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LIE GROUP CLASSIFICATION OF DIFFUSION-TYPE EQUATIONS

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Abstract

Group analysis of differential equations, was originally developed by the Norwegian mathematician Marious Sophus Lie in the latter part of 19th century [44, 45]. Lie formally defined and initiated the mathematical study of continuous groups of transformations, now known as Lie groups, and showed that the order of an ordinary differential equation (ODE) can be reduced by one if it is invariant under a one-parameter Lie group of point transformations.

Following the work of Lie, Ovsiannikov in the late 1950's and 1960's and Bluman in the late 1960's and 1970's developed a major revival of interest in symmetry methods for differential equations. With the publications of the texts of Ovsiannikov [55], Olver [53], and Bluman and Kumei [10], there are now several comprehensive accounts of the basic theory as well as more recent applications and generalizations.

Perhaps the most powerful tools currently available in the area of nonlinear partial differential equations (PDEs) are transformation methods. While there is no existing general theory for solving such equations, many cases have yielded to appropriate changes of variables. Point transformations are the ones which are mostly used. These are transformations in the space of the dependent and independent variables of a PDE. Probably the most useful point transformations of PDE's are those which form a continuous Lie group of transformations which leave the equation invariant.

The classical method of finding Lie symmetries is first to find the infinitesimal transformations and with the benefit of linearization to extend these to groups of finite transformations. This method is easy to apply and well-established in the last few years [10, 12, 28, 29, 35]. Symmetries of a PDE can enable new solutions to be found directly using known solutions or via similarity solutions. The similarity solutions arise from transformations which yield invariants that allow one to obtain solutions through

reducing the number of independent variables of a PDE by at least one.

Lie [45] was the first to study the complete group classification of $(1+1)$-dimensional linear parabolic equations (i.e., the complete description of their Lie symmetries up to the equivalence relation generated by the corresponding equivalence group). This was done as a part of the more general group classification of linear second-order partial differential equations in two independent variables. A modern treatment of the subject is given by Ovsiannikov [55]. There exist also a number of recent papers partially rediscovering the classical results of Lie and Ovsiannikov.

The idea of group classification of nonlinear equations was introduced by Ovsiannikov [56] who studied the Lie symmetries of the well known nonlinear diffusion equation $u_t = [D(u)u_x]_x$. Since then similar problems have been considered. For example, the problem of group classification of diffusion-convection equation was considered by many authors [24, 39, 54, 60, 74]. In [67] a group classification of the variable coefficient diffusion equation was carried out and in [60] the Lie symmetries of variable coefficient diffusion-convection equations were classified.

A new class of symmetries, nonlocal symmetries was also introduced by Bluman and al. [10, 11]. The method for finding these nonlocal symmetries, called potential symmetries, is first to write the system in conserved form and by introducing new variables, the potentials, to find the infinitesimal generators of local symmetries admitted by the auxiliary system of PDEs. The extension of local symmetries to potential symmetries widens the applicability of symmetry methods to the construction of solutions of ordinary and partial differential equations.

This thesis is organized as follows. In chapter 2 we present the theoretical background needed for the subsequent chapters and in chapter 3 we present the known results of the (1+1)-dimensional diffusion-convection equation. Following the idea of group classification for nonlinear equations we present in chapters 4 and 5, a complete group classification of the (2+1) and the (3+1) nonlinear diffusion-convection equations respectively. Another problem considered in chapter 6, is the classification of a class of systems of diffusion equations. Also, the method for finding a new class of symmetries, potential symmetries, is applied in chapter 7. Finally, in chapter 8 we suggest certain problems that can be considered in the future.

Περίληψη

Ο Marious Sophous Lie, Νορβηγός μαθηματικός, ανάπτυξε και εδραίωσε τη μαθηματική μελέτη των συνεχών ομάδων μετασχηματισμών, γνωστή ως ομάδες Lie, δείχνοντας ότι η τάξη μιας συνήθης διαφορικής εξίσωσης μπορεί να μειωθεί κατά ένα εάν είναι αναλλοίωτη κάτω από μια μονοπαραμετρική ομάδα τοπικών μετασχηματισμών Lie.

Συνεχίζοντας τη δουλειά του Lie, ο Ovsiannikov στα τέλη του 1950 και 1960, και ο Bluman στα τέλη του 1960 και 1970, ανέπτυξαν μεγάλο ενδιαφέρον στις μεθόδους συμμετριών για διαφορικές εξισώσεις. Με τις δημοσιεύσεις των Ovsiannikov [55], Olver [53], και Bluman και Kumei [10] τώρα υπάρχουν διάφορες εκτιμήσεις της βασικής θεωρίας καθώς και πρόσφατες εφαρμογές και γενικεύσεις.

Οι μεθόδοι μετασχηματισμών είναι ίσως ένα από τα πιο χρήσιμα εργαλεία που είναι διαθέσιμα στην περιοχή των μη-γραμμικών μερικών διαφορικών εξισώσεων (ΜΔΕ). Παρόλο που δεν υπάρχει γενική θεωρία για την επίλυση τέτοιων εξισώσεων, εντούτοις με τις κατάλληλες αλλαγές των μεταβλητών οδηγούμαστε σε πολλές ειδικές περιπτώσεις. Αυτό έχει ως αποτέλεσμα την απλοποίηση της ΜΔΕ. Οι τοπικοί μετασχηματισμοί είναι αυτοί που χρησιμοποιούνται περισσότερο. Αυτοί είναι οι μετασχηματισμοί στο χώρο των εξαρτημένων και ανεξάρτητων μεταβλητών μιας ΜΔΕ. Ίσως οι πιο χρήσιμοι τοπικοί μετασχηματισμοί μιας ΜΔΕ είναι αυτοί που σχηματίζουν μια συνεχή ομάδα μετασχηματισμών Lie και αφήνουν την εξίσωση αναλλοίωτη.

Η κλασσική μέθοδος εύρεσης συμμετριών Lie προκύπτει πρώτα με την εύρεση των απειροστών μετασχηματισμών και μετά χρησιμοποιώντας ως πλεονέκτημα τη γραμμικότητά τους επεκτείνοντάς τους σε ομάδες πεπερασμένων μετασχηματισμών. Η μέθοδος αυτή εφαρμόζεται εύκολα και έχει καθιερωθεί τα τελευταία χρόνια [10,12,28,29,35]. Οι συμμετρίες αυτών των ΜΔΕ χρησιμοποιούνται στη συνέχεια για την εύρεση νέων λύσεων είτε απευθείας με τη χρήση γνωστών λύσεων, είτε μέσω των λύσεων ομοιότητας. Οι λύσεις ομοιότητας ορίζονται

ως οι μετασχηματισμοί που μειώνουν τον αριθμό των ανεξαρτήτων μεταβλητών μιας ΜΔΕ κατά ένα.

Η ιδέα της ταξινόμησης των ομάδων μιας μη-γραμμικής εξίσωσης πρωτοεμφανίστηκε από τον Ovsiannikov ο οποίος μελέτησε τις συμμετρίες Lie της πολύ γνωστής μη-γραμμικής εξίσωσης της διάχυσης

$$u_t = [D(u)u_x]_x \,.$$

Από τότε παρόμοια προβλήματα έχουν θεωρηθεί. Για παράδειγμα, με την ταξινόμηση ομάδων της

$$u_t = [D(u)u_x]_x + K(u)u_x,$$

έχουν ασχοληθεί πολλοί συγγραφείς [24, 39, 54, 60, 74]. Επίσης στο [67] παρουσιάζεται η ταξινόμηση της εξίσωσης

$$f(x)u_t = [g(x)D(u)u_x]_x \,,$$

και στο [60] ταξινομούνται οι συμμετρίες Lie της

$$f(x)u_t = [g(x)D(u)u_x]_x + K(u)u_x.$$

Σε αυτή τη διατριβή παρουσιάζουμε μια πλήρη ταξινόμηση των συμμετριών Lie της $(2+1)$ και $(3+1)$ μη-γραμμικής εξίσωσης διάχυσης της θερμότητας κατασκευάζοντας ειδικές λύσεις για κάποιες ειδικές περιπτώσεις αυτών των εξισώσεων.

Παρουσιάζουμε επίσης ταξινόμηση για συστήματα εξισώσεων της διάχυσης, βρίσκοντας τις συμμετρίες Lie καθώς και μια νέα τάξη μη-τοπικών συμμετριών, γνωστές ως συμμετρίες δυναμικού. Για τη μέθοδο εύρεσης αυτών των μη-τοπικών συμμετριών, η οποία παρουσιάστηκε από τον Bluman [10, 11], πρώτα το σύστημα χρείαζεται να γραφτεί σε μορφή διατήρησης και στη συνέχεια με την εισαγωγή μιας νέας μεταβλητής, της μεταβλητής δυναμικού, να βρεθούν οι απειροστοί γεννήτορες του βοηθητικού συστήματος από ΜΔΕ που δημιουργείται. Η επέκταση των τοπικών συμμετριών στις συμμετρίες δυναμικού διευρύνει την εφαρμογή των μεθόδων συμμετριών στην εύρεση λύσεων για συνήθεις και μερικές διαφορικές εξισώσεις.

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Contents

List of Tables

Chapter 1

Introduction

Group analysis furnishes a universal and effective method for analytical investigations of nonlinear mathematical models in physics, mathematical biology and engineering sciences. It was Lie, the Norwegian mathematician, who discovered that symmetries of differential equations can be found and exploited systematically [44, 45]. Although Lie's methods for determining and using symmetries for many years were largely neglected, fairly recently with the advent of powerful computation packages it has become possible to apply Lie's method to explore the symmetries of a wide range of physical systems.

Lie group methods are perhaps the most powerful currently available in finding exact solutions of nonlinear partial differential equations. Probably the most useful method is the application of Lie point transformations which are those that form a continuous Lie group of transformations, that leave the PDE invariant. Symmetries of this PDE are then revealed, perhaps enabling new solutions to be found directly or via similarity reductions.

The idea of group classification of nonlinear equations was introduced by Ovsiannikov [56] who studied the Lie symmetries of the well known nonlinear diffusion equation

$$u_t = [D(u)u_x]_x \, .$$

Since then similar problems have been considered. For example, many authors [24, 39, 54, 60, 74] studied the problem of group classification of diffusion-convection equation

$$u_t = [D(u)u_x]_x + K(u)u_x.$$

Also, in [67] a group classification of the variable coefficient diffusion equation

$$f(u)u_t = [g(x)D(u)u_x]_x \, ,$$

was carried out and in [60] the Lie symmetries of

$$f(x)u_t = [g(x)D(u)u_x]_x + K(u)u_x,$$

were classified. The Lie symmetries of many other physically important systems have been classified, the first two volumes of a handbook of symmetry analysis edited by Ibragimov [33, 34] are excellent sources for such classifications.

In this thesis following the idea of group classification for nonlinear equations we present in chapter 4 a complete group classification of the (2+1) nonlinear diffusion-convection equations

$$u_t = (D(u)u_x)_x + (F(u)u_y)_y + K(u)u_x,$$

and in chapter 5 the (3+1) nonlinear diffusion-convection equations

$$u_t = (D(u)u_x)_x + (F(u)u_y)_y + (G(u)u_z)_z + K(u)u_x.$$

The method for doing this is first to find the infinitesimal transformations and with the benefit of linearization to extend these to groups of finite transformations. Using then the derived Lie symmetries we construct similarity reductions and exact solutions of certain equations.

Another problem considered in this thesis in chapter 6, is the classification of a class of systems of diffusion equations of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left[f(u,v)\frac{\partial u}{\partial x}\right], \\ \frac{\partial v}{\partial t} &= \frac{\partial}{\partial x}\left[g(u,v)\frac{\partial v}{\partial x}\right]. \end{aligned} \tag{1.1}$$

We apply also in chapter 7, the method for finding a new class of symmetries for systems of diffusion equation of the form (1.1). Bluman and al. [10, 11] introduced this method for a system of PDEs $\Delta(x, u)$ in the case when at least one of the PDEs can be written in conserved form. If we introduce potential variables v for the equations written in conserved forms as further unknown functions, we obtain a system $Z(x, u, v)$. Any Lie symmetry for $Z(x, u, v)$ induces a symmetry $\Delta(x, u)$. When at least one of the infinitesimals correspond to the variables x and u depends explicitly on potentials, then the local symmetry of $Z(x, u, v)$ induces a nonlocal symmetry of $\Delta(x, u)$. These nonlocal symmetries are called potential symmetries.

As a final application of symmetry methods we use infinite-dimensional potential symmetries of (1.1) to derive linearizing mappings.

All calculations have been greatly facilitated by the computer algebraic package REDUCE [30].

Chapter 2

Lie Groups of Transformations

Before developing the theory of group classification, we first give the necessary theoretical background needed for the subsequent chapters. Analytically we define the Lie groups of transformations, the infinitesimal transformations, we examine when a PDE is invariant under the infinitesimal transformations and how with the use of Lie symmetries we construct the similarity solutions. We also define the potential symmetries and we present an example of invertible mappings and an example of equivalence transformation.

2.1 Groups

Definition 2.1. *A group is a set G with a binary operation $* : G \times G \to G$ satisfying the following axioms:*

(i) Closure property: For any element g and h of G, $g * h$ is an element of G.

(ii) Associative property: For any elements g, h and k of G

$$g * (h * k) = (g * h) * k.$$

(iii) Identity element: There exists a unique identity element e of G such that for any element g of G

$$e * g = g * e = g.$$

(iv) Inverse element: For any element g of G there exists a unique inverse element g^{-1} in G such that

$$g^{-1} * g = g * g^{-1} = e.$$

2.2 Groups of Transformations

Here and below we assume that all constants to be smooth.

With the term transformation of the space, we mean a function $T : R^3 \to R^3$ which is defined by:

$$
\begin{aligned}
x' &= \psi(x, t, u), \\
t' &= \phi(x, t, u), \\
u' &= \omega(x, t, u),
\end{aligned} \tag{2.1}
$$

where ψ, ϕ and ω are known functions.

Geometrically, T transforms the point (x, t, u) to another point (x', t', u') at the same plane of coordinates.

If the equations that define the transformation T can be solved in terms of x, t, u, then the transformation that appears is the inverse transformation T^{-1} which is defined by:

$$
\begin{aligned}
x &= \Psi(x', t', u'), \\
t &= \Phi(x', t', u'), \\
u &= \Omega(x', t', u').
\end{aligned}
$$

From the composition of those two transformations we obtain the identity transformation which is defined by:

$$
\begin{aligned}
x' &= x, \\
t' &= t, \\
u' &= u.
\end{aligned}
$$

Now we consider the transformations where the functions ψ, ϕ and ω in the equations (2.1) depend on a real parameter, ϵ. The parameter ϵ continuously changes in an open interval such that $\mid \epsilon \mid < \epsilon_0$. Then the whole of transformations compose the family transformation T_ϵ which is defined by:

$$
\begin{aligned}
x' &= \psi(x, t, u, \epsilon), \\
t' &= \phi(x, t, u, \epsilon), \\
u' &= \omega(x, t, u, \epsilon),
\end{aligned} \tag{2.2}
$$

where ψ, ϕ, and ω, are analytic functions.

Definition 2.2. *A group of transformations of the form (2.2) defines a one-parameter Lie group of transformations if the following axioms are satisfied:*

(i) $T_0 = I$ $(T_{\epsilon_0} = I)$ *(existence of the identity)*

(ii) $T_\epsilon^{-1} = T_{\epsilon^{-1}}$ *(existence of the inverse element)*

(iii) $T_\gamma(T_\delta T_\epsilon) = (T_\gamma T_\delta)T_\epsilon$ *(associativity of a group multiplication)*

(iv) $T_\delta T_\epsilon = T_{\phi(\epsilon,\delta)}$ *(closure)*

Every value of parameter ϵ *corresponds to a special part of the family transformations. The transformations* T_ϵ *belongs to the one-parameter family group of transformations.*

Below we present some examples of one-parameter Lie Groups of transformations.

Example 2.1. *Group of Translations*

The group of translations is given by

$$x' = x,$$

$$t' = t + \epsilon,$$

$$u' = u,$$

where $\epsilon_0 = 0,\ \epsilon^{-1} = -\epsilon$ *and* $\phi(\epsilon, \delta) = \epsilon + \delta$.

Example 2.2. *Group of Rotations*

The group of rotations is given by

$$x' = x\cos\epsilon - t\sin\epsilon,$$

$$t' = x\sin\epsilon + t\cos\epsilon,$$

$$u' = u,$$

where $\epsilon_0 = 0,\ \epsilon^{-1} = -\epsilon$ *and* $\phi(\epsilon, \delta) = \epsilon + \delta$. *These transformations describe the rotation of a point in the xt plane by an angle* ϵ.

Example 2.3. *Group of Scalings*

A group of scalings is given by

$$x' = \epsilon x,$$

$$t' = \epsilon^2 t,$$

$$u' = u,$$

where $\epsilon_0 = 1,\ \epsilon^{-1} = 1/\epsilon$ *and* $\phi(\epsilon, \delta) = \epsilon\delta$.

2.3 Infinitesimal Transformations

We consider the one-parameter group of transformations T_ϵ with identity $\epsilon = 0$. Then we can expand into Taylor series every right part of the equations that define the one-parameter group of transformations T_ϵ in the neighborhood of $\epsilon = 0$. So,

$$\begin{aligned} x' &= x + \epsilon X(x,t,u) + O(\epsilon^2), \\ t' &= t + \epsilon T(x,t,u) + O(\epsilon^2), \\ u' &= u + \epsilon U(x,t,u) + O(\epsilon^2), \end{aligned} \tag{2.3}$$

where $X = \frac{\partial \psi}{\partial \epsilon}|_{\epsilon=0}$, $T = \frac{\partial \phi}{\partial \epsilon}|_{\epsilon=0}$ and $U = \frac{\partial \omega}{\partial \epsilon}|_{\epsilon=0}$.

The first order transformation is known as *infinitesimal transformation* and X, T, U are called the *infinitesimals* of the transformation.

By knowing the infinitesimals X, T, U of the transformation, we can find the Lie group of transformations of the form (2.2) by solving the following system of first order differential equations

$$\begin{aligned} \frac{\mathrm{d}x'}{\mathrm{d}\epsilon} &= X(x', t', u'), \\ \frac{\mathrm{d}t'}{\mathrm{d}\epsilon} &= T(x', t', u'), \\ \frac{\mathrm{d}u'}{\mathrm{d}\epsilon} &= U(x', t', u'), \end{aligned} \tag{2.4}$$

with initial conditions

$$x' = x, \quad t' = t, \quad u' = u, \qquad \text{when } \epsilon = 0.$$

The linear differential operator

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u}, \tag{2.5}$$

is called the *infinitesimal generator* of the infinitesimal transformation.

Example 2.4. *We consider the rotation group*

$$\begin{aligned} x' &= x\cos\epsilon - t\sin\epsilon, \\ t' &= x\sin\epsilon + t\cos\epsilon, \\ u' &= u. \end{aligned}$$

Hence, the system (2.4) has the following form

$$\begin{aligned}
\frac{\mathrm{d}x'}{\mathrm{d}\epsilon} &= -x'\sin\epsilon - t'\cos\epsilon,\\
\frac{\mathrm{d}t'}{\mathrm{d}\epsilon} &= x'\cos\epsilon - t'\sin\epsilon,\\
\frac{\mathrm{d}u'}{\mathrm{d}\epsilon} &= 0,
\end{aligned}$$

with initial conditions

$$x' = x, \quad t' = t, \quad u' = u, \qquad \text{when } \epsilon = 0.$$

The infinitesimals for the rotation group are

$$\begin{aligned}
\left.\frac{\mathrm{d}x'}{\mathrm{d}\epsilon}\right|_{\epsilon=0} &= (-x'\sin\epsilon - t'\cos\epsilon)|_{\epsilon=0} = -t',\\
\left.\frac{\mathrm{d}t'}{\mathrm{d}\epsilon}\right|_{\epsilon=0} &= (x'\cos\epsilon - t'\sin\epsilon)|_{\epsilon=0} = x',\\
\left.\frac{\mathrm{d}u'}{\mathrm{d}\epsilon}\right|_{\epsilon=0} &= 0.
\end{aligned}$$

So, the infinitesimal generator is

$$\Gamma = -t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}.$$

2.4 Invariant Functions

An infinitely differentiable function $F(x', t', u')$ is an invariant function of the Lie group of transformations (2.3) if and only if for any group of transformations of the form (2.3)

$$F(x', t', u') = F(x, t, u).$$

Theorem 2.1. *A function $F(x', t', u')$ is invariant if and only if is the solution of the PDE*

$$\Gamma F(x', t', u') = 0,$$

where Γ is defined by equation (2.5).

The proof of the theorem can be found in [10].

Example 2.5. *As we have seen above, a function* $F(x', t', u')$ *is invariant if and only if*

$$\Gamma F(x', t', u') = 0.$$

So, for the group of rotations a function $F(x', t', u')$ *is invariant if and only if*

$$\Gamma F = -t\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial t} = 0. \tag{2.6}$$

Equation (2.6) is a first order linear partial differential equation which can be solved by the method of characteristics. That is,

$$\frac{\mathrm{d}x}{-t} = \frac{\mathrm{d}t}{x} = \frac{\mathrm{d}u}{0},$$

from which we deduce that

$$F = \Phi(x^2 + t^2).$$

Hence, any function of the form $\Phi(x^2 + t^2)$ *remains invariant under the rotation group.*

2.5 Invariant PDEs

In this section we will examine when a second order (1+1)-dimensional PDE is invariant under the transformation of the form (2.3). So we need to know how the derivatives are transformed.

We define the extended infinitesimal transformation:

$$\begin{aligned}
u'_{x'} &= u_x + \epsilon U^x(x, t, u, u_x, u_t) + O(\epsilon^2),\\
u'_{t'} &= u_t + \epsilon U^t(x, t, u, u_x, u_t) + O(\epsilon^2),\\
u'_{x'x'} &= u_{xx} + \epsilon U^{xx}(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) + O(\epsilon^2),\\
u'_{x't'} &= u_{xt} + \epsilon U^{xt}(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) + O(\epsilon^2),\\
u'_{t't'} &= u_{tt} + \epsilon U^{tt}(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) + O(\epsilon^2),
\end{aligned}$$

with the prolongation formula:

$$\begin{aligned}
U^x &= D_x(U) - u_x D_x(X) - u_t D_x(T), \dots\\
U^{xx} &= D_x(U^x) - u_{xx} D_x(X) - u_{xt} D_t(T), \dots
\end{aligned} \tag{2.7}$$

and similarly for the others. Here D_x and D_t are the total derivatives with respect to x and t, respectively.

The extended infinitesimal generators are defined by:

$$\Gamma^{(1)} = \Gamma + U^x \frac{\partial}{\partial u_x} + U^t \frac{\partial}{\partial u_t},$$
$$\Gamma^{(2)} = \Gamma^{(1)} + U^{xx} \frac{\partial}{\partial u_{xx}} + U^{xt} \frac{\partial}{\partial u_{xt}} + U^{tt} \frac{\partial}{\partial u_{tt}}.$$

A transformation is said to be *symmetry* of a second order PDE

$$E(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0,$$

if the PDE has the same form in the new variables $x',\ t',\ u'$. That is,

$$E(x', t', u', u'_{x'}, u'_{t'}, u'_{x'x'}, u'_{x't'}, u'_{t't'}) = 0.$$

The PDE $E = 0$ admits a symmetry of the infinitesimal transformation if and only if

$$\Gamma^{(2)} E|_{E=0} = 0. \tag{2.8}$$

Equation (2.8) is a polynomial in $u_x, u_t, u_{xx}, u_{xt}, u_{tt}$. We impose the condition that (2.8) is an identity in seven variables of $x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}$, since one can be eliminated using equations $E = 0$. Equating coefficients of these variables leads to an over-determined system of differential equations for the functions $X(x, t, u),\ T(x, t, u)$ and $U(x, t, u)$. Its solution defines the Lie symmetries of the under-examined PDE.

In the case of a single evolution equation of the form $u_t = H$, where H is a function of x, t, u and derivatives of u with respect to x, it can be shown that $T = T(t)$ [40, 72]. We generalize this result for systems of evolution equations.

Theorem 2.2. *For the infinitesimal invariant transformation*

$$\begin{aligned} x^* &= x + \epsilon X(x, t, u, v),\\ t^* &= t + \epsilon T(x, t, u, v),\\ u^* &= u + \epsilon U(x, t, u, v),\\ v^* &= v + \epsilon V(x, t, u, v), \end{aligned} \tag{2.9}$$

with $x,\ t$ be the independent variables and $u,\ v$ the dependent variables of the following system of PDE's

$$u_t = H_1(x,t,u,v,u_i,v_j), \tag{2.10}$$
$$v_t = H_2(x,t,u,v,u_i,v_j),$$

($u_i,\ v_j$ are the x derivatives of u and v, respectively, $i = 1, ...n, \quad j = 1, ..., m$ with $n \geq m > 1$) where $H_{1u_n}^2 + H_{1v_n}^2 + H_{2u_n}^2 + H_{2v_n}^2 \neq 0$, it follows that $T = T(t)$.

Proof. The system of PDE's (2.10) admits infinitesimal generator of the form

$$X = X(x,t,u,v)\frac{\partial}{\partial x} + T(x,t,u,v)\frac{\partial}{\partial t} + U(x,t,u,v)\frac{\partial}{\partial u} + V(x,t,u,v)\frac{\partial}{\partial v},$$

with nth extension

$$X^{(n)} = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} + \sum_{i=1}^{n} U^i\frac{\partial}{\partial u_i} + \sum_{j=1}^{m} V^j\frac{\partial}{\partial v_j}, \quad m \leq n$$

if and only if

$$X^{(n)}(u_t - H_1) = 0,$$
$$X^{(n)}(v_t - H_2) = 0.$$

The nth extended infinitesimals $U^{(n)}$ is given by

$$U^{(n)} = D_x(U^{n-1}) - u_n D_x(X) - u_{n-1,t}D_x(T).$$

Taking coefficients of $u_{n-1,t}$ that occurs from the total derivatives $D_x(U^{n-1})$ and $D_x(T)$ we get

$$nH_{1u_n}(T_u u_1 + T_v v_1 + T_x) = 0,$$

that is

$$T_u = T_v = T_x = 0.$$

Hence,

$$T = T(t).$$

□

Furthermore, if the right hand side of an evolution equation is a polynomial in the pure derivatives with respect to x then $X = X(x,t)$ [41]. This result is generalized for system of evolution equations.

Theorem 2.3. *For the infinitesimal transformations (2.9) of the system of PDEs (2.10) where u_i, v_j are the x derivatives of u and v respectively, $i = 1, ...n$, $j = 1, ..., m$ with $n \geq m > 1$ and $H_{1u_n}^2 + H_{1v_n}^2 + H_{2u_n}^2 + H_{2v_n}^2 \neq 0$, if H_1 and H_2 are polynomial in the pure derivatives with respect to x, it follows that $X = X(x,t)$ [71].*

2.6 Similarity Transformations

The similarity solutions that arise from transformations which yield invariants, allow one to obtain solutions through reducing the number of independent variables of a PDE by at least one. For example, a PDE with two independent variables can be reduced into an ordinary differential equation (ODE). The similarity transformations are constructed from the solution of the *invariant surface condition*

$$Xu_x + Tu_t = U. \tag{2.11}$$

Now if $\frac{X}{T}$ is independent of u, then the solution of (2.11) has the form

$$\begin{aligned} &\eta(x,t) = constant, \\ &u(x,t) = F(x,t,\eta,f(\eta)), \end{aligned} \tag{2.12}$$

where F is a known function. Equation (2.12) is the *similarity solution* and the function $\eta(x,t)$ is called the *similarity variable* that constitute the independent variable of the ODE that we get from the transformation. The function $f(\eta)$ is the unknown function of the ODE.

2.7 Potential Symmetries

Bluman and al. [10, 11] introduced a method for finding a new class of symmetries, non-local symmetries, for a system of PDEs $\Delta(x,t,u)$ with independent variables x, t and dependent variables u in the case when at least one of the PDEs can be written in conserved form.

If we introduce new variables v which are potentials for the PDEs written in conserved forms as further unknown functions, we obtain a system $Z(x,t,u,v)$. By construction, any solution $u(x,t)$, $v(x,t)$ of $Z(x,t,u,v)$ defines a solution $u(x,t)$ of $\Delta(x,t,u)$. The given system $\Delta(x,t,u)$ is then said to be embedded in the auxiliary system $Z(x,t,u,v)$, so any Lie group of transformation for $Z(x,t,u,v)$ induces a symmetry for $\Delta(x,t,u)$. When at least one of the infinitesimals which correspond to the variables x, t and u depends explicitly on the potential v, then the local symmetry of $Z(x,t,u,v)$ induces a nonlocal symmetry of $\Delta(x,t,u)$. These nonlocal symmetries are called *potential symmetries*. More details about potential symmetries and their uses can be found in [13, 14].

2.8 Invertible Mappings of nonlinear systems of PDEs to linear systems of PDEs

Another application of symmetry methods to differential equations is to discover related differential equations of simpler form. By comparing the Lie groups admitted by a given differential equation and another differential equation (target equation), one can find constructively, necessary conditions for a mapping of the given equation to the target equation. If the target equation, which is a member of a class of equations, is characterized completely in terms of a Lie symmetry group then one can algorithmically determine if an invertible mapping exists between the equations. In [10] it is shown that an invertible mapping that transforms a nonlinear PDE does not exist if the nonlinear PDE does not admit an infinite-parameter Lie group of contact transformations. Also such mappings do not exist for a nonlinear system of PDEs if the system does not admit an infinite-parameter Lie group of transformations. If such infinite-parameter groups exist then the nonlinear PDE (or system of nonlinear PDEs) can be transformed into a linear PDE (or into a system of linear PDEs) provided that these groups satisfy certain criteria [10].

For example, we consider the generalized nonlinear heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K(u)\frac{\partial u}{\partial x}\right). \tag{2.13}$$

If we introduce the potential variable v we obtain the auxiliary system of (2.13)

$$\begin{aligned}\frac{\partial v}{\partial x} &= u, \\ \frac{\partial v}{\partial t} &= K(u)\frac{\partial u}{\partial x}.\end{aligned} \tag{2.14}$$

In the case when $K(u) = u^{-2}$ equations (2.14) admit the following infinite-dimensional symmetry

$$\Gamma_\infty = h(t,v)\frac{\partial}{\partial x} - u^2\frac{\partial h(t,v)}{\partial v}\frac{\partial}{\partial u},$$

where $h(t,v)$ is an arbitrary solution of the linear heat equation $h_{vv} - h_t = 0$.

The above infinite-dimensional symmetry admitted by (2.14) now written as

$$\begin{aligned}v_x &= u, \\ v_t &= u^{-2}u_x,\end{aligned} \tag{2.15}$$

satisfies the required criteria for linearization. Hence, system (2.15) can be linearized. The procedure for determining invertible mappings with the employment of infinite-parameter Lie groups of transformations is well explained in [10]. The above infinite-dimensional Lie symmetry leads to the mapping

$$t' = t, \quad x' = v, \quad v' = x, \quad u' = \frac{1}{u}, \tag{2.16}$$

that transforms any solution $(u'(x',t'), v'(x',t'))$ of the linear system of PDEs

$$v'_{x'} = u', \quad v'_{t'} = u'_{x'}, \tag{2.17}$$

to a solution $(u(x,t), v(x,t))$ of the nonlinear system (2.15). In turn, the mapping (2.16) produces the one-to-one contact transformation [15]

$$\mathrm{d}x' = u\mathrm{d}x + u^{-2}u_x\mathrm{d}t, \quad \mathrm{d}t' = \mathrm{d}t, \quad u' = \frac{1}{u},$$

which transforms the linear diffusion equation

$$\frac{\partial u'}{\partial t'} = \frac{\partial^2 u'}{\partial x'^2}, \tag{2.18}$$

into the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right). \tag{2.19}$$

Using the inverse mapping of (2.16)

$$t = t', \quad x = v', \quad v = x', \quad u = \frac{1}{u'},$$

we deduce the contact transformation

$$\mathrm{d}x = u'\mathrm{d}x' + u'_{x'}\mathrm{d}t', \quad \mathrm{d}t = \mathrm{d}t', \quad u = \frac{1}{u'},$$

which also connects the linear diffusion equation (2.18) and the nonlinear PDE (2.19).

2.9 Equivalence Transformations

An equivalence transformation is a transformation which has the property that it transforms any member of a class of PDEs to an equation which is also a member of the same class. The set of all equivalence transformations of a given family of equations forms a group called the equivalence group. There exist two methods for calculations of equivalence transformations: the direct method which was used by Lie [44] for calculation of equivalence transformations and group classification of family of second-order ordinary differential equations. The second method was suggested by Ovsiannikov [55] for determining generators of continuous equivalence groups which is a subgroup of equivalence group. Although, the direct method involves considerable computational difficulties, it has the benefit of finding the most general equivalence group. For recent applications of the direct method one can refer to references [41, 60, 61, 73]. More detailed description and examples of both methods can be found in [32].

As an example of equivalence transformations we consider the class of diffusion equations

$$u_t = k(u_x)u_{xx}, \quad k(u_x) \neq \text{const}.$$

The equivalence transformations for this class of equation is [3]

$$\tilde{t} = at + \gamma_1, \quad \tilde{x} = \beta_1 x + \beta_2 u + \gamma_2, \quad \tilde{u} = \beta_3 x + \beta_4 u + \gamma_3, \quad \tilde{k} = (\beta_1 + \beta_2 u_x)^2 k/a,$$

where a, β_i and γ_i are arbitrary constants, $a \neq 0$ and $\beta_1\beta_4 - \beta_2\beta_3 \neq 0$. A special case of this, is the hodograph transformation

$$\tilde{x} = u, \quad \tilde{u} = x,$$

that connects the nonlinear equation

$$u_t = \frac{u_{xx}}{u_x^2},$$

and the linear equation

$$\tilde{u}_t = \tilde{u}_{xx}.$$

Chapter 3

Group analysis of (1+1)-dimensional diffusion-convection equation

The idea of group analysis of the nonlinear diffusion-convection equations

$$u_t = (D(u)u_x)_x + K(u)u_x, \tag{3.1}$$

was considered by many authors. These equations are used to model a wide variety of phenomena in mathematical physics, mathematical chemistry, mathematical biology, fluid mechanics etc. For example equation (3.1) describes vertical one-dimensional transport of water in homogeneous non-deformable porous media [42, 66]. In the case when $K(u) = 0$ it describes stationary motion of a boundary layer of fluid over a flat plate and a vortex of incompressible fluid in a porous medium polytropic relations of gas density and pressure. The outstanding representative of the class (3.1) is the Burgers equation that is the mathematical model for a large number of physical phenomena [6–9, 47].

In this chapter we present the group classification of point symmetries admitted by $(1 + 1)$ diffusion-convection equations [24, 39, 54, 60, 74] and also the potential symmetries for this equation [38, 68, 69]. A continuing interest exists in finding exact solutions for these equations [16, 58]. Motivated by the results of this chapter, we present a complete classification of Lie symmetries for the $(2 + 1)$ and $(3 + 1)$ diffusion-convection equations in the chapters 4 and 5, respectively.

