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DIFFERENTIAL INVARIANTS OF HYPERBOLIC EQUATIONS

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Abstract

This thesis consists of two parts: in the major part we calculate the infinitesimal generators of families of partial differential equations which are used for derivation of differential invariants. In the other part we have drawn our attention to point transformations which preserve the general form of partial differential equations.

In the applied group analysis, one-parameter Lie groups of transformations are determined by infinitesimal transformations or infinitesimal generators. Using the infinitesimal generator of a one-parameter Lie group of transformations one can construct various kinds of invariants (invariant surfaces, invariant points, invariant families of surfaces). A one-parameter Lie group of transformations acting on the space of independent and dependent variables is naturally extended (prolonged) to one-parameter Lie group of transformations acting on an enlarged space that includes all derivatives of the dependent variables up to a fixed finite order. Consequently, one-parameter extended Lie groups of transformations are characterized completely by their infinitesimals. This allows one to establish an algorithm to determine the infinitesimal transformations admitted by a given differential equation.

There exist two methods for calculation of equivalence transformations, the direct method which was used first by Lie and the Lie infinitesimal method which was introduced by Ovsyannikov. Although, the direct method involves considerable computational difficulties, it has the benefit of finding the most general equivalence group.

Recently, Ibragimov developed a simple method for constructing invariants for families of differential equations. The method is based in the theory of equivalence groups in the infinitesimal form. Basically, the method consists of two steps: classification of equivalence groups and then use these groups (and extended groups) to derive the desired differential invariants. Ibragimov used his method to solve the Laplace problem. That is, to derive

all invariants for the linear hyperbolic equation

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0.$$

To achieve this, he constructed a basis for the invariants and then using this basis and invariant differentiation all invariants, of any order, can be derived. The idea of Ibragimov was adopted by a number of authors who derived differential invariants for ordinary differential equations, linear and non-linear partial differential equations.

Differential invariants of the Lie groups of continuous transformations can be used in wide fields: classification of invariant differential equations and variational problems arising in the construction of physical theories, solution methods for ordinary and partial differential equations, equivalence problems for geometric structures. First it was noted by Lie (see [33]), who proved that every system of differential equations (see [34]), and every variational problem (see [36]), could be directly expressed in terms of differential invariants. Lie also showed (see [34]) how differential invariants play an important role to integrate ordinary differential equations and succeeded in completely classifying all differential invariants for all possible finite-dimensional Lie groups of point transformations in the case of one independent and one dependent variable. Tresse (see [56]) and Ovsyannikov [44] generalized the Lie's preliminary results on invariant differentiations and existence of finite bases of differential invariants. The general theory of differential invariants of Lie groups together with algorithms of construction of differential invariants can be found in [42, 44].

Also, there is merit in studying point transformations directly in finite form with the ultimate dual goals of finding the complete set of point transformations of systems of two partial differential equations and discovering new links between these systems.

Relationships between partial derivatives are considerably more cumbersome than the corresponding relationships for infinitesimal transformations which themselves expand rapidly with increasing order. However several results are presented. These results help us achieve the second aim which is to discover the nature of point transformations connecting systems of two partial differential equations belonging to given classes. Thus, we look at systems with one partial derivative of $u(t, x)$ and $v(t, x)$ of any order, possibly mixed, related to lower-order derivatives of u and v, u and v themselves and t and x.

In this thesis, firstly we develop the basic concepts of Lie groups of transformations, infinitesimal transformations and invariance of partial differential equations that are necessary in the following chapters. In the beginning we start with known results. That is, we use the Lie infinitesimal method for calculating the continuous group of equivalence transformations, for the non-linear diffusion equation. Also, we apply this method to derive differential invariants for the linear hyperbolic equation in two variables. Finally, we describe the method which used by Ibragimov to solve the Laplace problem.

The second step is to calculate equivalence transformations for given families of equations. In the spirit of the recent work of Ibragimov (see [19]), who adopted the infinitesimal method for calculation of invariants of families of differential equations using the infinitesimal groups, we apply the method to several partial differential equations. In this thesis, we derive the equivalence group for hyperbolic equations of general class and for two special cases of it. Also, we calculate equivalence transformations for n-dimensional hyperbolic equations, for n-dimensional wave-type equations and finally for hyperbolic equations with two dependent variables. For these families of equations, we find the forms of differential invariants of first or/and second order. In certain cases, we will use the derived invariants or/and invariant equations to find the form of those equations that can be mapped into an equation with particular form.

Furthermore, we work on form-preserving point transformations for partial differential equations. We present some known results (see [29]) for three classes of equations restricted to one dependent variable and two independent variables concerning the nature of connecting point transformations. We will generalize these results for forms of point transformations connecting two systems of two partial differential equations. The aim of this part is first to present results concerning the relation of the transformed partial derivatives to the original partial derivatives and secondly to exploit these results to reduce the general range of point transformations connecting systems of two partial differential equations belonging to restricted classes.

Περίληψη

Αυτή η διατριβή αποτελείται από δύο μέρη: στο πρώτο μέρος υπολογίζουμε απειροστούς γεννήτορες για διάφορες κατηγορίες μερικών διαφορικών εξισώσεων οι οποίοι χρησιμοποιούνται για την εύρεση αναλλοίωτων συναρτήσεων και στο δεύτερο μέρος μελετούμε σημειακούς μετασχηματισμούς οι οποίοι διατηρούν την γενική μορφή των μερικών διαφορικών εξισώσεων αναλλοίωτη.

Στην εφαρμοσμένη ανάλυση, οι μονοπαραμετρικές ομάδες μετασχηματισμών Lie, καθορίζονται από τους απειροστούς μετασχηματισμούς ή τους απειροστούς γεννήτορες. Χρησιμοποιώντας τον απειροστό γεννήτορα μιας μονοπαραμετρικής ομάδας μετασχηματισμών Lie, μπορούμε να κατασκευάσουμε ποικίλες μορφές αναλλοίωτων συναρτήσεων (όπως αναλλοίωτες επιφάνειες, αναλλοίωτα σημεία, αναλλοίωτες οικογένειες επιφανειών). Μια μονοπαραμετρική ομάδα μετασχηματισμών Lie που δρα πάνω στον χώρο των ανεξάρτητων και εξαρτημένων μεταβλητών είναι επεκτεταμένη σε μια μονοπαραμετρική ομάδα μετασχηματισμών Lie που δρα σε μεγαλύτερο χώρο, ο οποίος περιλαμβάνει όλες τις παραγώγους των εξαρτημένων μεταβλητών μέχρι κάποιας πεπερασμένης τάξης. Συμπερασματικά, μια μονοπαραμετρική επεκτεταμένη ομάδα μετασχηματισμών Lie χαρακτηρίζεται από τα infinitesimal της. Αυτό μας επιτρέπει να κατασκευάσουμε ένα αλγόριθμο για να καθορίσουμε τους απειροστούς μετασχηματισμούς τους οποίους επιδέχεται μια δοσμένη διαφορική εξίσωση.

Υπάρχουν δύο τρόποι για υπολογισμό των ισοδύναμων μετασχηματισμών, ο άμεσος τρόπος ο οποίος χρησιμοποιήθηκε πρώτα από τον Lie και η απειροστή μέθοδος Lie η οποία κατασκευάστηκε από τον Ovsyannikov. Αν και ο άμεσος τρόπος περιλαμβάνει πολλές υπολογιστικές δυσκολίες, εντούτοις έχει το πλεονέκτημα εύρεσης πιο γενικής ισοδύναμης ομάδας.

Πρόσφατα, ο Ibragimov ανέπτυξε μια απλή μέθοδο για κατασκευή αναλλοίωτων συναρτήσεων. Αυτή η μέθοδος βασίζεται στην θεωρία των ισοδύναμων ομάδων σε απειροστή μορφή.

Αυτή η μέθοδος αποτελείται από δύο βήματα: πρώτα την ταξινόμηση των ισοδύναμων ομάδων και έπειτα τη χρησιμοποίησή τους για εύρεση των αναλλοίωτων συναρτήσεων. Ο Ibragimov χρησιμοποίησε αυτή την μέθοδο για να λύσει το πρόβλημα του Laplace. Δηλαδή, χρησιμοποίησε αυτή την μέθοδο για να βρει όλες τις αναλλοίωτες συναρτήσεις για την γραμμική υπερβολική εξίσωση:

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0.$$

Για να το πετύχει, κατασκεύασε μια βάση για τις αναλλοίωτες συναρτήσεις και χρησιμοποιώντας αυτή τη βάση και με αναλλοίωτη παραγώγιση βρήκε όλες τις αναλλοίωτες συναρτήσεις οποιασδήποτε τάξης. Η ιδέα του Ibragimov υιοθετήθηκε από πολλούς συγγραφείς, οι οποίοι υπολόγισαν αναλλοίωτες συναρτήσεις για συνήθεις διαφορικές εξισώσεις, γραμμικές και μη-γραμμικές μερικές διαφορικές εξισώσεις.

Οι αναλλοίωτες συναρτήσεις των ομάδων μετασχηματισμών Lie μπορούν να χρησιμοποιηθούν σε διάφορα πεδία: όπως για την ταξινόμηση των αναλλοίωτων διαφορικών εξισώσεων, σε προβλήματα μεταβολών, σε μεθόδους για τη λύση συνήθων και μερικών διαφορικών εξισώσεων. Πρώτος ο Lie απέδειξε ότι κάθε αναλλοίωτο σύστημα μερικών διαφορικών εξισώσεων και κάθε πρόβλημα μεταβολών, μπορεί να εκφραστεί μέσω των αναλλοίωτων συναρτήσεων. Επίσης, έδειξε, πώς οι αναλλοίωτες συναρτήσεις παίζουν σημαντικό ρόλο στην ολοκλήρωση συνήθων διαφορικών εξισώσεων, για την ταξινόμηση όλων των αναλλοίωτων συναρτήσεων για όλες τις πεπερασμένης -διάστασης ομάδες Lie σημειακών μετασχηματισμών στην περίπτωση μιας ανεξάρτητης και μιας εξαρτημένης μεταβλητής. Η Tresse και ο Ovsyannikov γενίκευσαν τα αποτελέσματα του Lie σε αναλλοίωτη παραγώγιση και για την ύπαρξη πεπερασμένων βάσεων αναλλοίωτων συναρτήσεων. Η γενική θεωρία των αναλλοίωτων συναρτήσεων των ομάδων Lie μαζί με αλγορίθμους για κατασκευή των αναλλοίωτων συναρτήσεων μπορεί να βρεθεί στα [42, 44].

Επίσης, είναι σημαντικό να ασχοληθούμε και με σημειακούς μετασχηματισμούς, με σκοπό την εύρεση πλήρους ομάδας σημειακών μετασχηματισμών για συστήματα που αποτελούνται από δύο μερικές διαφορικές εξισώσεις.

Οι σχέσεις μεταξύ των μερικών παραγώγων είναι πιο πολύπλοκες από τις αντίστοιχες σχέσεις για τους απειροστούς μετασχηματισμούς. Εντούτοις θα παρουσιάσουμε κάποια αποτελέσματα. Αυτά τα αποτελέσματα μας βοηθούν να πετύχουμε τον δεύτερο σκοπό μας που είναι να βρούμε τους σημειακούς μετασχηματισμούς που συνδέουν συστήματα μερικών διαφορικών εξισώσεων συγκεκριμένης μορφής. Γι΄ αυτό ασχολούμαστε με συστήματα που αποτελούνται από δύο μερικές διαφορικές εξισώσεις και περιλαμβάνουν μια μερική παράγωγο των εξαρτημένων μεταβλητών $u(t,x)$ και $v(t,x)$ και έπειτα με συστήματα που περιλαμβάνουν μεικτές παραγώγους των εξαρτημένων μεταβλητών $u(t,x)$ και $v(t,x)$, οι οποίες σχετίζονται με μικρότερης τάξης παραγώγους των $u(t,x)$ και $v(t,x)$, με τα ίδια τα $u(t,x)$ και $v(t,x)$ και τα $t,\ x$.

Γι΄ αυτό σε αυτή την διατριβή, πρώτα θα αναφερθούμε σε βασικούς ορισμούς των ομάδων μετασχηματισμών Lie , των απειροστών μετασχηματισμών και το αναλλοίωτο των μερικών διαφορικών εξισώσεων, τα οποία είναι χρήσιμα για τα επόμενα κεφάλαια. Καταρχήν, θα ξεκινήσουμε παρουσιάζοντας γνωστά αποτελέσματα. Δηλαδή, θα εφαρμόσουμε την απειροστή μέθοδο του Lie για τον υπολογισμό ομάδων ισοδύναμων μετασχηματισμών για τη μη γραμμική εξίσωση διάχυσης. Επίσης, θα χρησιμοποιήσουμε τη μέθοδο για το υπολογισμό των αναλλοίωτων συναρτήσεων της γραμμικής υπερβολικής εξίσωσης. Επιπλέον, θα περιγράψουμε τον τρόπο με τον οποίο ο Ibragimov έλυσε το πρόβλημα του Laplace.

Σαν δεύτερο βήμα, ακολουθώντας την ιδέα του Ibragimov, για εύρεση ισοδύναμων μετασχηματισμών, θα υπολογίσουμε ισοδύναμους μετασχηματισμούς για δοσμένες οικογένειες μερικών διαφορικών εξισώσεων. Συγκεκριμένα, θα υπολογίσουμε τις ομάδες ισοδυναμίας για τη γενική μορφή της υπερβολικής εξίσωσης και δύο ειδικών περιπτώσεων της. Επίσης, θα υπολογίσουμε τις ομάδες ισοδυναμίας για την υπερβολική εξίσωση διάστασης n, για την κυμματική εξίσωση διάστασης n και τέλος για σύστημα που αποτελείται από δύο υπερβολικές εξίσωσεις. Γι΄ αυτές τις οικογένειες εξισώσεων, βρίσκουμε τις αναλλοίωτες συναρτήσεις πρώτης ή/και δεύτερης τάξης. Σε περισσότερες από αυτές, θα χρησιμοποιήσουμε τις αναλλοίωτες συναρτήσεις ή/και τις αναλλοίωτες εξισώσεις για να βρούμε τη μορφή αυτών των εξισώσεων οι οποίες μπορούν να απεικονισθούν σε εξισώσεις συγκεκριμένης μορφής.

Τέλος, θα δώσουμε κάποια γνωστά αποτελέσματα που αφορούν τους σημειακούς μετασχηματισμούς μιας μερικής διαφορικής εξίσωσης. Συγκεκριμένα, θα παρουσιάσουμε την μορφή των σημειακών μετασχηματισμών οι οποίοι συνδέουν τρεις ομάδες μερικών διαφορικών εξισώσεων που αποτελούνται από μια εξαρτημένη και δύο ανεξάρτητες μεταβλητές. Σκοπός μας είναι να γενικεύσουμε αυτά τα αποτελέσματα για συστήματα που αποτελούνται από δύο διαφορικές εξισώσεις. Δηλαδή, πρώτα θα παρουσιάσουμε αποτελέσματα που αφορούν τις σχέσεις των μετασχηματισμένων μερικών παραγώγων με τις αρχικές μερικές παραγώγους και δεύτερον χρησιμοποιώντας αυτά τα αποτελέσματα θα προσδιορίσουμε τη μορφή των σημειακών μετασχηματισμών που συνδέουν συγκεκριμένα συστήματα μερικών διαφορικών εξισώσεων.

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Contents

Chapter 1

Introduction

Modern mathematics has over 300 years history. From the very beginning it was focused on differential equations as a major tool for the mathematical modeling. Most of mathematical models in physics, engineering sciences, biomathematics, etc. lead to nonlinear differential equations.

The theory of differential equations is one of the most important disciplines in modern mathematics. It would be correct to say that the notions of derivative and integral, whose origin goes back to Archimedes, were in fact introduced later in works of Kepler, Descartes, Cavalieri, Fermat and Wallis. Later, Newton and Leibnitz realized that differentiation and integration are inverse operations and developed the appropriate algorithms.

The brothers Jacob and Johann Bernoulli (1654-1705, 1667-1748) made further contribution to the theory of differential equations. Especially, famous are their investigations of geodesic curves and isoperimetric problems that are considered to be the origin of variational calculus.

The Italian mathematician Riccati (1676-1754) paid attention to particular cases of the following equation which later became popular:

$$\frac{dy}{dx} = X(x) + X_1(x)y + X_2(x)y^2.$$

This equation should certainly be considered as the simplest and the most significant among non-integrable differential equations. In particular, new group-theoretic investigations show that this equation can be interpreted as an analogue of the algebraic equation of fifth degree.

A further important contribution to the theory of differential equations was made by d' Alembert (1717-1783). By formulating the general mechanical principle, he reduced all problems of dynamics to differential equations and furnished Newton's revolutionary mechanical ideas with a general and definite form.

The first category of all investigations on partial differential equations (PDEs) of the first order started by Euler, Lagrange and Monge, and continued by Pfaff, Cauchy, Hamilton, Jacobi, A. Mayer and others. Research on PDEs of second and higher order started by Monge and Laplace. Among followers of Laplace and Monge in this field are Ampére, Darboux and some other French mathematicians who ensured a considerable advance in the theory of differential equations. The notion of characteristics introduced by Monge played implicitly or explicitly an important role.

The linear wave equation $u_{xy} = 0$ for vibrating strings, was formulated and solved by d'Alembert in 1747. In 1769/1770, Euler (see [7]) and later, in 1773, Laplace (see [32]) derived the two invariant quantities

$$h = a_x + ab - c, \quad k = b_y + ab - c. \tag{1.1}$$

These fundamental invariant quantities are known today as the *Laplace invariants*.

We owe to Leonard Euler the first significant results in integration theory of general hyperbolic equations with two independent variables $x,\ y$:

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{1.2}$$

In his "Integral calculus " (see [7]), Euler introduced what is known as the Laplace invariants h and k. Namely, he generalized d' Alembert's solution and showed that equation (1.2) is factorable, and hence integrable by solving two first-order ordinary differential equations, if and only if its coefficients $a,\ b,\ c$ obey one of the following equations:

$$h \equiv a_x + ab - c = 0$$

or

$$k \equiv b_y + ab - c = 0.$$

If $h = 0$, equation (1.2) is factorable in the form

$$\left(\frac{\partial}{\partial x} + b\right)\left(\frac{\partial u}{\partial y} + au\right) = 0,$$

and if $k = 0$, equation (1.2) is factorable in the form

$$\left(\frac{\partial}{\partial y} + a\right)\left(\frac{\partial u}{\partial x} + bu\right) = 0.$$

In the 1773s, Laplace developed a new method, known as Laplace's "cascade method", in his fundamental paper "Studies on integral calculus of partial differences". The central role in his method play the semi-invariants h and k. His method is used to solve many hyperbolic equations.

In the 1890s, Darboux discovered the invariance of h and k and called them the Laplace invariants. He also simplified and improved Laplace's method, and the method became widely known due to Darboux's excellent presentation. Since the quantities h and k are invariant only under a subgroup of the equivalence group, Ibragimov proposed to call h and k the *semi-invariants* in accordance with Cayley's theory of algebraic invariants.

Louise Petren, in her PhD thesis defended at Lund Univarsity in 1911, extended Laplace's method and the Laplace invariants to higher-order equations.

Semi-invariants for linear ordinary differential equations were intensely discussed in the 1870-1880's by J.Cockle, E. Laguerre, J.C. Malet, G.H. Halphen, R. Harley and A.R. Forsyth. The restriction to linear equations was essential in their approach. They used calculations following directly from the definition of invariants. These calculations would be extremely lengthly in the case of non-linear equations.

In the second half of the 19th century, the Norwegian mathematician Sophus Lie began to create a remarkable work that unified all known methods of solving differential equations. In 1871 Lie had started examining PDEs, hoping that he could find a theory which was analogous to Galois's theory of equations. He applied his contact transformations to extend a method, due to Jacobi, of generating further solutions from a particular set. This led Lie to define what he called a continuous transformation group. He discovered that symmetries of differential equations can be found and exploited systematically. Over many years, considerable research effort has been directed at understanding the elegant algebraic structure of symmetry groups, but Lie's methods for determining and using symmetries were largely neglected until fairly recently. With the advent of powerful symbolic computation packages, it has become possible to apply Lie's methods to explore the symmetries and conservation laws of a wide range of physical systems.

It was during the winter of 1873-1874 that Lie began to develop systematically his theory of continuous transformation groups, later called Lie groups, leaving behind his original intention of studying PDEs. Later Killing worked on the Lie algebras associated with Lie groups. He did this, quite independently of Lie, and it was Cartan who completed the classification of semi-simple Lie algebras in 1900.

Lie' s work related a miscellany of topics in ODEs including: integrating factor, separable equation, homogeneous equation reduction of order, the methods of undetermined coefficients and variation of parameters for linear equations, solution of the Euler equation, and the use of the Laplace transform. Lie (1881) also indicated that for linear PDEs, invariance under a Lie group, leads directly to superpositions of solutions in terms of transforms.

Recently, Ibragimov (see [15,16,19]) developed a simple method for constructing invariants of families of differential equations. The method is based in the theory of equivalence groups in the infinitesimal form. Basically, the method consists of two steps: classification of equivalence groups and then use of these groups (and extended groups) to derive the desired differential invariants. Ibragimov (see [20]) used his method to solve the Laplace problem. That is, to derive all invariants for the linear hyperbolic equations (1.2). To achieve this, he constructed a basis for the invariants. Using this basis and invariant differentiation, all invariants, of any order, can be derived. The idea of Ibragimov was adopted by a number of authors who derived differential invariants for ordinary differential equations, linear and non-linear PDEs (see [21–26, 48, 50, 52–55, 58]).

Different approaches of calculating differential invariants have also been applied. See, for example, references [9, 39, 40, 66–68].

In this thesis, in the spirit of Ibragimov's work, we consider families of PDEs with the ultimate goal to derive differential invariants. In order to achieve it, we firstly need to derive the equivalence transformations. The method is based on the infinitesimal approach and is applicable to algebraic and differential equations possessing finite or infinite equivalence group. Our first aim is to discuss the main principles of the method.

A brief description of the method used to derive equivalence transformations is presented. In particular, we apply the method for the families of non-linear diffusion equations:

$$u_t = f(u)u_{xx}.$$

As a second example, we calculate the equivalence transformations for linear hyperbolic equations in two variables:

$$u_{tx} + a(t,x)u_t + b(t,x)u_x + c(t,x)u = 0. \tag{1.3}$$

These results can be also found in [15, 16].

Next, motivated by these results, we present our work for several families of hyperbolic equations (see [57–61]). In particular, in chapter 6, we derive equivalence transformations for the class

$$u_{tx} = F(t,x,u,u_x,u_t)$$

and for two subclasses of it:

$$u_{tx} = f(x,t,u)u_x u_t + g(x,t,u)u_x + h(x,t,u)u_t + l(x,t,u),$$

$$u_{tx} = m_u(x,t,u)u_x u_t + m_t u_x + m_x u_t + k(x,t,u).$$

Furthermore, in chapter 7, we calculate equivalence transformations for the n-dimensional hyperbolic equations

$$u_{tt} = \sum_{i=1}^{n} u_{x_i x_i} + \sum_{i=1}^{n} X_i(x_1,x_2,...,x_n,t)u_{x_i} + T(x_1,x_2,...,x_n,t)u_t + U(x_1,x_2,...,x_n,t)u$$

and in chapter 8, for n-dimensional wave type equations:

$$u_{tt} = \sum_{i=1}^{n} F_i(x_1,x_2,\ldots,x_n)u_{x_i x_i}.$$

Finally, in chapter 10, we use this method to calculate equivalence transformations for systems that consist of two linear hyperbolic equations

$$u_{xt} = a_1(t,x)u_x + b_1(t,x)v_x + c_1(t,x)u_t + d_1(t,x)v_t + f_1(t,x)u + g_1(t,x)v,$$

$$v_{xt} = a_2(t,x)u_x + b_2(t,x)v_x + c_2(t,x)u_t + d_2(t,x)v_t + f_2(t,x)u + g_2(t,x)v.$$

For these equations, we employ these equivalence transformations in order to derive differential invariants. We adopt the idea of Ibragimov, who derived differential invariants using the infinitesimal method. The derivation of differential invariants enable us to classify forms of PDEs that can be linearized via local mappings. In particular, we find those equations that can be mapped into one of the four linear forms of equation (1.3), described in the applications of Laplace invariants (see [17]). Some examples are given to illustrate our results.

Another important tool that enables one to calculate differential invariants of higher order is the derivation of operators of invariant differentiation. This method was applied by Ibragimov in order to solve the Laplace problem. That is, to find all invariants for the family of the linear hyperbolic equations (1.3) (see [20]).

Another task of the present work, is to consider point transformations of general form. Motivated by the existing work (see [29]) for point transformations of the form

$$t' = Q(t, x, u), \quad x' = P(t, x, u), \quad u' = R(t, x, u)$$

admitted by classes of single PDEs, in chapter 9, we generalize certain results for systems of PDEs. In chapter 3, we present existing results (see [29]). In particular, we explain the notation and summarize the basic theory. These results are useful to find a complete set of point transformations connecting PDEs belonging to given classes of equations. Using this approach, equivalence transformations for a given PDE can be derived in finite form.

In chapter 9, following this approach for a single PDE, we consider point transformations of the form

$$t' = Q(t, x, u, v), \quad x' = P(t, x, u, v), \quad u' = R(t, x, u, v), \quad v' = S(t, x, u, v).$$

General results are presented for the restricted forms of point transformations that connect classes of systems of PDEs with two dependent variables and two independent variables.

The calculations involved in this thesis have been facilitated by the computer algebraic package "REDUCE" (see [11]).

This thesis is organized as follows: in chapter 2, we give the basic definitions which are needed for the remaining chapters. In chapter 3, we introduce the notion of point transformations of PDEs. In chapter 4, we present the idea of equivalence transformations. In chapter 5, the new method determining the differential invariants is described. Chapters 6-10 are new contributions. Motivated, by the existence results of chapter 5, in chapter 6 we derive differential invariants for the hyperbolic equations of general class and for two subclasses of it. Also, we use the derived invariants to construct equations that can be linearized via local mappings. In chapter 7, we consider n-dimensional linear hyperbolic equations. We derive equivalence transformations which are used to obtain differential invariants for the cases $n = 2, 3$. Motivated by these results, we present the general results for the n-dimensional case. In chapter 8, we consider the n-dimensional wave type equations. We determine differential invariants of first order. For the cases $n = 1, 2, 3$ we determine differential invariants of order two. In chapter 9, we generalize the idea of point transformations as presented in chapter 3. Finally, in chapter 10, we derive the differential invariants of a system of hyperbolic equations.

Chapter 2

Basic definitions

2.1 Introduction

In this chapter are developed basic results for continuous groups (Lie groups of transformations) that are generated by a free parameter, hereafter denoted by ε. Therefore, each element of the group corresponds to a specific value of this parameter. This group is continuous because ε can vary continuously over the real numbers. Furthermore, a general idea of transformations is given and a variety of transformations groups are exhibited. This chapter provides a presentation of the infinitesimal theory of one-parameter (ε) Lie groups, their invariants, invariant functions and invariant solutions. Finally, we are concerned with applications to PDEs. We find admitted point symmetries and how to construct invariant solutions. More details about Lie groups of transformations (Lie symmetries) and their applications to differential equations can be found in a number of recent textbooks. See, for example [4, 5, 12, 13, 16, 41, 44].

2.2 Lie group of transformations

Sophus Lie introduced the notion of continuous group of transformations to put order to the hodgepodge of techniques for solving ordinary differential equations. Our discussion begins by first defining arbitrary groups, then we consider a group of transformations and more specifically, a one-parameter Lie group of transformations. Here, the transformations act on $\mathbb{R}^n$.

2.2.1 Groups

Definition 2.1. A *group* G is a set of elements with a law of composition ϕ between elements satisfying the following axioms:

(i) *Closure property*: For any elements a and b of G, $\phi(a, b)$ is an element of G.

(ii) *Associative property*: For any elements a, b, c of G:

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c).$$

(iii) *Identity element*: There exists a unique identity element e of G such that for any element a of G:

$$\phi(a, e) = \phi(e, a) = a.$$

(iv) *Inverse element*: For any element a of G there exists a unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = e.$$

Definition 2.2. A group G is *abelian* if $\phi(a, b) = \phi(b, a)$ for any elements a and b in G.

Definition 2.3. A *subgroup* of G is a subset of G, which is also a group with the same law of composition ϕ.

2.2.2 One-parameter Lie group of transformations

Definition 2.4. : Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ lie in region $D \subset \mathbb{R}^n$. The set of transformations

$$\mathbf{x}' = \Gamma(\mathbf{x}, \varepsilon)$$

defined for each $\mathbf{x}$ in D and parameter ε in $S \subset \mathbb{R}$, with $\phi(\varepsilon, \delta)$ defining a law of composition of parameters ε and δ in S, forms a *one-parameter group of transformations* on D if the following hold:

(i) For each ε in S the transformations are one-to-one onto D.

(ii) S with the law of composition ϕ forms a group G.

(iii) For each $\mathbf{x}$ in D, $\mathbf{x}' = \mathbf{x}$ when $\varepsilon = \varepsilon_0$ corresponds to the identity e, i.e.,

$$\Gamma(\mathbf{x}, \varepsilon_0) = \mathbf{x}.$$

(iv) If $\mathbf{x}' = \Gamma(\mathbf{x}, \varepsilon)$, $\mathbf{x}'' = \Gamma(\mathbf{x}', \delta)$, then

$$\mathbf{x}'' = \Gamma(\mathbf{x}, \phi(\varepsilon, \delta)).$$

Definition 2.5. A one parameter group of transformations defines a *one-parameter Lie group of transformations* if, in addition to axioms (i)-(iv), the following hold:

(v) ε is a continuous parameter, i.e. S is an interval in $\mathbb{R}$.

(vi) Γ is infinitely differentiable with respect to $\mathbf{x}$ in D and an analytic function of ε in S.

(vii) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ, $\varepsilon \in S, \ \delta \in S$.

A Lie group of transformations admitted by a differential equation corresponds to a mapping of each of its solutions to another solution of the same differential equation.

2.3 Infinitesimal transformations

Consider a one- parameter (ε) Lie group of transformations

$$\mathbf{x}' = \Gamma(\mathbf{x}, \varepsilon) \tag{2.1}$$

with the identity $\varepsilon = 0$ and law of composition ϕ. Expanding (2.1) about $\varepsilon = 0$, in some neighborhood of $\varepsilon = 0$, we get

$$\mathbf{x}' = \mathbf{x} + \varepsilon \left(\left. \frac{\partial \Gamma(x, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} \right) + \frac{1}{2}\varepsilon^2 \left(\left. \frac{\partial^2 \Gamma(x, \varepsilon)}{\partial \varepsilon^2} \right|_{\varepsilon=0} \right) + \cdots = \mathbf{x} + \varepsilon \left(\left. \frac{\partial \Gamma(x, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} \right) + O\left(\varepsilon^2\right).$$

Let

$$\xi(\mathbf{x}) = \left. \frac{\partial \Gamma(x, \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}.$$

The transformation $\mathbf{x} + \varepsilon \xi(\mathbf{x})$ is called the *infinitesimal transformation* of the Lie group of transformation (2.1). The components of $\xi(\mathbf{x})$ are called the *infinitesimals* of (2.1).

2.3.1 First Fundamental Theorem of Lie

Theorem 2.1. *There exists a parametrization $\tau(\varepsilon)$, such that the Lie group of transformations (2.1) is equivalent to the solution of an initial value for the system of first order ODEs given by*

$$\frac{d\mathbf{x}'}{d\tau} = \xi(\mathbf{x}'),$$

with

$$\mathbf{x}' = \mathbf{x} \quad when \quad \tau = 0.$$

In particular,

$$\tau(\varepsilon) = \int_0^{\varepsilon} F(\varepsilon')d\varepsilon',$$

where

$$F(\varepsilon) = \left.\frac{\partial\phi(a,b)}{\partial b}\right|_{(a,b)=(\varepsilon^{-1},\varepsilon)}$$

and

$$F(0) = 1.$$

2.3.2 Infinitesimal generators

Lie groups of transformations are characterized by infinitesimal generators. Lie gave an algorithm to find all infinitesimal generators of point transformations. Significantly, for a given differential equation, the basic applications of Lie groups of transformations only require knowledge of the admitted infinitesimal generators.

Definition 2.6. The *infinitesimal generator* of the one-parameter Lie group of transformations (2.1) is the operator

$$\Gamma = \Gamma(\mathbf{x}) = \xi(\mathbf{x}).\nabla = \sum_{i=1}^{n} \xi_i(\mathbf{x})\frac{\partial}{\partial x_i},$$

where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right).$$

For any differentiable function $F(\mathbf{x}) = F(x_1, x_2, \ldots, x_n)$, we have

$$\Gamma F(\mathbf{x}) = \xi(\mathbf{x}).\nabla F(\mathbf{x}) = \sum_{i=1}^{n} \xi_i(\mathbf{x})\frac{\partial F(\mathbf{x})}{\partial x_i}.$$

2.3.3 Invariants Functions

Definition 2.7. An infinitely differentiable function $F(\mathbf{x})$ is an *invariant function* of the Lie group of transformations (2.1) if and only if, for any group transformation (2.1),

$$F(\mathbf{x}') \equiv F(\mathbf{x}).$$

Theorem 2.2. *$F(\mathbf{x})$ is invariant under a Lie group of transformation (2.1) if and only if*

$$\Gamma F(\mathbf{x}) \equiv 0.$$

Remark 2.1. Given an invariant $F(\mathbf{x})$, any function $\Phi(F(\mathbf{x}))$ is also invariant.

2.3.4 Point transformations

In this subsection, we are interested in determining one-parameter Lie groups of point transformations admitted by a given system S of differential equations.

Definition 2.8. A *one parameter* (ε) *Lie group of point transformations* is a group of transformations of the form

$$x' = X(x, u, \varepsilon),$$

$$u' = U(x, u, \varepsilon),$$

acting on the space of $n + m$ variables

$$x = (x_1, x_2, \dots, x_n),$$

$$u = (u^1, u^2, \dots, u^m);$$

x represents n independent variables and u represents m dependent variables.

A Lie group of point transformation admitted by S maps any solution $u = \Theta(x)$ of S into a one-parameter family of solutions $u = \phi(x, \varepsilon)$ of S. Equivalently, a Lie group of point transformations leaves S invariant in the sense that the form of S is unchanged in terms of the transformed variables for any solution $u = \Theta(x)$ of S.

Theorem 2.3. *The kth extension of the one-parameter Lie group of point transformations*

$$x' = X(x, y, \varepsilon),$$

$$y' = Y(x, y, \varepsilon),$$

$k \geq 2$, is the following one-parameter Lie group of transformations acting on $(x, y, y_1, \dots, y_k)$-space:

$$\begin{aligned} x' &= X(x, y, \varepsilon), \\ y' &= Y(x, y, \varepsilon), \\ y'_1 &= Y_1(x, y, y_1, \varepsilon), \\ &. \\ &. \\ y'_k &= Y_k(x, y, y_1, \dots, y_k, \varepsilon) = \frac{\frac{\partial Y_{k-1}}{\partial x} + y_1 \frac{\partial Y_{k-1}}{\partial y} + \cdots + y_k \frac{\partial Y_{k-1}}{\partial y_{k-1}}}{\frac{\partial X(x,y,\varepsilon)}{\partial x} + y_1 \frac{\partial X(x,y,\varepsilon)}{\partial y}}, \end{aligned}$$

where $y'_1 = Y_1(x, y, y_1, \varepsilon)$ is defined by

$$y'_1 = Y_1(x, y, y_1, \varepsilon) = \frac{\frac{\partial Y(x,y,\varepsilon)}{\partial x} + y_1 \frac{\partial Y(x,y,\varepsilon)}{\partial y}}{\frac{\partial X(x,y,\varepsilon)}{\partial x} + y_1 \frac{\partial X(x,y,\varepsilon)}{\partial y}}$$

and $Y_i = Y_i(x, y, y_1, \dots, y_i, \varepsilon)$, $i = 1, 2, \dots, k$.

2.4 Invariance of PDEs

Similar to the case for an ordinary differential equation, we will see that the infinitesimal criterion for the invariance of a PDE leads directly to an algorithm to determine the infinitesimal generators of the Lie group of point transformations admitted by a given PDE.

Firstly, we have a kth-order scalar PDE

$$F(x, u, \partial u, \partial^2 u, \dots, \partial^k u) = 0, \tag{2.2}$$

where $x = (x_1, x_2, \dots, x_n)$ denotes the coordinates corresponding to its n independent variables, u denotes the coordinate corresponding to its dependent variable, and $\partial^j u$ denotes the coordinates with components

$$\frac{\partial^j u}{\partial x_{i_1} \partial x_{i_2} \dots \partial_{x_{i_j}}} = u_{i_1 i_2 \dots i_j}, \quad i_j = 1, 2, \dots, n \text{ for } j = 1, 2, \dots, k,$$

corresponding to all jth-order partial derivatives of u with respect to x.

Definition 2.9. The one parameter Lie group of point transformations

$$x' = X(x, u, \varepsilon), \tag{2.3}$$

$$u' = U(x, u, \varepsilon), \tag{2.4}$$

leaves invariant the PDE (2.2), i.e. *is a point symmetry admitted by PDE* (2.2), if and only if its kth extension, leaves invariant the surface (2.2).

A solution $u = \Theta(x)$ of PDE (2.2) lies on the surface (2.2) with $u_{i_1 i_2 \ldots i_j} = \frac{\partial^j \Theta}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_j}}$, $i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, k$. The invariance of surface (2.2) under the kth-extension of (2.3)-(2.4) means that any solution $u = \Theta(x)$ of PDE (2.2) maps into another solution $\Phi(x, \varepsilon)$ of (2.2) under the action of the one-parameter group (2.3)-(2.4) for any ε.

Theorem 2.4. *(Infinitesimal Criterion for the Invariance of a PDE) Let*

$$\Gamma = \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u} \tag{2.5}$$

be the infinitesimal generator of the Lie group of point transformation (2.3), (2.4). Let

$$\begin{aligned} \Gamma^{(k)} &= \xi_i(x, u)\frac{\partial}{\partial x_i} + \eta(x, u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x, u, \partial u)\frac{\partial}{\partial u} + \ldots \\ &+ \eta^{(k)}_{i_1 i_2 \ldots, i_k}(x, u, \partial u, \partial^2 u, \ldots, \partial^k u)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}} \end{aligned}$$

be the kth-extended infinitesimal generator of (2.5), where $\eta_i^{(1)}$ given by

$$\eta_i^{(1)} = D_i \eta - (D_i \xi_j) u_j, \quad i = 1, 2, \ldots, n$$

and $\eta^{(j)}_{i_1 i_2 \ldots i_j}$ by

$$\eta^{(k)}_{i_1 i_2 \ldots i_k} = D_{i_k} \eta^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} - (D_{i_k} \xi_j) u_{i_1 i_2 \ldots i_{k-1} j},$$

where $i_j = 1, 2, \ldots, n,\ j = 1, 2, \ldots, k$, in terms of $\xi(x, u) = (\xi_1(x, u), \xi_2(x, u), \ldots, \xi_n(x, u))$, $\eta(x, u)$. Then the one-parameter Lie group of point transformations (2.3), (2.4) is admitted by PDE (2.2), i.e. is a point symmetry of PDE (2.2), if and only if

$$\Gamma^{(k)} F(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0 \text{ when } F(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0.$$

2.4.1 Invariant solutions

Consider a kth-order PDE (2.2) ($k \geq 2$) that admits a one parameter Lie group of point transformation with the infinitesimal generator (2.5). We assume that $\xi(x, u) \neq 0$.

Definition 2.10. $u = \Theta(x)$ is an *invariant solution* of PDE (2.2) resulting from its admitted point symmetry with the infinitesimal generator (2.5) if and only if:

(i) $u = \Theta(x)$ is an invariant surface of (2.5). Namely,

$$\Gamma(u - \Theta(x)) = 0 \quad \text{when} \quad u = \Theta(x)$$

i.e.,

$$\xi_i(x, \Theta(x)) \frac{\partial \Theta(x)}{\partial x_i} = \eta(x, \Theta(x));$$

and

(ii) $u = \Theta(x)$ solves (2.2). Namely,

$$F(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0 \quad \text{when} \quad u = \Theta(x)$$

i.e.,

$$F(x, \Theta(x), \partial \Theta(x), \partial^2 \Theta(x), \ldots, \partial^k \Theta(x)) = 0.$$

Invariant solutions for PDEs were first considered by Lie (1881).

Chapter 3

Point transformations of PDEs

3.1 Introduction

Probably the most useful point transformations of PDEs are those which form a continuous Lie group of transformations, each member of which leaves an equation invariant. The method of finding these transformations consists of two steps: first to find infinitesimal transformations, with the benefit of linearization, and second to extend these groups of finite transformations. The use of point transformations, is significant to relate a nonlinear PDE with a linear PDE for which the solution exists. In this case, we can derive the solution of the first PDE. The infinitesimal transformations are not appropriate for directly linking a PDE with an equation of different form.

Hence, there is merit in studying point transformations directly in finite forms with the ultimate goal of finding the complete set of point transformations of PDEs and discovering new links between different equations.

The aim of this chapter is first to present results concerning the relation of the transformed partial derivatives to the original partial derivatives and secondly to exploit these results in order to find the form of the point transformations connecting PDEs belonging to restricted classes of equations. More details and the proofs of the theorems below can be found in [29].

3.2 Point transformations: Notation and the basic theory

In this section we explain the notation and summarize the basic theory on which the work is based.

