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Reversible Computation in Petri Nets

Kyriaki Psara

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Examination Committee:

Research Supervisor

Associate Professor Anna Philippou

Committee Chair

Associate Professor Chryssis Georgiou

Committee Member

Professor Yannis Dimopoulos

Committee Member

Professor Maciej Koutny

Committee Member

Associate Professor Ivan Lanese

DECLARATION OF DOCTORAL CANDIDATE

The present Doctoral Dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy of the University of Cyprus. It is a product of original work of my own, unless otherwise mentioned through references, notes, or any other statements.

Kyriaki Psara

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Περίληψη

Ο αναστρέψιμος υπολογισμός είναι μια μη συμβατική μορφή υπολογισμού που επεκτείνει τον τυπικό τρόπο υπολογισμού με τη δυνατότητα αντίστροφης εκτέλεσης λειτουργιών. Η αναστρεψιμότητα προσέλκυσε πρόσφατα αυξανόμενη προσοχή σε διάφορες ερευνητικές κοινότητες καθώς από τη μία υπόσχεται υπολογισμούς χαμηλής ισχύος και, από την άλλη, είναι εφαρμόσιμη σε μια ποικιλία εφαρμογών.

Η διερεύνηση της αναστρεψιμότητας μέσω τυπικών μοντέλων καθορίζει τα θεωρητικά θεμέλια για το τι είναι η αναστρεψιμότητα, ποιο σκοπό εξυπηρετεί, και πως ωφελεί τα φυσικά και τεχνητά συστήματα. Ως εκ τούτου, προτείνουμε μια αναστρέψιμη προσέγγιση για τα δίκτυα Πέτρι, εισάγοντας μηχανισμούς και σχετική λειτουργική σημασιολογία για την αντιμετώπιση των προκλήσεων που έχουν οι κύριες μορφές αναστρεψιμότητας.

Τα δίκτυα Πέτρι είναι μια μαθηματική γλώσσα για μοντελοποίηση και συλλογισμό κατανεμημένων συστημάτων. Η πρόταση μας αφορά μία παραλλαγή των δικτύων Πέτρι, που ονομάζεται Αναστρέψιμα Δίκτυα Πέτρι, όπου τα διακριτικά ενός δικτύου ξεχωρίζουν μεταξύ τους με μοναδικές ταυτότητες. Δείχνουμε τη δυνατότητα εφαρμογής της προσέγγισής μας σε ένα μοντέλο μεταβολικής διαδρομής και ένα σύστημα επεξεργασίας συναλλαγών όπου και τα δύο εκδηλώνουν αναστρέψιμη συμπεριφορά.

Μια άμεση επέκταση του αρχικού μοντέλου συμπεριλαμβάνει την παροχή πολλαπλών διακριτικών που εκπροσωπούν τον ίδιο τύπο. Μία τέτοια επέκταση σε ένα μοντέλο όπως τα δίκτυα Πέτρι, έχει ως αποτέλεσμα αντίστροφες συγκρούσεις όπου ένα διακριτικό μπορεί να έχει τοποθετηθεί σε μία θέση από διαφορετικές μεταβάσεις. Προτείνουμε λοιπόν μια επέκταση των αναστρέψιμων δικτύων Πέτρι που επιτρέπει πολλαπλά διακριτικά του ίδιου τύπου σε ένα μοντέλο, ενώ παράλληλα διασφαλίζεται ο ντετερμινισμός κατά την αναστρο-

φή. Συγκεκριμένα, στην προσέγγιση την οποία διερευνούμε, διαφορετικά διακριτικά που βρίσκονται στην ίδια θέση μπορούν να διακριθούν με βάση την πορεία που έχουν ακολουθήσει στο δίκτυο. Αποδεικνύουμε ότι η εκφραστική ισχύς των αναστρέψιμων δικτύων Πέτρι με πολλαπλά διακριτικά είναι ισοδύναμη με εκείνη των αναστρέψιμων δικτύων Πέτρι με μοναδικά διακριτικά. Προτείνουμε επίσης την αντίθετη προσέγγιση, η οποία θεωρεί ότι όλα τα διακριτικά ενός συγκεκριμένου τύπου είναι πανομοιότυπα, αγνοώντας την πορεία που ακολούθησαν κατά την εκτέλεση του δικτύου. Δείχνουμε την ευρωστία αυτής της προσέγγισης ως τεχνική μοντελοποίησης συστημάτων που αφορούν πόρους μέσω ενός παραδείγματος από τη βιοχημεία, γνωστό ως αυτοπροτόλυση του νερού.

Και τα δύο προτεινόμενα μοντέλα αναστρέψιμων δικτύων Πέτρι (με μοναδικά ή πολλαπλά διακριτικά) επιτρέπουν την αναστροφή μεταβάσεων χωρίς περιορισμούς ως προς το πότε και αν θα αναστραφεί η εκτέλεση ή όχι. Με στόχο να περιορίσουμε την αναστρεψιμότητα, επεκτείνουμε τη σημασιολογία μας συσχετίζοντας τις μεταβάσεις με συνθήκες των οποίων η ικανοποίηση επιτρέπει την εκτέλεση μεταβάσεων προς τα εμπρός/πίσω.

Καταλήγοντας, για να διευκολύνουμε την ανάλυση της συμπεριφοράς μοντέλων αναστρέψιμου υπολογισμού διατυπώνουμε στο πλαίσιο μας βασικές ιδιότητες όπως η ασφάλεια και η προσβασημότητα όταν εφαρμόζονται διαφορετικές στρατηγικές αναστρεψιμότητας. Παρουσιάζουμε το πλαίσιο μαζί με τις σχετικές ιδιότητες με ένα μοντέλο ενός καινοτόμου, κατανεμημένου αλγορίθμου που επιλέγει κεραίες σε κατανεμημένες σειρές κεραιών.

Abstract

Reversible computation is an unconventional form of computing that extends the standard forward-only mode of computation with the ability to execute a sequence of operations in reverse at any point during computation. Reversibility has recently been attracting increasing attention in various research communities, as on the one hand it promises low-power computation, and on the other hand it is inherent or of interest in a variety of applications.

Exploring reversibility through formal models formulates the theoretical foundations of what reversibility is, what purpose it serves, and how it benefits natural and artificial systems. As such, in this thesis we propose a reversible approach to Petri nets by introducing machinery and associated operational semantics to tackle the challenges of the main forms of reversibility. Petri nets are a mathematical language for modelling and reasoning about distributed systems. Our proposal concerns a variation of cyclic Petri nets, called Reversing Petri Nets (RPNs) where tokens are persistent and distinguished from each other by an identity. We demonstrate the applicability of our approach with a model of the ERK signalling pathway and an example of a transaction-processing system both featuring reversible behaviour.

An immediate extension of the original model includes allowing multiple tokens of the same base/type to occur in a model. The addition of token multiplicity into a model like Petri nets results in various backward conflicts where a token can be generated in a place because of different transition firings. We therefore propose an extension of reversing Petri nets that allows multiple tokens of the same base/type to occur in a model while still ensuring backward determinism. Specifically, we explore the individual token interpretation where one distinguishes different tokens residing in the same place by keeping track of where they

come from. We prove that the expressive power of RPNs with multi tokens is equivalent to that of RPNs with single tokens, and we measure the expressiveness in terms of Labelled Transition Systems (LTSs) up to isomorphism of reachable parts that can be denoted by nets of the respective RPN models. We also propose the collective token interpretation, as the opposite approach to token ambiguity, which considers all tokens of a certain type to be identical, disregarding their history during execution. We show the robustness of this approach as a modelling technique for resource-aware systems by modelling an example from biochemistry, known as the autoprotolysis of water.

Both of the proposed models of RPNs (with single or multi tokens) implement the notion of uncontrolled reversibility, meaning that it specifies how to reverse an execution and allows to do so freely, yet it places no restrictions as to when and whether to prefer backward execution over forward execution or vice versa. In this respect, a further aim is to control reversibility by extending our formal semantics where transitions are associated with conditions whose satisfaction allows the execution of transitions in the forward/reversed direction.

Finally, in order to facilitate the analysis of the behaviour of reversible models, we formulate the basic properties of our framework such as safety, reachability, precedence and exception when different notions and strategies of reversibility are applied. We illustrate the framework along with the associated properties with a model of a novel, distributed algorithm for antenna selection in distributed antenna arrays.

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Thesis Contributions

The following papers were published as a result of the research done for the requirements of this dissertation and are its primary contributing sources.

1. Philippou A. and **Psara K.**, 2018. Reversible computation in Petri nets. In Proceedings of the 10th International Conference on Reversible Computation (pp. 84-101). Lecture Notes in Computer Science volume 11497. Springer.

2. Barylska K., Gogolinska A., Mikulski L., Philippou A., Piatkowski, M. and **Psara. K.**, 2018. Reversing computations modelled by coloured Petri nets. In Proceedings of the International Workshop on Algorithms & Theories for the Analysis of Event Data 2018 (pp. 91–111). CEUR Workshop Proceedings volume 2115. CEUR-WS.org.

3. Philippou A., **Psara K.** and Siljak H., 2019. Controlling reversibility in reversing Petri nets with application to wireless communications. In Proceedings of the 11th International Conference on Reversible Computation (pp. 238-245). Lecture Notes in Computer Science volume 11497. Springer.

4. Siljak H., **Psara K.** and Philippou A., 2019. Distributed antenna selection for massive MIMO using reversing Petri nets. IEEE Wireless Communications Letters, volume 8(5), pp.1427-1430.

5. Dimopoulos Y., Kouppari E., Philippou A. and **Psara K.**, 2020. Encoding Reversing Petri Nets in Answer Set Programming. In Proceedings of the 12th International Conference on Reversible Computation (pp. 264-271). Lecture Notes in Computer Science volume 12227. Springer.

6. Kuhn S., Aman B., Ciobanu G., Philippou A., **Psara K.** and Ulidowski I., 2020. Reversibility in Chemical Reactions. In Reversible Computation: Extending Horizons of Computing - Selected Results of the COST Action IC1405 (pp. 151-176), Lecture Notes in Computer Science volume 1270. Springer.

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Chapter 1

Introduction

1.1 Motivation

Reversible computation bi [40] is an unconventional form of computing where computation can be executed in the backward direction as effortlessly as it can be executed in the standard forward direction. In particular, individual operations can be carried out reversibly and thus, at any point of the execution we are able to uniquely identify the forward or backward state. Hence, every reversible computation process can be traced backward uniquely from end to start whilst exhibiting both forward and backward determinism. Its characteristics make reversibility a very promising paradigm that extends the current irreversible mode of computation by delivering novel computing devices and software.

The study of reversibility originated in the 1960s when scientists and mathematicians started to be concerned with energy efficient computation. In particular, Landauer [75] answered many questions by proving that logically irreversible operations result in bit erasure that causes heat dissipation and, in general, loss of energy. In particular, a proportion of electrical power consumed by current computers is lost in the form of heat because every time a computer throws away bits of information it generates at least $kTln2$ (k is the Boltzmann constant which is approximately 1.38×10^{-23} J/K, T is the temperature of the heat sink in kelvins, and $ln2$ is the natural logarithm of 2 which is approximately 0.69315) of entropy for each bit of information it erases.

Reversibility offers the potential for computationally proceeding in the forward direction as well as in reverse resulting in going back to states visited before or even states that cannot be reached by going forwards alone. It is encountered in a wide range of systems. For instance, it is a property of biochemical systems [118], where reactions can be executed in both the forward and backward direction based on the imposed physical conditions. Another application is quantum computing which in contrast to classical computing is always reversible [44]. At the same time, it is on many occasions a desirable system property. To begin with, reversible computing comes to solve the miniaturisation limitations of current technology that aim to increase the speed and capacity of circuits. recovery from failures such as corrupted data, deadlocked programs and breached security is crucial and could be effortlessly obtained in the presence of reversibility. Further applications are encountered in programming languages and concurrent transactions.

One line of work in the study of reversible computation has been the investigation of its theoretical foundations [83]. Understanding the role that reversibility plays in natural systems calls for the development of realistic formal models for concurrent and distributed systems. Reversible models can be based on already existing abstract formalisms or specially proposed languages, and can be used not only for modelling reversible systems but also for investigating suitable notions of behavioural equivalences, logics and other analysis techniques. Furthermore, the study of reversible formalisms may aid the understanding of the foundation of reversibility and can help towards the understanding, modelling and implementing reversible actions as a feature of computation.

In the context of the theoretical study of reversible computation, the different strategies of reversing and their relationships are being investigated and have led to the definition of different forms of reversibility: While in the sequential setting reversibility is generally understood as the ability to execute past actions in the exact inverse order in which they have occurred, a process commonly referred to as *backtracking*, in a concurrent scenario it can be argued that reversal of actions can take place in a more liberal fashion. The main alternatives proposed are those of *causal order reversibility* [83], a form of reversing where an action can be undone provided that all of its effects (if any) have been undone beforehand, and *out-of-causal order reversibility* [119], a form of reversing featured most notably in biochemical systems.

1.2 Previous Work

Even though reversing computational processes in concurrent and distributed systems has many promising applications, it also has many technical and conceptual challenges. The main challenge being the ability to identify the legitimate backward moves by maintaining the information needed to reverse executed computation, e.g., to keep track of the history of execution and the choices that have not been made. In contrast to the sequential setting that is well understood, the concurrent setting poses the conceptual question of how do we define a causally-respecting order of execution. *Causal-consistent reversibility* [83] is the most common notion of reversibility in the concurrent and distributed setting. Since its definition, various approaches in formal models and applications of causal-consistent reversibility have been considered. The first works handling reversibility in process calculi are the Chemical Abstract Machine [19], a calculus inspired by reactions between molecules whose operational semantics define both forward and reverse computations, and RCCS [31], an extension of the Calculus of Communicating Systems (CCS) [97] equipped with a reversible mechanism that uses memory stacks for concurrent threads, further developed in [32, 33]. This mechanism was represented at an abstract level using categories with an application to Petri nets [34]. Subsequently, a general method for reversing process calculi with CCSK being a special instance of the methodology was proposed in [117]. This proposal introduced the use of communication keys to bind together communication actions as needed for isolating communicating partners during action reversal. Reversible versions of the π-calculus include $\rho\pi$ [79] and $R\pi$ [29].

While all the above concentrate on the notion of causal reversibility, approaches considering other forms of reversibility have also been proposed. Consider every state of the execution to be a result of a series of actions that have causally contributed to the existence of the current state. If the actions were to be reversed in a causally-respecting manner then we would only be able to move back and forth through previously visited states. Therefore, one might wish to apply *out-of-causal-order reversibility* in order to create fresh alternatives of current states that were formerly inaccessible by any forward-only execution path. This has been achieved in [70] by introducing a new operator for modelling local reversibility, a form of out-of-causal-order reversibility, whereas a mechanism for controlling out-of-causal reversibility has also been considered in [118]. The modelling of bonding within reversible processes and event structures was considered in [119], whereas a reversible computational calculus for modelling chemical systems composed of signals and gates was proposed in [27]. The study of reversible process calculi has also triggered research on various other models of concurrent computation such as reversible event structures [136].

A distinguishing feature between the cited approaches is that of *controlling reversibility*: while various frameworks make no restriction as to when an action can be reversed (uncontrolled reversibility), it can be argued that some means of controlling the conditions of reversal is often useful in practice. For instance, when dealing with fault recovery, reversal should only be triggered when a fault is encountered. Based on this observation, a number of strategies for controlling reversibility have been proposed: [32] introduces the concept of irreversible actions, and [80] introduces compensations to deal with these irreversible actions and to avoid repeating past errors. Another approach is that of [118] which proposes the usage of an external entity for capturing the order in which transitions can be executed in the forward or the backward direction. In another line of work, [78] defines a roll-back primitive for reversing computation, and in [76] roll-back is extended with the possibility of specifying the alternatives to be taken on resuming the forward execution. Finally, in [9] the authors associate the direction of action reversal with energy parameters capturing environmental conditions of the modelled systems.

Research on reversible models from process calculi continues in Petri nets, the first approach being that of [15, 16] which implemented a liberal way of reversing computation in Petri nets by introducing additional reversed transitions. In these works, the authors investigate the effects of adding reversed versions of selected transitions in a Petri net and they explore decidability problems regarding reachability and coverability in the resulting Petri nets. Towards examining causal consistent reversibility in Petri nets, the work in [96] investigates whether it is possible to add a complete set of effect-reverses for a given transition without changing the set of reachable markings. The authors show that this problem is in general undecidable however it can be decidable in *cyclic Petri nets* where with the addition of new places these non-reversible Petri nets can become reversible while preserving their behaviour. Recently, an alternative approach [93, 94] on reversing Petri nets introduces reversibility in Petri nets by unfolding the original Petri net into *occurrence nets* and *coloured Petri nets*. The authors encode causal memories while preserving the original computation by adding for each transition its reversible counterpart. In [35], the authors examine the possibility of reversing the effect of the execution of groups of various transitions (steps). They then present a number of properties which arise in this context and show that there is a crucial difference between reversing steps which are sets and those which are true multisets.

1.3 Thesis Aims

In this thesis, we shall consider a particular model of computation, known as *Petri nets*, that will be extended to a reversible variant. Petri nets are a basic model of parallel and distributed systems, designed by Carl Adam Petri in 1962 in his PhD Thesis: "Kommunikation mit Automaten” [110, 122]. They constitute a graphical mathematical language that can be used for the specification and analysis of discrete event systems and they support both action-based and state-based modelling and reasoning.

In contrast to the extensive research carried out in process calculi and event structures, work done on reversing Petri nets is still at an initial stage. Thus, a first aim of this thesis is exploring the several results discussed in process calculi, such as the flexibility of reversible actions allowed in causal reversibility, within Petri nets. This enables us to investigate how and whether these can be embedded within the Petri net model. At the same time, while understanding the theoretical properties of reversibility within Petri Nets, an extension of Petri nets with reversibility offers an added benefit. Petri nets can be applied informally to any area or system that needs some means of representing parallel or concurrent activities as well as systems that can be described graphically like flow charts. The easy applicability of Petri nets is inherent due to their generality and permissiveness [110]. Since Petri nets are visually comprehensible and simple in their application, they can be used for modelling by both practitioners and theoreticians.

However, classical Petri nets are not reversible by nature, in the sense that every transition cannot be executed in both directions. The reason being the nondeterministic nature of Petri nets. Specifically, by observing the state of a Petri net with tokens scattered along its places it is not possible to discern the history that led to the specific state and consequently the precise transitions that can be undone. Therefore, an inverse action in classical Petri nets, needs to be added as a supplementary forward transition for achieving the undoing of a previous action. This explicit approach of modelling both forward and reverse transitions can prove cumbersome in systems that express multiple reversible patterns of execution, resulting in larger and more complex systems. Furthermore, it fails to capture reversibility as a mode of computation. Motivated by this, our intention is to study an approach for modelling reversible computation that does not require the addition of new, reversed transitions but instead allows to execute transitions in both the forward as well as the backward direction. This framework should be able to identify at each point in time the history of execution, a necessary aspect for all forms of reversibility. As such, this thesis aims to propose a reversible approach to Petri nets which introduces machinery and associated operational semantics where executed transitions can be reversed according to three different semantic relations capturing the notions of backtracking, causal reversibility and out-of-causal-order reversibility.

Reversible formal models used to model reversible systems need to be able to identify the legitimate backward moves according to forward execution. When it comes to Petri nets, the ability to formally express causal dependencies based on an appropriate causality based concept is one of the most well-known concepts of Petri net theory [138]. As such, when proposing a reversible variant of Petri nets the interplay between reversibility and concurrency should be investigated. Specifically, investigating the notion of causal dependence in Petri nets equipped with the ability to reverse is one of the primitive aims of this thesis.

At the same time, out-of-causal reversibility has been observed in many important reversible examples where concurrent systems violate causality. Nonetheless, this body of research is still at a preliminary stage, and while interesting ideas have been discovered, a systematic study of the related problems and of the possible application areas is still missing. This means that research on how to generalise causality or how it relates to out-of-causal reversibility, in order to deal with such systems, deserves much further investigation. For example, studying the properties of out-of-causal reversibility could potentially prove that both backtracking and causal reversibility are in some essence subsets of out-of-causal-order reversibility, which could potentially yield a universal approach on the strategy used for reversing. Since out-of-order reversibility comes with its own peculiarities that need to be taken into consideration while modelling reversible systems, this thesis aims to understand these peculiarities and obtain an approach that addresses out-of-causal reversibility within Petri nets.

Understanding the basics of reversibility through reversible models of concurrent computation is useful but it is not directly suitable for most applications, since they do not determine when and whether to prefer a forward over a backward action. One of the objectives of this thesis is to consider a strategy for controlling reversibility in Petri nets which along with the wide use of Petri nets can find application in various domains. The resulting framework will enable us to study and understand reversibility through various case studies thus overcoming some limitations in the understanding, modelling and implementing reversible actions as a feature of computation.

1.4 Obtained Results

In Chapter 3 we propose the *first reversible approach to Petri nets*which introduces reversing Petri nets (RPNs), a variation of cyclic Petri nets where executed transitions can be reversed according to three different semantic relations capturing the notions of *backtracking*, *causal reversibility* and *out-of-causal-order reversibility*. Furthermore, during a transition firing, tokens can be bonded with each other. The creation of bonds is considered to be the effect of a transition, whereas their destruction is the effect of the transition's reversal.

Causality. When it comes to causality, cyclic structures make causality quite non-trivial since the presence of cycles exposes the need to define causality of actions within repeated executions of transitions. Indeed there are different ways of introducing reversible behaviour depending on how causality is defined. In our approach, we follow the notion of causality as defined by Carl Adam Petri for one-safe nets that provides the notion of a run of a system where causal dependencies are reflected in terms of a partial order [110]. A causal link is considered to exist between two transitions if one produces tokens that are used to fire the other. In this partial order, a causal dependence relation is explicitly defined as an unfolding of an occurrence net which is an acyclic net that does not have backward conflicts. Based on this notion of causality we handle cyclic structures by adopting “lists of histories" associated with each transition recording all of its previous occurrences. Also, additional machinery has been necessary that captures the causal dependencies in the presence of cycles. We prove that the amount of flexibility allowed in causal reversibility indeed yields causally consistent semantics.

Out-of-causal order. Until the proposal of RPNs, it had yet to be proposed a reversible approach to Petri nets which introduces machinery and associated operational semantics to tackle the challenges of all three forms of reversibility. We therefore propose the reversible semantics of out-of-causal-order reversibility and demonstrate that this form of reversing is able to create new states unreachable by forward-only execution. Additionally, we establish the relationship between the three forms of reversing and define a transition relation that can capture each of the three strategies modulo the enabledness condition for each strategy. This allows us to provide a uniform treatment of the basic theoretical results.

Multiple tokens. The proposed model of reversing Petri nets considers tokens to be distinct from each other and assigns unique names to them. Hence, a natural extension is to allow multiple tokens of the same base/type to occur in a model. Allowing multiple instances of identical tokens results in ambiguities when it comes to causal dependencies. Depending on how we treat these ambiguities we define two different approaches when it comes to causal-order reversibility, the first one being the individual token interpretation and the second one the collective token interpretation [24, 137, 139].

In chapter 4 we explore the individual token interpretation of reversing Petri nets where tokens of the same type are distinguished as individual. The model keeps track of where the tokens come from and therefore causal dependencies between transitions are reflected in terms of a partial order similar to the partial order of reversing Petri nets with single tokens. Thus, we allow identical tokens to fire the same transition when going forward, however when going backwards tokens are able to reverse only the transitions that they have fired. Additionally we provide the reversible semantics for out-of-causal-order reversibility in the presence bond destruction. We then proceed to translate reversing Petri nets into labelled transition systems (LTSs) as an event-oriented representation of the operational behaviour of the model. We compare the expressive power offered by multi tokens against that of single tokens, in terms of the associated Labelled Transition Systems, denoted up to isomorphism of reachable parts. As a result, we find that reversing Petri nets with single tokens are equally expressive as reversing Petri nets with multi tokens. As an alternative direction, we then propose the collective token interpretation of reversing Petri nets which is inspired from biochemical reactions and resource allocation systems. In this approach all tokens of a certain type are identical, disregarding their history during execution and therefore assuming the ambiguities between them to be equivalent.

Controlled reversibility. The framework of reversing Petri nets has been extended with a mechanism for *controlling reversibility* in Chapter 5. In our model control is enforced with the aid of conditions associated with transitions, whose satisfaction acts as a guard for executing the transition in the forward/backward direction. The conditions are enunciated within a simple logical language expressing properties relating to available tokens. The mechanism may capture environmental conditions, e.g., changes in temperature, or the presence of faults. We present a reversible semantics of the resulting framework. The resulting model is general enough to capture a wide range of systems, in this context we give an overview of several properties of reversing Petri nets that could be used to analyse the behaviour of these systems.

Case studies. Our approach is motivated by applications from biochemistry, but it can be applied to a wide range of problems featuring reversibility. Specifically, we demonstrate the original RPN framework with various examples including a model of the ERK pathway, and a model of a transaction processing system, examples that inherently feature (out-of-causal-order) reversibility. We then show the same transaction processing system modelled by RPNs with multiple tokens under the individual token interpretation. Multi tokens are also demonstrated under the collective token interpretation by modelling a case study from biochemistry, known as the autoprotolysis of water, where instances of the same atom are indistinguishable. Finally, we show the robustness of our control mechanism by modelling an example from telecommunications of a distributed algorithm for antenna selection illustrating the ability of RPNs to not only formalise complex distributed systems but also naturally capture reversible, controlled execution and conservation of information in a system.

1.5 Document Outline

In Chapter 2 we present the basic background theory for reversible computation, Petri nets, and reversible models of concurrency. This chapter presents an overview of the main approaches, results, potential benefits, and applications of reversible computation. In particular, we focus on reversible formal models of concurrency used for modelling reversible systems or developing techniques to analyse descriptions of reversible protocols. Furthermore, we provide an overview of the traditional model of Petri nets. We present the main extensions of Petri nets and discuss techniques for system validation and verification for the model. Finally, we discuss causality, a concept of high relevance in the context of Petri net semantics.

In Chapter 3 we propose the formalism of reversing Petri nets by introducing machinery, associated operational semantics and results for transition enabledness as captured by forward execution, backtracking, causal and out-of-causal-order reversibility. We illustrate the RPN framework with a model of the ERK pathway and a transaction processing system.

In Chapter 4 we extend this formalism by allowing multiple tokens of the same type to occur in a system as well as the ability of forward transitions to break bonds. We then compare the two models and show that the expressive power of RPNS with multi tokens is equivalent to the expressive power of RPNs with single tokens. The final contribution of this chapter is a demonstration of a biological case study, namely the autoprotolysis of water reaction.

In Chapter 5 we introduce another extension of the RPN model with a mechanism for controlling reversibility. We use the resulting formalism of Controlled RPNs to model a novel distributed algorithm for antenna selection.

The last part of the thesis, Chapter 6, concludes this work by comparing the results of this proposal with the related literature and proposing the current and future work after the completion of this thesis.

Chapter 2

Background

2.1 Reversible Computation

Mechanical Computing essentially dates back to the 1800s, followed by Electronic Computing, starting at least six decades ago, and ever since both of them have been supporting and enhancing all aspects of our lives [109]. Forward-only computing has been extensively researched, developed and analysed in academia, industry and government across the globe. Forward direction is the standard kind of execution whereas backward direction is the ability to go back to previous states by undoing previously executed actions.

Since current technology is not invertible, it leads to loss of information where previous states cannot be recovered from the current state. It follows an irreversible computation paradigm where ordinary computer chips do not qualify for reversible operations. For example, a simple standard operation, like the logical AND, illustrates that given the output 0 it is impossible to determine the input values as one of combinations of 1 and 0, 0 and 1 or 0 and 0 [40]. Therefore, such logically irreversible operations lack determinism due to the fact that the partial function that maps each machine state onto its successor generates more than one inverse values.

This entails that, if there is a logic gate that generates an output from a given input, the gate is reversible only when there is an inverse operation that performs a bijective transformation of its local configuration space. Crucially, the ability to have two-way determinism requires an one-to-one mapping, where each input produces a unique output. These bijective reversible operations have at most one previous configuration giving the ability to uniquely identify the forward or backward state at any point of the execution. Such backward deterministic systems are the foundations of an alternative computation paradigm, called *Re-*

versible Computation [40]. As such, a computation is logically reversible if it is always possible to efficiently reconstruct the previous state of the computation from its current state.

Reversible computation is an emerging paradigm that extends the standard forward-only mode of computation by allowing one to execute programs in the backward direction as effortlessly as it can be executed in the standard forward direction. In particular, individual operations become time reversible that can easily and exactly be reversed, or undone at any point of the execution. Computer scientists believe that reversible computation is an unconventional but promising form of computing, which is able to deliver novel computing devices and software [40]. More importantly, reversibility is emerging as one of the most exciting new dimensions in computing for the future, positioned for inevitable progress and expansion in the coming decades.

2.1.1 Motivation

Reversible computing combines thermodynamics and information theory in order to reflect physical reversibility one of the fundamental microscopic physical properties of Nature. Since all successful fundamental physical theories share the property of reversibility, future computing could also follow rather trivially certain basic facts of fundamental physics. Such properties can be effectively used in computing in order to create an interface between computation and the laws of physics where logical reversibility implements physical reversibility [99].

Back to the 1960's, three foundational studies of information lossless computations made their appearance. Having different motivations, Huffman studied finite state machines that do not erase information [62], Lecerf studied the theoretical properties of reversible Turing machines [86] and Landauer studied the thermodynamics of reversible logics. Of these studies, Landauer's work is the most prominent, but the other two have laid the foundation of the theoretical study of reversible computing.

Physicist Rolf Landauer was the first to argue the relation between thermodynamics and the irreversible character of conventional computers. Specifically, reversible computation originated in the 60's when Landauer published a paper titled "Irreversibility and Heat Generation in the Computing Process" [75], where he attempted to apply the most fundamental, reversible laws of physics to digital computers. Landauer's key insight follows directly as an immediate logical consequence of our most thorough, battle-tested understanding of fundamental physics. He noted that, while classical mechanics and quantum mechanics are fundamentally reversible [1] by obeying the laws of motion, their logical state often evolves irreversibly since it is not backward deterministic. This means that since all of the fundamental laws of physical dynamics are reversible, then conceptually any machine should be able to run the laws of physics backwards and thus be able to determine the system's backward states.

Specifically, Landauer observed the direct implications on the thermodynamic behaviour of a device that is carrying out irreversible operations. His reasoning can be understood by realising that reversibility at the lowest level of physics means that we can never truly erase information in a computer. He notes that for the entire history of computers, our computing machines have been erasing bits of information in the process of performing irreversible computations.

If we return to the example of the logic gate AND, we can observe that given the output 0, the input 1 has been erased after the execution of the operation. Whenever a bit of information gets overwritten by a new value or whenever a logic gate produces several unused outputs, the previous information might get lost but it will not be physically destroyed. Instead, bit erasure pushes bits out into the computer's thermal environment, where they become entropy causing heat dissipation and, in general, loss of energy. This is known as the von Neumann-Landauer (VNL) principle [133] where one bit's worth of lost logical information always leads to at least $kTln2$ (k is the Boltzmann constant which is approximately 1.38×10^{-23} J/K, T is the temperature of the heat sink in kelvins, and $ln2$ is the natural logarithm of 2 which is approximately 0.69315) amount of physical energy dissipation.

This result is of great interest because it makes plausible the existence of thermodynamically reversible computers which could perform computations while dissipating considerably less energy per logical step. The energy used in reversible bit operations can be fully recovered and reused for subsequent operations so that every computation can be performed without bit erasures. This means that a computation is physically reversible when it can be carried out without loss of energy or, more formally, with no increase in physical entropy and it is thus energy efficient.

Landauer's theoretical lower bound has since been experimentally confirmed and it has been argued many times that efficient operations of future computers require them to be reversible [20]. In the scenario of low-energy computing, the gap between computation and reality needs to be bridged by introducing the reversible or possibly the quantum mechanical mode of computation. Currently, computers are commonly irreversible with their technology rapidly approaching the elementary particle level extrapolating towards Landauer's limit.

In particular, based on Moore's law, computer power roughly doubles every 18 months for the last half century [25]. The miniaturisation of transistors increases their per-area leakage current and standby power; meanwhile, the reduction of signal energies, causes significant thermal fluctuations which eventually prevent any further progress within the traditional computing paradigm [47]. Efforts are being made within the semiconductor industry in order to try to reduce and forestall these problems, but the solutions are becoming ever more expensive to deploy where eventually no level of spending can ever defeat the laws of physics. Smaller transistors in new conventionally-designed computers would no longer be any cheaper, faster, or more energy-efficient than any predecessors, and at that point, the progress of conventional semiconductor technology will stop being any longer economically justifiable. Landaure's limit threatens to end improvements in practical computer performance within the next few decades and to avoid this a solution could be to avoid losing track of logical information.

However, for several decades now, we have known that reversible computing is a theoretically possible alternative paradigm which is in fact the only possible way, within the laws of physics, to have energy and cost efficient computers. However, when reversible computations are implemented on the right hardware, they should be able to circumvent Landauer's limit. The technical motivation given by Landauer has inspired theoretical work in the area of computational models. Lecerf [86] was the first to describe reversible computations executed on reversible Turing machines, and invented the Lecerf reversal technique to uncompute histories, but he was unaware of Landauer's thermodynamic applications, and therefore his machines did not save their outputs. Bennett [18] then reinvented Lecerf's reversal based on Landauer's point of view that any desired logically irreversible computational operation could be embedded in a reversible one, by simply saving aside any information that it would otherwise erase. For example, the machine might be given an extra tape to record each operation as it was being performed, in order to be able to uniquely determine the preceding state by the present state and the last record on the tape. However, Landauer noted that this method was only going to postpone the inevitable, because the tape would still need to be erased eventually, when the available memory filled up.

Bennet managed to prove that it is possible to construct fully reversible Turing machines capable of performing any computation whilst erasing garbage information on its tape when it halts and therefore leaving behind only the desired output and the originally furnished input. The trick is to decompute the operations that produced the intermediate results and therefore erasing the temporary data from the memory. This would allow any temporary memory to be reused for subsequent computations without ever having to erase or overwrite it. He also pointed out the possibility of a physically reversible computer where dissipation of energy is arbitrarily small.

However Bennett's construction only addressed the logical level where any arguments based on thermodynamics needed to be applied on specific hardware technologies. Toffoli and Fredkin [49] were the first to address precisely how to construct a practical physical mechanism for computation that would also be physically reversible. They have reinvented reversible computing in the form of conservative logic circuits, and proved their universality. Toffoli [134] invented the Toffoli gate which is perhaps the most convenient universal reversible logic primitive. All these pioneering developments together incrementally set the stage for the field of reversible computing.

As a result, reversible computing has the potential to alleviate the ever-increasing demand for electricity by designing revolutionary reversible logic gates and circuits that lead to low-power computing. Hardware-wise, the potential benefits of reversible computing come to solve the miniaturisation limitations of current technology that aim to increase the speed and capacity of circuits. On the other hand, there already exist various occasions where reversibility is naturally embedded in computation. For example, recovery from failures such as corrupted data, deadlocked programs and breached security is crucial and could be effortlessly obtained in a reversible manner. Hence, such a mode of computation aligns naturally with many computational tasks such as the treatment of faults and recovery in distributed systems, coding and decoding and many others.

So far, it is considered to be highly challenging to implement reversibility effectively, because it comes with many underlying problems and the alternative of advancing conventional technology is much easier. Even though it comes with many promising benefits and applications it also comes with its own limitations. Since, the theory of reversible computation is based on the idea of computing and uncomputing operations, it means that arbitrarily large computations executed in reverse would result in almost twice as many steps as an ordinary computation and may require a large amount of temporary storage. This means that there is an underlying trade-off between the efficiency of such recovery and the speed of computation.

On a practical level achieving efficient reversible computing will likely require new hardware materials, new design tools and device structures, new hardware description languages with the supporting software and overall a thorough remodelling of the entire computer design infrastructure. This also means that a large part of computer scientists and digital engineering workforce will have to be trained to use novel reversible design methodologies.

Nevertheless, the upside potential of reversible computing has attracted many researchers that made significant conceptual progress over the past few decades. This effort of addressing the challenges of reversible computation is highly worthwhile, because with the current rapidly advancing technology, it is now time to focus on reversible computing, and begin collaborative effort to materialise this idea. Committed attention can eventually improve current information technology by making it many orders of magnitude greater than any existing irreversible technology [135].

2.1.2 General Overview

As discussed above, computer scientists, mathematicians and physicians believe that reversible computation will be a key technique in the not so distant future of computer models. As such, it attracts much interest for its potential in a growing number of application areas ranging from cellular automata, software architectures, reversible programming languages, digital circuit design to quantum computing. Below we present the main advances in these fields.

There exist several notions of logical reversibility on computing models with a finite number of discrete internal states that evolve in discrete time. Their precise impact on the computational capacities and decidability properties of devices has been considered from different points of view. In the literature there exist various models, including the massively parallel model of cellular automata, the weakly parallel model of multi-head finite automata [8,72,100,120] as well as sequential models such as Turing machines [7,18], push-down [71,74] and queue automata [73], and finite state machines [60]. In order to examine whether reversibility increases computational capacities it is useful to study the properties and the impact on suitable models when different notions of reversibility are applied. These models have been equipped with additional resources or structural properties in order to examine whether a computational model can be made reversible and the associated costs.

A number of interesting reversible programming languages have been developed since 1986. The first reversible programming language follows the imperative paradigm [142,143] followed by another simple reversible imperative languages named R [46]. Other general purpose functional programming languages are RFUN [144], muOz [88] the causal-consistent reversible extension of Oz and Theseus [63]. The family of quantum programming languages consists of languages based on the imperative paradigm such as QCL (Quantum

Computation Language) [104], LanQ [98] and languages based on the functional paradigm such as cQPL [92], and QML [2]. Research on compiler technology for reversible languages has also progressed in the last several years [6, 46]. The main challenge in this area is that these languages are still prototype languages. Thus, the code base for each of these languages is limited, and the languages do not offer many of the usual programming abstractions. This in turn has hindered the developments of reversible algorithms and useful data structures. Persistent (immutable) data structures [103] offer more efficient storing of multiple versions of a data structure, sharing structure where possible.

The idea of using reversibility for the development of reliable software is quite natural since backward recovery is an instance of reversible computation in which errors trigger inverse actions. In case of trouble fault treatment seeks to handle certain system errors after their occurrence and therefore stop them from causing a failure. Then the system can go backwards to a past safe state of the system and try to explore new directions, avoiding the troublesome actions and therefore bringing the system to a consistent state. If a fault is detected, checkpointing can be used as a recovery mechanism that restores the system to a previously saved state which is essentially a snapshot of the entire system that is safe from errors [22, 45, 147]. The past 40 years reversibility has also been naturally applied in the area of debugging [56, 146] because it gives the ability to explore the computation in both forward and backward, and therefore assisting the programmer in the search of possible misbehaviours. Indeed, many reversible debugging tools exist [28, 42, 67, 87] and some reversible debugging features are available in mainstream debuggers. There have also been proposed reversible process calculi used to build constructs for reliability, and in particular communicating transactions with compensations [36] where interacting transactions with compensations have been mapped into a reversible calculus with alternatives in [76]. Behavioural equivalences for communicating transactions with compensation have been studied in [37, 68].

Another area that can benefit from reversibility is that of control systems and robotics. Robots are generic mechatronic devices controlled by a computer that can essentially be made reversible. Reversibility plays vital role in different programming paradigms that are used to operate robots. Many operations in the field of robotics are naturally reversible both for single robots as well as multi-robot swarms. These operations assist systems that can autonomously accumulate and revise knowledge from their own experience via self-programming [124, 125]. Control systems operate concurrently during forward execution in order to predict the behaviour of the environment under constrains of limited time and computational resources. Whereas, during backward execution they retrieve the goals the system was designed to achieve. Another operation is that of reversing the forward execution of an assembly sequence in order to generate the backward disassembly process as well as changing the direction of a mobile robot from forward to backward. The increase likelihood of errors in industrial robots can be addressed using reverse execution in order to withdraw an erroneous situation and thereafter automatically retry the assembly operation [85, 126].

As motivated by Landauer [75], reversible circuits have several promising applications such as low-power circuit design and quantum computing. The inherent properties of reversibility can be exploited in the design of conventional circuits with many advantages. One of them is the ability to undo operations in case the system reaches an erroneous state as well as full connectivity which detects errors by applying randomly generated stimuli. Conventional computing also benefits from reversibility by achieving perfect observability and controllability which provides easy testability. Another application of reversible circuits is quantum computing [66] which is inherently reversible and allows reversible computation to be exploited as a subset of quantum operations. The goal of conventional circuit design is to find a logic circuit that implements the Boolean function and minimises the number of gates or the circuit depth. Reversible circuit synthesis is a special case of conventional circuit design where all gates are made reversible by disallowing any fanout. In order to avoid any fanout, it must be that the number of input wires of a gate is the same as the number of output wires [108]. The most well known gates in reversible and quantum computing are the Fredkin gate [48] together with Toffoli [134] and Feynman [44] gates.

2.1.3 Reversible Models of Concurrency

There exist many questions that need to be addressed when it comes to reversible computing, including what are the main approaches, results, potential benefits, and applications. Exploring reversibility through formal models formulates the theoretical foundations of what reversibility is, what purpose it serves, and how it benefits natural and artificial systems.

In particular, reversible formal models can be used for modelling reversible systems or developing techniques to analyse them. In order to comprehend the way reversibility works, it is useful to study the properties of these models when different notions and strategies of reversibility are applied. Understanding reversibility through various case studies could potentially propose a unified theory for reversibility in distributed systems, including behavioural and logic semantics, and explore how reversibility can help in specification, verification, and testing.

Creating expressive reversible formalisms that can be easily understood and simulated, even by scientists with expertise outside Computer Science, can prove very useful to understand, model, and design complex systems. The expressive power and descriptive nature offered by formal models coupled with reversible computation has the potential of providing an attractive setting for studying, analysing, or even imagining alternatives in a wide range of systems. For example, reversibility-inspired theories and formal methods will enable the software industry to deliver safer and more reliable distributed software and systems.

Such reversible formalisms will also assist scientists from other disciplines -for example biochemistry, mathematics and material science (superconductors)- since there exists various systems in the world of artificial and natural sciences where reversibility is inherent or it could be of interest. For instance, biochemical reactions, such as the isomerisation of glucose to fructose, are typically bidirectional, meaning that the direction of the computational system is fixed by an appropriate injection of energy or a change of entropy from environmental conditions like temperature or pressure [118, 119] Similarly, quantum computations are also inherently reversible because many of the components in quantum computers, such as databases or modular exponentiation, are reversible [44]. Reversibility is also used in software engineering to better explore a computation and analyse different possibilities, as in the exploration of a program state-space toward a solution, or in constructions of mechanisms for system reliability. In the same category belong systems of industrial robots often used in production for assembly and disassembly and are normally controlled by a single host computer.

Even though the physical implementations of the computational steps of such systems are naturally reversible, most abstract computation frameworks usually model the progress of computations through a sequence of forward irreversible steps. Therefore, the construction of reversible modelling languages can indicate how to capture the behaviour of reversible actions in order to implement or even extend the primitive processes of biological reactions, quantum computation, reliable systems, and movement in robotics.

These abstract computation models can be based on existing abstract formalisms and can be used not only for modelling reversible systems but also for investigating suitable notions of behavioural equivalences. The natural and artificial processes in these formalisms can be made reversible in order to facilitate more efficient model checking of new formulations of useful properties such as reachability, safety, exception and precedence. We can also explore whether adding the reversibility feature to these abstract models can increase considerably their computational and descriptional complexities. Hence, research on suitable behavioural semantics and modal logics for reversibility can result in sound foundations to commercial reversible modelling, debugging and testing software tools.

Challenges

Even though reversing computational processes in concurrent and distributed systems has many promising applications it also has many technical and conceptual challenges. In order to create the theoretical foundations of reversible formal models and to discover their purpose and benefits in natural and artificial systems we have to ascertain the costs and limitations that come with reversibility, and to explore the challenges and open problems. A formal model for concurrent systems that embeds reversible computation needs to address two challenges. The first one being the ability to identify the legitimate backward moves at any point during execution and the second one is the ability to compute without forgetting.

The first challenge depends on the choice of the computation's semantics that determine the order of forward and reverse actions. There are several forms of undoing computation that have been studied in the literature over the past years. In the sequential scenario, the legitimate backtracking moves can be trivially determined based on the order of execution. The computational steps are reversed based on the time of their execution and hence are undone in the exact inverse order of the forward execution.

Although, in the concurrent scenario speaking about backtracking in time is immaterial and the interplay with reversibility is no longer trivial. Therefore understanding this interplay is fundamental in many of the areas above, e.g., for biological or reliable distributed systems, which are usually naturally concurrent. In such concurrent systems we do not want to reverse the actions precisely in the opposite order than the one in which they were executed during forward computation, as this order is irrelevant. The concurrency relation between forward actions has to be taken into account and independent threads of execution should be reversed independently, whereas causal dependencies between related threads should remain protected.

There are however, many important examples, such as mechanisms driving long-running transactions [32,45] and biochemical reactions [118,119], where concurrent systems violate causality. Causally dependent threads are allowed to freely backtrack in an out-of-causal order which in a way would result in losing the initial computation structure. This means that reversing in out-of-causal order will not return a thread into a previously executed state but it would give it the ability to reach computation states which were formerly inaccessible.

The second challenge, of forgetting previously executed actions, applies to both concurrent and sequential systems. Since processes do not remember their past states if we want to reverse a standard process we will generate multiple possibilities. This challenge can be addressed by making the system exactly reversible using memories that remember the position and momentum of each action. When building or extending a reversible variant of a formal models, the syntax can be extended to allow the appropriate syntactic representations for computation memories that allow processes to keep track of everything that has been executed. The resulting mechanism should be light in terms of memory without the need of a global control.

Forms of Reversibility

The first challenge of identifying legitimate backward moves during reversal calls to identify possible strategies for going backwards. A large amount of work focused on identifying such strategies within process calculi [31,78,117,118]. Behind the insights of the theoretical study of reversible computation lie the challenging quest of understanding the nature of reversibility while formally representing various computational concepts. Understanding the role that reversibility plays in natural systems, helps in the development of realistic formal models for concurrent and distributed systems. Reversibility could initially be divided into two main categories: *Rigid* and *Uncontrolled* [83].

Rigid means that the execution of a forward step followed by the corresponding backward step leads back to the starting state, where an identical computation can restart. However rigid reversibility may not always be the best choice especially in the case of reliable systems. If the error that we are trying to recover was a transient fault then going back to the state that the error occurred and retrying the computations might solve the problem. Although, if the failure was permanent going back and, forth by following the same computational steps will infinitely result in the same error.

Uncontrolled means that there is no hint as to when to go forward and backward. Uncontrolled reversibility defines how to reverse a process execution by determining the necessary history and the associated causal transformation yet it does not specify when and whether to prefer backward execution over forward execution or vice versa. Uncontrolled reversibility gives good understanding on how reversible computation works, but it does not exploit it into applications because different application areas need different mechanisms to control reversibility.

Uncontrolled reversibility can be further subcategorised into several approaches for performing and undoing steps, which differ in the order in which steps are taken backwards and forwards. The most prominent of these are *backtracking*, *causal reversibility* and *out-of-causal-order reversibility*.

Backtracking is the process of rewinding one's computation trace, that is, computation steps are undone in the exact inverse order to the one in which they occurred. It does not allow any thread to freely backtrack because it might result in losing the initial computation structure and reaching computation states which were formerly inaccessible. This form of reversing ensures that at any state in a computation there is at most one predecessor state, yielding the property of *backwards determinism*. In the context of concurrent systems, this form of reversibility can be thought of as overly restrictive since, undoing moves only in the order in which they were taken, induces fake causal dependencies on backward sequences of actions: actions, which could have been reversed in any order are forced to be undone in the precise order in which they occurred.

Figure 2.1: Causal dependencies

Consider the following example with order of forward execution t_1, t_2, t_3. As indicated in Figure 2.1 let us assume that the action t_1 occurs independently of action t_2 and when both t_1 and t_2 occur they cause the execution of action t_3. Figure 2.2 shows that in backtracking mode there exists only one order of reverse execution which is the exact opposite direction of the forward one t_3, t_2, t_1.

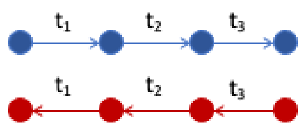


Figure 2.2: Backtracking

Relaxing the rigidity of backtracking, a second approach to reversibility, ***causal reversibility***, allows a more flexible form of reversing by allowing events to reverse in an arbitrary order, assuming that they respect the causal dependencies that hold between them.

Thus, in the context of causal reversibility, reversing does not have to follow the exact inverse order for independent events as long as caused actions, also known as effects, are undone before the reversal of the actions that have caused them. This form of reversibility is called causal, meaning that it respects causality a binary irreflexive relation of events that identifies which events cause others, and therefore need to be reversed last. Thus, causally backtracking a trace could be allowed along any path that respects causality also known as a causally equivalent path. A main feature of causal reversibility is that reversing an action returns a thread into a previously executed state, thus, any continuation of the computation after the reversal would also be possible in a forward-only execution where the specific step was not taken in the first place.

Consider the same example as before with forward execution t_1, t_2, t_3. Since t_1 occurs independently of action t_2 we can now reverse t_1 and t_2 in any order we want, although we can never reverse them before t_3. As indicated in Figure 2.3, causal order reversal gives an additional reverse path which is the execution of t_3, t_1, t_2, as well as, the backtracking path t_3, t_2, t_1.

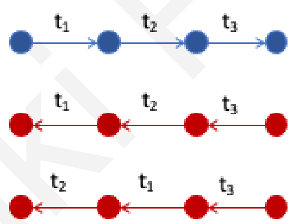


Figure 2.3: Causal reversing

Both backtracking and causal reversing are cause-respecting. There are however, many important examples where undoing events in an ***out-of-causal*** order is either inherent or could be beneficial. In fact, this form of undoing plays vital role on mechanisms driving long-running transactions and biochemical reactions [119]. This flexible notion of reversibility cancels out soundness since some backtracking computations could give access to formerly unreachable states. Consider every state of the execution to be a result of a series of actions that have causally contributed to the existence of the current state. If the actions were to be reversed in a causally-respecting manner then we would only be able to move back and forth through previously visited states. Therefore, one might wish to apply out-of-order reversibility in order to create fresh alternatives of current states that were formerly inaccessible by any forward-only execution path.

Again, consider the above example where now actions can be reversed in any possible execution path. Given the forward execution of t_1, t_2, t_3, six alternative reversing paths are produced as potential reverse executions based on out-of-causal reversibility. In Figure 2.4 can be observed that these paths include paths produced by backtracking execution as well as, paths produced during causal reversal.

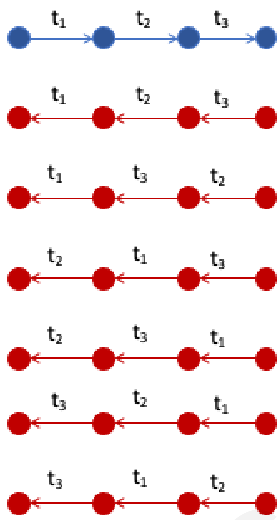


Figure 2.4: Out-of-causal reversing

Reversible Formalisms

Both challenges have been addressed within various computational models ranging from process calculi [31, 117] to event structures [27, 139]. The second challenge, of forgetting previously executed actions, has been addressed using external mechanisms such as memories or identifiers that remember the position and momentum of each action. Since processes do not remember their past states, various reversible formalisms make a system exactly reversible by extending the syntax to include mechanisms that serve as histories of executed actions of the corresponding processes.

Research on reversible formal languages can be traced back to a publication from Berry and Boudol titled “Chemical Abstract Machine" [19]. The authors propose a calculus, inspired by chemical reactions, whose operational semantics define forward and the corresponding reverse reduction relation. They have introduced the notion of a chemical abstract machine called “cham" which is based on the chemical metaphor used in the Γ language. They have illustrated the descriptive powers of the chemical abstract machine by showing that it is suited to model concurrent systems and reversible computations.

The first attempt to reverse classical process calculi was explored by Vincent Danos and Jean Krivine [33] who built a notion of distributed backtracking on top of Milner’s CCS [97].

The proposed process calculus was named CCS-R which is essentially a reversible extension of CCS motivated by the desire to represent reversible biological systems as the evolution of biological processes. Reversibility is embedded in the syntax of CCS as a distributed monitoring system and meshes well with the forward only syntax of the host calculus.

Some of the limitations of the model of CCS-R have been later addressed on a newer version named RCCS [31] where CCS-R is extended to deal with recursion, and uses unique names to identify threads. RCCS is again a process algebra in the style of CCS where processes have the ability to backtrack. Their calculus is essentially Milner's CCS with the added bonus that some observable actions in the standard labelled transition system semantics can be understood to be reversible. Their seminal paper was the first to discuss the notion of causality as a suitable requirement for reversibility in a concurrent scenario and paved the way to the definition of causal-consistent reversibility. RCCS is a causal-consistent reversible extension of CCS that uses memory stacks in order to keep track of past communications, further developed in [32].

RCCS [31] and the work of [1] on mapping functional programs into reversible automata inspired I. Phillips and I. Ulidowski to proposed another approach on reversing CCS, named CCSK [117]. CSSK is a reversible version of CCS based on the use of communication keys. It can be used to model and analyse the bidirectional behaviour of systems that are able to choose the direction of execution spontaneously, for example the binding and unbinding of molecules in biochemical reactions. Given a forward transition relation their reversible algebraic process calculi is able to obtain the inverse because it has the ability to remember previously executed actions. To achieve this they have introduced a method for converting the standard irreversible operators of CCS into reversible operators, while preserving their operational semantics. Similarly to RCCS, they use a memory mechanism which contains a history of past communication keys and it can be used to reverse computation in a causally preserving manner. Contrary to the global control and extensive record keeping of RCCS, their motivation was to produce a reversible process calculus that does not rely heavily on external devices such as memories. The most crucial component of their procedure is the notion of communication keys which are a more expressive form of past actions. These are unique identifiers that are used to "mark" a previously executed action a by a fresh identifier k and record it in the syntax of the action's occurrence as $a[k]$.

The creators of RCCS continued their work by employing their reversible mechanism to π-calculus by proposing a reversible labelled transition semantics called $R\pi$ [29]. They introduce the syntax and semantics for the reversible π-calculus and they prove similar results to the ones proved for RCCS, such as equivalence between any backtracking path and forward computation as well as causal equivalence up-to permutation which means that computations are maximally liberal with respect to the structural causality of the reduction semantics.

This work continues in [82] where Lanese et al. presented a reversible asynchronous higher-order π-calculus, called $\rho\pi$, which has been shown to be causally consistent and gives two original contributions. The first one is a novel reversible machinery which, in the contrary of the previously proposed machineries in CCS, preserves the classical structural congruence laws of the π-calculus, and relies on simple name tags for identifying threads and explicit memory processes. The second contribution is a faithful encoding on $\rho\pi$ calculus into a variant of HOπ, showing that adding reversibility does not change substantially the expressive power of HOπ. The work on reversible π-calculus continues from $\rho\pi$ to roll-π [78], which is a fine-grained rollback primitive for higher-order π-calculus, that builds on the reversibility apparatus of $\rho\pi$ in order to adopt the ability to undo every single step in a concurrent execution. In [76] Lanese et al. continued their work on reversible π-calculus by proposing a new concurrent process calculus, named croll-π, as a framework for flexible reversibility and compensating roll-π. Croll-π features flexible reversibility, where it is possible to specify alternatives to a computation, that can be used upon explicit rollback.

On a more general note, the work in [77] proposes a general and automatic technique which defines a causal-consistent reversible extension for forward models. These models include a variety of formalisms studied in the literature on causal-consistent reversibility such as Higher-Order π-calculus and Core Erlang. Another work aiming to generalise causal reversibility in formalisms, is that of [84]. This work examines the various properties that a reversible system should enjoy and shows how they relate to the already suggested properties such as the parabolic lemma and the causal consistency property. Specifically, a generic labelled transition system has been used to capture these properties as a set of axioms which can then be used by reversible formalisms in order to verify their properties. Additionally, two new notions of causal consistent reversibility are derived from these axioms, namely safety and causal liveness.

Most recently, the study of out-of-causal-order reversibility continued with the introduction of a new operator for modelling local reversibility in [70]. The authors here also consider controlled reversibility in CCS, in the form of a reversible process calculus called Calculus of Covalent Bonding (CCB). Their reversible calculus has a novel and purely local in character prefixing operator. This operator has been inspired by the mechanism of covalent bonding, which is the most common type of chemical bonds between atoms, that allows modelling of locally controlled reversibility. In their proposal actions can be undone spontaneously or as pairs of concerted actions, where performing a weak action forces reversing of another action. The new operator in a restricted version of their calculus preserves causal consistency, however in its full generality it also allows modelling in out-of-causal order, where effects are undone before their causes.

Reversibility has also been extended to quantum process calculi, that are used to describe and model the behaviour of systems that combine classical and quantum communication and computation the most prominent being qCCS [43] and CQP [52]. qCCS is a natural quantum extension of CCS which can deal with input and output of quantum states, and unitary transformations and measurements on quantum systems. The operational semantics of qCCS is given in terms of probabilistic labeled transition system. CQP (Communicating Quantum Processes) has been defined for modelling systems which combine quantum and classical communication and computation. CQP combines the communication primitives of the pi-calculus with primitives for measurement and transformation of quantum state; in particular, it has a static type system which classifies channels, distinguishes between quantum and classical data, and controls the use of quantum state.

The study of reversible process calculi triggered also research on more abstract models for describing concurrent systems such as event structures [115, 119]. In particular, the work on CCSK has also continued in a paper titled “Reversibility and models of Concurrency" [115] where the authors studied the impact of allowing events to be undone in prime event structures. They proposed prime graphs to prove that transition systems associated with CCSK and other reversible process algebras are equivalent as models to labelled prime event structures. This study continued in [119] where they proposed how to model reaction systems that consist of objects that are combined together by the means of bonds or dissolved via reduction-style semantics. Motivated from the initial study of [119] research on reversible event structures continued with introducing reversible forms of prime event structures and asymmetric event structures [116]. In order to control the manner in which events are reversed, the authors focused on analysing asymmetric conflict and causation of events in the reversible, and not necessarily causal, setting. Ulidowski et al. continued their research on reversible event structures in a publication titled “Concurrency and reversibility" [136], where they have shown how to model reversibility in concurrent computation as realised abstractly in terms of event structures. The authors have introduced two different forms of events structures: event structures defined in terms of the causation and precedence relations, and event structures defined by the enabling relation. The proposed forms of event structures have been illustrated in various examples that demonstrate how to model causally consistent reversibility as well as out-of-causal-order reversibility.

In the literature there exists another line of research concerning reversible process calculi that focuses on dealing with reduction semantics describing the evolution of processes in isolation. This approach is usually simpler and hence more easily applicable to expressive calculi such as CCS and π-calculus. In this line of research we are able to find Reversible structures a reversible computational calculus for modelling chemical systems, composed of signals and gates [27]. Reversible structures are computational units that may progress in forward and backward direction. They are amenable to biological implementations in terms of DNA circuits and are expressive enough to encode a reversible process calculus such as asynchronous RCCS.

The first study of reversible computation within Petri nets was proposed in [15, 16]. In these works, the authors investigate the effects of adding *reversed* versions of selected transitions in a Petri net, where these transitions are obtained by reversing the directions of a transition's arcs. They then explore decidability problems regarding reachability and coverability in the resulting Petri nets. However, non-deterministically deciding to reverse any of the transitions causes the reversal of the "wrong" transition which might lead to new states that have not been reached through forward execution only. The reason behind this is that the addition of reversibility into a model like Petri nets results in various backward conflicts where a token can be generated in a place because of different transition firings. The marking of that particular place is not enough to deduce whether the token has been produced because of a particular transition. This approach on reversible computation violates causality which is more challenging than randomly selecting reversed transitions since in a concurrent setting there is no natural way for totally ordering events.

Towards examining causal consistent reversibility in Petri nets, the work in [96] investigates whether it is possible to add a complete set of effect-reverses for a given transition without changing the set of reachable markings. The authors show that this problem is in general undecidable however it can be decidable in cyclic Petri nets where with the addition of new places these non-reversible Petri nets can become reversible while preserving their behaviour. Moreover, the works of [94, 95] propose a causal semantics for P/T nets by identifying the causalities and conflicts of a P/T net through unfolding it into an equivalent occurrence net and subsequently introducing appropriate reverse transitions to create a coloured Petri net that captures a causal-consistent reversible semantics. The colours in this coloured Petri net capture causal histories. On a similar note, [93] introduces the notion of reversible occurrence nets and associate a reversible occurrence net to a causal reversible prime event structure, and vice versa. In [35] the authors examine the possibility of reversing the effect of the execution of groups of various transitions (steps). They then present a number of properties which arise in this context and show that there is a crucial difference between reversing steps which are sets and those which are true multisets.

On another note, having models that express controlled reversibility is more useful in real life applications. For instance, various biological phenomena control the direction of the computation based on physical conditions such as temperature, pressure and reaction rates. Therefore, a distinguishing feature of reversible computation is that of *c*ontrolling reversibility: while various frameworks make no restriction as to when a transition can be reversed (uncontrolled reversibility), it can be argued that some means of controlling the conditions of transition reversal is often useful in practice. For instance, when dealing with fault recovery, reversal should only be triggered when a fault is encountered. Based on this observation, a number of strategies for controlling reversibility have been proposed: [32] introduces the concept of irreversible actions, and [80] introduces compensations to deal with irreversible actions in the context of programming abstractions for distributed systems. Another approach for controlling reversibility is proposed in [118] where an external entity is employed for capturing the order in which transitions can be executed in the forward or the backward direction. In another line of work, [78] defines a roll-back primitive for reversing computation, and in [76] roll-back is extended with the possibility of specifying the alternatives to be taken on resuming the forward execution. Finally, in [9] the authors associate the direction of action reversal with energy parameters capturing environmental conditions of the modelled systems.

Reversible calculi were born with mainly biological motivation. Since many biological phenomena are naturally reversible, a reversible formalism seems to be suitable to model such systems. Indeed, efforts have been made to model biological systems [27, 33, 118, 119] as well as chemical reactions [70] using reversible process calculi. We highlight [27] which illustrates a compilation from asynchronous CCS to DNA circuits. Given their formal definition, process calculi are suitable to formally verify properties of systems. There is a strong line of work concerned with applications of reversible process calculi such as session types, contracts, biological phenomena, and constructs for reliability. [131] shows how the session type discipline of π-calculus extends to its reversible variants. In [132], (binary and multiparty) session type systems are used to restrict the study of reversibility in π-calculus to single sessions. Instead, [11, 12] study the compliance of a client and a server when both of them have the ability to backtrack to past states. In a different setting, reversible process calculi have also been used to build constructs for reliability, and in particular communicating transactions with compensations [36]. Transactions with compensations are computations that either succeed, or their effects are compensated by a dedicated ad-hoc piece of code. In [36], the effect of the transaction is first undone, and then a compensation is executed. Behavioural equivalences for communicating transactions with compensation have also been studied in [37, 68]. In [76], interacting transactions with compensations are mapped into a reversible calculus with alternatives.