3.1 Classification of Lie Symmetries

We consider the (1+1)-dimensional diffusion-convection equations (3.1). The equivalence transformation $G_1^{\sim}$ of class (3.1) consists of the 7-parameter group of transformations

$$\begin{aligned} &\tilde{t} = \varepsilon_4^2 \varepsilon_5 t + \varepsilon_1, \quad \tilde{x} = \varepsilon_4 x + \varepsilon_7 t + \varepsilon_2, \quad \tilde{u} = \varepsilon_6 u + \varepsilon_3, \\ &\tilde{D} = \varepsilon_5^{-1} D, \quad \tilde{K} = \varepsilon_4^{-1} \varepsilon_5^{-1} K - \varepsilon_7, \end{aligned} \tag{3.2}$$

where $\varepsilon_1, \ldots, \varepsilon_7$ are arbitrary constants, $\varepsilon_4 \varepsilon_5 \varepsilon_6 \neq 0$.

We have seen that a PDE of second order admits Lie symmetries if and only if

$$\Gamma^{(2)} E|_{E=0} = 0,$$

where $\Gamma^{(2)}$ is the second extended generator of

$$\Gamma = X \frac{\partial}{\partial x} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u},$$

which is given by the relation

$$\begin{aligned} \Gamma^{(2)} &= \Gamma + [D_x U - (D_x X) u_x - (D_x T) u_t] \frac{\partial}{\partial u_x} + [D_t U - (D_t X) u_x - (D_t T) u_t] \frac{\partial}{\partial u_t} \\ &+ [D_x(U^x) - u_{xx} D_x(X) - u_{xt} D_t(T)] \frac{\partial}{\partial u_{xx}}. \end{aligned}$$

Here D_x and D_t are the total derivatives with respect to x and t respectively and U^x is the extended transformation.

In this case

$$E = u_t - D_u u_x^2 - D u_{xx} - K u_x = 0. \tag{3.3}$$

So, equation (3.1) admits Lie symmetries if and only if

$$\Gamma^{(2)} \left[u_t - D_u u_x^2 - D u_{xx} - K u_x \right] = 0, \tag{3.4}$$

where $u_t = D_u u_x^2 + D u_{xx} + K u_x$.

After elimination of u_t using the above expression, equation (3.4) becomes an identity in the variables $t,\ x,\ u,\ u_x,\ u_{xx}$. The coefficients of different powers of these variables must be zero giving the determining equations for the unknown functions $T,\ X,\ U,\ D$ and K. From [41] we can assume that $T = T(t)$ and $X = X(x,t)$. Also from coefficients of u_x^2 and u_{xx} we deduce that $U_{uu} = 0$. So,

$$U = m_1(x,t) u + m_2(x,t).$$

Using the fact that $T = T(t)$, $X = X(x,t)$ and the above expression of U from coefficients of u_{xx}, u_x and the term independent of derivatives of (3.4) we have the following determining equations respectively:

$$D_u(m_1 u + m_2) + D(T_t - 2X_x) = 0, \tag{3.5}$$

$$2D_u(m_{1x}u + m_{2x}) + K_u(m_1 u + m_2) + D(2m_{1x} - X_{xx}) \tag{3.6}$$

$$+K(T_t - X_x) + X_t = 0,$$

$$-m_{1t}u + m_{1xx}D_u + m_{1x}K_u - m_{2t} + m_{2xx}D + m_{2x}K = 0. \tag{3.7}$$

Equation(3.5) suggests the following forms of $D(u)$:

(1) $D(u)$ arbitrary;

(2) $D(u) = e^{\mu u}$;

(3) $D(u) = u^{\mu}$.

However in the subsequent analysis, these forms of D lead to further cases. Summarizing we have the following forms of D:

(1) $D(u)$ arbitrary;

(2) $D(u) = e^{\mu u}$;

(3) $D(u) = e^{u}$;

(4) $D(u) = u^{\mu}$;

(5) $D(u) = u^{-2}$;

(6) $D(u) = u^{-\frac{4}{3}}$;

(7) $D(u) = 1$.

Case 1. $D(u)$ arbitrary.

Using the fact that $D(u)$ is arbitrary from (3.5) we have $m_1(x,t) = m_2(x,t) = 0$. So, from equation (3.6) we have the following forms of $K(u)$:

Subcase 1.1: $K(u)$ arbitrary.

In this case from equations (3.6) and (3.7) we deduce that

$$X = c_1, \quad T = c_2, \quad U = 0.$$

The Lie algebra is two-dimensional spanned by

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}.$$

Subcase 1.2: $K(u) = 0$.

If $K(u) = 0$ then from equation (3.6) we get that

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3, \quad U = 0.$$

The Lie algebra is three-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

Case 2. $D(u) = e^{\mu u}$.

In this case from equation (3.5) we deduce that $m_1 = 0$ and $m_2 = \frac{1}{\mu}(2X_x - T_t)$. Then from equations (3.6) and (3.7) we get that $K(u) = e^u$ and after some calculations we have

$$X = c_1(\mu - 1)x + c_2, \quad T = c_1(\mu - 2)t + c_3, \quad U = c_1.$$

Therefore, the Lie algebra is three-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_4 = (\mu - 2)t\frac{\partial}{\partial t} + (\mu - 1)x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

Case 3. $D(u) = e^u$.

If $D(u) = e^u$ then from equation (3.5) we have $m_1 = 0$ and $m_2 = 2X_x - T_t$. Equations (3.6) and (3.7) deduce the following forms of $K(u)$:

Subcase 3.1: $K(u) = u$.

In this case from equations (3.6) and (3.7) we have

$$X = c_1(x - t) + c_2, \quad T = c_1 t + c_3, \quad U = c_1.$$

So, the Lie algebra is spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_5 = t\frac{\partial}{\partial t} + (x - t)\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

Subcase 3.2: $K(u) = 0$.

Equations (3.6) and (3.7) deduce that

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 t + c_4, \quad U = -c_3.$$

In this subcase the Lie algebra is four-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_6 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}.$$

Case 4. $D(u) = u^{\mu}$.

In this case from (3.5) we get $m_2 = 0$ and $m_1 = \frac{1}{\mu}(2X_x - T_t)$. From equations (3.6) and (3.7) we have the following different forms of K:

Subcase 4.1: $K(u) = u^{\nu},\ \nu \neq 0$.

In this case after some calculations we get

$$X = c_1(\mu - \nu)x + c_2, \quad T = c_1(\mu - 2\nu)t + c_3, \quad U = c_1 u.$$

So, the Lie algebra is three-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_7 = (\mu - 2\nu)t\frac{\partial}{\partial t} + (\mu - \nu)x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Subcase 4.2: $K(u) = 0$.

Here from equations (3.6) and (3.7) we deduce that

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 \mu t + c_4, \quad U = -c_3 u.$$

In this subcase the Lie algebra is

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_8 = \mu t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

Subcase 4.3: $K(u) = \ln u$.

In this subcase using equations (3.6) and (3.7) we have

$$X = c_1(\mu x - t) + c_2, \quad T = c_1 \mu t + c_3, \quad U = c_1 u.$$

So, the Lie algebra is three-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_9 = \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Case 5. $D(u) = u^{-2}$.

Here equation (3.5) gives $m_2 = 0$ and $m_1 = \frac{1}{2}(T_t - 2X_x)$. If we substitute these expressions into equations (3.6), (3.7) we deduce that $K(u) = u^{-2}$ and finally we have

$$X = c_1 e^{-x} + c_2, \quad T = 2c_3 t + c_4, \quad U = c_1 e^{-x} u + c_3 u.$$

Therefore, the Lie algebra is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_{10} = 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \quad \Gamma_{11} = e^{-x}\left(\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\right).$$

Case 6. $D(u) = -\frac{4}{3}$.

From equation (3.5) we get that $m_2 = 0$ and $m_1 = \frac{1}{4}(3T_t - 6X_x)$. Then, after some calculations using equations (3.6) and (3.7) we deduce that $K(u) = 0$ and

$$X = c_1 x + c_2 x^2 + c_3, \quad T = 4c_4 t + 2c_1 t + c_5, \quad U = 3c_4 u - 3c_2 x u.$$

Hence, the Lie algebra is five-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{12} = 4t\frac{\partial}{\partial t} + 3u\frac{\partial}{\partial u}, \quad \Gamma_{13} = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}.$$

Case 7. $D(u) = 1$.

From equation (3.5) we get that $X = \frac{x}{2}T_t + g_1(t)$ and then from equations (3.6) and (3.7) we deduce that $K(u) = u$. So, we have the Burger's equation with

$$X = c_1 t x + c_2 x + c_3 t + c_4, \quad T = c_1 t^2 + 2c_2 t + c_5, \quad U = -c_1(tu + x) - c_2 u - c_3.$$

Therefore, the Lie algebra is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_6, \quad \Gamma_{14} = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (tu + x)\frac{\partial}{\partial u}, \quad \Gamma_{15} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

In Table 3.1 we give briefly the Lie symmetries for the different forms of $D(u)$ and $K(u)$.

Table 3.1: Classification of equation $u_t = (D(u)u_x)_x + K(u)u_x$

N	$D(u)$	$K(u)$	Basis of $A^{\max}$
1	$\forall$	$\forall$	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x}$
2	$\forall$	0	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
3	$e^{\mu u}$	e^u	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ (\mu-2)t\frac{\partial}{\partial t} + (\mu-1)x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$
4	e^u	u	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ t\frac{\partial}{\partial t} + (x-t)\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$
5	e^u	0	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
6	u^μ	u^ν	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ (\mu-2\nu)t\frac{\partial}{\partial t} + (\mu-\nu)x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$
7a	u^μ	0	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ \mu t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$
7b	u^{-2}	u^{-2}	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, e^{-x}(\frac{\partial}{\partial x} + u\frac{\partial}{\partial u})$
8	$u^{-4/3}$	0	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ 4t\frac{\partial}{\partial t} + 3u\frac{\partial}{\partial u}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\ x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}$
9	u^μ	$\ln u$	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$
10	1	u	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - (tu+x)\frac{\partial}{\partial u},\ 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},\ t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}$

Here $\mu, \nu = \text{const.}$ $(\mu,\nu) \neq (-2,-2)$, $(0,1)$ and $\nu \neq 0$ for N = 6., $\mu \neq -4/3$ for N = 7a. Case 7b can be reduced to 7a ($\mu = -2$) by means of the conditional equivalence transformation $\tilde{t} = t$, $\tilde{x} = e^x$, $\tilde{u} = e^{-x}u$, (that do not belong to the equivalence group of class 3.1).

3.2 Classification of Potential Symmetries

We consider the nonlinear diffusion-convection equations of the type

$$u_t = (D(u)u_x)_x + K(u)u_x,$$

which we write in the following form

$$u_t = (D(u)u_x)_x - k_u u_x, \tag{3.8}$$

where $k(u) = -\int K(u)\mathrm{d}u$.

If we introduce the potential v, equation (3.8) can be written as a system of two PDE's

$$\begin{aligned} v_x &= u, \\ v_t &= D(u)u_x - k(u). \end{aligned} \tag{3.9}$$

We determine infinitesimal transformations of the form

$$
\begin{aligned}
x' &= x + \epsilon X(x,t,u,v) + O(\epsilon^2),\\
t' &= t + \epsilon T(x,t,u,v) + O(\epsilon^2),\\
u' &= u + \epsilon U(x,t,u,v) + O(\epsilon^2),\\
v' &= v + \epsilon V(x,t,u,v) + O(\epsilon^2),
\end{aligned} \tag{3.10}
$$

admitted by system (3.9).

These transformations induce potential and point symmetries for (3.8) and point symmetries for the integrated form of (3.8) $v_t = D(v_x)v_{xx} - k(v_x)$ where $u = v_x$.

As we have seen, a PDE of first order, admits Lie symmetries if and only if

$$\Gamma^{(1)}E|_{E=0} = 0.$$

So, the system (3.9) admits Lie transformations of the form (3.10) if and only if

$$
\begin{aligned}
&\Gamma^{(1)}[v_x - u] = 0,\\
&\Gamma^{(1)}[v_t - D(u)u_x + k(u)] = 0,
\end{aligned} \tag{3.11}
$$

where $v_x = u,\ v_t = D(u)u_x - k(u)$ and $\Gamma^{(1)}$ is the first extended generator of

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v},$$

which is given by the relation

$$
\begin{aligned}
\Gamma^{(1)} = \Gamma &+ [D_xU - (D_xX)u_x - (D_xT)u_t]\frac{\partial}{\partial u_x} + [D_tU - (D_tX)u_x - (D_tT)u_t]\frac{\partial}{\partial u_t}\\
&+[D_xV - (D_xX)v_x - (D_xT)v_t]\frac{\partial}{\partial v_x} + [D_tV - (D_tX)v_x - (D_tT)v_t]\frac{\partial}{\partial v_t},
\end{aligned}
$$

and D_x and D_t are the total derivatives with respect to x and t respectively.

Eliminating v_x and v_t from equation (3.9), equations (3.11) take the form

$$E_1(x,t,u,v,u_x,u_t) = 0, \quad E_2(x,t,u,v,u_x,u_t) = 0,$$

where E_1 and E_2 are determined polynomials in u_x and u_t.

Now if we take coefficients of $E_{1u_xu_x}$ and $E_{2u_t} - E_{1u_x}$ we get respectively:

$$
\begin{aligned}
&DT_u = 0,\\
&2D(uT_v + T_x) = 0,
\end{aligned}
$$

from which we get that T is a function of T only.

Calculation of $E_{2u_xu_x}$ and E_{1u_x} respectively gives,

$$X_u = V_u = 0.$$

Finally from E_1 and E_2 (the coefficient of u_x and the term independent of u_x) we get the following determining equations of $D(u),\ k(u)$ and the generators $X,\ T,\ U$ and V:

$$U = -X_v u^2 + (V_v - X_x)u + V_x, \tag{3.12}$$

$$[X_v u^2 + (X_x - V_v)u - V_x]\frac{\mathrm{d}D}{\mathrm{d}u} + [2X_v u + 2X_x - T_t]D = 0, \tag{3.13}$$

$$\begin{aligned}&\left[X_v u^2 + (X_x - V_v)u\right]\frac{\mathrm{d}k}{\mathrm{d}u} + \left[-X_v u + V_v - T_t\right]k = \\ &\left[X_{vv}u^3 + (2X_{xv} - V_{vv})u^2 + (X_{xx} - 2V_{xv})u - V_{xx}\right]D + V_t - X_t u.\end{aligned} \tag{3.14}$$

From equations (3.12), (3.13) and (3.14) we deduce the different forms of $D(u),\ k(u)$ and the generators $X,\ T,\ U$ and V. We only classify the Lie symmetries of (3.9) which are such that $X_v^2 + T_v^2 + U_v^2 \neq 0$ and induce potential symmetries of (3.8). So, from equation (3.13) we conclude that the function $D(u)$ satisfies an ODE of the form

$$(\lambda_1 u^2 + \lambda_2 u + \lambda_3)\frac{\mathrm{d}D}{\mathrm{d}u} + (2\lambda_1 u + \lambda_4)D = 0.$$

where λ_i are constants.

So, we get the following forms of $D(u)$:

(1) $D = \frac{p}{(u+q)^2}$, with $p^2 - 4q - r^2 = 0$;

(2) $D = \frac{1}{u^2+pu+q}\exp[r\int\frac{\mathrm{d}u}{u^2+pu+q}]$, with $p^2 - 4q - r^2 \neq 0$;

(3) D =constant;

where in all three cases $p = \frac{\lambda_2}{\lambda_1},\ q = \frac{\lambda_3}{\lambda_1}$ and $r = \frac{\lambda_4-\lambda_2}{\lambda_1}$.

Case 1. $D = \frac{p}{(u+q)^2}$, with $p^2 - 4q - r^2 = 0$.

From (3.14) we get the following forms of $k(u)$ which produce the following potential symmetries.

Subcase 1.1: $k(u) = \frac{r(u+q)^m}{(u+s)^{m-1}},\ (q \neq s)$.

Equation (3.8) admits the potential symmetry

$$\begin{aligned}\Gamma_1 &= 2m(q-s)t\frac{\partial}{\partial t} + ((mq - ms - s)x - v)\frac{\partial}{\partial x} + (u+q)(u+s)\frac{\partial}{\partial u} \\ &+ (qsx + (mq - ms + q)v)\frac{\partial}{\partial v}.\end{aligned}$$

Subcase 1.2: $k(u) = r(u+q)\exp(\frac{s}{(u+q)}),\ (s \neq 0)$.

Here equation (3.8) admits the potential symmetry

$$\Gamma_2 = 2st\frac{\partial}{\partial t} + ((s-q)x - v)\frac{\partial}{\partial x} + (u+q)^2\frac{\partial}{\partial u} + \left(q^2 + (q+s)v\right)\frac{\partial}{\partial v}.$$

Subcase 1.3: $k(u) = \frac{r}{u+q}$.

Equation (3.8) admits the following potential symmetries

$$\begin{aligned}
\Gamma_3 &= 4rt^2\frac{\partial}{\partial t} - \left[2pt + (v+qx)^2\right]\frac{\partial}{\partial x} + 2(u+q)\left[(u+q)(v+qx) + 2rt\right]\frac{\partial}{\partial u} \\
&+ \left[q(v+qx)^2 + 4rt(v+qx) + 2pqt\right]\frac{\partial}{\partial v}, \\
\Gamma_4 &= (v+qx)\frac{\partial}{\partial x} - (u+q)^2\frac{\partial}{\partial u} - \left[q(v+qx) + 2rt\right]\frac{\partial}{\partial v}, \\
\Gamma_{1\infty} &= e^{-\frac{rx}{p}}\left[p\phi\frac{\partial}{\partial x} - (u+q)\left(p(u+q)\phi_\xi - r\phi\right)\frac{\partial}{\partial u} - pq\phi\frac{\partial}{\partial v}\right],
\end{aligned}$$

where in $\Gamma_{1\infty},\ y = \phi(t,\xi),\ \xi = v + qx$ is an arbitrary solution of the linear heat equation

$$p\frac{\partial^2 y}{\partial \xi^2} - \frac{\partial y}{\partial t} = 0. \tag{3.15}$$

Subcase 1.4: $k(u) = r(u+q)$.

Here we have the following potential symmetries

$$\begin{aligned}
\Gamma_5 &= (v+qrt)\frac{\partial}{\partial x} - u(u+q)\frac{\partial}{\partial u} - q(v+qrt)\frac{\partial}{\partial v}, \\
\Gamma_6 &= 12pqt^2\frac{\partial}{\partial t} + \left[(v+qx)^3 + 3q(rt-x)(v+qx)^2 + 6(ptv + 3pqrt^2)\right]\frac{\partial}{\partial x} \\
&+ 3(u+q)\left[-u(v+qx)^2 + 2q(u+q)(qx^2 - rtv - qrtx + xv) + 4pqt - 2ptu\right]\frac{\partial}{\partial u} \\
&+ q\left[-(v+qx)^3 + 3q(x-rt)(v+qx)^2 + 6(ptv + 2pqxt - 3pqrt^2)\right]\frac{\partial}{\partial v}, \\
\Gamma_7 &= \left[(v+qx)(v-qx) + 2(qrtv + pt + q^2rtx)\right]\frac{\partial}{\partial x} + 2(u+q)\left[-uv + q^2x - qrt(u+q)\right]\frac{\partial}{\partial u} \\
&+ q\left[-(v+qx)(v-qx) + 2(pt - qrtv - q^2rtx)\right]\frac{\partial}{\partial v}, \\
\Gamma_{2\infty} &= \phi\frac{\partial}{\partial x} - (u+q)^2\phi_\xi\frac{\partial}{\partial u} - q\phi\frac{\partial}{\partial v},
\end{aligned}$$

where $y = \phi(t,\xi)$ satisfies (3.15).

Case 2. $D = \frac{1}{u^2+pu+q}\exp\left[r\int\frac{du}{u^2+pu+q}\right]$, with $p^2 - 4q - r^2 \neq 0$.

From equation (3.13) and (3.14) we deduce that X and V are linear in x and v. In this case we obtain the following results:

Subcase 2.1: $k(u) = \sqrt{u^2+pu+q}\exp\left[s\int\frac{du}{u^2+pu+q}\right]$.

Equation (3.8) admits the potential symmetry

$$\Gamma_8 = (r+2s)t\frac{\partial}{\partial t} + \left[(r+s-\frac{p}{2})x - v\right]\frac{\partial}{\partial x} + (u^2+pu+q)\frac{\partial}{\partial u} + \left[qx + (r+s+\frac{p}{2})v\right]\frac{\partial}{\partial v}.$$

Subcase 2.2: $k(u) = \frac{1}{I(u)}\int \frac{(\lambda_1 u+\lambda_2)I(u)\mathrm{d}u}{u^2+pu+q}$.

Here the function $I(u)$ is given by

$$I(u) = \frac{1}{\sqrt{u^2+pu+q}}\exp\left(s\int\frac{\mathrm{d}u}{u^2+pu+q}\right),$$

where $s = \frac{\lambda_4-\lambda_2-\lambda_5}{2\lambda_1}$. For this case we have

$$\begin{aligned}\Gamma_9 &= (r+2s)t\frac{\partial}{\partial t} + \left[(r+s-\frac{p}{2})x + \lambda_1 t - v\right]\frac{\partial}{\partial_x} + (u^2+pu+q)\frac{\partial}{\partial u} \\ &+ \left[qx - \lambda_2 t + (r+s+\frac{p}{2})v\right]\frac{\partial}{\partial v}.\end{aligned}$$

Subcase 2.3: $k(u) = \lambda(u+q)$.

Equation (3.8) admits the following potential symmetries

$$\begin{aligned}\Gamma_{10} &= (p^2-4q-r^2)t\frac{\partial}{\partial t} - \left[(p+r)v + 2qx + \lambda(r^2+pq+qr+2q-p^2)t\right]\frac{\partial}{\partial x} \\ &+ \left[(p^2+pr-2q)v + q(p+r)x + \lambda q(pr-p+2q+r^2-r)t\right]\frac{\partial}{\partial v} \\ &+ (p+r)(u^2+pu+q)\frac{\partial}{\partial u}, \\ \Gamma_{11} &= \left[2v + (p-r)x + \lambda(2q-p+r)t\right]\frac{\partial}{\partial x} - 2(u^2+pu+q)\frac{\partial}{\partial u} \\ &- \left[(p+r)v + 2qx + \lambda q(p+r-2)t\right]\frac{\partial}{\partial v}.\end{aligned}$$

Case 3. D =constant= p.

From equation (3.13) we get that $X = \frac{1}{2}T_t + \theta(t)$ and from (3.14) we deduce that function $k(u)$ satisfies an ODE of the form:

$$(\lambda_1 u + \lambda_2)\frac{\mathrm{d}k}{\mathrm{d}u} + \lambda_3 k = \lambda_4 u^2 + \lambda_5 u + \lambda_6.$$

Equation (3.8) admits a potential symmetry only when $k = r(u+s)^2$. Any other form of $k(u)$ which satisfies the above ODE leads to point symmetries. If $k = r(u+s)^2$ then equation (3.8) becomes the well known Burger's equation which admits the potential symmetry

$$\Gamma_{3\infty} = e^{\frac{rv}{p}}(ph_x + rhu)\frac{\partial}{\partial u} + pe^{\frac{rv}{p}}h\frac{\partial}{\partial v},$$

where the function $h(x,t)$ satisfies the linear PDE

$$h_t - ph_{xx} + 2rsh_x - \frac{r^2s^2}{p}h = 0.$$

As one can see, the obtained results are very cumbersome. However, one can easily simplify them using equivalence transformations. It is known, that equivalence group of a PDE can be (trivially) prolonged to equivalence group of any potential system of the equation. Here we adduce classification of potential system (3.9) up to such (trivial) prolongation [61].

Table 3.2: Results of group classification for systems (3.9) with respect to $G^{\sim}_{triv.pot}$-equivalence

N	$D(u)$	$k(u)$	Basis of $A^{\max}$
1	$u^{-2}e^{\mu/u}$	$ue^{1/u}$	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial v}, (\mu-2)t\frac{\partial}{\partial t} + ((\mu-1)x+v)\frac{\partial}{\partial x} - u^2\frac{\partial}{\partial u} + (\mu-1)v\frac{\partial}{\partial v}$
2	$u^{-2}e^{1/u}$	u^{-1}	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial v}, t\frac{\partial}{\partial t} + (x+v)\frac{\partial}{\partial x} - u^2\frac{\partial}{\partial u} + (v-2t)\frac{\partial}{\partial v}$
3	$u^{-2}e^{1/u}$	0	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial v}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}, t\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}$
4	$\dfrac{u^{\mu}}{(u+1)^{\mu+2}}$	$\dfrac{u^{\nu+1}}{(u+1)^{\nu}}$	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial v}, (\mu-2\nu)t\frac{\partial}{\partial t} + ((\mu-\nu)x-v)\frac{\partial}{\partial x} + u(u+1)\frac{\partial}{\partial u}+$ $+(\mu-\nu+1)v\frac{\partial}{\partial v}$
5	$\dfrac{u^{\mu}}{(u+1)^{\mu+2}}$	$u\ln\dfrac{u}{u+1}$	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial v}, \mu t\frac{\partial}{\partial t} + (\mu x+v-t)\frac{\partial}{\partial x} + u(u+1)\frac{\partial}{\partial u} + (\mu+1)v\frac{\partial}{\partial v}$
6	$\dfrac{u^{\mu}}{(u+1)^{\mu+2}}$	0	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial v}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}, \mu t\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - u(u+1)\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$
7	$\dfrac{e^{\mu\arctan u}}{u^2+1}$	$\sqrt{u^2+1}\,e^{\nu\arctan u}$	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial v}, (\mu-2\nu)t\frac{\partial}{\partial t} + (u^2+1)\frac{\partial}{\partial u}+$ $+(x+(\mu-\nu)v)\frac{\partial}{\partial v}$
8	$\dfrac{e^{\mu\arctan u}}{u^2+1}$	0	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial v}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}, \mu t\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - (u^2+1)\frac{\partial}{\partial u} - x\frac{\partial}{\partial v}$
9	u^{-2}	0	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial v},\ 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ -vx\frac{\partial}{\partial x} + u(ux+v)\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial v},$ $4t^2\frac{\partial}{\partial t} - (v^2+2t)x\frac{\partial}{\partial x} + u(v^2+6t+2xuv)\frac{\partial}{\partial u} + 4tv\frac{\partial}{\partial v},$ $x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},\ \phi\frac{\partial}{\partial x} - \phi_v u^2\frac{\partial}{\partial u}$
10	u^{-2}	u^{-1}	$\frac{\partial}{\partial t},\ \frac{\partial}{\partial v},\ 2t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},\ -v\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u} + 2t\frac{\partial}{\partial v},$ $4t^2\frac{\partial}{\partial t} - (v^2+2t)\frac{\partial}{\partial x} + 2u(uv+2t)\frac{\partial}{\partial u} + 4tv\frac{\partial}{\partial v},$ $\frac{\partial}{\partial x},\ e^{-x}\phi\frac{\partial}{\partial x} + e^{-x}(\phi-u\phi_v)u\frac{\partial}{\partial u}$
11	1	$-u^2$	$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, 2t\frac{\partial}{\partial x} - \frac{\partial}{\partial u} - x\frac{\partial}{\partial v},$ $4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - 2(x+2ut)\frac{\partial}{\partial u} - (x^2+2t)\frac{\partial}{\partial v},$ $\frac{\partial}{\partial v},\ e^{-v}(h_x-hu)\frac{\partial}{\partial u} + e^{-v}h\frac{\partial}{\partial v}$

In the above consideration we have seen that the auxiliary system of the nonlinear equation

$$u_t = [u^{-2}u_x]_x + u^{-2}u_x,$$

admits an infinite dimensional Lie symmetry which is equivalent to Lie symmetry algebra of potential system corresponding to the linear heat equation. This observation shows that this nonlinear equation can be mapped into the linear heat equation $u_t = u_{xx}$ by contact transformations [61, 68].

Chapter 4

Group analysis of (2+1) nonlinear diffusion-convection equations

We consider the (2+1) nonlinear diffusion-convection equations

$$u_t = (D(u)u_x)_x + (F(u)u_y)_y + K(u)u_x, \tag{4.1}$$

where $D(u)$, $F(u)$ and $K(u)$ are arbitrary smooth functions. These equations generalize the well-known Richard's equation and arise naturally in certain physical applications. Thus, for example, superdiffusivities of this type have been proposed [20] as a model for long-range Van der Waals interactions in thin films spreading on solid surfaces. Equations of such form may describe the flow of particles in a lattice fluid past an impenetrable obstacle [4, 5]. This equation also appears in the study of cellular automata and interacting particle systems with self-organized criticality (see [17] and references therein) and describes a model of water flow in unsaturated soil [63]. For other applications of many special cases of these classes of equations we refer the reader to [18, 48, 49, 57].

Our goal in this chapter is to carry out a group classification of (4.1) [21,22]. In the case when $K(u) = 0$, the corresponding results can be found in [34]. This classification generalizes and completes existing results in the literature. Using the derived Lie symmetries we also construct similarity reductions and exact solutions of certain equations.

4.1 Classification of Lie Symmetries

The equivalence group of (4.1) consists of the 9-parameter group of point transformations

$$\tilde{t} = \varepsilon_5 t + \varepsilon_1, \quad \tilde{x} = \varepsilon_6 x + \varepsilon_9 t + \varepsilon_2, \quad \tilde{y} = \varepsilon_7 y + \varepsilon_3, \quad \tilde{u} = \varepsilon_8 u + \varepsilon_4,$$
$$\tilde{D} = \varepsilon_5^{-1}\varepsilon_6^2 D, \quad \tilde{F} = \varepsilon_5^{-1}\varepsilon_7^2 F, \quad \tilde{K} = \varepsilon_5^{-1}\varepsilon_6 K - \varepsilon_9,$$

where ε_i are arbitrary constants, $\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8 \neq 0$. This means that scalings and translations of x, t, y and u may be used to simplify the analysis with the understanding that these equivalence transformations are included in the conclusions. In particular, $D(u)(\neq 0)$ may be scaled and also u can be translated in order to simplify the form of $D(u)$ without any loss of generality. For example, if $D(u)$ is a non-zero constant it may be assumed that $D(u) = 1$.

We classify the Lie symmetries of equation (4.1) that are not equivalent. For example, we exclude the case K =constant, since it can be mapped into the case with $K = 0$ using a special form of the equivalence transformations.

Equation (4.1) admits Lie transformations of the form

$$\begin{aligned} x' &= x + \epsilon X(x,t,y,u) + O(\epsilon^2), \\ t' &= t + \epsilon T(x,t,y,u) + O(\epsilon^2), \\ y' &= y + \epsilon Y(x,t,y,u) + O(\epsilon^2), \\ u' &= u + \epsilon U(x,t,y,u) + O(\epsilon^2), \end{aligned} \tag{4.2}$$

if and only if

$$\Gamma^{(2)} E|_{E=0} = 0, \tag{4.3}$$

where $\Gamma^{(2)}$ is the second extended generator of

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + Y\frac{\partial}{\partial y} + U\frac{\partial}{\partial u},$$

which is given by the relation

$$\begin{aligned}\Gamma^{(2)} &= \Gamma + [D_x(U) - (D_xX)u_x - (D_xT)u_t - (D_xY)u_y]\frac{\partial}{\partial u_x}\\ &+ [D_t(U) - (D_tX)u_x - (D_tT)u_t - (D_tY)u_y]\frac{\partial}{\partial u_t}\\ &+ [D_y(U) - (D_yX)u_x - (D_yT)u_t - (D_yY)u_y]\frac{\partial}{\partial u_y}\\ &+ [D_x(U^x) - u_{xx}D_x(X) - u_{xt}D_x(T) - u_{xy}D_x(Y)]\frac{\partial}{\partial u_{xx}}\\ &+ [D_y(U^y) - u_{yx}D_y(X) - u_{yt}D_y(T) - u_{yy}D_y(Y)]\frac{\partial}{\partial u_{yy}}.\end{aligned}$$

Here $D_x,\ D_t$ and D_y are the total derivatives with respect to $x,\ t$ and y, respectively and $U^x,\ U^y$ the extended transformations. We point out that in $\Gamma^{(2)}$ we did not include the terms that give no contribution.

In this case $E = u_t - D_u u_x^2 - Du_{xx} - F_u u_y^2 - Fu_{yy} - Ku_x$. So, equation (4.1) admits Lie symmetries if and only if

$$\Gamma^{(2)}[u_t - D_u u_x^2 - Du_{xx} - F_u u_y^2 - Fu_{yy} - Ku_x] = 0, \tag{4.4}$$

where $u_t = D_u u_x^2 + Du_{xx} + F_u u_y^2 + Fu_{yy} + Ku_x$.

If we take coefficients of u_{xt} we have

$$D(T_u u_x + T_x) = 0, \tag{4.5}$$

from we get two cases: (A) $D \neq 0$ and (B) $D = 0$.

Case A. $D \neq 0$.

In this case from (4.5) we have $T_u = T_x = 0$ and also from coefficient of u_{yt} in (4.4) $T_y = 0$. So, function T is only function of t. If we take then coefficient of u_{xy} in (4.4) we have

$$\begin{aligned}&X_u = Y_u = 0,\\ &X_yD + Y_xF = 0.\end{aligned} \tag{4.6}$$

From (4.6) we deduce two cases: (I) $F \neq \epsilon D$ and (II) $F = \epsilon D$.

Case A(I): $F \neq \epsilon D$.

If $F \neq \epsilon D$ then from (4.6) we get that $X_y = Y_x = 0$, hence $X = X(x,t)$ and $Y = y(t,y)$. Also from coefficients of u_{yy} and u_y^2 we get that $U_{uu} = 0$, so $U = a_1(x,t,y)u + a_2(x,t,y)$.

Using the fact that $T = T(t)$, $X = X(x,t)$, $Y = Y(t,y)$ and the form of U, then from equation (4.4) we obtain the following determining equations of the functional forms of D, F, K, X, T, Y and U:

$$(a_1 u + a_2)D_u + (T_t - 2X_x)D = 0, \tag{4.7}$$

$$(a_1 u + a_2)F_u + (T_t - 2Y_y) = 0, \tag{4.8}$$

$$(2a_{1x}u + 2a_{2x})D_u + (2a_{1x} - X_{xx})D + (a_1 u + a_2)K_u + \tag{4.9}$$

$$(T_t - X_x)K + X_t = 0, \tag{4.10}$$

$$(2a_{1y}u + 2a_{2y})F_u + (2a_{1y} - Y_{yy})F + Y_t = 0, \tag{4.11}$$

$$(a_{1xx}u + a_{2xx})D + (a_{1yy}u + a_{2yy})F + (a_{1x}u + a_{2x})K - a_{1t}u - a_{2t} = 0. \tag{4.12}$$

From equation (4.7) we conclude that function $D(u)$ satisfies an ordinary differential equation (ODE) of the form

$$(\lambda_1 u + \lambda_2)D_u + \lambda_3 D = 0, \tag{4.13}$$

where λ_i are constants. Equation (4.13) suggests the following forms of $D(u)$:

(i) $D(u)$ arbitrary;

(ii) $D(u) = e^{\mu u}$;

(iii) $D(u) = u^{\mu}$.