We consider the point transformation

$$x' = P(x,t,u), \quad t' = Q(x,t,u), \quad u' = R(x,t,u), \tag{3.1}$$

relating $x,\ t,\ u(x,t)$ and $x',\ t',\ u'(x',t')$, and assume that this is non-degenerate in the sense that the Jacobian

$$J = \frac{\partial(P,Q,R)}{\partial(x,t,u)} \neq 0 \tag{3.2}$$

and also that

$$\delta = \frac{\partial(P(x,t,u(x,t)),Q(x,t,u(x,t)))}{\partial(x,t)} \neq 0. \tag{3.3}$$

In (3.3) P and Q are expressed as functions of x and t whereas in (3.2) $P,\ Q$ and R are to be regarded as functions of the independent variables x,t,u.

The derivatives of $u(x,t)$ and $u'(x',t')$ will be denoted by

$$u_{ij} = \frac{\partial^{i+j}u}{\partial x^i \partial t^j}, \qquad u'_{ij} = \frac{\partial^{i+j}u'}{\partial x'^i \partial t'^j}. \tag{3.4}$$

If Ψ is a function of $x,\ t,\ u$ and the derivatives of u, the total derivatives of Ψ with respect to x and t will be denoted by

$$\Psi_X = \Psi_x + \sum\sum u_{i+1j}\frac{\partial\Psi}{\partial u_{ij}}, \tag{3.5}$$

$$\Psi_T = \Psi_t + \sum\sum u_{ij+1}\frac{\partial\Psi}{\partial u_{ij}}, \tag{3.6}$$

where the double summations are to be taken over the values of i and j which cover all derivatives u_{ij} and v_{ij} occurring in Ψ.

With this notation δ may be expressed as

$$\begin{aligned} \delta &= \frac{\partial(P,Q)}{\partial(X,T)} = P_X Q_T - P_T Q_X \\ &= u_{10}(P_u Q_t - P_t Q_u) + u_{01}(P_x Q_u - P_u Q_x) + (P_x Q_t - P_t Q_x) \\ &= \frac{\partial(P,Q)}{\partial(u,t)}u_{10} + \frac{\partial(P,Q)}{\partial(x,u)}u_{01} + \frac{\partial(P,Q)}{\partial(x,t)}. \end{aligned} \tag{3.7}$$

Also, under the point transformation (3.1),

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \begin{pmatrix} P_X & P_T \\ Q_X & Q_T \end{pmatrix} \begin{pmatrix} dx \\ dt \end{pmatrix}, \quad \begin{pmatrix} dx \\ dt \end{pmatrix} = \frac{1}{\delta} \begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix} \begin{pmatrix} dx' \\ dt' \end{pmatrix} \tag{3.8}$$

and

$$d\Psi = \Psi_X dx + \Psi_T dt = \frac{1}{\delta}(\Psi_X \quad \Psi_T) \begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix} \begin{pmatrix} dx' \\ dt' \end{pmatrix}. \tag{3.9}$$

Hence, taking $\Psi = u'_{ij-1},\ u'_{i-1j}$ respectively, gives

$$u'_{ij} = \delta^{-1} \left(P_X (u'_{ij-1})_T - P_T (u'_{ij-1})_X \right), \qquad j \geq 1, \quad i \geq 0, \tag{3.10}$$

$$u'_{ij} = \delta^{-1} \left(Q_T (u'_{i-1j})_X - Q_X (u'_{i-1j})_T \right), \qquad i \geq 1, \quad j \geq 0. \tag{3.11}$$

Also,

$$u'_{00} = u' = R. \tag{3.12}$$

Equations (3.10)-(3.12) furnish recurrence relations which enable u'_{ij} to be expressed in terms of x, t, u and the derivatives of u for any $i \geq 0,\ j \geq 0$. The factor δ^{-1} makes the expressions for u'_{ij} grow with i and j in a very cumbersome manner.

In the case of infinitesimal Lie point transformations in which:

$$\begin{aligned} P(x,t,u) &= x + \varepsilon P^*(x,t,u) + O(\varepsilon^2), \\ Q(x,t,u) &= t + \varepsilon Q^*(x,t,u) + O(\varepsilon^2), \\ R(x,t,u) &= u + \varepsilon R^*(x,t,u) + O(\varepsilon^2), \end{aligned} \tag{3.13}$$

the forms of J and δ in (3.2) and (3.3) simplify to

$$J = 1 + \varepsilon(P^*_x + Q^*_t + R^*_u), \tag{3.14}$$

$$\delta = 1 + \varepsilon(P^*_x + Q^*_t), \tag{3.15}$$

to the first order of ε. In this case the recurrence relations corresponding to (3.10)-(3.12) are

$$u'_{ij} = (u'_{ij-1})_T - \varepsilon[P^*_T (u'_{ij-1})_X + Q^*_T (u'_{ij-1})_T], \quad j \geq 1, \quad i \geq 0, \tag{3.16}$$

$$u'_{ij} = (u'_{i-1j})_X - \varepsilon[P^*_X (u'_{i-1j})_X + Q^*_X (u'_{i-1j})_T], \quad i \geq 1, \quad j \geq 0, \tag{3.17}$$

$$u'_{00} = u + \varepsilon R^*, \tag{3.18}$$

to the first order in ε. These relations of course lead to considerably less cumbersome forms of u'_{ij} than those obtained from (3.10)-(3.12).

In the following sections, some results are presented for the point transformations (3.1). These results help us achieve the second aim which is to discover the nature of point transformations connecting PDEs belonging to given classes of equations.

3.3 Properties of transformations

Under the point transformation (3.1) each derivative of $u'(x', t')$, that is $u'_{ij},\ i \geq 0,\ j \geq 0$, may be expressed, via the recurrence relations (3.10)-(3.12), as functions of x, t, u and the derivatives of u. A number of results concerning the functional form of $u'_{pq}(x, t, u, v, \ldots, u_{ij}, \ldots)$ are presented in this section. In the next section, the results of this section are necessary in order to study the nature of point transformations which perform specific changes to PDEs. Of particular interest, for example, are the cases of no change which correspond to symmetries of the equations. The proofs of the results are generally inductive and use the recurrence relations (3.10)-(3.12).

Lemma 3.1. *If* $x' = P(x, t, u),\ t' = Q(x, t, u),\ u' = R(x, t, u)$

$$\sum_{i=0}^{n} z^i \frac{\partial u'_{pq}}{\partial u_{ij}} = \begin{cases} (-1)^p (Q_X - zQ_T)^p (P_X - zP_T)^q J \delta^{-p-q-1}, & n > 0 \\ R_u, & n = 0 \end{cases},$$

where $i + j = p + q = n \geq 0$.

Corollary 3.1. *The coefficients of* z^n *and* z^0 *in lemma 3.1 give, respectively*

$$\frac{\partial u'_{pq}}{\partial u_{p+q0}} = (-1)^q P_T^q Q_T^p J \delta^{-p-q-1}, \qquad p + q \geq 1,$$

$$\frac{\partial u'_{pq}}{\partial u_{0p+q}} = (-1)^p P_X^q Q_X^p J \delta^{-p-q-1}, \qquad p + q \geq 1.$$

Lemma 3.2. *If* $x' = P(x, t, u),\ t' = Q(x, t, u),\ u' = R(x, t, u)$ *then*

$$\frac{\partial^{m+n} u'_{10}}{\partial u_{10}^m \partial u_{01}^n} = (-1)^n C_{mn} \left(n\alpha Q_X + m\beta Q_T\right) \delta^{-m-n-1},$$

$$\frac{\partial^{m+n} u'_{01}}{\partial u_{10}^m \partial u_{01}^n} = (-1)^n C_{mn} \left(n\alpha P_X + m\beta P_T\right) \delta^{-m-n-1},$$

where $m + n \geq 1,\ C_{mn} = (m + n - 1)!\alpha^{m-1}\beta^{n-1}J$, *depends only on* $x,\ t$ *and* u *and where* $\alpha = P_t Q_u - P_u Q_t$ *and* $\beta = P_x Q_u - P_u Q_x$.

The proofs of lemmas 3.1 and 3.2 can be found in [29].

3.4 Form-preserving transformations of PDEs

In this section we first look at PDEs with one derivative of $u(x,t)$ of any order, possibly mixed, related to lower-order derivatives of u, u itself and x and t. Subsequently, we consider three classes of equations.

3.4.1 Basic results

We start with a wide class of PDEs for which general deductions about the forms of $P(x,t,u)$ and $Q(x,t,u)$ can be made. These will be useful when discussing more restricted classes of equations.

Theorem 3.1. *The PDE* $u_{pq} = H(x,t,u,\{u_{ij}\})$ *is related to* $u'_{pq} = H'(x',t',u',\{u'_{ij}\})$, *where* $\{u_{ij}\}$ *and* $\{u'_{ij}\}$ *respectively denote all derivatives of* u *and* u' *of order* $i+j < p+q$, *by the point transformation* $x' = P(x,t,u),\ t' = Q(x,t,u),\ u' = R(x,t,u)$ *in the cases: (a)* $p \neq 0,\ q \neq 0$, *(b)* $p \neq 0,\ q = 0$, *(c)* $p = 0,\ q \neq 0$ *only if (a)* $\{P = P(x),\ Q = Q(t)\}$, *or* $\{P = P(t),\ Q = Q(x)\}$, *(b)* $Q = Q(t)$, *(c)* $P = P(x)$, *respectively.*

3.4.2 Equations of the form $u_{01} = H(x,t,u,\ldots,u_{n0})$

Two evolution equations are considered of the form $u_{01} = H(x,t,u,\ldots,u_{n0})$. Tu (see [62]) proved that for evolution equations of this form the time transformation takes the simple form $t' = t + \varepsilon f(t) + O(\varepsilon^2)$, the interesting feature being that Q is independent of both x and u. This is a striking result and has been exploited for example by Doyle and Englefield (see [6]) who used the result to simplify the analysis of infinitesimal transformations of generalized Burger's equations. Using the fact that all point transformations connecting two different Burgers-type equations (Kingston and Sophocleous (see [30])) were also of this form, Kingston (see [28]) generalized Tu's result and he showed that for a wide subclass of these equations it is necessary $x' = P(x,t)$ (no u dependency).

Theorem 3.2. *The point transformation* $x' = P(x,t,u),\ t' = Q(x,t,u),\ u' = R(x,t,u)$ *transforms*

$$u'_{01} = H'(x',t',u',\ldots,u'_{n0})$$

to

$$u_{01} = H(x, t, u, \ldots, u_{n0}),$$

where $n \geq 2$*, if and only if* $Q = Q(t)$ *and*

$$H = J^{-1} Q_t \left(P_X Q_t H' + P_t R_X - P_X R_t \right).$$

Theorem 3.3. *If, in the theorem 3.2,* H *and* H' *are polynomials (non-negative integral powers) in* $u_{10}, \ldots, u_{n0}$ *and* $u'_{10}, \ldots, u'_{n0}$ *respectively (dependency on* x, t, u *and* x', t', u' *unspecified) then* $P = P(x, t)$*.*

These results have been used, for example, to aid the classification of point transformations within the following classes of PDEs: generalized Burgers equations (see [30]), radially symmetric non-linear diffusion equation (see [49]), generalized non-linear diffusion equations (see [45]).

3.4.3 Equations of the form $u_{11} = H(x, t, u, \ldots, u_{n0})$

This class of PDEs includes, for example, Liouville's equation $u_{xt} = e^x$, sine-Gordon equation $u_{xt} = \sin u$ and $u_{xt} = u\sqrt{1 - u_x^2}$.

Theorem 3.4. *($n \geq 3$) The point transformation* $x' = P(x, t, u)$*,* $t' = Q(x, t, u)$*,* $u' = R(x, t, u)$ *transforms*

$$u'_{11} = H'(x', t', u', \ldots, u'_{n0})$$

into

$$u_{11} = H(x, t, u, \ldots, u_{n0}),$$

where $n \geq 3$*, if and only if* $P = P(x, t)$*,* $Q = Q(t)$*,* $R = A(t)u + B(x, t)$ *and*

$$H = A^{-1} P_x Q_t H' + u_{20} P_x^{-1} P_t + u_{10}((P_x^{-1} P_t)_x - A^{-1} A_t) - A^{-1}(B_t - P_x^{-1} P_t B_x)_x.$$

Theorem 3.5. *($n = 2$) The point transformations* $x' = P(x, t, u)$*,* $t' = Q(x, t, u)$*,* $u' = R(x, t, u)$ *which transform*

$$u'_{11} = H'(x', t', u', u'_{10}, u'_{20})$$

to

$$u_{11} = H(x, t, u, u_{10}, u_{20}),$$

belong to one of the two categories:

(a) $P,\ Q,\ R$ *and* H *restricted as in the conditions for the theorem 3.4;*

(b) $P = P(x,t),\ Q = Q(x,t),\ R = A(x,t)u + B(x,t),\ H' = -P_x Q_x^{-1} u'_{20} - A\delta^{-1}(A^{-1}Q_x^{-1}\delta)_x u'_{10} + G'(x', t', u'),\ H = Q_x^{-1}Q_t u_{20} + A^{-1}((AQ_x^{-1}Q_t)_x - A_t)u_{10} + G(x,t,u)$. *For any* $G'(x',t',u')$ *the form of* $G(x,t,u)$ *is then determined by the transformation without further condition. Also,* $\delta = P_x Q_t - P_t Q_x$.

Theorem 3.6. *(*$n = 0, 1$*) The point transformations* $x' = P(x,t,u),\ t' = Q(x,t,u),\ u' = R(x,t,u)$ *which transform*

$$u'_{11} = H'(x', t', u', u'_{10})$$

to

$$u_{11} = H(x, t, u, u_{10}),$$

belong to one of the two categories (when $n = 0$ *set* A *constant in (a) and (b)):*

(a) $P = P(x),\ Q = Q(t),\ R = A(t)u + B(x,t),\ H = A^{-1}P_x Q_t H' - A^{-1}A_t u_{10} - A^{-1}B_{xt}$*;*

(b)$P = P(t),\ Q = Q(x),\ R = A(x,t)u + B(x,t),\ H' = A^{-1}A_x Q_x^{-1} u'_{10} + G'(x', t', u'),\ H = -A^{-1}A_t u_{10} + A^{-1}P_t Q_x G' - u(A^{-1}A_t)_x - (A^{-1}B_t)_x$.

3.4.4 Equations of the form $u_{02} = H(x, t, u, \ldots, u_{n0})$

These equations include many models of physical phenomena, especially wave-type motions, for example the equation $u_t = -u_x u_{xx}$, which arises as a model of steady transonic gas-dynamic flow, the family of non-linear equations $u_{tt} = (f(u)u_x)_x$ and the Boussinesq-type equation $u_{tt} = u_{xx} - 2(u^3)_{xx} + u_{xxxx}$.

Theorem 3.7. *(*$n \geq 3$*) The point transformation* $x' = P(x,t,u),\ t' = Q(x,t,u),\ u' = R(x,t,u)$ *transforms*

$$u'_{02} = H'(x', t', u', \ldots, u'_{n0})$$

to

$$u_{02} = H(x, t, u, \dots, u_{n0}),$$

where $n \geq 3$ *if and only if* $P = P(x),\ Q = Q(t)$ *and* $R = A(x)Q_t^{\frac{1}{2}}u + B(x,t)$*. Also,*

$$H = A(x)^{-1}Q_t^{-\frac{3}{2}}\left(Q_t^3 H' + Q_{tt}R_t - Q_t R_{tt}\right).$$

Theorem 3.8. *($n = 2$) The point transformation* $x' = P(x,t,u),\ t' = Q(x,t,u),\ u' = R(x,t,u)$ *transforms*

$$u'_{02} = H'(x', t', u', u'_{10}, u'_{20})$$

to

$$u_{02} = H(x, t, u, u_{10}, u_{20}),$$

where $H'_{u'_{20}} \neq 0$*, belong to one of the three categories:*

(a) $P,\ Q,\ R$ *and* H *restricted as in the conditions for the theorem 3.7;*

(b) $P = P(t),\ Q = Q(x),\ H' = H'(x', t', u', u'_{20} + \lambda u'^2_{10} + \mu u'_{10})$ *where* $\lambda = -R_{uu}R_u^{-2},\ \mu = P_t^{-2}R_u^{-2}(2P_tR_tR_{uu} - 2P_tR_uR_{ut} + P_{tt}R_u^2)$, $H = H(x, t, u, u_{20} + R_{uu}R_u^{-1}u_{10}^2 + (2R_{ux}R_u^{-1} - Q_{xx}Q_x^{-1})u_{10})$*;*

(c) $P = P(x,t),\ Q = Q(x,t),\ R = A(x,t)u + B(x,t),\ H' = P_xP_tQ_x^{-1}Q_t^{-1}u'_{20} + G'_1(x', t')u'_{10} + G'_2(x', t', u'),\ H = P_x^{-1}P_tQ_x^{-1}Q_tu_{20} + G_1(x,t)u_{10} + G_2(x,t,u).$

The proofs of the theorems in this section can be found in [29]. The results of this subsection were employed in [51] to classify form preserving transformations for three classes of non-linear wave-type equations.

The results of the present chapter will be generalized in chapter 9 for systems of two PDEs.

Chapter 4

Equivalence groups for differential equation

4.1 Introduction

Equivalence transformations which play the central role in the theory of invariants are discussed in the present chapter. The set of all equivalence transformations of a given family of equations forms a group called the equivalence group. There exist two methods for the calculation of equivalence transformations: the direct method which was used by Lie (see [35]) and the Lie infinitesimal method suggested by Ovsyannikov (see [44]). Although, the direct method involves considerable computational difficulties, it has the benefit of finding the most general equivalence group. For recent applications of direct method one can refer to [29, 46, 47, 63]. More detailed description and examples of both methods can be found in [17].

Here we present the Lie infinitesimal method for calculating the continuous group of equivalence transformations. The method is described by applying it to the non-linear diffusion equation.

4.2 Equivalence groups for the non-linear diffusion equation

In this section, we consider the class of non-linear diffusion equation

$$u_t = f(u)u_{xx}. \tag{4.1}$$

We call *equivalence transformation* of the family of equations (4.1), a change of variables:

$$t' = Q(t, x, u), \quad x' = P(t, x, u), \quad u' = R(t, x, u), \tag{4.2}$$

taking any equation of the form (4.1) into an equation of the same form, generally, with different function f.

In order to find the continuous group $\mathcal{E}_C$ of equivalence transformations (4.2) for the equation (4.1), we search for the operators of the group $\mathcal{E}_C$, in the following form:

$$\Gamma = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \mu \frac{\partial}{\partial f}. \tag{4.3}$$

The generator Γ defines the group $\mathcal{E}_C$ of equivalence transformations

$$t' = Q(t, x, u), \quad x' = P(t, x, u), \quad u' = R(t, x, u), \quad f' = F(t, x, u, u_t, u_x, f),$$

for the family of equations (4.1) if and only if Γ obeys the condition of invariance of the following system:

$$u_t - f(u)u_{xx} = 0, \tag{4.4}$$

$$f_x = f_t = 0. \tag{4.5}$$

In order to write the infinitesimal invariance test for the system (4.4)-(4.5), we should extend the action of the operator (4.3) to all variables involved in (4.4)-(4.5), i.e. take

$$\widetilde{\Gamma} = \Gamma + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \mu_1 \frac{\partial}{\partial f_t} + \mu_2 \frac{\partial}{\partial f_x} + \mu_3 \frac{\partial}{\partial f_u}. \tag{4.6}$$

Here, u and f are considered as differential variables: u on the space (t, x) and f on the space (t, x, u, u_t, u_x). The coordinates $\xi^1,\ \xi^2,\ \eta$ of operator (4.3) depend on $t,\ x,\ u$, while coordinate μ depends on $x,\ t,\ u,\ f$. The coefficients $\zeta_1,\ \zeta_2,\ \zeta_{22}$ are given by:

$$\zeta_1 = D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2),$$
$$\zeta_2 = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2),$$
$$\zeta_{22} = D_x(\zeta_2) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2),$$

whereas the coefficients μ are obtained by applying the prolongation procedure to the differential variables f_t and f_x with dependent variables (t, x, u, u_t, u_x). Accordingly, we use the total differentiations:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x},$$
$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x}.$$

The infinitesimal $\mu_i,\ i = 1, 2, 3$ has the form:

$$\mu_1 = \widetilde{D}_t(\mu) - f_t \widetilde{D}_t(\xi^1) - f_x \widetilde{D}_t(\xi^2) - f_u \widetilde{D}_t(\eta) - f_{u_t} \widetilde{D}_t(\zeta_1) - f_{u_x} \widetilde{D}_t(\zeta_2),$$
$$\mu_2 = \widetilde{D}_x(\mu) - f_t \widetilde{D}_x(\xi^1) - f_x \widetilde{D}_x(\xi^2) - f_u \widetilde{D}_x(\eta) - f_{u_t} \widetilde{D}_x(\zeta_1) - f_{u_x} \widetilde{D}_x(\zeta_2),$$
$$\mu_3 = \widetilde{D}_u(\mu) - f_t \widetilde{D}_u(\xi^1) - f_x \widetilde{D}_u(\xi^2) - f_u \widetilde{D}_u(\eta) - f_{u_t} \widetilde{D}_u(\zeta_1) - f_{u_x} \widetilde{D}_u(\zeta_2),$$

where $\widetilde{D}_i,\ i = t, x, u$, denote the new total differentiations:

$$\widetilde{D}_i = \frac{\partial}{\partial i} + f_i \frac{\partial}{\partial f} + f_{iu} \frac{\partial}{\partial f_u},$$

where $i = t, x, u$.

The infinitesimal invariance for the system (4.4)-(4.5) has the form:

$$\widetilde{\Gamma}\left(u_t - f(u) u_{xx}\right) = 0, \tag{4.7}$$

$$\widetilde{\Gamma}(f_x) = \widetilde{\Gamma}(f_t) = 0. \tag{4.8}$$

In view of equations (4.5) we have

$$\widetilde{D}_t = \frac{\partial}{\partial t}, \qquad \widetilde{D}_x = \frac{\partial}{\partial x}$$

and

$$\widetilde{D}_u = \frac{\partial}{\partial u} + f_u \frac{\partial}{\partial f} + f_{uu} \frac{\partial}{\partial f_u}.$$

So we have the following prolongation formula:

$$
\begin{aligned}
\mu_1 &= \mu_t - \eta_t f_u, \\
\mu_2 &= \mu_x - \eta_x f_u, \\
\mu_3 &= \mu_u - (\eta_u - \mu_f) f_u.
\end{aligned}
\tag{4.9}
$$

Using (4.6), the invariance conditions

$$\widetilde{\Gamma}(f_x) = \widetilde{\Gamma}(f_t) = 0$$

give:

$$\mu_1 = \mu_2 = 0.$$

So, taking into account equations (4.9) and the fact that $\mu_1 = \mu_2 = 0$ must hold for every f, we obtain:

$$\mu_t = \mu_x = 0,$$

$$\eta_t = \eta_x = 0.$$

Integrations yield:

$$\mu = \mu(u, f, f_u), \quad \eta = \eta(u). \tag{4.10}$$

The remaining invariance condition (4.4), can be written as:

$$\zeta_1 - \mu u_{xx} - \zeta_{22} f = 0. \tag{4.11}$$

From (4.11), taking into account (4.8), (4.10), introducing the relation $u_t = f u_{xx}$ to eliminate u_t and using the fact that the quantities u_x, u_t, u_{xt}, u_{xx} are considered to be independent variables, it follows:

$$
\begin{aligned}
\xi^1 &= c_1 t + c_2, \\
\xi^2 &= c_3 x + c_4, \\
\eta &= c_5 u + c_6, \\
\mu &= f(2c_3 - c_1),
\end{aligned}
$$

where $c_i,\ i = 1, \ldots, 6$ are arbitrary constants. Thus we have the following results.

Theorem 4.1. *The equivalence algebra $L_{\mathcal{E}}$ for equations $u_t = f(u)u_{xx}$ is an 6-dimensional Lie algebra spanned by the following infinitesimals operators:*

$$\begin{aligned}\Gamma_1 &= \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial u}, \\ \Gamma_4 &= u\frac{\partial}{\partial u}, \quad \Gamma_5 = t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f}, \quad \Gamma_6 = x\frac{\partial}{\partial x} + 2f\frac{\partial}{\partial f}.\end{aligned}$$

The above equivalence transformations in finite form, can be derived by using First Fundamental theorem of Lie (2.1). Without presenting any calculations, this transformation have the following finite form:

$$\begin{aligned}t' &= c_1 t + c_2, \\ x' &= c_3 x + c_4, \\ u' &= c_5 u + c_6, \\ f' &= \frac{c_3^2}{c_1} f.\end{aligned}$$

Alternatively, the above transformation can be obtained using the results of chapter 3, and in particular theorems (3.2) and (3.3).

Chapter 5

Invariants of hyperbolic linear partial differential equations in two variables

5.1 Introduction

In this chapter, we derive the differential invariants for the scalar linear hyperbolic PDE in two variables by the infinitesimal method. In fact, our intention is to present the infinitesimal method for determining differential invariants. Firstly, we calculate the equivalence transformations which are used to derive differential invariants. After that, we present Ibragimov's work on finding a basis for the invariants. That is, the solution of the Laplace problem (see [19, 20]). The general invariant-differentiation operator is computed and a basis of all invariants is constructed. Furthermore, all invariants of any order are combinations of the coefficients of the equation and their derivatives. A detailed description of the method can be found in [15, 16].

We give some basic definitions using equation

$$u_{tx} + a(t,x)u_t + b(t,x)u_x + c(t,x)u = 0. \tag{5.1}$$

The same definitions follow for any other class of PDEs.

Let a class of PDEs (5.1) admit a continuous group $\mathcal{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$. As we will see later, this algebra is spanned by 3 operators, say $\Gamma_1,\ \Gamma_2$ and Γ_3.

Definition 5.1. A function

$$J = J(t, x, u, a, b, c, a_i, b_i, c_i, a_{ij}, b_{ij}, c_{ij}, \dots), \qquad i, j, \dots = t, x$$

is called an *invariant* of the family of equations (5.1) if J is invariant under the equivalence group $\mathcal{E}$. That is,

$$\Gamma_i(J) = 0, \qquad i = 1, 2, 3.$$

We call J a *semi-invariant* if it is invariant only under the subgroup of equivalence transformations. For example, if it is invariant only under Γ_1, $\Gamma_1(J) = 0$. The *order of the invariant* is equal to the order of the highest derivative that appear in the form of J. If no derivatives appear, we say that we have *invariant of zero order*.

Definition 5.2. Any system of equations

$$E_i(t, x, u, a, b, c, a_j, b_j, c_j, \dots) = 0$$

that satisfies the condition

$$\Gamma_k^{(s)}(E_i)|_{E_1=0, E_2=0, \dots} = 0, \quad i = 1, 2, \dots$$

is called an *invariant system*.

Definition 5.3. If for $i = j$, we have

$$\Gamma_k^{(s)}(E_j)|_{E_j=0},$$

then $E_j = 0$ is called an *invariant equation*.

These definitions will be used throughout in the present and in the next chapters.

5.2 Equivalence transformations

Consider the general hyperbolic equation written in the characteristic variables t, x, i.e. in the following standard form:

$$u_{tx} + a(t, x)u_t + b(t, x)u_x + c(t, x)u = 0. \tag{5.2}$$

Recall that an equivalence transformation of equation (5.2) is defined as an invertible transformation

$$t' = Q(t, x, u), \quad x' = P(t, x, u), \quad u' = R(t, x, u),$$

which preserves the order of equation (5.2) as well as the properties of linearity and homogeneity. In general, the transformed equations can have new coefficients $a',\ b',\ c'$.

In order to find the continuous group of equivalence transformations of equation (5.2) by means Lie infinitesimal invariance criterion (see [44]), we need the equivalent operator:

$$\Gamma = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_3 \frac{\partial}{\partial u_{tx}} + \mu^1 \frac{\partial}{\partial a} + \mu^2 \frac{\partial}{\partial b} + \mu^3 \frac{\partial}{\partial c}, \quad (5.3)$$

where $\xi^i = \xi^i(t, x, u),\ i = 1, 2,\ \eta = \eta(t, x, u)$ and $\mu^i,\ i = 1, 2, 3$ are functions of t, x, u, a, b and c. If we solve $\Gamma\ (u_{tx} + au_t + bu_x + cu)|_{(5.2)} = 0$, we easily get that:

$$\xi^1 = \tau(t), \quad \xi^2 = \phi(x), \quad \eta = \alpha(t, x)u, \quad \mu^1 = -a\phi_x - \alpha_x, \quad \mu^2 = -b\tau_t - \alpha_t,$$

$$\mu^3 = -(c\tau_t + c\phi_x + \alpha_{tx} + \alpha_t a + \alpha_x b),$$

where the functions $\tau(t),\ \phi(x)$ and $\alpha(t, x)$ are arbitrary.

We find that equation (5.2) admits an infinite continuous group $\mathcal{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$ spanned by the operators:

$$\Gamma_\tau = \tau(t)\frac{\partial}{\partial t} - \tau' b \frac{\partial}{\partial b} - \tau' c \frac{\partial}{\partial c},$$

$$\Gamma_\phi = \phi(x)\frac{\partial}{\partial x} - \phi' a \frac{\partial}{\partial a} - \phi' c \frac{\partial}{\partial c},$$

$$\Gamma_\alpha = \alpha(t, x)u\frac{\partial}{\partial u} - \alpha_x \frac{\partial}{\partial a} - \alpha_t \frac{\partial}{\partial b} - (\alpha_{tx} + a\alpha_t + b\alpha_x)\frac{\partial}{\partial c}.$$

5.3 Calculation of differential invariants

In this section, we consider the problem of finding differential invariants of the class of equations (5.2), using the equivalence transformations which are derived in the previous section.

Firstly, we seek for differential invariants of zero order, i.e. invariants of the form:

$$J = J(x, t, u, a, b, c).$$

Applying the invariant test $\Gamma(J) = 0$ to the operators $\Gamma_\tau,\ \Gamma_\phi$ and Γ_α and using the fact that functions $\tau,\ \phi$ and α are arbitrary, we easily obtain that $J = constant$. Hence, equations (5.2) do not have differential invariants of zero order.

In order to obtain differential invariants of first order,

$$J = J(x, t, u, a, b, c, a_t, a_x, b_t, b_x, c_t, c_x),$$

we need to consider the first prolongation of the operator Γ defined by (5.3):

$$\Gamma^{(1)} = \Gamma + \mu^{11}\frac{\partial}{\partial a_t} + \mu^{12}\frac{\partial}{\partial a_x} + \mu^{21}\frac{\partial}{\partial b_t} + \mu^{22}\frac{\partial}{\partial b_x} + \mu^{31}\frac{\partial}{\partial c_t} + \mu^{32}\frac{\partial}{\partial c_x}. \tag{5.4}$$

We introduce the local notation $f_1 = a,\ f_2 = b,\ f_3 = c$. The coefficients $\mu^{i1},\ \mu^{i2},\ i = 1, 2, 3$ are given by:

$$\mu^{i1} = D_t(\mu^i) - f_{i_t} D_t(\xi^1) - f_{i_x} D_t(\xi^2),$$

$$\mu^{i2} = D_x(\mu^i) - f_{i_t} D_x(\xi^1) - f_{i_x} D_x(\xi^2),$$

and the operators $D_t,\ D_x$ denote the total derivatives with respect to t and x:

$$\begin{aligned} D_t = \frac{\partial}{\partial t} \quad &+ \quad a_t\frac{\partial}{\partial a} + a_{tt}\frac{\partial}{\partial a_t} + a_{tx}\frac{\partial}{\partial a_x} + \dots \\ &+ \quad b_t\frac{\partial}{\partial b} + b_{tt}\frac{\partial}{\partial b_t} + b_{tx}\frac{\partial}{\partial b_x} + \dots \\ &+ \quad c_t\frac{\partial}{\partial c} + c_{tt}\frac{\partial}{\partial c_t} + c_{tx}\frac{\partial}{\partial c_x} + \dots\ , \end{aligned}$$

$$\begin{aligned} D_x = \frac{\partial}{\partial x} \quad &+ \quad a_x\frac{\partial}{\partial a} + a_{tx}\frac{\partial}{\partial a_t} + a_{xx}\frac{\partial}{\partial a_x} + \dots \\ &+ \quad b_x\frac{\partial}{\partial b} + b_{tx}\frac{\partial}{\partial b_t} + b_{xx}\frac{\partial}{\partial b_x} + \dots \\ &+ \quad c_x\frac{\partial}{\partial c} + c_{tx}\frac{\partial}{\partial c_t} + c_{xx}\frac{\partial}{\partial c_x} + \dots\ . \end{aligned}$$

After calculations we obtain the following form for the coefficients:

$$\mu^{11} = -a_t(\tau_t + \phi_x) + \alpha_{tx}, \qquad \mu^{12} = -a\phi_{xx} - 2a_x\alpha_x + \alpha_{xx},$$

$$\mu^{21} = -b\tau_{tt} - 2b_t\tau_t + \alpha_{tt}, \qquad \mu^{22} = -b_x(\tau_t + \phi_x) + \alpha_{tx},$$

$$\mu^{31} = -c\tau_{tt} - 2c_t\tau_t - c_t\phi_x - b\alpha_{tx} + \alpha_{ttx} - a\alpha_{tt} - a_t\alpha_t - b_t\alpha_x,$$

$$\mu^{32} = -c_x\tau_t - c\phi_{xx} - 2c_x\phi_x + \alpha_{txx} - a\alpha_{tx} - a_x\alpha_t - b\alpha_{xx} - b_x\alpha_x,$$

where $\alpha = \alpha(t, x)$ is an arbitrary function.

The infinitesimal test $\Gamma^{(1)}(J) = 0$ for invariants $J(x, t, u, a, b, c, a_t, a_x, b_t, b_x, c_t, c_x)$ give straightforward that

$$J = J(a, b, c, a_t, b_x).$$

The first prolongation of generator Γ_τ is

$$\Gamma_\tau^{(1)} = -\tau_t \left(a_t \frac{\partial}{\partial a_t} + b\frac{\partial}{\partial b} + b_x \frac{\partial}{\partial b_x} + c\frac{\partial}{\partial c} \right).$$

Applying generator $\Gamma_\tau^{(1)}$ to the differential invariant, we have

$$-\tau_t \left(a_t \frac{\partial J}{\partial a_t} + b\frac{\partial J}{\partial b} + b_x \frac{\partial J}{\partial b_x} + c\frac{\partial J}{\partial c} \right) = 0.$$

The characteristic equations:

$$\frac{da_t}{a_t} = \frac{db}{b} = \frac{db_x}{b_x} = \frac{dc}{c}$$

yield that $J = J(p_1, p_2, p_3, p_4)$, where

$$p_1 = \frac{b_x}{b}, \quad p_2 = \frac{a_t}{b}, \quad p_3 = \frac{c}{b}, \quad p_4 = a.$$

Now first prolongation of the operator Γ_α becomes:

$$\begin{aligned} \Gamma_\alpha^{(1)} &= -\alpha_{xt} \left(\frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2} + \frac{\partial}{\partial p_3} \right) + \alpha_t \left(p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} - (p_4 - p_3)\frac{\partial}{\partial p_3} \right) \\ &- b\alpha_x \left(\frac{\partial}{\partial p_3} + \frac{\partial}{\partial p_4} \right). \end{aligned}$$

The invariant test $\Gamma_\alpha^{(1)}(J) = 0$ is written:

$$\alpha_{xt} \left(\frac{\partial J}{\partial p_1} + \frac{\partial J}{\partial p_2} + \frac{\partial J}{\partial p_3} \right) - \alpha_t \left(p_1 \frac{\partial J}{\partial p_1} + p_2 \frac{\partial J}{\partial p_2} - (p_4 - p_3)\frac{\partial J}{\partial p_3} \right) + b\alpha_x \left(\frac{\partial J}{\partial p_3} + \frac{\partial J}{\partial p_4} \right) = 0. \tag{5.5}$$

Since $\alpha(x, t)$ is an arbitrary function, (5.5) splits into the following equations:

$$\frac{\partial J}{\partial p_1} + \frac{\partial J}{\partial p_2} + \frac{\partial J}{\partial p_3} = 0,$$

$$p_1 \frac{\partial J}{\partial p_1} + p_2 \frac{\partial J}{\partial p_2} - (p_4 - p_3)\frac{\partial J}{\partial p_3} = 0,$$

$$\frac{\partial J}{\partial p_3} + \frac{\partial J}{\partial p_4} = 0.$$

The solution of the third equation gives:

$$J = J(p_1, p_2, m_1),$$

where $m_1 = p_3 - p_4 = \frac{c-ab}{b}$. Now the first equation takes the form:

$$\frac{\partial J}{\partial p_1} + \frac{\partial J}{\partial p_2} + \frac{\partial J}{\partial m_1} = 0$$

and its characteristic equation yields

$$J = J(l_1, l_2),$$

where $l_1 = p_2 - p_1 = \frac{a_t - b_x}{b}$, $l_2 = m_1 - p_1 = \frac{c-ab-b_x}{b}$. Finally, the second equation and operator $\Gamma_\phi^{(1)}$ become identical. That is,

$$l_1 \frac{\partial J}{\partial l_1} + l_2 \frac{\partial J}{\partial l_2} = 0.$$

Solving this characteristic equation, we arrive to the following first order differential invariant:

$$\overline{p} = \frac{l_1}{l_2} = \frac{b_x - a_t}{b_x + ab - c}.$$

Denoting

$$h = l_2 - l_1, \qquad k = l_2$$

we obtain the two independent *semi-invariants* of equation (5.2):

$$h = a_t + ab - c, \qquad k = b_x + ab - c$$

known as the *Laplace invariants.* Now,

$$h = 0 \quad \text{and} \quad k = 0$$

are invariant equations. To show this, we need to apply the first prolongation $\Gamma^{(1)}$ to these equations. That is, we have to show the following:

$$\Gamma^{(1)}(h)|_{(h=0)} = 0 \quad \text{and} \quad \Gamma^{(1)}(k)|_{(k=0)} = 0.$$

The Laplace invariants are useful in various problems, for example in the group classification of differential equations (see [43]) and the solution of initial value problems for hyperbolic equations by Riemann's method (see [14]).

Finally, we recall the following simple but fundamental applications of the Laplace invariants:

1. A hyperbolic equation of the form (5.2) can be transformed into $u_{tx} = 0$ iff $h = k = 0$.
2. A hyperbolic equation of the form (5.2) can be transformed into $u_{tx} + c(t, x)u = 0$ iff $h = k$.
3. A hyperbolic equation of the form (5.2) can be transformed into $u_{tx} + cu = 0$, $c = constant$ iff $h = k = f(t)g(x)$.
4. A hyperbolic equation of the form (5.2) can be factorized iff $h = 0$ or $k = 0$. That is, the second order operator $L = D_tD_x + a(t, x)D_t + b(t, x)D_x + c(t, x)$ can be expressed as a product of two operators of first order iff one of the Laplace invariants vanishes. Namely,

$$L = [D_t + \alpha(t, x)]\,[D_x + \beta(t, x)] \quad \text{iff} \quad h = 0$$

and

$$L = [D_x + \beta(t, x)]\,[D_t + \alpha(t, x)] \quad \text{iff} \quad k = 0.$$

The proofs of the above statements can be found in [18, 19]. Motivated by the results of this section, we derive the corresponding results for systems of hyperbolic equations in chapter 10.

5.4 Invariant Differentiation

The famous Laplace invariants h and k appeared in Laplace's paper (1773) on the theory of integration of linear hyperbolic equations with two independent variables. But, the question of the presence or absence of other invariants remained open.

Nearly 200 years had passed before Ovsyannikov (see [43]), studying the problem of group classification of hyperbolic equations, found two invariants

$$p = \frac{k}{h}, \qquad q = \frac{1}{h}\frac{\partial^2 |h|}{\partial t \partial x},$$

which do not change under all equivalence transformation. At that time, the general approach of constructing invariants of systems of equations with an infinite equivalence transformation group had not been developed, and, hence, the problem of whether all invariants are exhausted by the quantities found remained open.

A general method for constructing invariants of systems of linear and non-linear equations using infinite equivalence transformation groups was recently developed in [15, 16]. This method is applied to several linear and non-linear equations.

In the present section, we give a description of the method that Ibragimov used to solve the Laplace problem. More detailed description of the method can be found in [20]. This problem consists of finding all invariants of the linear hyperbolic equations (5.2) and constructing a basis of invariants. To construct a basis of invariants, one first computes all invariants up to second order, inclusive, and then finds the next three new invariants:

$$I = \frac{p_t p_x}{h}, \qquad N = \frac{1}{p_t}\frac{\partial}{\partial t}\ln\left|\frac{p_t}{h}\right|, \qquad H = \frac{1}{p_x}\frac{\partial}{\partial x}\ln\left|\frac{p_x}{h}\right|.$$

After that, the general invariant-differentiation operator:

$$\mathcal{D} = F(p, I)\frac{1}{p_t}D_t + G(p, I)\frac{1}{p_x}D_x \tag{5.6}$$

can be computed. It is proved that, it is possible to construct of the new invariants and Ovsyannikov invariants, a basis of all invariants. Any invariant of any order is a function of the basis invariants and their invariant derivatives.

Collecting together invariants, Ibragimov arrived at the following complete set of second-order invariants for equations (5.2):

$$p = \frac{k}{h}, \qquad q = \frac{1}{h}\frac{\partial^2 \ln|h|}{\partial t \partial x}, \qquad \widetilde{q} = \frac{1}{k}\frac{\partial^2 \ln|k|}{\partial t \partial x},$$

$$N = \frac{1}{p_t}\frac{\partial}{\partial t}\ln\left|\frac{p_t}{h}\right|, \qquad H = \frac{1}{p_x}\frac{\partial}{\partial x}\ln\left|\frac{p_x}{h}\right|, \qquad I = \frac{p_t p_x}{h}.$$

In addition, there are the following invariant equations:

$$h = 0, \qquad k = 0, \qquad k_t - p h_t = 0, \qquad k_x - p h_x = 0.$$

Now, we will find the invariant differentiation operator of the form (5.6), that transforms each invariant of (5.2) into invariants of the same equation. Recall that an operator Γ is said to be an *operator of invariant differentiation* for a group $\mathcal{E}$ if for any differential invariant J of the group $\mathcal{E}$, $\Gamma(J)$ is also a differential invariant of this group.