2.2 Petri Nets

In this work, we shall consider a particular model of concurrency, known as Petri nets, that in this thesis will be extended to its reversible variant. It is a basic model of parallel and distributed systems, designed by Carl Adam Petri in 1962 in his PhD Thesis:"Kommunikation mit Automaten" [110, 122]. Petri Nets are a graphical mathematical language that can be used for the specification and analysis of discrete event systems. Petri nets are a formal model of concurrent systems which supports both action-based and state-based modelling and reasoning where the basic idea behind it is to describe state changes in a system with transitions.

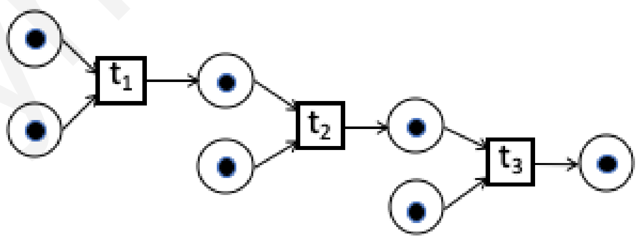


Figure 2.5: Petri net

Petri nets are extensions of directed, finite, bipartite graphs, typically without isolated nodes as seen in Figure 2.5. They are also known as place/transition (PT) nets based on their four main components: places, transitions, arcs and tokens. Formally:

Definition 1. A *Petri Net* is a tuple $N = (P, T, F, W, M_0)$ where:

1. P is a finite set of *places*.

2. T is a finite set of *transitions* such that $P \cap T = \emptyset$.

3. F is a set of arcs (or flow relations) $F \subset (P \times T) \cup (T \times P)$

4. $W : ((P \times T) \cup (T \times P)) \rightarrow \mathbb{N}$ is the arc weight mapping where $W(f) = 0$ for all $f \notin F$, and $W(f) > 0$ for all $f \in F$, and

5. $M_0 : P \rightarrow \mathbb{N}$ is the initial marking representing the initial distribution of tokens

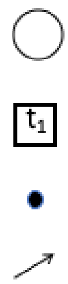


Figure 2.6: Petri net components

Main Components. As seen in Figure 2.6 the first component is places which are illustrated by circles and refer to as conditions or states. They are passive nodes used to store, accumulate or show tokens to indicate the current state of the execution. Places are used to define conditions that need to be satisfied in order to execute a specific action. Therefore, if a place has an incoming directed arrow then it is called a post-place and if it has outgoing directed arrows then it is called a pre-place. Places filled by tokens indicate the current state of the execution and the overall distribution of tokens across places is known as the marking M. The input state is indicated by the initial marking M_0 and every other consecutive state can be reached by relocating the required tokens.

The second component is transitions which are indicated by bars or boxes containing the respective label and model activities which can occur when a transition fires. They are active nodes that when fired they can produce, consume transport or change tokens, indicating the execution of the corresponding event. A transition is enabled to be fired only when the number of tokens in each of its input places is at least equal to the number of arcs going from the place to the transition meaning that all of its pre places need to be filled with tokens. Firing a transition means consuming tokens from its pre-places and then distributing them to each output place. Thus, after the firing of a transition the marking of the net is changed to a new reachable marking, where some transitions are no longer enabled while others become enabled.

Places and transitions are connected to each other by directed arcs. Graphically an arc is represented by an arrow indicating the relation between the components such as logical connections, access rights, spatial proximities or immediate linkings. An arc never connects two places or two transitions. It rather runs from a place to a transition or from a transition to a place. The order in which transitions and places appear amongst the directed arcs defines which places are pre-places and therefore are required in order to fire the transition and which are post-places. Labelled incoming arcs may be drawn from a place into a transition labelled with a natural number indicating the finite number of tokens in the associated place that need to be consumed by the occurrence of the transition. Labelled outgoing arcs may be drawn from a transition to a place labelled with a natural number indicating the number of tokens to be deposited in the place by the occurrence of an event.

A Petri net is a particular kind of directed graph, together with an initial state called the initial marking. A marking assigns to each place a non-negative integer indicating the is the configuration of tokens distributed over an entire Petri net diagram. Pictorially, we place black bullets representing tokens in various places of a net where semantically a marking is denoted by M, an m-vector, where m is the total number of places. If a marking assigns to place p a non-negative integer k, we say that p is marked with k tokens. A marking is depicted by placing a dot (token) in each of its places. The dynamic behaviour of the represented system, in terms of system state and its evolution, is defined by describing the possible moves between markings based on the marking evolution rule. The marking of the net changes through the occurrence of transitions according to what is commonly called the token game for nets.

As in many formal models the concept of conditional events can be used in order to represent the dynamics of a system. In Petri net modelling places represent conditions, and transitions represent events. A transition has a certain number of input and output places representing the pre-conditions and post-conditions of the event, respectively. The presence of a required number of tokens in a place is interpreted as holding the truth of the condition associated with the incoming arc. A transition is said to be enabled if all input places have sufficient number of tokens for the firing consumptions to be possible. Meaning that the number of tokens in an input place should be at least equal to the weight of the arc joining such a place with the considered transition. Formally, a transition t is said to be enabled if each input place p of t is marked with at least $w(p, t)$ tokens, where $w(p, t)$ is the label of the arc from p to t.

Once a transition is enabled it may or may not fire depending on whether or not the event actually takes place. The firing of an enabled transition is an instantaneous operation which evolves a marking into a new marking. In that case, a number of tokens is consumed by removing from each input place of the transition a number of tokens equal to the weight of the incoming arc leading to a transition. Tokens are consumed by the firing, but also new tokens are produced in its outgoing places, namely a number of tokens are created to each output place equal to the weight of the arc joining the considered transition with such a place.

Extensions. There are many extensions of Petri nets. Some of them are completely backwards-compatible with the original Petri nets, while some add properties that cannot be modelled in the original Petri net formalism. Although backwards-compatible models do not extend the computational power of Petri nets, they may have more succinct representations and may be more convenient for modelling. Extensions that cannot be transformed into Petri nets are sometimes very powerful, but usually lack the full range of mathematical tools available to analyse ordinary Petri nets. An extension of Petri nets is the addition of new types of arcs; such as inhibitor arcs which impose the precondition that the transition may only fire when the place is empty and reset arcs [41] which do not impose a precondition on firing, and empties the place when the transition fires. Reset arcs make reachability undecidable, while some other properties, such as termination, remain decidable [5]; whereas inhibitor arcs allow arbitrary computations on numbers of tokens to be expressed, which makes the formalism Turing complete and implies existence of a universal net [145].

The term *high-level Petri net* is used for many extensions of Petri nets although, the term is mostly used for the type of Coloured Petri nets [64] supported by CPN Tools. In a standard Petri net, tokens are indistinguishable whereas in a coloured Petri nets, every token has a colour. This allows tokens to have a data value attached to them which can be of arbitrarily complex type where places in CPNs usually contain tokens of one type, which is called a coloured set. In popular tools for coloured Petri nets such as CPN Tools, the values of tokens are typed, and can be tested using guard expressions and manipulated with a functional programming language. Coloured Petri nets preserve useful properties of Petri nets and at the same time extend the initial formalism to allow the distinction between tokens.

Another extension is that of *timed Petri nets* [141] used to model the timing of a model, where time constraints restrict the causal behaviour of the system and limit its state space by forcing events to occur and keep others from happening following the constraints. In time Petri nets, there is an upper and a lower bound for the time an event can remain enabled without occurring after its preconditions are met. The upper time bound of one potential event can limit the time when another conflicting event can occur, creating dependencies not seen in the simple causal view of the system.

The qualitative notion of time is implicitly represented in Petri nets in the sense that each firing of a transition is associated with a timestamp or clock cycle. In such a representation, the firing of transitions depends not only on the marking, but also on the elapsed time since the occurrence of some other events. The elapsed time is not represented by the number of internal ticks since the start of the clock but is represented based on the configuration of a marking at the given clock cycle.

There are several variations of timed Petri nets incorporating the notion of time to virtually every component of the Petri nets framework, namely transition, tokens, arcs, and, places. A subsidiary of timed Petri nets are the *stochastic Petri nets* [90] that add nondeterministic time through adjustable randomness of the transitions. The exponential random distribution is usually used to time these nets. In this case, the nets' reachability graph can be used as a continuous time Markov chain (CTMC).

Modelling and Analysis. Various kinds of Petri nets are applied in different disciplines, including Computer Science (formal languages [123], logic programs [130]), business process modelling (decision models [129]), information management (distributed-database systems [140]), software engineering (concurrent and parallel programs [101]), and systems engineering (multiprocessor memory systems [91], asynchronous circuits and structures [38, 65], compiler and operating systems [10]). The reason is that Petri nets are a promising tool for describing and studying information processing systems, that are characterised as being concurrent, asynchronous, distributed, parallel, nondeterministic, and/or stochastic. Specifically, executable modelling languages, applied with proper tool support, are expected to automate system validation and verification, simulation and code generation from the modelling language representation. Another advantage, is that formal modelling is more rigorous by nature because it explores every possibility to ensure correctness and completeness. Formal techniques mainly include process algebras, temporal logic, automata theoretic techniques, Petri nets and partial order models.

As a mathematical tool, it is possible to set up state equations, algebraic equations, and other mathematical models governing the behaviour of systems. The simplicity of the basic user interface of Petri nets has easily enabled extensive tool support over the years, particularly in the areas of model checking, graphically oriented simulation, and software verification. As a graphical tool, Petri nets can be used as a visual-communication aid similar to flow charts, block diagrams, and networks. The use of computer-aided tools is a necessity for practical applications of Petri nets and thus most Petri-net research groups have their own software packages and tools to assist the drawing, analysis, and simulation of various applications.

A major strength of Petri nets is their support for analysis of many properties and problems associated with concurrent systems [102]. They can be used to study the reachability and coverability problems as well as study properties such as liveness, boundedness, invariance and conservativeness. Reachability is a fundamental basis for studying the dynamic properties of any system and is essentially the ability to identify whether a given state is reachable from the initial state. However, coverability represents precisely the coverable sets rather than the reachable sets and is closely related to liveness which is the possibility to ultimately fire any transition of the net by progressing through some further firing sequence. A Petri net is also said to be k-bounded or simply bounded if the number of tokens in each place does not exceed a finite number k for any marking reachable from the initial marking. It is also said to be conservative if there exists a positive integer for every place such that the weighted sum of tokens is the same for every marking and for any fixed initial marking. Another important feature of Petri nets is that their structural properties can be obtained by linear algebraic techniques. Such properties are called invariants because they are the properties that depend on only the topological structure of a Petri net and are independent of the initial marking. One of the properties studied in the context of Petri nets is that of Petri net reversibility which describes the ability of a system to return to the initial state from any reachable state. This, however, is in contrast to the notion of reversible computation as discussed in this work where the intention is not to return to a state via arbitrary execution but to reverse the effect of already executed transitions.

Petri Net Causality. *Causality* is one of the most interesting notions in Petri net theory since it allows the explicit representation of causal dependencies between action occurrences when modelling reactive systems. In fact, how to formalise causal dependencies based on an appropriate causality based concept is a well-known topic in Petri net theory [138]. The investigation of Petri nets has given rise into two different approaches when it comes to causality, one of them being disjunctive causality implemented by the collective token interpretation and the other one being partial order causality implemented by the individual token interpretation [24, 137, 139]. Many different semantics have been proposed in the literature for both views, all of them aiming to remain abstract enough while doing justice to the truly concurrent nature of Petri nets. Each philosophy can be justified either by the theoretical properties of the modelled systems, or by the implementation of possible applications.

A common concern between most of the theoretical models of computation is expressing causality in concurrent systems. In contrast to the sequential setting that is well understood, the concurrent setting poses the conceptual question of how do we define the causal order of execution. When it comes to Petri nets the ability to formally express causal dependencies based on an appropriate causality based concept is one of the most well-known problems of Petri nets but also one of the most interesting properties [138].

Most of the behavioural models for Petri nets have firing rules that embody the collective token philosophy rather than the individual token philosophy. The collective token philosophy has been investigated in [23] and is considered to be the standard firing rule of Petri nets were tokens residing in the same place are indistinguishable. In the collective token interpretation when multiple tokens of the same type reside in the same place then these tokens are not distinguished. This means that all that is known by the model is the amount of token occurrences of a specific type and their location in the marking. In the collective token philosophy were we assume unambiguous tokens to be equivalent because when focussing on the net behaviour these tokens are operationally equivalent. The collective approach fits well with resource allocation systems where tokens represent resources and their identity is indistinguishable since their behavioural capabilities are identical.

The computational interpretation of the collective token philosophy has been extended to the individual token approach, where tokens residing in the same place are distinguished based on their causal path [55,121]. As such, the individual token interpretation distinguishes tokens of the same type as individual and it has been formally described by the notion of a process in [55, 121]. In this approach the model keeps track of where the tokens come from and therefore identifies the causal links between transitions as means of partial order. The semantics of the individual token interpretation are more complicated since this approach requires precise correspondence between the token instances and their past. This approach solves the ambiguity between tokens of the same type by allowing tokens to carry information about their mappings. This distinction between tokens allows us to give a precise account of the causal and distributed nature of the net as a partial order. The causal relations between the transitions in a distributed run of a net can also be described by means of causal net [138]. In the standard approach to causality [110] a causal link is considered to exist between two transitions if one produces tokens that are used to fire the other. This relation is used to define “causal order" which is transitive so that if a transition t_1 causally precedes t_2 and t_2 causally precedes t_3 then t_1 also causally precedes t_3. Furthermore, it is an irreflexive relation, i.e., no transition causally precedes itself.

Chapter 3

Reversible Computation in Petri Nets

During the last few years a number of formal models have been developed aiming to provide understanding of the basic principles of reversibility along with its costs and limitations, and to explore how it can be used to support the solution of complex problems. In this chapter, we set out to study reversible computation in the context of Petri Nets and to explore the modelling of the main strategies for reversing computation. We aim to address the challenges of capturing the notions of backtracking, causal reversibility and out-of-causal-order reversibility within the Petri Net framework, thus proposing a novel, graphical methodology for studying reversibility in a model where transitions can be taken in either direction. Our proposal is motivated by applications from biochemistry where out-of- causal-order reversibility is inherent, and it supports all the forms of reversibility that have been discussed above.

Adding reversibility to Petri nets turns out to be quite nontrivial since the presence of cycles exposes the need to define causality of actions within a cyclic structure. Indeed, there are different ways of introducing reversible behaviour depending on how causality is defined. In our approach, we follow the notion of causality as defined by Carl Adam Petri for one-safe nets that provides the notion of a run of a system where causal dependencies are reflected in terms of a partial order [110]. A causal link is considered to exist between two transitions if one produces tokens that are used to fire the other. In this partial order, causal dependencies are explicitly defined as an unfolding of an occurrence net which is an acyclic net that does not have backward conflicts. We prove that the amount of flexibility allowed in causal reversibility indeed yields causally consistent semantics. We also demonstrate that out-of-causal-order reversibility is able to create new states unreachable by forward-only execution. Additionally, we establish the relationship between the three forms of reversing and define a transition relation that can capture each of the three strategies modulo the en-

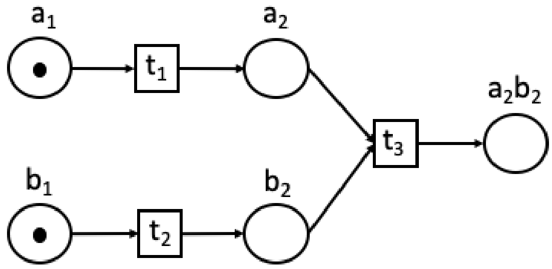


Figure 3.1: Causal reversibility

abledness condition for each strategy. This allows us to provide a uniform treatment of the basic theoretical results. We demonstrate the framework with a model of the Ras-Raf-MEK-ERK pathway [118], and a transaction processing system, examples that inherently feature (out-of-causal-order) reversibility.

3.1 Forms of Reversibility and Petri Nets

Reversing computational processes in concurrent and distributed systems has many promising applications but also presents some technical and conceptual challenges. In particular, a formal model for concurrent systems that embeds reversible computation needs to be able to compute without forgetting and to identify the legitimate backward moves at any point during computation.

The first challenge applies to both concurrent and sequential systems. Since processes typically do not remember their past states, reversing their execution is not directly supported. This challenge can be addressed with the use of memories. When building a reversible variant of a formal language, its syntax can be extended to include appropriate representations for computation memories to allow processes to keep track of past execution.

The second challenge regards the strategy to be applied when going backwards. As already mentioned, the most prominent approaches for performing and undoing steps are *backtracking*, *causal reversibility*, and *out-of-causal-order reversibility*. *Backtracking* is well understood as the process of rewinding one's computation trace, whereas in causal reversibility, reversing does not have to follow the exact inverse order for events as long as caused actions, also known as effects, are undone before the reversal of the actions that have caused them.

For example, consider the Petri net in Figure 3.1. We may observe that transitions t_1 and t_2 are independent from each other as they may be taken in any order, and they are both prerequisites for transition t_3. Backtracking the sequence of transitions $\langle t_1, t_2, t_3 \rangle$ would require that the three transitions should be reversed in exactly the reverse order, i.e. $\langle t_3, t_2, t_1 \rangle$.

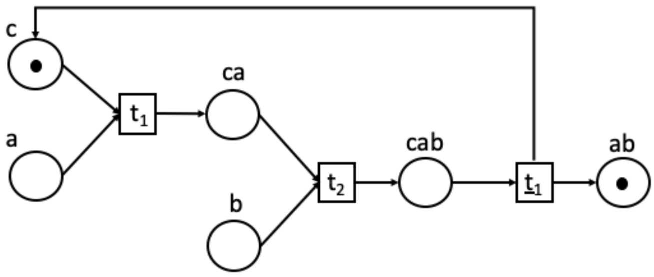


Figure 3.2: Catalysis in classic Petri nets

Instead, causal flexibility allows the inverse computation to rewind t_3 and then t_1 and t_2 in any order, but never t_1 or t_2 before t_3.

Both backtracking and causal reversing are cause-respecting. There are, however, many important examples where concurrent systems execute in out-of-causal-order reversibility in order to allow a system to discover states that are inaccessible in any forward-only execution. This can be achieved since, reversing in out-of-causal order allows reversing an action before its effects are undone, and subsequently exploring new computations while the effects of the reversed action are still present. As such, out-of-order reversibility can create new alternatives of current states that were formerly inaccessible by any forward-only execution path.

Since out-of-order reversibility contradicts program order, it comes with its own peculiarities that need to be taken into consideration while designing reversible systems. To appreciate these peculiarities and obtain insights towards our approach on addressing reversibility within Petri nets, consider the process of catalysis from biochemistry, whereby a substance called *catalyst* enables a chemical reaction between a set of other elements. Specifically consider a catalyst c that helps the otherwise inactive molecules a and b to bond. This is achieved as follows: catalyst c initially bonds with a which then enables the bonding between a and b. Finally, the catalyst is no longer needed and its bond to the other two molecules is released. A Petri net model of this process is illustrated in Figure 3.2. The Petri net executes transition t_1 via which the bond ca is created, followed by action t_2 to produce cab. Finally, action $\underline{t_1}$ "reverses" the bond between a and c, yielding ab and releasing c. (The figure portrays the final state of the execution assuming that initially exactly one token existed in places a, b, and c.)

This example illustrates that Petri nets are not reversible by nature, in the sense that every transition cannot be executed in both directions. Therefore an inverse action (e.g., transition $\underline{t_1}$ for undoing the effect of transition t_1), needs to be added as a supplementary forward transition for achieving the undoing of a previous action. This explicit approach of modelling reversibility can prove cumbersome in systems that express multiple reversible patterns of execution, resulting in larger and more intricate systems. Furthermore, it fails to capture reversibility as a mode of computation. The intention of our work is to study an approach for modelling reversible computation that does not require the addition of new, reversed transitions but instead offers as a basic building block transitions that can be taken in both the forward as well as the backward direction, and, thereby, explore the theory of reversible computation within Petri nets.

However, when attempting to model the catalysis example while executing transitions in both the forward and the backward directions, we may observe a number of obstacles. At an abstract level, the behaviour of the system should exhibit a sequence of three transitions: execution of t_1 and t_2, followed by the reversal of transition t_1. The reversal of transition t_1 should implement the release of c from the bond cab and make it available for further instantiations of transitions, if needed, while the bond ab should remain in place. This implies that a reversing Petri net model should provide resources a, b and c as well as ca, cab and ab and implement the reversal of action t_1 as the transformation of resource cab into c and ab. Note that resource ab is inaccessible during the forward execution of transitions t_1 and t_2 and only materialises after the reversal of transition t_1, i.e., only once the bond between a and c is broken. Given the static nature of a Petri net, this suggests that resources such as ab should be represented at the token level (as opposed to the place level). As a result, the concept of token individuality is of particular relevance to reversible computation in Petri nets while other constructs/functions at token level are needed to capture the effect and reversal of a transition.

Indeed, reversing a transition in an out-of-causal order may imply that while some of the effects of the transition can be reversed (e.g., the release of the catalyst back to the initial state), others must be retained due to computation that succeeded the forward execution of the next transition (e.g., token a cannot be released during the reversal of t_1 since it has bonded with b in transition t_2). This latter point is especially challenging since it requires to specify a model in a precise manner so as to identify which effects are allowed to be “undone” when reversing a transition. Thus, as highlighted by the catalysis example, reversing transitions in a Petri net model requires close monitoring of token manipulation within a net and clear enunciation of the effects of a transition.

As already mentioned, the concept of token individuality can prove useful to handle these challenges. This concept has also been handled in various works, e.g., [128, 137, 138],

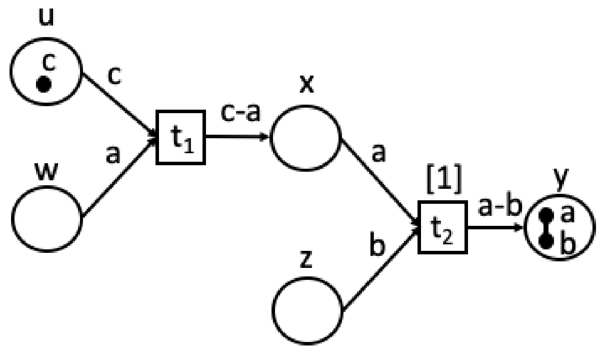


Figure 3.3: Catalysis in reversing Petri nets

where each token is associated with information regarding its causal path, i.e., the places and transitions it has traversed before reaching its current state. In our approach, we also implement the notion of token individuality where instead of maintaining extensive histories for recording the precise evolution of each token through transitions and places, we employ a novel approach inspired by out-of-causal reversibility in biochemistry as well as approaches from related literature [119]. The resulting framework is light in the sense that no memory needs to be stored per token to retrieve its causal path while enabling reversible semantics for the three main types of reversibility. Specifically, we introduce two notions that intuitively capture tokens and their history: the notion of a *base* and a new type of tokens called *bonds*. A base is a persistent type of token which cannot be consumed and therefore preserves its individuality through various transitions. For a transition to fire, the incoming arcs identify the required tokens/bonds and the outgoing arcs may create new bonds or transfer already existing tokens/bonds along the places of a Petri Net. Therefore, the effect of a transition is the creation of new *bonds* between the tokens it takes as input and the reversal of such a transition involves undoing the respective bonds. In other words, a token can be a base or a coalition of bases connected via bonds into a structure.

Based on these ideas, we may describe the catalysis example in our proposed framework as shown in Figure 3.3. In this setting a and c are bases which are connected via a bond into place x during transition t_1, while transition t_2 brings into place a new bond between a and b. In Figure 3.3 we see the state that arises after the execution of t_1 and t_2 and the reversal of transition t_1. In this state, base c has returned to its initial place u whereas bond $a-b$ has remained in place y. A thorough explanation of the notation is given in the next section.

Finally, in order to identify at each point in time the history of execution, thus to discern the transitions that can be reversed given the presence of backward nondeterminism of Petri nets, we associate transitions with history storing keys in increasing order each time an instance of the transition is executed. This allows to backtrack computation as well as to extract the causes of bonds as needed in causal and out-of-causal-order reversibility.

3.2 Reversing Petri Nets

We define reversing Petri nets as follows:

Definition 2. A *Reversing Petri net* (RPN) is a tuple (A, P, B, T, F) where:

1. A is a finite set of *bases* or *tokens* ranged over by a, b, ... $\overline{A} = \{\overline{a} \mid a \in A\}$ contains a *negative instance* for each token and we write $\mathcal{A} = A \cup \overline{A}$.

2. P is a finite set of *places*.

3. $B \subseteq A \times A$ is a set of undirected *bonds* ranged over by β, γ, ... We use the notation $a{-}b$ for a bond $(a, b) \in B$. $\overline{B} = \{\overline{\beta} \mid \beta \in B\}$ contains a *negative instance* for each bond and we write $\mathcal{B} = B \cup \overline{B}$.

4. T is a finite set of *transitions*.

5. $F : (P \times T \cup T \times P) \to 2^{\mathcal{A} \cup \mathcal{B}}$ defines a set of directed *arcs* each associated with a subset of $\mathcal{A} \cup \mathcal{B}$.

A Reversing Petri net is built on the basis of a set of *bases* or *tokens*. We consider each token to have a unique name. In this way, tokens may be distinguished from each other, their persistence can be guaranteed and their history inferred from the structure of a Petri net (as implemented by function F, discussed below). Tokens correspond to the basic entities that occur in a system. They may occur as stand-alone elements but they may also merge together to form *bonds*. *Places* and *transitions* have the standard meaning.

Directed arcs connect places to transitions and vice versa and are labelled by a subset of $\mathcal{A} \cup \mathcal{B}$ where $\overline{A} = \{\overline{a} \mid a \in A\}$ is the set of *negative* tokens expressing token absence, and $\overline{B} = \{\overline{\beta} \mid \beta \in B\}$ is the set of *negative* bonds expressing bond absence. For a label $\ell = F(x, t)$ or $\ell = F(t, x)$, we assume that each token a can appear in ℓ at most once, either as a or as $\overline{a}$, and that if a bond $(a, b) \in \ell$ then $a, b \in \ell$. Furthermore, for $\ell = F(t, x)$, it must be that $\ell \cap (\overline{A} \cup \overline{B}) = \emptyset$, that is, negative tokens/bonds may only occur on arcs incoming to a transition. Intuitively, these labels express the requirements for a transition to fire when placed on arcs incoming the transition, and the effects of the transition when placed on the outgoing arcs. Thus, if $a \in F(x, t)$ this implies that token a is required for the transition t to fire, and similarly for a bond $\beta \in F(x,t)$. On the other hand, $\overline{a} \in F(x,t)$ expresses that token a should not be present in the incoming place x of t for the transition to fire and similarly for a bond β, $\overline{\beta} \in F(x,t)$. Note that negative tokens/bonds are close in spirit to the inhibitor arcs of extended Petri nets. Finally, note that $F(x,t) = \emptyset$ implies that there is no arc from place x to transition t and similarly for $F(t,x) = \emptyset$.

We introduce the following notations. We write $\circ t = \{x \in P \mid F(x,t) \neq \emptyset\}$ and $t\circ = \{x \in P \mid F(t,x) \neq \emptyset\}$ for the incoming and outgoing places of transition t, respectively. Furthermore, we write $\mathsf{pre}(t) = \bigcup_{x\in P} F(x,t)$ for the union of all labels on the incoming arcs of transition t, and $\mathsf{post}(t) = \bigcup_{x\in P} F(t,x)$ for the union of all labels on the outgoing arcs of transition t.

Definition 3. A Reversing Petri net (A, P, B, T, F) is *well-formed* if it satisfies the following conditions for all $t \in T$:

1. $A \cap \mathsf{pre}(t) = A \cap \mathsf{post}(t)$,
2. If $a{-}b \in \mathsf{pre}(t)$ then $a{-}b \in \mathsf{post}(t)$,
3. $F(t,x) \cap F(t,y) = \emptyset$ for all $x, y \in P$, $x \neq y$.

According to the above we have that: (1) transitions do not erase tokens or create new ones, (2) transitions do not destroy bonds, that is, if a bond $a{-}b$ exists in an input place of a transition, then it is maintained in some output place, and (3) tokens/bonds cannot be cloned into more than one outgoing place.

As with standard Petri nets, we employ the notion of a *marking*. A marking is a distribution of tokens and bonds across places, $M : P \to 2^{A\cup B}$ where $a{-}b \in M(x)$, for some $x \in P$, implies $a, b \in M(x)$. In addition, we employ the notion of a *history*, which assigns a memory to each transition of a reversing Petri net as $H : T \to 2^{\mathbb{N}}$. Intuitively, a history of $H(t) = \emptyset$ for some $t \in T$ captures that the transition has not taken place, and a history of $H(t) = \{k_1, \ldots, k_n\}$ captures that the transition was executed and not reversed n times where k_i, $1 \leq i \leq n$, indicates the order of execution of the i^{th} instance amongst non-reversed actions. Note that this machinery, is needed to accommodate the presence of cycles, which yield the possibility of repeatedly executing the same transitions. H_0 denotes the initial history where $H_0(t) = \emptyset$ for all $t \in T$. A pair of a marking and a history describes a *state* of a reversing Petri net based on which execution is determined. We use the notation $\langle M, H\rangle$ to denote states.

In a graphical representation, tokens are indicated by •, places by circles, transitions by boxes, and bonds by lines between tokens. Furthermore, histories are presented over the respective transitions as the list $[k_1, ..., k_n]$ when $H(t) = \{k_1, \ldots, k_n\}$, $n > 0$, and omitted when $H(t) = \emptyset$.

As the last piece of our machinery, we define a notion that identifies connected components of tokens and their associated bonds within a place. Note that more than one connected component may arise in a place due to the fact that various unconnected tokens may be moved to a place simultaneously by a transition, while the reversal of transitions, which results in the destruction of bonds, may break down a connected component into various subcomponents. We define $\mathsf{con}(a, C)$, where a is a base and $C \subseteq A \cup B$ to be the tokens connected to a via sequences of bonds as well as the bonds creating these connections according to set C.

$$\mathsf{con}(a, C) = (\{a\} \cap C) \cup \{\beta, b, c \mid \exists w \text{ s.t. } \mathsf{path}(a, w, C), \beta \in w, \text{ and } \beta = (b, c)\}$$

where $\mathsf{path}(a, w, C)$ if $w = \langle \beta_1, \ldots, \beta_n \rangle$, and for all $1 \leq i \leq n$, $\beta_i = (a_{i-1}, a_i) \in C \cap B$, $a_i \in C \cap A$, and $a_0 = a$.

Returning to the example of Figure 3.3, we may see a reversing net with three tokens a, b, and c, transition t_1, which bonds tokens a and c within place x, and transition t_2, which bonds the a of bond $c-a$ with token b into place y. Note that to avoid overloading figures, we omit writing the bases of bonds on the arcs of RPNs, so, e.g., on the arc between t_1 and x, we write $a-b$ as opposed to $\{a-b, a, b\}$. (The marking depicted in the figure is the one arising after the execution of transitions t_1 and t_2 and subsequently the reversal of transition t_1 by the semantic relations to be defined in the next section.)

We may now define the various types of execution for reversing Petri nets. In what follows we restrict our attention to well-formed RPNs (A, P, B, T, F) with initial marking M_0 such that for all $a \in A$, $|\{x \mid a \in M_0(x)\}| = 1$.

3.2.1 Forward Execution

In this section we consider the standard, forward execution of RPNs.

Definition 4. Consider a reversing Petri net (A, P, B, T, F), a transition $t \in T$, and a state $\langle M, H \rangle$. We say that t is *forward-enabled* in $\langle M, H \rangle$ if the following hold:

1. if $a \in F(x, t)$, for some $x \in \circ t$, then $a \in M(x)$, and if $\overline{a} \in F(x, t)$ for some $x \in \circ t$, then $a \notin M(x)$,

2. if $\beta \in F(x,t)$, for some $x \in \circ t$, then $\beta \in M(x)$, and if $\overline{\beta} \in F(x,t)$ for some $x \in \circ t$, then $\beta \notin M(x)$,

3. if $a \in F(t,y_1)$, $b \in F(t,y_2)$, $y_1 \neq y_2$, then $b \notin \mathsf{con}(a, M(x))$ for all $x \in \circ t$, and

4. if $\beta \in F(t,x)$ for some $x \in t\circ$ and $\beta \in M(y)$ for some $y \in \circ t$, then $\beta \in F(y,t)$.

Thus, t is enabled in state $\langle M, H\rangle$ if (1), (2), all tokens and bonds required for the transition to take place are available in the incoming places of t and none of the tokens/bonds whose absence is required exists in an incoming place of the transition, (3) if a transition forks into outgoing places y_1 and y_2 then the tokens transferred to these places are not connected to each other in the incoming places of the transition, and (4) if a pre-existing bond appears in an outgoing arc of a transition, then it is also a precondition of the transition to fire. Contrariwise, if the bond appears in an outgoing arc of a transition ($\beta \in F(t,x)$ for some $x \in t\circ$) but is not a requirement for the transition to fire ($\beta \notin F(y,t)$ for all $y \in \circ t$), then the bond should not be present in an incoming place of the transition ($\beta \notin M(y)$ for all $y \in \circ t$).

We observe that the new bonds created by a transition are exactly those that occur in the outgoing edges of a transition but not in the incoming edges. Thus, we define the effect of a transition as

$$\mathsf{eff}(t) = \mathsf{post}(t) - \mathsf{pre}(t)$$

This will subsequently enable the enunciation of transition reversal by the destruction of exactly the bonds in $\mathsf{eff}(t)$.

Definition 5. Given a reversing Petri net (A, P, B, T, F), a state $\langle M, H\rangle$, and a transition t enabled in $\langle M, H\rangle$, we write $\langle M, H\rangle \xrightarrow{t} \langle M', H'\rangle$ where:

$$M'(x) = \begin{cases} M(x) - \bigcup_{a \in F(x,t)} \mathsf{con}(a, M(x)) & \text{if } x \in \circ t \\ M(x) \cup F(t,x) \cup \bigcup_{a \in F(t,x) \cap F(y,t)} \mathsf{con}(a, M(y)) & \text{if } x \in t\circ \\ M(x), & \text{otherwise} \end{cases}$$

and

$$H'(t') = \begin{cases} H(t') \cup \{\max(\{0\} \cup \{k \mid k \in H(t''), t'' \in T\}) + 1\}, & \text{if } t' = t \\ H(t'), & \text{otherwise} \end{cases}$$

Thus, when a transition t is executed in the forward direction, all tokens and bonds occurring in its incoming arcs are relocated from the input places to the output places along with their connected components. An example of forward transitions can be seen in Figure 3.4

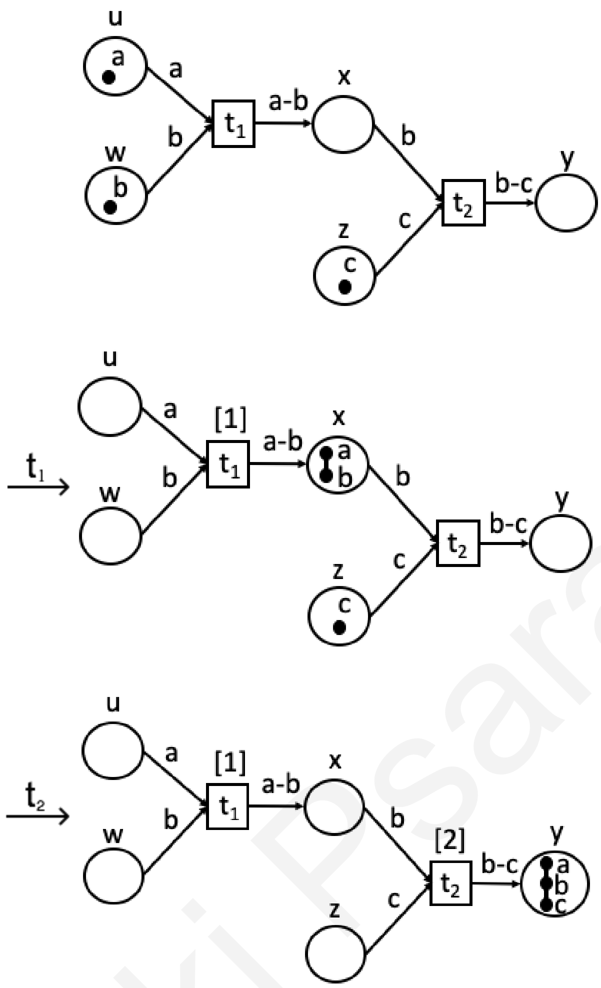


Figure 3.4: Forward execution

where transitions t_1 and t_2 take place with the histories of the two transitions becoming **[1]** and **[2]**, respectively.

We may prove the following result, which verifies that bases are preserved during forward execution in the sense that transitions neither erase nor clone them. As far as bonds are concerned, the proposition states that forward execution may create but not destroy bonds.

Proposition 1 (Token and bond preservation)**.** Consider a reversing Petri net (A, P, B, T, F), a state $\langle M, H \rangle$ such that for all $a \in A$, $|\{x \in P \mid a \in M(x)\}| = 1$, and a transition $\langle M, H \rangle \xrightarrow{t} \langle M', H' \rangle$. Then:

1. for all $a \in A$, $|\{x \in P \mid a \in M'(x)\}| = 1$, and

2. for all $\beta \in B$, $|\{x \in P \mid \beta \in M(x)\}| \leq |\{x \in P \mid \beta \in M'(x)\}| \leq 1$.

Proof. The proof of the result follows the definition of forward execution and relies on the well-formedness of RPNs. Consider a reversing Petri net (A, P, B, T, F), a state $\langle M, H \rangle$ such that $|\{x \in P \mid a \in M(x)\}| = 1$ for all $a \in A$, and suppose $\langle M, H\rangle \xrightarrow{t} \langle M', H'\rangle$.

For the proof of clause (1) let $a \in A$. Two cases exist:

1. $a \in \mathsf{con}(b, M(x))$ for some $b \in F(x,t)$. Note that x is unique by the assumption that $|\{x \in P \mid a \in M(x)\}| = 1$. Furthermore, according to Definition 5, we have that $M'(x) = M(x) - \{\mathsf{con}(b, M(x)) \mid b \in F(x,t)\}$, which implies that $a \notin M'(x)$. On the other hand, by Definition 3(1), $b \in \mathsf{post}(t)$. Thus, there exists $y \in t\circ$, such that $b \in F(t,y)$. Note that this y is unique by Definition 3(3). As a result, by Definition 5, $M'(y) = M(y) \cup F(t,y) \cup \{\mathsf{con}(b, M(x)) \mid b \in F(t,y), x \in \circ t\}$. Since $b \in F(x,t) \cap F(t,y)$, $a \in \mathsf{con}(b, M(x))$, this implies that $a \in M'(y)$.

   Now suppose that $a \in \mathsf{con}(c, M(x))$ for some $c \neq b$, $c \in F(t,y')$. Then, by Definition 4(3), it must be that $y = y'$. As a result, we have that $\{z \in P \mid a \in M'(z)\} = \{y\}$ and the result follows.

2. $a \notin \mathsf{con}(b, M(x))$ for all $b \in F(x,t)$, $x \in P$. This implies that $\{x \in P \mid a \in M'(x)\} = \{x \in P \mid a \in M(x)\}$ and the result follows.

To prove clause (2) of the proposition, consider a bond $\beta \in B$, $\beta = (a,b)$. We observe that, since $|\{x \in P \mid a \in M(x)\}| = 1$ for all $a \in A$, $|\{x \in P \mid \beta \in M(x)\}| \leq 1$. The proof follows by case analysis as follows:

1. Suppose $|\{x \in P \mid \beta \in M(x)\}| = 0$. Two cases exist:

   - Suppose $\beta \notin F(t,x)$ for all $x \in P$. Then, by Definition 5, $\beta \notin M'(x)$ for all $x \in P$. Consequently, $|\{x \in P \mid \beta \in M'(x)\}| = 0$ and the result follows.

   - Suppose $\beta \in F(t,x)$ for some $x \in P$. Then, by Definition 3(3), x is unique, and by Definition 5, $\beta \in M'(x)$. Consequently, $|\{x \in P \mid \beta \in M'(x)\}| = 1$ and the result follows.

2. Suppose $|\{x \in P \mid \beta \in M(x)\}| = 1$. Two cases exist:

   - $\beta \notin \mathsf{con}(c, M(x))$ for all $c \in F(x,t)$. This implies that $\{x \in P \mid \beta \in M'(x)\} = \{x \in P \mid \beta \in M(x)\}$ and the result follows.

   - $\beta \in \mathsf{con}(c, M(x))$ for some $c \in F(x,t)$. Then, according to Definition 5, we have that $M'(x) = M(x) - \{\mathsf{con}(c, M(x)) \mid c \in F(x,t)\}$, which implies that $\beta \notin M'(x)$. On the other hand, by the definition of well-formedness, Definition 3(1), $c \in \mathsf{post}(t)$. Thus, there exists $y \in t\circ$, such that $c \in F(t,y)$. Note that this y is unique by Definition 3(3). As a result, by Definition 5, $M'(y) = M(y) \cup F(t,y) \cup \{\mathsf{con}(c, M(x)) \mid c \in F(t,y), x \in \circ t\}$. Since $c \in F(x,t) \cap F(t,y)$, $\beta \in \mathsf{con}(c, M(x))$, this implies that $\beta \in M'(y)$.

Now suppose that $\beta \in \mathsf{con}(d, M(x))$ for some $d \neq c$, $c \in F(d, y')$. Then, by Definition 4, and since $\mathsf{con}(c, M(x)) = \mathsf{con}(d, M(x))$, it must be that $y = y'$. As a result, we have that $\{z \in P \mid \beta \in M'(z)\} = \{y\}$ and the result follows.

□

3.2.2 Backtracking

Let us now proceed to the simplest form of reversibility, namely, backtracking. We define a transition to be *bt*-enabled (backtracking-enabled) if it was the last executed transition:

Definition 6. Consider a state $\langle M, H\rangle$ and a transition $t \in T$. We say that t is *bt-enabled* in $\langle M, H\rangle$ if $k \in H(t)$ with $k \geq k'$ for all $k' \in H(t')$, $t' \in T$.

Thus, a transition t is *bt*-enabled if its history contains the highest value among all transitions. The effect of backtracking a transition in a reversing Petri net is as follows:

Definition 7. Given a reversing Petri net (A, P, B, T, F), a state $\langle M, H\rangle$, and a transition t that is *bt*-enabled in $\langle M, H\rangle$, we write $\langle M, H\rangle \overset{t}{\rightsquigarrow}_b \langle M', H'\rangle$ where:

$$M'(x) \;=\; \begin{cases} M(x) \cup \bigcup_{y \in t\circ, a \in F(x,t) \cap F(t,y)} \mathsf{con}(a, M(y) - \mathsf{eff}(t)), & \text{if } x \in \circ t \\ M(x) - \bigcup_{a \in F(t,x)} \mathsf{con}(a, M(x)), & \text{if } x \in t\circ \\ M(x) & \text{otherwise} \end{cases}$$

and

$$H'(t') \;=\; \begin{cases} H(t') - \{k\}, & \text{if } t' = t, k = \max(H(t)) \\ H(t') & \text{otherwise} \end{cases}$$

When a transition t is reversed in a backtracking fashion all tokens and bonds in the postcondition of the transition, as well as their connected components, are transferred to the incoming places of the transition and any newly-created bonds are broken. Furthermore, the largest key in the history of the transition is removed.

An example of backtracking extending the example of Figure 3.4 can be seen in Figure 3.5 where we observe transitions t_2 and t_1 being reversed with the histories of the two transitions being eliminated. A further example can be seen in Figure 3.6 where after the execution of transition sequence $\langle t_1, t_2, t_3, t_4, t_3\rangle$, only transition t_3 is *bt*-enabled since it was

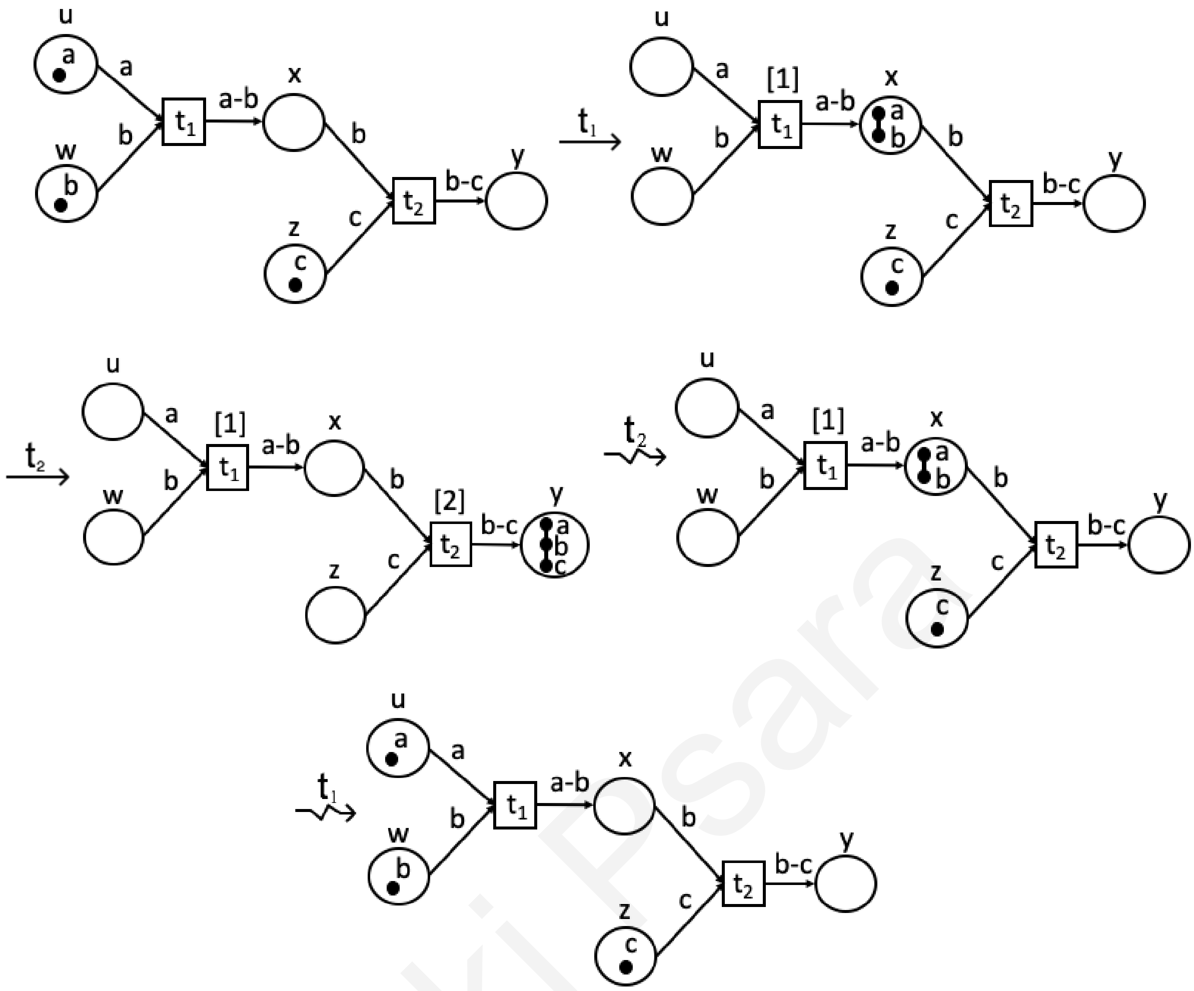


Figure 3.5: Backtracking execution

the last transition to be executed. During its reversal, the component $a-b-c$ is returned to place u. Furthermore, the largest key of the history of t_3 becomes empty.

We may prove the following result, which verifies that bases are preserved during backtracking execution in the sense that there exists exactly one instance of each base and backtracking transitions neither erase nor clone them. As far as bonds are concerned, the proposition states that at any time there may exist at most one instance of a bond and that backtracking transitions may only destroy bonds.

Proposition 2 (Token preservation and bond destruction)**.** Consider a reversing Petri net (A, P, B, T, F), a state $\langle M, H\rangle$ such that for all $a \in A$, $|\{x \in P \mid a \in M(x)\}| = 1$, and a transition $\langle M, H\rangle \overset{t}{\rightsquigarrow}_b \langle M', H'\rangle$. Then:

1. for all $a \in A$, $|\{x \in P \mid a \in M'(x)\}| = 1$, and

2. for all $\beta \in B$, $1 \geq |\{x \in P \mid \beta \in M(x)\}| \geq |\{x \in P \mid \beta \in M'(x)\}|$.

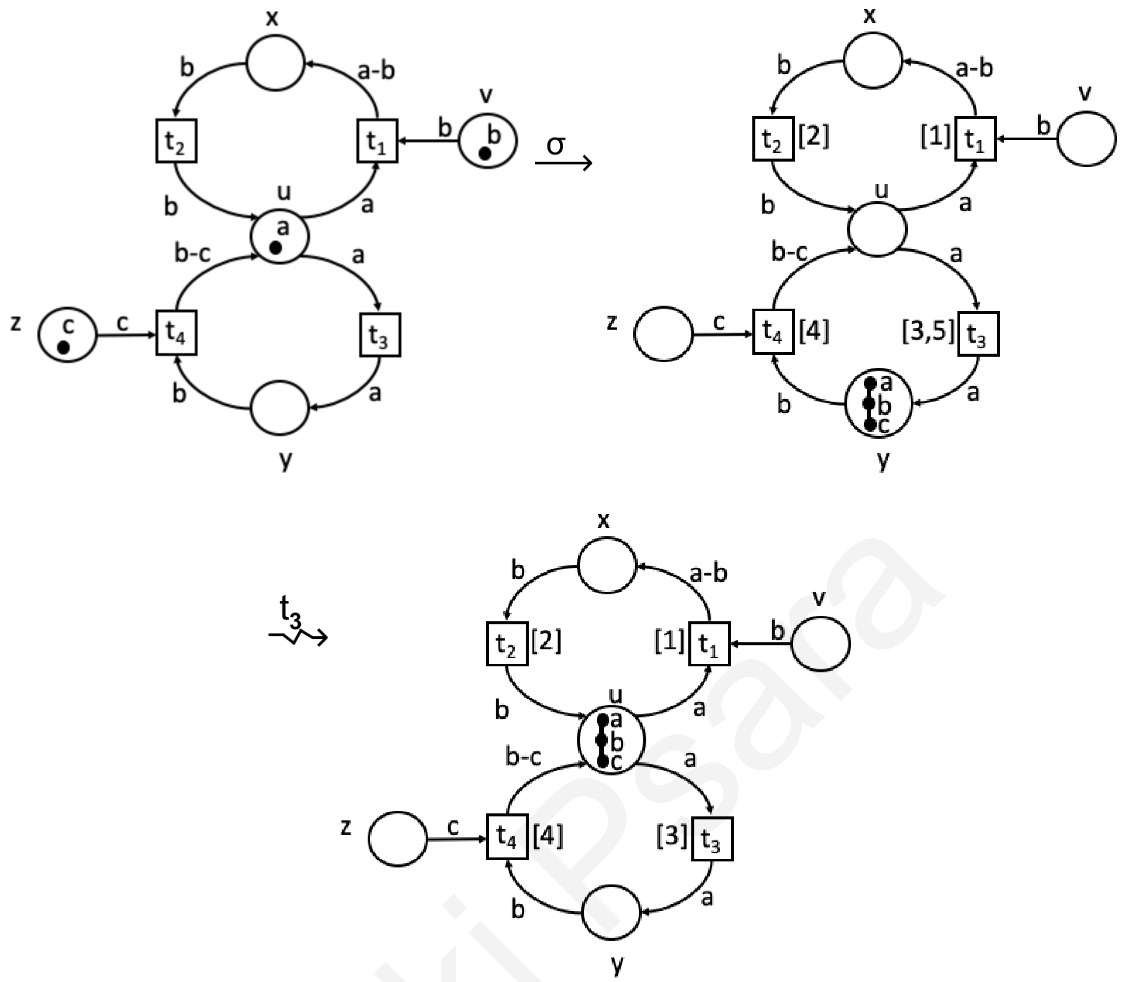


Figure 3.6: Backtracking execution where $\sigma = \langle t_1, t_2, t_3, t_4, t_3 \rangle$

Proof. The proof of the result follows the definition of backward execution and relies on the well-formedness of reversing Petri nets. Consider RPN (A, P, B, T, F), a state $\langle M, H \rangle$ such that $|\{x \in P \mid a \in M(x)\}| = 1$ for all $a \in A$, and suppose $\langle M, H \rangle \overset{t}{\rightsquigarrow}_b \langle M', H' \rangle$.

We begin with the proof of clause (1) and let $a \in A$. Two cases exist:

1. $a \in \mathsf{con}(b, M(x))$ for some $b \in F(t, x)$. Note that by the assumption of $|\{x \in P \mid a \in M(x)\}| = 1$, x must be unique. Let us choose b such that, additionally, $a \in \mathsf{con}(b, M(x) - \mathsf{eff}(t))$. Note that such a b must exist, otherwise the forward execution of t would not have transferred a along with b to place x.

   According to Definition 7, we have that $M'(x) = M(x) - \{\mathsf{con}(b, M(x)) \mid b \in F(t, x)\}$, which implies that $a \notin M'(x)$. On the other hand, note that by the definition of well-formedness, Definition 3(1), $b \in \mathsf{pre}(t)$. Thus, there exists $y \in \circ t$, such that $b \in F(y, t)$. Note that this y is unique. If not, then there exist y and y' such that $y \neq y'$ with $b \in F(y, t)$ and $b \in F(y', t)$. By the assumption, however, that there exists at most one token of each base, and Proposition 1, t would never be enabled, which leads to a contradiction. As a result, by Definition 7, $M'(y) = M(y) \cup \{\mathsf{con}(b, M(x) - \mathsf{eff}(t)) \mid b \in F(y,t) \cap F(t,x)\}$. Since $b \in F(y,t) \cap F(t,x)$, $a \in \mathsf{con}(b, M(x) - \mathsf{eff}(t))$, this implies that $a \in M'(y)$.

Now suppose that $a \in \mathsf{con}(c, M(x) - \mathsf{eff}(t))$, $c \neq b$, and $c \in F(y', t)$. Since $a \in \mathsf{con}(b, M(x)-\mathsf{eff}(t))$, it must be that $\mathsf{con}(b, M(x)-\mathsf{eff}(t)) = \mathsf{con}(c, M(x)-\mathsf{eff}(t))$. Since b and c are connected to each other but the connection was not created by transition t (the connection is present in $M(x) - \mathsf{eff}(t)$), it must be that the connection was already present before the forward execution of t and, by token uniqueness, we conclude that $y = y'$.

2. $a \notin \mathsf{con}(b, M(x))$ for all $b \in F(t,x)$, $x \in P$. This implies that $\{x \in P \mid a \in M'(x)\} = \{x \in P \mid a \in M(x)\}$ and the result follows.

Let us now prove clause (2) of the proposition. Consider a bond $\beta \in B$, $\beta = (a, b)$. We observe that, since $|\{x \in P \mid a \in M(x)\}| = 1$ for all $a \in A$, $|\{x \in P \mid \beta \in M(x)\}| \leq 1$. The proof follows by case analysis as follows:

1. $\beta \in \mathsf{con}(c, M(x))$ for some $c \in F(t,x)$, $x \in P$. By the assumption of $|\{x \in P \mid \beta \in M(x)\}| = 1$, x must be unique. Then, according to Definition 7, we have that $M'(x) = M(x) - \{\mathsf{con}(c, M(x)) \mid c \in F(x,t)\}$, which implies that $\beta \notin M'(x)$. Two cases exist:

   - If $\beta \in \mathsf{eff}(t)$, then $\beta \notin M'(y)$ for all places $y \in P$.
   - If $\beta \notin \mathsf{eff}(t)$ then let us choose c such that $\beta \in \mathsf{con}(c, M(x)-\mathsf{eff}(t))$. Note that such a c must exist, otherwise the forward execution of t would not have connected β with c. By the definition of well-formedness, Definition 3(1), $c \in \mathsf{pre}(t)$. Thus, there exists $y \in \circ t$, such that $c \in F(y,t)$. Note that this y is unique (if not, t would not have been enabled). As a result, by Definition 7, $\beta \in M'(y)$.

     Now suppose that $\beta \in \mathsf{con}(d, M(x) - \mathsf{eff}(t))$, $d \neq c$, and $d \in M'(y')$. Since $\beta \in \mathsf{con}(c, M(x)-\mathsf{eff}(t))$, it must be that $\mathsf{con}(c, M(x)-\mathsf{eff}(t)) = \mathsf{con}(d, M(x)-\mathsf{eff}(t))$. Since c and d are connected to each other but the connection was not created by transition t (the connection is present in $M(x) - \mathsf{eff}(t)$), it must be that the connection was already present before the forward execution of t and, by token uniqueness, we conclude that $y = y'$. This implies that $\{z \in P \mid \beta \in M'(z)\} = \{y\}$.

The above imply that $\{z \in P \mid \beta \in M(z)\} = \{x\}$ and $\{z \in P \mid \beta \in M'(z)\} \subseteq \{y\}$ and the result follows.

2. $\beta \notin \mathsf{con}(c, M(x))$ for all $c \in F(t, x)$, $x \in P$. This implies that $\{x \in P \mid \beta \in M'(x)\} = \{x \in P \mid \beta \in M(x)\}$ and the result follows. □

Let us now consider the combination of forward and backward moves in executions. We write $\longmapsto_b$ for $\longrightarrow \cup \rightsquigarrow_b$. The following result establishes that in an execution beginning in the initial state of a reversing Petri net, bases are preserved, bonds can have at most one instance at any time and a new occurrence of a bond may be created during a forward transition that features the bond as its effect whereas a bond can be destroyed during the backtracking of a transition that features the bond as its effect. This last point clarifies that the effect of a transition characterises the bonds that are newly-created during the transition's forward execution and the ones that are destroyed during its reversal.

Proposition 3. Given a reversing Petri net (A, P, B, T, F), an initial state $\langle M_0, H_0\rangle$ and an execution $\langle M_0, H_0\rangle \overset{t_1}{\longmapsto}_b \langle M_1, H_1\rangle \overset{t_2}{\longmapsto}_b \ldots \overset{t_n}{\longmapsto}_b \langle M_n, H_n\rangle$, the following hold:

1. For all $a \in A$ and i, $0 \le i \le n$, $|\{x \in P \mid a \in M_i(x)\}| = 1$.

2. For all $\beta \in B$ and i, $0 \le i \le n$,

   (a) $0 \le |\{x \in P \mid \beta \in M_i(x)\}| \le 1$,

   (b) if t_i is executed in the forward direction and $\beta \in \mathsf{eff}(t_i)$, then $\beta \in M_i(x)$ for some $x \in P$ where $\beta \in F(t_i, x)$, and $\beta \notin M_{i-1}(y)$ for all $y \in P$,

   (c) if t_i is executed in the forward direction, $\beta \in M_{i-1}(x)$ for some $x \in P$, and $\beta \notin \mathsf{eff}(t_i)$ then, if $\beta \in \mathsf{con}(a, M_{i-1}(x))$ and $a \in F(t_i, y)$, then $\beta \in M_i(y)$, otherwise $\beta \in M_i(x)$,

   (d) if t_i is executed in the reverse direction and $\beta \in \mathsf{eff}(t_i)$ then $\beta \in M_{i-1}(x)$ for some $x \in P$ where $\beta \in F(t_i, x)$, and $\beta \notin M_i(y)$ for all $y \in P$, and

   (e) if t_i is executed in the reverse direction, $\beta \in M_{i-1}(x)$ for some $x \in P$, and $\beta \notin \mathsf{eff}(t_i)$ then, if $\beta \in \mathsf{con}(a, M_{i-1}(x))$ and $a \in F(y, t_i)$, then $\beta \in M_i(y)$, otherwise $\beta \in M_i(x)$.

Proof. To begin with, we observe that the proofs of clauses (1) and (2)(a) follow directly from clauses (1) and (2) of Propositions 1 and 2. Clause (2)(b) follows from Definition 4(4) and Definition 5. Clause (2)(c) follows from Definition 5 and the condition refers to whether the bond is part of a component manipulated by the forward execution of t_i. Similarly,to (2)(a) clause (2)(d) stems from Definition 7. Finally, Clause (2)(e) follows from Definition 7 and the condition refers to whether the bond is part of a component manipulated by the reverse execution of t_i. □

In this setting we may establish a loop lemma:

Lemma 1 (Loop)**.** For any forward transition $\langle M, H\rangle \xrightarrow{t} \langle M', H'\rangle$ there exists a backward transition $\langle M', H'\rangle \overset{t}{\rightsquigarrow}_b \langle M, H\rangle$ and vice versa.

Proof. Suppose $\langle M, H\rangle \xrightarrow{t} \langle M', H'\rangle$. Then t is clearly bt-enabled in H'. Furthermore, $\langle M', H'\rangle \overset{t}{\rightsquigarrow}_b \langle M'', H''\rangle$ where $H'' = H$. In addition, all tokens and bonds involved in transition t (except those in $\mathsf{eff}(t)$) will be returned from the outgoing places of transition t back to its incoming places. Specifically, for all $a \in A$, it is easy to see by the definition of $\rightsquigarrow_b$ that $a \in M''(x)$ if and only if $a \in M(x)$. Similarly, for all $\beta \in B$, $\beta \in M''(x)$ if and only if $\beta \in M(x)$. The opposite direction can be argued similarly. □

3.2.3 Causal-Order Reversibility

We now move on to consider causal-order reversibility in RPNs. To define such as reversible semantics in the presence of cycles, a number of issues need to be resolved. To begin with, consider a sequence of transitions pertaining to the repeated execution of a cycle. Adopting the view that reversible computation has the ability to rewind *every* executed action of a system, we require that each of these transitions is executed in reverse as many times as it was executed in the forward direction. Furthermore, the presence of cycles raises questions about the causal relationship between transitions of a cycle as well as of overlapping or even structurally distinct cycles. In the next subsection we discuss our adopted notion of transition causality. Subsequently, we develop a theory for causal-order reversibility in RPNs.

Causality in cyclic reversing Petri nets

A cycle in a reversing Petri net is associated with a cyclic path in the net's graph structure. It contains a sequence of transitions where an outgoing place of the last transition coincides with an incoming place of the first transition. Note that a cycle in the graph of a reversing Petri net does not necessarily imply the repeated execution of its transitions since, for instance, entrance to the cycle may require a token or a bond that has been directed into a different part of the net during execution of the cycle.

