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DUALITY FOR HARDY SPACES ON TUBE DOMAINS IN $\mathbb{C}^2$ AND APPLICATIONS

DOCTOR OF PHILOSOPHY DISSERTATION

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Abstract

Let $T_{B_1} = \mathbb{R}^2 \times i\{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1\}$, $T_{B_2} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \times i\mathbb{R}^2$ be tubes in $\mathbb{C}^2$ and $H^2(T_{B_j})$, $j = 1, 2$, be the spaces of holomorphic functions $f(z) = \int\limits_{\mathbb{R}^2} f(t)e^{2\pi iz\cdot t}dt$, $z \in T_{B_1}$ and $g(z) = \int\limits_{\mathbb{R}^2} f(t)e^{2\pi z\cdot t}dt$, $z \in T_{B_2}$. The main result of the present thesis is a separation of singularity type theorem allowing to express a function $f \in H^2(T_{B_1})$ as a difference of two holomorphic functions $f_1 \in H^2(T_{(S_H^-)^{int}})$ and $f_2 \in H^2(T_{(S_H^+)^{int}})$, defined on suitable tubes $T_{(S_H^-)^{int}}$ and $T_{(S_H^+)^{int}}$, whose base contains a cone, and satisfying $T_{B_1} = T_{(S_H^-)^{int}} \cap T_{(S_H^+)^{int}}$. It is proven that every function $f_1 \in H^2(T_{(S_H^-)^{int}})$ or $f_2 \in (T_{(S_H^+)^{int}})$ is representable by Cauchy-Fantappie formula (and conversely). As a direct consequence of separation of singularities theorem it is shown also that every function $f \in H^2(T_{B_1})$ is represented by Cauchy-Fantappie formula supported on the boundary ∂T_{B_1}. Actually, if $\Phi_1(\zeta, \bar{\zeta}) = \left(\frac{\zeta_1 - \bar{\zeta}_1}{2i}\right)^2 + \left(\frac{\zeta_2 - \bar{\zeta}_2}{2i}\right)^2 - 1$ is the defining function of the tube T_{B_1} then for every function $f \in H^2(T_{B_1})$

$$f(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times iS^1} \frac{f(\zeta)\left(\partial\Phi_1(\zeta, \bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1(\zeta, \bar{\zeta})\right)}{\left(< \nabla_\zeta \Phi_1(\zeta, \bar{\zeta}), \zeta - z >\right)^2}$$

Similar results are valid for the tube $H^2(T_{B_2})$.

Περίληψη

Έστω $T_{B_1} = \mathbb{R}^2 \times i\{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1\}$, $T_{B_2} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \times i\mathbb{R}^2$ κυλινδρικά χωρία στο $\mathbb{C}^2$ και $H^2(T_{B_j})$, $j = 1, 2$, είναι οι χώροι ολόμορφων συναρτήσεων $f(z) = \int\limits_{\mathbb{R}^2} f(t)e^{2\pi i z\cdot t}dt$, $z \in T_{B_1}$ και $g(z) = \int\limits_{\mathbb{R}^2} f(t)e^{2\pi z\cdot t}dt$, $z \in T_{B_2}$. Το κύριο αποτέλεσμα της παρούσας εργασίας είναι ένα θεώρημα τύπου διαχωρισμού των ιδιαζόντων σημείων το οποίο επιτρέπει την έκφραση συνάρτησης $f \in H^2(T_{B_1})$ ώς διαφορά δύο ολόμορφων συναρτήσεων $f_1 \in H^2(T_{(S_H^-)^{int}})$ και $f_2 \in H^2(T_{(S_H^+)^{int}})$, ορισμένων σε κατάλληλα επιλεγμένα, μή φραγμένα κυλινδρικά χωρία $T_{(S_H^-)^{int}}$ και $T_{(S_H^+)^{int}}$, των οποίων η βάση περιέχει κώνο, τέτοια ώστε να ικανοποιούν τη συνθήκη $T_{B_1} = T_{(S_H^-)^{int}} \cap T_{(S_H^+)^{int}}$. Επίσης, αποδεικνύεται ότι κάθε συνάρτηση $f_1 \in H^2(T_{(S_H^-)^{int}})$ ή $f_2 \in H^2(T_{(S_H^+)^{int}})$, αντίστοιχα, εκφράζεται μέσω ολοκληρωτικής αναπαράστασης τύπου Cauchy-Fantappie (και αντίστροφα). Ώς άμεσο αποτέλεσμα λαμβάνουμε, επίσης, ότι κάθε συνάρτηση $f \in H^2(T_{B_1})$ μπορεί να αναπαρασταθεί μέσω ολοκληρωτικής αναπαράστασης τύπου Cauchy-Fantappie με φορέα στο σύνορο ∂T_{B_1}. Ειδικότερα, άν $\Phi_1(\zeta, \bar{\zeta}) = \left(\frac{\zeta_1 - \bar{\zeta}_1}{2i}\right)^2 + \left(\frac{\zeta_2 - \bar{\zeta}_2}{2i}\right)^2 - 1$ είναι η ορίζουσα συνάρτηση που αντιστοιχεί στο κυλινδρικό χωρίο T_{B_1} τότε για κάθε συνάρτηση $f \in H^2(T_{B_1})$ ισχύει

$$f(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times iS^1} \frac{f(\zeta)\left(\partial\Phi_1(\zeta, \bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1(\zeta, \bar{\zeta})\right)}{\left(< \nabla_\zeta \Phi_1(\zeta, \bar{\zeta}), \zeta - z >\right)^2}$$

Όμοια αποτελέσματα ισχύουν και για τον χώρο Hardy $H^2(T_{B_2})$.

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Chapter 1

Introduction

The theory of Hardy spaces and their integral representations is rather well developed for bounded domains $\Omega \subset \mathbb{C}^n$ of different types of convexity (convex and pseudo-convex domains with reasonably smooth boundaries). However, almost nothing is known about the related questions for Hardy spaces on unbounded domains $G \subset \mathbb{C}^n$, the main reason being the absence of suitable integral representation for holomorphic functions $f \in \mathcal{H}(G)$ on such domains, even if one assumes high degree of smoothness of the boundary ∂G and continuity of f on the closure $\overline{G}$. The main obstacle to be able to obtain such integral formulas is the lack of Stoke's theorem for unbounded domains.

The main result of the present thesis is a Cauchy-Fantappie formula for the Hardy spaces $H^2(T_{B_j})$, $j = 1, 2$, on tubular domains consisting of holomorphic functions

$$F(z) = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi i z\cdot t} dt, z \in T_{B_1} \text{ and } G(z) = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi z\cdot t} dt, z \in T_{B_2}$$

correspondingly. The tubes are described by their defining functions as follows

$$\begin{aligned} T_{B_1} &= \{z \in \mathbb{C}^2 : (\frac{z_1 - \bar{z}_1}{2i})^2 + (\frac{z_2 - \bar{z}_2}{2i})^2 - 1 < 0\} \\ T_{B_2} &= \{z \in \mathbb{C}^2 : (\frac{z_1 + \bar{z}_1}{2})^2 + (\frac{z_2 + \bar{z}_2}{2})^2 - 1 < 0\}. \end{aligned}$$

Our approach is based on that of Aizenberg-Martineau ([1, 2, 4],[21, 22]) with the use of the notion of the "exterior of a domain " not in a topological sense, but rather through a generalized complement (or dual complement in ([10])). To be more specific, if $G \subset \mathbb{C}^n$, then

G^* denotes its generalized complement, that is, the set of points through which there exists a complex hyperplane that does not intersect G. One of the main results of Aizenberg-Martineau theory is that the Cauchy-Fantappie Transform $\mathcal{F}_C : (\mathcal{O}(G))' \longrightarrow \mathcal{O}(G^*)$, mapping the analytic functional $\mu \in (\mathcal{O}(G))'$ into the space of analytic functions $\mathcal{O}(G^*)$ via $\mathcal{F}_C(\mu)(\zeta) = \mu(\frac{1}{(1+<\zeta,\cdot>)^2})$, is an isomorphism whenever G is an open (or compact) $\mathbb{C}$-convex set ([10, 29]). Such approach was used to obtain duality results for Hardy spaces on bounded domains with suitably smooth boundary. Namely, it was proved in ([6, 7]) that $(H^p(G))' = H^q(G^*)$, whenever $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and G is a bounded convex domain with smooth enough boundary, where a crucial step was the knowledge of the boundary values of the Cauchy-Fantappie integral from ([27]). One should note here a string of recent papers concerning the boundary values behavior of the Cauchy-Fantappie kernel and the description of the corresponding Hardy spaces ([19, 18, 26]). This cycle of ideas breaks down when $G = T_{B_1}$ or $G = T_{B_2}$, because no Stokes theorem can be applied. Instead, we prove separation of singularity (Aronsajn type theorem ([11]) for functions spaces equipped with norm, using the approach developed by L.Aizenberg in [5] via duality arguments for Hardy spaces on the generalized exteriors (generalized dual complement) and then using the reflexivity of spaces obtained we return back to the original space. The outline of the thesis is as follows: in Chapter 2 we present the results describing precisely the "exterior" (generalized dual complement) of suitable tubes $T_{S_H^-}$ and $T_{S_H^+}$ with convex , unbounded base containing a cone, whose intersection is T_{B_1}, in order to obtain the fact that $T^*_{B_1}$ is the envelope of holomorphy of the union of the compacts $T^*_{S_H^-}$ and $T^*_{S_H^+}$. In Chapter 3 we formulate and prove some results concerning $H^2(T_{B_1})$ (similarly arguing results concerning $H^2(T_{B_2})$). In Chapter 4 we prove that $f \in H^2(T_{S_H^-})$ if and only if it is representable by Cauchy-Fantappie formula. In Chapter 5 we develop sort of duality theory in the spirit of Martineau-Aizenberg for the spaces $H^2(T_{S_H^-})$ and $H^2(T_{B_1})$. The main result of this chapter describes the general form of $F \in \left(H^2(T_{S_H^-})\right)'$. Finally, in Chapter 6 we derive the separation of singularities theorem and its consequence: the Cauchy-Fantappie integral representation for $f \in H^2(T_{B_1})$.

Chapter 2

Convexity and exterior

We begin by recalling some basic notions and facts from the theory of real and complex convexity emphasizing on the linear (lineal in ([20])) convexity theory for domains in $\mathbb{C}^n$ that will be used throughout the thesis. Next, we introduce the notion of generalized dual complement of a domain in $\mathbb{C}^n$ and then we describe explicitly the generalized dual complement of particular convex sets in $\mathbb{C}^2$.

2.1 Notions of convexity

A domain Ω in $\mathbb{C}^n$ is a non-empty, open and connected subset of $\mathbb{C}^n$. A domain $\Omega \subset \mathbb{C}^n$ has a boundary of class $\mathcal{C}^k$ for $k \geq 1$ ([17]) if $\Omega = \{z \in \mathbb{C}^n : \Phi_\Omega(z, \overline{z}) < 0\}$ where Φ_Ω is a real-valued function at least k times continuously differentiable in some neighborhood of the closure of Ω so that the complex gradient $\nabla_z \Phi_\Omega = (\frac{\partial \Phi_\Omega}{\partial z_1}, \cdots, \frac{\partial \Phi_\Omega}{\partial z_n})$ is assumed to be non-vanishing at all points of the boundary $\partial\Omega$. We write $\partial\Omega \in \mathcal{C}^k$. It is clear that the boundary of Ω corresponds to the set $\partial\Omega = \{z \in \mathbb{C}^n : \Phi_\Omega(z, \bar{z}) = 0\}$. Thus, the boundary $\partial\Omega$ has real dimension $2n - 1$. If $k = 1$, then one says that Ω is a domain with smooth boundary. The function Φ_Ω is called the defining function for the domain Ω and in general is not uniquely determined. This notation for Ω will be used throughout the thesis. Furthermore, notice that if $U \supset \bar{\Omega}$ is a neighborhood of the closure of Ω then $U \cap \partial\Omega = \{z \in \mathbb{C}^n : \Phi_\Omega(z, \overline{z}) = 0\}$. Note that once a defining function is given on a neighborhood of the boundary of a domain Ω, then using a partition of unity, the defining function is extended to the whole domain.

A domain Ω has a piece-wise smooth boundary ([17]) if $\Omega = \{z \in \mathbb{C}^n : \Phi_{\Omega_j}(z) < 0, j = 1, \cdots, m\}$ where the real-valued functions Φ_{Ω_j} are of class $\mathcal{C}^1$ in some neighborhood of $\bar{\Omega}$ and for every set of distinct indices $j_1, \cdots, j_l$ where $1 \leq l \leq m$, the condition $d\Phi_{\Omega_{j_1}} \wedge \cdots \wedge d\Phi_{\Omega_{j_l}} \neq 0$ is valid on the set $\{z \in \mathbb{C}^n : \Phi_{\Omega_{j_1}}(z, \bar{z}) = \cdots = \Phi_{\Omega_{j_l}}(z, \bar{z}) = 0\}$. In the particular case when $m = 1$, a domain $\Omega \subset \mathbb{C}^n$ has a smooth boundary if $d\Phi_\Omega \neq 0$ on $\partial\Omega$.

We are going to study particular type of domains and compact subsets of $\mathbb{C}^n$ that are described in terms of notions analogous to those of real convexity theory. Recall that $A \subset \mathbb{R}^{2n}$ is called geometrically convex set if and only if its intersection with every line is connected or equivalently if and only if the line segment connecting any two points of A lies entirely in A. An alternative and equivalent description of convexity consists in the study of the exterior of a set. Particularly, $A \subset \mathbb{R}^{2n}$ is convex if and only if for every $\alpha \in A^c$ there is a real hyperplane $\{x \in \mathbb{R}^{2n} :< x, y > \leq \alpha\}$ which does not intersect A, where $< x, y >= x_1y_1 + \cdots + x_{2n}y_{2n}$ is the usual inner product in $\mathbb{R}^{2n}$. Thus, through every point in the topological complement of a convex set $A \subset \mathbb{R}^{2n}$ there passes a real hyperplane which does not intersect A. The topological dimension of such a hyperplane is $2n-1$.

Actually, assume that $A \subset \mathbb{R}^{2n}$ has the property that through any point of its exterior there passes a real hyperplane not intersecting A. If A is not geometrically convex then there are points $x, y \in A$ so that for some $a \in \overline{xy}$, implies that $a \in \mathbb{R}^{2n} \setminus A$. Then there is a point $b \in \partial A$ (which may be a itself) on the line segment connecting x and b and a real hyperplane through b which is disjoint from A. Since all points sufficiently near to x, y belong to A, this is a contradiction. Conversely, assume A is convex according to the line segment definition and take $x \in \partial A$. Assume $c \in A^c$. Then there is a point $a \in \overline{A}$ so that the distance $|\overline{ac}|$ is minimum for points ranging over $\overline{A}$. Then the real hyperplane $< a, x >= |\overline{ac}|$ is passing through c and is disjoint from A. Indeed, if this hyperplane contains $b \in \overline{A}$, then since the line segment ab is consisted of points of $\overline{A}$, there will be a point $x \in \overline{A}$ such that $|\overline{cx}|$ is less than $|\overline{ca}|$, a contradiction. Thus, geometric convexity of a set $A \subset \mathbb{R}^{2n}$ is equivalently described through the conditions stated above.

Thus, it is natural to define the polar set of a domain $\Omega \subset \mathbb{C}^n$ (not necessarily convex) as the set of all real hyperplanes passing through some point $\zeta \in \Omega$ that do not intersect Ω. The polar set is denoted by Ω°. Straightforward reasoning shows that one always has that $\Omega \subset (\Omega^\circ)^\circ$. The equality $\Omega = (\Omega^\circ)^\circ$ is valid only when the domain Ω is convex. The difference $(\Omega^\circ)^\circ \setminus \Omega$ is measuring then how far from being convex is the set Ω.

At this point, we remark that the inner product is a linear map from $\mathbb{R}^{2n}$ to $\mathbb{R}$. Thus, by Riesz Representation Theorem, being linear it is an element of the dual space $(\mathbb{R}^{2n})'$. Thus, points in $\mathbb{R}^{2n}$ correspond to hyperplanes in $(\mathbb{R}^{2n})'$. In order to derive further convexity information we may exploit the duality stated above.

Next, let us recall the notions of real and complex tangent space ([25]).
The real tangent space of a domain $\Omega \subset \mathbb{C}^n \approx \mathbb{R}^{2n}$ with $\mathcal{C}^k$ boundary for $k \geq 1$ at a point $p \in \partial\Omega$ is the $(2n-1)$- real dimensional hyperplane

$$\mathrm{T_p}(\partial\Omega) = \left\{ w \in \mathbb{C}^n : \Re\left(\sum_{\jmath=1}^{n} \frac{\partial \Phi_\Omega}{\partial z_\jmath}(p) w_\jmath \right) = 0 \right\}$$

Analogously, the complex tangent space of a domain $\Omega \subset \mathbb{C}^n$ at the point $p \in \partial\Omega$ is the $(n-1)$- complex dimensional hyperplane

$$\mathfrak{T}_p(\partial\Omega) = \left\{ w \in \mathbb{C}^n : \sum_{\jmath=1}^{n} \frac{\partial \Phi_\Omega}{\partial z_\jmath}(p) w_\jmath = 0 \right\}$$

It is clear that $\mathfrak{T}_p(\partial\Omega)$ is a real $(2n-2)-$ dimensional subspace of $\mathrm{T_p}(\partial\Omega)$.

For domains with smooth enough boundary its convexity (or its variations) is often deduced by the boundary behavior of the gradient of its defining function.

For simplicity, we first remark that for a real-valued twice continuously differentiable function Φ_Ω the quadratic form

$$H_{\Phi_\Omega}(p,w) = \Re\left(\sum_{j,k} \frac{\partial^2 \Phi_\Omega}{\partial z_j \partial z_k}(p) w_j w_k\right) + \sum_{j,k} \frac{\partial^2 \Phi_\Omega}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \tag{2.1.1}$$

is called the Hessian of Φ_Ω at p, whereas its hermitian part

$$L_{\Phi_\Omega}(p,w) = \sum_{j,k} \frac{\partial^2 \Phi_\Omega}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \tag{2.1.2}$$

is called the Levi form.

If Φ'_Ω is another defining function for the domain Ω then $\Phi'_\Omega = h\Phi_\Omega$ where the function h is strictly positive in a neighborhood of the boundary $\partial\Omega$. If $p \in \partial\Omega$ and $w \in \mathfrak{T}_p(\partial\Omega)$ we therefore have that $H_{\Phi_\Omega}(p,w) = h(p)H_{\Phi'_\Omega}(p,w)$. In other words, the value of the Hessian is independent of the selection of the defining function.

It is a complicated problem to describe the geometric convexity of a domain Ω in $\mathbb{C}^n$ by using its definition. Thus, one is looking for alternative approaches for testing it. Such is the approach for bounded domains $\Omega = \{z \in \mathbb{C}^n : \Phi_\Omega(z,\bar{z}) < 0\}$ with $\mathcal{C}^2$ boundary by using the Hessian of the defining function. The following definition is from ([25]).

Definition 2.1.1 *Let $\Omega = \{z \in \mathbb{C}^n : \Phi_\Omega(z,\overline{z}) < 0\} \subset \mathbb{C}^n$ is a domain with $\mathcal{C}^2$ boundary then it is convex if the value of the Hessian when restricted to the real tangent space is positive semi-definite:*

$$H_{\Phi_\Omega}(p,w) \geq 0 \textit{ whenever } p \in \partial\Omega \textit{ and } w \in \mathrm{T}_p(\partial\Omega) \tag{2.1.3}$$

When the inequality (2.1.3) is proper for any $w \in \mathrm{T_p}(\partial\Omega)\backslash\{0\}$, the domain Ω is strictly convex at $p \in \partial\Omega$ and p is called a point of strict convexity for the domain Ω. If every point of the boundary $\partial\Omega$ is a point of strict convexity then Ω is called strictly convex domain.

It is well known that the usual geometric definition of convexity is equivalent to the above analytic requirement that the Hessian of the defining function Φ_Ω be positive, semidefinite, whenever it is restricted to the real tangent space at every $p \in \partial\Omega$ ([10]).

E. Levi introduced another notion of convexity for domains with $\mathcal{C}^2$ boundary satisfying a complex analogue of (2.1.3), by formulating the following ([20], [25])

Definition 2.1.2 *Let $\Omega \subset \mathbb{C}^n$ be a domain with $\mathcal{C}^2$ boundary and Φ_Ω be its corresponding defining function. Then $\partial\Omega$ is called Levi pseudoconvex at the point $p \in \partial\Omega$ if the restriction of the Levi form to the complex tangent plane is positive semi-definite. A domain is Levi pseudoconvex if every boundary point is a point of Levi pseudoconvexity.*

A domain is called strictly Levi pseudoconvex if in the neighborhood of each of its boundary points the domain is strictly convex for a suitable choice of coordinates.

Every domain in $\mathbb{C}$ is Levi pseudoconvex. However, that is not the case for $n > 1$. It is elementary to verify directly that convex domains are Levi pseudoconvex but the converse is not always true. Simply observe that $\mathbb{C}^n$ minus a hyperplane is Levi pseudoconvex. More precisely, if $\wp$ is a complex hyperplane in $\mathbb{C}^n$ then $\mathbb{C}^n \setminus \{\wp\}$ is Levi pseudoconvex but not convex.

An alternative way to describe the convexity of the domain Ω is to involve directly its boundary. To be more specific, one observes that for a convex domain $\Omega \subset \mathbb{C}^n$ there is a real hyperplane through every point $\zeta \in \mathbb{C}^n \setminus \Omega$ which does not meet the domain. Remark that a real hyperplane in $\mathbb{R}^{2n}$ is of real dimension $2n-1$, while a complex hyperplane in $\mathbb{C}^n$ is of real dimension $2(n-1)$.

The complex analogue of this was introduced by A. Martineau and L. Aizenberg and is formulated as follows :

Definition 2.1.3 *A domain $\Omega \subset \mathbb{C}^n$ is said to be linearly convex (or weakly lineally convex in ([10])) if for every $\zeta \in \partial\Omega$ there exists a complex hyperplane $\wp = \{z \in C^n : \alpha_1 z_1 + ... + \alpha_n z_n + \beta = 0\}$ through ζ that does not intersect Ω. A domain $\Omega \subset \mathbb{C}^n$ is called strictly linearly convex if through every point $\zeta \in \Omega^c$ of its exterior there passes an $(n-1)-$ dimensional complex hyperplane not intersecting Ω.*

Thus, strict linear convexity amounts to the condition that through any boundary point there should pass a complex tangent hyperplane intersecting $\partial\Omega$ at precisely one point.

We will say that a set $\Omega \subset \mathbb{C}^n$ is approximated from the outside (inside) by the sequence of domains $\{\Omega_k\}_{k\in\mathbb{N}}$ if $\bar{\Omega}_{k+1} \subset \Omega_k$ ($\bar{\Omega}_k \subset \Omega_{k+1}$ respectively) and $\Omega = \cap_k \Omega_k$ ($\Omega = \cup_k \Omega_k$ correspondingly) where $\Omega_k = \{z \in C^n : \Phi_{\Omega_k}(z, \overline{z}) < 0\}$, $\Phi_{\Omega_k} \in C^2(\overline{\Omega}_k)$ and $\nabla_z \Phi_{\Omega_k}(z, \bar{z}) \neq 0$ for all $z \in \partial\Omega_k$. A compact set $M \subset \mathbb{C}^n$ is said to be linearly convex if there exists a sequence of linearly convex domains approximating M from the outside ([3], [5]).

Linear convexity is preserved under intersections. Furthermore, all cartesian products $\Omega_1 \times \Omega_2$ of linearly convex sets are also linearly convex ([10]). Since every real hyperplane contains a complex hyperplane, it is clear that every convex domain $\Omega \subset \mathbb{C}^n$ is linearly convex. However, the converse claim is not always true. For example, the set $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2, |z_2| < 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 2\}$ is linearly convex, but not convex.

The above approach has a natural realization for domains Ω with $\mathcal{C}^1$ boundary (so that it has unique complex tangent hyperplane at every boundary point). Indeed, for a linearly convex domain $\Omega \subset \mathbb{C}^n$ and $\zeta \in \Omega^c$ we consider the complex hyperplane

$$\{z \in \mathbb{C}^n :< \nabla_\zeta \Phi_\Omega(\zeta, \bar{\zeta}), \zeta - z >= 0\},$$

that is, the complex tangent hyperplane passing through ζ. Whenever $< \nabla_\zeta \Phi_\Omega(\zeta, \bar{\zeta}), \zeta > \neq 0, \zeta \in \mathbb{C}^n \setminus \Omega$ the tangent hyperplane above can be written as

$$\left\{ z \in \mathbb{C}^n : \left\langle \frac{\nabla_\zeta \Phi_\Omega(\zeta, \bar{\zeta})}{< \nabla_\zeta \Phi_\Omega(\zeta, \bar{\zeta}), \zeta >}, z \right\rangle = 1 \right\}$$

Particularly, let us take $n = 2$ and assume that $\Omega = \{z \in \mathbb{C}^2 : \Phi_\Omega(z, \bar{z}) < 0\} \subset \mathbb{C}^2$ is a linearly convex domain with smooth boundary, that is $\nabla_\zeta \Phi_\Omega(\zeta, \bar{\zeta}) = (\frac{\partial \Phi_\Omega(\zeta,\bar{\zeta})}{\partial \zeta_1}, \frac{\partial \Phi_\Omega(\zeta,\bar{\zeta})}{\partial \zeta_2}) \neq 0$ for $\zeta \in \partial\Omega$. Furthermore, assume that $0 \in \Omega$. For a $(1,0)-$ form $q = \sum_{j=1}^2 q_j dz_j$ and $z \in \mathbb{C}^2$ we will write $< q, z >= \sum_{j=1}^2 q_j z_j$. For convenience, we will sometimes identify $(1,0)-$ forms with vectors, i.e. we identify $\partial_\zeta \Phi_\Omega(\zeta, \bar{\zeta})$ with $(\frac{\partial \Phi_\Omega(\zeta,\bar{\zeta})}{\partial \zeta_1}, \frac{\partial \Phi_\Omega(\zeta,\bar{\zeta})}{\partial \zeta_2})$. Identifying the slope

$w = \frac{\nabla_\zeta \Phi_\Omega(\zeta,\bar{\zeta})}{<\nabla_\zeta \Phi_\Omega(\zeta,\bar{\zeta}),\zeta>}$ with the hyperplane above we have the correspondence

$$\mathbb{C}^2 \ni w \longrightarrow \{z \in \mathbb{C}^2 :< z, w >= 1\} \in \Omega^c$$

Thus, we obtain a description of the exterior of such a domain.

A. Martineau in [21] introduced a way to measure how far away is a set of being linearly convex. He measured the linear convexity of a domain through the existence of certain hyperplanes contained in its topological complement. More specifically, he defined the notion of the generalized dual complement (dual complement in ([10]))

Definition 2.1.4 *Let $\Omega \subset C^n$ be a domain. The generalized dual complement Ω^* of Ω is defined to be the set*

$$\Omega^* = \{\zeta \in \mathbb{C}^n : z_1\zeta_1 + ... + z_n\zeta_n \neq 1, \forall z \in \Omega\} \tag{2.1.4}$$

The notion of the generalized dual complement is geometric in spirit and it plays the role of the exterior for the domain Ω. For the domain $\Omega \subset \mathbb{C}$ one obtains that $\Omega^* = \Omega^c$, since there is no difference between a complex hyperplane and a point. However in higher dimensions this is not in general the case. In particular, for $n = 1$ the dual complement can be expressed as $\Omega^* = \{\frac{1}{\zeta} : \zeta \in \mathbb{C} \setminus \Omega\} \cap \{0\}$. The whole space $\mathbb{C}^n$ and the singleton $\{0\}$ are generalized dual complements the one of the other.

Even though the concept of polar of a set and the concept of its dual complement appear to be similar, there is no equality between the sets Ω° and Ω^*, even though $\Omega^\circ \subset \Omega^*$ is always valid, unless some additional geometric characteristics of Ω are present.

2.2 Generalized dual complement for particular linearly convex domains

2.2.1 Basic properties of generalized dual complement

We now briefly recall some properties of the generalized dual complement and the notion of linear (weakly linear) convexity (linearly convex in ([10])) to be found in ([1],[2], [10]). For a domain $\Omega \subset \mathbb{C}^n$ the generalized dual complement of its closure, $\left(\overline{\Omega}\right)^*$, is the interior of Ω^*, that is $\left(\overline{\Omega}\right)^* = (\Omega^*)^{int}$. The interior of a linearly convex set is linearly convex while its closure is not in general (i.e. Hartogs triangle). In addition if Ω is compact then Ω^* is open and vice versa. Furthermore, if Ω_1, Ω_2 are domains in $\mathbb{C}^n$ so that the inclusion $\Omega_1 \subset \Omega_2$ is hold, then $\Omega_2^* \subset \Omega_1^*$. The generalized dual complement of a domain $\Omega = \cup_{k\in\mathbb{N}}\Omega_k$ is $\Omega^* = \cap_{k\in\mathbb{N}}\Omega_k^*$. Furthermore, $\left(\cap_{k\in\mathbb{N}}\Omega_k\right)^* \supseteq \cup_{k\in\mathbb{N}}\Omega_k^*$. On the other hand, if a compact set $M \subset \mathbb{C}^n$ is defined as $M = \cap M_k$ and M_{k+1} is relatively compact in M_k for every $k \in \mathbb{N}$ (we denote this by $M_{k+1} \subset\subset \bar{\Omega}_k$) then $M^* = \cap_{k\in\mathbb{N}}M_k^*$ and $\left(\cap_{k\in\mathbb{N}} M_k\right)^{**} = \cup_{k\in\mathbb{N}}M_k^{**}$. Subsequently, any linearly convex and compact set admits a basis of linearly convex open neighborhoods.

Some examples are in order.

Example 2.2.1 *Let $B_n(0,r) = \{(z_1,\cdots,z_n) \in \mathbb{C}^n : |z_1|^2+\cdots+|z_n|^2 < r^2\}$ be the open ball in $\mathbb{C}^n$. Assume, now, that $\zeta \in (B_n(0,r))^*$. Then $z_1\zeta_1+\cdots+z_n\zeta_n \neq 1$ for all $z = (z_1,\cdots,z_n) \in B_n(0,r)$. Without loss of generality take $|\zeta_n| \neq 0$. Thus the hyperplane*

$$z_n \neq \frac{1}{\zeta_n} - \left(\frac{\zeta_{n-1}}{\zeta_n}z_{n-1}+\cdots+\frac{\zeta_1}{\zeta_n}z_1\right)$$

does not meet the ball $B_n(0,r)$ for $\frac{\zeta_{n-1}}{\zeta_n}z_{n-1}+\cdots+\frac{\zeta_1}{\zeta_n}z_1 = z_n$. Follows that $(0,\cdots,0,\frac{1}{\zeta_n}) \notin B_n(0,r)$. Hence $|\frac{1}{\zeta_n}|^2 \geq r^2$ or equivalently $|\zeta_n|^2 \leq \frac{1}{r^2}$. Similarly, take $|\zeta_j| \neq 0$ for $j = 1,\cdots,n-1$ and conclude that $(B_n(0,r))^* \subset \bar{B}_n(0,\frac{1}{r})$. For the converse inclusion, assume $\zeta \in \bar{B}_n(0,\frac{1}{r})$. Then $|\zeta_1|^2+\cdots+|\zeta_n|^2 \leq \frac{1}{r^2}$. Therefore for every $z \in B_n(0,r)$ we have that

$$|z_1\zeta_1+\cdots+z_n\zeta_n| \leq \left(\sum_{i=1}^{n}|z_i|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}|\zeta_j|^2\right)^{\frac{1}{2}} \leq \left(\frac{1}{r^2}r^2\right)^{\frac{1}{2}} = 1$$

Thus $z_1\zeta_1 + \cdots + z_n\zeta_n \neq 1$ *for every* $z \in B_n(0,r)$ *and the inclusion* $\bar{B}_n(0,\frac{1}{r}) \subset (B_n(0,r))^*$ *follows. Hence* $(B_n(0,r))^* = \bar{B}_n(0,\frac{1}{r})$. *Note that both* $B_n(0,r)$ *and* $\bar{B}_n(0,\frac{1}{r})$ *have smooth boundary, but that is not the case in general. For* $r = 1$ *we obtain that* $(B_n(0,1))^* = \bar{B}_n(0,1)$.

One might be tempted, motivated by the previous example, to assume that $\Omega^{**} = (\Omega^*)^* = \Omega$, where Ω^{**} is the union of all complex hyperplanes that intersect Ω. In general, however, only the inclusion $\Omega \subset \Omega^{**}$ is valid and the difference $(\Omega^*)^* \setminus \Omega$ quantifies how far away is the set Ω from being linearly convex. The domains for which $\Omega = \Omega^{**}$ were introduced and studied by A. Martineau in [22] in relation to the solution of the duality problem in several complex variables. These domains are known also as Martineau linearly convex domains. However, ([9]) provides examples of linearly convex domains which are not Martineau linearly convex domains. Therefore one can define for any domain Ω its linearly convex hull to be the smallest linearly convex domain containing it, that is, the generalized dual complement of its generalized dual complement, Ω^{**} ([10]). Hence, linear convexity is a type of convexity defined more conveniently in terms of the generalized dual complement of a domain in $\mathbb{C}^n$. Specifically, if $\Omega = \Omega^{**}$ ([10]) then the domain Ω is linearly convex.

Of particular interest to us is the following domain with non-smooth boundary and its generalized dual complement.