For any family of infinitesimal operators:

$$\Gamma_\nu = \xi^i_\nu(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha_\nu(x, u)\frac{\partial}{\partial u^\alpha},$$

with n independent variables $x = (x^1, \dots, x^n)$, there exist n invariant differentiations of the form (see [16, 44])

$$\mathcal{D} = f^i D_i. \tag{5.7}$$

Their coefficients have the form:

$$f^i = f^i(x, t, u, u_{(1)}, u_{(2)}, \dots)$$

and are found by solving the differential equations:

$$\Gamma_\nu(f^i) = f^j D_j(\xi^i_\nu), \qquad i = 1, \dots, n. \tag{5.8}$$

In our case, the operators Γ_ν are the second extension of the operators

$$\Gamma_1 = -\xi(t)\frac{\partial}{\partial t} + \xi'(t)\left[h\frac{\partial}{\partial h} + k\frac{\partial}{\partial k}\right]$$

and

$$\Gamma_2 = -\eta(x)\frac{\partial}{\partial x} + \eta'(x)\left[h\frac{\partial}{\partial h} + k\frac{\partial}{\partial k}\right],$$

using the general procedure. The invariant differentiation operator (5.7) can be written as

$$\mathcal{D} = f D_t + g D_x, \tag{5.9}$$

and the equations (5.8) for the coefficients can be written as:

$$\begin{aligned} &\Gamma_1(f) = f D_t(\xi(t)) + g D_x(\xi(t)) \equiv -\xi'(t) f, \qquad \Gamma_1(g) = 0, \\ &\Gamma_2(g) = f D_t(\eta(x)) + g D_x(\eta(x)) \equiv -\eta'(x) g, \quad \Gamma_2(f) = 0. \end{aligned} \tag{5.10}$$

Here f and g are unknown functions of $t, x, h, k, h_t, h_x, k_t, k_x, h_{tt}, \dots$. The operators Γ_1 and Γ_2 are extended to all derivatives of h and k.

We begin with case where $f = f(x, t, h, k)$ and $g = g(x, t, h, k)$. Then, equations (5.10) give the following system of equations for f:

$$\xi(t)\frac{\partial f}{\partial t} - \xi'(t)\left[h\frac{\partial f}{\partial h} + k\frac{\partial f}{\partial k}\right] = \xi'(t) f, \quad \eta(x)\frac{\partial f}{\partial x} - \eta'(x)\left[h\frac{\partial f}{\partial h} + k\frac{\partial f}{\partial k}\right] = 0.$$

Using the fact that $\xi,\ \xi',\ \eta,\ \eta'$ are arbitrary functions, we arrive at the following four equations:

$$\frac{\partial f}{\partial t}=0,\qquad h\frac{\partial f}{\partial h}+k\frac{\partial f}{\partial k}=-f,\qquad \frac{\partial f}{\partial x}=0,\qquad h\frac{\partial f}{\partial h}+k\frac{\partial f}{\partial k}=0,$$

which yield that $f=0$. Similarly, equations (5.10), for $g=g(x,t,h,k)$, give $g=0$. This means that there are no invariant differentiations of (5.9) with the coefficients $f=f(x,t,h,k)$ and $g=g(x,t,h,k)$.

Therefore, we continue the search by setting:

$$f=f(x,t,h,k,h_x,h_t,k_x,k_t),\quad g=g(x,t,h,k,h_x,h_t,k_x,k_t).$$

The extended operators Γ_1 and Γ_2 lead to the following operators:

$$\Gamma_{1_\xi}=\frac{\partial}{\partial t},\qquad \Gamma_{1\xi''}=h\frac{\partial}{\partial h_t}+k\frac{\partial}{\partial k_t},$$

$$\Gamma_{1_{\xi'}}=h\frac{\partial}{\partial h}+k\frac{\partial}{\partial k}+2h_t\frac{\partial}{\partial h_t}+h_x\frac{\partial}{\partial h_x}+2k_t\frac{\partial}{\partial k_t}+k_x\frac{\partial}{\partial k_x}$$

and, hence, to the operators:

$$\Gamma_{2_\eta}=\frac{\partial}{\partial x},\qquad \Gamma_{2_{\eta''}}=h\frac{\partial}{\partial h_x}+k\frac{\partial}{\partial k_x},$$

$$\Gamma_{2_{\eta'}}=h\frac{\partial}{\partial h}+k\frac{\partial}{\partial k}+h_t\frac{\partial}{\partial h_t}+2h_x\frac{\partial}{\partial h_x}+k_t\frac{\partial}{\partial k_t}+2k_x\frac{\partial}{\partial k_x}.$$

The existence operators Γ_{1_ξ} and Γ_{2_η} leads to the fact that f and g do not depend on x and t. Next, equations (5.10) split into the equations:

$$\Gamma_{1_{\xi'}}(f)=-f,\ \ \Gamma_{1_{\xi''}}(f)=0,\ \ \Gamma_{2_{\eta'}}(f)=0,\ \ \Gamma_{2_{\eta''}}(f)=0,$$

for the function $f(h,k,h_x,h_t,k_x,k_t)$ and the equations:

$$\Gamma_{1_{\xi'}}(g)=0,\ \ \Gamma_{1_{\xi''}}(g)=0,\ \ \Gamma_{2_{\eta'}}(g)=-g,\ \ \Gamma_{2_{\eta''}}(g)=0,$$

for the function $g(h,k,h_x,h_t,k_x,k_t)$. From these, the pair of equations $\Gamma_{1_{\xi''}}(f)=0,\ \Gamma_{2_{\eta''}}(f)=0$ for f and the pair of equations $\Gamma_{1_{\xi''}}(g)=0,\ \Gamma_{2_{\eta''}}(g)=0$ for g, show that f and g depend only on the following four variables:

$$h,\quad k,\quad \lambda=k_t-ph_t=hp_t,\quad \mu=k_x-ph_x=hp_x.$$

Now we rewrite the operators $\Gamma_{1_{\xi'}}$ and $\Gamma_{2_{\eta'}}$ in the variables $h,\ \lambda,\ \mu$ and $p = k/h$:

$$\Gamma_{1_{\xi'}} = h\frac{\partial}{\partial h} + 2\lambda\frac{\partial}{\partial \lambda} + \mu\frac{\partial}{\partial \mu}, \quad \Gamma_{2_{\eta'}} = h\frac{\partial}{\partial h} + \lambda\frac{\partial}{\partial \lambda} + 2\mu\frac{\partial}{\partial \mu},$$

and integrate the equations:

$$\Gamma_{1_{\xi'}}(f) = -f, \quad \Gamma_{2_{\eta'}}(f) = 0,$$

for the function $f(h, p, \lambda, \mu)$ and similar equations:

$$\Gamma_{1_{\xi'}}(g) = 0, \quad \Gamma_{2_{\eta'}}(g) = -g,$$

for the function $g(h, p, \lambda, \mu)$. As a result, we obtain:

$$f = \frac{h}{\lambda}F(p, I), \quad g = \frac{h}{\mu}G(p, I),$$

where $\lambda = hp_t,\ \mu = hp_x$, and p and I are invariants:

$$p = \frac{k}{h}, \quad I = \frac{\lambda\mu}{h^3} = \frac{p_t p_x}{h}.$$

Substitution of expression f and g into (5.9) leads to the invariant-differentiation operator:

$$\mathcal{D} = F(p, I)\frac{1}{p_t}D_t + G(p, I)\frac{1}{p_x}D_x,$$

with arbitrary function $F(p, I)$ and $G(p, I)$.

Setting $F = 1$ and $G = 0$ and then $F = 0$ and $G = 1$ in above operator, we obtain the following simple invariant differentiations in t and x directions:

$$\mathcal{D}_t = \frac{1}{p_t}D_t, \qquad \mathcal{D}_x = \frac{1}{p_x}D_x.$$

It is now possible to construct higher-order invariants using the above invariant differentiations and to prove the following statement.

Theorem 5.1. *The basis of invariants of arbitrary order for (5.2) consists of the invariants*

$$p = \frac{k}{h}, \qquad I = \frac{p_t p_x}{h}, \qquad q = \frac{1}{h}\frac{\partial^2 \ln|h|}{\partial t \partial x}, \qquad \widetilde{q} = \frac{1}{k}\frac{\partial^2 \ln|k|}{\partial t \partial x},$$

or the alternative basis invariants

$$p = \frac{k}{h}, \qquad I = \frac{p_t p_x}{h}, \qquad N = \frac{1}{p_t}\frac{\partial}{\partial t}\ln\left|\frac{p_t}{h}\right|, \qquad q = \frac{1}{h}\frac{\partial^2 \ln|h|}{\partial t \partial x}.$$

Therefore, we described how Ibragimov derived the complete set of differential invariants for the scalar linear hyperbolic equation (5.2). This completes the Ovsyannikov invariants obtained in [43, 44].

Chapter 6

Hyperbolic equations of general class

6.1 Introduction

In this chapter, we consider the general class of hyperbolic equations

$$u_{xt} = F(x, t, u, u_x, u_t).$$

We use equivalence transformations to derive differential invariants for this class and for two subclasses:

$$u_{xt} = f(x, t, u)u_x u_t + g(x, t, u)u_x + h(x, t, u)u_t + l(x, t, u),$$

$$u_{xt} = m_u(x, t, u)u_x u_t + m_t(x, t, u)u_x + m_x(x, t, u)u_t + k(x, t, u).$$

Then we employ these invariants to construct equations that can be linearized via local mappings. Furthermore, we give applications of the differential invariants.

The approach used here is similar to the one used in [52] for the class of equations

$$u_{tt} - u_{xx} = f(u, u_x, u_t).$$

We point out that, we can alternatively use the direct method (see [29]) to determine equivalence transformations in finite form. These can be expressed in the infinitesimal form using Lie's method. Therefore the results of chapter 3 are useful to derive equivalence transformations if finite form.

Hyperbolic type second-order non-linear PDEs in two independent variables are used in mathematical physics. They can describe various type of wave propagation and model several phenomena in various fields of hydro and gas dynamics, chemical technology, super conductivity, crystal dislocation. Well-known equations of this type are the Liouville, sine-Gordon, Goursat, d'Alembert and Tzitzeica equations. These models are integrable by the inverse problem methods (see [2, 38, 65]) or linearizable (see [1, 10, 27, 64, 69]).

6.2 Invariants for the general class of hyperbolic equations

6.2.1 Equivalence transformations

In this section, we consider hyperbolic differential equations of general class

$$u_{xt} = F(x, t, u, u_x, u_t). \tag{6.1}$$

In order to find the continuous group of equivalence transformations for the class (6.1) by means of the Lie infinitesimal invariance criterion, we follow Ovsyannikov's method (see [44]). That is, we search for equivalent operator in the following form:

$$\Gamma = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \eta \frac{\partial}{\partial F},$$

where

$$\xi^1 = \xi^1(t, x, u), \quad \xi^2 = \xi^2(t, x, u), \quad \xi^3 = \xi^3(t, x, u),$$

and η is function of t, x, u, u_t, u_x, F. The infinitesimals ζ_t and ζ_x are given by:

$$\zeta_t = D_t(\xi^3) - u_t D_t(\xi^1) - u_x D_t(\xi^2) \quad \text{and} \quad \zeta_x = D_x(\xi^3) - u_t D_x(\xi^1) - u_x D_x(\xi^2).$$

The operators D_t and D_x denote the total derivatives with respect to t and x, respectively.

The equivalence transformations for the similar class of equations:

$$u_{tt} - u_{xx} = f(x, t, u, u_x)$$

were derived in [31].

In order to determine the coefficients that appear in operator Γ, we have to solve the equation:

$$\Gamma \left[u_{tx} - F(t, x, u, u_x, u_t)\right]\big|_{(6.1)} = 0.$$

Solution of the equation gives

$$\xi^1 = \tau(t), \quad \xi^2 = \varphi(x), \quad \xi^3 = \psi(t, x, u),$$

$$\zeta_t = \psi_t + (\psi_u - \tau_t)u_t, \quad \zeta_x = \psi_x + (\psi_u - \varphi_x)u_x,$$

$$\eta = (\psi_u - \tau_t - \varphi_x)F + \psi_{tx} + \psi_{tu}u_x + \psi_{xu}u_t + \psi_{uu}u_x u_t,$$

where $\tau = \tau(t)$, $\varphi = \varphi(x)$, $\psi = \psi(x, t, u)$ are arbitrary functions. Therefore, the generator takes the form:

$$\begin{aligned}\Gamma &= \tau\frac{\partial}{\partial t} + \varphi\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial u} + [\psi_t + (\psi_u - \tau_t)u_t]\frac{\partial}{\partial u_t} + [\psi_x + (\psi_u - \varphi_x)u_x]\frac{\partial}{\partial u_x} \\ &+ [(\psi_u - \tau_t - \varphi_x)F + \psi_{tx} + (\psi_{tu}u_x + \psi_{xu}u_t + \psi_{uu}u_x u_t)]\frac{\partial}{\partial F}.\end{aligned}$$

Therefore, equations (6.1) have a continuous group $\mathcal{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$ which is spanned by the operators:

$$\begin{aligned}\Gamma_\tau &= \tau\frac{\partial}{\partial t} - \tau_t u_t\frac{\partial}{\partial u_t} - \tau_t F\frac{\partial}{\partial F}, \\ \Gamma_\varphi &= \varphi\frac{\partial}{\partial x} - \varphi_x u_x\frac{\partial}{\partial u_x} - \varphi_x F\frac{\partial}{\partial F}, \\ \Gamma_\psi &= \psi\frac{\partial}{\partial u} + (\psi_x + \psi_u u_x)\frac{\partial}{\partial u_x} + (\psi_t + \psi_u u_t)\frac{\partial}{\partial u_t} \\ &+ (\psi_{tx} + \psi_{xu}u_t + \psi_{tu}u_x + \psi_{uu}u_t u_x + \psi_u F)\frac{\partial}{\partial F}.\end{aligned} \tag{6.2}$$

Also, one can show that the equivalence transformations (6.2) can be written in the finite form, using theorems (3.2) and (3.3):

$$x' = P(x), \quad t' = Q(t), \quad u' = R(t, x, u). \tag{6.3}$$

Alternatively, (6.3) can be obtained using the First Fundamental theorem of Lie (2.1).

6.2.2 Differential invariants and invariant equations

In this subsection, we consider the problem of finding differential invariants of the class of equations (6.1). Firstly, we search for invariants of zero order. That is, we look for functions of the form $J(t, x, u, u_x, u_t, F)$ that satisfy the invariance criterion

$$\Gamma_\tau(J) = 0, \quad \Gamma_\varphi(J) = 0, \quad \Gamma_\psi(J) = 0.$$

To this end, we look for functions that satisfy the following equations:

$$\tau\frac{\partial J}{\partial t}-\tau_t u_t\frac{\partial J}{\partial u_t}-\tau_t F\frac{\partial J}{\partial F}=0,$$
$$\varphi\frac{\partial J}{\partial x}-\varphi_x u_x\frac{\partial J}{\partial u_x}-\varphi_x F\frac{\partial J}{\partial F}=0,$$
$$\psi\frac{\partial J}{\partial u}+(\psi_u u_x+\psi_x)\frac{\partial J}{\partial u_x}+(\psi_t+\psi_u u_t)\frac{\partial J}{\partial u_t}$$
$$+(\psi_u F+\psi_{tu}u_x+\psi_{tx}+\psi_{xu}u_t+\psi_{uu}u_t u_x)\frac{\partial J}{\partial F}=0.$$

Since the functions $\tau,\ \varphi,\ \psi$ are arbitrary, these identities lead to linear first order PDEs for J. Straightforward calculations lead to the trivial solution, i.e. $J=$constant. Hence, equations (6.1) do not admit differential invariants of order zero.

So, it is necessary to consider first-order differential invariants, of the form:

$$J(t,x,u,u_x,u_t,F,F_t,F_x,F_u,F_{u_t},F_{u_x}).$$

To find such invariants, one needs to calculate the first prolongation of the operator Γ

$$\Gamma^{(1)}=\Gamma+\eta^{i_1}\frac{\partial}{\partial F_{i_1}},\quad i_1=x,t,u,u_x,u_t,$$

where

$$\begin{aligned}\eta^{i_1} &= \widetilde{D}_{i_1}(\eta)-F_t\widetilde{D}_{i_1}(\xi^1)-F_x\widetilde{D}_{i_1}(\xi^2)-F_u\widetilde{D}_{i_1}(\xi^3)-F_{u_t}\widetilde{D}_{i_1}(\zeta_t)\\ &- F_{u_x}\widetilde{D}_{i_1}(\zeta_x)-F_{u_{tx}}\widetilde{D}_{i_1}(\zeta_{tx}),\end{aligned}$$

and $\widetilde{D}_{i_1}$ denote the total derivatives with respect to i_1:

$$\widetilde{D}_{i_1}=\frac{\partial}{\partial i_1}+F_{i_1}\frac{\partial}{\partial F}+F_{i_1x}\frac{\partial}{\partial F_x}+F_{i_1t}\frac{\partial}{\partial F_t}+F_{i_1u}\frac{\partial}{\partial F_u}+F_{i_1u_t}\frac{\partial}{\partial F_{u_t}}+F_{i_1u_x}\frac{\partial}{\partial F_{u_x}}+\dots.$$

Similarly, the first prolongation of the operators (6.2) lead to the invariance criterion:

$$\Gamma_\tau^{(1)}(J)=0,\quad \Gamma_\varphi^{(1)}(J)=0,\quad \Gamma_\psi^{(1)}(J)=0.$$

The fact that τ, φ and ψ are arbitrary functions, leads to linear first order PDEs. Without giving any details, we obtain the trivial invariant. Hence, equations (6.1) also do not admit differential invariants of order one.

In order to find differential invariants of order two, i.e. that depend on the second derivatives of F, we need the second prolongation of the operators (6.2), which can be derived using the formula:

$$\Gamma^{(2)}=\Gamma^{(1)}+\eta^{i_1i_2}\frac{\partial}{\partial F_{i_1i_2}},\quad i_1,\ i_2=x,t,u,u_x,u_t,$$

where

$$\begin{aligned}\eta^{i_1 i_2} &= \widetilde{D}_{i_2}(\eta^{i_1}) - F_{i_1 t}\widetilde{D}_{i_2}(\xi^1) - F_{i_1 x}\widetilde{D}_{i_2}(\xi^2) - F_{i_1 u}\widetilde{D}_{i_2}(\xi^3) - F_{i_1 u_t}\widetilde{D}_{i_2}(\zeta_t)\\ &- F_{i_1 u_x}\widetilde{D}_{i_2}(\zeta_x) - F_{i_1 u_{tx}}\widetilde{D}_{i_2}(\zeta_{tx}),\end{aligned}$$

and $\widetilde{D}_{i_2}$ denote the total derivatives with respect to i_2:

$$\widetilde{D}_{i_2} = \frac{\partial}{\partial i_2} + F_{i_2}\frac{\partial}{\partial F} + F_{i_2 x}\frac{\partial}{\partial F_x} + F_{i_2 t}\frac{\partial}{\partial F_t} + F_{i_2 u}\frac{\partial}{\partial F_u} + F_{i_2 u_t}\frac{\partial}{\partial F_{u_t}} + F_{i_2 u_x}\frac{\partial}{\partial F_{u_x}} + \dots.$$

From the differential invariant test

$$\Gamma_k^{(2)}(J) = 0, \quad k = \tau, \varphi, \psi,$$

we state that equations (6.1) do not admit differential invariants of second order.

However, equations (6.1) admit the following invariant equations:

$$F_{u_t u_t} = 0, \quad F_{u_x u_x} = 0. \tag{6.4}$$

That is, we have to show:

$$\Gamma_k^{(2)}[F_{u_t u_t}]|_{(F_{u_t u_t}=0)} = 0, \quad \Gamma_k^{(2)}[F_{u_x u_x}]|_{(F_{u_x u_x}=0)} = 0, \quad k = \tau, \varphi, \psi.$$

Furthermore, the quantity

$$J = \frac{F_{u_x u_x}}{F_{u_t u_t}}$$

is a semi-invariant of second order. In this case J satisfies the equation $\Gamma_\psi^{(2)}(J) = 0$. That is, in (6.3) $P = x$ and $Q = t$ which means that (6.1) is invariant only under the transformation of the dependent variable.

In order to find differential invariants of third order, we follow the same procedure as before. We get that equations (6.1) admit 13 differential invariants of third order:

$$\begin{aligned}J_1 &= \frac{F^2_{u_x u_x u_x} F_{u_t u_t}}{F_{u_x u_x} F^2_{u_x u_x u_t}}, \quad J_2 = \frac{F_{u_x u_x u_x} F_{u_t u_t}}{F_{u_x u_x} F_{u_x u_x u_t}}, \quad J_3 = \frac{F_{u_x u_x} F_{u_t u_t u_t}}{F_{u_t u_t} F_{u_x u_x u_t}},\\ J_4 &= \frac{F_{u_x u_x u_t}}{F^2_{u_x u_x} F_{u_x u_x u_t}}\left[(F_{u_x} F_{u_x u_t})_{u_x} + F_{u_x u_x u_x} F_{u_t} + F_{u u_x u_x}\right],\\ J_5 &= \frac{F^3_{u_x u_x u_x}}{F^2_{u_x u_x} F^3_{u_x u_x u_t}}\left[(F_{u_x} F_{u_t u_t})_{u_t} + F_{u_x u_t u_t} F_{u_t} + F_{u u_t u_t}\right],\end{aligned}$$

$$
\begin{aligned}
J_6 &= \frac{F^3_{u_x u_x u_x}}{F^4_{u_x u_x} F^2_{u_x u_x u_t}} [F F_{u_x u_x} F_{u_x u_t u_t} - F F_{u_x u_x u_t} F_{u_x u_t} + F_u F_{u_x u_x u_t} - F_{u_x} F_{u_x u_x} F_{u_t u_t} \\
&- F_{u_x u_x} F_{u u_t} + F_{u_x} F_{u_x u_x u_t} F_{u_t} + F_{u_x u_x} F_{u u_x u_t} u_x + F_{u_x u_x} F_{x u_x u_t} - F_{u_x u_x u_t} F_{u u_x} u_x \\
&- F_{u_x u_x u_t} F_{x u_x}], \\
J_7 &= \frac{F^4_{u_x u_x u_x}}{F^4_{u_x u_x} F^3_{u_x u_x u_t}} [F F_{u_x u_x u_t} F_{u_t u_t} - F F_{u_x u_t} F_{u_x u_t u_t} + F_{t u_x u_t} F_{u_t u_t} - F_{t u_t} F_{u_x u_t u_t} \\
&+ F_{u_x} F_{u_x u_t u_t} F_{u_t} - F_{u_x u_x} F_{u_t} F_{u_t u_t} - F_{u_x u_t u_t} F_{u u_t} u_t - F_{u_t u_t} F_{u u_x} + F_{u_t u_t} F_{u u_x u_t} u_t \\
&+ F_u F_{u_x u_t u_t}], \\
J_8 &= \frac{F^5_{u_x u_x u_x}}{F^5_{u_x u_x} F^4_{u_x u_x u_t}} [(F_{u_x u_x} F_{u_t u_t})_{u_t} \{-F F_{u_x u_t} - F_{t u_t} + F_u + F_{u_x} F_{u_t} - F_{u u_t} u_t\} \\
&+ F F_{u_t u_t} (F_{u_x u_x} F_{u_t u_t})_{u_x} + F_{u_t u_t} u_t (F_{u_x u_x} F_{u_t u_t})_u + 2 F_{u_x} F_{u_x u_x} F^2_{u_t u_t} \\
&+ F_{u_t u_t} (F_{u_x u_x} F_{u_t u_t})_t], \\
J_9 &= \frac{F^4_{u_x u_x u_x}}{F^5_{u_x u_x} F^3_{u_x u_x u_t}} [(F_{u_x u_x} F_{u_t u_t})_{u_x} \{-u_x F_{u u_x} - F F_{u_x u_t} + F_u + F_u F_{u_t} - F_{x u_x}\} \\
&+ F F_{u_x u_x} (F_{u_x u_x} F_{u_t u_t})_{u_t} + F_{u_x u_x} u_x (F_{u_x u_x} F_{u_t u_t})_u + 2 F_{u_t} F^2_{u_x u_x} F_{u_t u_t} \\
&+ F_{u_x u_x} (F_{u_x u_x} F_{u_t u_t})_x], \\
J_{10} &= \frac{F^6_{u_x u_x u_x}}{F^6_{u_x u_x} F^4_{u_x u_x u_t}} (-F F_{u_x} F_{u_x u_t} F_{u_t u_t u_t} + F F_{u_x} F_{u_x u_t u_t} F_{u_t u_t} + F F_{u_x u_x} F^2_{u_t u_t} \\
&- F F^2_{u_x u_t} F_{u_t u_t} + F F_{u_x u_x u_t} F_{u_t} F_{u_t u_t} - F F_{u_x u_t} F_{u_x u_t u_t} F_{u_t} - F F_{u_x u_t} F_{u u_t u_t} \\
&+ F F_{u_t u_t} F_{u u_x u_t} + F_{t u_x} F^2_{u_t u_t} + F_{t u_x u_t} F_{u_t} F_{u_t u_t} - F_{t u_t} F_{u_x} F_{u_t u_t u_t} \\
&- F_{t u_t} F_{u_x u_t} F_{u_t u_t} - F_{t u_t} F_{u_x u_t u_t} F_{u_t} - F_{u u_t} F_{u u_t u_t} u_t + F_{t u_t u_t} F_{u_x} F_{u_t u_t} \\
&+ F_{t u u_t} F_{u_t u_t} + F_u F_{u_x} F_{u_t u_t u_t} + 2 F_u F_{u_x u_t} F_{u_t u_t} + F_u F_{u_x u_t u_t} F_{u_t} \\
&+ F_u F_{u u_t u_t} + F^2_{u_x} F_{u_t} F_{u_t u_t u_t} + F_{u_x} F_{u_x u_t} F_{u_t} F_{u_t u_t} + F_{u_x} F_{u_x u_t u_t} F^2_{u_t} \\
&+ F_{u_x} F_{u_t} F_{u u_t u_t} - F_{u_x u_t u_t} F_{u_t} F_{u u_t} u_t + F_{u_x} F_{u_t u_t} F_{u u_t u_t} u_t - F_{u_x} F_{u_t u_t u_t} F_{u u_t} u_t \\
&- F_{u_x u_x} F^2_{u_t} F_{u_t u_t} - F_{u_x u_t} F_{u_t u_t} F_{u u_t} u_t - F_{u_x} F_{u_t u_t} F_{u u_t} - F_{u_t u_t} F_{uu} - 2 F_{u_t} F_{u_t u_t} F_{u u_x} \\
&+ F_{u_t} F_{u_t u_t} F_{u u_x u_t} u_t - F_{t u_t} F_{u u_t u_t} + F^2_{u_t u_t} F_{u u_x} u_t + F_{u_t u_t} F_{u u u_t} u_t), \\
J_{11} &= \frac{F^4_{u_x u_x u_x}}{F^6_{u_x u_x} F^2_{u_x u_x u_t}} (-F F_{u_x} F_{u_x u_x} F_{u_x u_t u_t} + F F_{u_x} F_{u_x u_x u_t} F_{u_x u_t} - F F^2_{u_x u_x} F_{u_t u_t} \\
&+ F F_{u_x u_x} F^2_{u_x u_t} - F F_{u_x u_x} F_{u_x u_x u_t} F_{u_t} - F F_{u_x u_x} F_{u u_x u_t} + F F_{u_x u_x u_x} F_{u_x u_t} F_{u_t} \\
&+ F F_{u_x u_t} F_{u u_x u_x} - F_u F_{u_x} F_{u_x u_x u_t} - 2 F_u F_{u_x u_x} F_{u_x u_t} - F_u F_{u_x u_x u_x} F_{u_t} - F_u F_{u u_x u_x} \\
&+ F^2_{u_x} F_{u_x u_x} F_{u_t u_t} - F^2_{u_x} F_{u_x u_x u_t} F_{u_t} - F^2_{u_x u_x} F_{u u_t} u_x - F_{u_x} F_{u_x u_x} F_{u_x u_t} F_{u_t} \\
&- F_{u_x} F_{u_x u_x} F_{u u_x u_t} u_x + 2 F_{u_x} F_{u_x u_x} F_{u u_t} - F_{u_x} F_{u_x u_x} F_{x u_x u_t} + F_{u u_x u_x} F_{x u_x} \\
&- F_{u_x} F_{u_x u_x u_x} F^2_{u_t} + F_{u_x} F_{u_x u_x u_t} F_{u u_x} u_x + F_{u_x} F_{u_x u_x u_t} F_{x u_x} - F_{u_x} F_{u_t} F_{u u_x u_x}
\end{aligned}
$$

$$
\begin{aligned}
&+ && F_{u_xu_xu_x}F_{u_t}F_{xu_x} - F_{u_xu_x}F_{u_t}F_{uu_xu_x}u_x + F_{u_xu_x}F_{u_xu_t}F_{uu_x}u_x + F_{u_xu_x}F_{u_xu_t}F_{xu_x} \\
&+ && F_{u_xu_x}F_{u_t}F_{uu_x} + F_{uu_x}F_{uu_xu_x}u_x - F^2_{u_xu_x}F_{xu_t} - F_{u_xu_x}F_{u_t}F_{xu_xu_x} - F_{u_xu_x}F_{uuu_x}u_x \\
&- && F_{u_xu_x}F_{xuu_x} + F_{u_xu_xu_x}F_{u_t}F_{uu_x}u_x + F_{u_xu_x}F_{uu}), \\
J_{12} &= && \frac{F^6_{u_xu_xu_x}}{F^8_{u_xu_x}F^3_{u_xu_xu_t}}(-F^2F_{u_xu_x}F_{u_xu_xu_t}F_{u_tu_t} + F^2F_{u_xu_x}F_{u_xu_t}F_{u_xu_tu_t} \\
&+ && F^2F_{u_xu_xu_x}F_{u_xu_t}F_{u_tu_t} - F^2F_{u_xu_xu_t}F^2_{u_xu_t} + FF_{tu_xu_x}F_{u_xu_t}F_{u_tu_t} - FF_{tu_xu_t}F_{u_xu_x}F_{u_tu_t} \\
&+ && FF_{tu_t}F_{u_xu_x}F_{u_xu_tu_t} - F^2_uF_{u_xu_xu_t} - FF_{tu_t}F_{u_xu_xu_t}F_{u_xu_t} - FF_uF_{u_xu_x}F_{u_xu_tu_t} \\
&- && FF_uF_{u_xu_xu_x}F_{u_tu_t} + 2FF_uF_{u_xu_xu_t}F_{u_xu_t} + F_uF_{u_xu_x}F_{uu_t} - FF_{u_x}F_{u_xu_x}F_{u_xu_tu_t}F_{u_t} \\
&- && FF_{u_x}F_{u_xu_xu_x}F_{u_t}F_{u_tu_t} + 2FF_{u_x}F_{u_xu_xu_t}F_{u_xu_t}F_{u_t} - FF_{u_xu_x}F_{u_xu_t}F_{uu_t} \\
&+ && FF_{u_xu_x}F_{u_xu_t}F_{uu_xu_t}u_x + FF_{u_xu_x}F_{u_xu_t}F_{xu_xu_t} + FF_{u_xu_x}F_{u_xu_tu_t}F_{uu_t}u_t \\
&- && FF_{u_xu_x}F_{u_tu_t}F_{uu_xu_x}u_x - FF_{u_xu_x}F_{u_tu_t}F_{uu_xu_t}u_t - FF_{u_xu_x}F_{u_tu_t}F_{xu_xu_x} \\
&+ && FF_{u_xu_xu_x}F_{u_tu_t}F_{xu_x} - FF_{u_xu_xu_t}F_{u_xu_t}F_{uu_x}u_x - FF_{u_xu_xu_t}F_{u_xu_t}F_{uu_t}u_t \\
&+ && FF_{u_xu_t}F_{u_tu_t}F_{uu_xu_x}u_t - F_tF_{u_xu_x}F_{u_xu_t}F_{u_tu_t} + F_{tu}F_{u_xu_x}F_{u_tu_t} + F_{tu_x}F_{u_xu_x}F_{u_t}F_{u_tu_t} \\
&- && F_{tu_xu_x}F_uF_{u_tu_t} + F_{tu_t}F_uF_{u_xu_xu_t} + FF_{u_xu_xu_x}F_{u_tu_t}F_{uu_x}u_x - FF_{u_xu_xu_t}F_{u_xu_t}F_{xu_x} \\
&- && F_{tu_xu_x}F_{u_x}F_{u_t}F_{u_tu_t} + F_{tu_xu_x}F_{u_tu_t}F_{uu_x}u_x + F_{tu_xu_x}F_{u_tu_t}F_{xu_x} + F_{tu_t}F_{u_x}F_{u_xu_xu_t}F_{u_t} \\
&+ && F_{tu_t}F_{u_xu_x}F_{uu_xu_t}u_x + F_{tu_t}F_{u_xu_x}F_{xu_xu_t} - F_{tu_t}F_{u_xu_xu_t}F_{uu_x}u_x - F_{tu_t}F_{u_xu_xu_t}F_{xu_x} \\
&- && 2F_uF_{u_x}F_{u_xu_xu_t}F_{u_t} - F_uF^2_{u_xu_x}F_{u_tu_t}u_x - F_uF_{u_xu_x}F_{u_xu_t}F_{u_tu_t}u_t - F_uF_{u_xu_x}F_{uu_xu_t}u_x \\
&- && F_uF_{u_xu_x}F_{xu_xu_t} + F_uF_{u_xu_xu_t}F_{uu_x}u_x - F_{u_x}F_{u_xu_x}F_{u_t}F_{uu_xu_t}u_x + F_uF_{u_xu_xu_t}F_{xu_x} \\
&- && F_{u_x}F_{u_t}F_{u_tu_t}F_{uu_xu_x}u_t + F_{u_x}F_{u_xu_x}F_{u_t}F_{uu_t} - F_{u_x}F_{u_xu_x}F_{u_t}F_{xu_xu_t} \\
&+ && F_{u_x}F_{u_xu_x}F_{u_tu_t}F_{xu_x} + F_{u_x}F_{u_xu_xu_t}F_{u_t}F_{uu_x}u_x + F_{u_x}F_{u_xu_xu_t}F_{u_t}F_{uu_t}u_t \\
&- && F_{tuu_x}F_{u_xu_x}F_{u_tu_t}u_x + F_{u_xu_x}F_{u_t}F_{u_tu_t}F_{uu_x}u_t + F_uF_{u_xu_xu_t}F_{uu_t}u_t \\
&- && F_{u_xu_x}F_{u_tu_t}F_{uuu_x}u_xu_t - F_{u_xu_x}F_{u_tu_t}F_{xtu_x} - F_{u_xu_x}F_{u_tu_t}F_{xuu_x}u_t \\
&+ && F_{u_xu_x}F_{uu_t}F_{xu_xu_t}u_t - F_{u_xu_xu_t}F_{uu_x}F_{uu_t}u_xu_t - F_{u_xu_xu_t}F_{uu_t}F_{xu_x}u_t \\
&+ && F_{u_tu_t}F_{uu_xu_x}F_{xu_x}u_t - F_{tu_t}F_{u_xu_x}F_{uu_t} - F^2_{u_x}F_{u_xu_xu_t}F^2_{u_t} - F_uF_{u_tu_t}F_{uu_xu_x}u_t \\
&- && F_{u_xu_x}F^2_{uu_t}u_t + F_{u_x}F_{u_xu_xu_t}F_{u_t}F_{xu_x} + F_{u_x}F_{u_xu_x}F_{u_tu_t}F_{uu_x}u_x + F_{u_xu_x}F_{u_tu_t}F_{uu}u_t \\
&+ && F_{u_xu_x}F_{uu_xu_t}F_{uu_t}u_xu_t - F^2_{u_xu_x}F_{u_tu_t}F_x + F_{u_tu_t}F_{uu_x}F_{uu_xu_x}u_xu_t), \\
J_{13} &= && \frac{F^7_{u_xu_xu_x}}{F^8_{u_xu_x}F^4_{u_xu_xu_t}}(-F^2F_{u_xu_x}F_{u_xu_t}F_{u_tu_tu_t} + F^2F_{u_xu_x}F_{u_xu_tu_t}F_{u_tu_t} - F^2F_{u_xu_xu_t}F_{u_xu_t}F_{u_tu_t} \\
&+ && F^2F^2_{u_xu_t}F_{u_xu_tu_t} - FF_{tu_xu_t}F_{u_xu_t}F_{u_tu_t} - FF_{tu_t}F_{u_xu_x}F_{u_tu_tu_t} + FF_{tu_t}F_{u_xu_t}F_{u_xu_tu_t} \\
&+ && F^2_uF_{u_xu_tu_t} + FF_uF_{u_xu_x}F_{u_tu_tu_t} + FF_{tu_tu_t}F_{u_xu_x}F_{u_tu_t} + FF_uF_{u_xu_xu_t}F_{u_tu_t}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad F F_{u_x} F_{u_x u_x} F_{u_t} F_{u_t u_t u_t} + F F_{u_x} F_{u_x u_x u_t} F_{u_t} F_{u_t u_t} - 2F F_{u_x} F_{u_x u_t} F_{u_x u_t u_t} F_{u_t} \\
&- \quad 2F F_u F_{u_x u_t} F_{u_x u_t u_t} - F F_{u_x u_x} F_{u_x u_t} F_{u u_t u_t} u_x - F F_{u_x u_x} F_{u_x u_t} F_{x u_t u_t} \\
&+ \quad F F_{u_x u_x} F_{u_t u_t} F_{u u_x u_t} u_x + F F_{u_x u_x} F_{u_t u_t} F_{u u_t u_t} u_t + F F_{u_x u_x} F_{u_t u_t} F_{x u_x u_t} \\
&- \quad F F_{u_x u_x} F_{u_t u_t u_t} F_{u u_t} u_t - F F_{u_x u_x u_t} F_{u_t u_t} F_{u u_x} u_x - F F_{u_x u_x u_t} F_{u_t u_t} F_{x u_x} \\
&+ \quad F F_{u_x u_t} F_{u_x u_t u_t} F_{u u_x} u_x + F F_{u_x u_t} F_{u_x u_t u_t} F_{u u_t} u_t + F_u F_{u_x u_x} F_{u_x u_t} F_{u_t u_t} u_x \\
&+ \quad F F_{u_x u_t} F_{u_t u_t} F_{u u_x} - F F_{u_x u_t} F_{u_t u_t} F_{u u_x u_t} u_t + F_u F_{u_x u_x} F_{x u_t u_t} \\
&+ \quad F_t F_{u_x u_x} F^2_{u_t u_t} + F_{t u_x u_t} F_u F_{u_t u_t} + F_{t u_x u_t} F_{u_x} F_{u_t} F_{u_t u_t} - F_{t u_x u_t} F_{u_t u_t} F_{u u_x} u_x \\
&- \quad F_u F_{u_x u_t u_t} F_{u u_t} u_t - F_{t u_t} F_u F_{u_x u_t u_t} - F_{t u_t} F_{u_x} F_{u_x u_t u_t} F_{u_t} + F_{t u_t} F_{u_x u_t u_t} F_{x u_x} \\
&- \quad F_{t u_t} F_{u_x u_x} F_{u_t} F_{u_t u_t} - F_{t u_t} F_{u_x u_x} F_{u u_t u_t} u_x - F_{t u_t} F_{u_x u_x} F_{x u_t u_t} \\
&+ \quad F F_{u_x u_t} F_{u_x u_t u_t} F_{x u_x} + F_{t u u_t} F_{u_x u_x} F_{u_t u_t} u_x + 2F_u F_{u_x} F_{u_x u_t u_t} F_{u_t} - F_{t u_x u_t} F_{u_t u_t} F_{x u_x} \\
&+ \quad F_{t u_t} F_{u_x u_t u_t} F_{u u_x} u_x + F_u F_{u_x u_x} F^2_{u_t u_t} u_t + F_u F_{u_x u_x} F_{u u_t u_t} u_x - F_u F_{u_x u_t u_t} F_{u u_x} u_x \\
&- \quad F_u F_{u_x u_t u_t} F_{x u_x} - F_u F_{u_t u_t} F_{u u_x} + F_u F_{u_t u_t} F_{u u_x u_t} u_t + F^2_{u_x} F_{u_x u_t u_t} F^2_{u_t} \\
&+ \quad F_{u_x} F_{u_x u_x} F_{u_t} F_{u u_t u_t} u_x + F_{u_x} F_{u_x u_x} F_{u_t} F_{x u_t u_t} - F_{u_x} F_{u_x u_x} F_{u_t u_t} F_{u u_t} u_x \\
&- \quad F_{u_x} F_{u_x u_x} F_{u_t u_t} F_{x u_t} - F_{u_x} F_{u_x u_t u_t} F_{u_t} F_{u u_x} u_x - F_{u_x} F_{u_x u_t u_t} F_{u_t} F_{u u_t} u_t \\
&- \quad F_{u_x} F_{u_x u_t u_t} F_{u_t} F_{x u_x} + F_{u_x} F_{u_t} F_{u_t u_t} F_{u u_x u_t} u_t + F_{u_x u_x} F_{u_t u_t} F_{u u u_t} u_x u_t \\
&+ \quad F_{u_x u_x} F_{u_x u_t} F_{u_t u_t} F_x - F_{u_x} F_{u_t} F_{u_t u_t} F_{u u_x} - F_{u_x u_x} F_{u_t u_t} F_{u u} u_x - F_{u_x u_x} F_{u_t} F_{u_t u_t} F_{u u_t} u_t \\
&+ \quad F_{u_t u_t} F^2_{u u_x} u_x + F_{u_x u_x} F_{u_t u_t} F_{x t u_t} - F_{u_x u_x} F_{u_t u_t} F_{x u} + F_{u_x u_x} F_{u_t u_t} F_{x u u_t} u_t \\
&- \quad F_{u_x u_x} F_{u u_t} F_{u u_t u_t} u_x u_t + F_{u_t u_t} F_{u u_x} F_{x u_x} + F_{u_x u_t u_t} F_{u u_x} F_{u u_t} u_x u_t + F_{u_x u_t u_t} F_{u u_t} F_{x u_x} u_t \\
&- \quad F_{u_t u_t} F_{u u_x} F_{u u_x u_t} u_x u_t - F_{u_x u_x} F_{u u_t} F_{x u_t u_t} u_t - F_{u_t u_t} F_{u u_x u_t} F_{x u_x} u_t). \qquad (6.5)
\end{aligned}
$$

6.2.3 Linearization

Now, we classify all equations of class (6.1) that can be mapped into the linear wave equation

$$u_{tx} = 0. \qquad (6.6)$$

Since the most general linear equation

$$u_{tx} + a(t,x)u_t + b(t,x)u_x + c(t,x)u = 0$$

satisfies the invariant equations, we conclude that these invariant equations provide necessary conditions for linearization. The solution of the system of invariant equations (6.4) gives us the form of equation of the class (6.1) that can be mapped into linear equation. Therefore an equation of the class (6.1) that can be linearized by local mapping must be also a member of the class of hyperbolic equations. Now, the solution of the system (6.4) gives

$$F = f(x,t,u)u_x u_t + g(x,t,u)u_x + h(x,t,u)u_t + l(x,t,u), \tag{6.7}$$

where $f,\ g,\ h,\ l$ are arbitrary functions.