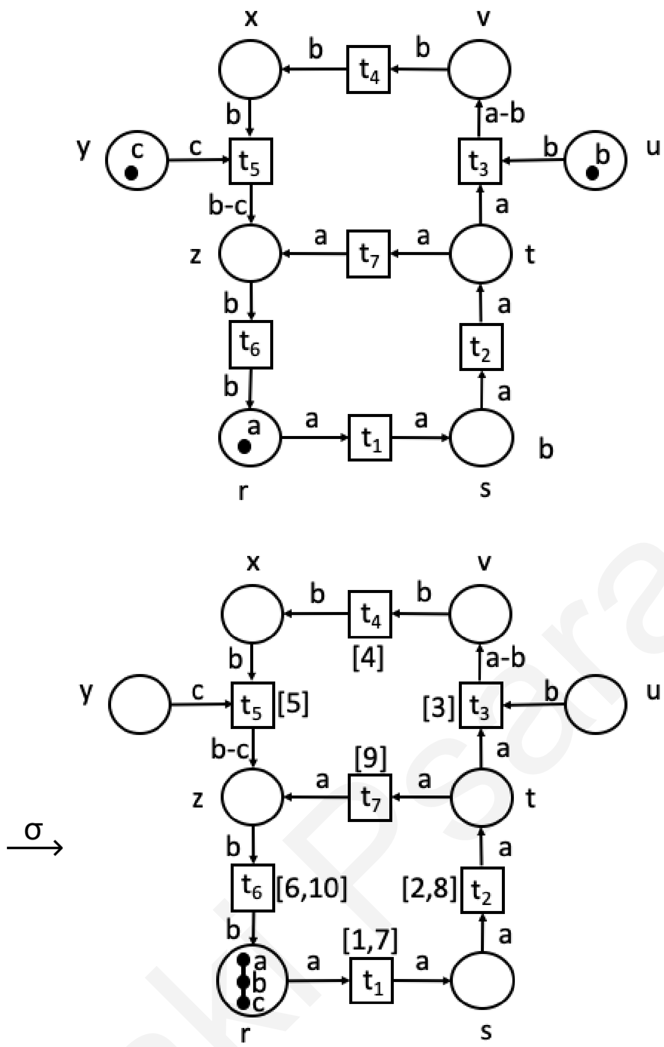


Figure 3.7: RPN with overlapping cycles $\sigma_1 = \langle t_1, t_2, t_3, t_4, t_5, t_6 \rangle$ and $\sigma_2 = \langle t_1, t_2, t_7, t_6 \rangle$, and the state arising after the forward execution of $\sigma = \sigma_1\sigma_2$

In the standard approach to causality in classical Petri nets [110], a causal link is considered to exist between two transitions if one produces tokens that are used to fire the other. This relation is used to define a “causal order", $\prec$, which is transitive so that if a transition t_1 causally precedes t_2 and t_2 causally precedes t_3, then t_1 also causally precedes t_3.

Adapting this notion in the context of cycle execution, consider a cycle with transitions t_1 and t_2, executed twice yielding the transition instances t_1^1, t_2^1, t_1^2, t_2^2, where t_i^j denotes the j-th execution of transition t_i. Furthermore, suppose that t_1 produces tokens that are consumed by t_2 and vice versa. This implies the causal order relation $\prec$, such that $t_1^1 \prec t_2^1 \prec t_1^2 \prec t_2^2$, allowing us to conclude that each execution of the cycle causally precedes any subsequent executions. This is a natural conclusion in the case of the consecutive execution of cycles, since a second execution of a cycle cannot be initiated before the first one is completed. This is because the tokens manipulated by the first transition of the cycle need to return to its input places before the transition can be repeated.

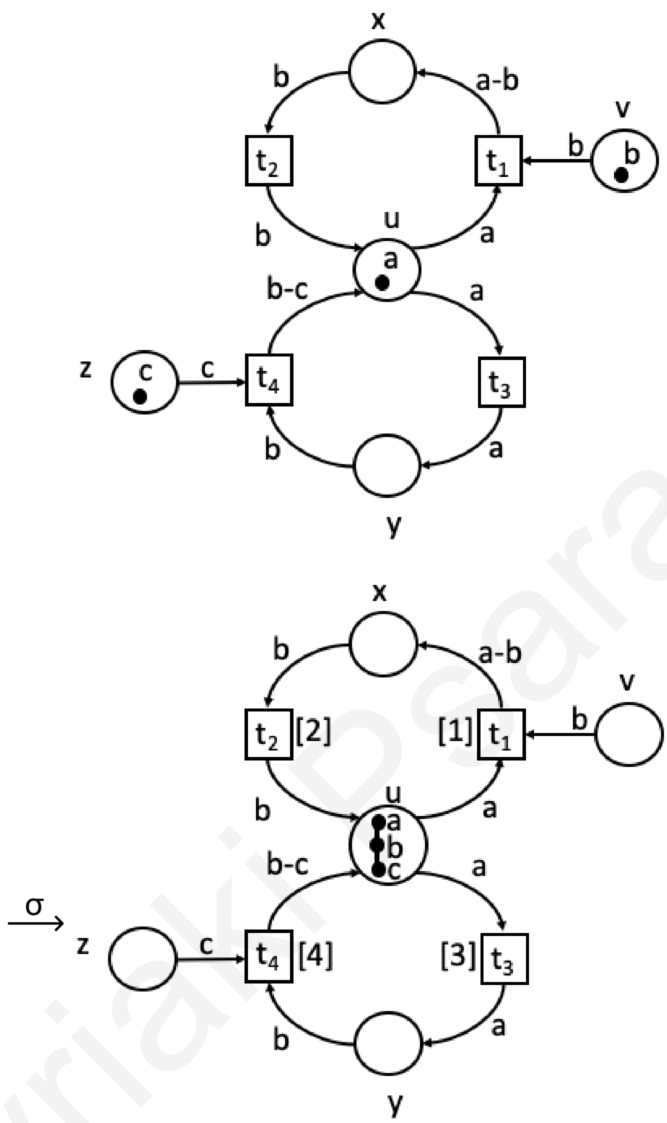


Figure 3.8: Causally dependent cycles, where $\sigma = \langle t_1, t_2, t_3, t_4 \rangle$

Let us now move on to determining when a token produced by a transition is consumed by another. In RPNs this concept acquires an additional complexity due to the fact that tokens are distinguished by names and the fact that the creation of bonds between tokens may disguise the causal relation between transitions. For instance, consider the example of Figure 3.7. This RPN features two overlapping cycles, which can be executed sequentially. Suppose we execute the outer cycle (transition sequence $\langle t_1, t_2, t_3, t_4, t_5, t_6 \rangle$) followed by the inner cycle (transition sequence $\langle t_1, t_2, t_7, t_6 \rangle$).

Observing the token manipulation of the transition instances as captured by the arcs of the transition, we obtain the order $t_1^1 \prec t_2^1 \prec t_3^1 \prec t_4^1 \prec t_5^1 \prec t_6^1$ and $t_1^2 \prec t_2^2 \prec t_7^1 \prec t_6^2$. However by simply observing the structure of the RPN there is no evidence that t_1 consumes tokens produced by t_6. Nonetheless, in this scenario transition instance t_3^1 has bonded tokens a and b and, thus, transition instance t_1^2 requires bond $a - b$ to be produced and placed at r by t_6 before transition t_1 can be executed for the second time. Thus, $t_6^1 \prec t_1^2$ also holds.

Note that, if the two cycles were not considered to be causally dependent and were allowed to reverse in any order, then, reversal of the first before the second one would disable the reversal of the second cycle. This is because reversing transition t_3 would return token b to place u, thus disabling a second reversal of transition t_6 (and consequently the reversal of the inner cycle).

Similarly, in the example of Figure 3.8 we observe two cycles that are structurally independent but where the presence of common tokens between the two cycles creates a dependence between their executions. For instance, suppose that the upper cycle is initially selected via execution of transition t_1. This choice disables the lower cycle, which is only re-enabled once the upper cycle is completed and token a is returned to place u. As a result, the execution of t_3, and thus the lower cycle, following an execution of the upper cycle, is considered to be causally dependent on the execution of t_2.

The above examples highlight that syntactic token independence between two transitions or cycles does not preclude their causal dependence. Instead, causal dependence is determined by the path that tokens follow: two transition occurrences are causally dependent, if a token produced by the one occurrence was subsequently used to fire the other occurrence. To capture this type of dependencies, we adopt the following definitions.

Definition 8. Consider a state $\langle M, H\rangle$ and a transition t. We refer to (t,k) as a *transition occurrence* in $\langle M, H\rangle$ if $k \in H(t)$.

Definition 9. Consider a state $\langle M, H\rangle$ and suppose $\langle M, H\rangle \xrightarrow{t} \langle M', H'\rangle$ with (t,k), (t',k') transition occurrences in $\langle M', H'\rangle$, $k = \max(H(t))$. We say that (t,k) *causally depends* on (t',k') denoted by $(t',k') \prec (t,k)$, if $k' < k$ and there exists $a \in F(x,t)$ where $\mathsf{con}(a, M(x)) \cap \mathsf{post}(t') \neq \emptyset$.

Thus, a transition occurrence (t,k) causally depends on a preceding transition occurrence (t',k') if one or more tokens used during the firing of (t,k) was produced by (t',k'). Note that the tokens employed during a transition in a specific marking are determined by the connected components of $F(x,t)$ in the marking. For example, in Figure 3.7 we have $(t_5, 5) \prec (t_7, 9)$ and in Figure 3.8 $(t_1, 1) \prec (t_4, 4)$, where in each case token a has been transferred from its initial place through $(t_5, 5)$ to $(t_7, 9)$ and through $(t_1, 1)$ to $(t_4, 4)$.

Causal reversing

Following this approach to causality, we now move on to define causal-order reversibility in reversing Petri nets. As expected, we consider a transition t to be enabled for causal-order reversal only if all transitions that are causally dependent on it have either been reversed or not executed. To this respect, relation $\prec$ becomes an important piece of machinery and we extend the notion of a *state* for the purposes of causal dependence to a triple $\langle M, H, \prec\rangle$ where $\prec$ captures the causal dependencies that have formed up to the creation of the state. We assume that in the initial state $\prec = \emptyset$ and we extend the definition of forward execution as follows:

Definition 10. Given a reversing Petri net (A, P, B, T, F), a state $\langle M, H, \prec\rangle$, and a transition t forward-enabled in $\langle M, H\rangle$, we write $\langle M, H, \prec\rangle \xrightarrow{t} \langle M', H', \prec'\rangle$ where M' and H' are defined as in Definition 5, and

$$\prec' = \prec \cup \{((t',k'),(t,k)) \mid k = \max(H'(t)), (t,k) \text{ causally depends on } (t',k')\}$$

We may now define that a transition is enabled for causal-order reversal as follows:

Definition 11. Consider a state $\langle M, H, \prec\rangle$ and a transition $t \in T$. Then t, $H(t) \neq \emptyset$, is c-enabled (causal-order reversal enabled) in $\langle M, H, \prec\rangle$ if

1. for all $x \in t\circ$, if $a \in F(t, x)$ then $a \in M(x)$ and if $\beta \in F(t, x)$ then $\beta \in M(x)$, and
2. there is no transition occurrence $(t', k') \in \langle M, H, \prec\rangle$ with $(t, k) \prec (t', k')$, for $k = max(H(t))$.

According to the definition, an executed transition is c-enabled if all tokens and bonds required for its reversal (i.e., in $\mathsf{post}(t)$) are available in its outgoing places and there are no transitions which depend on it causally. Note that the second condition becomes relevant in the presence of cycles since it is possible that, while more than one transitions simultaneously have available the tokens required for their reversal, only one of them is c-enabled. Such an example can be seen in the final state of Figure 3.8 and transitions t_2 and t_4.

Reversing a transition in a causally-respecting manner is implemented similarly to backtracking, i.e. the tokens are moved from the outgoing places to the incoming places of the transition and all bonds created by the transition are broken. In addition, the history function is updated in the same manner as in backtracking, where we remove the key of the reversed transition. Finally, the causal dependence relation removes all references to the reversed transition occurrence.

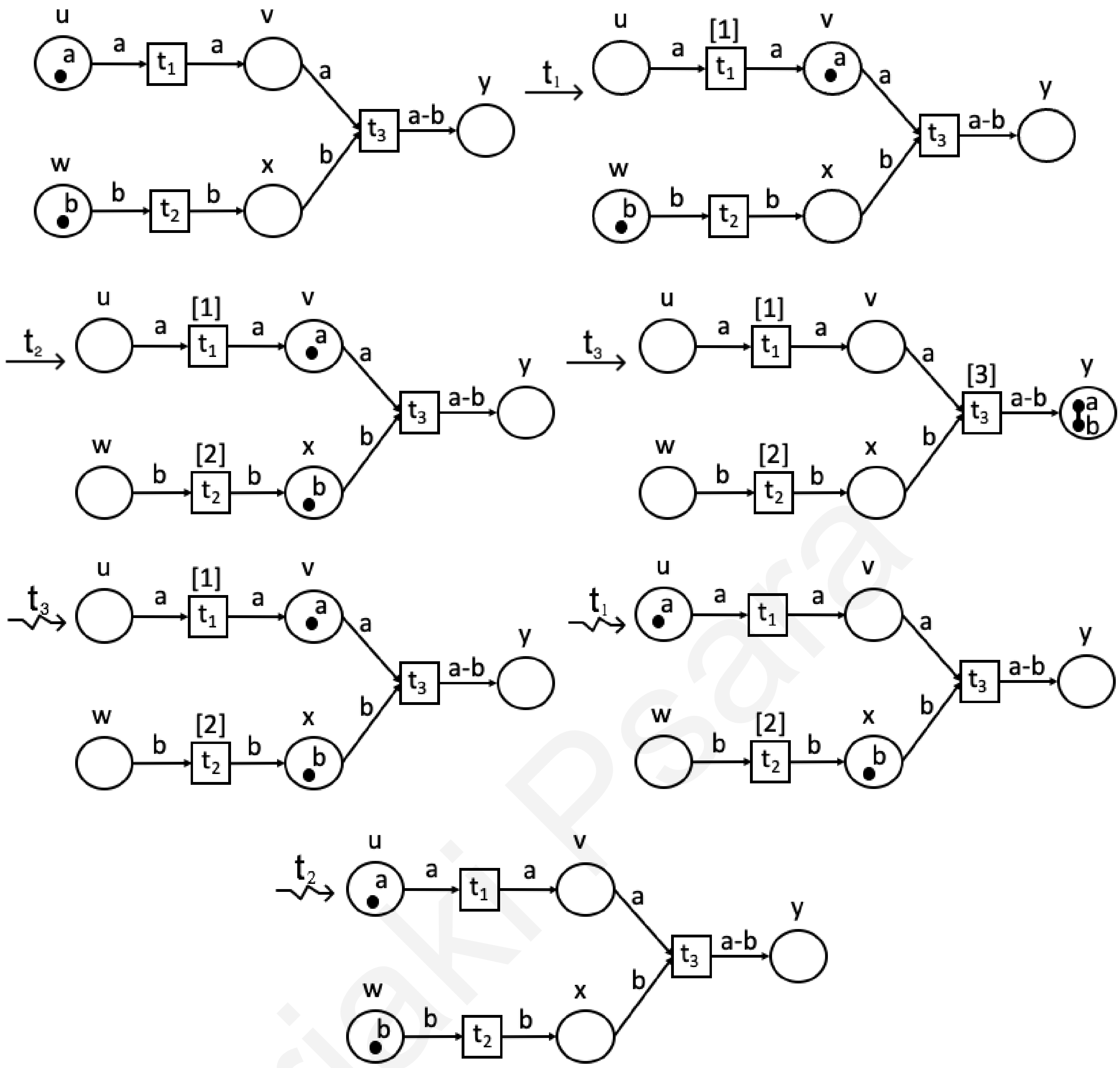


Figure 3.9: Causal-order example

Definition 12. Given a state $\langle M, H, \prec \rangle$ and a transition t c-enabled in $\langle M, H, \prec \rangle$, we write $\langle M, H, \prec \rangle \overset{t}{\rightsquigarrow}_c \langle M', H', \prec' \rangle$ for M' and H' as in Definition 7, and $\prec'$ such that

$$\prec' \quad = \quad \{((t_1, k_1), (t_2, k_2)) \in \prec \mid \ k_2 \neq k, k = \max(H(t))\}$$

An example of causal-order reversibility can be seen in Figure 3.9. Here we have two independent transitions, t_1 and t_2 causally preceding transition t_3. Once the transitions are executed in the order t_1, t_2, t_3, the example demonstrates a causally-ordered reversal where t_3 is (the only transition that can be) reversed, followed by the reversal of its two causes t_1 and t_2. In general t_1 and t_2 can be reversed in any order although in the example t_1 is reversed before t_2. Whenever a transition occurrence is reversed its key is eliminated from the history of the transition.

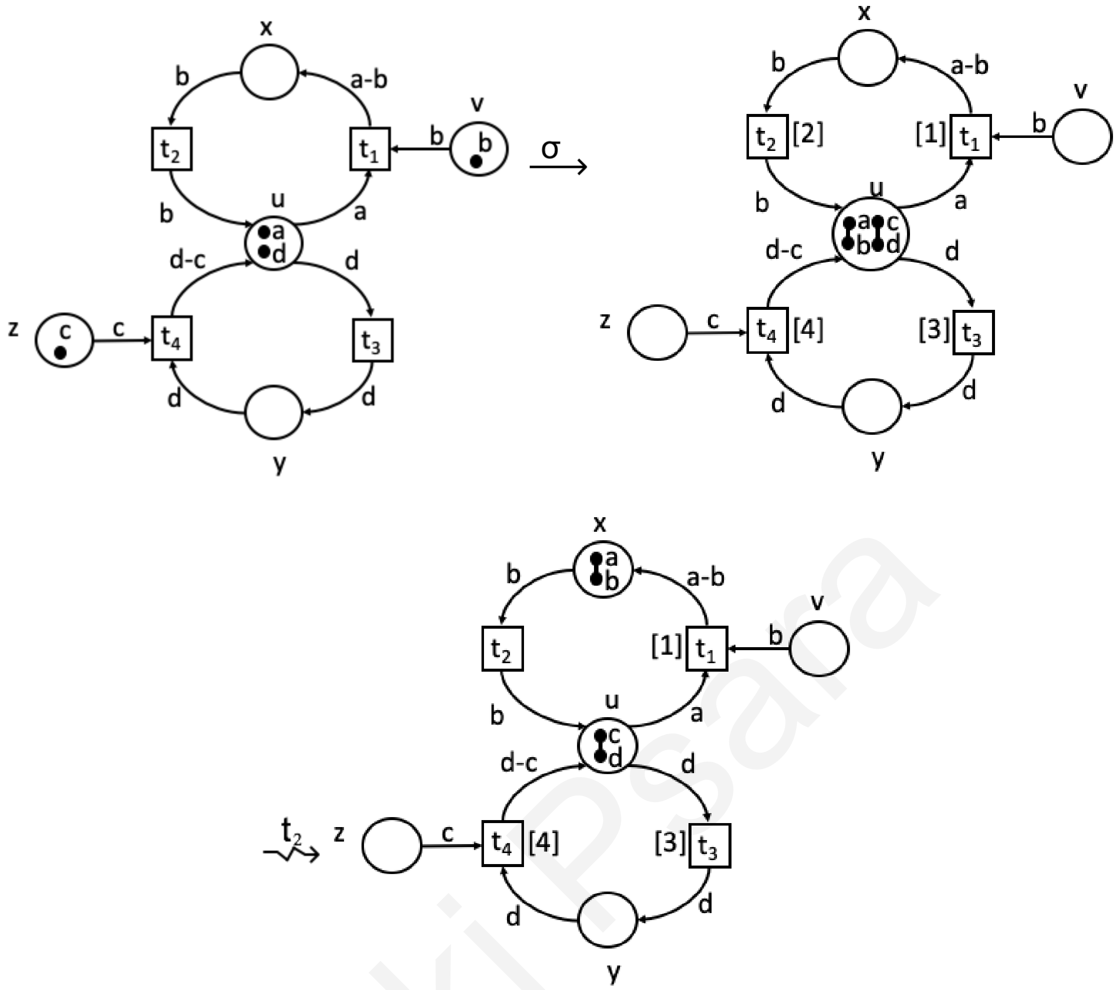


Figure 3.10: Causal execution where $\sigma = \langle t_1, t_2, t_3, t_4 \rangle$

As a further example consider the example in Figure 3.10 demonstrating a cyclic RPN. Assume that $\sigma = \langle t_1, t_2, t_3, t_4 \rangle$, i.e. from the initial state of the RPN the upper cycle is executed followed by the lower cycle. The transitions of the two cycles are causally independent since they manipulate different sets of tokens and therefore they can be reversed in any order. The figure illustrates the reversal of transition t_2 before transition t_4, which returns the bond between $a\!-\!b$ to place x.

In what follows we write $\longmapsto_c$ for $\longrightarrow \cup \rightsquigarrow_c$. The following result, similarly to Proposition 3, establishes that under the causal-order reversibility semantics, tokens are unique and preserved, bonds are unique, and they can only be created during forward execution and destroyed during reversal. Note that in what follows we will often omit the causal dependence relation and simply write $\langle M, H \rangle$ for states when it is not relevant to the discussion.

Proposition 4. Given a reversing Petri net (A, P, B, T, F), an initial state $\langle M_0, H_0 \rangle$ and an execution $\langle M_0, H_0 \rangle \xmapsto{t_1}_c \langle M_1, H_1 \rangle \xmapsto{t_2}_c \ldots \xmapsto{t_n}_c \langle M_n, H_n \rangle$, the following hold:

1. For all $a \in A$ and i, $0 \le i \le n$, $|\{x \in P \mid a \in M_i(x)\}| = 1$.
2. For all $\beta \in B$ and i, $0 \le i \le n$,
   (a) $0 \le |\{x \in P \mid \beta \in M_i(x)\}| \le 1$,
   (b) if t_i is executed in the forward direction and $\beta \in \mathsf{eff}(t_i)$, then $\beta \in M_i(x)$ for some $x \in P$ where $\beta \in F(t_i, x)$, and $\beta \notin M_{i-1}(y)$ for all $y \in P$,
   (c) if t_i is executed in the forward direction, $\beta \in M_{i-1}(x)$ for some $x \in P$, and $\beta \notin \mathsf{eff}(t_i)$, then, if $\beta \in \mathsf{con}(a, M_{i-1}(x))$ and $a \in F(t_i, y)$ then $\beta \in M_i(y)$, otherwise $\beta \in M_i(x)$
   (d) if t_i is executed in the reverse direction and $\beta \in \mathsf{eff}(t_i)$, then $\beta \in M_{i-1}(x)$ for some $x \in P$ where $\beta \in F(t_i, x)$, and $\beta \notin M_i(y)$ for all $y \in P$, and
   (e) if t_i is executed in the reverse direction, $\beta \in M_{i-1}(x)$ for some $x \in P$, and $\beta \notin \mathsf{eff}(t_i)$, then, if $\beta \in \mathsf{con}(a, M_{i-1}(x))$ and $a \in F(y, t_i)$ then $\beta \in M_i(y)$, otherwise $\beta \in M_i(x)$.

Proof. The proof follows along the same lines as that of Proposition 3 with $\rightsquigarrow_b$ replaced by $\rightsquigarrow_c$. □

We may now establish the causal consistency of our semantics. First we define some auxiliary notions. Given a transition $\langle M, H\rangle \stackrel{t}{\longmapsto}_c \langle M', H'\rangle$, we say that the *action* of the transition is t if $\langle M, H\rangle \stackrel{t}{\longrightarrow} \langle M', H'\rangle$ and $\underline{t}$ if $\langle M, H\rangle \stackrel{t}{\rightsquigarrow}_c \langle M', H'\rangle$ and we may write $\langle M, H\rangle \stackrel{\underline{t}}{\longmapsto}_c \langle M', H'\rangle$. We use α to range over $\{t, \underline{t} \mid t \in T\}$ and write $\underline{\underline{\alpha}} = \alpha$. We extend this notion to sequences of transitions and, given an execution $\langle M_0, H_0\rangle \stackrel{t_1}{\longmapsto}_c \ldots \stackrel{t_n}{\longmapsto}_c \langle M_n, H_n\rangle$, we say that the *trace* of the execution is $\sigma = \langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle$, where α_i is the action of transition $\langle M_{i-1}, H_{i-1}\rangle \stackrel{t_i}{\longmapsto}_c \langle M_i, H_i\rangle$, and write $\langle M, H\rangle \stackrel{\sigma}{\longmapsto}_c \langle M_n, H_n\rangle$. Given $\sigma_1 = \langle \alpha_1, \ldots, \alpha_k\rangle$, $\sigma_2 = \langle \alpha_{k+1}, \ldots, \alpha_n\rangle$, we write $\sigma_1;\sigma_2$ for $\langle \alpha_1, \ldots, \alpha_n\rangle$. We may also use the notation $\sigma_1;\sigma_2$ when σ_1 or σ_2 is a single transition.

An execution of a Petri net can be partitioned as a set of independent flows of execution running through the net. We capture these flows by the notion of causal paths:

Definition 13. Given a state $\langle M, H, \prec\rangle$ and transition occurrences (t_i, k_i) in $\langle M, H, \prec\rangle$, $1 \le i \le n$, we say that $(t_1, k_1), \ldots, (t_n, k_n)$ is a *causal path* in $\langle M, H, \prec\rangle$, if $(t_i, k_i) \prec (t_{i+1}, k_{i+1})$, for all $0 \le i < n$.

As an example, consider the reversing Petri net in Figure 3.11 where we denote the first execution by $\langle M_0, H_0, \emptyset\rangle \stackrel{\sigma_1}{\longrightarrow} \langle M_4, H_4, \prec\rangle$ for $\sigma_1 = \langle t_1, t_2, t_3, t_4\rangle$, and the second execution

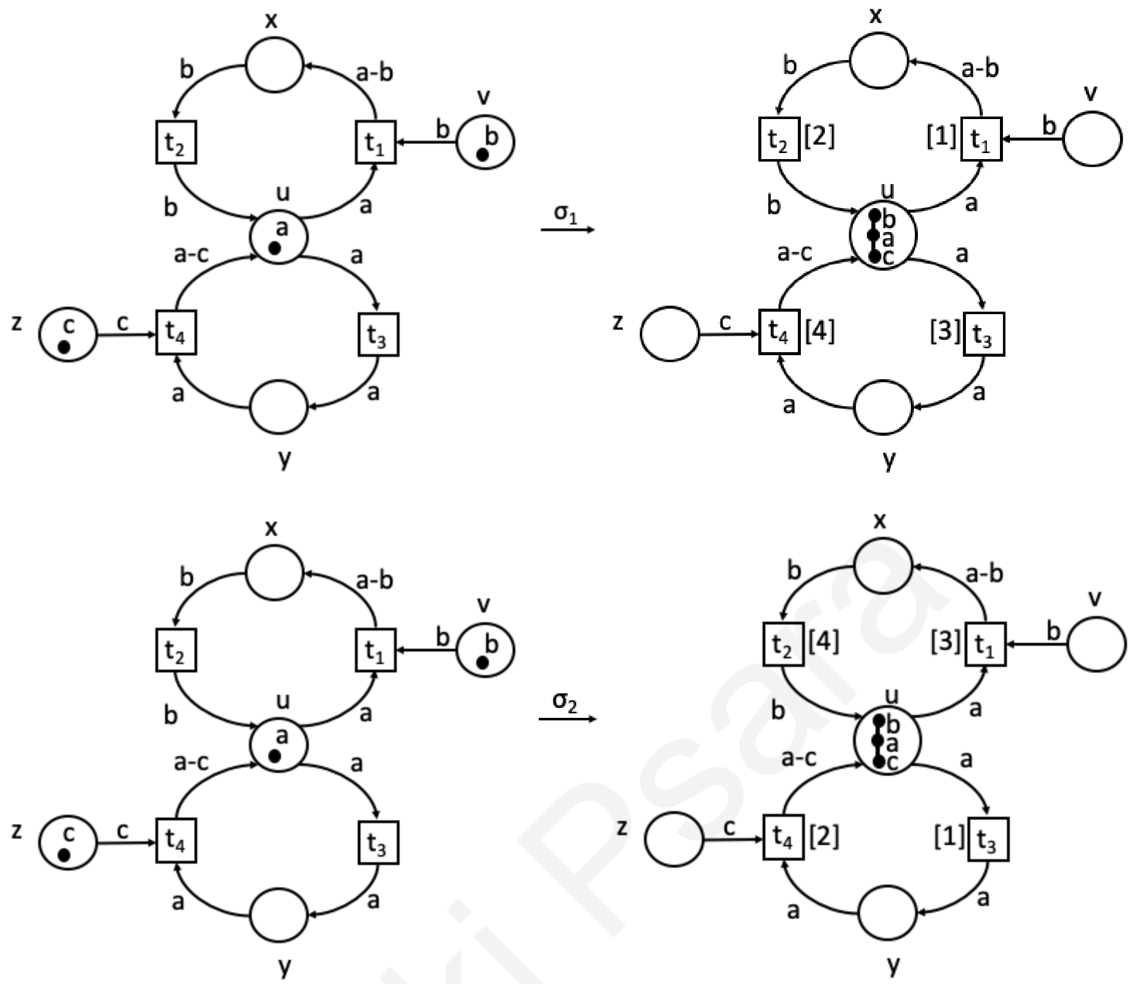


Figure 3.11: Causal paths in the context of dependent cycles, where $\sigma_1 = \langle t_1, t_2, t_3, t_4 \rangle$ and $\sigma_2 = \langle t_3, t_4, t_1, t_2 \rangle$

by $\langle M_0, H_0, \emptyset \rangle \xrightarrow{\sigma_2} \langle M'_4, H'_4, \prec' \rangle$ for $\sigma_2 = \langle t_3, t_4, t_1, t_2 \rangle$. In the case of σ_1 we have $\prec$ to be the transitive closure of $\{((t_1, 1), (t_2, 2)), ((t_2, 2), (t_3, 3)), ((t_3, 3), (t_4, 4))\}$, which results in the causal path $(t_1, 1), (t_2, 2), (t_3, 3), (t_4, 4)$. In the case of σ_2 where the cycles are executed in the opposite order, $\prec'$ is the transitive closure of $\{((t_3, 1), (t_4, 2)), ((t_4, 2), (t_1, 3)), ((t_1, 3), (t_2, 4))\}$, and the corresponding causal path is $(t_3, 1), (t_4, 2), (t_1, 3), (t_2, 4)$.

This comes in contrast to the RPN of Figure 3.12, which contains two independent cycles. Here, the causal dependencies of the first execution (trace σ_1) are constructed as $(t_1, 1) \prec (t_2, 2)$ and $(t_3, 3) \prec (t_4, 4)$, which results in the two independent causal paths $\langle (t_1, 1), (t_2, 2) \rangle$ and $\langle (t_3, 3), (t_4, 4) \rangle$. Similarly, after execution of σ_2, the causal dependencies are $(t_1, 3) \prec (t_2, 4)$ and $(t_3, 1) \prec (t_4, 2)$, which results in the causal paths $\langle (t_1, 3), (t_2, 4) \rangle$ and $\langle (t_3, 1), (t_4, 2) \rangle$.

As seen from the examples in Figures 3.11 and 3.12, the causal paths of an execution capture its causal behaviour. Based on this concept, we define the notion of causal equiv-

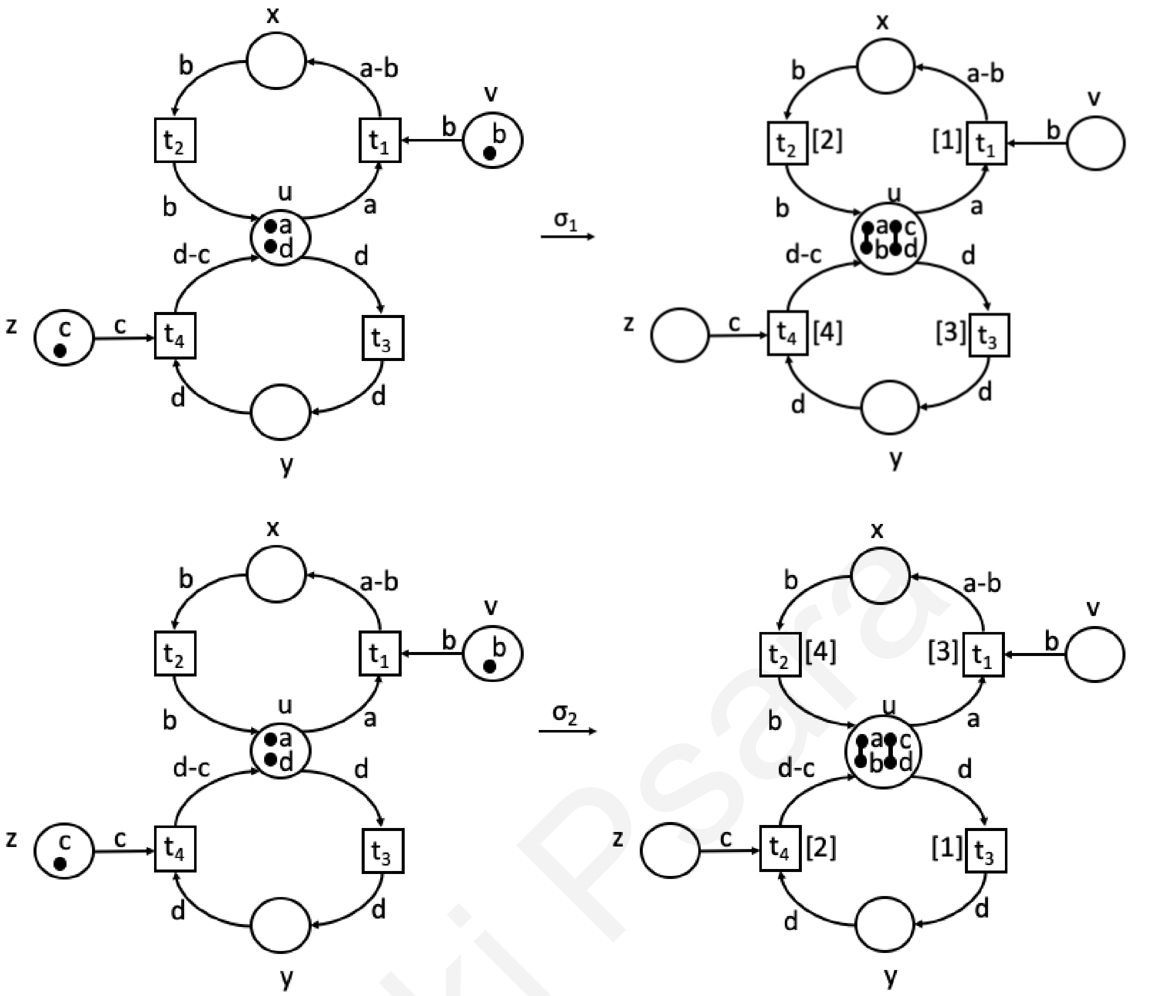


Figure 3.12: Causal paths in the context of independent cycles, where $\sigma_1 = \langle \pi_1, \pi_2 \rangle$ *such that* $\pi_1 = t_1, t_2$, $\pi_2 = t_3, t_4$ *and* $\sigma_2 = \langle \pi_1, \pi_2 \rangle$ *such that* $\pi_1 = t_3, t_4$, $\pi_2 = t_1, t_2$

alence for histories by requiring that two histories H and H' are causally equivalent if and only if they contain the same causal paths:

Definition 14. Consider a reversing Petri net (A, P, B, T, F) and two executions $\langle M, H, \prec \rangle \xrightarrow{\sigma}_c \langle M', H', \prec' \rangle$ and $\langle M, H, \prec \rangle \xrightarrow{\sigma'}_c \langle M'', H'', \prec'' \rangle$. Then the histories H' and H'' are *causally equivalent*, denoted by $H' \asymp H''$, if for each causal path $(t_1, k_1), \ldots, (t_n, k_n)$ in $\langle M', H', \prec' \rangle$, there is a causal path $(t_1, k_1'), \ldots, (t_n, k_n')$ in $\langle M'', H'', \prec'' \rangle$, and vice versa.

We extend this notion and write $\langle M, H, \prec \rangle \asymp \langle M', H', \prec' \rangle$ if and only if $M = M'$ and $H \asymp H'$.

Returning to the example in Figure 3.11 we observe that while the two executions result in the same marking, the resulting states do not have the same causal paths and, as such, they are not considered as causally equivalent.

We may now establish the Loop lemma.

Lemma 2 (Loop)**.** For any forward transition $\langle M,H\rangle \xrightarrow{t} \langle M',H'\rangle$ there exists a backward transition $\langle M',H'\rangle \overset{t}{\rightsquigarrow}_c \langle M,H\rangle$ and for any backward transition $\langle M,H\rangle \overset{t}{\rightsquigarrow}_c \langle M',H'\rangle$ there exists a forward transition $\langle M',H'\rangle \xrightarrow{t} \langle M,H''\rangle$ where $H \asymp H''$.

Proof. The proof of the first direction follows along the same lines as that of Lemma 1 with $\rightsquigarrow_b$ replaced by $\rightsquigarrow_c$. For the other direction suppose $\langle M,H\rangle \overset{t}{\rightsquigarrow}_c \langle M',H'\rangle \xrightarrow{t} \langle M,H''\rangle$. To begin with, we may observe that, as with Lemma 1, $M = M''$. To show that $H \asymp H''$, we observe that $H = H''$ with the exception of t, where, if $k = \max(H(t))$, and $k' = \max(\{0\} \cup \{k''|(t',k'') \in H'(t'), t' \in T\}) + 1$, then $H''(t) = (H(t) - \{k\}) \cup \{k'\})$. Furthermore, since t is c-enabled in $\langle M,H\rangle$, (t,k) must be the last transition occurrence in all the causal paths it occurs in, and we may observe that H'' contains the same causal paths with (t,k) replaced by (t,k'). As a result it must be that $H \asymp H''$ and the result follows. □

We now proceed to define causal equivalence on traces, a notion that employs the concept of concurrent transitions:

Definition 15. Actions α_1 and α_2 are *concurrent* in state $\langle M,H,\prec\rangle$, if whenever $\langle M,H,\prec\rangle \overset{\alpha_1}{\longmapsto}_c \langle M_1,H_1,\prec_1\rangle$ and $\langle M,H,\prec\rangle \overset{\alpha_2}{\longmapsto}_c \langle M_2,H_2,\prec_2\rangle$ then $\langle M_1,H_1,\prec_1\rangle \overset{\alpha_2}{\longmapsto}_c \langle M',H',\prec'\rangle$ and $\langle M_2,H_2,\prec_2\rangle \overset{\alpha_1}{\longmapsto}_c \langle M'',H'',\prec''\rangle$, where $\langle M',H',\prec'\rangle \asymp \langle M'',H'',\prec''\rangle$.

Thus, two actions are concurrent if the execution of the one does not preclude the other and the two execution orderings lead to causally equivalent states. The condition on final states being equivalent is required to rule out transitions constituting self-loops to/from the same place that are causally dependent on each other.

Definition 16. *Causal equivalence on traces*, denoted by $\asymp$, is the least equivalence relation closed under composition of traces such that (i) if α_1 and α_2 are concurrent actions then $\alpha_1;\alpha_2 \asymp \alpha_2;\alpha_1$ and (ii) $\alpha;\underline{\alpha} \asymp \epsilon$.

The first clause states that in two causally-equivalent traces concurrent actions may occur in any order and the second clause states that it is possible to ignore transitions that have occurred in both the forward and the reverse direction.

The following proposition establishes that two transition instances belonging to distinct causal paths are in fact concurrent transitions and thus can be executed in any order.

Proposition 5. Consider a reversing Petri net (A,P,B,T,F) and suppose $\langle M,H,\prec\rangle \xrightarrow{t_1} \langle M_1,H_1,\prec_1\rangle \xrightarrow{t_2} \langle M_2,H_2,\prec_2\rangle$, where the executions of t_1 and t_2 correspond to transition instances (t_1,k_1) and (t_2,k_2) in $\langle M_2,H_2,\prec_2\rangle$. If there is no causal path π in $\langle M_2,H_2,\prec_2\rangle$ with

$(t_1, k_1) \in \pi$ and $(t_2, k_2) \in \pi$, then (t_1, k_1) and (t_2, k_2) are concurrent transition occurrences in $\langle M, H, \prec \rangle$.

Proof. Since there is no causal path containing both (t_1, k_1) and (t_2, k_2) in $\langle M_2, H_2, \prec_2 \rangle$, we conclude that $(t_1, k_1) \not\prec_2 (t_2, k_2)$. This implies that the two transition occurrences do not handle any common tokens and they can be executed in any order leading to the same marking. Thus, they are concurrent in $\langle M, H, \prec \rangle$. □

We note that causally-equivalent states can execute the same transitions.

Proposition 6. Consider a reversing Petri net (A, P, B, T, F) with causally-equivalent states $\langle M, H_1, \prec_1 \rangle \asymp \langle M, H_2, \prec_2 \rangle$. Then $\langle M, H_1, \prec_1 \rangle \xmapsto{\alpha}_c \langle M_1, H_1', \prec_1' \rangle$ if and only if $\langle M, H_2, \prec_2 \rangle \xmapsto{\alpha}_c \langle M_2, H_2', \prec_2' \rangle$, where $\langle M_1, H_1', \prec_1' \rangle \asymp \langle M_2, H_2', \prec_2' \rangle$.

Proof. It is easy to see that if a transition α is enabled in $\langle M, H_1, \prec_1 \rangle$ it is also enabled in $\langle M, H_2, \prec_2 \rangle$. Therefore, if $\langle M, H_1, \prec_1 \rangle \xmapsto{\alpha}_c \langle M_1, H_1', \prec_1' \rangle$ then $\langle M, H_2, \prec_2 \rangle \xmapsto{\alpha}_c \langle M_2, H_2', \prec_2' \rangle$ where $M_1 = M_2$, and vice versa. In order to show that $H_1' \asymp H_2'$ two cases exist:

- Suppose α is a forward transition corresponding to transition occurrence (t, k_1) in $\langle M_1, H_1', \prec_1' \rangle$ and transition occurrence (t, k_2) in $\langle M_2, H_2', \prec_2' \rangle$. Suppose that $(t', k_1') \prec_1' (t, k_1)$. Then, $\mathsf{post}(t') \cap \mathsf{con}(a, M(x)) \neq \emptyset$ for some $a \in F(x, t)$. Since $H_1 \asymp H_2$, this implies that $(t', k_2') \prec_2' (t, k_2)$ where $k_2' = \max(H_2(t'))$. Therefore, for all causal paths π in $\langle M, H_1, \prec_1 \rangle$, if the last transition occurrence of π causes (t, k_1) then $\pi; (t, k_1)$ is a causal path of $\langle M_1, H_1', \prec_1' \rangle$ and, if not, then π is a causal path in $\langle M_1, H_1', \prec_1' \rangle$. The same holds for causal paths in $\langle M_2, H_2', \prec_2' \rangle$ and (t, k_2). Consequently, we deduce that $H_1' \asymp H_2'$, as required.

- Suppose that α is a reverse transition, i.e. $\alpha = \underline{t}$ for some t, and consider the causal paths of H_1' and H_2'. Since α is a reverse transition, there exists no transition occurrence caused by $(t, \max(H_1(t)))$ in $\langle M, H_1, \prec_1 \rangle$ and no transition occurrence caused by $(t, \max(H_2(t)))$ in $\langle M, H_2, \prec_2 \rangle$. As such, $(t, \max(H_1(t)))$ and $(t, \max(H_2(t)))$ are the last transition occurrences in all paths in $\langle M, H_1, \prec_1 \rangle$ and $\langle M, H_2, \prec_2 \rangle$, respectively, in which they belong. Reversing the transition occurrences results in their elimination from these causal paths. Therefore, we observe that for each causal path in $\langle M_1, H_1', \prec_1' \rangle$ there is an equivalent causal path in $\langle M_2, H_2', \prec_2' \rangle$, and vice versa. Thus $H_1' \asymp H_2'$ as required. □

Note that the above result can be extended to sequences of transitions:

Corollary 1. Consider a reversing Petri Net (A, P, B, T, F) with causally-equivalent states $\langle M, H_1, \prec_1\rangle \asymp \langle M, H_2, \prec_2\rangle$. Then $\langle M, H_1, \prec_1\rangle \xmapsto{\sigma}_c \langle M_1, H_1', \prec_1'\rangle$ if and only if $\langle M, H_2, \prec_2\rangle \xmapsto{\sigma}_c \langle M_2, H_2', \prec_2'\rangle$, where $\langle M_1, H_1', \prec_1'\rangle \asymp \langle M_2, H_2', \prec_2'\rangle$.

The main result, Theorem 1 below, states that two computations beginning in the same initial state lead to equivalent states if and only if the sequences of executed transitions of the two computations are causally equivalent. This guarantees the consistency of the approach since reversing transitions in causal order is in a sense equivalent to not executing the transitions in the first place. Reversal does not give rise to previously unreachable states, on the contrary, it gives rise to exactly the same markings and causally-equivalent histories due to the different keys being possibly assigned because of the different ordering of transitions. The proof structure along with the intermediate results follow those initially presented in [31].

Theorem 1. *[31]*Consider executions $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle$ and $\langle M, H\rangle \xmapsto{\sigma_2}_c \langle M_2, H_2\rangle$. Then, $\sigma_1 \asymp \sigma_2$ if and only if $\langle M_1, H_1\rangle \asymp \langle M_2, H_2\rangle$.

For the proof of Theorem 1 we employ some intermediate results. To begin with, the lemma below states that causal equivalence allows the permutation of reverse and forward transitions that have no causal relations between them. Therefore, computations are allowed to reach for the maximum freedom of choice going backward and then continue forward.

Lemma 3. *[31]* Let σ be a trace. Then there exist traces r, r' both forward such that $\sigma \asymp \underline{r}; r'$ and if $\langle M, H\rangle \xmapsto{\sigma} \langle M', H'\rangle$ then $\langle M, H\rangle \xmapsto{\underline{r}; r'} \langle M', H''\rangle$, where $H' \asymp H''$.

Proof. We prove this by induction on the length of σ and the distance from the beginning of σ to the earliest pair of transitions that contradicts the property $\underline{r}; r'$. If there is no such contradicting pair then the property is trivially satisfied. If not, we distinguish the following cases:

1. If the first contradicting pair is of the form $t; \underline{t}$ then we have $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{t}_c \langle M_2, H_2\rangle \xmapsto{\underline{t}}_c \langle M_3, H_3\rangle \xmapsto{\sigma_2}_c \langle M', H'\rangle$ where $\sigma = \sigma_1; t; \underline{t}; \sigma_2$. By the Loop Lemma $\langle M_1, H_1\rangle = \langle M_3, H_3\rangle$, which yields $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{\sigma_2}_c \langle M', H'\rangle$. Thus we may remove the two transitions from the sequence, the length of σ decreases, and the proof follows by induction.

2. If the first contradicting pair is of the form $t; \underline{t'}$ then we observe that the specific occurrences of t and $\underline{t'}$ must be concurrent. Specifically we have $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{t}_c$

$\langle M_2, H_2\rangle \stackrel{\underline{t'}}{\longmapsto}_c \langle M_3, H_3\rangle \stackrel{\sigma_2}{\longmapsto}_c \langle M', H'\rangle$ where $\sigma = \sigma_1; t; \underline{t'}; \sigma_2$. Since action t' is being reversed, all transition occurrences that are causally dependent on it have either not been executed up to this point or they have already been reversed. This implies that in $\langle M_2, H_2\rangle$ it was not the case that $(t, max(H_2(t))$ was causally dependent on $(t', max(H_2(t'))$. As such, by Proposition 5, $\underline{t'}$ and t are concurrent transitions and t' can be reversed before the execution of t to yield $\langle M, H\rangle \stackrel{\sigma_1}{\longmapsto}_c \langle M_1, H_1\rangle \stackrel{\underline{t'}}{\longmapsto}_c \langle M_2', H_2'\rangle \stackrel{t}{\longmapsto}_c \langle M_3, H_3'\rangle \stackrel{\sigma_2}{\longmapsto}_c \langle M', H''\rangle$, where $H_3' \asymp H_3$ and $H' \asymp H''$. This results in a later earliest contradicting pair and by induction the result follows. □

From the above lemma we conclude the following corollary establishing that causal-order reversibility is consistent with standard forward execution in the sense that causal execution will not generate states that are unreachable in forward execution:

Corollary 2. *[31]* Suppose that H_0 is the initial history. If $\langle M_0, H_0\rangle \stackrel{\sigma}{\longmapsto}_c \langle M, H\rangle$, and σ is a trace with both forward and backward transitions then there exists a transition $\langle M_0, H_0\rangle \stackrel{\sigma'}{\longmapsto}_c \langle M, H'\rangle$, where $H \asymp H'$ and σ' a trace of forward transitions.

Proof. According to Lemma 3, $\sigma \asymp \underline{r}; r'$ where both r and r' are forward traces. Since, however, H_0 is the initial history it must be that r is empty. This implies that $\langle M_0, H_0\rangle \stackrel{r'}{\longmapsto}_c \langle M, H'\rangle$, $H \asymp H'$ and r' is a forward trace. Consequently, writing σ' for r', the result follows. □

Lemma 4. *[31]* Suppose $\langle M, H\rangle \stackrel{\sigma_1}{\longmapsto}_c \langle M', H_1\rangle$ and $\langle M, H\rangle \stackrel{\sigma_2}{\longmapsto}_c \langle M', H_2\rangle$, where $H_1 \asymp H_2$ and σ_2 is a forward trace. Then, there exists a forward trace σ_1' such that $\sigma_1 \asymp \sigma_1'$.

Proof. If σ_1 is forward then $\sigma_1 = \sigma_1'$ and the result follows trivially. Otherwise, we may prove the lemma by induction on the length of σ_1. We begin by noting that, by Lemma 3, $\sigma_1 \asymp \underline{r}; r'$ and $\langle M, H\rangle \stackrel{\underline{r}; r'}{\longmapsto}_c \langle M', H_1\rangle$. Let $\underline{t}$ be the last action in $\underline{r}$. Given that σ_2 is a forward execution that simulates σ_1, it must be that r' contains a forward execution of transition t so that $\langle M', H_1\rangle$ and $\langle M', H_2\rangle$ contain the same causal paths involving transition t (if not we would have $|H_1(t)| < |H_2(t)|$ leading to a contradiction). Consider the earliest occurrence of t in r'. If t is the first transition in r', by the Loop Lemma we may remove the pair of opposite transitions and the result follows by induction. Otherwise, suppose $\langle M, H\rangle \stackrel{\underline{r_1}}{\longmapsto}_c \stackrel{\underline{t}}{\longmapsto}_c \stackrel{r_1'}{\longmapsto}_c \langle M_1, H_3\rangle \stackrel{t*}{\longmapsto}_c \stackrel{t}{\longmapsto}_c \langle M_1', H_4\rangle \stackrel{r_2'}{\longmapsto}_c \langle M', H_1\rangle$, where $r = r_1; t$ and $r' = r_1'; t*; t; r_2$. Two cases exist:

1. Suppose $t* \in \sigma_2$. Let us denote by $num(t, \sigma)$, the number of executions of transition t in a sequence of transitions σ. We observe that since σ_2 contains no reverse executions of t, it must be that $num(t, r') = num(t, \sigma_2) + num(t, r)$. Suppose that the transition occurrences of $t*$ and t as shown in the execution belong to a common causal path. We may extend this path with the succeeding occurrences of t and obtain a causal path such that $t*$ is succeeded by $num(t, \sigma_2) + num(t, r)$ occurrences of t. We observe that it is impossible to obtain such a causal path in $\langle M', H_2\rangle$, since $t*$ is followed by fewer occurrences of t in σ_2. This contradicts the assumption that $H_1 \asymp H_2$. We conclude that the transition occurrences of t and $t*$ above do not belong to any common causal path and therefore, by Proposition 5, the two transition occurrences are concurrent in $\langle M_1, H_3\rangle$.

2. Now suppose that $t* \notin \sigma_2$. Since $H_1(t*) \neq \emptyset$ it must be that $H_2(t*) \neq \emptyset$ and $|H(t*)| = |H_1(t*)| = |H_2(t*)|$. As such, it must be that $t* \in r$ and that its reversal has preceded the reversal of t. Let us suppose that the transition occurrences of $t*$ and t as shown in the execution belong to a common causal path. This implies that a causal path with $t*$ preceding t also occurs in H_2 as well as in H. If we observe that $t*$ has reversed before t we conclude that $t*$ does not cause the preceding occurrence of t. As such there is no causal path within $\langle M, H\rangle$ or $\langle M', H_2\rangle$ containing both t and $t*$, which results in a contradiction. We conclude that the forward occurrences of t and $t*$ are, by Proposition 5, concurrent in $\langle M_1, H_3\rangle$.

Given the above, since the occurrences of t and $t*$ are concurrent the two occurrences may be swapped to yield $\langle M, H\rangle \stackrel{\underline{r_1}}{\longmapsto}_c \stackrel{\underline{t}}{\longmapsto}_c \stackrel{r'_1}{\longmapsto}_c \langle M_1, H_3\rangle \stackrel{t}{\longmapsto}_c \stackrel{t*}{\longmapsto}_c \langle M'_1, H'_4\rangle \stackrel{r'_2}{\longmapsto}_c \langle M', H'_1\rangle$ where $H_4 \asymp H'_4$ and, by Corollary 1, $H_1 \asymp H'_1$. By repeating the process for the remaining transition occurrences in r'_1, this implies that we may permute t with transitions in r'_1 to yield the sequence $\underline{t}; t$. By the Loop Lemma we may remove the pair of opposite transitions and obtain a shorter equivalent trace, also equivalent to σ_2 and conclude by induction. □

We now proceed with the proof of Theorem 1:

Proof of Theorem 1. Suppose $\langle M, H\rangle \stackrel{\sigma_1}{\longmapsto}_c \langle M_1, H_1\rangle$, $\langle M, H\rangle \stackrel{\sigma_2}{\longmapsto}_c \langle M_2, H_2\rangle$ with $\langle M_1, H_1\rangle \asymp \langle M_2, H_2\rangle$. We prove that $\sigma_1 \asymp \sigma_2$ by using a lexicographic induction on the pair consisting of the sum of the lengths of σ_1 and σ_2 and the depth of the earliest disagreement between them. By Lemma 3 we may suppose that σ_1 and σ_2 satisfy the property $\underline{r}; r'$. Call t_1 and t_2 the earliest actions where they disagree. There are three cases in the argument depending on whether these are forward or backward.

1. If t_1 is backward and t_2 is forward, we have $\sigma_1 = \underline{r};\underline{t_1};u$ and $\sigma_2 = \underline{r};t_2;v$ for some r, u, v. Lemma 4 applies to $t_2;v$, which is forward, and $\underline{t_1};u$, which contains both forward and backward actions and thus, by the lemma, it has a shorter forward equivalent. Thus, σ_1 has a shorter forward equivalent and the result follows by induction.

2. If t_1 and t_2 are both forward then it must be the case that $\sigma_1 = \underline{r};r';t_1;u$ and $\sigma_2 = \underline{r};r';t_2;v$, for some r, u, v. Note that it must be that $t_1 \in v$ and $t_2 \in u$. If not, we would have $|H_1(t_1)| \neq |H_2(t_1)|$, and similarly for t_2, which contradicts the assumption that $H_1 \asymp H_2$. As such, we may write $\sigma_1 = \underline{r};r';t_1;u_1;t_2;u_2$, where $u = u_1;t_2;u_2$ and t_2 is the first occurrence of t_2 in u. Consider $t*$ the action immediately preceding t_2. We observe that $t*$ and t_2 cannot belong to a common causal path in $\langle M_1, H_1\rangle$, since an equivalent causal path is impossible to exist in $\langle M_2, H_2\rangle$. This is due to the assumption that σ_1 and σ_2 coincide up to transition sequence $\underline{r};r'$. Thus, we conclude by Proposition 5 that $t*$ and t_2 are in fact concurrent and can be swapped. The same reasoning may be used for all transitions preceding t_2 up to and including t_1, which leads to the conclusion that $\sigma_1 \asymp \underline{r};r';t_2;t_1;u_1;u_2$. This results in an equivalent execution of the same length with a later earliest divergence with σ_2 and the result follows by the induction hypothesis.

3. If t_1 and t_2 are both backward, we have $\sigma_1 = \underline{r};\underline{t_1};u$ and $\sigma_2 = \underline{r};\underline{t_2};v$ for some r, u, v. Two cases exist:

   (a) If $\underline{t_1}$ occurs in v, then we have that $\sigma_2 = \underline{r};\underline{t_2};\underline{v_1};\underline{t_1};v_2$. Given that t_1 reverses right after $\underline{r}$ in σ_1, we may conclude that there is no transition occurrence at this point that causally depends on t_1. As such it cannot have caused the transition occurrences of t_2 and v_1 whose reversal precedes it in σ_2. This implies that the reversal of t_1 may be swapped in σ_2 with each of the preceding transitions, to give $\sigma_2 \asymp \underline{r};\underline{t_1};\underline{t_2};\underline{v_1};v_2$. This results in an equivalent execution of the same length with a later earliest divergence with σ_1 and the result follows by the induction hypothesis.

   (b) If $\underline{t_1}$ does not occur in v, this implies that t_1 occurs in the forward direction in u, i.e. $\sigma_1 = \underline{r};\underline{t_1};u_1;t_1;u_2$, where $u = u_1;t_1;u_2$ with the specific occurrence of t_1 being the first such occurrence in u. Using similar arguments as those in

Lemma 4, we conclude that $\sigma_1 \asymp \underline{r}; \underline{t_1}; t_1; u_1; u_2 \asymp \underline{r}; u_1; u_2$, an equivalent execution of shorter length for σ_1 and the result follows by the induction hypothesis.

We may now prove the opposite direction. Suppose that $\sigma_1 \asymp \sigma_2$ and $\langle M, H\rangle \xrightarrow{\sigma_1}_c \langle M_1, H_1\rangle$ and $\langle M, H\rangle \xrightarrow{\sigma_2}_c \langle M_2, H_2\rangle$. We will show that $\langle M_1, H_1\rangle \asymp \langle M_2, H_2\rangle$. The proof is by induction on the number of rules, k, applied to establish the equivalence $\sigma_1 \asymp \sigma_2$. For the base case we have $k = 0$, which implies that $\sigma_1 = \sigma_2$ and the result trivially follows. For the inductive step, let us assume that $\sigma_1 \asymp \sigma_1' \asymp \sigma_2$, where σ_1 can be transformed to σ_1' with the use of $k = n - 1$ rules and σ_1' can be transformed to σ_2 with the use of a single rule. By the induction hypothesis, we conclude that $\langle M, H\rangle \xrightarrow{\sigma_1'}_c \langle M_1, H_1'\rangle$, where $H_1 \asymp H_1'$. We need to show that $\langle M_1, H_1'\rangle \asymp \langle M_2, H_2\rangle$. Let us write $\sigma_1' = u; w; v$ and $\sigma_2 = u; w'; v$, where w, w' refer to the parts of the two executions where the equivalence rule has been applied. Furthermore, suppose that $\langle M, H\rangle \xrightarrow{u}_c \langle M_u, H_u\rangle \xrightarrow{w}_c \langle M_w, H_w\rangle \xrightarrow{v}_c \langle M_1, H_1'\rangle$ and $\langle M, H\rangle \xrightarrow{u}_c \langle M_u, H_u\rangle \xrightarrow{w'}_c \langle M_w', H_w'\rangle \xrightarrow{v}_c \langle M_2, H_2\rangle$. Three cases exist:

(a) $w = t_1; t_2$ and $w' = t_2; t_1$ with t_1 and t_2 concurrent

(b) $w = t; \underline{t}$ and $w' = \epsilon$

(c) $w = \underline{t}; t$ and $w' = \epsilon$

In all the cases above, we have that $\langle M_w, H_w\rangle \asymp \langle M_w', H_w'\rangle$: for (a) this follows by the definition of concurrent transitions, whereas for (b) and (c) by the Loop Lemma. Given the equivalence of these two states, by Corollary 2, we have that $\langle M_w, H_w\rangle \xrightarrow{v}_c \langle M_1, H_1'\rangle$ and $\langle M_w', H_w'\rangle \xrightarrow{v}_c \langle M_2, H_2\rangle$, where $\langle M_1, H_1'\rangle \asymp \langle M_2, H_2\rangle$, as required. This completes the proof. □

We note that the causal-consistency theorem has been proved using the standard approach of [31]. An alternative approach, stemming from he recent work of [84] could also be possible, whereby the study of various properties within a general framework for reversible systems is established. More precisely, causal consistency can be guaranteed by proving a set of axioms relating to the parabolic Lemma and the Square property.

3.2.4 Out-of-Causal-Order Reversibility

While in backtracking and causal-order reversibility reversing is cause respecting, there are many examples of systems where undoing actions in an out-of-causal order is either inherent

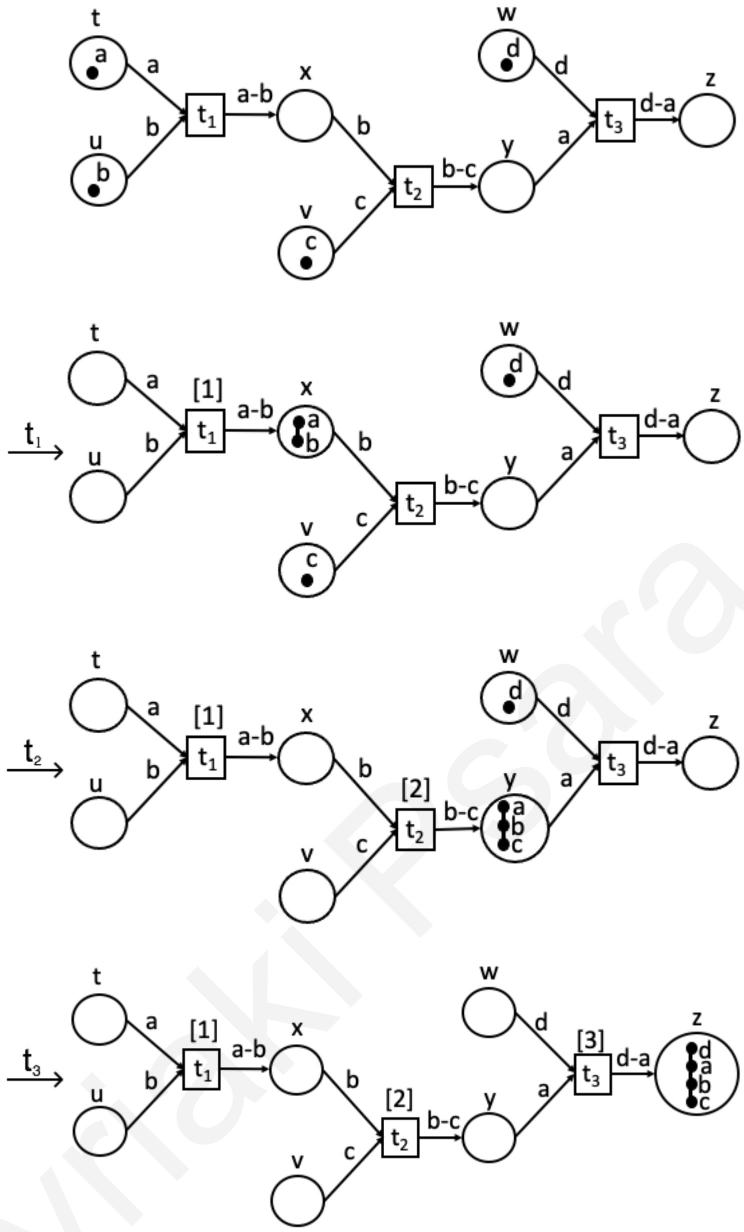


Figure 3.13: Forward execution of out-of-causal-order example

or desirable. In this section we consider this type of reversibility in the context of RPNs. We begin by specifying that in out-of-causal-order reversibility any executed transition can be reversed at any time.