However in the following analysis, these forms of D lead to further special cases. Summarizing we have the following forms of $D(u)$:

1. $D(u)$ arbitrary;
2. $D(u) = e^{\mu u}$;
3. $D(u) = u^{\mu}$;
4. $D(u) = u^{-2}$;
5. $D(u) = 1$.

Case 1. $D(u)$ arbitrary.

In the case when $D(u)$ is arbitrary (4.7) we deduce that $a_1 = a_2 = 0$ and $X = \frac{x}{2}T_t + g_1(t)$. Substituting the above expressions into (4.8) and (4.11) we have $Y = \frac{y}{2}T_t + g_2(t)$, $T = c_1 t + c_2$, $g_2(t) = c_3$. Equation (4.10) then, suggests the following forms of K:

(i) $K(u)$ arbitrary;

(ii) $K(u) = 0$.

Subcase 1.1: $K(u)$ arbitrary.

Here, from equation (4.10) we get

$$X = c_1, \quad T = c_2, \quad Y = c_3, \quad U = 0.$$

So, the Lie algebra is three-dimensional and is spanned by

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}.$$

Subcase 1.2: $K(u) = 0$.

From equation (4.10) we have

$$X = c_4 x + c_1, \quad T = 2c_4 t + c_2, \quad Y = c_4 y + c_3, \quad U = 0.$$

Hence, the Lie algebra is four dimensional and is given by

$$\Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \Gamma_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Case 2. $D(u) = e^{\mu u}$.

If $D(u) = e^{\mu u}$ then from equation (4.7) we get that $a_1 = 0$ and $a_2 = \frac{1}{mu}(2X_x - T_t)$. Also from (4.8) we deduce that $F(u) = \epsilon e^{\nu u}$ and substituting the expression of F into equations (4.8), (4.10) and after some calculations we get

$$Y = yg_1(t) + g_2(t), \quad X = xg_3(t) + g_4(t), \quad g_3(t) = \frac{1}{2}T_t + c_1, \quad g_1 = \frac{1}{2\mu}(T_t\mu + 2c_1\nu).$$

Substituting the above expressions into (4.11) we have

$$T = c_2 t + c_3, \quad g_2 = c_4.$$

So, equation (4.10) can be written in the form

$$\mu_1 K_u + \mu_2 K = \mu_3,$$

where μ_i are constants, that gives the following forms of $K(u)$:

(i) $K(u) = e^{pu}$;

(ii) $K(u) = u$;

(iii) $K(u) = 0$.

Subcase 2.1: $K(u) = e^{pu}$.

In this subcase using equation (4.10) we have

$$X = 2(p-\mu)c_1 x + c_5, \quad T = 2(2p-\mu)c_1 t + c_3, \quad Y = (2p-\mu-\nu)c_1 y + c_4, \quad U = 2c_1.$$

So, the Lie algebra is spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_5 = 2(2p-\mu)t\frac{\partial}{\partial t} + 2(p-\mu)x\frac{\partial}{\partial x} + (2p-\mu-\nu)y\frac{\partial}{\partial y} - 2\frac{\partial}{\partial u}.$$

Subcase 2.2: $K(u) = u$.

Here from equation (4.10) we get

$$X = 2c_1\mu x - 2c_1 t + c_2, \quad T = 2c_1\mu t + c_3, \quad Y = (\mu+\nu)c_1 y + c_4, \quad U = 2c_1.$$

The Lie algebra is four dimensional and is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_6 = 2\mu t\frac{\partial}{\partial t} + 2(\mu x - t)\frac{\partial}{\partial x} + (\mu+\nu)\frac{\partial}{\partial y} + 2\frac{\partial}{\partial u}.$$

Subcase 2.3: $K(u) = 0$.

Using the fact that $K(u) = 0$ then from (4.10) we have

$$X = c_1(\nu-\mu)x + c_2\mu x + c_3, \quad T = 2c_1\nu t + c_4, \quad Y = c_2\nu y + c_5, \quad U = -2c_1 + 2c_2.$$

So, the Lie algebra is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_7 = 2\mu t\frac{\partial}{\partial t} + x(\nu-\mu)\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}, \quad \Gamma_8 = \mu x\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial u}.$$

Case 3. $D(u) = u^{\mu}$

In this case from equation (4.7) we have $a_2 = 0$ and $a_1 = \frac{1}{\mu}(2X_x - T_t)$ and from (4.8) we deduce that function $F(u)$ has the form $F(u) = \epsilon u^{\nu}$. So, from (4.8) we get that

$$Y = yg_1(t) + g_2(t), \quad X = xg_3(t) + g_4(t), \quad g_1 = \frac{1}{2\mu}\left(2g_3\nu + T_t(\mu - nu)\right).$$

Substituting the above expressions into (4.11) and (4.12) we get

$$g_2 = c_1, \quad g_3 = \frac{1}{2}T_t + c_2, \quad T = c_3 t + c_4.$$

We can then suppose that equation (4.10) can be written in the form

$$\mu_1 k_u + \mu_2 k = \mu_3,$$

from which we get the following forms of $K(u)$:

(i) $K(u) = u^p,\ p \neq 0$;

(ii) $K(u) = \log u$;

(iii) $K = 0$.

Subcase 3.1: $K(u) = u^p$.

From (4.10) we have

$$X = 2c_2(\mu - p)x + c_3,\quad T = 2c_2(\mu - 2p)t + c_4,\quad Y = c_2(\mu + \nu - 2p)y + c_1,\quad U = 2c_2u.$$

So the Lie algebra is four-dimensional spanned by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_9 = 2(2p - \mu)t\frac{\partial}{\partial t} + 2(p - \mu)x\frac{\partial}{\partial x} + (2p - \mu - \nu)y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}.$$

Subcase 3.2: $K(u) = \log u$.

In this subcase from equations (4.10) and (4.11) we have

$$X = 2c_2\mu x - 2c_2t + c_1,\quad T = 2c_2\mu t + c_3,\quad Y = c_2y(\mu + \nu) + c_4,\quad U = 2c_2u.$$

Hence, the Lie algebra is given by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_{10} = 2\mu t\frac{\partial}{\partial t} + 2(\mu x - t)\frac{\partial}{\partial x} + (\mu + \nu)y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}.$$

Subcase 3.3: $K(u) = 0$.

From equations (4.10) and (4.11) we deduce that

$$X = c_2\mu x + c_3x + c_1,\quad T = 2c_3t + c_4,\quad Y = c_2\nu y + c_3y + c_5,\quad U = 2c_2u.$$

So, the Lie algebra is five-dimensional given by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_{11} = \mu x\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}.$$

Case 4. $D(u) = u^{-2}$.

From equation (4.7) we have $a_2 = 0$ and $a_1 = \frac{1}{2}(T_t - 2X_x)$. Also from (4.8) we deduce that $F(u) = \epsilon$ and $Y = \frac{y}{2}T_t + g_1(t)$. Using these expressions, from (4.11) we get $T = c_1t + c_2$ and $g_1(t) = c_3$. Finally from (4.12) we deduce that $K(u) = u^{-2}$. So,

$$X = c_4 + c_5e^{-x},\quad T = 2c_1t + c_2,\quad Y = c_1y + c_3,\quad U = (c_1 + c_5e^{-x})u.$$

Hence, the Lie algebra is spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{12} = 2t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}, \quad \Gamma_{13} = e^{-x}\frac{\partial}{\partial x} + e^{-x}u\frac{\partial}{\partial u}.$$

Case 5. $D(u) = 1$.

In this case using the fact that $D(u) = 1$ from (4.7) we get $X = \frac{x}{2}T_t + g_1(t)$ and from (4.8) we deduce that $F(u) = u^{-\frac{4}{3}}$. So, $a_2 = 0$ and $a_1 = \frac{1}{4}(3T_t - 6Y_x)$. Substituting the above expressions into (4.10)-(4.12) we have $K(u) = 0$ and

$$X = 2c_1x + c_6, \quad T = 4c_1t + c_2, \quad Y = c_4y^2 - 2c_5y + c_6, \quad U = 3(c_1 + c_5 + c_4y)u.$$

The Lie algebra is six-dimensional spanned by

$$\Gamma_1, \ \Gamma_2, \ \Gamma_3, \ \Gamma_{14} = 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3u\frac{\partial}{\partial u}, \ \Gamma_{15} = -2y\frac{\partial}{\partial y} - 3u\frac{\partial}{\partial u}, \ \Gamma_{16} = -y^2\frac{\partial}{\partial y} - 3yu\frac{\partial}{\partial u}.$$

Case A(II): $F = \epsilon D$.

In the case when $F = \epsilon D$, from the coefficients u_{xx}, u_x^2 of (4.4) we deduce that $U_{uu} = 0$, so $U = a_1(x,t,y)u + a_2(x,t,y)$. Substituting the expression of $D(u)$ into (4.4) we obtain the following determining equations of the functional forms of D, K, X, T, Y and U:

$$(a_1u + a_2)D_u + (T_t - 2X_x)D = 0, \tag{4.14}$$

$$(a_1u + a_2)D_u + (T_t - 2Y_y)D = 0, \tag{4.15}$$

$$\epsilon X_y + Y_x = 0, \tag{4.16}$$

$$(2a_{1x}u + 2a_{2x})D_u + (2a_{1x} - X_{xx} - \epsilon X_{yy})D + (a_1u + a_2)K_u + (T_t - X_x)K + X_t = 0, \tag{4.17}$$

$$(2\epsilon a_{1y}u + 2\epsilon a_{2y})D_u + (2\epsilon a_{1y} - Y_{xx} - \epsilon Y_{yy})D - Y_xK + Y_t = 0, \tag{4.18}$$

$$((a_{1xx} + \epsilon a_{1yy})\,u + a_{2xx} + \epsilon a_{2yy})D + (a_{1x}u + a_{2x})K - a_{1t}u - a_{2t} = 0. \tag{4.19}$$

From equation (4.14), we conclude that function $D(u)$ satisfies an ODE of the form

$$(\mu_1u + \mu_2)D_u + \mu_3D = 0,$$

where μ_i are constants. The solution of the above ODE gives us the following different forms of function $D(u)$:

(1) $D(u)$ arbitrary;

(2) $D(u) = e^u$;

(3) $D(u) = u^\mu$;

(4) $D(u) = 1$.

Case 1. $D(u)$ arbitrary.

Solving equations (4.14)-(4.19) we deduce that $K = 0$ and

$$X = c_1 x + c_3 y + c_5, \quad T = 2c_1 t + c_2, \quad Y = c_1 y - c_3 \epsilon x + c_4, \quad U = 0.$$

Hence, the Lie algebra is five-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{17} = y\frac{\partial}{\partial x} - \epsilon x\frac{\partial}{\partial y}.$$

Case 2. $D(u) = e^u$.

From (4.14)-(4.19) we get that $K = 0$ and

$$X = c_3 x + c_4 y + c_5, \quad T = c_1 t + c_2, \quad Y = c_3 y - c_4 \epsilon x + c_6, \quad U = -c_1 + 2c_3.$$

So, the Lie algebra is six-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{17}, \quad \Gamma_{18} = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}, \quad \Gamma_{19} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial u}.$$

Case 3. $D(u) = u^\mu$.

In this case from equations (4.14)-(4.19) we deduce that $K = 0$ and

$$X = c_1 x + c_2 \mu x + c_3 y + c_4, \quad T = 2c_1 t + c_5, \quad Y = c_1 y + c_2 \mu y - c_3 \epsilon x + c_6, \quad U = 2c_2 u.$$

Hence, the Lie algebra is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{17}, \quad \Gamma_{18} = \mu x\frac{\partial}{\partial x} + \mu y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}.$$

Case 4. $D(u) = 1$.

Here after some calculations using equations (4.14)-(4.19) we deduce that $K = u$ and

$$X = c_1 x + c_4 t + c_5, \quad T = 2c_1 t + c_2, \quad Y = c_1 y + c_3, \quad U = -c_4 - c_1 u.$$

Therefore, the Lie algebra is five-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{19} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \quad \Gamma_{20} = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

Case B. $D(u) = 0$.

In the case when $D(u) = 0$ then from coefficients of u_{yt}, $u_{yy}u_y$, and u_{xy} in equation (4.4) we get respectively

$$T_u = T_y = 0,$$
$$Y_u = 0,$$
$$X_u = X_y = 0.$$

Using the fact that $T = T(x,t)$, $Y = Y(x,t,y)$ and $X = X(x,t)$ then from (4.4) we get the following determined equations for the functional forms of F, K, X, T, Y and U:

$$UF_u + (T_t - T_xK - 2Y_y)F = 0, \tag{4.20}$$
$$UF_{uu} + [T_t - T_xK + U_u - 2Y_y]F_u + U_{uu}F = 0, \tag{4.21}$$
$$UK_u + (T_t - KT_x - X_x)K + X_t = 0, \tag{4.22}$$
$$U_{yy}F + U_xK - U_t = 0, \tag{4.23}$$
$$2U_yF_u + (2U_{uy} - Y_{yy})F - Y_xK + Y_t = 0. \tag{4.24}$$

Using the above equations we get the following forms of $F(u)$

(1) $F(u)$ arbitrary;

(2) $F(u) = e^u$;

(3) $F(u) = u^{\nu}$;

(4) $F(u) = u^{-\frac{1}{2}}$;

(5) $F(u) = \frac{e^{\int \frac{\mu \mathrm{d}u}{(\phi + \frac{p}{2} - r)u}}}{(\phi + \frac{p}{2} - r)u}$;

(6) $F(u) = 1$.

Case 1. $F(u)$ arbitrary.

In this case K is arbitrary and

$$X = 2c_1x + c_4, \quad T = 2c_1t + c_2, \quad Y = c_1y + c_3, \quad U = 0.$$

So, the Lie algebra is four-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4.$$

Case 2. $F(u) = e^u$.

Here $K(u)$ has the following forms:

(i) $K(u) = u$;

(ii) $K(u) = e^{\mu u}$.

Subcase 2.1: $K(u) = u$.

In this subcase we have

$$X = 2c_1 x - 2c_2 t + c_3, \quad T = 2c_1 t + c_4, \quad Y = c_1 y + c_2 y + c_5, \quad U = 2c_2.$$

The Lie algebra is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{21} = -2t\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial u}.$$

Subcase 2.2: $K(u) = e^{\mu u}$.

Here after some calculations we deduce that

$$X = 2c_1 x + 2c_2 \mu x + c_3, \quad T = 2c_1 t + c_4, \quad Y = c_1 y + c_2 y + c_5, \quad U = 2c_2.$$

The Lie algebra is spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{22} = 2\mu x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial u}.$$

Case 3. $F(u) = u^\nu$.

In this case K has the following forms:

(i) $K(u) = u^p$;

(ii) $K(u) = u^{-1}$.

Subcase 3.1: $K(u) = u^p$.

Using the fact that $K(u) = u^p$ we have

$$X = 2c_1 x + c_2 x(p - \nu) + c_3, \quad T = 2c_1 t - c_2 \nu t + c_4, \quad Y = c_1 y + c_5, \quad U = c_2 u.$$

So, the Lie algebra is

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{23} = -\nu t\frac{\partial}{\partial t} + x(p - \nu)\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Subcase 3.2: $K(u) = u^{-1}$.

In this subcase

$$X = 2c_1 x - 2c_2 x + c_3, \quad T = 2c_1 t + c_4, \quad Y = c_1 y + c_2 \nu y + c_5, \quad U = 2c_2 u.$$

Therefore, the Lie algebra is five-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{24} = -2x\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u}.$$

Case 4. $F(u) = u^{-\frac{1}{2}}$.

In this case after some calculations we deduce that $K(u) = u^{-\frac{1}{2}}$ and

$$X = 2c_1x + 2c_2x + c_3, \quad T = 2c_1t + c_4x + c_5, \quad Y = c_1y + c_2y + c_6, \quad U = -4c_2u - 2c_4\sqrt{u}.$$

So, the Lie algebra is six-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{25} = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 4u\frac{\partial}{\partial u}, \quad \Gamma_{26} = x\frac{\partial}{\partial t} - 2\sqrt{u}\frac{\partial}{\partial u}.$$

Case 5. $F(u) = \dfrac{e^{\int \frac{\mu \mathrm{du}}{(\phi + \frac{p}{2} - r)u}}}{(\phi + \frac{p}{2} - r)u}$.

In this case we deduce that $K(u) = [u\phi(u)]'$ where $u = (\phi^2 + p\phi + q)^{-1/2} e^{\int \frac{r \mathrm{du}}{\phi^2 + p\phi + q}}$ and

$$X = 2c_1x + \left[\left(\frac{p}{2} + r + \mu\right)x - qt\right]c_2 + c_3, \quad T = 2c_1t + \left[x + \left(r + \mu - \frac{p}{2}\right)t\right]c_2 + c_4,$$

$$Y = c_1y + c_5, \quad U = c_2\left(r - \frac{p}{2} - \phi\right)u.$$

Hence, the Lie algebra is five-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4,$$

$$\Gamma_{27} = \left[x + \left(r + \mu - \frac{p}{2}\right)t\right]\frac{\partial}{\partial t} + \left[\left(\frac{p}{2} + r + \mu\right)x - qt\right]\frac{\partial}{\partial x} + \left(r - \frac{p}{2} - \phi\right)u\frac{\partial}{\partial u}.$$

Case 6. $F(u) = 1$.

Using the fact that $F(u) = 1$ we deduce that $K(u)$ has the following forms:

(i) $K(u) = e^u$;

(ii) $K(u) = u$.

Subcase 6.1: $K(u) = e^u$.

In this subcase after some calculations we get

$$X = 2c_1x + c_2x + c_3, \quad T = 2c_1t + c_4, \quad Y = c_1y + c_5, \quad U = c_2.$$

So, the Lie algebra is

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{28} = x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

Subcase 6.2: $K(u) = u$.

In this subcase we have

$$X = 2c_1x + c_2x + c_3t + c_4, \quad T = 2c_1t + c_5, \quad Y = c_1y + c_6, \quad U = c_2u - c_3.$$

Hence the Lie algebra is six-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{20}, \quad \Gamma_{28} = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

In Table 4.1 we summarize the Lie symmetries for the different forms of $D(u)$, $F(u)$ and $K(u)$.

Table 4.1: Group classification of $u_t = (D(u)u_x) + \left(F(u)u_y\right)_y + K(u)u_x$

N	$D(u)$	$F(u)$	$K(u)$	$A^{\max}$
1	$\forall$	$\forall$	$\forall$	$A^{\ker} = \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$
2	$\forall$	$\forall$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \rangle$
3	$\forall$	εD	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} - \varepsilon x\frac{\partial}{\partial y} \rangle$
4	0	$\forall$	$\forall$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \rangle$
5	0	e^u	$e^{\mu u}$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, 2\mu x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial u} \rangle$
6	0	e^u	u	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, -2t\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2\frac{\partial}{\partial u} \rangle$
7	0	u^{ν}	u^p	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x(p-\nu)\frac{\partial}{\partial x} - \nu t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \rangle$
8	0	u^n	u^{-1}	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, -2x\frac{\partial}{\partial x} + \nu\frac{\partial}{\partial y} + 2u\frac{\partial}{\partial u} \rangle$
9	0	$u^{-\frac{1}{2}}$	$u^{-\frac{1}{2}}$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - 4u\frac{\partial}{\partial u},$
				$x\frac{\partial}{\partial t} - 2\sqrt{u}\frac{\partial}{\partial u} \rangle$
10	0	$\frac{e^{\int \frac{\mu \mathrm{d}u}{(\phi+\frac{p}{2}-r)u}}}{(\phi+\frac{p}{2}-r)u}$	$[u\phi(u)]'$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, [(\frac{p}{2} + r + \mu)x - qt)\frac{\partial}{\partial x} +$
				$+[x + (r + \mu - \frac{p}{2})t]\frac{\partial}{\partial t} + (r - \frac{p}{2} - \phi)u\frac{\partial}{\partial u} \rangle$
11	0	1	e^u	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$
12	0	1	u	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, t\frac{\partial}{\partial x} - \frac{\partial}{\partial u} \rangle$

13	$e^{\mu u}$	$\varepsilon e^{\nu u}$	e^{pu}	$A^{\mathrm{ker}}+\langle 2t(2p-\mu)\frac{\partial}{\partial t}+2x(p-\mu)\frac{\partial}{\partial x}+$ $+y(2p-\mu-\nu)y\frac{\partial}{\partial y}-2\frac{\partial}{\partial u}\rangle$
14	$e^{\mu u}$	$\varepsilon e^{\nu u}$	u	$A^{\mathrm{ker}}+\langle 2\mu t\frac{\partial}{\partial t}+2(\mu x-t)\frac{\partial}{\partial x}+(\mu+\nu)y\frac{\partial}{\partial y}+2\frac{\partial}{\partial u}\rangle$
15	$e^{\mu u}$	$\varepsilon e^{\nu u}$	0	$A^{\mathrm{ker}}+\langle 2\nu t\frac{\partial}{\partial t}+x(\nu-\mu)\frac{\partial}{\partial x}-2\frac{\partial}{\partial u},\mu x\frac{\partial}{\partial x}+\nu y\frac{\partial}{\partial y}+2\frac{\partial}{\partial u}\rangle$
16	e^{u}	εe^{u}	0	$A^{\mathrm{ker}}+\langle t\frac{\partial}{\partial t}-\frac{\partial}{\partial u},x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+2\frac{\partial}{\partial u},y\frac{\partial}{\partial x}-\varepsilon x\frac{\partial}{\partial y}\rangle$
17	u^{μ}	εu^{ν}	u^{p}	$A^{\mathrm{ker}}+\langle 2t(2p-\mu)\frac{\partial}{\partial t}+2x(p-\mu)\frac{\partial}{\partial x}+$ $+y(2p-\nu-\mu)\frac{\partial}{\partial y}-2u\frac{\partial}{\partial u}\rangle$
18	u^{μ}	εu^{ν}	$\log u$	$A^{\mathrm{ker}}+\langle 2\mu t\frac{\partial}{\partial t}+2(\mu x-t)\frac{\partial}{\partial x}+(\nu+\mu)y\frac{\partial}{\partial y}+2u\frac{\partial}{\partial u}\rangle$
19	u^{μ}	εu^{ν}	0	$A^{\mathrm{ker}}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\mu x\frac{\partial}{\partial x}+\nu y\frac{\partial}{\partial y}+2u\frac{\partial}{\partial u}\rangle$
20	u^{μ}	εu^{μ}	0	$A^{\mathrm{ker}}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\mu x\frac{\partial}{\partial x}+\mu y\frac{\partial}{\partial y}+2u\frac{\partial}{\partial u},$ $y\frac{\partial}{\partial x}-\varepsilon x\frac{\partial}{\partial y}\rangle$
21	u^{-2}	ε	u^{-2}	$A^{\mathrm{ker}}+\langle 2t\frac{\partial}{\partial t}+y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u},e^{-x}\frac{\partial}{\partial x}+ue^{-x}\frac{\partial}{\partial u}\rangle$
22	1	$u^{-\frac{4}{3}}$	0	$A^{\mathrm{ker}}+\langle 4t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}+3u\frac{\partial}{\partial u},$ $-2y\frac{\partial}{\partial y}+3u\frac{\partial}{\partial u},-y^2\frac{\partial}{\partial y}+3yu\frac{\partial}{\partial u}\rangle$
23	1	ε	u	$A^{\mathrm{ker}}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-u\frac{\partial}{\partial u},t\frac{\partial}{\partial x}-\frac{\partial}{\partial u}\rangle$

Here $\varepsilon=\pm1$, $p\neq0$, in case 10 $u=(\phi^2+p\phi+q)^{-1/2}e^{\int\frac{r\mathrm{d}u}{\phi^2+p\phi+q}}$.

4.2 Similarity reductions of $u_t = u_{yy} - 2uu_x$

We consider the two-dimensional diffusion-advection equation of the form

$$u_t = Du_{yy} - v_0 \frac{\mathrm{d}}{\mathrm{d}u}[u(1-u)]u_x. \tag{4.25}$$

This equation describes the flow of particles in a lattice fluid past an impenetrable obstacle [4,5], where $u(x, y, t)$ is the particle concentration and D and v_0 are constants representing the diffusion coefficient and the drift velocity, respectively.

Using the transformation

$$x' = v_0(t - x), \quad y' = \sqrt{D}y, \quad t' = t, \quad u' = u,$$

equation (4.25) can be mapped into the following equation

$$u_t = u_{yy} - 2uu_x. \tag{4.26}$$

This transformation is a special case of the equivalence transformations of

$$u_t = Du_{yy} - K(u)u_x,$$

given by

$$x' = c_1x + c_2t + c_3, \quad y' = \epsilon c_4\sqrt{\frac{D'}{D}}y + c_5, \quad t' = c_4^2t + c_6, \quad u' = c_7u + c_8.$$

In this section we construct all possible similarity solutions of (4.26). Similarity solutions are obtained by solving the invariant surface condition

$$Xu_x + Yu_y + Tu_t = U.$$

These are transformations that reduce the number of independent variables by one. Hence, in the present work similarity solutions will transform (4.26) into a PDE with two independent variables using its Lie symmetries

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \quad \Gamma_5 = 2t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

In order to achieve this we need to construct the optimal system following the method of Ovsiannikov [53,55]. First we make the commutator table for the Lie algebra Γ_i and then using the Lie series we construct a table showing the separate adjoint actions for each element Γ_i acting on all the rest of elements. Finally, this table enables us to derive the optimal system that provides all possible similarity solutions. So, in the next tables we present the commutator table for the Lie algebra Γ_i, the adjoint table for the Lie algebra and the table for the infinitesimal generator Δ_i of the optimal system, the similarity variables and the similarity solutions.

Table 4.2: Commutator table for the Lie algebra $\{\Gamma_i\}$ of $u_t = u_{yy} - 2uu_x$

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	0	$2\Gamma_1$	0	$2\Gamma_2$
Γ_2	0	0	0	$2\Gamma_2$	Γ_2	0
Γ_3	0	0	0	Γ_3	0	0
Γ_4	$-2\Gamma_1$	$-2\Gamma_2$	$-\Gamma_3$	0	0	0
Γ_5	0	$-\Gamma_2$	0	0	0	$-\Gamma_6$
Γ_6	$-2\Gamma_2$	0	0	0	Γ_6	0

Table 4.3: Adjoint table for the Lie algebra $\{\Gamma_i\}$ of $u_t = u_{yy} - 2uu_x$

Ad	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	Γ_1	Γ_2	Γ_3	$\Gamma_4 - 2\epsilon\Gamma_1$	Γ_5	$\Gamma_6 - 2\epsilon\Gamma_2$
Γ_2	Γ_1	Γ_2	Γ_3	$\Gamma_4 - 2\epsilon\Gamma_2$	$\Gamma_5 - \epsilon\Gamma_2$	Γ_6
Γ_3	Γ_1	Γ_2	Γ_3	$\Gamma_4 - \epsilon\Gamma_3$	Γ_5	Γ_6
Γ_4	$e^{2\epsilon}\Gamma_1$	$e^{2\epsilon}\Gamma_2$	$e^{\epsilon}\Gamma_3$	Γ_4	Γ_5	Γ_6
Γ_5	Γ_1	$e^{\epsilon}\Gamma_2$	Γ_3	Γ_4	Γ_5	$e^{\epsilon}\Gamma_6$
Γ_6	$\Gamma_1 + 2\epsilon\Gamma_2$	Γ_2	Γ_3	Γ_4	$\Gamma_5 - \epsilon\Gamma_6$	Γ_6

Table 4.4: Infinitesimal generators $\langle\Delta_i\rangle$ of the optimal system, similarity variables, similarity solutions

	$\langle\Delta_i\rangle$	η	ξ	similarity solution
1	$\langle\Gamma_4 + a\Gamma_5\rangle$	$yt^{-\frac{1}{2}}$	$xt^{-\frac{2+a}{2}}$	$u = t^{\frac{a}{2}}\phi(\eta,\xi)$
2	$\langle\Gamma_4 + \delta\Gamma_6\rangle$	$yt^{-\frac{1}{2}}$	$\frac{x}{t} - \delta\ln t$	$u = -\frac{\delta}{2}\ln t + \phi(\eta,\xi)$
3	$\langle\Gamma_5 + \delta\Gamma_3 + a\Gamma_1\rangle$	$t - \delta ay$	$x\mathrm{e}^{-\delta y}$	$u = x\phi(\eta,\xi)$
4	$\langle\Gamma_5 + \varepsilon\Gamma_1\rangle$	y	$t - \varepsilon\log x$	$u = x\phi(\eta,\xi)$
5	$\langle\Gamma_1 + a\Gamma_6 + \varepsilon\Gamma_3\rangle$	$y - \varepsilon t$	$x - at^2$	$u = -at + \phi(\eta,\xi)$
6	$\langle\Gamma_6 + \varepsilon\Gamma_3\rangle$	$y - \frac{\varepsilon x}{2t}$	t	$u = -\frac{x}{2t} + \phi(\eta,\xi)$
7	$\langle\Gamma_2 + \varepsilon\Gamma_3\rangle$	$y - \varepsilon x$	t	$u = \phi(\eta,\xi)$
8	$\langle\Gamma_3\rangle$	x	t	$u = \phi(\eta,\xi)$

$\varepsilon = -1, 0, 1$ and $\delta = \pm 1$.

We use the optimal system in Table 4.4 to derive the reduced PDE and then we classify the Lie symmetries for this PDE. We employ the Lie symmetries of the reduced PDE to derive similarity transformations that map the reduced PDEs into ODEs. Solutions of these ODEs lead to similarity solutions of (4.26). Next we present some examples and we ignore the trivial case that corresponds to the generator $\langle\Gamma_3\rangle$. The solutions of those that can be solved analytically are presented in the next section.

Case 1. The similarity solution that corresponds to the generator $\Gamma_4 + a\Gamma_5$ reduces (4.26) to

$$2\phi_{\eta\eta} - 4\phi\phi_\xi + \eta\phi_\eta + (2+a)\xi\phi_\xi - a\phi = 0. \tag{4.27}$$

Equation (4.27) admits the Lie symmetry

$$X_1 = \xi\frac{\partial}{\partial\xi} + \phi\frac{\partial}{\partial\phi},$$

which produces the similarity solution

$$\phi = \xi\psi(z), \quad z = \eta,$$

that reduces (4.27) to

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \frac{1}{2}z\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2\psi^2 + \psi = 0. \tag{4.28}$$

A second symmetry exists if $a=0$ or $a=-2$. If $a=0$ equation (4.27) admits the Lie symmetries X_1 and

$$X_2 = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \phi}.$$

The generator $X_2 + \mu X_1,\ \mu \neq 0$ produces the similarity solution

$$\phi = \frac{1}{\mu}\left(2+\mu\xi\right)\psi(z) - \frac{1}{\mu}, \quad z = \eta,$$

that maps (4.27) into ODE (4.28). The generator X_2 leads to the similarity solution

$$\phi = \frac{1}{2}\xi + \psi(z), \quad z = \eta,$$

that reduces (4.27) into

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \frac{1}{2}z\frac{\mathrm{d}\psi}{\mathrm{d}z} - \psi = 0. \tag{4.29}$$

If $a=-2$ equation (4.27) admits the Lie symmetries X_1 and

$$X_2 = \frac{\partial}{\partial \xi}.$$

The generator $X_2 + \mu X_1$ leads to the similarity solution

$$\phi = \left(1+\mu\xi\right)\psi(z), \quad z = \eta,$$

that reduces (4.27) to

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \frac{1}{2}z\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2\mu\psi^2 + \psi = 0. \tag{4.30}$$

Case 2. The similarity solution that corresponds to the generator $\Gamma_4 + \delta\Gamma_6$ reduces (4.26) to

$$2\phi_{\eta\eta} + (2\xi + 2\delta - 4\phi)\phi_\xi + \eta\phi_\eta = \delta, \tag{4.31}$$

which admits the Lie symmetry

$$X_1 = 2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \phi}.$$

The above Lie symmetry leads to the similarity solution

$$\phi = \frac{1}{2}\xi + \psi(\eta),$$

that transforms (4.31) into the ordinary differential equation (4.29). It turns out that the similarity solution obtained here is the same as the one produced by the generator $X_2 + \mu X_1$ in the case 1, where $a = 0$.

Case 3. The similarity solution that corresponds to the generator $\Gamma_5 + \delta\Gamma_3 + a\Gamma_1$ reduces (4.26) to

$$a^2\phi_{\eta\eta} + 2a\xi\phi_{\eta\xi} + \xi^2\phi_{\xi\xi} + (1-2\phi)\xi\phi_\xi - \phi_\eta - 2\phi^2 = 0,$$

and using the change of variables

$$\xi' = \xi - a\log\eta, \quad \eta' = \xi + a\log\eta,$$

where $a \neq 0$, this parabolic PDE simplifies to

$$4a^2\phi_{\eta'\eta'} + (a-1)(\phi_{\eta'} + \phi_{\xi'}) - 2a\phi\phi_{\eta'} - 2\phi^2 = 0. \tag{4.32}$$

Equation (4.32) admits two Lie symmetries

$$X_1 = \frac{\partial}{\partial\eta'}, \quad X_2 = \frac{\partial}{\partial\xi'},$$

which are simply translations in the independent variables. The optimal system is $\langle X_1 + \mu X_2,\ X_2\rangle$. The generator $X_1 + \mu X_2$ produces the similarity solution

$$\phi(\eta', \xi') = \psi(z), \quad z = \xi' - \mu\eta',$$

that reduces (4.32) to the ODE

$$4a^2\mu^2\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + (a-1)(1-\mu)\frac{\mathrm{d}\psi}{\mathrm{d}z} + 2a\mu\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2\psi^2 = 0. \tag{4.33}$$

The generator X_2 produces the similarity solution

$$\phi(\eta', \xi') = \psi(z), \quad z = \eta',$$

that reduces (4.32) to the ODE

$$4a^2\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + (a-1)\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2a\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2\psi^2 = 0. \tag{4.34}$$

In the case where $a = 0$, we have the following PDE

$$\xi^2\phi_{\xi\xi} + (1-2\phi)\xi\phi_\xi - \phi_\eta - 2\phi^2 = 0, \tag{4.35}$$

which admits the Lie symmetries

$$X_1 = \frac{\partial}{\partial \eta}, \quad X_2 = \xi \frac{\partial}{\partial \xi}.$$

The generator $X_2 + \mu X_1$ leads to the similarity solution

$$\phi(\eta, \xi) = \psi(z), \quad z = \eta - \mu \log \xi,$$

that reduces (4.35) to

$$\mu^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + (\mu - 1)\frac{\mathrm{d}\psi}{\mathrm{d}z} - \mu\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2\psi^2 = 0, \tag{4.36}$$

and from the generator X_1 we obtain

$$\phi(\eta, \xi) = \psi(z), \quad z = \xi,$$

which reduces (4.35) to

$$z^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + z\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2z\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2\psi^2 = 0,$$

and using the transformation $z = \mathrm{e}^w$, this ODE becomes a constant coefficient equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}w^2} - 2\psi\frac{\mathrm{d}\psi}{\mathrm{d}w} - 2\psi^2 = 0. \tag{4.37}$$

Case 4. The similarity solution that corresponds to the generator $\Gamma_5 + \epsilon\Gamma_1$ reduces (4.26) to

$$\phi_{\eta\eta} + 2\varepsilon\phi\phi_\xi - \phi_\xi - 2\phi^2 = 0. \tag{4.38}$$

If $\varepsilon \neq 0$, equation (4.38) admits two Lie symmetries

$$X_1 = \frac{\partial}{\partial \eta}, \quad X_2 = \frac{\partial}{\partial \xi},$$

with the optimal system: $\langle X_1 + \mu X_2,\ X_2 \rangle$. The generator $X_1 + \mu X_2$ produces the similarity solution

$$\phi(\eta, \xi) = \psi(z), \quad z = \xi - \mu\eta,$$

that reduces (4.38) to the ODE

$$\mu^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} - \frac{\mathrm{d}\psi}{\mathrm{d}z} + 2\varepsilon\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2\psi^2 = 0. \tag{4.39}$$

The generator X_2 produces the similarity solution

$$\phi(\eta,\xi)=\psi(z),\quad z=\eta,$$

that reduces (4.38) to the ODE

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}-2\psi^2=0. \tag{4.40}$$

If $\varepsilon=0$, then the symmetry Lie algebra is three-dimensional and is spanned by the generators

$$X_1=\frac{\partial}{\partial\eta},\quad X_2=\frac{\partial}{\partial\xi},\quad X_3=\eta\frac{\partial}{\partial\eta}+2\xi\frac{\partial}{\partial\xi}-2\phi\frac{\partial}{\partial\phi}.$$

Here the optimal system is: $\langle X_3,\ X_2{+}\alpha X_1,\ X_1\rangle$. The generator X_3 produces the similarity solution

$$\phi(\eta,\xi)=\frac{1}{\xi}\psi(z),\quad z=\eta\xi^{-\frac{1}{2}},$$

that reduces (4.38) to the ODE (4.28). The generator $X_2+\alpha X_1$ produces the similarity solution

$$\phi(\eta,\xi)=\psi(z),\quad z=\eta-\alpha\xi,$$

that reduces (4.38) to the ODE

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}+\alpha\frac{\mathrm{d}\psi}{\mathrm{d}z}-2\psi^2=0. \tag{4.41}$$

The generator X_1 produces the similarity solution

$$\phi(\eta,\xi)=\psi(z),\quad z=\xi,$$

that reduces (4.38) to the ODE

$$\frac{\mathrm{d}\psi}{\mathrm{d}z}+2\psi^2=0.$$

Solving this ODE leads to the similarity solution $u(x,y,t)=\frac{x}{2t+c}$ of (4.26). This solution can be obtained, using the similarity transformation that corresponds to the generator Γ_3 (Table 4.4, entry 8).