Hence, an equation of the class (6.1) is linked with (6.6) only if F is of the form (6.7). Therefore, an equation of the class (6.1) is mapped into (6.6) only if it is of the form

$$u_{xt} = f(x,t,u)u_x u_t + g(x,t,u)u_x + h(x,t,u)u_t + l(x,t,u). \tag{6.8}$$

Now, our goal is to find the differential invariants for the family of equations (6.8).

6.3 Invariants for equation (6.8)

6.3.1 Equivalence transformations

We employ the same procedure used in the previous section, to derive equivalence transformations and then the differential invariants.

We use Lie's infinitesimal method for calculating the equivalence transformations of the class of equations (6.8). We find that equation (6.8) admits an infinite continuous group of equivalence transformations generated by the Lie algebra spanned by the operators:

$$\begin{aligned}
\Gamma_\tau &= \tau\frac{\partial}{\partial t} - \tau_t(g\frac{\partial}{\partial g} + l\frac{\partial}{\partial l}),\\
\Gamma_\varphi &= \varphi\frac{\partial}{\partial x} - \varphi_x(h\frac{\partial}{\partial h} + l\frac{\partial}{\partial l}),\\
\Gamma_\psi &= \psi\frac{\partial}{\partial u} + (\psi_{uu} - \psi_u f)\frac{\partial}{\partial f} + (\psi_{tu} - \psi_t f)\frac{\partial}{\partial g} + (\psi_{xu} - \psi_x f)\frac{\partial}{\partial h}\\
&+ (\psi_{tx} - \psi_t h - \psi_x g + \psi_u l)\frac{\partial}{\partial l},
\end{aligned} \tag{6.9}$$

where $\tau = \tau(t),\ \varphi = \varphi(x),\ \psi = \psi(t,x,u)$ are arbitrary functions. Also, equivalence transformations can be written in the finite form (6.3).

6.3.2 Differential invariants and invariant equations

Using the operators (6.9) and their suitable prolongations, we find that (6.8) do not admit invariants of zero and first order.

However, the expressions

$$f_t - g_u = 0, \quad f_x - h_u = 0 \tag{6.10}$$

are invariant equations of first order. Hence, we have shown that

$$\Gamma^{(1)}[f_t - g_u]|_{(f_t - g_u = 0)} = 0, \quad \text{and} \quad \Gamma^{(1)}[f_x - h_u]|_{(f_x - h_u = 0)} = 0.$$

We also point out that

$$J = \frac{f_t - g_u}{f_x - h_u}$$

is a semi-invariant of first order. That is, it is invariant only under Γ_ψ, $\Gamma_\psi^{(1)}(J) = 0$.

6.3.3 Linearization

Now, any equation of the linear class of hyperbolic equations

$$u_{tx} + a(t,x)u_t + b(t,x)u_x + c(t,x)u = 0$$

satisfies invariant equations (6.10). Therefore equations (6.10) provide necessary conditions for linearization. Solving the system (6.10) we obtain the following forms for functions f, g, h:

$$f = m_u, \quad g = m_t + \alpha(x,t), \quad h = m_x + \beta(x,t), \quad m = m(x,t,u).$$

For the sake of simplicity we take $\alpha = \beta = 0$ and hence (6.8) takes the form

$$u_{xt} = m_u(x,t,u)u_x u_t + m_t u_x + m_x u_t + k(x,t,u). \tag{6.11}$$

Next step is to study the class (6.11).

6.4 Invariants for equation (6.11)

6.4.1 Equivalence transformations

We use Lie infinitesimal method for calculating the equivalence transformations of the class of equations (6.11).

We find that, equations (6.11) admits an infinite continuous group of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$ spanned by the operators:

$$\begin{aligned}
\Gamma_\tau &= \tau\frac{\partial}{\partial t} - \tau_t k\frac{\partial}{\partial k} - \tau_t m_t\frac{\partial}{\partial m_t}, \\
\Gamma_\varphi &= \varphi\frac{\partial}{\partial x} - \varphi_x k\frac{\partial}{\partial k} - \phi_x m_x\frac{\partial}{\partial m_x}, \\
\Gamma_\psi &= \psi\frac{\partial}{\partial u} + \psi_u\frac{\partial}{\partial m} + (\psi_{tx} - \psi_t m_x - \psi_x m_t + \psi_u k)\frac{\partial}{\partial k} \\
&- (\psi_x m_u - \psi_{xu})\frac{\partial}{\partial m_x} - (\psi_t m_u - \psi_{tu})\frac{\partial}{\partial m_t} - (\psi_u m_u - \psi_{uu})\frac{\partial}{\partial m_u},
\end{aligned} \tag{6.12}$$

where $\tau = \tau(t),\ \varphi = \varphi(x),\ \psi = \psi(t, x, u)$ are arbitrary functions. Now using the results of chapter 3, equivalence transformations (6.12), can be written in the finite form (6.3).

6.4.2 Differential invariants and invariant equations

We use equivalence transformations (6.12) to derive differential invariants for the class (6.11). We find no invariants of zero and first order. However, we derive the invariant equation

$$m_t m_x - m_u k - m_{tx} + k_u = 0. \tag{6.13}$$

Further calculations produce the following invariant of second order:

$$J = \frac{e^m(m_t m_x - m_u k - m_{tx} + k_u)_u}{m_t m_x - m_u k - m_{tx} + k_u}. \tag{6.14}$$

That is, we have shown that J is such that:

$$\Gamma_\tau^{(2)}(J) = 0,\quad \Gamma_\phi^{(2)}(J) = 0,\quad \Gamma_\psi^{(2)}(J) = 0.$$

6.4.3 Linearization

Now, we use invariant equation (6.13) and the differential invariant (6.14) to classify hyperbolic equations that can be linearized, i.e. we can list equations that can be mapped into a linear hyperbolic equation. We note that the linear wave equation

$$u_{tx} = 0$$

satisfies invariant equation (6.13), while any other linear hyperbolic equation substituted in (6.14) gives

$$J = 0.$$

Therefore the expression

$$(m_t m_x - m_u k - m_{tx} + k_u)_u = 0$$

provides a necessary condition for linearization. Solving for k, we find:

$$k = e^m \int (m_{tx} - m_t m_x + \varphi(t,x))\, e^{-m} du + \psi(t,x) e^m, \tag{6.15}$$

where $\varphi(t,x)$ and $\psi(t,x)$ are arbitrary functions. We also point out that using invariant equation (6.13), which is a necessary condition for mapping a hyperbolic equation into

$$u_{tx} = 0,$$

we obtain (6.15) with $\varphi(t,x) = 0$. Expression (6.15) implies that equation (6.11) takes the form

$$u_{tx} = m_u u_t u_x + m_t u_x + m_x u_t + e^m \int (m_{tx} - m_t m_x + \varphi(t,x))\, e^{-m} du + \psi(t,x) e^m. \tag{6.16}$$

Therefore, an equation of the form (6.11) is linked with the linear hyperbolic equation

$$u_{tx} = 0$$

only if is of the form (6.16).

6.5 Applications

In this section, we turn into the problem of finding point transformations of the form (6.3) that map (6.16) into the linear hyperbolic equations

$$u'_{x't'} = a(x',t') u'_{x'} + b(x',t') u'_{t'} + c(x',t') u'. \tag{6.17}$$

Details of how such transformations are constructed can be found in [29]. Using the results of chapter 3, we find that equations (6.16) and (6.17) are connected by the local mapping

$$x' = P(x), \quad t' = Q(t), \quad u' = \gamma(x,t) \int \mathrm{e}^{-m} \mathrm{d}u + \delta(x,t), \tag{6.18}$$

providing the following identities are satisfied:

$$\gamma_t - \gamma a Q_t = 0, \tag{6.19}$$

$$\gamma_x - \gamma b P_x = 0, \tag{6.20}$$

$$\gamma_{xt} - (2ab + c)\gamma P_x Q_t + \gamma\varphi = 0, \tag{6.21}$$

$$\delta_{xt} - a\delta_x Q_t - b\delta_t P_x - c\delta P_x Q_t + \gamma\psi = 0. \tag{6.22}$$

We note from identities (6.19) and (6.20) that $a_{x'} = b_{t'}$. This relation restricts the form of linear hyperbolic equation (6.17). This restriction can be eliminated if in the construction of (6.11) the functions $\alpha(x,t)$ and $\beta(x,t)$ do not vanish. Such case is example 6.4 given below.

Motivated by the applications of Laplace invariants, we use the above results to classify those hyperbolic equations that can be mapped into simple linear equations. We use equations (6.16) - (6.22) to construct the following examples:

Example 6.1. An equation of the class (6.1) can be mapped into the linear equation

$$u_{tx} = 0 \tag{6.23}$$

by the mapping

$$u' = \gamma \int \mathrm{e}^{-m} \mathrm{d}u + \delta(x,t), \quad \gamma = constant$$

if and only if it is of the form (6.16) with $\varphi = 0$ and

$$\delta_{xt} + \gamma\psi = 0.$$

Example 6.2. An equation of the class (6.1) can be mapped into the linear equation

$$u'_{x't'} = c(x',t')u'$$

by the mapping

$$u' = \gamma \int \mathrm{e}^{-m} \mathrm{d}u + \delta(x,t), \quad \gamma = constant$$

if and only if it is of the form (6.16) with

$$\varphi(x,t) = c(x,t), \quad \delta_{xt} - c(x,t)\delta + \psi(x,t)\gamma = 0.$$

Example 6.3. An equation of the class (6.1) can be mapped into the factorized equation

$$u'_{x't'} = a_{t'}(t')u'_{x'} + b_{x'}(x')u'_{t'} + a_{t'}b_{x'}u'$$

by the mapping

$$x' = P(x), \quad t' = Q(t), \quad u' = c\mathrm{e}^{a(Q(t))+b(P(x))}\int \mathrm{e}^{-m}\mathrm{d}u + \delta(x,t)$$

if and only if it is of the form (6.16) where $\varphi(x,t) = 2a_{t'}(Q(t))b_{x'}(P(x))P_xQ_t$ and

$$\delta_{xt} - a_{t'}Q_t\delta_x - b_{x'}P_x\delta_t - a_{t'}b_{x'}\delta P_xQ_t + c\mathrm{e}^{a+b}\psi(x,t) = 0.$$

In the following example we use equation (6.11) with $\alpha \neq 0$. In this case, equation (6.11) takes the form

$$u_{xt} = m_u(x,t,u)u_xu_t + (m_t + \alpha(x,t))u_x + m_xu_t + k(x,t,u).$$

Example 6.4. We consider the first Lie canonical equation (see [34])

$$u'_{x't'} = \alpha(x')u'_{x'} - u'. \tag{6.24}$$

It can be shown that an equation of the class (6.1) can be mapped into (6.24) if and only if it is of the form

$$u_{xt} = m_uu_xu_t + (m_t + \alpha(x))\,u_x + m_xu_t + \mathrm{e}^m\int(m_{xt} - m_xm_t - \alpha m_x - 1)\mathrm{e}^{-m}\mathrm{d}u + \psi(x,t)\mathrm{e}^m$$

by the point transformation

$$x' = x, \quad t' = t, \quad u' = c\int \mathrm{e}^{-m}\mathrm{d}u + \delta(x,t),$$

where

$$\delta_{xt} - \alpha\delta_x + \delta + c\psi = 0.$$

Similar results can be obtained for the other three Lie canonical equations.

6.6 Further applications

In this section we employ differential invariants to derive local mappings that connect equations of the class (6.1) to known equations. We present two examples.

Example 6.5. We consider the Liouville equation (see [37])

$$u_{xt} = \mathrm{e}^u.$$

Its general solution is:

$$u(x,t) = f(x) + g(t) - 2\ln\left|a\int \mathrm{e}^{f(x)}\mathrm{d}x + \tfrac{1}{2k}\int \mathrm{e}^{g(t)}\mathrm{d}t\right|,$$

where $f(x)$ and $g(t)$ are arbitrary functions and a is an arbitrary constant. This general solution can be found using the Bäcklund transformations that connect Liouville equation and the linear wave equation (6.23). Since Liouville equation satisfies invariant equations (6.4) and (6.10), we deduce that any equation of the class (6.1) that can be linked to it, has to be of the form (6.11). Therefore in sections 6.4.1-6.4.2, if we set $m = constant$, $k = \mathrm{e}^u$ in (6.14), we find that $J = \lambda$, where λ is an arbitrary constant. Hence, the expression

$$\frac{e^m(m_t m_x - m_u k - m_{tx} + k_u)_u}{m_t m_x - m_u k - m_{tx} + k_u} = \lambda$$

provides necessary conditions for an equation of the class (6.11), and consequently of the class (6.1), to be connected to Liouville equation. Solving the above expression for η, we find:

$$k(x,t,u) = \mathrm{e}^m \int \left(m_{xt} - m_x m_t + \varphi(x,t)\mathrm{e}^{\lambda\int \mathrm{e}^{-m}\mathrm{d}u}\right)\mathrm{e}^{-m}\mathrm{d}u + \psi(x,t)\mathrm{e}^m.$$

Finally, using point transformation of the form (6.3), we conclude that an equation of the class (6.1) can be connected with Liouville equation if and only if it is of the form

$$\begin{aligned} u_{xt} &= m_u(x,t,u)u_x u_t + m_t u_x + m_x u_t + \mathrm{e}^m \int \left(m_{xt} - m_x m_t + \varphi(x,t)\mathrm{e}^{\lambda\int \mathrm{e}^{-m}\mathrm{d}u}\right)\mathrm{e}^{-m}\mathrm{d}u \\ &+ \psi(x,t)\mathrm{e}^m. \end{aligned}$$

The above equation and Liouville equation are connected by the local mapping

$$x' = P(x), \quad t' = Q(t), \quad u' = c\int \mathrm{e}^{-m}\mathrm{d}u + \delta(x,t),$$

where

$$\delta_{xt} + c\psi = 0, \quad \mathrm{e}^\delta Q_t P_x + \varphi = 0.$$

Example 6.6. In this example we consider the Goursat equation

$$u_{xt} = \sqrt{u_x u_t}.$$

Using nonlocal mappings, this equation is connected with the linear hyperbolic equation $u_{xt} = \frac{1}{4}u$ (see [10]). In order to derive equations of the class (6.1) that are linked with Goursat equation, we need to use the differential invariants of third order (6.5).

Setting $F = \sqrt{u_x u_t}$, in the forms of the 13 differential invariants, we find

$$J_1 = 9, \quad J_2 = -3, \quad J_3 = -3, \quad J_4 = -3, \quad J_5 = -27, \quad J_6 = 0, \quad J_7 = 0, \quad J_8 = 0,$$

$$J_9 = 0, \quad J_{10} = 0, \quad J_{11} = 0, \quad J_{12} = -729, \quad J_{13} = -2187.$$

We note that all differential invariants are constants. Solving the above 13 equations, where $J_1, \ldots, J_{13}$ are the invariants (6.5), we find necessary conditions for an equation of the class (6.1) to be connected with Goursat equation. That is, any equation that can be mapped into Goursat equation must satisfy the above 13 equations. Clearly, to solve the above system with 13 equations is a very difficult task. We give one simple example. In the case where $F = \frac{2u_t u_x}{u} - \frac{\sqrt{u_t u_x}}{xt}$, it can be shown that all 13 equations are satisfied. Furthermore, we state that the reciprocal transformation (twice application of it gives the identity transformation)

$$t' = \frac{1}{t}, \quad x' = \frac{1}{x}, \quad u' = \frac{1}{u}$$

maps

$$u_{xt} = \frac{2u_t u_x}{u} - \frac{\sqrt{u_t u_x}}{xt}$$

into Goursat equation (with primed variables). Since Goursat equation can be linearized by the nonlocal mapping, the above hyperbolic equation can be mapped into a linear hyperbolic equation using nonlocal mapping.

The results of this chapter are contained in [58].

6.7 Conclusion

In this chapter our main goal was to classify hyperbolic equations of the class (6.1) that can be transformed into linear equations by local mappings. To achieve this, we used its differential invariants. We applied an infinitesimal technique developed by Ibragimov (see [15, 16, 23]) and we point out that the class of equations (6.1) has no differential invariants up to order two, inclusive. The knowledge of semi-invariants was useful for the linearization of equation (6.1). Also, we calculated equivalence transformations and differential invariants using Lie's infinitesimal method for two subclasses of it.

Motivated by the applications of Laplace invariants, we classify those hyperbolic equations (6.1) that can be mapped into simple linear equations. Furthermore, in the last section, we presented two examples in which the knowledge of differential invariants can be used to derive local mappings that connect equations of the class (6.1) to known equations.

Chapter 7

Differential invariants for n-dimensional hyperbolic equations

7.1 Introduction

In this chapter we consider n-dimensional hyperbolic equations. In the spirit of Ibragimov's work (see [17]), we construct differential invariants with the employment of the derived equivalence transformations for the cases $n = 2$ and $n = 3$. Motivated by these results, we present the corresponding results for the n-dimensional case of hyperbolic equations. For the case $n = 2$ we obtain one invariant of first order, while for the case $n = 3$ we find two invariants. We present the corresponding results for the one-dimensional equation. Finally, we employ the derived invariants to get certain mappings that connect equivalent equations.

7.2 Invariants for two-dimensional hyperbolic equations

7.2.1 Equivalence transformations

Firstly, we consider the two-dimensional linear hyperbolic equations of the form:

$$u_{tt} = u_{xx} + u_{yy} + X(t,x,y)u_x + Y(t,x,y)u_y + T(t,x,y)u_t + U(t,x,y)u. \tag{7.1}$$

We employ the same procedure used in the previous chapter, to derive equivalence transformations and then differential invariants for the class (7.1).

Using the Lie infinitesimal method for calculating the equivalence transformations of the class of equation (7.1), we find that equation (7.1) admit an infinite continuous group $\mathcal{E}$ of equivalence transformations generated by Lie algebra $L_{\mathcal{E}}$ spanned by operators:

$$
\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + T\frac{\partial}{\partial X} + X\frac{\partial}{\partial T}, \\
\Gamma_5 &= y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y} + T\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial T}, \quad \Gamma_6 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + Y\frac{\partial}{\partial X} - X\frac{\partial}{\partial Y}, \\
\Gamma_7 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y} - T\frac{\partial}{\partial T} - 2U\frac{\partial}{\partial U}, \\
\Gamma_8 &= \tfrac{1}{2}\left(t^2 + x^2 + y^2\right)\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + (xT - tX)\frac{\partial}{\partial X} + (yT - tY)\frac{\partial}{\partial Y} \\
&+ (xX + yY - tT + 1)\frac{\partial}{\partial T} - 2tU\frac{\partial}{\partial U}, \\
\Gamma_9 &= xt\frac{\partial}{\partial t} + \tfrac{1}{2}\left(t^2 + x^2 - y^2\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} - (xX + yY - tT + 1)\frac{\partial}{\partial X} \\
&+ (yX - xY)\frac{\partial}{\partial Y} + (tX - xT)\frac{\partial}{\partial T} - 2xU\frac{\partial}{\partial U}, \\
\Gamma_{10} &= ty\frac{\partial}{\partial t} + xy\frac{\partial}{\partial x} + \tfrac{1}{2}\left(t^2 - x^2 + y^2\right)\frac{\partial}{\partial y} + (xY - yX)\frac{\partial}{\partial X} \\
&- (xX + yY - tT + 1)\frac{\partial}{\partial Y} + (tY - yT)\frac{\partial}{\partial T} - 2yU\frac{\partial}{\partial U}, \\
\Gamma_\alpha &= \alpha u\frac{\partial}{\partial u} - 2\alpha_x\frac{\partial}{\partial X} - 2\alpha_y\frac{\partial}{\partial Y} + 2\alpha_t\frac{\partial}{\partial T} \\
&+ (\alpha_{tt} - \alpha_{xx} - \alpha_{yy} - \alpha_x X - \alpha_y Y - \alpha_t T)\frac{\partial}{\partial U},
\end{aligned}
\tag{7.2}
$$

where $\alpha = \alpha(t, x, y)$ is an arbitrary function.

7.2.2 Differential invariants and invariant equations

We consider the problem of finding differential invariants of the class of equations (7.1).

Firstly, we seek for differential invariants of zero order. That is, invariants of the form:

$$J = J(t, x, y, u, X, Y, T, U).$$

Using the operators (7.2), the invariant criterion $\Gamma(J) = 0$ gives the following identities:

$$\Gamma_i(J) = 0, \quad i = 1, 2, \ldots, 10, \alpha.$$

It is straightforward that $J = constant$. Hence, the family of equations (7.1) does not admit differential invariants of zero order.

Now, the next step is to consider the problem of existence of differential invariants of first order, that depend on the first derivatives of the functions X, Y, T, U. That is, for invariants of the form

$$J = J(t, x, y, u, X, Y, T, U, X_i, Y_i, T_i, U_i), \quad i = t, x, y.$$

In order to achieve that, we must calculate the first prolongations of the operators. For more details of how the operators $\Gamma_i,\ i = 1, 2, \ldots, 10, \alpha$ can be extended, we refer to the previous chapter or to [16, 44].

First, we consider the problem of calculating semi-invariants of first order. In this case J only satisfies the invariant criterion

$$\Gamma_{\alpha}^{(1)}(J) = 0. \tag{7.3}$$

That is, using the results of chapter 3, $P = x$ and $Q = t$ which means that (7.1) is invariant only under the transformation of the dependent variable.

Equation (7.3) is a polynomial in the derivatives of $\alpha(t, x, y)$. Using the fact that $\alpha(t, x, y)$ is arbitrary, we set the coefficients of the derivatives of it equal to zero. This leads to a system of linear first order partial differential equations. First, we note that $\Gamma_i^{(1)} = \Gamma_i,\ i = 1, 2, 3$ and therefore $\Gamma_i^{(1)}(J) = 0$ implies that J is independent of $t,\ x,\ u$.

Furthermore, the coefficients of $\alpha,\ \alpha_{xxy},\ \alpha_{xxt},\ \alpha_{xyy}$ in (7.3) give

$$\frac{\partial J}{\partial u} = \frac{\partial J}{\partial U_y} = \frac{\partial J}{\partial U_t} = \frac{\partial J}{\partial U_x} = 0.$$

Hence,

$$J = J(X, Y, T, U, X_t, X_x, X_y, Y_t, Y_x, Y_t, T_t, T_x, T_y).$$

Now, coefficients of $\alpha_x,\ \alpha_y,\ \alpha_t,\ \alpha_{xx},\ \alpha_{xy},\ \alpha_{xt},\ \alpha_{yy},\ \alpha_{yt}$ and α_{tt} in (7.3) give:

$$\begin{aligned}
&2\frac{\partial J}{\partial X} + X\frac{\partial J}{\partial U} = 0, && 2\frac{\partial J}{\partial Y} + Y\frac{\partial J}{\partial U} = 0, && 2\frac{\partial J}{\partial T} - T\frac{\partial J}{\partial U} = 0,\\
&2\frac{\partial J}{\partial X_x} + \frac{\partial J}{\partial U} = 0, && \frac{\partial J}{\partial X_y} + \frac{\partial J}{\partial Y_x} = 0, && \frac{\partial J}{\partial X_t} - \frac{\partial J}{\partial T_x} = 0,\\
&2\frac{\partial J}{\partial Y_y} + \frac{\partial J}{\partial U} = 0, && \frac{\partial J}{\partial Y_t} - \frac{\partial J}{\partial T_y} = 0, && 2\frac{\partial J}{\partial T_t} + \frac{\partial J}{\partial U} = 0.
\end{aligned}$$

Solving this system we obtain four independent integrals which form the set of semi-invariants of first order for the class of equations (7.1):

$$J_1 = Y_x - X_y,\ \ J_2 = X_t + T_x,\ \ J_3 = Y_t + T_y,\ \ J_4 = X^2 + Y^2 - T^2 + 2X_x + 2Y_y + 2T_t - 4U. \tag{7.4}$$

Now, in order to obtain a complete set of differential invariants, in addition to (7.3), we apply the invariance criterion to the other operators:

$$\Gamma_j^{(1)}(J) = 0, \quad j = 4, 5, \ldots, 10$$

and we obtain the following list of equations:

$$\begin{aligned}
\Gamma_4^{(1)}(J) = 0 \quad \Leftrightarrow \quad & T\frac{\partial J}{\partial X} - X_x\frac{\partial J}{\partial X_t} + T_t\frac{\partial J}{\partial X_t} - X_t\frac{\partial J}{\partial X_x} + T_x\frac{\partial J}{\partial X_x} + T_y\frac{\partial J}{\partial X_y} - Y_x\frac{\partial J}{\partial Y_t} \\
& -Y_t\frac{\partial J}{\partial Y_x} + X\frac{\partial J}{\partial T} + X_t\frac{\partial J}{\partial T_t} - T_x\frac{\partial J}{\partial T_t} + X_x\frac{\partial J}{\partial T_x} - T_t\frac{\partial J}{\partial T_x} + X_y\frac{\partial J}{\partial T_y} = 0, \\
\Gamma_5^{(1)}(J) = 0 \quad \Leftrightarrow \quad & -X_y\frac{\partial J}{\partial X_t} - X_t\frac{\partial J}{\partial X_y} + T\frac{\partial J}{\partial Y} - Y_y\frac{\partial J}{\partial Y_t} + T_t\frac{\partial J}{\partial Y_t} + T_x\frac{\partial J}{\partial Y_x} - Y_t\frac{\partial J}{\partial Y_y} \\
& +T_y\frac{\partial J}{\partial Y_y} + Y\frac{\partial J}{\partial T} + Y_t\frac{\partial J}{\partial T_t} - T_y\frac{\partial J}{\partial T_t} + Y_x\frac{\partial J}{\partial T_x} + Y_y\frac{\partial J}{\partial T_y} - T_t\frac{\partial J}{\partial T_y} = 0, \\
\Gamma_6^{(1)}(J) = 0 \quad \Leftrightarrow \quad & -Y\frac{\partial J}{\partial X} - Y_t\frac{\partial J}{\partial X_t} - X_y\frac{\partial J}{\partial X_x} - Y_x\frac{\partial J}{\partial X_x} + X_x\frac{\partial J}{\partial X_y} - Y_y\frac{\partial J}{\partial X_y} + X\frac{\partial J}{\partial Y} \\
& +X_t\frac{\partial J}{\partial Y_T} + X_x\frac{\partial J}{\partial Y_x} - Y_y\frac{\partial J}{\partial Y_x} + X_y\frac{\partial J}{\partial Y_y} + Y_x\frac{\partial J}{\partial Y_y} - T_y\frac{\partial J}{\partial T_x} + T_x\frac{\partial J}{\partial T_y} = 0, \\
\Gamma_7^{(1)}(J) = 0 \quad \Leftrightarrow \quad & -X\frac{\partial J}{\partial X} - 2X_t\frac{\partial J}{\partial X_t} - 2X_x\frac{\partial J}{\partial X_x} - 2X_y\frac{\partial J}{\partial X_y} - Y\frac{\partial J}{\partial Y} - 2Y_t\frac{\partial J}{\partial Y_t} - 2Y_x\frac{\partial J}{\partial Y_x} \\
& -2Y_y\frac{\partial J}{\partial Y_y} - T\frac{\partial J}{\partial T} - 2T_t\frac{\partial J}{\partial T_t} - 2T_x\frac{\partial J}{\partial T_x} - 2T_y\frac{\partial J}{\partial T_y} - 2U\frac{\partial J}{\partial U} = 0, \\
\Gamma_8^{(1)}(J) = 0 \quad \Leftrightarrow \quad & t\Gamma_7^{(1)}(J) + x\Gamma_4^{(1)}(J) + y\Gamma_5^{(1)}(J) - X\frac{\partial J}{\partial X_t} + T\frac{\partial J}{\partial X_x} - Y\frac{\partial J}{\partial Y_t} + T\frac{\partial J}{\partial Y_y} \\
& +\frac{\partial J}{\partial T} - T\frac{\partial J}{\partial T_t} + X\frac{\partial J}{\partial T_x} + Y\frac{\partial J}{\partial T_y} = 0, \\
\Gamma_9^{(1)}(J) = 0 \quad \Leftrightarrow \quad & t\Gamma_4^{(1)}(J) + x\Gamma_7^{(1)}(J) + y\Gamma_6^{(1)}(J) - \frac{\partial J}{\partial X} + T\frac{\partial J}{\partial X_t} - X\frac{\partial J}{\partial X_x} - Y\frac{\partial J}{\partial X_y} \\
& -Y\frac{\partial J}{\partial Y_x} + X\frac{\partial J}{\partial Y_y} + X\frac{\partial J}{\partial T_t} - T\frac{\partial J}{\partial T_x} = 0, \\
\Gamma_{10}^{(1)}(J) = 0 \quad \Leftrightarrow \quad & t\Gamma_5^{(1)}(J) - x\Gamma_6^{(1)}(J) + y\Gamma_7^{(1)}(J) + Y\frac{\partial J}{\partial X_x} - X\frac{\partial J}{\partial X_y} - \frac{\partial J}{\partial Y} + T\frac{\partial J}{\partial Y_t} \\
& -X\frac{\partial J}{\partial Y_x} - Y\frac{\partial J}{\partial Y_y} + Y\frac{\partial J}{\partial T_t} - T\frac{\partial J}{\partial T_Y} = 0.
\end{aligned}$$

Using the semi-invariants (7.4), the above equations take the form:

$$\Gamma_4^{(1)}(J) = 0 \quad \Leftrightarrow \quad J_1\frac{\partial J}{\partial J_3} + J_3\frac{\partial J}{\partial J_1} = 0, \tag{7.5}$$

$$\Gamma_5^{(1)}(J) = 0 \quad \Leftrightarrow \quad J_1\frac{\partial J}{\partial J_2} + J_2\frac{\partial J}{\partial J_1} = 0, \tag{7.6}$$

$$\Gamma_6^{(1)}(J)=0 \quad \Leftrightarrow \quad J_3\frac{\partial J}{\partial J_2}-J_2\frac{\partial J}{\partial J_3}=0, \tag{7.7}$$

$$\Gamma_7^{(1)}(J)=0 \quad \Leftrightarrow \quad 2\left(J_1\frac{\partial J}{\partial J_1}+J_2\frac{\partial J}{\partial J_2}+J_3\frac{\partial J}{\partial J_3}+J_4\frac{\partial J}{\partial J_4}\right)=0, \tag{7.8}$$

$$\Gamma_8^{(1)}(J)=0 \quad \Leftrightarrow \quad t\Gamma_7^{(1)}(J)+x\Gamma_4^{(1)}(J)+y\Gamma_5^{(1)}(J)=0, \tag{7.9}$$

$$\Gamma_9^{(1)}(J)=0 \quad \Leftrightarrow \quad t\Gamma_4^{(1)}(J)+x\Gamma_7^{(1)}(J)+y\Gamma_6^{(1)}(J)=0, \tag{7.10}$$

$$\Gamma_{10}^{(1)}(J)=0 \quad \Leftrightarrow \quad t\Gamma_5^{(1)}(J)-x\Gamma_6^{(1)}(J)+y\Gamma_7^{(1)}(J)=0. \tag{7.11}$$

Solving the system (7.5)-(7.8), we obtain

$$J=\frac{J_2^2+J_3^2-J_1^2}{J_4^2},$$

which also satisfies the remaining equations $\Gamma_j^{(1)}(J)=0,\ j=8,9,10$. Therefore, we have derived the differential invariant of first order

$$J=\frac{(X_t+T_x)^2+(Y_t+T_y)^2-(Y_x-X_y)^2}{(X^2+Y^2-T^2+2X_x+2Y_y+2T_t-4U)^2}. \tag{7.12}$$

Furthermore, we obtain the invariant system

$$X_t+T_x=0, \qquad Y_t+T_y=0, \qquad Y_x-X_y=0, \tag{7.13}$$

and the invariant equation

$$X^2+Y^2-T^2+2X_x+2Y_y+2T_t-4U=0. \tag{7.14}$$

That is,

$$\Gamma_j^{(1)}\left(X_t+T_x\right)\big|_{(7.13)}=0,\quad \Gamma_j^{(1)}\left(Y_t+T_y\right)\big|_{(7.13)}=0,\quad \Gamma_j^{(1)}\left(Y_x-X_y\right)\big|_{(7.13)}=0$$

and

$$\Gamma_j^{(1)}\left(X^2+Y^2-T^2+2X_x+2Y_y+2T_t-4U\right)\big|_{(7.14)}=0,$$

where $j=1,2,\ldots,10,\alpha$, respectively.

Now, in order to derive differential invariants of second order we need to consider the invariant criterion

$$\Gamma_j^{(2)}(J)=0, \qquad j=1,2,\ldots,10,\alpha,$$

where $\Gamma_j^{(2)}$ is the second order extension of Γ_j. Without presenting any calculations we state that we only re-obtained the differential invariant (7.12). That is, they do not exist differential invariants of second order.

7.3 Invariants for three-dimensional hyperbolic equations

7.3.1 Equivalence transformations

Using the same procedure used in the previous section, we calculate the equivalence transformations of the three-dimensional linear hyperbolic equations

$$\begin{aligned} u_{tt} &= u_{xx} + u_{yy} + u_{zz} + X(t,x,y,z)u_x + Y(t,x,y,z)u_y + Z(t,x,y,z)u_z \\ &+ T(t,x,y,z)u_t + U(t,x,y,z)u. \end{aligned} \tag{7.15}$$

We find that the family of equations (7.15) admits an infinite continuous group $\mathcal{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$ spanned by the operators:

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = \frac{\partial}{\partial z}, \\
\Gamma_5 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y} - T\frac{\partial}{\partial T} - Z\frac{\partial}{\partial Z} - 2U\frac{\partial}{\partial U}, \\
\Gamma_6 &= x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + T\frac{\partial}{\partial X} + X\frac{\partial}{\partial T}, \quad \Gamma_7 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y} + T\frac{\partial}{\partial Y} + Y\frac{\partial}{\partial T}, \\
\Gamma_8 &= z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z} + Z\frac{\partial}{\partial T} + T\frac{\partial}{\partial Z}, \quad \Gamma_9 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - Y\frac{\partial}{\partial X} + X\frac{\partial}{\partial Y}, \\
\Gamma_{10} &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + Z\frac{\partial}{\partial Y} - Y\frac{\partial}{\partial Z}, \quad \Gamma_{11} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + Z\frac{\partial}{\partial X} - X\frac{\partial}{\partial Z}, \\
\Gamma_{12} &= \frac{1}{2}\left(t^2 + x^2 + y^2 + z^2\right)\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + tz\frac{\partial}{\partial z} + (xT - tX)\frac{\partial}{\partial X} \\
&+ (yT - tY)\frac{\partial}{\partial Y} + (zT - tZ)\frac{\partial}{\partial Z} + (xX + yY - tT + zZ + 2)\frac{\partial}{\partial T} - 2tU\frac{\partial}{\partial U}, \\
\Gamma_{13} &= tx\frac{\partial}{\partial t} + \frac{1}{2}\left(t^2 + x^2 - y^2 - z^2\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z} \\
&- (xX + yY - tT + zZ + 2)\frac{\partial}{\partial X} + (yX - xY)\frac{\partial}{\partial Y} + (zX - xZ)\frac{\partial}{\partial Z} \\
&+ (tX - xT)\frac{\partial}{\partial T} - 2xU\frac{\partial}{\partial U}, \\
\Gamma_{14} &= ty\frac{\partial}{\partial t} + xy\frac{\partial}{\partial x} + \frac{1}{2}\left(t^2 - x^2 + y^2 - z^2\right)\frac{\partial}{\partial y} + yz\frac{\partial}{\partial z} \\
&+ (xY - yX)\frac{\partial}{\partial X} - (xX + yY - tT + zZ + 2)\frac{\partial}{\partial Y} + (zY - yZ)\frac{\partial}{\partial Z} \\
&+ (tY - yT)\frac{\partial}{\partial T} - 2yU\frac{\partial}{\partial U},
\end{aligned}$$

$$
\begin{aligned}
\Gamma_{15} &= tz\frac{\partial}{\partial t} + xz\frac{\partial}{\partial x} + yz\frac{\partial}{\partial y} + \frac{1}{2}(t^2 - x^2 - y^2 + z^2)\frac{\partial}{\partial z} + (xZ - zX)\frac{\partial}{\partial X} \\
&+ (yZ - zY)\frac{\partial}{\partial Y} + (tZ - zT)\frac{\partial}{\partial T} - (xX + yY - tT + zZ + 2)\frac{\partial}{\partial Z} - 2zU\frac{\partial}{\partial U}, \\
\Gamma_{\alpha} &= \alpha u\frac{\partial}{\partial u} - 2\alpha_x\frac{\partial}{\partial X} - 2\alpha_y\frac{\partial}{\partial Y} - 2\alpha_z\frac{\partial}{\partial Z} + 2\alpha_t\frac{\partial}{\partial T} \\
&+ (\alpha_{tt} - \alpha_{xx} - \alpha_{yy} - \alpha_{zz} - \alpha_x X - \alpha_y Y - \alpha_t T - \alpha_z Z)\frac{\partial}{\partial U},
\end{aligned}
$$

where $\alpha = \alpha(x, t, y, z)$.

7.3.2 Differential invariants and invariant equations

In order to find semi-invariants we have to apply operator $\Gamma_{\alpha}^{(1)}$ onto invariants of first order, i.e. of the form

$$J = J(t, x, y, z, u, X, Y, Z, T, U, X_i, Y_i, Z_i, T_i, U_i), \quad i = t, x, y, z.$$

The invariant criterion $\Gamma_{\alpha}^{(1)}(J) = 0$ leads to seven semi-invariants:

$$J_1 = Y_x - X_y, \quad J_2 = X_t + T_x, \quad J_3 = Y_t + T_y,$$

$$J_4 = Z_x - X_z, \quad J_5 = T_z + Z_t, \quad J_6 = Z_y - Y_z,$$

$$J_7 = X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U.$$

Now using the complete equivalence group we find that equations (7.15) admit two differential invariants of first order:

$$J_1 = \frac{(T_x + X_t)^2 + (T_y + Y_t)^2 + (T_z + Z_t)^2 - (Y_x - X_y)^2 - (Z_x - X_z)^2 - (Z_y - Y_z)^2}{(X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U)^2},$$

$$J_2 = \frac{(T_x + X_t)(Y_z - Z_y) - (T_y + Y_t)(X_z - Z_x) + (T_z + Z_t)(X_y - Y_x)}{(X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U)^2}.$$

In addition we have an invariant system with six equations:

$$X_t + T_x = 0, \quad T_z + Z_t = 0, \quad Y_t + T_y = 0,$$

$$Y_x - X_y = 0, \quad Z_x - X_z = 0, \quad Z_y - Y_z = 0,$$

and the invariant equation

$$X^2 + Y^2 + Z^2 - T^2 + 2X_x + 2Y_y + 2Z_z + 2T_t - 4U = 0.$$

We point out that, this results are more general from the two-dimensional equation, with the exception that the three-dimensional equation admits two differential invariants.

7.4 Invariants for n-dimensional hyperbolic equations

7.4.1 Equivalence transformations

In this section, we consider the n-dimensional ($n \geq 3$) linear hyperbolic equations of the form:

$$u_{tt} = \sum_{i=1}^{n} u_{x_i x_i} + \sum_{i=1}^{n} X_i(x_1, x_2, \ldots, x_n, t) u_{x_i} + T(x_1, x_2, \ldots, x_n, t) u_t + U(x_1, x_2, \ldots, x_n, t) u. \tag{7.16}$$

Motivated by the results of the previous sections, we can generalize them to n dimensions.