Definition 17. Consider a reversing Petri net (A, P, B, T, F), a state $\langle M, H\rangle$, and a transition $t \in T$. We say that t is *o-enabled* in $\langle M, H\rangle$, if $H(t) \neq \emptyset$.

Let us begin to consider out-of-causal-order reversibility via the example of Figures 3.13 and 3.14. The first Figure 3.13 presents the forward execution of the transition sequence $\langle t_1, t_2, t_3\rangle$. The second Figure 3.14 represents the out-of-causal-order reversal of transition sequence $\langle t_1, t_2, t_3\rangle$. Suppose that transition t_1 is to be reversed out of order. The effect of

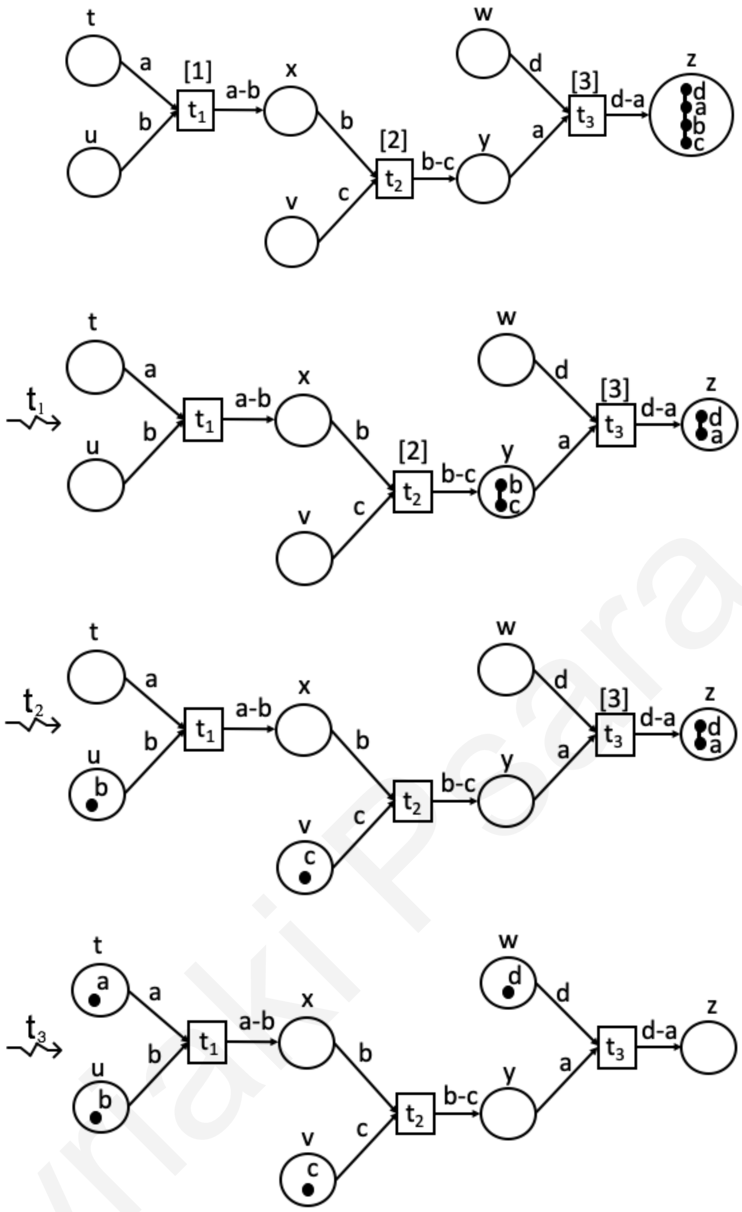


Figure 3.14: Out-of-causal-order example

this reversal should be the destruction of the bond between a and b. This means that the component $d{-}a{-}b{-}c$ is broken into the bonds $d{-}a$ and $b{-}c$, which should backtrack within the net to capture the reversal of the transition. Nonetheless, the tokens of $d{-}a$ must remain at place z. This is because a bond exists between them that has not been reversed and was the effect of the immediately preceding transition t_3. However, in the case of $b{-}c$, the bond can be returned to place y, which is the place where the two tokens were connected and from where they could continue to participate in any further computation requiring their coalition. Once transition t_2 is subsequently reversed, the bond between b and c is destroyed and thus the two tokens are able to return to their initial places as shown in the third net in the figure. Finally, when subsequently transition t_3 is reversed, the bond between d and a breaks and, given that neither d nor a are connected to other elements, the tokens return to their initial places. As with the other types of reversibility, when reversing a transition histories are updated by removing the greatest key identifier of the executed transition.

Summing up, the effect of reversing a transition in out-of-causal order is that all bonds created by the transition are undone. This may result in tokens backtracking in the net. Further, if the reversal of a transition causes a coalition of bonds to be broken down into a set of subcomponents due to the destruction of bonds, then each of these coalitions should flow back, as far back as possible, after the last transition in which this sub-coalition participated. To capture this notion of "as far backwards as possible" we introduce the following:

Definition 18. Given a reversing Petri net (A, P, B, T, F), an initial marking M_0, a history H, and a set of bases and bonds $C \subseteq A \cup B$ we write:

$$\mathsf{last}_T(C, H) = \begin{cases} t, & \text{if } \exists t,\ \mathsf{post}(t) \cap C \neq \emptyset,\ H(t) \neq \emptyset, \text{ and} \\ & \nexists t',\ \mathsf{post}(t') \cap C \neq \emptyset,\ H(t') \neq \emptyset, \\ & max(H(t')) \geq max(H(t)) \\ \bot, & \text{otherwise} \end{cases}$$

$$\mathsf{last}_P(C, H) = \begin{cases} x, & \text{if } t = \mathsf{last}_T(C, H), \{x\} = \{y \in t\circ \mid F(t, y) \cap C \neq \emptyset\} \\ & \text{or, if } \bot = \mathsf{last}_T(C, H), C \subseteq M_0(x) \\ \bot, & \text{otherwise} \end{cases}$$

Thus, if component C has been manipulated by some previously-executed transition, then $\mathsf{last}_T(C, H)$ is the last executed such transition. Otherwise, if no such transition exists (e.g., because all transitions involving C have been reversed), then $\mathsf{last}_T(C, H)$ is undefined ($\bot$). Similarly, $\mathsf{last}_P(C, H)$ is the outgoing place connected to $\mathsf{last}_T(C, H) \neq \bot$ having common tokens with C, assuming that such a place is unique, or the place in the initial marking in which C existed if $\mathsf{last}_T(C, H) = \bot$, and undefined otherwise.

Transition reversal in an out-of-causal order can thus be defined as follows:

Definition 19. Given a reversing Petri net (A, P, B, T, F), an initial marking M_0, a state $\langle M, H\rangle$ and a transition t that is o-enabled in $\langle M, H\rangle$, we write $\langle M, H\rangle \overset{t}{\rightsquigarrow}_o \langle M', H'\rangle$ where H' is defined as in Definition 7 and we have:

$$\begin{aligned} M'(x) = &\Big(M(x) \cup \bigcup_{a \in M(y) \cap \mathsf{post}(t), \mathsf{last}_P(C_{a,y}, H') = x} C_{a,y}\Big) \\ &-\Big(\mathsf{eff}(t) \cup \bigcup_{a \in M(x) \cap \mathsf{post}(t), \mathsf{last}_P(C_{a,x}, H') \neq x} C_{a,x}\Big) \end{aligned}$$

where we use the shorthand $C_{b,z} = \mathsf{con}(b, M(z) - \mathsf{eff}(t))$ for $b \in A$, $z \in P$.

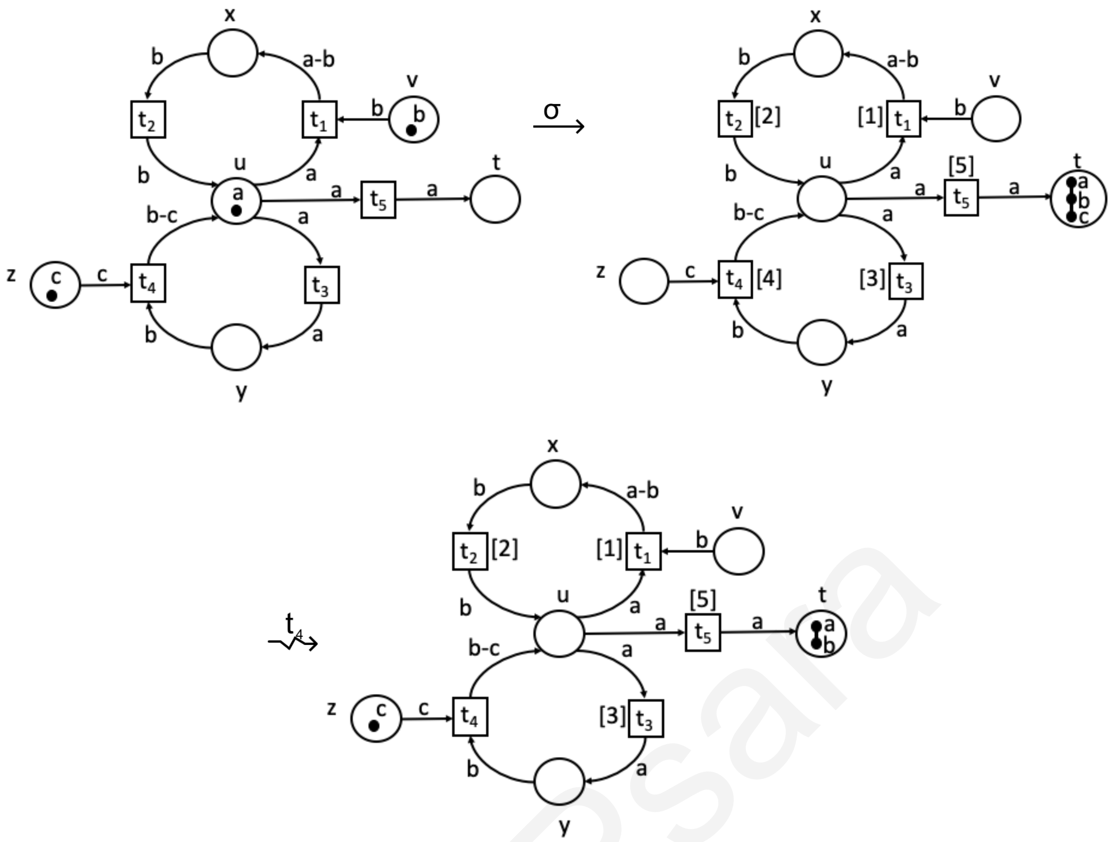


Figure 3.15: Out-of-causal-order reversing where $\sigma = \langle t_1, t_2, t_3, t_4, t_5 \rangle$

Thus, when a transition t is reversed in an out-of-causal-order fashion all bonds that were created by the transition in $\mathsf{eff}(t)$ are undone. Furthermore, tokens and bonds involved in the transition are relocated back to the place where they would have existed if transition t never took place, as defined by $\mathsf{last}_P(C, H')$. Note that if the destruction of a bond divides a component into smaller connected sub-components then each of these sub-components is relocated separately. Specifically, the definition states that: if a token a and its connected components involved in transition t, last participated in some transition with outgoing place y other than x, then the sub-component is removed from place x and returned to place y, otherwise it is returned to the place where it occurred in the initial marking.

An example of out-of-causal-order reversibility in a cyclic RPN can be seen in Figure 3.15. Here the cycles $\langle t_1, t_2 \rangle$ and $\langle t_3, t_4 \rangle$ are executed in this order followed by transition t_5. We reverse in out-of-causal order transition t_4, which breaks the bond between $b-c$ and returns token c back to its original place z. Moreover, the bond between a–b remains in place t, which is the outgoing place of the last transition of token a. Note that this state did not occur during the forward execution of the RPN.

The following results describe how tokens and bonds are manipulated during out-of-causal-order reversibility, where we write $\longmapsto_o$ for $\longrightarrow \cup \rightsquigarrow_o$.

Proposition 7. Suppose $\langle M, H\rangle \overset{t}{\longmapsto}_o \langle M', H'\rangle$ and let $a \in A$ where $a \in M(x)$ and $a \in M'(y)$. Then, $\mathsf{con}(a, M'(y)) = \mathsf{con}(a, M(x) \cup C)$, where $C = \mathsf{eff}(t) \cup \{\mathsf{con}(b, M(u)) \mid a{-}b \in \mathsf{eff}(t), b \in M(u)\}$), if t is a forward transition, and $\mathsf{con}(a, M'(y)) = \mathsf{con}(a, M(x) - \mathsf{eff}(t))$, if t is a reverse transition.

Proof. The proof is straightforward by the definition of the firing rules. □

Proposition 8. Given a reversing Petri net (A, P, B, T, F), an initial state $\langle M_0, H_0\rangle$, and an execution $\langle M_0, H_0\rangle \overset{t_1}{\longmapsto}_o \langle M_1, H_1\rangle \overset{t_2}{\longmapsto}_o \ldots \overset{t_n}{\longmapsto}_o \langle M_n, H_n\rangle$ the following hold for all $0 \leq i \leq n$:

1. For all $a \in A$, $|\{x \in P \mid a \in M_i(x)\}| = 1$, and $a \in M_i(x)$ where $x = \mathsf{last}_P(\mathsf{con}(a, M_i(x)), H_i)$.
2. For all $\beta \in B$,
   (a) $0 \leq |\{x \in P \mid \beta \in M_i(x)\}| \leq 1$.
   (b) if $|\{x \in P \mid \beta \in M_{i-1}(x)\}| = 0$ and $|\{x \in P \mid \beta \in M_i(x)\}| = 1$, then t_i is a forward transition and $\beta \in \mathsf{eff}(t_i)$,
   (c) if $|\{x \in P \mid \beta \in M_{i-1}(x)\}| = 1$ and $|\{x \in P \mid \beta \in M_i(x)\}| = 0$, then t_i is a reverse transition and $\beta \in \mathsf{eff}(t_i)$,
   (d) if $|\{x \in P \mid \beta \in M_{i-1}(x)\}| = |\{x \in P \mid \beta \in M_i(x)\}|$, then $\beta \notin \mathsf{eff}(t_i)$.

Proof. Consider a reversing Petri net (A, P, B, T, F), an initial state $\langle M_0, H_0\rangle$, and an execution $\langle M_0, H_0\rangle \overset{t_1}{\longmapsto}_o \langle M_1, H_1\rangle \overset{t_2}{\longmapsto}_o \ldots \overset{t_n}{\longmapsto}_o \langle M_n, H_n\rangle$. The proof is given by induction on n.

Base Case. For $n = 0$, by our assumption of token uniqueness and the definitions of last_P and last_T the claim follows trivially.

Induction Step. Suppose the claim holds for all but the last transition and consider transition t_n. Two cases exist, depending on whether t_n is a forward or a reverse transition:

- Suppose that t_n is a forward transition. Then by Proposition 1, for all $a \in A$, $|\{x \in P \mid a \in M_n(x)\}| = 1$. Additionally, we may see that if $a \in M_n(x)$ two cases exists. If $a \in \mathsf{con}(b, M_{n-1}(y))$, for some $b \in F(t_n, z)$ then $x = z = \mathsf{last}_P(\mathsf{con}(a, M_n(x)), H_n)$.

Otherwise, it must be that $a \in M_{n-1}(x)$ where, by the induction hypothesis, $x = \mathsf{last}_P(\mathsf{con}(a, M_{n-1}(x)), H_{n-1})$. Since $a \notin \mathsf{eff}(t_n)$, by clause 2(b) we may deduce that $\mathsf{con}(a, M_{n-1}(x)) = \mathsf{con}(a, M_n(x))$, which leads to $x = \mathsf{last}_P((\mathsf{con}(a, M_{n-1}(x)), H_{n-1}) = \mathsf{last}_P(\mathsf{con}(a, M_n(x)), H_n)$. Thus, the result follows.

Now let $\beta \in B$. To begin with, clause (2)(a) follows by Proposition 1. Furthermore, we may see that the forward transition t_n may only create exactly the bonds in $\mathsf{eff}(t_n)$ and it maintains all remaining bonds. Thus, clauses 2(b) and 2(d) follow.

- Suppose that t_n is a reverse transition. Consider $a \in A$ with $a \in M_{n-1}(x)$ for some $x \in P$. Two cases exist:

  – Suppose $\mathsf{last}_T(\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)), H_n) = \bot$. It must be that $\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)) \subseteq M_0(y)$ for some y such that $a \in M_0(y)$. Suppose that this is not the case. Then there must exist some $\beta \in \mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n))$ with $\beta \notin M_0(y)$. By the induction hypothesis, there exists some t_i in the execution such that $\beta \in \mathsf{eff}(t_i)$ which was not reversed, i.e. $H_n(t_i) \neq \emptyset$. This however implies that t_i is a transition that has manipulated the connected component $\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n))$, which contradicts our assumption of $\mathsf{last}_T(\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)), H_n) = \bot$. Therefore, $a \in M_n(y)$, where $a \in M_0(y)$ and by Proposition 7 $\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)) = \mathsf{con}(a, M_n(y))$ which gives $y = \mathsf{last}_P(\mathsf{con}(a, M_n(y)), H_n)$ and the result follows.

  – Suppose $\mathsf{last}_T(\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)), H_n) = t_k$. Then, it must be that there exists a unique $y \in t_k\circ$ such that $\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)) \cap F(t_k, z) \neq \emptyset$. Suppose that this is not the case. Then there must exist some $\beta = (a, b) \in \mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n))$ with $a \in F(t_k, y_1)$, $b \in F(t_k, y_2)$, and $y_1 \neq y_2$. Since $\beta \in M_n(y)$, by the induction hypothesis, there exists some t_i in the execution such that $\beta \in \mathsf{eff}(t_i)$, $i > k$ which was not reversed, i.e. $H_n(t_i) \neq \emptyset$. This however implies that t_i is a transition that has manipulated the connected component $\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n))$ later than t_k, which contradicts our assumption of $\mathsf{last}_T(\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)), H_n) = t_k$. Therefore, there exists a unique $y \in t_k\circ$ such that $\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)) \cap F(t_k, z) \neq \emptyset$, $a \in M_n(y)$. Furthermore, by Proposition 7 $\mathsf{con}(a, M_{n-1}(x) - \mathsf{eff}(t_n)) = \mathsf{con}(a, M_n(y))$ which gives $y = \mathsf{last}_P(\mathsf{con}(a, M_n(y)), H_n)$ and the result follows.

Now consider $\beta \in B$. By clause 1, we may deduce clause 2(a). Finally, we may observe that the reverse transition t_n may only remove exactly the bonds in $\mathsf{eff}(t_n)$ and it maintains all remaining bonds, thus, clauses 2(b)-2(d) follow. □

As we have already discussed (e.g., see Figures 3.2 and 3.15), unlike causal-order reversibility, out-of-causal-order reversibility may give rise to states that cannot be reached by forward-only execution. Nonetheless, note that the proposition establishes that during out-of-causal-order reversing it is not the case that tokens and bonds may reach places they have not previously occurred in. On the contrary, a component will always return to the place following the last transition that has manipulated it. This observation also gives rise to the following corollary, which characterises the marking of a state during computation.

Corollary 3. Given a reversing Petri net (A, P, B, T, F), an initial state $\langle M_0, H_0\rangle$, and an execution $\langle M_0, H_0\rangle \xrightarrow{t_1}_o \langle M_1, H_1\rangle \xrightarrow{t_2}_o \ldots \xrightarrow{t_n}_o \langle M_n, H_n\rangle$, then for all $x \in P$ we have

$$M_n(x) = \bigcup_{a \in M_n(y), \mathsf{last}_P(C_{a,y}, H_n) = x} C_{a,y}$$

where $C_{a,y} = \mathsf{con}(a, M_n(y))$.

Proof. According to Proposition 8 clauses (1) and 2(a) the result follows. □

The dependence of the position of a connected component and a transition sequence can be exemplified by the following proposition.

Proposition 9. Consider executions $\langle M_0, H_0\rangle \xrightarrow{\sigma_1}_o \langle M_1, H_1\rangle$, $\langle M_0, H_0\rangle \xrightarrow{\sigma_2}_o \langle M_2, H_2\rangle$, and a token a such that $a \in M_1(x)$, $a \in M_2(y)$, for some x, $y \in P$, and $\mathsf{con}(a, M_1(x)) = \mathsf{con}(a, M_2(x))$. Then, $\mathsf{last}_T(\mathsf{con}(a, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a, M_2(y)), H_2)$ implies $x = y$.

Proof. Consider executions $\langle M_0, H_0\rangle \xrightarrow{\sigma_1}_o \langle M_1, H_1\rangle$, $\langle M_0, H_0\rangle \xrightarrow{\sigma_2}_o \langle M_2, H_2\rangle$ and a token a such that $a \in M_1(x)$, $a \in M_2(x)$. Further, let us assume that $\mathsf{last}_T(\mathsf{con}(a, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a, M_2(y)), H_2)$. Two cases exist:

- $\mathsf{last}_T(\mathsf{con}(a, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a, M_2(y)), H_2) = \bot$. This implies that no transition has manipulated any of the tokens and bonds of the two connected components. As such, by Proposition 8, $\mathsf{con}(a, M_1(x)) \subseteq M_0(x)$ and $\mathsf{con}(a, M_2(y)) \subseteq M_0(y)$, and by the uniqueness of tokens we conclude that $x = y$ as required.

- $\mathsf{last}_T(\mathsf{con}(a, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a, M_2(y)), H_2) = t$. This implies that there is $b \in \mathsf{con}(a, M_1(x)) = \mathsf{con}(a, M_2(y))$ such that $b \in F(t, z)$ for some place z. By definition, we deduce that $\mathsf{last}_P(\mathsf{con}(a, M_1(x)), H_1) = z = \mathsf{last}_P(\mathsf{con}(a, M_2(y)), H_2)$, thus, $x = y$ as required. □

From the above result we may prove the following proposition establishing that executing two causally equivalent sequences of transitions in the out-of-causal-order setting will give rise to causally equivalent states.

Proposition 10. Suppose $\langle M_0, H_0\rangle \overset{\sigma_1}{\longmapsto}_o \langle M_1, H_1\rangle$ and $\langle M_0, H_0\rangle \overset{\sigma_2}{\longmapsto}_o \langle M_2, H_2\rangle$. If $\sigma_1 \asymp \sigma_2$ then $\langle M_1, H_1\rangle \asymp \langle M_2, H_2\rangle$.

Proof. Suppose $\langle M_0, H_0\rangle \overset{\sigma_1}{\longmapsto}_o \langle M_1, H_1\rangle$, $\langle M_0, H_0\rangle \overset{\sigma_2}{\longmapsto}_o \langle M_2, H_2\rangle$ and $\sigma_1 \asymp \sigma_2$. Since $\sigma_1 \asymp \sigma_2$ it must be that the two executions contain the same causal paths, therefore, $H_1 \asymp H_2$. To show that $M_1 = M_2$ consider token a such that $a \in M_1(x) \cap M_2(y)$. Since $\sigma_1 \asymp \sigma_2$, we may conclude that the two executions contain the same set of executed and not reversed transitions. Thus, by Proposition 8(2), we have $\mathsf{con}(a, M_1(x)) = \mathsf{con}(a, M_2(y))$. Furthermore, it must be that $t_1 = \mathsf{last}_T(\mathsf{con}(a, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a, M_2(y)), H_2) = t_2$. If not, since $\sigma_1 \asymp \sigma_2$, we would have that t_1 and t_2 are concurrent, which is not possible since they manipulate the same connected component and thus a causal relation exists between them. Therefore, by Proposition 9, $x = y$. This implies by Corollary 3 that $M_1(x) = M_2(x)$, for all places x, which completes the proof. □

We finally establish a Loop Lemma for out-of-causal reversibility.

Lemma 5 (Loop)**.** For any forward transition $\langle M, H\rangle \xrightarrow{t} \langle M', H'\rangle$ there exists a reverse transition $\langle M', H'\rangle \overset{t}{\rightsquigarrow}_o \langle M, H\rangle$.

Proof. Suppose $\langle M, H\rangle \xrightarrow{t} \langle M', H'\rangle$. Then t is clearly o-enabled in H'. Furthermore, $\langle M', H'\rangle \overset{t}{\rightsquigarrow}_o \langle M'', H''\rangle$ where $H'' = H$ by the definition of $\rightsquigarrow_o$. In addition, for all $a \in A$, we may prove that $a \in M''(x)$ if and only if $a \in M(x)$. Suppose $a \in M(y)$, we distinguish two cases. If $\mathsf{con}(a, M(y)) \cap \mathsf{pre}(t) = \emptyset$, then we may see that $a \in M'(y)$ and $a \in M''(y)$, and the result follows. Otherwise, if $\mathsf{con}(a, M(y)) \cap \mathsf{pre}(t) \neq \emptyset$, then $a \in M'(z)$, where $F(t, z) \cap \mathsf{con}(a, M(y)) \neq \emptyset$. Furthermore, suppose that $a \in M''(w)$. By Proposition 7 we have that $\mathsf{con}(a, M'(z)) = \mathsf{con}(a, M(y) \cup C)$, $C = \mathsf{eff}(t) \cup \{\mathsf{con}(b, M(u)) \mid a - b \in \mathsf{eff}(t), b \in M(u)\}$, and $\mathsf{con}(a, M''(w)) = \mathsf{con}(a, M'(z) - \mathsf{eff}(t)) = \mathsf{con}(a, (M(y) \cup C) - \mathsf{eff}(t)) = \mathsf{con}(a, M(y))$. Furthermore, $y = \mathsf{last}_P(\mathsf{con}(a, M(y)), H)$, by Corollary 3. Since $H = H''$, we have $w = \mathsf{last}_P(\mathsf{con}(a, M''(w)), H'')) = \mathsf{last}_P(\mathsf{con}(a, M(y)), H)) = y$, and the result follows. □

Note that in the case of out-of-causal-order reversibility, the opposite direction of the lemma does not hold. This is because reversing a transition in an out-of-causal-order fashion may bring a system to a state not reachable by forward-only transitions, and where the transition is not enabled in the forward direction. As an example, consider the RPN of Figure 3.14

and after the reversal of transition t_2. In this state, transition t_2 is not forward enabled since token b is not available in place x, as required for the transition to fire.

3.2.5 Relationship Between Reversibility Notions

We continue to study the relationship between the three forms of reversibility. Our first result confirms the relationship between the enabledness conditions for each of backtracking, causal-order, and out-of-causal-order reversibility.

Proposition 11. Consider a state $\langle M, H\rangle$, and a transition t. Then, if t is bt-enabled in $\langle M, H\rangle$ it is also c-enabled. Furthermore, if t is c-enabled in $\langle M, H\rangle$ then it is also o-enabled.

Proof. The proof is immediate by the respective definitions. □

We next demonstrate a "universality" result of the $\rightsquigarrow_o$ transition relation by showing that it manipulates the state of a reversing Petri net in an identical way to $\rightsquigarrow_c$, in the case of c-enabled transitions, and to $\rightsquigarrow_b$, in the case of bt-enabled transitions. Central to the proof is the following result establishing that during causal-order reversibility a component is returned to the place following the last transition that has manipulated it or, if no such transition exists, in the place where it occurred in the initial marking.

Proposition 12. Given a reversing Petri net (A, P, B, T, F), an initial state $\langle M_0, H_0\rangle$, and an execution $\langle M_0, H_0\rangle \xmapsto{t_1}_c \langle M_1, H_1\rangle \xmapsto{t_2}_c \ldots \xmapsto{t_n}_c \langle M_n, H_n\rangle$. Then for all $a \in A$, $a \in M_n(x)$ where $x = \mathsf{last}_P(\mathsf{con}(a, M_n(x)), H_n)$.

Proof. The proof is by induction on n and it follows along similar lines to the proof of Proposition 8(1). □

Propositions 8 and 12 yield the following corollary for forward-only execution.

Corollary 4. Given a reversing Petri net (A, P, B, T, F), an initial state $\langle M_0, H_0\rangle$, and an execution $\langle M_0, H_0\rangle \xrightarrow{t_1} \langle M_1, H_1\rangle \xrightarrow{t_2} \ldots \xrightarrow{t_n} \langle M_n, H_n\rangle$, for all $a \in A$, $a \in M_n(x)$ where $x = \mathsf{last}_P(\mathsf{con}(a, M_n(x)), H_n)$.

We may now verify that the causal-order and out-of-causal-order reversibility have the same effect when reversing a c-enabled transition.

Proposition 13. Consider a state $\langle M, H\rangle$ and a transition t c-enabled in $\langle M, H\rangle$. Then, $\langle M, H\rangle \overset{t}{\rightsquigarrow}_c \langle M', H'\rangle$ if and only if $\langle M, H\rangle \overset{t}{\rightsquigarrow}_o \langle M', H'\rangle$.

Proof. Let us suppose that transition t is c-enabled and $\langle M, H\rangle \overset{t}{\rightsquigarrow}_c \langle M_1, H_1\rangle$. By Proposition 11, t is also o-enabled. Suppose $\langle M, H\rangle \overset{t}{\rightsquigarrow}_o \langle M_2, H_2\rangle$. It is easy to see that in fact $H_1 = H_2$ (the two histories are as H with the exception that $H_1(t) = H_2(t) = H(t) - \{\max(H(t))\}$).

To show that $M_1 = M_2$ first we observe that for all $a \in A$, by Proposition 12 we have $a \in M_1(x)$ where $x = \mathsf{last}_P(\mathsf{con}(a, M_1(x)), H_1)$ and by Proposition 8 we have $a \in M_1(y)$ where $y = \mathsf{last}_P(\mathsf{con}(a, M_2(y)), H_2)$. We may also see that $\mathsf{con}(a, M_1(x)) = \mathsf{con}(a, M(z) - \mathsf{eff}(t)) = \mathsf{con}(a, M_2(y))$, where $a \in M(z)$. Since in addition we have $H_1 = H_2$ the result follows.

Now let $\beta \in B$. We must show that $\beta \in M_1(x)$ if and only if $\beta \in M_2(x)$. Two cases exist:

- If $\beta \in \mathsf{eff}(t)$ then by Propositions 4 and 8, $\beta \notin M_1(x)$ and $\beta \notin M_2(x)$ for all $x \in P$.
- if $\beta \notin \mathsf{eff}(t)$ then by Propositions 4 and 8, $|\{x \in P \mid \beta \in M_1(x))\}| = |\{x \in P \mid \beta \in M_2(x))\}| = 1$ and by the analysis on tokens $\beta \in M_1(x)$ if and only if $\beta \in M_2(x)$ and the result follows.

This completes the proof. □

An equivalent result can be obtained for backtracking.

Proposition 14. Consider a state $\langle M, H\rangle$, and a transition t, bt-enabled in $\langle M, H\rangle$. Then, $\langle M, H\rangle \overset{t}{\rightsquigarrow}_b \langle M', H'\rangle$ if and only if $\langle M, H\rangle \overset{t}{\rightsquigarrow}_o \langle M', H'\rangle$.

Proof. Consider a state $\langle M, H\rangle$ and suppose that transition t is bt-enabled and $\langle M, H\rangle \overset{t}{\rightsquigarrow}_b \langle M', H'\rangle$. Then, by Proposition 11, there exists $k \in H(t)$, such that for all $t' \in T$, $k' \in H(t')$, it holds that $k \geq k'$. This implies that t is also c-enabled, and by the definition of $\rightsquigarrow_c$, we conclude that $\langle M, H\rangle \overset{t}{\rightsquigarrow}_c \langle M', H'\rangle$. Furthermore, by Proposition 13 $\langle M, H\rangle \overset{t}{\rightsquigarrow}_o \langle M', H'\rangle$, and the result follows. □

We obtain the following corollary confirming the expectation that backtracking is an instance of causal reversing, which in turn is an instance of out-of-causal-order reversing. It is easy to see that both inclusions are strict, as for example illustrated in Figures 3.5, 3.9, and 3.14.

Corollary 5. $\rightsquigarrow_b \subset \rightsquigarrow_c \subset \rightsquigarrow_o$.

Proof. The proof follows from Propositions 13 and 14. □

We note that in addition to establishing the relationship between the three notions of reversibility, the above results provide a unification of the different reversal strategies, in the sense that a single firing rule, $\rightsquigarrow_o$, may be paired with the three notions of transition enabledness to provide the three different notions of reversibility. This fact may be exploited in the proofs of results that span the three notions of reversibility. Such a proof follows in the following proposition that establishes a reverse diamond property for RPNs. According to this property, the execution of a reverse transition does not preclude the execution of another reverse transition and their execution leads to the same state. In what follows we write $\longmapsto$ for $\longrightarrow \cup \rightsquigarrow$ where $\rightsquigarrow$ could be an instance of one of $\rightsquigarrow_b$, $\rightsquigarrow_c$, and $\rightsquigarrow_o$.

Proposition 15 (Reverse Diamond)**.** Consider a state $\langle M, H\rangle$, and reverse transitions $\langle M, H\rangle \overset{t_1}{\rightsquigarrow} \langle M_1, H_1\rangle$ and $\langle M, H\rangle \overset{t_2}{\rightsquigarrow} \langle M_2, H_2\rangle$, $t_1 \neq t_2$. Then $\langle M_1, H_1\rangle \overset{t_2}{\rightsquigarrow} \langle M', H'\rangle$ and $\langle M_2, H_2\rangle \overset{t_1}{\rightsquigarrow} \langle M', H'\rangle$.

Proof. Let us suppose that $\langle M, H\rangle \overset{t_1}{\rightsquigarrow} \langle M_1, H_1\rangle$ and $\langle M, H\rangle \overset{t_2}{\rightsquigarrow} \langle M_2, H_2\rangle$, $t_1 \neq t_2$. First we note that $\rightsquigarrow$ may be an instance of $\rightsquigarrow_c$ or $\rightsquigarrow_o$ but not $\rightsquigarrow_b$, since in the case of $\rightsquigarrow_b$ the backward transition is uniquely determined as the transition with the maximum key. Furthermore, we observe that t_2 remains backward-enabled in $\langle M_1, H_1\rangle$ and likewise t_1 in $\langle M_2, H_2\rangle$. Specifically, if $\rightsquigarrow = \rightsquigarrow_c$, since t_1 and t_2 are $c-$enabled in $\langle M, H\rangle$, by Definition 11 we conclude that $(t_2, max(H(t_2)))$ is not causally dependent on $(t_1, max(H(t_1)))$ and vice versa, which continues to hold after the reversal of each of these transitions. In the case of $\rightsquigarrow = \rightsquigarrow_o$ this is straightforward from the definition of o-enabledness.

So, let us suppose that $\langle M_1, H_1\rangle \overset{t_2}{\rightsquigarrow}_c \langle M_1', H_1'\rangle$ and $\langle M_2, H_2\rangle \overset{t_1}{\rightsquigarrow}_c \langle M_2', H_2'\rangle$. It is easy to see that $H_1' = H_2'$ since both of these histories are identical with H with the maximum keys of t_1 and t_2 removed.

To show that $M_1' = M_2'$ first we observe that for all $a \in A$, by Propositions 8 and 12 we have $a \in M_1'(x)$, $a \in M_2'(y)$ where $x = \mathsf{last}_P(\mathsf{con}(a, M_1'(x)), H_1')$, $y = \mathsf{last}_P(\mathsf{con}(a, M_2'(y)), H_2')$. We may see that $\mathsf{con}(a, M_1'(x)) = \mathsf{con}(a, M(z) - (\mathsf{eff}(t_1) \cup \mathsf{eff}(t_2)) = \mathsf{con}(a, M_2'(y))$, where $a \in M(z)$. Since in addition we have $H_1' = H_2'$ the result follows.

Now let $\beta \in B$. We must show that $\beta \in M_1'(x)$ if and only if $\beta \in M_2'(x)$. Two cases exist:

- If $\beta \in \mathsf{eff}(t_1) \cup \mathsf{eff}(t_2)$ then by Propositions 4 and 8, $\beta \notin M_1'(x)$ and $\beta \notin M_2'(x)$ for all $x \in P$.

- if $\beta \notin \mathsf{eff}(t_1) \cup \mathsf{eff}(t_2)$ then by Propositions 4 and 8, $|\{x \in P \mid \beta \in M_1'(x))\}| = |\{x \in P \mid \beta \in M_2'(x))\}|$ and, by the analysis on tokens, $\beta \in M_1'(x)$ if and only if $\beta \in M_2'(x)$.

This completes the proof. □

Corollary 6. Consider a state $\langle M, H\rangle$, and traces σ_1, σ_2 permutations of the same reverse transitions where $\langle M, H\rangle \overset{\sigma_1}{\longmapsto} \langle M', H'\rangle$ and $\langle M, H\rangle \overset{\sigma_2}{\longmapsto} \langle M'', H''\rangle$. Then $\langle M', H'\rangle = \langle M'', H''\rangle$.

Proof. The proof follows by induction on the sum of the length $|\sigma_1| = |\sigma_2|$ and the depth of the earliest disagreement between the two traces, and uses similar arguments to those found in the proof of Proposition 15. □

We note that the analogue of Proposition 15 for forward transitions, i.e. the Forward Diamond property, does not hold for RPNs. To begin with t_1 and t_2 may be in conflict. The proposition fails to hold even in the case of joinable transitions (i.e. transitions that may yield the same marking after a sequence of forward moves) due to the case of co-initial, independent cycles: Even though such cycles can be executed in any order, it is impossible to complete the square for their initial transitions.

3.3 Case Studies

The framework of reversing Petri nets could be applied in fields outside Computer Science, since the expressive power and visual nature offered by Petri nets coupled with reversible computation has the potential of providing an attractive setting for analysing systems, for instance in biology, chemistry or hardware engineering. The construction of reversible modelling languages can indicate how to capture the behaviour of reversible actions in order to implement or even extend the primitive processes of biological reactions, movement in robotics, quantum computation and reliable systems. Implementing several applications ranging from biochemistry to long-running transactions would give a better understanding on reversible computation especially when it comes to out-of-causal modelling which is still not very well understood in the area of Computer Science.

3.3.1 ERK Signalling Pathway

Biochemical systems, such as covalent bonds, constitute the ideal setting to study reversible computation especially in its out-of-causal-order form. In particular, the MAPK/ERK pathway (also known as the Ras-Raf-MEK-ERK pathway)is one of many real-life examples that naturally feature reversibility that violates the causal ordering established by forward execution. This pathway has been modelled in various formalisms including *CCSK* [118], PEPA [26], BioNetGen [21], and Kappa [30].

In this section we illustrate how reversing Petri nets allow us to capture naturally this form of out-of-causal-order reversible system. Specifically, our configuration follows that of CCSK in [118] where out-of-causal reversibility is triggered to release tokens form connected component so that the tokens can proceed to participate in other transitions. Additionally, in [118] an execution control operator is used to enforce a particular order of execution between forward and reverse actions. In RPNs this is achieved by using negative tokens that require the reversal of specific transitions in order to reverse negative tokens and therefore allow the forward execution of following transitions. However, in RPNs the execution of concurrent forward transitions can be executed in any order unlike the control operator of CCSK which is able to require a specific order among forward transitions.

In Figure 3.16 we demonstrate the extracellular-signal-regulated kinase (*ERK*) pathway, also known as *Ras/Raf-1, MEK, ERK* pathway, which is one of the major signalling cassettes of the mitogen activated protein kinase (*EMAPK*) signalling pathway. The ERK pathway is a chain of proteins in the cell that delivers mitogenic and differentiation signals from the membrane of a cell to the *DNA* in the nucleus, and is regulated by the protein *RKIP*. The starting point of the pathway is when a signalling molecule binds to a receptor on the cell surface and is spatially organised so that, when a signal arrives at the membrane, it can be transmitted to the nucleus via a cascade of biological reactions that involves protein kinases. A kinase is an enzyme that catalyses the transfer of a phosphate group from a donor molecule to an acceptor. The main $MAPK/ERK$ kinase kinase (*MEKK*) component is the kinase component *Raf-1* that phosphorylates the serine residue on the *MAPK/ERK* kinase *MEK*. We denote *Raf*-1* with *F*, *MEK* with *M*, *ERK* with *E*, *RKIP* with *R*, and the phosphorylation of the bonded molecule is denoted by P.

The pathway begins with the activation of the protein kinase of *Raf-1* by the G protein *Ras* that has been activated near a receptor on the cell's membrane. *Ras* activates a kinase *Raf-1* to become *Raf*-1*, which is generally known as a mitogen-activated protein kinase kinase kinase (*MAP-KKK*) and can be inhibited by *RKIP*. Subsequently, as we may see in Figure 3.16, *Raf*-1* (F) may bind with *MEK* (F–M) by facilitating the next step in the cascade ($MAPKK$), which is the phosphorylation of the *MEK* ($F-M-P$) protein and the release of *Raf*-1* (M–P). The phosphorylated *MEK* (M–P) activates a mitogen-activated protein kinase, *ERK* (E–M–P), which in turn becomes phosphorylated and releases *MEK* (P–E). Finally, the phosphorylation of *MAPK* allows the phosphorylated *ERK* ($P-E$) to function as an enzyme and translocate in order to signal the nucleus. Now the regulation sequence consumes the phosphorylated *ERK* ($P-E$) in order to deactivate *RKIP* (R) from regulating *Raf*-1* (F).

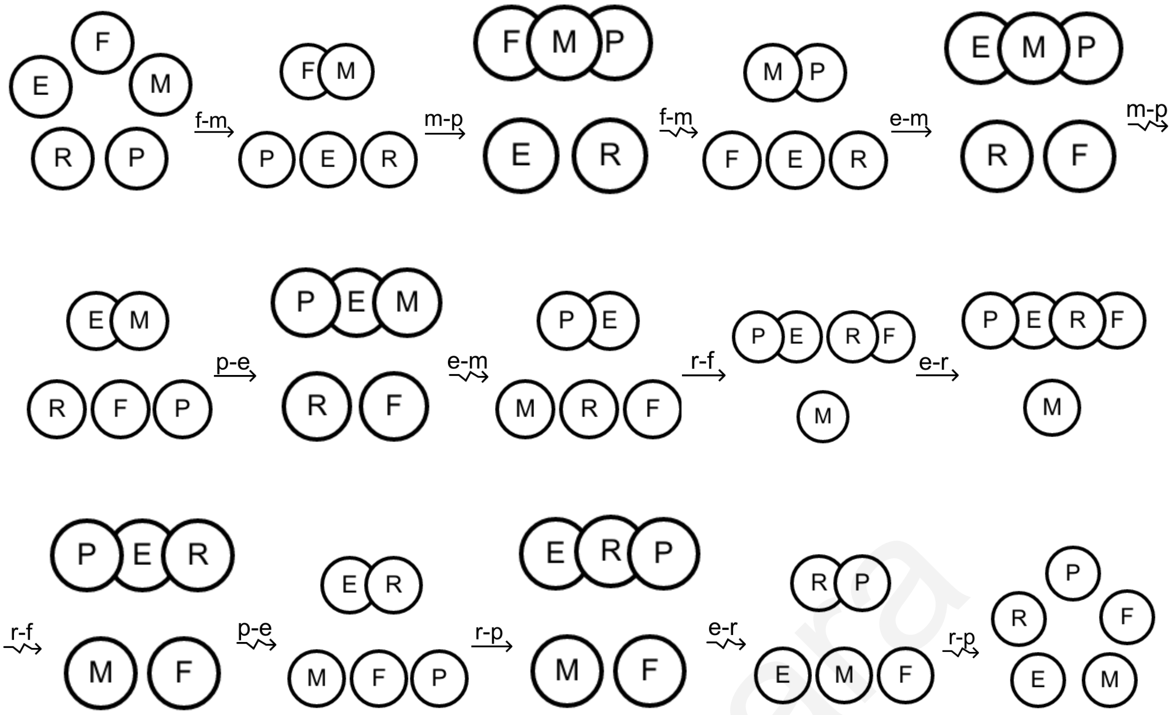


Figure 3.16: Reactions in the ERK*-pathway where* F *denotes* Raf*-1*,* M *denotes* MEK*,* E *denotes* ERK*,* R *denotes* RKIP*, and* P *denotes the phosphorylation of the bonded molecule*

Therefore, when *RKIP* binds *Raf*-1* ($R-F$), the resulting complex binds to a phosphorylated *ERK* ($P-E-R-F$). In the end, the complex breaks releasing *Raf*-1* ($P-E-R$), which can get involved in the cascade and after the phosphorylation of *RKIP* ($E-R-P$) the system releases *ERK* ($R-P$) and the phosphorylated *RKIP*.

We now describe the biochemical reactions of the ERK signalling pathway as the RPN demonstrated in Figure 3.17. On this RPN we represent molecules as tokens that can bond with each other, thus creating more complex molecules, and these composite molecules can be dissolved back to single tokens. The building blocks of the system are the base tokens representing the associated molecules.

We begin our execution from the already activated *Raf-1* kinase that has become *Raf*-1*. The molecule of *Raf*-1* is represented by base token f and resides in place F. The token availability of base m, which represents *MEK* in place M enables the firing of transition a_2 denoting that f has bonded with m and thus creating molecule f–m. The firing of transition a_2 facilitates the next step in the cascade, which is transition p_1 representing the phosphorylation of m as the binding between p and m. Since transition a_2 has enabled the execution of p_1 the transition a_2 is now reversed and therefore releases f back to place F. This reversal in necessary for the next step of the execution where the absence of f is a condition for transition c to fire. Indeed, in transition c, the phosphorylated m is now able to activate the

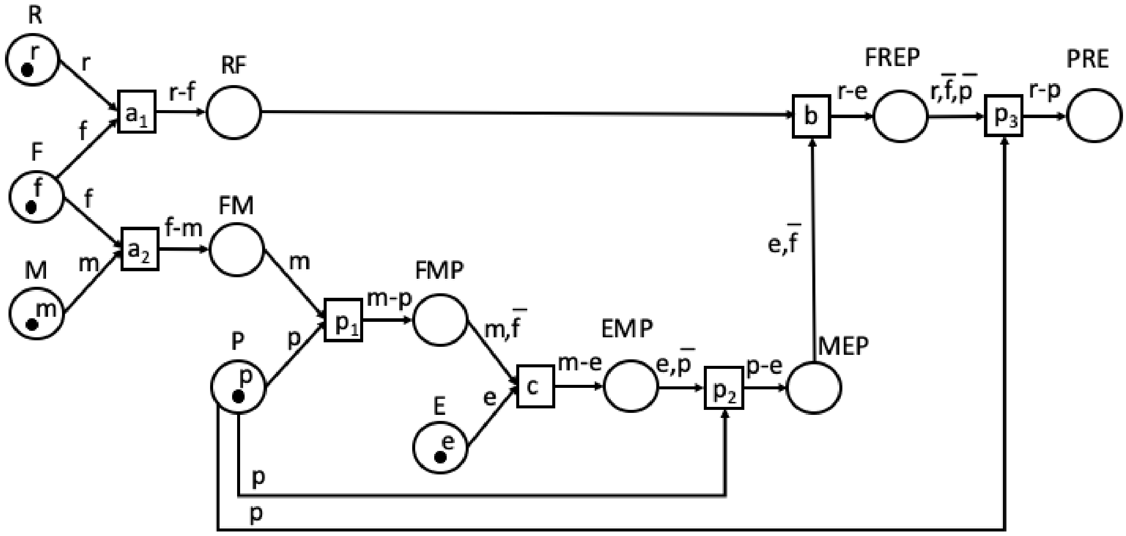


Figure 3.17: ERK-pathway example in reversing Petri nets

kinase *ERK* denoted by base e and thus creating a bond between m–e along with p that shows that m is already phosphorylated. In the next step, transition p_1 reverses in order to release p, which can then be used in the firing of transition p_2 to phosphorylate e and therefore creating the molecule p–e. After transition p_2, transition c is reversed in order to release m back to M. Finally, after the phosphorylation of $p-e$, transition a_1 executes in order to bond $f-r$ where r represents molecule *RKIP*. Base r functions as an enzyme and by enabling transition b represents the passing of the signal to the nucleus which can then consume p–e by creating a connected component between $f-r-e-p$. In the end, the complex breaks by reversing a_1 in order to release f and p_2 to release p which then in action p_3 phosphorylates r. Finally, the system reverses b to release e followed by the reversal of p_3, which releases both r and p and therefore returns the system back to its initial marking.

We show below an execution of the reversing Petri net that illustrates the process until the signal that arrived at the membrane is transmitted to the nucleus. The following states of the net (with histories omitted) represent a cascade of reactions that involve protein kinases F, M, E, P, R, with initial marking M_0 such that $M_0(R) = \{r\}$, $M_0(F) = \{f\}$, $M_0(M) = \{m\}$, $M_0(P) = \{p\}$, $M_0(E) = \{e\}$, and $M_0(p) = \emptyset$ for all remaining places. (In the following, the markings of places with no tokens are omitted.)

$$
\begin{aligned}
M_0 \xrightarrow{a_2} M_1, \quad & \text{where} \quad M_1(R) = \{r\}, M_1(P) = \{p\}, M_1(E) = \{e\}, \\
& \qquad\quad M_1(FM) = \{f-m\} \\
M_1 \xrightarrow{p_1} M_2, \quad & \text{where} \quad M_2(R) = \{r\}, M_2(E) = \{e\}, \\
& \qquad\quad M_2(FMP) = \{f-m, m-p\} \\
M_2 \overset{a_2}{\rightsquigarrow} M_3, \quad & \text{where} \quad M_3(R) = \{r\}, M_3(F) = \{f\}, M_3(E) = \{e\},
\end{aligned}
$$

$$
\begin{aligned}
& & & M_3(FMP) = \{m-p\} \\
M_3 &\xrightarrow{c} M_4, & \text{where} \quad & M_4(R) = \{r\}, M_4(F) = \{f\}, \\
& & & M_4(EMP) = \{m-e, m-p\} \\
M_4 &\overset{p_1}{\rightsquigarrow} M_5, & \text{where} \quad & M_5(R) = \{r\}, M_5(F) = \{f\}, M_5(P) = \{p\}, \\
& & & M_5(EMP) = \{m-e\} \\
M_5 &\xrightarrow{p_2} M_6, & \text{where} \quad & M_6(R) = \{r\}, M_6(F) = \{f\}, \\
& & & M_6(MEP) = \{m-e, e-p\} \\
M_6 &\overset{c}{\rightsquigarrow} M_7, & \text{where} \quad & M_7(R) = \{r\}, M_7(F) = \{f\}, M_7(M) = \{m\}, \\
& & & M_7(MEP) = \{e-p\} \\
M_7 &\xrightarrow{a_1} M_8, & \text{where} \quad & M_8(M) = \{m\}, M_8(RF) = \{r-f\}, \\
& & & M_8(MEP) = \{e-p\} \\
M_8 &\xrightarrow{b} M_9, & \text{where} \quad & M_9(M) = \{m\}, M_9(FREP) = \{r-f, r-e, e-p\} \\
M_9 &\overset{a_1}{\rightsquigarrow} M_{10}, & \text{where} \quad & M_{10}(M) = \{m\}, M_{10}(F) = \{f\}, \\
& & & M_{10}(FREP) = \{r-e, e-p\} \\
M_{10} &\overset{p_2}{\rightsquigarrow} M_{11}, & \text{where} \quad & M_{11}(M) = \{m\}, M_{11}(F) = \{f\}, M_{11}(P) = \{p\}, \\
& & & M_{11}(FREP) = \{r-e\} \\
M_{11} &\xrightarrow{p_3} M_{12}, & \text{where} \quad & M_{12}(M) = \{m\}, M_{12}(F) = \{f\}, \\
& & & M_{12}(PRE) = \{r-e, p-r\} \\
M_{12} &\overset{b}{\rightsquigarrow} M_{13}, & \text{where} \quad & M_{13}(M) = \{m\}, M_{13}(F) = \{f\} M_{13}(E)\{e\}, \\
& & & M_{13}(PRE) = \{p-r\} \\
M_{13} &\overset{p3}{\rightsquigarrow} M_{14}, & \text{where} \quad & M_{14}(R) = \{r\}, M_{14}(F) = \{f\}, M_{14}(M) = \{m\}, \\
& & & M_{14}(P) = \{p\}, M_{14}(E) = \{e\}
\end{aligned}
$$

3.3.2 Transactions with Compensations

Transaction processing manages sequences of operations, also called transactions, that can either succeed or fail as a complete unit. Specifically, a long-running transaction aggregates smaller atomic transactions, and typically use a coordinator to complete or abort the transaction. An atomic transaction is an indivisible and irreducible series of operations such that either all occur, or nothing occurs [106].

Long-running transactions consist of a sequence of steps that avoid locks on non-local resources and use compensation to handle failures. Each of these steps may either succeed, in which case the flow of control moves on to the next atomic step in the sequence, or it may fail, in which case a compensating transaction is often used to undo failed transactions

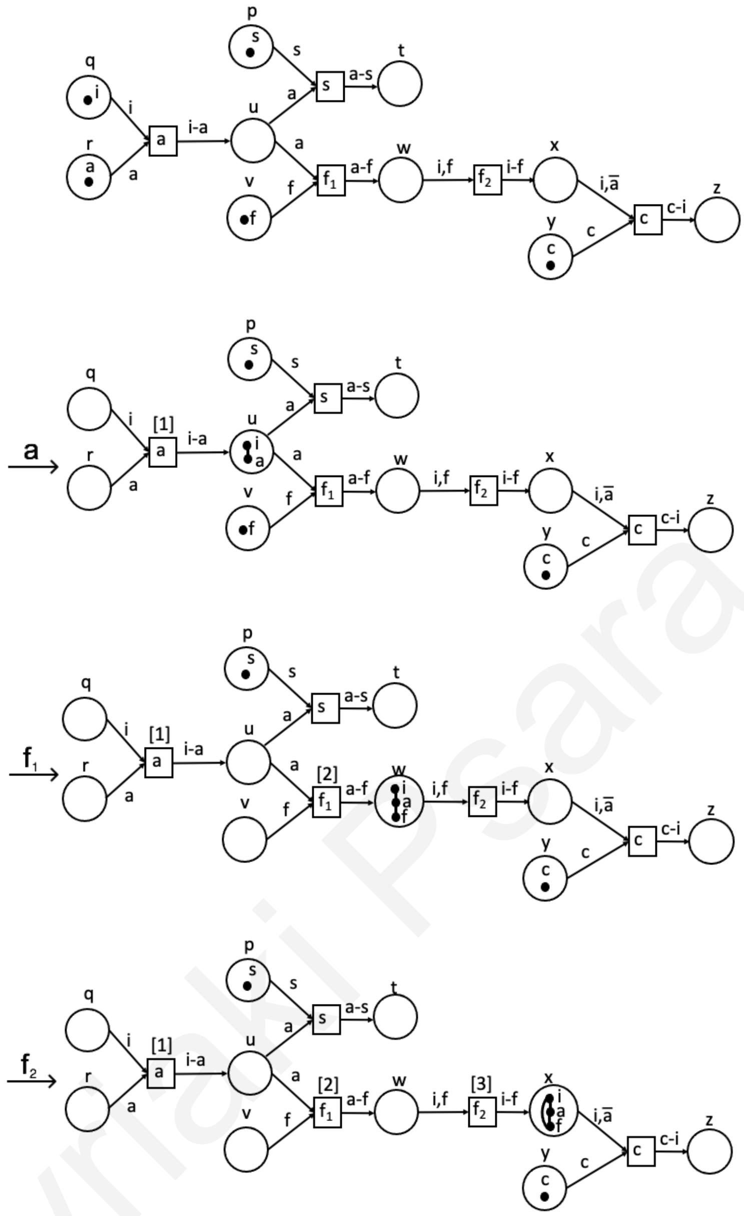


Figure 3.18: Transaction processing - forward execution

and restore the system to a previous state. In contrast to rollback in atomic transactions, compensation restores the original state, or an equivalent, and it is business-specific. If all steps of the transaction execute successfully then the transaction is considered as successful and it is committed.

The definition of causal reversibility has spawned various reversible extensions of concurrent languages that are used for validating formal connections between causal-consistent reversibility and reliability as well as studying its consequences. It enables a new strategy for debugging concurrent systems, where the different speed of processes that are replaying an execution looking for a bug may cause different behaviours. There have been proposed reversible process calculi used to build constructs for reliability, and in particular communicat-

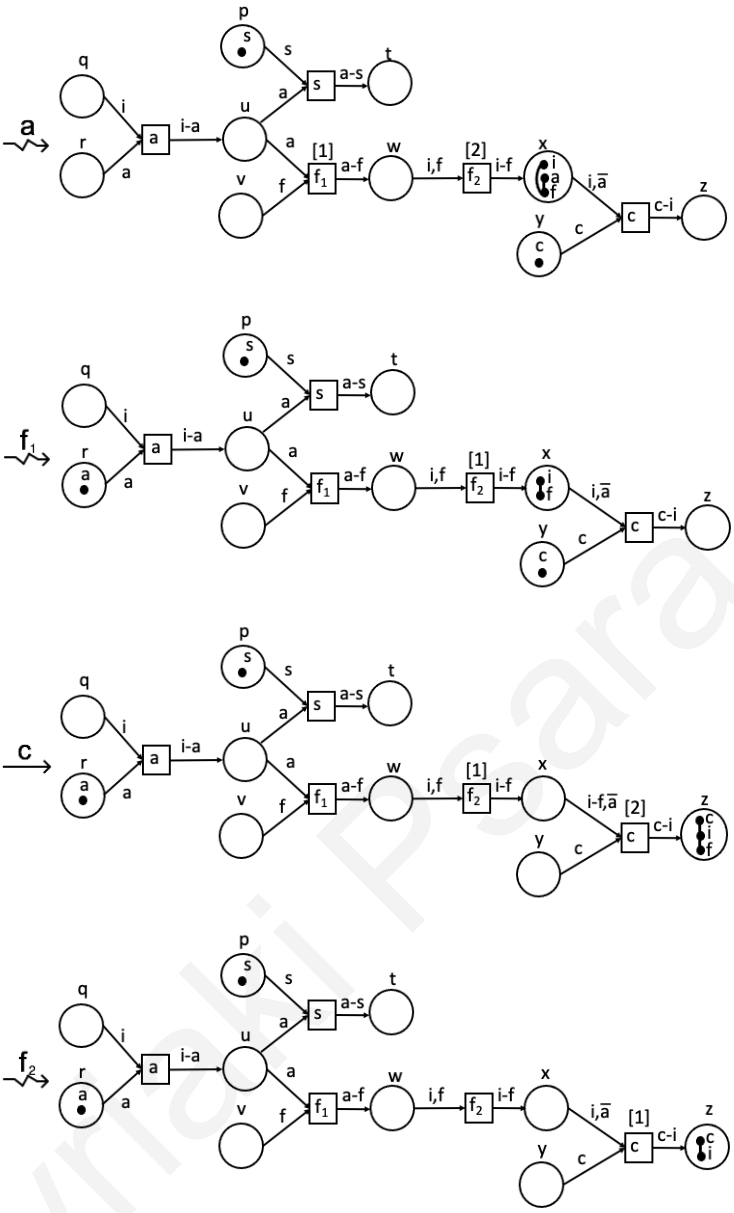


Figure 3.19: Transaction processing: out-of-causal-order execution

ing transactions with compensations [36] where interacting transactions with compensations have been mapped into a reversible calculus with alternatives in [76]. [36] uses transactions with compensations, which are computations that either succeed, or their effects are reversed and then a compensation is executed by a dedicated ad-hoc piece of code. Behavioural equivalences for communicating transactions with compensation have been studied in [37, 68].

In Figures 3.18 and 3.19 we consider a model of such a transaction. Specifically in Figure 3.18 we demonstrate the forward execution of a failed transaction and in 3.19 we demonstrate the compensation part of the transaction execution which follows the strategy of out-of-causal-order reversing. Due to the size of the net we restrict our attention to a transaction with only one step. The computation starts with the initialisation step *a* which never fails. The intuition is as follows: for the execution of the transaction to commence it is necessary for token i to be available. This token is bonded with token a in which case transition a can be executed with the effect of creating the bond $i-a$ in place u. At this stage there are two possible continuations. The first possibility is that the bond $i-a$ will participate in transition s which models the successful completion of step a as well as the transaction, yielding the bond $i-a-s$. The second possibility that a transaction can fail at any stage after step a. In this case, token f comes in place and the failure is implemented via transitions f_1 and f_2 as follows: To begin with in action f_1, token f is bonded with token a repressing that the transaction has failed, whereas in action f_2 token i is bonded with token f indicating that the initialisation has failed thus triggering reversal. At this stage the compensation comes in place (token c) where the intention is that step a should be undone by undoing transition a. Note that this will have to be done according to our out-of-causal-order definition since transition a was followed by f_1 and f_2 which have not been undone. Only once this is accomplished, will the precondition of transition c, namely $\overline{a}$, be enabled. In this case, transition c can be executed leading to the creation of bond $i-c$ in place z.

3.4 Concluding Remarks

This chapter proposes a reversible approach to Petri nets [111,112] that allows the modelling of reversibility as realised by backtracking, causal-order reversing and out-of-causal-order reversing. To the best of our knowledge, this is the first such proposal in the context of Petri nets. For instance, the works of [15, 16] introduce reversed transitions in a Petri net and study various decidability problems in this setting. This approach, however, does not precisely capture reversible behaviour due to the property of backward conflict in PNs. On the contrary, [96] is concerned with causal order reversal where a subclass of Petri nets can be restructured by adding effect-reversals that do not affect the computational behaviour of the model. Moreover, the works of [94,95] propose a causal semantics for P/T nets by identifying the causalities and conflicts of a P/T net through unfolding it into an equivalent occurrence net and subsequently introducing appropriate reverse transitions to create a coloured Petri net that captures a causal-consistent reversible semantics. The colours in this net capture causal histories.