Example 2.2.2 *Let* $\mathcal{A}_2 = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$ *be the hyper-cone. We claim that its generalized dual complement* $\mathcal{A}_2^*$ *is the closed complex bi-disk* $\overline{\mathbb{D}}_2(0,1) = \{(\zeta_1,\zeta_2) \in \mathbb{C}^2 : |\zeta_1| \leq 1, |\zeta_2| \leq 1\}$. *Let* $\zeta = (\zeta_1,\zeta_2) \in \overline{\mathbb{D}}_2(0,1)$. *Then* $|\zeta_1| \leq 1$, $|\zeta_2| \leq 1$. *Therefore, for every* $z \in \mathcal{A}_2$ *we have*

$$|\zeta_1 z_1 + \zeta_2 z_2| \leq |\zeta_1||z_1| + |\zeta_2||z_2| < |z_1| + |z_2| < 1$$

Thus $\zeta_1 z_1 + \zeta_2 z_2 \neq 1$ *for every* $z \in \mathcal{A}_2$. *Hence* $\overline{\mathbb{D}}_2(0,1) \subset \mathcal{A}_2^*$. *In order to prove the converse inclusion, let us assume that* $\zeta' = (\zeta_1',\zeta_2') \in \mathcal{A}_2^*$. *It means that* $\zeta_1' z_1 + \zeta_2' z_2 \neq 1$ *for every* $z = (z_1,z_2) \in \mathcal{A}_2$. *Without loss of generality, we may assume that* $|\zeta_2'| = \varrho_2 \neq 0$. *Hence* $z_2 \neq \frac{1}{\zeta_2'} - \frac{\zeta_1'}{\zeta_2'} z_1$. *Now, the line* $\frac{1}{\zeta_2'} - \frac{\zeta_1'}{\zeta_2'} z_1$ *does not intersect the hyper-cone for* $z_1 = 0$

also, thus $(0, \frac{1}{\zeta_2'}) \notin \mathcal{A}_2$ *or* $|\frac{1}{\zeta_2'}| > 1$*. Thus* $|\zeta_2'| < 1$*. Similarly, we treat the other cases. Thus* $\mathcal{A}_2^* \subset \overline{\mathbb{D}}_2(0,1)$*. Thus* $\bar{\mathbb{D}}_2(0,1) = \mathcal{A}_2^*$*. Note that both domains are circular (Reinhardt), but do have a non smooth boundary.*

The above example can be generalized for the hyper-cone

$$\mathcal{A}_n = \{(z_1, ..., z_n) \in \mathbb{C}^n : r|z_1| + ... + r|z_n| < 1\}$$

whose generalized dual complement is

$$\mathcal{A}_n{}^* = \{(\zeta_1, ..., \zeta_n) \in \mathbb{C}^n : |\zeta_1| \le \frac{1}{r}, ..., |\zeta_n| \le \frac{1}{r}\} = \overline{\mathbb{D}}_n(0, \frac{1}{r})$$

If the bounded domain $\Omega \subset \mathbb{C}^n$ is convex containing the origin $0 \in \Omega$ then $\lambda\overline{\Omega} \subset \Omega$, for all $|\lambda| < 1$ and thus, $\overline{\Omega}$ and $(\overline{\Omega})^*$ are star shaped domains. Recall that a set $\Omega \subset \mathbb{C}^n$ is called star shaped with respect to some point $z_0 \in \Omega$ if $\lambda z_0 + (1-\lambda)z \in \Omega$ for every $z \in \Omega$ and $0 \le \lambda \le 1$. Observe that all convex sets are star shaped with respect to any point $z_0 \in \Omega$, but a star shaped set is not convex in general, (i.e. consider the star set). Furthermore, if Ω is a convex domain containing the origin with smooth boundary then Ω^* is star-shaped ([7]). The converse claim is not true in general.

Example 2.2.3 *The domains* $\mathcal{A}_2$ *and* $\bar{D}_2(0,1)$ *considered in Example (2.2.2) are both star-shaped sets with a non-smooth boundary.*

Definition 2.2.1 *A domain* $\Omega \subset \mathbb{C}^n$ *is called* $\mathbb{C}-$ *convex if* $\Omega \cap l$ *is connected and simply connected for every complex line* $l = \{\lambda z : \lambda \in \mathbb{C}\}$*.*

In order that the Cauchy-Fantappie Transform $\widetilde{\mathcal{F}} : \mathcal{H}(E) \longrightarrow \mathcal{H}(E^*)$ is isomorphism it is necessary and sufficient the set $E \subset \mathbb{C}^n$ being convex ([10]). In the one variable case a non-empty set E is $\mathbb{C}-$ convex precisely if its complement is. For $n > 1$ however, the complement of a $\mathbb{C}-$ convex set, $E \subset \mathbb{C}^n$, is never $\mathbb{C}-$ convex. Every convex domain $\Omega \subset \mathbb{C}^n$ is strictly linearly convex since through each point in Ω^c there is a $(2n-1)$ real hyperplane α that does not intersect Ω containing a complex hyperplane of dimension $n-1$. A linearly convex domain $\Omega \subset \mathbb{C}^n$ is not necessarily a $\mathbb{C}-$ convex domain. In fact the two notions coincide when the domain is smoothly bounded ([10]).

Example 2.2.4 *Let $\Omega \subset \mathbb{R}^{2n} \subset \mathbb{C}^n$. Then Ω is $\mathbb{C}-$ convex if and only if Ω is convex. Notice that $\Omega \subset \mathbb{R}^{2n}$ is linearly convex if its complement in $\mathbb{R}^{2n}$ is a union of real $(n-2)-$ dimensional planes. This happens because any complex hyperplane's intersection with $\mathbb{R}^{2n}$ represents either a real $(n-2)-$ plane or a real hyperplane. For example, a circle in $\mathbb{R}^2$ is complex linearly convex, but a sphere in $\mathbb{R}^3$ is not.*

2.2.2 On the generalized dual $\mathbf{T}^*_{\mathbf{B_1}}$ for the tube $\mathbf{T_{B_1}}$

Consider the tube domains $T_B = \mathbb{R}^n \times iB \subset \mathbb{C}^n$ (or $T_B = B \times i\mathbb{R}^n$), where the set B is called the base of T_B. A tubular domain is convex whenever its base is convex. Thus, for the rest of the thesis, we assume that the base B of T_B is open and convex.

At this point we turn our attention to the tube domains

$$\begin{aligned} T_{B_1} &= \mathbb{R}^2 \times i\{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1\} \\ T_{B_2} &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \times i\mathbb{R}^2 \end{aligned} \tag{2.2.1}$$

having a convex base (namely a disk). Actually, $B_1 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1\}$ and $B_2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$. Observe that the intersection of the tubes (2.2.1) is realizing the topological bi-disk:

$$\mathcal{U}_\tau = \{(x_1, x_2, iy_1, iy_2) \in \mathbb{R}^2 \times i\mathbb{R}^2 : \sum_{\imath=1}^{2} {x_\imath}^2 < 1, \sum_{\imath=1}^{2} {y_\imath}^2 < 1\} = T_{B_1} \cap T_{B_2} \tag{2.2.2}$$

Notice that the tubes T_{B_1}, T_{B_2} are star shaped with respect to the origin.
It is straight-forward to verify that the defining functions for the tubes T_{B_i} $i = 1, 2$ are correspondingly the $\mathcal{C}^2$ maps

$$\begin{aligned} \Phi_1(\zeta, \bar{\zeta}) &= \left(\frac{\zeta_1 - \bar{\zeta}_1}{2i}\right)^2 + \left(\frac{\zeta_2 - \bar{\zeta}_2}{2i}\right)^2 - 1 \\ \Phi_2(\zeta, \bar{\zeta}) &= \left(\frac{\zeta_1 + \bar{\zeta}_1}{2}\right)^2 + \left(\frac{\zeta_2 + \bar{\zeta}_2}{2}\right)^2 - 1 \end{aligned} \tag{2.2.3}$$

for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$. Furthermore, $\partial T_{B_i} = \{\zeta \in \mathbb{C}^2 : \Phi_i(\zeta, \bar{\zeta}) = 0\}$, $i = 1, 2$. We use, for simplicity, the notation $\partial_{\zeta_j}\Phi_i(\zeta, \bar{\zeta})$, $i, j = 1, 2$ for the partial derivatives of the defining functions at the point $(\zeta, \bar{\zeta})$ and the corresponding 1-form as well. Furthermore, notation $\nabla_\zeta \Phi_i(\zeta, \bar{\zeta}) = (\frac{\partial \Phi_i(\zeta,\bar{\zeta})}{\partial \zeta_1}, \frac{\partial \Phi_i(\zeta,\bar{\zeta})}{\partial \zeta_2})$ is used for the complex gradient at the point $(\zeta, \bar{\zeta}) \in \overline{T}_{B_i}$. The tubes T_{B_1} and T_{B_2} are convex sets by definition and thus are linearly convex sets also. Let us consider the tube T_{B_1}. Calculating the partial derivatives of Φ_1 in order to verify convexity condition (2.1.1) we obtain for $j = 1, 2$

$$\frac{\partial \Phi_1(\zeta, \bar{\zeta})}{\partial \zeta_j} = \frac{2}{(2i)^2}(\zeta_j - \bar{\zeta}_j) = -\frac{\partial \Phi_1(\zeta, \bar{\zeta})}{\partial \bar{\zeta}_j}$$

Furthermore,

$$\frac{\partial^2 \Phi_1(\zeta, \bar{\zeta})}{\partial \zeta_i \partial \zeta_j} = -\frac{\partial^2 \Phi_1(\zeta, \bar{\zeta})}{\partial \zeta_i \partial \bar{\zeta}_j} = \begin{cases} \frac{2}{(2i)^2}, & \text{i=j;} \\ 0, & \text{i} \neq \text{j} \end{cases}$$

for $i, j = 1, 2$. Hence, $\nabla_\zeta \Phi_i(\zeta, \bar{\zeta}) \neq 0$ for any $\zeta \in \partial T_{B_1} = \mathbb{R}^2 \times iS^1$ where $S^1 = \partial B_1 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1\}$. Condition (2.1.1) is equal to

$$-\frac{1}{2}\Re(w_1^2 + w_2^2) + \frac{1}{2}(|w_1|^2 + |w_2|^2) = \Im^2 w_1 + \Im^2 w_2 \geq 0$$

valid for all $w \in \mathrm{T}_p(\partial T_{B_1})$.
Setting $\zeta_j = x_j + iy_j$ for $j = 1, 2$ one obtains

$$\frac{\partial \Phi_1(\zeta, \bar{\zeta})}{\partial \zeta_j} = \frac{1}{2}\left(\frac{\partial \Phi_1(x, y)}{\partial x_j} - i\frac{\partial \Phi_1(x, y)}{\partial y_j}\right) = -iy_j$$

and thus,

$$\frac{\partial \Phi_1(x, y)}{\partial x_j} = 0 \text{ and } \frac{\partial \Phi_1(x, y)}{\partial y_j} = 2y_j$$
$$\frac{\partial^2 \Phi_1(x, y)}{\partial x_j^2} = \frac{\partial^2 \Phi_1(x, y)}{\partial x_j y_i} = 0 \text{ and } \frac{\partial^2 \Phi_1(x, y)}{\partial y_j^2} = 2$$

An essential characteristic of the domains T_{B_1}, T_{B_2} is that they do not contain any complex line. In particular, if $\epsilon : \alpha\zeta_1 + \beta\zeta_2 = 1$ is a complex line then for $\beta \neq 0$ we have that $\zeta_2 = \frac{1}{\beta} - \lambda\zeta_1$ for $\lambda = \frac{\alpha}{\beta}$. Thus ϵ cannot lie entirely in T_{B_1} considering the fact that the values of $\Re\zeta_1$ and $\Re\zeta_2$ can be arbitrarily large although $\Im\zeta_1, \Im\zeta_2$ are restricted inside the unit disc. Similarly, one derives that T_{B_2} can not contain an entire complex line.

Our purpose is to describe the generalized dual complement of the tubes introduced in (2.2.1). Since $\nabla_\zeta \Phi_i(\zeta,\bar{\zeta}) \neq 0$ is valid for any $\zeta \in \partial T_{B_i}$ we may consider the complex hyperplane passing through $\zeta \in \partial T_{B_i}$ without intersecting T_{B_i}. This hyperplane is given by the equation

$$\big\langle \nabla_\zeta \Phi_i(\zeta,\bar{\zeta}), \zeta - z \big\rangle \neq 0$$

for every $z \in T_{B_i}$ or equivalently by the equation

$$\Big\langle \frac{\nabla_\zeta \Phi_i(\zeta,\bar{\zeta})}{\big\langle \nabla_\zeta \Phi_i(\zeta,\bar{\zeta}), \zeta \big\rangle}, z \Big\rangle = \frac{\partial_{\zeta_1} \Phi_i(\zeta,\bar{\zeta})}{\big\langle \nabla_\zeta \Phi_i(\zeta,\bar{\zeta}), \zeta \big\rangle} z_1 + \frac{\partial_{\zeta_2} \Phi_i(\zeta,\bar{\zeta})}{\big\langle \nabla_\zeta \Phi_i(\zeta,\bar{\zeta}), \zeta \big\rangle} z_2 \neq 1$$

Conclude that

$$w = \frac{\nabla_\zeta \Phi_i(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_i(\zeta,\bar{\zeta}), \zeta >} = \left(\frac{\partial_{\zeta_1} \Phi_i(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_i(\zeta,\bar{\zeta}), \zeta >}, \frac{\partial_{\zeta_2} \Phi_i(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_i(\zeta,\bar{\zeta}), \zeta >} \right) \in T^*_{B_1}$$

for $\zeta \in \mathbb{C}^2 \setminus T_{B_i}, i = 1, 2$.

In order to define the generalized dual complement of the tubular domains $T_{B_i}, i = 1, 2$ we introduce increasing families of unbounded tubular domains $T_{B_{i,r}}$ for $i = 1, 2$ and $1 \leq r < \infty$. Precisely, we consider the tubular sets

$$\begin{aligned} T_{B_{1,r}} &= \left\{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 : \Phi_{1,r}(\zeta,\bar{\zeta}) = \left(\frac{\zeta_1 - \bar{\zeta}_1}{2i} \right)^2 + \left(\frac{\zeta_2 - \bar{\zeta}_2}{2i} \right)^2 - r < 0 \right\} \\ T_{B_{2,r}} &= \left\{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 : \Phi_{2,r}(\zeta,\bar{\zeta}) = \left(\frac{\zeta_1 + \bar{\zeta}_1}{2} \right)^2 + \left(\frac{\zeta_2 + \bar{\zeta}_2}{2} \right)^2 - r < 0 \right\} \end{aligned}$$

It is straightforward to verify that $\Phi_{i,r}(\zeta,\bar{\zeta})$ are $\mathcal{C}^2$ functions for every $i = 1, 2$ and every $r \in [1, \infty)$. Particularly, for $1 \leq i, j \leq 2$

$$\frac{\partial \Phi_{i,r}(\zeta,\bar{\zeta})}{\partial \zeta_j} = \frac{\partial \Phi_i(\zeta,\bar{\zeta})}{\partial \zeta_j} \text{ and } \frac{\partial^2 \Phi_{i,r}(\zeta,\bar{\zeta})}{\partial \zeta_i \partial \zeta_j} = \frac{\partial^2 \Phi_i(\zeta,\bar{\zeta})}{\partial \zeta_i \partial \zeta_j}$$

Furthermore, $\nabla\Phi_{i,r}(\zeta,\bar{\zeta}) \neq 0$ for all $\zeta \in \partial T_{B_{i,r}}$ and $i = 1, 2$.

Observe that for every $\zeta \in \mathbb{C}^2 \setminus T_{B_1}$ there exists unique $1 \leq r < \infty$ so that $\zeta \in \partial T_{B_{1,r}}$. The tubes $T_{B_{i,r}}$ are all star shaped, linearly convex sets so that $T_{B_i} \subseteq T_{B_{i,r}} \subset T_{B_{i,r'}}$ whenever $r, r' \in [1,\infty)$, $r \leq r'$ and $i = 1, 2$. Thus, $T^*_{B_{i,r'}} \subset T^*_{B_{i,r}}$ when $i = 1, 2$ and $r, r' \in [1, \infty)$ for $r \leq r'$.

Following ([1]) we define the generalized dual complements of the tubular domains $T_{B_i}, i = 1, 2$.

Lemma 2.2.1 *Let T_{B_i}, $i = 1, 2$ be the tubes defined by* (2.2.1). *Then*

$$\begin{aligned}
T^*_{B_1} &= \{(\omega_1,\omega_2) \in \mathbb{C}^2 : \omega_i = \frac{\partial_{\zeta_i}\Phi_{1,r}(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}), \zeta >},\ i = 1,2,\ 1 \leq r < \infty\} \\
T^*_{B_2} &= \{(\omega_1,\omega_2) \in \mathbb{C}^2 : \omega_i = \frac{\partial_{\zeta_i}\Phi_{2,r}(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_{2,r}(\zeta,\bar{\zeta}), \zeta >},\ i = 1,2,\ 1 \leq r < \infty\},
\end{aligned}$$

Furthermore $T^*_{B_i} \subset \bar{B}(0,1) \subset \mathbb{C}^2$, $i = 1, 2$.

Proof: First we observe that $(0,0) \in T^*_{B_i}$, $i = 1, 2$. For the nontrivial elements of the generalized dual sets it is enough to prove only the first claim, the proof of the second follows along the same lines. For every point $\zeta \in T^c_{B_1}$ there exists unique $1 \leq r < \infty$ so that $\zeta \in \partial T_{B_{1,r}}$. Fix $1 \leq r < \infty$. Since $\nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}) \neq (0,0)$ for every $\zeta \in \partial T_{B_{1,r}}$, there exists complex analytic hyperplane passing through $\zeta \in \partial T_{B_{1,r}}$ and not intersecting $T_{B_{1,r}}$. This hyperplane is given by the equation $< \nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}), \zeta - z > \neq 0$ for every $z \in T_{B_{1,r}}$ or equivalently, for the same z, by the equation

$$< \frac{\nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}), \zeta >}, z >= \frac{\partial_{\zeta_1}\Phi_{1,r}(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}), \zeta >} z_1 + \frac{\partial_{\zeta_2}\Phi_{1,r}(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}), \zeta >} z_2 \neq 1.$$

Hence

$$\left(\frac{\partial_{\zeta_1}\Phi_{1,r}(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}), \zeta >}, \frac{\partial_{\zeta_2}\Phi_{1,r}}{< \nabla_\zeta \Phi_{1,r}, \zeta >}\right) \in T^*_{B_{1,r}},$$

whenever $\zeta \in \partial T_{B_{1,r}}$. Thus, if we put $y_i = \frac{w_i - \bar{w}_i}{2i}$, $x_i = \frac{w_i + \bar{w}_i}{2}$, $i = 1, 2$, with $w \in \partial T_{B_1}$, then at $\zeta = r \cdot w$ we have

$$\begin{aligned}
\omega_1 &= \frac{\partial_{\zeta_1}\Phi_{1,r}(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}),\zeta >} = \frac{r^3(y_1(x_1y_1+x_2y_2) - iy_1(y_1^2+y_2^2))}{r^4((y_1^2+y_2^2)^2+(x_1y_1+x_2y_2)^2)} \\
&= \frac{1}{r}\frac{\partial_{w_1}\Phi_1(w,\bar{w})\overline{< \nabla_w \Phi_1(w,\bar{w}),w >}}{| < \nabla_w \Phi_1(w,\bar{w}),w > |^2} \qquad (2.2.4)
\end{aligned}$$

and

$$\begin{aligned}
\omega_2 &= \frac{\partial_{\zeta_2}\Phi_{1,r}(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_{1,r}(\zeta,\bar{\zeta}),\zeta >} = \frac{r^3(y_2(x_1y_1+x_2y_2) - iy_2(y_1^2+y_2^2))}{r^4((y_1^2+y_2^2)^2+(x_1y_1+x_2y_2)^2)} \\
&= \frac{1}{r}\frac{\partial_{w_2}\Phi_1(w,\bar{w})\overline{< \nabla_w \Phi_1(w,\bar{w}),w >}}{| < \nabla_w \Phi_1(w,\bar{w}),w > |^2} \qquad (2.2.5)
\end{aligned}$$

This essentially means that

$$T^*_{B_{1,r}} = \frac{1}{r} T^*_{B_1} \text{ whenever } 1 \leq r < \infty \qquad (2.2.6)$$

Furthermore, elementary computations show that

$$\begin{aligned}
(\Re\omega_1)^2 + (\Re\omega_2)^2 + (\Im\omega_1)^2 + (\Im\omega_2)^2 &= \frac{1}{r^2}\frac{y_1^2+y_2^2}{(y_1^2+y_2^2)^2+(x_1y_1+x_2y_2)^2} \qquad (2.2.7) \\
&= \frac{1}{r^2} - \frac{1}{r^2}\frac{(x_1y_1+x_2y_2)^2}{(y_1^2+y_2^2)^2+(x_1y_1+x_2y_2)^2} \leq \frac{1}{r^2}
\end{aligned}$$

where the equality occurs in the case when $x_1y_1 + x_2y_2 = 0$. Hence the mapping defined by (2.2.4) and (2.2.5) maps the exterior of tubes into the unit ball. We observe that $T^*_{B_1}$ contains a disk. Actually, since $T^*_{B_1}$ is a star compact and the circumference $y_1^2 + y_2^2 = 1$ (when $x_1y_1 + x_2y_2 = 0$) is subset of the generalized dual we have the desired result. Thus $T^*_{B_1} \subset \bar{B}(0,1) \subset \mathbb{C}^2$. Actually

$$T^*_{B_1} \cap \partial B(0,1) = \{(0,0,-iy_1,-iy_2) \in \mathbb{R}^2 \times i\mathbb{R}^2 : y_1^2 + y_2^2 = 1\}$$

Similarly, $T^*_{B_2} \subset \bar{B}(0,1) \subset \mathbb{C}^2$. $\diamondsuit$

Lemma 2.2.2 *The generalized dual $T^*_{B_1}$ cannot contain a ball in its interior.*

Proof : Assume that $B(q,\varrho) \subset (T^*_{B_1})^{int}$ for some $q, \varrho > 0$. Hence $B(q,\varrho) \subset T^*_{B_1}$. Thus,

$(B(q,\varrho))^* \supset (T^*_{B_1})^* = T_{B_1}$ because T_{B_1} is strictly linearly convex. But $(B(q,\varrho))^* = B(q,\frac{1}{\varrho})$. Thus the tube T_{B_1} (unbounded set) is contained in a ball, contradiction. $\diamondsuit$

Since $\langle \nabla_\zeta \Phi_1(\zeta,\bar\zeta),\zeta\rangle \neq 0$ for $\zeta \in \partial T_{B_1}$, one considers the mapping $\sigma : T^c_{B_1} \longrightarrow T^*_{B_1}$

$$\begin{aligned}
\sigma(\zeta,\bar\zeta) &= \frac{\nabla_\zeta \Phi_1(\zeta,\bar\zeta)}{\langle \nabla_\zeta \Phi_1(\zeta,\bar\zeta),\zeta\rangle} = \left(\frac{\partial_{\zeta_1}\Phi_1(\zeta,\bar\zeta)}{< \nabla_\zeta \Phi_1(\zeta,\bar\zeta),\zeta >}, \frac{\partial_{\zeta_2}\Phi_1(\zeta,\bar\zeta)}{< \nabla_\zeta \Phi_1(\zeta,\bar\zeta),\zeta >}\right)\\
&= \left(\frac{\zeta_1-\bar\zeta_1}{(\zeta_1-\bar\zeta_1)\zeta_1+(\zeta_2-\bar\zeta_2)\zeta_2}, \frac{\zeta_2-\bar\zeta_2}{(\zeta_1-\bar\zeta_1)\zeta_1+(\zeta_2-\bar\zeta_2)\zeta_2}\right)
\end{aligned}$$

for every $\zeta = (\zeta_1,\zeta_2) \in T^c_{B_1}$. It is straightforward to see that σ is onto but it is neither $\mathbb{R}-$ linear nor $\mathbb{C}-$ linear (σ is $\mathbb{C}-$ anti-linear). Next, notice that σ is non invertible for any $\zeta \in T^c_{B_1}$. More precisely, we formulate the following.

Lemma 2.2.3 *The real Jacobian of σ is identically zero.*

Proof: One can do the computations directly to verify the claim, however we can have the following short argument. Assume that $J_\sigma(x,y) \neq 0$ for some $\omega = (x,iy) \in T^c_{B_1}$. From the inverse function theorem follows that there exist open neighborhoods $B(\omega,r) \subset T^c_{B_1}$ and $B(\sigma(\omega),\rho) \subset T^*_{B_1}$ for some $r,\rho > 0$ such that $\sigma(B(\omega,r)) \subset B(\sigma(\omega),\rho)$ and the restriction $\sigma_{|_{B(\omega,r)}} : B(\omega,r) \to B(\sigma(\omega),\rho)$ is biholomorphic. Lemma 2.2.2 leads to a contradiction and the desired result follows. $\diamondsuit$

2.3 Holomorphic extension in tubular domains

A function f defined on an open set $U \subset \mathbb{C}^n$ belongs to the space $\mathcal{C}^k(U)$, where k is a non-negative integer, if f is k times continuously differentiable in U. More precisely, $f \in \mathcal{C}^k(U)$ if all of the partial derivatives $\frac{\partial^k f_i}{\partial z_{i_1}^{k_1} \cdots \partial z_{i_l}^{k_l}}$ exist and are continuous, where $1 \leq i_1, \cdots, i_l \leq n, k = (k_1, \cdots, k_l) \in \mathbb{N}^l$ is a non-negative integer and $k = k_1 + \cdots + k_l$. We then write $f \in \mathcal{C}^k(U)$. In the special case of $k = 0$ one obtains the space of continuous functions, $\mathcal{C}^0(U) = \mathcal{C}(U)$. If M is a closed set in $\mathbb{C}^n$ then $f \in \mathcal{C}^k(M)$ when f extends to some neighborhood U of M as a function of class $\mathcal{C}^k(U)$. Similarly, the space $\mathcal{H}(U)$ is consisted of all functions f that are holomorphic on the open set U. Whenever M is a closed set, $\mathcal{H}(M)$ consists of all functions f that are holomorphic in some neighborhood of M. Recall that if $\Omega \subset \mathbb{C}^n$ is a domain then the vector space $\mathcal{H}(\Omega)$ of all holomorphic functions on Ω is equipped with the usual topology of uniform convergence on compact subsets.

For every domain $\Omega \subset \mathbb{C}$ there exists a holomorphic function $f \in \mathcal{H}(\Omega)$ which cannot be holomorphically extended to a strictly larger domain. Take, for instance, $\zeta_0 \in \partial\Omega \subset \mathbb{C}$ and then consider the function $f(z) = \frac{1}{\zeta - \zeta_0}$ which is holomorphic in a neighborhood U of Ω but is not holomorphically extendable at ζ_0. A domain for which this simultaneous extension phenomenon does not occur is called a domain of holomorphy. Recall that $f \in \mathcal{H}(\Omega)$ can be holomorphically extended to a larger domain $\widetilde{\Omega} \supset \Omega$ if there is a function $F \in \mathcal{H}(\widetilde{\Omega})$ whose restriction to Ω coincides with f, i.e., $F_{|\Omega} = f$. We recall the notion of the domain of holomorphy from ([16]).

Definition 2.3.1 *A domain Ω in $\mathbb{C}^n$ is called a domain of holomorphy if the following property holds : There do not exist non-empty open sets Ω_1, Ω_2 with Ω_2 connected, $\Omega_2 \nsubseteq \Omega, \Omega_1 \subseteq \Omega_2 \cap \Omega$, such that for every holomorphic function f on Ω there is a function f_2 holomorphic on Ω_2 such that $f = f_2$ on Ω_1.*

A domain Ω is a domain of holomorphy if every $f \in \mathcal{H}(\Omega)$ cannot be holomorphically extended to a strictly larger domain. In the complex plane $\mathbb{C}$ every open set is a domain of holomorphy (i.e. $f(\zeta) = \frac{1}{\zeta - \zeta_0}$ cannot holomorphically extend at ζ_0). Every linearly convex domain in $\mathbb{C}^n$ is a domain of holomorphy. To be more precise, if $< \zeta - \zeta_0, \zeta_0 >= 0$ is a hyperplane passing through $\zeta_0 \in \partial\Omega$, then the function $f(\zeta) = \frac{1}{<\zeta - \zeta_0, \zeta_0>} \in \mathcal{H}(\Omega)$ cannot be holomorhically extended at ζ_0. Particularly, for convex tube domains the notions domain of holomorphy, Levi pseudoconvex domain and geometrically convex domain are all equivalent ([16]). Any domain of holomorphy can be approximated from the inside by strictly pseudoconvex domains ([4]).

Next, we recall the notion of envelope of holomorphy.

Definition 2.3.2 *Let $\Omega \subset \mathbb{C}^n$ be a domain. The envelope of holomorphy of Ω is a domain E_Ω with the following properties:*

(i) Every holomorphic function in Ω can be extended holomorphically to E_Ω

(ii) For every boundary point $z_0 \in E_\Omega$ there exists a function holomorphic in E_Ω which has no holomorphic extension to a neighborhood of z_0.

Thus, the envelope of holomorphy E_Ω of a domain $\Omega \subset \mathbb{C}^n$ is the largest domain to which all holomorphic functions on Ω may holomorphically extend.
Next, we recall some facts connecting the envelope of holomorphy to convexity.

For arbitrary domains $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$ the inclusions $\Omega_1 \subset E_{\Omega_1}$ and $E_{\Omega_1} \subset E_{\Omega_2}$ are always true. Furthermore, the envelope of holomorphy of any domain is a domain of holomorphy, while condition $E_\Omega = \Omega$ is valid for any domain of holomorphy Ω. Fundamental results of Oka and Cartan show that a necessary and sufficient condition for a domain $\Omega \subset \mathbb{C}^n$ to be a domain of holomorphy is that each function holomorphic on a domain $\Omega' \subset \Omega \subset \mathbb{C}^n$ is the restriction of some function holomorphic on the whole domain Ω. Furthermore, the theorem of Bochner ([16]), for tubular domains $T_B = \mathbb{R}^n \times iB$ states that the notion of the envelope of holomorphy coincides with the one of the convex envelope (convex hull), $conv(T_B)$. Thus, $E_{T_B} = conv(T_B) = T_{conv(B)}$. Recall, at this point, that the convex hull of a set is the smallest convex set containing it and it is defined to be the intersection of all convex sets containing a given set. It is clear that for convex sets one simply has that $conv(\Omega) = \Omega$. Thanks to Caratheodory, the convex hull of a set $\Omega \subset \mathbb{R}^{2n}$ is given by

$$conv(\Omega) = \{\sum_{j=0}^{2n} \lambda_j x_j, \text{ whenever } x_j \in \Omega, \lambda_j \in [0,1], \sum_{j=0}^{2n} \lambda_j = 1\}$$

It is easy to observe that if Ω is open, then so is its convex hull, $conv(\Omega)$. Thus, the smallest convex set containing a tubular domain T_B is exactly the largest set in which all holomorphic functions defined on T_B may holomorphically extend.

It is known that the relation $T^*_{B_1} \cup T^*_{B_2} \subset (T_{B_1} \cap T_{B_2})^*$ always holds. However, for the intersection of the above tubes one has that the generalized dual complement of the topological bi-disk is a compact in $\mathbb{C}^2$, which is similar to the closed hyper-cone.

Lemma 2.3.1 *Let $\mathcal{U}_\tau = T_{B_1} \cap T_{B_2}$ be the topological bi-disk and $\mathcal{G}_2 \subset \mathbb{C}^2$ be the compact* *defined as the closure of the convex hull of star compact $T^*_{B_1} \cup T^*_{B_2}$, that is*

$$\mathcal{G}_2 = \{(1-\lambda)w + \lambda u,\ \lambda \in [0,1],\ w, u \in T^*_{B_1} \cup T^*_{B_2}\}. \tag{2.3.1}$$

Then $\mathcal{G}_2 = \{(w_1, w_2) \in \mathbb{R}^2 \times i\mathbb{R}^2 : \|w_1\|_2 + \|w_2\|_2 \leq 1,\}$, where $\|\cdot\|_2$ is the standard Euclidean norm, is compact of holomorphy and $\mathcal{U}^*_\tau = \mathcal{G}_2$.

Proof: First we observe that $\mu T^*_{B_i} \subset T^*_{B_i}$, $\mu \in [0,1]$, $i = 1,2$, because the compacts $T^*_{B_i}$, $i = 1,2$ are star. Thus the set $\mathcal{G}_2$ is convex and hence linearly convex compact set. Thus $\mathcal{G}_2 = E_{(T^*_{B_1} \cup T^*_{B_2})}$ is a hull of holomoprhy. Furthermore, if $(z_1, z_2) = (x_1, x_2, iy_1, iy_2) \in \mathcal{U}_\tau$ and $(w, u) \in \mathcal{U}^*_\tau$ then $< (z_1, z_2), (w, u) > \neq 1$ implies that either $|x_1 \Re w + x_2 \Re u - y_1 \Im w - y_2 \Im u| > 1$ or $|x_1 \Re w + x_2 \Re u - y_1 \Im w - y_2 \Im u| \leq 1$. The first case is excluded, because $\mathcal{U}^*_\tau$ is star with respect to the origin. Rewriting the above equation in the standard norm as $\|(x_1, x_2)\|_2 \cdot \|(\Re w, \Re u)\|_2 \cos\theta - \|(y_1, y_2)\|_2 \cdot \|(\Im w, \Im u)\|_2 \cos\alpha \neq 0$ and assuming that $\|(\Re w, \Re u)\|_2 \cos\theta \neq 0$ we observe that the line $\|(x_1, x_2)\|_2 = \frac{1}{\|(\Re w, \Re u)\|_2 \cos\theta} - \frac{\|(\Im w, \Im u)\|_2 \cos\alpha}{\|(\Re w, \Re u)\|_2 \cos\theta} \|(y_1, y_2)\|_2$ does not intersect $\mathcal{U}_\tau$ for any value $(z_1, z_2) \in \mathcal{U}_\tau$. In particular for $z_2 = 0$ we deduce that $|\frac{1}{\|(\Re w, \Re u)\|_2 \cos\theta}| > 1$. This implies that $(\Re w, \Re u, 0, 0) \in \mathcal{G}_2$ for particular choice of θ . Similar argument shows that $(0, 0, -\Im w, -\Im u) \in \mathcal{G}_2$. Convexity of $\mathcal{G}_2$ implies that $\mathcal{U}^*_\tau \subset \mathcal{G}_2$. On the other hand, if $(w, u) \in \mathcal{G}_2$, then $|(z_1 w + z_2 u)| \leq |z_1 w| + |z_2 u| \leq |w| + |u| \leq 1$. Thus $\mathcal{G}_2 \subset \mathcal{U}^*_\tau$. This concludes the proof of the lemma. $\diamondsuit$

Furthermore, we show that the generalized dual complement of the topological bi-disk is the envelope of holomorphy of the star compact $T^*_{B_1} \cup T^*_{B_2}$.