We state that equations (7.16) admit an infinite continuous group $\mathcal{E}$ of equivalence transformations generated by the Lie algebra $L_{\mathcal{E}}$ spanned by the operators:

$$\Gamma_{1_i} = \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, n, \qquad \Gamma_{1_{n+1}} = \frac{\partial}{\partial t},$$

$$\Gamma_2 = t\frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} X_i \frac{\partial}{\partial X_i} - T\frac{\partial}{\partial T} - 2U\frac{\partial}{\partial U},$$

$$\Gamma_{3_{ij}} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} + X_i \frac{\partial}{\partial X_j} - X_j \frac{\partial}{\partial X_i}, \quad i = 1, 2, \ldots, n-1, \quad j = i+1, \ldots n,$$

$$\Gamma_{4_i} = x_i \frac{\partial}{\partial t} + t\frac{\partial}{\partial x_i} + T\frac{\partial}{\partial X_i} + X_i \frac{\partial}{\partial T}, \quad i = 1, 2, \ldots, n,$$

$$\Gamma_{5_i} = x_i t\frac{\partial}{\partial t} + \frac{1}{2}\left(t^2 + x_i^2 - \sum_{j=1,\ j\neq i}^{n} x_j^2\right)\frac{\partial}{\partial x_i} + \sum_{j=1,\ j\neq i}^{n} x_i x_j \frac{\partial}{\partial x_j}$$

$$+ \sum_{j=1,\ j\neq i}^{n} (x_j X_i - x_i X_j)\frac{\partial}{\partial X_j} + (tX_i - x_i T)\frac{\partial}{\partial T} - \left(\sum_{j=1}^{n} x_j X_j - tT + n - 1\right)\frac{\partial}{\partial X_i}$$

$$-2x_i U\frac{\partial}{\partial U}, \quad i = 1, 2, \ldots, n,$$

$$\Gamma_{5_{n+1}} = \frac{1}{2}\left(\sum_{i=1}^{n} x_i^2 + t\right)\frac{\partial}{\partial t} + \sum_{i=1}^{n} t x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n}(x_i T - tX_i)\frac{\partial}{\partial X_i}$$

$$+ \left(\sum_{i=1}^{n} x_i X_i - tT + n - 1\right)\frac{\partial}{\partial T} - 2tU\frac{\partial}{\partial U},$$

$$\Gamma_\alpha = \alpha u\frac{\partial}{\partial u} - 2\sum_{i=1}^{n} \alpha_{x_i}\frac{\partial}{\partial X_i} + 2\alpha_t \frac{\partial}{\partial T} + \left(\alpha_{tt} - \sum_{i=1}^{n} \alpha_{x_i x_i} - \sum_{i=1}^{n} \alpha_{x_i} X_i - \alpha_t T\right)\frac{\partial}{\partial U},$$

where $\alpha = \alpha(t, x_1, x_2, \ldots, x_n)$ is an arbitrary function.

7.4.2 Differential invariants and invariant equations

Motivated by the results about differential invariants, we have that equations (7.16) admit the one differential invariant of first order, namely,

$$J = \frac{\sum_{i=1}^{n}(T_{x_i} + X_{i_t})^2 - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(X_{i_{x_j}} - X_{j_{x_i}})^2}{(\sum_{i=1}^{n} X_i^2 - T^2 + 2\sum_{i=1}^{n} X_{i_{x_i}} + 2T_t - 4U)^2}.$$

We note that for the case where $n = 3$, two differential invariants exist.

The invariant criterion $\Gamma_\alpha^{(1)}(J) = 0$, leads to $\frac{1}{2}n(n+1) + 1$ semi-invariants:

$$J_i = T_{x_i} + X_{i_t}, \quad i = 1, 2, \dots n,$$

$$J_{ij} = X_{i_{x_j}} - X_{j_{x_i}}, \quad i = 1, 2, \dots n-1, \ j = i+1, \dots n,$$

$$J_{\frac{1}{2}n(n+1)+1} = \sum_{i=1}^{n} X_i^2 - T^2 + 2\sum_{i=1}^{n} X_{i_{x_i}} + 2T_t - 4U.$$

Furthermore, we point out that the $\frac{1}{2}n(n+1)$ equations

$$T_{x_i} + X_{i_t} = 0, \quad i = 1, 2, \dots n,$$

$$X_{i_{x_j}} - X_{j_{x_i}} = 0, \quad i = 1, 2, \dots n-1, \ j = i+1, \dots n,$$

form an invariant system and

$$\sum_{i=1}^{n} X_i^2 - T^2 + 2\sum_{i=1}^{n} X_{i_{x_i}} + 2T_t - 4U = 0$$

is an invariant equation. Semi-invariants, invariant system and invariant equation generalize naturally with no exceptions.

7.5 Invariants for one-dimensional hyperbolic equations

7.5.1 Equivalence transformations

In this section we consider the one-dimensional linear hyperbolic equation of the form:

$$u_{tt} = u_{xx} + X(t,x)u_x + T(t,x)u_t + U(t,x)u. \tag{7.17}$$

From the elementary study of partial differential equations, it is known that canonical variables connect the linear hyperbolic equations

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0 \tag{7.18}$$

and (7.17). Therefore the results of (7.18) (see [19, 20]) can be mapped into results of (7.17) using canonical variables. In fact, this procedure was carried out in [17]. For completeness, we present the results for the one-dimensional linear hyperbolic (7.17).

We find that the family of equations (7.17) has an infinite equivalence group $\mathcal{E}$. The corresponding Lie algebra $L_{\mathcal{E}}$ is spanned by the operators:

$$\Gamma_\phi = -\phi\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial x} - \phi'(X+T)\frac{\partial}{\partial X} - \phi'(X+T)\frac{\partial}{\partial T} - 2\phi' U\frac{\partial}{\partial U},$$

$$\Gamma_\psi = \psi\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial x} - \psi'(X-T)\frac{\partial}{\partial X} + \psi'(X-T)\frac{\partial}{\partial T} - 2\psi' U\frac{\partial}{\partial U},$$

$$\Gamma_\alpha = \alpha u\frac{\partial}{\partial u} - 2\alpha_x\frac{\partial}{\partial X} + 2\alpha_t\frac{\partial}{\partial T} + (\alpha_{tt} - \alpha_{xx} - \alpha_x X - \alpha_t T)\frac{\partial}{\partial U},$$

where $\phi = \phi(x-t),\ \psi = \psi(x+t),\ \alpha = \alpha(x,t)$ are arbitrary functions. We note that the above equivalence group is not a special form of the equivalence group of the family of n-dimensional linear hyperbolic equations (7.16).

7.5.2 Differential invariants and invariant equations

In order to find semi-invariants for equation (7.17) we have to solve $\Gamma_\alpha^{(1)}(J) = 0$. The invariant criterion $\Gamma_\alpha^{(1)}(J) = 0$ leads to two semi-invariants:

$$J_1 = X_t + T_x,$$

$$J_2 = X^2 - T^2 + 2(X_x + T_t) - 4U.$$

These semi-invariants can be transformed into Laplace invariants, using canonical variables. We also point out that $J_1 = 0$ and $J_2 = 0$ are invariant equations.

Also, we obtain one differential invariant of first order

$$J = \frac{X_t + T_x}{X^2 - T^2 + 2(X_x + T_t) - 4U}.$$

The above differential invariant can be obtained from the general case by setting $n = 1$. However, the family (7.17) admits differential invariants of higher order (see [20]).

The results of this chapter are contained in [59].

7.6 Applications

Two given partial differential equations are called *equivalent* if one can be transformed into the other by a change of variables. The equivalence problem consists of two parts: deciding if there exists equivalence and then determining a transformation that connects the partial differential equations. The motivation for considering this problem is to translate a known solution of a partial differential equation to solutions of others which are equivalent to this one.

In general, the equivalence problem is considered to be solved when a complete set of invariants has been found. In practice, using invariants to solve the equivalence problem for a given class of partial differential equations may require substantial computational effort. However any set of invariants can provide necessary conditions for deriving equivalent equations.

Here we consider the problem of finding those forms of the class (7.1) that can be mapped to an equation of the same class with constant coefficients. That is, we determine the forms of the functions $X(t,x,y)$, $Y(t,x,y)$, $T(t,x,y)$ and $U(t,x,y)$ such that equations (7.1) is mapped into

$$u_{tt} = u_{xx} + u_{yy} + c_1 u_x + c_2 u_y + c_3 u_t + c_4 u, \tag{7.19}$$

where $c_1, \ldots c_4$ are constants. Firstly, we note that the mapping

$$t' = at, \quad x' = \varepsilon_1 ax, \quad y' = \varepsilon_2 ay, \quad u' = e^{\frac{1}{2}(c_1 x + c_2 y - c_3 t)} u,$$

where a is an arbitrary constant, $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, transforms

$$u'_{t't'} = u'_{x'x'} + u'_{y'y'} + \frac{4c_4 - c_1^2 - c_2^2 + c_3^2}{4a^2} u'$$

into (7.19). Hence, choosing the appropriate value of the parameter a, equation (7.19) is equivalent with

$$u'_{t't'} = u'_{x'x'} + u'_{y'y'} + u'. \tag{7.20}$$

Therefore we can, equivalently, consider the problem of finding those forms of the class (7.1) that can be mapped into (7.20) instead of those forms that can be mapped into (7.19).

In the special case $c_4 = \frac{1}{4}(c_1^2 + c_2^2 - c_3^2)$, equation (7.19) can be mapped into the two-dimensional linear wave equation

$$u'_{t't'} = u'_{x'x'} + u'_{y'y'}. \tag{7.21}$$

We point out that equations (7.20) and (7.21) are inequivalent. Hence, there is merit to consider additionally the problem of finding those forms of the class (7.1) that can be mapped into (7.21).

Note 7.1. For equivalent equations (7.19) and (7.20) the differential invariant J in equation (7.12) is equal to zero. Equations (7.19) and (7.21) satisfy the invariant system (7.13) and the invariant equation (7.14) only if $c_4 = \frac{1}{4}(c_1^2 + c_2^2 - c_3^2)$.

We state the results of this section in the following theorem. The proof can be carried out using first that two equivalent equations have the same invariants or/and satisfy the invariant equations. This fact provides necessary conditions for connecting two equations. The second step is to find a point transformation that connects these equations (or special cases). Details of how such transformations are constructed can be found in [29, 46].

Theorem 7.1. *(i) An equation of the class (7.1) can be mapped into the two-dimensional linear wave equation (7.21) by the point transformation*

$$t' = ct, \quad x' = \varepsilon_1 cx, \quad y' = \varepsilon_2 cy, \quad u' = e^{-\frac{1}{2}F} u, \tag{7.22}$$

where $\varepsilon_1 = \pm 1,\ \varepsilon_2 = \pm 1$, *and* c *is an arbitrary constant, if and only if it is of the form*

$$\begin{aligned} u_{tt} &= u_{xx} + u_{yy} - F_x(t,x,y)u_x - F_y(t,x,y)u_y + F_t(t,x,y)u_t \\ &+ \tfrac{1}{4}\left[F_x^2 + F_y^2 - F_t^2 - 2(F_{xx} + F_{yy} - F_{tt})\right] u, \end{aligned} \tag{7.23}$$

where $F(t,x,y)$ *is an arbitrary function. Transformation (7.22) is a member of the equivalence transformations admitted by the class (7.1).*

(ii) An equation of the class (7.1) can be mapped into the constant coefficient equation (7.20) by the point transformation (7.22) if and only if it is of the form

$$\begin{aligned} u_{tt} &= u_{xx} + u_{yy} - F_x(t,x,y)u_x - F_y(t,x,y)u_y + F_t(t,x,y)u_t \\ &+ \tfrac{1}{4}\left[F_x^2 + F_y^2 - F_t^2 - 2(F_{xx} + F_{yy} - F_{tt}) + 4c^2\right] u. \end{aligned} \tag{7.24}$$

Note 7.2. Equation (7.21) and equation (7.23) satisfy the invariant system (7.13) and invariant equation (7.14). Equation (7.20) and equation (7.24) are such that the invariant (7.12) vanishes. This is the starting point for proving the above theorem.

Note 7.3. The results derived in this section can easily be generalized to n-dimensional equations of the class (7.16).

7.7 Conclusion

In this chapter, we used Lie infinitesimal method for calculating the equivalence transformations of the class of two- and three-dimensional hyperbolic equations. We have derived the differential invariants up to first order. Motivated by these results, we generalized them to n-dimensions. For one-dimensional hyperbolic equations a different form of equivalence transformations have been derived. In the last section, we used the fact that the knowledge of differential invariants can be useful to find the forms of those equations of the form (7.1), that can be mapped into an equation of the same class with constant coefficients.

Chapter 8

Differential Invariants for n-dimensional wave-type equations

8.1 Introduction

In this chapter, equivalence transformations and differential invariants of first order for the n-dimensional wave type equations of the form: $u_{tt} = \sum_{i=1}^{n} F_i(t, x_1, x_2, \ldots, x_n) u_{x_i x_i}$, are given. These equations have considerable interest in Mathematical Physics and Biology (see [3,8,65,69]). They have a number of applications, for example, in population dynamics, tides and waves, chemical reactors, flame and combustion problems and problems in transonic aerodynamics. Also, for the cases where $n = 1, 2, 3$ we present differential invariants of second order. In order to produce higher order invariants, we need to consider higher order prolongations. Finally, we employ the derived invariants to find the form of those equations that can be mapped into an equation with constant coefficients.

8.2 Differential Invariants for n-dimensional wave-type equations

8.2.1 Equivalence Transformations

We consider the n-dimensional wave-type class of equations

$$u_{tt} = \sum_{i=1}^{n} F_i(x_1, x_2, \ldots, x_n) u_{x_i x_i}. \tag{8.1}$$

In the spirit of Ibragimov's work (see [19]), we classify differential invariants of first order for the class (8.1). In order to achieve this goal, we firstly need to derive the equivalence transformations for the class (8.1).

We use infinitesimal method for calculating the equivalence transformations of the class (8.1). We find that the class of equations (8.1) admits a $(3n+4)$-dimensional continuous group $\mathcal{E}$ of equivalence transformations generated by Lie algebra $L_{\mathcal{E}}$ given by the operators:

$$\begin{aligned}
\Gamma_{1_i} &= \frac{\partial}{\partial x_i}, \quad i = 1, 2, \ldots, n,\\
\Gamma_{1_{n+1}} &= \frac{\partial}{\partial t},\\
\Gamma_{1_{n+2}} &= \frac{\partial}{\partial u},\\
\Gamma_{2_i} &= x_i \frac{\partial}{\partial x_i} + 2F_i \frac{\partial}{\partial F_i}, \quad i = 1, 2, \ldots, n,\\
\Gamma_{2_{n+1}} &= t\frac{\partial}{\partial t} - 2\sum_{i=1}^{n} F_i \frac{\partial}{\partial F_i},\\
\Gamma_{3_i} &= x_i^2 \frac{\partial}{\partial x} + x_i u \frac{\partial}{\partial u} + 4x_i F_i \frac{\partial}{\partial F_i}, \quad i = 1, 2, \ldots, n,\\
\Gamma_{3_{n+1}} &= t^2 \frac{\partial}{\partial t} + tu\frac{\partial}{\partial u} - 4t\sum_{i=1}^{n} F_i \frac{\partial}{\partial F_i}.
\end{aligned} \tag{8.2}$$

8.2.2 Differential invariants and invariant equations

Firstly, we consider the problem of finding differential invariants of the class of equations (8.1). Firstly, we seek for differential invariants of order zero. Using the $3n+4$ operators given by relations (8.2), the invariance criterion $\Gamma(J) = 0$, lead to the trivial invariant, $J = constant$.

In order to find differential invariants of first order, that depend of first derivatives of the functions $F_i,\ i = 1, 2, \dots, n$,

$$J = J(t,\ x_i,\ u,\ F_i,\ F_{i_t},\ F_{i_{x_j}}), \quad i, j = 1, 2, \dots, n, \tag{8.3}$$

we need to consider the first prolongation of the operators (8.2). Using the formula:

$$\Gamma^{(1)} = \Gamma + \mu_i^{j_1} \frac{\partial}{\partial F_{i_{j_1}}},$$

where:

$$\mu_i^{j_1} = \widetilde{D}_{j_1}(\mu_i) - \sum_{k=1}^{n} F_{k_t} \widetilde{D}_{j_1}(\nu) - \sum_{k=1}^{n} \sum_{l=1}^{n} F_{k_{x_l}} \widetilde{D}_{j_1}(\xi_l),$$

$i = 1, 2, \dots, n,\ j_1 = t, x_1, x_2, \dots, x_n$ and $\widetilde{D}_j$ denote the total derivative with respect to $j = t, x_1, x_2, \dots, x_n$

$$\widetilde{D}_j = \frac{\partial}{\partial j} + F_{i_j} \frac{\partial}{\partial F_i} + F_{i_{jx}} \frac{\partial}{\partial F_{i_x}} + F_{i_{jt}} \frac{\partial}{\partial F_{i_t}} + \dots,$$

we obtain the first extension of the generators (8.2):

$$\Gamma^{(1)}_{1_i} = \frac{\partial}{\partial x_i}, \quad \Gamma^{(1)}_{1_{n+1}} = \frac{\partial}{\partial t}, \quad \Gamma^{(1)}_{1_{n+2}} = \frac{\partial}{\partial u}, \tag{8.4}$$

$$\Gamma^{(1)}_{2_i} = \Gamma_{2_i} + F_{i_{x_i}} \frac{\partial}{\partial F_{i x_i}} + 2F_{i_t} \frac{\partial}{\partial F_{i_t}} + 2 \sum_{\substack{j=1 \\ j \neq i}}^{n} F_{i_{x_j}} \frac{\partial}{\partial F_{i_{x_j}}} - \sum_{\substack{j=1 \\ j \neq i}}^{n} F_{j_{x_i}} \frac{\partial}{\partial F_{j_{x_i}}}, \tag{8.5}$$

$$\Gamma^{(1)}_{2_{n+1}} = \Gamma_{2_{n+1}} - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} F_{i_{x_j}} \frac{\partial}{\partial F_{i_{x_j}}} - 3 \sum_{i=1}^{n} F_{i_t} \frac{\partial}{\partial F_{i_t}}, \tag{8.6}$$

$$\Gamma^{(1)}_{3_i} = 2x_i \Gamma^{(1)}_{2_i} - x_i^2 \frac{\partial}{\partial x_i} + x_i u \frac{\partial}{\partial u} + 4F_i \frac{\partial}{\partial F_{i_{x_i}}}, \tag{8.7}$$

$$\Gamma^{(1)}_{3_{n+1}} = 2t \Gamma^{(1)}_{2_{n+1}} - t^2 \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u} - 4 \sum_{i=1}^{n} F_i \frac{\partial}{\partial F_{i_t}}, \tag{8.8}$$

where $i = 1, 2, \dots, n$. Applying the operators (8.4), differential invariant (8.3) simplifies to

$$J = J(F_1, F_2, \dots, F_n, F_{1_t}, F_{1_{x_1}}, \dots, F_{1_{x_n}}, \dots, F_{n_t}, F_{n_{x_1}}, \dots, F_{n_{x_n}}).$$

Using the operators (8.7) we deduce, from the terms independent of x_i, that

$$\frac{\partial J}{\partial F_{i_{x_i}}} = 0, \quad i = 1, 2, \dots, n.$$

Hence,

$$J = J(F_1, F_2, \ldots, F_n, F_{i_t}, F_{i_{x_j}}), \quad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{8.9}$$

Now, operators (8.7) and operators (8.5) become identical.

Applying the operator (8.8) to differential invariant given by (8.9), and the vanishing coefficients that are independent of t, lead to the following characteristic system of equations:

$$\frac{dF_{1_t}}{F_1} = \frac{dF_{2_t}}{F_2} = \cdots = \frac{dF_{n_t}}{F_n},$$

which produces the $n-1$ integrals

$$p_k = F_{1_t} F_k - F_1 F_{k_t}, \quad k = 2, 3, \ldots, n.$$

Hence,

$$J = J(F_1, F_2, \ldots, F_n, F_{i_{x_j}}, p_2, \ldots, p_n), \; i, j = 1, 2, \ldots, n, \; i \neq j. \tag{8.10}$$

This implies that operators (8.8) and (8.6) are the same.

Now, we have to employ operators (8.5) and (8.6). Application of these operators to differential invariants given by (8.10), lead to a system of $n+1$ PDEs. Solving this system, we arrive to

$$(n-1)\left(\tfrac{3}{2}n - 1\right) \tag{8.11}$$

first order differential invariants:

$$\begin{aligned}
I_{i-1} &= \frac{F_1 F_{i_t} - F_{1_t} F_i}{F_1^{\frac{3}{2}} F_{i_{x_1}}}, \quad i = 2, 3, \ldots, n, \\
J_{i-1} &= \frac{F_1^{\frac{3}{2}} F_{i_{x_1}}}{F_i^{\frac{3}{2}} F_{1_{x_i}}}, \quad i = 2, 3, \ldots, n, \\
K_{i-1j-1} &= \frac{F_1 F_{i_{x_j}}}{F_{1_{x_j}} F_i}, \quad i \neq j, \quad i, \; j = 2, 3, \ldots, n, \\
L_{i-2j-1} &= \frac{F_i^{\frac{1}{2}} F_{1_{x_i}}}{F_j^{\frac{1}{2}} F_{1_{x_j}}}, \quad i > j, \quad i = 3, 4, \ldots, n, \quad j = 2, 3, \ldots, n-1.
\end{aligned} \tag{8.12}$$

Furthermore, we point out that the following $(n-1)(n+1)$ expressions:

$$\begin{aligned}
&F_{i_{x_j}} = 0, \quad i, j = 1, 2, \ldots, n, \quad i \neq j, \\
&F_1 F_{i_t} - F_{1_t} F_i = 0, \quad i = 2, 3, \ldots, n,
\end{aligned} \tag{8.13}$$

are invariant equations for the class of equations (8.1). That is, they satisfy the relations:

$$\Gamma^{(1)}_{l_m}\left(F_{i_{x_j}}\right)\Big|_{\left[F_{i_{x_j}}=0\right]}=0,\quad i,j=1,2,\ldots,n,\quad i\neq j$$

and

$$\Gamma^{(1)}_{l_m}\left(F_1F_{i_t}-F_{1_t}F_i\right)\big|_{\left[F_1F_{i_t}-F_{1_t}F_i=0\right]}=0,\quad i=2,3,\ldots,n,$$

where $l=1,2,3,\ m=1,\ldots,n+2$.

8.3 Differential invariants for the case $n=1$

We consider the one-dimensional wave-type equation

$$u_{tt}=F_1(x,t)u_{xx}. \tag{8.14}$$

We set $n=1$ in operators (8.2) to deduce that the class (8.14) admits a 7-dimensional continuous group of equivalence transformations. From (8.11) we deduce that the class of equation (8.14) does not admit differential invariants of order one.

In order to determine differential invariants of order two, we need to apply the invariant test

$$\Gamma^{(2)}_k(J)=0,\quad k=1,2,\ldots,7.$$

The second prolongation of operators (8.2), can be calculated using the formulas

$$\Gamma^{(2)}_k \;=\; \Gamma^{(1)}_k+\mu_i^{j_1j_2}\frac{\partial}{\partial F_{i_{j_1j_2}}},$$

where:

$$\mu_i^{j_1j_2} \;=\; \widetilde{D}_{j_2}(\mu_i^{j_1})-\sum_{k=1}^{n}F_{k_{j_1t}}\widetilde{D}_{j_2}(\nu)-\sum_{k=1}^{n}\sum_{l=1}^{n}F_{k_{j_1x_l}}\widetilde{D}_{j_2}(\xi_l),$$

$i=1,2,\ldots,n,\ j_1,j_2=t,x_1,x_2,\ldots,x_n$ and $\widetilde{D}_j$ denote the total derivative with respect to $j=t,x_1,x_2,\ldots,x_n$.

We state that the class of equations (8.14) admits two differential invariants of second order:

$$I_1=\frac{2F_1^{\frac{1}{2}}(F_{1_t}F_{1_x}-F_1F_{1_{xt}})}{4F_1F_{1_{tt}}-5F_{1_t}^2},\qquad I_2=\frac{F_1(3F_{1_x}^2-4F_1F_{1_{xx}})}{4F_1F_{1_{tt}}-5F_{1_t}^2}.$$

Furthermore, we have the following three invariant equations:

$$F_{1_t}F_{1_x}-F_1F_{1_{xt}}=0,\qquad 3F_{1_x}^2-4F_1F_{1_{xx}}=0,\qquad 4F_1F_{1_{tt}}-5F_{1_t}^2=0.$$

8.4 Differential Invariants for the case $n = 2$

Now, we consider the two-dimensional wave-type equation

$$u_{tt} = F_1(x, y, t)u_{xx} + F_2(x, y, t)u_{yy}. \tag{8.15}$$

For the case $n = 2$, it follows that equation (8.15) admits a 10- dimensional continuous group of equivalence transformations. Now, from (8.11), we have that there exist two differential invariants of first order, which are given from relations (8.12), by setting $n = 2$. In order to find differential invariants of second order, we need to calculate the second prolongations of generators. The invariant test

$$\Gamma_k^{(2)}(J), \quad k = 1, 2, \ldots, 10$$

leads to a system of PDEs. The solution of the system produce the following 12 differential invariants of second order:

$$\begin{aligned}
I_1 &= \frac{4F_1F_{1_{tt}} - 5F_{1_t}^2}{F_{1_y}^2 F_2}, & I_2 &= \frac{4F_1F_{1_{xx}} - 3F_{1_x}^2}{F_{1_y}^{\frac{4}{3}} F_{2_x}^{\frac{2}{3}}}, \\
I_3 &= \frac{4F_2F_{2_{yy}} - 3F_{2_y}^2}{F_{1_y}^{\frac{2}{3}} F_{2_x}^{\frac{4}{3}}}, & I_4 &= \frac{F_1F_{2_{xt}} - F_{1_t}F_{2_x}}{F_2^{\frac{1}{2}} F_{1_y} F_{2_x}}, \\
I_5 &= \frac{F_2F_{2_{yt}} - F_{2_t}F_{2_y}}{F_2^{\frac{1}{2}} F_{1_y}^{\frac{2}{3}} F_{2_x}^{\frac{4}{3}}}, & I_6 &= \frac{F_1F_{1_{yt}} - F_{1_t}F_{1_y}}{F_2^{\frac{1}{2}} F_{1_y}^2}, \\
I_7 &= \frac{F_1F_{1_{xt}} - F_{1_t}F_{1_x}}{F_2^{\frac{1}{2}} F_{1_y}^{\frac{5}{3}} F_{2_x}^{\frac{1}{3}}}, & I_8 &= \frac{F_1F_{1_{xy}} - F_{1_x}F_{1_y}}{F_{1_y}^{\frac{5}{3}} F_{2_x}^{\frac{1}{3}}}, \\
I_9 &= \frac{F_2F_{2_{xy}} - F_{2_x}F_{2_y}}{F_{1_y}^{\frac{1}{3}} F_{2_x}^{\frac{5}{3}}}, & I_{10} &= \frac{2F_1F_{2_{xx}} + F_{1_x}F_{2_x}}{F_{1_y}^{\frac{2}{3}} F_{2_x}^{\frac{4}{3}}}, \\
I_{11} &= \frac{F_{1_y}F_{2_y} + 2F_{1_{yy}}F_2}{F_{1_y}^{\frac{4}{3}} F_{2_x}^{\frac{2}{3}}}, & I_{12} &= \frac{4F_1^2F_{2_{tt}} - 10F_1F_{1_t}F_{2_t} + 5F_{1_t}^2F_2}{F_{1_y}^2 F_2^2}.
\end{aligned}$$

8.5 Differential Invariants for the case $n = 3$

Finally, we consider the 3-dimensional wave-type equation of the form

$$u_{tt} = F_1(x, y, z, t)u_{xx} + F_2(x, y, z, t)u_{yy} + F_3(x, y, z, t)u_{zz}. \tag{8.16}$$

From operators (8.2), by setting $n = 3$, we obtain the 13-dimensional continuous group of equivalence transformations of equations (8.16). Also, from (8.11) we deduce that it admits seven differential invariants of first order. These invariants can be obtained from relations (8.12) by setting $n = 3$.

Now, we determine differential invariants of second order, that depend on the second derivatives of $F_1,\ F_2,\ F_3$. Therefore we need the second prolongation of the operators (8.2). The invariance criterion

$$\Gamma_k^{(2)}(J) = 0, \quad k = 1, 2, \ldots, 13,$$

leads to the following differential invariants:

$$
\begin{aligned}
I_1 &= \frac{F_1 F_{1_{yz}}}{F_{1_y} F_{1_z}}, \qquad I_2 = \frac{F_1 F_{2_{xz}} F_{1_z}^{\frac{1}{2}}}{F_{2_z}^{\frac{1}{2}} F_{1_y} F_{2_z}}, \qquad I_3 = \frac{F_1^2 F_{1_z}^{\frac{1}{2}} F_{3_{xy}}}{F_{2_z}^{\frac{1}{2}} F_{1_y}^2 F_3}, \\
I_4 &= \frac{4F_1 F_{1_{tt}} - 5F_{1_t}^2}{2F_{1_y}^2 F_2}, \quad I_5 = \frac{F_1 F_{1_{yt}} - F_{1_t} F_{1_y}}{F_{1_y}^2 F_2^{\frac{1}{2}}}, \quad I_6 = \frac{F_1 F_{2_{zt}} - F_{1_t} F_{2_z}}{F_{2_z} F_3^{\frac{1}{2}} F_{1_z}}, \\
I_7 &= \frac{F_1 F_{1_{zt}} - F_{1_t} F_{1_z}}{F_{1_y} F_{1_z} F_2^{\frac{1}{2}}}, \qquad I_8 = \frac{F_1 F_{1_{xz}} - F_{1_x} F_{1_z}}{F_{1_z}^{\frac{1}{2}} F_{1_y} F_{2_z}^{\frac{1}{2}}}, \\
I_9 &= \frac{F_1^2 (F_{1_y} F_{3_{yy}} - F_{1_{yy}} F_{3_y})}{F_{1_y}^3 F_3}, \qquad I_{10} = \frac{F_1^2 (F_{1_z} F_{2_{zz}} - F_{1_{zz}} F_{2_z})}{F_{1_z}^3 F_2}, \\
I_{11} &= \frac{F_1^2 (F_{1_z} F_{3_{yz}} + 2F_{1_{zz}} F_{3_y})}{F_{1_y}^3 F_2}, \qquad I_{12} = \frac{F_1 (F_{1_y} F_{2_{yz}} + 2F_{1_{yy}} F_{2_z})}{F_{1_y}^2 F_{2_z}}, \\
I_{13} &= \frac{F_1 (F_{1_y} F_{2_y} + 2F_{1_{yy}} F_2)}{F_{1_z}^2 F_3}, \qquad I_{14} = \frac{F_1 (F_{1_z} F_{3_z} + 2F_{1_{zz}} F_3)}{F_2 F_{1_y}^2}, \\
I_{15} &= \frac{F_{1_z}^4 (F_1 F_{3_{xt}} - F_{1_t} F_{3_x})}{F_1^{\frac{1}{2}} F_{1_y}^4 F_{2_z}^2}, \qquad I_{16} = \frac{F_1^{\frac{5}{2}} (F_{1_z} F_{3_{xz}} + 2F_{1_{zz}} F_{3_x})}{F_{1_y}^3 F_2^{\frac{3}{2}}}, \\
I_{17} &= \frac{F_1^{\frac{3}{2}} (F_1 F_{2_{xt}} - F_{1_t} F_{2_x})}{F_{1_y}^2 F_2^2}, \qquad I_{18} = \frac{F_1 (F_1 F_{3_{yt}} - F_{1_t} F_{3_y})}{F_{1_y}^2 F_3 F_2^{\frac{1}{2}}}, \\
I_{19} &= \frac{F_1^{\frac{1}{2}} (F_1 F_{1_{xt}} - F_{1_t} F_{1_x})}{F_{1_y}^2 F_2}, \qquad I_{20} = \frac{F_1 (4F_1 F_{1_{xx}} - 3F_{1_x}^2)}{2F_{1_y}^2 F_2}, \\
I_{21} &= \frac{F_1^{\frac{1}{2}} (F_1 F_{1_{xy}} - F_{1_x} F_{1_y})}{F_2^{\frac{1}{2}} F_{1_y}^2}, \qquad I_{22} = \frac{F_1 F_{1_z}^{\frac{3}{2}} (F_{1_y} F_{2_{xy}} + 2F_{1_{yy}} F_{2_x})}{F_{2_z}^{\frac{3}{2}} F_{1_y}^3}, \\
I_{23} &= \frac{F_1^2 (2F_1 F_{2_{xx}} + F_{1_x} F_{2_x})}{F_2^2 F_{1_y}^2}, \qquad I_{24} = \frac{F_1 F_{1_z} (2F_1 F_{3_{xx}} + F_{1_x} F_{3_x})}{F_{1_y}^2 F_{2_z} F_3}, \\
I_{25} &= \frac{4F_1^2 F_{2_{tt}} - 10 F_1 F_{1_t} F_{2_t} + 5F_{1_t}^2 F_2}{2F_{1_y}^2 F_2^2},
\end{aligned}
$$

$$
\begin{aligned}
I_{26} &= \frac{4F_1^2 F_{3_{tt}} - 10F_1 F_{1_t} F_{3_t} + 5F_{1_t}^2 F_3}{2F_1^{\frac{2}{3}} F_{1_y}^{\frac{4}{3}} F_{2_z}^{\frac{2}{3}} F_3^{\frac{4}{3}}}, \\
I_{27} &= \frac{F_1^2 (F_{1_y}^2 F_{2_{yy}} + 3F_{1_y} F_{2_y} F_{1_{yy}} + 3F_2 F_{1_{yy}}^2)}{F_2 F_{1_y}^4}, \\
I_{28} &= \frac{F_1 (F_{1_z}^2 F_{3_{zz}} + 3F_{1_z} F_{1_{zz}} F_{3_z} + 3F_{1_{zz}}^2 F_3)}{F_{1_y}^2 F_{1_z} F_{2_z}}, \\
I_{29} &= \frac{F_1 F_{1_z} F_{3_{zt}} + 2F_1 F_{1_{zz}} F_{3_t} - F_{1_t} F_{1_z} F_{3_z} - 2F_{1_t} F_{1_{zz}} F_3}{F_{1_y}^2 F_{2_z} F_3^{\frac{1}{2}}}, \\
I_{30} &= \frac{F_1^{\frac{1}{2}} F_{1_z}^{\frac{1}{2}} (F_1 F_{1_y} F_{2_{yt}} + 2F_1 F_{1_{yy}} F_{2_t} - F_{1_t} F_{1_y} F_{2_y} - 2F_{1_t} F_{1_{yy}} F_2)}{F_{2_z}^{\frac{1}{2}} F_{1_y}^3 F_2}.
\end{aligned}
$$

8.6 Applications

We recall that, two given partial differential equations are called *equivalent* if one can be transformed into the other by a change of variables. However a complete set of invariants can provide necessary conditions for deriving equivalent equations.

In this section, we use invariants to classify equivalent PDEs. In particular, we aim to derive all equations of the form (8.1) that can be linked with the constant coefficient equation

$$
u_{tt} = \sum_{i=1}^{n} \varepsilon_i u_{x_i x_i}, \quad \varepsilon_i = \pm 1. \tag{8.17}
$$

Equation (8.17) is a member of the class (8.1). If we set $F_i = \varepsilon_i$ the invariant equations (8.13) are satisfied. Hence, any equation of the class (8.1) that is connected with equation (8.17) must satisfy the invariant equations. Consequently, the solution of the invariant equations will provide necessary conditions for an equation of the class (8.1) to be mapped into equation (8.17).

Solving the invariant equations, we find that

$$
F_i(t, x_1, x_2, \ldots, x_n) = \Phi(t) A_i(x_i),
$$

where $\Phi(t)$ and $A_i(x_i)$, $i = 1, 2, \ldots, n$ are arbitrary functions. Hence, an equation of the class (8.1) is linked with equation (8.17) only if is of the form

$$
u_{tt} = \Phi(t) \sum_{i=1}^{n} A_i(x_i) u_{x_i x_i}. \tag{8.18}
$$

Now, we will use the results of chapter 3 to derive the equivalence transformation in finite form. In this case we consider transformations of the form

$$x_i' = P_i(\mathbf{x}, t, u), \quad t' = Q(\mathbf{x}, t, u), \quad u' = R(\mathbf{x}, t, u), \quad \mathbf{x} = (x_1, \dots, x_n).$$

It can be shown that equation (8.18) (consequently equation (8.1)) can be mapped into equation (8.17) if and only if it is of the form

$$u_{tt} = Q^2(t) \sum_{i=1}^{n} \frac{\varepsilon_i}{P_i^2(x_i)} u_{x_i x_i}, \tag{8.19}$$

where the functions $P_i(x_i)$ and $Q(t)$ are solutions of the third order ordinary differential equation

$$f' f''' - \tfrac{3}{2} f''^2 = 0.$$

The transformation that connects equation

$$u_{tt} = Q^2(t) \sum_{i=1}^{n} \frac{\varepsilon_i}{P_i^2(x_i)} u_{x_i x_i}$$

and equation

$$u'_{t't'} = \sum_{i=1}^{n} \varepsilon_i u'_{x_i' x_i'}, \quad \varepsilon_i = \pm 1$$

is given by

$$t' = Q(t), \quad x_i' = P_i(x_i), \quad u' = \sqrt{Q_t \prod_i^n P_{i_{x_i}}} u.$$

The results of this chapter are appeared in a recent paper [57].

8.7 Conclusion

In this chapter, we have derived the complete set of differential invariants and invariant equations for the n-dimensional wave-type equations (8.1) up to order one by the infinitesimal method. Also, we have determined differential invariants of second order, for the cases where $n = 1, 2, 3$. As an application of the differential invariants, in the last section, we find the form of those equations (8.1) that can be mapped into an equation with constant coefficients.

Chapter 9

Point Transformations: Notations and basic theory

9.1 Introduction

In the spirit of chapter 3, we generalize the results of point transformations for systems of two partial differential equations. Similar as in chapter 3, we start with presenting identities relating arbitrary order partial differential derivatives of $u(x,t)$, $v(x,t)$ and $u'(x',t')$, $v'(x',t')$. These identities are useful to study the nature of those point transformations which preserve specific types of systems of two PDEs. We study three common classes of systems of PDEs restricted to two dependent variables and two independent variables and deduce results, summarized in theorems. These classes of systems are such that $\{u_t,\ v_t\}$, $\{u_{xt},\ v_{xt}\}$, $\{u_{tt},\ v_{tt}\}$ are functions of x,t,u,v and x-derivatives of u and v.

9.2 Notations and basic theory

In this section, we generalize the notation, that we had in chapter 3, in notation with two dependent variable, and summarize the basic theory on which the work in the sections below is based.

We consider the point transformation

$$x' = P(x,t,u,v), \quad t' = Q(x,t,u,v), \quad u' = R(x,t,u,v), \quad v' = S(x,t,u,v), \tag{9.1}$$

relating $x,\ t,\ u(x,t),\ v(x,t)$ and $x',\ t',\ u'(x',t'),\ v'(x',t')$, and assume that this is non-degenerate in the sense that the Jacobian

$$J = \frac{\partial(P,Q,R,S)}{\partial(x,t,u,v)} \neq 0 \tag{9.2}$$

and also that

$$\delta = \frac{\partial(P(x,t,u(x,t),v(x,t)),Q(x,t,u(x,t),v(x,t)))}{\partial(x,t)} \neq 0. \tag{9.3}$$

In (9.3) P and Q are expressed as functions of x and t whereas in (9.2) $P,\ Q,\ R$ and S are to be regarded as functions of the independent variables x,t,u,v.

The derivatives of $u(x,t),\ v(x,t)$ and $u'(x',t'),\ v'(x',t')$ will be denoted by

$$u_{ij} = \frac{\partial^{i+j}u}{\partial x^i \partial t^j}, \qquad u'_{ij} = \frac{\partial^{i+j}u'}{\partial x'^i \partial t'^j}, \tag{9.4}$$

$$v_{ij} = \frac{\partial^{i+j}v}{\partial x^i \partial t^j}, \qquad v'_{ij} = \frac{\partial^{i+j}v'}{\partial x'^i \partial t'^j}. \tag{9.5}$$

If Ψ is a function of $x,\ t,\ u,\ v$ and the derivatives of $u,\ v$, the total derivatives of Ψ with respect to x and t will be denoted by

$$\Psi_X = \Psi_x + \sum\sum u_{i+1j}\frac{\partial\Psi}{\partial u_{ij}} + \sum\sum v_{i+1j}\frac{\partial\Psi}{\partial v_{ij}}, \tag{9.6}$$

$$\Psi_T = \Psi_t + \sum\sum u_{ij+1}\frac{\partial\Psi}{\partial u_{ij}} + \sum\sum v_{ij+1}\frac{\partial\Psi}{\partial v_{ij}}, \tag{9.7}$$

where the double summations are to be taken over the values of i and j which cover all derivatives u_{ij} and v_{ij} occurring in Ψ.