On the other hand, our proposal consists of a reversible approach to Petri nets, where the formalism supports the reversible semantics without explicitly introducing reverse transitions. This is achieved with the use of bonds of tokens, which can be thought of as colours and, combined with the history function of the semantics, capture the memory of an execution as needed to implement reversibility. Furthermore, the approach allows to implement both causal-order and out-of-causal-order reversibility.

As in [94,95], our goal has been to allow a causally-consistent semantics reflecting causal dependencies as a partial order, and allowing an event to be reversed only if all its consequences have already been undone. To achieve this goal we have defined a causal dependence relation that resorts to the *marking* of a net. As illustrated via examples (e.g. see Figures 3.7 and 3.8), this is central in capturing causal dependencies and the intended causal-consistent semantics. Our dependence relation is strong enough to capture partial order causality even in the absence of bonds. Specifically, the introduction of bonds can be handled by representing tokens as colours similarly to coloured Petri nets. Therefore, a simplification of the model can be proposed without including bonds that will still preserve the causal-consistent semantics of RPNs. The resulting framework would be closely related to coloured Petri nets thus possibly inheriting various theoretical results proven for the traditional model.

In a related line of work, we are also investigating the expressiveness relationship between RPNs and Coloured Petri Nets. Specifically, in [13] a subclass of RPNs with trans-acyclic structures has been encoded in coloured PNs. Currently, we are extending this work with ultimate objective to provide and prove the correctness of the translation between the two formalisms and analyse the associated trade-offs in terms of Petri net size.

Another possible direction for future work would be the extension of RPNs with directed bonds. This would enable the framework to model double bonds as defined in biochemistry, where a covalent bond between two atoms involves four bonding electrons as opposed to two in a single bond.

Chapter 4

Token Multiplicity in Reversing Petri Nets

In the previous chapter we have introduced a form of reversing Petri nets that assumes tokens to be unique and does not allow transitions to break bonds. In this chapter we focus on relaxing these restrictions, to develop reversible semantics in the presence of bond destruction, and to allow multiple tokens of the same base/type to occur in a model.

By allowing the destruction of bonds in forward transitions we alter our perception of what the effect of a transition is, as additionally to the addition of new bonds the destruction of existing bonds should also be considered as the effect of a transition. We show the associated semantics of all three forms of reversible computing in this setting.

We also enhance the modelling flexibility of reversing Petri nets by introducing token multiplicity. We study how the partial order of causality in the original RPN model is affected and explain what modifications to the causal semantics are needed in order to provide a satisfactory treatment of reversibility. The causal semantics of such extended nets can often no longer be described solely in terms of a partial order, thus we introduce a new form of reversibility that follows disjunctive causality and we justify our proposal by modelling an example of a biochemical reaction.

4.1 Destruction of Bonds in Reversing Petri Nets.

As our original RPN model is inspired by biochemistry and since concerted chemical reactions involve the breaking and making of bonds in a single step, a consideration of transitions that allow the simultaneous bonding and destructing of molecules is essential to this understanding. We motivate our design decisions by giving an example of a concerted reversible chemical reaction that allows the simultaneous creation and destruction of bonds.

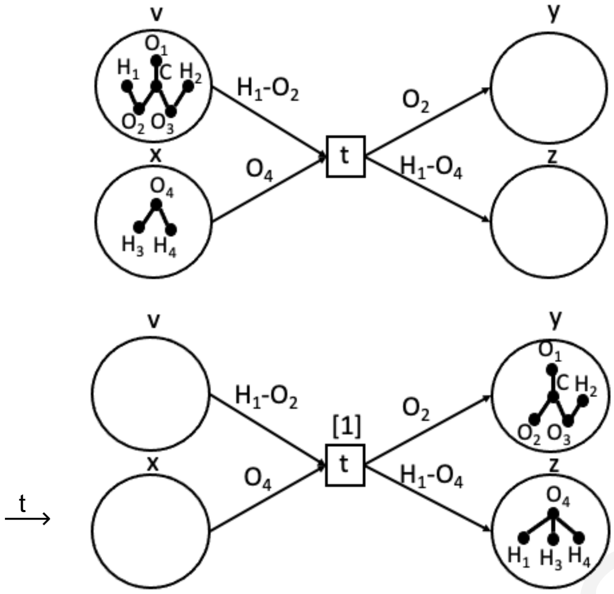


Figure 4.1: Reversible chemical reaction

A reversible chemical reaction is a reaction where the reactants and products react together to give the reactants back. Weak acids, such as carbonic acid, and bases, such as water, undertake reversible reactions. Carbonic acid is a chemical compound with the chemical formula H_2CO_3. It is also a name sometimes given to solutions of carbon dioxide in water also known as carbonated water, because such solutions contain small amounts of H_2CO_3. In the example presented in Figure 4.1, carbonic acid H_2CO_3 and water H_2O react to form bicarbonate HCO_3 and hydronium H_3O.

We may now proceed to define *Single Reversing Petri Nets (SRPNs)* that extend the model of Chapter 3 by allowing transitions to break bonds during forward execution. As with the original RPN model we use the tuple (A, P, B, T, F) to define SRPN structures that consist of bases, places, bonds, transitions and labelled directed arcs. In this section we only consider the extension of destructing bonds thus we still consider each token to have a unique name. Allowing connected tokens to fork in different outgoing places requires close attention to the existing connected components so that we do not clone tokens. To avoid duplicating tokens we still require directed arcs to ensure token preservation as defined by the well-formedness of RPNs. However, compared to the original model, we have eliminated one condition as we no longer require existing bonds to be preserved on the outgoing arcs of a transition.

Definition 20. A SRPN (A, P, B, T, F) is *well-formed* if it satisfies the following conditions for all $t \in T$:

1. $A \cap \mathsf{pre}(t) = A \cap \mathsf{post}(t)$,

2. $F(t, x) \cap F(t, y) = \emptyset$ for all $x, y \in P$, $x \neq y$.

According to the above we have that: (1) transitions do not erase tokens, and (2) tokens and bonds cannot be cloned into more than one outgoing place.

We may now define the various types of execution for single reversing Petri nets where forward transitions are able to break bonds. As with the original RPNs in this extension we restrict our attention to well-formed SRPNs (A, P, B, T, F) with initial marking M_0 such that for all $a \in A$, $|\{x \mid a \in M_0(x)\}| = 1$.

4.1.1 Forward Execution

In this section we consider the standard, forward execution of SRPNs. As before, for a transition to fire in the forward direction we require the corresponding token and bond availability. As we now allow connected tokens to break their bonds during forward execution we require the bonds that connect them to be a requirement for the transition to fire. Formally:

Definition 21. Consider a SRPN (A, P, B, T, F), a transition $t \in T$, and a state $\langle M, H \rangle$. We say that t is *forward-enabled* in $\langle M, H \rangle$ if the following hold:

1. if $a \in F(x, t)$, for some $x \in \circ t$, then $a \in M(x)$, and if $\beta \in F(x, t)$, for some $x \in \circ t$, then $\beta \in M(x)$,

2. for all $a, b \in F(x, t)$, $x \in \circ t$ where $(a, b) \in M(x)$ then $(a, b) \in F(x, t)$, and

3. if $a \in F(t, y_1)$, $b \in F(t, y_2)$, $y_1, y_2 \in t\circ$, $y_1 \neq y_2$ then $a \notin \mathsf{con}(b, (M(x) - \mathsf{pre}(t)) \cup \mathsf{post}(t)), x \in \circ t$.

Thus, t is enabled in state $\langle M, H \rangle$ if (1) all tokens and bonds required for the transition to take place are available in the incoming places of t, (2) if a pre-existing bond appears in an incoming place of the transition and its tokens are required for the transition to fire then the bond should also appear as a requirement on the incoming arcs (we do not recreate bonds), and (3) if two tokens are transferred by a transition to different outgoing places then these tokens should not remain connected when removing the incoming arcs and adding the outgoing arcs. Transferring tokens that are connected either directly or indirectly to different places without breaking their bonds it would result in token duplication. As such, we require for the tokens that fork to different places not to be connected when executing the transitions, thus any bonds that exist between them in the incoming places of the transition will be destructed by the specifications on the arcs of the transition.

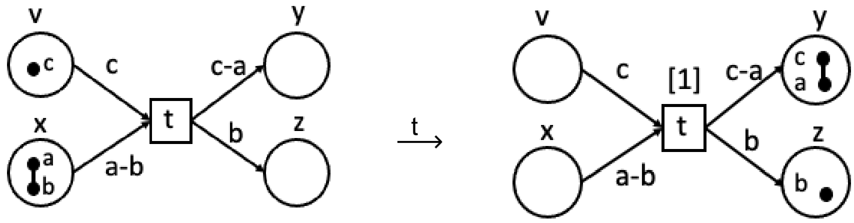


Figure 4.2: Forward execution

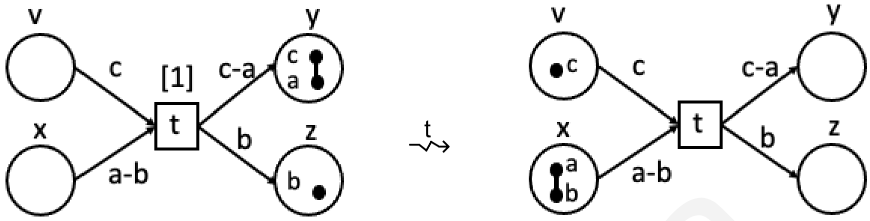


Figure 4.3: Backtracking execution

During forward execution the new bonds created by a transition are exactly those that occur in the outgoing edges of a transition but not in the incoming edges and the bonds that are broken are those that occur in the incoming edges of a transition but not in the outgoing edges. Thus, firing a transition in the forward direction recreates the marking by removing the bonds that occur on the incoming arcs but adding the bonds that occur in the outgoing arcs. Specifically, firing a transition in the forward direction is defined as follows:

Definition 22. Given a SRPN (A,P,B,T,F), a state $\langle M,H\rangle$, and a transition t enabled in $\langle M,H\rangle$, we write $\langle M,H\rangle \xrightarrow{t} \langle M',H'\rangle$ where H' is updated as in Definition 5 and:

$$M'(x) = \begin{cases} M(x) - \bigcup_{a\in F(x,t)} \mathsf{con}(a, M(x)) & \text{if } x \in \circ t \\ M(x) \cup \bigcup_{a\in F(t,x)\cap F(y,t)} \mathsf{con}(a, (M(y) - F(y,t)) \cup F(t,x)) & \text{if } x \in t\circ \\ M(x), & \text{otherwise} \end{cases}$$

Thus, when a transition t is executed in the forward direction, all tokens and bonds occurring in its outgoing arcs are relocated from the input places to the output places along with their connected components. The SRPN in Figure 4.2 represents the destruction of bond $a{-}b$ and the creation of bond $c{-}a$ where the new bond $c{-}a$ is relocated in place y and token b is relocated in place z. The history is updated as usual.

4.1.2 Backtracking

Let us now proceed to backtracking. As with the original model the destruction of bonds does not affect *bt*-enabledness thus we define a transition to be *bt*-enabled if it was the last executed transition:

Definition 23. Consider a SRPN (A, P, B, T, F), a state $\langle M, H\rangle$, and a transition $t \in T$. We say that t is *bt-enabled* in $\langle M, H\rangle$ as in Definition 6.

The effect of backtracking a transition in a single reversing Petri net with bond destruction is shown in Figure 4.3 which reverses the execution of the reversing Petri net by recreating the bond $a{-}b$ and returning it to its initial place x as well as breaking the bond $c{-}a$ and returning c to place v. Thus backtracking execution is defined as follows:

Definition 24. Given a SRPN (A, P, B, T, F), a state $\langle M, H\rangle$, and a transition t with history $k = max(H(t))$ that is bt-enabled in $\langle M, H\rangle$, we write $\langle M, H\rangle \overset{t}{\rightsquigarrow}_b \langle M', H'\rangle$ where H' is updated as in Definition 7 and:

$$M'(x) = \begin{cases} M(x) \cup \bigcup_{y \in t\circ, a \in F(x,t)\cap F(t,y)} \mathsf{con}(a, (M(y) - F(t,y)) \cup F(x,t)), & \text{if } x \in \circ t \\ M(x) - \bigcup_{a \in F(t,x)} \mathsf{con}(a, M(x)), & \text{if } x \in t\circ \\ M(x) & \text{otherwise} \end{cases}$$

4.1.3 Causal Reversal

We introduce destruction of bonds in causal reversibility and show that no modifications are needed to the notions of causal dependence and causal enabledness of the original reversing Petri nets. As expected, we consider a transition t to be enabled for causal-order reversal only if all transitions that are causally dependent on it have either been reversed or not executed. We may now define that a transition is enabled for causal-order reversal as follows:

Definition 25. Consider a SRPN (A, P, B, T, F), a state $\langle M, H, \prec\rangle$, and a transition $t \in T$. Then t is c-enabled in $\langle M, H, \prec\rangle$ as in Definition 11.

Reversing a transition in a causally-respecting order is implemented similarly to backtracking, i.e. the tokens are moved from the outgoing places to the incoming places of the transition, all bonds created by the transition are broken and all bonds destructed by the transition are reconstructed. In addition, the history function is updated in the same manner as in backtracking, where we remove the key of the reversed transition, and the causal dependence relation removes all references to the reversed transition occurrence. The example in Figure 4.4 represents two concurrent transitions that have been executed and reversed in different orders.

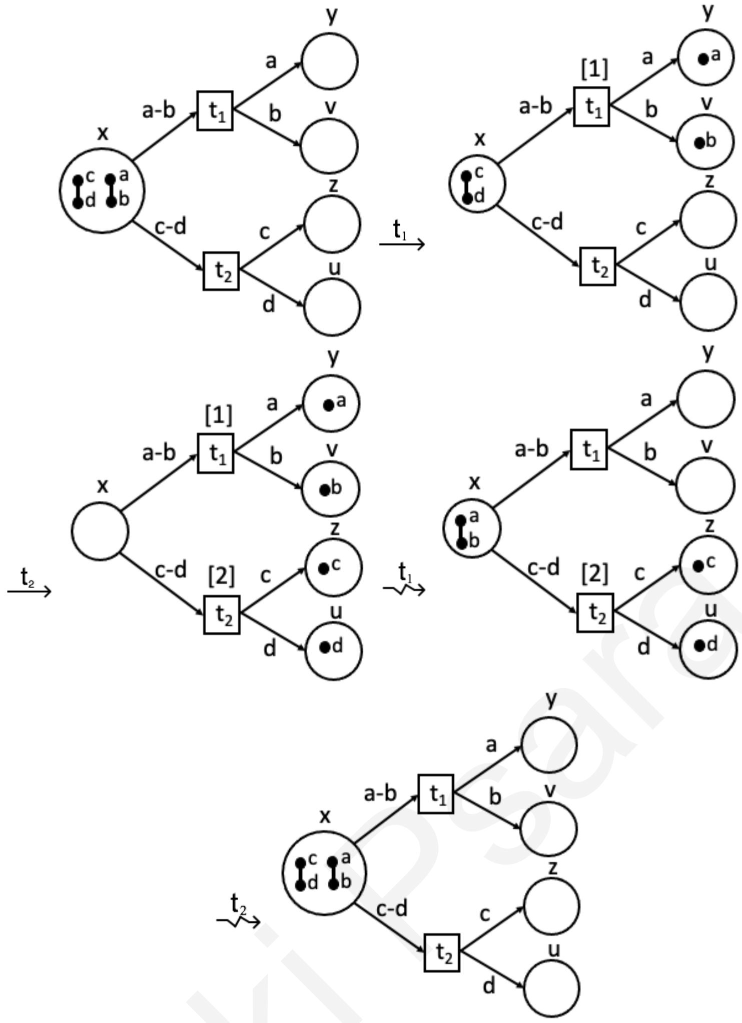


Figure 4.4: Causal-order execution

Definition 26. Given a SRPN (A, P, B, T, F), a state $\langle M, H, \prec \rangle$, and a transition t c-enabled in $\langle M, H \rangle$, we write $\langle M, H, \prec \rangle \overset{t}{\rightsquigarrow}_c \langle M', H', \prec' \rangle$ for M' and H' as in Definition 24, and $\prec'$ such that

$$\prec' \;=\; \{((t_1, k_1), (t_2, k_2)) \in \prec \mid k_2 \neq k\}$$

Theorem 2. Consider executions $\langle M, H \rangle \overset{\sigma_1}{\longmapsto}_c \langle M_1, H_1 \rangle$ and $\langle M, H \rangle \overset{\sigma_2}{\longmapsto}_c \langle M_2, H_2 \rangle$. Then, $\sigma_1 \asymp \sigma_2$ if and only if $\langle M_1, H_1 \rangle \asymp \langle M_2, H_2 \rangle$.

Proof. The proof of the theorem follows as a corollary of Theorem 3, which will be presented in Section 4.2 since SRPNs are a special instance of Multi Reversing Petri Nets. □

4.1.4 Out-of-Causal Order

We may now proceed to out-of-causal order reversibility which in the original model of RPNs allows any transition to reverse as long as it is an executed transition. When allowing the destruction of bonds during forward execution, this form of reversing presents a peculiarity since we allow bonds to be broken during forward execution then there is the possibility to execute two forward transitions that have the opposite effect. Consider the example in Figure 4.5 where the first transition creates a bond and the second transition destructs the same bond. When reversing the first transition in out-of-causal order, then we try to reverse a bond that has already been broken by the second transition. In this way we negate a bond that was necessary for the effect of the second transition and thus create inconsistencies such as the token duplication presented in the final net of Figure 4.5. In this case reversing a bond that was required for the already executed following transition leads to inconsistencies in regards to token preservation, an important feature of reversing Petri nets and reversible computation in general. As such, we assume transitions like these to be irreversible and we only allow out-of-causal reversal in transitions that do not generate these paradoxes.

We begin by noting that in out-of-causal-order reversibility any executed transition can be reversed at any time as long as its effect has not been reversed by a forward transition.

Definition 27. Consider a SRPN (A, P, B, T, F), a state $\langle M, H\rangle$, and a transition $t \in T$. We say that transition t is *o-enabled* in $\langle M, H\rangle$, if (1) $H(t) \neq \emptyset$ and:

a for all $(a, b) \in \mathsf{pre}(t)$, $(a, b) \notin \mathsf{post}(t)$ then $\nexists t', k' \in H(t')$ where $k' > k$ such that $(a, b) \in \mathsf{post}(t')$, $(a, b) \notin \mathsf{pre}(t')$

b for all $(a, b) \in \mathsf{post}(t)$, $(a, b) \notin \mathsf{pre}(t)$ then $\nexists t', k' \in H(t')$ where $k' > k$ such that $(a, b) \in \mathsf{pre}(t)$, $(a, b) \notin \mathsf{post}(t)$

The definition states that for t to be $o-$enabled then (1) the transition should be executed, (2)(a) if the transition breaks a bond during forward execution then it should not be followed by an executed transition that has destructed the same bond, and (2)(b) if the transition creates a bond during forward execution then it should not be followed by a transition that has destructed the same bond. Requirements (2)(a) and (2)(b) are used to avoid token duplication as in the case of 4.5. As in the original model we define the last transition that a connected component has participated and the last place where it should be relocated.

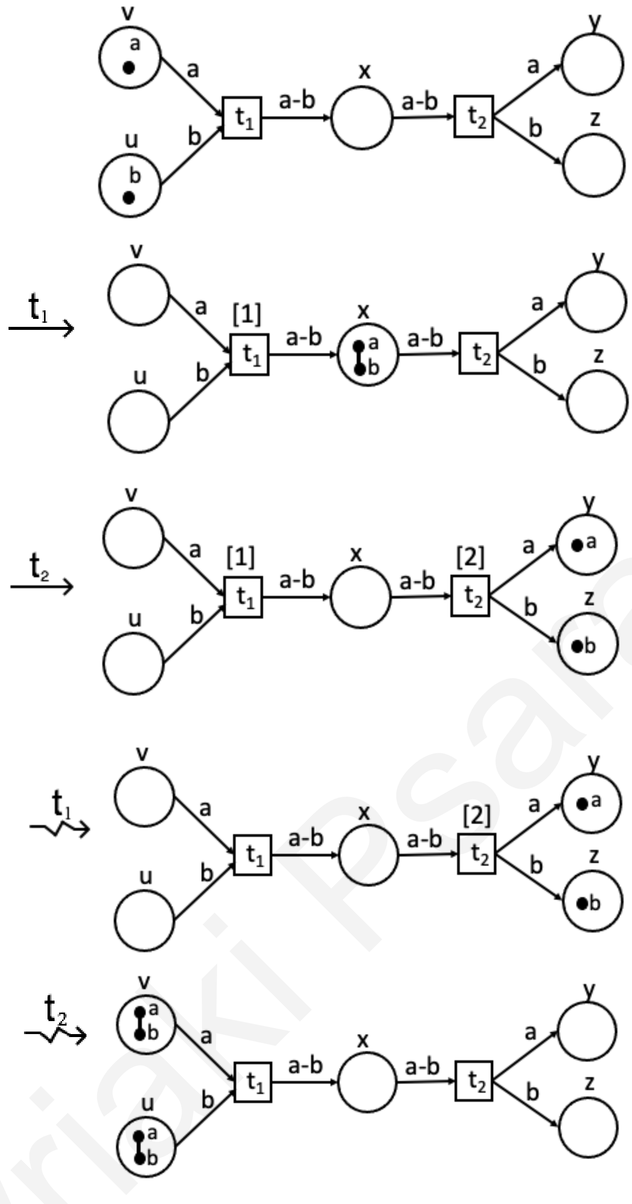


Figure 4.5: Out-of-causal order paradox

Definition 28. Given a SRPN (A, P, B, T, F), an initial marking M_0, a history H, and a set of bases and bonds $C \subseteq A \cup B$ we write:

$$\mathsf{last}_T(C, H) = \begin{cases} t, & \text{if } \exists t,\ \mathsf{post}(t) \cap C \neq \emptyset,\ H(t) \neq \emptyset, \text{ and} \\ & \nexists t',\ \mathsf{post}(t') \cap C \neq \emptyset,\ H(t') \neq \emptyset, \\ & max(H(t')) \geq max(H(t)) \\ \bot, & \text{otherwise} \end{cases}$$

$$\mathsf{last}_P(C, H) = \begin{cases} x, & \text{if } t = \mathsf{last}_T(C, H), \{x\} = \{y \in t\circ \mid F(t, y) \cap C \neq \emptyset\} \\ & \text{or, if } \bot = \mathsf{last}_T(C, H), C \subseteq M_0(x) \\ \bot, & \text{otherwise} \end{cases}$$

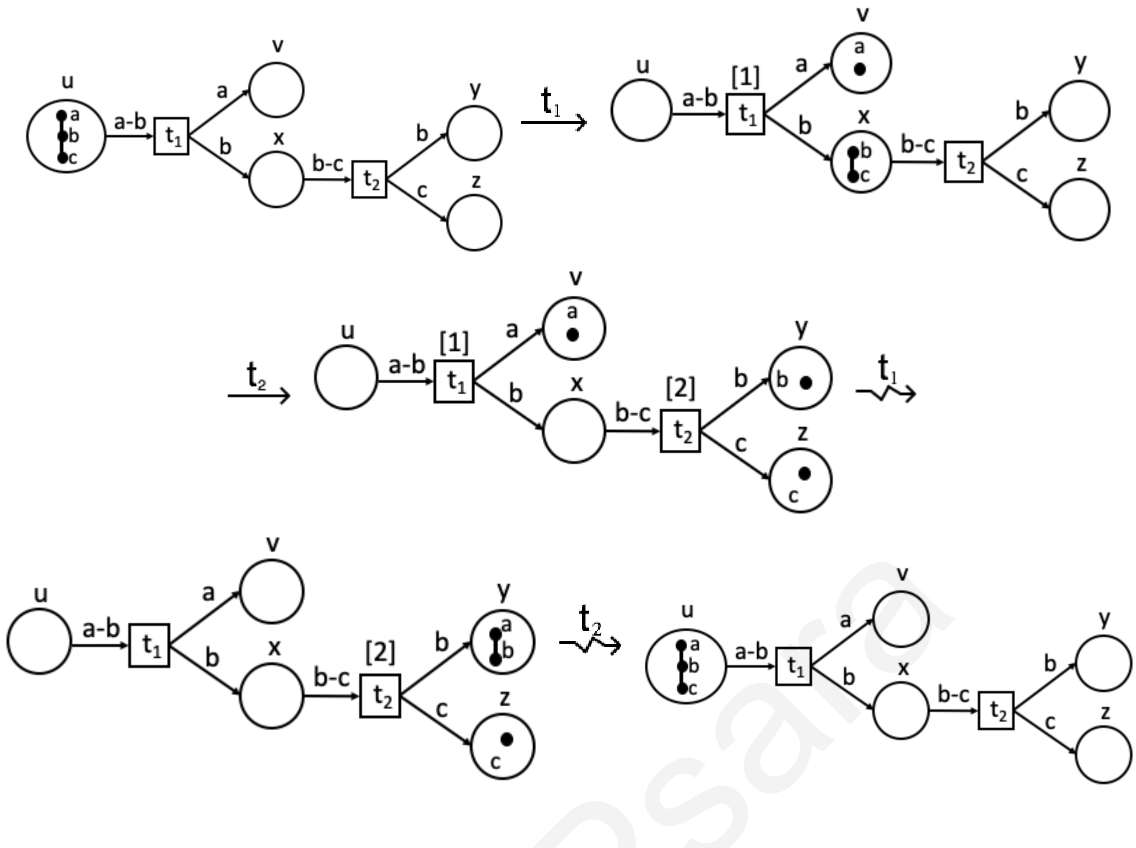


Figure 4.6: Out-of-causal order execution

Note that similarly to backtracking and causal order we recreate broken bonds and re-break created bonds by removing the bonds in the outgoing arcs and adding the bonds in the incoming arcs. Transition reversal in an out-of-causal order can thus be defined as follows:

Definition 29. Given a SRPN (A, P, B, T, F), an initial marking M_0, a state $\langle M, H\rangle$ and a transition t with history k that is o-enabled in $\langle M, H\rangle$, we write $\langle M, H\rangle \overset{t}{\rightsquigarrow}_o \langle M', H'\rangle$ where H' is defined as in Definition 26 and we have:

$$M'(x) \;=\; \Big(M(x) \cup \bigcup_{a \in M(y) \cap \mathsf{post}(t), \mathsf{last}_P(C_{a,y}, H') = x} C_{a,y}\Big) - \Big(\mathsf{eff}(t) \cup \bigcup_{a \in M(x) \cap \mathsf{post}(t), \mathsf{last}_P(C_{a,x}, H') \neq x} C_{a,x}\Big)$$

where we use the shorthand $C_{b,z} = \mathsf{con}(b, \{\mathsf{con}(c, M(z)) | c \in A, z \in P\} - \mathsf{post}(t)) \cup \mathsf{pre}(t))$ for $b \in A, z \in P$.

Thus, when a transition t is reversed in an out-of-causal-order fashion all bonds that were created by the transition are undone and all bonds destructed are recreated. If the destruction of a bond divides a component into smaller connected sub-components then each of these sub-components should be relocated (if needed) back to the place where the sub-complex would have existed if transition t never took place, i.e., exactly after the last transition that involves tokens from the sub-complex. Otherwise if the recreation of a bond creates a larger component then this component should be moved (if needed) to the place where the complex would have existed if transition t never took place, i.e. exactly after the last transition that involves tokens from the bigger complex. The example in Figure 4.6 represents the ou-of-causal reversal of two bond breaking transitions t_1 and t_2. By initially reversing t_1 the bond $a-b$ is reconstructed and the history of t_1 is eliminated. Since reversing a transition is equivalent to skipping the transition in the net, then the bond $a-b$ is transferred to place y as if transition t_1 has never been executed.

4.2 Token Multiplicity

We now proceed to explore token multiplicity in reversing Petri nets. Allowing multiple tokens of the same type to occur in a model entails that tokens of the same type are allowed to execute the same transition. As a transition can be fired by different sets of tokens this introduces possible nondeterminism when going backwards. This nondeterminism is also known as backward conflicts since multiple different tokens are allowed to reverse the same transition resulting in different states.

In order to define reversible semantics for RPNs in the presence of backward conflict we have identified two different approaches. The first approach is inspired by the individual token interpretation presented in [137] and the second by the collective token interpretation presented in [24, 139]. The two approaches differ on the way they handle backward conflicts, however, both of them remain abstract enough while doing justice to the truly concurrent nature of Petri nets. We observe that the individual token interpretation is accompanied by a set of desirable theoretical properties while the collective token interpretation is well suited for modelling a variety of possible applications.

According to the individual token philosophy, tokens are distinguished based on their causal path [55, 121]. The approach distinguishes tokens as individual by providing precise correspondence between the token instances and their past. Specifically, the model keeps track of where the tokens come from and therefore identifies the causal links between transitions in terms of a partial order. In this partial order, causal dependencies are explicitly defined as an unfolding of an occurrence net which is an acyclic net that does not have backward conflicts. This approach ensures backward determinism which is a crucial property of reversible systems.

Let us consider the example in Figure 4.7, which illustrates backward determinism as understood by the individual token interpretation. As already discussed in Chapter 3 the

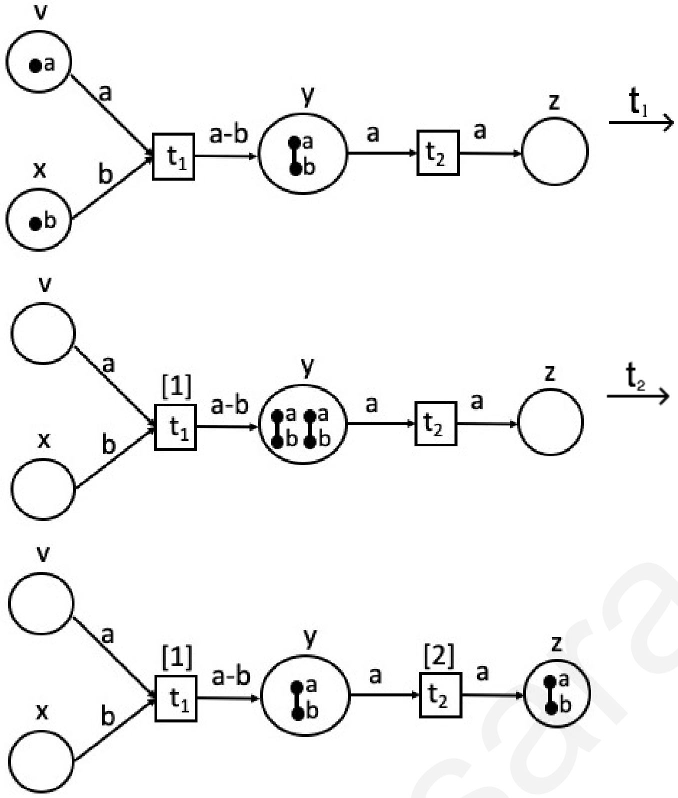


Figure 4.7: Individual token interpretation

causal relationship between transitions is defined as the manipulation of common tokens. Based on this relationship we are able to uniquely identify the transition that can be reversed by a particular token. If we consider the example in Figure 4.7 after the execution of t_1 there are two identical connected components in the middle place. If the component that was already there was used to fire t_2, then there is no causal link between the two transitions. If the component produced by t_1 was used to fire t_2, then t_2 is causally dependent on t_1. Depending on how the causal relationships are defined the behaviour of reversible actions changes, as a causal link between t_1 and t_2 means that transition t_1 is unable to reverse until t_2 has also reversed.

On the other hand, based on the collective token philosophy, when multiple tokens of the same type reside in the same place then these tokens are indistinguishable. The rationality behind this approach is that in the example of Figure 4.7 the preconditions for firing transition t_2 do not change and consequently t_2 is always independent of t_1. This means that all that is known by the model is the amount of token occurrences of a specific type and their location in the marking.

4.3 Multi Reversing Petri Nets

In this section we propose an extension of the SRPN model, multi reversing Petri nets, by allowing multiple tokens of the same type as well as the possibility for transitions to break bonds under the individual token interpretation. Thus, we allow tokens of the same type to fire the same transition when going forward, however when going backwards tokens will be able to reverse only the transitions that they have fired. Therefore, the individuality of tokens of the same type is imposed by their causal path.

We formulate four firing rules for multi reversing Petri nets under the individual token interpretation with multiple tokens, namely forward, backtracking, causal-order reversing, and out-of-causal-order reversing. We then proceed to translate multi reversing Petri nets and single reversing Petri nets into Labelled Transition Systems (LTSs) [54]. We compare the expressive power offered by multi tokens against that of single tokens, in terms of the associated Labelled Transition Systems. We conclude that reversing Petri nets with single tokens are equally expressive as reversing Petri nets with multiple tokens.

We present multi reversing Petri nets (MRPNs) which are Petri net structures with multiple tokens of the same type, which we refer to as multi-tokens that allow transitions to be reversed. Formally, a MRPN is defined as follows:

Definition 30. A *multi reversing Petri net* (MRPN) is a tuple (P, T, A, A_V, B, F) where:

1. P is a finite set of *places* and T is a finite set of *transitions*.
2. A is a finite set of *base* or *token types* ranged over by $a, b, \ldots$
3. A_V is a finite set of *token variables*. We write *type*(v) for the type of variable v and assume that *type*$(v) \in A$ for all $v \in A_V$.
4. $B \subseteq A \times A$ is a finite set of undirected *bond* types ranged over by $\beta, \gamma, \ldots$ We use the notation $a{-}b$ for a bond $(a, b) \in B$.
5. $F : (P \times T \cup T \times P) \rightarrow \mathcal{P}(A_V \cup (A_V \times A_V))$ is a set of directed labelled *arcs*.

A multi reversing Petri net is built on the basis of a set of *tokens* or *bases*. These are organized in a set of token types A, where each token type is associated with a set of token instances. A_I defined as follows:

Definition 31. Given a multi reversing Petri net (P, T, A, A_V, B, F) the set of token instances A_I is recursively defined by:

- $(a, *, i)$, $i \in \mathbb{N}$, $a \in A$, and
- (a_i, k, u) where $a_i \in A_I$, $k \in \mathbb{N}$ and $u \in \{*\} \cup \{A_V\}$.

For $a_i, a_j \in A_I$ we use the notation $a_i \overline{\in} a_j$ if (i) $a_i = a_j$ or (ii) $a_j = (a'_j, k, u)$, $a_i \overline{\in} a'_j$. Moreover, we define

$$a_j \downarrow a_i = \begin{cases} a_i & \text{if } a_j = (a_i, k, u) \\ (a'_j \downarrow a_i, k_j, u_j) & \text{if } a_j = (a'_j, k_j, u_j), a'_j \neq a_i \end{cases}$$

The set of token instances A_I corresponds to the basic entities that occur in a system. Initially tokens of type $a \in A$ are denoted by $(a, *, i)$ where $type((a, *, i)) = a$ and $i \in \mathbb{N}$ is a unique number that distinguishes tokens from each other. As computation proceeds the tokens evolve by extending their memory whenever they fire a transition in the forward direction or decreasing their memory whenever they execute a transition in reverse. Note that in a token of the form (a_i, k, u), k is the key of the last transition the token has engaged in and u the variable to which the token was assigned, with $u = *$ for tokens that participate in the transition but do not correspond to any variables. Token instances may occur as stand-alone elements but they may also merge together to form *bonds*. Bond instances are denoted by the set B_I and are formed similarly to the other variations of RPNs. For $a_i, a_j \in A_I$ we use the notation $a_i \overline{\in} a_j$ to denote that token a_i has evolved to a_j. As such, the memories of a_i are also part of the memory of a_j. For $a_i, a_j \in A_I$ we use the notation $a_j \downarrow a_i$ to denote the removal of a specific memory in token a_j. Specifically, by replacing (a_i, k, u) with a_i we remove from a_j the memory of transition occurrence (t, k) where a_i, and as a result a_j, has participated in.

As with the original RPN model, places and transitions are connected via labelled directed arcs. These labels are derived from $A_V \cup (A_V \times A_V)$. They express the requirements and the effects of the transition based solely on the type of tokens consumed. Thus, any token corresponding to the same type as the variable on the labelled arc is able to participate in the transition. More precisely, if $F(x, t) = U \cup V$, where $U \subseteq A_V$, $V \subseteq A_V \times A_V$, this implies that for the transition to fire for each $u \in U$ a distinct token instance of type $type(u)$ is required. These instances should be bonded together according to V. Similarly, if $F(t, x) = U \cup V$, where $U \subseteq A_V$, $V \subseteq A_V \times A_V$, this implies that during the forward execution of the transition for each $u \in U$ a token instance of type $type(u)$ will be transmitted to place x by the transition, in addition to the bonds specified by V, some of which will be created as an effect of the transition. We make the assumption that if $(u, v) \in V$ then $u, v \in U$.

We restrict our attention to well-formed MRPNs defined as follows:

Definition 32. A multi reversing Petri net (P, T, A, A_V, B, F) is *well-formed*, if for all $t \in T$:

1. $A_V \cap \mathsf{pre}(t) = A_V \cap \mathsf{post}(t)$, and
2. $F(t, x) \cap F(t, y) \cap A_V = \emptyset$ for all $x, y \in P$, $x \neq y$.

Thus, a multi reversing Petri net is well-formed if (1) whenever a variable exists in the incoming arcs of a transition then it also exists on the outgoing arcs, which implies that transitions do not erase tokens, and (2) tokens/bonds cannot be cloned into more than one outgoing places.

As with RPNs the association of token/bond instances to places is called a *marking* such that $M : P \rightarrow 2^{A_I \cup B_I}$, where we assume that if $(u, v) \in M(x)$ then $u, v \in M(x)$. In addition, we employ the notion of a *history*, which assigns a memory to each transition $H : T \rightarrow 2^{\mathbb{N}}$. Intuitively, a history of $H(t) = \emptyset$ for some $t \in T$ captures that the transition has not taken place, or every execution of it has been reversed, and a history of $k \in H(t)$, captures that the transition was executed as the k^{th} transition occurrence. Note that $|H(t)| > 1$ may arise due to cycles but also due to the consecutive execution of the transition by different token instances. A pair of a marking and a history, $\langle M, H\rangle$, describes a *state* of a MRPN with $\langle M_0, H_0\rangle$ the initial state, where $H_0(t) = \emptyset$ for all $t \in T$ and if $a_i \in M_0(x), x \in P$, then $a_i = (a, *, i), i \in \mathbb{N}, a \in A$. Graphically, token variables $u \in F(x, t) \cap A_V$ of type $type(v) = a$ are denoted by $u : a$ over the corresponding arc $F(x, t)$ (respectively for $F(t, x)$).

Finally, we define $\mathsf{con}(a_i, C)$, where $a_i \in A_I$ and $C \subseteq 2^{A_I \cup B_I}$, to be the tokens connected to a_i as well as the bonds creating these connections according to set C, in the usual way.

SRPNs are a special case of MRPNs as tokens in SRPNs correspond to tokens in MRPNs with the associated memories ignored. Additionaly, tokens are explicitly requested on the directed arcs of SRPN transitions where in MRPNs a variable is used to represent tokens of the same type.

4.4 Semantics Under the Individual Token Interpretation

We may now define the forward and backward execution within multi reversing Petri nets. Note that as in Section 4.1 we allow transitions to break bonds and we restrict our attention to well-formed MRPNs (P, T, A, A_V, B, F) with initial marking M_0 such that for all $a_i \in A_I$, $|\{x \mid a_i \in M_0(x)\}| = 1$.

4.4.1 Forward Execution

During the forward execution of a transition in a MRPN, a set of tokens and bonds, as specified by the incoming arcs of the transition, are selected and moved to the outgoing places of the transition, as specified by the transition's outgoing arcs, possibly forming or destructing bonds, as necessary. Due to the presence of multiple instances of the same token type, it is possible that different token instances are selected during the transition's execution.

A transition is forward-enabled in a state $\langle M, H \rangle$ of a MRPN if there exists a selection of token instances available at the incoming places of the transition matching the requirements on the transitions incoming arcs. Also the transition should not recreate bonds or clone tokens. Formally:

Definition 33. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H \rangle$, and a transition t, we say that t is *forward-enabled* in $\langle M, H \rangle$ if there exists an injective function $U_f : \mathsf{pre}(t) \cap A_V \rightarrow A_I$ such that:

1. for all $u \in F(x, t)$, $x \in \circ t$, then $U_f(u) \in M(x)$ where $type(u) = type(U_f(u))$, and for all $(u, v) \in F(x, t)$, for some $x \in \circ t$, then $(U_f(u), U_f(v)) \in M(x)$,
2. for all $u, v \in F(x, t), x \in \circ t$ and $(U_f(u), U_f(v)) \in M(x)$, then $(u, v) \in F(x, t)$, and
3. if $u \in F(t, y_1)$, $v \in F(t, y_2)$, $y_1, y_2 \in t\circ$, $y_1 \neq y_2$ then $U_f(u) \notin \mathsf{con}(U_f(v), (M(x) - {}^\bullet\mathsf{pre}(t, U_f)) \cup \mathsf{post}^\bullet(t, U_f)), x \in \circ t$.

where ${}^\bullet\mathsf{pre}(t, U) = \{U(u) | u \in F(x, t), x \in \circ t\} \cup \{(U(u), U(v)) | (u, v) \in F(x, t), x \in \circ t\}$ and $\mathsf{post}^\bullet(t, U) = \{U(u) | u \in F(t, y), y \in t\circ\} \cup \{(U(u), U(v)) | (u, v) \in F(t, y), y \in t\circ\}$.

Thus, t is enabled in state $\langle M, H \rangle$ if (1) there is a type-respecting assignment of token instances in the incoming places of the transition to the variables on the incoming edges, with the token instances originating from the appropriate input places and where tokens are connected with bonds as required by the transition's incoming edges, (2) if the selected token instances are bonded together in an incoming place of the transition then the bond should also exist on the variables labelling the incoming arcs (thus transitions do not recreate bonds), and (3) if two token instances are transferred by a transition to different outgoing places then these tokens should not remain connected when removing the selected incoming tokens and adding the selected outgoing tokens (we do not clone tokens). We use ${}^\bullet\mathsf{pre}(t, U)$ and $\mathsf{post}^\bullet(t, U)$ to help us identify the effect of the transition t on the particular selection of token instances U. We refer to U_f as a forward enabling assignment.

We now define the incoming token/bond instances as:

Definition 34. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition t and an enabling assignment U_f, we define ${}^{\bullet}U_f : P \rightarrow 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of incoming token and bond instances that are used for the firing of t:

${}^{\bullet}U_f(x) = \bigcup_{u \in F(x,t)} \mathsf{con}(U_f(u), M(x))$

We now define the outgoing token/bond instances as:

Definition 35. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition t, and an enabling assignment U_f, we define $U_f^{\bullet} : P \rightarrow 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of outgoing token/bond instances of t:

$U_f^{\bullet}(x) = \bigcup_{u \in F(t,x), U_f(u) \in M(y)} \mathsf{con}(U_f(u), (M(y) - {}^{\bullet}\mathsf{pre}(t, U_f)) \cup \mathsf{post}^{\bullet}(t, U_f))$

To execute a transition t according to an enabling assignment U_f, the selected token instances, along with their connected components, are relocated to the outgoing places of the transition as specified by the outgoing arcs, with bonds created and destructed accordingly. Furthermore, the history of the executed transition is updated in the standard way. As the same transition can be executed by different tokens of the same type we indicate transition firings by (t, k) in order to be able to identify the set of tokens that have participated in this specific transition occurrence. In Figure 4.8 we observe the change in history of transition t, as well as, the change in the name of the token instances $(a, *, 1)$, and $(b, *, 1)$ to $((a, *, 1), 1, u)$ and $((b, *, 1), 1, v)$, respectively. Thus, the memory of token instance $(a, *, 1)$ is extended to indicate that $(a, *, 1)$ has participated in transition t with history identifier 1 corresponding to variable u. Similarly, for token instance $(b, *, 1)$ and variable v. Specifically, we define:

Definition 36. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition t that is enabled in state $\langle M, H\rangle$, and an enabling assignment U_f, we write $\langle M, H\rangle \xrightarrow{(t,k)} \langle M', H'\rangle$ where $k = max(\{0\} \cup \{k' | k' \in H(t'), t' \in T\}) + 1$ and for all $x \in P$:

$$M'(x) = (M(x) - {}^{\bullet}U_f(x)) \cup (\bigcup_{a_i \in U_f^{\bullet}(x)} (a_i, k, V(a_i)) \cup \bigcup_{(a_i,b_i) \in U_f^{\bullet}(x)} ((a_i, k, V(a_i)), (b_i, k, V(b_i)))$$

where

$$V(a_i) = \begin{cases} u & \text{if } U_f(u) = a_i \\ * & \text{otherwise} \end{cases}$$

$$\text{and} \quad H'(t') = \begin{cases} H(t') \cup \{k\} & \text{if } t' = t \\ H(t'), & \text{otherwise} \end{cases}$$

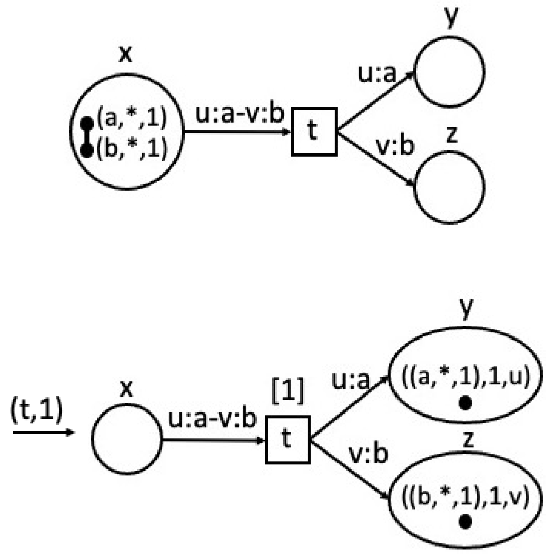


Figure 4.8: Forward execution

The following proposition states that tokens are preserved throughout forward execution such that the amount of tokens of the same type remains the same.

Proposition 16 (Token preservation)**.** Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$ and a transition $\langle M, H\rangle \xrightarrow{(t,k)} \langle M', H'\rangle$. Then for all $a_i \in A_I \cap M_0(z), z \in P$ we have $|\{a_j \mid a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j\}| = |\{a'_j \mid a'_j \in M'(y) \cap A_I, y \in P, a_i \overline{\in} a'_j\}| = 1$.

Proof. The proof follows from the definition of forward execution and relies on the well-formedness of MRPNs. Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$ such that $|\{a_j \mid a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j\}| = 1$ for some $a_i \in A_I \cap M_0(z), z \in P$, and suppose $\langle M, H\rangle \xrightarrow{(t,k)} \langle M', H'\rangle$ such that $|\{a'_j \mid a'_j \in M'(y) \cap A_I, y \in P, a_i \overline{\in} a'_j\}| = n$. Let $a_j \in A_I$. Two cases exist:

1. $a_j \in \mathsf{con}(b_j, M(x))$ for some $b_j = U_f(v), v \in F(x, t)$. According to Definition 34, we have that $a_j \in^{\bullet} U_f(x)$, which by Definition 36 implies that $a_j \notin M'(x)$. On the other hand, by Definition 32(1), $v \in \mathsf{post}(t)$. Thus, there exists $y \in t\circ$, such that $v \in F(t, y)$. Note that this y is unique by Definition 32(2). As a result, by Definition 35, $a_j \in U_f^{\bullet}(y)$ which by Definition 36 yields $a_j \overline{\in} a'_j$, $a'_j \in M'(y)$ such that $a_i \overline{\in} a'_j$.
Now suppose that $a_j \in \mathsf{con}(c_j, M(x))$ for some $c_j \neq b_j$, $u \in F(t, y'), U_f(u) = c_j$. Then, by Definition 32(2), it must be that $y = y'$. As a result, we have that $n = |\{a'_j \mid a'_j \in M'(y') \cap A_I, y' \in P, a_i \overline{\in} a'_j\}| = |\{a_j \mid a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j\}| = 1$ and the result follows.

2. $a_j \notin \mathsf{con}(b_j, M(x))$ for all $v \in F(x, t), U_f(v) = b_j$, $x \in P$. This implies that $1 = |\{a_j \mid$

$a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j\}| = |\{a_j \mid a_j \in M'(x), x \in P, a_i \overline{\in} a_j\}| = n$ and the result follows.

□

4.4.2 Backtracking

Let us now proceed to backtracking. A transition can be reversed in a certain state if it was the last executed transition and there exist token instances in its output places that match the requirements on its outgoing arcs. To capture this, we define transition occurrence as follows:

Definition 37. Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, and a transition t. We refer to (t, k) as a *transition occurrence* in $\langle M, H\rangle$ if $k \in H(t)$.

We now define the notion of backtracking enabledness as follows.

Definition 38. Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, and a transition occurrence (t, k). We say that (t, k) is *bt-enabled* in $\langle M, H\rangle$ if (1) $k \in H(t)$ with $k \geq k'$ for all $k' \in H(t')$, $t' \in T$, and (2) there exists an injective function $U_b : \mathsf{post}(t) \cap A_V \to A_I$ such that:

(a) for all $u \in F(t, x)$, $x \in t\circ$, we have $U_b(u) = (a_i, k, u)$, $U_b(u) \in M(x)$ where $type(u) = type(U_b(u))$, and

(b) for all $(u, v) \in F(t, x)$, $x \in t\circ$, we have $(U_b(u), U_b(v)) = ((a_i, k, u), (b_i, k, v))$, $(U_b(u), U_b(v)) \in M(x)$.

Thus, a transition t is bt-enabled in $\langle M, H\rangle$ if (1) it was the last transition to be executed, and (2) there exists a type-respecting assignment of token instances in the outgoing places of the transition, to the variables on the outgoing edges of the transition, and where the tokens are connected with bonds as required by the transition's outgoing edges. We refer to U_b as a backtracking enabling assignment.

Similarly to forward execution, the following definition selects the incoming connected components and the outgoing connected components. Note that the incoming connected components are selected based on the outgoing arcs of the transition and the outgoing connected components are selected based on the incoming arcs. We now define the incoming token/bond instances as:

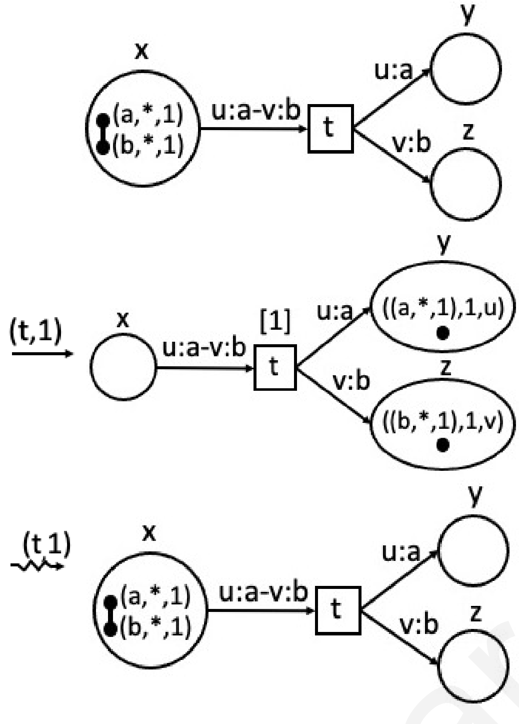


Figure 4.9: Backtracking execution

Definition 39. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition t and an enabling assignment U_b, we define ${}^{\bullet}\mathcal{U}_b : P \to 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of incoming token and bond instances that are used for the backtracking of t:

$${}^{\bullet}\mathcal{U}_b(x) = \bigcup_{u \in F(t,x)} \mathsf{con}(U_b(u), M(x))$$

We now define the outgoing token/bond instances as:

Definition 40. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition t, and an enabling assignment U_b, we define $\mathcal{U}_b^{\bullet} : P \to 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of outgoing token/bond instances of t:

$$\mathcal{U}_b^{\bullet}(x) = \bigcup_{u \in F(x,t), U_b(u) \in M(y)} \mathsf{con}(U_b(u), (M(y) - \mathsf{post}^{\bullet}(t, U_b)) \cup {}^{\bullet}\mathsf{pre}(t, U_b))$$

To implement the reversal of a transition t according to a backtracking enabling assignment U_b, the selected instances are relocated from the outgoing places of the transition to the incoming places, as specified by the incoming arcs of the transition, with bonds created and destructed accordingly. In Figure 4.9 the backtracking execution of Figure 4.8 is illustrated, where we can observe the history of the reversing transition being eliminated and the token instances returning to their initial place. Specifically we define:

Definition 41. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition occurrence (t, k) that is $bt-$enabled and an enabling assignment U_b, we write $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_b \langle M', H'\rangle$ where for all $x \in P$:

$$M'(x) = (M(x) -^{\bullet} U_b(x)) \cup (\bigcup_{(a_i,k,u)\in U_b^{\bullet}(x)} a_i \cup \bigcup_{((a_i,k,u),(b_i,k,v))\in U_b^{\bullet}(x)} (a_i, b_i))$$

$$\text{and} \quad H'(t') = \begin{cases} H(t') - \{k\}, & \text{if } t' = t \\ H(t'), & \text{otherwise} \end{cases}$$

The following proposition states that tokens are preserved throughout backtracking execution such that the amount of tokens of the same type remains the same.

Proposition 17 (Token preservation)**.** Consider a multi reversing Petri net (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, and a transition $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_b \langle M', H'\rangle$. Then for all $a_i \in A_I \cap M_0(z)$, $z \in P$ we have $|\{a_j \mid a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j| = |\{a'_j \mid a'_j \in M'(y) \cap A_I, y \in P, a_i \overline{\in} a'_j\}| = 1$.

Proof. The proof of the result follows the definition of backward execution and relies on the well-formedness of multi reversing Petri nets. Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$ such that for all $a_i \in A_I \cap M_0(z)$, $z \in P$ we have $|\{a_j \mid a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j\}| = 1$, and suppose $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_b \langle M', H'\rangle$ such that $|\{a'_j \mid a'_j \in M'(y) \cap A_I, y \in P, a_i \overline{\in} a'_j\}| = n$. Two cases exist:

1. $a_j \in \mathsf{con}(b_j, M(x))$ for some $b_j = U_b(v), v \in F(t, x)$. Let us choose b_j such that $a_j \in \mathsf{con}(b_j, (M(x) - \mathsf{post}^{\bullet}(t, U_b)) \cup^{\bullet} \mathsf{pre}(t, U_b))$. Note that such a b_j must exist, otherwise the forward execution of t would not have transferred a_j along with b_j to place x.

   According to Definition 39, we have that $a_j \in^{\bullet} U_b(x)$, which implies that $a_j \notin M'(x)$. On the other hand, note that by the definition of well-formedness, Definition 32(1), $v \in \mathsf{pre}(t)$. Thus, there exists $y \in \circ t$, such that $v \in F(y, t)$. Note that this y is unique by Definition 32(2). As a result, by Definition 40, $a_j \in U_b^{\bullet}(y)$. Since $v \in F(y, t) \cap F(t, x)$, $a_j \in \mathsf{con}(b_j, (M(x) - \mathsf{post}^{\bullet}(t, U_b)) \cup^{\bullet} \mathsf{pre}(t, U_b))$, this implies that $a'_j \overline{\in} a_j, a'_j \in M'(y)$ where $a_i \overline{\in} a'_j$.

   Now suppose that $a_j \in \mathsf{con}(c_j, (M(x) - \mathsf{post}^{\bullet}(t, U_b)) \cup^{\bullet} \mathsf{pre}(t, U_b))$, $c_j \neq b_j$, and $c_j \in F(y', t)$. Since $a_j \in \mathsf{con}(b_j, (M(x) - \mathsf{post}^{\bullet}(t, U_b)) \cup^{\bullet} \mathsf{pre}(t, U_b))$, it must be that $\mathsf{con}(b_j, (M(x) - \mathsf{post}^{\bullet}(t, U_b)) \cup^{\bullet} \mathsf{pre}(t, U_b)) = \mathsf{con}(c_j, (M(x) - \mathsf{post}^{\bullet}(t, U_b)) \cup^{\bullet} \mathsf{pre}(t, U_b))$. Since b_j and c_j are connected to each other but the connection was not created by transition (t, k) (the connection is present in $(M(x) - \mathsf{post}^{\bullet}(t, U_b)) \cup^{\bullet} \mathsf{pre}(t, U_b)$), it must be that the connection was already present before the forward execution of t

and, by token uniqueness, we conclude that $y = y'$ and therefore $1 = |\{a_j \mid a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j\}| = |\{a'_j \mid a'_j \in M'(y') \cap A_I, y' \in P, a_i \overline{\in} a'_j\}| = n$.

2. $a_j \notin \mathsf{con}(b_j, M(x))$ for all $b_j = U_b(v), v \in F(t, x)$. This implies that $|\{a_j \mid a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j\}| = |\{a'_j \mid a'_j \in M'(x) \cap A_I, x \in P, a_i \overline{\in} a'_j\}| = 1$ and the result follows.

□

We may establish a loop lemma:

Lemma 6 (Loop)**.** For any forward transition $\langle M, H\rangle \xrightarrow{(t,k)} \langle M', H'\rangle$ there exists a backward transition $\langle M', H'\rangle \overset{(t,k)}{\rightsquigarrow}_b \langle M, H\rangle$ and vice versa.

Proof. Suppose $\langle M, H\rangle \xrightarrow{(t,k)} \langle M', H'\rangle$. Then t is clearly bt-enabled in H'. Furthermore, $\langle M', H'\rangle \overset{(t,k)}{\rightsquigarrow}_b \langle M'', H''\rangle$ where $H'' = H$. In addition, all tokens and bonds involved in transition t (except those that have been created but including those that have been broken by t) will be returned from the outgoing places of transition t back to its incoming places. Specifically, for all $a_i \in A_I$, it is easy to see by the definition of $\rightsquigarrow_b$ that $a_i \in M''(x)$ if and only if $a_i \in M(x)$. Similarly, for all $\beta_i \in B_I$, $\beta_i \in M''(x)$ if and only if $\beta_i \in M(x)$. The opposite direction can be argued similarly, only this time tokens and bonds involved in transition t will be moved from the incoming places to the outgoing places of transition t. □

4.4.3 Causal Order Reversing

We now move on to reversing transitions in causal order. Causal dependence is determined by the path that tokens follow: two transition occurrences are causally dependent, if a token produced by the one occurrence was subsequently used to fire the other. To capture this type of dependencies, we adopt the following definition of causal dependence.

Definition 42. Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$ and suppose (t, k) and (t', k') are transition occurrences in $\langle M, H\rangle$. We say that (t', k') *causally depends* on (t, k) denoted by $(t, k) \prec (t', k')$, if $k < k'$ and there exists $a_i \in M(x), x \in P$, such that $(a_j, k, u) \overline{\in} a_i$ and $(a'_j, k', u') \overline{\in} a_i$.

As tokens in multi reversing Petri nets are associated with their causal path, we are able to identify the transitions that each token has participated in by observing the memory of the token. When the keys of two transitions belong to the memory of the same token then it means that this token has participated in both transitions. Thus, a transition occurrence

(t', k') causally depends on a preceding transition occurrence (t, k) if one or more tokens used during the firing of (t', k') was also used for the firing of (t, k).

A transition can be reversed in a certain state if there are no transitions causally following it and there exist token instances in its output places that match the requirements on its outgoing arcs. Specifically, we define the notion of reverse enabledness as follows.

Definition 43. Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, and a transition occurrence (t, k). We say that transition occurrence (t, k) is *c-enabled* in $\langle M, H\rangle$ if (1) there is no transition occurrence $(t', k') \in \langle M, H\rangle$ with $(t, k) \prec (t', k')$, and (2) there exists an injective function $U_c : \mathsf{post}(t) \cap A_V \rightarrow A_I$ such that:

(a) for all $u \in F(t, x)$, $x \in t\circ$, we have $U_c(u) = (a_i, k, u), U_c(u) \in M(x)$ where $type(u) = type(U_c(u))$, and

(b) for all $(u, v) \in F(t, x)$, $x \in t\circ$, we have $(U_c(u), U_c(v)) = ((a_i, k, u), (b_i, k, v))$, $(U_c(u), U_c(v)) \in M(x)$.

Thus, a transition occurrence (t, k) is c-enabled in $\langle M, H\rangle$ if (1) there are no transitions causally dependent on it, and (2) there exists a type-respecting assignment of token instances in the outgoing places of the transition, to the variables on the outgoing edges of the transition, and where the tokens are connected with bonds as required by the transition's outgoing edges. We refer to U_c as a causal enabling assignment.

We now define the incoming token/bond instances as:

Definition 44. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition t and an enabling assignment U_c, we define ${}^{\bullet}U_c : P \rightarrow 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of incoming token and bond instances that are used for the reversing of t where for all $x \in P$, ${}^{\bullet}U_c(x)$ is defined as ${}^{\bullet}U_b(x)$ in Definition 39 with U_b replaced by U_c.

We now define the outgoing token/bond instances as:

Definition 45. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition t, and an enabling assignment U_c, we define $U_c^{\bullet} : P \rightarrow 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of outgoing token/bond instances of t where for all $x \in P$, $U_c^{\bullet}(x)$ is defined as $U_b^{\bullet}(x)$ in Definition 40 with U_b replaced by U_c.