Lemma 2.3.2 *Let $\mathcal{U}_\tau = T_{B_1} \cap T_{B_2}$ be the topological bi-disk in $\mathbb{C}^2$. Then*

$$\mathcal{G}_2 = \mathcal{U}^*_\tau = (T_{B_1} \cap T_{B_2})^* = E_{(T^*_{B_1} \cup T^*_{B_2})}$$

where $E_{T^*_{B_1} \cup T^*_{B_2}}$ denotes the envelope of holomorphy of the normal compact $T^*_{B_1} \cup T^*_{B_2}$ and $\mathcal{G}_2$ is defined in (2.3.1)*.*

Proof: Observe that the compact $T^*_{B_1} \cup T^*_{B_2} \subset \mathbb{C}^2$ is also star with respect to the origin. Thus $T^*_{B_1} \cup T^*_{B_2} \subset \mathbb{C}^2$ has an envelope of holomorphy $E_{(T^*_{B_1} \cup T^*_{B_2})}$. Furthermore,

$T^*_{B_1} \cup T^*_{B_2} \subset (T_{B_1} \cap T_{B_2})^* = \mathcal{G}_2$ where $\mathcal{G}_2$, is a compact of holomorphy. Consequently, $E_{(T^*_{B_1} \cup T^*_{B_2})} \subset \mathcal{G}_2$. At the same time, the envelope of holomorphy, $E_{(T^*_{B_1} \cup T^*_{B_2})}$ is by definition the maximal set in which all holomorphic functions in $T^*_{B_1} \cup T^*_{B_2}$ can holomorphically extend. The converse inclusion follows and $E_{(T^*_{B_1} \cup T^*_{B_2})} = \mathcal{G}_2.\diamondsuit$

2.4 Tubular domains of type one

We begin this section by recalling the notion of an open cone.

Definition 2.4.1 *A non-empty set $\Gamma \subset \mathbb{R}^n$ satisfying*

(i) $0 \notin \Gamma$

(ii) *For every $x_1, x_2 \in \Gamma$ and $\lambda_1, \lambda_2 > 0$ then $\lambda_1 x_1 + \lambda_2 x_2 \in \Gamma$*

is an open cone in $\mathbb{R}^n$.

A closed cone is an open cone's closure. We define the dual cone Γ^* to be the set

$$\Gamma^* = \{y \in \mathbb{R}^n : y_1 t_1 + \cdots + y_n t_n \geq 0,\ t \in \Gamma\}$$

Obviously, whenever Γ is open then Γ^* is closed. If $n = 1$ is the case then the open cones are just the half-lines $\Gamma_{11} = \{y \in \mathbb{R} : y > 0\}$ and $\Gamma_{12} = \{y \in \mathbb{R} : y < 0\}$. For $n = 2$ open cones are angular regions of two rays meeting at the origin and forming an angle less or equal to π.

Consider the tubular domain T_B having a base containing a cone. Notice that B may contain cones of different dimensions. Assume, for instance, that $T_B \subset \mathbb{R}^2 \times i\mathbb{R}^2$. Then its base, B, may contain one-dimensional cones, two-dimensional cones, or both. It is natural therefore to define the type of the tube T_B to be the dimension of the maximal cone for which a displacement lies in B([13], [14]), i.e. the type is equal to 2. With the term displacement of B we mean the parallel transfer of all vectors lying in B. The simplest tubular domain T_B having type the dimension of a cone Γ contained in B is taken when $B = \Gamma$.

We restrict our attention to particular tubular domains. In order to realize T_{B_1} as an intersection of tubular domains whose base contain a cone, we introduce the tubular domains with bases contained in iR^2. We define

$$\begin{aligned} S_H^- &= \{(iy_1, iy_2) \in i\mathbb{R}^2 : y_1^2 + y_2^2 \le 1,\ -1 \le y_1 < 0\} \\ &\cup \{(iy_1, iy_2) \in i\mathbb{R}^2 : -1 \le y_2 \le 1,\ y_1 \ge 0\} \\ S_H^+ &= \{(iy_1, iy_2) \in i\mathbb{R}^2 : y_1^2 + y_2^2 \le 1,\ 0 < y_1 \le 1\} \\ &\cup \{(iy_1, iy_2) \in i\mathbb{R}^2 : -1 \le y_2 \le 1,\ y_1 \le 0\} \end{aligned} \tag{2.4.1}$$

and then

$$\begin{aligned} T_{(S_H^-)^{int}} &= \mathbb{R}^2 \times (S_H^-)^{int} \\ T_{(S_H^+)^{int}} &= \mathbb{R}^2 \times (S_H^+)^{int} \end{aligned} \tag{2.4.2}$$

Thus,

$$T_{B_1} = T_{(S_H^-)^{int}} \cap T_{(S_H^+)^{int}} = \mathbb{R}^2 \times \{(iy_1, iy_2) \in i\mathbb{R}^2 : y_1^2 + y_2^2 < 1\} \tag{2.4.3}$$

The tubular domains (2.4.2) with bases the convex open sets $(S_H^-)^{int}$, $(S_H^+)^{int}$ (interiors of the sets S_H^-, S_H^+ correspondingly) are open, convex sets and hence linearly convex. Thus both tubular domains defined by (2.4.2) are hulls of holomorphy. Furthermore, notice that the bases $(S_H^-)^{int}, (S_H^+)^{int}$ of the half tubes $T_{(S_H^-)^{int}}, T_{(S_H^+)^{int}}$ have been selected in order not to contain any entire straight line, but they both contain a cone.

One observes that both $T_{(S_H^-)^{int}}$ and $T_{(S_H^+)^{int}}$ contain only one-dimensional cone, that is $\{(y_1, 0),\ y_1 \ge 0\} \subset \mathbb{R}^2$ for the first and $\{(y_1, 0),\ y_1 \le 0\} \subset \mathbb{R}^2$ for the latter. Obviously, they cannot contain angular regions of two rays meeting at the origin and forming an angle less or equal to π. Following ([14], [13]), we observe that the type of $T_{(S_H^-)^{int}}$ is the dimension of the cone $\{(y_1, 0),\ y_1 \ge 0\} \subset \mathbb{R}^2$ and for $T_{(S_H^+)^{int}}$ its type is the dimension of the cone $\{(y_1, 0),\ y_1 \le 0\} \subset \mathbb{R}^2$. According to ([14], [13]) we will say that $T_{(S_H^-)^{int}}$ and $T_{(S_H^+)^{int}}$ are tubular domains of type one.

It is straightforward from (2.4.1) that the semi-tubes are defined via smooth defining functions. One corresponding to the strictly convex part of them and another for the one of the strip. It follows from (2.4.1) and (2.4.2) that

$$\begin{aligned} T_{(S_H^-)^{int}} &= \{(x_1,x_2,iy_1,iy_2)\in\mathbb{R}^2\times i\mathbb{R}^2:\ \Phi_1(\zeta,\bar\zeta)=\Phi_1^-(\zeta,\bar\zeta),\text{when }-1\le y_1<0\} \\ &\cup\ \{(x_1,x_2,iy_1,iy_2)\in\mathbb{R}^2\times i\mathbb{R}^2:\Phi_1^-(\zeta,\bar\zeta)=(\frac{\zeta_2-\bar\zeta_2}{2i})^2-1,\ \text{when } y_1\ge 0\} \\ T_{(S_H^+)^{int}} &= \{(x_1,x_2,iy_1,iy_2)\in\mathbb{R}^2\times i\mathbb{R}^2:\ \Phi_1(\zeta,\bar\zeta)=\Phi_1^+(\zeta,\bar\zeta),\text{when } 0<y_1\le 1,\ \} \\ &\cup\ \{(x_1,x_2,iy_1,iy_2)\in\mathbb{R}^2\times i\mathbb{R}^2:\Phi_1^+(\zeta,\bar\zeta)=(\frac{\zeta_2-\bar\zeta_2}{2i})^2-1,\ \text{when } y_1\le 0\} \end{aligned} \tag{2.4.4}$$

The tubes defined equivalently by (2.4.2) and (2.4.4) are piece-wise smooth, in the sense that each one of their boundaries $\partial T_{(S_H^-)^{int}}$ and $\partial T_{(S_H^+)^{int}}$ is the union of disjoint, smooth hyper-surfaces. Namely, the connected part $\mathbb{R}^2\times iS^-$ and the part consisting of two connected components $\mathbb{R}^2\times i\{(y_1,y_2)\in\mathbb{R}^2 : y_1>0,\ y_2=1\}$ and $\mathbb{R}^2\times i\{(y_1,y_2)\in\mathbb{R}^2 : y_1>0,\ y_2=-1\}$ where S^- denotes the left hand-side half unit circle. Similarly, the boundary of the tube $T_{(S_H^+)^{int}}$ is the union of the connected part $\mathbb{R}^2\times iS^+$ and the part consisting of two connected components $\mathbb{R}^2\times i\{(y_1,y_2)\in\mathbb{R}^2 : y_1<0,\ y_2=1\}$ and $\mathbb{R}^2\times i\{(y_1,y_2)\in\mathbb{R}^2 : y_1>0,\ y_2=-1\}$, where S^+ denotes the right hand-side half unit circle. At every point $\zeta\in\partial T_{(S_H^-)^{int}}$ ($\partial T_{(S_H^+)^{int}}$ respectively) of strict convexity there is a unique analytic hyperplane (complex line) passing through ζ without intersecting the interior of $T_{(S_H^-)^{int}}(T_{(S_H^+)^{int}}$ respectively). Furthermore, the existence of (algebraic) tangent line at the non-smoothness points $\mathbb{R}^2\times i\{(0,\pm1)\}$ of the boundary $\partial T_{(S_H^-)^{int}}$ is guaranteed by the convexity of the tubes (2.4.4).

Observe that for every point of strict convexity $(\zeta_1,\zeta_2)\in T_{(S_H^-)^{int}}$, that is $-1<\Im\zeta_1<0$, the defining function of the semi-tube $T_{(S_H^-)^{int}}$ coincides with the defining function of T_{B_1} (2.4.4). Hence, the partial derivatives of first and second order of Φ_1^- coincide with the corresponding partial derivatives of Φ_1 as they were calculated in (2.2.4). Actually,

$$\frac{\partial\Phi_1^-(\zeta,\bar\zeta)}{\partial\zeta_j} = \frac{2}{(2i)^2}(\zeta_j-\bar\zeta_j)=-\frac{\partial\Phi_1^-(\zeta,\bar\zeta)}{\partial\bar\zeta_j}$$

and

$$\frac{\partial^2 \Phi_1^-(\zeta,\bar{\zeta})}{\partial \zeta_i \partial \zeta_j} = -\frac{\partial^2 \Phi_1^-(\zeta,\bar{\zeta})}{\partial \zeta_i \partial \bar{\zeta_j}} = \begin{cases} \frac{2}{(2i)^2}, & \text{i=j;} \\ 0, & \text{i} \neq \text{j} \end{cases}$$

whenever $-1 < \Im\zeta_1 < 0$ and $j = 1, 2$. However, that is not the case for $(\zeta_1, \zeta_2) \in T_{(S_H^-)^{int}}$ with $\Im\zeta_1 > 0$. Since $\Phi_1^-(\zeta,\bar{\zeta}) = \left(\frac{\zeta_2 - \bar{\zeta}_2}{2i}\right)^2 - 1$ for $\Im\zeta_1 > 0$ it is clear that all partial derivatives with respect to ζ_1 or $\bar{\zeta_1}$ are equal to zero on these points. Furthermore,

$$\begin{aligned}
\frac{\partial \Phi_1^-(\zeta,\bar{\zeta})}{\partial \zeta_2} &= \frac{2}{(2i)^2}(\zeta_2 - \bar{\zeta_2}) = -\frac{\partial \Phi_1^-(\zeta,\bar{\zeta})}{\partial \bar{\zeta_2}} \\
\frac{\partial^2 \Phi_1^-(\zeta,\bar{\zeta})}{\partial \zeta_2^2} &= \frac{2}{(2i)^2} = \frac{\partial^2 \Phi_1^-(\zeta,\bar{\zeta})}{\partial \bar{\zeta}_2^2} \\
\frac{\partial^2 \Phi_1^-(\zeta,\bar{\zeta})}{\partial \zeta_2 \partial \bar{\zeta_2}} &= -\frac{2}{(2i)^2} = \frac{\partial^2 \Phi_1^-(\zeta,\bar{\zeta})}{\partial \bar{\zeta_2} \partial \zeta_2}
\end{aligned}$$

whenever $\Im\zeta_1 > 0$.

Rewriting in real variables, taking into account that $\Phi_1^-(x_1, x_2, y_1, y_2) = y_1^2 + y_2^2 - 1$ whenever $-1 < y_1 < 0$ and $\Phi_1^-(x_1, x_2, y_1, y_2) = y_2^2 - 1$ for $y_1 \geq 0$, one obtains

$$\begin{aligned}
\frac{\partial \Phi_1^-(x,y)}{\partial x_i} &= \frac{\partial \Phi_1^-(\zeta,\bar{\zeta})}{\partial \zeta_i} + \frac{\partial \Phi_1^-(\zeta,\bar{\zeta})}{\partial \bar{\zeta_i}} = 0 \text{ for } i = 1, 2 \\
\frac{\partial \Phi_1^-(x,y)}{\partial y_1} &= \frac{1}{i}\left(\frac{\partial \Phi_1^-(\zeta,\bar{\zeta})}{\partial \bar{\zeta_1}} - \frac{\partial \Phi_1^-(\zeta,\bar{\zeta})}{\partial \zeta_1}\right) = \begin{cases} 2y_1, -1 < y_1 < 0; \\ 0, \ \ y_1 > 0. \end{cases} \\
\frac{\partial \Phi_1^-(x,y)}{\partial y_2} &= \frac{1}{i}\left(\frac{\partial \Phi_1^-(\zeta,\bar{\zeta})}{\partial \bar{\zeta_2}} - \frac{\partial \Phi_1^-(\zeta,\bar{\zeta})}{\partial \zeta_2}\right) = 2y_2
\end{aligned}$$

It is clear, now, that all partial derivatives of second order are equal to zero except from the cases $\frac{\partial^2 \Phi_1^-(x,y)}{\partial y_1^2}$ for $-1 < y_1 < 0$ and $\frac{\partial^2 \Phi_1^-(x,y)}{\partial y_2^2}$ for $y_1 \in \mathbb{R}$ that are both equal to 2.

Particularly,

$$\begin{aligned}
\frac{\partial^2\Phi_1^-(x,y)}{\partial x_i^2} &= \frac{\partial^2\Phi_1^-(x,y)}{\partial x_i y_j} = \frac{\partial^2\Phi_1^-(x,y)}{\partial y_i y_j} = 0 \text{ for } i,j=1,2 \\
\frac{\partial^2\Phi_1^-(x,y)}{\partial y_j^2} &= \begin{cases} 2, & \text{j=1;} \\ 2, & \text{j=2.} \end{cases} \text{ whenever } -1 < y_1 < 0 \\
\frac{\partial^2\Phi_1^-(x,y)}{\partial y_j^2} &= \begin{cases} 0, & \text{j=1;} \\ 2, & \text{j=2.} \end{cases} \text{ whenever } y_1 > 0
\end{aligned}$$

It is straightforward that Φ_1^- is smooth except from the points $(x_1, x_2, 0, \pm 1) \in \partial T_{(S_H^-)^{int}}$.

Assume $\zeta \in \mathbb{C}^2 \setminus T_{(S_H^-)^{int}}$, i.e. $\Phi_1^-(\zeta,\bar{\zeta}) > 0$. The complex gradient of $\Phi_1^-(\zeta,\bar{\zeta})$ is the vector $\nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}) = \left(\frac{\partial\Phi_1^-}{\partial\zeta_1}, \frac{\partial\Phi_1^-}{\partial\zeta_2}\right)$. For $\zeta \in \mathbb{R}^2 \times iS^-$ we obtain that

$$\nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}) = -\frac{1}{2}\left((\zeta_1 - \bar{\zeta}_1), (\zeta_2 - \bar{\zeta}_2)\right)$$

As expected $\nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}) \neq 0$ is valid for any ζ in the strictly convex part of the boundary, $\mathbb{R}^2 \times iS^-$. Actually, every $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2 \times iS^-$ satisfies

$$\left(\frac{\zeta_1 - \bar{\zeta}_1}{2i}\right)^2 + \left(\frac{\zeta_2 - \bar{\zeta}_2}{2i}\right)^2 = 1 \text{ or equivalently } (\zeta_1 - \bar{\zeta}_1)^2 + (\zeta_2 - \bar{\zeta}_2)^2 = -4$$

Since $\nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}) \neq 0$ is valid for any $\zeta \in \mathbb{R}^2 \times iS^-$ we consider the analytic hyperplane passing through $\zeta \in \mathbb{R}^2 \times iS^-$ without intersecting $T_{(S_H^-)^{int}}$. This hyperplane is given by the equation

$$\left\langle \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - z \right\rangle \neq 0$$

for every $z \in T_{(S_H^-)^{int}}$ or equivalently by the equation

$$\left\langle \frac{\nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta})}{\left\langle \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta \right\rangle}, z \right\rangle = \frac{\partial_{\zeta_1}\Phi_1^-(\zeta,\bar{\zeta})}{\left\langle \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta \right\rangle} z_1 + \frac{\partial_{\zeta_2}\Phi_1^-(\zeta,\bar{\zeta})}{\left\langle \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta \right\rangle} z_2 \neq 1$$

In order to describe the generalized dual complements of the semi-tubes as they were introduced in (2.4.4) we consider more "narrow" tubes. More precisely, we consider the tubular domains :

$$\begin{aligned}
T_{(S^-_{H_r})^{int}} &= \{(x_1,x_2,iy_1,iy_2)\in\mathbb{R}^2\times i\mathbb{R}^2:\ \Phi^-_{1,r}(\zeta,\bar\zeta)=\left(\frac{\zeta_1-\bar\zeta_1}{2i}\right)^2+\left(\frac{\zeta_2-\bar\zeta_2}{2i}\right)^2-r,\\
&\quad \text{when } -r\le y_1<0,\ \Phi^-_{1,r}(\zeta,\bar\zeta)=(\frac{\zeta_2-\bar\zeta_2}{2i})^2-r,\ \text{when } y_1\ge 0\}\\
T_{(S^+_{H_r})^{int}} &= \{(x_1,x_2,iy_1,iy_2)\in\mathbb{R}^2\times i\mathbb{R}^2:\ \Phi^+_{1,r}(\zeta,\bar\zeta)=\left(\frac{\zeta_1-\bar\zeta_1}{2i}\right)^2+\left(\frac{\zeta_2-\bar\zeta_2}{2i}\right)^2-r,\\
&\quad \text{when } 0<y_1\le r,\ \Phi^+_{1,r}(\zeta,\bar\zeta)=(\frac{\zeta_2-\bar\zeta_2}{2i})^2-r,\ \text{when } y_1\le 0\} \qquad (2.4.5)
\end{aligned}$$

Observe that $\Phi^-_{1,r}(\zeta,\bar\zeta)$ and $\Phi^+_{1,r}(\zeta,\bar\zeta)$ are $\mathcal{C}^1$ functions, for every $r\in[1,\infty)$, except from the points $\mathbb{R}^2\times i\{(0,\pm1)\}$ which are points of non-smoothness. Particularly,

$$\begin{aligned}
\nabla_\zeta\Phi^-_{1,r}(\zeta,\bar\zeta) &= \nabla_\zeta\Phi^-_1(\zeta,\bar\zeta) \text{ and}\\
\frac{\partial^2\Phi^-_{1,r}(\zeta,\bar\zeta)}{\partial\zeta_i\partial\zeta_j} &= \frac{\partial^2\Phi^-_1(\zeta,\bar\zeta)}{\partial\zeta_i\partial\zeta_j} \text{ for } i,j=1,2\\
\frac{\partial^2\Phi^-_{1,r}(\zeta,\bar\zeta)}{\partial\zeta_i\partial\bar\zeta_j} &= \frac{\partial^2\Phi^-_1(\zeta,\bar\zeta)}{\partial\zeta_i\partial\bar\zeta_j} \text{ for } i,j=1,2
\end{aligned}$$

Thus, $\nabla\Phi^-_{1,r}(\zeta,\bar\zeta)\neq 0$ for $\zeta\in\partial T_{(S^-_{H_r})^{int}}\setminus\{\mathbb{R}^2\times i(0,\pm1)\}$.

2.5 Generalized dual complement of the semi-tubes

The tubes $T_{S^-_H}$ and $T_{S^+_H}$ were suitably chosen in order to have a convex, unbounded base containing a cone, whose intersection is T_{B_1}, in order to obtain the fact that $T^*_{B_1}$ is the envelope of holomorphy of the union of the compacts $T^*_{S^-_H}$ and $T^*_{S^+_H}$.
Next, we turn to the precise description of the generalized dual of the domains (2.4.2), by using the method of lemma (2.2.1).

Lemma 2.5.1 *Let $T_{(S^-_H)^{int}}$, $T_{(S^+_H)^{int}}$ be the unbounded domains in $\mathbb{C}^2$ defined by* (2.4.2).

Then $T^*_{(S_H^-)^{int}} = \overline{\mathcal{V}_2^-}$, $T^*_{(S_H^+)^{int}} = \overline{\mathcal{V}_2^+}$, *where* $(x, iy) \in \partial T_{(S_H^-)^{int}}$ *or* $\partial T_{(S_H^+)^{int}}$ *correspondingly*

$$\begin{aligned}
\mathcal{V}_2^- &= \{(\omega_1,\omega_2)\in\mathbb{C}^2 : \omega_i = \frac{r^3\left(y_i(x_1y_1+x_2y_2) - iy_i(y_1^2+y_2^2)\right)}{r^4\left((y_1^2+y_2^2)^2 + (x_1y_1+x_2y_2)^2\right)}, \\
&\quad (x,iy)\in\mathbb{R}^2\times i\mathbb{R}^2,\ 1\le r<\infty, i=1,2,\ -1<y_1<0\} \\
&\cup\ \{(\omega_1,\omega_2)\in\mathbb{C}^2 : (0,\omega_2) = (0, \frac{1}{r}\frac{x_2y_2^2 - iy_2^3}{y_2^4+(x_2y_2)^2}), \\
&\quad (x,iy)\in\mathbb{R}^2\times i\mathbb{R}^2,\ \textit{whenever}\ y_1>0,\ r\ge 1\} \\
\mathcal{V}_2^+ &= \{(\omega_1,\omega_2)\in\mathbb{C}^2 : \omega_i = \frac{r^3\left(y_i(x_1y_1+x_2y_2) - iy_i(y_1^2+y_2^2)\right)}{r^4\left((y_1^2+y_2^2)^2 + (x_1y_1+x_2y_2)^2\right)}, \\
&\quad (x,iy)\in\mathbb{R}^2\times i\mathbb{R}^2,\ i=1,2,\ 1\le r<\infty,\ 0<y_1<1\} \\
&\cup\ \{(\omega_1,\omega_2)\in\mathbb{C}^2 : (0,\omega_2) = (0, \frac{1}{r}\frac{x_2y_2^2 - iy_2^3}{y_2^4+(x_2y_2)^2}), \\
&\quad (x,iy)\in\mathbb{R}^2\times i\mathbb{R}^2,\ \textit{whenever}\ y_1<0,\ r\ge 1\}
\end{aligned}\tag{2.5.1}$$

The closure $\overline{\mathcal{V}_2^\pm}$ *corresponds to the case when we add to* $\mathcal{V}_2^\pm$ *the slopes of complex tangent hyperplanes to the tubes* $T_{(S_H^-)^{int}}$ *and* $T_{(S_H^+)^{int}}$ *at the points* $(x_1, x_2, 0, \pm i) \in \partial T_{(S_H^\pm)^{int}}$. *Thus, we are leading to* $(\omega_1, \omega_2) = (0, \frac{1}{r}\frac{x_2 \pm i}{1+x_2^2}) \in T^*_{(S_H^\pm)^{int}}$. *These slopes exist because* $\nabla\Phi_1^\pm(x_1, x_2, 0, \pm i)$ *is well defined. Furthermore,*

$$T^*_{(S_H^-)^{int}} \cup T^*_{(S_H^+)^{int}} = T^*_{B_1} \subset \bar{B}(0,1) \subset \mathbb{C}^2. \tag{2.5.2}$$

and

$$\begin{aligned}
&T^*_{(S_H^-)^{int}} \cap \partial B(0,1) = T^*_{(S_H^-)^{int}} \cap \mathbb{R}^2 \times i\overline{S^+} \\
= \ &\{(0,0,-iy_1,-iy_2)\in\mathbb{R}^2\times i\mathbb{R}^2 :\ y_1^2+y_2^2 = 1,\ -1\le y_1\le 0\} \subset \bar{B}(0,1) \\
&T^*_{(S_H^+)^{int}} \cap \partial B(0,1) = T^*_{(S_H^+)^{int}} \cap \mathbb{R}^2 \times i\overline{S^-} \\
= \ &\{(0,0,-iy_1,-iy_2)\in\mathbb{R}^2\times i\mathbb{R}^2 :\ y_1^2+y_2^2 = 1,\ 0\le y_1\le 1\} \subset \bar{B}(0,1)
\end{aligned}\tag{2.5.3}$$

Proof: It is enough to prove the first claim in (2.5.1), since the proof of the second follows along the same lines. Recall that the boundary of the tube $T_{(S_H^-)^{int}}$ is union of a half circle with parallel lines. Following the method of Lemma (2.2.1) we consider the tubular domains $T_{(S_{H_r}^-)^{int}}$ whenever $r \in [1, \infty)$. For every $\zeta \in T^c_{(S_H^-)^{int}}$ there exist $r \in [1, \infty)$ so that $\overline{T}_{(S_H^-)^{int}} \subseteq \overline{T}_{(S_{H_r}^-)^{int}}$. Note that at every point $(\zeta_1, \zeta_2) \in \partial T_{(S_{H_r}^-)^{int}}$ where the boundary is strictly convex the tangent line (hyperplane) is uniquely determined. If $\zeta \in \partial T_{(S^-_{H_r})^{int}}$ is a point of strict convexity then $\Phi^-_1(\zeta,\bar\zeta) = \left(\frac{\zeta_1-\bar\zeta_1}{2i}\right)^2 + \left(\frac{\zeta_2-\bar\zeta_2}{2i}\right)^2 - r = 0$ where $-1 < \Im\zeta_1 < 0$ and $r \in [1,\infty)$. The complex line passing through ζ without intersecting $T_{(S^-_H)^{int}}$ or $T_{(S^-_{H_r})^{int}}$ is described by the equation $\langle \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta - z\rangle \neq 0$ or equivalently,

$$\Big\langle \frac{\nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta)}{\langle \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta\rangle}, z\Big\rangle = \frac{\partial_{\zeta_1}\Phi^-_{1,r}(\zeta,\bar\zeta)}{\langle \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta\rangle} z_1 + \frac{\partial_{\zeta_2}\Phi^-_{1,r}(\zeta,\bar\zeta)}{\langle \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta\rangle} z_2 \neq 1$$

since $< \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta > \neq 0$ for every $\zeta \in \partial T_{(S^-_{H_r})^{int}}$ and $-1 < \Im\zeta_1 < 0$. Thus, for $\zeta \in \partial T_{(S^-_{H_r})^{int}}$ such that $-1 < \Im\zeta_1 < 0$ one has that

$$\left(\frac{\partial_{\zeta_1}\Phi^-_{1,r}(\zeta,\bar\zeta)}{\langle \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta\rangle}, \frac{\partial_{\zeta_2}\Phi^-_{1,r}(\zeta,\bar\zeta)}{\langle \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta\rangle}\right) \in T^*_{(S^-_{H_r})^{int}}$$

Setting $y_i = \frac{w_i-\bar w_i}{2i}$, $x_i = \frac{w_i+\bar w_i}{2}$, $i=1,2$, with $w \in \partial T_{(S^-_H)^{int}}$, at $\zeta = r\cdot w$ we have

$$\begin{aligned}\omega_i &= \frac{\partial_{\zeta_i}\Phi^-_{1,r}(\zeta,\bar\zeta)}{< \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta >} = \frac{r^3(y_i(x_1y_1+x_2y_2) - iy_i(y_1^2+y_2^2))}{r^4((y_1^2+y_2^2)^2 + (x_1y_1+x_2y_2)^2)} \\ &= \frac{1}{r}\frac{\partial_{w_i}\Phi^-_1(w,\bar w)\overline{< \nabla_w \Phi^-_1(w,\bar w), w >}}{| < \nabla_w \Phi^-_1(w,\bar w), w > |^2}\end{aligned}$$

for $i=1,2$, $-1 < \Im\zeta_1 < 0$ and $r \in [1,\infty)$.

Thus, $T^*_{(S^-_{H_r})^{int}} = \frac{1}{r}T^*_{(S^-_H)^{int}}$ for $1 \le r < \infty$. Similarly one obtains that $T^*_{(S^+_{H_r})^{int}} = \frac{1}{r}T^*_{(S^+_H)^{int}}$ for $1 \le r < \infty$.

The elements of the generalized duals at the smooth points of $r\cdot T_{(S^\pm_H)^{int}}$, $1 \le r < \infty$ are described in (2.5.1). For every $\zeta = (\zeta_1,\zeta_2) = (x_1,x_2,iy_1,iy_2) \in \partial T_{(S^-_H)^{int}}$ satisfying $\Phi^-_1(\zeta,\bar\zeta) = y_2^2 - 1 = 0$ the existence of a complex (algebraic) line passing through ζ without intersecting $T_{(S^-_H)^{int}}$ is guaranteed from the convexity of the tube. Actually, if $\zeta = (x_1,x_2,iy_1,\pm i) \in \partial T_{(S^-_H)^{int}}$ whenever $y_1 > 0$ then for $r \ge 1$

$$\begin{aligned}\omega_1 &= \frac{\partial_{\zeta_1}\Phi^-_{1,r}(\zeta,\bar\zeta)}{< \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta >} = 0 \\ \omega_2 &= \frac{\partial_{\zeta_2}\Phi^-_{1,r}(\zeta,\bar\zeta)}{< \nabla_\zeta \Phi^-_{1,r}(\zeta,\bar\zeta), \zeta >} = \frac{r(\zeta_2-\bar\zeta_2)}{r^2(\zeta_2-\bar\zeta_2)\zeta_2} = \frac{1}{r}\frac{1}{x_2 \pm i}\end{aligned}$$

Taking the closure $\overline{\mathcal{V}_2^-}$ by adding to $\mathcal{V}_2^-$ the slopes of complex tangent hyperplanes to the tube $T_{(S_H^-)^{int}}$ at the points $(x_1, x_2, 0, \pm i) \in \partial T_{(S_H^-)^{int}}$ we are leading to $(\omega_1, \omega_2) = (0, \frac{1}{r}\frac{x_2 \pm i}{1+x_2^2}) \in T^*_{(S_H^-)^{int}}$. These slopes exist because $\nabla\Phi_1^-(x_1, x_2, 0, \pm i)$ is well defined.
It is straightforward to observe that if $(\omega_1, \omega_2) \in$ then

$$T^*_{(S_H^-)^{int}} \cup T^*_{(S_H^+)^{int}} = \{(\omega_1, \omega_2) \in \mathbb{C}^2 : \omega_i \quad = \quad \frac{r^3 y_i((x_1y_1 + x_2y_2) - i)}{r^4(1 + (x_1y_1 + x_2y_2)^2)} \text{ for } i = 1, 2,\ r \geq 1\}.$$

Elementary computations as in Lemma (2.2.1) show that if $(\omega_1, \omega_2) \in T^*_{(S_H^-)^{int}} \cup T^*_{(S_H^+)^{int}}$ then

$$(\Re\omega_1)^2 + (\Re\omega_2)^2 + (\Im\omega_1)^2 + (\Im\omega_2)^2 \quad = \quad \frac{1}{r^2}\frac{y_1^2 + y_2^2}{(y_1^2 + y_2^2)^2 + (x_1y_1 + x_2y_2)^2} \leq \frac{1}{r^2}$$

Recall that

$$T^*_{B_1} = \left(T_{(S_H^-)^{int}} \cap T_{(S_H^+)^{int}}\right)^* \supset T^*_{(S_H^-)^{int}} \cup T^*_{(S_H^+)^{int}}$$

valid for any intersection of sets. In order to derive the converse inclusion it is sufficient to see what happens with the points $(y_1, y_2) = (0, \pm 1)$. Now, it remains to show that the compact sets $\overline{\mathcal{V}_2^-}, \overline{\mathcal{V}_2^-}$ are contained in the closure of the unit ball. It is enough to verify that the norm of the regular points of the form $(0, \omega)$ from (2.5.1) have norm smaller or equal to 1. But this indeed is the case because in this case $y_2 = 1$. Thus, the topological closure of the set of regular points cannot generate points in the sets $\overline{\mathcal{V}_2^-}, \overline{\mathcal{V}_2^+}$, whose norms are strictly larger than 1. In order to proof the last claim one observes that $\|(0, \omega_2)\|^2 = \frac{1}{x_2^2+1} < 1$ for every $(0, \omega_2) \in \mathcal{V}_2^-$, since for $y_2 = 1$ (and hence $y_1 = 0$) we deduce that $\|(0, \omega_2)\| = \frac{1}{1+x_2^2}$, for every element $(0, \omega_2) \in T^*_{B_1}$ and (2.5.2) has been proved. In order to prove (2.5.3), it is enough to observe that in (2.5.4) one has equality only when either $r = 1$, $x_1y_1 + x_2y_2 = 0$ or $r = 1$ and $x_2 = 0$. Similarly the other case. Thus the claim follows. $\diamondsuit$

An immediate observation is that the only elements $w \in T^*_{(S_H^-)^{int}}$ satisfying $||w|| = 1$ are of the form $w = (w_1, w_2) = (0 - iy_1, 0 - iy_2)$ whenever $y_1^2 + y_2^2 = 1, -1 \leq y_1 \leq 0$. Furthermore, if $w \in T^*_{(S_H^-)^{int}}$ is such that $||w|| < 1$ then either $w = (0, w_2) \in \mathbb{C}^2$ or w belongs to the generalized dual complement of the circular part of $T_{(S_H^-)^{int}}$.