With this notation δ may be expressed as

$$\begin{aligned}
\delta &= \frac{\partial(P,Q)}{\partial(X,T)} = P_XQ_T - P_TQ_X \\
&= u_{10}(P_uQ_t - P_tQ_u) + u_{01}(P_xQ_u - P_uQ_x) + v_{10}(P_vQ_t - P_tQ_v) \\
&+ v_{01}(P_xQ_v - P_vQ_x) + (u_{10}v_{01} - u_{01}v_{10})\,(P_uQ_v - P_vQ_u) + (P_xQ_t - P_tQ_x) \\
&= \frac{\partial(P,Q)}{\partial(u,t)}u_{10} + \frac{\partial(P,Q)}{\partial(x,u)}u_{01} + \frac{\partial(P,Q)}{\partial(v,t)}v_{10} + \frac{\partial(P,Q)}{\partial(x,v)}v_{01} \\
&+ \frac{\partial(P,Q)}{\partial(u,v)}\,(u_{10}v_{01} - u_{01}v_{10}) + \frac{\partial(P,Q)}{\partial(x,t)}.
\end{aligned} \tag{9.8}$$

Also, under the point transformation (9.1),

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \begin{pmatrix} P_X & P_T \\ Q_X & Q_T \end{pmatrix}\begin{pmatrix} dx \\ dt \end{pmatrix}, \quad \begin{pmatrix} dx \\ dt \end{pmatrix} = \frac{1}{\delta}\begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix}\begin{pmatrix} dx' \\ dt' \end{pmatrix}$$

(9.9)

and

$$d\Psi = \Psi_X dx + \Psi_T dt = \frac{1}{\delta}(\Psi_X \quad \Psi_T) \begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix} \begin{pmatrix} dx' \\ dt' \end{pmatrix}. \tag{9.10}$$

Hence, taking $\Psi = u'_{ij-1},\ u'_{i-1j},\ v'_{ij-1},\ v'_{i-1j}$ respectively, gives

$$u'_{ij} = \delta^{-1}\left(P_X(u'_{ij-1})_T - P_T(u'_{ij-1})_X\right), \qquad j \geq 1, \quad i \geq 0, \tag{9.11}$$

$$u'_{ij} = \delta^{-1}\left(Q_T(u'_{i-1j})_X - Q_X(u'_{i-1j})_T\right), \qquad i \geq 1, \quad j \geq 0, \tag{9.12}$$

$$v'_{ij} = \delta^{-1}\left(P_X(v'_{ij-1})_T - P_T(v'_{ij-1})_X\right), \qquad j \geq 1, \quad i \geq 0, \tag{9.13}$$

$$v'_{ij} = \delta^{-1}\left(Q_T(v'_{i-1j})_X - Q_X(v'_{i-1j})_T\right), \qquad i \geq 1, \quad j \geq 0. \tag{9.14}$$

Also,

$$u'_{00} = u' = R, \quad v'_{00} = v' = S. \tag{9.15}$$

Equations (9.11)-(9.15) furnish recurrence relations which enable u'_{ij} and v'_{ij} to be expressed in terms of x, t, u, v and the derivatives of u and v for any $i \geq 0,\ j \geq 0$. The factor δ^{-1} makes the expressions for $u'_{ij},\ v'_{ij}$ grow with i and j in a very cumbersome manner.

In the case of infinitesimal Lie point transformations in which:

$$\begin{aligned} P(x,t,u,v) &= x + \varepsilon P^*(x,t,u,v) + O(\varepsilon^2), \\ Q(x,t,u,v) &= t + \varepsilon Q^*(x,t,u,v) + O(\varepsilon^2), \\ R(x,t,u,v) &= u + \varepsilon R^*(x,t,u,v) + O(\varepsilon^2), \\ S(x,t,u,v) &= v + \varepsilon S^*(x,t,u,v) + O(\varepsilon^2), \end{aligned}$$

the forms of J and δ in (9.2) and (9.3) simplify to

$$J = 1 + \varepsilon(P^*_x + Q^*_t + R^*_u + S^*_v), \tag{9.16}$$

$$\delta = 1 + \varepsilon(P^*_x + Q^*_t), \tag{9.17}$$

to the first order of ε. In this case the recurrence relations corresponding to (9.11)-(9.15) are

$$
\begin{aligned}
u'_{ij} &= (u'_{ij-1})_T - \varepsilon[P^*_T(u'_{ij-1})_X + Q^*_T(u'_{ij-1})_T], \quad j \geq 1, \quad i \geq 0, && (9.18)\\
u'_{ij} &= (u'_{i-1j})_X - \varepsilon[P^*_X(u'_{i-1j})_X + Q^*_X(u'_{i-1j})_T], \quad i \geq 1, \quad j \geq 0, && (9.19)\\
v'_{ij} &= (v'_{ij-1})_T - \varepsilon[P^*_T(v'_{ij-1})_X + Q^*_T(v'_{ij-1})_T], \quad j \geq 1, \quad i \geq 0, && (9.20)\\
v'_{ij} &= (v'_{i-1j})_X - \varepsilon[P^*_X(v'_{i-1j})_X + Q^*_X(v'_{i-1j})_T], \quad i \geq 1, \quad j \geq 0, && (9.21)\\
u'_{00} &= u + \varepsilon R^*, && (9.22)\\
v'_{00} &= v + \varepsilon S^*, && (9.23)
\end{aligned}
$$

to the first order in ε. These relations of course lead to considerably less cumbersome forms of u'_{ij} and v'_{ij} than those obtained from (9.11)-(9.15).

9.3 Properties of the transformations

Under the point transformation (9.1) each derivative of $u'(x', t')$ and $v'(x', t')$, that is u'_{ij} and v'_{ij}, $i \geq 0$, $j \geq 0$, may be expressed, via the recurrence relations (9.11)-(9.15), as functions of x, t, u, v and the derivatives of u and v. A number of results concerning the functional form of $u'_{pq}(x, t, u, v, \ldots, u_{ij}, \ldots, v_{ij}, \ldots)$ and $v'_{pq}(x, t, u, v, \ldots, u_{ij}, \ldots, v_{ij}, \ldots)$ are presented in this section. The proofs of the results are generally inductive and use the recurrence relations (9.11)-(9.15).

Lemma 9.1. *If* $x' = P(x, t, u, v)$, $t' = Q(x, t, u, v)$, $u' = R(x, t, u, v)$, $v' = S(x, t, u, v)$

$$
\sum_{i=0}^{n} z^i \frac{\partial u'_{pq}}{\partial u_{ij}} = \begin{cases} (-1)^p (Q_X - zQ_T)^p (P_X - zP_T)^q J_1 \delta^{-p-q-1}, & n > 0 \\ R_u, & n = 0 \end{cases}
$$

$$
\sum_{i=0}^{n} z^i \frac{\partial u'_{pq}}{\partial v_{ij}} = \begin{cases} (-1)^p (Q_X - zQ_T)^p (P_X - zP_T)^q J_2 \delta^{-p-q-1}, & n > 0 \\ R_v, & n = 0 \end{cases}
$$

$$
\sum_{i=0}^{n} z^i \frac{\partial v'_{pq}}{\partial u_{ij}} = \begin{cases} (-1)^p (Q_X - zQ_T)^p (P_X - zP_T)^q J_3 \delta^{-p-q-1}, & n > 0 \\ S_u, & n = 0 \end{cases}
$$

$$
\sum_{i=0}^{n} z^i \frac{\partial v'_{pq}}{\partial v_{ij}} = \begin{cases} (-1)^p (Q_X - zQ_T)^p (P_X - zP_T)^q J_4 \delta^{-p-q-1}, & n > 0 \\ S_v, & n = 0 \end{cases}
$$

where $i+j=p+q=n\geq 0$, *and*

$$
\begin{aligned}
J_1 &= \frac{\partial(P,Q,R)}{\partial(t,u,v)}v_x - \frac{\partial(P,Q,R)}{\partial(x,u,v)}v_t + \frac{\partial(P,Q,R)}{\partial(x,t,u)}, &(9.24)\\
J_2 &= -\frac{\partial(P,Q,R)}{\partial(t,u,v)}u_x + \frac{\partial(P,Q,R)}{\partial(x,u,v)}u_t + \frac{\partial(P,Q,R)}{\partial(x,t,v)}, &(9.25)\\
J_3 &= \frac{\partial(P,Q,S)}{\partial(t,u,v)}v_x - \frac{\partial(P,Q,S)}{\partial(x,u,v)}v_t + \frac{\partial(P,Q,S)}{\partial(x,t,u)}, &(9.26)\\
J_4 &= -\frac{\partial(P,Q,S)}{\partial(t,u,v)}u_x + \frac{\partial(P,Q,S)}{\partial(x,u,v)}u_t + \frac{\partial(P,Q,S)}{\partial(x,t,v)}. &(9.27)
\end{aligned}
$$

We point out that $J_i,\ i=1,2,3,4$ *are non-zero quantities.*

Corollary 9.1. *The coefficient of* z^n *and* z^0 *in lemma 9.1 give, respectively*

$$\frac{\partial u'_{pq}}{\partial u_{p+q0}} = (-1)^q P_T^q Q_T^p J_1 \delta^{-p-q-1}, \qquad p+q\geq 1, \tag{9.28}$$

$$\frac{\partial u'_{pq}}{\partial u_{0p+q}} = (-1)^p P_X^q Q_X^p J_1 \delta^{-p-q-1}, \qquad p+q\geq 1, \tag{9.29}$$

$$\frac{\partial u'_{pq}}{\partial v_{p+q0}} = (-1)^q P_T^q Q_T^p J_2 \delta^{-p-q-1}, \qquad p+q\geq 1, \tag{9.30}$$

$$\frac{\partial u'_{pq}}{\partial v_{0p+q}} = (-1)^p P_X^q Q_X^p J_2 \delta^{-p-q-1}, \qquad p+q\geq 1, \tag{9.31}$$

$$\frac{\partial v'_{pq}}{\partial u_{p+q0}} = (-1)^q P_T^q Q_T^p J_3 \delta^{-p-q-1}, \qquad p+q\geq 1, \tag{9.32}$$

$$\frac{\partial v'_{pq}}{\partial u_{0p+q}} = (-1)^p P_X^q Q_X^p J_3 \delta^{-p-q-1}, \qquad p+q\geq 1, \tag{9.33}$$

$$\frac{\partial v'_{pq}}{\partial v_{p+q0}} = (-1)^q P_T^q Q_T^p J_4 \delta^{-p-q-1}, \qquad p+q\geq 1, \tag{9.34}$$

$$\frac{\partial v'_{pq}}{\partial v_{0p+q}} = (-1)^p P_X^q Q_X^p J_4 \delta^{-p-q-1}, \qquad p+q\geq 1. \tag{9.35}$$

Lemma 9.2. *If* $x'=P(x,t,u,v),\ t'=Q(x,t,u,v),\ u'=R(x,t,u,v,),\ v'=S(x,t,u,v)$ *then*

$$
\begin{aligned}
\frac{\partial^{m_1+n_1+m_2+n_2}u'_{10}}{\partial u_{10}^{m_1}\partial u_{01}^{n_1}\partial v_{10}^{m_2}\partial v_{01}^{n_2}} &= [(-1)^{n_1}C_{m_1n_1}(n_1\alpha_1Q_X+m_1\beta_1Q_T)\\
&+ (-1)^{n_2}D_{m_2n_2}(n_2\alpha_2Q_X+m_2\beta_2Q_T)]\delta^{-m_1-n_1-m_2-n_2-1},\\
\frac{\partial^{m_1+n_1+m_2+n_2}u'_{01}}{\partial u_{10}^{m_1}\partial u_{01}^{n_1}\partial v_{10}^{m_2}\partial v_{01}^{n_2}} &= [(-1)^{m_1}C_{m_1n_1}(n_1\alpha_1P_X+m_1\beta_1P_T)\\
&+ (-1)^{m_2}D_{m_2n_2}(n_2\alpha_2P_X+m_2\beta_2P_T)]\delta^{-m_1-n_1-m_2-n_2-1},
\end{aligned}
$$

where

$$\begin{aligned}
\alpha_1 &= \frac{\partial(P,Q)}{\partial(t,u)} - \frac{\partial(P,Q)}{\partial(u,v)} v_t, \\
\beta_1 &= \frac{\partial(P,Q)}{\partial(x,u)} - \frac{\partial(P,Q)}{\partial(u,v)} v_x, \\
\alpha_2 &= \frac{\partial(P,Q)}{\partial(t,v)} + \frac{\partial(P,Q)}{\partial(u,v)} u_t, \\
\beta_2 &= \frac{\partial(P,Q)}{\partial(x,v)} + \frac{\partial(P,Q)}{\partial(u,v)} u_x.
\end{aligned}$$

Lemma 9.3. *If* $x' = P(x),\ t' = Q(t),\ u' = R(x,t,u,v),\ v' = S(x,t,u,v)$ *then*

$$\frac{\partial^2 u'_{pq}}{\partial u_{ij}\partial u_{kl}} = \begin{cases} \begin{pmatrix} p \\ i \end{pmatrix} \begin{pmatrix} q \\ j \end{pmatrix} P_x^{-p} Q_t^{-q} R_{uu}, & i+k=p, \quad j+l=q \\ 0, & i+k>p \ \ or \ \ j+l>q \end{cases}$$

$$\frac{\partial^2 u'_{pq}}{\partial u_{ij}\partial v_{kl}} = \begin{cases} \begin{pmatrix} p \\ i \end{pmatrix} \begin{pmatrix} q \\ j \end{pmatrix} P_x^{-p} Q_t^{-q} R_{uv}, & i+k=p, \quad j+l=q \\ 0, & i+k>p \ \ or \ \ j+l>q \end{cases}$$

$$\frac{\partial^2 u'_{pq}}{\partial v_{ij}\partial v_{kl}} = \begin{cases} \begin{pmatrix} p \\ i \end{pmatrix} \begin{pmatrix} q \\ j \end{pmatrix} P_x^{-p} Q_t^{-q} R_{vv}, & i+k=p, \quad j+l=q \\ 0, & i+k>p \ \ or \ \ j+l>q. \end{cases}$$

For the derivatives of v'_{pq} *we simply* $R \to S$ *in the above relations.*

9.4 Form-preserving transformations of systems of PDEs

9.4.1 Basic results

Here, we will use the results of the previous section in order to study the nature of point transformations which perform specific changes to systems of PDEs. We start with a general class of systems of PDEs for which general deductions about the forms of $P(x,t,u,v)$ and $Q(x,t,u,v)$ can be made. These will be useful for the discussion of restricted classes of systems.

We give a similar theorem as theorem 3.1 for systems of two PDEs.

Theorem 9.1. *The system of PDEs*

$$u_{pq} = H\left(x, t, u, v, \{u_{ij}\}, \{v_{kl}\}\right), \quad v_{\mu\nu} = F\left(x, t, u, v, \{u_{\alpha\beta}\}, \{v_{\gamma\delta}\}\right)$$

is related to

$$u'_{pq} = H'\left(x', t', u', v', \{u'_{ij}\}, \{v'_{kl}\}\right), \quad v'_{\mu\nu} \;=\; F'\left(x', t', u', v', \{u'_{\alpha\beta}\}, \{v'_{\gamma\delta}\}\right),$$

where $\{u_{ij}\},\ \{u_{\alpha\beta}\},\ \{v_{kl}\},\ \{v_{\gamma\delta}\}$ *and* $\{u'_{ij}\},\ \{u'_{\alpha\beta}\},\ \{v'_{kl}\},\ \{v'_{\gamma\delta}\}$ *respectively denote all derivatives of* $u,\ v,\ u'$ *and* v' *of order* $i+j < p+q,\ \ k+l < p+q,\ \alpha+\beta < \mu+\nu,\ \gamma+\delta < \mu+\nu$ *by the point transformation* $x' = P(x,t,u,v),\ t' = Q(x,t,u,v),\ u' = R(x,t,u,v),\ v' = S(x,t,u,v)$ *in the cases:*

(*a*) $(p \neq 0,\ \ q \neq 0)$ *or* $(\mu \neq 0,\ \ \nu \neq 0)$,

(*b*)$\{(p \neq 0,\ \ q \neq 0)$ *and* $(\mu \neq 0,\ \ \nu = 0)\}$ *or* $\{(p \neq 0,\ \ q \neq 0)$ *and* $(\mu = 0,\ \ \nu \neq 0)\}$ *or* $\{(p \neq 0,\ \ q = 0)$ *and* $(\mu = 0,\ \ \nu \neq 0)\}$ *or* $\{(p \neq 0,\ \ q = 0)$ *and* $(\mu \neq 0,\ \ \nu \neq 0)\}$ *or* $\{(p = 0,\ \ q \neq 0)$ *and* $(\mu \neq 0,\ \ \nu \neq 0)\}$ *or* $\{(p = 0,\ \ q \neq 0)$ *and* $(\mu \neq 0,\ \ \nu = 0)\}$,

(*c*) $(p = 0,\ \ q \neq 0)$ *or* $(\mu = 0,\ \ \nu \neq 0)$,

(*d*) $(p \neq 0,\ \ q = 0)$ *or* $(\mu \neq 0,\ \ \nu = 0)$

only if

(*a*) $\{P = P(x),\ \ Q = Q(t)\}$ *or* $\{P = P(t),\ \ Q = Q(x)\}$,

(*b*) $P = P(x),\ \ Q = Q(t)$,

(*c*) $P = P(x)$,

(*d*) $Q = Q(t)$,

respectively.

Proof. For the proof of the theorem 9.1, we consider the fate of the highest-order derivative of $u'_{pq} = H',\ v'_{\mu\nu} = F'$ under the point transformation. Consider the lemma 9.1, corollary 9.1, $p + q \geq 1$ and $\mu + \nu \geq 1$, that is equations (9.28)-(9.35).

In case (a) neither $(p = 0,\ q = 0)$ nor $(\mu = 0,\ \nu = 0)$ so that the expressions (9.28)-(9.35) must vanish in order for u'_{pq} and $v'_{\mu\nu}$ to generate u_{pq} and $v_{\mu\nu}$ alone of order $p + q$ and $\mu + \nu$, respectively. Any lower-order derivatives of u' and v' which occur in H' and F' transform to derivatives of u and v of order less than $p + q$ and $\mu + \nu$. Hence,

$Q_T P_T = Q_X P_X = 0$. Hence, either $\{P = P(x),\ Q = Q(t)\}$ or $\{P = P(t),\ Q = Q(x)\}$ as required.

In the first case of case (b), where $\{p \neq 0$ and $q \neq 0\}$, from case (a) we deduce that $\{P = P(x),\ Q = Q(t)\}$ or $\{P = P(t),\ Q = Q(x)\}$. Since, ($\mu \neq 0$ and $\nu = 0$) only the expressions (9.33) and (9.35) must vanish. So that $Q_X = 0$ and hence $Q = Q(t)$ as required. Therefore, $\{P = P(x),\ Q = Q(t)\}$. In the second case, first we have $\{P = P(x),\ Q = Q(t)\}$ or $\{P = P(t),\ Q = Q(x)\}$ and since ($\mu = 0$ and $\nu \neq 0$) only the expressions (9.32) and (9.34) must vanish. Therefore, $P_T = 0$ and hence $\{P = P(x),\ Q = Q(t)\}$. Finally, in the case where $\{(p \neq 0$ and $q = 0)$ and $(\mu = 0$ and $\nu \neq 0)\}$, only the expressions (9.29), (9.31), (9.32) and (9.34), must vanish together. Therefore, $P = P(x),\ Q = Q(t)$. The other cases can be proved in a similar way.

Case (c), where $\{(p = 0,\ q \neq 0)$ and $(\mu = 0,\ \nu \neq 0)\}$, the expressions (9.28), (9.30), (9.32) and (9.34) must vanish. Therefore $P = P(x)$.

Case (d) follows by symmetry ($x \leftrightarrow t,\ P \leftrightarrow Q,\ X \leftrightarrow T,\ p \leftrightarrow q,\ \mu \leftrightarrow \nu$) from case (c). □

9.4.2 System of two equations of the form $u_{01} = H(x, t, u, v, \ldots, u_{n_1 0}, v_{m_1 0})$

In this subsection, we are interested in system of two equations where $u_t,\ v_t$ are related to $x,\ t,\ u,\ v$ and derivatives of u and v with respect to x. We will generalize the theorems 3.2 and 3.3. That is, we will show that point transformations for systems of this type with $n_1 \geq 2,\ m_2 \geq 2$ must take the form $t' = Q(t)$ (no $x,\ u,\ v$ dependency). Also, for restricted classes of these systems it is necessary for $x' = P(x, t)$.

Theorem 9.2. *The point transformations* $x' = P(x, t, u, v),\ t' = Q(x, t, u, v),\ u' = R(x, t, u, v),\ v' = S(x, t, u, v)$ *transform*

$$u_{01} = H(x, t, u, v, \ldots, u_{n_1 0}, v_{m_1 0}), \quad v_{01} = F(x, t, u, v, \ldots, u_{n_2 0}, v_{m_2 0}) \tag{9.36}$$

to

$$u'_{01} = H'(x', t', u', v', \ldots, u'_{n_1 0}, v'_{m_1 0}), \quad v'_{01} = F'(x', t', u', v', \ldots, u'_{n_2 0}, v'_{m_2 0}), \tag{9.37}$$

where at least one of $n_1,\ m_2 \geq 2$ *if and only if* $Q = Q(t)$ *and*

$$H' = \frac{P_X R_t - P_t R_X + H\left[-v_{10}\frac{\partial(P,R)}{\partial(u,v)} + \frac{\partial(P,R)}{\partial(x,u)}\right] + F\left[u_{10}\frac{\partial(P,R)}{\partial(u,v)} + \frac{\partial(P,R)}{\partial(x,v)}\right]}{P_X Q_t}, \tag{9.38}$$

$$F' = \frac{P_X S_t - P_t S_X + H\left[-v_{10}\frac{\partial(P,S)}{\partial(u,v)} + \frac{\partial(P,S)}{\partial(x,u)}\right] + F\left[u_{10}\frac{\partial(P,S)}{\partial(u,v)} + \frac{\partial(P,S)}{\partial(x,v)}\right]}{P_X Q_t}. \tag{9.39}$$

Proof. Without loss of generality, we assume that $n_1 \geq 2$. Theorem 9.1 applies with $\{(p = n_1,\ q = 0)$, so that $Q = Q(t)$. Each u'_{i0} and v'_{i0} in H' and F' transforms to an expression in $x, t, u, v, u_{10}, v_{10}, \ldots, u_{i0}, v_{i0}$, that is no t derivatives of u and v are introduced. System (9.36) thus transforms to the form (9.37) and the form of H' and F' are determined, with no further conditions on $P,\ Q,\ R$ and S, from (9.38) and (9.39) for any H and G. □

Note 9.1. In theorem 9.2, the identity $H^2_{u_{n_1 0}} + H^2_{v_{m_1 0}} + F^2_{u_{n_2 0}} + F^2_{v_{m_2 0}} \neq 0$, holds.

The following theorem is a generalization of theorem 3.3, where $u_t,\ v_t$ and $u'_{t'},\ v'_{t'}$ are polynomials in $\{u_{i0}\}, \{v_{j0}\}$ and $\{u'_{i0}\}, \{v'_{j0}\}$, respectively.

Theorem 9.3. *If, in the above theorem,* $H,\ F$ *and* $H',\ F'$ *are polynomials (non-negative integral powers) in* $\{u_{i0}\}, \{v_{j0}\}$ *and* $\{u'_{i0}\}, \{v'_{j0}\}$ *respectively, then* $P = P(x,t)$.

The following lemmas will be needed for the proof of theorem 9.3.

Lemma 9.4. *If* u'_{r0} *and* v'_{r0} *are expressed in terms of* $x,\ t,\ u,\ v$ *and the* $x,\ t$*-derivatives of* $u,\ v$ *then*

$$\frac{\partial u'_{r0}}{\partial u_{0r}} = (-1)^r \frac{J_1 Q_X^r}{\delta^{r+1}},$$

$$\frac{\partial v'_{r0}}{\partial v_{0r}} = (-1)^r \frac{J_4 Q_X^r}{\delta^{r+1}},$$

where $r \geq 1$ *and* $J_1,\ J_4$ *are given by relations (9.24) and (9.27).*

Proof. The proof is by induction on r.

$$\begin{aligned} \frac{\partial u'_{r+10}}{\partial u_{0r+1}} &= \frac{\partial}{\partial u_{0r+1}}\left\{\frac{\partial}{\partial x'}u'_{r0}\right\} \\ &= \frac{\partial}{\partial u_{0r+1}}\left\{\left((u'_{r0})_X \quad (u'_{r0})_T\right)\frac{1}{\delta}\begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \end{aligned}$$

using (9.10) with $\psi = u'_{r0}$

$$= \frac{1}{\delta} \begin{pmatrix} 0 & \frac{\partial u'_{r0}}{\partial u_{0r}} \end{pmatrix} \begin{pmatrix} Q_T \\ -Q_X \end{pmatrix}$$

using (9.6) and (9.7) and noting that for $r \geq 1$ the term u_{0r+1} only appears in the second term of the row vector,

$$= (-1)^{r+1} \frac{J_1 Q_X^{r+1}}{\delta^{r+2}}$$

from the induction hypothesis. For the basis of the induction consider firstly, from (9.10), for $\Psi = u$:

$$du' = \frac{1}{\delta}(R_X \quad R_T) \begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix} \begin{pmatrix} dx' \\ dt' \end{pmatrix}, \tag{9.40}$$

we have

$$u'_{x'} = \frac{1}{\delta}(R_X \quad R_T) \begin{pmatrix} Q_T & -P_T \\ -Q_X & P_X \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{R_X Q_T - R_T Q_X}{P_X Q_T - Q_X P_T}. \tag{9.41}$$

Noting from

$$\phi_X = \phi_x + u_x \phi_u + v_x \phi_v, \qquad \phi_T = \phi_t + u_t \phi_u + v_t \phi_v$$

that

$$\frac{\partial \phi_T}{\partial u_t} = \phi_u, \quad \frac{\partial \phi_X}{\partial u_t} = 0,$$

(9.41) may be differentiated to give

$$\frac{\partial u'_{10}}{\partial u_{01}} \equiv \frac{\partial u'_{x'}}{\partial u_t} = -\frac{J_1 Q_X}{\delta^2},$$

which is relation with $r = 1$, completing the induction and proof of lemma 9.4. □

Using the result of lemma 9.4, we can proof the following relations.

With $Q = Q(t)$,

$$\delta = P_X Q_T - P_T Q_X = P_X Q_t \neq 0$$

and

$$J = -Q_t \left((R_u S_x - R_x S_u) P_v - (R_v S_x - R_x S_v) P_u - (R_u S_v - R_v S_u) P_x \right) \neq 0.$$

Equation (9.10) simplifies to

$$d\Psi = \frac{1}{P_X Q_t}(\Psi_X \quad \Psi_T) \begin{pmatrix} Q_T & -P_T \\ 0 & P_X \end{pmatrix} \begin{pmatrix} dx' \\ dt' \end{pmatrix},$$

so that

$$\Psi_{x'} = \frac{1}{P_X}\Psi_X, \quad \Psi_{t'} = -\frac{1}{P_X Q_t}(P_T\Psi_X - P_X\Psi_T). \tag{9.42}$$

In particular,

$$\begin{aligned} u'_{t'} &= u'_{01} = -\frac{1}{P_X Q_t}(P_T R_X - P_X R_T) \\ &= -\frac{1}{P_X Q_t}[(P_t R_x - P_x R_t) + u_x(P_t R_u - P_u R_t) + v_x(P_t R_v - P_v R_t) \\ &+ u_t(P_u R_x - P_x R_u) + v_t(P_v R_x - P_x R_v) + (u_t v_x - u_x v_t)(P_u R_v - P_v R_u)], \end{aligned} \tag{9.43}$$

$$\begin{aligned} v'_{t'} &= v'_{01} = -\frac{1}{P_X Q_t}(P_T S_X - P_X S_T) \\ &= -\frac{1}{P_X Q_t}[(P_t S_x - P_x S_t) + u_x(P_t S_u - P_u S_t) + v_x(P_t S_v - P_v S_t) \\ &+ u_t(P_u S_x - P_x S_u) + v_t(P_v S_x - P_x S_v) + (u_t v_x - u_x v_t)(P_u S_v - P_v S_u)], \end{aligned} \tag{9.44}$$

$$u'_{x'} = u'_{10} = \frac{R_X}{P_X} = \left(\frac{1}{P_X}D\right)R, \tag{9.45}$$

$$v'_{x'} = v'_{10} = \frac{S_X}{P_X} = \left(\frac{1}{P_X}D\right)S, \tag{9.46}$$

denoting R_X by DR,

$$u'_{x'x'} = u'_{20} = \left(\frac{1}{P_X}D\right)^2 R, \quad u'_{n0} = \left(\frac{1}{P_X}D\right)^n R, \quad n \geq 1, \tag{9.47}$$

$$v'_{x'x'} = v'_{20} = \left(\frac{1}{P_X}D\right)^2 S, \quad v'_{n0} = \left(\frac{1}{P_X}D\right)^n S, \quad n \geq 1. \tag{9.48}$$

The lemma below will be needed in order to find the coefficients of the terms (9.43) and (9.44) which contain the highest power of the highest-order derivatives. These coefficients is found to contain non-zero factors.

Lemma 9.5. *If u'_{r0} and v'_{r0} are expressed in terms of $x,\ t,\ u,\ v$ and the $x,\ t$-derivatives of $u,\ v$ then*

$$\frac{\partial u'_{r0}}{\partial u_{r0}} = \frac{(P_v R_u - P_u R_v)v_{10} + P_x R_u - P_u R_x}{P_X^{r+1}}, \qquad r \geq 1,$$

$$\frac{\partial v'_{r0}}{\partial v_{r0}} = \frac{(P_u S_v - P_v S_u)u_{10} + P_x S_v - P_v S_x}{P_X^{r+1}}, \qquad r \geq 1,$$

where $P_X = P_x + u_x P_u + v_x P_v$.

Proof. The proof of this lemma is by induction on r.

$$\begin{aligned}
\frac{\partial u'_{r+10}}{\partial u_{r+10}} &= \frac{\partial}{\partial u_{r+10}}\left\{\frac{\partial}{\partial x'} u'_{r0}\right\} \\
&= \frac{\partial}{\partial u_{r+10}}\left\{\frac{1}{P_X}(u'_{r0})_X\right\}, \qquad \text{from} \quad (9.42) \\
&= \frac{1}{P_X}\frac{\partial u'_{r0}}{\partial u_{r0}}, \qquad \text{from} \quad (9.6), \quad r \geq 1 \\
&= \frac{(P_v R_u - P_u R_v)v_{10} + P_x R_u - P_u R_x}{P_X^{r+2}},
\end{aligned}$$

from the induction hypothesis. Similarly, we can prove the other expression. □

Now, using lemmas 9.4 and 9.5 we are ready to give the proof of theorem 9.3.

Proof. of theorem 9.3: Suppose that the leading term in $H(x, t, u, v, u_{10}, v_{10}, \ldots, u_{n_1 0}, v_{m_1 0})$ is

$$F_1(x, t, u, v) u_{n_1 0}^{\alpha_{n_1}} u_{n_1 - 10}^{\alpha_{n_1 - 1}} \ldots u_{10}^{\alpha_1} v_{m_1 0}^{\beta_{m_1}} v_{m_1 - 10}^{\beta_{m_1 - 1}} \ldots v_{10}^{\beta_1}, \tag{9.49}$$

and the corresponding term in $F(x, t, u, v, u_{10}, v_{10}, \ldots, u_{n_2 0}, v_{m_2 0})$ is

$$F_2(x, t, u, v) u_{n_2 0}^{c_{n_2}} u_{n_2 - 10}^{c_{n_2 - 1}} \ldots u_{10}^{c_1} v_{m_2 0}^{d_{m_2}} v_{m_2 - 10}^{d_{m_2 - 1}} \ldots v_{10}^{d_1}, \tag{9.50}$$

where $F_1(x, t, u, v) \neq 0$, $F_2(x, t, u, v) \neq 0$, $n_1, n_2 \geq 2$, $\alpha_{n_1} \geq 1$, $c_{n_2} \geq 1$ is the highest power of the highest-order derivative. Similarly, the leading term in $H'(x', t', u', v', u'_{10}, v'_{10}, \ldots, u'_{n_1 0}, v'_{m_1 0})$ is

$$G_1(x', t', u', v') u'^{A_{n_1}}_{n_1 0} u'^{A_{n_1 - 1}}_{n_1 - 10} \ldots u'^{A_1}_{10} v'^{B_{m_1}}_{m_1 0} v'^{B_{m_1 - 1}}_{m_1 - 10} \ldots v'^{B_1}_{10}, \tag{9.51}$$

and the corresponding term in $F'(x', t', u', v', u'_{10}, v'_{10}, \ldots, u'_{n0}, v'_{m0})$ is

$$G_2(x', t', u', v') u'^{C_{n_2}}_{n_2 0} u'^{C_{n_2 - 1}}_{n_2 - 10} \ldots u'^{C_1}_{10} v'^{D_{m_2}}_{m_2 0} v'^{D_{m_2 - 1}}_{m_2 - 10} \ldots v'^{D_1}_{10}, \tag{9.52}$$

where $G_1(x,t,u,v) \neq 0,\ G_2(x,t,u,v) \neq 0,\ n_1, n_2 \geq 2,\ A_{n_1} \geq 1,\ C_{n_2} \geq 1$.

Substituting for u_{01} and v_{01} by

$$H(x, t, u, v, u_{10}, v_{10}, \ldots, u_{n_1 0}, v_{m_1 0})$$

and

$$F(x, t, u, v, u_{10}, v_{10}, \ldots, u_{n_2 0}, v_{m_2 0}),$$

respectively, in the transformed form of

$$u'_{01} = H'(x', t', u', v', u'_{10}, v'_{10}, \ldots, u'_{n_1 0}, v'_{m_1 0})$$

and

$$v'_{01} = F'(x', t', u', v', u'_{10}, v'_{10}, \ldots, u'_{n_2 0}, v'_{m_2 0}),$$

and using the identities (9.43)-(9.48) we arrive:

$$\begin{aligned}
u'_{t'} &= u'_{01} = -\frac{1}{P_X Q_t}(P_T R_X - P_X R_T) \\
&= -\frac{1}{P_X Q_t}\left[u_{10}(P_t R_u - P_u R_t) + v_{10}(P_t R_v - P_v R_t) + (P_t R_x - P_x R_t)\right. \\
&- H[v_{10}(P_v R_u - P_u R_v) + P_x R_u - P_u R_x] - F[u_{10}(P_u R_v - P_v R_u) + P_x R_v - P_v R_x] \\
&\equiv H'\left(P, Q, R, S, \left(\frac{1}{P_X}D\right)R, \left(\frac{1}{P_X}D\right)S, \ldots, \left(\frac{1}{P_X}D\right)^{n_1}R, \left(\frac{1}{P_X}D\right)^{m_1}S\right) \qquad (9.53)
\end{aligned}$$

and

$$\begin{aligned}
v'_{t'} &= v'_{01} = -\frac{1}{P_X Q_t}(P_T S_X - P_X S_T) \\
&= -\frac{1}{P_X Q_t}\left[u_{10}(P_t S_u - P_u S_t) + v_{10}(P_t S_v - P_v S_t) + (P_t S_x - P_x S_t)\right. \\
&- H[v_{10}(P_v S_u - P_u S_v) + P_x S_u - P_u S_x] - F[u_{10}(P_u S_v - P_v S_u) + P_x S_v - P_v S_x] \\
&\equiv F'\left(P, Q, R, S, \left(\frac{1}{P_X}D\right)R, \left(\frac{1}{P_X}D\right)S, \ldots, \left(\frac{1}{P_X}D\right)^{n_2}R, \left(\frac{1}{P_X}D\right)^{m_2}S\right), \qquad (9.54)
\end{aligned}$$

respectively.

Retaining the leading terms on both the left and the right sides of (9.53), (9.54) and making use of (9.49)-(9.52), lemma 9.5 and the fact that $J \neq 0$, produces the following terms:

from (9.49):

$$\frac{1}{P_X Q_t}\Big[F_1(x,t,u,v)u_{n_1 0}^{\alpha_{n_1}}u_{n_1-10}^{\alpha_{n_1-1}}\dots u_{10}^{\alpha_1}v_{m_1 0}^{\beta_{m_1}}v_{m_1-10}^{\beta_{m_1-1}}\dots v_{10}^{\beta_1+1}$$

$$-F_2(x,t,u,v)u_{n_2 0}^{c_{n_2}}u_{n_2-10}^{c_{n_2-1}}\dots u_{10}^{c_1+1}v_{m_2 0}^{d_{m_2}}v_{m_2-10}^{d_{m_2-1}}\dots v_{10}^{d_1}\Big](P_vR_u-P_uR_v), \qquad (9.55)$$

from (9.50):

$$\frac{1}{P_X Q_t}\Big[F_1(x,t,u,v)u_{n_1 0}^{\alpha_{n_1}}u_{n_1-10}^{\alpha_{n_1-1}}\dots u_{10}^{\alpha_1}v_{m_1 0}^{\beta_{m_1}}v_{m_1-10}^{\beta_{m_1-1}}\dots v_{10}^{\beta_1+1}$$

$$-F_2(x,t,u,v)u_{n_2 0}^{c_{n_2}}u_{n_2-10}^{c_{n_2-1}}\dots u_{10}^{c_1+1}v_{m_2 0}^{d_{m_2}}v_{m_2-10}^{d_{m_2-1}}\dots v_{10}^{d_1}\Big](P_vS_u-P_uS_v), \qquad (9.56)$$

from (9.51):

$$G_1(P,Q,R,S)\frac{1}{P_X^{a_1}}(P_vR_u-P_uR_v)^{b_1}(P_uS_v-P_vS_u)^{b_2}u_{n_1 0}^{A_{n_1}}\dots u_{20}^{A_2}u_{10}^{b_2+A_1}v_{m_1 0}^{B_{m_1}}\dots v_{20}^{B_2}v_{10}^{b_1+B_1}, \qquad (9.57)$$

where

$$a_1=(n_1+1)A_{n_1}+(m_1+1)B_{m_1}+n_1A_{n_1-1}+m_1B_{m_1-1}+\dots+2A_1+2B_1,$$

$$b_1=A_{n_1}+A_{n_1-1}+\dots+A_1,$$

$$b_2=B_{m_1}+B_{m_1-1}+\dots+B_1,$$

and from (9.52):

$$G_2(P,Q,R,S)\frac{1}{P_X^{a_2}}(P_vR_u-P_uR_v)^{b_3}(P_uS_vP_vS_u)^{b_4}u_{n_2 0}^{C_{n_2}}\dots u_{20}^{C_2}u_{10}^{b_4+C_1}v_{m_2 0}^{D_{m_2}}\dots v_{20}^{C_2}v_{10}^{b_3+D_1}, \qquad (9.58)$$

where

$$a_2=(n_2+1)C_{n_2}+(m_2+1)D_{m_2}+n_2C_{n_2-1}+m_2D_{m_2-1}+\dots+2C_1+2D_1,$$

$$b_3=C_{n_2}+C_{n_2-1}+\dots+C_1,$$

$$b_4=D_{m_2}+D_{m_2-1}+\dots+D_1.$$

Multiplying by $P_X^{a_1}$ and $P_X^{a_2}$ equations (9.55) and (9.56), respectively, the leading terms gives

$$\frac{(u_x^{a_1-1}P_u^{a_1-1}+v_x^{a_1-1}P_v^{a_1-1})}{Q_t}\Big[F_1(x,t,u,v)u_{n_10}^{\alpha_{n_1}}u_{n_1-10}^{\alpha_{n_1-1}}\dots u_{10}^{\alpha_1}v_{m_10}^{\beta_{m_1}}v_{m_1-10}^{\beta_{m_1-1}}\dots v_{10}^{\beta_1+1}$$

$$-F_2(x,t,u,v)u_{n_20}^{c_{n_2}}u_{n_2-10}^{c_{n_2-1}}\dots u_{10}^{c_1+1}v_{m_20}^{d_{m_2}}v_{m_2-10}^{d_{m_2-1}}\dots v_{10}^{d_1}\Big](P_vR_u-P_uR_v) \tag{9.59}$$

and

$$\frac{(u_x^{a_2-1}P_u^{a_2-1}+v_x^{a_2-1}P_v^{a_2-1})}{Q_t}\Big[F_1(x,t,u,v)u_{n_10}^{\alpha_{n_1}}u_{n_1-10}^{\alpha_{n_1-1}}\dots u_{10}^{\alpha_1}v_{m_10}^{\beta_{m_1}}v_{m_1-10}^{\beta_{m_1-1}}\dots v_{10}^{\beta_1+1}$$

$$-F_2(x,t,u,v)u_{n_20}^{c_{n_2}}u_{n_2-10}^{c_{n_2-1}}\dots u_{10}^{c_1+1}v_{m_20}^{d_{m_2}}v_{m_2-10}^{d_{m_2-1}}\dots v_{10}^{d_1}\Big](P_vS_u-P_uS_v). \tag{9.60}$$

Similarly, multiplying by $P_X^{a_1}$ and $P_X^{a_2}$ equations (9.57) and (9.58), respectively, the leading terms gives

$$G_1(P,Q,R,S)(P_vR_u-P_uR_v)^{b_1}(P_uS_v-P_vS_u)^{b_2}u_{n_10}^{A_{n_1}}\dots u_{10}^{b_2+A_1}v_{m_10}^{B_{m_1}}\dots v_{10}^{b_1+B_1} \tag{9.61}$$

and

$$G_2(P,Q,R,S)(P_vR_u-P_uR_v)^{b_3}(P_uS_v-P_vS_u)^{b_4}u_{n_20}^{C_{n_2}}\dots u_{10}^{b_4+C_1}v_{m_20}^{D_{m_2}}\dots v_{10}^{b_3+D_1}. \tag{9.62}$$

The equation (9.59) must be matched by equation (9.61) and equation (9.60) must be matched by equation (9.62). Therefore

$$P_vS_u-P_uS_v=0,$$

$$P_vR_u-P_uR_v=0.$$

The solution of the above system is $P_u=P_v=0$, otherwise $J=0$. Therefore $P=P(x,t)$. □

9.4.3 System of two equations of the form $u_{11}=H(x,t,u,v,\dots,u_{n_10},v_{m_10})$

In this subsection, we are working on systems where u_{xt} and v_{xt} are related with $x,\ t,\ u,\ v$ and x-derivatives of u and v. Firstly, we consider that the order of derivatives is bigger or equal to 3. Then, we consider that the order of derivatives is 2 and finally we have lower-order derivatives.

Now, we give a similar theorem as theorem 3.4.