Case 5. The similarity solution that corresponds to the generator $\Gamma_1 + a\Gamma_6 + \varepsilon\Gamma_3$ reduces (4.26) to

$$\phi_{\eta\eta} + \varepsilon\phi_\eta - 2\phi\phi_\xi = a. \tag{4.42}$$

If $\varepsilon \neq 0$, that is $\varepsilon = \delta = \pm 1$, equation (4.42) admits two Lie symmetries

$$X_1 = \frac{\partial}{\partial \eta}, \quad X_2 = \frac{\partial}{\partial \xi}.$$

The optimal system is: $\langle X_1 + \mu X_2,\ X_2\rangle$. The generator $X_1 + \mu X_2$ produces the similarity solution

$$\phi(\eta, \xi) = \psi(z), \quad z = \xi - \mu\eta,$$

that reduces (4.42) to the ODE

$$\mu^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} - \delta\mu\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} = a. \tag{4.43}$$

The generator X_2 produces the similarity solution

$$\phi(\eta, \xi) = \psi(z), \quad z = \eta,$$

that reduces (4.42) to the ODE

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + \delta\frac{\mathrm{d}\psi}{\mathrm{d}z} = a,$$

which produces the similarity solution $u(x, y, t) = A\mathrm{e}^{-\delta(y-\delta t)} + \delta a y + B$ of (4.26).
If $\varepsilon = 0$, then the symmetry Lie algebra is three-dimensional and is spanned by

$$X_1 = \frac{\partial}{\partial \eta}, \quad X_2 = \frac{\partial}{\partial \xi}, \quad X_3 = \eta\frac{\partial}{\partial \eta} + 4\xi\frac{\partial}{\partial \xi} + 2\phi\frac{\partial}{\partial \phi}.$$

The optimal system for these Lie symmetries is: $\langle X_3,\ X_2 + \alpha X_1,\ X_1\rangle$. The generator X_3 produces the similarity solution

$$\phi(\eta, \xi) = \eta^2\psi(z), \quad z = \frac{\eta^4}{\xi},$$

that reduces (4.42) to the ODE

$$8z^2\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + 14z\frac{\mathrm{d}\psi}{\mathrm{d}z} + z^2\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} + \psi = \frac{a}{2}. \tag{4.44}$$

The generator $X_2 + \alpha X_1$ produces the similarity solution

$$\phi(\eta, \xi) = \psi(z), \quad z = \eta - \alpha\xi,$$

that reduces (4.42) to the ODE

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + 2\alpha\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} = a. \tag{4.45}$$

The generator X_1 produces the similarity solution

$$\phi(\eta, \xi) = \psi(z), \quad z = \xi,$$

that reduces (4.42) to the ODE

$$2\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} = a,$$

which leads to the similarity solution $u(x, y, t) = \sqrt{a(x - at)^2 + c} + at$ of (4.26).

Case 6. The similarity transformation that corresponds to the generator $\Gamma_6 + \varepsilon\Gamma_3$ transforms (4.26) to

$$\phi_{\eta\eta} + \frac{\varepsilon}{\xi}\phi\phi_\eta - \phi_\xi - \frac{1}{\xi}\phi = 0. \tag{4.46}$$

If $\varepsilon \neq 0$, then equation (4.46) admits three Lie symmetries

$$X_1 = \frac{\partial}{\partial\eta}, \quad X_2 = \frac{\varepsilon}{\xi}\frac{\partial}{\partial\eta} + \frac{1}{\xi}\frac{\partial}{\partial\phi}, \quad X_3 = \eta\frac{\partial}{\partial\eta} + 2\xi\frac{\partial}{\partial\xi} + \phi\frac{\partial}{\partial\phi},$$

with optimal system: $\langle X_3,\ X_1 + \alpha X_2,\ X_2\rangle$. The generator X_3 produces the similarity solution

$$\phi(\eta, \xi) = \eta\psi(z), \quad z = \eta\xi^{-\frac{1}{2}},$$

that reduces (4.46) to the ODE

$$z\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + (2 + \frac{1}{2}z^2)\frac{\mathrm{d}\psi}{\mathrm{d}z} + \epsilon z^2\psi\frac{\mathrm{d}\psi}{\mathrm{d}z} + \epsilon z\psi^2 - z\psi = 0. \tag{4.47}$$

The generator $X_1 + \alpha X_2$ produces the similarity solution

$$\phi(\eta, \xi) = \frac{\xi}{\xi + \epsilon\alpha}\left(\frac{\alpha}{\xi}\eta + \psi(z)\right), \quad z = \xi,$$

that reduces (4.46) to the ODE

$$z\frac{\mathrm{d}\psi}{\mathrm{d}z} + \psi = 0. \tag{4.48}$$

The generator X_2 produces the similarity solution

$$\phi(\eta,\xi) = \frac{1}{\epsilon}\eta + \psi(z), \quad z = \xi,$$

that reduces (4.46) to the ODE

$$\frac{\mathrm{d}\psi}{\mathrm{d}z} = 0,$$

which gives $u(x,y,t) = \frac{1}{2}y + c$.

If $\varepsilon = 0$, then equation (4.46) becomes a linear PDE that admits an infinite dimensional Lie group

$$X_1 = \frac{\partial}{\partial\eta}, \quad X_2 = \frac{\partial}{\partial\xi} - \frac{\phi}{\xi}\frac{\partial}{\partial\phi}, \quad X_3 = 2\xi\frac{\partial}{\partial\eta} - \phi\eta\frac{\partial}{\partial\phi}, \quad X_4 = 2\xi\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\eta},$$
$$X_5 = 4\xi\eta\frac{\partial}{\partial\eta} + 4\xi^2\frac{\partial}{\partial\xi} - (\eta^2 + 6\xi)\phi\frac{\partial}{\partial\phi}, \quad X_6 = \phi\frac{\partial}{\partial\phi}, \quad X_h = h(\eta,\xi)\frac{\partial}{\partial\phi},$$

where $h(\xi,\eta)$ satisfies the linear equation $h_{\eta\eta} - h_\xi - h/\xi = 0$. We point out that if $\varepsilon = 0$, then the transformation

$$\phi = \frac{\psi}{\xi},$$

maps (4.46) into the linear heat equation

$$\psi_\xi - \psi_{\eta\eta} = 0.$$

Hence, we conclude that

$$u(x,y,t) = \frac{1}{2}\frac{x}{t} + \frac{1}{t}\psi(y,t),$$

is a solution of (4.26), where $\psi(y,t)$ is a solution of the linear diffusion equation

$$\psi_t - \psi_{yy} = 0.$$

Case 7. Finally, the generator $\Gamma_2 + \varepsilon\Gamma_3$ produces a similarity solution that maps (4.26) into the well-known Burgers equation

$$\phi_{\eta\eta} + 2\varepsilon\phi\phi_\eta - \phi_\xi = 0, \tag{4.49}$$

which among other applications is of considerable interest in nonlinear acoustics [19]. In the next section we use known solutions of Burgers equation to derive similarity solutions for equation (4.26). In fact, the similarity solutions of (4.26) presented in [27] can be obtained from known solutions of Burgers equation which can be found, for example, in [33, 59].

4.3 Exact solutions for $u_t = u_{yy} - 2uu_x$

In this section we derive exact similarity solutions for equation (4.26) by considering certain ODEs which have been obtained in the previous section. Solution of this ODE yields exact solutions of (4.26).

Multiplying equation (4.40) by $\frac{\mathrm{d}\psi}{\mathrm{d}z}$ and integrating twice, we obtain its solution in an implicit form. This in turn, leads to the steady state similarity solution of (4.26),

$$y = \pm \int \left(\frac{4}{3}\chi^3 + c_1 \right)^{-\frac{1}{2}} \mathrm{d}\chi + c_2, \quad \chi = \frac{u}{x}.$$

The solution of the ODE (4.48) produces the similarity solution

$$u(x, y, t) = \frac{x + 2\alpha y + 2\alpha c}{2(t + \alpha\varepsilon)}.$$

Now we consider the ODE (4.43). If $a \neq 0$, we obtain the similarity solution

$$u(x, y, t) = at - \frac{\mu^2 W'(z)}{W(z)}, \qquad z = x - \mu y + \varepsilon\mu t - at^2,$$

where

$$W(z) = \mathrm{e}^{\frac{\delta z}{2\mu}} \sqrt{\chi} \left[c_1 J_{1/3} \left(\frac{2}{3\mu} \sqrt{\frac{a}{\mu}} \chi^{3/2} \right) + c_2 Y_{1/3} \left(\frac{2}{3\mu} \sqrt{\frac{a}{\mu}} \chi^{3/2} \right) \right], \quad \chi = z + \frac{\mu(4c - 1)}{4a}$$

where $J_{1/3}$ and $Y_{1/3}$ are Bessel functions of first and second kind, respectively. In the case where $a = 0$, from the solution of the ODE (4.43) we have the following similarity solutions of (4.26), depending on the form of the first constant of integration

$$u(x, y, t) = \frac{\mu^2}{B - x + \mu y - \varepsilon\mu t} - \frac{1}{2}\delta\mu,$$
$$u(x, y, t) = A \tan \left(\frac{A}{\mu^2}(x - \mu y + \varepsilon\mu t) + B \right),$$
$$u(x, y, t) = A \tanh \left(\frac{A}{\mu^2}(x - \mu y + \varepsilon\mu t) + B \right).$$

We can obtain similar results to the above, using ODE (4.45). However the solutions are of the steady state form.

The general solution of (4.29) is expressed in terms of the degenerate hypergeometric functions [2]. We derive the similarity solution

$$u(x,y,t) = \frac{1}{2}\frac{x}{t} + c_1\Phi\left(-1, \frac{1}{2}; -\frac{1}{4}\frac{y^2}{t}\right) + c_2\Psi\left(-1, \frac{1}{2}; -\frac{1}{4}\frac{y^2}{t}\right),$$

where $\Phi(a, b; \chi)$ and $\Psi(a, b; \chi)$ are the degenerate hypergeometric functions

$$\Phi(a, b; \chi) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k}\frac{\chi^k}{k!}, \quad \Psi(a, b; \chi) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\Phi(a, b; \chi) + \frac{\Gamma(b-1)}{\Gamma(a)}\chi^{1-b}\Phi(a-b+1, 2-b; \chi),$$

where $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$ and $\Gamma(a) = \int_0^\infty \mathrm{e}^{-t}t^{a-1}\mathrm{d}t$ is the gamma function.

Now we turn into case 7 of the previous section. We use known solutions of Burgers equation [59] and the corresponding similarity transformation, to derive similarity solutions for equation (4.26). We obtain the following list of solutions:

$$u(x,y,t) = \frac{2\varepsilon(y-\varepsilon x) + A\varepsilon}{(y-\varepsilon x)^2 + 2A(y-\varepsilon x) + 2t + B},$$

$$u(x,y,t) = \frac{3\varepsilon[(y-\varepsilon x)^2 + 2t + A]}{(y-\varepsilon x)^3 + 6(y-\varepsilon x)t + 3A(y-\varepsilon x) + B},$$

$$u(x,y,t) = \frac{\varepsilon\lambda}{1 + A\exp(-\lambda^2 t - \lambda(y-\varepsilon x))},$$

$$u(x,y,t) = \frac{\varepsilon\lambda}{2(\lambda^2 t + A)}\left[2\tanh\left(\frac{\lambda(y-\varepsilon x) + B}{\lambda^2 t + A}\right) - \lambda(y-\varepsilon x) - B\right],$$

$$u(x,y,t) = \frac{\varepsilon\lambda\cos[\lambda(y-\varepsilon x) + A]}{B\exp(\lambda^2 t) + \sin[\lambda(y-\varepsilon x) + A]},$$

$$u(x,y,t) = \frac{\varepsilon A}{\sqrt{\pi(t+\lambda)}}\exp\left[-\frac{(y-\varepsilon x + B)^2}{4(t+\lambda)}\right]\left[A\mathrm{erf}\left(\frac{y-\varepsilon x + B}{2\sqrt{t+\lambda}}\right) + C\right]^{-1},$$

where in the last solution, $\mathrm{erf}z = \frac{2}{\pi}\int_0^z \exp(-\xi^2)\mathrm{d}\xi$ is the error function. Furthermore, if we consider the Cauchy problem with initial condition of the form $u(x, y, 0) = f(\eta)$, $\eta = y - \varepsilon x$, then using the corresponding solution of Burgers equation [31], we find the solution of (4.26)

$$u(x,y,t) = \varepsilon\frac{\partial}{\partial\eta}\ln F(\eta, t),$$

where

$$F(\eta, t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\exp\left[-\frac{(\eta-\chi)^2}{4t} - \frac{1}{2}\int_0^{\chi} f(\chi')\mathrm{d}\chi'\right]\mathrm{d}\chi.$$

Note 4.1. *The transformation*

$$\theta(\chi) = \frac{\mathrm{d}\psi}{\mathrm{d}w}, \quad \chi = \psi^2,$$

maps ODE (4.37) into an Abel equation of the second kind

$$\theta\frac{\mathrm{d}\theta}{\mathrm{d}\chi} - \theta = \sqrt{\chi}.$$

Similar transformations exist for equations (4.33), (4.34), (4.36), (4.39) and (4.41) which map them into an Abel equation of the second kind.

Note 4.2. *Some of the above similarity solutions do no depend on the time t. That is, these solutions satisfy the PDE*

$$u_x = \frac{u_{yy}}{2u},$$

which can be transformed into a nonlinear diffusion type equation

$$v_x = \left[v^{-\frac{1}{2}}v_y\right]_y,$$

by the mapping $u \mapsto \frac{1}{2}\sqrt{v}$. This diffusion type equation admits four Lie symmetries [33,56] which can be employed to derive further steady state similarity solutions for equation (4.26). If $u = \theta(x,y)$ is a steady state solution of (4.26), then using note 4.1 we deduce that $u = \theta(x + 2at, y) - a$ is also a solution of (4.26).

Note 4.3. *Any solution of the linear heat equation $u_t = u_{yy}$ yields a solution of the equation (4.26).*

4.4 Similarity reductions of (2+1)-dimensional Burgers equation

Consider the 2-dimensional Burgers equation (Table 4.1 equation 23, $\varepsilon = 1$)

$$u_t = u_{xx} + u_{yy} + uu_x. \tag{4.50}$$

We derive similarity reductions using one- and two-dimensional subalgebras of its maximal Lie invariance algebra. One-dimensional subalgebras enable us to derive similarity reductions that reduce (4.50) into a partial differential equations in two independent variables. These equations are also studied from Lie's point of view. That is, we classify their Lie symmetries. The symmetries lead to reductions that transform the reduced partial differential equations into ordinary differential equations. Two-dimensional subalgebras enable us to derive similarity reductions that reduce (4.50) directly into an ordinary differential equation. Both approaches are examined in the following two subsections. The first approach was also employed in [25, 62].

4.4.1 One-dimensional subalgebras

Equation (4.50) admits the Lie symmetry algebra spanned by the following generators

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \quad \Gamma_5 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

We classify the similarity reductions that reduce (4.50) into a partial differential equation with two independent variables. In Tables 4.5 and 4.6 we give commutation relations and adjoint representations for the Lie symmetries of equation (4.50). Using these, then we derive the optimal system of one-dimensional subalgebras and the Lie symmetries of the reduced equations.

Table 4.5: Commutator table for the Lie algebra $\{\Gamma_i\}$ of $u_t = u_{xx} + u_{yy} + uu_x$

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	0	0	0	$2\Gamma_1$	Γ_2
Γ_2	0	0	0	Γ_2	0
Γ_3	0	0	0	Γ_3	0
Γ_4	$-2\Gamma_1$	$-\Gamma_2$	$-\Gamma_3$	0	Γ_5
Γ_5	$-2\Gamma_2$	0	0	$-\Gamma_5$	0

Table 4.6: Adjoint table for the Lie algebra $\{\Gamma_i\}$ of $u_t = u_{xx} + u_{yy} + uu_x$

Ad	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	Γ_1	Γ_2	Γ_3	$\Gamma_4 - 2\epsilon\Gamma_1$	$\Gamma_5 - 2\epsilon\Gamma_2$
Γ_2	Γ_1	Γ_2	Γ_3	$\Gamma_4 - 2\epsilon\Gamma_2$	Γ_5
Γ_3	Γ_1	Γ_2	Γ_3	$\Gamma_4 - \epsilon\Gamma_3$	Γ_5
Γ_4	$e^{2\epsilon}\Gamma_1$	$e^{\epsilon}\Gamma_2$	$e^{\epsilon}\Gamma_3$	Γ_4	$e^{-\epsilon}\Gamma_5$
Γ_5	$e^{2\epsilon}\Gamma_1$	Γ_2	Γ_3	$\Gamma_4 + \epsilon\Gamma_5$	Γ_5

Table 4.7: Subalgebras $\langle\Delta_i\rangle$, similarity variables, similarity solutions and reduced equations of (4.50)

	$\langle\Delta_i\rangle$	η	ξ	similarity solution	reduced equations
1	$\langle\Gamma_4\rangle$	$xt^{-\frac{1}{2}}$	$yt^{-\frac{1}{2}}$	$u = \frac{1}{x}\phi(\eta,\xi)$	$2\eta^2\phi_{\eta\eta} + 2\eta^2\phi_{\xi\xi} + (2\eta\phi - 4\eta + \eta^3)\phi_\eta$ $+\eta^2\xi\phi_\xi + 4\phi - 2\phi^2 = 0$
2	$\langle\Gamma_5 + \delta_1\Gamma_1 + \delta_3\Gamma_3\rangle$	$t^2 - 2\delta_1 x$	$y - \delta_1\delta_3 t$	$u = \phi(\eta,\xi) - \delta_3 y$	$4\phi_{\eta\eta} + \phi_{\xi\xi} - 2(\xi + \delta_1\phi)\phi_\eta$ $+\delta_1\delta_3\phi_\xi = 0$
3	$\langle\Gamma_5 + \varepsilon\Gamma_3\rangle$	t	$y - \frac{\varepsilon x}{t}$	$u = \phi(\eta,\xi) - \frac{x}{t}$	$(\varepsilon^2 + \eta^2)\phi_{\xi\xi} - \varepsilon\eta\phi\phi_\xi - \eta^2\phi_\eta - \eta\phi = 0$
4	$\langle\Gamma_5 + \delta_1\Gamma_1\rangle$	y	$t^2 - 2\delta_1 x$	$u = \phi(\eta,\xi) - \delta_1 t$	$4\delta_1\phi_{\xi\xi} + \delta_1\phi_{\eta\eta} - 2\phi\phi_\xi + 1 = 0$
5	$\langle\Gamma_2 + \delta_1\Gamma_1 + \delta_3\Gamma_3\rangle$	$x - \delta_1 t$	$y - \delta_3 x$	$u = \phi(\eta,\xi)$	$\phi_{\eta\eta} + 2\phi_{\xi\xi} - 2\delta_3\phi_{\eta\xi} + (\phi + \delta_1)\phi_\eta$ $-\delta_3\phi\phi_\xi = 0$
6	$\langle\Gamma_2 + \delta\Gamma_1\rangle$	y	$x - \delta_1 t$	$u = \phi(\eta,\xi)$	$\phi_{\xi\xi} + \phi_{\eta\eta} + (\phi + \delta_1)\phi_\xi = 0$
7	$\langle\Gamma_2 + \varepsilon\Gamma_3\rangle$	t	$y - \varepsilon x$	$u = \phi(\eta,\xi)$	$(\epsilon^2 + 1)\phi_{\xi\xi} - \varepsilon\phi\phi_\xi - \phi_\eta = 0$
8	$\langle\Gamma_1 + \varepsilon\Gamma_3\rangle$	x	$y - \varepsilon t$	$u = \phi(\eta,\xi)$	$\phi_{\eta\eta} + \phi_{\xi\xi} + \phi\phi_\eta + \varepsilon\phi_\xi = 0$
9	$\langle\Gamma_3\rangle$	t	x	$u = \phi(\eta,\xi)$	$\phi_{\xi\xi} + \phi\phi_\xi - \phi_\eta = 0$

$\varepsilon = -1, 0, 1$ and $\delta_1, \delta_3 = \pm 1$.

We note that reduced equations in Table 4.8 in the entries 7a and 9 are the well known Burgers equations. Also equations in the entries 3b and 7b are linear equations. The lists of known exact solutions of these equations can be found in [36, 59]. Therefore, in the subsequent analysis we do not consider the latter equations and also equation in the entry 1 since it does not admit Lie symmetries. We use the symmetries of the remaining equations in Table 4.8 to reduce them to ordinary equations. Exact solutions of the ordinary equations provide, in turn, exact solutions for the original equation (4.50).

Table 4.8: Symmetries of the reduced equations

N	Reduced equation	Symmetry algebra
1	$2\eta^2\phi_{\eta\eta} + 2\eta^2\phi_{\xi\xi} + (2\eta\phi - 4\eta + \eta^3)\phi_\eta$ $+\eta^2\xi\phi_\xi + 4\phi - 2\phi^2 = 0$	No symmetries
2	$4\phi_{\eta\eta} + \phi_{\xi\xi} - 2(\xi + \delta_1\phi)\phi_\eta + \delta_1\delta_3\phi_\xi = 0$	$\langle \frac{\partial}{\partial\eta}, \delta_1\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\phi} \rangle$
3a	$(\varepsilon^2 + \eta^2)\phi_{\xi\xi} - \varepsilon\eta\phi\phi_\xi - \eta^2\phi_\eta - \eta\phi = 0$ $(\varepsilon \neq 0)$	$\langle \frac{\partial}{\partial\xi}, \eta^{-1}(\varepsilon\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\phi}) \rangle$
3b	$\eta^2\phi_{\xi\xi} - \eta^2\phi_\eta - \eta\phi = 0$ $(\varepsilon = 0)$	$\langle \frac{\partial}{\partial\xi}, 2\eta\frac{\partial}{\partial\eta} + \xi\frac{\partial}{\partial\xi}, \phi\frac{\partial}{\partial\phi}, 2\eta\frac{\partial}{\partial\xi} - \xi\phi\frac{\partial}{\partial\phi}, \lambda^1\frac{\partial}{\partial\phi},$ $\frac{\partial}{\partial\eta} - \frac{\phi}{\eta}\frac{\partial}{\partial\phi}, 4\eta^2\frac{\partial}{\partial\eta} + 4\eta\xi\frac{\partial}{\partial\xi} - (6\eta\phi + \xi^2\phi)\frac{\partial}{\partial\phi} \rangle$
4	$4\delta_1\phi_{\xi\xi} + \delta_1\phi_{\eta\eta} - 2\phi\phi_\xi + 1 = 0$	$\langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta} \rangle$
5	$\phi_{\eta\eta} + 2\phi_{\xi\xi} - 2\delta_3\phi_{\eta\xi} + (\phi + \delta_1)\phi_\eta - \delta_3\phi\phi_\xi = 0$	$\langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta} \rangle$
6	$\phi_{\xi\xi} + \phi_{\eta\eta} + (\phi + \delta_1)\phi_\xi = 0$	$\langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta} - (\phi + \delta_1)\frac{\partial}{\partial\phi} \rangle$
7a	$(\varepsilon^2 + 1)\phi_{\xi\xi} - \varepsilon\phi\phi_\xi - \phi_\eta = 0$ $(\varepsilon \neq 0)$	$\langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \xi\frac{\partial}{\partial\xi} + 2\eta\frac{\partial}{\partial\eta} - \phi\frac{\partial}{\partial\phi}, \varepsilon\eta\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\phi},$ $\varepsilon\eta^2\frac{\partial}{\partial\eta} + \varepsilon\eta\xi\frac{\partial}{\partial\xi} + (\xi - \varepsilon\eta\phi)\frac{\partial}{\partial\phi} \rangle$
7b	$\phi_{\xi\xi} - \phi_\eta = 0$ $(\varepsilon = 0)$	$\langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \xi\frac{\partial}{\partial\xi} + 2\eta\frac{\partial}{\partial\eta}, \phi\frac{\partial}{\partial\phi}, 2\eta\frac{\partial}{\partial\xi} - \xi\phi\frac{\partial}{\partial\phi},$ $4\eta^2\frac{\partial}{\partial\eta} + 4\eta\xi\frac{\partial}{\partial\xi} - (\xi^2 + 2\eta)\phi\frac{\partial}{\partial\phi}, \lambda^2\frac{\partial}{\partial\phi} \rangle$
8	$\phi_{\eta\eta} + \phi_{\xi\xi} + \phi\phi_\eta + \varepsilon\phi_\xi = 0$	$\langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta} \rangle$
9	$\phi_{\xi\xi} + \phi\phi_\xi - \phi_\eta = 0$	$\langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \xi\frac{\partial}{\partial\xi} + 2\eta\frac{\partial}{\partial\eta} - \phi\frac{\partial}{\partial\phi}, \eta\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\phi},$ $\eta^2\frac{\partial}{\partial\eta} + \eta\xi\frac{\partial}{\partial\xi} - (\xi + \eta\phi)\frac{\partial}{\partial\phi} \rangle$

Here $\lambda^i = \lambda^i(\eta, \xi)$, $i = 1, 2$ are solutions of the linear equations $\lambda^1 = \eta(\lambda^1_{\xi\xi} - \lambda^1)$ and $\lambda^2_\eta = \lambda^2_{\xi\xi}$ correspondingly.

Equation 2 of Table 4.8 admits a two-dimensional symmetry algebra spanned by symmetry generators $X_1 = \frac{\partial}{\partial\eta}$, $X_2 = \delta_1\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\phi}$. Its optimal system of one-dimensional subalgebras consists of $\langle X_1 + aX_2\rangle$ and $\langle X_2\rangle$. If $a \neq 0$, generator $X_1 + aX_2$ leads to the similarity solution

$$\phi = -a\eta + \psi(\nu), \quad \nu = \xi - \delta_1 a\eta,$$

which reduces equation 4.8.2 to

$$(4a^2+1)\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} + (2\delta_1 a\nu + 2a\psi + \delta_1\delta_3)\frac{\mathrm{d}\psi}{\mathrm{d}\nu} + 2a\nu + 2\delta_1 a\psi = 0.$$

Making the substitution $w = \psi + \delta_1\nu$ and integrating once we obtain

$$\frac{\mathrm{d}w}{\mathrm{d}\nu} = Aw^2 + Bw + C\nu + D, \tag{4.51}$$

where $A = \frac{-a}{4a^2+1}$, $B = -\frac{\delta_1\delta_3}{4a^2+1}$, $C = \frac{\delta_3}{4a^2+1}$, and D is an integration constant. In order to solve (4.51), we make a further transformation

$$w = -\frac{\theta_\nu}{A\theta},$$

which maps (4.51) into

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\nu^2} - B\frac{\mathrm{d}\theta}{\mathrm{d}\nu} + (AC\nu + AD)\theta = 0,$$

having the following solution

$$\theta = \exp\left(\frac{B\nu}{2}\right)\sqrt{p}\left[C_1 J_{\frac{1}{3}}\left(\tfrac{2}{3}\sqrt{AC}p^{3/2}\right) + C_2 Y_{\frac{1}{3}}\left(\tfrac{2}{3}\sqrt{AC}p^{3/2}\right)\right].$$

Here $p = \nu + \frac{4AD-B^2}{4AC}$ and $J_{\frac{1}{3}}$ and $Y_{\frac{1}{3}}$ are the Bessel functions of first and second kind, respectively. Here and below C_1, C_2, c_1 and c_2 are arbitrary constants. Collecting all the subsequent results, we derive the similarity solution for equation (4.50),

$$\begin{aligned} u(t,x,y) = \delta_3 t - (\delta_1+\delta_3)y - \frac{B}{2A} - \frac{1}{2pA} - \\ - \sqrt{\frac{Cp}{A}}\left[\frac{C_1 J'_{\frac{1}{3}}(\frac{2}{3}\sqrt{AC}p^{3/2}) + C_2 Y'_{\frac{1}{3}}(\frac{2}{3}\sqrt{AC}p^{3/2})}{C_1 J_{\frac{1}{3}}(\frac{2}{3}\sqrt{AC}p^{3/2}) + C_2 Y_{\frac{1}{3}}(\frac{2}{3}\sqrt{AC}p^{3/2})}\right], \end{aligned} \tag{4.52}$$

where

$$A = -\frac{a}{4a^2+1}, \quad B = -\frac{\delta_1\delta_3}{4a^2+1}, \quad C = \frac{\delta_3}{4a^2+1},$$

$$p = y + 2ax - \delta_1 at^2 - \delta_1\delta_3 t + \frac{4AD - B^2}{4AC}.$$

Solution (4.52) was also obtained in [25] in terms of Airy functions.

In the case where $a = 0$, we obtain the solution

$$u(t, x, y) = c_1 e^{-\delta_1\delta_3 y + t} - \delta_3 y + c_2.$$

The second subalgebra, $\langle X_2 \rangle$, leads to similarity reduction $\phi = \delta_1\xi + \psi(\nu),\ \nu = \eta$ which reduces 4.8.2 to

$$4\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} - 2\delta_1\psi\frac{\mathrm{d}\psi}{\mathrm{d}\nu} - \delta_3 = 0.$$

Integration gives an equation of form (4.51), with $A = \frac{\delta_1}{4},\quad B = 0,\quad C = \frac{\delta_3}{4}$, which can be solved to obtain a solution similar to (4.52), where $p = t^2 - 2\delta_1 x + D/C$.

The Lie symmetry algebra of equation 3a of Table 4.8 is spanned by the Lie symmetries $X_1 = \partial\xi,\ X_2 = \eta^{-1}(\varepsilon\partial\xi - \partial\phi)$. Its inequivalent one-dimensional subalgebras $\langle X_1 + aX_2 \rangle$ and $\langle X_2 \rangle$ lead to exact solutions

$$u = \frac{-x + ay + c}{t - \varepsilon a} \quad \text{and} \quad u = \frac{-x - \varepsilon yt + c_1 t + \varepsilon^2}{t},$$

respectively.

The optimal system of one-dimensional subalgebras of the maximal Lie invariance algebra of equation 4 of Table 4.8 leads to two exact solutions of form (4.52) and to the solution

$$u = -\delta_1(t - cy + \tfrac{1}{2}y^2 + c_1).$$

Equation 5 of Table 4.8 admits two Lie symmetry algebra $\langle X_1 = \frac{\partial}{\partial\xi},\ X_2 = \frac{\partial}{\partial\eta}\rangle$. Its inequivalent one-dimensional subalgebras are $\langle X_1 + aX_2 \rangle$ and $\langle X_2 \rangle$. The first subalgebra leads to the similarity reduction $\phi = \psi(\nu)$, $\nu = \eta - a\xi$ that transforms equation 5 of Table 4.8 into

$$\frac{\mathrm{d}\psi}{\mathrm{d}\nu} = A\psi^2 + B\psi + C, \tag{4.53}$$

where $A = -\frac{(1+a\delta_3)}{2(1+2a^2+2\delta_3 a)}$, $B = -\frac{\delta_1}{1+2a^2+2\delta_3 a}$ and C is the integration constant. If $A = 0$ we obtain exact solution of form

$$u = c_1 \exp\left(t + \delta_1 a y\right).$$

If $A \neq 0$, we solve (4.53) and using that

$$u(t, x, y) = \phi(x - \delta_1 t, y - \delta_3 x) = \psi((1 + a\delta_3)x - ay - \delta_1 t)$$

we obtain the similarity solution

$$u(t,x,y) = \begin{cases} \frac{r_1 - Dr_2 \exp(A(r_1-r_2)\nu)}{1 - D\exp(A(r_1-r_2)\nu)}, & r_1 \neq r_2,\ r_1,\ r_2 \in \mathbb{R} \\ -\frac{1}{A\nu + D} + r_1, & r_1 = r_2,\ r_1,\ r_2 \in \mathbb{R} \\ \frac{1}{2A}\left[-B + \sqrt{4AC - B^2}\tan\left(\frac{\sqrt{4AC-B^2}\nu}{2} + D\right)\right], & r_1, r_2 \text{ complex numbers,} \end{cases} \tag{4.54}$$

where $r_1,\ r_2$ are the roots of the quadratic $Ar^2 + Br + C = 0$, $\nu = (1 + a\delta_3)x - ay - \delta_1 t$ and D is an arbitrary constant of integration. The first two branches of the solution (4.54) also appear in [25].

The second subalgebra, $\langle X_2 \rangle$, leads to a steady state similarity solution of form (4.54) with $\nu = y - \delta_3 x$.