Observe that equation (2.5.2) follows directly from the linear convexity of the tubular domains $T_{B_1}, T_{(S_H^-)^{int}}$ and $T_{(S_H^+)^{int}}$. Actually,

$$\begin{aligned}\left(T^*_{B_1}\right)^* \subset \left(T^*_{(S_H^-)^{int}} \cup T^*_{(S_H^+)^{int}}\right)^* &= \left(T^*_{(S_H^-)^{int}}\right)^* \cap \left(T^*_{(S_H^+)^{int}}\right)^* = T_{(S_H^-)^{int}} \cap T_{(S_H^+)^{int}} \\ &= T_{B_1} \subset \left(T^*_{B_1}\right)^*\end{aligned}$$

Recall, now, that for a compact $K \subset \mathbb{C}^n$, which is also a star, there exists a sequence of star domains $\{\mathcal{D}_k\}_{k\in\mathbb{N}}$ so that $K = \cap_k \mathcal{D}_k$, $\mathcal{D}_{k+1} \subset\subset \mathcal{D}_k$, $\forall k \in \mathbb{N}$. Since every star domain $\mathcal{D}_k$ is Runge, there exists its holomorphic envelope $E_{\mathcal{D}_k}$, for every $k \in \mathbb{N}$. Thus the compact K, which is a star, is a normal compact (a compact is called normal if it can be approximated from outside by a sequence of compactly contained in each other domains having envelope of holomorphy) ([3]). The envelope of holomorphy E_K of a normal compact K is defined to be the intersection of envelopes of holomoprhy $E_{\mathcal{D}_k}$ of domains $\mathcal{D}_k$ approximating K, that is, $E_K = \cap_{k\in\mathbb{N}} E_{\mathcal{D}_k}$ ([2]). A normal compact K is called a compact of holomorphy if $E_K = K$. Furthermore, every function holomorphic on K is also holomorphic on E_K. Thus, the envelope of holomorphy $E_{(\overline{\mathcal{V}_2^-} \cup \overline{\mathcal{V}_2^+})}$ of the compact $\overline{\mathcal{V}_2^-} \cup \overline{\mathcal{V}_2^+}$ is defined. We have the following lemma.

Lemma 2.5.2 *Let $\overline{\mathcal{V}_2^-}, \overline{\mathcal{V}_2^+}$ be the compacts defined in* (2.5.1). *Then* $E_{(\overline{\mathcal{V}_2^-} \cup \overline{\mathcal{V}_2^+})} = T^*_{B_1}$.

Proof: Since $T^*_{B_1}$ is a strictly linearly convex set it is a compact of holomorphy. The last claim follows from Proposition 2.1.5 in ([10]), because $T^*_{B_1}$ is connected (being star) and taking into account that there linear convexity means strict linear convexity in the present paper. Thus from equality (2.5.2) the claim of the lemma is valid.$\diamondsuit$

In order to realize the tube T_{B_2} as an intersection of tubular domains whose base contain a cone, we define the closed half-strips contained in $\mathbb{R}^2$:

$$\begin{aligned} R_H^- &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1,\ -1 \leq x_1 \leq 0\} \\ &\cup\ \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_2 \leq 1,\ x_1 \geq 0\} \\ R_H^+ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1,\ 0 \leq x_1 \leq 1\} \\ &\cup\ \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_2 \leq 1,\ x_1 \leq 0\} \end{aligned} \tag{2.5.4}$$

Similarly to (2.4.2) and using the interiors $(R_H^-)^{int}, (R_H^+)^{int}$ of the closed half-strips from (2.5.4) we introduce tubular domains with unbounded base contained in $\mathbb{R}^2$:

$$\begin{aligned} T_{(R_H^-)^{int}} &= (R_H^-)^{int} \times i\mathbb{R}^2 \\ T_{(R_H^+)^{int}} &= (R_H^+)^{int} \times i\mathbb{R}^2 \end{aligned} \tag{2.5.5}$$

The tube domains introduced above in (2.5.5) are defined via piece-wise smooth defining functions similarly to (2.4.2) as follows:

$$\begin{aligned} T_{(R_H^-)^{int}} &= \{(x_1, x_2, iy_1, iy_2) \in \mathbb{R}^2 \times i\mathbb{R}^2 : \ \Phi_2(\zeta, \bar{\zeta}) = \Phi_2^-(\zeta, \bar{\zeta}), \text{when } -1 \le x_1 < 0\} \\ &\cup \ \{(x_1, x_2, iy_1, iy_2) \in \mathbb{R}^2 \times i\mathbb{R}^2 : \Phi_2^-(\zeta, \bar{\zeta}) = (\frac{\zeta_2 + \bar{\zeta}_2}{2i})^2 - 1, \text{ when } x_1 \ge 0\} \\ T_{(R_H^+)^{int}} &= \{(x_1, x_2, iy_1, iy_2) \in \mathbb{R}^2 \times i\mathbb{R}^2 : \ \Phi_2(\zeta, \bar{\zeta}) = \Phi_2^+(\zeta, \bar{\zeta}), \text{when } 0 < x_1 \le 1, \ \} \\ &\cup \ \{(x_1, x_2, iy_1, iy_2) \in \mathbb{R}^2 \times i\mathbb{R}^2 : \Phi_2^+(\zeta, \bar{\zeta}) = (\frac{\zeta_2 + \bar{\zeta}_2}{2i})^2 - 1, \text{ when } x_1 \le 0\} \end{aligned} \tag{2.5.6}$$

where $\Phi_2(\zeta, \bar{\zeta}) = (\frac{\zeta_1 + \bar{\zeta}_1}{2i})^2 + (\frac{\zeta_2 + \bar{\zeta}_2}{2i})^2 - 1$ is smooth. One deduces that the domains defined in (2.5.5) are convex tube domains whose bases contain the one-dimensional cones $\{(x_1, 0) \in \mathbb{R}^2 : x_1 > 0\}$ and $\{(x_1, 0) \in \mathbb{R}^2 : x_1 < 0\}$ respectively. Thus, a tube domain with convex base contained in $\mathbb{R}^2$ is now realized as the intersection of the tubes defined in (2.5.5). Actually,

$$T_{B_2} = T_{(R_H^+)^{int}} \cup T_{(R_H^-)^{int}} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \times i\mathbb{R}^2 \tag{2.5.7}$$

Analogously to the case of the tube T_{B_1} one obtains the generalized dual complement $T_{B_2}^*$ of T_{B_2}. We formulate the following lemma without proof.

Lemma 2.5.3 *Let* $T_{(R_H^-)^{int}}$, $T_{(R_H^+)^{int}}$ *be the unbounded domains defined in* (2.5.5). *Then*

$T^*_{(R_H^-)^{int}} = \overline{\mathcal{V'}_2^-}$, $T^*_{(R_H^+)^{int}} = \overline{\mathcal{V'}_2^+}$, *where* $(x, iy) \in \partial T_{(R_H^-)^{int}}$ *or* $\partial T_{(R_H^+)^{int}}$, *correspondingly.*

$$
\begin{aligned}
\mathcal{V'}_2^- &= \{(\omega_1,\omega_2) \in \mathbb{C}^2 : \omega_i = \frac{r^3(x_i(x_1y_1+x_2y_2) - ix_i(y_1^2+y_2^2))}{r^4((x_1^2+x_2^2)^2 + (x_1y_1+x_2y_2)^2)}, \\
&\quad (x,iy) \in \mathbb{R}^2 \times i\mathbb{R}^2,\ 1 \le r < \infty, i = 1,2,\ -1 < x_1 < 0\} \\
&\cup\ \{(\omega_1,\omega_2) \in \mathbb{C}^2 : (0,\omega_2) = (0, \frac{1}{r}\frac{x_2^2y_2 - ix_2^3}{x_2^4 + (x_2y_2)^2}), \\
&\quad (x,iy) \in \mathbb{R}^2 \times i\mathbb{R}^2,\ \textit{when}\ x_1 > 0,\ r \ge 1\} \\
\mathcal{V'}_2^+ &= \{(\omega_1,\omega_2) \in \mathbb{C}^2 : \omega_i = \frac{r^3(x_i(x_1y_1+x_2y_2) - ix_i(x_1^2+x_2^2))}{r^4((x_1^2+x_2^2)^2 + (x_1y_1+x_2y_2)^2)}, \\
&\quad (x,iy) \in \mathbb{R}^2 \times i\mathbb{R}^2,\ i = 1,2,\ 1 \le r < \infty,\ 0 < x_1 < 1\} \\
&\cup\ \{(\omega_1,\omega_2) \in \mathbb{C}^2 : (0,\omega_2) = (0, \frac{1}{r}\frac{x_2^2y_2 - ix_2^3}{x_2^4 + (x_2y_2)^2}), \\
&\quad (x,iy) \in \mathbb{R}^2 \times i\mathbb{R}^2,\ \textit{whenever}\ x_1 < 0,\ r \ge 1\} \qquad (2.5.8)
\end{aligned}
$$

The closures $\overline{\mathcal{V'}_2^\pm}$ *correspond to the case when we add to* $\mathcal{V'}_2^\pm$ *the slopes of complex tangent hyperplanes to the tubes* $T_{(R_H^\pm)^{int}}$ *at the points* $(0, \pm 1, iy_1, iy_2) \in \partial T_{(R_H^\pm)^{int}}$. *Thus, we are leading to* $(\omega_1, \omega_2) = (0, \frac{1}{r}\frac{x_2 \pm i}{1+x_2^2}) \in T^*_{(R_H^\pm)^{int}}$. *These slopes exist because* $\nabla\Phi_2^\pm(0, \pm 1, iy_1, iy_2)$ *is well defined, where* $\Phi_2^\pm$ *are the defining functions of the tubes involved.*

Furthermore,

$$
T^*_{B_2} = T^*_{(R_H^+)^{int}} \cup T^*_{(R_H^-)^{int}} \;=\; \overline{\mathcal{V}_2'^-} \cup \overline{\mathcal{V}_2'^+} \subset \bar{B}(0,1) \qquad (2.5.9)
$$

and

$$
\begin{aligned}
& T^*_{(R_H^-)^{int}} \cap \partial B(0,1) = T^*_{(R_H^-)^{int}} \cap \overline{S^+} \times i\mathbb{R}^2 \\
=\ & \{(x_1,x_2,0,0) \in \mathbb{R}^2 \times i\mathbb{R}^2 :\ x_1^2 + x_2^2 = 1,\ -1 \le x_1 \le 0\} \subset \bar{B}(0,1) \\
& T^*_{(R_H^+)^{int}} \cap \partial B(0,1) = T^*_{(S_H^+)^{int}} \cap \overline{S^-} \times i\mathbb{R}^2 \\
=\ & \{(x_1,x_2,0,0) \in \mathbb{R}^2 \times i\mathbb{R}^2 :\ x_1^2 + x_2^2 = 1,\ 0 \le x_1 \le 1\} \subset \bar{B}(0,1) \qquad (2.5.10)
\end{aligned}
$$

Similarly to lemma 2.5.2 one obtains the following equality for the envelope of holomorphy of the compact $T^*_{(R_H^+)^{int}} \cup T^*_{(R_H^-)^{int}}$.

Lemma 2.5.4 *Let* $T_{(R_H^-)^{int}}, T_{(R_H^+)^{int}}$ *be the unbounded domains defined in* (2.5.5). *If* $T^*_{(R_H^-)^{int}}$

and $T^*_{(R_H^+)^{int}}$ *are like in Lemma* 2.5.3 *then*

$$T^*_{B_2} = E_{\overline{(\mathcal{V}_2'^-} \cup \overline{\mathcal{V}_2'^+})}$$

Corollary 2.5.1 *Let* $T_{(S_H^-)^{int}}, T_{(S_H^+)^{int}}$ *and* $T_{(R_H^-)^{int}}, T_{(R_H^+)^{int}}$ *be the unbounded domains defined in* (2.4.2) *and* (2.5.5) *correspondingly. It follows directly from* (2.5.2) *and* (2.5.9) *that*

$$\begin{aligned} T^*_{B_1} \cup T^*_{B_2} &= (T_{(S_H^+)^{int}} \cap T_{(S_H^-)^{int}})^* \cup (T_{(R_H^+)^{int}} \cap T_{(R_H^-)^{int}})^* \\ &= \left(T^*_{(S_H^+)^{int}} \cup T^*_{(S_H^-)^{int}}\right) \cup \left(T^*_{(R_H^+)^{int}} \cup T^*_{(R_H^-)^{int}}\right) \end{aligned}$$

Chapter 3

The Hardy space $\mathbf{H^2(T_{B_1})}$ and its dual

We begin this chapter by recalling some basic facts about Hardy spaces $H^2(T_D)$ on tubular domains $T_D = \mathbb{R}^2 \times D$, taken from [27]. One considers then the Hardy space of holomorphic functions $F \in \mathcal{H}(T_D)$ defined by

$$H^2(T_D) = \{F \in \mathcal{H}(T_D) : \int\limits_{\mathbb{R}^2} |F(x+iy)|^2 dx \leq A^2 < +\infty,\ \forall y \in D\} \tag{3.0.1}$$

The space defined by (3.0.1) becomes a normed vector space when

$$\|F\|_{H^2(T_D)} = inf A \text{ where the constant } A \text{ is satisfying } (3.0.1). \tag{3.0.2}$$

The main result in Chap.3, §2, [27] states that $F \in H^2(T_D)$ if and only if

$$F(z) = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi i z\cdot t} dt,\ z \in T_D, \tag{3.0.3}$$

whenever f satisfies

$$\sup_{y\in D} \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi y\cdot t} dt \leq A^2 < +\infty \tag{3.0.4}$$

Plancherel's Theorem implies then

$$\left(\sup_{y\in D}\int\limits_{\mathbb{R}^2}|f(t)|^2e^{-4\pi y\cdot t}dt\right)^{\frac{1}{2}}=\|F\|_{H^2(T_D)}.$$

3.1 Basic properties of the Hardy space $\mathbf{H^2(T_{(S_H^-)^{int}})}$

If $y_0\in\partial D$ is such that it can be approached by a sequence $\{y_n\}_n$, $y_n\in D$, non-tangentially, then the function

$$F_0(x)=F(x+iy_0)=\int\limits_{\mathbb{R}^2}f(t)e^{2\pi i(x+iy_0)\cdot t}dt,$$

defined for almost all x, is the $L^2(\mathbb{R}^2)$-limit of functions $F_n(x)=F(x+iy_n)$. We remark that this definition of the function $F(x+iy_0)$ is independent from the sequence $\{y_n\}_n$.

We look at the particular tube $T_{(S_H^-)^{int}}$ with base the unbounded, symmetric, convex set $(S_H^-)^{int}$, introduced in the previous chapter, focusing on the values of $F\in H^2(T_{(S_H^-)^{int}})$ at the boundary $\partial T_{(S_H^-)^{int}}=\mathbb{R}^2\times\partial S_H^-$. As was pointed out in Cor.2.10 in [27] these values in general exist almost everywhere. Their existence is proven by using the analytic continuation of Fourier Transform and then its inversion. However, it is also stated in Th.2.11 in [27] that such limits do exist at every point of the boundary $\partial T_{(S_H^-)^{int}}$ provided the point is polygonal (that is, vertex of bounded, convex polygon contained entirely in the tube $T_{(S_H^-)^{int}}$), which is exactly the setting in our case. However, for completeness of the presentation of the results, we will prove the existence of our limits at every $z_0\in\mathbb{R}^2\times\partial S_H^-$ by simple, classical means.

Actually, consider the sequence $\{r_n\}\subset(0,1]$ such that $r_n\uparrow 1$. Define the sequence of functions $g_n(t)=|f(t)|^2e^{-4\pi r_n y\cdot t}$ converging point-wise in $t\in\mathbb{R}^2$ as $n\longrightarrow\infty$ (i.e $r_n\uparrow 1$) to $g(t)=|f(t)|^2e^{-4\pi y\cdot t}$. Now, $y\cdot t=\|y\|\|t\|\cos\theta$, where θ is the angle between the corresponding vectors. Fix $y\in\mathbb{R}^2$, $y\neq 0$, its direction defines the horizontal "axis". Then, we split the plane $\mathbb{R}^2$ into two closed half-planes, intersecting along a line. Namely, the closed half-plane $\Pi_+=\{(t_1,t_2):y\cdot t=\|y\|\|t\|\cos\theta\geq 0\}$ and the closed half-plane $\Pi_-=\{(t_1,t_2):y\cdot t=\|y\|\|t\|\cos\theta\leq 0\}$. Observe that $\Pi_+\cap\Pi_-=\{(t_1,t_2):y\cdot t=\|y\|\|t\|\cos\theta=0\}$. If $\cos\theta\geq 0$, then, after changing coordinates in Π_+, $g_n(\|t\|\cos\theta, \|t\|\sin\theta) \leq |f(\|t\|\cos\theta, \|t\|\sin\theta)|$, $g(\|t\|\cos\theta, \|t\|\sin\theta) \leq |f(\|t\|\cos\theta, \|t\|\sin\theta)|$. Since

$$\int\limits_{\Pi_+} |f(t_1,t_2)|^2 dt_1 dt_2 = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_0^{\infty} |f(\theta, r)|^2 r d\theta dr,\ r = \|t\|$$

Lebesgue dominated convergence theorem implies then that $\int\limits_{\Pi_+} |f(t)|^2 e^{-4\pi r_n y\cdot t} dt \longrightarrow \int\limits_{\Pi_+} |f(t)|^2 e^{-4\pi y\cdot t} dt$, as $n \longrightarrow \infty$. Similarly, in the half-plane Π_-, $\cos\theta \leq 0$, and thus Monotone convergence theorem combined with the fact that $\int\limits_{\Pi_-} |f(t)|^2 e^{-4\pi r_n y\cdot t} dt \leq A^2$ imply that $\int\limits_{\Pi_-} |f(t)|^2 e^{-4\pi r_n y\cdot t} dt \longrightarrow \int\limits_{\Pi_-} |f(t)|^2 e^{-4\pi y\cdot t} dt$, as $n \longrightarrow \infty$. The case $\cos\theta = 0$ corresponds to line $\Pi_+ \cap \Pi_-$. Now, it is straightforward to show that $\|g_n - g\|_{L^2(\mathbb{R}^2)} \longrightarrow 0$ in $L^2(\mathbb{R}^2)$ also. Thus, every function $F \in H^2(T_{(S_H^-)^{int}})$ has boundary values on $\mathbb{R}^2 \times \partial S_H^-$ everywhere.

Furthermore, one can also show, using the particular form of boundary of the base of the tube, that whenever $y_n \longrightarrow y_0$, $y_n, y_0 \in \partial S_H^-$ for every $n \in \mathbb{N}$ one has that $\mathcal{F}(y_n) \longrightarrow \mathcal{F}(y)$, $n \longrightarrow \infty$, where

$$\mathcal{F}(y) = \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi y\cdot t} dt,\ y \in \partial S_H^-. \tag{3.1.5}$$

Thus, we have the following

Lemma 3.1.1 *Let $F(z) = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi i z\cdot t} dt \in H^2(T_{(S_H^-)^{int}})$. Then the function $\mathcal{F}(y)$, $y \in \partial S_H^-$ defined by (3.1.5) is continuous.*

Reasoning along the same lines, when the tube in question is the tube T_{B_1} and $y \in \partial T_{B_1}$, $y_1^2 + y_2^2 < 1$ leads to the formulation of the following

Lemma 3.1.2 *Let $F_1(z) = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi i z\cdot t} dt \in H^2(T_{B_1})$. Then the function $\mathcal{F}_1(y)$, defined by (3.1.5) for $y \in S^1 \subset \mathbb{R}^2$ is continuous.*

Direct implications of the above lemmas are the facts:

1) if $F \in H^2(T_{(S_H^-)^{int}})$ then its norm $\|F\|_{H^2(T_{(S_H^-)^{int}})}$ is realized on the part of the boundary ∂S_H^- satisfying $\|y\| = \sqrt{y_1^2 + y_2^2} = 1$;

2) if $G \in H^2(T_{B_j})$, $j = 1, 2$, then its norm $\|G\|_{H^2(T_{B_j})}$ is realized on the boundary $\mathbb{R}^2 \times iS^1$

$(S^1 \times i\mathbb{R}^2)$ or at $\|y\| = 0$, using the concavity of the functions in question, after proving similar results for the tube T_{B_2}. Direct comparison of the integrals implies then that $\|G\|_{H^2(T_{B_j})}$ appears on the boundary of the tubes $\mathbb{R}^2 \times iS^1$ $(S^1 \times i\mathbb{R}^2)$.
The proof of the above claims is subject to the following comparison of norms corollary.

Corollary 3.1.1 1) *If $F \in H^2(T_{(S_H^-)^{int}})$, then the following is valid*

$$\frac{\pi}{2}\|F\|^2_{H^2(T_{(S_H^-)^{int}})} < \|F\|^2_{L^2_\nu(\mathbb{R}^2 \times iS^-)} < 2\pi\|F\|^2_{H^2(T_{(S_H^-)^{int}})}, \tag{3.1.6}$$

where the measure ν is equivalent to the Lebesgue measure λ on $\mathbb{C}^2$ restricted to the $\mathbb{R}^2 \times iS^-$, that is, any measure ν that satisfies $\frac{d\Phi(\zeta,\bar{\zeta})}{\|\nabla_\zeta \Phi(\zeta,\bar{\zeta})\|} \wedge \nu = |K| dx_1 dx_2 dy_1 dy_2$
2) *If $G \in H^2(T_{B_j})$, then the following is valid*

$$\frac{\pi}{2}\|F\|^2_{H^2(T_{B_j})} < \|F\|^2_{L^2_{\nu_j}(\mathcal{V})} < 2\pi\|F\|^2_{H^2(T_{B_j})}, \tag{3.1.7}$$

where $\mathcal{V}$ is $\mathbb{R}^2 \times iS^1$ or $S^1 \times i\mathbb{R}^2$ and the measure ν_j is equivalent to the Lebesgue measure λ on $\mathbb{C}^2$ restricted to the $\mathbb{R}^2 \times iS^1$ (or to $S^1 \times i\mathbb{R}^2$), that is, any measure ν_j that satisfies $\frac{d\Phi(\zeta,\bar{\zeta})}{\|\nabla_\zeta \Phi(\zeta,\bar{\zeta})\|} \wedge \nu_j = |K_j| dx_1 dx_2 dy_1 dy_2$

Proof: In order to proof the first part we begin by remarking that the continuity of the function $\mathcal{F} : y \longrightarrow \int\limits_{\mathbb{R}^2} |F(x+iy)|^2 dx$ for all $y \in \partial S_H^-$ implies that it is integrable on S^-.
Furthermore, for every point $y \in S_H^-$ one has that $\mathcal{F}(y) = \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi\|y\|\|t\|\cos\theta} dt_1 dt_2$, where θ is the angle between vectors y and t. It is evident that $\theta \in [0, 2\pi]$. Now, by taking the direction of y as our new "x" axis one has that $\|y\|\|t\|\cos\theta = \|y\|\tau_1$. Thus the integral defining $\mathcal{F}(y)$ is transformed, after rotation of the coordinate system, to $\mathcal{F}(y) = \int\limits_{\mathbb{R}^2} |f(\tau_1, \tau_2)|^2 e^{-4\pi\|y\|\tau_1} d\tau_1 d\tau_2$, taking into account that the determinant of the rotation matrix in the plane is equal to 1. Thus $\mathcal{F}(y) = \mathcal{F}(\|y\|)$. It is straightforward to see now that the norm $\|F\|_{H^2(T_{(S_H^-)^{int}})}$ is realized on the arc S^-, where we have $\|F\|_{H^2(T_{(S_H^-)^{int}})} = \int\limits_{\mathbb{R}^2} |f(\tau)|^2 e^{-4\pi\tau_1} d\tau_1 d\tau_2$, since $\|y\| = 1$. Actually, differentiating twice $\mathcal{F}(r)$ with respect to $r = \|y\|$, one has on the horizontal part of ∂S_H^-, except at the points $(0, \pm 1)$, that this function is concave up at every point (which means that possible changes of its monotony on every horizontal component of ∂S_H^- determines only local minima). Since at $r = +\infty$

the value of $\mathcal{F}$ is finite one has the desired result. It is easy now to deduce that since $\int\limits_{\mathbb{R}^2} |F(x+iy)|^2 dx = \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi y \cdot t} dt_1 dt_2$ for $y \in (S_H^-)^{int}$ we have, after taking limits and integrating with respect to y, that

$$\frac{\pi}{2}\|F\|^2_{H^2(T_{(S_H^-)^{int}})} \leq \int\limits_{S^-}\int\limits_{\mathbb{R}^2} |F(x+iy)|^2 dx d\nu_{S^-} \leq 2\pi \|F\|^2_{H^2(T_{(S_H^-)^{int}})},$$

where $d\nu_{iS^-}$ is a measure on the arc iS^- that makes the measure $d\nu = dx d\nu_{iS^-}$ equivalent to the Lebesgue measure in $\mathbb{C}^2$. This implies the required inequality in the first part.

To prove the second part , we argue along the same line, but the justification that supremum appears on the circumference or at the center of the base of the tube has to do with the concavity of the map $\mathcal{F}_j$ involved. $\diamondsuit$

An example of such a measure is in order.

Lemma 3.1.3 *The Lebesgue measure $m(\zeta, \bar{\zeta})$ in $\mathbb{C}^2$ restricted to the sets $\mathbb{R}^2 \times iS^- \subset \partial T_{S_H^-}$, $\mathbb{R}^2 \times iS^+ \subset \partial T_{S_H^+}$ or $\partial T_{B_1} = \mathbb{R}^2 \times iS^1$ is equivalent to any one of the following measures*

$$\begin{aligned}
\mu_{\Phi_1}(\zeta,\bar{\zeta}) &= \partial\Phi_1(\zeta,\bar{\zeta}) \wedge (\partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta})) \\
\mu_{\Phi_1^+}(\zeta,\bar{\zeta}) &= \partial\Phi_1^+(\zeta,\bar{\zeta}) \wedge (\partial\bar{\partial}\Phi_1^+(\zeta,\bar{\zeta})) \\
\mu_{\Phi_1^-}(\zeta,\bar{\zeta}) &= \partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge (\partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta}))
\end{aligned}$$

induced by the defining functions of the tubes involved.

Proof: The proof consists of direct verification by computation. We will verify the claim only for the defining function Φ_1, the rest of the cases are proved analogously. Recall that the Lebesgue measure $m(\zeta,\bar{\zeta})|_{\mathbb{R}^2\times iS^1}$ is equivalent to the measure $\partial\Phi_1(\zeta,\bar{\zeta}) \wedge (\partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta}))$ if there exists a positive constant K satisfying

$$(\frac{1}{2i})^2 \frac{d\Phi_1(\zeta,\bar{\zeta})}{\|\nabla_\zeta \Phi_1(\zeta,\bar{\zeta})\|} \wedge \partial\Phi_1(\zeta,\bar{\zeta}) \wedge (\partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta})) = Km(\zeta,\bar{\zeta}),\ z \in \mathbb{R}^2 \times iS^1.$$

Elementary calculations show that

$$\bar{\partial}\Phi_1(\zeta,\bar{\zeta}) = \bar{\partial}((\frac{\zeta_1-\bar{\zeta}_1}{2i})^2 + (\frac{\zeta_2-\bar{\zeta}_2}{2i})^2 - 1) = \frac{1}{2}(\zeta_1-\bar{\zeta}_1)d\bar{\zeta}_1 + \frac{1}{2}(\zeta_2-\bar{\zeta}_2)d\bar{\zeta}_2$$

Similarly,

$$\partial\Phi_1(\zeta,\bar{\zeta}) \;=\; \partial((\frac{\zeta_1-\bar{\zeta}_1}{2i})^2-(\frac{\zeta_2-\bar{\zeta}_2}{2i})^2-1)=-\frac{1}{2}(\zeta_1-\bar{\zeta}_1)d\zeta_1-\frac{1}{2}(\zeta_2-\bar{\zeta}_2)d\zeta_2$$

Furthermore,

$$\partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta}) \;=\; \partial\bar{\partial}((\frac{\zeta_1-\bar{\zeta}_1}{2i})^2-(\frac{\zeta_2-\bar{\zeta}_2}{2i})^2-1)=\frac{1}{2}d\zeta_1\wedge d\bar{\zeta}_1+\frac{1}{2}d\zeta_2\wedge d\bar{\zeta}_2$$

Finally, after substitution, we get

$$\begin{aligned}\partial\Phi_1(\zeta,\bar{\zeta})\wedge\partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta}) \;&=\; \frac{1}{4}(((\zeta_1-\bar{\zeta}_1)d\zeta_1+(\zeta_2-\bar{\zeta}_2)d\zeta_2))\wedge(-d\zeta_1\wedge d\bar{\zeta}_1-d\zeta_2\wedge d\bar{\zeta}_2)\\ &=\; -\frac{1}{4}(\zeta_1-\bar{\zeta}_1)d\zeta_1\wedge d\zeta_2\wedge d\bar{\zeta}_2-\frac{1}{4}(\zeta_2-\bar{\zeta}_2)d\zeta_2\wedge d\zeta_1\wedge d\bar{\zeta}_1\\ &=\; -\frac{1}{4}(\zeta_1-\bar{\zeta}_1)d\zeta_1\wedge d\zeta_2\wedge d\bar{\zeta}_2-\frac{1}{4}(\zeta_2-\bar{\zeta}_2)d\zeta_1\wedge d\bar{\zeta}_1\wedge d\zeta_2\end{aligned}$$

On the other hand

$$\frac{d\Phi_1(\zeta,\bar{\zeta})}{\|\nabla_\zeta\Phi_1(\zeta,\bar{\zeta})\|} \;=\; \frac{\partial\Phi_1(\zeta,\bar{\zeta})}{\|\nabla_\zeta\Phi_1(\zeta,\bar{\zeta})\|}+\frac{\bar{\partial}\Phi_1(\zeta,\bar{\zeta})}{\|\nabla_\zeta\Phi_1(\zeta,\bar{\zeta})\|}=\partial\Phi_1(\zeta,\bar{\zeta})+\bar{\partial}\Phi_1(\zeta,\bar{\zeta})$$

Thus, for $(\zeta,\bar{\zeta})\in\mathbb{R}^2\times iS^1$, we deduce that

$$\begin{aligned}(\frac{1}{2i})^2\frac{d\Phi_1(\zeta,\bar{\zeta})}{\|\nabla_\zeta\Phi_1(\zeta,\bar{\zeta})\|}\wedge\big(\partial\Phi_1(\zeta,\bar{\zeta})\wedge\partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta})\big) \;&=\; -\bar{\partial}\Phi_1(\zeta,\bar{\zeta})\wedge\big(\partial\Phi_1(\zeta,\bar{\zeta})\wedge\partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta})\big)\\ &=\; (\frac{1}{2i})^2(\frac{4y_1^2+4y_2^2}{8})d\zeta_1\wedge d\bar{\zeta}_1\wedge d\zeta_2\wedge d\bar{\zeta}_2\\ &=\; \frac{1}{2}dx_1\wedge dy_1\wedge dx_2\wedge dy_2,\end{aligned}$$

that is, the measures in question are equivalent. ♢

Corollary 3.1.2 1) *The space* $(H^2(T_{(S_H^-)^{int}}),\|\cdot\|_{H^2(T_{(S_H^-)^{int}})})$ *is Banach.*
2) *The space* $(H^2(T_{B_1}),\|\cdot\|_{H^2(T_{B_1})})$ *is Banach.*

Proof: We will give the proof of the first part only, because the second one is proved similarly. The completeness of the space in question is proven as follows: let $\{F_n\}_n\subset$

$H^2(T_{(S_H^-)^{int}})$ be Cauchy. Then, for every $z = x + iy \in T_{(S_H^-)^{int}}$ we have by Plancherel Theorem

$$\begin{aligned}\int\limits_{\mathbb{R}^2} |F_n(x+iy) - F_m(x+iy)|^2 dx_1 dx_2 &= \int\limits_{\mathbb{R}^2} |f_n(t) - f_m(t)|^2 e^{-4\pi y\cdot t} dt_1 dt_2 \\ &\leq ||F_n - F_m||_{H^2(T_{(S_H^-)^{int}})} < \epsilon,\end{aligned}$$

for every $n, m \geq n_0$. Thus, for $y = 0$ one is led to $\int\limits_{\mathbb{R}^2} |f_n(t) - f_m(t)|^2 dt_1 dt_2 \longrightarrow 0$. Completeness of the space $L^2(\mathbb{R}^2)$ implies that $\|f_n - f\|_{L^2(\mathbb{R}^2)} \longrightarrow 0$, where $f \in L^2(\mathbb{R}^2)$. We claim that $F(z) = \int\limits_{\mathbb{R}^2} f(t)e^{2\pi i z\cdot t}dt$, $z \in T_{(S_H^-)^{int}}$, belongs to the space $H^2(T_{(S_H^-)^{int}})$. Actually, $\|f_n - f\|_{L^2(\mathbb{R}^2)} \longrightarrow 0$ implies that $\|f_n\|_{L^2(\mathbb{R}^2)}, \|f\|_{L^2(\mathbb{R}^2)} < C$. Hence we deduce that $f_n^2(t) \longrightarrow f^2(t)$ a.e and $\int\limits_{\mathbb{R}^2} ||f_n(t)|^2 - |f(t)|^2|dt \longrightarrow 0$. Thus, Fatou's lemma implies that for every $z \in T_{(S_H^-)^{int}}$ one has

$$\int\limits_{\mathbb{R}^2} |f^2(t)|e^{4\pi y\cdot t}dt \leq \liminf_{n\longrightarrow\infty} \int\limits_{\mathbb{R}^2} |f_n^2(t)|e^{-4\pi y\cdot t}dt \leq \liminf_{n\longrightarrow\infty} \|F_n\|_{H^2(T_{(S_H^-)^{int}})}.$$

This completes the proof of the claim. The other part is proved similarly. ♢

The following proposition is similar to convergence in mean results for the classical Hardy spaces to be found in ([12], [24]).