Theorem 9.4. ($n_i \geq 3$, $m_j \geq 3$) *The point transformation* $x' = P(x,t,u,v)$, $t' = Q(x,t,u,v)$, $u' = R(x,t,u,v)$, $v' = S(x,t,u,v)$ *transforms*

$$u_{11} = H(x,t,u,v,\ldots,u_{n_10},v_{m_10}), \qquad v_{11} = F(x,t,u,v,\ldots,u_{n_20},v_{m_20}) \tag{9.63}$$

into

$$u'_{11} = H'(x',t',u',v',\ldots,u'_{n_10},v'_{m_10}), \qquad v'_{11} = F'(x',t',u',v',\ldots,u'_{n_20},v'_{m_20}), \tag{9.64}$$

where at least $n_1 \geq 3$ *or* $m_2 \geq 3$ *if and only if*

$$\begin{aligned} P &= P(x,t), \quad Q = Q(t), \\ R &= A_1(t)u + A_2(t)v + B_1(x,t), \quad S = A_3(t)u + A_4(t)v + B_2(x,t), \end{aligned}$$

$$\begin{aligned} H' &= P_x^{-1}Q_t^{-1}(A_1H + A_2F) - P_x^{-2}P_tQ_t^{-1}(A_1u_{20} + A_2v_{20}) + u_{10}P_x^{-1}Q_t^{-1}(A_{1_t} - (P_x^{-1}P_t)_xA_1) \\ &+ v_{10}P_x^{-1}Q_t^{-1}(A_{2_t} - (P_x^{-1}P_t)_xA_2) + P_x^{-1}Q_t^{-1}(B_{1_t} - P_x^{-1}P_tB_{1_x})_x, \end{aligned} \tag{9.65}$$

$$\begin{aligned} F' &= P_x^{-1}Q_t^{-1}(A_3H + A_4F) - P_x^{-2}P_tQ_t^{-1}(A_3u_{20} + A_4v_{20}) + u_{10}P_x^{-1}Q_t^{-1}(A_{3_t} - (P_x^{-1}P_t)_xA_3) \\ &+ v_{10}P_x^{-1}Q_t^{-1}(A_{4_t} - (P_x^{-1}P_t)_xA_4) + P_x^{-1}Q_t^{-1}(B_{2_t} - P_x^{-1}P_tB_{2_x})_x. \end{aligned} \tag{9.66}$$

Proof. From the theorem 9.1 with $\{(p = n_1,\ q = 0)$ and $(\mu = m_2,\ \nu = 0)\}$ it follows that $Q = Q(t)$. Relations (9.12) and (9.14) simplifies to

$$u'_{i0} = P_X^{-1}(u'_{i-10})_X, \quad v'_{i0} = P_X^{-1}(v'_{i-10})_X, \qquad i \geq 1,$$

so that no t derivatives of u and v arise from u'_{i0} and v'_{i0}, $i \geq 0$ and H, F transform to the forms H', F'.

Hence, system (9.63) only transform to (9.64) if u'_{11} and v'_{11} give rise to no terms of u_{02}, u_{01}, v_{02} and v_{01}. Thus $\frac{\partial u'_{11}}{\partial u_{01}} \equiv 0$, $\frac{\partial u'_{11}}{\partial v_{01}} \equiv 0$, $\frac{\partial v'_{11}}{\partial u_{01}} \equiv 0$ and $\frac{\partial v'_{11}}{\partial v_{01}} \equiv 0$, so that

$$\begin{aligned} \frac{\partial}{\partial u_{01}}\left(\frac{\partial u'_{11}}{\partial u_{20}}\right) &= -Q_t^2P_u\left[v_{10}(P_vR_u - P_uR_v) + P_xR_u - P_uR_x\right]\delta^{-3} \equiv 0, \\ \frac{\partial}{\partial u_{01}}\left(\frac{\partial v'_{11}}{\partial u_{20}}\right) &= -Q_t^2P_u\left[v_{10}(P_vS_u - P_uS_v) + P_xS_u - P_uS_x\right]\delta^{-3} \equiv 0, \\ \frac{\partial}{\partial u_{01}}\left(\frac{\partial u'_{11}}{\partial v_{20}}\right) &= -Q_t^2P_u\left[u_{10}(P_uR_v - P_vR_u) + P_xR_v - P_vR_x\right]\delta^{-3} \equiv 0, \end{aligned}$$

$$\frac{\partial}{\partial u_{01}}\left(\frac{\partial v'_{11}}{\partial v_{20}}\right) = -Q_t^2 P_u \left[u_{10}(P_u S_v - P_v S_u) + P_x S_v - P_v S_x\right] \delta^{-3} \equiv 0,$$

and

$$\frac{\partial}{\partial v_{01}}\left(\frac{\partial u'_{11}}{\partial u_{20}}\right) = -Q_t^2 P_v \left[v_{10}(P_v R_u - P_u R_v) + P_x R_u - P_u R_x\right] \delta^{-3} \equiv 0,$$

$$\frac{\partial}{\partial v_{01}}\left(\frac{\partial v'_{11}}{\partial u_{20}}\right) = -Q_t^2 P_v \left[v_{10}(P_v S_u - P_u S_v) + P_x S_u - P_u S_x\right] \delta^{-3} \equiv 0,$$

$$\frac{\partial}{\partial v_{01}}\left(\frac{\partial u'_{11}}{\partial v_{20}}\right) = -Q_t^2 P_v \left[u_{10}(P_u R_v - P_v R_u) + P_x R_v - P_v R_x\right] \delta^{-3} \equiv 0,$$

$$\frac{\partial}{\partial v_{01}}\left(\frac{\partial v'_{11}}{\partial v_{20}}\right) = -Q_t^2 P_v \left[u_{10}(P_u S_v - P_v S_u) + P_x S_v - P_v S_x\right] \delta^{-3} \equiv 0.$$

Hence, $P(x,t)$. Then the following system:

$$\frac{\partial u'_{11}}{\partial u_{01}} = \delta^{-1}(R_{ux} + u_{10}R_{uu} + v_{10}R_{uv}) \equiv 0,$$

$$\frac{\partial u'_{11}}{\partial v_{01}} = \delta^{-1}(R_{vx} + u_{10}R_{uv} + v_{10}R_{vv}) \equiv 0,$$

$$\frac{\partial v'_{11}}{\partial u_{01}} = \delta^{-1}(S_{ux} + u_{10}S_{uu} + v_{10}S_{uv}) \equiv 0,$$

$$\frac{\partial v'_{11}}{\partial v_{01}} = \delta^{-1}(S_{vx} + u_{10}S_{uv} + v_{10}S_{vv}) \equiv 0,$$

give the form of $R,\ S$. Now, system (9.63) transform to system (9.64) and $H',\ G'$ are given in terms of $H,\ G$ by equations (9.65) and (9.66), respectively. □

In the theorem below, we give a generalization of theorem 3.5.

Theorem 9.5. $(n_1 = n_2 = m_1 = m_2 = 2)$ *The point transformations* $x' = P(x,t,u,v),\ t' = Q(x,t,u,v),\ u' = R(x,t,u,v),\ v' = S(x,t,u,v)$ *which transform*

$$u_{11} = H(x,t,u,v,u_{10},v_{10},u_{20},v_{20}), \quad v_{11} = F(x,t,u,v,u_{10},v_{10},u_{20},v_{20}) \tag{9.67}$$

into

$$u'_{11} = H'(x',t',u',v',u'_{10},v'_{10},u'_{20},v'_{20}), \quad v'_{11} = F'(x',t',u',v',u'_{10},v'_{10},u'_{20},v'_{20}) \tag{9.68}$$

belongs to one of the two categories:

a) *$P,\ Q,\ R,\ S,\ H$ and F restricted as in the condition for theorem 9.4;*

b) $P = P(x,t), \quad Q = Q(x,t),$

$$R = H_1(x,t)u + H_2(x,t)v + H_3(x,t), \quad S = H_4(x,t)u + H_5(x,t)v + H_6(x,t),$$

$$H' = -P_x Q_x^{-1} u'_{20} + D_1(x',t',u',v')u'_{10} + D_2(x',t',u',v')v'_{10} + D_3(x',t',u',v'),$$

$$F' = -P_x Q_x^{-1} v'_{20} + D_4(x',t',u',v')u'_{10} + D_5(x',t',u',v')v'_{10} + D_6(x',t',u',v'),$$

$$H = Q_t Q_x^{-1} u_{20} + f_1(x,t,u,v)u_{10} + f_2(x,t,u,v)v_{10} + f_3(x,t,u,v),$$

$$F = Q_t Q_x^{-1} v_{20} + f_4(x,t,u,v)u_{10} + f_5(x,t,u,v)v_{10} + f_6(x,t,u,v).$$

Proof. Let $E_1 = u'_{11} - H', \quad E_2 = v'_{11} - F'$, apply the transformation and then substitute $u_{11} = H$ and $v_{11} = F$. E_1 and E_2 will now, possibly, depend on $x, t, u, v, u_{10}, v_{10}, u_{01}, v_{01}, u_{20}, v_{20}, u_{02}$ and v_{02}, but for system (9.67) to transform into system (9.68), we require that $E_1 \equiv 0$ and $E_2 \equiv 0$.

In particular,

$$\frac{\partial E_1}{\partial u_{02}} = \frac{\partial u'_{11}}{\partial u_{02}} - \frac{\partial H'}{\partial u'_{20}}\frac{\partial u'_{20}}{\partial u_{02}} - \frac{\partial H'}{\partial v'_{20}}\frac{\partial v'_{20}}{\partial u_{02}} \equiv 0,$$

$$\frac{\partial E_1}{\partial v_{02}} = \frac{\partial u'_{11}}{\partial v_{02}} - \frac{\partial H'}{\partial u'_{20}}\frac{\partial u'_{20}}{\partial v_{02}} - \frac{\partial H'}{\partial v'_{20}}\frac{\partial v'_{20}}{\partial v_{02}} \equiv 0,$$

$$\frac{\partial E_2}{\partial u_{02}} = \frac{\partial v'_{11}}{\partial u_{02}} - \frac{\partial F'}{\partial u'_{20}}\frac{\partial u'_{20}}{\partial u_{02}} - \frac{\partial F'}{\partial v'_{20}}\frac{\partial v'_{20}}{\partial u_{02}} \equiv 0,$$

$$\frac{\partial E_2}{\partial v_{02}} = \frac{\partial v'_{11}}{\partial v_{02}} - \frac{\partial F'}{\partial u'_{20}}\frac{\partial u'_{20}}{\partial v_{02}} - \frac{\partial F'}{\partial v'_{20}}\frac{\partial v'_{20}}{\partial v_{02}} \equiv 0,$$

and from the lemma 9.1, corollary 9.1, equations (9.29), (9.31), (9.33), (9.35) corresponding to $\{p = q = 1\}$ and $\{p = 2,\ q = 0\}$, we arrive to the following system:

$$\delta^{-3} Q_X \left(\frac{\partial H'}{\partial u'_{20}} Q_X J_1 + \frac{\partial H'}{\partial v'_{20}} Q_X J_3 + P_X J_1 \right) = 0,$$

$$\delta^{-3} Q_X \left(\frac{\partial H'}{\partial u'_{20}} Q_X J_2 + \frac{\partial H'}{\partial v'_{20}} Q_X J_4 + P_X J_2 \right) = 0,$$

$$\delta^{-3} Q_X \left(\frac{\partial F'}{\partial u'_{20}} Q_X J_1 + \frac{\partial F'}{\partial v'_{20}} Q_X J_3 + P_X J_3 \right) = 0,$$

$$\delta^{-3} Q_X \left(\frac{\partial F'}{\partial u'_{20}} Q_X J_2 + \frac{\partial F'}{\partial v'_{20}} Q_X J_4 + P_X J_4 \right) = 0.$$

Hence, either (a) $Q_X = 0$, so that $Q = Q(t)$, or (b) $Q_X \neq 0$, $H' = -P_X Q_X^{-1} u'_{20} + A_1(x', t', u', v', u'_{10}, v'_{10})$ and $F' = -P_X Q_X^{-1} v'_{20} + A_2(x', t', u', v', u'_{10}, v'_{10})$.

For case (a) the same analysis applies as for the theorem 9.4.

For case (b) system (9.68) is linear in the second-order derivatives of u' and v' and this will transform into a system which is also linear in second -order derivatives. Thus

$$H = B_1(x, t, u, v, u_{10}, v_{10})u_{20} + B_2(x, t, u, v, u_{10}, v_{10}),$$

$$F = B_3(x, t, u, v, u_{10}, v_{10})v_{20} + B_4(x, t, u, v, u_{10}, v_{10}).$$

Since

$$\frac{\partial E_1}{\partial u_{20}} = -\delta^{-1} Q_x^{-1} R_u (B_1 Q_x - Q_t) \equiv 0,$$

$$\frac{\partial E_1}{\partial v_{20}} = -\delta^{-1} Q_x^{-1} R_v (B_3 Q_x - Q_t) \equiv 0,$$

$$\frac{\partial E_2}{\partial u_{20}} = -\delta^{-1} Q_x^{-1} S_u (B_1 Q_x - Q_t) \equiv 0,$$

$$\frac{\partial E_2}{\partial v_{20}} = -\delta^{-1} Q_x^{-1} S_v (B_3 Q_x - Q_t) \equiv 0,$$

it follows that $B_1 = Q_x^{-1} Q_t$, $B_3 = Q_x^{-1} Q_t$. Next,

$$\frac{\partial^2 E_1}{\partial u_{01}^2} = -\delta^{-2} Q_x^2 (R_u^2 A_{1_{u'_{10}u'_{10}}} + 2R_u S_u A_{1_{u'_{10}v'_{10}}} + S_u^2 A_{1_{v'_{10}v'_{10}}}) \equiv 0,$$

$$\frac{\partial^2 E_1}{\partial v_{01}^2} = -\delta^{-2} Q_x^2 (R_v^2 A_{1_{u'_{10}u'_{10}}} + 2R_v S_v A_{1_{u'_{10}v'_{10}}} + S_v^2 A_{1_{v'_{10}v'_{10}}}) \equiv 0,$$

$$\frac{\partial^2 E_1}{\partial u_{01} v_{01}} = -\delta^{-2} Q_x^2 (R_u R_v A_{1_{u'_{10}u'_{10}}} + (R_u S_v + R_v S_u) A_{1_{u'_{10}v'_{10}}} + S_u S_v A_{1_{v'_{10}v'_{10}}}) \equiv 0.$$

The solution of this system is:

$$A_1 = D_1(x', t', u', v')u'_{10} + D_2(x', t', u', v')v'_{10} + D_3(x', t', u', v').$$

Similarly, from equation E_2:

$$A_2 = D_4(x', t', u', v')u'_{10} + D_5(x', t', u', v')v'_{10} + D_6(x', t', u', v').$$

Then,

$$\frac{\partial^2 E_1}{\partial u_{10} u_{01}} = -\delta^{-1} R_{uu} \equiv 0,$$

$$\frac{\partial^2 E_1}{\partial u_{10} v_{01}} = -\delta^{-1} R_{uv} \equiv 0,$$
$$\frac{\partial^2 E_1}{\partial u_{01} v_{10}} = -\delta^{-1} R_{vv} \equiv 0,$$

so that

$$R = H_1(x,t)u + H_2(x,t)v + H_3(x,t).$$

Similarly, from equation E_2:

$$S = H_4(x,t)u + H_5(x,t)v + H_6(x,t).$$

Also,

$$\frac{\partial^2 E_1}{\partial u_{10}^2} = -\delta^{-1}(B_{2_{u_{10}u_{10}}} H_1 + B_{4_{u_{10}u_{10}}} H_2) \equiv 0,$$
$$\frac{\partial^2 E_1}{\partial v_{10}^2} = -\delta^{-1}(B_{2_{v_{10}v_{10}}} H_1 + B_{4_{v_{10}v_{10}}} H_2) \equiv 0,$$
$$\frac{\partial^2 E_1}{\partial u_{10} v_{10}} = -\delta^{-1}(B_{2_{u_{10}v_{10}}} H_1 + B_{4_{u_{10}v_{10}}} H_2) \equiv 0,$$
$$\frac{\partial^2 E_2}{\partial u_{10}^2} = -\delta^{-1}(B_{2_{u_{10}u_{10}}} H_4 + B_{4_{u_{10}u_{10}}} H_5) \equiv 0,$$
$$\frac{\partial^2 E_2}{\partial v_{10}^2} = -\delta^{-1}(B_{2_{v_{10}v_{10}}} H_4 + B_{4_{v_{10}v_{10}}} H_5) \equiv 0,$$
$$\frac{\partial^2 E_2}{\partial u_{10} v_{10}} = -\delta^{-1}(B_{2_{u_{10}v_{10}}} H_4 + B_{4_{u_{10}v_{10}}} H_5) \equiv 0.$$

Since $H_1 H_5 - H_2 H_4 \neq 0$ (otherwise $J = 0$), from the first and the fourth equations, we have

$$B_{2_{u_{10}u_{10}}} = 0, \quad B_{4_{u_{10}u_{10}}} = 0.$$

Using the second and the fifth equations, we lead to

$$B_{2_{v_{10}v_{10}}} = 0, \quad B_{4_{v_{10}v_{10}}} = 0.$$

Finally, using the third and the sixth equations,

$$B_{2_{u_{10}v_{10}}} = 0, \quad B_{4_{u_{10}v_{10}}} = 0.$$

Solving the above system for B_2 and B_4 we take

$$B_2 = f_1(x,t,u,v)u_{10} + f_2(x,t,u,v)v_{10} + f_3(x,t,u,v),$$

$$B_4 = f_4(x,t,u,v)u_{10} + f_5(x,t,u,v)v_{10} + f_6(x,t,u,v).$$

Solving the system of equations

$$\frac{\partial E_1}{\partial u_{10}} \equiv 0, \qquad \frac{\partial E_1}{\partial v_{10}} \equiv 0,$$

$$\frac{\partial E_2}{\partial u_{10}} \equiv 0, \qquad \frac{\partial E_2}{\partial v_{10}} \equiv 0,$$

and then

$$\frac{\partial E_1}{\partial u_{01}} \equiv 0, \qquad \frac{\partial E_1}{\partial v_{01}} \equiv 0,$$

$$\frac{\partial E_2}{\partial u_{01}} \equiv 0, \qquad \frac{\partial E_2}{\partial v_{01}} \equiv 0,$$

give the form of $f_1,\ f_2,\ f_4,\ f_5$ and $D_1,\ D_2,\ D_4,\ D_5$, respectively, in term of functions $H_1,\ H_2,\ H_4,\ H_5$.

Finally, $E_1 \equiv 0$ and $E_2 \equiv 0$ provides a length relation between $f_3,\ f_6$ and $D_3,\ D_6$. $\square$

In the following theorem, we relate $u_{xt},\ v_{xt}$ and $u'_{x't'},\ v'_{x't'}$ with lower-order x-derivatives of $u,\ v$ and $u',\ v'$, respectively. That is, we generalize theorem 3.6.

Theorem 9.6. $(n_1, n_2, m_1, m_2 = 0, 1)$ *The point transformations* $x' = P(x,t,u,v),\ t' = Q(x,t,u,v),\ u' = R(x,t,u,v),\ v' = S(x,t,u,v)$ *which transform*

$$u_{11} = H(x,t,u,v,u_{10},v_{10}), \qquad v_{11} = F(x,t,u,v,u_{10},v_{10}) \tag{9.69}$$

into

$$u'_{11} = H'(x',t',u',v',u'_{10},v'_{10}), \qquad v'_{11} = F'(x',t',u',v',u'_{10},v'_{10}) \tag{9.70}$$

belongs to one of the two categories:

$$\begin{aligned} a)\ P &= P(x), \quad Q = Q(t), \\ R &= A_1(t)u + A_2(t)v + B_1(x,t), \\ S &= A_3(t)u + A_4(t)v + B_2(x,t), \\ H' &= P_x^{-1}Q_t^{-1}(A_1H + A_2F + A_{1_t}u_{10} + A_{2_t}v_{10} + B_{1_{xt}}), \\ F' &= P_x^{-1}Q_t^{-1}(A_3H + A_4F + A_{3_t}u_{10} + A_{4_t}v_{10} + B_{2_{xt}}); \end{aligned}$$

$$
\begin{aligned}
b)\ P &= P(t), \quad Q = Q(x),\\
R &= A_1(x,t)u + A_2(x,t)v + A_3(x,t),\\
S &= A_4(x,t)u + A_5(x,t)v + A_6(x,t),\\
H &= (A_1A_5 - A_2A_4)^{-1}\left[(-A_{1_t}A_5 + A_2A_{4_t})u_{10} + (A_2A_{5_t} - A_{2_t}A_5)v_{10}\right]\\
&+ D_1(x,t,u,v),\\
F &= (A_1A_5 - A_2A_4)^{-1}\left[(-A_1A_{4_t} + A_{1_t}A_4)u_{10} + (A_4A_{2_t} - A_1A_{5_t})v_{10}\right]\\
&+ D_2(x,t,u,v),\\
H' &= Q_x^{-1}(A_1A_5 - A_2A_4)^{-1}\left[(A_{1_x}A_5 - A_{2_x}A_4)u'_{10} + (A_1A_{2_x} - A_{1_x}A_2)v'_{10}\right]\\
&+ H_1(x',t',u',v'),\\
F' &= Q_x^{-1}(A_1A_5 - A_2A_4)^{-1}\left[(A_{4_x}A_5 - A_4A_{5_x})u'_{10} + (A_1A_{5_x} - A_2A_{4_x})v'_{10}\right]\\
&+ H_2(x',t',u',v'),
\end{aligned}
$$

where

$$
\begin{aligned}
D_1 &= \left((A_1A_5 - A_2A_4)^{-1}(-A_{1_t}A_5 + A_2A_{4_t})\right)_x u + \left((A_1A_5 - A_2A_4)^{-1}(A_2A_{5_t} - A_{2_t}A_5)\right)_x v\\
&- A_5\left((A_1A_5 - A_2A_4)^{-1}A_{3_t}\right)_x + A_2\left((A_1A_5 - A_2A_4)^{-1}A_{6_t}\right)_x\\
&+ (A_1A_5 - A_2A_4)^{-1}(A_{2_x}A_{6_t} - A_{3_t}A_{5_x}) + P_tQ_x(A_1A_5 - A_2A_4)^{-1}(A_5H_1 - A_2H_2),\\
D_2 &= \left((A_1A_5 - A_2A_4)^{-1}(A_{1_t}A_4 - A_1A_{4_t})\right)_x u + \left((A_1A_5 - A_2A_4)^{-1}(A_{2_t}A_4 - A_1A_{5_t})\right)_x v\\
&- A_1\left((A_1A_5 - A_2A_4)^{-1}A_{6_t}\right)_x + A_4\left((A_1A_5 - A_2A_4)^{-1}A_{3_t}\right)_x\\
&+ (A_1A_5 - A_2A_4)^{-1}(A_{3_t}A_{4_x} - A_{1_x}A_{6_t}) + P_tQ_x(A_1A_5 - A_2A_4)^{-1}(A_1H_2 - A_4H_1).
\end{aligned}
\tag{9.71}
$$

Proof. From the theorem 9.1 with $\{(p = 1,\ q = 1)$ and $(\mu = 1,\ \nu = 1)\}$ we have two cases to consider: (a) $P = P(x),\ Q = Q(t)$ and (b) $P = P(t),\ Q = Q(x)$.

For case (a) H' and F' transforms into functions of $x, t, u, v, u_{10}, v_{10}$ so we required that u'_{11} and v'_{11} transforms into functions of the same variables, having replaced u_{11} and v_{11} by H and G, respectively. Hence,

$$
\begin{aligned}
\frac{\partial u'_{11}}{\partial u_{01}} &= \delta^{-1}(R_{uu}u_{10} + R_{uv}v_{10} + R_{ux}) \equiv 0,\\
\frac{\partial u'_{11}}{\partial v_{01}} &= \delta^{-1}(R_{uv}u_{10} + R_{vv}v_{10} + R_{vx}) \equiv 0,\\
\frac{\partial v'_{11}}{\partial u_{01}} &= \delta^{-1}(S_{uu}u_{10} + S_{uv}v_{10} + S_{ux}) \equiv 0,
\end{aligned}
$$

$$\frac{\partial v'_{11}}{\partial v_{01}} = \delta^{-1}(S_{uv}u_{10} + S_{vv}v_{10} + S_{vx}) \equiv 0,$$

giving

$$R = A_1(t)u + A_2(t)v + B_1(x,t),$$

$$S = A_3(t)u + A_4(t)v + B_1(x,t).$$

System (9.69) now transform into system (9.70) with H' and F' as stated in relation

$$\begin{aligned} H' &= P_x^{-1}Q_t^{-1}(A_1H + A_2F + A_{1_t}u_{10} + A_{2_t}v_{10} + B_{1_{xt}}), \\ F' &= P_x^{-1}Q_t^{-1}(A_3H + A_4F + A_{3_t}u_{10} + A_{4_t}v_{10} + B_{2_{xt}}). \end{aligned}$$

In case (b) let $E_1 = u'_{11} - H'$ and $E_2 = v'_{11} - F'$ with H and F substituted for u_{11} and v_{11}, respectively. Thus $E_1 \equiv 0$ and $E_2 \equiv 0$ for the given transformation to exist. Hence,

$$\frac{\partial E_1}{\partial u_{10}} = \delta^{-1}(R_{uu}u_{01} + R_{uv}v_{01} + R_{ut} + H_{u_{10}}R_u + F_{u_{10}}R_v) \equiv 0,$$

$$\frac{\partial E_1}{\partial v_{10}} = \delta^{-1}(R_{uv}u_{01} + R_{vv}v_{01} + R_{vt} + H_{v_{10}}R_u + F_{v_{10}}R_v) \equiv 0,$$

$$\frac{\partial E_2}{\partial u_{10}} = \delta^{-1}(S_{uu}u_{01} + S_{uv}v_{01} + S_{ut} + H_{u_{10}}S_u + F_{u_{10}}S_v) \equiv 0,$$

$$\frac{\partial E_2}{\partial v_{10}} = \delta^{-1}(S_{uv}u_{01} + S_{vv}v_{01} + S_{vt} + H_{v_{10}}S_u + F_{v_{10}}S_v) \equiv 0,$$

giving

$$R = A_1(x,t)u + A_2(x,t)v + A_3(x,t),$$

$$S = A_4(x,t)u + A_5(x,t)v + A_6(x,t),$$

and

$$\begin{aligned} H &= (A_1A_5 - A_2A_4)^{-1}\left[(-A_{1_t}A_5 + A_2A_{4_t})u_{10} + (A_2A_{5_t} - A_{2_t}A_5)v_{10}\right] \\ &+ D_1(x,t,u,v), \\ F &= (A_1A_5 - A_2A_4)^{-1}\,(-A_1A_{4_t} + A_{1_t}A_4)u_{10} + (-A_1A_{5_t} + A_{2_t}A_4)v_{10}] \\ &+ D_2(x,t,u,v). \end{aligned}$$

Also,

$$\frac{\partial E_1}{\partial u_{01}} = \delta^{-1}\left(A_{1_x} - Q_xH'_{u'_{10}}A_1 - Q_xH'_{v'_{10}}A_4\right) \equiv 0,$$

$$
\begin{aligned}
\frac{\partial E_1}{\partial v_{01}} &= \delta^{-1}\left(A_{2_x} - Q_x H'_{u'_{10}} A_2 - Q_x H'_{v'_{10}} A_5\right) \equiv 0, \\
\frac{\partial E_2}{\partial u_{01}} &= \delta^{-1}\left(A_{4_x} - Q_x F'_{u'_{10}} A_1 - Q_x F'_{v'_{10}} A_4\right) \equiv 0, \\
\frac{\partial E_2}{\partial v_{01}} &= \delta^{-1}\left(A_{5_x} - Q_x F'_{u'_{10}} A_2 - Q_x F'_{v'_{10}} A_5\right) \equiv 0,
\end{aligned}
$$

so that

$$
\begin{aligned}
H' &= Q_x^{-1}(A_1A_5 - A_2A_4)^{-1}\left[(A_{1_x}A_5 - A_{2_x}A_4)u'_{10} + (A_1A_{2_x} - A_{1_x}A_2)v'_{10}\right] \\
&+ H_1(x', t', u', v'), \\
F' &= Q_x^{-1}(A_1A_5 - A_2A_4)^{-1}\left[(-A_4A_{5_x} + A_{4_x}A_5)u'_{10} + (A_1A_{5_x} - A_2A_{4_x})v'_{10}\right] \\
&+ H_2(x', t', u', v').
\end{aligned}
$$

System (9.69) now transforms into system (9.70) with D_1 and D_2 being determined by H_1 and H_2 as (9.71) and the proof of case (b) is complete for $n_1 = n_2 = m_1 = m_2 = 1$. When $n_1 = n_2 = m_1 = m_2 = 0$, $H,\ F$ and $H',\ F'$ contain no derivatives of $u,\ v$ and $u',\ v'$, respectively and the further restriction $A_i,\ i = 1, \ldots, 6$ must apply. □

9.4.4 System of two equations of the form $u_{02} = H(x, t, u, v, \ldots, u_{n_10}, v_{m_10})$

In the third class of systems, we restrict to relations of u_{tt} and v_{tt} with $x,\ t,\ u,\ v$ and x-derivatives of u and v.

In the following theorem, we give similar result as theorem 3.7.

Theorem 9.7. *($n_1 \geq 3,\ m_2 \geq 3$) The point transformation $x' = P(x,t,u,v),\ t' = Q(x,t,u,v),\ u' = R(x,t,u,v),\ v' = S(x,t,u,v)$ transforms*

$$
u_{02} = H(x, t, u, v, \ldots, u_{n_10}, v_{m_10}), \qquad v_{02} = F(x, t, u, v, \ldots, u_{n_20}, v_{m_20}) \tag{9.72}
$$

to

$$
u'_{02} = H'(x', t', u', v', \ldots, u'_{n_10}, '_{m_10}), \qquad v'_{02} = F'(x', t', u', v', \ldots, u'_{n_20}, v'_{m_20}) \tag{9.73}
$$

if

$$\begin{aligned}
P &= P(x), \quad Q = Q(t), \\
R &= Q_t^{1/2}\left(c_1(x)u + c_2(x)v\right) + B_1(x,t), \quad S = Q_t^{1/2}(c_3(x)u + c_4(x)v) + B_2(x,t), \\
H' &= Q_t^{-7/2}\left((c_1H + c_2F)Q_t^2 - (c_1u + c_2v)\left(\frac{3}{4}Q_{tt}^2 - \frac{1}{2}Q_{ttt}Q_t\right)\right. \\
&- \left. Q_t^{1/2}(B_{1_t}Q_{tt} - B_{1_{tt}}Q_t)\right), \qquad (9.74) \\
F' &= Q_t^{-7/2}\left((c_3H + c_4F)Q_t^2 - (c_3u + c_4v)\left(\frac{3}{4}Q_{tt}^2 - \frac{1}{2}Q_{ttt}Q_t\right)\right. \\
&- \left. Q_t^{1/2}(B_{2_t}Q_{tt} - B_{2_{tt}}Q_t)\right). \qquad (9.75)
\end{aligned}$$

Proof. From theorem 9.1 with $(p = n_1,\ q = 0)$ it follows that $Q = Q(t)$. Relations (9.12) and (9.14) simplify to $u'_{i0} = P_X^{-1}(u'_{i-10})_X$ and $v'_{i0} = P_X^{-1}(v'_{i-10})_X,\ i \geq 1$ respectively, so it is evident that the transformed u'_{i0} and $v'_{i0},\ i \geq 1$, involve no t derivatives of u and v. Hence, system (9.72) can only be transformed into system (9.73) if u'_{02} and v'_{02} do not give rise to either of terms $u_{11}, u_{01}, v_{11}, v_{01}$. However, lemma 9.1, corollary 9.1 give

$$\frac{\partial u'_{02}}{\partial u_{11}} = 2\delta^{-2}P_T\left[(P_uR_v - P_vR_u)v_{10} + P_uR_x - P_xR_u\right] \equiv 0,$$

$$\frac{\partial u'_{02}}{\partial v_{11}} = -2\delta^{-2}P_T\left[(P_uR_v - P_vR_u)u_{10} + P_xR_v - P_vR_x\right] \equiv 0,$$

$$\frac{\partial v'_{02}}{\partial u_{11}} = 2\delta^{-2}P_T\left[(P_uS_v - P_vS_u)v_{10} + P_uS_x - P_xS_u\right] \equiv 0,$$

$$\frac{\partial v'_{02}}{\partial v_{11}} = -2\delta^{-2}P_T\left[(P_uS_v - P_vS_u)u_{10} + P_xS_v - P_vS_x\right] \equiv 0.$$

Therefore, it follows that $P_T = 0$. So that $P = P(x)$.

Lemma 9.3, now gives

$$\frac{\partial^2 u'_{02}}{\partial u_{01}^2} = 2R_{uu}Q_t^{-2} \equiv 0, \qquad \frac{\partial^2 u'_{02}}{\partial v_{01}^2} = 2R_{vv}Q_t^{-2} \equiv 0, \qquad \frac{\partial^2 u'_{02}}{\partial u_{01}v_{01}} = 2R_{uv}Q_t^{-2} \equiv 0,$$

$$\frac{\partial^2 v'_{02}}{\partial u_{01}^2} = 2S_{uu}Q_t^{-2} \equiv 0, \qquad \frac{\partial^2 v'_{02}}{\partial v_{01}^2} = 2S_{vv}Q_t^{-2} \equiv 0, \qquad \frac{\partial^2 v'_{02}}{\partial u_{01}v_{01}} = 2S_{uv}Q_t^{-2} \equiv 0,$$

showing that R and S are of the form $R = A_1(x,t)u + A_2(x,t)v + B_1(x,t)$, $S = A_3(x,t)u + A_4(x,t)v + B_2(x,t)$. Further

$$\frac{\partial u'_{02}}{\partial u_{01}} = Q_t^{-3}\left[2(R_{uv}v_{01} + R_{uu}u_{01} + R_{ut})Q_t - Q_{tt}R_u\right],$$

$$\frac{\partial u'_{02}}{\partial v_{01}} = Q_t^{-3}\left[2(R_{uv}u_{01} + R_{vv}v_{01} + R_{vt})Q_t - Q_{tt}R_v\right],$$

$$\frac{\partial v'_{02}}{\partial u_{01}} = Q_t^{-3}\left[2(S_{uv}v_{01} + S_{uu}u_{01} + S_{ut})Q_t - Q_{tt}S_u\right],$$
$$\frac{\partial v'_{02}}{\partial v_{01}} = Q_t^{-3}\left[2(S_{uv}u_{01} + S_{vv}v_{01} + S_{vt})Q_t - Q_{tt}S_v\right],$$

so that R and S are of the form $R = Q_t^{1/2}(c_1(x)u + c_2(x)v) + B_1(x,t)$, $S = Q_t^{1/2}(c_3(x)u + c_4(x)v) + B_2(x,t)$. With these form of P, Q, R and S system (9.72) is transformed to system (9.73) and H', F' are given by relations (9.74) and (9.75). □

Finally, we give a generalization of theorem 3.8, where u_{tt} and v_{tt} are related with x, t, u, v and second-order x-derivatives of u and v.

Theorem 9.8. *($n_1 = n_2 = m_1 = m_2 = 2$) Point transformations $x' = P(x,t,u,v)$, $t' = Q(x,t,u,v)$, $u' = R(x,t,u,v)$, $v' = S(x,t,u,v)$ which transform*

$$u_{02} = H(x,t,u,v,u_{10},v_{10},u_{20},v_{20}), \qquad v_{02} = F(x,t,u,v,u_{10},v_{10},u_{20},v_{20}) \tag{9.76}$$

into

$$u'_{02} = H'(x',t',u',v',u'_{10},v'_{10},u'_{20},v'_{20}), \qquad v'_{02} = F'(x',t',u',v',u'_{10},v'_{10},u'_{20},v'_{20}) \tag{9.77}$$

belongs to one of the three categories:

a) P, Q, R *and* S *restricted as in the conditions for theorem 9.7;*

b) $P = P(t)$, $Q = Q(x)$;

c) $P = P(x,t)$, $Q = Q(x,t)$,

$$R = c_1(x,t)u + c_2(x,t)v + c_3(x,t), \quad S = c_4(x,t)u + c_5(x,t)v + c_6(x,t),$$
$$H' = P_tP_xQ_t^{-1}Q_x^{-1}u'_{20} + G_1(x',t')u'_{10} + G_2(x',t')v'_{10} + G_3(x',t',u',v'),$$
$$F' = P_tP_xQ_t^{-1}Q_x^{-1}v'_{20} + G_4(x',t')u'_{10} + G_5(x',t')v'_{10} + G_6(x',t',u',v'),$$
$$H = P_tQ_tP_x^{-1}Q_x^{-1}u_{20} + F_1(x,t)u_{10} + F_2(x,t)v_{10} + F_3(x,t,u,v),$$
$$F = P_tQ_tP_x^{-1}Q_x^{-1}v_{20} + F_4(x,t)u_{10} + F_5(x,t)v_{10} + F_6(x,t,u,v).$$

Proof. The expressions $E_1 = u'_{02} - H'$ and $E_2 = v'_{02} - F'$ become an expressions in x,t,u,v and the derivatives of u and v up to order 2. This expressions (=0) is identified with system (9.76). That is, if u_{02} and v_{02} are replaced by H and F in E_1 and E_2, respectively, then the resulting expression is required to be identically zero in terms of the remaining variables $x, t, u, v, u_{10}, v_{10}, u_{01}, v_{01}, u_{20}, v_{20}, u_{11}, v_{11}$. In particular,

$$\frac{\partial E_1}{\partial u_{11}} = 0, \qquad \frac{\partial E_1}{\partial v_{11}} = 0, \qquad \frac{\partial E_2}{\partial u_{11}} = 0, \qquad \frac{\partial E_2}{\partial v_{11}} = 0,$$

give

$$\begin{aligned}
P_T P_X J_1 &= Q_T Q_X \left(J_1 H'_{u'_{20}} + J_3 H'_{v'_{20}} \right), \\
P_X P_T J_2 &= Q_T Q_X \left(J_2 H'_{u'_{20}} + J_4 H'_{v'_{20}} \right), \\
P_T P_X J_3 &= Q_T Q_X \left(J_1 F'_{u'_{20}} + J_3 F'_{v'_{20}} \right), \\
P_T P_X J_4 &= Q_T Q_X \left(J_2 F'_{u'_{20}} + J_4 F'_{v'_{20}} \right).
\end{aligned}$$

More complicated conditions give the following system:

$$\frac{\partial E_1}{\partial u_{20}} = 0, \qquad \frac{\partial E_1}{\partial v_{20}} = 0, \qquad \frac{\partial E_2}{\partial u_{20}} = 0, \qquad \frac{\partial E_2}{\partial v_{20}} = 0.$$

These conditions show that all possibilities are included in the three cases:

$(a) \qquad P = P(x), \quad Q = Q(t);$

$(b) \qquad P = P(t), \quad Q = Q(x);$

(c)

$$\begin{aligned}
H' &= P_X P_T Q_X^{-1} Q_T^{-1} u'_{20} + A'_1(x', t', u', v', u'_{10}, v'_{10}), \\
F' &= P_X P_T Q_X^{-1} Q_T^{-1} v'_{20} + A'_2(x', t', u', v', u'_{10}, v'_{10})
\end{aligned} \tag{9.78}$$

and

$$\begin{aligned}
H &= P_X^{-1} P_T Q_X^{-1} Q_T u_{20} + B_1(x, t, u, v, u_{10}, v_{10}), \\
F &= P_X^{-1} P_T Q_X^{-1} Q_T v_{20} + B_2(x, t, u, v, u_{10}, v_{10}).
\end{aligned} \tag{9.79}$$

Case (a) follows exactly as in the proof of theorem 9.7 following the stage at which $P = P(x)$ and the results here are exactly as in the sole case of that theorem.

