepsilon)$, $x^2 + y^2 = 3 + O(\varepsilon)$ are unstable and the limit cycles $x^2 + y^2 = 2 + O(\varepsilon)$, $x^2 + y^2 = (504/43) + O(\varepsilon)$ are stable.

From Theorem 9.1.1 we have for the system (9.10) that, if we change τ from 43/8 to $-43/8$ the unstable limit cycles $x^2 + y^2 = 1 + O(\varepsilon)$, $x^2 + y^2 = 3 + O(\varepsilon)$ become stable and the stable limit cycles $x^2+y^2 = 2+O(\varepsilon)$, $x^2+y^2 = (504/43)+O(\varepsilon)$ become unstable.

Example 9.4.4. We consider $\lambda_i = i$ for $i = 1, 2, \ldots, 6$. These $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ satisfy our assertions, since $\lambda_i \neq \lambda_j$ for all $i \neq j$ where $i, j = 1, 2, 3, 4, 5, 6$ and are positive. It is easy to show, after some calculations, that $(1, 2, 3, 4, 5, 6) \in V^6$. We calculate the dependent radius λ_7, associated to the $1, 2, 3, 4, 5, 6$, which from Proposition 9.2.1 is positive and we have that $\lambda_7 = 13337/690$. So, we have the Λ-point $(1, 2, 3, 4, 5, 6, 13337/690)$ which belongs to the set V_7^6.

Therefore, from Theorem 9.1.1 exists a system of the form (6.2), where $0 < \varepsilon \ll 1$ and f is an odd polynomial of degree 13, which has exactly the limit cycles: $x^2 + y^2 = 1+O(\varepsilon)$, $x^2+y^2 = 2+O(\varepsilon)$, $x^2+y^2 = 3+O(\varepsilon)$, $x^2+y^2 = 4+O(\varepsilon)$, $x^2+y^2 = 5+O(\varepsilon)$, $x^2 + y^2 = 6 + O(\varepsilon)$, $x^2 + y^2 = (13337/690) + O(\varepsilon)$.

Example 9.4.5. We consider $\lambda_1 = 7, \lambda_2 = 701/100$. It is easy to show, after some calculations, that $(7, 701/100) \in V^2$. We calculate the dependent radius λ_3, associated to the $7, 701/100$, which from Proposition 9.2.1 is positive and we have that $\lambda_3 = 5204/1703$. So, we have the Λ-point $(7, 701/100, 5204/1703)$ which belongs to the set V_3^2.

Now, using Theorem 9.1.1, for $\tau = -2179840$, we have that the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\big(- 2179840y^5 + 12351224y^3 - 25536028y\big)\big(1 - x^2\big), \end{cases} \tag{9.11}$$

with $0 < \varepsilon \ll 1$ has exactly, the limit cycles $x^2+y^2 = 7+O(\varepsilon)$, $x^2+y^2 = (701/100) + O(\varepsilon)$ and $x^2 + y^2 = (5204/1703) + O(\varepsilon)$.

Since $5204/1703 < 7 < 701/100$, the limit cycles $x^2 + y^2 = (5204/1703) + O(\varepsilon)$, $x^2+y^2 = (701/100)+O(\varepsilon)$ are unstable and the limit cycle $x^2+y^2 = 7+O(\varepsilon)$ is stable.

Since $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ have very small difference ($|\sqrt{\lambda_1}-\sqrt{\lambda_2}| = |\frac{10\sqrt{7}-\sqrt{701}}{10}| \simeq 0.0019$), the qualitative and quantitative image that one gets using a program, may give the misimpression that system (9.11) has a semistable limit cycle. This happens because the stable limit cycle $x^2 + y^2 = 7 + O(\varepsilon)$ lies close enough to the unstable limit cycle $x^2+y^2 = (701/100)+O(\varepsilon)$. This of course is prospective since a priori we have chosen the λ_1 and λ_2 so as to be close enough the one to the other. So, the two limit cycles $x^2 + y^2 = 7 + O(\varepsilon)$ and $x^2 + y^2 = (701/100) + O(\varepsilon)$, create "one system with one pseudosemistable limit cycle" as we can say, since the two limit cycles together behave like a semistable limit cycle.

Remark 9.4.1. It is easy to see, according to Remark 9.2.3, that system (6.2) with $n = 1$ can't have "a system with a pseudosemistable limit cycle" as we mean above "the system with a pseudosemistable limit cycle".

Example 9.4.6. We consider $\lambda_1 = 7$. This λ_1 belongs to S_2^1. We know from Remark 9.2.3 that the point $(7, \lambda_2) \in S_{2,2}^1$, where λ_2 is the dependent radius associated to the 7.

We calculate the dependent radius λ_2, associated to the 7, (which from Proposition 9.2.1 is positive) and we have that

$$\lambda_2 := \frac{24 - 4\lambda_1}{4 - \lambda_1} = \frac{24 - 28}{4 - 7} = \frac{4}{3}.$$

So, we have the Λ-point $(7, 4/3)$ which belongs to the set V_2^1.

Now, using Theorem 9.1.3, for $\tau = 6$, we have that the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\big(6y^3 - 7y\big)\big(1 - x^2\big), \end{cases} \tag{9.12}$$

with $0 < \varepsilon \ll 1$ has exactly, the limit cycles $x^2+y^2 = 7+O(\varepsilon)$ and $x^2+y^2 = (4/3)+O(\varepsilon)$.

Since $4/3 < 7$, the limit cycle $x^2 + y^2 = (4/3) + O(\varepsilon)$ is unstable and the limit cycle $x^2 + y^2 = 7 + O(\varepsilon)$ is stable.

From Theorem 9.1.3 we have for the system (9.12) that, if we change τ from 6 to -6 the unstable limit cycle $x^2 + y^2 = (4/3) + O(\varepsilon)$ becomes stable and the stable limit cycle $x^2 + y^2 = 7 + O(\varepsilon)$ becomes unstable.

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