Proposition 3.1.1 *Let $F \in H^2(T_{(S_H^-)^{int}})$ and $r \in (0,1]$. Then $\lim\limits_{r\uparrow 1} F(rz_1, rz_2) = F(z_1, z_2)$ in $\|\cdot\|_{H^2(T_{(S_H^-)^{int}})}$ norm. Consequently, there exists a subsequence $\{r_k\}_k \subset (0,1]$, $r_k \uparrow 1$ so that $F_{r_k}(z) = F(r_k z_1, r_k z_2) \longrightarrow F(z_1, z_2)$ for almost all $(z_1, z_2) \in \mathbb{R}^2 \times iS^-$. Similarly, the same claim is valid for $G \in H^2(T_{B_1})$ with respect to $\|.\|_{H^2(T_{B_1})}-$ norm.*

Proof: First we observe that if $(z_1, z_2) \in T_{(S_H^-)^{int}}$ then $(rz_1, rz_2) \in T_{(r\cdot S_H^-)^{int}}$, that is, $(iry_1, iry_2) \in (rS_H^-)^{int} \subset (S_H^-)^{int}$ for every $r \in (0,1]$. For simplicity of the reasoning, we consider the sequence $F_n(z) = F(r_n z)$, $r_n \in (0,1]$, $F \in H^2(T_{(S_H^-)^{int}})$. Since $\int\limits_{\mathbb{R}^2} |F_n(x + iy)|^2 dx = \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4r_n\pi y\cdot t}dt \leq A^2$, $A^2 = \|F\|^2_{H^2(T_{(S_H^-)^{int}})}$, for every r_n, we deduce that $F_n \in H^2(T_{(S_H^-)^{int}})$ for every $n = 1, 2, \ldots$. Furthermore,

$$\int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4r_n\pi y\cdot t}dt \leq A^2,$$

for all $y \in S^-$. Thus, taking into account that the functions above have values everywhere on the boundary of the tube, one has there for $t \in \Pi_-$

$$\begin{aligned} |f(t)e^{-2\pi r_n y\cdot t} - f(t)e^{-2\pi y\cdot t}|^2 &\leq |f(t)|^2 e^{-4\pi r_n y\cdot t} + |f(t)|^2 e^{-4\pi y\cdot t} \\ &+ 2|f(t)|^2 e^{-2\pi(1+r_n)y\cdot t} \leq 4|f(t)|^2 e^{-4\pi y\cdot t}. \end{aligned}$$

Similarly, whenever $t \in \Pi_+$, one has that

$$\begin{aligned} |f(t)e^{-2\pi r_n y\cdot t} - f(t)e^{-2\pi y\cdot t}|^2 &\leq |f(t)|^2 e^{-4\pi r_n y\cdot t} + |f(t)|^2 e^{-4\pi y\cdot t} \\ &+ 2|f(t)|^2 e^{-2\pi(1+r_n)y\cdot t} \leq 4|f(t)|^2 e^{-2\pi y\cdot t}, \end{aligned}$$

if one assumes (without loss of generality) that $r_1 = \frac{1}{2}$. Thus, by keeping y fixed, Lebesgue dominated convergence theorem implies that

$$\int\limits_{\mathbb{R}^2} |f(t)e^{-2\pi r_n y\cdot t} - f(t)e^{-2\pi y\cdot t}|^2 dt \longrightarrow 0, \tag{3.1.8}$$

as $r_n \longrightarrow 1$. Since (3.1.8) is valid in particular at the points y of the boundary where the norm $\|.\|_{H^2(T_{(S_H^-)^{int}})}$ is realized one deduces the convergence in $\|.\|_{H^2(T_{(S_H^-)^{int}})}$ norm. The second claim follows. The other case is proved analogously. $\diamondsuit$

3.2 Duality results

At this point we recall the Schwarz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < +\infty, \forall \alpha, \beta \in \mathbb{N}_0\}$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$.
The natural topology on $\mathcal{S}(\mathbb{R}^n)$ is defined as follows :

$$f_n \to f \text{ if and only if } \lim_{n\to\infty} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta (f_n - f)(x)| = 0 \text{ for all } \alpha, \beta$$

If $f \in \mathcal{S}(\mathbb{R}^n)$ then its derivative and $x_j f(x)$ are elements of $\mathcal{S}(\mathbb{R}^n)$. Direct computations show that

$$F(x^\alpha \partial_x^\beta f)(x) \;=\; \frac{1}{(2\pi i)^\beta}(\frac{\partial}{\partial x})^\alpha (x^\beta F(x))$$

The inclusion $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ is valid for any $p \geq 1$. If $C_0^\infty(\mathbb{R}^n)$ is the space of smooth functions with compact support on $\mathbb{R}^n$ then $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, follows that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for any $p \geq 1$. Furthermore, the Fourier transform is an automorphism of the Schwarz space.

The following result is of importance, because using the Corollary 3.1.1 it allows to connect the regular L^2- approximants of functions to approximants in the spaces with *sup*-norms.

Theorem 3.2.1 *Every* $F \in H^2(T_{(S_H^-)^{int}})$ *is a* $\|\cdot\|_{H^2(T_{(S_H^-)^{int}})}$ *limit of a sequence* $\{F_n\} \subset H^2(T_{(S_H^-)^{int}})$, *where*

$$F_n(x+iy) = \int\limits_{\mathbb{R}^2} f_n(t) e^{2\pi i z\cdot t} = \int\limits_{\mathbb{R}^2} f_n(t) e^{2\pi i x\cdot t} e^{-2\pi y\cdot t} dt, \;\; z \in T_{(S_H^-)^{int}},$$

whenever $z = x+iy \in T_{(S_H^-)^{int}}$, $f_n \in \mathcal{S}(\mathbb{R}^2)$ *and* $\|f_n - f\|_{L^2(\mathbb{R}^2)} \longrightarrow 0$. *Similarly, the same claim is valid for every* $G \in H^2(T_{B_1})$ *with respect to the* $\|\cdot\|_{H^2(T_{B_1})}$*-norm and* $z \in T_{B_1}$.

Proof: Recall that $F \in H^2(T_{(S_H^-)^{int}})$ is equivalent by definition to the fact that $\int\limits_{\mathbb{R}^2} |F(x+iy)|^2 dx \leq A^2$, $\forall y \in (S_H^-)^{int}$. As it was pointed above, for any $y_0 \in \partial \left(S_H^-\right)^{int} = \partial S_H^-$ one has that

$$F_0(x) = F(x+iy_0) = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi i z\cdot t} dt = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi i x\cdot t} e^{-2\pi y_0\cdot t} dt,$$

is the $L^2(\mathbb{R}^2)$-limit of functions

$$F(x+iy_n) = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi i z_n\cdot t} dt = \int\limits_{\mathbb{R}^2} f(t) e^{2\pi i x\cdot t} e^{-2\pi y_n\cdot t} dt, \; y_n \in (S_H^-)^{int},$$

whenever $y_n \longrightarrow y_0$. Therefore $F(x+iy_0) \in L^2(\mathbb{R}^2)$ as a function of $x = (x_1, x_2)$.

Hence $\int\limits_{\mathbb{R}^2} |F(x+iy_0)|^2 dx \le A^2$ for all $y_0 \in S^-$. Then, Corollary 3.1.1 implies that we can integrate the last inequality with respect to y_0 on S^-. Thus, we imply that $F \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$.

Next , we are going to show that $F(x+iy)$ is a limit of rapidly decreasing functions for y-fixed in $L^2(\mathbb{R}^2)$, for every $y \in (S_H^-)^{int}$. To this end we need to recall the constructions of such approximants . Here again one takes into account the particular form of F for y-fixed:

$$F(x+iy) = \int\limits_{\mathbb{R}^2} f(t)e^{2\pi i z\cdot t}dt = \int\limits_{\mathbb{R}^2} f(t)e^{2\pi i x\cdot t}e^{-2\pi y\cdot t}dt$$

It implies that $f(t)$ decreases sufficiently rapidly at infinity when $-2\pi y \cdot t > 0$. Now, since $f \in L^2(\mathbb{R}^2)$ (this is the case when $y = 0$), we know that there exists a sequence $\{f_n\}_n \subset \mathcal{S}(\mathbb{R}^2)$ converging to f in $L^2(\mathbb{R}^2)$-norm. We claim that there exists a subsequence $\{f_{n_k}\}_{n_k} \subset \{f_n\}_n \subset \mathcal{S}(\mathbb{R}^2)$ such that

$$\begin{aligned} \lim_{n_k \longrightarrow \infty} \int\limits_{\mathbb{R}^2} |f_{n_k}(t)|^2 e^{-4\pi y\cdot t}dt &= \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi y\cdot t}dt, \text{ while} \\ \int\limits_{\mathbb{R}^2} |f_{n_k}(t)|^2 e^{-4\pi y\cdot t}dt &\le K \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi y\cdot t}dt, \ \forall n_k, \end{aligned} \tag{3.2.9}$$

for some constant $K > 0$. Actually, given any $M > 0$, define $g_M(t) = f(t)$, whenever $|f(t)| \le M$, $|t| \le M$ and $g_M(t) = 0$ otherwise. If $K_\delta(t) = \frac{1}{\delta^2}\phi(\frac{x}{\delta})$ is an approximation to the identity, then $g_M * K_\delta(t) \le \sup_{t\in\mathbb{R}^2} |g_M(t)|$, meaning that $g_M * K_\delta$ is uniformly bounded with respect to δ. It is known that $|g_M * K_\delta(t) - g_M(t)| \longrightarrow 0$ for almost all $t \in \mathbb{R}^2$, as $\delta \longrightarrow 0$. From bounded convergence theorem it follows that $\|g_M * K_\delta - g_M\|_{L^2(\mathbb{R}^2)} \longrightarrow 0$. But $\|g_M * K_\delta(t)\|_{L^2(\mathbb{R}^2)} \le M \sup_{t\in\mathbb{R}^2} |g_M(t)|$, $\|g_M(t)\|_{L^2(\mathbb{R}^2)} \le M \sup_{t\in\mathbb{R}^2} |g_M(t)|$, thus

$$\begin{aligned} \int\limits_{\mathbb{R}^2} ||g_M * K_\delta(t)|^2 - |g_M(t)|^2| dt &\le 4M \sup_{t\in\mathbb{R}^2} |g_M(t)| \|g_M * K_\delta - g_M\|_{L^2(\mathbb{R}^2)} \\ &\le 4M^2 \|g_M * K_\delta - g_M\|_{L^2(\mathbb{R}^2)} \end{aligned}$$

Then, for sufficiently small $\delta > 0$, one has that

$$\int_{\mathbb{R}^2} |g_M * K_\delta(t)|^2 dt \le 2 \int_{\mathbb{R}^2} |g_M(t)|^2 dt \tag{3.2.10}$$

Relation (3.2.10) implies then that

$$\int_{\mathbb{R}^2} |g_M * K_\delta(t)|^2 dt \le 2 \int_{\mathbb{R}^2} |g_M(t)|^2 dt \le 2 \int_{\mathbb{R}^2} |f(t)|^2 dt, \tag{3.2.11}$$

where the function $g_M * K_\delta(t) \in \mathcal{S}(\mathbb{R}^2)$ by construction. Appropriate choice of $M > 0$ and $\delta > 0$ leads to the construction of sequence $f_{n_k} = g_{M_{n_k}} * K_\delta$ satisfying (3.2.9) taking into account that

$$\|g_M(t) - f(t)\|^2_{L^2(\mathbb{R}^2)} \longrightarrow 0, \text{ while } M \longrightarrow \infty$$

Relation (3.2.11) implies that for every $n_k \in \mathbb{N}$ we have

$$sup_{y \in (S_H^-)^{int}} \int_{\mathbb{R}^2} |f_{n_k}(t)|^2 e^{-4\pi y \cdot t} dt \le K sup_{y \in (S_H^-)^{int}} \int_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi y \cdot t} dt,$$

where K is a positive constant independent of y. Hence, we deduce that

$$F_{n_k}(x + iy) = \int_{\mathbb{R}^2} f_{n_k}(t) e^{2\pi i z \cdot t} dt \in H^2(T_{(S_H^-)^{int}}) \tag{3.2.12}$$

Direct computations show that for every $y \in \partial S_H^-$ fixed, $F_{n_k}(x + iy) \in \mathcal{S}(\mathbb{R}^2)$.
It remains to show that F_{n_k} converges to F in $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$. Actually, it follows from (3.2.12) and the fact that $F \in H^2(T_{(S_H^-)^{int}})$ that for any $y \in iS^- \subset \partial S_H^-$ fixed, one has that

$$\int_{\mathbb{R}^2} |f_{n_k}(t) - f(t)|^2 e^{-4\pi y \cdot t} dt \longrightarrow 0$$

and thus $\|F_{n_k} - F\|_{H^2(T_{(S_H^-)^{int}})} \longrightarrow 0$, since the norm $\| \cdot \|_{H^2(T_{(S_H^-)^{int}})}$ is realized on iS^-. Integrating with respect to y on S^-, using Lemma 3.1.1, and applying the comparison of the norms Corollary 3.1.1 one concludes the proof of the first claim. The other case is proved analogously.$\diamondsuit$

The last Theorem gives us a first, rather simplistic, description of the dual space $\left(H^2(T_{(S_H^-)^{int}})\right)'$, which is the following

Corollary 3.2.1 *Consider the subspace of the space* $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$ *defined by*

$$\mathcal{G}^- = \{h \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-) : \ \textit{every restriction} \ \ h|_{\mathbb{R}^2 \times i\{(y_1,y_2)\}} \in \mathcal{S}(\mathbb{R}^2)\} \qquad (3.2.13)$$

Then $\left(H^2(T_{(S_H^-)^{int}})\right)' = \widetilde{\mathcal{G}^-}$, *where* $\widetilde{\mathcal{G}^-}$ *is* $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$*-closure of the space* (3.2.13). *Furthermore, the space* $H^2(T_{(S_H^-)^{int}})$ *is reflexive.*

Proof: It follows from the above that every $F \in H^2(T_{(S_H^-)^{int}})$ has boundary values on $\mathbb{R}^2 \times iS^-$ denoted also by F and it is the $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$-limits of a sequence $\{F_{n_k}\} \subset H^2(T_{(S_H^-)^{int}})$ satisfying (3.2.12), where $f_{n_k} \in \mathcal{S}(\mathbb{R}^2)$. Thus, in order to determine $L \in \left(H^2(T_{(S_H^-)^{int}})\right)'$, it is enough to determine it on elements of $H^2(T_{(S_H^-)^{int}})$ of the form (3.2.12). In this case $F_{n_k}(x+iy) \in \mathcal{S}(\mathbb{R}^2)$, for y-fixed, while remaining $F_{n_k} \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$. Hence, if $g \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$ represents the functional L, then for almost all $(iy_1, iy_2) \in iS^-$ the mapping $(y_1, y_2) \longrightarrow \int\limits_{\mathbb{R}^2} |g(x_1,x_2,y_1,y_2)|^2 dx_1 dx_2$ is almost everywhere continuous (with respect to the measure $\mu_{\Phi_1^-}$) on iS^-. Denote by $\Delta \subset iS^-$ the metric space of continuity points of the mapping. Note that Δ is a paracompact. Following, [23], we consider the carrier $\varphi_g : \Delta \longrightarrow 2^Y$, where $Y = L^2_{dx_1dx_2}(\mathbb{R}^2)$ is a Banach space. The carrier φ_g maps every element $(iy_1, iy_2) \in \Delta$ into a ball of positive radius centered at the element

$$\frac{iy_2}{2} g_{(y_1,y_2)}(x_1,x_2) = \frac{iy_2}{2} g(x_1,x_2,y_1,y_2) \in L^2_{dx_1dx_2}(\mathbb{R}^2).$$

This ball a convex set. The factor $\frac{iy_2}{2}$ that appears above is there to justify the compatibility of the measures: the measure $\frac{iy_2}{2} dx_1 dx_2$ is the measure $\mu_{\Phi_1^-}$ on the surface $\mathbb{R}^2 \times iS^-$ taking into account that $dy_1 dy_2 = 0$ on it. It is straightforward to check that the carrier ϕ_g is lower semi-continuous and thus admits a continuous selection $G : \Delta \longrightarrow Y$, satisfying $G(y_1, y_2) \in \phi_g^{-1}(y_1, y_2)$. We choose our selection to be an element (depending on (y_1, y_2)) $G_{(y_1,y_2)} \in \mathcal{S}(\mathbb{R}^2)$ which is sufficiently close to $g_{(y_1,y_2)}(x_1, x_2)$ in $L^2_{dx_1dx_2}(\mathbb{R}^2)$ norm. We extend our selection to the set $S^- \setminus \Delta$ by corresponding to every $(y_1, y_2) \in S^- \setminus \Delta$ the value 0 considered now as an element of $\mathcal{S}(\mathbb{R}^2)$. Since $(y_1, y_2) \longrightarrow \int\limits_{\mathbb{R}^2} |g(x_1, x_2, y_1, y_2)|^2 \frac{iy_2}{2} dx_1 dx_2$ is integrable with respect to (y_1, y_2) we deduce that the extended selection $G_{(y_1,y_2)}(x_1, x_2) \in \mathcal{S}(\mathbb{R}^2)$ defines an integrable mapping with respect to y_1 via the formula

$$(y_1, y_2) \in S^- \longrightarrow \int\limits_{\mathbb{R}^2} |G_{(y_1,y_2)}(x_1, x_2)|^2 \frac{iy_2}{2} dx_1 dx_2.$$

Thus, every functional acting on a function defined by (3.2.12), slice-wise gives rise to the space $\mathcal{G}^-$. Taking the closures of both spaces with respect to the $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$-norm and the application of the relation (3.1.6) lead to the desired conclusion. The only thing that remains to show is the reflexivity of the space $H^2(T_{(S_H^-)^{int}})$. Let $\{g_n\}_n \subset H^2(T_{(S_H^-)^{int}})$ be a bounded sequence. Then (3.1.6) implies that this sequence is also bounded $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$. Reflexivity of the space $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$ implies that there exists a weakly convergent subsequence of $\{g_{n_k}\}_{n_k}$ of $\{g_n\}_n$. Thus, it is weakly convergent with respect to $\widetilde{\mathcal{G}^-}$ too as a subset of $H^2(T_{(S_H^-)^{int}})$, taking into account (3.1.6).$\diamondsuit$
Similarly, we have the following

Corollary 3.2.2 *Consider the subspace of the space* $L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1)$, *where* $\Phi_1(\zeta_1, \zeta_2) = (\frac{\zeta_1 - \bar{\zeta}_1}{2i})^2 + (\frac{\zeta_2 - \bar{\zeta}_2}{2i})^2 - 1$ *is the defining function for the tube* T_{B_1}, *defined by*

$$\mathcal{G}_{S^1} = \{h \in L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1) : \ \textit{every restriction} \ \ h|_{\mathbb{R}^2 \times i\{(y_1,y_2)\}} \in \mathcal{S}(\mathbb{R}^2)\} \qquad (3.2.14)$$

Then $(H^2(T_{B_1}))' = \widetilde{\mathcal{G}}_{S^1}$, *where* $\widetilde{\mathcal{G}}_{S^1}$ *is* $L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1)$*-closure of the space* (3.2.14). *Furthermore, the space* $H^2(T_{B_1})$ *is reflexive.*

Proof: The proof follows along the line of the previous corollary and is given here for the completeness of presentation. Actually, every $F \in H^2(T_{B_1})$ has boundary values on $\mathbb{R}^2 \times iS^1$ denoted also by F and it is the $L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1)$-limits of a sequence $\{F_{n_k}\} \subset H^2(T_{B_1})$ satisfying

$$F_{n_k}(x + iy) = \int\limits_{\mathbb{R}^2} f_{n_k}(t) e^{2\pi i z \cdot t} dt \in H^2(T_{B_1}) \qquad (3.2.15)$$

(3.2.15), where $f_{n_k} \in \mathcal{S}(\mathbb{R}^2)$. The proof of this claim is the repetition of the one presented in the proof of the previous lemma. Thus, in order to determine $L \in (H^2(T_{B_1}))'$, it is enough to determine it on elements of $H^2(T_{B_1})$ of the form (3.2.15). In this case $F_{n_k}(x+iy) \in \mathcal{S}(\mathbb{R}^2)$, for y-fixed, while remaining $F_{n_k} \in L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1)$. Thus, following the reasoning of the previous corollary with respect to the Michael selection principle, every functional acting on such a function, slice-wise gives rise to the space $\mathcal{G}_{S^1}$. Taking the closures of both spaces with respect to the $L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1)$-norm and the application of the relation (3.1.6) lead to the desired conclusion. The only thing that remains to show is the reflexivity of the space $H^2(T_{B_1})$. Let $\{g_n\}_n \subset H^2(T_{B_1})$ be a bounded sequence. Then (3.1.7) implies that this sequence is also bounded $L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1)$. Reflexivity of the space $L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1)$ implies that there exists a weakly convergent subsequence of $\{g_{n_k}\}_{n_k}$ of $\{g_n\}_n$. Thus, it is weakly convergent with respect to $\widetilde{\mathcal{G}}_{S^1}$ too as a subset of $H^2(T_{B_1})$, taking into account (3.1.7).$\diamondsuit$

We note however, that this description of the dual space is rather restrictive, because it does not provide us with the exact information about the structure of the space $\mathcal{G}^-$ and the space $\mathcal{G}_{S^1}$.

Chapter 4

Integral representation for the space $\mathbf{H^2(T_{(S_H^-)^{int}})}$

Our purpose is to describe this space through Cauchy-Fantappie type formulas to be found in [13]. To be more specific, we recall the closed half-strips (which are convex sets) in $i\mathbb{R}^2$ defined by (2.4.1). Using the interiors $(S_H^-)^{int}, (S_H^+)^{int}$ of the closed half-strips from (2.4.1) we consider the tubular domains defined in (2.4.2).

Following ([14], [13]), we have derived in Chapter 2 that the tubular domains $T_{(S_H^-)^{int}}$ and $T_{(S_H^+)^{int}}$ defined by (2.4.2) are both tubular domains of type one. The corresponding (to the type of the domain cones) conjugate cones are the half-planes

$$\begin{aligned} P(S_H^-)^{int} &= \{(t_1,t_2)\in\mathbb{R}^2 : t_1>0\} \\ &= \{(t_1,t_2)\in\mathbb{R}^2 : t_1y_1+t_2 0\geq 0,\ y_1\geq 0\} \\ P(S_H^+)^{int} &= \{(t_1,t_2)\in\mathbb{R}^2 : t_1<0\} \\ &= \{(t_1,t_2)\in\mathbb{R}^2 : t_1y_1+t_2 0\geq 0,\ y_1\leq 0\} \end{aligned} \tag{4.0.1}$$

It is easy to see that the half planes (4.0.1) are cones of the corresponding half-circles. That is, if S^+ and S^- denote the closed half circles on the unit circle $S=\{(t_1,t_2)\in\mathbb{R}^2 : t_1^2+t_2^2=1\}$, corresponding to $t_1>0$ and $t_1<0$ correspondingly, then

$$\begin{aligned} P(S_H^-)^{int} &= \{\alpha S^+,\ \alpha>0,(t_1,t_2)\in S^+\} \\ P(S_H^+)^{int} &= \{\alpha S^-,\ \alpha>0,(t_1,t_2)\in S^-\} \end{aligned} \tag{4.0.2}$$

In particular, $P(S_H^{\pm})^{int}$ are the sets of generators on S^1 for the conjugate cones of the cones defining the type of the tubes.

We now observe that the boundary ∂S_H^- is smooth and is the union of half circle with parallel lines. For every vector $t \in \partial S_H^-$, where the boundary is strictly convex, one can correspond a unique unit vector $\xi \in S^1$ so that $< \xi, y >= p$, for some $p = a(\xi)$- the supporting hyperplane to S_H^- at t and satisfying $< \xi, y >< a(\xi)$ for $y \in (S_H^-)^{int}$. Near the point of strict convexity of ∂S_H^-, that is , where ∂S_H^- is described by $\Phi_1^-(y) = y_1^2 + y_2^2 - 1$, $-1 < y_1 < 0$, the domain S_H^- is characterized by $\Phi_1^-(y) < 0$ and its boundary by $\Phi_1^-(y) = 0$. Thus, for (y_1, y_2) satisfying $y_1^2 + y_2^2 - 1 = 0$, $-1 \leq y_1 < 0$, one has that the vector ξ realizing the supporting hyperplane (line) at the present point is the vector

$$\xi(y) = \frac{-\nabla\Phi_1^-(y)}{||\nabla\Phi_1^-(y)||} \in S^+ \subset P(S_H^-)^{int}$$

The points $\{(y_1, y_2) \in \partial S_H^- : y_1 \in [0, \infty),\ y_2 = \pm 1\}$ are not points of strict convexity. The above correspondence means in this case that to the vector $(0, 1) \in S^1$ there correspond the points $(y_1, 1) \in \partial S_H^-$ and to the vector $(0, -1) \in S^1$ there correspond the points $(y_1, -1) \in \partial S_H^-$.

This leads us to consider the skeleton $\Omega_{S_H^-}$ ([13]).

Definition 4.0.1 *The skeleton $\Omega_{(S_H^-)^{int}}$ of the base of the tube $T_{(S_H^-)^{int}}$ is defined to be the set of $\xi(y) \in S^1$ realizing the supporting hyperplane to ∂S_H^- at a unique $y \in \partial S_H^-$.*

Direct computation shows that

$$\Omega_{S_H^-} = S^-.$$

Furthermore, one observes that $T_{(S_H^-)^{int}} = \{z \in \mathbb{C}^2 : \Im(< \xi, z >) < a(\xi),\ \forall \xi \in S^+\}$.

Now we are ready to formulate and prove the following proposition following closely the ideas from ([14], [13]).

Proposition 4.0.1 *Let f be function holomorphic in $T_{(S_H^-)^{int}}$ so that for every $(y_1, y_2) \in \partial S_H^-$ the limit $\lim_{r \longrightarrow 1^-} f(x_1 + iry_1, x_2 + iry_2) = f(x_1 + iy_1, x_2 + iy_2)$. Assume further that for every $(z_1, z_2) \in \overline{T}_{(S_H^-)^{int}} \subset \mathbb{C}^2$ with $(y_1, y_2) = (\Im z_1, \Im z_2)$ fixed, the restriction $f|_{\mathbb{R}^2 \times i\{(y_1, y_2)\}}$ is in $\mathcal{S}(R^2)$. If $f|_{\mathbb{R}^2 \times i\{(y_1, y_2)\}}$ is continuous with respect to $(y_1, y_2) = (\Im z_1, \Im z_2) \in S^-$ and* *is also in* $L^2(\partial T_{(S_H^-)^{int}})$*, then*

$$f(w) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{f(z)\|\nabla\Phi_1^-(y)\|^2 d\theta dx_1 dx_2}{(<\nabla\Phi_1^-(y), z-w>)^2},\ w \in T_{(S_H^-)^{int}}. \tag{4.0.3}$$

Proof: The Radon transform of the function f is $\check{f}(\xi,p) = \int\limits_{<\xi,z>=p} f(z)d\omega_\xi$, where the integral is over the hyperplane $<\xi,z>=p$, $\xi = \frac{-\nabla\Phi_1^-(y)}{||\nabla\Phi_1^-(y)||} \in S^+$, chosen in a such a way that $\Im z = y \in (S_H^-)^{int}$ and $<\xi,y>=\Im p$. The value $\check{f}(\xi,p)$ is independent from the choice of y satisfying $<\xi,y>=\Im p$. Thus the resulting function $\check{f}(\xi,p)$ is holomorphic function of p in the strip $-a(-\xi) < \Im p < a(\xi)$. Furthermore, $\check{f}(\xi,p) \neq 0$, since both, ξ and $-\xi$ do not belong to S^+. We observe that $\check{f}(\xi,p) \neq 0$ for $\Im z = y \in (S_H^-)^{int}$ fixed is identical to the Radon Transform $\check{f}(\xi,\Re p) \neq 0$, as a function of $\Re z = x \in \mathbb{R}^2$. Radon Transform Inversion Formula for $n=2$ implies that one has for every $w \in T_{(S_H^-)^{int}}$ the following

$$\begin{aligned}
f(w) &= \frac{(-1)^2(2-1)!}{(2\pi i)^2} \int\limits_{S^+} d\xi \int\limits_{\Im p=a(\xi)} \frac{\check{f}(\xi,p)dp}{(p-<\xi,w>)^2} \\
&= \frac{1}{(2\pi i)^2} \int\limits_{S^+} d\xi \int\limits_{\Im z=y(\xi)} \frac{f(z)dz}{(<\xi,z-w>)^2} \\
&= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2\times\Omega_{S_H^-}} \frac{f(z)\|\nabla\Phi_1^-(y)\|^2 d\theta dx_1 dx_2}{(<\nabla\Phi_1^-(y), z-w>)^2} \\
&= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2\times iS^-} \frac{f(z)\|\nabla\Phi_1^-(y)\|^2 d\theta dx_1 dx_2}{(<\nabla\Phi_1^-(y), z-w>)^2},
\end{aligned} \tag{4.0.4}$$

where $d\theta = -d\xi$ is a measure on $\overline{S^-}$, making the measure $d\theta dx_1 dx_2$ equivalent to a three dimensional Lebesgue measure. We remark here that the first equality in (4.0.4) is just the Radon inversion formula involving the first derivative with respect to p of $\check{f}(\xi,\Re p)$, represented by the inner integral. The second equality is realized by replacing $\check{f}(\xi,\Re p)$ with its equal $\check{f}(\xi,\Re p) = \int\limits_{<\xi,\Re z>=\Re p} f(\Re z)d\omega_\xi$, taking into account that 1-form $d\omega_\xi$ is chosen, by the definition of the Radon transform, to satisfy $d\omega_\xi dp = dx_1 dx_2$. The last equality follows from the discussion that preceded the formulation of the proposition. $\diamondsuit$

Recall that the measure $\mu_{\Phi_1^-}(\zeta,\bar\zeta) = \partial\Phi_1^-(\zeta,\bar\zeta) \wedge \partial\bar\partial\Phi_1^-(\zeta,\bar\zeta)$ is equivalent to the Lebesgue measure also. This observation and the concluding remarks of the last proof allow to rewrite (4.0.3) as a classic Cauchy-Fantappie formula for $w \in T_{(S_H^-)^{int}}$:

$$f(w) = \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{f(\zeta)\left(\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - w >)^2},\ w \in T_{(S_H^-)^{int}} \tag{4.0.5}$$

$$= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{f(\zeta)}{(1- < \frac{\nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta})}{<\nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}),\zeta>}, w >)^2} \frac{\left(\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta >)^2}.$$

One more observation is needed to clarify the relation between the functions satisfying (4.0.5) and functions satisfying (4.0.3): whenever $z \in T_{(S_H^-)^{int}}$, the restriction of (4.0.3) to the plane $\mathbb{R}^2 \times i(\Im z_1, \Im z_2)$ is a rapidly decreasing function. The crucial information available to us here is that (4.0.3) is holomorphic for $z \in T_{(S_H^-)^{int}}$. Therefore it is not the case when $z \in \partial T_{(S_H^-)^{int}}$. That is, the theorem below has to be understood in the context of understanding the boundary behavior of the function in question. Therefore when we say that a function F is representable by (4.0.5) then it means that F is holomorphic in the tube and has directional boundary values with respect to $y \in \partial T_{(S_H^-)^{int}}$, i.e. $\lim\limits_{r \longrightarrow 1^-} F(x_1 + iry_1, x_2 + iry_2) = F(x_1 + iy_1, x_2 + iy_2)$. Furthermore, in addition to $F \in L^2(\mathbb{R}^2 \times iS^-)$ we require

$$(y_1, y_2) \longrightarrow \int\limits_{\mathbb{R}^2} |F(x_1, x_2, y_1, y_2)|^2 dx_1 dx_2$$

to be continuous when $y \in \partial T_{(S_H^-)^{int}}$. We say, for simplicity, that F is continuous along the boundary of the base of the tube. Before proceeding any further, we set $S_{H_r}^- = r \cdot S_H^-$, $0 < r \leq 1$. Then $(S_{H_r}^-)^{int} = r \cdot (S_H^-)^{int}$ and we have the following

Theorem 4.0.2 *Consider the tube domain* $T_{(S_H^-)^{int}} \subset \mathbb{C}^2$. *If* $F \in H^2(T_{(S_H^-)^{int}})$ *then* F *satisfies* (4.0.5). *If* F *satisfies* (4.0.5) *then* $F \in H^2(T_{(S_H^-)^{int}})$.

Proof : If $F \in H^2(T_{(S_H^-)^{int}})$, then its Cauchy-Fantappie $C_{Fa}(F)$ transform is given by

$$C_{Fa}(F)(w) = \frac{(2-1)!}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{F(\zeta)\|\nabla\Phi_1^-(y)\|^2 d\theta dx_1 dx_2}{(< \nabla\Phi_1^-(y), \zeta - w >)^2}$$

$$= \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{F(\zeta)\left(\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - w >)^2},\ w \in T_{(S_H^-)^{int}}, \tag{4.0.6}$$

because of the relation (4.0.5). We want to prove that $C_{F_a}(F)(w) = F(w)$ for $w \in T_{(S_H^-)^{int}}$.

Prop.4.0.1 and relation (4.0.6) imply that

$$\begin{aligned}
C_{F_a}(F)(w) - C_{F_a}(F_{n_k})(w) &= C_{F_a}(F)(w) - F_{n_k}(w) \\
&= \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{(F - F_{n_k})(\zeta)\left(\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - w >)^2},
\end{aligned}$$

whenever $w \in T_{(S_H^-)^{int}}$ and $\{F_{n_k}\}$ is a sequence such that $F_{n_k} \longrightarrow F$ in $\|\cdot\|_{H^2(T_{(S_H^-)^{int}})}$-norm. Since $F_{n_k} \longrightarrow F$ in $\|\cdot\|_{H^2(T_{(S_H^-)^{int}})}$-norm, taking the limit when $n_k \longrightarrow \infty$ leads to the desired conclusion.