In case (c), we have that H', F' and H, F given by (9.78) and (9.79). H and F are independent of u_{01} and v_{01}, so that (9.79) implies that $P_X^{-1} P_T Q_X^{-1} Q_T$ is independent of u_{01} and v_{01}. It readily follows that $P = P(x,t)$, $Q = Q(x,t)$. Considering again $E_1 = u'_{02} - H'$ and $E_2 = v'_{02} - F'$, transformed, with u_{02}, v_{02} replaced by H, F, respectively, we have:

$$\frac{\partial^2 E_1}{\partial u_{10} u_{01}} = Q_t Q_x (A_{1_{u'_x u'_x}} R_u^2 + 2 R_u S_u A_{1_{u'_x v'_x}} + S_u^2 A_{1_{v'_x v'_x}}) \delta^{-2} \equiv 0,$$

$$\frac{\partial^2 E_2}{\partial u_{10} u_{01}} = Q_t Q_x (A_{2_{u'_x u'_x}} R_u^2 + 2 R_u S_u A_{2_{u'_x v'_x}} + S_u^2 A_{2_{v'_x v'_x}}) \delta^{-2} \equiv 0,$$

$$\frac{\partial^2 E_1}{\partial v_{10} v_{01}} = Q_t Q_x (A_{1_{u'_x u'_x}} R_v^2 + 2 R_v S_v A_{1_{u'_x v'_x}} + S_v^2 A_{1_{v'_x v'_x}}) \delta^{-2} \equiv 0,$$

$$\frac{\partial^2 E_2}{\partial v_{10} v_{01}} = Q_t Q_x (A_{2_{u'_x u'_x}} R_v^2 + 2 R_v S_v A_{2_{u'_x v'_x}} + S_v^2 A_{2_{v'_x v'_x}}) \delta^{-2} \equiv 0,$$

$$\frac{\partial^2 E_1}{\partial u_{01} v_{10}} = \frac{\partial^2 E_1}{\partial u_{10} v_{01}} = Q_t Q_x (A_{1_{u'_x u'_x}} R_u R_v + R_u S_v A_{1_{u'_x v'_x}} + R_v S_u A_{1_{u'_{10} v'_{10}}} + S_u S_v A_{1_{v'_x v'_x}}) \delta^{-2} \equiv 0,$$

$$\frac{\partial^2 E_2}{\partial u_{01} v_{10}} = \frac{\partial^2 E_2}{\partial u_{10} v_{01}} = Q_t Q_x (A_{2_{u'_x u'_x}} R_u R_v + R_u S_v A_{2_{u'_x v'_x}} + R_v S_u A_{2_{u'_{10} v'_{10}}} + S_u S_v A_{2_{v'_x v'_x}}) \delta^{-2} \equiv 0,$$

giving

$$A_1 = G_1(x', t', u', v') u'_{10} + G_2(x', t', u', v') v'_{10} + G_3(x', t', u', v'),$$

$$A_2 = G_4(x', t', u', v') u'_{10} + G_5(x', t', u', v') v'_{10} + G_6(x', t', u', v').$$

Now,

$$\frac{\partial^2 E_1}{\partial u_{01}^2} \equiv 0, \qquad \frac{\partial^2 E_1}{\partial v_{01}^2} \equiv 0,$$

$$\frac{\partial^2 E_2}{\partial u_{01}^2} \equiv 0, \qquad \frac{\partial^2 E_2}{\partial v_{01}^2} \equiv 0,$$

and

$$\frac{\partial^2 E_1}{\partial u_{01} v_{01}} \equiv 0, \qquad \frac{\partial^2 E_2}{\partial u_{01} v_{01}} \equiv 0,$$

give

$$R = D_2(x,t) u + D_3(x,t) v + D_4(x,t), \qquad S = D_6(x,t) u + D_7(x,t) v + D_8(x,t).$$

The relations:

$$\frac{\partial^2 E_1}{\partial u_{10}^2} = P_x Q_t^{-1} (B_{1_{u_{10} u_{10}}} D_2 + B_{2_{u_{10} u_{10}}} D_3) \delta^{-1} \equiv 0,$$

$$\frac{\partial^2 E_1}{\partial v_{10}^2} = P_x Q_t^{-1} (B_{1_{v_{10} v_{10}}} D_2 + B_{2_{v_{10} v_{10}}} D_3) \delta^{-1} \equiv 0,$$

$$\frac{\partial^2 E_2}{\partial u_{10}^2} = P_x Q_t^{-1} (B_{1_{u_{10} u_{10}}} D_6 + B_{2_{u_{10} u_{10}}} D_7) \delta^{-1} \equiv 0,$$

$$\frac{\partial^2 E_2}{\partial v_{10}^2} = P_x Q_t^{-1} (B_{1_{v_{10} v_{10}}} D_6 + B_{2_{v_{10} v_{10}}} D_7) \delta^{-1} \equiv 0,$$

$$\frac{\partial^2 E_1}{\partial u_{10} v_{10}} = P_x Q_t^{-1} (B_{1_{u_{10}v_{10}}} D_2 + B_{2_{u_{10}v_{10}}} D_3) \delta^{-1} \equiv 0,$$
$$\frac{\partial^2 E_2}{\partial u_{10} v_{10}} = P_x Q_t^{-1} (B_{1_{u_{10}v_{10}}} D_6 + B_{2_{u_{10}v_{10}}} D_7) \delta^{-1} \equiv 0,$$

where $D_2 D_7 - D_3 D_6 \neq 0$ (otherwise $J = 0$), give the form of B_1 and B_2:

$$B_1 = F_1(x, t, u, v) u_{10} + F_2(x, t, u, v) v_{10} + F_3(x, t, u, v),$$

$$B_2 = F_6(x, t, u, v) u_{10} + F_7(x, t, u, v) v_{10} + F_8(x, t, u, v).$$

Finally,

$$\frac{\partial^2 E_1}{\partial u_{01} u} \equiv 0, \qquad \frac{\partial^2 E_1}{\partial u_{01} v} \equiv 0, \qquad \frac{\partial^2 E_1}{\partial v_{01} u} \equiv 0, \qquad \frac{\partial^2 E_1}{\partial v_{01} v} \equiv 0,$$

give $G_{1_{u'}} = G_{1_{v'}} = G_{2_{u'}} = G_{2_{v'}} = 0$ and

$$\frac{\partial^2 E_2}{\partial u_{01} u} \equiv 0, \qquad \frac{\partial^2 E_2}{\partial u_{01} v} \equiv 0, \qquad \frac{\partial^2 E_2}{\partial v_{01} u} \equiv 0, \qquad \frac{\partial^2 E_2}{\partial v_{01} v} \equiv 0,$$

give $G_{4_{u'}} = G_{4_{v'}} = G_{5_{u'}} = G_{5_{v'}} = 0$. Similarly,

$$\frac{\partial^2 E_1}{\partial u_{10} u} \equiv 0, \qquad \frac{\partial^2 E_1}{\partial u_{10} v} \equiv 0, \qquad \frac{\partial^2 E_1}{\partial v_{10} u} \equiv 0, \qquad \frac{\partial^2 E_1}{\partial v_{10} v} \equiv 0$$

and

$$\frac{\partial^2 E_2}{\partial u_{10} u} \equiv 0, \qquad \frac{\partial^2 E_2}{\partial u_{10} v} \equiv 0, \qquad \frac{\partial^2 E_2}{\partial v_{10} u} \equiv 0, \qquad \frac{\partial^2 E_2}{\partial v_{10} v} \equiv 0,$$

give $F_{2_u} = F_{2_v} = F_{3_u} = F_{3_v} = F_{6_u} = F_{6_v} = F_{6_u} = F_{6_v} = 0$ which, completes the proof of case (c) of the theorem. □

9.5 Applications

In this section, as application, we present the form of point transformation which connect restricted form of system of two PDEs, in which u_{tt} and v_{tt} is a linear combinations of u_{xx} and v_{xx}, respectively.

Theorem 9.9. *The point transformation* $x' = P(x, t, u, v)$, $t' = Q(x, t, u, v)$, $u' = R(x, t, u, v)$, $v' = S(x, t, u, v)$ *transforms*

$$u_{02} = \varepsilon u_{20} + H(x, t, u, v, u_{10}, v_{10}), \qquad v_{02} = \varepsilon v_{20} + F(x, t, u, v, u_{10}, v_{10})$$

to

$$u'_{02} = \varepsilon u'_{20} + H'(x', t', u', v', u'_{10}, v'_{10}), \quad v'_{02} = \varepsilon v'_{20} + F'(x', t', u', v', u'_{10}, v'_{10})$$

if

$$\begin{aligned}
P &= \varepsilon_1 c_1 x + c_2, \qquad \varepsilon_1 = \pm 1, \\
Q &= c_1 t + c_3, \\
R &= \phi_1(x)u + \phi_2(x)v + B_1(x,t), \\
S &= \phi_3(x)u + \phi_4(x)v + B_2(x,t), \\
H' &= \frac{1}{c_1^2}\left(H\phi_1 + F\phi_2 - \varepsilon(\phi_{1_{xx}}u + 2\phi_{1_x}u_x + \phi_{2_{xx}}v + 2\phi_{2_x}v_x) + B_{1_{tt}} - \varepsilon B_{1_{xx}}\right), \\
F' &= \frac{1}{c_1^2}\left(H\phi_3 + F\phi_4 - \varepsilon(\phi_{3_{xx}}u + 2\phi_{3_x}u_x + \phi_{4_{xx}}v + 2\phi_{4_x}v_x) + B_{2_{tt}} - \varepsilon B_{2_{xx}}\right),
\end{aligned}$$

or

$$\begin{aligned}
P &= c_1 t + c_2, \\
Q &= \frac{\varepsilon_1 c_1}{\varepsilon} x + c_3, \qquad \varepsilon_1 = \pm 1, \\
R &= \phi_1(x)u + \phi_2(x)v + B_1(x,t), \\
S &= \phi_3(x)u + \phi_4(x)v + B_2(x,t), \\
H' &= \frac{\varepsilon}{c_1^2}\left(-H\phi_1 - F\phi_2 + \varepsilon(\phi_{1_{xx}}u + 2\phi_{1_x}u_x + \phi_{2_{xx}}v + 2\phi_{2_x}v_x) - B_{1_{tt}} + \varepsilon B_{1_{xx}}\right), \\
F' &= \frac{\varepsilon}{c_1^2}\left(-H\phi_3 - F\phi_4 + \varepsilon(\phi_{3_{xx}}u + 2\phi_{3_x}u_x + \phi_{4_{xx}}v + 2\phi_{4_x}v_x) - B_{2_{tt}} + \varepsilon B_{2_{xx}}\right),
\end{aligned}$$

or

$$\begin{aligned}
P &= P(x,t), \qquad Q = Q(x,t), \\
R &= \phi_1(x)u + \phi_2(x)v + B_1(x,t), \\
S &= \phi_3(x)u + \phi_4(x)v + B_2(x,t), \\
H' &= \frac{1}{Q_t^2 - \varepsilon Q_x^2}\left(H\phi_1 + F\phi_2 - \varepsilon(\phi_{1_{xx}}u + 2\phi_{1_x}u_x + \phi_{2_{xx}}v + 2\phi_{2_x}v_x) + B_{1_{tt}} - \varepsilon B_{1_{xx}}\right), \\
F' &= \frac{1}{Q_t^2 - \varepsilon Q_x^2}\left(H\phi_3 + F\phi_4 + \varepsilon(\phi_{3_{xx}}u + 2\phi_{3_x}u_x + \phi_{4_{xx}}v + 2\phi_{4_x}v_x) + B_{2_{tt}} - \varepsilon B_{2_{xx}}\right),
\end{aligned}$$

where $Q_{tt} = \varepsilon Q_{xx}$ *and* $P_{tt} = \varepsilon P_{xx}$.

9.6 Conclusion

We have drawn attention to form-preserving point transformations. In particular, we have generalized the results of chapter 3, into systems of two equations. We have studied three special classes of systems restricted to two independent and dependent variables. The work of this chapter is the subject of a forthcoming article [60].

Chapter 10

System of hyperbolic equations

10.1 Introduction

Finally, in this chapter, we consider the system of linear hyperbolic equations. In the spirit of Ibragimov's work, who adopted the infinitesimal method for calculating invariants of families of PDEs using the equivalence groups, we apply the method to system of two hyperbolic equations. We will show that this system admits five differential invariants of first order. As applications, we use the semi-invariants to determine systems that can be transformed into simpler systems.

10.2 Equivalence transformations

In this chapter, we consider the system of linear hyperbolic equations of the form

$$u_{xt} = a_1(t,x)u_x + b_1(t,x)v_x + c_1(t,x)u_t + d_1(t,x)v_t + f_1(t,x)u + g_1(t,x)v,$$
$$v_{xt} = a_2(t,x)u_x + b_2(t,x)v_x + c_2(t,x)u_t + d_2(t,x)v_t + f_2(t,x)u + g_2(t,x)v. \quad (10.1)$$

In order to find continuous group of equivalence transformations of a class of system (10.1) by means of the Lie infinitesimal invariance criterion, we search for the equivalence operator Γ in the following form:

$$\begin{aligned} \Gamma &= \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \nu^1\frac{\partial}{\partial u} + \nu^2\frac{\partial}{\partial v} + \zeta_{11}\frac{\partial}{\partial u_t} + \zeta_{12}\frac{\partial}{\partial u_x} + \zeta_{21}\frac{\partial}{\partial v_t} + \zeta_{22}\frac{\partial}{\partial v_x} \\ &+ \mu^{1i}\frac{\partial}{\partial a_i} + \mu^{2i}\frac{\partial}{\partial b_i} + \mu^{3i}\frac{\partial}{\partial c_i} + \mu^{4i}\frac{\partial}{\partial d_i} + \mu^{5i}\frac{\partial}{\partial f_i} + \mu^{6i}\frac{\partial}{\partial g_i}, \end{aligned}$$

where $\xi^1,\ \xi^2,\ \nu^1,\ \nu^2$ depend on $t,\ x,\ u$ and v, while $\mu^{ji},\ j=1,\ldots,6,\ i=1,2$ depend on $t,\ x,\ u,\ v,\ a_i,\ b_i,\ c_i,\ d_i,\ f_i,\ g_i$. The infinitesimals $\zeta_{ik},\ i,k=1,2$ are given by

$$\zeta_{11}=D_t(\nu^1)-u_tD_t(\xi^1)-u_xD_t(\xi^2),\quad \zeta_{12}=D_x(\nu^1)-u_tD_x(\xi^1)-u_xD_x(\xi^2),$$

$$\zeta_{21}=D_t(\nu^2)-v_tD_t(\xi^1)-v_xD_t(\xi^2),\quad \zeta_{22}=D_x(\nu^2)-v_tD_x(\xi^1)-v_xD_x(\xi^2).$$

The operators D_t and D_x are the total derivatives with respect to t and x, respectively.

By using the same procedure used in the previous chapters, we find that system (10.1) admits an infinite continuous group $\mathcal{E}$ of equivalence transformations generated by Lie algebra $L_{\mathcal{E}}$ spanned by the operators:

$$\begin{aligned}
&\Gamma_\phi=\phi\frac{\partial}{\partial x}-\phi'\left[c_1\frac{\partial}{\partial c_1}+d_1\frac{\partial}{\partial d_1}+f_1\frac{\partial}{\partial f_1}+g_1\frac{\partial}{\partial g_1}+c_2\frac{\partial}{\partial c_2}+d_2\frac{\partial}{\partial d_2}+f_2\frac{\partial}{\partial f_2}+g_2\frac{\partial}{\partial g_2}\right],\\
&\Gamma_\tau=\tau\frac{\partial}{\partial t}-\tau'\left[a_1\frac{\partial}{\partial a_1}+b_1\frac{\partial}{\partial b_1}+f_1\frac{\partial}{\partial f_1}+g_1\frac{\partial}{\partial g_1}+a_2\frac{\partial}{\partial a_2}+b_2\frac{\partial}{\partial b_2}+f_2\frac{\partial}{\partial f_2}+g_2\frac{\partial}{\partial g_2}\right],\\
&\Gamma_{\phi_1}=\phi_1u\frac{\partial}{\partial u}+\phi_{1_t}\frac{\partial}{\partial a_1}+\phi_1b_1\frac{\partial}{\partial b_1}+\phi_{1_x}\frac{\partial}{\partial c_1}+\phi_1d_1\frac{\partial}{\partial d_1}+\left(\phi_{1_{tx}}-\phi_{1_t}c_1-\phi_{1_x}a_1\right)\frac{\partial}{\partial f_1}\\
&-\phi_1g_1\frac{\partial}{\partial g_1}-a_2\phi_1\frac{\partial}{\partial a_2}-\phi_1c_2\frac{\partial}{\partial c_2}-\left(\phi_{1_t}c_2+\phi_{1_x}a_2+\phi_1f_2\right)\frac{\partial}{\partial f_2},\\
&\Gamma_{\phi_2}=\phi_2v\frac{\partial}{\partial v}-\phi_2b_1\frac{\partial}{\partial b_1}-\phi_2d_1\frac{\partial}{\partial d_1}-\left(\phi_{2_t}d_1+\phi_{2_x}b_1+\phi_2g_1\right)\frac{\partial}{\partial g_1}\\
&+\phi_2a_2\frac{\partial}{\partial a_2}+\phi_{2_t}\frac{\partial}{\partial b_2}+\phi_2c_2\frac{\partial}{\partial c_2}+\phi_{2_x}\frac{\partial}{\partial d_2}+\phi_2f_2\frac{\partial}{\partial f_2}+\left(\phi_{2_{tx}}-\phi_{2_t}d_2-\phi_{2_x}b_2\right)\frac{\partial}{\partial g_2},\\
&\Gamma_{\phi_3}=\phi_3v\frac{\partial}{\partial u}+\phi_3a_2\frac{\partial}{\partial a_1}+\left(\phi_{3_t}-\phi_3a_1+\phi_3b_2\right)\frac{\partial}{\partial b_1}+\phi_3c_2\frac{\partial}{\partial c_1}\\
&+\left(\phi_{3_x}-\phi_3c_1+\phi_3d_2\right)\frac{\partial}{\partial d_1}+\phi_3f_2\frac{\partial}{\partial f_1}+\left(\phi_{3_{tx}}-\phi_{3_t}c_1-\phi_{3_x}a_1-\phi_3f_1+\phi_3g_2\right)\frac{\partial}{\partial g_1}\\
&-\phi_3a_2\frac{\partial}{\partial b_2}-\phi_3c_2\frac{\partial}{\partial d_2}-\left(\phi_{3_t}c_2+\phi_{3_x}a_2+\phi_3f_2\right)\frac{\partial}{\partial g_2},\\
&\Gamma_{\phi_4}=\phi_4u\frac{\partial}{\partial v}-\phi_4b_1\frac{\partial}{\partial a_1}-\phi_4d_1\frac{\partial}{\partial c_1}-\left(\phi_{4_t}d_1+\phi_{4_x}b_1+\phi_4g_1\right)\frac{\partial}{\partial f_1}\\
&+\left(\phi_{4_t}+\phi_4a_1-\phi_4b_2\right)\frac{\partial}{\partial a_2}+\phi_4b_1\frac{\partial}{\partial b_2}+\left(\phi_{4_x}+\phi_4c_1-\phi_4d_2\right)\frac{\partial}{\partial c_2}\\
&+\phi_4d_1\frac{\partial}{\partial d_2}+\left(\phi_{4_{tx}}-\phi_{4_t}d_2-\phi_{4_x}b_2+\phi_4f_1-\phi_4g_2\right)\frac{\partial}{\partial f_2}\\
&+\phi_4g_1\frac{\partial}{\partial g_2},
\end{aligned}\tag{10.2}$$

where $\phi=\phi(x),\ \tau=\tau(t),\ \phi_i=\phi_i(t,x),\ i=1,2,3,4,$ are arbitrary functions.

10.3 Differential invariants and invariant equations

We consider the problem of finding differential invariants of the system (10.1). Using the operators (10.2), the invariance criterion $\Gamma(J) = 0$ gives the six identities

$$\Gamma_k(J) = 0, \quad k = \phi, \tau, \phi_1, \phi_2, \phi_3, \phi_4.$$

Since $\phi(x), \tau(t), \phi_1(t,x), \phi_2(t,x), \phi_3(t,x)$ and $\phi_4(t,x)$ are arbitrary functions, these identities lead to the trivial solution, J =constant. Hence, the system (10.1) does not admit differential invariants of order zero.

In order to calculate the differential invariants of order one, we need the first prolongation of the operators (10.2). The first prolongation lead to the invariant criterion

$$\Gamma_k^{(1)}(J) = 0, \quad k = \phi, \tau, \phi_1, \phi_2, \phi_3, \phi_4.$$

Using the fact that $\phi(x), \tau(t), \phi_1(t,x), \phi_2(t,x), \phi_3(t,x)$ and $\phi_4(t,x)$ are arbitrary functions, these leads to a system of linear first-order PDEs for J. Without presenting any calculations, we state that the differential invariants of first order are the following:

$$J_1 = \frac{I_2}{I_1}, \qquad J_2 = \frac{I_3}{I_1^2}, \qquad J_3 = \frac{I_4}{I_1^2}, \qquad J_4 = \frac{I_5}{I_1^2}, \qquad J_5 = \frac{I_6}{I_1^4},$$

where

$$\begin{aligned}
I_1 &= K_5 + K_8, \\
I_2 &= K_1 + K_4, \\
I_3 &= K_1K_4 - K_2K_3, \\
I_4 &= K_5K_8 - K_6K_7, \\
I_5 &= K_1K_5 + K_2K_7 + K_3K_6 + K_4K_8, \\
I_6 &= K_2K_3(K_5 - K_8)^2 + K_6K_7(K_1 - K_4)^2 - (K_2K_7 - K_3K_6)^2 \\
&+ (K_2K_7 + K_3K_6)(K_1 - K_4)(K_8 - K_5),
\end{aligned}$$

and

$$K_1 = a_1c_1 + a_2d_1 - a_{1_x} + f_1,$$

$$K_2 = a_1c_2 + a_2d_2 - a_{2_x} + f_2,$$

$$K_3 = b_1c_1 + b_2d_1 - b_{1_x} + g_1,$$

$$K_4 = b_1c_2 + b_2d_2 - b_{2_x} + g_2,$$

$$K_5 = a_1c_1 + b_1c_2 - c_{1_t} + f_1,$$

$$K_6 = a_2c_1 + b_2c_2 - c_{2_t} + f_2,$$

$$K_7 = a_1d_1 + b_1d_2 - d_{1_t} + g_1,$$

$$K_8 = a_2d_1 + b_2d_2 - d_{2_t} + g_2.$$

Furthermore, it can be shown that the quantities

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = 0, \quad I_4 = 0, \quad I_5 = 0, \quad I_6 = 0 \qquad (10.3)$$

are invariant equations of system (10.1). That is,

$$\Gamma_k^{(1)} (I_m)|_{I_m=0} = 0,$$

where $k = \phi, \tau, \phi_1, \phi_2, \phi_3, \phi_4,\ m = 1, \ldots, 6$.

Also, the following quantities:

$$I_1, \quad I_2, \quad I_3, \quad I_4, \quad I_5, \quad I_6 \qquad (10.4)$$

are semi-invariants for the system (10.1).

Also, any three of following quantities:

$$K_1 = 0, \quad K_2 = 0, \quad K_3 = 0, \quad K_4 = 0$$

or, any three of the following quantities:

$$K_5 = 0, \quad K_6 = 0, \quad K_7 = 0, \quad K_8 = 0$$

are invariant systems. That is,

$$\Gamma_k^{(1)} (K_i)|_{K_i=0,\ K_j=0,\ K_m=0} = 0,$$

where $k = \phi, \tau, \phi_1, \phi_2, \phi_3, \phi_4$ and $\{i, j, m = 1, \ldots, 4\}$ or $\{i, j, m = 5, \ldots, 8\}$.

10.4 Applications

In this section we present examples in which we derive systems of the class (10.1) that can be linked with a known system. Consider the system of hyperbolic equations of the form

$$u_{xt} = a_1(t,x)u_x + b_1(t,x)v_x + c_1(t,x)u_t + d_1(t,x)v_t + f_1(t,x)u + g_1(t,x)v,$$
$$v_{xt} = a_2(t,x)u_x + b_2(t,x)v_x + c_2(t,x)u_t + d_2(t,x)v_t + f_2(t,x)u + g_2(t,x)v. \quad (10.5)$$

First of all, we identify the most general form of changes of variables that can be utilized without loss of linearity and homogeneity of system (10.5) as well as their standard form. Using the results of chapter 9, to derive the equivalence transformations in the finite form, we deduce that these changes of variables have the following form:

$$t' = Q(t), \quad x' = P(x), \quad u' = k_1(x,t)u + k_2(x,t)v, \quad v' = k_3(x,t)u + k_4(x,t)v. \quad (10.6)$$

Systems (10.5) related by an equivalence transformation (10.6) are said to be equivalent.

(1) We consider the linear system

$$u_{xt} = 0, \quad v_{xt} = 0. \quad (10.7)$$

System (10.7) is a member of the class (10.5). If we set $a_i = b_i = c_i = d_i = f_i = g_i = 0$, $i = 1,2$ the invariant equations (10.3) are all satisfied. Hence any system of the form (10.5) that is connected with the linear system (10.7) satisfies the invariant equations. Consequently, the solution of the system (10.3) will provide necessary conditions for a system of the form (10.5) to be mapped into (10.7).

It can be shown that system (10.5) can be mapped into (10.7) by transformation (10.6) providing the following 12 identities are satisfying:

$$k_{i_t} + a_1k_i + a_2k_j = 0,$$
$$k_{j_t} + b_1k_i + b_2k_j = 0,$$
$$k_{i_x} + c_1k_i + c_2k_j = 0,$$
$$k_{j_x} + d_1k_i + d_2k_j = 0,$$
$$k_{i_{xt}} + f_1k_i + f_2k_j = 0,$$
$$k_{j_{xt}} + g_1k_i + g_2k_j = 0,$$

where $\{i=1,\ j=2\}$ and $\{i=3,\ j=4\}$ and k_1, k_2, k_3, k_4 are arbitrary functions. These identities lead $K_i = 0,\ i = 1, \ldots, 8$.

Hence we have proved that invariant equations provided necessary and sufficient conditions for linking system of the form (10.5) and the linear system (10.7).

Example 10.1. The transformation

$$x' = x, \quad t' = t, \quad u' = xtu, \quad v' = xu + tv$$

maps

$$u'_{x't'} = 0, \quad v'_{x't'} = 0$$

into

$$\begin{aligned} u_{xt} &= -\frac{1}{t}u_x - \frac{1}{x}u_t - \frac{1}{xt}u, \\ v_{xt} &= \frac{x}{t^2}u_x - \frac{1}{t}v_x + \frac{1}{t^2}u. \end{aligned} \tag{10.8}$$

Hence the general solution of system (10.8) is

$$u = \frac{f_1(x) + g_1(t)}{xt}, \quad v = \frac{f_2(x) + g_2(t)}{t} - \frac{f_1(x) + g_1(t)}{t^2}.$$

(2) Now we consider the linear system

$$\begin{aligned} &u_{xt} + u = 0, \\ &v_{xt} + v = 0. \end{aligned} \tag{10.9}$$

The system of the form (10.5) can be mapped into system (10.9), by the point transformation (10.6) providing the following 12 identities are satisfying:

$$\begin{aligned} &k_{i_t} + a_1 k_i + a_2 k_j = 0, \\ &k_{j_t} + b_1 k_i + b_2 k_j = 0, \\ &k_{i_x} + c_1 k_i + c_2 k_j = 0, \\ &k_{j_x} + d_1 k_i + d_2 k_j = 0, \\ &k_{i_{xt}} + P_x Q_t k_i + f_1 k_i + f_2 k_j = 0, \\ &k_{j_{xt}} + P_x Q_t k_j + g_1 k_i + g_2 k_j = 0, \end{aligned}$$

where $\{i = 1,\ j = 2\}$ and $\{i = 3,\ j = 4\}$ and k_1, k_2, k_3, k_4 are arbitrary functions. These identities lead to the following results:

$$K_1 = K_4 = K_5 = K_8 = -P_x Q_t, \quad K_2 = K_3 = K_6 = K_7 = 0.$$

It can be shown that the system of the form (10.5) can be mapped into system (10.9), if and only if

$$I_1 = I_2 = -2H(x)G(t), \quad I_3 = I_4 = I_5 = H^2(x)G^2(t), \quad I_6 = 0.$$

Example 10.2. The transformation

$$x' = x, \quad t' = t, \quad u' = u + xv, \quad v' = xtv$$

maps

$$u'_{x't'} + u' = 0, \quad v'_{x't'} + v' = 0$$

into

$$\begin{aligned} u_{xt} &= \frac{x}{t}v_x - u + \frac{1}{t}v, \\ v_{xt} &= -\frac{1}{t}v_x - \frac{1}{x}v_t - \left(1 + \frac{1}{tx}\right)v. \end{aligned} \tag{10.10}$$

(3) Now, we consider the linear system

$$\begin{aligned} &u_{xt} + v = 0, \\ &v_{xt} + u = 0. \end{aligned} \tag{10.11}$$

The system of the form (10.5) can be mapped into system (10.11), by the point transformation (10.6) providing the following 12 identities are satisfying:

$$k_{1_t} + a_1 k_1 + a_2 k_2 = 0,$$

$$k_{2_t} + b_1 k_1 + b_2 k_2 = 0,$$

$$k_{1_x} + c_1 k_1 + c_2 k_2 = 0,$$

$$k_{2_x} + d_1 k_1 + d_2 k_2 = 0,$$

$$k_{1_{xt}} + P_x Q_t k_3 + f_1 k_1 + f_2 k_2 = 0,$$

$$k_{2_{xt}} + P_x Q_t k_4 + g_1 k_1 + g_2 k_2 = 0$$

and

$$k_{3_t} + a_1 k_3 + a_2 k_4 = 0,$$
$$k_{4_t} + b_1 k_3 + b_2 k_4 = 0,$$
$$k_{3_x} + c_1 k_3 + c_2 k_4 = 0,$$
$$k_{4_x} + d_1 k_3 + d_2 k_4 = 0,$$
$$k_{3_{xt}} + P_x Q_t k_1 + f_1 k_3 + f_2 k_4 = 0,$$
$$k_{4_{xt}} + P_x Q_t k_2 + g_1 k_3 + g_2 k_4 = 0,$$

where k_1, k_2, k_3, k_4 are arbitrary functions.

It can be shown that the system of the form (10.5) can be mapped into system (10.11), if and only if

$$I_1 = I_2 = 0, \quad I_3 = I_4 = I_5 = -H^2(x)G^2(t), \quad I_6 = 0.$$

Example 10.3. The transformation

$$x' = x, \quad t' = t, \quad u' = tu, \quad v' = xu + xtv$$

maps

$$u'_{x't'} + v' = 0, \quad v'_{x't'} + u' = 0$$

into

$$\begin{aligned} u_{xt} &= -\frac{1}{t}u_x - \frac{x}{t}u - xv, \\ v_{xt} &= \frac{1}{t^2}u_x - \frac{1}{t}v_x - \frac{1}{tx}u_t - \frac{1}{x}v_t + \left(\frac{x^2-t^2}{t^2x}\right)u + \left(\frac{x^2-1}{tx}\right)v. \end{aligned} \tag{10.12}$$

(4) System

$$\begin{aligned} u_{xt} &= a_1(t,x)u_x + c_1(t,x)u_t + f_1(t,x)u, \\ v_{xt} &= b_2(t,x)v_x + d_2(t,x)v_t + g_2(t,x)v \end{aligned} \tag{10.13}$$

is factorable, viz. the differential operators of the second order

$$L_1 = D_t D_x - a_1 D_x - c_1 D_t - f_1, \quad L_2 = D_t D_x - b_2 D_x - d_2 D_t - g_2$$

can be expressed as a product of two operators of the first order if and only if the semi-invariants vanishes. Namely,

$$L_1 = [D_x + m_1(x,t)][D_t + m_2(x,t)], \quad L_2 = [D_x + m_3(x,t)][D_t + m_4(x,t)] \quad \text{iff} \;\; I_2 = I_3 = I_5 = I_6 = 0, \tag{10.14}$$

$$L_1 = [D_t + m_1(x,t)][D_x + m_2(x,t)], \quad L_2 = [D_t + m_3(x,t)][D_x + m_4(x,t)] \quad \text{iff} \;\; I_1 = I_4 = I_6 = 0, \tag{10.15}$$

$$L_1 = [D_t + m_1(x,t)][D_x + m_2(x,t)], \quad L_2 = [D_x + m_3(x,t)][D_t + m_4(x,t)] \tag{10.16}$$

or iff $I_3 = I_4 = I_6 = 0$.

$$L_1 = [D_x + m_1(x,t)][D_t + m_2(x,t)], \quad L_2 = [D_t + m_3(x,t)][D_x + m_4(x,t)] \tag{10.17}$$

The proof of the both statements (10.14) and (10.15) are similar, therefore let us prove only one of them, e.g. (10.14). Let

$$L_1 = [D_x + m_1(x,t)][D_t + m_2(x,t)]$$

and

$$L_2 = [D_x + m_3(x,t)][D_t + m_4(x,t)].$$

If we compare this operators with the linear system (10.5), the coefficients of L_1 and L_2 have the form

$$a_1 = -m_2, \quad b_1 = 0, \quad c_1 = -m_1, \quad d_1 = 0, \quad f_1 = -m_{2_x} - m_1 m_2, \quad g_1 = 0$$

and

$$a_2 = 0, \quad b_2 = -m_4, \quad c_2 = 0, \quad d_2 = -m_3, \quad f_2 = 0, \quad g_2 = -m_{4_x} - m_3 m_4,$$

respectively. Therefore the semi-invariants $I_2, \; I_3, \; I_5, \; I_6$ vanish.

Conversely, if

$$I_2 = 0, \quad I_3 = 0, \quad I_5 = 0, \quad I_6 = 0,$$

and solve this system for f_1 and g_2, we arrive to the following form of f_1 and g_2:

$$f_1 = a_{1_x} - a_1 c_1, \quad g_2 = b_{2_x} - b_2 d_2.$$

Hence, L_1 and L_2 are factorable

$$L_1 = D_t D_x - a_1 D_x - c_1 D_t - a_{1_x} + a_1 c_1 \equiv [D_x - c_1] [D_t - a_1],$$

$$L_2 = D_t D_x - b_2 D_x - d_2 D_t - b_{2_x} + b_2 d_2 \equiv [D_x - d_2] [D_t - b_2].$$

Also, the proof of the both statements (10.16) and (10.17) arise together. Let

$$\{L_1 = [D_t + m_1(x,t)][D_x + m_2(x,t)] \quad \text{and} \quad L_2 = [D_x + m_3(x,t)][D_t + m_4(x,t)]\}$$

or

$$\{L_1 = [D_x + m_1(x,t)][D_t + m_2(x,t)] \quad \text{and} \quad L_2 = [D_t + m_3(x,t)][D_x + m_4(x,t)]\}.$$

If we compare these operators with the linear system (10.5), the coefficients of L_1 and L_2 have the form

$$\{a_1 = -m_1,\ b_1 = 0,\ c_1 = -m_2,\ d_1 = 0,\ f_1 = -m_{2_t} - m_1 m_2,\ g_1 = 0\}$$

and

$$\{a_2 = 0, \quad b_2 = -m_4, \quad c_2 = 0, \quad d_2 = -m_3, \quad f_2 = 0, \quad g_2 = -m_{4_x} - m_3 m_4\},$$

or

$$\{a_1 = -m_2, \quad b_1 = 0, \quad c_1 = -m_1, \quad d_1 = 0, \quad f_1 = -m_{2_x} - m_1 m_2, \quad g_1 = 0\}$$

and

$$\{a_2 = 0, \quad b_2 = -m_3, \quad c_2 = 0, \quad d_2 = -m_4, \quad f_2 = 0, \quad g_2 = -m_{4_t} - m_3 m_4\},$$

respectively. In the both cases, the semi-invariants $I_3,\ I_4,\ I_6$ vanish.

Conversely, if $I_3 = 0,\ I_4 = 0,\ I_6 = 0$, and solve this system for f_1 and g_2, we arrive to the following forms of f_1 and g_2:

$$f_1 = c_{1_t} - a_1 c_1, \quad g_2 = b_{2_x} - b_2 d_2$$

or

$$f_1 = a_{1_x} - a_1 c_1, \quad g_2 = d_{2_t} - b_2 d_2.$$

For the first solution of the system, L_1 and L_2 are factorable as:

$$L_1 = D_t D_x - a_1 D_x - c_1 D_t - c_{1_t} + a_1 c_1 \equiv [D_t - a_1]\,[D_x - c_1]\,,$$

$$L_2 = D_t D_x - b_2 D_x - d_2 D_t - b_{2_x} + b_2 d_2 \equiv [D_x - d_2]\,[D_t - b_2]\,.$$

For the second solution of the system, L_1 and L_2 are factorable as

$$L_1 = D_t D_x - a_1 D_x - c_1 D_t - a_{1_x} + a_1 c_1 \equiv [D_x - c_1]\,[D_t - a_1]\,,$$

$$L_2 = D_t D_x - b_2 D_x - d_2 D_t - d_{2_t} + b_2 d_2 \equiv [D_t - b_2]\,[D_x - d_2]\,.$$

For illustration, we consider the following examples.

Example 10.4. We consider the following system:

$$\begin{aligned} u_{xt} &= t u_x + \left(\frac{x-t}{t}\right) u_t - \left(\frac{x}{t^2} + x - t\right) u, \\ v_{xt} &= \left(\frac{t}{x}\right) v_x + x v_t - \left(\frac{t}{x^2} + t\right) v. \end{aligned} \tag{10.18}$$

This system is a member of the class of system (10.13). Comparing system (10.18) with (10.13), we have the following forms of coefficients:

$$a_1 = t, \quad c_1 = \frac{x-t}{t}, \quad f_1 = -\left(\frac{x}{t^2} + x - t\right),$$

$$b_2 = \frac{t}{x}, \quad d_2 = x, \quad g_2 = -\left(\frac{t}{x^2} + t\right).$$

The semi-invariants (10.4) are

$$I_3 = 0, \quad I_4 = 0, \quad I_6 = 0.$$

Hence, the system (10.18) is factorable. It is easy to show that system (10.18) is written in the following form:

$$[D_t - t]\left[D_x - \left(\frac{x-t}{t}\right)\right] u = 0,$$

$$[D_x - x]\left[D_t - \frac{t}{x}\right] v = 0.$$

Example 10.5. We consider the system:

$$
\begin{aligned}
u_{xt} &= xtu_x - tu_t + (t + xt^2)u, \\
v_{xt} &= -xv_x + xv_t + (1 - x^2)v.
\end{aligned} \tag{10.19}
$$

Comparing the system (10.19) with system (10.13), we have that:

$$a_1 = xt, \quad c_1 = -t, \quad f_1 = t(1 + x^2),$$

$$b_2 = -x, \quad d_2 = x, \quad g_2 = 1 - x^2.$$

Its semi-invariants $I_2,\ I_3,\ I_5,\ I_6$ are vanish. Hence the system (10.19) is factorable and can be written in the form:

$$
\begin{aligned}
&[D_x + t]\,[D_t - xt]\,u = 0, \\
&[D_x - x]\,[D_t + x]\,v = 0.
\end{aligned}
$$

Example 10.6. Now, we consider the following system:

$$
\begin{aligned}
u_{xt} &= \left(\frac{tx - 1}{t}\right) u_x - \left(\frac{tx + 1}{x}\right) u_t + \left(\frac{t^2x^2 - tx - 1}{tx}\right) u, \\
v_{xt} &= -\left(\frac{t + 1}{t}\right) v_x + xv_t + \left(\frac{x(t + 1)}{t}\right) v.
\end{aligned} \tag{10.20}
$$

The system (10.20) is also a member of system (10.13) with coefficients:

$$
\begin{aligned}
&a_1 = -\frac{tx - 1}{t}, \quad c_1 = \frac{tx + 1}{x}, \quad f_1 = \frac{t^2x^2 - tx - 1}{tx}, \\
&b_2 = \frac{t + 1}{t}, \quad d_2 = x, \quad g_2 = \frac{x(t + 1)}{t}.
\end{aligned}
$$

Substituting these coefficients into semi-invariants (10.4) we arrive

$$I_1 = 0, \quad I_4 = 0, \quad I_6 = 0.$$

Therefore, the system (10.20) is factorable. It is straightforward to show that system (10.20) takes the following form:

$$
\begin{aligned}
&\left[D_t - \frac{tx - 1}{t}\right] \left[D_x + \frac{tx + 1}{x}\right] u = 0, \\
&\left[D_t + \frac{t + 1}{t}\right] [D_x - x]\, v = 0.
\end{aligned}
$$

Example 10.7. Finally, we consider the following system:

$$
\begin{aligned}
u_{xt} &= -\frac{x}{t^2}u_x + \left(\frac{t-2}{x}\right)u_t + \left(\frac{2x}{t^3} + \frac{t-2}{t^2}\right)u, \\
v_{xt} &= \left(\frac{tx-1}{t}\right)v_x + tv_t + (2-tx)v.
\end{aligned} \tag{10.21}
$$

The system (10.21) has the form (10.13) with coefficients:

$$
a_1 = -\frac{x}{t^2}t, \quad c_1 = \frac{t-2}{x}, \quad f_1 = \frac{2x}{t^3} + \frac{t-2}{t^2},
$$

$$
b_2 = \frac{tx-1}{t}, \quad d_2 = t, \quad g_2 = 2 - tx.
$$

Then the semi-invariants $I_3,\ I_4,\ I_6$ vanish. Therefore, the system (10.21) is factorable and is given by:

$$
\begin{aligned}
&\left[D_x - \frac{t-2}{x}\right]\left[D_t + \frac{x}{t^2}\right]u = 0, \\
&\left[D_t - \frac{tx-1}{t}\right]\left[D_x - t\right]v = 0.
\end{aligned}
$$

10.5 Conclusion

In this chapter, we work on invariants for systems of hyperbolic equations. We have shown that the class of systems (10.1) has no differential invariants of order zero. We determined five independent differential invariants of first order. Also, we have derived invariant equations and two invariant systems for (10.1). Motivated by the applications of Laplace invariants, we use the forms of the semi-invariants to classify those systems of hyperbolic equations that can be mapped into simple linear systems. We used these results to construct some examples.

The work of this chapter is the subject of a forthcoming article [61].

Chapter 11

Final remarks

Recently, Ibragimov developed a systematic method for determining invariants of families of equations. This method is based on the infinitesimal approach and is applicable to algebraic and differential equations possessing finite or infinite equivalence groups. The method was applied to a number of ordinary and partial differential equations.

The present thesis is aimed at discussing the main principles of the method and its applications to more general hyperbolic equations. In particular, we apply it to non-linear hyperbolic equations and two subclasses of it, to n-dimensional hyperbolic equations, to n-dimensional wave-type equations and to system of two hyperbolic equations. Also, known identities are presented relating arbitrary order partial derivatives of $u(x,t)$ and $u'(x',t')$ for the general point transformation $x' = P(x,t,u), t' = Q(x,t,u),\ u' = R(x,t,u)$. These identities are used to study the nature of those point transformations which preserve the general form of wide class of 1+1 PDEs. These results are generalized to system of two equations.

The work here opens the way on certain other problems that can be considered in the near future. For example, the work on differential invariants for hyperbolic equation of general class

$$u_{xt} = f(x,t,u,u_t,u_x) \tag{11.1}$$

is incomplete. We can use invariant differentiation to construct a basis for the invariants in the same way as Ibragimov did for the linear hyperbolic equation (see [20]).

Another problem is to find equivalence transformations and differential invariants for the following general class of equations:

$$u_{xt} = f(x, t, u, u_t, u_x)u_{xx} + g(x, t, u, u_t, u_x).$$

Further study, along the lines of the chapter 3, of a single equation with more than two independent variables, can be carried out.

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