Equation 6 of Table 4.8 admits three-dimensional Lie symmetry algebra spanned by $X_1 = \frac{\partial}{\partial \xi}$, $X_2 = \frac{\partial}{\partial \eta}$, and $X_3 = \xi\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial \eta} - (\phi + \delta_1)\frac{\partial}{\partial \phi}$. The optimal system of one-dimensional subalgebras consists of $\langle X_3 \rangle$, $\langle X_2 + aX_1 \rangle$, $\langle X_1 \rangle$. The first subalgebra, $\langle X_3 \rangle$, leads to the reduction $\phi = \frac{1}{\eta}\psi(\nu) - \delta_1,\ \nu = \frac{\xi}{\eta}$ that maps equation 6 of Table 4.8 into

$$(1 + \nu^2)\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} + (4\nu + \psi)\frac{\mathrm{d}\psi}{\mathrm{d}\nu} + 2\psi = 0.$$

Integrating once, we get the Riccati equation

$$(1 + \nu^2)\frac{\mathrm{d}\psi}{\mathrm{d}\nu} + \tfrac{1}{2}\psi^2 + 2\nu\psi = c_1.$$

In the case where $c_1 = 0$, its general solution has the form

$$\psi = \frac{4}{\nu + (\arctan\nu + c_2)(\nu^2 + 1)}\,.$$

This leads to the solution of the Burgers equation of form

$$u = \frac{4}{y(\nu + (\arctan\nu + c_2)(\nu^2 + 1))} - \delta_1,$$

where $\nu = (x - \delta_1 t)/y$.

The second subalgebra, $\langle X_2 + aX_1 \rangle$, leads to a solution of the form (4.54), while the third subalgebra, $\langle X_1 \rangle$, produces a solution of form

$$u = c_1 y + c_0.$$

Finally, equation 8 of Table 4.8 admits Lie symmetry algebra $\langle X_1 = \frac{\partial}{\partial \xi}, X_2 = \frac{\partial}{\partial \eta}\rangle$. The corresponding reductions lead to a similarity solution of the form (4.54) and two solutions of form

$$u = c_1 \exp\{-\varepsilon(y - \varepsilon t) + c_2\} \quad \text{and} \quad u = c_1(y - \varepsilon t) + c_2.$$

Note 4.4. *Some of the derived classes of exact solutions can be extended by symmetry transformations of the corresponding reduced equations, more precisely by scaling and translation transformations.*

4.4.2 Two-dimensional subalgebras

In this subsection we consider the optimal set of the two-dimensional subalgebras of the Lie symmetry algebra of the two-dimensional Burgers equation. These enable us to derive similarity reductions that reduce the two-dimensional Burgers equation into ordinary differential equation. The optimal goal is to derive solutions that cannot be obtained by the first approach. In each of the following cases we give the reduction and the corresponding reduced equation. We present only the similarity solutions that are different from subsection 4.4.1.

1. $\langle \Gamma_4, \Gamma_5 \rangle$: $u = -x/t + \psi(\nu)/y$, $\nu = yt^{-1/2}$ reduces (4.50) to

$$2\nu^2 \frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} + \nu(\nu^2 - 4)\frac{\mathrm{d}\psi}{\mathrm{d}\nu} + 2(2 - \nu^2)\psi = 0,$$

and making the substitution $\psi = \nu^2 w$ this equation changes to

$$2\nu \frac{\mathrm{d}^2\mathrm{w}}{\mathrm{d}\nu^2} + (4 + \nu^2)\frac{\mathrm{dw}}{\mathrm{d}\nu} = 0,$$

which has the solution $w(\nu) = c_1 \int \nu^{-2} \exp(-\frac{\nu^2}{4})\mathrm{d}\nu + \mathrm{c}_2$. Therefore the similarity solution for (4.50) leads to

$$u(t, x, y) = -xt^{-1} + yt^{-1}\left(c_1 \int \nu^{-2} \exp\left(-\frac{\nu^2}{4}\right)\mathrm{d}\nu + \mathrm{c}_2\right),$$

where $\nu = yt^{-\frac{1}{2}}$.

2. $\langle \Gamma_4, \varepsilon_2\Gamma_2 + \mu_3\Gamma_3 \rangle$: If $\varepsilon = \delta = \pm 1$ then $u = t^{-1/2}\psi(\nu)$, $\nu = t^{-1/2}(y - \delta\mu_3 x)$ reduces (4.50) to

$$2(\mu_3^2 + 1)\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} + (\nu - 2\delta\mu_3\psi)\frac{\mathrm{d}\psi}{\mathrm{d}\nu} + \psi = 0.$$

Its general solution is

$$\psi(\nu) = \frac{1}{\mu_3\nu(c_1M_1+W_1)}[(2c_1+2c_1\mu_3^2-c_1c_2\mu_3)M_3-4(\mu_3^2+1)W_3]+\frac{c_2}{\nu},$$

where

$$W_1 = W\left(-\frac{\mu_3^2+1+c_2\mu_3}{4(\mu_3^2+1)},\frac{1}{4},\frac{\nu^2}{4(\mu_3^2+1)}\right),\quad M_1 = M\left(-\frac{\mu_3^2+1+c_2\mu_3}{4(\mu_3^2+1)},\frac{1}{4},\frac{\nu^2}{4(\mu_3^2+1)}\right),$$
$$W_3 = W\left(\frac{3(\mu_3^2+1)-c_2\mu_3}{4(\mu_3^2+1)},\frac{1}{4},\frac{\nu^2}{4(\mu_3^2+1)}\right),\ M_3 = M\left(\frac{3(\mu_3^2+1)-c_2\mu_3}{4(\mu_3^2+1)},\frac{1}{4},\frac{\nu^2}{4(\mu_3^2+1)}\right).$$

are Whittaker functions [2]. In particular, if $c_2=0$ this expression can be simplified to

$$\psi(\nu) = \frac{2\sqrt{\mu_3^2+1}\exp\left(-\frac{\nu^2}{4(\mu_3^2+1)}\right)}{\operatorname{erf}\left(\frac{\nu}{2\sqrt{\mu_3^2+1}}\right)\sqrt{\pi}\mu_3+2c_1\sqrt{\mu_3^2+1}},$$

where $\operatorname{erf}z = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}d\xi$ is the error function (also called the probability integral). In fact, this latter expression is the solution of the ordinary differential equation

$$2(\mu_3^2+1)\frac{\mathrm{d}\psi}{\mathrm{d}\nu}+\mu_3\psi^2+\nu\psi=0.$$

This solution gives a solution of the Burgers equation of the form

$$u(x,y,t) = t^{-1/2}\frac{2\sqrt{\mu_3^2+1}\exp\left(-\frac{t(y-\delta\mu_3x)^2}{4(\mu_3^2+1)}\right)}{\operatorname{erf}\left(\frac{t^{-1/2}(y-\delta\mu_3x)}{2\sqrt{\mu_3^2+1}}\right)\sqrt{\pi}\mu_3+2c_1\sqrt{\mu_3^2+1}}.$$

If $\varepsilon=0$ then $u=\psi(\nu)/x$, $\nu=xt^{-1/2}$ that reduces (4.50) to

$$2\nu^2\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2}+(2\nu\psi-4\nu+\nu^3)\frac{\mathrm{d}\psi}{\mathrm{d}\nu}+4\psi-2\psi^2=0.$$

The above equation can be solved numerically.

3. $\langle\Gamma_5+\varepsilon_1\Gamma_1+\varepsilon_3\Gamma_3,\Gamma_3\rangle$: If $\varepsilon_1\neq 0$, then $u=-\varepsilon_1t+\psi(\nu)$, $\quad\nu=2\varepsilon_1x-t^2$, that reduces (4.50) to

$$4\varepsilon_1\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2}-2\psi\frac{\mathrm{d}\psi}{\mathrm{d}\nu}+1=0,$$

which leads to results that obtained using one-dimensional subalgebras. If $\varepsilon_1=0$ then $u=-x/t+\psi(t)$ that leads to the similarity solution

$$u=(c-x)/t.$$

4. $\langle \Gamma_5 + \varepsilon_1\Gamma_1 + \varepsilon_3\Gamma_3, \Gamma_2 + b_3\Gamma_3\rangle$: If $\varepsilon_1 = \delta = \pm 1$, then $u = -\delta t + \psi(\nu)$, $\nu = y - b_3x + b_3\delta t^2 - \delta\varepsilon_3 t$, that reduces (4.50) to

$$\delta(b_3^2 + 1)\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} + (\varepsilon_3 - \delta b_3\psi)\frac{\mathrm{d}\psi}{\mathrm{d}\nu} + 1 = 0.$$

If $\varepsilon_1 = 0$, then $u = \frac{y-b_3x}{b_3t-\varepsilon_3} + \psi(\nu)$, $\nu = t$ that reduces (4.50) to

$$(b_3\nu - \varepsilon_3)\frac{\mathrm{d}\psi}{\mathrm{d}\nu} + b_3\psi = 0.$$

The solutions of these two latter ordinary differential equations lead to similarity solutions obtained in subsection 4.4.1.

5. $\langle \Gamma_1 + \varepsilon_2\Gamma_2 + a_3\Gamma_3, \Gamma_2 + b_3\Gamma_3\rangle$: If $(\varepsilon_2, a_3) \neq (0, 0)$ then $u = \psi(\nu)$, $\nu = b_3(x - \varepsilon_2 t) - y + \alpha_3 t$, that reduces (4.50) to

$$(b_3^2 + 1)\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} + (b_3\psi + \varepsilon_2 b_3 - \alpha_3)\frac{\mathrm{d}\psi}{\mathrm{d}\nu} = 0.$$

6. $\langle \Gamma_1, \Gamma_2 + \varepsilon_3\Gamma_3\rangle$: Here $u = \psi(\nu)$, $\nu = y - \varepsilon_3 x$, that reduces (4.50) to

$$(1 + \varepsilon_3^2)\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} - \varepsilon_3\psi\frac{\mathrm{d}\psi}{\mathrm{d}\nu} = 0.$$

7. $\langle \Gamma_1, \Gamma_3\rangle$: Here $u = \psi(\nu)$, $\nu = x$, that reduces (4.50) to

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\nu^2} + \psi\frac{\mathrm{d}\psi}{\mathrm{d}\nu} = 0.$$

Integration of the ordinary differential equations obtained in cases 5, 6 and 7 lead to similarity solutions of the form (4.54).

8. $\langle \Gamma_2 + a_3\Gamma_3, \Gamma_3\rangle$: Produces the solution $u = \text{const}$.

4.5 Hidden Symmetries for the two-dimensional Burgers equation

We consider the two-dimensional Burgers equation

$$u_t = u_{xx} + u_{yy} + uu_x. \tag{4.55}$$

As we have seen before in this section the Lie group generators of (4.55) are

$$\Gamma_1 = \frac{\partial}{\partial t},\ \ \Gamma_2 = \frac{\partial}{\partial x},\ \ \Gamma_3 = \frac{\partial}{\partial y},\ \ \Gamma_4 = 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-u\frac{\partial}{\partial u},\ \ \Gamma_5 = t\frac{\partial}{\partial x}-\frac{\partial}{\partial u}. \tag{4.56}$$

We reduce the number of variables of the two-dimensional Burgers equation using the following similarity solution and similarity variables

$$u = \phi(\eta, \xi), \quad \eta = t, \quad \xi = y - \epsilon x,$$

found using symmetry $\Gamma_2 + \epsilon\Gamma_3$ (Table 4.7). The reduced PDE is the one-dimensional Burgers equation

$$(\epsilon^2 + 1)\phi_{\xi\xi} - \epsilon\phi\phi_\xi - \phi_\eta = 0, \quad \epsilon \neq 0. \tag{4.57}$$

The symmetries of (4.57) are

$$X_1 = \frac{\partial}{\partial \xi}, \quad X_2 = \frac{\partial}{\partial \eta}, \quad X_3 = \xi\frac{\partial}{\partial \xi} + 2\eta\frac{\partial}{\partial \eta} - \phi\frac{\partial}{\partial \phi},$$

$$X_4 = \epsilon\eta\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \phi}, \quad X_5 = \epsilon\eta^2\frac{\partial}{\partial \eta} + \epsilon\eta\xi\frac{\partial}{\partial \xi} + (\xi - \epsilon\eta\phi)\frac{\partial}{\partial \phi}.$$

The inherited symmetries are $\Gamma_1 \to X_2,\ \Gamma_2 \to X_1,\ \Gamma_3 \to X_1,\ \Gamma_4 \to X_3,\ \Gamma_5 \to X_5$, all of which can be inferred by looking at the Lie algebra of (4.56). The symmetry X_5 is known as hidden symmetry of Type II [1].

Chapter 5

Group analysis of (3+1) nonlinear diffusion-convection equations

We consider the (3+1)-dimensional nonlinear diffusion convection equations

$$u_t = (D(u)u_x)_x + (F(u)u_y)_y + (G(u)u_z)_z + K(u)u_x. \tag{5.1}$$

Here $D(u)$, $F(u)$, $G(u)$ and $K(u)$ are arbitrary smooth functions, $F(u) \neq 0$, $G(u) \neq 0$ and $D^2(u) + K^2(u) \neq 0$ (otherwise the problem is reduced to that of lower dimension) and $D_u^2 + F_u^2 + G_u^2 + K_u^2 \neq 0$ (i.e, the equation is nonlinear). A complete Lie group classification of (3+1)-dimensional nonlinear diffusion convection equations is presented in this chapter. We derive its Lie symmetries [22] and we give an example of similarity reduction of a PDE of four independent variables into an ODE using a three-dimensional subalgebra.

5.1 Classification of Lie Symmetries

The equivalence group of (5.1) consists of the 11-parameter group of point transformations

$$\tilde{t} = \varepsilon_6 t + \varepsilon_1, \quad \tilde{x} = \varepsilon_7 x + \varepsilon_{11} t + \varepsilon_2, \quad \tilde{y} = \varepsilon_8 y + \varepsilon_3, \quad \tilde{z} = \varepsilon_9 z + \varepsilon_4, \quad \tilde{u} = \varepsilon_{10} u + \varepsilon_5,$$
$$\tilde{D} = \varepsilon_6^{-1}\varepsilon_7^2 D, \quad \tilde{F} = \varepsilon_6^{-1}\varepsilon_8^2 F, \quad \tilde{G} = \varepsilon_6^{-1}\varepsilon_9^2 G, \quad \tilde{K} = \varepsilon_6^{-1}\varepsilon_3 K - \varepsilon_{11},$$

where $\varepsilon_1, \ldots, \varepsilon_{11}$ are arbitrary constants, $\varepsilon_6\varepsilon_7\varepsilon_8\varepsilon_9\varepsilon_{10} \neq 0$, and a discrete transformation

$$\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{y} = z, \quad \tilde{z} = y, \quad \tilde{u} = u,$$
$$\tilde{D} = D, \quad \tilde{F} = G, \quad \tilde{G} = F, \quad \tilde{K} = K,$$

of interchanging the variables y and z. The equivalence transformations are used to simplify the forms of $D(u),\ F(u),\ G(u)$ and $K(u)$ in the subsequent analysis.

Equation (5.1) admits Lie transformations of the form

$$\begin{aligned}
x' &= x + \epsilon X(x,t,y,z,u) + O(\epsilon^2),\\
t' &= t + \epsilon T(x,t,y,z,u) + O(\epsilon^2),\\
y' &= y + \epsilon Y(x,t,y,z,u) + O(\epsilon^2),\\
z' &= z + \epsilon Z(x,t,y,z,u) + O(\epsilon^2),\\
u' &= u + \epsilon U(x,t,y,z,u) + O(\epsilon^2),
\end{aligned}$$

if and only if

$$\Gamma^{(2)} E|_{E=0} = 0,$$

where $\Gamma^{(2)}$ is the second extended generator of

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} + U\frac{\partial}{\partial u},$$

which is given by the relation

$$\begin{aligned}
\Gamma^{(2)} &= \Gamma + [D_x(U) - (D_xX)u_x - (D_xT)u_t - (D_xY)u_y - (D_xZ)u_z]\frac{\partial}{\partial u_x}\\
&+ [D_t(U) - (D_tX)u_x - (D_tT)u_t - (D_tY)u_y - (D_tZ)u_z]\frac{\partial}{\partial u_t}\\
&+ [D_y(U) - (D_yX)u_x - (D_yT)u_t - (D_yY)u_y - (D_yZ)u_z]\frac{\partial}{\partial u_y}\\
&+ [D_z(U) - (D_zX)u_x - (D_zT)u_t - (D_zY)u_y - (D_zZ)u_z]\frac{\partial}{\partial u_z}\\
&+ [D_x(U^x) - u_{xx}D_x(X) - u_{xt}D_x(T) - u_{xy}D_x(Y) - u_{xz}D_x(Z)]\frac{\partial}{\partial u_{xx}}\\
&+ [D_y(U^y) - u_{yx}D_y(X) - u_{yt}D_y(T) - u_{yy}D_y(Y) - u_{yz}D_y(Z)]\frac{\partial}{\partial u_{yy}}\\
&+ [D_z(U^z) - u_{zx}D_z(X) - u_{zt}D_z(T) - u_{zy}D_z(Y) - u_{zz}D_z(Z)]\frac{\partial}{\partial u_{zz}}.
\end{aligned}$$

Here $D_x,\ D_t,\ D_y$ and D_z are the total derivatives with respect to $x,\ t,\ y$ and z respectively, and $U^x,\ U^y,\ U^z$ are the extended transformations. We point out that in $\Gamma^{(2)}$ we did not include terms with no contribution.

In this case $E = u_t - [D(u)u_x]_x - [F(u)u_y]_y - [G(u)u_z]_z - K(u)u_x$. So, equation (5.1) admits Lie symmetries if and only if

$$\Gamma^{(2)}\left[u_t - [D(u)u_x]_x - [F(u)u_y]_y - [G(u)u_z]_z - K(u)u_x\right] = 0, \tag{5.2}$$

where $u_t = [D(u)u_x]_x + [F(u)u_y]_y + [G(u)u_z]_z + K(u)u_x$.

If we take coefficients of u_{xt} we have two cases: (A) $D \neq 0$ and (B) $D = 0$.

Case A. $D \neq 0$.

In the case when $D \neq 0$ from the coefficients of u_{xt}, u_{yt} and u_{zt} in (5.2) we deduce that $T_x = T_y = T_z = T_u = 0$, so, T is only function of t. Also from the coefficients of $u_{xy}u_y$, $u_{xy}u_x$ we get that $X_u = Y_u = 0$ and from the coefficients of u_{xx} and u_x^2 we get $U_{uu} = 0$. Hence, $U = a_1(x,t,y,z)u + a_2(x,t,y,z)$.

Using the fact that $T = T(t)$, $X = X(x,t,y,z)$, $Y = Y(x,t,y,z)$ and the form of U into (5.2) we obtain the following determining equations of the functional forms of D, F, G, K, X, T, Y, Z and U:

$$(a_1u + a_2)D_u + (T_t - 2X_x)D = 0, \tag{5.3}$$

$$(a_1u + a_2)F_u + (T_t - 2Y_y)F = 0, \tag{5.4}$$

$$(a_1u + a_2)G_u + (T_t - 2Z_z)G = 0, \tag{5.5}$$

$$Y_xD + X_yF = 0, \tag{5.6}$$

$$Z_xD + X_zG = 0, \tag{5.7}$$

$$Y_zG + Z_yF = 0, \tag{5.8}$$

$$(2a_{1x}u + 2a_{2x})D_u + (2a_{1x} - X_{xx})D - X_{yy}F - X_{zz}G + (a_1u + a_2)K_{uu} \tag{5.9}$$
$$+(T_t - X_x)K_u + X_t = 0,$$

$$-Y_{xx}D + (2a_{1y}u + 2a_{2y})F_u + (2a_{1y} - Y_{yy})F - Y_{zz}G - Y_xK_u + Y_t = 0, \tag{5.10}$$

$$-Z_{xx}D - Z_{yy}F + (2a_{1z}u + 2a_{2z})G_u + (2a_{1z} - Z_{zz})G - Z_xK_u + Z_t = 0, \tag{5.11}$$

$$(a_{1xx}u + a_{2xx})D + (a_{1yy}u + a_{2yy})F + (a_{1zz}u + a_{2zz})G + (a_{1x}u + a_{2x})K_u \tag{5.12}$$
$$-a_{1t}u - a_{2t} = 0.$$

Equation (5.3) suggests the following forms of $D(u)$:

(1) $D(u)$ arbitrary;

(2) $D(u) = e^{\mu u}$;

(3) $D(u) = u^{\mu}$.

However in the analysis, these forms of $D(u)$ lead to further special cases. Summarizing we have the following forms of $D(u)$:

(1) $D(u)$ arbitrary;

(2) $D(u) = e^{\mu u}$;

(3) $D(u) = e^{u}$;

(4) $D(u) = u^{\mu}$;

(5) $D(u) = u^{-2}$;

(6) $D(u) = u^{-\frac{4}{3}}$;

(7) $D(u) = u^{-\frac{4}{5}}$;

(8) $D(u) = u^{-1}$;

(9) $D(u) = 1$.

Case 1. $D(u)$ arbitrary.

In this case from equations (5.3)-(5.12) we have the following subcases for the functions of $F,\ G,\ K$ and the Lie algebra for each subcase.

Subcase 1.1: $F(u),\ G(u),\ K(u)$ arbitrary.

From (5.3)-(5.12) we have

$$X = c_1, \quad T = c_2, \quad Y = c_3, \quad Z = c_4, \quad U = 0.$$

So, the Lie algebra is four-dimensional spanned by

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial y}, \quad \Gamma_4 = \frac{\partial}{\partial z}.$$

Subcase 1.2: $F(u),\ G(u)$ arbitrary, $K(u) = 0$.

Using equations (5.3)-(5.12) we deduce that

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3, \quad Y = c_1 y + c_4, \quad Z = c_1 z + c_5, \quad U = 0.$$

The Lie algebra is five-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Subcase 1.3: $F(u)$ arbitrary, $G(u) = \varepsilon_z F$, $K(u)$ arbitrary.

From (5.3)-(5.12) we have

$$X = c_1, \quad T = c_2, \quad Y = -c_3\varepsilon_z z + c_4, \quad Z = c_3 y + c_5, \quad U = 0.$$

Therefore, the Lie algebra is spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_6 = y\frac{\partial}{\partial z} - \varepsilon_z z\frac{\partial}{\partial y}.$$

Subcase 1.4: $F(u)$ arbitrary, $G(u) = \varepsilon_z F$, $K(u) = 0$.

In this subcase from (5.3)-(5.12) we have

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3, \quad Y = c_1 y - c_4\varepsilon_z z + c_5,$$

$$Z = c_1 z + c_4 y + c_6, \quad U = 0.$$

The Lie algebra in this subcase is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_6.$$

Subcase 1.5: $F(u) = \varepsilon_y D$, $G(u)$ arbitrary, $K(u) = 0$.

From (5.3)-(5.12) we deduce that

$$X = c_1 x + c_2 y + c_3, \quad T = 2c_1 t + c_4, \quad Y = c_1 y - c_2\varepsilon_y x + c_5,$$

$$Z = c_1 z + c_6, \quad U = 0.$$

So, the Lie algebra is spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_7 = y\frac{\partial}{\partial x} - \varepsilon_y x\frac{\partial}{\partial y}.$$

Note 5.1. *It is obvious that subcase 1.5 is equivalent to 1.4 with respect to the rotation transformation. However this transformation does not belong to equivalence group of the class (5.1). We adduce it after the Table 5.1. The same is true for all similar cases below.*

Subcase 1.6: $F(u) = \varepsilon_y D$, $G(u) = \varepsilon_z D$, $K(u) = 0$.

Here after some calculations using equations (5.3)-(5.12) we have

$$X = c_1 x + c_2 y + c_3 z + c_4, \quad T = 2c_1 t + c_5, \quad Y = c_1 y - c_2\varepsilon_y x - c_6\varepsilon_y z + c_7,$$

$$Z = c_1 z + c_6\varepsilon_z y - c_3\varepsilon_z x + c_8, \quad U = 0.$$

The Lie algebra is eight-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_7, \quad \Gamma_8 = \varepsilon_z y\frac{\partial}{\partial z} - \varepsilon_y z\frac{\partial}{\partial y}, \quad \Gamma_9 = z\frac{\partial}{\partial x} - \varepsilon_z x\frac{\partial}{\partial z}.$$

Case 2. $D(u) = e^{\mu u}$.

Using the fact that $D(u) = e^{\mu u}$ from equations (5.3)-(5.12) we have the following subcases for the functions of $F,\ G,\ K$ and the Lie algebra for each subcase.

Subcase 2.1: $F(u) = \varepsilon_y e^{\nu u},\ G(u) = \varepsilon_z e^{\lambda u},\ K(u) = u.$

In this subcase from (5.3)-(5.12) we deduce that

$$X = c_1(\mu x - t) + c_2,\quad T = c_1\mu t + c_3,\quad Y = \frac{c_1}{2}(\nu + \mu)y + c_4,$$

$$Z = \frac{c_1}{2}(\lambda + \mu)z + c_5,\quad U = c_1.$$

The Lie algebra is five-dimensional spanned by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_{10} = \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x} + \frac{1}{2}(\nu + \mu)y\frac{\partial}{\partial y} + \frac{1}{2}(\lambda + \mu)z\frac{\partial}{\partial z} + \frac{\partial}{\partial u}.$$

Subcase 2.2: $F(u) = \varepsilon_y e^{\nu u},\ G(u) = \varepsilon_z e^{\lambda u},\ K(u) = e^{pu}.$

From (5.3)-(5.12) we have

$$X = c_1(p - \mu)x + c_2,\quad T = c_1(2p - \mu)t + c_3,\quad Y = \frac{c_1}{2}(2p - \nu - \mu)y + c_4,$$

$$Z = -\frac{c_1}{2}(2p - \lambda - \mu)z + c_5,\quad U = -c_1 u.$$

The Lie algebra in this subcase is five-dimensional given by

$$\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_{11} = (2p-\mu)t\frac{\partial}{\partial t} + (p-\mu)x\frac{\partial}{\partial x} + \frac{1}{2}(2p-\nu-\mu)y\frac{\partial}{\partial y} - \frac{1}{2}(2p-\lambda-\mu)z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}.$$

Subcase 2.3: $F(u) = \varepsilon_y e^{\nu u},\ G(u) = \varepsilon_z e^{\lambda u},\ K(u) = 0.$

Here, using equations (5.3)-(5.12) we deduce that

$$X = c_1 x + \frac{c_2}{2}(\lambda - \mu)x + c_3,\quad T = 2c_1 t + c_2\lambda t + c_4,\quad Y = c_1 y + \frac{c_2}{2}(\lambda - \nu)y + c_5,$$

$$Z = c_1 z + c_6,\quad U = -c_1.$$

The Lie algebra is given by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_5\quad \Gamma_{12} = \lambda t\frac{\partial}{\partial t} + \frac{1}{2}(\lambda - \mu)x\frac{\partial}{\partial x} + \frac{1}{2}(\lambda - \nu)y\frac{\partial}{\partial y} - \frac{\partial}{\partial u}.$$

Subcase 2.4: $F(u) = \varepsilon_y e^{\nu u},\ G(u) = \varepsilon_z e^{\nu u},\ K(u) = u.$

In this subcase from (5.3)-(5.12) we have

$$X = c_1(\mu x - t) + c_2,\quad T = c_1\mu t + c_3,\quad Y = \frac{c_1}{2}(\nu + \mu)y - c_4\varepsilon_y\varepsilon_z z + c_5,$$

$$Z = \frac{c_1}{2}(\nu + \mu)z + c_4 y + c_6, \quad U = c_1.$$

Hence, the Lie algebra is six-dimensional spanned by

$$\Gamma_1, \; \Gamma_2, \; \Gamma_3, \; \Gamma_4, \; \Gamma_{13} = y\frac{\partial}{\partial z} - \varepsilon_y \varepsilon_z z\frac{\partial}{\partial y}, \; \Gamma_{14} = \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x} + \frac{1}{2}(\nu+\mu)\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) + \frac{\partial}{\partial u}.$$

Subcase 2.5: $F(u) = \varepsilon_y e^{\nu u}, \; G(u) = \varepsilon_z e^{\nu u}, \; K(u) = e^{pu}.$

In this subcase from (5.3)-(5.12) we get

$$X = c_1(p - \mu)x + c_2, \quad T = c_1(2p - \mu)t + c_3, \quad Y = \frac{c_1}{2}(\nu + \mu)y - c_4 \varepsilon_y \varepsilon_z z + c_5,$$

$$Z = \frac{c_1}{2}(2p - \nu - \mu)z + c_4 y + c_6, \quad U = -c_1.$$

The Lie algebra is six-dimensional given by

$$\Gamma_1, \; \Gamma_2, \; \Gamma_3, \; \Gamma_4, \; \Gamma_{13}, \; \Gamma_{15} = (2p-\mu)t\frac{\partial}{\partial t} + (p-\mu)x\frac{\partial}{\partial x} + \frac{1}{2}(2p-\nu-\mu)\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial u}.$$

Subcase 2.6: $F(u) = \varepsilon_y e^{\nu u}, \; G(u) = \varepsilon_z e^{\nu u}, \; K(u) = 0.$

Using equations (5.3)-(5.12) we have

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 \mu t + c_4, \quad Y = c_1 y - c_5 \varepsilon_y \varepsilon_z z + \frac{c_3}{2}(\nu - \mu)y + c_6,$$

$$Z = c_1 z + c_5 y + \frac{c_3}{2}(\nu - \mu)y + c_7, \quad U = -c_3.$$

In this subcase the Lie algebra is

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_{13}, \quad \Gamma_{16} = \mu t\frac{\partial}{\partial t} + \frac{1}{2}(\nu - \mu)\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) - \frac{\partial}{\partial u}.$$

Subcase 2.7: $F(u) = \varepsilon_y e^{\mu u}, \; G(u) = \varepsilon_z e^{\lambda u}, \; K(u) = 0.$

Using the forms of F, G and K from (5.3)-(5.12) we get

$$X = c_1 x + \frac{c_2}{2}(\lambda - \mu)x - c_3 \varepsilon_y y + c_4, \; T = 2c_1 t + c_2 \lambda t + c_5, \; Y = c_1 y + c_3 x + \frac{c_2}{2}(\lambda - \mu)y + c_6,$$

$$Z = c_1 z + c_7, \quad U = -c_2.$$

Therefore, the Lie algebra is seven-dimensional spanned by

$$\Gamma_1, \; \Gamma_2, \; \Gamma_3, \; \Gamma_4, \; \Gamma_5, \; \Gamma_{17} = x\frac{\partial}{\partial y} - \varepsilon_y y\frac{\partial}{\partial x}, \; \Gamma_{18} = \lambda t\frac{\partial}{\partial t} + \frac{1}{2}(\lambda - \mu)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial u}.$$

Case 3. $D(u) = e^u$.

In this case using the fact that $D(u) = e^u$ from equations (5.3)-(5.12) we deduce that $F(u) = \varepsilon_y e^u$, $G(u) = \varepsilon_z e^u$ and $K(u) = 0$. So,

$$X = c_1 x + c_2 z - c_4 \varepsilon_y y + c_5, \quad T = 2c_1 t + c_6 t + c_7, \quad Y = c_1 y - c_3 \varepsilon_y \varepsilon_z z + c_4 x + c_8,$$

$$Z = c_1 z - c_2 \varepsilon_z x + c_3 y + c_9, \quad U = -c_6,$$

and the Lie algebra is nine-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_9, \quad \Gamma_{13}, \quad \Gamma_{17}, \quad \Gamma_{19} = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}.$$

Case 4. $D(u) = u^\mu$.

From equations (5.3)-(5.12) we have the following subcases for the different forms of F, G, K, X, T, Y, Z and U and its Lie algebra respectively.

Subcase 4.1: $F(u) = \varepsilon_y u^\nu,\ G(u) = \varepsilon_z u^\lambda,\ K(u) = \ln u$.

In this subcase from (5.3)-(5.12) we have

$$X = c_1(\mu x - t) + c_2, \quad T = c_1 \mu t + c_3, \quad Y = \frac{c_1}{2}(\nu + \mu) y + c_4,$$

$$Z = \frac{c_1}{2}(\lambda + \mu) z + c_5, \quad U = c_1 u.$$

The Lie algebra is five-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{20} = \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x} + \frac{1}{2}(\nu + \mu) y\frac{\partial}{\partial y} + \frac{1}{2}(\lambda + \mu) z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}.$$

Subcase 4.2: $F(u) = \varepsilon_y u^\nu,\ G(u) = \varepsilon_z u^\lambda,\ K(u) = u^p$.

Using equations (5.3)-(5.12) we have

$$X = c_1(p - \mu)x + c_2, \quad T = c_1(2p - \mu)t + c_3, \quad Y = \frac{c_1}{2}(2p - \nu - \mu)y + c_4,$$

$$Z = -\frac{c_1}{2}(2p - \lambda - \mu)z + c_5, \quad U = -c_1 u.$$

Therefore, the Lie algebra is five-dimensional given by

$$\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_{21} = (2p-\mu)t\frac{\partial}{\partial t} + (p-\mu)x\frac{\partial}{\partial x} + \frac{1}{2}(2p-\nu-\mu)y\frac{\partial}{\partial y} - \frac{1}{2}(2p-\lambda-\mu)z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}.$$

Subcase 4.3: $F(u) = \varepsilon_y u^\nu,\ G(u) = \varepsilon_z u^\lambda,\ K(u) = 0$.