In order to prove the second claim, we assume first that F satisfies the conditions of Prop.4.0.1 and thus is expressed by (4.0.3) or , equivalently, by (4.0.5). Hence, F being holomorphic in the tube $T_{(S_H^-)^{int}}$, implies that it is holomorphic in the closure of a more "narrow" tube $T_{(S_{H_r}^-)^{int}}$, $0 < r < 1$. The convexity of S^- and straightforward calculations imply the inequality

$$2\Re < \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - z > \geq \Phi_1^-(\zeta) - \Phi_1^-(z) + \gamma\|\zeta - z\|^2, \tag{4.0.7}$$

whenever $\zeta \in \mathbb{R}^2 \times iS^-$ and $z \in T_{(S_{H_r}^-)^{int}}$ for $0 < r \longrightarrow 1^-$. Using (4.0.7) and (4.0.5) one deduces, using Holder inequality, that the function

$$\begin{aligned}
|F(z)| &\leq \frac{1}{\gamma} \int\limits_{\mathbb{R}^2 \times iS^-} |F(\zeta)| \frac{1}{\|\zeta - z\|^4} |\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})| \\
&\leq \|F\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)} \left\|\frac{1}{\gamma\|z - \zeta\|^4}\right\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)} = \alpha(r),
\end{aligned}$$

is bounded by a constant $\alpha(r)$ dependent on r, whenever $z \in T_{(S_{H_r}^-)^{int}}$. Thus, for $(\Im z_1, \Im z_2)$ fixed, using (4.0.3) one has from Tonelli's Theorem,

$$\begin{aligned}
&\int\limits_{\mathbb{R}^2} |F(\Re z_1, \Re z_2, \Im z_1, \Im z_2)|^2 d\Re z_1 d\Re z_2 \leq \alpha(r) \int\limits_{\mathbb{R}^2} |F(\Re z_1, \Re z_2, \Im z_1, \Im z_2)| d\Re z_1 d\Re z_2 \\
\leq\ & \alpha(r) \int\limits_{\mathbb{R}^2} \left(\int\limits_{\mathbb{R}^2 \times iS^-} \frac{|F(\zeta,\bar{\zeta})||\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})|}{(\sum\limits_{i=1}^{2}(\Im z_i - \Im\zeta_i)^2 + \sum\limits_{i=1}^{2}(\Re z_i - x_i)^2)^4} \right) d\Re z_1 d\Re z_2
\end{aligned}$$

$$\leq \alpha(r) \int\limits_{\mathbb{R}^2\times iS^-} \left(\int\limits_{\mathbb{R}^2} \frac{|F(\zeta,\bar{\zeta})| d\Re z_1 d\Re z_2}{(\sum_{i=1}^{2}(\Im z_i - \Im\zeta_i)^2 + \sum_{i=1}^{2}(\Re z_i - x_i)^2)^4} \right) |\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \bar{\partial}\partial\Phi_1^-(\zeta,\bar{\zeta})|, \tag{4.0.8}$$

where $x_i = \Re\zeta_i$, $i = 1, 2$, and $(\Im z_1 - \Im\zeta_1)^2 + (\Im z_2 - \Im\zeta_2)^2 \geq (1-r)^2 > 0$. Now, in order to compute the inner integral we make the change of variables $\Re z_1 - x_1 = \rho\cos\theta$, $\Re z_2 - x_2 = \rho\sin\theta$, $0 \leq \rho < +\infty$, $0 \leq \theta \leq 2\pi$ and obtain

$$\int\limits_{\mathbb{R}^2} \frac{d\Re z_1 d\Re z_2}{((1-r)^2 + (\Re z_1 - x_1)^2 + (\Re z_2 - x_2)^2)^4} \leq \int\limits_0^\infty \int\limits_0^{2\pi} \frac{\rho d\rho d\theta}{((1-r)^2+\rho^2)^4} \leq \frac{\pi}{3(1-r)^6}. \tag{4.0.9}$$

Thus, using (4.0.9), (4.0.8) becomes

$$\int\limits_{\mathbb{R}^2} |F(\Re z_1, \Re z_2, \Im z_1, \Im z_2)|^2 d\Re z_1 d\Re z_2 \leq \frac{\pi\alpha(r)}{3(1-r)^6} \int\limits_{\mathbb{R}^2\times iS^-} |F(\zeta,\bar{\zeta})||\partial\Phi_1^-(\zeta) \wedge \partial\bar{\partial}\Phi_1^-(\zeta)| \tag{4.0.10}$$

On the other hand, if the point $(\Im\zeta_{10}, \Im\zeta_{10}) = (y_{10}, y_{20})$ realizes the maximum of the continuous function

$$S^- \ni (\Im\zeta_1, \Im\zeta_2) \longrightarrow \int\limits_{\mathbb{R}^2} |F(x_1 + i\Im\zeta_1, x_2 + i\Im\zeta_2)| dx_1 dx_2,$$

then

$$\begin{aligned} \int\limits_{\mathbb{R}^2\times iS^-} |F(\zeta,\bar{\zeta})||\partial\Phi_1^-(\zeta) \wedge \bar{\partial}\partial\Phi_1^-(\zeta)| &= \int\limits_{S^-}\int\limits_{\mathbb{R}^2} |F(\zeta,\bar{\zeta})|| - y_1 dx_1 dx_2 dy_2 + y_2 dx_1 dx_2 dy_1| \\ &\leq C \int\limits_{\mathbb{R}^2} |F(x_1 + i\Im\zeta_{10}, x_2 + i\Im\zeta_{20})| dx_1 dx_2 \end{aligned} \tag{4.0.11}$$

Thus, for some constant C, (4.0.11) implies that

$$\int\limits_{\mathbb{R}^2} |F(\Re z_1, \Re z_2, \Im z_1, \Im z_2)|^2 d\Re z_1 d\Re z_2 \leq \frac{C\pi^2\alpha(r)}{(1-r)^6} \int\limits_{\mathbb{R}^2} |F(x_1, x_2, y_{10}, y_{20})| dx_1 dx_2, \tag{4.0.12}$$

where the last integral is finite because the restriction of $F(x_1+iy_{10}, x_2+iy_{20})|_{\mathbb{R}^2} \in \mathcal{S}(\mathbb{R}^2)$. Combining (4.0.8) and (4.0.12) we deduce that $F \in H^2(T_{(S^-_{H_r})^{int}})$. We now rewrite (3.0.3) for $F \in H^2(T_{(S^-_{H_r})^{int}})$ as follows: for any sequence $\{r_k\} \subset [0,1]$, $r_k \uparrow 1^-$ one has that $S^-_{H_{r_k}} = r_k \cdot S^-_H$. Thus, the restriction of $F(z)$ from (4.0.5) to the closure $\overline{T}_{(S^-_{H_{r_k}})^{int}}$ is given by

$$\begin{aligned} F_{r_k}(w) &= \int\limits_{\mathbb{R}^2} f(t) e^{2r_k \pi i w\cdot t} dt, \quad \text{whenever } w \in T_{(S^-_H)^{int}}, \\ &\text{and} \sup_{y\in (S^-_H)^{int}} \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4r_k\pi y\cdot t} dt < \beta^2(r_k), \end{aligned} \tag{4.0.13}$$

where the constant $\beta^2(r_k)$ depends on the base of the tube $S^-_{H_{r_k}}$. Since the function $F_{r_k}(z)$ (an element of $H^2(T_{(S^-_H)^{int}})$) is a restriction of the function $F(z)$ to the tube $T_{(S_{H^-_{r_k}})^{int}}$ the directional limit (along iy) exists and we denote it by $\int\limits_{\mathbb{R}^2} f(t)e^{2r_k\pi i z\cdot t}dt$, thus defining

$$F(z) = \int\limits_{\mathbb{R}^2} f(t) e^{2r_k\pi i z\cdot t} dt = \lim_{r_k \longrightarrow 1^-} \int\limits_{\mathbb{R}^2} f(t) e^{2r_k \pi i z \cdot t} dt$$

for every $z \in T_{(S^-_H)^{int}}$. Now the crucial step is to show that (4.0.13) is extended to the case when $r_k = 1$. We achieve it by proving first that the claim is valid since $f \in \mathcal{S}(\mathbb{R}^2)$ (inverse Fourier transform is a mapping from $\mathcal{S}(\mathbb{R}^2)$ into $\mathcal{S}(\mathbb{R}^2)$). The desired conclusion will follow then from the fact that every element $f \in L^2(\mathbb{R}^2)$ is a $L^2(\mathbb{R}^2)$ limit of sequence of functions $\{f_k\} \subset \mathcal{S}(\mathbb{R}^2)$ and from the way one extends the definition of the Fourier transform in the space $L^2(\mathbb{R}^2)$.

Let us begin by considering the integral

$$\begin{aligned} \int\limits_{\mathbb{R}^2} |F_{r_k}(x_1+iy_1, x_2+iy_2)|^2 dx_1 dx_2 &= \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi r_k y\cdot t} dt \\ &= \int\limits_0^{2\pi} \int\limits_0^{\infty} |f(\varrho,\theta)|^2 e^{-4\pi r_k \|y\| \varrho \cos\theta} \varrho \, d\varrho \, d\theta, \end{aligned}$$

where $f \in \mathcal{S}(\mathbb{R}^2)$ and y-fixed is assumed to be defining the "horizontal axis" for the use of polar coordinates. This integral cannot be bounded above (independently of $\|y\|$), when $r_k = 1$ and $\cos\theta < 0$. Thus, it is enough to consider the integral over the "left" hand-side half-plane:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{\infty} |f(\varrho,\theta)|^2 e^{-4\pi r_k \|y\| \varrho \cos\theta} \varrho d\varrho d\theta.$$

In general this is a non-solvable problem. However, in the present setting we do the following. Choosing sufficiently large $y_1^0 > 0$ we split the strip $(S_H^-)^{int}$ into two parts: the part $S_H^- \cap i\{(y_1, y_2) \in \mathbb{R}^2 : -1 < y_1 < y_1^0\}$ whose closure is compact and the unbounded part $S_H^- \cap i\{(y_1, y_2) \in \mathbb{R}^2 : y_1^0 < y_1\}$. Over the unbounded part of the strip one expresses an estimate like (4.0.11) in terms of a constant like $\frac{C\pi\alpha(r)}{(1-r)^6}$ but now written in terms of $(\Im z_1, \Im z_2) = (y_1, y_2) \in \partial S_H^- \cap i\{(y_1, y_2) \in \mathbb{R}^2 : -1 \leq y_1^0 < y_1\}$. That is , we rewrite (4.0.8) as $(y_1 - \Im\zeta_1)^2 - (y_2 - \Im\zeta_2)^2 \geq \frac{1}{2}\|y\|^2 > 0$. This is possible for suitably chosen $y_1^0 > 0$, because $(\Im\zeta_1, \Im\zeta_2) \in \overline{S^-}$. This way one can have the estimate $|F(z)| \leq \frac{2\|F\|_{L^2(\mathbb{R}^2 \times iS^-)}}{\|y\|^4}$ whenever $(y_1, y_2) \in \partial S_H^- \cap i\{(y_1, y_2) \in \mathbb{R}^2 : y_1^0 < y_1\}$. Furthermore, the bound on the right of (4.0.9) becomes $\frac{2\pi}{\|y\|^8}$. Thus, modified inequalities (4.0.10) and (4.0.11) give

$$\int_{\mathbb{R}^2} |F(\Re z_1, \Re z_2, \Im z_1, \Im z_2)|^2 d\Re z_1 d\Re z_2 \leq \frac{C_1 2\pi^2}{\|y\|^8} \int_{\mathbb{R}^2} |F(x_1, x_2, y_{10}, y_{20})| dx_1 dx_2, \quad (4.0.14)$$

whenever $(\Im z_1, \Im z_2) = (y_1, y_2) \in \partial S_H^- \cap \{(y_1, y_2) \in \mathbb{R}^2 : y_1^0 < y_1\}$, for $y_1^0 > 0$ large enough. It remains to show the boundedness of the integrals over the rest of the tube. Since point-wise

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{\infty} |f(\varrho,\theta)|^2 e^{-4\pi r_k \|y\| \varrho \cos\theta} \varrho d\varrho d\theta \leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{\infty} |f(\varrho,\theta)|^2 e^{-4\pi \|y\| \varrho \cos\theta} \varrho d\varrho d\theta,$$

on the "left" half-plane. The continuity of F along the boundary of the base of the tube implies the desired result. That is, we have shown that $F \in H^2(T_{(S_H^-)^{int}})$, with $F(z) = \int_{\mathbb{R}^2} f(t) e^{2\pi i z \cdot t} dt$, $z \in T_{(S_H^-)^{int}}$ whenever $f \in \mathcal{S}(\mathbb{R}^2)$. Using the definition of the Fourier Transform on $L^2(\mathbb{R}^2)$ we deduce that

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{\infty} |f(r,\theta)|^2 e^{-4\pi \|y\| r \cos\theta} r dr d\theta < +\infty, \quad (4.0.15)$$

for every $f \in L^2(\mathbb{R}^2)$. Finally, for every $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in T_{(S_H^-)^{int}}$ we have, using Plancherel's Theorem, that

$$\int\limits_{\mathbb{R}^2} |F(x_1, x_2, y_1, y_2)|^2 dx = \int\limits_{\mathbb{R}^2} |f(t)|^2 e^{-4\pi y \cdot t} dt < +\infty, \tag{4.0.16}$$

whenever $y \in S_H^-$ is fixed. Furthermore, using previously developed argument, one can deduce know that $F(z) \in H^2(T_{(S_H^-)^{int}})$, by splitting the set ∂S_H^- into non compact part $\partial S_H^- \cap i\{(y_1, y_2) \in \mathbb{R}^2 : \ y_1^0 < y_1\}$, $y_1^0 > 0$ and into compact part $\partial S_H^- \cap i\{(y_1, y_2) \in \mathbb{R}^2 : -1 \leq y_1 < y_1^0\}$. $\diamondsuit$

Next, we are going to examine the boundary behavior of the Cauchy-Fantappie type integral over $\mathbb{R}^2 \times iS^-$ of a function $f \in L^2(\mathbb{R}^2 \times iS^-)$. We begin by considering the other form of (4.0.5) but for a function $f \in L^2(\mathbb{R}^2 \times iS^-)$ so that for every the restriction $f|_{\mathbb{R}^2 \times i\{(y_1, y_2)\}} \in \mathcal{S}(\mathbb{R}^2)$. Namely, the identity

$$C_{F_a}(f)(z) = \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{f(\zeta)}{(1- < \frac{\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta})}{<\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta}), \zeta>}, z >)^2} \frac{\left(\partial \Phi_1^-(\zeta, \bar{\zeta}) \wedge \partial \bar{\partial} \Phi_1^-(\zeta, \bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta}) \zeta >)^2}, \tag{4.0.17}$$

whenever $z \in T_{(S_H^-)^{int}}$, describing the Cauchy-Fantappie type integral. It was shown previously that $\frac{\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta})}{<\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta}), \zeta>}$ is an element of the generalized dual complement $T^*_{(S_H^-)^{int}}$. For every $(\zeta, \bar{\zeta}) \in \mathbb{R}^2 \times iS^-$, the denominator $1- < \frac{\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta})}{<\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta}), \zeta>}, z >$ is the equation of hyperplane, which does not intersect the closure of the tube $T_{(S_H^-)^{int}}$ whenever $\| \frac{\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta})}{<\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta}), \zeta>} \| < 1$. Therefore, the singularities of the integral occur when $\| \frac{\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta})}{<\nabla_\zeta \Phi_1^-(\zeta, \bar{\zeta}), \zeta>} \| = 1$ and $\|\zeta\| = 1$ or, taking into account the endpoints of the semicircle S^-, when $w = (w_1, w_2) \in T^*_{(S_H^-)^{int}}$, $\|w\| = 1$ and $\zeta \in \partial T_{(S_H^-)^{int}}$. Analytically, the above are expressed by the realization of the points of generalized dual having length equal to one, that is $((0, -iy_1), (0, -iy_2))$ satisfying $y_1^2 + y_2^2 = 1$, $-1 \leq y_1 \leq 0$ while $z \in T_{(S_H^-)^{int}}$ and tends to the boundary. This means $C_{F_a}(f)(z)$ is holomorphic in a neighborhood of the part of the tube $T_{(S_H^-)^{int}}$ whose imaginary part of the base is described by the $\{(y_1, y_2) \in S_H^- : y_1 > 0\}$. Thus it remains only to examine the existence of the boundary values of $C_{F_a}(f)(z)$ at the singular points $((0, -iy_1), (0, -iy_2))$ satisfying $y_1^2 + y_2^2 = 1$, $-1 \leq y_1 \leq 0$. In our particular setting this problem is reduced (see the proof of next corollary) to the extension of the domain of convergence of the Fourier Transform.$\diamondsuit$

Lemma 4.0.1 *The value of the integral*

$$\int\limits_{\mathbb{R}^2\times iS^-} \frac{\Omega(\Phi_1^-(\zeta,\bar{\zeta}))}{< \nabla\Phi_1^-(\zeta,\bar{\zeta}),\zeta >^2} = \int\limits_{\mathbb{R}^2\times iS^-} \frac{\partial\Phi_1^-(\zeta,\bar{\zeta})\wedge\partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})}{< \nabla\Phi_1^-(\zeta,\bar{\zeta}),\zeta >^4} \tag{4.0.18}$$

is finite.

Proof: We have the following

$$\frac{1}{|(y_1^2+y_2^2-i(x_1y_1+x_2y)^4|} = \frac{1}{(1+\|x\|^2\|y\|^2\cos^2\theta)^2} = \frac{1}{(1+\|x\|^2y_1^2)^2},$$

where $\|\cdot\|$ is the usual norm in $\mathbb{R}^2$ and θ is the angle between the vectors x and y in $\mathbb{R}^2$ and we assume that $y_1 = \|y\|\cos\theta$. Furthermore, the integration on $\mathbb{R}^2\times iS^-$, for $(\zeta,\bar{\zeta})\in\mathbb{R}^2\times iS^-$, is reduced to the integration of the forms $y_1dx_1dx_2dy_2$ and $y_2dx_1dx_2dy_1$, while $S^- = \lim\limits_{\epsilon\longrightarrow 0} S_\epsilon^-$, where $S_\epsilon^- = \{(-\sqrt{1-y_2^2},y_2),\ y_2\in[-1+\epsilon,1-\epsilon]\}$. Thus, we have that (understanding the integrals on the arc as generalized ones)

$$\begin{aligned}\int\limits_{\mathbb{R}^2\times iS_\epsilon^-} \left|\frac{\partial\Phi_1^-(\zeta,\bar{\zeta})\wedge\partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})}{< \nabla_\zeta\Phi_1^-(\zeta,\bar{\zeta}),\zeta >^4}\right| &\leq \left|\int\limits_{\mathbb{R}^2}\int\limits_{-1+\epsilon}^{1-\epsilon}(-\sqrt{1-y_2^2})\frac{dx_1dx_2dy_2}{(1+\|x\|^2y_1^2)^2}\right| \\ &+ \left|\int\limits_{\mathbb{R}^2}\int\limits_{-1+\epsilon}^{1-\epsilon}\frac{y_2^2}{\sqrt{1-y_2^2}}\frac{dx_1dx_2dy_2}{(1+\|x\|^2y_1^2)^2}\right|\end{aligned}$$

We turn now to the evaluation of every integral in the last relation. We have, since

$$\left|\int\limits_{\mathbb{R}^2}\int\limits_{-1+\epsilon}^{1-\epsilon}(-\sqrt{1-y_2^2})\frac{dx_1dx_2dy_2}{(1+\|x\|^2y_1^2)^2}\right| = 2\pi\int\limits_{[-1+\epsilon,1-\epsilon]}\frac{dy_2}{2\sqrt{1-y_2^2}} = \pi\theta_\epsilon,$$

where $\pm\theta_\epsilon \longrightarrow \pm\frac{\pi}{2}$. Similarly, the other integral is

$$\begin{aligned}
|\int_{\mathbb{R}^2}\int_{-1+\epsilon}^{1-\epsilon} \frac{y_2^2}{\sqrt{1-y_2^2}}\frac{dx_1dx_2dy_2}{(1+\|x\|^2y_1^2)^2}| &= |\int_{\mathbb{R}^2}\int_{-1+\epsilon}^{1-\epsilon} -\frac{1-y_2^2-1}{\sqrt{1-y_2^2}}\frac{dx_1dx_2dy_2}{(1+\|x\|^2y_1^2)^2}| \\
&\leq \pi\int_{-1+\epsilon}^{1-\epsilon}\sqrt{(1-y_2^2)}dy_2+\pi|\int_{-1+\epsilon}^{1-\epsilon}\frac{1}{\sqrt{(1-y_2^2)}}dy_2| \\
&\leq 2\pi^2.
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^2\times iS^-} |\frac{\partial\Phi_1^-(\zeta,\bar\zeta)\wedge\partial\bar\partial\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>^4}| = \lim_{\epsilon\longrightarrow 0}\int_{\mathbb{R}^2\times iS_\epsilon^-} |\frac{\partial\Phi_1^-(\zeta,\bar\zeta)\wedge\partial\bar\partial\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>^4} \leq 2\pi^2$$

Thus we have the desired result. $\diamondsuit$

The proof of the last Theorem, the concept of generalized dual $T^*_{(S_H^-)^{int}}$ introduced in Chapter 2 and the above lemma lead to the following

Corollary 4.0.3 *Let us assume that the function $h \in L^2(\mathbb{R}^2\times iS^-)$ so that for every the restriction $h|_{\mathbb{R}^2\times i\{(y_1,y_2)\}} \in \mathcal{S}(\mathbb{R}^2)$. Assume further that $\Phi_1^-(\zeta,\bar\zeta)$ is the defining function for the tube $T_{(S_H^-)^{int}}$. Then the Cauchy-Fantappie transform of the function h defined by*

$$C_{Fa}(h)(z) = \int_{\mathbb{R}^2\times iS^-} \frac{h(\zeta,\bar\zeta)\Omega(\Phi_1^-(\zeta,\bar\zeta))}{(1-<\frac{\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>},z>)^2},\ z\in T_{(S_H^-)^{int}} \tag{4.0.19}$$

is holomorphic in $T_{(S_H^-)^{int}}$. The function $C_{Fa}(h) \in H^2(T_{(S_H^-)^{int}})$, provided that the ratio $\frac{|h(\zeta,\bar\zeta)|}{(1-r)^2}$ remains bounded, whenever $(\Im z_1, \Im z_2) \longrightarrow (0,\pm i)$ within an angle $\measuredangle\alpha \leq \frac{\pi}{4}$ and $(\zeta,\bar\zeta) \in \partial T_{(S_H^-)^{int}}$ are near $(0,0,0,\pm i)$. The differential form-measure is

$$\Omega(\Phi_1^-(\zeta,\bar\zeta)) = \frac{\partial\Phi_1^-(\zeta,\bar\zeta)\wedge\partial\bar\partial\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>^2} \tag{4.0.20}$$

Proof: The analyticity of $C_{Fa}(h)(z)$ follows from the fact that $\frac{\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>} \in T^*_{(S_H^-)^{int}}$ for any $\zeta=(\zeta_1,\zeta_2)\in\mathbb{R}^2\times iS^-$, thus implying that $1-<\frac{\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>},z>\neq 0$ for every

$\zeta \in T_{(S_H^-)^{int}}$. Hence, we can rewrite the desired formula as

$$C_{Fa}(h)(z) = \int\limits_{\mathbb{R}^2 \times iS^-} \frac{h(\zeta,\bar{\zeta})\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})}{< \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - z >^2},\ z \in T_{(S_H^-)^{int}} \tag{4.0.21}$$

Using (4.0.7) and Holder inequality, like in the proof of the Theorem 4.0.2, one can show that for any $0 < r < 1$,

$$|C_{Fa}(h)(z)| \leq \beta(r)\ \forall\ z \in T_{(S_{H_r}^-)^{int}}, \tag{4.0.22}$$

where the constant $\beta(r)$ depends on the tube $T_{(S_{H_r}^-)^{int}}$. Now, using the same line of arguments that led to (4.0.8), (4.0.9),(4.0.10),(4.0.11), we deduce that for $\Im z = (\Im z_1, \Im z_2)$-fixed the inequality

$$\int\limits_{\mathbb{R}^2} |C_{Fa}(h)(\Re z_1, \Re z_2, \Im z)|^2 d\Re z_1 d\Re z_2 \leq \frac{K\pi^2\delta(r)}{(1-r)^8} \int\limits_{\mathbb{R}^2} |h(x_1, x_2, y_{10}, y_{20})| dx_1 dx_2 \tag{4.0.23}$$

holds for some constant $K > 0$, since the last integral is finite because the restriction of $h(x + iy_{10}, x + iy_{20})|_{\mathbb{R}^2} \in \mathcal{S}(\mathbb{R}^2)$. The inequality (4.0.23) is similar to (4.0.12). Combining (4.0.8) and (4.0.23) we deduce that $C_{Fa}(h) \in H^2(T_{(S_{H_r}^-)^{int}})$. We now rewrite (3.0.3) for $C_{Fa}(h) \in H^2(T_{(S_{H_r}^-)^{int}})$ as follows. For any sequence $\{r_k\} \subset [0,1]$, $r_k \uparrow 1^-$ one has that $S_{H_{r_k}}^- = r_k \cdot S_H^-$. Thus, the restriction of $C_{Fa}(h)(z)$ from (4.0.5) to the closure $\overline{T}_{(S_{H_{r_k}}^-)^{int}}$ is given by

$$\begin{aligned} C_{Fa}^{r_k}(h)(z) \quad &= \quad \int\limits_{\mathbb{R}^2} j(t) e^{2r_k\pi i z\cdot t} dt, \quad \text{whenever } z \in T_{(S_H^-)^{int}}, \\ &\text{and} \quad \sup_{y \in (S_H^-)^{int}} \int\limits_{\mathbb{R}^2} |j(t)|^2 e^{-4r_k\pi y\cdot t} dt < \gamma^2(r_k), \end{aligned} \tag{4.0.24}$$

where the constant $\gamma^2(r_k)$ depends on the base of the tube $S_{H_{r_k}}^-$. The difference here with respect to the proof of Theorem 4.0.2 is that we do not know that the function $j(t) \in \mathcal{S}(\mathbb{R}^2)$, we know that $j(t) \in L^2(\mathbb{R}^2)$ instead. The discussion that preceded the present corollary has shown that $C_{Fa}^{r_k}(h)(z) = C_{Fa}(h)(z)$, $z \in T_{(S_{H_{r_k}}^-)^{int}}$ for every r_k has directional boundary values for every $z_0 = (z_{01}, z_{02}) \in \partial T_{(S_H^-)^{int}}$, with $Imz_{01} > 0$, but we do not know how it is realized. In other words, if it is realized as a Fourier transform of

j at the boundary point z_0, with $Imz_{01} > 0$. At any such boundary point consider the sub-tube $T_{\Pi_{z_0}} \subset T_{(S_H^-)^{int}}$, having as a base the convex polygon with one of its vertices at $z_0 \in \partial T_{(S_H^-)^{int}}$. The fact that the boundary values of $C_{Fa}(h)(z)$ in the tube $T_{(S_H^-)^{int}}$ are bounded (locally) (especially in $\{(0,0)\} \times iS^-$) we deduce that $C_{Fa}(h)(z) \in H^2(T_{\Pi_{z_0}})$. Thus one can define $C_{Fa}(h)(z_0) = \int\limits_{\mathbb{R}^2} j(t)e^{2\pi i z_0 \cdot t}dt$, whenever $z_0 \in \partial T_{(S_H^-)^{int}}$, $Imz_{01} > 0$. Since the arc S^+ (the right half of the circumference) is open, is contained in the strip $(S_H^-)^{int}$ and the integral $\int\limits_{\mathbb{R}^2} |j(t)|^2 e^{-4\pi y \cdot t}dt$ converges whenever $y \in S^-$. Thus, the only points where we do not know the behavior of $C_{F_a}(h)(z)$ are the boundary points whose imaginary parts are $(y_1, y_2) = (0, \pm i)$. These cases are covered by the assumptions of the corollary. Once we have established the existence of boundary values, the boundedness of integrals follows as in the Theorem 4.0.2. Thus $C_{Fa}(h) \in H^2(T_{(S_H^-)^{int}})$. Note however, that in this case $C_{F_a}(h)(z_0) \neq h(z_0)$ in general for $z_0 \in \mathbb{R}^2 \times iS^- \subset \partial T_{(S_H^-)^{int}}$. $\diamondsuit$

Chapter 5

Version of Aizenberg-Martineau duality for the space $\mathbf{H^2(T_{(S_H^-)^{int}})}$

In the present section we will formulate and prove duality results for spaces $H^2(T_{(S_H^-)^{int}})$ in the spirit of results of Aizenberg-Martineau, [1, 3, 21, 22]. Similar results are valid for space $H^2(T_{(S_H^+)^{int}})$. Using the relation (2.5.1) from Chapter 2 we consider the star compact $T^*_{(S_H^-)^{int}} = \overline{\mathcal{V}_2^-}$ and denote by $\mathcal{H}(T^*_{(S_H^-)^{int}})$ the space of holomorphic functions in a neighborhood U of the compact $T^*_{(S_H^-)^{int}}$.

The notation of the $(n, n-1)$-form $\Omega(\Phi_1^-(\zeta,\bar{\zeta})) = \frac{\partial\Phi_1^-(\zeta,\bar{\zeta})\wedge\partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})}{(<\nabla_\zeta\Phi_1^-(\zeta,\bar{\zeta}),\zeta>)^2}$ and the vector function $\tau(\Phi_1^-) = (\tau_1(\Phi_1^-), \ldots, \tau_n(\Phi_1^-))$, $\tau_i(\Phi_1^-) = \frac{\partial_{\zeta_i}\Phi_1^-(\zeta,\bar{\zeta})}{<\nabla_\zeta\Phi_1^-(\zeta,\bar{\zeta}),\zeta>}$, $i = 1, 2$ is consistent with notation in ([1]).

We begin by observing that in the integral representation of $f \in H^2(T_{(S_H^-)^{int}})$, given by (4.0.5),where the only part of the boundary $\partial T_{(S_H^-)^{int}}$ present is the set $\mathbb{R}^2 \times iS^-$. Thus for every $\zeta \in \mathbb{R}^2 \times iS^-$ there exists a tangent complex line

$$\{z \in \mathbb{C}^2 :< \nabla_\zeta\Phi_1^-(\zeta,\bar{\zeta}), \zeta - z >= 0\} = \{z \in \mathbb{C}^2 : \tau_1(\Phi_1^-)z_1 + \tau_2(\Phi_1^-)z_2 = 1\}$$

where $\tau_i(\Phi_1^-) = \frac{\partial_{\zeta_i}\Phi_1^-(\zeta,\bar{\zeta})}{<\nabla_\zeta\Phi_1^-(\zeta,\bar{\zeta}),\zeta>}$, $i = 1, 2$, not intersecting $T_{(S_H^-)^{int}}$. Observe that $< \nabla_\zeta\Phi_1^-(\zeta,\bar{\zeta}), \zeta >\neq 0$, since $0 \in T_{(S_H^-)^{int}}$. Furthermore, for $\zeta \in \mathbb{R}^2 \times iS^-$ and from Lemma

2.5.1 we have that

$$\begin{aligned}\tau_1(\Phi_1^-) &= \frac{\partial_{\zeta_1}\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>} = \frac{y_1(x_1y_1+x_2y_2)-iy_1(y_1^2+y_2^2)}{(y_1^2+y_2^2)^2+(x_1y_1+x_2y_2)^2}\\ \tau_2(\Phi_1^-) &= \frac{\partial_{\zeta_2}\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>} = \frac{y_2(x_1y_1+x_2y_2)-iy_2(y_1^2+y_2^2)}{(y_1^2+y_2^2)^2+(x_1y_1+x_2y_2)^2}\end{aligned}$$

Using the fact that the compact $T^*_{(S_H^-)^{int}}$ is star, the same computations for the vector

$$\left(\frac{r\partial_{\zeta_1}\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>},\frac{r\partial_{\zeta_2}\Phi_1^-(\zeta,\bar\zeta)}{<\nabla_\zeta\Phi_1^-(\zeta,\bar\zeta),\zeta>}\right),$$

where $0<r\leq 1$, as in Lemma 2.5.1 imply that

$$r^2\tau_1^2(\Phi_1^-)+r^2\tau_2^2(\Phi_1^-)=\frac{y_1^2+y_2^2}{(y_1^2+y_2^2)^2+(x_1y_1+x_2y_2)^2}\leq\frac{1}{r},$$

with equality taking place when $x_1y_1+x_2y_2=0$. This last inequality implies that $(r\tau_1(\Phi_1^-),r\tau_2(\Phi_1^-))\in T^*_{(S^-_{H_{\frac{1}{r}}})^{int}}$, whenever $r\in(0,1]$.

Recall the fact that the measure $\mu_{\Phi_1^-}(\zeta,\bar\zeta)=\partial\Phi_1^-(\zeta,\bar\zeta)\wedge\partial\bar\partial\Phi_1^-(\zeta,\bar\zeta)$ is equivalent to the Lebesgue measure on the boundary of the tube $\mathbb{R}^2\times iS^-$. Furthermore, any linear continuous functional $F\in(H^2(T_{(S_H^-)^{int}}))'$ is represented by an element $h\in\widetilde{\mathcal{G}^-}$, as we have seen in Corollary 3.2.1. However, this description gives us an implicit description of the dual space, because there is no knowledge of the limit points. Our goal is to give a sharper description for the space $(H^2(T_{(S_H^-)^{int}}))'$, that is, for $h\in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2\times iS^-)$

$$\begin{aligned}F(f) &= \int_{\mathbb{R}^2\times iS^-} f(\zeta)h(\zeta)\partial\Phi_1^-(\zeta,\bar\zeta)\wedge\partial\bar\partial\Phi_1^-(\zeta,\bar\zeta)=\int_{\mathbb{R}^2\times iS^-} f(\zeta)h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar\zeta),\\ |F(f)| &\leq \|f\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2\times iS^-)}\|h\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2\times iS^-)}\leq A\|h\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2\times iS^-)}\|f\|_{H^2(T_{(S_H^-)^{int}})},\end{aligned}\tag{5.0.1}$$

where the constant A depends on constants from comparison of norms corollary.