To implement the reversal of a transition t according to a causal enabling assignment U_c, the selected instances are relocated from the outgoing places of the transition to the incoming

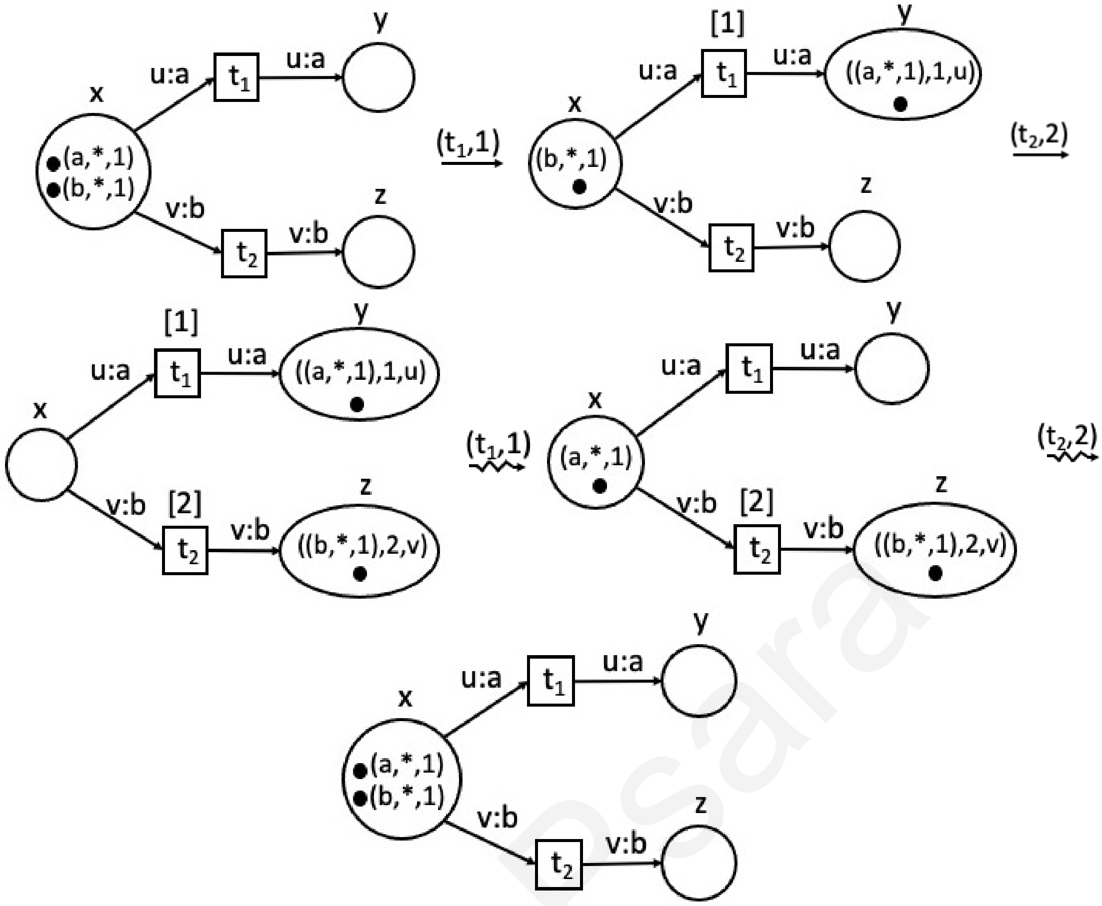


Figure 4.10: Causal-order execution

places, as specified by the incoming arcs of the transition, with bonds created and destructed accordingly. In Figure 4.10 we can observe causal order reversal of transitions t_1 and t_2 where the history of transitions and the memories of token/bond instances are updated as defined by the definition bellow:

Definition 46. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition occurrence (t, k) that is $c-$enabled and an enabling assignment U_c, we write $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M', H'\rangle$ where H is updated as in Definition 41 and for all $x \in P$:

$$M'(x) = (M(x) -^{\bullet} U_c(x)) \cup (\bigcup_{(a_i,k,u)\in U_c^{\bullet}(x)} a_i \cup \bigcup_{((a_i,k,u),(b_i,k,v))\in U_c^{\bullet}(x)} (a_i, b_i))$$

The following proposition states that tokens are preserved throughout backtracking execution such that the amount of tokens of the same type remains the same.

Proposition 18 (Token preservation)**.** Consider a multi reversing Petri net (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, and a transition $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M', H'\rangle$. Then for all $a_i \in A_I \cap M_0(z)$, $z \in P$ we have $|\{a_j \mid a_j \in M(x) \cap A_I, x \in P, a_i \overline{\in} a_j| = |\{a'_j \mid a'_j \in M'(y) \cap A_I, y \in P, a_i \overline{\in} a'_j\}| = 1$.

Proof. The proof follows along the same lines as that of Proposition 17 with $\rightsquigarrow_b$ replaced by $\rightsquigarrow_c$. $\square$

We may now establish the causal consistency of our semantics. First, we define some auxiliary notions. Given a transition $\langle M,H\rangle \xmapsto{(t,k)}_c \langle M',H'\rangle$, we say that the *action* of the transition is (t,k) if $\langle M,H\rangle \xrightarrow{(t,k)} \langle M',H'\rangle$ and $\underline{(t,k)}$ if $\langle M,H\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M',H'\rangle$ and we may write $\langle M,H\rangle \xmapsto{\underline{(t,k)}}_c \langle M',H'\rangle$. We use α to range over $\{(t,k),\mid t \in T\}$, $\underline{\alpha}$ to range over $\{\underline{(t,k)},\mid t \in T\}$. Given an execution $\langle M_0,H_0\rangle \xmapsto{\alpha_1}_c \ldots \xmapsto{\alpha_n}_c \langle M_n,H_n\rangle$, we say that the *trace* of the execution is $\sigma = \langle \alpha_1,\alpha_2,\ldots,\alpha_n\rangle$, and write $\langle M,H\rangle \xmapsto{\sigma}_c \langle M_n,H_n\rangle$. Given $\sigma_1 = \langle \alpha_1,\ldots,\alpha_k\rangle$, $\sigma_2 = \langle \alpha_{k+1},\ldots,\alpha_n\rangle$, we write $\sigma_1;\sigma_2$ for $\langle \alpha_1,\ldots,\alpha_n\rangle$. We may also use the notation $\sigma_1;\sigma_2$ when σ_1 or σ_2 is a single transition.
As in RPNs, the execution of a MRPN can be partitioned as a set of independent flows of executions running through the net. We capture these flows by the notion of causal paths:

Definition 47. Given a MRPN (P,T,A,A_V,B,F), a state $\langle M,H\rangle$ and transition occurrences (t_i,k_i) in $\langle M,H\rangle$, $1 \le i \le n$, we say that $(t_1,k_1),\ldots,(t_n,k_n)$ is a *causal path* in $\langle M,H\rangle$, if $(t_i,k_i) \prec (t_{i+1},k_{i+1})$, for all $0 \le i < n$.

Based on this concept, we define the notion of causal equivalence for histories by requiring that two histories H and H' are causally equivalent if and only if they contain the same causal paths:

Definition 48. Consider a MRPN (P,T,A,A_V,B,F) and two executions $\langle M,H\rangle \xmapsto{\sigma}_c \langle M',H'\rangle$ and $\langle M,H\rangle \xmapsto{\sigma'}_c \langle M'',H''\rangle$. Then the histories H' and H'' are *causally equivalent*, denoted by $H' \asymp H''$, if for each causal path $(t_1,k_1),\ldots,(t_n,k_n)$ in $\langle M',H'\rangle$, there is a causal path $(t_1,k_1'),\ldots,(t_n,k_n')$ in $\langle M'',H''\rangle$, and vice versa.

Now we define causal equivalence of markings as the equivalence where markings consist of identical token instances participating in the same transitions that have been assigned different keys. Two equivalent markings can be observed in Figure 4.11, where in the first execution we fire t with $((a,*,1),(c,*,1))$ first and then with $((a,*,2),(b,*,1))$, and in the second execution we fire with $((a,*,2),(b,*,1))$ first and then $((a,*,1),(c,*,1))$. This results in equivalent markings, i.e. markings consisting of connected components that have tokens of the same type used to fire the same transitions.

Definition 49. Consider a MRPN (P,T,A,A_V,B,F) and two executions $\langle M,H\rangle \xmapsto{\sigma}_c \langle M',H'\rangle$ and $\langle M,H\rangle \xmapsto{\sigma'}_c \langle M'',H''\rangle$. Then the markings M' and M'' are *causally equivalent*, denoted by $M' \asymp M''$, if for each $a_i \in M'(x)$ where $(a_j,k,u)\overline{\in}a_i$, $k \in H'(t)$, $u \in \mathsf{pre}(t), t \in T$ there

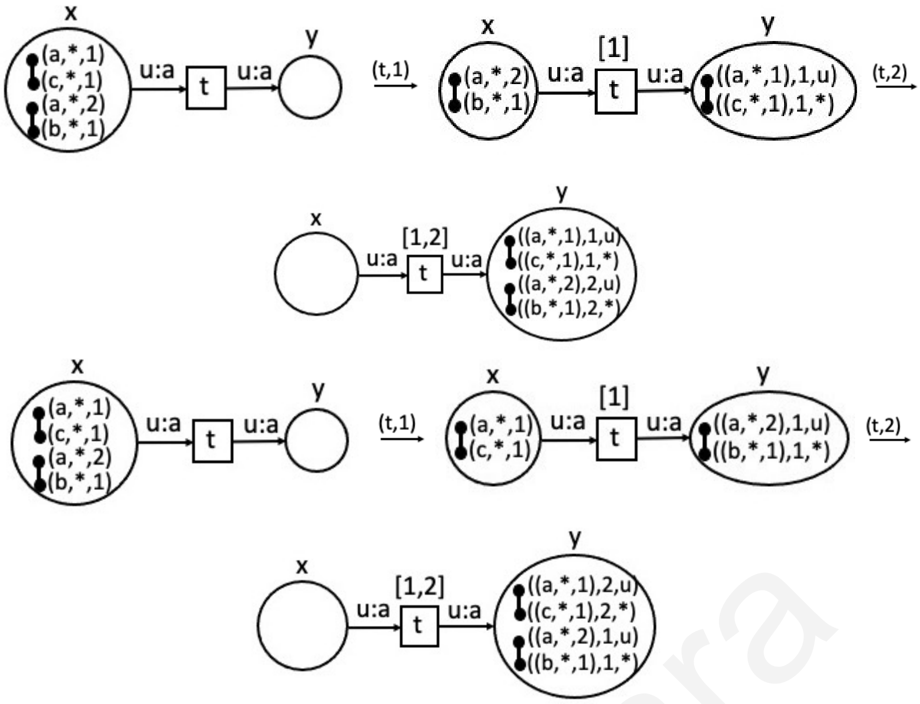


Figure 4.11: Equivalent markings

exists $a'_i \in M''(x)$ where$(a'_j, k', u)\overline{\in}a'_i$ such that $k' \in H''(t)$ and vice versa.
We extend this notion and write $\langle M, H\rangle \asymp \langle M', H'\rangle$ if and only if $M \asymp M'$ and $H \asymp H'$.

We may now establish the Loop lemma.

Lemma 7 (Loop)**.** For any forward transition $\langle M, H\rangle \xrightarrow{(t,k)} \langle M', H'\rangle$ there exists a backward transition $\langle M', H'\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M, H\rangle$ and for any backward transition $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M', H'\rangle$ there exists a forward transition $\langle M', H'\rangle \xrightarrow{(t,k')} \langle M'', H''\rangle$ where $\langle M, H\rangle \asymp \langle M'', H''\rangle$.

Proof. The proof of the first direction follows along the same lines as that of Lemma 6 with $\rightsquigarrow_b$ replaced by $\rightsquigarrow_c$. For the other direction, suppose $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M', H'\rangle \xrightarrow{(t,k')} \langle M'', H''\rangle$. To begin with, we may observe that, as with Lemma 6, by Definitions 36 and 46, the tokens involved in transition t will be transferred to the incoming places of t and then back to the outgoing places leading to $M \asymp M''$. To show that $H \asymp H''$, we observe that $H = H''$ with the exception of t, where, if $k \in H(t)$, and $k' = \max(\{0\} \cup \{k'' | (t', k'') \in H'(t'), t' \in T\}) + 1$, then $H''(t) = (H(t) - \{k\}) \cup \{k'\})$. Furthermore, since t is c-enabled in $\langle M, H\rangle$, (t, k) must be the last transition occurrence in all the causal paths it occurs in, and we may observe that H'' contains the same causal paths with (t, k) replaced by (t, k'). As a result it must be that $H \asymp H''$ and the result follows. □

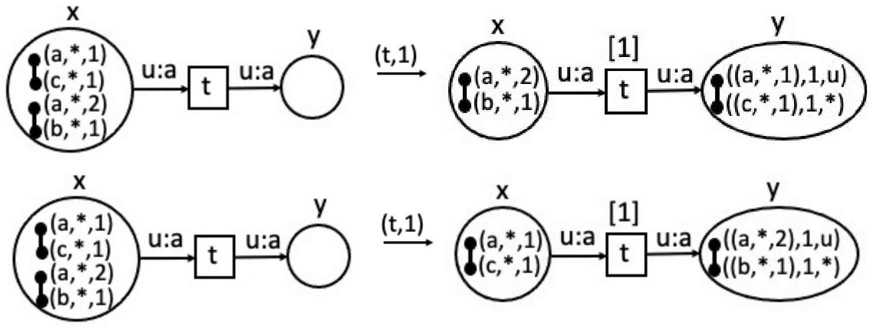


Figure 4.12: Non-equivalent transition firings

Definition 50. Consider a MRPN (P, T, A, A_V, B, F), two actions α_1 and α_2, and a state $\langle M, H\rangle$. Then α_1 and α_2 are said to be *concurrent* in state $\langle M, H\rangle$, if whenever $\langle M, H\rangle \overset{\alpha_1}{\longmapsto}_c \langle M_1, H_1\rangle$ and $\langle M, H\rangle \overset{\alpha_2}{\longmapsto}_c \langle M_2, H_2\rangle$ then $\langle M_1, H_1\rangle \overset{\alpha_2}{\longmapsto}_c \langle M', H'\rangle$ and $\langle M_2, H_2\rangle \overset{\alpha_1}{\longmapsto}_c \langle M'', H''\rangle$, and $\langle M', H'\rangle \asymp \langle M'', H''\rangle$.

As in the original RPNs two actions are concurrent when they can be executed in any order while preserving path equivalence.

Definition 51. Consider a MRPN (P, T, A, A_V, B, F), and two actions $\langle M_1, H_1\rangle \xrightarrow{(t,k_1)} \langle M_1', H_1'\rangle$ and $\langle M_2, H_2\rangle \xrightarrow{(t,k_2)} \langle M_2', H_2'\rangle$. Then (t, k_1) and (t, k_2) are said to be *equivalent* if for all $(a_i, k_1, u) \in M_1'(x)$ there exists $(a_i, k_2, u) \in M_2'(x)$ for some $u \in \mathsf{pre}(t)$.

Since a transition can be executed in the forward direction by different token instances, as long as they respect the arc requirements, then it is possible for the same transition to fire using different connected components. As these connected components might consist of different token instances then it is possible to fire the same transition resulting in markings that are not equivalent. Consider the example in Figure 4.12, where firing transition t with bond $((a, *, 1), (c, *, 1))$ will result in a different marking than firing the transition with bond $((a, *, 2), (b, *, 1))$. Thus, two transition occurrences are said to be equivalent when they execute the same transition by manipulating the same token instances.

Definition 52. Consider a multi reversing Petri nets (P, T, A, A_V, B, F) and two executions $\langle M_1, H_1\rangle \xrightarrow{\sigma_1} \langle M_1', H_1'\rangle$ and $\langle M_2, H_2\rangle \xrightarrow{\sigma_2} \langle M_2', H_2'\rangle$. *Causal equivalence on executions*, is the least equivalence relation closed under composition of traces such that if (i) $\sigma_1 = (t_1, k_1); (t_2, k_2)$ and $\sigma_2 = (t_2, k_1); (t_1, k_2)$ where (t_1, k_1) and (t_2, k_2) are concurrent actions in state $\langle M_1, H_1\rangle = \langle M_2, H_2\rangle$, (ii) $\sigma_1 = (t, k); (\underline{t, k})$ and $\sigma_2 = \epsilon$, and (iii) $\sigma_1 = (\underline{t, k_1}); (t, k_2)$ and $\sigma_2 = \epsilon$ where (t, k_1) and (t, k_2) are equivalent actions according to states $\langle M_1', H_1'\rangle$ and

$\langle M'_2, H'_2 \rangle$. If the executions $\langle M_1, H_1 \rangle \xrightarrow{\sigma_1} \langle M'_1, H'_1 \rangle$ and $\langle M_2, H_2 \rangle \xrightarrow{\sigma_2} \langle M'_2, H'_2 \rangle$ are causally equivalent then we say that traces σ_1 and σ_2 are also causally equivalent denoted by $\sigma_1 \asymp \sigma_2$.

The first clause states that in two causally-equivalent executions concurrent actions may occur in any order and the second clause states that it is possible to ignore transitions that have occurred in both the forward and the reverse direction. The third clause states that it is possible to ignore equivalent transitions that have occurred in both the reverse and forward direction. Note that unlike $(t,k);\underline{(t,k)} \asymp \epsilon$, we require these transitions to be equivalent as with token multiplicity it is possible to fire again a reversed transition by manipulating different connected tokens of the same type. These two transitions should be equivalent in order to be ignored so that they will produce the same marking, as explained for Figure 4.12.

The following proposition establishes that two transition instances belonging to distinct causal paths are in fact concurrent transitions, and thus can be executed in any order.

Proposition 19. Consider a MRPN (P,T,A,A_V,B,F) and suppose $\langle M, H \rangle \xrightarrow{(t_1,k_1)} \langle M_1, H_1 \rangle \xrightarrow{(t_2,k_2)} \langle M_2, H_2 \rangle$. If there is no causal path π in $\langle M_2, H_2 \rangle$ with $(t_1,k_1) \in \pi$ and $(t_2,k_2) \in \pi$, then (t_1,k_1) and (t_2,k_2) are concurrent transition occurrences in $\langle M, H \rangle$.

Proof. Since there is no causal path containing both (t_1,k_1) and (t_2,k_2) in $\langle M_2, H_2 \rangle$, we conclude that $(t_1,k_1) \nprec (t_2,k_2)$. This implies that there is no token that has participated in both transition occurrences and they can be executed in any order, leading to the same marking. Thus, they are concurrent in $\langle M, H \rangle$. □We note that causally-equivalent states can execute the same transitions.

Proposition 20. Consider a MRPN (P,T,A,A_V,B,F) and states $\langle M_1, H_1 \rangle \asymp \langle M_2, H_2 \rangle$. Then $\langle M_1, H_1 \rangle \overset{(t,k_1)}{\longmapsto}_c \langle M'_1, H'_1 \rangle$ if and only if $\langle M_2, H_2 \rangle \overset{(t,k_2)}{\longmapsto}_c \langle M'_2, H'_2 \rangle$, where $\langle M'_1, H'_1 \rangle \asymp \langle M'_2, H'_2 \rangle$.

Proof. It is easy to see that if a transition (t,k_1) is enabled in $\langle M_1, H_1 \rangle$ it is also enabled in $\langle M_2, H_2 \rangle$. Specifically, there exists an enabling assignment U_1 for (t,k_1) and U_2 for (t,k_2) such that they manipulate the same components that have been assigned different keys. Therefore if $\langle M_1, H_1 \rangle \overset{(t,k_1)}{\longmapsto}_c \langle M'_1, H'_1 \rangle$ then $\langle M_2, H_2 \rangle \overset{(t,k_2)}{\longmapsto}_c \langle M'_2, H'_2 \rangle$ where $M'_1 \asymp M'_2$, and vice versa. In order to show that $H'_1 \asymp H'_2$ two cases exist:

- Suppose t is a forward transition corresponding to transition occurrences (t,k_1) and (t,k_2) in each state respectively. Suppose that $(t',k'_1) \prec (t,k_1)$. Then, $\exists a_i, (a'_1, k'_1, u) \overline{\in} a_i$

and $(a_1, k_1, v)\overline{\in} a_i$, $a_i \in M'_1(x)$. Since $H_1 \asymp H_2$ this implies that $(t', k'_2) \prec (t, k_2)$. Therefore, for all causal paths π in $\langle M_1, H_1\rangle$, if the last transition occurrence of π causes (t, k_1) then $\pi; (t, k_1)$ is a causal path of $\langle M'_1, H'_1\rangle$ and, if not, then π is a causal path in $\langle M'_1, H'_1\rangle$. The same holds for causal paths in $\langle M_2, H'_2\rangle$ and (t, k_2). Consequently, we deduce that $H'_1 \asymp H'_2$, as required.

- Suppose that t is a reverse transition and consider the causal paths of H'_1 and H'_2. Since t is a reverse transition, there exists no transition occurrence in $\langle M_1, H_1\rangle$ caused by (t, k_1) and no transition occurrence in $\langle M_2, H_2\rangle$ caused by (t, k_2). As such, (t, k_1) and (t, k_2) are the last transition occurrences in all paths in $\langle M_1, H_1\rangle$ and $\langle M_2, H_2\rangle$, respectively, in which they belong. Reversing the transition occurrences results in their elimination from these causal paths. Therefore, we observe that for each causal path in $\langle M'_1, H'_1\rangle$ there is an equivalent causal path in $\langle M'_2, H'_2\rangle$, and vice versa. Thus $H'_1 \asymp H'_2$ as required. □

Note that the above result can be extended to sequences of transitions:

Corollary 7. Consider a MRPN (P, T, A, A_V, B, F) and states $\langle M_1, H_1\rangle \asymp \langle M_2, H_2\rangle$. Then $\langle M_1, H_1\rangle \overset{\sigma}{\longmapsto}_c \langle M'_1, H'_1\rangle$ if and only if $\langle M_2, H_2\rangle \overset{\sigma}{\longmapsto}_c \langle M'_2, H'_2\rangle$, where $\langle M'_1, H'_1\rangle \asymp \langle M'_2, H'_2\rangle$.

The main result, Theorem 3 below, states that two computations beginning in the same initial state lead to equivalent states if and only if the two computations are causally equivalent. Specifically, if two executions from the same state reach causally-equivalent states by executing transitions σ_1 and σ_2, then the two executions are causally equivalent and vice versa. This guarantees the consistency of the approach since reversing transitions in causal order is in a sense equivalent to not executing the transitions in the first place. Reversal does not give rise to previously unreachable states, on the contrary, it gives rise to causally-equivalent markings and histories due to the different keys being possibly assigned because of the different ordering of transitions.

Theorem 3. Consider executions $\langle M, H\rangle \overset{\sigma_1}{\longmapsto}_c \langle M_1, H_1\rangle$ and $\langle M, H\rangle \overset{\sigma_2}{\longmapsto}_c \langle M_2, H_2\rangle$. Then, $\langle M, H\rangle \overset{\sigma_1}{\longmapsto}_c \langle M_1, H_1\rangle$ and $\langle M, H\rangle \overset{\sigma_2}{\longmapsto}_c \langle M_2, H_2\rangle$ are causally equivalent executions if and only if $\langle M_1, H_1\rangle \asymp \langle M_2, H_2\rangle$.

For the proof of Theorem 3 we employ some intermediate results. To begin, the lemma below states that causal equivalence allows the permutation of reverse and forward transitions that have no causal relations between them. Therefore, computations are allowed to reach for the maximum freedom of choice going backward and then continue forward.

Lemma 8. Let $\langle M, H\rangle \xmapsto{\sigma} \langle M', H'\rangle$ be an execution. Then there exist traces r, r' both forward such that $\langle M, H\rangle \xmapsto{\sigma} \langle M', H'\rangle$ and $\langle M, H\rangle \xmapsto{\underline{r};r'} \langle M'', H''\rangle$ are causally equivalent executions where $\langle M', H'\rangle \asymp \langle M'', H''\rangle$.

Proof. We prove this by induction on the length of σ and the distance from the beginning of σ to the earliest pair of transitions that contradicts the property $\underline{r}; r'$. If there is no such contradicting pair, then the property is trivially satisfied. If not, we distinguish the following cases:

1. If the first contradicting pair is of the form $(t,k); \underline{(t,k)}$ then we have $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{(t,k)}_c \langle M_2, H_2\rangle \xmapsto{\underline{(t,k)}}_c \langle M_3, H_3\rangle \xmapsto{\sigma_2}_c \langle M', H'\rangle$ where $\sigma = \sigma_1; (t,k); \underline{(t,k)}; \sigma_2$. By the Loop Lemma 2 $\langle M_1, H_1\rangle = \langle M_3, H_3\rangle$, which yields $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{\sigma_2}_c \langle M', H'\rangle$. Thus we may remove the two transitions from the sequence, the length of σ decreases, and the proof follows by induction.

2. If the first contradicting pair is of the form $(t,k); \underline{(t',k')}$, then we observe that the specific occurrences of (t,k) and $\underline{(t',k')}$ must be concurrent. Specifically, we have $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{(t,k)}_c \langle M_2, H_2\rangle \xmapsto{\underline{(t',k')}}_c \langle M_3, H_3\rangle \xmapsto{\sigma_2}_c \langle M', H'\rangle$ where $\sigma = \sigma_1; (t,k); \underline{(t',k')}; \sigma_2$. Since action (t',k') is being reversed it implies that all transition occurrences that are causally dependent on it have either not been executed up to this point or they have already been reversed. This implies that in $\langle M_2, H_2\rangle$ it was not the case that (t,k) was causally dependent on (t',k'). As such, by Proposition 19 $\underline{(t',k')}$ and (t,k) are concurrent transitions and (t',k') can be reversed before the execution of t to yield $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{\underline{(t',k')}}_c \langle M_2', H_2'\rangle \xmapsto{(t,k'')}_c \langle M_3', H_3'\rangle \xmapsto{\sigma_2}_c \langle M'', H''\rangle$ where $\langle M_3, H_3\rangle \asymp \langle M_3', H_3'\rangle$ and (t,k'') is an equivalent transition to (t,k) as in Definition 4.12. Note that it is possible for $k'' = k$ if (t',k') was not last the transition to be executed in the forward direction before (t,k), otherwise $k'' \neq k$. This results in a later earliest contradicting pair and by induction the result follows.

3. If the first contradicting pair is of the form $(t,k); \underline{(t,k')}$, then we have $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{(t,k)}_c \langle M_2, H_2\rangle \xmapsto{\underline{(t,k')}}_c \langle M_3, H_3\rangle \xmapsto{\sigma_2}_c \langle M', H'\rangle$, where $\sigma = \sigma_1; (t,k); \underline{(t,k')}; \sigma_2$. Then (t,k) and $\underline{(t,k')}$ are not the same transition occurrence as transition (t,k') reverses with a different key value than the forward execution (t,k) thus they do not cancel each other out. As (t,k') reverses before (t,k) this means that (t,k) and $\underline{(t,k')}$ must be concurrent and by applying similar arguments as those in (2) we observe that the specific occurrences of (t,k) and $\underline{(t,k')}$ can be swapped to yield $\langle M, H\rangle \xmapsto{\sigma_1}_c \langle M_1, H_1\rangle \xmapsto{\underline{(t,k')}}_c$

$\langle M_2', H_2'\rangle \overset{(t,k'')}{\longmapsto}_c \langle M_3', H_3'\rangle \overset{\sigma_2}{\longmapsto}_c \langle M'', H''\rangle$ where $\langle M_3, H_3\rangle \asymp \langle M_3', H_3'\rangle$ and (t, k'') is an equivalent transition to (t, k) as in Definition 4.12. Note that it is possible for $k'' = k$ if (t, k') was not the last transition to be executed in the forward direction before (t, k), otherwise $k'' \neq k$. This results in a later earliest contradicting pair and by induction the result follows. □

From the above lemma we may conclude the following corollary. The result establishes that causal-order reversibility is consistent with standard forward execution in the sense that causal execution will not generate states that are unreachable in forward execution:

Corollary 8. Suppose that H_0 is the initial history. If $\langle M_0, H_0\rangle \overset{\sigma}{\longmapsto}_c \langle M, H\rangle$, and σ is a trace with both forward and backward transitions then there exists a transition $\langle M_0, H_0\rangle \overset{\sigma'}{\longmapsto}_c \langle M', H'\rangle$ where $\langle M, H\rangle \asymp \langle M', H'\rangle$, and σ' a trace of forward transitions.

Proof. According to Lemma 8, $\sigma \asymp \underline{r}; r'$ where both r and r' are forward traces. Since, however, H_0 is the initial history it must be that r is empty. This implies that $\langle M_0, H_0\rangle \overset{r'}{\longmapsto}_c \langle M', H'\rangle$, $\langle M, H\rangle \asymp \langle M', H'\rangle$ and r' is a forward trace. Consequently, writing σ' for r', the result follows. □

Lemma 9. Suppose $\langle M, H\rangle \overset{\sigma_1}{\longmapsto}_c \langle M_1, H_1\rangle$ and $\langle M, H\rangle \overset{\sigma_2}{\longmapsto}_c \langle M_2, H_2\rangle$, where $\langle M_1, H_1\rangle \asymp \langle M_2, H_2\rangle$ and σ_2 is a forward trace. Then, there exists a forward trace σ_1' such that $\langle M, H\rangle \overset{\sigma_1'}{\longmapsto}_c \langle M_1', H_1'\rangle$ and $\langle M, H\rangle \overset{\sigma_1}{\longmapsto}_c \langle M_1, H_1\rangle$ are causally equivalent executions.

Proof. If σ_1 is forward, then $\sigma_1 = \sigma_1'$ and the result follows trivially. Otherwise, we may prove the lemma by induction on the length of σ_1. We begin by noting that, by Lemma 8, $\sigma_1 \asymp \underline{r}; r'$ and $\langle M, H\rangle \overset{\underline{r}; r'}{\longmapsto}_c \langle M_1, H_1\rangle$. Let $\underline{(t, k)}$ be the last action in $\underline{r}$. Given that σ_2 is a forward execution that simulates σ_1, it must be that r' contains a forward execution of transition t manipulating the same tokens since $\langle M_1, H_1\rangle$ and $\langle M_2, H_2\rangle$ contain the same causal paths involving transition t (if not we would have $\langle M_1, H_1\rangle \not\asymp \langle M_2, H_2\rangle$ leading to a contradiction). Consider the earliest such occurrence in r' to be (t, k') an equivalent transition to (t, k). If (t, k') is the first transition in r' and as it is equivalent to (t, k) the Loop Lemma 2 can be applied to remove the pair of opposite transitions and the result follows by induction. Otherwise, suppose $\langle M, H\rangle \overset{\underline{r_1}}{\longmapsto}_c \overset{\underline{(t,k)}}{\longmapsto}_c \overset{r_1'}{\longmapsto}_c \langle M_3, H_3\rangle \overset{(t*,k*)}{\longmapsto}_c \overset{(t,k')}{\longmapsto}_c \langle M_4, H_4\rangle \overset{r_2'}{\longmapsto}_c \langle M_1, H_1\rangle$, where $r = r_1; (t, k)$ and $r' = r_1'; (t*, k*); (t, k'); r_2$. Two cases exist:

1. Suppose $(t*, k*) \in \sigma_2$. Let us denote by $num(\alpha, \sigma)$, the number of executions of action α in a sequence of transitions σ where α represents all transition occurrences of tran-

sition t manipulating the same connected components. We observe that since σ_2 contains no reverse executions of t, it must be that $num(\alpha, r') = num(\alpha, \sigma_2) + num(\underline{\alpha}, r)$. Suppose that the transition occurrences of $(t*, k*)$ and (t, k') as shown in the execution belong to a common causal path. We may extend this path with the succeeding occurrences of α and obtain a causal path such that $(t*, k*)$ is succeeded by $num(\alpha, \sigma_2) + num(\alpha, r)$ occurrences of α. We observe that it is impossible to obtain such a causal path in $\langle M_2, H_2\rangle$, since $(t*, k*)$ is followed by fewer occurrences of α in σ_2. This contradicts the assumption that $H_1 \asymp H_2$. We conclude that the transition occurrences of (t, k') and $(t*, k*)$ above do not belong to any common causal path and, therefore, by Proposition 19, the two transition occurrences are concurrent in $\langle M_3, H_3\rangle$.

2. Now suppose that $(t*, k*) \notin \sigma_2$. Since $k* \in H_1(t*)$ it must be that $H_2(t*) \neq \emptyset$ and $|H(t*)| = |H_1(t*)| = |H_2(t*)|$. As such, it must be that $(t*, k'*) \in r$ and that its reversal has preceded the reversal of (t, k). Let us suppose that the transition occurrences of $(t*, k*)$ and (t, k') as shown in the execution belong to a common causal path. This implies that a causal path with $(t*, k'*)$ preceding (t, k) also occurs in H_2 as well as in H. If we observe that $(t*, k'*)$ has reversed before (t, k) we conclude that $(t*, k'*)$ does not cause the preceding occurrence of (t, k). As such there is no causal path within $\langle M, H\rangle$ or $\langle M_2, H_2\rangle$ containing both (t, k) and $(t*, k*)$, which results in a contradiction. We conclude that the forward occurrences of (t, k') and $(t*, k*)$ are, by Proposition 19, concurrent in $\langle M_3, H_3\rangle$.

Given the above, we conclude that we may swap the occurrences of (t, k') and $(t*, k*)$ to obtain $\langle M, H\rangle \xrightarrow{\underline{r_1}}_c \xrightarrow{\underline{(t,k)}}_c \xrightarrow{r'_1}_c \langle M_3, H_3\rangle \xrightarrow{(t,k'')}_c \xrightarrow{(t*,k''*)}_c \langle M'_4, H'_4\rangle \xrightarrow{r'_2}_c \langle M''_1, H''_1\rangle$ where $\langle M_4, H_4\rangle \asymp \langle M'_4, H'_4\rangle$ and, by Corollary 7, $\langle M_1, H_1\rangle \asymp \langle M''_1, H''_1\rangle$. By repeating the process for the remaining transition occurrences in r'_1, this implies that we may permute (t, k') with transitions in r'_1 to yield the sequence $\underline{(t, k)}; (t, k')$. By the Loop Lemma 2 we may remove the pair of opposite transitions and obtain a shorter equivalent trace, also equivalent to σ_2 and conclude by induction. □

We may now proceed with the proof of Theorem 3:

Proof of Theorem 3. Suppose that we have $\langle M, H\rangle \xrightarrow{\sigma_1}_c \langle M_1, H_1\rangle$, $\langle M, H\rangle \xrightarrow{\sigma_2}_c \langle M_2, H_2\rangle$ with $\langle M_1, H_1\rangle \asymp \langle M_2, H_2\rangle$. We prove that $\langle M, H\rangle \xrightarrow{\sigma_1}_c \langle M_1, H_1\rangle$ and $\langle M, H\rangle \xrightarrow{\sigma_2}_c \langle M_2, H_2\rangle$ are causally equivalent executions thus giving $\sigma_1 \asymp \sigma_2$ by using a lexicographic induction on the pair consisting of the sum of the lengths of σ_1 and σ_2 and the depth of the earliest disagreement between them. By Lemma 8 we may suppose that σ_1 and σ_2 satisfy the property $\underline{r}; r'$. Call (t_1, k_1) and (t_2, k_2) the earliest actions where they disagree. There are three cases in the argument depending on whether these are forward or backward.

1. If $\underline{(t_1, k_1)}$ is backward and (t_2, k_2) is forward, we have $\sigma_1 = \underline{r}; \underline{(t_1, k_1)}; u$ and $\sigma_2 = \underline{r}; (t_2, k_2); v$ for some r, u, v. Lemma 9 applies to $(t_2, k_2); v$, which is forward, and $\underline{(t_1, k_1)}; u$, which contains both forward and backward actions and thus, by the lemma, it has a shorter forward equivalent. Thus, σ_1 has a shorter forward equivalent and the result follows by induction.

2. If (t_1, k) and (t_2, k) are both forward then it must be the case that $\sigma_1 = \underline{r}; r'; (t_1, k); u$ and $\sigma_2 = \underline{r}; r'; (t_2, k); v$, for some r, u, v. Note that it must be that an equivalent transition to t_1 appears in v and an equivalent transition to t_2 appears in u. If not, we would have $H_1 \not\asymp H_2$, which contradicts the assumption that $H_1 \asymp H_2$. As such, we may write $\sigma_1 = \underline{r}; r'; (t_1, k); u_1; (t_2, k_2); u_2$, where $u = u_1; (t_2, k_2); u_2$ and (t_2, k_2) is the first occurrence of t_2 in u manipulating the same tokens as (t_2, k). Consider $(t*, k*)$ the action immediately preceding (t_2, k_2). We may observe that $(t*, k*)$ and (t_2, k_2) cannot belong to a common causal path in $\langle M_1, H_1 \rangle$, since an equivalent causal path is impossible to exist in $\langle M_2, H_2 \rangle$. This is due to the assumption that σ_1 and σ_2 coincide up to transition sequence $\underline{r}; r'$. Thus, we may conclude by Proposition 19 that $(t*, k*)$ and (t_2, k_2) are in fact concurrent and can be swapped. The same reasoning may be used for all transitions preceding (t_2, k_2) up to and including (t_1, k), which leads to the conclusion that $\sigma_1 \asymp \underline{r}; r'; (t_2, k); (t_1, k_1); u_1; u_2$. This results in an equivalent execution of the same length with a later earliest divergence with σ_2 and the result follows by the induction hypothesis.

3. If (t_1, k_1) and (t_2, k_2) are both backward, we have $\sigma_1 = \underline{r}; \underline{(t_1, k_1)}; u$ and $\sigma_2 = \underline{r}; \underline{(t_2, k_2)}; v$ for some r, u, v. Two cases exist:

   (a) If $\underline{(t_1, k_1')}$ occurs in v, then we have that $\sigma_2 = \underline{r}; \underline{(t_2, k_2)}; \underline{v_1}; \underline{(t_1, k_1')}; v_2$. Given that t_1 reverses right after $\underline{r}$ in σ_1, we may conclude that there is no transition occurrence at this point that causally depends on (t_1, k_1'). As such it cannot have caused the transition occurrences of (t_2, k_2) and v_1 whose reversal precedes it in σ_2. This implies that the reversal of (t_1, k_1') may be swapped in σ_2 with each of the preceding transitions, to give $\sigma_2 \asymp \underline{r}; \underline{(t_1, k_1')}; \underline{(t_2, k_2)}; \underline{v_1}; v_2$. This results in an equivalent execution of the same length with a later earliest divergence with σ_1

and the result follows by the induction hypothesis.

(b) If $\underline{(t_1,k_1')}$ does not occur in v, this implies that (t_1,k_1'), an equivalent transition of (t_1,k_1) occurs in the forward direction in u, i.e. $\sigma_1 = \underline{r};\underline{(t_1,k_1)};u_1;(t_1,k_1');u_2$, where $u = u_1;(t_1,k_1');u_2$ with the specific occurrence of (t_1,k_1') being the first such occurrence in u. Using similar arguments as those in Lemma 9, we conclude that $\sigma_1 \asymp \underline{r};\underline{(t_1,k_1)};(t_1,k_1');u_1;u_2 \asymp \underline{r};u_1;u_2$, an equivalent execution of shorter length for σ_1 and the result follows by the induction hypothesis.

We may now prove the opposite direction. Suppose that $\langle M,H\rangle \stackrel{\sigma_1}{\longmapsto}_c \langle M_1,H_1\rangle$ and $\langle M,H\rangle \stackrel{\sigma_2}{\longmapsto}_c \langle M_2,H_2\rangle$ are causally equivalent executions thus $\sigma_1 \asymp \sigma_2$. We will show that $\langle M_1,H_1\rangle \asymp \langle M_2,H_2\rangle$. The proof is by induction on the number of rules, k, applied to establish the equivalence $\sigma_1 \asymp \sigma_2$. For the base case we have $k = 0$, which implies that $\sigma_1 = \sigma_2$ and the result trivially follows. For the inductive step, let us assume that $\langle M,H\rangle \stackrel{\sigma_1}{\longmapsto}_c \langle M_1,H_1\rangle$, $\langle M,H\rangle \stackrel{\sigma_2}{\longmapsto}_c \langle M_2,H_2\rangle$, and $\langle M,H\rangle \stackrel{\sigma_1'}{\longmapsto}_c \langle M_1',H_1'\rangle$ are causally equivalent executions thus $\sigma_1 \asymp \sigma_1' \asymp \sigma_2$, where σ_1 can be transformed to σ_1' with the use of $k = n-1$ rules and σ_1' can be transformed to σ_2 with the use of a single rule. By the induction hypothesis, we conclude that $\langle M,H\rangle \stackrel{\sigma_1'}{\longmapsto}_c \langle M_1',H_1'\rangle$, where $\langle M_1,H_1\rangle \asymp \langle M_1',H_1'\rangle$. We need to show that $\langle M_1',H_1'\rangle \asymp \langle M_2,H_2\rangle$. Let us write $\sigma_1' = u;w;v$ and $\sigma_2 = u;w';v$, where w, w' refer to the parts of the two executions where the equivalence rule has been applied. Furthermore, suppose that $\langle M,H\rangle \stackrel{u}{\longmapsto}_c \langle M_u,H_u\rangle \stackrel{w}{\longmapsto}_c \langle M_w,H_w\rangle \stackrel{v}{\longmapsto}_c \langle M_1',H_1'\rangle$ and $\langle M,H\rangle \stackrel{u}{\longmapsto}_c \langle M_u,H_u\rangle \stackrel{w'}{\longmapsto}_c \langle M_w',H_w'\rangle \stackrel{v}{\longmapsto}_c \langle M_2,H_2\rangle$. Three cases exist:

(a) $w = (t_1,k_1);(t_2,k_2)$ and $w' = (t_2,k_1);(t_1,k_2)$ with (t_1,k_1) and (t_2,k_2) concurrent

(b) $w = (t,k);\underline{(t,k)}$ and $w' = \epsilon$

(c) $w = \underline{(t,k)};(t,k')$ and $w' = \epsilon$ with (t,k) and (t,k') equivalent.

In all the cases above, we have that $\langle M_w,H_w\rangle \asymp \langle M_w',H_w'\rangle$: for (a) this follows by the definition of concurrent transitions, whereas for (b) and (c) by the Loop Lemma. Given the equivalence of these two states, by Corollary 8, we have that $\langle M_w,H_w\rangle \stackrel{v}{\longmapsto}_c \langle M_1',H_1'\rangle$ and $\langle M_w',H_w'\rangle \stackrel{v}{\longmapsto}_c \langle M_2,H_2\rangle$, where $\langle M_1',H_1'\rangle \asymp \langle M_2,H_2\rangle$, as required. This completes the proof. □

4.4.4 Out-of-Causal Order

In this form of reversibility we allow events to reverse without the need to respect causality as long as the transition is executed and its effect (creation/destruction of a bond) has not been undone.

Definition 53. Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, and a transition occurrence (t, k) in $\langle M, H\rangle$. We say that (t, k) is *o-enabled* in $\langle M, H\rangle$ if there exists an injective function $U_o : \mathsf{post}(t) \cap A_V \rightarrow A_I$ such that:

1. for all $u \in F(t, x)$, $x \in t\circ$,we have $U_o(u) = a_j$, $(a_i, k, u)\overline{\in}a_j$, $U_o(u) \in M(y)$ for some y where $type(u) = type(U_o(u))$,

2. for all $(u, v) \in F(t, x)$, $x \in t\circ$ we have $(U_o(u), U_o(v)) = (a_j, b_j), (a_i, k, u)\overline{\in}a_j, (b_i, k, v)$ $\overline{\in}b_j, (U_o(u), U_o(v)) \in M(y)$,

3. for all $(u, v) \in \mathsf{pre}(t)$, $(u, v) \notin \mathsf{post}(t)$ and $U_o(u) = a_j$, $U_o(v) = b_j$, $(a'_i, k', u')\overline{\in}a_j$, $(b'_i, k', v')\overline{\in}b_j$ then $\nexists t'$, $k' \in H(t')$ where $k' > k$ such that $(u', v') \in \mathsf{post}(t')$, $(u', v') \notin \mathsf{pre}(t')$, and

4. for all $(u, v) \in \mathsf{post}(t)$, $(u, v) \notin \mathsf{pre}(t)$ and $U_o(u) = a_j$, $U_o(v) = b_j$, $(a'_i, k', u')\overline{\in}a_j$, $(b'_i, k', v')\overline{\in}b_j$ then $\nexists t'$, $k' \in H(t')$ where $k' > k$ such that $(u', v') \in \mathsf{pre}(t)$, $(u', v') \notin \mathsf{post}(t)$.

Thus, a transition occurrence (t, k) is o-enabled in $\langle M, H\rangle$ if (1) and (2) there exists a type-respecting assignment of token instances in the outgoing places of the transition, to the variables on the outgoing edges of the transition, and where the instances are connected with bonds as required by the transition's outgoing edges. Finally, (3) and (4) require the effect of the transition, i.e. breaking or creating a bond, not to have been undone by a following forward transition. We refer to U_o as an out-of-causal-order enabling assignment.

Summing up, the effect of reversing a transition in out-of-causal order is that all bonds created by the transition are undone and all bonds broken by the transition are redone. This may result in tokens backtracking in the net, in the case where the reversal of a transition causes a coalition of bonds to be broken down into a set of subcomponents and moving forward in the net, in the case where the reversal of a transition recreates a coalition into a larger component. In both cases the component should be relocated (if needed) after the last transition in which this sub-coalition participated. To capture this we introduce the following:

Definition 54. Given a MRPN (P, T, A, A_V, B, F), an initial marking M_0, a history H, and a set of bases and bonds $C \subseteq A_I \cup B_I$ we write:

$$\mathsf{last}_T(C, H) = \begin{cases} (t,k), & \text{if } \exists t,\ (a_j, k, u) \overline{\in} a_i, a_i \in C,\ k \in H(t), u \neq * \text{ and} \\ & \quad \nexists t',\ (b_j, k', u') \overline{\in} b_i, b_i \in C,\ k' \in H(t'), u' \neq *, k' > k \\ \bot, & \text{otherwise} \end{cases}$$

$$\mathsf{last}_P(C, H) = \begin{cases} x, & \text{if } (t,k) = \mathsf{last}_T(C,H), \{x\} = \{y \in t\circ \mid (a_j, k, u) \overline{\in} a_i, a_i \in C, u \in F(t, y)\} \\ & \quad \text{or, if } \bot = \mathsf{last}_T(C, H), C \in M_0(x) \\ \bot, & \text{otherwise} \end{cases}$$

Thus, if the tokens from component C have been manipulated by some previously-executed transition, then $\mathsf{last}_T(C, H)$ is the last executed such transition. Otherwise, if no such transition exists (e.g., because all transitions involving C have been reversed), then $\mathsf{last}_T(C, H)$ is undefined ($\bot$). Similarly, $\mathsf{last}_P(C, H)$ is the outgoing place connected to t with common tokens with C, if $\mathsf{last}_T(C, H) \neq \bot$ assuming that such a place is unique, or the place in the initial marking in which C existed if $\mathsf{last}_T(C, H) = \bot$, and undefined otherwise.

The following definition defines all tokens to be removed from a place because their last transition has changed and their current place is not the outgoing place of last.

Definition 55. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, an o-enabled transition occurrence (t, k), a history H' as in Definition 46, and an enabling assignment U_o, we define ${}^\bullet U_o : P \to 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of incoming token and bond instances:

$${}^\bullet U_o(x) = \mathsf{post}^\bullet(t, U_o) \cup \{C_{a_i,x} \mid \exists a_i \in M(x), x \neq \mathsf{last}_P(C_{a_i,x}, H')\}$$

where we use the shorthand $C_{b_i,z} = \mathsf{con}(b_i, (\{\mathsf{con}(c_i, M(z)) | c_i \in A_I, z \in P\} - \mathsf{post}^\bullet(t, U_o)) \cup {}^\bullet\mathsf{pre}(t, U_o))$ for $b_i \in A_I$.

We now define the outgoing tokens as the tokens that remain or move to these places because it is an outgoing place of their last transition.

Definition 56. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, an o-enabled transition occurrence (t, k), a history H' updated as in Definition 46 and an enabling assignment U_o we define $U_o^\bullet : P \to 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of outgoing token/bond instances:

$$\begin{aligned} U_o^\bullet(x) = \; & \{\mathsf{last}_{A_I}(a_i, L, k) | \exists a_i \in M(y), L = \mathsf{last}_T(C_{a_i,y}, H'), x = \mathsf{last}_P(C_{a_i,y}, H')\} \cup \\ & \{(\mathsf{last}_{A_I}(a_i, L, k), \mathsf{last}_{A_I}(b_i, L, k)) | \exists (a_i, b_i) \in M(y), L = \mathsf{last}_T(C_{a_i,y}, H'), x = \mathsf{last}_P(C_{a_i,y}, H')\} \end{aligned}$$

where

$$\mathsf{last}_{A_I}(a_i, L, k) = \begin{cases} (a_m, k_m, u_m) \downarrow a_k, & \text{if } \nexists (a'_m, k'_m, u'_m) \mathrel{\overline{\in}} a_i, k' \geq k'_m > k_m \text{ where} \\ & (a_m, k_m, u_m) \mathrel{\overline{\in}} a_i, L = (t', k') \text{ and } (a_k, k, u) \mathrel{\overline{\in}} (a_m, k_m, u_m) \\ (a, *, i), & \text{if } (a, *, i) \mathrel{\overline{\in}} a_i, L = \bot \end{cases}$$

and we use the shorthand $C_{b_i,z} = \mathsf{con}(b_i, (\{\mathsf{con}(c_i, M(z)) | c_i \in A_I, z \in P\} - \mathsf{post}^\bullet(t, U_o)) \cup {}^\bullet\mathsf{pre}(t, U_o))$ for $b_i \in A_I$.

The above definition reconstructs connected components by undoing the effect of the transition and by removing from tokens the memory of the transition along with the memories that where recorded later than their last transition. The definition uses $C_{a_i,y}$ to reconstruct the component as a result of breaking or creating bonds during reversal. By $\mathsf{last}_{A_I}(a_i, L, k)$ we indicate the updated memory of token a_i by removing transition (t, k) and all memories executed later than its last transition $L = (t', k')$. Specifically, (a_m, k_m, u_m) is the latest memory taken before and including its last transition $L = (t', k')$. $(a_m, k_m, u_m) \downarrow a_k$ indicates that the memory of transition (t, k) has been removed from (a_m, k_m, u_m). In the case that there is no last transition the initial token $(a, *, i)$ is returned. In this way we remove the implicit memories where the transition has not actively participated in. As demonstrated in the example of Figure 4.13 after the reversal of $(t_1, 1)$ the implicit memory of transition $(t_2, 2)$ is removed from the token $(((a, *, 1), 1, u), 2, *)$ along with the memory of transition $(t_1, 1)$ as defined below:

Definition 57. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition occurrence (t, k) that is o-enabled and an enabling assignment U_o, we write $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_o \langle M', H'\rangle$ where H' is updated as in Definition 46 and for all $x \in P$:

$$M'(x) = (M(x) - {}^\bullet U_o(x)) \cup U_o^\bullet(x)$$

Thus, when a transition t is reversed in an out-of-order fashion all bonds that were created by the transition are undone and all bonds broken by the transition are reconstructed. If the destruction of a bond divides a component into smaller connected components then each of these components should again be relocated (if needed) back to the place where the complex would have existed if transition t never took place, i.e., exactly after the last transition that involves tokens from the sub-complex. Otherwise when a recreation of a bond creates a larger connected component then this component should be relocated (if needed) to the place where the complex would have existed if transition t, never took place, i.e., exactly after the

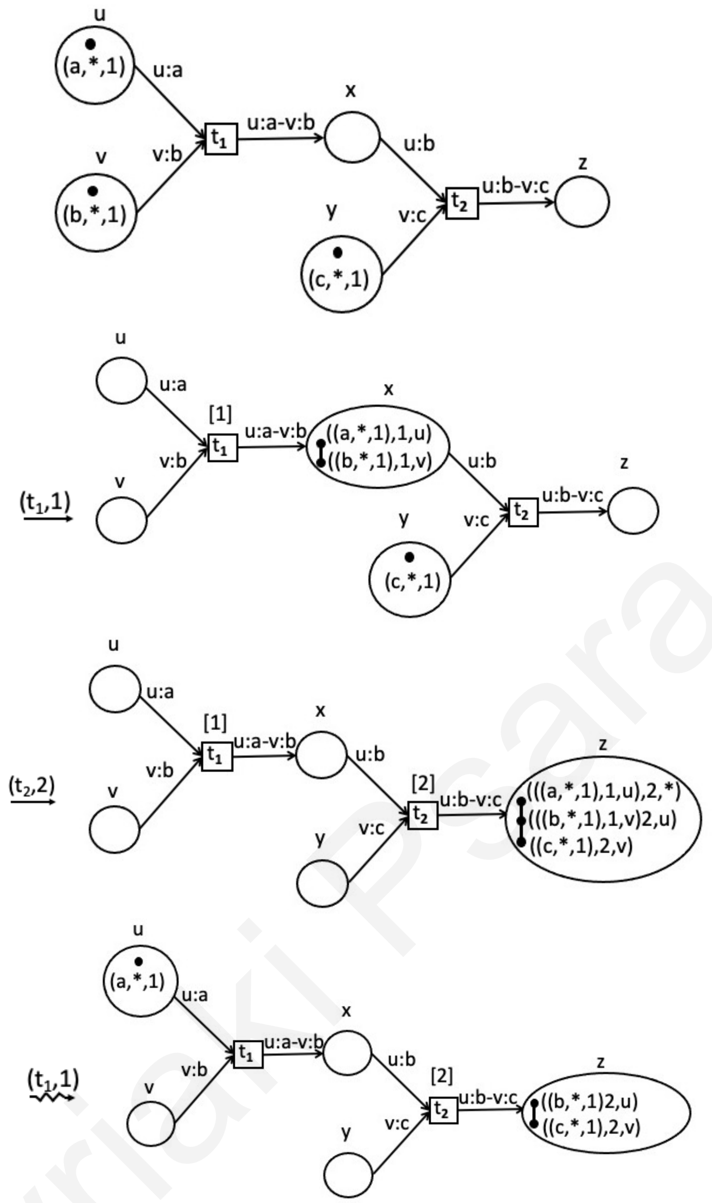


Figure 4.13: Updating the memories of tokens in out-of-causal-order reversibility

last transition that involves tokens from the bigger complex. Token memories are updated by removing the memory of the reversed transition (t,k) and removing all implicit memories of transitions executed later than their last transition. Also the history is update as defined in Definition 46.

From the example in Figure 4.14 we observe that after the execution of transitions t_1 and t_2, the component a–b–c has been broken to three parts a, b, and c located in different places. The reversal of t_1 recreates the bond between $a-b$ and since b last participated in t_2 then $a-b$ is moved to the outgoing place of t_2 as it would have happened if we had skipped the execution of t_1. However when t_1 and t_2 are reversed then our system resets and there are no transitions holding the tokens further down the execution of the RPN and therefore the

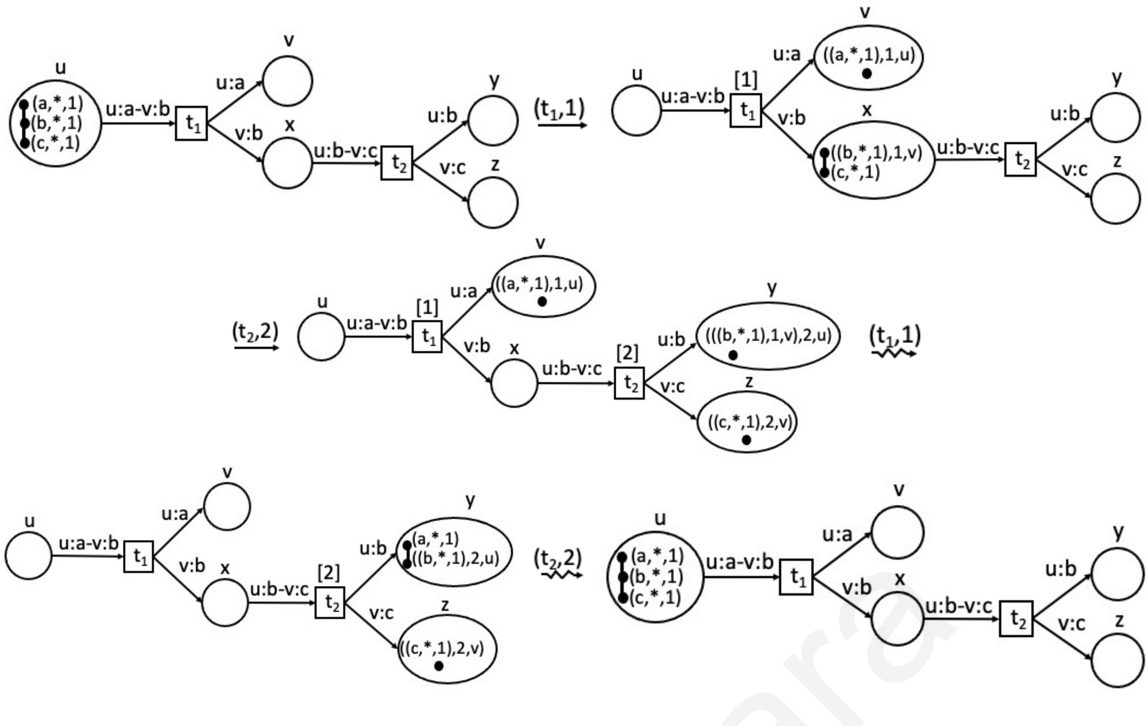


Figure 4.14: Out-of-causal order execution

component $a-b-c$ returns to its initial place. Note that the history and the token instances are updated accordingly.

The following results describe how tokens and bonds are manipulated during out-of-causal-order reversibility, where we write $\longmapsto_o$ for $\longrightarrow \cup \rightsquigarrow_o$.

Proposition 21. Suppose $\langle M,H\rangle \overset{(t,k)}{\longmapsto}_o \langle M',H'\rangle$ and let $a_i, a_i' \in A_I$ where $a_i \in M(x)$ and $a_i' \in M'(y)$. If (t,k) is a forward occurrence with U_f then $C = \mathsf{con}(a_i, (\{\mathsf{con}(b_i, M(z)) | b_i \in A_I, z \in P\} -{}^\bullet\mathsf{pre}(t,U_f)) \cup \mathsf{post}^\bullet(t,U_f))$ and $C' = \mathsf{con}(a_i', M'(y))$ such that for all $a_i \in C$ and $a_i' \in C'$, $a_i \overline{\in} a_i'$ and if (t,k) is a reverse transition with U_o then $C' = \mathsf{con}(a_i', M'(y))$ and $C = \mathsf{con}(a_i, (\{\mathsf{con}(b_i, M(z)) | b_i \in A_I, z \in P\}) - \mathsf{post}^\bullet(t,U_o)) \cup {}^\bullet\mathsf{pre}(t,U_o))$ such that for all $a_i \in C$ and $a_i' \in C'$, $a_i' \overline{\in} a_i$.

Proof. The proof is straightforward by the definition of the firing rules. □

Proposition 22. Given a MRPN (P,T,A,A_V,B,F), an initial state $\langle M_0,H_0\rangle$, and an execution $\langle M_0,H_0\rangle \overset{(t_1,k_1)}{\longmapsto}_o \langle M_1,H_1\rangle \overset{(t_2,k_2)}{\longmapsto}_o \ldots \overset{(t_n,k_n)}{\longmapsto}_o \langle M_n,H_n\rangle$ the following hold for all $0 \le i \le n$ where $a_i \in A_I \cap M_0(z)$, $z \in P$, $|\{a_i \mid a_i \in M_i(x) \cap A_I, x \in P, a_0 \overline{\in} a_i\}| = |\{a_{i+1} \mid a_{i+1} \in M_{i+1}(y) \cap A_I, y \in P, a_0 \overline{\in} a_{i+1}\}| = 1$, and $a_i \in M_i(x)$ where $x = \mathsf{last}_P(\mathsf{con}(a_i, M_i(x)), H_i)$.

Proof. Consider a MRPN (P,T,A,A_V,B,F), an initial state $\langle M_0,H_0\rangle$, and an execution $\langle M_0,H_0\rangle \overset{(t_1,k_1)}{\longmapsto}_o \langle M_1,H_1\rangle \overset{(t_2,k_2)}{\longmapsto}_o \ldots \overset{t_n,k_n}{\longmapsto}_o \langle M_n,H_n\rangle$. The proof is by induction on n.

Base Case. For $n = 0$, by our assumption of token uniqueness and the definitions of last_P and last_T the claim follows trivially.

Induction Step. Suppose the claim holds for all but the last transition and consider transition (t_n, k_n). Two cases exist, depending on whether t_n is a forward or a reverse transition:

- Suppose that (t_n, k_n) is a forward transition. Then by Proposition 16, for all $a_0 \in A_I \cap M_0(z), z \in P$, $|\{a_n \mid a_n \in M_n(x) \cap A_I, x \in P, a_o \overline{\in} a_n\}| = 1$. Additionally, we may see that if $a_n \in M_n(x)$ two cases exists. If $a_{n-1} \in \mathsf{con}(b_{n-1}, M_{n-1}(y))$, for some b_{n-1} where $U_f(v) = b_{n-1}, v \in F(t_n, z)$ then $x = z = \mathsf{last}_P(\mathsf{con}(a_{n-1}, M_{n-1}(x)), H_n)$ and $a_{n-1} \overline{\in} a_n$ then $\mathsf{last}_P(\mathsf{con}(a_n, M_n(x)), H_n) = x = z$. Otherwise, it must be that $a_{n-1} \in M_{n-1}(x)$ where, by the induction hypothesis, $x = \mathsf{last}_P(\mathsf{con}(a_{n-1}, M_{n-1}(x)), H_{n-1})$. Since $a_{n-1} \notin \mathsf{con}(b_{n-1}, M_{n-1}(y))$ we may deduce that $\mathsf{con}(a_{n-1}, M_{n-1}(x)) = \mathsf{con}(a_n, M_n(x))$, leading to $x = \mathsf{last}_P(\mathsf{con}(a_n, M_n(x)), H_n) = \mathsf{last}_P(\mathsf{con}(a_{n-1}, M_{n-1}(x)), H_n)$. Thus, the result follows.

- Suppose that (t_n, k_n) is a reverse transition. Consider $a_{n-1} \in A_I$ with $a_{n-1} \in M_{n-1}(x)$ for some $x \in P$. Two cases exist:

  - Suppose $C = \mathsf{con}(a_{n-1}, (\{\mathsf{con}(b_{n-1}, M(z)) | b_{n-1} \in A_I, z \in P\} - \mathsf{post}^{\bullet}(t, U_o)) \cup^{\bullet} \mathsf{pre}(t, U_o))$ where $\mathsf{last}_T(C, H_n) = \perp$. Then, it must be that for $C' = \mathsf{con}(a_n, M_n(y))$ where $a_n \overline{\in} a_{n-1}$ by Proposition 21 where $C' \subseteq M_0(y)$. Suppose that this is not the case. By the induction hypothesis, there exists some t_i in the execution such that $\exists \beta_i \in C'$ and $\beta_i \notin M_0(y)$, if β_i is produced by t_i, or $\exists \beta_i \in M_0(y)$ and $\beta_i \notin C'$, if β_i is destructed by t_i. This however implies that t_i is a transition that has manipulated the connected component C, which contradicts our assumption of $\mathsf{last}_T(C, H_n) = \perp$. Therefore, $a_n \in M_n(y)$, where $a_n \in M_0(y)$ and by Proposition 21 $a_{n-1} \overline{\in} a_n$ which gives $y = \mathsf{last}_P(C', H_n)$ and the result follows.
  - Suppose $C = \mathsf{con}(a_{n-1}, (\{\mathsf{con}(b_{n-1}, M(z)) | b_{n-1} \in A_I, z \in P\} - \mathsf{post}^{\bullet}(t, U_o)) \cup^{\bullet} \mathsf{pre}(t, U_o))$ where $\mathsf{last}_T(C, H_n) = (t_k, k)$. Then, it must be that there exists a unique $y \in t_k \circ$ such that $c_{n-1} \in C$ where $(c_k, k, v) \overline{\in} c_{n-1}, v \in F(t_k, z)$. Suppose that this is not the case. Then for $C' = \mathsf{con}(a_n, M_n(x))$ there must exist some $\beta_n = (a_n, c_n) \in C'$ with $(a_k, k, u) \overline{\in} a_n$, $u \in F(t_k, y_1)$, $v \in F(t_k, y_2)$, and $y_1 \neq y_2$. By the induction hypothesis, there exists some t_i in the execution such that $(a_i, k_i, u_i) \overline{\in} a_n$ and $(c_i, k_i, v_i) \overline{\in} c_n$, where $(u_i, v_i) \in F(t_i, y_i)$ and $k_i > k$ which was not reversed. This however implies that t_i is a transition that has manipulated the

connected component C later than (t_k, k), which contradicts our assumption of $\mathsf{last}_T(C, H_n) = t_k$. Therefore, there exists a unique $y \in t_k\circ$ such that $a_n \in M_n(y)$. Furthermore, by Proposition 21 $a_n \bar{\in} a_{n-1}$ which gives $y = \mathsf{last}_P(C', H_n)$ and the result follows.