Here after some calculations using (5.3)-(5.12) we have

$$X = c_1 x + \frac{c_2}{2}(\lambda - \mu)x + c_3, \quad T = 2c_1 t + c_2 \lambda t + c_4, \quad Y = c_1 y + \frac{c_2}{2}(\lambda - \nu)y + c_5,$$

$$Z = c_1 z + c_6, \quad U = -c_2 u.$$

Hence, the Lie algebra is six-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_{22} = \lambda t\frac{\partial}{\partial t} + \frac{1}{2}(\lambda - \mu)x\frac{\partial}{\partial x} + \frac{1}{2}(\lambda - \nu)y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}.$$

Subcase 4.4: $F(u) = \varepsilon_y u^\nu, \; G(u) = \varepsilon_z u^\nu, \; K(u) = \ln u.$

In this subcase from (5.3)-(5.12) we deduce that

$$X = c_1(\mu x - t) + c_2, \quad T = c_1 \mu t + c_3, \quad Y = \frac{c_1}{2}(\nu + \mu)y - c_4 \varepsilon_y \varepsilon_z z + c_5,$$

$$Z = \frac{c_1}{2}(\nu + \mu)z + c_4 y + c_6, \quad U = c_1 u.$$

The Lie algebra is six-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{13}, \quad \Gamma_{23} = \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x} + \frac{1}{2}(\nu + \mu)\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) + u\frac{\partial}{\partial u}.$$

Subcase 4.5: $F(u) = \varepsilon_y u^\nu, \; G(u) = \varepsilon_z u^\nu, \; K(u) = u^p.$

In this subcase we have

$$X = c_1(p - \mu)x + c_2, \quad T = c_1(2p - \mu)t + c_3, \quad Y = \frac{c_1}{2}(2p - \nu - \mu)y - c_4 \varepsilon_y \varepsilon_z z + c_5,$$

$$Z = \frac{c_1}{2}(2p - \nu - \mu)z + c_4 y + c_6, \quad U = -c_1 u.$$

Hence, the Lie algebra is

$$\Gamma_1, \; \Gamma_2, \; \Gamma_3, \; \Gamma_4, \; \Gamma_{13}, \; \Gamma_{24} = (2p-\mu)t\frac{\partial}{\partial t} + (p-\mu)x\frac{\partial}{\partial x} + \frac{1}{2}(2p-\nu-\mu)\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) - u\frac{\partial}{\partial u}.$$

Subcase 4.6: $F(u) = \varepsilon_y u^\nu, \; G(u) = \varepsilon_z u^\nu, \; K(u) = 0.$

Here from (5.3)-(5.12) we have

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 \mu t + c_4, \quad Y = c_1 y + \frac{c_3}{2}(\nu - \mu)y - c_5 \varepsilon_y \varepsilon_z z + c_6,$$

$$Z = c_1 z + c_7, \quad U = -c_3 u,$$

and the Lie algebra is seven-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_{13}, \quad \Gamma_{25} = \mu t\frac{\partial}{\partial t} + \frac{1}{2}(\nu - \mu)\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) - u\frac{\partial}{\partial u}.$$

Subcase 4.7: $F(u) = \varepsilon_y u^\mu, \; G(u) = \varepsilon_z u^\lambda, \; K(u) = 0.$

In this subcase

$$X = c_1x - c_2\varepsilon_y y + \frac{c_3}{2}(\lambda-\mu)x + c_4,\;\; T = 2c_1t + c_3\lambda t + c_5,\;\; Y = c_1y + c_2x + \frac{c_3}{2}(\lambda-\mu)y + c_6,$$

$$Z = c_1z + c_7,\quad U = -c_3u.$$

So, the Lie algebra is spanned by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_5,\quad \Gamma_{17},\quad \Gamma_{26} = \lambda t\frac{\partial}{\partial t} + \frac{1}{2}(\lambda-\mu)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - u\frac{\partial}{\partial u}.$$

Subcase 4.8: $F(u) = \varepsilon_y u^{\mu},\; G(u) = \varepsilon_z u^{\mu},\; K(u) = 0.$

Using equations (5.3)-(5.12) we have

$$X = c_1x + c_2z - c_3\varepsilon_y y + c_4,\quad T = 2c_1t + c_5\mu t + c_6,\quad Y = c_1y - c_7\varepsilon_y\varepsilon_z z + c_3x + c_8,$$

$$Z = c_1z - c_2\varepsilon_z x + c_7y + c_3x + c_9,\quad U = -c_5u.$$

The Lie algebra is nine-dimensional given by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_5,\quad \Gamma_9,\quad \Gamma_{13},\quad \Gamma_{17},\quad \Gamma_{27} = \mu t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

Case 5. $D(u) = u^{-2}$.

In this case from equations (5.3)-(5.12) we get that $F(u) = \varepsilon_y,\; G(u) = \varepsilon_z$ and $K(u) = u^{-2}$. So,

$$X = c_1e^{-x} + c_2,\quad T = 2c_3t + c_4,\quad Y = c_3y - c_5\varepsilon_y\varepsilon_z z + c_6,$$

$$Z = c_5y + c_3z + c_7,\quad U = c_1e^{-x}u,$$

and the Lie algebra is seven-dimensional spanned by

$$\Gamma_1,\;\; \Gamma_2,\;\; \Gamma_3,\;\; \Gamma_4,\;\; \Gamma_{13},\;\; \Gamma_{28} = 2t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u},\;\; \Gamma_{29} = e^{-x}\left(\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}\right).$$

Case 6. $D(u) = u^{-4/3}$.

From equations (5.3)-(5.12) we have $F(u) = \varepsilon_y,\;\; G(u) = \varepsilon_z,\; K(u) = 0$ and

$$X = c_1x + 2c_2x + c_3x^2 + c_4,\quad T = 2c_1t + c_5\;\; Y = c_1y - c_6\varepsilon_y\varepsilon_z z + c_7,$$

$$Z = c_1z + c_6y + c_8,\quad U = -3c_2u - 3c_3xu.$$

So, the Lie algebra in this case is

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_5,\quad \Gamma_{13},\quad \Gamma_{30} = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u},\quad \Gamma_{31} = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}.$$

Case 7. $D(u) = u^{-4/5}$.

From equations (5.3)-(5.12) we get $F(u) = \varepsilon_y u^{-4/5}$, $G(u) = \varepsilon_z u^{-4/5}$ and $K(u) = 0$. Therefore,

$$X = c_1 x + c_2 z - c_3\varepsilon_y y + \frac{c_4}{2}(\varepsilon_y\varepsilon_z x^2 - \varepsilon_y z^2 - \varepsilon_z y^2) + c_5\varepsilon_z x y^2 + c_6\varepsilon_y x z + c_7,$$

$$T = 2c_1 t + 4c_{11} t + c_8,$$

$$Y = c_1 y - c_9\varepsilon_y\varepsilon_z z + c_3 x + c_4\varepsilon_y\varepsilon_z x y + \frac{c_5}{2}(\varepsilon_z y^2 - \varepsilon_y\varepsilon_z x^2 - \varepsilon_y z^2) + c_6\varepsilon_y z y + c_{10},$$

$$Z = c_1 z - c_2\varepsilon_z x + c_9 y + c_4\varepsilon_y\varepsilon_z x z + c_5\varepsilon_z y z + \frac{c_6}{2}(\varepsilon_y z^2 - \varepsilon_y\varepsilon_z x^2 - \varepsilon_z y^2) + c_{12},$$

$$U = 5c_{11} u - \frac{5}{2}(c_4\varepsilon_y\varepsilon_z x + c_5\varepsilon_z y + c_6\varepsilon_y z)u.$$

Hence, the Lie algebra is twelve-dimensional spanned by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_5,\quad \Gamma_9,\quad \Gamma_{13},\quad \Gamma_{17},\quad \Gamma_{32} = 4t\frac{\partial}{\partial t} + 5u\frac{\partial}{\partial u},$$

$$\Gamma_{33} = \frac{1}{2}\left(\varepsilon_y\varepsilon_z x^2 - \varepsilon_y z^2 - \varepsilon_z y^2\right)\frac{\partial}{\partial x} + \varepsilon_y\varepsilon_z x\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - \frac{5}{2}u\frac{\partial}{\partial u}\right),$$

$$\Gamma_{34} = \frac{1}{2}\left(\varepsilon_z y^2 - \varepsilon_y\varepsilon_z x^2 - \varepsilon_y z^2\right)\frac{\partial}{\partial y} + \varepsilon_z y\left(xy\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - \frac{5}{2}u\frac{\partial}{\partial u}\right),$$

$$\Gamma_{35} = \frac{1}{2}\left(\varepsilon_y z^2 - \varepsilon_y\varepsilon_z x^2 - \varepsilon_z y^2\right)\frac{\partial}{\partial z} + \varepsilon_y z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{5}{2}u\frac{\partial}{\partial u}\right).$$

Case 8. $D(u) = u^{-1}$.

From equations (5.3)-(5.12) we have $F(u) = \varepsilon_y u^{-1}$, $G(u) = \varepsilon_z$ and $K(u) = 0$. So,

$$X = c_1\phi(x,y),\ \ T = 2c_3 t + c_2,\ \ Y = c_1\psi(x,y),\ \ Z = c_3 z + c_4,\ \ U = -2c_3 u - 2c_1\phi_x(x,y)u,$$

and the Lie algebra is given by

$$\Gamma_1,\quad \Gamma_4,\quad \Gamma_{36} = 2t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u},\quad \Gamma_{37} = \phi\frac{\partial}{\partial x} + \psi\frac{\partial}{\partial y} - 2\phi_x u\frac{\partial}{\partial u},$$

where $\phi = \phi(x,y)$, $\psi = \psi(x,y)$ run through the solution set of the system $\phi_x = \psi_y$, $\psi_x = -\varepsilon_y\phi_y$.

Case 9. $D(u) = 1$.

From (5.3)-(5.12) we have the following subcases:

Subcase 9.1: $F(u) = \varepsilon_y, \quad G(u) = \varepsilon_z, \quad K(u) = u.$

In this subcase we have

$$X = c_1 x + c_2 t + c_3, \quad T = 2c_1 t + c_4, \quad Y = c_1 y - c_5 \varepsilon_y \varepsilon_z z + c_6,$$

$$Z = c_1 z + c_5 y + c_7, \quad U = -c_1 u - c_2.$$

Hence, the Lie algebra is seven-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{13}, \quad \Gamma_{38} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}, \quad \Gamma_{39} = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

Subcase 9.2: $F(u) = \varepsilon_y, \quad G(u) = \varepsilon_z u^{-4/3}, \quad K(u) = 0.$

In this subcase from (5.3)-(5.12) we have

$$X = c_1 x - c_2 \varepsilon_y y + c_3, \quad T = 2c_1 t + c_4, \quad Y = c_1 y - c_2 x + c_5,$$

$$Z = c_1 x + 2c_6 z + c_7 z^2 + c_8, \quad U = -3(c_6 + c_7 z)u.$$

The Lie algebra is eight-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_{17}, \quad \Gamma_{40} = 2z\frac{\partial}{\partial z} - 3u\frac{\partial}{\partial u}, \quad \Gamma_{41} = z^2\frac{\partial}{\partial z} - 3zu\frac{\partial}{\partial u}.$$

Subcase 9.3: $F(u) = \varepsilon_y u^{-1}, \quad G(u) = \varepsilon_z u^{-1}, \quad K(u) = 0.$

Using equations (5.3)-(5.12) we deduce that

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3, \quad Y = c_2 \tilde{\psi}(x, y), \quad Z = c_2 \tilde{\phi}(x, y), \quad U = 2\left(c_1 - c_2 \tilde{\phi}_z(x, y)\right) u.$$

The Lie algebra in this subcase is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_{42} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \quad \Gamma_{43} = \tilde{\phi}\frac{\partial}{\partial z} + \tilde{\psi}\frac{\partial}{\partial y} - 2\tilde{\phi}_z u\frac{\partial}{\partial u},$$

where $\tilde{\phi} = \tilde{\phi}(z, y)$, $\tilde{\psi} = \tilde{\psi}(z, y)$ run the solution set of the system $\tilde{\phi}_z = \tilde{\psi}_y$, $\varepsilon_z \tilde{\psi}_z = -\varepsilon_y \tilde{\phi}_y$.

In Table 5.1 we give briefly the Lie symmetries for the different forms of $D(u)$, $F(u)$, $G(u)$ and $K(u)$.

Table 5.1: Group classification of class (5.1), $D \neq 0$

N	$D(u)$	$F(u)$	$G(u)$	$K(u)$	$A^{\max}$
1	$\forall$	$\forall$	$\forall$	$\forall$	$A^{\ker} = \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$
2	$\forall$	$\forall$	$\forall$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \rangle$
3	$\forall$	$\forall$	$\varepsilon_z F$	$\forall$	$A^{\ker} + \langle y\frac{\partial}{\partial z} - \varepsilon_z z\frac{\partial}{\partial y} \rangle$
4a	$\forall$	$\forall$	$\varepsilon_z F$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y\frac{\partial}{\partial z} - \varepsilon_z z\frac{\partial}{\partial y} \rangle$
4b	$\forall$	$\varepsilon_y D$	$\forall$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - \varepsilon_y x\frac{\partial}{\partial y} \rangle$
5	$\forall$	$\varepsilon_y D$	$\varepsilon_z D$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y\frac{\partial}{\partial x} - \varepsilon_y x\frac{\partial}{\partial y}, \varepsilon_z y\frac{\partial}{\partial z} - \varepsilon_y z\frac{\partial}{\partial y},$ $z\frac{\partial}{\partial x} - \varepsilon_z x\frac{\partial}{\partial z} \rangle$
6	$e^{\mu u}$	$\varepsilon_y e^{\nu u}$	$\varepsilon_z e^{\lambda u}$	u	$A^{\ker} + \langle \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x} + \frac{1}{2}(\nu+\mu)y\frac{\partial}{\partial y}$ $+\frac{1}{2}(\lambda+\mu)z\frac{\partial}{\partial z} + \frac{\partial}{\partial u} \rangle$
7	$e^{\mu u}$	$\varepsilon_y e^{\nu u}$	$\varepsilon_z e^{\lambda u}$	e^{pu}	$A^{\ker} + \langle (2p-\mu)t\frac{\partial}{\partial t} + (p-\mu)x\frac{\partial}{\partial x}$ $+\frac{1}{2}(2p-\nu-\mu)y\frac{\partial}{\partial y} - \frac{1}{2}(2p-\lambda-\mu)z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} \rangle$
8	$e^{\mu u}$	$\varepsilon_y e^{\nu u}$	$\varepsilon_z e^{\lambda u}$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$ $\lambda t\frac{\partial}{\partial t} + \frac{1}{2}(\lambda-\mu)x\frac{\partial}{\partial x} + \frac{1}{2}(\lambda-\nu)y\frac{\partial}{\partial y} - \frac{\partial}{\partial u} \rangle$
9	$e^{\mu u}$	$\varepsilon_y e^{\nu u}$	$\varepsilon_z e^{\nu u}$	u	$A^{\ker} + \langle y\frac{\partial}{\partial z} - \varepsilon_y\varepsilon_z z\frac{\partial}{\partial y}, \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x}$ $+\frac{1}{2}(\nu+\mu)(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}) + \frac{\partial}{\partial u} \rangle$
10	$e^{\mu u}$	$\varepsilon_y e^{\nu u}$	$\varepsilon_z e^{\nu u}$	e^{pu}	$A^{\ker} + \langle y\frac{\partial}{\partial z} - \varepsilon_y\varepsilon_z z\frac{\partial}{\partial y}, (2p-\mu)t\frac{\partial}{\partial t} + (p-\mu)x\frac{\partial}{\partial x}$ $+\frac{1}{2}(2p-\nu-\mu)(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}) - \frac{\partial}{\partial u} \rangle$
11a	$e^{\mu u}$	$\varepsilon_y e^{\nu u}$	$\varepsilon_z e^{\nu u}$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, y\frac{\partial}{\partial z} - \varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},$ $\mu t\frac{\partial}{\partial t} + \frac{1}{2}(\nu-\mu)(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}) - \frac{\partial}{\partial u} \rangle$
11b	$e^{\mu u}$	$\varepsilon_y e^{\mu u}$	$\varepsilon_z e^{\lambda u}$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - \varepsilon_y y\frac{\partial}{\partial x},$ $\lambda t\frac{\partial}{\partial t} + \frac{1}{2}(\lambda-\mu)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) - \frac{\partial}{\partial u} \rangle$
12	e^{u}	$\varepsilon_y e^{u}$	$\varepsilon_z e^{u}$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - \varepsilon_y y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - \varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},$ $z\frac{\partial}{\partial x} - \varepsilon_z x\frac{\partial}{\partial z}, t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} \rangle$
13	u^{μ}	$\varepsilon_y u^{\nu}$	$\varepsilon_z u^{\lambda}$	$\ln u$	$A^{\ker} + \langle \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x}$ $+\frac{1}{2}(\nu+\mu)y\frac{\partial}{\partial y} + \frac{1}{2}(\lambda+\mu)z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} \rangle$
14	u^{μ}	$\varepsilon_y u^{\nu}$	$\varepsilon_z u^{\lambda}$	u^{p}	$A^{\ker} + \langle (2p-\mu)t\frac{\partial}{\partial t} + (p-\mu)x\frac{\partial}{\partial x}$ $+\frac{1}{2}(2p-\nu-\mu)y\frac{\partial}{\partial y} - \frac{1}{2}(2p-\lambda-\mu)z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} \rangle$
15	u^{μ}	$\varepsilon_y u^{\nu}$	$\varepsilon_z u^{\lambda}$	0	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$ $\lambda t\frac{\partial}{\partial t} + \frac{1}{2}(\lambda-\mu)x\frac{\partial}{\partial x} + \frac{1}{2}(\lambda-\nu)y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} \rangle$
16	u^{μ}	$\varepsilon_y u^{\nu}$	$\varepsilon_z u^{\nu}$	$\ln u$	$A^{\ker} + \langle y\frac{\partial}{\partial z} - \varepsilon_y\varepsilon_z z\frac{\partial}{\partial y}, \mu t\frac{\partial}{\partial t} + (\mu x - t)\frac{\partial}{\partial x}$ $+\frac{1}{2}(\nu+\mu)(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}) + u\frac{\partial}{\partial u} \rangle$

17	u^{μ}	$\varepsilon_y u^{\nu}$	$\varepsilon_z u^{\nu}$	u^p	$A^{\ker}+\langle y\frac{\partial}{\partial z}-\varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},(2p-\mu)t\frac{\partial}{\partial t}+(p-\mu)x\frac{\partial}{\partial x}$ $+\frac{1}{2}(2p-\nu-\mu)(y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})-u\frac{\partial}{\partial u}\rangle$
18a	u^{μ}	$\varepsilon_y u^{\nu}$	$\varepsilon_z u^{\nu}$	0	$A^{\ker}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},y\frac{\partial}{\partial z}-\varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},$ $\mu t\frac{\partial}{\partial t}+\frac{1}{2}(\nu-\mu)(y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})-u\frac{\partial}{\partial u}\rangle$
18b	u^{μ}	$\varepsilon_y u^{\mu}$	$\varepsilon_z u^{\lambda}$	0	$A^{\ker}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},x\frac{\partial}{\partial y}-\varepsilon_y y\frac{\partial}{\partial x},$ $\lambda t\frac{\partial}{\partial t}+\frac{1}{2}(\lambda-\mu)(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})-u\frac{\partial}{\partial u}\rangle$
18c	u^{-2}	ε_y	ε_z	u^{-2}	$A^{\ker}+\langle y\frac{\partial}{\partial z}-\varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},2t\frac{\partial}{\partial t}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+u\frac{\partial}{\partial u},e^{-x}(\frac{\partial}{\partial x}+u\frac{\partial}{\partial u})\rangle$
19	u^{μ}	$\varepsilon_y u^{\mu}$	$\varepsilon_z u^{\mu}$	0	$A^{\ker}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},x\frac{\partial}{\partial y}-\varepsilon_y y\frac{\partial}{\partial x},y\frac{\partial}{\partial z}-\varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},$ $z\frac{\partial}{\partial x}-\varepsilon_z x\frac{\partial}{\partial z},\mu t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}\rangle$
20	1	ε_y	ε_z	u	$A^{\ker}+\langle y\frac{\partial}{\partial z}-\varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}-u\frac{\partial}{\partial u},t\frac{\partial}{\partial x}-\frac{\partial}{\partial u}\rangle$
21a	$u^{-4/3}$	ε_y	ε_z	0	$A^{\ker}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},y\frac{\partial}{\partial z}-\varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},2x\frac{\partial}{\partial x}-3u\frac{\partial}{\partial u},$ $x^2\frac{\partial}{\partial x}-3xu\frac{\partial}{\partial u}\rangle$
21b	1	ε_y	$\varepsilon_z u^{-4/3}$	0	$A^{\ker}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},x\frac{\partial}{\partial y}-\varepsilon_y y\frac{\partial}{\partial x},2z\frac{\partial}{\partial z}-3u\frac{\partial}{\partial u},$ $z^2\frac{\partial}{\partial z}-3zu\frac{\partial}{\partial u}\rangle$
22	$u^{-4/5}$	$\varepsilon_y u^{-4/5}$	$\varepsilon_z u^{-4/5}$	0	$A^{\ker}+\langle 2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z},x\frac{\partial}{\partial y}-\varepsilon_y y\frac{\partial}{\partial x},y\frac{\partial}{\partial z}-\varepsilon_y\varepsilon_z z\frac{\partial}{\partial y},$ $z\frac{\partial}{\partial x}-\varepsilon_z x\frac{\partial}{\partial z},4t\frac{\partial}{\partial t}+5u\frac{\partial}{\partial u},$ $\frac{1}{2}(\varepsilon_y\varepsilon_z x^2-\varepsilon_y z^2-\varepsilon_z y^2)\frac{\partial}{\partial x}+\varepsilon_y\varepsilon_z x(y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}-\frac{5}{2}u\frac{\partial}{\partial u}),$ $\frac{1}{2}(\varepsilon_z y^2-\varepsilon_y\varepsilon_z x^2-\varepsilon_y z^2)\frac{\partial}{\partial y}+\varepsilon_z y(xy\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}-\frac{5}{2}u\frac{\partial}{\partial u}),$ $\frac{1}{2}(\varepsilon_y z^2-\varepsilon_y\varepsilon_z x^2-\varepsilon_z y^2)\frac{\partial}{\partial z}+\varepsilon_y z(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-\frac{5}{2}u\frac{\partial}{\partial u})\rangle$
23a	1	$\varepsilon_y u^{-1}$	$\varepsilon_z u^{-1}$	0	$\langle\frac{\partial}{\partial t},\frac{\partial}{\partial x},2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u},\tilde{\phi}\frac{\partial}{\partial z}+\tilde{\psi}\frac{\partial}{\partial y}-2\tilde{\phi}_z u\frac{\partial}{\partial u}\rangle$
23b	u^{-1}	$\varepsilon_y u^{-1}$	ε_z	0	$\langle\frac{\partial}{\partial t},\frac{\partial}{\partial z},2t\frac{\partial}{\partial t}+z\frac{\partial}{\partial z}+2u\frac{\partial}{\partial u},\phi\frac{\partial}{\partial x}+\psi\frac{\partial}{\partial y}-2\phi_x u\frac{\partial}{\partial u}\rangle$

Here $\varepsilon_y,\varepsilon_z=\pm1$, $p\neq0$; in case 23b, $\phi=\phi(x,y)$, $\psi=\psi(x,y)$ are arbitrary solutions of the system $\phi_x=\psi_y$, $\psi_x=-\varepsilon_y\phi_y$; in case 23a, $\tilde{\phi}=\tilde{\phi}(z,y)$, $\tilde{\psi}=\tilde{\psi}(z,y)$ are arbitrary solutions of the system $\tilde{\phi}_z=\tilde{\psi}_y$, $\varepsilon_z\tilde{\psi}_z=-\varepsilon_y\tilde{\phi}_y$.

Additional equivalence transformations

$4b\longmapsto 4a,\ 11b\longmapsto 11a,\ 18b\longmapsto 18a,\ 21b\longmapsto 21a,\ 23b\longmapsto 23a:\tilde{t}=\epsilon t,\ \tilde{x}=z,\ \tilde{z}=x,\ \tilde{D}=F,\ \tilde{F}=\epsilon D;$

$18c\longmapsto 18a\mid_{\mu=-2}$: $\tilde{x}=e^x,\ \tilde{u}=e^{-x}u.$

Case B. $D=0$.

In this case when $D=0$ from the coefficients of u_{yt} and u_{zt} of equation (5.2) we get that $T_y=T_z=T_u=0$. Also from the coefficients of $u_{yz}u_z$, $u_{yz}u_z$, u_{xz} and u_{xy} we deduce that $Y_u=Z_u=X_y=X_z=X_u=0$ and from the coefficients of u_{yy} and u_y^2 we get the following form of U

$$U=-T_xK+a_1(x,t,y,z)u+a_2(x,t,y,z).$$

Using the fact that $T=T(x,t)$, $X=X(x,t)$, $Y=Y(x,t,y,z)$, $Z=Z(x,t,y,z)$ and the form of U then from (5.2) we obtain the following determining equations of the functional forms of $F,\ G,\ K,\ X,\ T,\ Y,\ Z$ and U:

$$(a_1u+a_2-T_xK)F_u+(T_t-T_xK_u-2Y_y)F=0, \tag{5.13}$$

$$(a_1u+a_2-T_xK)G_u+(T_t-T_xK_u-2Z_z)G=0, \tag{5.14}$$

$$Z_yF+Y_zG=0, \tag{5.15}$$

$$(2a_{1y}u+2a_{2y})F_u+(2a_{1y}-Y_{yy})F-Y_{zz}G-Y_xK_u+Y_t=0, \tag{5.16}$$

$$-Z_{yy}F+(2a_{1z}u+2a_{2z})G_u+(2a_{1z}-Z_{zz})G-Z_xK_u+Z_t=0, \tag{5.17}$$

$$(T_xK-a_2-a_1u)K_{uu}+T_xK_u^2+(-T_t+X_x)K_u-X_t=0, \tag{5.18}$$

$$(a_{1yy}u+a_{2yy})F+(a_{1zz}u+a_{2zz})G+(a_{1x}u+a_{2x}-T_{xx}K)K_u+ \tag{5.19}$$

$$T_{tx}K-a_{1t}u-a_{2t}=0.$$

From equations (5.13) we get the following different forms of $F(u)$:

(1) $F(u)$ arbitrary;

(2) $F(u)=e^{\nu u}$;

(3) $F(u)=e^{u}$;

(4) $F(u)=u^{\nu}$;

(5) $F(u)=u^{-1/2}$;

(6) $F(u)=\dfrac{e^{\int\frac{\mu_1 \mathrm{d}u}{(\kappa+p-r)u}}}{(\kappa+p-r)u}$, where $u=(\kappa^2+2p\kappa+q)^{-1/2}e^{\int\frac{r\mathrm{d}u}{\kappa^2+2p\kappa+q}}$;

(7) $F(u)=1$.

Case 1. $F(u)$ arbitrary

Solving equations (5.13)-(5.19) we get the following subcases for functions G and K:

Subcase 1.1: $G(u),\ K(u)$ arbitrary.

In this subcase we have

$$X=2c_1x+c_2,\quad T=2c_1t+c_3,\quad Y=c_1y+c_4,\quad Z=c_1z+c_5,\quad U=0.$$

So, the Lie symmetries are

$$\Gamma_1=\frac{\partial}{\partial t},\quad \Gamma_2=\frac{\partial}{\partial x},\quad \Gamma_3=\frac{\partial}{\partial y},\quad \Gamma_4=\frac{\partial}{\partial z},\quad \Gamma_5=2t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}.$$

Subcase 1.2: $G(u)=\varepsilon_zF$, $K(u)$ arbitrary.

Using equations (5.13)-(5.19) we have

$$X = 2c_1x + c_2, \quad T = 2c_1t + c_3, \quad Y = c_1y + c_4z + c_5, \quad Z = c_1z - c_4\varepsilon_z y + c_6, \quad U = 0.$$

So, the Lie algebra is six-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_5, \quad \Gamma_6 = z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}.$$

Case 2. $F(u) = e^{\nu u}$

From equations (5.13)-(5.19) we get the following different forms for functions G and K.

Subcase 2.1: $G(u) = \varepsilon_z e^{\lambda u}, \ K(u) = e^{pu}$.

In this subcase after some calculations we get

$$X = 2c_1px + 2c_2(\nu - p)x + c_3, \quad T = 2c_2\nu t + c_4, \quad Y = c_1\nu y + c_5,$$

$$Z = c_1\lambda z + c_2(\nu - \lambda)z + c_6, \quad U = 2(c_1 - c_2).$$

Hence, the Lie algebra is six-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_7 = 2px\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \lambda z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u},$$

$$\Gamma_8 = 2\nu t\frac{\partial}{\partial t} + 2(\nu - p)x\frac{\partial}{\partial x} + (\nu - \lambda)z\frac{\partial}{\partial z} - 2\frac{\partial}{\partial u}.$$

Subcase 2.2: $G(u) = \varepsilon_z e^{\lambda u}, \ K(u) = u$.

Substituting the above forms of G and K into (5.13)-(5.19) we have

$$X = -2c_1t + 2c_2(\nu x + t) + c_3, \quad T = 2c_2\nu t + c_4, \quad Y = c_1\nu y + c_5z + c_6,$$

$$Z = c_1\lambda z + c_2(\nu - \lambda)z + c_6, \quad U = 2(c_1 - c_2).$$

So, we get the following Lie algebra

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_9 = -2t\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \lambda z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u},$$

$$\Gamma_{10} = 2\nu t\frac{\partial}{\partial t} + 2(\nu x + t)\frac{\partial}{\partial x} + (\nu - \lambda)z\frac{\partial}{\partial z} - 2\frac{\partial}{\partial u}.$$

Subcase 2.3: $G(u) = \varepsilon_z e^{\nu u}, \ K(u) = e^{pu}$.

Using equations (5.13)-(5.19) we have

$$X = 2c_1px + c_2(\nu - p)x + c_3, \quad T = c_2\nu t + c_4, \quad Y = c_1\nu y + c_5z + c_6,$$

$$Z = c_1\nu z - c_5\varepsilon_z y + c_7, \quad U = 2c_1 - c_2.$$

Therefore, the Lie algebra is seven-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_6, \quad \Gamma_{11} = 2px\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \nu z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u},$$

$$\Gamma_{12} = \nu t\frac{\partial}{\partial t} + (\nu - p)x\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

Case 3. $F(u) = e^u$.

From equations (5.13)-(5.19) we deduce that $G(u) = \varepsilon_z e^u$, $K(u) = u$ and

$$X = -2c_1t + c_2(x + t) + c_3, \quad T = c_2t + c_4, \quad Y = c_1y + c_5z + c_6,$$

$$Z = c_1z - c_5\varepsilon_z y + c_7, \quad U = 2c_1 - c_2.$$

The Lie algebra is seven-dimensional given by

$$\Gamma_1, \; \Gamma_2, \; \Gamma_3, \; \Gamma_4, \; \Gamma_6, \; \Gamma_{13} = -2t\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u}, \; \Gamma_{14} = t\frac{\partial}{\partial t} + (x+t)\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

Case 4. $F(u) = u^\nu$.

Substituting the above form of F from equations (5.13)-(5.19) we deduce the following subcases:

Subcase 4.1: $G(u) = \varepsilon_z u^\lambda, \; K(u) = u^p$

In this subcase we have

$$X = 2c_1(\lambda - p)x + 2c_2(\nu - p)x + c_3, \quad T = 2c_1\lambda t + 2c_2\nu t + c_4, \quad Y = c_1(\lambda - \nu)y + c_5,$$

$$Z = c_2(\nu - \lambda)x + c_6, \quad U = -2(c_1 + c_2)u.$$

Therefore, the Lie algebra is six-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{15} = 2\lambda t\frac{\partial}{\partial t} + 2(\lambda - p)x\frac{\partial}{\partial x} + (\lambda - \nu)y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u},$$

$$\Gamma_{16} = 2\nu t\frac{\partial}{\partial t} + 2(\nu - p)x\frac{\partial}{\partial x} + (\nu - \lambda)\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u}.$$

Subcase 4.2: $G(u) = \varepsilon_z u^{\lambda},\ K(u) = \ln u$.

In this subcase using equations (5.13)-(5.19) we have

$$X = 2c_1(\lambda x + t) - 2c_2(\nu x + t) + c_3, \quad T = 2c_1\lambda t - 2c_2\nu t + c_4, \quad Y = c_1(\lambda - \nu)y + c_5,$$

$$Z = c_2(\lambda - \nu)z + c_6, \quad U = -2(c_1 - c_2)u.$$

Hence, the Lie algebra is six-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_{17} = 2\lambda t\frac{\partial}{\partial t} + 2(\lambda x + t)\frac{\partial}{\partial x} + y(\lambda - \nu)\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u},$$

$$\Gamma_{18} = -2\nu t\frac{\partial}{\partial t} - 2(\nu x + t)\frac{\partial}{\partial x} + z(\lambda - \nu)\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u}.$$

Subcase 4.3: $G(u) = \varepsilon_z u^{\nu},\ K(u) = u^{p}$.

Here from (5.13)-(5.19) we get

$$X = 2c_1 px + c_2(\nu - p)x + c_3, \quad T = c_2\nu t + c_4, \quad Y = c_1\nu y + c_5 z + c_6,$$

$$Z = c_1\nu z - c_5\varepsilon_z y + c_7, \quad U = 2(c_1 - c_2)u.$$

The Lie algebra is seven-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_6, \quad \Gamma_{19} = 2px\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \nu z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u},$$

$$\Gamma_{20} = \nu t\frac{\partial}{\partial t} + x(\nu - p)\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

Subcase 4.4: $G(u) = \varepsilon_z u^{\nu},\ K(u) = \ln u$.

Substituting the above forms of G and K we deduce that

$$X = -2c_1 t + c_2(\nu x + t) + c_3, \quad T = c_2\nu t + c_4, \quad Y = c_1\nu y + c_5 z + c_6,$$

$$Z = c_1\nu z - c_5\varepsilon_z y + c_7, \quad U = 2(c_1 - c_2)u.$$

So, the Lie algebra is

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_6, \quad \Gamma_{21} = -2t\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \nu z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u},$$

$$\Gamma_{22} = \nu t\frac{\partial}{\partial t} + (\nu x + t)\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}.$$

Case 5. $F(u) = u^{-1/2}$.

In this case from equations (5.13)-(5.19) we get that $G(u)=\varepsilon u^{-1/2},\ K(u)=u^{-1/2}$ and

$$X=c_1x+c_2x+c_3,\quad T=2c_1t+c_4t+c_5,\quad Y=c_1y+c_6z+c_7,$$

$$Z=c_1z-c_6\varepsilon_z y+c_8,\quad U=(-c_1+2c_4)u-2c_2u^{\frac{1}{2}}.$$

Hence, the Lie algebra is eight-dimensional spanned by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_6,\quad \Gamma_{23}=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}-u\frac{\partial}{\partial u},$$

$$\Gamma_{24}=t\frac{\partial}{\partial t}+2u\frac{\partial}{\partial u},\quad \Gamma_{25}=x\frac{\partial}{\partial t}-2u^{1/2}\frac{\partial}{\partial u}.$$

Case 6. $F(u)=\frac{e^{\int\frac{\mu_1\mathrm{d}u}{(\kappa+p-r)u}}}{(\kappa+p-r)u}$, where $u=(\kappa^2+2p\kappa+q)^{-1/2}e^{\int\frac{r\mathrm{d}u}{\kappa^2+2p\kappa+q}}$.

Substituting the above form of function F we get the following subcases:

Subcase 6.1: $G(u)=\varepsilon_z\frac{e^{\int\frac{\mu_2\mathrm{d}u}{(\kappa+p-r)u}}}{(\kappa+p-r)u},\ K(u)=[u\kappa(u)]'$.

In this subcase we have

$$X=2c_1x+c_2\left[(p+r+\mu_1)x-qt\right]+c_3,\quad T=2c_1t+c_2\left[x+(r+\mu_1-p)t\right]+c_4,$$

$$Y=c_1y+c_5,\quad Z=c_1z+\frac{c_2}{2}(\mu_1-\mu_2)z+c_6,\quad U=c_2(r-p-\kappa)u.$$

The Lie algebra is six-dimensional given by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_5,$$

$$\Gamma_{26}=\left[x+(r+\mu_1-p)t\right]\frac{\partial}{\partial t}+\left[(p+r+\mu_1)x-qt\right]\frac{\partial}{\partial x}+\frac{1}{2}(\mu_1-\mu_2)z\frac{\partial}{\partial z}+(r-p-\kappa)u\frac{\partial}{\partial u}.$$

Subcase 6.2: $G(u)=\varepsilon_z\frac{e^{\int\frac{\mu_1\mathrm{d}u}{(\kappa+p-r)u}}}{(\kappa+p-r)u},\ K(u)=[u\kappa(u)]'$.