Thus, using (4.0.5) for $f\in H^2(T_{(S_H^-)^{int}})$ we have the following formula representing the functional F:

$$\begin{aligned}F(f) &= \int_{\mathbb{R}^2\times iS^-} f(\zeta)h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar\zeta) &(5.0.2)\\ &= \int_{\mathbb{R}^2\times iS^-}\lim_{r_k\longrightarrow 1^-} f(r_k\zeta)h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar\zeta), &(5.0.3)\end{aligned}$$

where the limit denotes the L^2-limit in the sense of Prop. 3.1.1. But, for every $0 < r_k < 1$ the point $r_k\zeta \in T_{(S_H^-)^{int}}$, whenever $\zeta \in \mathbb{R}^2 \times iS^-$. Theorem 4.0.2, formal application of Fubini's Theorem and the properties of the inner product imply then

$$\begin{aligned}(2\pi)^2 F(f) &= \int_{\mathbb{R}^2\times iS^-} \Big(\lim_{r_k \longrightarrow 1^-} \int\limits_{\mathbb{R}^2\times iS^-} \frac{f(\omega)\Omega(\Phi_1^-(\omega,\bar{\omega}))}{(1- < \frac{r_k \nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}, \zeta >)^2} \Big) h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar{\zeta}) \\ &= \int_{\mathbb{R}^2\times iS^-} \Big(\lim_{r_k \longrightarrow 1^-} \int\limits_{\mathbb{R}^2\times iS^-} \frac{h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar{\zeta})}{(1- < \frac{r_k \nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}, \zeta >)^2} \Big) f(\omega)\Omega(\Phi_1^-(\omega,\bar{\omega})).\end{aligned} \tag{5.0.4}$$

This approach has meaning provided that the denominator $\frac{1}{(1-<\frac{r_k \nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>},\zeta>)^2}$ in the inner integral does not vanish. Actually, the line $l = 1- < \frac{r_k \nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}, \zeta >$ is tangent to the tube $T_{(\frac{1}{r_k}S_H^-)^{int}}$ only at a point of its boundary, and thus does not vanish at any point inside this tube. Thus, in particular, it does not vanish for any $\zeta \in \mathbb{R}^2 \times iS^- \subset T_{(\frac{1}{r_k}S_H^-)^{int}}$. Thus, assuming that h is also in $L^1_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$, the inner integral has a meaning.
This approach brings forward the following two problems with respect to the inner integral:
1) the inner integral

$$\int\limits_{\mathbb{R}^2\times iS^-} \frac{h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar{\zeta})}{(1- < \frac{r_k \nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}, \zeta >)^2}, \tag{5.0.5}$$

defines a function of $w = \frac{r_k \nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}$, $\omega \in \mathbb{R}^2 \times iS^-$, whose behavior at $r = 1$ has to be investigated
2) what is the meaning of the limit in the front of it.
Furthermore, one observes that there is a question related to the outer integral in (5.0.4) (its existence).

One observes that for every $0 < r < 1$, the function $\phi_{h,r} : T^*_{(S_H^-)^{int}} \longrightarrow \mathbb{C}$ defined by

$$\phi_{h,r}(w) = \phi_h(rw) = \int\limits_{\mathbb{R}^2\times iS^-} \frac{h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar{\zeta})}{(1- < rw, \zeta >)^2} \tag{5.0.6}$$

is well defined since $1- < rw, \zeta > \neq 0$. The relation (5.0.5) is particular case of (5.0.6) in the case when $w = \frac{\nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}$, $\omega \in \mathbb{R}^2 \times iS^-$. One can also interpret the limiting case $r \longrightarrow 1^-$ of (5.0.6) as the analogue of the Cauchy-Fantappie transform of the element of the dual space of the normed space of analytic functions $H^2(T_{(S_H^-)^{int}})$, i.e analytic functional, defined by $h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar{\zeta})$, $\zeta \in \mathbb{R}^2 \times iS^-$. Recall that in the Aizenberg-Martineau setting for an open domain $U \subset \mathbb{C}^n$, the Cauchy-Fantappie transform $\mathcal{F}_C(\mu)(w) = \mu(\frac{1}{(1+<w,>)^2})$ maps an element of $(\mathcal{H}(U))'$ into the function belonging to $\mathcal{H}(U^*)$. Furthermore, the main result of their approach is the fact that the Cauchy-Fantappie transform is an isomorphism whenever U is $\mathbb{C}$-convex ([1, 2],[10, 21, 22],[29]).

It is clear therefore, that the integral in (5.0.6) might have singularities when $r = 1$ among the points $w \in T^*_{S_H^-}$ having $\|w\| = 1$ and $\|w\| < 1$. Recall that, according to results of Chapter 2, $w \in T^*_{(S_H^-)^{int}}$ and $\|w\| = 1$ happens only when $w = (w_1, w_2) = ((0, -iy_1), (0, -iy_2))$ satisfying $y_1^2 + y_2^2 = 1$, $-1 \leq y_1 \leq 0$. The following lemma, describing the values of the pseudo-analogue of the Cauchy-Fantappie transform of an element of the dual space $(H^2(T_{(S_H^-)^{int}}))'$ is of importance.

Lemma 5.0.2 *Let us assume that the function $h \in L^2_{\mu_\Phi}(\mathbb{R}^2 \times iS^-)$ and satisfies the property: for every $(y_1, y_2) \in S^-$ the restriction $h|_{\mathbb{R}^2\times i(y_1,y_2)} \in \mathcal{S}(\mathbb{R}^2)$. Let*

$$\mathcal{B} = \{w = \frac{r\nabla\Phi_1^-(\omega,\bar{\omega})}{< \nabla\Phi_1^-(\omega,\bar{\omega}),\omega >} \in T^*_{(S_H^-)^{int}},\ 0 \leq r < 1\}.$$

Then the function $\phi_h : \mathcal{B} \longrightarrow \mathbb{C}$ defined by

$$\phi_h(\frac{r\nabla\Phi_1^-(\omega,\bar{\omega})}{< \nabla\Phi_1^-(\omega,\bar{\omega}),\omega >}) = \int\limits_{\mathbb{R}^2\times iS^-} \frac{h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar{\zeta})}{(1- < \frac{r\nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}, \zeta >)^2}$$

extends in a unique way to a holomorphic function $\widetilde{\phi}_h$ defined on the open set $\mathcal{E} \subset \mathbb{C}^2$ containing the set $T^*_{(S_H^-)^{int}} \setminus \mathcal{B}$. Furthermore, this extension is bounded on $T^*_{S_H^-}$ and continuous on $T^*_{S_H^-} \setminus \{((0,0),(0,\pm i))\}$.

Proof: We begin by pointing out that the assumption on h implies also that $h \in L^1_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$. Furthermore, one observes that when $x_r = \frac{r\nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>} \in \mathcal{B} \subset T^*_{S_H^-}$, $0 < r < 1$, then $\Re\omega_1\Im\omega_1 + \Re\omega_2\Im\omega_2 = 0$ and thus its norm is equal to $\|x_r\| = r$ taking into account Lemma 2.5.1. Since the set $T^*_{(S_H^-)^{int}}$ is star with respect to the origin the point $\frac{r\nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>} \in T^*_{(S_H^-)^{int}}$ for every $0 < r < 1$. Thus, for every $0 < r < 1$ one obtains the function

$$\phi_h(r\tau_1(\Phi_1^-), r\tau_2(\Phi_1^-)) = \int\limits_{\mathbb{R}^2\times iS^-} \frac{h(\zeta)\mu_{\Phi_1^-}(\zeta,\bar{\zeta})}{(1- < \frac{r\nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}, \zeta >)^2}$$

defined on a segment contained in $T^*_{(S_H^-)^{int}}$. Even though the above integral at first glance is similar to Cauchy-Fantappie type integral they differ crucially: the direction of the element $(w_1, w_2) \in T^*_{(S_H^-)^{int}}$ does not change with $\zeta \in \partial T_{(S_H^-)^{int}}$. The claim is that it extends to a full dimensional open set $\mathcal{E} \subset \mathbb{C}^2$. Actually, for $r_0 \in (0,1)$, the convexity of the tube in w implies that $< \frac{r_0\nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>}, \zeta >\neq 1$, for every $\zeta \in \mathbb{R}^2 \times iS^-$, since $\zeta \in \mathbb{R}^2 \times iS^- \subset T_{(\frac{1}{r_0}S_H^-)^{int}}$. Thus, there exists $\delta(r_0, w_{r_0}) > 0$ so that $\zeta \in \mathbb{R}^2 \times iS^- \subset T_{(\frac{1}{r}S_H^-)^{int}}$, whenever $r \in (r_0 - \delta, r_0 + \delta)$. Thus the function (5.0.6) has holomorphic extension in a neighborhood of every point $rw \in T^*_{(S_H^-)^{int}}$ defined by the same formula, whenever $r \in [0, 1)$. Thus, it remains to see what happens in the case $r = 1$. In this setting there are two possibilities:

i) $w \in T^*_{(S_H^-)^{int}}$ and $\|w\| = 1$ or

ii) $w \in T^*_{(S_H^-)^{int}}$ and $\|w\| < 1$.

If $w \in T^*_{(S_H^-)^{int}}$ and $\|w\| < 1$, then according to (2.5.1), $w = (w_1, w_2) = (0, w_2) \in \mathbb{C}^2$ or w belongs to generalized dual of the circular part of the strip $T_{(S_H^-)^{int}}$.

In the first case it represents the line that is passing through the point "above " the horizontal part of the strip and thus does not intersect the tube $\mathbb{R}^2 \times iS^-$. In the second case w coincides (through a different parametrization) with a point "outside" the cylinder $\mathbb{R}^2 \times iS^-$. In both cases one can consider a small ball centered at w so that for every w' in this ball the line passing through this point does not intersect the cylinder, that is $1- < w', \zeta >\neq 0$ for every $\zeta \in \mathbb{R}^2 \times iS^-$. Thus (5.0.6) extends holomorphically into a ball around every such point through the same formula. The union of all the balls provides us with the open set $\mathcal{E}$.

Thus, the only case left to examine about the behavior of the function in question is on the points of $w \in T^*_{S_H^-}$ satisfying $\|w\| = 1$. As pointed out above, these are exactly the points with coordinates $((0, -iy_1), (0, -iy_2))$ satisfying $y_1^2 + y_2^2 = 1$, $0 \leq y_1 \leq 1$. That is, whenever $(w_1, w_2) \neq (0,0,0,\pm i) \in T^*_{S^-_H}$ and $\|w\| = 1$, then $(w_1, w_2) \in \mathbb{R}^2 \times iS^+$,

$$\begin{aligned} |(1- < rw, \zeta >)^2| &= |(1- < (-riy_1, -riy_2), (\zeta_1, \zeta_2) >)|^2 \\ &= |(1 - r(y_1 \Im\zeta_1 + y_2 \Im\zeta_2) - i(ry_1 \Re\zeta_1 + ry_2 \Re\zeta_2)|^2 \\ &\geq (1 - r\|y\|\|\Im\zeta\| \cos\beta)^2 = (1 - r\cos\beta)^2, \end{aligned}$$

since $\|y\| = \sqrt{y_1^2 + y_2^2} = 1$, $\|\Im\zeta\| = \sqrt{\Im\zeta_1^2 + \Im\zeta_2^2} = 1$ and β is the angle between the vectors y and $\Im\zeta$. The only solutions to $1 - r\cos\beta = 0$ occur when $r = 1$ and $\beta = 0$, which is impossible since $\beta \neq 0$. Thus the function $\tilde{\phi}_h(w)$ is well defined and holomorphic in a neighborhood of any point $w \in T^*_{S^-_H}$ that satisfies $\|w\| < 1$ and is continuous at any $w \in T^*_{S^-_H} \setminus \{((0,0),(0,\pm i))\}$. If $w = (w_1, w_2) = ((0,0),(0,\pm i))$ then the denominator in (5.0.6) vanishes $1- < ((0,0),(0,i)), r(\zeta_1, \zeta_2) >= 0$ if and only if $1 - ri\zeta_2 = 0$, or, which is equivalent, $(1 + r\Im\zeta_2) + ir\Re\zeta_2 = 0$. Hence, the only singularity that appears in the integral (5.0.6) corresponds to the case when $r = 1$, $\Im\zeta_2 = \mp 1$ and $\Re\zeta_2 = 0$. Naturally, these points provide the sets $\{(\Re\zeta_1, 0, \Im\zeta_1, \mp 1)\}$ when viewed as points on $\mathbb{R}^2 \times iS^-$. Thus, even though the denominator in the integral (5.0.6) does not vanish (recall that S^- is an open arc), it is not evident that its value is finite. We have

$$|\phi_{h,r}(0 + i0, 0 + i)| = |\phi_h(0 + i0, 0 + ri)| \leq \int\limits_{\mathbb{R}^2 \times iS^-} |\frac{h(\zeta)\mu_{\Phi_1^-}(\zeta, \bar{\zeta})}{(1- < (0,i), r\zeta >)^2}|, \qquad (5.0.7)$$

even if we do not know yet that the right hand-side of (5.0.7) is finite when $r = 1$. Observe, that the denominator of (5.0.7) is estimated from below in the context of the observation above (approaching the singularity at $\Im\zeta_2 = -1$ within a cone located in $\mathbb{C}_{\zeta_2}$ coordinate), leads in the case $r = 1$, to

$$\begin{aligned} |(1- < ((0,0),(0,i)), (\zeta_1, \zeta_2) > |^2 &= |(1 - i\zeta_2)|^2 \\ &= ((1 + \Im\zeta_2)^2 + (\Re\zeta_2)^2) \\ &\geq (1 + \Im\zeta_2)^2, \end{aligned}$$

where $(1 + \Im\zeta_2)^2$ becomes very small, whenever $\Im\zeta_2 \longrightarrow -1^-$. Similar estimate from below one obtains in the case

$$\begin{aligned}
|(1-<((0,0),(0,-i)),(\zeta_1,\zeta_2)>|^2 &= |(1+i\zeta_2)|^2 \\
&= ((1-\Im\zeta_2)^2+(\Re\zeta_2)^2) \\
&\geq (1-\Im\zeta_2)^2,
\end{aligned}$$

where $(1-\Im\zeta_2)^2$ becomes very small, whenever $\Im\zeta_2 \longrightarrow 1^-$. Recall now that

$$\partial\Phi_1^-(\zeta,\bar{\zeta})\wedge\partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta}) = -\Im\zeta_1 d\Re\zeta_1 d\Re\zeta_2 d\Im\zeta_2 + \Im\zeta_2 d\Re\zeta_1 d\Re\zeta_2 d\Im\zeta_1.$$

Using the fact that $\Im\zeta_1 = -\sqrt{1-(\Im\zeta_2)^2}$, $\Im\zeta_2 \in (-1,1)$, we deduce that

$$\partial\Phi_1^-(\zeta,\bar{\zeta})\wedge\partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta}) = \sqrt{1-(\Im\zeta_2)^2}d\Re\zeta_1 d\Re\zeta_2 d\Im\zeta_2 + \frac{(\Im\zeta_2)^2}{\sqrt{1-(\Im\zeta_2)^2}}d\Re\zeta_1 d\Re\zeta_2 d\Im\zeta_2.$$

Thus, if $h \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2\times iS^-)\cap L^1_{\mu_{\Phi_1^-}}(\mathbb{R}^2\times iS^-)$, then the above implies that h tends to ∞ (in the worst case) to the order strictly smaller than that of $\frac{1}{\sqrt{1-(\Im\zeta_2)^2}}$ while approaching the points $((0,0),(0,\pm i)) \in \mathbb{R}^2\times iS^-$. Now, in a small neighborhood of $B((0,0,-i),\varrho_0)$ of $(0,0,-i)\in\mathbb{R}^2\times i\mathbb{R}$ we introduce the spherical coordinates $z=-i+i\varrho\cos\phi$, $x_1 = \varrho\sin\phi\cos\theta$, $x_2=\varrho\sin\phi\sin\theta$, where $0\leq\varrho\leq\varrho_0 << 1$, $\frac{\pi}{2}<\phi_0<\phi<\phi_1<\pi$, $0 < \phi_1-\phi_0 << 1$, $\theta\in[0,2\pi]$. Thus

$$\begin{aligned}
|(1-<((0,0),(0,i)),(\zeta_1,\zeta_2)>|^2 &= |(1-i\zeta_2)|^2 \\
&= |1-i((-i+i\varrho\cos\phi)+i\varrho\sin\phi\sin\theta)|^2 \\
&= |\varrho\cos\phi+i\varrho\sin\phi\sin\theta|^2 \geq \varrho^2\cos^2\phi.
\end{aligned}$$

Thus, approaching $\Im\zeta_2=-1$ non-tangentially, through a wedge in $\mathbb{R}^2\times iS^-$ with the apex at $((0,0),(0,-i))$ (or with the apex at $((0,0),(0,i))$), removed, described by

$$\begin{aligned}
W^\delta_{\phi,\varrho_0} &= \{(x_1,x_2,y_1): y_1=\varrho\cos\phi,\ x_1=\varrho\sin\phi\cos\theta,\ x_2=\varrho\sin\phi\sin\theta, \\
&\quad \varrho\in(\delta,\varrho_0),\phi\in(\phi_0,\phi_1),\theta\in[0,2\pi)\}
\end{aligned}$$

in the three dimensional space we are led to the following estimate, valid for every $\delta > 0$

$$
\begin{aligned}
\int_{\phi_0}^{\phi_1}\int_{\delta}^{\varrho_0}\int_{0}^{2\pi} \left|\frac{h(\varrho,\phi,\theta)}{\varrho^2\cos^2\phi}\right| \varrho^2 \sin\phi\, d\varrho d\phi d\theta &\leq A_{\phi_i,\phi_j}\int_{\phi_0}^{\phi_1}\int_{0}^{\varrho_0}\int_{0}^{2\pi} |h(\varrho,\phi,\theta)| d\varrho d\phi d\theta \\
&\leq A_{\phi_i,\phi_j}\int_{\mathbb{R}^2\times iS^-} |h(\zeta,\bar{\zeta})|\mu_{\Phi_1^-}(\zeta,\bar{\zeta}) < \infty,
\end{aligned}
\tag{5.0.8}
$$

where the positive constant $A_{\phi_i,\phi_j} = \frac{|\sin\phi_1|}{\cos^2\phi_0}$ depends only on $\cos\phi_i$ and $\sin\phi_j$, $i,j = 0,1$. The size of the radius ϱ_0 also does not depend on h and on ϕ_0, ϕ_1. The second inequality follows by comparison of integrals over subsets of $\mathbb{R}^2$ ("horizontal slices" of the wedge W_{ϕ,ϱ_0}) and of integrals over $\mathbb{R}^2$. Taking the limit $\delta \longrightarrow 0$ in (5.0.8) one is led to the estimate

$$
|\phi_h(0+i0, 0\pm i)| \leq (1 + A_{\phi_i,\phi_j}) \int_{\mathbb{R}^2\times iS^-} |h(\zeta,\bar{\zeta})|\mu_{\Phi}(\zeta,\bar{\zeta}) < \infty. \tag{5.0.9}
$$

Similarly one treats the case of the singular point $((0,0),(0,-i)) \in T^*_{(S_H^-)^{int}}$. That is, ϕ_h is bounded function on $T_{(S_H^-)^*}$ and continuous everywhere in $T_{(S_H^-)^*} \subset \mathbb{C}^2$, except at points $(0+i0, 0\pm i)$. This completes the proof of the lemma. $\diamondsuit$

We may assume that the open set $\mathcal{E} \subset \mathbb{C}^2$ from the last lemma is an open ellipsoid contained in the unit ball and containing the set $T^*_{(S_H^-)^{int}} \setminus \{((0,-iy_1),(0,-iy_2)) :\ y_1^2+y_2^2 = 1\}$ so that $\partial\mathcal{E}\cap T^*_{(S_H^-)^{int}} = \{((0,-iy_1),(0,-iy_2)) :\ y_1^2+y_2^2 = 1\}$. Then, $\widetilde{\phi}_h \in L^2(\overline{\mathcal{E}}) \subset L^1(\overline{\mathcal{E}})$, also it is holomorphic in $\mathcal{E}$. The open set $\mathcal{E}$ is independent of h.

Now, for a function $\widetilde{\phi}_h$ like in Lemma 5.0.2, we define the function

$$
\begin{aligned}
\widetilde{\phi}_{h,\chi} &: (\omega,\bar{\omega}) \in \mathbb{R}^2 \times iS^- \longrightarrow \mathbb{C} \\
\widetilde{\phi}_{h,\chi}(\omega,\bar{\omega}) &:= \widetilde{\phi}_h \circ \chi,
\end{aligned}
\tag{5.0.10}
$$

that is, $\widetilde{\phi}_{h,\chi}$ is composition of the mapping $\chi : \mathbb{R}^2 \times iS^- \longrightarrow T^*_{(S_H^-)^{int}}$ sending $(\omega,\bar{\omega}) \in \mathbb{R}^2 \times iS^-$ to $\frac{\nabla\Phi_1^-(\omega,\bar{\omega})}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>} \in T^*_{(S_H^-)^{int}}$ and of the function $\widetilde{\phi}_h$. It follows from the definition of χ that it is continuous. Thus we have the following result

Lemma 5.0.3 *Let us assume that the function $h \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$ and satisfies the property: for every $(y_1, y_2) \in S^-$ the restriction $h|_{\mathbb{R}^2 \times i\{(y_1,y_2)\}} \in \mathcal{S}(\mathbb{R}^2)$. Let*

$$A = \{f \in H^2(T_{(S_H^-)^{int}}) : \textit{every restriction } f|_{\mathbb{R}^2 \times i\{(y_1,y_2)\}} \in \mathcal{S}(\mathbb{R}^2)\}.$$

Then the functional $L_h : A \longrightarrow \mathbb{R}$ defined by

$$L_h(f) = \int\limits_{\mathbb{R}^2 \times iS^-} \widetilde{\phi}_{h,\chi}(\omega) f(\omega) \Omega(\Phi_1^-(\omega, \bar{\omega})), \tag{5.0.11}$$

for $f \in A$, is continuous and extends to continuous functional on $H^2(T_{(S_H^-)^{int}})$.

Proof: The functional defined by (5.0.11) is continuous. Actually the conditions on both functions f and h imply that both of them belong to space $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$, because being in $\mathcal{S}(\mathbb{R}^2)$ for fixed $(y_1, y_2) \in S^-$ implies integrability on $\mathbb{R}^2$ and then integrability (or square integrability) on the finite measure arc S^-. Furthermore, applying Holder inequality in (5.0.11), because $\widetilde{\phi}_{h,\chi}$ is bounded and because of the Lemma 4.0.1, we have that

$$\begin{aligned} |\int\limits_{\mathbb{R}^2 \times iS^-} \widetilde{\phi}_{h,\chi}(\omega) f(\omega) \Omega(\Phi_1^-(\omega, \bar{\omega}))| &= |\int\limits_{\mathbb{R}^2 \times iS^-} \frac{\widetilde{\phi}_{h,\chi}(\omega)}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} f(\omega) \mu_{\Phi_1^-}(\omega, \bar{\omega})| \\ &\leq \| \frac{\widetilde{\phi}_{h,\chi}(\omega)}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} \|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)} \|f\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)}. \end{aligned}$$

Now, every function $g \in H^2(T_{(S_H^-)^{int}})$, which on the boundary of the tube is $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$, is the $\| \cdot \|_{H^2(T_{(S_H^-)^{int}})}$-limit of the sequence $\{g_n\}_n \subset A$. Thus, using (3.1.6), the functional defined above is extended by continuity

$$\begin{aligned} L_h(g) &= \int\limits_{\mathbb{R}^2 \times iS^-} \widetilde{\phi}_{h,\chi}(\omega) g(\omega) \Omega(\Phi_1^-(\omega, \bar{\omega})) \\ &= \lim_{n \longrightarrow \infty} \int\limits_{\mathbb{R}^2 \times iS^-} \widetilde{\phi}_{h,\chi}(\omega) g_n(\omega) \Omega(\Phi_1^-(\omega, \bar{\omega})) \end{aligned}$$

and gives us an element of the space $\left(H^2(T_{(S_H^-)^{int}})\right)'$. $\diamondsuit$

Taking into account that $\mathcal{S}'(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ and the fact that every element $f \in A$ is as in the lemma above implies that on almost any slice $\mathbb{R}^2 \times i\{(y_1, y_2)\}$ an element $g \in (H^2(T_{(S_H^-)^{int}}))' = L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$ gives a rise to the same continuous functional realized by an element $h_{(y_1,y_2)} \in \mathcal{S}'(\mathbb{R}^2)$. This implies that $\{h_{(y_1,y_2)},$ for almost all $(y_1, y_2) \in S^-\}$ is also in $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$. Such a selection is possible by the use of Michael's selection principle as in the proof of the lemmas in Chapter 3. Thus, such a function gives rise to an element

$$\widetilde{\phi}_h,\ h \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-) :\ \text{every restriction}\ \ h|_{\mathbb{R}^2\times i\{(y_1,y_2)\}} \in \mathcal{S}(\mathbb{R}^2),$$

by completing $h|_{\mathbb{R}^2\times i\{(y_1,y_2)\}} \equiv 0$, when necessary.

If $\mathcal{A}(T^*_{(S_H^-)^{int}})$ denotes the space of functions holomorphic in a neighborhood of every point belonging to $T^*_{(S_H^-)^{int}} \setminus \{((0,0),(0,\pm i))\}$ and bounded on the compact $T^*_{(S_H^-)^{int}}$ then

$$\mathcal{V}_{T^*_{(S_H^-)^{int}}} = \{\widetilde{\phi}_h \in \mathcal{A}(T^*_{(S_H^-)^{int}}),\ h \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-) :\ \text{every restriction}\ \ h|_{\mathbb{R}^2\times i\{(y_1,y_2)\}} \in \mathcal{S}(\mathbb{R}^2)\}$$

is its subspace. Note also that the space $\mathcal{A}(T^*_{(S_H^-)^{int}})$ equipped with

$$\|\phi\|_{\mathcal{A}(T^*_{(S_H^-)^{int}})} = \sup_{w\in T^*_{(S_H^-)^{int}}} |\phi(w)|$$

becomes a normed space.

Lemma 5.0.4 *Let* $\mathcal{G}^-$ *be the space described by* (3.2.13). *The mapping* $\psi : \mathcal{G}^- \longrightarrow \mathcal{V}_{T^*_{(S_H^-)^{int}}}$ *defined by correspondence* $\psi(h) = \widetilde{\phi}_h$ *using* (5.0.6) *is monomorphism. Furthermore, if* $\{h_n\}_n \subset \mathcal{G}^-$ *is Cauchy, then so is the sequence* $\{\widetilde{\phi}_{h_n}\} \subset \mathcal{V}_{T^*_{(S_H^-)^{int}}}$.

Proof: The only non-obvious part of the first part of lemma is to show that the correspondence is one-to-one. Let us assume that $h_1 \neq h_2$ almost everywhere, but $\psi(h_1) = \psi(h_2)$. Thus the equality $\widetilde{\phi}_{h_1} = \widetilde{\phi}_{h_2}$ implies the equality of holomorphic $\phi_{h_1} = \phi_{h_2}$ on the set $\mathcal{B}$ defined in the Lemma 5.0.2 above. But this set is a set of uniqueness for the holomorphic functions involved. Thus $\phi_{h_1} = \phi_{h_2}$ everywhere on the set $T^*_{(S_H^-)^{int}}$. Thus the function $h_1 - h_2 \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$ defines the zero functional in (5.0.11). The action on the boundary $\mathbb{R}^2 \times iS^-$ implies that the restriction of $h_1 - h_2$ on the slice $\mathbb{R}^2 \times i\{(y_1, y_2)\}$ is $(h_1 - h_2)_{(y_1,y_2)} = 0$ for almost all $(y_1, y_2) \in S^-$. Thus $h_1 = h_2$ a.e. This concludes the proof of the first part of the lemma.

In order to prove the second part we first observe that the Jensen inequality

$$\left(\int\limits_{\mathbb{R}^2\times iS^-} |h_n(\zeta)-h_m(\zeta)|\mu_{\Phi_1^-}(\zeta,\bar{\zeta})\right)^2 \leq \int\limits_{\mathbb{R}^2\times iS^-} |h_n(\zeta)-h_m(\zeta)|^2\mu_{\Phi_1^-}(\zeta,\bar{\zeta})$$

holds. Actually, applying the classical Jensen inequality on the sets of finite measure $\overline{D}_R \times iS^-$, where $D_R = \{(x,y)\in\mathbb{R}^2 : \; x^2+y^2<R\}$, $R>0$ and then taking the limit $R\longrightarrow\infty$ one gets the Jensen inequality on $\mathbb{R}^2\times iS^-$ stated above.

Now for every compact subset $K\subset T^*_{(S_H^-)^{int}}\subset\overline{\mathcal{E}}$ so that $((0,0),(0,\pm i))\notin K$ there exists an open set $V\subset\mathcal{E}$ containing it, so that $\overline{V}\cap\partial B(0,1)=\emptyset$. Then for every $w\in V$, in particular for every $w\in K$, taking into account (5.0.6), one has

$$\begin{aligned}|\phi_{h_n}(w)-\phi_{h_m}(w)|^2 &\leq C_{w,\overline{V}}\left(\int\limits_{\mathbb{R}^2\times iS^-} |h_n(\zeta)-h_m(\zeta)|\mu_{\Phi_1^-}(\zeta,\bar{\zeta})\right)^2\\ &\leq C_{w,\overline{V}}\int\limits_{\mathbb{R}^2\times iS^-} |h_n(\zeta)-h_m(\zeta)|^2\mu_{\Phi_1^-}(\zeta,\bar{\zeta}),\end{aligned}$$

where $|(1-<w,\zeta>)^2|\geq C_{w,\overline{V}}>0$ for every $w\in\overline{V}$ and $\zeta\in\mathbb{R}^2\times iS^-$ and for some constant $C_{w,\overline{V}}$, which depends on $\overline{V}$ and w only (or, alternatively on K and w only). Thus, one has that the sequence $\{\phi_{h_n}\}$ is Cauchy over compact subsets (such as $\overline{K}$) under the topology of uniform convergence over compact set. K can be realized as $(B((0,0),(0,-i)),\varrho_0')\cup B((0,0),(0,i)),\varrho_0))^c\cap T^*_{(S_H^-)^{int}}$. Using the estimates (5.0.9) one can prove that $\{\phi_{h_n}(w)-\phi_{h_m}(w)\}\subset\mathcal{A}(T^*_{(S_H^-)^{int}})$ is Cauchy, in the sense that

$$|\phi_{h_n}(w)-\phi_{h_m}(w)|^2\leq(2+A_{\phi_i,\phi_j})\int\limits_{\mathbb{R}^2\times iS^-} |h_n(\zeta)-h_m(\zeta)|^2\mu_{\Phi_1^-}(\zeta,\bar{\zeta})<\epsilon$$

for every $w\in\overline{\mathcal{E}}$, whenever $n,m\geq n_0(\epsilon)$ or which is the same

$$|\phi_{h_n}(w)-\phi_{h_m}(w)|\leq\sqrt{(2+A_{\phi_i,\phi_j})}||h_n-h_m||_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2\times iS^-)}<\epsilon \tag{5.0.12}$$

The relation (5.0.12) shows that $\lim\limits_{n\longrightarrow\infty}\phi_{h_n}(w)=\phi$ belongs to the space $\mathcal{V}_{T^*_{(S_H^-)^{int}}}$. This concludes the proof of the lemma. $\diamondsuit$

Thus it is natural to state the following

Lemma 5.0.5 *Let $\widetilde{\mathcal{G}^-}$ be $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$-closure of the space $\mathcal{G}^-$. Then ψ has continuous extension $\widetilde{\psi} : \widetilde{\mathcal{G}^-} \longrightarrow \widetilde{\mathcal{V}}_{T^*_{(S_H^-)^{int}}}$, where $\widetilde{\mathcal{V}}_{T^*_{(S_H^-)^{int}}}$ is the closure of $\mathcal{V}_{T^*_{(S_H^-)^{int}}}$ under* (5.0.12).

Proof : Every element $h \in \widetilde{\mathcal{G}^-}$ is a limit of the sequence $\{h_n\}_n \subset \mathcal{G}^-$. Without loss of generality, passing to a subsequence if necessary, we may assume that $\lim_{n \longrightarrow \infty} h_n(\zeta, \bar{\zeta}) = h(\zeta, \bar{\zeta})$ for almost all $(\zeta, \bar{\zeta}) \in \mathbb{R}^2 \times iS^-$. Therefore the extension is defined by

$$\widetilde{\psi}(h) = \lim_{n \longrightarrow \infty} \psi(h_n) = \lim_{n \longrightarrow \infty} \phi_{h_n}.$$

But $\{\phi_{h_n}\} \subset \mathcal{V}_{T^*_{(S_H^-)^{int}}}$ is a Cauchy sequence, since the sequence $\{h_n\} \subset \mathcal{G}^-$ is. Thus $\widetilde{\psi}(h) = \phi$, where $\phi(w) = \lim_{n \longrightarrow \infty} \phi_{h_n}(w)$, that is ϕ is of the form ϕ_h. The existence of the limit above follows from (5.0.12).$\diamondsuit$

Lemma 5.0.6 1) *For every $h \in \mathcal{A}(T^*_{(S_H^-)^{int}})$ the function $\frac{(h \circ \chi)(\omega, \bar{\omega})}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$ and whose norm is $\| \frac{h \circ \chi(\omega, \bar{\omega})}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} \|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)}$.*
2) *For every $\zeta \in T_{(S_H^-)^{int}}$ fixed and $w \in T^*_{(S_H^-)^{int}}$, consider the function $\vartheta_\zeta(w) = \frac{1}{(1 - <w, \zeta>)^2}$. Then the function $\vartheta_\zeta \in \mathcal{A}(T^*_{(S_H^-)^{int}})$ and $\frac{(\vartheta_\zeta \circ \chi)(\omega, \bar{\omega})}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} = \frac{\vartheta_{\zeta, \chi}(\omega, \bar{\omega})}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$.*

Proof: The proof of the first part follows directly from the fact that $\frac{1}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$. In order to prove the second part, one observes first that for every $\zeta \in T_{(S_H^-)^{int}}$ the function $\vartheta_\zeta(w) = \frac{1}{(1 - <w, \zeta>)^2}$ is holomorphic in a neighborhood of the compact $T^*_{(S_H^-)^{int}}$. We may assume that this neighborhood contains the closure of the ellipsoid $\mathcal{E}$ considered above. Furthermore, direct computation shows that

$$\frac{\vartheta_{\zeta, \chi}(\omega, \bar{\omega})}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} = \frac{1}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega - \zeta >^2}$$

Thus, on a strictly convex part of the boundary $\partial T_{S_H^-}$ consisting of $\mathbb{R}^2 \times iS^-$, we have the estimate (4.0.7). Using the reasoning that led to (4.0.9) we deduce the desired conclusion. $\diamondsuit$

Now we are ready to formulate the following theorem, interpreted as boundary value analogue of the result from ([1]), describing the space $(H^2(T_{(S_H^-)^{int}}))'$.