□

Lemma 10 (Loop)**.** For any forward transition $\langle M, H\rangle \xrightarrow{(t,k)} \langle M', H'\rangle$ there exists a reverse transition $\langle M', H'\rangle \overset{(t,k)}{\rightsquigarrow}_o \langle M, H\rangle$.

Proof. Suppose $\langle M, H\rangle \xrightarrow{(t,k)} \langle M', H'\rangle$. Then t is clearly o-enabled in H'. Furthermore, $\langle M', H'\rangle \overset{(t,k)}{\rightsquigarrow}_o \langle M'', H''\rangle$ where $H'' = H$ by the definition of $\rightsquigarrow_o$. In addition, for all $a_i \in A_I$, we distinguish two cases. If for some $a_i \in M(x)$, $\nexists (a_i, k, u) = a'_i, a'_i \in M'(y)$, then we may see that $a_i \in M'(y)$ and $a_i \in M''(y)$, and the result follows. Otherwise, if $\exists (a_i, k, u) = a'_i$ then $a'_i \in M'(z)$ where $u \in F(t, z)$. Furthermore suppose $a''_i \in M''(w)$. By proposition 21 for $C = \mathsf{con}(a_i, \{\mathsf{con}(b_i, M(z)) | b_i \in A_I, z \in P\} -^{\bullet}\mathsf{pre}(t, U_f) \cup \mathsf{post}^{\bullet}(t, U_f))$ and $C'_1 = \mathsf{con}(a'_i, M'(y))$ we have $a_i \bar{\in} a'_i$ and for $C'_2 = \mathsf{con}(a'_i, \{\mathsf{con}(b'_i, M'(z)) | b'_i \in A_I, z \in P\} - \mathsf{post}^{\bullet}(t, U_o) \cup {}^{\bullet}\mathsf{pre}(t, U_o))$ and $C'' = \mathsf{con}(a''_i, M''(w))$ we have $a''_i \bar{\in} a'_i$. Since $H = H''$ we have $w = \mathsf{last}_P(C'', H'')) = \mathsf{last}_P(C, H)) = y$ and the result follows. □

As in the original RPN model, the opposite direction of the lemma does not hold. The following result establishes that the placement of a connected component is uniquely determined by the last transition to have manipulated it.

Proposition 23. Consider executions $\langle M_0, H_0\rangle \overset{\sigma_1}{\longmapsto}_o \langle M_1, H_1\rangle$ and $\langle M_0, H_0\rangle \overset{\sigma_2}{\longmapsto}_o \langle M_2, H_2\rangle$, and a token $a_i \in A_I, a_i \in M_1(x) \cap M_2(y)$ for some $x, y \in P$. Then, $\mathsf{last}_T(\mathsf{con}(a_i, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a_i, M_2(y)), H_2)$ implies $x = y$.

Proof. Consider executions $\langle M_0, H_0\rangle \overset{\sigma_1}{\longmapsto}_o \langle M_1, H_1\rangle$, $\langle M_0, H_0\rangle \overset{\sigma_2}{\longmapsto}_o \langle M_2, H_2\rangle$ and a token a_i as specified by the lemma. Further, let us assume that $\mathsf{last}_T(\mathsf{con}(a_i, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a_i, M_2(y)), H_2)$. Two cases exist:

- $\mathsf{last}_T(\mathsf{con}(a_i, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a_i, M_2(y)), H_2) = \bot$. This implies that no transition has manipulated any of the tokens and bonds in the two connected components. As such, by Proposition 22, $\mathsf{con}(a_i, M_1(x)) \subseteq M_0(x)$ and $\mathsf{con}(a_i, M_2(y)) \subseteq M_0(y)$, and we conclude that $x = y$ as required.

- $\mathsf{last}_T(\mathsf{con}(a_i, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a_i, M_2(y)), H_2) = (t, k)$. This implies that there exists $b_i \in \mathsf{con}(a_i, M_1(x)) \cap \mathsf{con}(a_i, M_2(y))$ such that $b_i = (b_j, k, u), u \in \mathsf{post}(t)$. By

Proposition 22, $x = \mathsf{last}_P(\mathsf{con}(b_i, M_1(x)), H_1)$, $y = \mathsf{last}_P(\mathsf{con}(b_i, M_2(y)), H_2)$. Since we have that $\mathsf{last}_T(\mathsf{con}(a_i, M_1(x)), H_1) = \mathsf{last}_T(\mathsf{con}(a_i, M_2(y)), H_2)$ we conclude that $\mathsf{last}_P(\mathsf{con}(b_i, M_1(x)), H_1) = \mathsf{last}_P(\mathsf{con}(b_i, M_2(y)), H_2)$, thus, $x = y$ as required. □

As in the original RPN model we confirm the relationship between the enabledness conditions for each of backtracking, causal-order, and out-of-causal-order reversibility.

Proposition 24. Consider a state $\langle M, H\rangle$, and a transition occurrence (t, k). Then, if (t, k) is bt-enabled in $\langle M, H\rangle$ it is also c-enabled. Furthermore, if (t, k) is c-enabled in $\langle M, H\rangle$ then it is also o-enabled.

Proof. The proof is immediate by the respective definitions. □

The following result establishing that during causal-order reversibility a component is returned to the place following the last transition that has manipulated it or, if no such transition exists, in the place where it occurred in the initial marking.

Proposition 25. Given a multi reversing Petri net (P, T, A, A_V, B, F), an initial state $\langle M_0, H_0\rangle$, and an execution $\langle M_0, H_0\rangle \xrightarrow{(t_1,k_1)}_c \langle M_1, H_1\rangle \xrightarrow{(t_2,k_2)}_c \ldots \xrightarrow{(t_n,k_n)}_c \langle M_n, H_n\rangle$. Then for all $a_i \in A_I$, $a_i \in M_n(x)$ where $x = \mathsf{last}_P(\mathsf{con}(a_i, M_n(x)), H_n)$.

Proof. The proof is by induction on n and it follows along similar lines to the proof of Proposition 22. □

We may now verify that the causal-order and out-of-causal-order reversibility have the same effect in MRPNs when reversing a c-enabled transition.

Proposition 26. Consider a state $\langle M, H\rangle$ and a transition occurrence (t, k) c-enabled in $\langle M, H\rangle$. Then, $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M', H'\rangle$ if and only if $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_o \langle M', H'\rangle$.

Proof. Let us suppose that (t, k) is c-enabled and $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M_1, H_1\rangle$. By Proposition 24, (t, k) is also o-enabled. Suppose $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_o \langle M_2, H_2\rangle$. It is easy to see that in fact $H_1 = H_2$ (the two histories are as H with the exception that $H_1(t) = H_2(t) = H(t) - \{k\}$).

To show that $M_1 = M_2$ first we observe that for all $a_i \in A_I$, by Proposition 25 we have $a_i \in M_1(x)$ where $x = \mathsf{last}_P(\mathsf{con}(a_i, M_1(x)), H_1)$ and by Proposition 22 we have $a_i \in M_1(y)$ where $y = \mathsf{last}_P(\mathsf{con}(a_i, M_2(y)), H_2)$. We may also see that $\mathsf{con}(a_i, M_1(x)) = \mathsf{con}(a_i, M_2(y))$. Since in addition we have $H_1 = H_2$ the result follows. □

An equivalent result can be obtained for backtracking.

Proposition 27. Consider a state $\langle M, H\rangle$, and a transition occurrence (t, k), bt-enabled in $\langle M, H\rangle$. Then, $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_b \langle M', H'\rangle$ if and only if $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_o \langle M', H'\rangle$.

Proof. Consider a state $\langle M, H\rangle$ and suppose that transition occurrence (t, k) is bt-enabled and $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_b \langle M', H'\rangle$. Then, by Proposition 24, there exists $k \in H(t)$, such that for all $t' \in T$, $k' \in H(t')$, it holds that $k \geq k'$. This implies that (t, k) is also c-enabled, and by the definition of $\rightsquigarrow_c$, we conclude that $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_c \langle M', H'\rangle$. Furthermore, by Proposition 26 $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_o \langle M', H'\rangle$, and the result follows. □

As in RPNS we obtain the following corollary confirming the "universality" of $\rightsquigarrow_o$.

Corollary 9. $\rightsquigarrow_b \subset \rightsquigarrow_c \subset \rightsquigarrow_o$.

4.4.5 Example of MRPNs under the Individual Token Interpretation

The individual token interpretation can be used to model systems that require an association between the modelled system and its processes. Specifically, in [138], the classical notion of a process is given as a run of the modelled system, obtained by choosing one of the alternatives in case of a conflict. As such, the individual token interpretation is able to represent such processes as token memories. These memories record all occurrences of the transitions and places visited during a run, together with the causal dependencies between them, which are given by the flow relation of the net. Causal semantics of the system are thus obtained by associating with tokens the processes running in a net. According to the individual token interpretation, causal dependencies are a central aspect in the dynamic evolution of a net. In this case, the actual order of execution of concurrent transitions in the net is invisible when reversing, but all the causal dependencies are preserved.

Let us consider the multicasting system of transaction processing in Figure 4.15. In this example we demonstrate a multi reversing Petri net that corresponds to the example in Figure 3.19 of the original RPN model. An agent can simultaneously execute multiple transactions in the same system, thus, several processes are running in parallel. In case one transaction fails whereas the rest of the transactions have been successfully completed, the system should be able to correspond transaction initialisations to failed transactions so that only failed initialisations can be reversed. This can be done by associating each transaction token with its process indicating which transitions the token has traversed and whether one of these transitions represents failure. In this way the individual token interpretation can be used to coordinate and synchronize a system consisting of multiple transactions in case

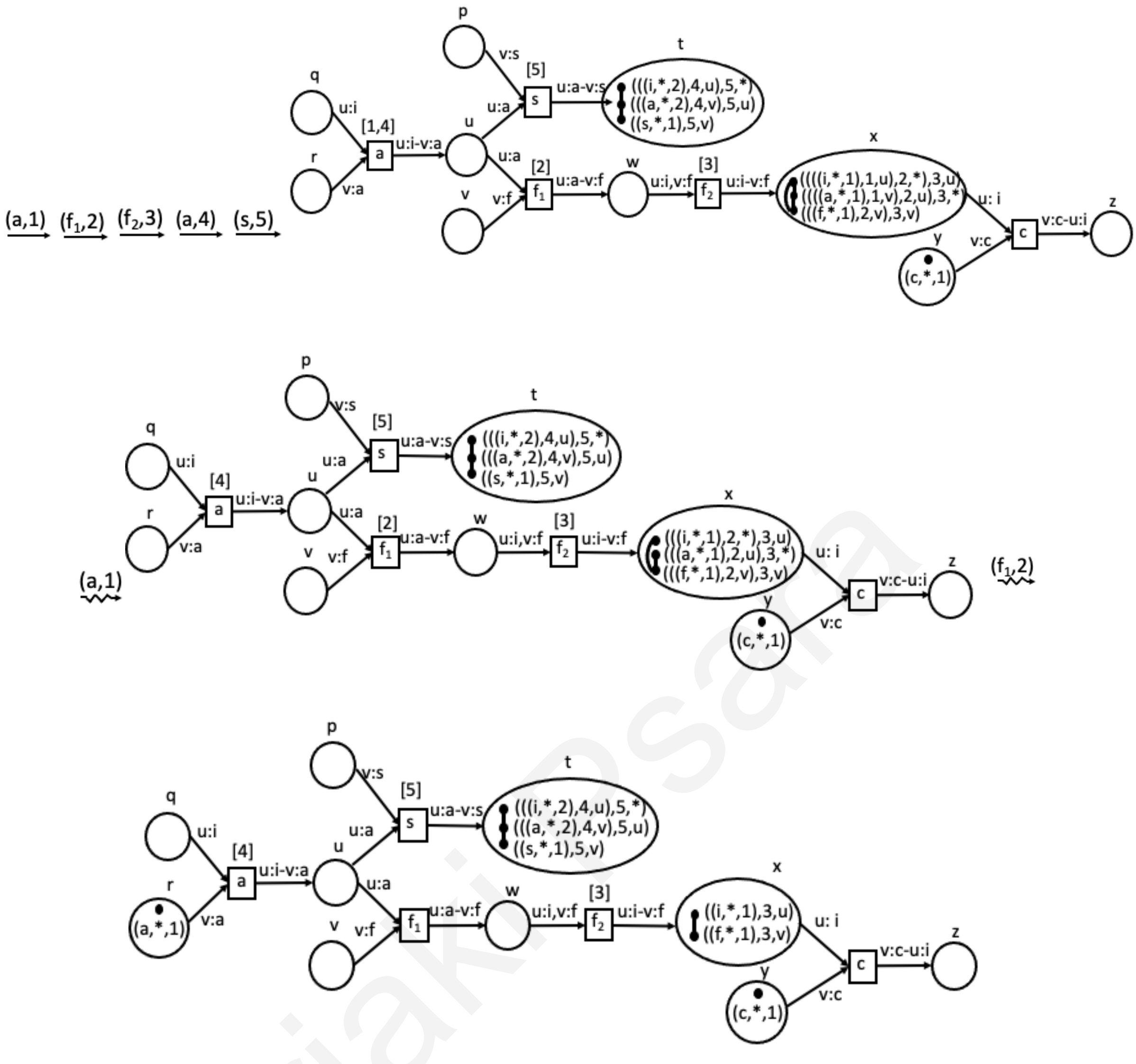


Figure 4.15: Transaction processing with multitokens

out-of-causal reversal is necessary due to failures. Multi reversing Petri nets rely on the memories of tokens as a mechanism which is used to express their behaviours.

Specifically, in this example we have two transaction tokens of type a, $(a,*,1)$ and $(a,*,2)$, which can participate in the same transitions. We randomly select one of the two transaction tokens, token $(a,*,1)$, to be involved in a sequence of failed transitions, whereas token $(a,*,2)$ will be executed successfully. As we have a failed transaction in the model we should reverse in out-of-causal order transition a to be able to proceed with the compensation transition c. In this example the approach of individual token interpretation plays an important role as it is essential to reverse the occurrence of transition a that is associated with the failed transaction rather than the transaction that has been completed successfully. Thus, by observing the memory of the tokens we are able to identify that the failed transaction corresponds to transition occurrence $(a, 1)$ and we may proceed by reversing $(a, 1)$ in order to release the failed transaction token.

4.5 Multi Reversing Petri Nets vs Reversing Petri Nets

In this section we present two translations from reversing Petri nets to Labelled Transition Systems (LTS), one from RPNs with single tokens and one from RPNs with multi tokens. This serves to establish the equivalence between the two models by showing that for every MRPN there is a SRPN which is equivalent in terms of the underlining LTS. Labelled Transition Systems (LTS) are defined as follows:

Definition 58. A labelled transition system is a tuple $(Q, E, \rightarrow, I)$ where:

- Q is a countable set of states,
- E is a countable set of actions,
- $\rightarrow \subseteq Q \times E \times Q$ is the step transition relation, and
- $I \in Q$ is the initial state.

Henceforth, we write $p \xrightarrow{u} q$ for $(p, u, q) \in \rightarrow$.

Here $p \xrightarrow{u} q$ means that the represented system can transition from state p to state q by performing action u.

When used for comparing systems, LTSs are considered modulo a suitable semantic equivalence. For our purposes, we employ the following notion of isomorphism of reachable parts, $\cong_{\mathcal{R}}$:

Definition 59. Two LTSs $A = (Q^A, E^A, \rightarrow, I^A)$ and $B = (Q^B, E^B, \rightarrow, I^B)$ are isomorphic, written $A \cong B$, if they differ only in the names of their states and events, i.e. if there are bijections $\beta : Q^A \rightarrow Q^B$ and $\eta : E^A \rightarrow E^B$ such that $\beta(I^A) = I^B$, and, for $p, q \in Q^A$, $u \in E^A : \beta(p) \xrightarrow{\eta(u)}_B \beta(q)$ iff $p \xrightarrow{u}_A q$.

The set $\mathcal{R}(Q)$ of reachable states in $A = (Q, E, \rightarrow, I)$ is the smallest set such that I is reachable and whenever p is reachable and $p \xrightarrow{u} q$ then q is reachable. We write $A \cong_{\mathcal{R}} B$ if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are isomorphic. To check $A \cong_{\mathcal{R}} B$ it suffices to restrict to subsets of Q^A and Q^B that contain all reachable states, and construct an isomorphism between the resulting LTSs.

We now give the translation from reversing Petri nets with multi and single tokens into labelled transition systems. In what follows we write $\longmapsto_s$ for $\longrightarrow \cup \rightsquigarrow$ where $\rightsquigarrow$ could be any of $\rightsquigarrow_b$, $\rightsquigarrow_c$, and $\rightsquigarrow_o$ with single tokens and $\longmapsto_m$ the equivalent for mutli tokens.

Definition 60. Let $N = (P, T, A, A_v, B, F)$ be a net with multi tokens and initial marking M_0 and $N' = (A', P, B', T', F')$ be a net with single tokens and initial marking M'_0 . Then $\mathcal{H}_m(N, M_0) = (2^{A_I \cup B_I}, (T \times \mathbb{N}), \longmapsto_m, M_0)$ is the LTS associated with N under the multi token interpretation, and $\mathcal{H}_s(N', M'_0) = (2^{A' \cup B'^P}, T', \longmapsto_s, M'_0)$ is the LTS associated with N' under the single token interpretation.

The following theorem says that reversing Petri nets under the single token interpretation are at least as expressive as reversing Petri nets under the multi token interpretation, in the sense that any LTS that can be denoted by a net under the latter interpretation can also be a denoted by a net under the former interpretation.

Theorem 4. For every multi reversing Petri net $N = (P, T, A, A_v, B, F)$ with initial markings M_0 there is a single reversing Petri net $N' = (A', P, B', T', F')$ with initial marking M'_0 such that $\mathcal{H}_s(N', M'_0) \cong_{\mathcal{R}} \mathcal{H}_m(N, M_0)$.

Proof. Let $N = (P, T, A, A_v, B, F)$ be a MRPN with initial marking M_0. We construct a SRPN as $N' = (A', P, B', T', F')$ with initial marking M'_0 as follows:

- $A' = \{a_i | (a, *, i) \in A_I \cap M_0(x), x \in P\}$
- $B' = \{(a_i, b_i) | (a, *, i), (b, *, i) \in A_I \cap M_0(x), x \in P\}$
- $T' = \{t_s | s \in S, S = \{(a_1, ..., a_n) \in (A')^n | type(a_i) = type(v_i), (v_1, ..., v_n) = \mathsf{pre}(t_m) \cap A_V, t_m \in T\}\}$
- $$F'(x,y) = \begin{cases} a_i, & \text{if } x \in P, y \in T', t_m \in T, v \in F(x, t_m) \cap A_V, type(v) = type(a_i), a_i \in A' \\ a_i, & \text{if } x \in T', y \in P, t_m \in T, v \in F(t_m, y) \cap A_V, type(v) = type(a_i), a_i \in A' \\ (a_i, b_i), & \text{if } x \in P, y \in T', t_m \in T, (u, v) \in F(x, t_m), u, v \in A_V, type(u) = \\ & \qquad type(a_i), type(v) = type(b_i), a_i, b_i \in A' \\ (a_i, b_i), & \text{if } x \in T', y \in P, t_m \in T, (u, v) \in F(t_m, y), u, v \in A_V, type(u) = \\ & \qquad type(a_i), type(v) = type(b_i), a_i, b_i \in A' \end{cases}$$
- $$M'_0(x) = \begin{cases} a_i, & \text{if } (a, *, i) \in M_0(x) \cap A_I, x \in P \\ \emptyset, & \text{otherwise} \end{cases}$$

We denote $M_s \in 2^{A' \cup B'}$ and $M_m \in 2^{A_I \cup B_I}$ for any possible marking in each respective RPN type, and $t_s \in T'$, $t_m \in T$ for transitions. For $\mathcal{H}_s(N', M'_0) \cong_{\mathcal{R}} \mathcal{H}_m(N, M_0)$ to hold it must be that $\beta(M_m) = M_s$ and $\eta(t_m, k_m) = t_s$. We define $\beta(M_m) = M_s$ for all $x \in P$ if there exists $a_i, b_i \in A_I \cap M_m(x)$ and $(a_i, b_i) \in B_I \cap M_m(x)$ then there exists $a, b \in A' \cap M_s(x)$ and $(a, b) \in B' \cap M_s(x)$ such that $type(a_i) = type(a)$ and $type(b_i) = type(b)$. We also define $\eta(t_m, k_m) = t_s$ if for all $v \in A_V$ where $v \in F(x, t_m) \cap F(t_m, y)$ for some $x \in \circ t_m$, $y \in t_m \circ$ then there exists $a \in A'$ where $a \in F'(x, t_s) \cap F'(t_s, y)$ such that $type(v) = type(a)$. Similarly for bonds, if $(u, v) \in F(x, t_m)$ for some $x \in \circ t_m$ then there exists$(a, b) \in B'$ where $(a, b) \in F'(x, t_s)$ such that $type(u) = type(a)$ and $type(v) = type(b)$ (respectively for $F(t_m, y)$).

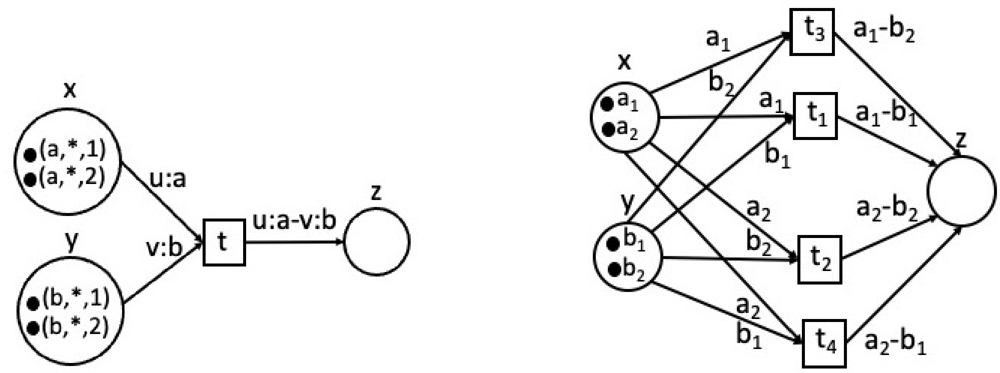
Now for the mapping of transition firings from N' to N we have two cases depending on the form of execution:

- $\langle M_m, H_m \rangle \xrightarrow{(t_m, k_m)} \langle M'_m, H'_m \rangle$ and $\langle M_s, H_s \rangle \xrightarrow{t_s} \langle M'_s, H'_s \rangle$ are both forward executions. During forward execution in both models tokens along with their connected components are transferred from the incoming to the outgoing places breaking or creating bonds according to the specifications on the incoming and outgoing arcs. Let us assume that $\beta(M_m) = M_s$. For $\beta(M'_m) = M'_s$ to hold by the respective definitions of $\longrightarrow$ of each RPN model it must be that $\eta(t_m, k_m) = t_s$. The bijections β and η constitute an isomorphism between the reachable parts of $\mathcal{H}_s(N', M'_0)$ and $\mathcal{H}_m(N, M_0)$, and the result follows.

- $\langle M_m, H_m \rangle \overset{(t_m, k_m)}{\rightsquigarrow} \langle M'_m, H'_m \rangle$ and $\langle M_s, H_s \rangle \overset{t_s}{\rightsquigarrow} \langle M'_s, H'_s \rangle$ are both reverse executions where $\rightsquigarrow$ represents either $\rightsquigarrow_b$, $\rightsquigarrow_c$, or $\rightsquigarrow_o$. In all three cases the transition is reversed by undoing the effect of the transition, i.e. breaking or creating a bond, and transferring the resulting components in the incoming places of the transition, for backtracking and causal order, or to the outgoing place of the last participating transition in out-of-causal order reversibility. We make again the same arguments as in forward execution and by assuming that $\beta(M_m) = M_s$ we show that $\beta(M'_m) = M'_s$ holds by the respective definitions of $\rightsquigarrow_b, \rightsquigarrow_c, \rightsquigarrow_o$ only if $\eta(t_m, k_m) = t_s$. Again the bijections β and η constitute an isomorphism between the reachable parts of $\mathcal{H}_s(N', M'_0)$ and $\mathcal{H}_m(N, M_0)$, and the result follows.

□

Theorem 5. For every single reversing Petri net $N = (A, P, B, T, F)$ with initial marking M_0 there is a multi reversing Petri net $N' = (P', T', A', A_v, B', F')$ with initial marking M'_0 such that $\mathcal{H}_m(N', M'_0) \cong_{\mathcal{R}} \mathcal{H}_s(N, M_0)$.



(a) Reversing Petri net with multi tokens

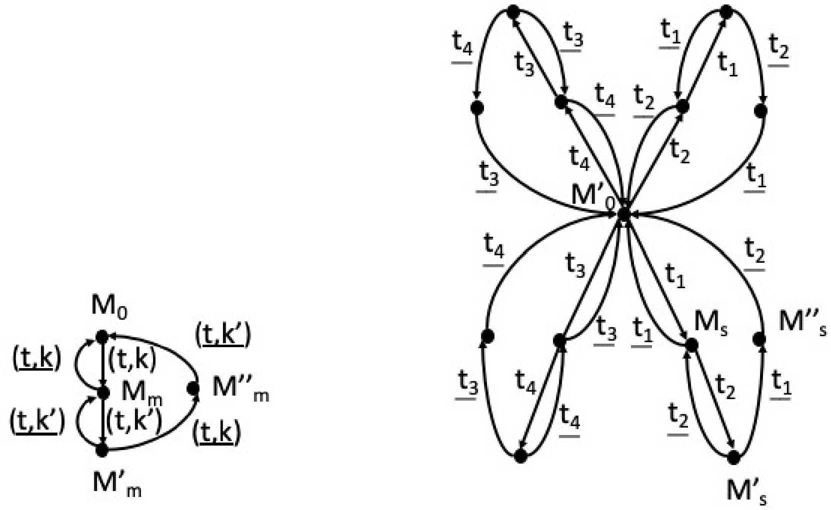
(b) Reversing Petri net with single tokens

Figure 4.16: Equivalent RPNs with multi and single tokens

Proof. The proof follows trivially as SRPNs are a special instance of MRPNs with single tokens. □

In Figure 4.16 we present a MRPN N and its respective SRPN N'. From N we are able to obtain the *SRPN* N' by constructing the unique tokens a_1, a_2, b_1 and b_2 each of them representing one of the tokens $(a, *, 1)$, $(a, *, 2)$, $(b, *, 1)$, and $(b, *, 2)$ respectively. The places are the same in both RPN models. The amount of transitions constructed for the SRPN is dependent on the type of variables required for each MRPN transition and the amount of tokens representing that type. Specifically for each token of type a associated with the variable $v, type(v) = a$ a respective transition is constructed in the SRPN. Thus, in this example two tokens of type a represent the token variable u and two tokens of type b represent the token variable v. As both variables u and v are required for the transition to fire four combinations of tokens of type a and b exist resulting in four different transitions. On that note, the arcs between the places and the constructed transitions follow the token/bond variable specifications in the MRPN expressed by the combinations of tokens in the SRPN.

Let the LTSs in Figure 4.17 capture the complete state space of the respective RPNs in Figure 4.16. The equivalence of N and N' manifests itself as an isomorphism of reachable parts of the associated LTSs. Letters like t and t_1 stand for different events labelled t. In fact the first step of $\mathcal{H}_s(N', M'_0)$ is (M'_0, t_1, M_s) where the first step of $\mathcal{H}_m(N, M_0)$ is $(M_0, (t, k), M_m)$. As we can see $\beta(M_0) = M'_0$ since $type(a_1) = type((a, *, 1))$, $type(a_2) = type((a, *, 2))$, $type(b_1) = type((b, *, 1))$, and $type(b_2) = type((b, *, 2))$. The same can be observed for all reachable markings in both RPNs. We also know that $\eta(t, k) = t_1$ since $type(u) = type(a_1)$ and $type(v) = type(b_1)$. The same equivalence applies between t and the



(a) LTS for reversing Petri net with multi tokens

(b) LTS for reversing Petri net with single tokens

Figure 4.17: Labelled transition systems for the reversing Petri nets in Figure 4.16

rest of the transitions in the SRPN. Therefore, $\mathcal{H}_s(N', M'_0) \cong_{\mathcal{R}} \mathcal{H}_m(N, M_0)$.

4.6 Semantics Under the Collective Token Interpretation

In this section we describe a new form of reversibility based on the collective token interpretation philosophy according to which tokens of the same type are not distinguished. We note that this philosophy is maintained by various application domains e.g. recourse aware systems or systems from biology. This approach is implemented as another firing rule for multi reversing Petri nets. Unlike forms of reversing under the individual token interpretation this approach focuses on the local nature of reversing Petri nets and introduces a new approach on reversing systems where the interest is on the location of tokens rather than the relations between transitions.

To better understand the purpose of the collective token interpretation consider the example in Figure 4.18, originally presented in [137]. This example illustrates the situation where two students want to buy a present for their teacher. In places S_1 and S_2 we are able to find their coins, indicated by tokens C_1, C_2. The actions t_1 and t_2 indicate the contribution of the coins where action t_3 indicates the act of buying the present that only costs one coin. After the contributions are made and the present has been bought one coin remains in place $S_{1,2}$ that needs to be returned. Based on the collective token interpretation this coin can be returned to any of the students since the purchase has been caused by the contribution of both

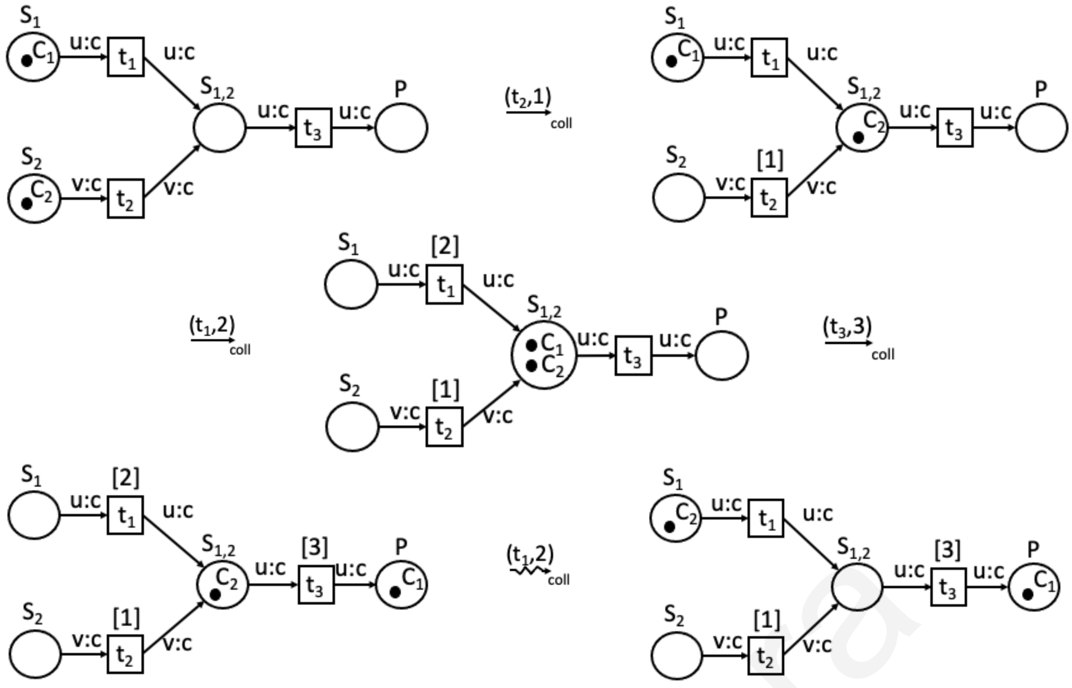


Figure 4.18: Students buying present for their teacher

of them. Whereas in the individual token interpretation we only have one option when going backwards that is predefined during forward execution. Reversal can therefore be identified by keeping track of the student whose coin has been used for the purchase.

In this case the collective token philosophy is a fairer description, in which the buying of the present is caused by a disjunction of the two contributions, whereas the individual philosophy suggests that the present is bought from the contribution of either one student or the other. Thus, we relax the requirement of backward determinism of reversible computation and propose a variation where we can reverse any of the transitions where students contribute their coins and return the coin to a randomly selected student. Therefore, the collective token philosophy gives rise to more subtle causal relationships between transitions that cannot be expressed by partial order. This relation where a transition could be causally dependent on either of two transitions is called disjunctive causality. In this case the system admits only one execution where the disjunctive causality is realised as $t_1 \vee t_2$ causing t_3. Whereas in the individual token interpretation causality is given as a partial order and we have two separate executions, one where t_1 causes t_3 and another execution where t_2 causes t_3.

Under the collective token interpretation, two transitions are considered to be causally independent when the preconditions for the execution of one do not change by the execution of the other. To understand the effect of this let us consider another example in Figure 4.19 which describes the process of creating two water molecules. In this example, transitions t_1

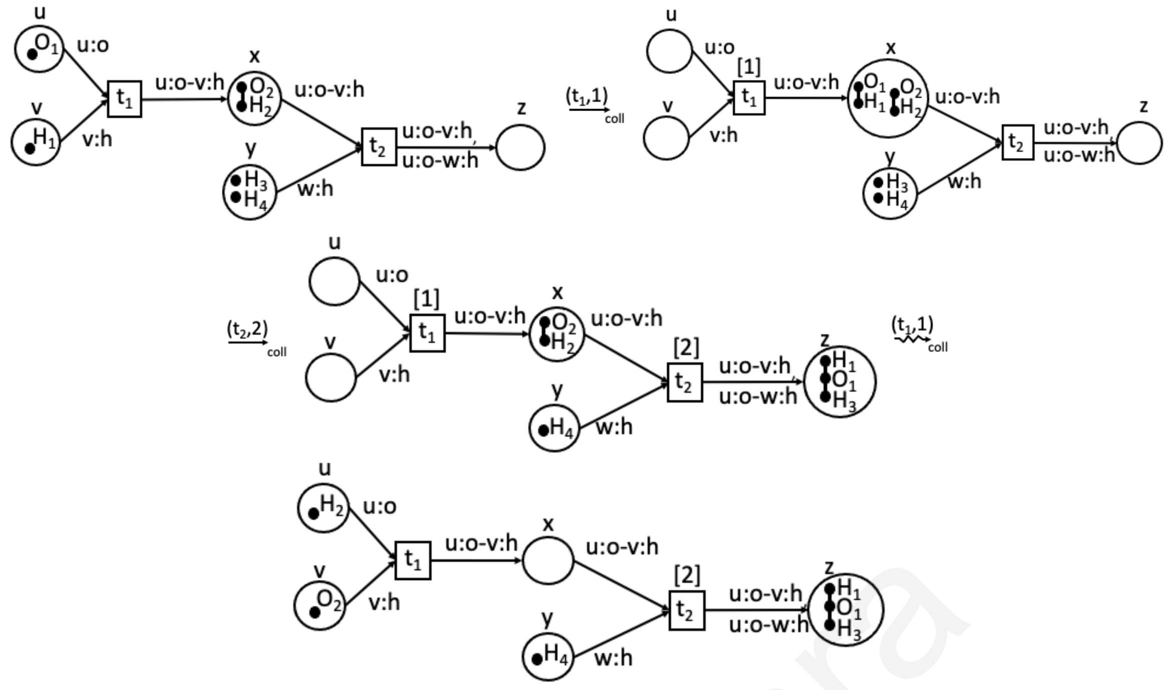


Figure 4.19: Chemical reaction for the creation of water molecules

and t_2 create a water molecule $H-O-H$. Both t_1 and t_2 are enabled in the initial marking since there already exists a hydroxide element O_2-H_2 in y and another one can be created by the execution of t_1. The execution of t_1 will place another hydroxide molecule O_1-H_1 in place y which results in two bonds of an identical type in the same place. According to the collective token interpretation, these transitions are always considered to be concurrent because the execution of the one does not preclude the execution of the other. Since the enabling condition of t_2 does not depend on the execution of t_1 but only on the existence of a token of type $o-h$, then there is no reason in distinguishing whether the molecule was a result of t_1 or not.

This frame of mind when it comes to causal relations also reflects on how we perceive reversibility. Lets us assume that during the execution of t_2 the O_1-H_1 molecule that has been produced by t_1 bonds with H_3 to create the water molecule H_1–O_1–H_3. In the collective token interpretation we are allowed to proceed with the reversal of t_1 since we already have the required o–h tokens in the outgoing place y. When reversing, we undo the effect of t_1 by breaking the bond O_2–H_2 without distinguishing whether this was the pre-existing molecule or the exact one that has been produced during forward execution. However, in the individual token interpretation, we keep track of the tokens that have executed each transition, which means that t_1 cannot be reversed by the pre-existing O_2-H_2 since it can only be reversed after the reversal of t_2.

In the following subsections we present an additional firing rule for multi reversing Petri nets under the collective interpretation. As we are no longer interested in explicitly distinguishing tokens that participate in specific transitions we no longer update the memories of token instances as triples of the form (a, k, u). We assume that for any token of type a there may exist a finite number of *token instances*. We denote initial token $(a, *, i)$ with a_i and we use A_I for the set of all token instances. Tokens are distinguished by their index i in order to avoid introducing multisets to the model. Tokens of the same type have identical capabilities on firing transitions and can participate only in transitions with variables of the same type.

The collective token interpretation is proposed as an additional form of reversibility because, due to its local nature, it allows reversing to states that cannot be reached through forward execution. For this reason it does not follow backtracking, causal, and non-causal semantics because it involves and proposes reversal based on a distinct notion of causality closely related to disjunctive causality (rather than the usual partial-order causality). Tokens of the same type are considered to be indistinguishable and when a transition involving a certain set of tokens is to be reversed, any set of tokens of the needed types can be employed to reverse the transition. As a result, different components may be involved in the forward and backward execution of a transition and reversing a transition may lead to states not reachable by forward-only execution. Furthermore, the approach is light in terms of memory and preserves the local nature of classical Petri nets. We note that an alternative approach could be followed to enable the definition of causal-order and out-causal-order reversibility by considering two tokens to be indistinguishable only if they belong to equivalent connected components. However, we have opted for the present approach, motivated by the applications at hand as well as the philosophy described above.

4.6.1 Forward Execution

We may now redefine the forward firing rule by ignoring the memories of token instances in order to allow tokens to reverse transitions that they have not participated in. Forward enabledness is defined in the same manner as in the individual token interpretation where token instances are selected non-deterministically as long as they respect the variable types required by the transition's incoming arcs. Based on the selected forward enabling assignment U_f we are able to identify the tokens that are removed from places as defined by ${}^{\bullet}U_f$ and the tokens that are added to places as defined by $U_f^{\bullet}$. Thus a transition firing executed in the forward direction will transfer a set of token and bond instances, as specified by the incoming

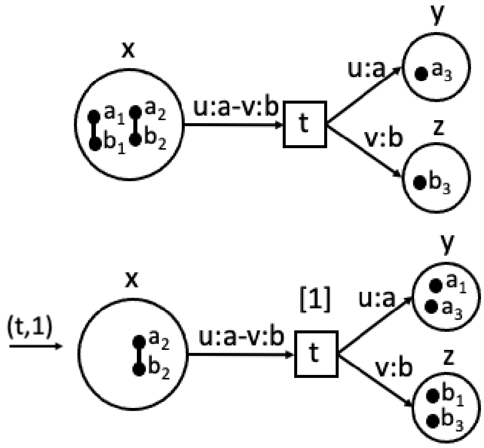


Figure 4.20: Forward execution under the collective token interpretation

arcs of the transition, to the outgoing places of the transition, as specified by the transition's outgoing arcs, possibly forming or destructing bonds, as necessary. In the collective token interpretation token instances are relocated without being updated with memories of their past transitions. Furthermore, the history of the executed transition is updated accordingly.

Definition 61. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition t that is enabled in state $\langle M, H\rangle$, and an enabling assignment U_f, we write $\langle M, H\rangle \xrightarrow{(t,k)}_{coll} \langle M', H'\rangle$ where $k = max(\{0\} \cup \{k' | k' \in H(t''), t'' \in T\}) + 1$ and for all $x \in P$:

$$M'(x) = (M(x) -^{\bullet} U_f(x)) \cup U_f^{\bullet}(x)$$

$$\text{and} \quad H'(t') = \begin{cases} H(t') \cup \{k\} & \text{if } t' = t \\ H(t'), & \text{otherwise} \end{cases}$$

As demonstrated in Figure 4.20 a bond of type a–b is selected from the incoming places of the transition as required by the variables labelled on the incoming arcs. The result of firing the transition is transferring the tokens into their respective outgoing places by breaking the bond between them. Unlike the individual token interpretation we do not update the memories of the token instances as we are only interested in the location of tokens, i.e. the effect of the transitions, as well as, the fact that a transition has been executed.

4.6.2 Reversing Execution

We now move on to reversing transitions. A transition can be reversed in a certain state if it has been previously executed and there exist token instances in its output places that match the requirements on its outgoing arcs. Note that compared to the individual token interpretation, in the collective approach we ignore the causal paths assigned to the tokens during forward execution. As such, tokens are allowed to reverse any transition as long as they respect the variable types, independently on whether the tokens were explicitly used for firing this particular transition occurrence. Specifically, we define the notion of collective reverse enabledness as follows:

Definition 62. Consider a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, and a transition occurrence (t, k). We say that (t, k) is *coll-enabled* in $\langle M, H\rangle$ if there exists an injective function $U_{coll} : \mathsf{post}(t) \cap A_V \rightarrow A_I$ such that:

1. for all $u \in F(t, x)$, $x \in t\circ$, we have $U_{coll}(u) \in M(x)$ where $type(u) = type(U_{coll}(u))$, and for all $(u, v) \in F(t, x)$, then $(U_{coll}(u), U_{coll}(v)) \in M(x)$,
2. If $u, v \in F(t, x), x \in t\circ$ and $(U_{coll}(u), U_{coll}(v)) \in M(x)$ then $(u, v) \in F(t, x)$, and
3. if $u \in F(y_1, t), v \in F(y_2, t), y_1, y_2 \in \circ t, y_1 \neq y_2$ then $U_f(u) \notin \mathsf{con}(\ U_f(v), (M(x) - \mathsf{post}^\bullet(t, U_{coll})) \cup {}^\bullet\mathsf{pre}(t, U_{coll})), x \in \circ t$.

Thus, a transition occurrence (t, k) is reverse-enabled based on the collective token interpretation in $\langle M, H\rangle$ if (1) there exists a type-respecting assignment of token instances, from the instances in the out-places of the transition, to the variables on the outgoing edges of the transition, and where the instances are connected with bonds as required by the transition's outgoing edges. Furthermore, (2) if the selected token instances are bonded together in an outgoing place of the transition then the bond should also exist on the variables labelling the outgoing arcs (thus we do not recreate existing bonds), and (3) if two tokens are transferred by a transition to different incoming places then these tokens should not remain connected when removing the selected outgoing tokens and adding the selected incoming tokens (we do not clone tokens). We refer to U_{coll} as a reversal enabling assignment.

We now define the incoming token/bond instances as:

Definition 63. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H\rangle$, a transition occurrence (t, k) and an enabling assignment U_{coll}, we define ${}^\bullet U_{coll} : P \rightarrow 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of incoming token and bond instances that are used for the firing of t:

${}^\bullet U_{coll}(x) = \bigcup_{u \in f(t,x)} \mathsf{con}(U_{coll}(u), M(x))$

We now define the outgoing token/bond instances as:

Definition 64. Given a MRPN (P, T, A, A_V, B, F), a transition t, and a state $\langle M, H \rangle$ and an enabling assignment U_{coll}, we define $U^{\bullet}_{coll} : P \to 2^{A_I \cup B_I}$ to be a function that assigns to each place a set of outgoing token/bond instances of t:

$$U^{\bullet}_{coll}(x) = \bigcup_{u \in f(x,t), U_{coll}(u) \in M(y)} \mathsf{con}(U_{coll}(u), (M(y) - \mathsf{post}^{\bullet}(t, U_{coll})) \cup {}^{\bullet}\mathsf{pre}(t, U_{coll})$$

To implement the reversal of a transition t according to a reversal enabling assignment U_{coll}, the selected instances are relocated from the outgoing places of the transition to the incoming places, as specified by the incoming arcs of the transition, with bonds created and destructed accordingly. Note that compared to the individual token interpretation in the collective approach we do not update the causal paths assigned to the tokens during forward execution.

Definition 65. Given a MRPN (P, T, A, A_V, B, F), a state $\langle M, H \rangle$, and a transition occurrence (t, k) that is reverse-enabled in $\langle M, H \rangle$ with U_{coll} a reversal enabling assignment, we write $\langle M, H \rangle \overset{(t,k)}{\rightsquigarrow}_{coll} \langle M', H' \rangle$ where for all x:

$$M'(x) = (M(x) -^{\bullet} U_{coll}(x)) \cup U^{\bullet}_{coll}(x)$$

$$\text{and} \qquad H'(t') = \begin{cases} H(t') - \{k\}, & \text{if } t' = t \\ H(t'), & \text{otherwise} \end{cases}$$

In Figure 4.21 we may observe the reverse execution of the net presented in Figure 4.20. As two tokens of type a and b are located in the outgoing places of the transition matching the requirements of the variables in the labelled outgoing arcs, we are able to non-deterministically select a pair to reverse transition t. As such it is not necessary to reverse the transition with the exact token instances that have contributed in its execution thus we select $a_3 - b_3$ to reverse.

4.6.3 Case Study for the Collective Token Interpretation

Biological reactions, pathways, and reaction networks have been extensively studied in the literature using various techniques, including process calculi and Petri nets. Initial research was mainly focused on reaction rates by modelling and simulating networks of reactions, in order to analyse or predict the common paths through the network. Reversibility was not considered explicitly. Later on reversibility started to be taken into account, since it plays a crucial role in many processes, typically by going back to a previous state in the system.

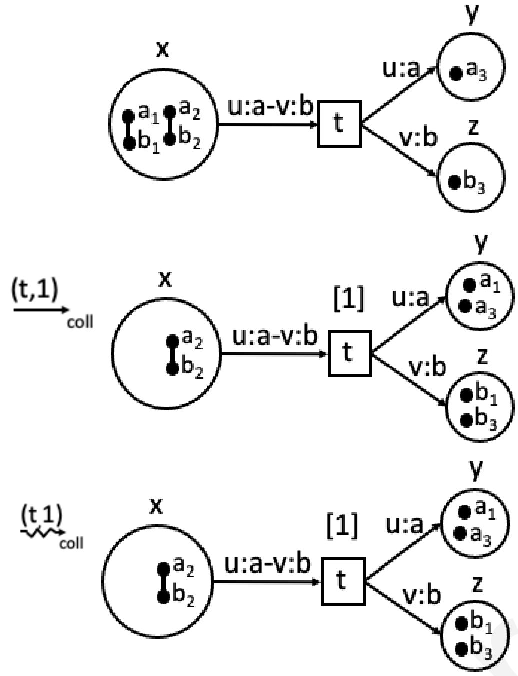


Figure 4.21: Forward execution under the collective token interpretation.

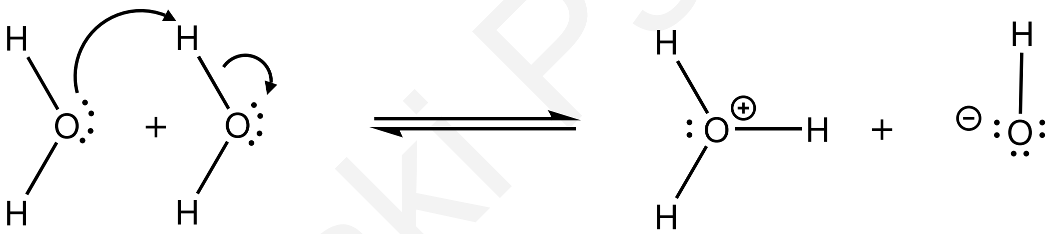


Figure 4.22: Autoprotolysis of water

Autoprotolysis of Water

In this case study we consider a chemical reaction that transfers a hydrogen atom between two water molecules. This reaction is known as the *autoprotolysis of water* and is shown in Figure 4.22. There, *O* indicates an oxygen atom and *H* a hydrogen atom. The lines indicate bonds. Positive and negative charges on atoms are shown by ⊕ and ⊖ respectively. The meaning of the curved arrows and the dots will be explained in the next paragraphs. The reaction is reversible and it takes place at a relatively low rate, making pure water slightly conductive. We have chosen this reaction as our example reaction, since it is non-trivial but manageable, and has some interesting aspects to be represented.

To model the reaction we need to understand why it takes place and what causes it. The main reason is that the oxygen in the water molecule is *nucleophilic*, meaning it has the tendency to bond to another atomic nucleus, which would serve as an *electrophile*. This is because oxygen has a high electro-negativity, therefore it attracts electrons and has an abundance of electrons around it. The electrons around the atomic nucleus are arranged on electron shells, where only those in the outer shell participate in bonding. Oxygen has four electrons in its outer shell, which are not involved in the initial bonding with hydrogen atoms. These electrons form two *lone pairs* of two electrons each, which can form new bonds (lone pairs are shown in Figure 4.22 by pairs of dots). All this makes oxygen nucleophilic: it tends to connect to other atomic nuclei by forming bonds from its lone pairs. Since oxygen attracts electrons, the hydrogen atoms in water have a positive partial charge and oxygen has a negative partial charge.

The reaction starts when an oxygen in one water molecule is attracted by a hydrogen in another water molecule due to their opposite charges. This results in a *hydrogen bond*. This bond is formed out of the electrons of one of the lone pairs of the oxygen. The large curved arrow in Figure 4.22 indicates the movements of the electrons. Since a hydrogen atom cannot have more than one bond, the creation of a new bond is compensated by breaking the existing hydrogen-oxygen bond (indicated by the small curved arrow). When this happens, the two electrons, which formed the original hydrogen-oxygen bond, remain with the oxygen. Since a hydrogen contains one electron and one proton, it is only the proton that is transferred, so the process can be called a proton transfer as well as a hydrogen transfer. The forming of the new bond and the breaking of the old bond are *concerted*, meaning that they happen together without a stable intermediate configuration. As a result we have reached the state where one oxygen atom has three bonds to hydrogen atoms and is positively charged, represented on the right side of the reaction in Figure 4.22. This molecule is called *hydronium* and is written as H_3O^+. The other oxygen atom bonds to only one hydrogen and is negatively charged, having an electron in surplus. This molecule is called a *hydroxide* and is written as OH^-.

Note that the reaction is reversible: the oxygen that lost a hydrogen can pull back one of the hydrogens from the other molecule, the H_3O^+ molecule. This is the case since the negatively charged oxygen is a strong nucleophile and the hydrogens in the H_3O^+ molecule are all positively charged. Thus, any of the hydrogens can be removed, making both oxygens formally uncharged, and restoring the two water molecules. In Figure 4.22 the curved arrows are given for the reaction going from left to right. Since the reaction is reversible (indicated by the double arrow) there are corresponding electron movements when going from right to left. These are not given in line with usual conventions, but can be inferred.

In this simple reaction, the forward and the reverse step consist of two steps each. The breaking of the old and the forming of the new bond occur simultaneously. This means that there is no strict causality of actions, since none of them can be called the cause of the overall reaction. Furthermore, the reverse step can be done with a different atom to the one used during the forward step because each of the molecules are in a sense identical and in practice there does not exist a single "reverse" path corresponding to a forward one.

It should be noted that there are two types of bonding modelled here. Firstly, we have the initial bonds where two atoms contribute an electron each. Secondly, the *dative* or *coordinate bonds* are formed where both electrons come from one atom (an oxygen in this case). Both are *covalent bonds*, and once formed they cannot be distinguished. Specifically, in the oxygen with three bonds all bonds are the same and no distinction can be made. If one of the bonds is broken by a deprotonation (as in the autoprotolysis of water) the two electrons are left behind and they form a lone pair. If the broken bond was not previously formed as a dative bond, the electrons changed their "role". This explains why any proton can be transferred in the reverse reaction and not just the one that was involved in the forward path.

Reversing Petri Net representation

Figure 4.23 shows the graphical representation of the forming of a water molecule as a RPN. In this model, we assume two token types, h for hydrogen and o for oxygen. They are instantiated via four token instances of h (H_1, H_2, H_3, and H_4) and two token instances of o, (O_1 and O_2). The net consists of five places and three transitions and the edges between them are associated with token variables and bonds, where we assume that $type(u) = type(q) = o$ and $type(v) = type(w) = type(r) = type(s) = h$. Looking at the transitions, transition t_1 models the formation of a bond between a hydrogen token and an oxygen token. Precisely, the transition stipulates a selection of two such molecules with the use of variables u and v on the incoming arcs of the transition which are bonded together, as described in the outgoing arc of the transition. Subsequently, transition t_2 completes the formation of a water molecule by selecting an oxygen token from place x and a hydrogen token from place v and forming a bond between them, placing the resulting component at place y. Note that the selected oxygen instance in this transition will be connected to a hydrogen token via a bond created by transition t_1; this bond is preserved and the component resulting from the creation of the new $o-h$ bond will be transferred to place y. Finally, transition t_3 models the autoprotolysis reaction: assuming the existence of two distinct oxygen instances, as required by the variables of type o and h on the incoming arc of the transition, connected with hydrogen instances as specified in $F(y, t_3)$, the transition breaks the bond $q-r$ and forms the bond $u-r$.

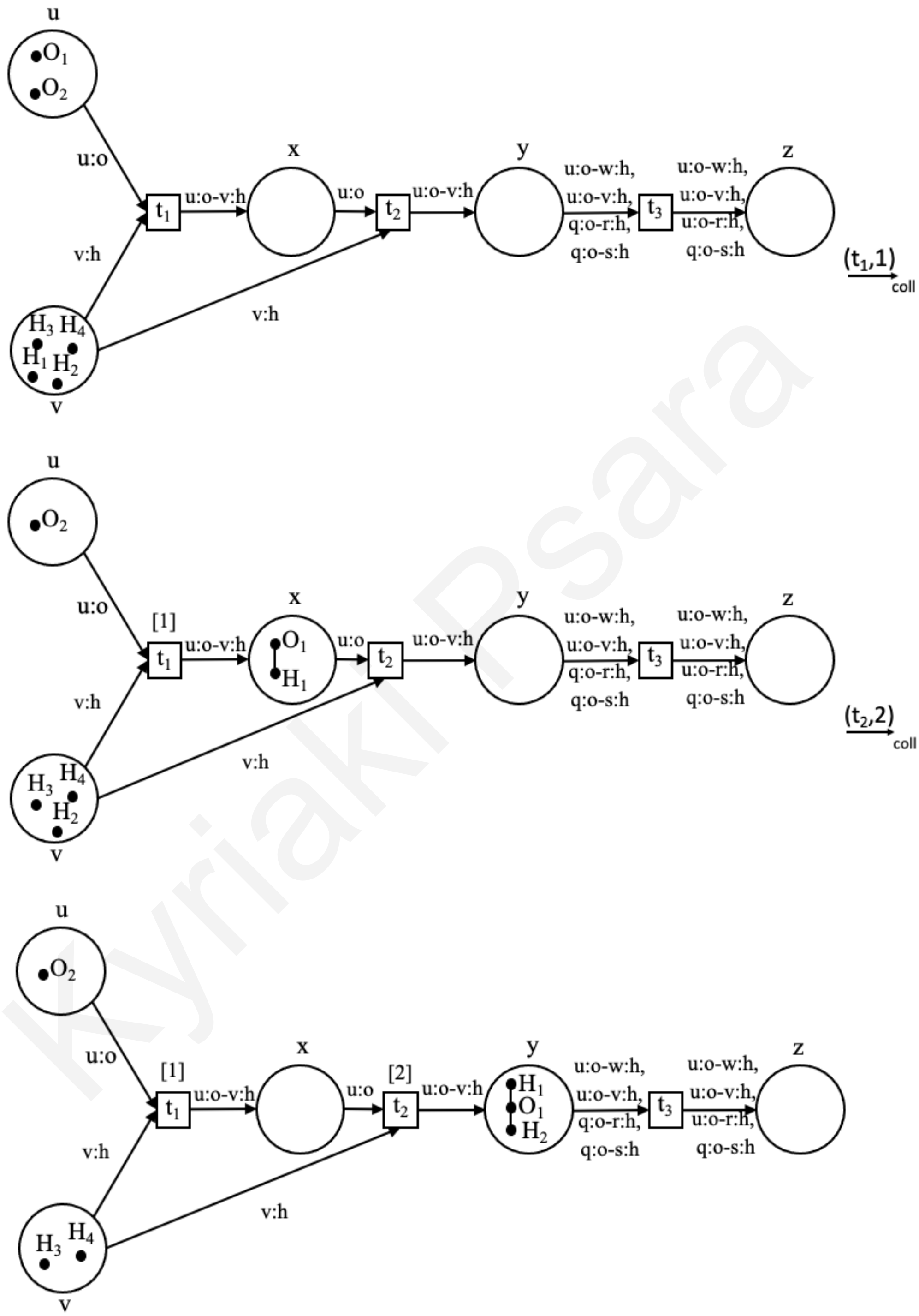


Figure 4.23: RPN model of the formation of a water molecule

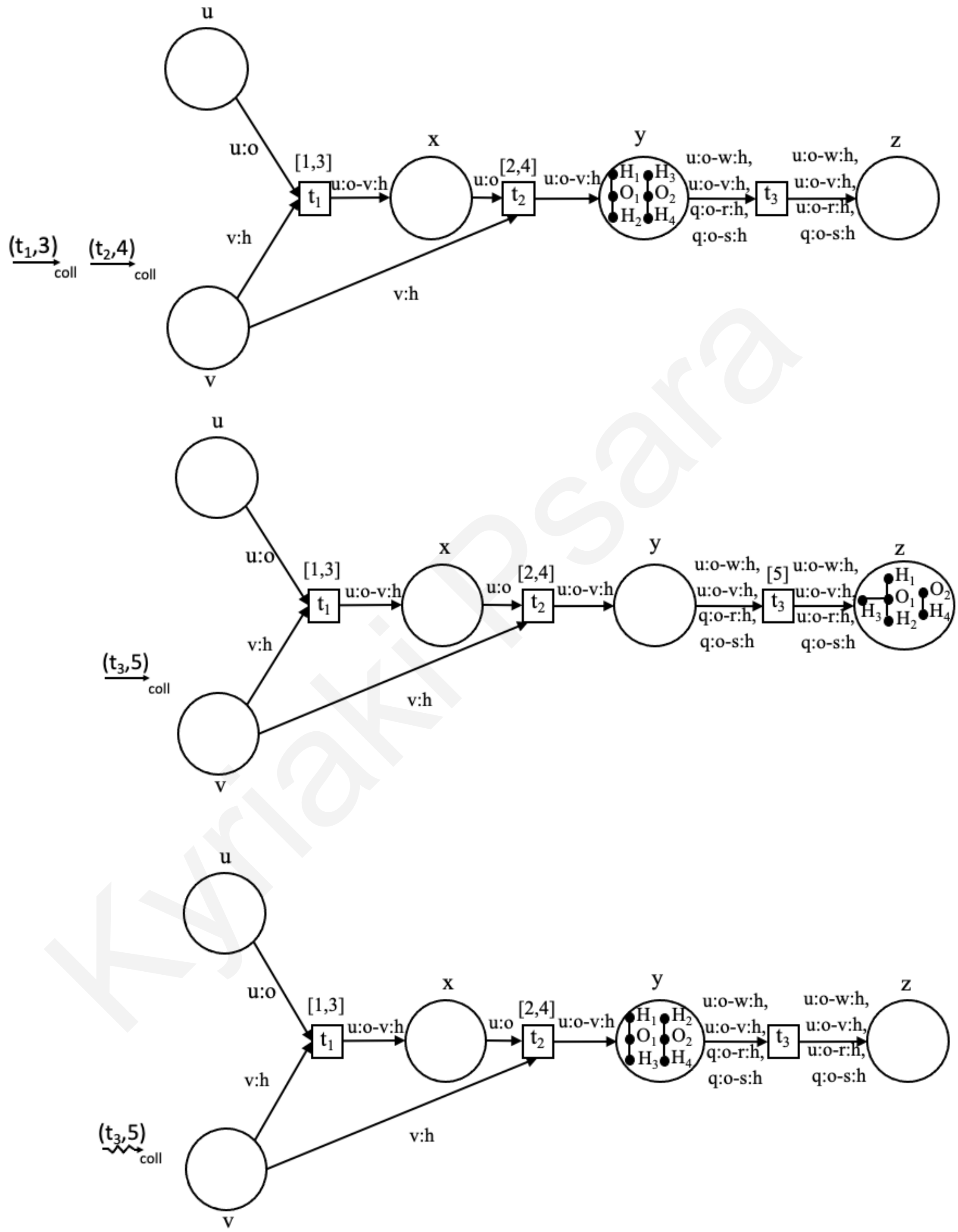


Figure 4.24: RPN model of the execution of the autoprotolysis of water

As such, assuming the existence of two water molecules at place y, the transition will form a hydronium (H_3^+O) and a hydroxin (OH^-) molecule in place z of the net. The reversibility semantics of RPNs ensures that reversing the transition t_3 will result in the re-creation of two water molecules placed at y, while the use of variables allows the formation of water molecules consisting of different bonds between the hydrogen and oxygen instances.

The first net in Figure 4.24 shows the system after the execution of transition t_1 with enabling assignment $U_f(v) = H_1$, $U_f(u) = O_1$. Subsequently, we have the model after execution of transition t_2 with enabling assignment $U_f(v) = H_2$, $U_f(u) = O_1$, creating the bond $O_1 - H_2$, thus forming the first water molecule. A second execution of transitions t_1 and t_2 results in the second molecule of water in the system, placed again at place y, as shown in the third net in the figure. At this state, transition t_3 is forward-enabled and, with enabling assignment $U_f(u) = O_1, U_f(q) = O_2, U_f(w) = H_1, U_f(v) = H_2, U_f(r) = H_3, U_f(s) = H_4$, we have the creation of the hydronium and hydroxide depicted at place z in the fourth net of the figure. At this stage, transition t_3 is now reverse-enabled and the last net in the figure illustrates the state resulting after reversing t_3 with reversal enabling assignment $U_{coll}(u) = O_1, U_{coll}(v) = O_2, U_{coll}(w) = H_1, U_{coll}(v) = H_2, U_{coll}(r) = H_3, U_{coll}(s) = H_4$.