Using equations (5.13)-(5.19) we get

$$X=2c_1x+c_2\left[(p+r+\mu_1)x-qt\right]+c_3,\quad T=2c_1t+c_2\left[x+(r+\mu_1-p)t\right]+c_4,$$

$$Y=c_1y+c_5z+c_6,\quad Z=c_1z-c_5\varepsilon_z y+c_7,\quad U=c_2(r-p-\kappa)u.$$

So, the Lie algebra is

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_4,\quad \Gamma_5,\quad \Gamma_6,$$

$$\Gamma_{27}=\left[x+(r+\mu_1-p)t\right]\frac{\partial}{\partial t}+\left[(p+r+\mu_1)x-qt\right]\frac{\partial}{\partial x}+(r-p-\kappa)u\frac{\partial}{\partial u}.$$

Case 7. $F(u) = 1$.

In this case from equations (5.13)-(5.19) we get that $G(u) = \varepsilon_z,\ K(u) = u$ and

$$X = c_1 x + c_2 t + c_3 t + c_4, \quad T = 2c_1 t + c_5, \quad Y = c_1 y + c_6 z + c_7,$$

$$Z = c_1 z - c_6 \varepsilon_z y + c_8, \quad U = -c_1 u - c_2 + c_3 u.$$

Therefore, the Lie algebra is eight-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4, \quad \Gamma_6, \quad \Gamma_{23}, \quad \Gamma_{28} = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, \quad \Gamma_{29} = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

In Table 5.2 we summarize the Lie symmetries for the different forms of F, G and K.

Table 5.2: Group classification of class (5.1), $D = 0$

N	$F(u)$	$G(u)$	$K(u)$	$A^{\max}$
1	$\forall$	$\forall$	$\forall$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\rangle$
2	$\forall$	$\varepsilon_z F$	$\forall$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}\rangle$
3	$e^{\nu u}$	$\varepsilon_z e^{\lambda u}$	e^{pu}	$A^{\ker} + \langle 2px\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \lambda z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u},$ $2\nu t\frac{\partial}{\partial t} + 2x(\nu - p)\frac{\partial}{\partial x} + z(\nu - \lambda)\frac{\partial}{\partial z} - 2\frac{\partial}{\partial u}\rangle$
4	$e^{\nu u}$	$\varepsilon_z e^{\lambda u}$	u	$A^{\ker} + \langle -2t\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \lambda z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u},$ $2\nu t\frac{\partial}{\partial t} + 2(\nu x + t)\frac{\partial}{\partial x} + z(\nu - \lambda)\frac{\partial}{\partial z} - 2\frac{\partial}{\partial u}\rangle$
5	$e^{\nu u}$	$\varepsilon_z e^{\nu u}$	e^{pu}	$A^{\ker} + \langle 2px\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \nu z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u},$ $\nu t\frac{\partial}{\partial t} + x(\nu - p)\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}\rangle$
6	e^{u}	$\varepsilon_z e^{u}$	u	$A^{\ker} + \langle -2t\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2\frac{\partial}{\partial u},$ $t\frac{\partial}{\partial t} + (x + t)\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}\rangle$
7	u^{ν}	$\varepsilon_z u^{\lambda}$	u^{p}	$A^{\ker} + \langle 2\lambda t\frac{\partial}{\partial t} + 2x(\lambda - p)\frac{\partial}{\partial x} + y(\lambda - \nu)\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u},$ $2\nu t\frac{\partial}{\partial t} + 2x(\nu - p)\frac{\partial}{\partial x} + z(\nu - \lambda)\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u}\rangle$
8	u^{ν}	$\varepsilon_z u^{\lambda}$	$\ln u$	$A^{\ker} + \langle 2\lambda t\frac{\partial}{\partial t} + 2(\lambda x + t)\frac{\partial}{\partial x} + y(\lambda - \nu)\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u},$ $-2\nu t\frac{\partial}{\partial t} - 2(\nu x + t)\frac{\partial}{\partial x} + z(\lambda - \nu)\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u}\rangle$
9	u^{ν}	$\varepsilon_z u^{\nu}$	u^{p}	$A^{\ker} + \langle 2px\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \nu z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u},$ $\nu t\frac{\partial}{\partial t} + x(\nu - p)\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}\rangle$
10	u^{ν}	$\varepsilon_z u^{\nu}$	$\ln u$	$A^{\ker} + \langle -2t\frac{\partial}{\partial x} + \nu y\frac{\partial}{\partial y} + \nu z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u},$ $\nu t\frac{\partial}{\partial t} + (\nu x + t)\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}, z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}\rangle$
11	$\frac{e^{\int \frac{\mu_1 \mathrm{du}}{(\kappa+p-r)u}}}{(\kappa+p-r)u}$	$\varepsilon_z \frac{e^{\int \frac{\mu_2 \mathrm{du}}{(\kappa+p-r)u}}}{(\kappa+p-r)u}$	$[u\kappa(u)]'$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, [x + (r + \mu_1 - p)t]\frac{\partial}{\partial t} +$ $+[(p + r + \mu_1)x - qt]\frac{\partial}{\partial x} + \frac{1}{2}(\mu_1 - \mu_2)z\frac{\partial}{\partial z} + (r - p - \kappa)u\frac{\partial}{\partial u}\rangle$
12	$\frac{e^{\int \frac{\mu_1 \mathrm{du}}{(\kappa+p-r)u}}}{(\kappa+p-r)u}$	$\varepsilon_z \frac{e^{\int \frac{\mu_1 \mathrm{du}}{(\kappa+p-r)u}}}{(\kappa+p-r)u}$	$[u\kappa(u)]'$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, [x + (r + \mu_1 - p)t]\frac{\partial}{\partial t} +$ $+[(p + r + \mu_1)x - qt]\frac{\partial}{\partial x} + (r - p - \kappa)u\frac{\partial}{\partial u}, z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}\rangle$
13a	1	ε_z	u	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}, t\frac{\partial}{\partial x} - \frac{\partial}{\partial u},$ $x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}\rangle$
13b	$u^{-1/2}$	$\varepsilon u^{-1/2}$	$u^{-1/2}$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}, t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u},$ $z\frac{\partial}{\partial y} - \varepsilon_z y\frac{\partial}{\partial z}, x\frac{\partial}{\partial t} - 2u^{1/2}\frac{\partial}{\partial u}\rangle$

Here $\varepsilon_z = \pm 1$; $p \neq 0$; $\nu = 0, 1$ in cases 3, 4, 5; in cases 11 and 12 $u = (\kappa^2 + 2p\kappa + q)^{-1/2} e^{\int \frac{r \mathrm{du}}{\kappa^2 + 2p\kappa + q}}$

Additional equivalence transformations

$13b \longmapsto 13a : \tilde{t} = -x,\ \tilde{x} = -2t,\ \tilde{y} = y,\ \tilde{z} = z,\ \tilde{u} = 2u^{1/2}.$

In order to reduce an equation from the class (5.1) into an ordinary differential equation, we need three-dimensional solvable subalgebras. For example, equation

$$u_t = (u^{-2}u_x)_x - (u^{-2}u_y)_y + (u^{-2}u_z)_z + u^{-1}u_x,$$

which is a special case of equation in Table 5.1.14, admits five Lie symmetries

$$\Gamma_1 = \frac{\partial}{\partial t},\ \Gamma_2 = \frac{\partial}{\partial x},\ \Gamma_3 = \frac{\partial}{\partial y},\ \Gamma_4 = \frac{\partial}{\partial z},\ \Gamma_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}.$$

The three-dimensional (solvable) subalgebra $\langle \Gamma_5, \Gamma_1, \Gamma_2 + \delta\Gamma_3 \rangle$ produces the similarity reduction

$$u = (y - \delta x)^{-1}\phi(\xi), \quad \xi = z(y - \delta x)^{-1},$$

that transforms the above partial differential equations in four independent variables to the ordinary differential equation

$$\phi\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} - 2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right)^2 + \xi\phi^2\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \phi^3 = 0.$$

Chapter 6

Lie Group Classification of Systems of Diffusion Equations

We consider the class of systems of diffusion equations of the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\left[f(u,v)\frac{\partial u}{\partial x}\right],\\ \frac{\partial v}{\partial t} &= \frac{\partial}{\partial x}\left[g(u,v)\frac{\partial v}{\partial x}\right],\end{aligned} \tag{6.1}$$

where f and g are arbitrary non-zero smooth functions in their arguments, both equations are nonlinear, and $f_v^2 + g_u^2 \neq 0$. Special cases of this class of equations have been used to model successfully physical situations, such as transport in porous media with variable transmissivity [26] and river pollution [46]. Special cases of (6.1) have also applications in Plasma Physics [64,65]. A number of diffusion type systems of equations has been studied using Lie group analysis by Nikitin [50–52]. Also the special case $f = f(u),\ g = g(v)$ can by found in [33,43]. In this chapter we perform the complete classification of the class of systems of diffusion equations (6.1). Then using some of the obtained Lie symmetries, we construct exact invariant solutions [23].

6.1 Classification of Lie Symmetries

For the class of systems of diffusion equations of the form (6.1) we determine infinitesimal transformations of the form

$$x' = x + \epsilon X(x,t,u,v) + O(\epsilon^2),$$

$$t' = t + \epsilon T(x,t,u,v) + O(\epsilon^2),$$

$$u' = u + \epsilon U(x,t,u,v) + O(\epsilon^2),$$

$$v' = v + \epsilon V(x,t,u,v) + O(\epsilon^2).$$

That is, we search for such transformations that leave system (6.1) invariant.

We have seen that a PDE of second order, admits Lie symmetries if and only if

$$\Gamma^{(2)} E|_{E=0} = 0.$$

So the system (6.1) admits Lie symmetries if and only if

$$\Gamma^{(2)} \left\{ u_t - f u_{xx} - f_u u_x^2 - f_v u_x v_x \right\} = 0, \qquad (6.2)$$
$$\Gamma^{(2)} \left\{ v_t - g v_{xx} - g_u u_x v_x - g_v v_x^2 \right\} = 0,$$

where $u_t = f u_{xx} + f_u u_x^2 + f_v u_x v_x$ and $v_t = g v_{xx} + g_u u_x v_x + g_v v_x^2$.

After elimination of u_t and v_t using the above expressions, equations (6.2) become identities in the variables $t, x, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xt}$ and v_{xt}. In fact, these identities are multi-variable polynomials in $u_x,\ v_x,\ u_{xx},\ v_{xx},\ u_{xt}$ and v_{xt}. The coefficients of different powers of these variables must be zero, giving the determining equations for the four unknown functions $T,\ X,\ U$ and V and also for the functions $f(u,v)$ and $g(u,v)$.

Since equations (6.1) are polynomial in the pure derivatives of u and v with respect to x, using Theorem 2.3 we deduce that $T = T(t)$ and $X = X(x,t)$. Therefore, equations (6.2) are simplified. Now the coefficient of v_{xx} in the first equation and u_{xx} in the second equation of Eqs. (6.2) give respectively,

$$U_v(f-g) = 0, \qquad V_u(f-g) = 0. \qquad (6.3)$$

Therefore, we can split the analysis into two cases:

A. $f \neq g$;

B. $f = g$.

Case A. $f \neq g$.

From equations (6.3) we deduce that $U = U(t,x,u)$ and $V = V(t,x,v)$ and from the first invariant condition in (6.2) that the coefficients of $u_x^2,\ u_x v_x,\ u_{xx},\ u_x,\ v_x$ and the term independent of derivatives give the following determining equations respectively:

$$U_{uu}f + (T_t + U_u - 2X_x)f_u + Vf_{uv} + Uf_{uu} = 0, \tag{6.4}$$

$$(T_t + V_v - 2X_x)f_v + Uf_{uv} + Vf_{vv} = 0, \tag{6.5}$$

$$Uf_u + Vf_v + (T_t - 2X_x)f = 0, \tag{6.6}$$

$$2U_xf_u + V_xf_v + (2U_{ux} - X_{xx})f + X_t = 0, \tag{6.7}$$

$$U_xf_v = 0, \tag{6.8}$$

$$U_t - U_{xx}f = 0. \tag{6.9}$$

Coefficients of v_x^2, u_xv_x, v_{xx}, v_x, u_x and the term independent of derivatives in the second equation in (6.2) give respectively:

$$V_{vv}g + (T_t + V_v - 2X_x)g_v + Ug_{uv} + Vg_{vv} = 0, \tag{6.10}$$

$$(T_t + U_u - 2X_x)g_u + Vg_{uv} + Ug_{uu} = 0, \tag{6.11}$$

$$(T_t - 2X_x)g + Ug_u + Vg_v = 0, \tag{6.12}$$

$$U_xg_u + 2V_xg_v + (2V_{vx} - X_{xx})g + X_t = 0, \tag{6.13}$$

$$V_xg_u = 0, \tag{6.14}$$

$$V_t - V_{xx}g = 0. \tag{6.15}$$

Using equations (6.4) and (6.6) we get $U_{uu} = 0$, so we can suppose that $U = uh_1(x,t) + h_2(x,t)$. Also from (6.10) and (6.12) we have $V_{vv} = 0$ that leads to $V = vh_3(x,t) + h_4(x,t)$. Using the form of V, equation (6.14) becomes

$$g_u(h_{3x}v + h_{4x}) = 0, \tag{6.16}$$

from we deduce the following two cases:

1. $g_u \neq 0$;
2. $g_u = 0$.

Case 1. $g_u \neq 0$.

Using the fact that $g_u \neq 0$ then from (6.16), from which we have $h_{3x} = h_{4x} = 0$, we deduce that functions h_3 and h_4 are only functions of t. Also from (6.15) we deduce that $h_{3t} = h_{4t} = 0$ so $h_3 = s_1$ and $h_4 = s_2$ where s_1 and s_2 are constants. Therefore equation (6.12) becomes

$$g_u(h_1u + h_2) + g_v(s_1v + s_2) = g(2X_x - T_t). \tag{6.17}$$

We can then suppose that g satisfies the following equation:

$$g_u(k_1u + k_2) + g_v(k_3v + k_4) = gk_5,$$

where $k_1,\ k_2,\ k_3,\ k_4,\ k_5$ are constants.

Therefore we have to solve the following system:

$$\frac{\mathrm{d}u}{k_1u + k_2} = \frac{\mathrm{d}v}{k_3v + k_4} = \frac{\mathrm{d}g}{k_5g}. \tag{6.18}$$

Equation (6.18) suggests the following forms of g:

(i) g arbitrary;

(ii) $g = u^nK(v + \epsilon \ln u)$;

(iii) $g = u^nK(\frac{v^m}{u})$;

(iv) $g = e^uK(v + \epsilon u)$.

However in the following analysis these forms of g lead to further special cases. Summarizing we have the following forms of g:

(i) g arbitrary;

(ii) $g = u^nK(v + \epsilon \ln u)$;

(iii) $g = u^nK(\frac{v^m}{u})$;

(iv) $g = e^uK(v + \epsilon u)$;

(v) $g = u^nv^m$;

(vi) $g = u^ne^v$;

(vii) $g = e^{u+v}$.

Subcase 1.1: g arbitrary.

In this subcase from (6.17) we get that $h_1 = h_2 = s_1 = s_2 = 0$ and $X_x = \frac{T_t}{2}$. So, $X = \frac{xT_t}{2} + a_1(t)$. From equation (6.13) we have $a_{1t} = T_{tt} = 0$, therefore $a_1 = s_3$ and $T = s_4t + s_5$. Finally,

$$X = c_1x + c_2, \quad T = 2c_1t + c_3, \quad U = 0, \quad V = 0.$$

So the Lie algebra is three-dimensional spanned by

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}.$$

Subcase 1.2: $g = u^nK(v + \epsilon \ln u)$.

Here from (6.12) after some calculations we deduce that $s_1 = h_2 = 0$, $h_1 = -\frac{s_2}{\epsilon}$ and $X = \frac{x}{2\epsilon}(\epsilon T_t - s_2 n) + a_1(t)$. Also from (6.13) we get that $a_1 = s_3$ and $T = s_4 t + s_5$. Finally from (6.6) we deduce that $f = u^n R(v + \epsilon \ln u)$. So,

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 n t + c_4, \quad U = -c_3 u, \quad V = c_3 \epsilon.$$

Therefore the Lie algebra is four-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_4 = nt\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + \epsilon\frac{\partial}{\partial v}.$$

Subcase 1.3: $g = u^n K\left(\frac{v^m}{u}\right)$.

In this subcase from equation (6.12) we deduce that $h_2 = s_2 = 0$, $h_1 = s_1 m$ and $X = \frac{x}{2}\left(T_t + s_1 m n\right) + a_1(t)$. From equation (6.13) we get that $a_1 = s_3$ and $T = s_4 t + s_5$. Also from (6.6) we deduce that $f = u^n R\left(\frac{v^m}{u}\right)$. So,

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 m n t + c_4, \quad U = -c_3 m u, \quad V = -c_3 v,$$

and the Lie algebra is four-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_5 = mnt\frac{\partial}{\partial t} - mu\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.$$

Subcase 1.4: $g = e^u K\left(e^{nu} v\right)$.

In this subcase from equation (6.12) we have that $s_2 = h1 = 0$, $h_2 = -\frac{s_1}{n}$ and $X = \frac{x}{2n}\left(nT_t - c_1\right) + a_1(t)$. From (6.13) we get $a_1 = s_3$ and $T = s_4 t + s_5$, while from (6.6) we deduce that $F = e^u R(e^{nu} v)$. Therefore,

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 t + c_4, \quad U = -c_3, \quad V = c_3 n v.$$

So, the Lie algebra is four-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_6 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} + nv\frac{\partial}{\partial v}.$$

Subcase 1.5: $g = e^u K(v + \epsilon u)$.

In this subcase from (6.12) we get $h_1 = s_1 = 0$, $h_2 = -\frac{s_2}{\epsilon}$ and $X = \frac{x}{2\epsilon}(\epsilon T_t - s_2) + a_1(t)$. Also from (6.13) we have $a_1 = s_3$ and $T = s_4 t + s_5$. Finally from equation (6.6) we deduce that $f = e^u R(v + \epsilon u)$. So,

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 t + c_4, \quad U = -c_3, \quad V = \epsilon c_3.$$

The Lie algebra is four-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_7 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} + \epsilon\frac{\partial}{\partial v}.$$

Subcase 1.6: $g = u^n v^m$.

Using the fact that $g = u^n v^m$ then from (6.12) we have $s_2 = h_2 = 0$, $h_1 = \frac{1}{n}(2X_x - T_t - s_1 m)$ and from (6.13) and (6.9) we get respectively, $X = s_3 x + s_4$, $T = s_5 t + s_6$. Also from (6.6) we deduce that $f = cu^n v^m$.

So, after some change in the constants we have

$$X = c_1 x + c_2 m x + c_3 n x + c_4, \quad T = 2c_1 t + c_5, \quad U = 2c_3 u, \quad V = 2c_2 v.$$

Therefore, the Lie algebra is five-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_8 = mx\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}, \quad \Gamma_9 = nx\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}.$$

Subcase 1.7: $g = u^n e^v$.

In this subcase from (6.12) we have $s_1 = h_2 = 0$, $h_1 = \frac{1}{n}(2X_x - T_t - s_2)$, from (6.13) we get $X = s_3 x + s_4$ and from (6.9) $T = s_5 t + s_6$. Also from (6.6) we deduce that $f = cu^n e^v$. So,

$$X = c_1 x + c_2 x + c_3 n x + c_4, \quad T = 2c_1 t + c_5, \quad U = 2c_3 u, \quad V = 2c_2,$$

and the Lie algebra is

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_9, \quad \Gamma_{10} = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial v}.$$

Subcase 1.8: $g = e^{u+v}$.

Using equations (6.12), (6.13) and (6.9) after some calculations we get $s_1 = h_1 = 0$, $h_2 = 2X_x - T_t - s_2$, $X = s_3 x + s_4$ and $T = s_5 t + s_6$. Also from (6.6) we deduce that $f = ce^{u+v}$. Therefore,

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 t + c_4 t + c_5, \quad U = -c_3 u, \quad V = -c_4 v,$$

and the Lie algebra is five-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{11} = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}, \quad \Gamma_{12} = t\frac{\partial}{\partial t} - \frac{\partial}{\partial v}.$$

Case 2. $g_u = 0$.

Since $g = g(v)$, equation (6.12) can be written as

$$(vh_3(x,t) + h_4(x,t))\, g_v + (T_t - 2X_x)g = 0. \tag{6.19}$$

We then suppose that equation (6.19) has the following form:

$$(k_1 v + k_2)g_v + k_3 k = 0. \tag{6.20}$$

where $k_1,\ k_2,\ k_3$ are constants.

Equation (6.20) suggests the following forms of $g(v)$:

(i) $g(v)$ arbitrary;

(ii) $g(v) = cv^m$;

(iii) $g(v) = ce^v$.

Subcase 2.1: $g(v)$ arbitrary.

From (6.19) we get that $h_3 = h_4 = 0$ and $X = \frac{x}{2}T_t + a_1(t)$. Also from (6.13) we deduce that $T = s_1 t + s_2$ and $a_1 = s_3$. Then from (6.7) we have $f = f(v)$ and from equations (6.8) and (6.9) we get that $h_1 = s_4$ and $h_2 = s_5$. So,

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_2, \quad U = c_4 u + c_5, \quad V = 0.$$

Therefore, the Lie algebra is five-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{13} = \frac{\partial}{\partial u}, \quad \Gamma_{14} = u\frac{\partial}{\partial u}.$$

Subcase 2.2: $g = v^m$.

In this subcase using equations (6.12), (6.13) and (6.15) we have $h_4 = 0$, $h_3 = \frac{1}{m}(2X_x - T_t)$, $X = s_1 x + s_2$, $T = s_3 t + s_4$. Also from (6.6) we deduce that $f = cv^m$. So, after some change of constants we have

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 m t + c_4, \quad U = c_5 u + c_6, \quad V = -c_3 v.$$

Hence, the Lie algebra is given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{13}, \quad \Gamma_{14}, \quad \Gamma_{15} = mt\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}.$$

Subcase 2.3: $g = e^v$.

Using the fact that $g = v^m$ then from equations (6.12), (6.13) and (6.15) we have $h_3 = 0$, $h_4 = 2X_x - T_t$, $X = s_1 x + s_2$ and $T = s_3 t + s_4$. Also from equations (6.6), (6.8) and (6.9) we deduce that $f = ce^v$, $h_1 = s_5$ and $h_2 = s_6$. So,

$$X = c_1 x + c_2, \quad T = 2c_1 t + c_3 t + c_4, \quad U = c_5 u + c_6, \quad V = -c_3,$$

and the Lie algebra is six-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{12}, \quad \Gamma_{13}, \quad \Gamma_{14}.$$

In the following table we summarize the Lie symmetries for the different forms of f and g in the case when $f \neq g$.

Table 6.1: Group classification of class (6.1) if $f \neq g$

N	$f(u,v)$	$g(u,v)$	$A^{\max}$
1	$\forall$	$\forall$	$A^{\ker} = \langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} \rangle$
2	$u^n R(v + \varepsilon \ln u)$	$u^n K(v + \varepsilon \ln u)$	$A^{\ker} + \langle nt\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + \varepsilon\frac{\partial}{\partial v} \rangle$
3	$u^n R(v^m/u)$	$u^n K(v^m/u)$	$A^{\ker} + \langle mnt\frac{\partial}{\partial t} - mu\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \rangle$
4	$e^u R(e^{nu} v)$	$e^u K(e^{nu} v)$	$A^{\ker} + \langle t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} + nv\frac{\partial}{\partial v} \rangle$
5	$e^u R(v + \varepsilon u)$	$e^u K(v + \varepsilon u)$	$A^{\ker} + \langle t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} + \varepsilon\frac{\partial}{\partial v} \rangle$
6	$u^n v^m$	$cu^n v^m$	$A^{\ker} + \langle mx\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}, nx\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} \rangle$
7	$u^n e^v$	$cu^n e^v$	$A^{\ker} + \langle x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial v}, nx\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} \rangle$
8	e^{u+v}	ce^{u+v}	$A^{\ker} + \langle t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}, t\frac{\partial}{\partial t} - \frac{\partial}{\partial v} \rangle$
9	$R(v)$	$K(v)$	$A^{\ker} + \langle \frac{\partial}{\partial u}, u\frac{\partial}{\partial u} \rangle$
10	v^m	cv^m	$A^{\ker} + \langle \frac{\partial}{\partial u}, u\frac{\partial}{\partial u}, mt\frac{\partial}{\partial t} - v\frac{\partial}{\partial v} \rangle$
11	e^v	ce^v	$A^{\ker} + \langle \frac{\partial}{\partial u}, u\frac{\partial}{\partial u}, t\frac{\partial}{\partial t} - \frac{\partial}{\partial v} \rangle$

Here $\varepsilon = 0, 1$, $c \neq 0, 1$, $n \neq 0$, $m \neq 0$; R, K are arbitrary functions of their variables, $K \neq R$.

Case B. $f = g$.

As in the previous case from the first invariant condition in (6.2) the coefficients of $u_x^2,\ u_xv_x,\ u_{xx},\ u_x,\ v_x^2,\ v_x$ and the term independent of derivatives give respectively:

$$U_{uu}f + (T_t + U_u - 2X_x)f_u + V_uf_v + Vf_{uv} + Uf_{uu} = 0, \tag{6.21}$$

$$2U_{uv}f + U_vf_u + (T_t + V_v - 2X_x)f_v + Uf_{uv} + Vf_{vv} = 0, \tag{6.22}$$

$$(T_t - 2X_x)f + Uf_u + Vf_v = 0, \tag{6.23}$$

$$2U_xf_u + V_xf_v + (2U_{ux} - X_{xx})f + X_t = 0, \tag{6.24}$$

$$U_{vv}f = 0, \tag{6.25}$$

$$2U_{vx}f + U_xf_v = 0, \tag{6.26}$$

$$U_t - U_{xx}f = 0. \tag{6.27}$$

Also, from the second invariant condition in (6.2) the coefficients of $v_x^2,\ u_xv_x,\ v_{xx},\ u_x^2,\ u_x$ and the term independent of derivatives give respectively:

$$V_{vv}f + U_vf_u + (T_t + V_v - 2X_x)f_v + Uf_{uv} + Vf_{vv} = 0, \tag{6.28}$$

$$2V_{uv}f + V_uf_v + (T_t + U_u - 2X_x)f_u + Vf_{uv} + Uf_{uu} = 0, \tag{6.29}$$

$$U_xf_u + 2V_vf_v + (2V_{vx} - X_{xx})f + X_t = 0, \tag{6.30}$$

$$V_{uu}f = 0, \tag{6.31}$$

$$2V_{ux}f + V_xf_u = 0, \tag{6.32}$$

$$V_t - V_{xx}f = 0. \tag{6.33}$$

The solution of the above system with thirteen equations provides the desired classification of Lie symmetries for the class (6.1) if $f = g$.

Specifically, using equations (6.21), (6.23) and (6.25) we deduce that $U_{uu} = U_{vv} = U_{uv} = 0$. So,

$$U = a_1(x,t)u + a_2(x,t)v + a_3(x,t).$$

Also from equations (6.28), (6.23) and (6.31) we have $V_{vv} = V_{uu} = V_{uv} = 0$. Therefore,

$$V = a_4(x,t)u + a_5(x,t)v + a_6(x,t).$$

Substituting the above expressions of U and V into equations (6.21)-(6.33), equation (6.26) becomes,

$$f_v(a_{1x}u + a_{2x}v + a_{3x}) + 2a_{2x}f = 0. \tag{6.34}$$

We can then suppose that (6.34) can be written in the following form:

$$f_v(\lambda_1 u + \lambda_2 v + \lambda_3) + 2\lambda_2 f = 0.$$

where $\lambda_1,\ \lambda_2,\ \lambda_3$ are constants.

In order to solve the above equation we just have to solve the following system:

$$\frac{\mathrm{d}u}{0} = \frac{\mathrm{d}}{\lambda_1 u + \lambda_2 v + \lambda_3} = \frac{\mathrm{d}f}{-2\lambda_2 f}. \tag{6.35}$$

The above system suggests that the analysis must be split into three cases:

1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$;
2. $\lambda_2 = 0$;
3. $\lambda_2 \neq 0$.

Case 1. $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

In this case from equation (6.27) we have $a_1 = s_2,\ a_2 = s_2,\ a_3 = s_3$ where $s_1,\ s_2,\ s_3$ are constants. So, equation (6.23) becomes

$$f_u(s_1 u + s_2 v + s_3) + f_v(a_4 u + a_5 v + a_6) + f(T_t - 2X_x) = 0. \tag{6.36}$$

Equation (6.36) suggests a PDE of the form

$$(au + bv + c)f_u + (mu + nv + r)f_v + pf = 0,$$

where $a,\ b,\ c,\ m,\ n,\ r$ and p are constants and $m,\ n \neq 0$.

Solution of this PDE leads to the following forms of f:

1. f arbitrary;

Certain forms of f admit additional symmetries.

2. $f = u^n(v + \epsilon u)^m$;
3. $f = e^u(v + \epsilon u)^m$;
4. $f = u^n e^{v+\epsilon u}$;
5. $f = e^{u^2} e^{v+\epsilon u}$;
6. $f = R(v + \epsilon u)$;
7. $f = (v + \epsilon u)^m$;
8. $f = e^{v+\epsilon u}$.

Subcase 1.1: f arbitrary solution of $(au + bv + c)f_u + (mu + nu + r)f_v + pf = 0$.

In this subcase using equations (6.21)-(6.33) we deduce that

$$X = c_1x + c_2, \quad T = 2c_1t + c_4pt + c_3, \quad U = c_4(au + bv + c), \quad V = c_4(mu + nv + r).$$

So, the Lie algebra is four-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_9 = pt\frac{\partial}{\partial t} + (au + bv + c)\frac{\partial}{\partial u} + (mu + nv + r)\frac{\partial}{\partial v}.$$

Subcase 1.2: $f = u^n(v + \epsilon u)^m$.

In this subcase from equation (6.23) we have $a_4 = \frac{\epsilon}{m}\left(-T_t + 2X_x - s_1m - s_1n\right)$, $s_2 = 0,\ a_5 = \frac{1}{m}\left(-T_t + 2X_x - s_1n\right)$ and $s_3 = a_6 = 0$. Finally from (6.24) and (6.33) we get respectively $X = s_4x + s_5$ and $T = s_6t + s_7$. After some changes in the constants $s_1, ..., s_7$ we have

$$X = c_1x + c_2mx + c_3, \quad T = 2c_1t + c_4, \quad U = c_5mu, \quad V = 2c_2(v + \epsilon u) - c_5(nv + \epsilon(m + n)u).$$

The Lie algebra is five-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{10} = mx\frac{\partial}{\partial x} + 2(v + \epsilon u)\frac{\partial}{\partial v}, \quad \Gamma_{11} = mu\frac{\partial}{\partial u} - (nv + \epsilon(m + n)u)\frac{\partial}{\partial v}.$$

Subcase 1.3: $f = e^u(v + \epsilon u)^m$.

Here from (6.23) we have $s_2 = 0,\ a_4 = \frac{1}{m}\left(-\epsilon T_t + 2\epsilon X_x - s_1\epsilon m - s_1v - \epsilon s_3m\right)$, $a_5 = \frac{1}{m}\left(-T_t + 2X_x - s_3\right)$ and $a_6 = -\epsilon s_3$. Also from (6.24) and (6.33) we get respectively $X = s_4x + s_5$ and $T = s_6t + s_7$ while from (6.29) we deduce that $s_1 = 0$. So,

$$X = c_1x + c_2mx + c_3x + c_4, \quad T = 2c_1t + c_5, \quad U = 2c_3, \quad V = 2c_2(v + \epsilon u) - 2c_3\epsilon.$$

Therefore, the Lie algebra is five-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{10}, \quad \Gamma_{12} = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u} - 2\epsilon\frac{\partial}{\partial v}.$$

Subcase 1.4: $f = u^ne^{v+\epsilon u}$.

In this subcase from (6.23) we deduce that $a_4 = -\epsilon s_1,\ a_5 = s_2 = s_3 = 0$, $a_6 = -T_t + 2X_x - s_1n$ and from (6.24) and (6.33) we get respectively $X = s_4x + s_5$ and $T = s_6t + s_7$. After a change in the constants $s_1, ...s_7$ we have

$$X = c_1x + c_2x + c_3nx + c_4, \quad T = 2c_1t + c_5, \quad U = 2c_3u, \quad V = 2c_2 - 2\epsilon c_3u.$$

The Lie algebra is five-dimensional given by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{13} = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial v}, \quad \Gamma_{14} = nx\frac{\partial}{\partial x} + 2u\left(\frac{\partial}{\partial u} - \epsilon\frac{\partial}{\partial v}\right).$$

Subcase 1.5: $f = e^{u^2}e^{v+\epsilon u}$.

In this subcase from (6.23) we have $s_1 = s_2 = 0,\ a_4 = -2s_3,\ a_5 = 0$ and $a_6 = -T_t + 2X_x - \epsilon s_3$. Also from (6.24) and (6.33) we deduce that $X = s_4x + s_5$ and $T = s_6t + s_7$. After a change of constants $s_3, ...s_7$ we deduce that

$$X = c_1x + c_2, \quad T = 2c_1t + \epsilon c_3t + c_4t + c_5, \quad U = -c_3, \quad V = 2c_3u - c_4,$$

and the Lie algebra is

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{15} = \epsilon t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} + 2u\frac{\partial}{\partial v}, \quad \Gamma_{16} = t\frac{\partial}{\partial t} - \frac{\partial}{\partial v}.$$

Subcase 1.6: $f = R(v + \epsilon u)$.

Using the fact that $f = R(v+\epsilon u)$ from (6.23) we have $a_4 = \epsilon a_5 - \epsilon s_1, +\epsilon^2 s_2,\ a_5 = -\epsilon s_2$, $a_6 = -\epsilon s_3$ and $X = \frac{x}{2}T_t + g_1(t)$. Finally from (6.24) we deduce that $T = s_4t + s_5$ and $g_1 = s_6$ where $s_1, ..., s_6$ are constants. After a change in the constants

$$X = c_1x + c_2, \quad T = 2c_1t + c_3, \quad U = c_5u + c_6v + c_4, \quad V = -\epsilon(c_5u + c_6v + c_4).$$

Therefore, the Lie algebra is six-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{17} = \frac{\partial}{\partial u} - \epsilon\frac{\partial}{\partial v}, \quad \Gamma_{18} = u\left(\frac{\partial}{\partial u} - \epsilon\frac{\partial}{\partial v}\right), \quad \Gamma_{19} = v\left(\frac{\partial}{\partial u} - \epsilon\frac{\partial}{\partial v}\right).$$

Subcase 1.7: $f = (v + \epsilon u)^m$.