Theorem 5.0.3 *Every linear continuous functional* $F \in (H^2(T_{(S_H^-)^{int}}))' = \widetilde{\mathcal{G}^-}$ *is represented by*

$$\begin{aligned} F(f) = F_h(f) &= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} f(\omega) h(\omega) \mu_{\Phi_1^-}(\omega, \bar{\omega}) \\ &= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} f(\zeta) \widetilde{\phi}_h(\tau(\Phi_1^-)) \Omega(\Phi_1^-(\zeta, \bar{\zeta})) \\ &= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} f(\zeta) \frac{\widetilde{\phi}_h(\tau(\Phi_1^-)) \mu_{\Phi_1^-}(\zeta, \bar{\zeta})}{< \nabla \Phi_1^-(\zeta, \bar{\zeta}), \zeta >^2} = F_{\phi_h}(f), \end{aligned} \tag{5.0.13}$$

where $h \in \widetilde{\mathcal{G}^-}$ *and* $\widetilde{\phi}_h \in \mathcal{A}(T^*_{(S_H^-)^{int}})$*. This mapping, corresponding to every* $h \in (H^2(T_{(S_H^-)^{int}}))' = \widetilde{\mathcal{G}^-}$ *the element* $\frac{\psi_h \circ \chi}{<\nabla \Phi_1^-(\omega, \bar{\omega}), \omega>^2} \in \mathcal{W}_{T^*_{(S_H^-)^{int}}}$*, where*

$$\mathcal{W}_{T^*_{(S_H^-)^{int}}} = \{ \frac{\psi \circ \chi}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2}, \quad \textit{where } \psi \in \mathcal{A}(T^*_{(S_H^-)^{int}}) \},$$

is subspace of the space $(L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-), \| \cdot \|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)})$*, induces a norm preserving monomorphism. Furthermore, every element* $\phi \in \mathcal{A}(T^*_{(S_H^-)^{int}})$ *induces an element of the space* $\mathcal{W}_{T^*_{(S_H^-)^{int}}}$ *defining an analytic functional.*

Proof: The relation (5.0.4), as we have shown in the above lemmas, is valid whenever $h \in \mathcal{G}^-$ and the holomorphic extension of ϕ_h, defined by (5.0.5), belongs to the space $\mathcal{A}(T^*_{(S_H^-)^{int}})$ as described by the corresponding lemma (5.0.2). The relation (5.0.4), as it follows through by using all the steps in between, shows that h and $\frac{\widetilde{\phi}_{h,\chi}}{<\nabla \Phi_1^-(\zeta, \bar{\zeta}), \zeta>^2}$ define the same analytic functional on $H^2(T_{(S_H^-)^{int}})$. Furthermore, for every $h \in \widetilde{\mathcal{G}^-}$ there exists an element $\widetilde{\phi}_h \in \mathcal{A}(T^*_{(S_H^-)^{int}})$ satisfying

$$\begin{aligned} (2\pi)^2 F(f) &= \int_{\mathbb{R}^2 \times iS^-} f(\zeta) h(\zeta) \mu_{\Phi_1^-}(\zeta, \bar{\zeta}) \\ &= \int_{\mathbb{R}^2 \times iS^-} \frac{\widetilde{\phi}_h(\omega)}{< \nabla \Phi_1^-(\omega, \bar{\omega}), \omega >^2} f(\omega) \mu_{\Phi_1^-}(\omega, \bar{\omega}), \end{aligned} \tag{5.0.14}$$

where $\frac{\widetilde{\phi}_h(\omega)}{<\nabla \Phi_1^-(\omega, \bar{\omega}), \omega>^2} \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$. The above identity follows from continuity of the correspondence L, without the intermediary steps to justify it. Now, if one interprets the first equality in (5.0.14) using Riesz Representation Theorem, then $\|F\| = \|h\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)}$.

On the other hand, the second equality in (5.0.14) and Riesz Representation Theorem imply that $\|F\| = \|\frac{\widetilde{\phi}_h(\omega)}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>^2}\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2\times iS^-)}$. Thus, the correspondence is norm preserving. To conclude the proof of the theorem we just remark that for every element $\widetilde{\phi} \in \mathcal{A}(T^*_{(S_H^-)^{int}})$, the function $\frac{\widetilde{\phi}(\omega)}{<\nabla\Phi_1^-(\omega,\bar{\omega}),\omega>^2} \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$ defines an element of the dual space. The proof of the theorem is complete. $\diamondsuit$

Naturally, results analogous to the proven ones in the present section for the space $H^2(T_{S_H^-})$, can be obtained for the space $H^2(T_{S_H^+})$ after suitable (and direct) reformulation.

Chapter 6

Separation of singularities and integral representation theorem for the space $\mathbf{H^2(T_{B_1})}$

Assume that a bounded domain $\Omega \subseteq \mathbb{C}^n$ is realized as the intersection of bounded domains Ω_1 and Ω_2. If f is a holomorphic function in Ω is it possible to define functions f_1 holomorphic in Ω_1 and f_2 holomorphic in Ω_2 such that $f = f_1 - f_2$? This is known as a separation of singularities problem for holomorphic functions on domains $\Omega \subset \mathbb{C}^n$. For $n = 1$ it was proved by Aronsajn ([11]).

However for $n > 1$ this is not valid in general. More precisely, recalling ([3]), consider the bi-disk $\mathbb{D}_{r,\rho} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < r, |z_2| < \rho\}$ and set $\Omega = \mathbb{D}_{1,1}, \Omega_1 = \mathbb{D}_{1,2}$ and $\Omega_2 = \mathbb{D}_{2,1}$. Then $\Omega = \Omega \cap \Omega_2$. The function

$$f(z_1, z_2) \quad = \quad \frac{1}{(1-z_1)(1-z_2)} = \sum_{n,m=0}^{\infty} z_1^m z_2^n$$

is holomorphic in Ω although it is not representable as a sum of holomorphic functions f_1 and f_2 defined in the domains Ω_1 and Ω_2 respectively. Simply, observe that $\Omega^* = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : |\zeta_1| + |\zeta_2| \leq 1\}$ while $\Omega_1^* \cup \Omega_2^* = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : |\zeta_1| + 2|\zeta_2| \leq 1\} \cup \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : 2|\zeta_1| + |\zeta_2| \leq 1\}$. The compact $\Omega_1^* \cup \Omega_2^*$ is star with respect to the origin. Thus, it has an envelope of holomorphy $E_{\Omega_1^* \cup \Omega_2^*}$. Furthermore, $\Omega_1^* \cup \Omega_2^* \subset$

$(\Omega_1 \cap \Omega_2)^* = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 : |\zeta_1| + |\zeta_2| \leq 1\} = \bar{\mathcal{A}}_2$ where $\mathcal{A}_2$ is the hyper-cone as in example 2.2.2. Recalling that $\bar{\mathcal{A}}_2$ is a compact of holomorphy one has that $E_{\Omega_1^* \cup \Omega_2^*} \subset \bar{\mathcal{A}}_2$ and thus $E_{\Omega_1^* \cup \Omega_2^*} = \bar{\mathcal{A}}_2$. Hence, $\Omega_1^* \cup \Omega_2^* \subset E_{\Omega_1^* \cup \Omega_2^*}$. Thus, f is not representable as a difference of holomorphic functions f_1 and f_2 defined in the domains Ω_1 and Ω_2 respectively.

However, in the opposite direction one has the following result ([3]).

Theorem 6.0.4 *Any holomorphic function f defined in a strictly linearly convex domain* $\Omega = \Omega_1 \cap \Omega_2$ *can be represented as* $f = f_1 + f_2$, *where* fi *is holomorphic in* Ω_i, $i =$ *1, 2 if and only if the compactum of holomorphy* $E_{\Omega_1^* \cup \Omega_2^*}$ *for the union* $\Omega_1^* \cup \Omega_2^*$ *satisfies* $E_{\Omega_1^* \cup \Omega_2^*} = (\Omega_1 \cap \Omega_2)^*$.

For classical Hardy spaces similar results were formulated by L. Aizenberg and G. Henkin ([3]). Their formulation is the following

Theorem 6.0.5 *Let* $\Omega = \Omega_1 \cap \cdots \cap \Omega_k$ *where all of the domains are strictly pseudoconvex with* $\mathcal{C}^3$ *boundary. Every* $f \in H^p(\Omega)$ *for* $1 < p < \infty$ *is written as* $f = f_1 + \cdots + f_k$ *with* $f_j \in H^p(\Omega_j)$ *for every* $j = 1, \cdots, k$.

Our purpose is to state and proof similar results for the space $H^2(T_{B_1})$.
Consider the space $H^2(T_{B_1})$, where the tube T_{B_1}, is defined by (2.2.1). Furthermore, we recall that $T_{B_1} = T_{S_H^-} \cap T_{S_H^+}$. The first result in the present section states that every $f \in H^2(T_{B_1})$ can be written in a unique way as $f = f_1 - f_2$, where $f_1 \in H^2(T_{S_H^-})$, $f_2 \in H^2(T_{S_H^-})$. It essentially means that while $f \in H^2(T_{B_1})$ cannot be expressed through Cauchy-Fantappie integral representation formula, it can be represented as a sum of functions which are represented in a such a manner. The first result of the present section is following separation of singularities theorem (Aronsajn type Theorem).

Theorem 6.0.6 *Let* $f \in H^2(T_{B_1})$. *Then there exist functions* $f_1 \in H^2(T_{S_H^-})$ *and* $f_2 \in H^2(T_{S_H^+})$ *that satisfy*

$$f(z) = f_1(z) - f_2(z),\ z \in T_{B_1}.$$

Furthermore, for every $z \in T_{B_1}$ one has that

$$
\begin{aligned}
f(z) &= \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{f_1(\zeta)\left(\partial \Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - z >)^2} \\
&+ \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^+} \frac{f_2(\zeta)\left(\partial \Phi_1^+(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^+(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^+(\zeta,\bar{\zeta}), \zeta - z >)^2}
\end{aligned} \tag{6.0.1}
$$

and

$$
f(z) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^1} \frac{f_(\zeta)\left(\partial \Phi_1(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta})\right)}{(\nabla_\zeta \Phi_1(\zeta,\bar{\zeta}), \zeta - z >)^2}. \tag{6.0.2}
$$

Proof: Recall that $E_{(T^*_{S_H^-} \cup T^*_{S_H^+})} = T^*_{B_1}$ because of the Lemma 2.5.2. We equip the space $\mathcal{A}(T^*_{B_1})$ consisting of functions which are holomorphic in a neighborhood of every point of $T^*_{B_1} \setminus \{((0,0),(0,\pm i))\}$ and bounded on the compact $T^*_{B_1}$, with the supremum norm $\|\phi\|_{\mathcal{A}(T^*_{B_1})} = \sup_{w \in T^*_{B_1}} |\phi(w)|$. Now, we consider the spaces

$$
\widetilde{\mathcal{W}}_{T^*_{(S_H^-)^{int}}} = \{\frac{\psi_h \circ \chi}{< \nabla\Phi_1^-(\omega,\bar{\omega}), \omega >^2}, \ \text{ where } \psi_h \in \mathcal{A}(T^*_{(S_H^-)^{int}}) \ \text{ for some } h \in \widetilde{\mathcal{G}^-}\}
$$

and

$$
\widetilde{\mathcal{W}}_{T^*_{(S_H^+)^{int}}} = \{\frac{\psi_h \circ \chi}{< \nabla\Phi_1^+(\omega,\bar{\omega}), \omega >^2}, \ \text{ where } \psi_h \in \mathcal{A}(T^*_{(S_H^+)^{int}}) \ \text{ for some } h \in \widetilde{\mathcal{G}^+}\}
$$

subspaces of $(L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-), \|\cdot\|_{L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)})$ and of $(L^2_{\mu_{\Phi_1^+}}(\mathbb{R}^2 \times iS^+), \|\cdot\|_{L^2_{\mu_{\Phi_1^+}}(\mathbb{R}^2 \times iS^+)})$ correspondingly. Here we used the notation $\Phi_1^\pm$ for the defining functions of $T_{(S_H^\pm)^{int}}$ correspondingly. Similarly, we consider the space, subspace of $(L^2_{\mu_{\Phi_1}(\mathbb{R}^2 \times iS^1)}, \|\cdot\|_{L^2_{\mu_{\Phi_1}}(\mathbb{R}^2 \times iS^1)})$, for Φ_1 being the defining function for the tube T_{B_1}:

$$
\begin{aligned}
\widetilde{\mathcal{W}}_{T^*_{B_1}} = \{ & \frac{\phi_{h^-} \circ \chi}{< \nabla\Phi_1(\omega,\bar{\omega}), \omega >^2}, \omega \in \mathbb{R}^2 \times iS^-, \ \text{ when } \phi_{h^-} \in \mathcal{A}(T^*_{(S_H^-)^{int}}), \\
& \frac{\phi_{h^+} \circ \chi'}{< \nabla\Phi_1(\omega,\bar{\omega}), \omega >^2}, \omega \in \mathbb{R}^2 \times iS^+, \text{ when } \phi_{h^+} \in \mathcal{A}(T^*_{(S_H^+)^{int}}), \text{ for } \ h \in \widetilde{\mathcal{G}}_{S^1}\},
\end{aligned}
$$

where χ' is the mapping defined on the complementary half-tube to the domain of definition of the mapping χ and $h^\mp$ denotes the restriction of $h \in \widetilde{\mathcal{G}}_{S^1}$ to the corresponding $\mathbb{R}^2 \times iS^\mp$. Here ψ_h stands for a function defined by a relation similar to (5.0.6) with integration taking place over $\mathbb{R}^2 \times iS^1$ and measure μ_{Φ_1} or by completion of limits like in a lemma before. Note that all measures of integration involved are equivalent to the Lebesgue measure. One can identify the space $\widetilde{\mathcal{W}}_{T^*_{B_1}}$ as a subspace of $\widetilde{\mathcal{G}}_{S^1}$. We claim that the topology corresponding to this norm coincides with the initial (projective) topology induced by the topologies of normed spaces $\widetilde{\mathcal{W}}_{T^*_{(S^-_H)^{int}}}$, $\widetilde{\mathcal{W}}_{T^*_{(S^+_H)^{int}}}$. To be more specific, by initial topology we mean the weakest (coarsest) topology that make the maps maps $\widetilde{p}^- : \widetilde{\mathcal{W}}_{T^*_{B_1}} \longrightarrow \widetilde{\mathcal{W}}_{T^*_{(S^-_H)^{int}}}$, $\widetilde{p}^+ : \widetilde{\mathcal{W}}_{T^*_{B_1}} \longrightarrow \widetilde{\mathcal{W}}_{T^*_{(S^+_H)^{int}}}$ continuous. The open sets of the initial topology are unions of finite intersections $(\widetilde{p}^+)^{-1}(V) \cap (\widetilde{p}^-)^{-1}(U)$, were U and V are open sets in the corresponding normed spaces. Naturally, it is meant that $(\widetilde{p}^{\pm})^{-1}(g) = \emptyset$ whenever $g \in \widetilde{\mathcal{W}}_{T^*_{(S^{\pm}_H)^{int}}} \setminus \widetilde{\mathcal{W}}_{T^*_{B_1}}$. Thus, it follows from Havin's lemma ([5]) every continuous functional $F \in (\widetilde{\mathcal{W}}_{T^*_{B_1}})'$ can be written as

$$F(\phi) = F_1(\phi) + F_2(\phi), \tag{6.0.3}$$

where the functionals $F_1 \in (\widetilde{\mathcal{W}}_{T^*_{(S^-_H)^{int}}})'$ and $F_2 \in (\widetilde{\mathcal{W}}_{T^*_{(S^-_H)^{int}}})'$ are continuous with respect to the initial topology induced by the corresponding spaces. Since $\widetilde{\mathcal{W}}_{T^*_{B_1}} \subset \widetilde{\mathcal{G}}_{S^1}$ one has that $(\widetilde{\mathcal{G}}_{S^1})' \subset (\widetilde{\mathcal{W}}_{T^*_{B_1}})'$. Since the space $H^2(T_{B_1})$ is reflexive (see Chapter 3) we have that $H^2(T_{B_1}) \subset (\widetilde{\mathcal{W}}_{T^*_{B_1}})'$ and thus the equation (6.0.7) becomes

$$F(\phi) = F_1(\phi) \bigoplus F_2(\phi), \tag{6.0.4}$$

taking into account that $(\widetilde{\mathcal{W}}_{T^*_{(S^-_H)^{int}}})' \subset H^2(T_{(S^-_H)^{int}})$, $(\widetilde{\mathcal{W}}_{T^*_{(S^+_H)^{int}}})' \subset H^2(T_{(S^+_H)^{int}})$, $H^2(T_{(S^-_H)^{int}}) \cap H^2(T_{(S^+_H)^{int}}) = \emptyset$. Hence we have

$$g = g_1 + g_2, \tag{6.0.5}$$

where the equality is understood as equality of functionals (elements of $L^2(\partial T_{B_1})$) on $H^2(T_{B_1})$. It implies that the boundary values of the function

$$h(z) = g(z) - g_1(z) - g_2(z) = \int\limits_{\mathbb{R}^2} f_h(t) e^{2\pi i z \cdot t} dt \in H^2(T_{B_1})$$

on the $\partial T_{B_1} = \mathbb{R}^2 \times iS^1$ defines a zero functional. Its slice-wise action (i.e $\Im z^0 = (\Im z^0_1, \Im z^0_2)$

fixed) on $h(\Im z_1^0, \Im z_2^0, x_1, x_2)$ and Parseval's Theorem imply that

$$\int\limits_{\mathbb{R}^2} |h(\Im z_1^0, \Im z_2^0, x_1, x_2)|^2 dx_1 dx_2 = \int\limits_{\mathbb{R}^2} |f_h(t) e^{-y^0 \cdot t}|^2 dt_1 dt_2 = 0.$$

This implies that $f_h(t) = 0$ a.e on $\mathbb{R}^2$ and hence $g(z^0) = g_1(z^0) + g_2(z^0)$ for almost all $z^0 \in \partial T_{B_1}$. Thus every element $g \in H^2(T_{B_1})$, is expressed (almost everywhere at the boundary) in a unique way as a difference on the boundary ∂T_{B_1} of the boundary values of two elements g_1 and g_2 from the spaces $H^2(T_{(S_H^-)^{int}})$ and $H^2(T_{(S_H^+)^{int}})$ correspondingly. Furthermore, for every $z \in T_{B_1}$ fixed, we have that the function $\phi_z(w_1, w_2) = \frac{1}{(1 - z_1 w_1 - z_2 w_2)^2}$ is holomorphic in a neighborhood of the compact $T^*_{B_1}$, and thus in a neighborhood of the compacts $T^*_{S_H^-}$, $T^*_{S_H^+}$ also. Then

$$(g_1 + g_2)(\phi_z) = g_1(z) + g_2(z), \ \forall z \in T_{B_1}, \tag{6.0.6}$$

because of the Theorem 4.0.2 and the equality

$$(w_1, w_2) = \frac{\nabla_\zeta \Phi_1(\zeta, \bar{\zeta})}{< \nabla_\zeta \Phi_1(\zeta, \bar{\zeta}), \zeta >} \in T^*_{B_1}.$$

Thus, taking into account that (6.0.5) holds for almost all $z_0 \in \partial T_{B_1}$ and the identity (6.0.6) one has firstly the relation (6.0.1) and secondly that

$$g(\phi_z) = (g_1 + g_2)(\phi_z) = g_1(z) + g_2(z), \ \forall z \in T_{B_1}$$

Thus, a Cauchy-Fantappie type integral $g(\phi_z)$, as a function of z, equals $g_1(z) + g_2(z), \ \forall z \in T_{B_1}$. Going to the boundary values on the left hand side of the last equality, we have that for almost all $z_0 \in \partial T_{B_1}$ the following identity

$$g_1(z_0) + g_2(z_0) = \int\limits_{\mathbb{R}^2} f_1(t) e^{2\pi i z_0 \cdot t} dt + \int\limits_{\mathbb{R}^2} f_2(t) e^{2\pi i z_0 \cdot t} dt$$

holds. Thus, for every $z \in T_{B_1}$

$$g(\phi_z) = (g_1 + g_2)(\phi_z) = g_1(z) + g_2(z) = g(z).$$

This identity concludes the proof of the theorem . $\diamondsuit$

The last proposition of the present describes the partial converse of the previous theorem. Its formulation and proof could be well located in Chapter 3, since the mathematical content of its proof is closer in spirit to results presented there.

Proposition 6.0.2 *Let f be holomorphic function in the tube T_{B_1}, defined also on ∂T_{B_1} and such that $f \in L^2(\partial T_{B_1})$. Assume also that f satisfies the property that every restriction $f|_{\mathbb{R}^2 \times i\{(y_1,y_2)\}}$ belongs to $\mathcal{S}(\mathbb{R}^2)$, whenever $(y_1, y_2) \in S^1$ and that the growth estimate at the endpoints present in Corollary 4.1 holds. If* (6.0.2) *is valid, then $f \in H^2(T_{B_1})$.*

Proof: Let us assume that for every $z \in T_{B_1}$ the integral representation formula (6.0.2) is valid. Then for the same $z \in T_{B_1}$ one defines the functions

$$\begin{aligned} f_1(z) &= \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{f(\zeta)\left(\partial\Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})\right)}{(\nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - z >)^2} \\ f_2(z) &= \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^+} \frac{f(\zeta)\left(\partial\Phi_1^+(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^+(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^+(\zeta,\bar{\zeta}), \zeta - z >)^2} \end{aligned} \qquad (6.0.7)$$

It is easy to see that application of Corollary 4.1 implies that $f_1 \in H^2(T_{(S_H^-)^{int}})$, $f_2 \in H^2(T_{(S_H^+)^{int}})$. Thus $f_1(z) = \int\limits_{\mathbb{R}^2} g_1(t) e^{2\pi i z \cdot t} dt$, $z \in T_{(S_H^-)^{int}}$, $f_2(z) = \int\limits_{\mathbb{R}^2} g_2(t) e^{2\pi i z \cdot t} dt$, $z \in T_{(S_H^+)^{int}}$. Therefore, one deduces that $\|f\|_{H^2(T_{B_1})} \leq \|f_1\|_{H^2(T_{(S_H^-)^{int}})} + \|f_2\|_{H^2(T_{(S_H^+)^{int}})}$. This concludes the proof of the proposition.$\diamondsuit$

Chapter 7

Conclusions and Further Research

7.1 Conclusions

The main objective of this thesis was to describe the Hardy space $H^2(T_{B_i})$, for $i = 1, 2$, through a Cauchy-Fantappie type formula, where T_{B_i}, $i = 1, 2$, are the tube domains (2.2.1). The main obstacle was the absence of Stoke's theorem for unbounded domains. In this direction, we realized each of tubes T_{B_i}, $i = 1, 2$, as an intersection of tube domains with convex, unbounded base which contains a cone. Namely, we defined $T_{B_1} = T_{(S_H^-)^{int}} \cap T_{(S_H^+)^{int}}$ and $T_{B_2} = T_{(R_H^-)^{int}} \cap T_{(R_H^+)^{int}}$ where the bases $S_H^{\pm}$ and $R_H^{\pm}$ are defined in (2.4.1) and (2.5.4), correspondingly. Following ([13], [14]) we derived a Cauchy-Fantappie formula for the space $H^2(T_{(S_H^-)^{int}})$. More specifically, we proved that a function is an element of the space $H^2(T_{(S_H^-)^{int}})$ if and only if it is representable by a Cauchy-Fantappie formula. Similarly arguing results are valid for the space $H^2(T_{(S_H^+)^{int}})$. In the spirit of Martineau-Aizenberg we obtained that every analytic functional $F \in \left(H^2(T_{(S_H^-)^{int}})\right)'$ is represented by

$$\begin{aligned} F(f) = F_h(f) &= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} f(\omega) h(\omega) \mu_{\Phi_1^-}(\omega, \bar{\omega}) \\ &= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} f(\zeta) \widetilde{\phi}_h(\tau(\Phi_1^-)) \Omega(\Phi_1^-(\zeta, \bar{\zeta})) \\ &= \frac{1}{(2\pi i)^2} \int\limits_{\mathbb{R}^2 \times iS^-} f(\zeta) \frac{\widetilde{\phi}_h(\tau(\Phi_1^-)) \mu_{\Phi_1^-}(\zeta, \bar{\zeta})}{< \nabla \Phi_1^-(\zeta, \bar{\zeta}), \zeta >^2} = F_{\phi_h}(f), \end{aligned}$$

where $h \in \widetilde{\mathcal{G}^-}$ and $\widetilde{\phi}_h \in \mathcal{A}(T^*_{(S_H^-)^{int}})$. Recall at this point that $\mathcal{A}(T^*_{(S_H^-)^{int}})$ is the space of holomorphic functions on a neighborhood of every point belonging to $T^*_{(S_H^-)^{int}} \backslash \{((0,0),(0,\pm i))\}$ and bounded on the compact $T^*_{(S_H^-)^{int}}$ and

$$\mathcal{G}^- = \{h \in L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-) : \text{ every restriction } \ h|_{\mathbb{R}^2 \times \{(y_1,y_2)\}} \in \mathcal{S}(\mathbb{R}^2)\}$$

where $\widetilde{\mathcal{G}^-}$ is its $L^2_{\mu_{\Phi_1^-}}(\mathbb{R}^2 \times iS^-)$-closure. Concluding, we derived the separation of singularities type theorem for the space $H^2(T_{B_1})$ providing that for every $f \in H^2(T_{B_1})$ there exist functions $f_1 \in H^2(T_{(S_H^-)^{int}})$ and $f_2 \in H^2(T_{(S_H^-)^{int}})$ that satisfy $f(z) = f_1(z) - f_2(z)$ for $z \in T_{B_1}$. Actually, for every $z \in T_{B_1}$ we have obtained that

$$\begin{aligned} f(z) &= \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^-} \frac{f_1(\zeta)\left(\partial \Phi_1^-(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^-(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^-(\zeta,\bar{\zeta}), \zeta - z >)^2} \\ &+ \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^+} \frac{f_2(\zeta)\left(\partial \Phi_1^+(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1^+(\zeta,\bar{\zeta})\right)}{(< \nabla_\zeta \Phi_1^+(\zeta,\bar{\zeta}), \zeta - z >)^2} \end{aligned}$$

and finally

$$f(z) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{R}^2 \times iS^1} \frac{f(\zeta)\left(\partial \Phi_1(\zeta,\bar{\zeta}) \wedge \partial\bar{\partial}\Phi_1(\zeta,\bar{\zeta})\right)}{(\nabla_\zeta \Phi_1(\zeta,\bar{\zeta}), \zeta - z >)^2}.$$

As a direct consequence we have derived an integral representation formula valid for any $f \in H^2(T_{B_1})$. Actually, if f is a holomorphic function in the tube T_{B_1}, defined also on ∂T_{B_1} and such that $f \in L^2(\partial T_{B_1})$ satisfying the property that every restriction $f|_{\mathbb{R}^2 \times i\{(y_1,y_2)\}}$ belongs to $\mathcal{S}(\mathbb{R}^2)$, whenever $(y_1, y_2) \in S^1$ and (6.0.2) is valid, then $f \in H^2(T_{B_1})$.

7.2 Further Research

In this thesis we exclusively focus on the tube domains $T_{(S_H^-)^{int}}$ and $T_{(S_H^+)^{int}}$ (2.4.2) the intersection of which defines the tube T_{B_1} (2.2.1). An obvious consideration consists in the determination of a class of convex tube domains $T_D = \mathbb{R}^2 \times iD$, $D \subset \mathbb{R}^2$ that admits a Cauchy-Fantappie integral representation formula. One considers that we may assume once again that the base of the tube is convex, does not contain an entire straight line but is intersection of tubes T_{D_i}, $i = 1, 2$, containing a cone. Furthermore, one has to ensure that whenever $f \in H^2(T_{D_i})$ then $\|f\|_{H^2(T_{D_i})}$ is realized on the strictly convex part of ∂D_i. At this point, we note that the existence of boundary values everywhere on $\partial T_{(S_H^{\pm})^{int}}$ was based to the fact that each point on $\partial T_{(S_H^{\pm})^{int}}$ was polygonal. Furthermore, an important fact is the form of the complex tangent hyperplanes when the strictly convex part of the boundary is not a part of a circle (half circle in our case) .

The problem of obtaining a Cauchy-Fantappie type integral formula for functions on Hardy spaces $H^2(T_D)$ over tube domains in higher dimensions is worth further investigation. Actually, if we assume that $T_D = \mathbb{R}^n \times iD$, $D \subset \mathbb{R}^n$ then in order to ensure the existence of a complex tangent hyperplane at every boundary point $\zeta \in \partial T_D$ one requires higher degree of smoothness of the boundary $\mathbb{R}^n \times i\partial D$, $D \subset \mathbb{R}^n$. If one decides to explore the duality for Hardy spaces $H^p(T_D)$ over tube domains for $p \neq 2$ then the loss of Plancherel's theorem will change an important part of the resulting method, thus the generalization of results to $H^p(T_D)$ for $p \neq 2$ are far from obvious.

Bibliography

[1] Aizenberg, L.: The general form of a continuous linear functional on the space of functions holomorphic in a convex region of C^n. (Russian) Dokl. Akad. Nauk SSSR 166, 1015-1018 (1966)

[2] Aizenberg, L.: The expansion of holomorphic functions of several complex variables in partial fractions. (Russian) Sibirsk. Mat. Z. 8 1124-1142, (1967)

[3] Aizenberg, L.: Linear convexity in $\mathbb{C}^n$ and separation of singularities of holomorphic functions. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 15 1967 487-495.

[4] Aizenberg, L.: The Duality in Complex Analysis. Proceedings of the Ashkelon Workshop on Complex Function Theory (1996), 27-35, Israel Math. Conf. Proc., 11, Bar-Ilan Univ., Ramat Gan, (1997)

[5] Aizenberg, L.: Separation of singularities for holomorphic functions. Anal. Math. Phys. 4 (2014), no.1-2.

[6] Aizenberg, L., Vidras, A.: Duality and the class of holomorphic functions representable by Carleman formula. Complex Analysis and Dynamical Systems V. Contemp. Math., AMS vol.591, p.24, (2013)

[7] Aizenberg, L., Gotlib, V., Vidras, A.: Duality for Hardy spaces in domains of $\mathbb{C}^n$ and some applications. Complex Anal. Oper. Theory 8 (2014), no. 6, 1341-1366

[8] Aizenberg, L., Yuzhakov, A.: Integral Representations and residues in Multidimensional Complex Analysis. p. 283 AMS, (1983)

[9] Aizenberg, L., Juzakov, A., Makarova, L. Ja : Linear convexity in $\mathbb{C}^n$. Sibirsk. Mat. Z. 9, 731-746, (1968)

[10] Andersson, M., Passare, M., Sigurdsson, R.: Complex convexity and Analytic functions. Progress in Mathematics 225, Birkäuser, (2004)

[11] Aronsajn, N.: Sur les decompositions des fonctions analytiques uniform et sur leur applications. Acta Mathem. 65, no.1, 1-156, (1935)

[12] Duren, P. L.: The theory of $\mathcal{H}^p$ spaces. Dover Publ., (2000)

[13] Gindikin, S: Radon Transformation for holomorphic functions of several complex variables and related integral formulas. Uspekhi Mat. Nauk 20, n0.3 (123), 237-239, (1965)

[14] Gindikin, S: Analytic functions in tubular regions. Dokl. Akad. Nauk SSSR 145, 1205-1208, (1962)

[15] Kerzman, N., Stein, E. M.: The Szego kernel in terms of Cauchy-Fantappie kernels. Duke Math. J. 145 (2) 197-224, (1978)

[16] Kranz, S.: Function theory of several complex variables. AMS, (2000)

[17] Kytmanov, A., Myslivets , S.: Multidimensional Integral Repfresentations. Springer, (2015)

[18] Lanzani, L., Stein, E.M.: The Cauchy-type integrals in $\mathbb{C}^n$ for domains with minimal smoothness. Adv. in Math. 264(2014), 776-830.

[19] Lanzani, L., Stein, E.M.: The Cauchy-type integrals in several complex varibables. Bull.Math.Sci.3(2013), no.2 241-285. Adv. in Math. 264(2014), 776-830.

[20] Lindholm, Niklas.: A Paley-Wiener theorem for convex sets in $\mathbb{C}^n$. Bull.Sc.Math.126(2002), no.4 289-314.

[21] Martineau, A.: Sur les fonctionelles analytiques et la transformation de Fourier-Borel. J.Anal. Math. 11, 1-164, (1963)

[22] Martineau, A.: Sur les topologies des espaces des fonctions holomorphes. Math.Ann. 103, 62-88, (1966)

[23] Michael, E.: Continuous selections. I Annals of Mathem. vol.63, no.2 pp.361-382, (1956)

[24] Privalov, I. I.: Randeigenschaften Analytischer Functionen. Deutscher V.der Wiss., (1956)

[25] Range, R. M.: Holomorphic Functions and Integral Representations in Several Complex Variables, Springer-Verlag, (1986)

[26] Rotkevich, A. S.: The Cauchy-Leray-Fantappie integral in linearly convex domains. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat.Inst. Steklov. (POMI) 401 (2012), Issledovaniya po Lineinym Operatoram i Teorii Funktsii. 40, 172–188, 201; translation in J. Math. Sci. (N. Y.) 194 (2013), no. 6, 693-702

[27] Stein, E., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces, Princeton University press, (1971)

[28] Stout, E. L.: Polynomial convexity, Springer-Verlag, (2007)

[29] Znamenskii, A.: A geometric criterion for strong linear convexity. Funkt.Anal. Appl. 13 , 224-225, (1979). 452-458, (1971)