In this subcase from (6.23) we get $a_6 = -\epsilon s_3,\ a_4 = \frac{1}{m}(-\epsilon T_t + 2\epsilon X_x - \epsilon s_1 m)$ and $a_5 = \frac{1}{m}(-T_t + 2X_x - \epsilon s_2 m)$. From (6.24) and (6.33) we get respectively $X = s_4x + s_5$ and $T = s_6t + s_7$ where $s_1, ..., s_7$ are constants. So,

$$X = c_1x+c_2, \ T = 2c_1t+c_3mt+c_4, \ U = c_5u+c_6v+c_7, \ V = -c_3(\epsilon u+v)-\epsilon(c_5u+c_6v+c_7).$$

The Lie algebra is seven-dimensional spanned by

$$\Gamma_1, \quad \Gamma_2, \quad \Gamma_3, \quad \Gamma_{17}, \quad \Gamma_{18}, \quad \Gamma_{19}, \quad \Gamma_{20} = mt\frac{\partial}{\partial t} - (\epsilon u + v)\frac{\partial}{\partial v}.$$

Subcase 1.8: $f = e^{v+\epsilon u}$.

Using the fact that $f = e^{v+\epsilon u}$ from (6.23) we have $a_5 = -\epsilon s_2,\ a_4 = -\epsilon s_1$ and $a_6 = -T_t + 2X_x - \epsilon s_3$. Equations (6.24) and (6.33) yield respectively $X = s_4 x + s_5$ and $T = s_6 t + s_7$ where $s_1, ... s_7$ are constants. So,

$$X = c_1 x + c_2,\ \ T = 2c_1 t + c_3 t + c_4,\ \ U = c_5 u + c_6 v + c_7,\ \ V = -\epsilon(c_5 u + c_6 v + c_7) - c_3,$$

and the seven-dimensional Lie algebra is spanned by

$$\Gamma_1,\ \ \Gamma_2,\ \ \Gamma_3,\ \ \Gamma_{16},\ \ \Gamma_{17},\ \ \Gamma_{18},\ \ \Gamma_{19}.$$

Case 2. $\lambda_1 = 0$.

In this case the results are the same as those of Case 1.

Case 3. $\lambda_2 \neq 0$.

From the system (6.35) we deduce that $f = \phi(u)(v + \epsilon u)^{-2}$. So, using equation (6.26) we get $a_3 = a_3(t)$ and $a_1 = \epsilon a_2 + g_1(t)$. Also from the coefficient of v^2 in equation (6.32) we deduce that $\phi(u) = \nu$ where ν is constant. So, using the fact that $f = \nu(v + \epsilon u)^{-2}$ then from equation (6.28) we have $a_5 = \frac{1}{2}(T_t - 2X_x - 2\epsilon a_2),\ a_6 = -\epsilon a_3$ and $a_4 = \frac{\epsilon}{2}(T_t - 2X_x - 2\epsilon a_2 - 2g_1)$. Also, from (6.30) we get $X = s_1 x + s_2$ and from (6.27) $a_3 = s_3,\ g_1 = s_4$ and $a_2 = s_5 x + s_6$. Finally equation (6.33) leads to $T = s_7 t + s_8$. Here $s_1, ..., s_8$ are constants and after some change of constants we have the following forms of $X,\ T,\ U$ and V:

$$X = c_1 x + c_2,\ \ T = 2c_1 t + 2c_3 + c_4,\ \ U = c_5 u + c_6(\epsilon u + v) + c_7 + c_8 x(\epsilon u + v),$$

$$V = c_3(\epsilon u + v) - c_5 \epsilon u - \epsilon c_6(\epsilon u + v) - \epsilon c_7 - \epsilon c_8 x(\epsilon u + v).$$

Therefore, the Lie algebra is eight-dimensional spanned by

$$\Gamma_1,\ \ \Gamma_2,\ \ \Gamma_3,\ \ \Gamma_{17},\ \ \Gamma_{18},\ \ \Gamma_{21} = 2t\frac{\partial}{\partial t} + (\epsilon u + v)\frac{\partial}{\partial v},$$

$$\Gamma_{22}(\epsilon u + v)\left(\frac{\partial}{\partial u} - \epsilon\frac{\partial}{\partial v}\right),\ \ \Gamma_{23} = x(\epsilon u + v)\left(\frac{\partial}{\partial u} - \epsilon\frac{\partial}{\partial v}\right).$$

In the following table we give briefly the Lie symmetries for the different forms of f in the case when $f = g$.

Table 6.2: Group classification of class (6.1) if $f = g$

N	$f(u,v)$	$A^{\max}$
1	$f^1(u,v)$	$A^{\ker} + \langle pt\frac{\partial}{\partial t} + (au+bv+c)\frac{\partial}{\partial u} + (mu+nv+r)\frac{\partial}{\partial v}\rangle$
2	$u^n(v+\varepsilon u)^m$	$A^{\ker} + \langle mx\frac{\partial}{\partial x} + 2(v+\varepsilon u)\frac{\partial}{\partial v}, mu\frac{\partial}{\partial u} - (nv+\varepsilon(m+n)u)\frac{\partial}{\partial v}\rangle$
3	$e^u(v+\varepsilon u)^m$	$A^{\ker} + \langle mx\frac{\partial}{\partial x} + 2(v+\varepsilon u)\frac{\partial}{\partial v}, x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u} - 2\varepsilon\frac{\partial}{\partial v}\rangle$
4	$u^n e^{v+\varepsilon u}$	$A^{\ker} + \langle x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial v}, nx\frac{\partial}{\partial x} + 2\left(u\frac{\partial}{\partial u} - \varepsilon u\frac{\partial}{\partial v}\right)\rangle$
5	$e^{u^2}e^{v+\epsilon u}$	$A^{\ker} + \langle \epsilon t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} + 2u\frac{\partial}{\partial v}, t\frac{\partial}{\partial t} - \frac{\partial}{\partial v}\rangle$
6	$R(v+\varepsilon u)$	$A^{\ker} + \langle \frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}, u\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right), v\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right)\rangle$
7	$(v+\varepsilon u)^m$	$A^{\ker} + \langle \frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}, u\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right), v\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right), mt\frac{\partial}{\partial t} - (\varepsilon u+v)\frac{\partial}{\partial v}\rangle$
8	$e^{v+\varepsilon u}$	$A^{\ker} + \langle \frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}, u\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right), v\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right), t\frac{\partial}{\partial t} - \frac{\partial}{\partial v}\rangle$
9	$(v+\varepsilon u)^{-2}$	$A^{\ker} + \langle 2t\frac{\partial}{\partial t} + (\varepsilon u+v)\frac{\partial}{\partial v}, u\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right), (\varepsilon u+v)\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right),$ $\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}, x(\varepsilon u+v)\left(\frac{\partial}{\partial u} - \varepsilon\frac{\partial}{\partial v}\right)\rangle$

Here $\varepsilon = 0,1$, $c \neq 0,1$, $n \neq 0$, $m \neq 0$;$f^1(u,v)$ is an arbitrary solution of the equation

$$(au+bv+c)f^1_u + (mu+nv+r)f^1_v + pf^1 = 0.$$

6.2 Similarity reductions

Lie symmetries of a system of differential equations can be used for the construction of exact solutions of the system. In particular, invariance with respect to a one-parameter group of symmetries leads to the reduction of the number of independent variables by one. For a case of two-dimensional systems in such way one obtains a reduced system of ordinary differential equations.

Consider the system

$$u_t = (v^m u_x)_x, \quad v_t = c(v^m v_x)_x, \tag{6.37}$$

which is a member of the class (6.1). This system admits nontrivial Lie symmetries found in previous section. Its Lie symmetry algebra is six-dimensional and spanned by the operators

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad \Gamma_4 = \frac{\partial}{\partial u}, \quad \Gamma_5 = u\frac{\partial}{\partial u}, \quad \Gamma_6 = mt\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}.$$

Any two conjugate subgroups of a Lie symmetry group of a system of differential equations give rise to reduced equations that are related by a conjugacy transformation in the point symmetry group of the system acting on the invariant solutions determined by each subgroup [55]. Hence, up to the action of the point symmetry transformations, all invariant solutions for a given system can be obtained by selecting a subgroup in each conjugacy class of all admitted one dimensional point symmetry subgroups. Such a selection is called an optimal set of one-dimensional subgroups. A set of subalgebras of the Lie algebra corresponding to the optimal set of subgroups consists of subalgebras inequivalent with respect to the actions of adjoint representation of the Lie symmetry group on its Lie algebra.

In Tables 6.3-6.6 we adduce results of classification of similarity reductions of system (6.37). More precisely, we give commutation relations for the Lie symmetries, present adjoint representations of the Lie symmetry group of system (6.37) on its Lie symmetry algebra, derive its optimal system of one-dimensional subalgebras and list the corresponding reduced systems.

Table 6.3: Commutator table for the Lie algebra

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	$2\Gamma_1$	0	0	$m\Gamma_1$
Γ_2	0	0	Γ_2	0	0	0
Γ_3	$-2\Gamma_1$	$-\Gamma_2$	0	0	0	0
Γ_4	0	0	0	0	Γ_4	0
Γ_5	0	0	0	$-\Gamma_4$	0	0
Γ_6	$-m\Gamma_1$	0	0	0	0	0

Table 6.4: Adjoint table for the Lie algebra

Ad	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	Γ_1	Γ_2	$\Gamma_3 - 2\epsilon\Gamma_1$	Γ_4	Γ_5	$\Gamma_6 - \epsilon m\Gamma_1$
Γ_2	Γ_1	Γ_2	$\Gamma_3 - \epsilon\Gamma_2$	Γ_4	Γ_5	Γ_6
Γ_3	$\Gamma_1\mathrm{e}^{2\epsilon}$	$\Gamma_2\mathrm{e}^{\epsilon}$	Γ_3	Γ_4	Γ_5	Γ_6
Γ_4	Γ_1	Γ_2	Γ_3	Γ_4	$\Gamma_5 - \epsilon\Gamma_4$	Γ_6
Γ_5	Γ_1	Γ_2	Γ_3	$\Gamma_4\mathrm{e}^{\epsilon}$	Γ_5	Γ_6
Γ_6	$\Gamma_1\mathrm{e}^{m\epsilon}$	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6

Table 6.5: Subalgebras $\langle\Delta_i\rangle$, similarity variables, similarity solutions

	$\langle\Delta_i\rangle$	ξ	similarity solution
1	$\langle\Gamma_3 + a_6\Gamma_6 + a_5\Gamma_5\rangle$	$x^{2+a_6m}t^{-1}$	$u = x^{a_5}\phi(\xi), \quad v = x^{-a_6}\psi(\xi)$
2	$\langle\Gamma_3 + a_6\Gamma_6 + \epsilon_4\Gamma_4\rangle$	$xt^{-\frac{1}{2+a_6m}}$	$u = \frac{\epsilon_4}{2+a_6m}\ln t + \phi(\xi), \quad v = t^{-\frac{a_6}{2+a_6m}}\psi(\xi)$
3	$\langle\Gamma_6 + a_5\Gamma_5 + \epsilon_2\Gamma_2\rangle$	$e^x t^{-\frac{\epsilon_2}{m}}$	$u = t^{\frac{a_5}{m}}\phi(\xi), \quad v = t^{-\frac{1}{m}}\psi(\xi)$
4	$\langle\Gamma_6 + \epsilon_2\Gamma_2 + \epsilon_4\Gamma_4$	$e^x t^{-\frac{\epsilon_2}{m}}$	$u = \frac{\epsilon_4}{m}\ln t + \phi(\xi), \quad v = t^{-\frac{1}{m}}\psi(\xi)$
5	$\langle\Gamma_5 + \delta_1\Gamma_1 + \epsilon_2\Gamma_2\rangle$	$x - \epsilon_2\delta_1 t$	$u = e^{\delta_1 t}\phi(\xi), \quad v = \psi(\xi)$
6	$\langle\Gamma_5 + \delta_2\Gamma_2\rangle$	t	$u = e^{\delta_2 x}\phi(\xi), \quad v = \psi(\xi)$
7	$\langle\Gamma_1 + \epsilon_2\Gamma_2 + \epsilon_4\Gamma_4\rangle$	$x - \epsilon_2 t$	$u = \epsilon_4 t + \phi(\xi), \quad v = \psi(\xi)$
8	$\langle\Gamma_2 + \epsilon_4\Gamma_4\rangle$	t	$u = \epsilon_4 x + \phi(\xi), \quad v = \psi(\xi)$

Table 6.6: Subalgebras $\langle\Delta_i\rangle$, reduced equations

	$\langle\Delta_i\rangle$	reduced equations
1	$\langle\Gamma_3 + a_6\Gamma_6 + a_5\Gamma_5\rangle$	$(2+a_6m)^2\xi^2\psi^m\phi_{\xi\xi} + a_5(a_5-1)\psi^m\phi +$ $(2a_5-1)(2+a_6m)\xi\psi^m\phi_\xi + \xi^2\phi_\xi + (2+a_6m)^2\xi\psi^m\phi_\xi = 0$ $c\xi^2\psi^m(2+a_6m)^2\psi_{\xi\xi} + cm\psi^{m-1}(-a_6\psi + (2+a_6m)\xi\psi_\xi)^2 + \xi^2\psi_\xi$ $+c\psi^m[a_6(a_6+1)\psi - (2a_6+1)(2+a_6m)\xi\psi_\xi + (2+a_6m)^2\xi\psi_\xi] = 0$
2	$\langle\Gamma_3 + a_6\Gamma_6 + \epsilon_4\Gamma_4\rangle$	$(2+a_6m)\psi^m\phi_{\xi\xi} + \xi\phi_\xi - \epsilon_4 = 0$ $(2+a_6m)c\psi^m\psi_{\xi\xi} + cm(2+a_6m)\psi^{m-1}\psi_\xi^2 + \xi\psi_\xi + a_6\psi = 0$
3	$\langle\Gamma_6 + a_5\Gamma_5 + \epsilon_2\Gamma_2\rangle$	$m\xi^2\psi^m\phi_{\xi\xi} + m\xi\psi^m\phi_\xi + \epsilon_2\xi\phi_\xi - a_5\phi = 0$ $cm\xi^2\psi^m\psi_{\xi\xi} + cm\xi\psi^m\psi_\xi + cm^2\xi^2\psi^{m-1}\psi_\xi^2 + \epsilon_2\xi\psi_\xi + \psi = 0$
4	$\langle\Gamma_6 + \epsilon_2\Gamma_2 + \epsilon_4\Gamma_4$	$m\xi^2\psi^m\phi_{\xi\xi} + m\xi\psi^m\phi_\xi + \epsilon_2\xi\phi_\xi - \epsilon_4 = 0$ $cm\xi^2\psi^m\psi_{\xi\xi} + cm\xi\psi^m\psi_\xi + cm^2\xi^2\psi^{m-1}\psi_\xi^2 - \epsilon_2\xi\psi_\xi + \psi = 0$
5	$\langle\Gamma_5 + \delta_1\Gamma_1 + \epsilon_2\Gamma_2\rangle$	$\psi^m\phi_{\xi\xi} + \epsilon_2\delta_1\phi_\xi - \delta_1\phi = 0$ $c\psi^m\psi_{\xi\xi} + cm\psi^{m-1}\psi_\xi^2 + \epsilon_2\delta_1\psi_\xi = 0$
6	$\langle\Gamma_5 + \delta_2\Gamma_2\rangle$	$\phi_\xi - \psi^m\phi = 0$ $\psi_\xi = 0$
7	$\langle\Gamma_1 + \epsilon_2\Gamma_2 + \epsilon_4\Gamma_4\rangle$	$\psi^m\phi_{\xi\xi} + \epsilon_2\phi_\xi - \epsilon_4 = 0$ $c\psi^m\psi_{\xi\xi} + cm\psi^{m-1}\psi_\xi^2 + \epsilon_2\psi_\xi = 0$
8	$\langle\Gamma_2 + \epsilon_4\Gamma_4\rangle$	$\psi^m\phi_{\xi\xi} - \phi_\xi = 0$ $\psi_\xi = 0$

Here a_i are arbitrary constants, $\epsilon_i = 0, \pm1$, $\delta_i = \pm1$.

Solutions of the reduced systems listed in Table 6.6 give rise to similarity solutions of the initial system. As examples we consider some cases in more detail.

Example 6.1. $\langle\Gamma_3 + a_6\Gamma_6 + \epsilon_4\Gamma_4\rangle$

For $a_6 = 1$ and $m = -1$, if we integrate once equation

$$c\psi^{-1}\psi_{\xi\xi} - c\psi^{-2}\psi_\xi^2 + \xi\psi_\xi + \psi = 0,$$

we get the following Riccati equation

$$\psi_\xi = A\xi\psi^2 + B\psi,$$

where $A = -\frac{1}{c}$, $B = \frac{c_1}{c}$ and c_1 is the integration constant. The solution of the Riccati equation is

$$\psi = \frac{B^2}{Ce^{B\xi} + AB\xi + A}, \quad B \neq 0; \qquad \psi = -\frac{2}{A\xi^2 + C}, \quad B = 0, \tag{6.38}$$

where C is an integration constant.

So the similarity solution of the initial system is

$$v = t^{-1}\psi(\xi), \quad u = \epsilon_4 \ln t + \phi(\xi),$$

where $\phi(\xi)$ is the solution of $\psi^m \phi_{\xi\xi} + \xi\phi_\xi - \epsilon_4 = 0$, $\psi(\xi)$ is given by (6.38) and $\xi = xt^{-1}$.

Example 6.2. $\langle \Gamma_5 + \delta_1\Gamma_1 + \epsilon_2\Gamma_2 \rangle$.

The similarity solution is $v = \psi(\xi)$, $u = e^{\delta_1 t}\phi(\xi)$, where $\phi(\xi)$ is the solution of

$$\psi^m \phi_{\xi\xi} + \epsilon_2\delta_1\phi_\xi - \delta_1\phi = 0,$$

and $\psi(\xi)$ is the solution of the following relation

$$\int \frac{c\psi^m \mathrm{d}\xi}{\epsilon_2\delta_1\psi + c_1} = \xi + c_2,$$

Here c_2 is the integration constant and $\xi = x - \epsilon_2\delta_1 t$.

Example 6.3. $\langle \Gamma_1 + \epsilon_2\Gamma_2 + \epsilon_4\Gamma_4 \rangle$.

If we integrate once equation

$$c\psi^m \psi_{\xi\xi} + cm\psi^{m-1}\psi_\xi^2 + \epsilon_2\psi_\xi = 0,$$

then we get $c\psi^m\psi_\xi + \epsilon_2\psi = c_2$, where c_2 is the integration constant. Taking $c_2 = 0$, we have

$$\psi(\xi) = \left[\frac{m}{c}(-\epsilon_2\xi + c_3)\right]^{\frac{1}{m}},$$

where c_3 is the integration constant. Substitution of this form of ψ into equation

$$\psi^m \phi_{\xi\xi} + \epsilon_2\phi_\xi - \epsilon_4 = 0,$$

leads us to two cases depending on the values that the constant c takes. In particular, we need to take $c \neq -m$ and $c = -m$.

If $c \neq -m$, then

$$\phi = \frac{\epsilon_4}{\epsilon_2}\xi - \frac{am}{\epsilon_2(c+1)}\left(-\epsilon_2\xi + c_3\right)^{\frac{c+m}{m}} + b,$$

where $c \neq -1$ and a, b are integration constants. So the similarity solution is

$$u = \epsilon_4 t + \frac{\epsilon_4}{\epsilon_2}(x - \epsilon_2 t) - \frac{am}{\epsilon_2(c+1)}\left[-\epsilon_2(x - \epsilon_2 t) + c_3\right]^{\frac{c+m}{m}} + b,$$
$$v = \left[\frac{m}{c}(-\epsilon_2(x - \epsilon_2 t) + c_3)\right]^{\frac{1}{m}}.$$

If $c = -m$, then

$$\phi = \frac{\epsilon_4}{\epsilon_2}\xi + \frac{(\epsilon_2 c_3 + a\epsilon_2)}{\epsilon_2^2}\log(\epsilon_2\xi - c_3) + b,$$

where a and b are integration constants. So the similarity solution is

$$u = \epsilon_4 t + \frac{\epsilon_4}{\epsilon_2}(x - \epsilon_2 t) + \frac{\epsilon_4 c_3 + a\epsilon_2}{\epsilon_2^2}\log\left[\epsilon_2(x - \epsilon_2 t) - c_3\right] + b,$$
$$v = \left[\epsilon_2(x - \epsilon_2 t) - c_3\right]^{\frac{1}{m}}.$$

Chapter 7

Potential Symmetries of Systems of Diffusion Equations

The problem of finding potential symmetries for the system of diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[f(u,v)\frac{\partial u}{\partial x}\right], \qquad (7.1)$$
$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[g(u,v)\frac{\partial v}{\partial x}\right],$$

is examined in this chapter. In order to achieve this we need to write system (7.1) as a system of three or four equations by introducing potential variables. These auxiliary systems can be constructed using the conservation laws of system (7.1). A complete classification of conservation laws for system (7.1) is presented in [37] where conservation laws were used, in particularly, to derive all possible auxiliary systems. Here we give the results only for arbitrary f and g and the case where $g = -f$.

$$\forall f, g$$

1. $u_t = (fu_x)_x$, $w_x = v$, $w_t = gv_x$;
2. $u_t = (fu_x)_x$, $w_x = u + v$, $w_t = fu_x + gv_x$;
3. $w_x = u$, $w_t = fu_x$, $z_x = v$, $z_t = gv_x$;

$$\forall f, g = -f$$

4. $u_t = (fu_x)_x$, $w_x = uv$, $w_t = fu_xv - fuv_x$;
5. $w_x = uv$, $w_t = fu_xv - fuv_x$, $z_x = v$, $z_t = -fv_x$;
6. $w_x = uv$, $w_t = fu_xv - fuv_x$, $z_x = u + v$, $z_t = fu_x - fv_x$;
7. $w_x = u$, $w_t = fu_x$, $z_x = v$, $z_t = gv_x$, $q_x = uv$, $q_t = fu_xv - fuv_x$.

All possible auxiliary systems that correspond to (7.1) can be found in [37].

In this chapter we consider one of the auxiliary systems of the systems of diffusion equations and we present some examples of potential symmetries. We also present an example of linearization using infinite-dimensional potential symmetries.

7.1 Examples of Potential Symmetries

We consider the following auxiliary system in the case when $g = -f$:

$$
\begin{aligned}
w_x &= uv, \\
w_t &= fu_xv - fuu_x, \\
z_x &= v, \\
z_t &= -fv_x.
\end{aligned}
\tag{7.2}
$$

We search for transformations admitted by system (7.2) of the form:

$$
\begin{aligned}
x' &= x + \epsilon X(x,t,u,v,w,z) + O(\epsilon^2), \\
t' &= t + \epsilon T(x,t,u,v,w,z) + O(\epsilon^2), \\
u' &= u + \epsilon U(x,t,u,v,w,z) + O(\epsilon^2), \\
v' &= v + \epsilon V(x,t,u,v,w,z) + O(\epsilon^2), \\
w' &= w + \epsilon W(x,t,u,v,w,z) + O(\epsilon^2), \\
z' &= z + \epsilon Z(x,t,u,v,w,z) + O(\epsilon^2),
\end{aligned}
$$

that leave system (7.2) invariant. The system (7.2) admits Lie symmetries if and only if

$$E_1 = \Gamma^{(1)}\left[w_x - uv\right] = 0, \tag{7.3}$$

$$E_2 = \Gamma^{(1)}\left[w_t - fu_xv + fuv_x\right] = 0, \tag{7.4}$$

$$E_3 = \Gamma^{(1)}\left[z_x - v\right] = 0, \tag{7.5}$$

$$E_4 = \Gamma^{(1)}\left[z_t + fv_x\right] = 0, \tag{7.6}$$

where $w_x = uv$, $w_t = fu_xv - fuv_x$, $z_x = v$ and $z_t = -fv_x$.

We only classify the Lie symmetries of (7.2) which are such that:

$$X_w^2 + X_z^2 + T_w^2 + T_z^2 + U_w^2 + U_z^2 + V_w^2 + V_z^2 \neq 0. \tag{7.7}$$

These symmetries induce potential symmetries of (6.1). If (7.7) is not satisfied then the symmetry of (7.2) projects into a point symmetry of (6.1).

Analytically, after elimination of w_x, w_t, z_x and z_t from equations (7.2), equations (7.3)-(7.6) become four identities in the variables $t,\ x,\ u,\ v,\ w,\ z,\ u_x,\ v_x,\ u_t,\ v_t,\ u_xv_x,\ u_tv_x,$ u_xv_t. These identities are multi-variable polynomials in $u_x,\ v_x,\ u_t,\ v_t,\ u_xv_x,\ u_xv_t,\ u_tv_x$; therefore the coefficients of different powers of these variables must be zero giving the determining equations for the unknown functions $T,\ X,\ U,\ V,\ W,\ Z$ and $f(u,v)$.

Taking the coefficients of $E_{4u_x^2,v_x^2}$, E_{3v_x} and E_{4v_t} we deduce that $T_x = T_u = T_v = T_w = T_z = 0$. Hence, $T = T(t)$. Also after some calculations using the coefficients of $E_{1u_xv_x,u_x,v_x}$, E_{3v_x,u_x}, E_{4u_x} we deduce that $X = X(x,t,z)$, $W = W(t,w,z)$ and $Z = Z(t,z)$ and from E_1 and E_3 we get the following expressions for functions U and V:

$$U = W_wu + W_z - Z_zu,$$

$$V = -X_xv - X_zv^2 + vZ_z.$$

Then from coefficient of u_x in E_2 we get the following equation:

$$f_u(uW_{ww} + W_{wz}) = 0. \tag{7.8}$$

So, the analysis is split into two cases:

1. $f_u = 0$;
2. $f_u \neq 0$.

Case 1. $f_u = 0$.

In this case using the fact that $f = f(v)$ from E_2 the only form of f that gives potentials is $f = \frac{1}{v^2}$. Therefore after some briefly calculations using equations (7.3)-(7.6) we deduce that

$$X = c_1x + c_2,\quad T = 2c_1t + c_3,\quad U = -c_1u + c_4u + h_{1z}(t,z),\quad V = 0,$$

$$W = c_4w + h_1(t,z),\quad Z = c_1z + c_5,$$

where function $h_1(t,z)$ is an arbitrary solution of $h_{1zz} - h_{1t} = 0$.

Therefore, the system (7.2) admits the following infinite Lie algebra spanned by

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial x}, \quad \Gamma_3 = \frac{\partial}{\partial z},$$
$$\Gamma_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} + z\frac{\partial}{\partial z}, \quad \Gamma_5 = u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w},$$
$$\Gamma_\infty = h_1(t,z)\frac{\partial}{\partial w} + h_{1z}(t,z)\frac{\partial}{\partial u}.$$

Symmetry Γ_∞ is a potential symmetry of (6.1).

Case 2. $f_u \neq 0$.

In this case from equation (7.8) and E_2 we deduce that $W = c_1 w + a_1(t,z)$ and from equations (7.3)-(7.6) we have the following determining equations

$$[(Z_z - c_1)u - a_{1z}]\, f_u + \left[X_z v^2 + (X_x - Z_z)v\right] f_v + (2vX_z + 2X_x - T_t)f = 0, \tag{7.9}$$
$$\left[-X_{zz}uv^2 + 2uv(Z_{zz} - X_{xz}) - a_{1zz}v - X_{xx}u\right] f - X_t uv + a_{1t} = 0, \tag{7.10}$$
$$\left[-v^3 X_{zz} + v^2(Z_{zz} - 2X_{xz} - vX_{xx}\right] f - vX_t + Z_t = 0. \tag{7.11}$$

We can then suppose that equation (7.9) can be written in the following form

$$(k_1 u + k_2)f_u + (k_3 v^2 + k_4 v)f_v + (2k_3 v + k_5)f = 0. \tag{7.12}$$

So, in order to find the different forms of f that induce potential symmetries for (6.1) we just have to solve the following system

$$\frac{\mathrm{du}}{k_1 u + k_2} = \frac{\mathrm{dv}}{k_3 v^2 + k_4 v} = \frac{\mathrm{df}}{f(-2k_3 v + k_5)},$$

with the method of characteristics.

Some of the forms of f that induce potentials symmetries are

(i) $f = v^{-2}\phi(u)$;

(ii) $f = v^{-2}e^{\nu u}$;

(iii) $f = v^{-2}u^{\nu}$.

Subcase 2.1: $f = v^{-2}\phi(u)$.

From equations (7.9)-(7.11) using the form of $f = v^{-2}\phi(u)$ after some calculations we deduce that

$$X = c_3 z + c_4 x + c_2, \;\; T = 2c_1 t + c_5, \;\; U = 0, \;\; V = v(c_1 - c_3 v - c_4), \;\; Z = c_1 z + c_6, \;\; W = c_1 w + c_7.$$

So, the Lie algebra is seven-dimensional spanned by

$$\Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_6=\frac{\partial}{\partial w},\ \Gamma_7=z\frac{\partial}{\partial x}-v^2\frac{\partial}{\partial v},\ \Gamma_8=x\frac{\partial}{\partial x}-v\frac{\partial}{\partial v},\ \Gamma_9=2t\frac{\partial}{\partial t}+v\frac{\partial}{\partial v}+z\frac{\partial}{\partial z}+w\frac{\partial}{\partial w}.$$

Here symmetry Γ_7 induce potential symmetry for equation (6.1).

Subcase 2.2: $f=v^{-2}e^{\nu u}$.

In this subcase from equations (7.9)-(7.11) we deduce that

$$X=c_3z+c_4x+c_5,\quad T=2c_1t-c_8\nu t+c_2,\quad U=c_8,\quad V=v(c_1-c_3v-c_4),$$

$$Z=c_1z+c_7,\quad W=c_1w+c_8z+c_6.$$

Therefore the Lie algebra is eight-dimensional spanned by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_6,\quad \Gamma_7,\quad \Gamma_8,\quad \Gamma_9,\quad \Gamma_{10}=\frac{\partial}{\partial u}+z\frac{\partial}{\partial w}.$$

So, Γ_7 induce potential symmetry for (6.1).

Subcase 2.3: $f=v^{-2}u^{\nu}$.

Here from equations (7.9)-(7.11) after some calculations we have

$$X=c_3z+c_4x+c_5,\quad T=(-c_1\nu+2c_6)t+c_2,\quad U=c_1u,\quad V=v(-c_3v-c_4+c_6),$$

$$Z=c_6z+c_7,\quad W=(c_1+c_6)w+c_8.$$

Hence the Lie algebra is eight-dimensional given by

$$\Gamma_1,\quad \Gamma_2,\quad \Gamma_3,\quad \Gamma_6,\quad \Gamma_7,\quad \Gamma_8,\quad \Gamma_9,\quad \Gamma_{11}=\nu t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}+w\frac{\partial}{\partial w}.$$

In this subcase symmetry Γ_7 induce potential symmetry for (6.1).

Although, the Lie algebra in the three subcases is different, all three equations admit the same potential symmetry.

7.2 Linearization using Potential Symmetries

We consider the following auxiliary system of (7.1)

$$\begin{aligned} &w_x=u,\\ &w_t=f(u,v)u_x, \qquad (7.13)\\ &v_t=[g(u,v)v_x]_x. \end{aligned}$$

In the case when $f = -u^{-2}$ and $g = -f$ system (7.13) is written as

$$\begin{aligned} &w_x = u, \\ &w_t = -u^{-2}u_x, \\ &v_t = [u^{-2}v_x]_x. \end{aligned} \tag{7.14}$$

The symmetries of (7.14) that induce potential symmetries of (7.1) are [70]

$$\begin{aligned} &\Gamma_1 = x(w^2-2t)\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} + u(6t-2xuw-w^2)\frac{\partial}{\partial u} - v(w^2+2t)\frac{\partial}{\partial v} + 4tw\frac{\partial}{\partial w}, \\ &\Gamma_2 = wx\frac{\partial}{\partial x} - u(w+ux)\frac{\partial}{\partial u} - vw\frac{\partial}{\partial v} + 2t\frac{\partial}{\partial w}, \\ &\Gamma_\phi = \phi(t,w)\frac{\partial}{\partial x} - u^2\phi_w\frac{\partial}{\partial u}, \\ &\Gamma_\psi = \psi(t,w)\frac{\partial}{\partial v}, \end{aligned}$$

where the function $\phi(t,w)$ satisfies the backward linear heat equation $\phi_t + \phi_{ww} = 0$ and $\psi(t,w)$ satisfies the linear heat equation $\psi_t - \psi_{ww} = 0$.

As mentioned in paragraph (2.8) if a nonlinear PDE (or a system of PDEs) admits infinite-parameter groups, then it can be transformed into a linear PDE (or into a linear system of PDEs) if these groups satisfy certain criteria. These criteria and the method for finding the linearizing mapping using the infinite-dimensional symmetries can be found in [10]. Hence, using the method described in [10] the infinite-dimensional Lie symmetries $\Gamma_\phi,\ \Gamma_\psi$ lead to the transformation

$$x' = w, \quad t' = t, \quad u' = \frac{1}{u}, \quad v' = v, \quad w' = x,$$

which maps the linear system

$$w'_{x'} = u', \quad w'_{t'} = -u'_{x'}, \quad v'_{t'} = v'_{x'x'},$$

into the nonlinear system (7.14). Consequently this mapping leads to the contact transformation

$$\mathrm{d}x' = u\mathrm{d}x + u^{-2}u_x\mathrm{d}t, \quad \mathrm{d}t' = \mathrm{d}t, \quad u' = \frac{1}{u}, \quad v' = u,$$

which maps the separable linear system

$$u'_{t'} = -u'_{x'x'}, \quad v'_{t'} = v'_{x'x'},$$

into the nonlinear system (7.1) now written as

$$u_t = -[u^{-2}u_x]_x, \quad v_x = [u^{-2}v_x]_x.$$

The question that arises here is what other equations of the class (7.1) can be linearized using similar linear approaches.

Chapter 8

Conclusions

The main goal of this thesis was the group classification of diffusion-type equations. The starting point was the known results of the (1+1) nonlinear diffusion equations and diffusion convection equations. In particular we made a complete classification of the (2+1) nonlinear diffusion-convection equations

$$u_t = (D(u)u_x)_x + (F(u)u_y)_y + K(u)u_x,$$

and the (3+1) nonlinear diffusion-convection equations

$$u_t = (D(u)u_x)_x + (F(u)u_y)_y + (G(u)u_z)_z + K(u)u_x.$$

In the literature appeared the results only in the cases $K(u) = 0$ and $D(u) = F(u) = G(u) = 1$ and $K(u) = u$ (Burgers' equation).

Furthermore, we classified the Lie symmetries for the systems of diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[f(u,v)\frac{\partial u}{\partial x}\right],$$
$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left[g(u,v)\frac{\partial v}{\partial x}\right],$$

which generalize Ovsiannikovs results of nonlinear diffusion equations.

The problem of classification of potential symmetries of a system is a very difficult and lengthy task. In addition to the existing results, we present some new cases of potential symmetries.

The work of the present thesis suggests certain problems that can be considered in the future. For example one can study the following problems:

- Similar to Theorem 2.2 find the restrictions of the infinitesimal functions for evolution equations with more than two independent variables.
- Classification of Lie symmetries for $(n+1)$ diffusion-convection equations.
- Classification of potential symmetries of the systems of diffusion equations (7.1) using all possible auxiliary systems.
- The Lie symmetries derived here can be employed to construct exact solutions.

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