4.7 Concluding Remarks

This chapter has focused on relaxing the restrictions of the RPN model presented in the previous chapter, by allowing multiple tokens of the same base/type to occur in a model and developing reversible semantics in the presence of bond destruction. We have extended our formalism with multi tokens by following the individual token philosophy, which defines the notion of causality in reversible systems as a partial order. The individuality of identical tokens can be imposed by their causal path, which allows identical tokens to fire the same transition when going forward, however, when going backwards tokens will be able to reverse only the transitions that they have fired. Additionally, our work provides the reversible semantics for out-of-causal-order reversibility and shows how the presence of bond destruction affects transition enabledness. Finally, we have shown that the expressive power of RPNs with multi tokens is equivalent to the expressive power of RPNs with single tokens.

Another approach on extending our formalism with multiple tokens is that of the collective token philosophy. Our experience strongly suggests that resource management systems can be studied and understood in terms of the collective token interpretation of RPNs [69]. Reversing Petri Nets are a natural choice to model and analyse biochemical reaction systems, such as the autoprotolysis of water, which by nature has multi-party interactions, is inherently concurrent, and features reversible behaviour.

The autoprotolysis of water has also been modelled by the Calculus of Covalent Bonding (CCB) as well as the Bonding Calculus [69]. All three models can perform the forward reaction using any of the hydrogens involved. In RPNs the feature of token multiplicity and the use of variables allows to non-deterministically select different combinations of atoms of a particular element when creating molecules. Unlike Bonding calculus, CCB and RPNs are able to express concerted actions, since a transition simultaneously destroys a water molecule and creates a hydronium whose reversal results in the opposite effect. CCB and RPNs can perform the reverse reaction by transferring arbitrary hydrogens, whereas the Bonding Calculus permits only the transfer of exactly those hydrogens that were used in the forward reaction.

The other criterion for comparing the formalisms for the modelling of chemical reactions is to ask if they enable the same actions as they appear in reality. Each of the three formalisms does not permit a H_3O molecule to be formed directly. Furthermore, CCB and the bonding calculus allow one reaction which is not realistic: If there are many water molecules and therefore several hydroxide and water molecules at the same time, it is possible that the remaining hydrogen is transferred from the hydroxide to a water. In reality, this is not possible since the hydroxide is strongly negatively charged and no hydrogen bond can form. However, this is not the case for RPNs since, on the one hand, a transition's conditions make restrictions on the types of molecules that will participate in a transition firing or its reversal and, on the other hand, places impose a form of locality for molecules. For instance, in the autoprotolysis example, each place is the location of specific types of molecules, e.g., transition t_3 modelling the autoprotolysis reaction is only applied on water molecules and its reversal only on pairs of a hydronium and a hydroxide molecule, as required.

Chapter 5

Controlling Reversibility in Reversing Petri Nets

In this chapter we extend the framework of reversing Petri nets with a mechanism for controlling reversibility [113, 114, 127]. This control is enforced with the aid of conditions associated with transitions, whose satisfaction acts as a guard for executing the transition in the forward/backward direction. The conditions are enunciated within a simple logical language expressing properties relating to available tokens. The mechanism may capture environmental conditions, e.g., changes in temperature, or the presence of faults. Note that conditional transitions can also be found in existing Petri net models, e.g., in [64], a Petri-net model that associates transitions and arcs with expressions. The resulting model is general enough to capture a wide range of systems, in this context we give an overview of several properties of reversing Petri nets that could be used to analyse the behaviour of these systems. We conclude this section with the model of a novel antenna selection (AS) algorithm which inspired the development of our framework.

5.1 Controlled Reversing Petri Nets

In this section we extend the multi reversing Petri nets of Chapter 4, by associating transitions with conditions that control their execution and reversal, enunciated on data values associated with tokens. We introduce controlled reversible semantics under the collective token interpretation where transitions are controlled and can break bonds, and tokens are indistinguishable and can curry data values. Specifically, we define:

Definition 66. A *Controlled Reversing Petri Net* (CRPN) is a tuple $(N, \Sigma, D, C_F, C_R, I)$ where:

1. N is a multi reversing Petri net.

2. Σ forms a finite set of data types with D the associated set of data values where $type_\Sigma(d) \in \Sigma, d \in D$.

3. $C_F : T \rightarrow COND_{A_V}$ is a function that assigns a forward condition to each transition $t \in T$.

4. $C_R : T \rightarrow COND_{A_V}$ is a function that assigns a reverse condition to each transition $t \in T$.

5. $I : A_I \rightarrow D$ is a function that associates a data value from D to each token instance $a_i \in A_I$ such that $type_\Sigma(I(a_i)) = type_\Sigma(a_i)$.

As in multi reversing Petri nets a controlled reversing Petri net is built on the basis of a set of *tokens* or *bases*. These are organized in a set of token types A, where each token type is associated with a set of token instances A_I. Variable tokens are associated with a data type such that for all $u \in A_V$, $type_\Sigma(u) \in \Sigma$. Places and transitions have the standard meaning and are connected via directed arcs which are labelled by a set of elements from $A_V \cup (A_V \times A_V)$. Finally, we define $C \subseteq A_I \cup B_I$ in the expected way according to MRPNs. We also assume that the CRPN model is well formed with distinct tokens.

In addidtion, in CRPNS token instances are associated with data values via function I. These data values have a type from the set Σ, and we write $type_\Sigma(d)$ to denote the type of a data value. Transitions are associated with conditions $COND_{A_V}$ which constitute additional preconditions for a transition to fire. Conditions are boolean expressions over a set of variables A_V that evaluate to either "TRUE" or "FALSE" determining the behaviour of the net. The function C_F assigns a forward condition to each transition that needs to be satisfied during forward execution, whereas, C_R assigns a reverse condition that needs to be satisfied during reverse execution. Graphically we indicate conditions below their respective transitions as $C_F(t)/C_R(t)$ and in case where $C_F(t) =!C_R(t)$ only $C_F(t)$ is presented.

Conditions are built via a simple propositional language whose basic building blocks are relations on the data types Σ applied on expressions involving token values of a CRPN model. An instantiation of such a language for arithmetic expressions follows, though this can be generalised for more complex types. Therefore, the grammar of the expression $COND_{A_V}$ is defined as follows:

$\phi := \;\; \neg\phi \;\mid\; \phi_1 \vee \phi_2 \;\mid e_1 > e_2$

$e := \;\; a_i.x \;\mid\; u \;\mid\; d \;\mid\; (e) \;\mid\; if \;\; \phi \;\; e_1 \;\; else \;\; e_2 \;\mid\; e_1 + e_2 \;\mid\; e_1 - e_2 \;\mid\; e_1 \times e_2 \;\mid\; e_1/e_2$

The free variables in a condition ϕ are denoted by $Free(\phi) \subseteq A_V$. Variable assignments $V : A_V \cap Free(\phi) \rightarrow A_I$ are the mappings of a token instance to a free variable. We require the variable assignments to respect the types of data values associated with the respective variable token instances such that for $V(u) = a_i$, $a_i \in A_I \cap M(x)$, $x \in P$ we have $type_\Sigma(u) = type_\Sigma(a_i)$. The variable assignment V of a transition covers (at least) all variables from $\mathsf{pre}(t)$ such that $u \in \mathsf{pre}(t) \cap Free(\phi)) = \emptyset$ ($\mathsf{post}(t) \cap Free(\phi) = \emptyset$ for reverse transitions).

Conditions are evaluated based on data values associated with the token instances of the model and functions/predicates over the associated data types. Given a transition t with condition ϕ, the corresponding variable assignment V, a marking M, and an assignment function I, we evaluate the condition ϕ as follows:

$$E(\phi, V, M, I) = \begin{cases} \neg E(\phi', V, M, I), & \text{if } \phi = \neg\phi' \\ E(\phi_1, V, M, I) \ \vee \ E(\phi_2, V, M, I), & \text{if } \phi = \phi_1 \ \vee \ \phi_2 \\ Eval(e_1, V, M, I) > Eval(e_2, V, M, I) & \text{if } \phi = e_1 \ > \ e_2 \end{cases}$$

$$Eval(e, V, M, I) = \begin{cases} Eval(e_1, V, M, I), & \text{if } e = if\ \phi\ then\ e_1\ else\ e_2,\ E(\phi, V, M, I) = T \\ Eval(e_2, V, M, I), & \text{if } e = if\ \phi\ then\ e_1\ else\ e_2,\ E(\phi, V, M, I) = F \\ Eval(e_1, V, M, I) \ \diamond \ Eval(e_2, V, M, I), & \text{if } e = e_1 \ \diamond \ e_2, \diamond \in \{+, -, \times, /\} \\ I(V(u)), & \text{if } e = u, u \in A_V \\ I(a_i), & \text{if } e = a_i.x, a_i \in M(x) \\ 0, & \text{if } e = a_i.x, a_i \notin M(x) \\ d, & \text{if } e = d, d \in D \end{cases}$$

The function $E : (COND_{A_V} \times A_I \times (2^{A_I \cup B_I}) \times D) \rightarrow BOOL$ evaluates the condition of a transition into a boolean value and the function $Eval : (COND_{A_V} \times A_I \times (2^{A_I \cup B_I}) \times D) \rightarrow D$ evaluates the data value of an arithmetic expression. The truth value of conditions depends on the interaction of the logical operators and their component conditions. Arithmetic conditions have the value "TRUE" if the relation exists between the two expressions and "FALSE" otherwise. We resolve nested conditions with recursive evaluation where variable assignments $V(v) = a_i$ are substituted by the data value of the selected token instance $I(a_i)$. Elements of the form $a_i.x$ are substituted by the data value of the token instance $I(a_i)$ if the token instance exists in place x, and 0 if not (0 is the identity element dependent on the application).

5.1.1 Controlled Forward Execution

A transition is forward-enabled in a MRPNs, if there exists a selection of token instances available at the incoming places of the transition matching the requirements on the transitions incoming arcs. Also the transition should not recreate bonds and duplicate tokens. The addition of conditions in CRPNs requires additionally for the forward condition of the transition to evaluate to TRUE according to the variable assignment V_f. Formally:

Definition 67. Given a CRPN $(N, \Sigma, D, C_F, C_R, I)$, a state $\langle M, H\rangle$, and a transition t, we say that t is *controlled-forward-enabled* in $\langle M, H\rangle$ if there exist two injective functions U_f : $\mathsf{pre}(t) \cap A_V \rightarrow A_I$ and $V_f : A_V \cap Free(COND_{A_V}) \rightarrow A_I$ such that:

1. transition t is forward-enabled in N (Definition 33),
2. for all $u \in Free(C_F(t))$ then $V_f(u) = a_i$, $a_i \in A_I \cap M(x)$, $x \in P$ such that $type_\Sigma(u) = type_\Sigma(a_i)$,
3. if $u \in F(x, t)$, $x \in P$, and $u \in Free(C_F(t))$ then $V_f(u) = U_f(u)$, and
4. $E(C_F(t), V_f, M, I)$= TRUE.

Thus, t is enabled in state $\langle M, H\rangle$ if (1) it is also forward-enabled in MRPNs, (2) there is a type respecting assignment of data instances to variables, (3) variables that appear on both the arcs and the condition should have the same variable assignment, (4) if the transition bears a forward condition $C_F(t)$, then by substituting the variables with the selected data values $E(C_F(t), V_f, M, I)$ evaluates to TRUE. Note that different selections of token and bond instances may yield different evaluations to a transition's condition. Thus, for some selections the transition may be enabled whereas for others not. Note that if $E(C_F(t), V_f, M, I)$ or $C_F(t)$ is not defined, then the validity of the input is only dependent on the first two conditions. We refer to V_f as a variable assignment.

As in MRPNs, when a transition t is executed in the forward direction, all tokens and bonds occurring in its outgoing arcs are relocated from the input to the output places along with their connected components. The history of t is extended accordingly:

Definition 68. Given a CRPN $(N, \Sigma, D, C_F, C_R, I)$, a state $\langle M, H\rangle$, and a transition t controlled-forward enabled in $\langle M, H\rangle$ with U_f an enabling assignment and V_f a variable assignment, we write $\langle M, H\rangle \xrightarrow{(t,k)}_{coll} \langle M', H'\rangle$ where M' and H' are updated as in N (Definition 61).

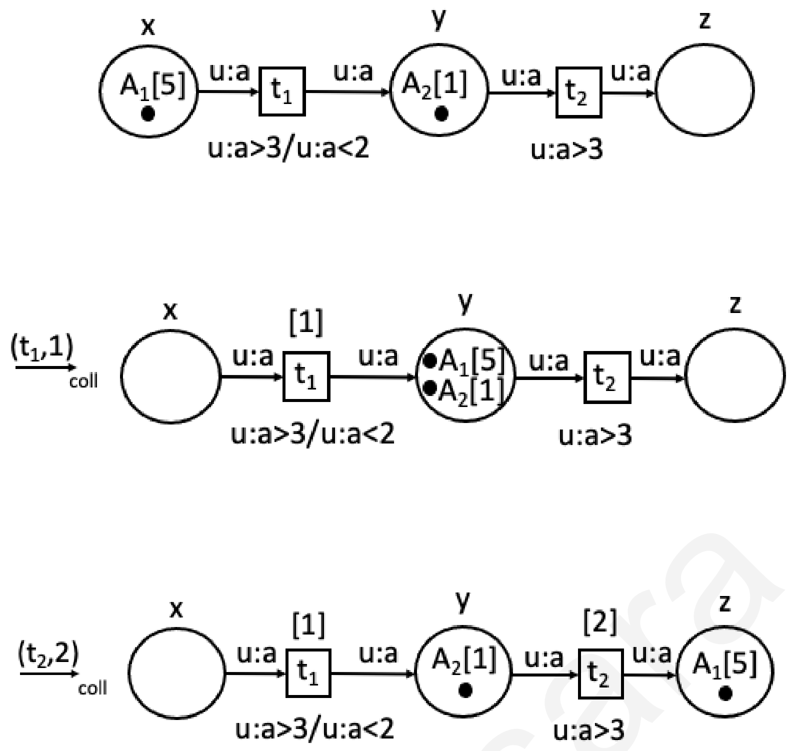


Figure 5.1: Forward execution in CRPNs

Figure 5.1 demonstrates the forward execution of a CRPN with two transitions. The forward conditions of both transitions require a variable assignment for variable u of type $type(u) = a$ to be greater than three ($u > 3$). Only token $A_1[5]$ is able to satisfy these conditions and thus fire both transitions. As such, token $A_1[5]$ has been selected as a variable assignment for variable u which is then used to fire transition t_1 followed by t_2.

5.1.2 Controlled Reversibility

We now move on to controlled reversibility. The following definition enunciates that a transition t is controlled-reverse enabled if it is also reverse-enabled in multi reversing Petri nets under the collective token interpretation. Furthermore, we require that the reverse condition of the transition is satisfied according to the variable assignment V_{coll}.

Definition 69. Consider a CRPN $(N, \Sigma, D, C_F, C_R, I)$, a state $\langle M, H \rangle$, and a transition occurrence (t, k). Then (t, k) is controlled-reverse-enabled in $\langle M, H \rangle$ if there exist two injective functions $U_{coll} : \mathsf{post}(t) \cap A_V \rightarrow A_I$ and $V_{coll} : A_V \cap Free(COND_{A_V}) \rightarrow A_I$ such that:

1. t is *coll*-enabled in in N (Definition 62),

2. for all $u \in Free(C_R(t))$ then $V_{coll}(u) = a_i$, $a_i \in A_I \cap M(x)$, $x \in P$ such that $type_\Sigma(u) = type_\Sigma(a_i)$,

3. if $u \in F(t, x)$, $x \in P$ and $u \in Free(C_R(t))$ then $V_{coll}(u) = U_{coll}(u)$, and

4. $E(C_R(t), V_{coll}, M, I)$= TRUE.

Thus, a transition occurrence (t, k) is reverse-enabled in $\langle M, H\rangle$ if (1) it is $coll-$enabled in MRPNs, (2) (3) the definition makes similar requirements to Definition 67 only this time in (4) it requires for the reverse condition of the transition to evaluate to TRUE. Note that while a selection of tokens may yield the reversal of the transition impossible by setting $E(C_R(t), V_{coll}, M, I)$ = FALSE, another selection, and its associated variable assignment V_{coll} may be such that $E(C_R(t), V_{coll}, M, I)$ = TRUE. This is determined by the data values associated to the token instances by the assignment I. We refer to V_{coll} as a reversal variable assignment.

When a transition occurrence (t, k) is reversed the marking and history of the controlled reversing Petri net are updated according to the respective definition of MRPNs under the collective token interpretation.

Definition 70. Given a CRPN $(N, \Sigma, D, C_F, C_R, I)$, a state $\langle M, H\rangle$, and a transition occurrence (t, k) controlled-reversed-enabled in $\langle M, H\rangle$ with U_{coll} a reversal enabling assignment and V_{coll} a reversal variable assignment, we write $\langle M, H\rangle \overset{(t,k)}{\rightsquigarrow}_{coll} \langle M', H'\rangle$ where M' and H' are updated as in N (Definition 65).

Figure 5.2 is a continuation of the forward execution of Figure 5.1. The reverse condition of transition t_1 requires for the variable assignment of token variable u of type $type(u) = a$ to be smaller than two ($u < 2$), whereas, the reverse condition of transition t_2 requires for a token variable u of type $type(u) = a$ to be smaller than three ($u < 3$). Note that in the case of transition t_2 we have $C_F(t_2) =!C_R(t_2)$, thus, the reverse condition is omitted from the figure. In this example, only transition t_1 is able to reverse as token $A_2[1]$ can be used for the variable assignment of u enabling the reversal of transition t_1.

Let us consider a more complicated example of a reversible chemical reaction that depends on environmental conditions. Ammonium chloride (NH_4Cl) is an inorganic compound that decomposes into ammonia (NH_3) and hydrogen chloride gas (HCl). This decomposition is a reversible reaction that occurs when ammonium chloride is heated to over 338 degrees Celsius. The two gases ammonia and hydrogen chloride can then react together in cooler temperatures to reform the solid ammonium chloride and therefore reverse the

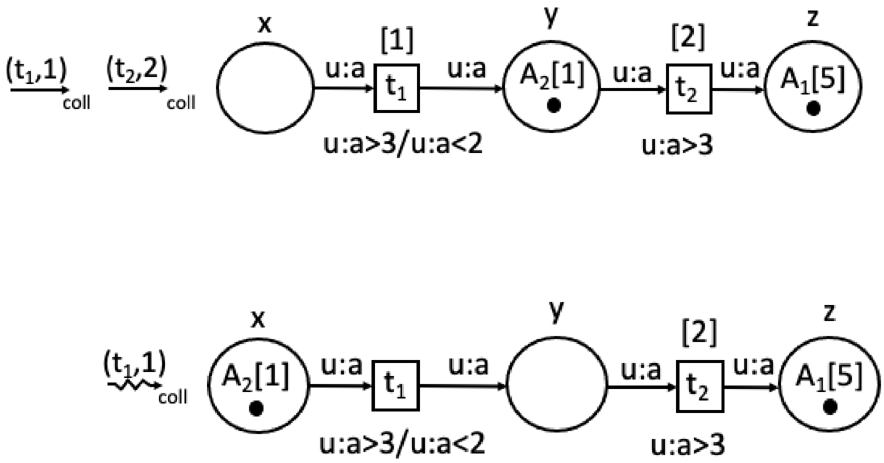


Figure 5.2: Reverse execution in CRPNs

decomposition. The recommended storing temperature of ammonium chloride is 15°C to 25°C.

The model of this reaction is shown in the initial marking of Figure 5.3. Here we assume the token types $A = \{H, N, Cl, T\}$, with the first three bearing the expected meaning and type T capturing different temperatures. In particular T has instances T_1 and T_2, bearing values $I(T_1) = 338$ and $I(T_2) = 20$. These are placed in places v and z, respectively. In place x, the initial marking contains the component NH_4Cl. In transition t_1, with condition $I(t) \geq 338$, the ammonium chloride decomposes into NH_3 and HCl, assuming that a T token with value at least 338 is present. This is the case, thus, the transition takes place as shown in the second marking of the figure. If the temperature decreases, as implemented in transition t_2 where token instance T_1 exchanges places with token instance T_2, $I(T_2) = 20$, then the reversal of the transition is enabled leading to the reversal of the decomposition, as shown in the last marking of the figure.

5.2 Behavioural Properties for Controlled Reversing Petri Nets

A major strength of Petri nets is their support for analysis of various properties and problems associated with concurrent systems [102]. Two types of properties can be studied within reversing Petri net models based on whether they are dependent on the initial marking, or are independent of the initial marking. The former type of properties is referred to as marking-

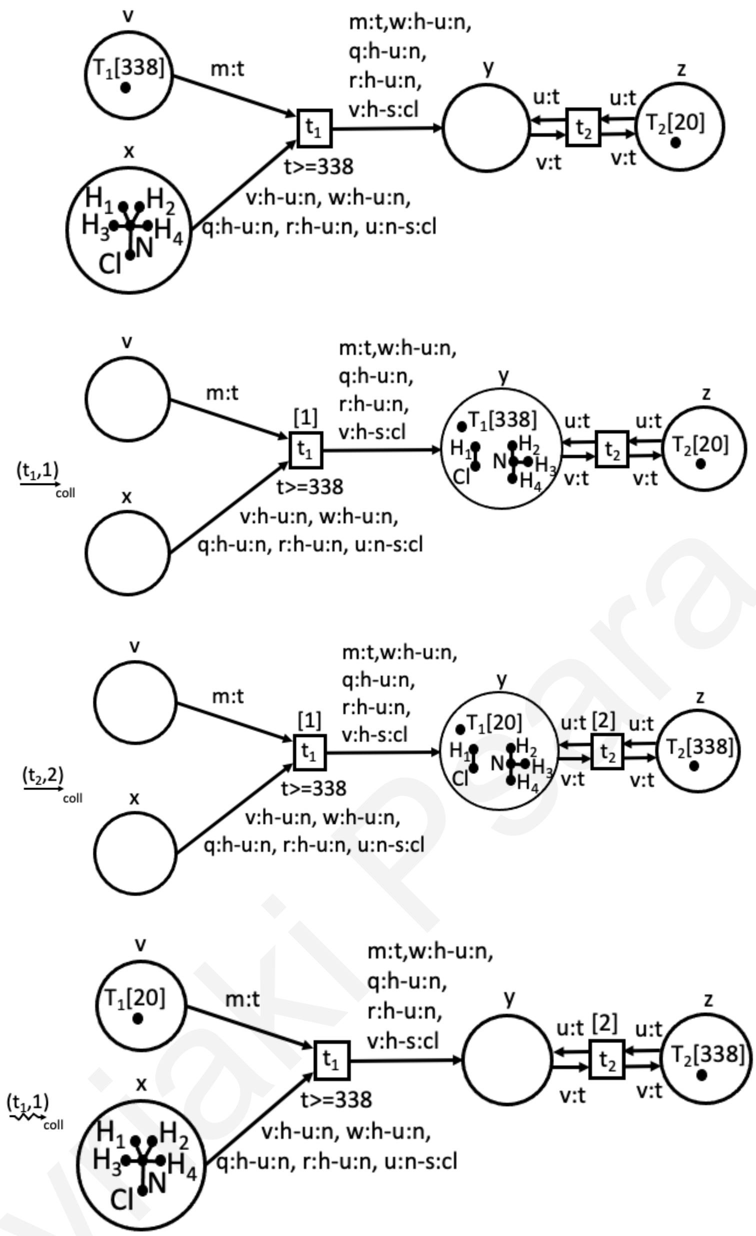


Figure 5.3: Ammonium chloride chemical reaction

dependent or behavioural properties, whereas the latter type of properties is called structural properties. In this section, we discuss only basic behavioural properties and their analysis.

Reachability. Reachability is a fundamental basis for studying the dynamic properties of systems. The firing of an enabled transition will change the token distribution in a net according to the firing rules. A sequence of firings will result in a sequence of states. A state $\langle M_n, H_n \rangle$, is said to be reachable from a state $\langle M_0, H_0 \rangle$ if there exists a sequence of firings that transforms $\langle M_0, H_0 \rangle$ to $\langle M_n, H_n \rangle$. A firing or transition sequence is denoted by $\sigma = t_1; t_2; ...; t_n$. If $\langle M_n, H_n \rangle$ is reachable from $\langle M_0, H_0 \rangle$ by σ we write $\langle M_0, H_0 \rangle \overset{\sigma}{\longmapsto} \langle M_n, H_n \rangle$. The set of all possible sates reachable from $\langle M_0, H_0 \rangle$ in a net N is denoted by $R(N, \langle M_0, H_0 \rangle)$

or simply $R(\langle M_0, H_0\rangle)$. The set of all possible firing sequences from $\langle M_0, H_0\rangle$ in a net N with initial state $\langle M_0, H_0\rangle$ is denoted by $L(N, \langle M_0, H_0\rangle)$ or simply $L(\langle M_0, H_0\rangle)$. Now, the reachability problem for controlled reversing Petri nets is the problem of finding if $\langle M_n, H_n\rangle \in R(\langle M_0, H_0\rangle)$ for a given state $\langle M_n, H_n\rangle$ in a net N with initial marking M_0. In some applications, one may be interested in the markings of a subset of places and not care about the rest of places in a net. This leads to a submarking reachability problem which is the problem of finding if $\langle M_j, H_j\rangle \in R(\langle M_0, H_0\rangle)$,where $\langle M_j, H_j\rangle$ is any marking whose restriction to a given subset of places agrees with that of a given marking $\langle M_n, H_n\rangle$.

Definition 71. Given a CRPN N an initial state $\langle M, H\rangle$ and an execution $\langle M, H\rangle \overset{\sigma}{\longmapsto} \langle M', H'\rangle$ then $\langle M', H'\rangle$ is reachable from $\langle M, H\rangle$ in the net N and the set of all states reachable from $\langle M, H\rangle$ is denoted by $R(N, \langle M, H\rangle)$ or simply $R(\langle M, H\rangle)$. The set of all possible firing sequences from $\langle M, H\rangle$ in N is denoted by $L(N, \langle M, H\rangle)$ or simply $L(\langle M, H\rangle)$.

As a state in CRPNs constitutes a combination of both a marking and a history most of the properties are defined based on both parameters. However, depending on the needs of the modelled system the reachability properties can be redefined to ignore the status of the history. The reachability property is redefined as follows:

Definition 72. Given a CRPN N an initial state $\langle M, H\rangle$ and an execution $\langle M, H\rangle \overset{\sigma}{\longmapsto} \langle M', H'\rangle$ then marking M' is reachable from $\langle M, H\rangle$ in the net N denoted by $M' \in R(\langle M, H\rangle)$.

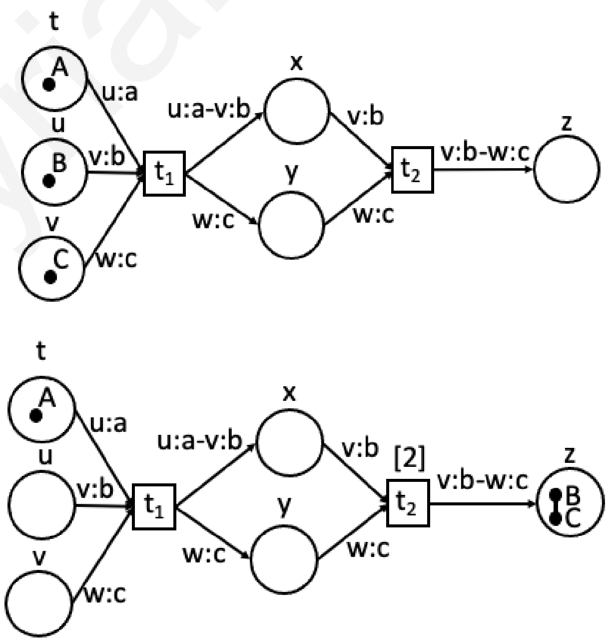


Figure 5.4: Reachability property

Let us consider the example in Figure 5.4. The figure on the top represents the initial marking $\langle M_0, H_0\rangle$. The figure on the bottom represents a desired marking $\langle M, H\rangle$ of the same controlled reversing Petri net. The reachability property questions whether $\langle M, H\rangle$ is reachable from the initial marking $\langle M_0, H_0\rangle$, such that $\langle M, H\rangle \in R(\langle M_0, H_0\rangle)$. We can see that indeed $\langle M, H\rangle$ is reachable from $\langle M_0, H_0\rangle$ through the firing sequence $\sigma = (t_1, 1); (t_2, 2); \underline{(t_1, 1)}$ such that $\langle M_0, H_0\rangle \xmapsto{\sigma} \langle M, H\rangle$.

Home state. In many applications, it is not necessary to get back to the initial state as long as one can get back to some (home) state. For example in various electronic devices, home states may be reached automatically after periods of inactivity, or may be forced to be reached by resetting the device. Also in self-stabilising systems, reaching a failed state can be recovered from automatically reaching a non-erroneous home state. A state $\langle M', H'\rangle$ is said to be a home state if, for each state $\langle M, H\rangle$ in $R(\langle M_0, H_0\rangle)$, $\langle M', H'\rangle$ is reachable from $\langle M, H\rangle$.

Definition 73. Given a CRPN N, a state $\langle M, H\rangle$ is a home state if $\langle M, H\rangle \in R(N, \langle M', H'\rangle)$ from every state $\langle M', H'\rangle \in R(N, \langle M_0, H_0\rangle)$ and N is reversible if $\langle M_0, H_0\rangle$ is a home state.

As with the reachability property, home state can be redefined to ignore the history. In this way we have a more flexible notion of a home state where only the location of tokens constitutes a home state.

Definition 74. Given a CRPN N, a marking M is a home state if $M \in R(N, \langle M', H'\rangle)$ from every marking $M' \in R(N, \langle M_0, H_0\rangle)$ and N is reversible if M_0 is a home state.

Now consider the controlled reversing Petri net in Figure 5.5. Given as initial state $\langle M_0, H_0\rangle$ the controlled reversing Petri net on the top we can observe that the state on the bottom $\langle M, H\rangle$ is a home state for that controlled reversing Petri net. Since only transitions t_1 and t_2 are irreversible then when executing any other transition we can reverse their execution and proceed with the firing sequence $(t_1, 1); (t_2, 2); (t_5, 3)$ leading to the home state $\langle M, H\rangle$. Also note that this controlled reversing Petri net is not reversible since t_1 and t_2 are not reversible and therefore after the are execution we cannot return to the initial marking $\langle M_0, H_0\rangle$.

Liveness. The concept of liveness is closely related to the complete absence of deadlocks in operating systems. A controlled reversing Petri net N with initial marking M_0 is said to be live (or equivalently $\langle M_0, H_0\rangle$ is said to be a live marking for N) if, no matter what state has been reached from $\langle M_0, H_0\rangle$, it is possible to ultimately fire any transition of the net by

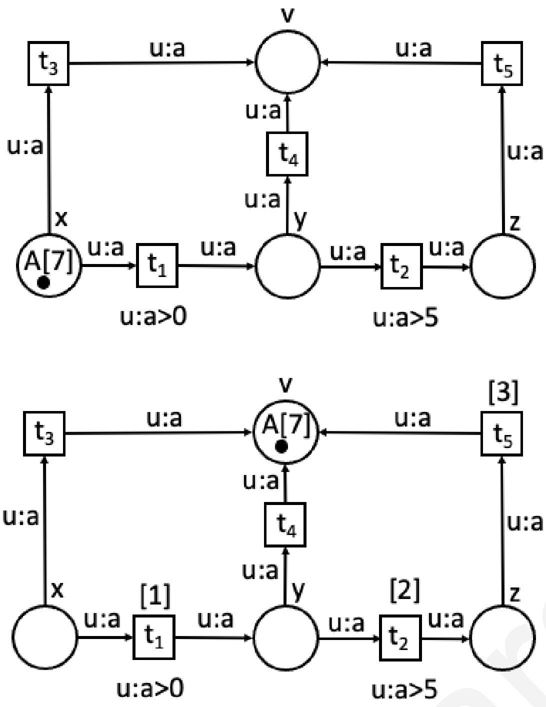


Figure 5.5: Home state property

progressing through some further firing sequence. This means that a live controlled reversing Petri net guarantees deadlock-free operation, no matter what firing sequence is chosen.

Definition 75. A CRPN N it is said to be live (or equivalently $\langle M_0, H_0\rangle$ is said to be a live state for N) if for all $\langle M, H\rangle \in R(N, \langle M_0, H_0\rangle)$ then there exists $t \in T$ such that t enabled in $\langle M, H\rangle$.

Liveness is an ideal property for many systems. However, it is impractical and too costly to verify this strong property for some systems such as the operating system of a large computer. Thus, we relax the liveness condition and define different levels of liveness.

Definition 76. Given a CRPN N and a transition $t \in T$ then t is said to be:

1. dead($L0-$live) if $t \notin L(\langle M_0, H_0\rangle)$,
2. $L1-$live (potentially fire-able) if $\mid \{t \mid t \in L(\langle M_0, H_0\rangle)\} \mid = 1$,
3. $L2-$live if given any positive integer $k \mid \{t \mid t \in L(\langle M_0, H_0\rangle)\} \mid = k$,
4. $L3-$live if $\mid \{t \in L(\langle M_0, H_0\rangle)\} \mid = \infty$,
5. $L4-$live or live if t is $L1-$live for all$\langle M, H\rangle$, $\langle M, H\rangle \in R(\langle M_0, H_0\rangle)$, and
6. Lk-live if every transition in the net is Lk-live, $k = 0, 1, 2, 3, 4$.

$L4-$liveness is the strongest and corresponds to the liveness defined earlier. It is easy to see the following implications: $L4$-liveness $\implies$ $L3-$liveness $\implies$ $L2-$liveness $\implies$ $L1-$liveness. We say that a transition is strictly $Lk-$live if it is $Lk-$live but not $L(k+1)-$live, $k = 1, 2, 3$.

In the case of liveness the history parameter of a state cannot be ignored as history in controlled reversing Petri nets plays an important role when deciding if a transition is reversed enabled in a specific state. For example non executed transitions indicated by $H(t) = \emptyset$ cannot be reversed.

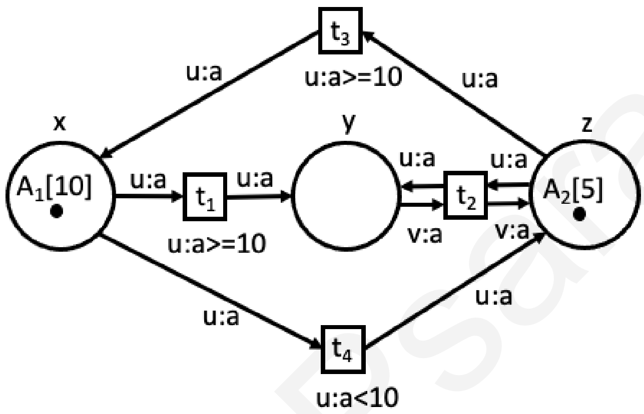


Figure 5.6: Liveness property

Consider the controlled reversing Petri net in Figure 5.6. We observe that the execution $(t_1, 1); (t_2, 2); (t_3, 3); \underline{(t_1, 1)}; (t_4, 4)$ can be repeated infinitely.

Deadlock. The concept of deadlock in controlled reversing Petri nets is the inability to proceed with the execution of a transition. Therefore, a controlled reversing Petri net has a deadlock when there are no other transitions that can be executed in the forward or reverse transition.

Definition 77. Given a CRPN N a deadlock is as state $\langle M, H\rangle$, $\langle M, H\rangle \in R(N, \langle M_0, H_0\rangle)$ such that there exists no $t \in T$ such that t (controlled-forward/controlled-reverse) enabled in $\langle M, H\rangle$.

As deadlock is equivalently defined as L0-liveness then the history as part of a state is an important element when deciding if a transition can fire.

Consider the CRPN in Figure 5.7. The controlled reversing Petri net on the top figure represents the initial marking $\langle M_0, H_0\rangle$ and the controlled reversing Petri net in the bottom figure represents the resulting marking $\langle M, H\rangle$ after the execution of t_1, such that

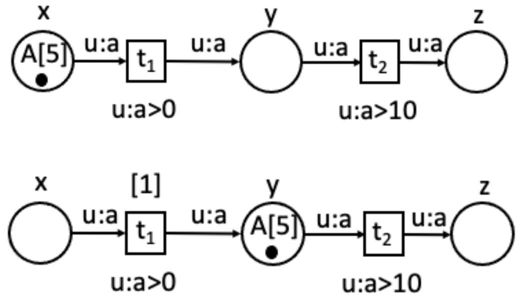


Figure 5.7: Deadlock property

$\langle M_0, H_0 \rangle \xrightarrow{(t_1,1)} \langle M, H \rangle$. Since transition t_2 requires $u > 10 = TRUE$, $type(u) = a$ in order to be executed in forward direction then it is not fireable by $A[5]$. Also t_1 requires $u < 0$ =TRUE in order to reverse then it is irreversible by $A[5]$. Therefore in state $\langle M, H \rangle$ there are no transitions fireable by the only token $A[5]$ and therefore the reversing Petri net has reached a deadlock.

Siphon. A non-empty subset of places P_S in a CRPN is called a siphon if for all $\langle M, H \rangle$ where $M(x) = \emptyset$, $x \in P_S$ and for all $\langle M', H' \rangle$ then $\langle M', H' \rangle \in R(\langle M, H \rangle)$ we have $M'(y) = \emptyset$, $y \in P_S$. A siphon has a behavioural property that if it is token-free under some marking, then it remains token-free under each successor marking. It is easy to verify that the union of two siphons is again a siphon. A siphon is called a basic siphon if it cannot be represented as a union of other siphons. All siphons in a CRPN can be generated by the union of some basis siphons. A siphon is said to be minimal if it does not contain any other siphon. A minimal siphon is a basis siphon, but not all basis siphons are minimal.

Exiting a siphon highly depends on whether a transition is executable in either the forward or reverse direction. The execution of transitions in controlled reversing Petri nets depends both on the satisfaction or violation of conditions but also on the form of execution i.e. whether we are firing in forward or reverse. As reversibility allows the execution of transitions in both forward and reverse execution this means that a fully reversible MRPN cannot have siphons as when exiting a siphon we always have the possibility of reversing in order to enter the siphon again. As such the use of conditions disables transitions from reversing when necessary and therefore irreversible transitions do not constitute entrance in a siphon area.

Trap. A non-empty subset of places P_T in a CRPN is called a trap if for all $\langle M, H \rangle$ where $M(x) \neq \emptyset$, $x \in P_T$ then for all $\langle M', H' \rangle$ where $\langle M', H' \rangle \in R(\langle M, H \rangle)$ we have $M'(y) \neq \emptyset$,

$y \in P_T$. A trap has a behavioural property that if it is marked (i.e., it has at least one token) under some marking, then it remains marked under each successor marking. It is easy to verify that the union of two traps is again a trap. A trap is called a basic trap if it cannot be represented as a union of other traps. All traps in a CRPN can be generated by the union of some basis traps. A trap is said to be minimal if it does not contain any other trap. A minimal trap is a basis trap, but not all basis traps are minimal.

Similarly to siphons, traps are a behavioural property in which when entering the trap region we are unable to execute transitions outside that region. As MRPNs can be fully reversible the introduction of conditions disables transitions from reversing and thus being able to exit the trap.

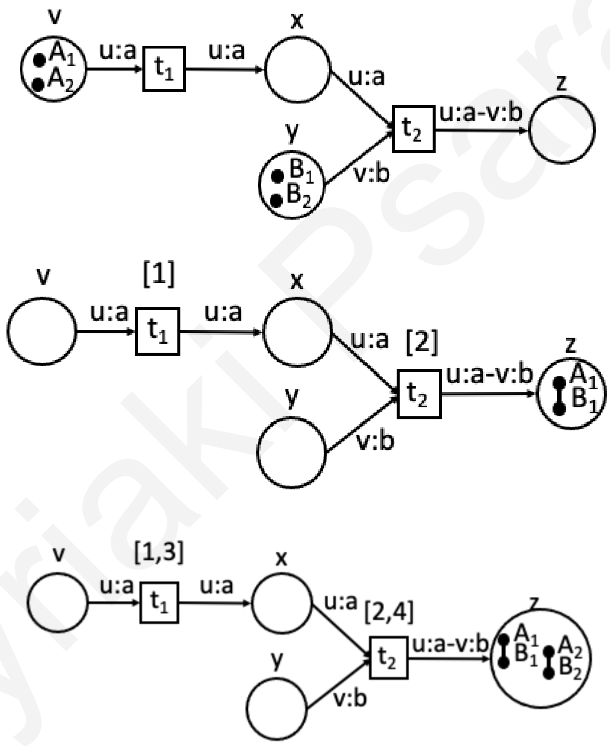


Figure 5.8: Coverability property

Coverability. A state $\langle M, H\rangle$ in a controlled reversing Petri net N with $\langle M_0, H_0\rangle$ is said to be coverable if there exists a marking $\langle M', H'\rangle$ in $R(\langle M_0, H_0\rangle)$ such that $M'(x) \subseteq M(x)$ for each $x \in P$ in the net and $H'(t) \subseteq H(t)$ for each $t \in T$ in the net. Coverability is closely related to $L1$- liveness (potential firability). Let $\langle M, H\rangle$ be the minimum marking needed to enable a transition t. Then t is dead (not $L1$- live) if and only if $\langle M, H\rangle$ is not coverable. That is, t is $L1$-live if and only if $\langle M, H\rangle$ is coverable.

Definition 78. Given a CRPN N and a state $\langle M, H\rangle$ is said to be coverable if there exists $\langle M', H'\rangle \in R(\langle M_0, H_0\rangle)$ such that $|\ \{a_i | a_i \in M(x), type(a_i) = a\}\ | \leqslant |\ \{a_i | a_i \in M(x), type(a_i) = a\}\ |$, for all $a \in A$ and if $k \in H(t)$ then $k \in H'(t)$, $t \in T$.

Coverable states can be similarly defined as:

Proposition 28. Given a CRPN N and a state $\langle M, H\rangle$ is said to be coverable if there exists $\langle M', H'\rangle \in R(\langle M_0, H_0\rangle)$ such that $\langle M', H'\rangle \geq \langle M, H\rangle$.

Proposition 29. Given a CRPN N and a state $\langle M, H\rangle$ is said to be coverable iff $\langle M, H\rangle \in R(\langle M_0, H_0\rangle)$.

Similarly, to the reachability property, coverability can be defined in terms of a marking and thus ignoring the history part of state.

Definition 79. Given a CRPN N and a marking M is said to be coverable if there exists $M' \in R(\langle M_0, H_0\rangle)$ such that $|\ \{a_i | a_i \in M(x), type(a_i) = a\}\ | \leqslant |\ \{a_i | a_i \in M(x), type(a_i) = a\}\ |$, for all $a \in A$.

Consider the CRPN in Figure 5.8. The first controlled reversing Petri net on the top corner is the initial marking $\langle M_0, H_0\rangle$. The second controlled reversing Petri net is the desired state $\langle M, H\rangle$ that we want to check if its coverable by some marking $\langle M', H'\rangle$ reachable from the initial marking such that $\langle M', H'\rangle \in R(\langle M_0, H_0\rangle)$. The final controlled reversing Petri net is the marking $\langle M', H'\rangle$ which is reachable from $\langle M_0, H_0\rangle$ by the execution $\sigma = (t_1, 1); (t_2, 2); (t_1, 3); (t_2, 4)$, $\langle M_0, H_0\rangle \xmapsto{\sigma} \langle M', H'\rangle$ which covers $\langle M, H\rangle$ such that $\langle M', H'\rangle \geq \langle M, H\rangle$. Note that the marking derived from the execution $\sigma' = (t_1, 1); (t_2, 2)$, $\langle M_0, H_0\rangle \xmapsto{\sigma'} \langle M'', H''\rangle$ also covers $\langle M, H\rangle$.

Persistence. A controlled reversing Petri net N with initial marking $\langle M_0, H_0\rangle$ is said to be persistent if, for any two enabled transitions, the firing of one transition will not disable the other. A transition in a persistent net, once it is enabled, will stay enabled until it fires. Persistence is closely related to conflict-free nets, and a safe persistent net can be transformed into a marked graph by duplicating some transitions and places.

Definition 80. Given a CRPN N and a state $\langle M, H\rangle \in R(\langle M_0, H_0\rangle)$ then N is said to be persistent if for all $t_1, t_2 \in T$, t_1, t_2 enabled in $\langle M, H\rangle$ and $\langle M, H\rangle \xrightarrow{t_1} \langle M', H'\rangle$ then t_2 enabled in $\langle M', H'\rangle$ and vice versa.

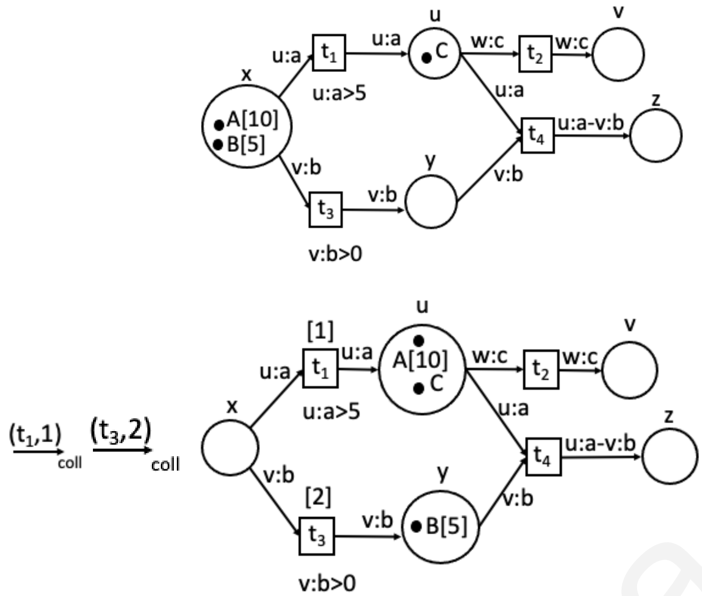


Figure 5.9: Persistence property

As persistence is dependent on the enabledness of transitions, it cannot be defined solely by the marking of a state.

Consider the controlled reversing Petri net in Figure 5.9. In the first controlled reversing Petri net we observed the initial marking $\langle M_0, H_0 \rangle$ where transitions t_1 and t_3 are simultaneously forward enabled and the execution of one does not preclude the execution of the other. On the second controlled reversing Petri net we observe the marking $\langle M, H \rangle$ after the execution of both t_1 and t_3 where transitions t_2 and t_4 are simultaneously forward enabled. Since transitions t_1 and t_3 are irreversible by $A[10]$ and $B[5]$ respectively then only transitions t_2 and t_4 can be executed and the execution of one of them does not preclude the execution of the other. Hence the controlled reversing Petri net is indeed persistent. Note that if transitions t_1 and t_3 were not irreversible then the reversal of transition t_1 would have precluded the execution of transition t_2 and t_4.

5.3 Case Study

Antenna selection in distributed Massive MIMO (Multiple Input Multiple Output) [50] antenna arrays is an important optimisation problem on a complex system comprised of a large number of simple, similar-behaving components. While a large number of antennas offers diversity, spatial multiplexing opportunities, interference suppression and redundance [105], not all antennas contribute the same, and powering all of them is not optimal [61]. Optimal transmit antenna selection for large antenna arrays is computationally demanding [51], so suboptimal approaches are pursued for real time use.

Petri nets are a convenient tool for modelling and control of networks, and have been applied in higher layers of ISO OSI model for wireless networks [58]. Centralized AS in DM MIMO (distributed, massive, multiple input, multiple output) systems is computationally complex, demands a large information exchange, and the communication channel between antennas and users changes rapidly. The reliability of distributed multiple-antenna systems depends on fault tolerance and recovery which align naturally with reversibility. We therefore introduce a CRPN-based, distributed, time-evolving solution with reversibility, asynchronous execution and local condition tracking for reliable performance and fault tolerance. In this setting, we use our expressive controlling mechanism in order to manage the pattern and the direction of computation in order to deal with error recovery or to provide the main focus of the computation. The internal control mechanism validates the conditions when the addition of an antenna improves the sum capacity and violates the condition in case an antenna is consider to be no longer useful triggering reversal which removes the antenna from the selected set.

Reversible models conserve quantities, both in the sense of energy and matter. In the case of our controlled reversing Petri net model, it preserves the number of tokens: in the particular implementation it means that a constant number of antennas will be used at all times, which is advantageous in terms of planning and hardware resource deployment and represents an improvement compared to the previous localised antenna selection in which the number of antennas was an emergent property. Conservation of the token count can also represent a fixed power budget, when we use the controlled reversing Petri net for power control, constant user count if we perform user selection, etc.

Reversibility allows the resource management algorithm to go back to a previous state and take a different execution route in case of an antenna becomes faulty in one or more places (in our presented case, antennas). Since there has been a fault and no forward transition is possible we should reverse the last transition and thus see the token returning to a properly-functioning antenna.

Reversing the evolution of a controlled reversing Petri net is a logical behaviour in some use cases. Without movement of users in the grid, the selected set of antennas is concentrated in a predefined state. With users moving, the antennas coordinate their tracking. Once there is no more need for their activity, the tokens return to the initial positions with simple reversal of their trajectories. At the same time, the whole network does not have to be reversed, as parts of it could still be engaged with serving users.

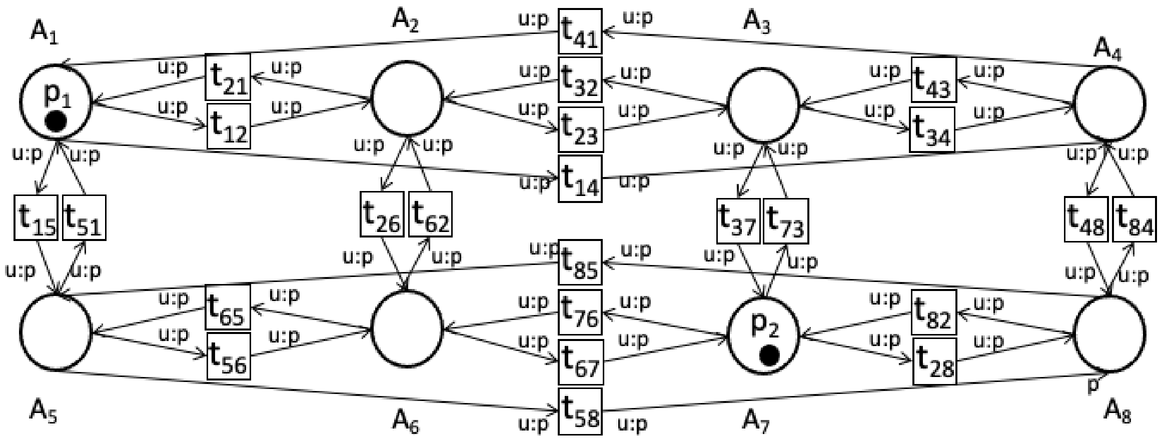


Figure 5.10: Antenna selection on massive-MIMO

5.3.1 Reversing Petri Net Representation

In this section, the semantics for collective reversibility as realised in the framework of controlled reversing Petri nets (CRPNs), dynamically illustrate antenna selection in massive-MIMO based on how the proposed algorithm is implemented and how it changes based on different operating scenarios. This section presents a new method for antenna selection which divides antennas into virtual sub-neighbourhoods whose dynamic behaviour can be observed by being simulated in controlled reversing Petri nets.

For real-life systems such as massive-MIMO the state space is too large to illustrate and therefore we decompose the whole system into smaller subsystems. This decomposition divides the whole model into two sub-models the first one being the power distribution among antennas and the second one being the memory mechanism that controls the execution of transitions. We call the power distribution model the high-level layer illustrating the exchanges of power between antennas as well as the neighbouring connections between them. The low-level layer consists of the memory mechanism that collects information about the powered antennas in a neighbourhood and thus executing transitions in forward or reverse direction depending on whether the required conditions are satisfied. Ideally, the two sub-models can be merged together resulting in a complete model, which then includes both the power allocation of the system and the controlled decision steps.

High-level layer. The model in Figure 5.10 illustrates the higher-level net of the antenna selection algorithm. We demonstrate a sample neighbourhood of eight base station antennas with random distributed topology. Every eight antennas are considered to belong in the same neighbourhood by allowing each antenna to be bidirectionally linked to four other antennas we create overlapping neighbourhoods. The maximum number of enabled antennas in this example is two and the power token is transferred from one antenna to the other through directly connected links. Note that resource allocation systems like antenna selection can greatly benefit from the collective token interpretation since the existence of any power token p in the corresponding place should be able to execute a transition in either the forward or reverse direction.

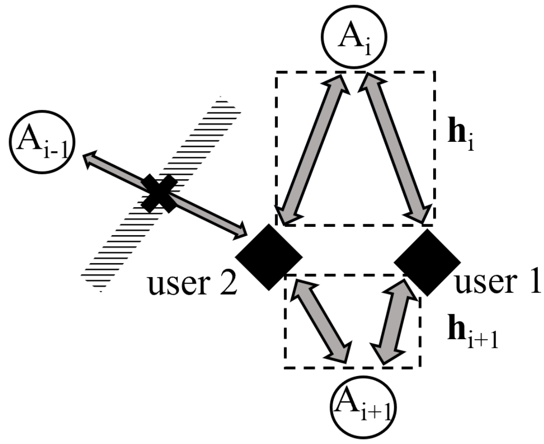
Low-level layer. The search for a suitable set of antennas is a sum capacity maximization problem:

$$C = \max_{\mathbf{P},\mathbf{H_c}} \log_2 \det\left(\mathbf{I} + \rho \frac{N_R}{N_{TS}} \mathbf{H_c} \mathbf{P} \mathbf{H_c}^H\right) \tag{5.1}$$

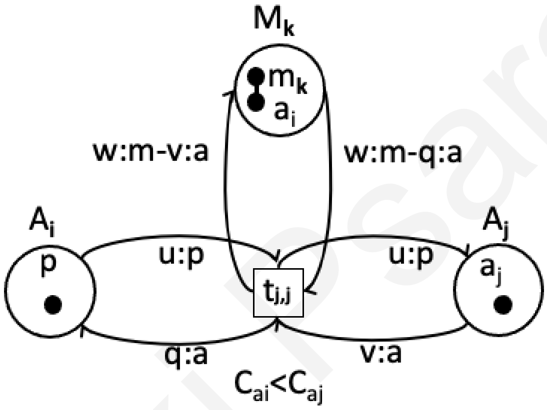
where ρ is the signal to noise ratio, N_{TS} the number of antennas selected from a total of N_T antennas, N_R the number of users, $\mathbf{I}$ the $N_{TS} \times N_{TS}$ identity matrix, $\mathbf{P}$ a diagonal $N_R \times N_R$ power matrix; $\mathbf{H_c}$ is the $N_{TS} \times N_R$ submatrix of $N_T \times N_R$ channel matrix $\mathbf{H}$. Instead of centralized AS, in our approach sum capacity is calculated locally for small sets of antennas (neighbourhoods), switching on only antennas which improve the capacity as for example in Figure 5.11(a), we demonstrate the case where antenna A_{i-1} decreases sum capacity and therefore it will not be selected.

In the CRPN interpretation, we present the antennas by places $A_1, \dots, A_n$, where $n = N_T$, and the overlapping neighbourhoods by places $M_1, \dots, M_h$. These places are connected together via transitions $t_{i,j}$, connecting A_i, A_j and M_k, whenever there is a connection link between antennas A_i and A_j. The transition captures that based on the neighbourhood knowledge in place M_k, antenna A_i may be preferred over A_j or vice versa (the transition may be reversed).

To implement the intended mechanism, we employ three types of tokens. First, we have the power tokens p_i, which are of the same type and therefore by the collective token interpretation are able to execute/reverse any transition that requires them. If token p_i is located on place A_i, antenna A_i is considered to be on. The transfer of these tokens results into new antenna selections, ideally converging to a locally optimal solution. Second, tokens $m_1, \dots, m_h$, each represent one neighbourhood. Finally, $a_1, \dots, a_n$, represent the antennas. The tokens are used as follows: Given transition $t_{i,j}$ between antenna places A_i and A_j in neighbourhood M_k, transition $t_{i,j}$ is enabled if token p is available on A_i, token a_j on A_j, and bond (a_i, m_k) on M_k, i.e., $F(A_i, t_{i,j}) = \{u\}$, $type(u) = p$, $F(A_j, t_{i,j}) = \{v\}$, $type(v) = a$, and $F(M_k, t_{i,j}) = \{(q, w), q, w\}$, $type(q) = a$ and $type(w) = m$. This configuration captures that antennas A_i and A_j are on and off, respectively. (Note that the bonds between token m_k and



(a) antennas and users



(b) a part of the CRPN model

Figure 5.11: CRPN for antenna selection in DM MIMO (large antenna array)

tokens of type a in M_k capture the active antennas in the neighbourhood.) Then, the effect of the transition is to break the bond (a_i, m_k), and release token a_i to place A_i, transferring the power token to A_j, and creating the bond (a_j, m_k) on M_k, i.e., $F(t_{i,j}, A_i) = \{q\}$, $F(t_{i,j}, A_j) = \{u\}$, and $F(t_{i,j}, M_k) = \{(v, w), v, w\}$. The mechanism achieving this for two antennas can be seen in Figure 5.11(b).

Finally, to capture the transition's condition, an antenna token a_i is associated with data vector $I(a_i) = \mathbf{h}_i$, $type_\Sigma(\mathbf{h}_i) = \mathbb{R}^2$ $(= \mathbb{C})$, i.e., the corresponding row of $\mathbf{H}$. The condition constructs the matrix $\mathbf{H}_c$ of (5.1) by collecting the data vectors $\mathbf{h}_i$ associated with the antenna tokens a_i in place M_k: $\mathbf{H}_c = (\mathbf{h}_1, ..., \mathbf{h}_n)^T$ where $\mathbf{h}_i = I(a_i)$ if $a_i \in M_k$, otherwise $\mathbf{h}_i = (0 \;\; \ldots \;\; 0)$. The transition $t_{i,j}$ will occur if the sum capacity calculated for all currently active antennas (including a_i), C_{a_i}, is less than the sum capacity calculated for the same neighbourhood with the antenna A_i replaced by A_j, C_{a_j}, i.e., $C_{a_i} < C_{a_j}$. Note that if the condition is violated, the transition may be executed in the reverse direction.

5.3.2 Property Analysis in Massive MIMO Systems

The behaviour of a modern wireless communications system (in our example, a massive MIMO system) can be thought as an aggregation of multiple networks employing varying levels of coordination and communication. Various resources (electromagnetic spectrum, power, physical infrastructure) are continuously managed, the network of users interacts with the network of base stations, computation and communication intertwine. Hardware faults happen, working modes change, and the modern networks are supposed to handle these unexpected events seamlessly. The general idea behind the aggregation method is to substitute complex Petri net structures by simple ones observing some important properties of the model like e.g. deadlock, siphons, traps, reachability etc. Intelligent monitoring of a massive distributed network requires careful specification and modelling in order to analyse its constraints and deliverables, as well as to avoid hazards, waste of resources and security threats and therefore be used by engineers and designers for what-if analysis and experimentation.

On that note, the modelled MIMO system can be used for formal verification of properties such as the deadlock property, siphons and traps. Optimisation processes in wireless communications in general should be converging fast to a steady state with minimal computational burden in order to enable real time application in high mobility scenarios. Therefore, we could use the deadlock property to define the final antenna selection where our algorithm converges. Similarly, in the case of siphons we know that once a token escapes the siphon region the token will never return to that region. As such once a power token escapes to an antenna outside a siphon region it will never consider that region as computationally better than the antenna that the token is already in. In this way we are able to define non critical antennas given a predefined set of specifications of our MIMO system. In the opposite manner once a token enters a trap region it will never escape that region, i.e. it will never consider antennas outside that region as better antennas than the already selected ones. As such we are able to identify critical antennas that improve sum capacity given a predefined set of specifications for our MIMO system. Finally, the ability of our model to allow tokens to carry data yields in customized performance properties which can be quantified by specific metrics that provide the average measure of the probability with which an error is encountered.

5.4 Concluding Remarks

In this chapter we have extended MRPNs with conditions that control reversibility [113, 127], and we have applied our framework in the context of wireless communications. Our formalism introduces conditional transitions that permit the system to manage the pattern and direction of computation. It allows systems to reverse under specified conditions leading to previously visited states or even new ones without the need of additional forward actions. A possibility to extend the model exists by introducing arc expressions that will perform operations on the data values associated with the manipulating tokens.

We have shown how the reversible structure of CRPNs is amenable to implementations from wireless communications in terms of distributed antenna selection and is expressive enough to encode reversible processes. This experience has illustrated that resource management can be studied and understood in terms of CRPNs as, along with their visual nature, they offer a number of features, such as token persistence, that is especially relevant in these contexts.

Chapter 6

Conclusions

6.1 Summary

This thesis proposes a reversible approach to Petri Nets [111, 112] that allows the modelling of reversibility as realised by backtracking, causal reversing, and out-of-causal-order reversing. Our proposal allows transitions to reverse at any time leading to previously visited states or even to new ones without the need of additional forward actions. Moreover, this interpretation of Petri Nets has the capability of reversing without the need of an extensive memory. To enable this, additional machinery has been necessary to capture causal dependencies in the presence of cycles. This machinery identifies a causal dependence relation that resorts to the marking of a net and is partnered along with stack histories for each transition that record all previous occurrences. To the best of our knowledge, this is the first such proposal, since the related previous work [15, 16], having a different aim, implemented a very liberal way of reversing computation in Petri nets by introducing additional reversed transitions. On the contrary, in [96], reversibility is achieved by adding new places to non-reversible Petri nets while preserving their computation, which however is only possible in a subclass of Petri nets and it is only focused on causal reversal. The works of [93–95] identify the causal memory of a Petri net by unfolding them into occurrence nets and coloured Petri nets. All of these approaches, including ours, are concerned with reversing single steps of transitions, unlike [35], which examines the possibility of reversing the effect of groups of actions.

Other than the technical scope of this work we are also concerned with the theoretical foundations of reversible computation, specifically the different strategies of reversing and their relationships. Through the aid of Petri nets we were able to examine the different strategies of reversing and focus mostly on causal reversing and out-of-causal-order reversing. Causality is one of the most interesting topics within models of concurrency where various interpretations have been proposed throughout time which can be justified either by theoretical properties, or by the implementation of possible applications. We focus on the approach where dependencies between transitions are determined by the token manipulation performed during an execution. We prove that the amount of flexibility allowed in causal reversibility indeed yields a causally consistent semantics.

On a similar note, research on out-of-causal reversibility is very limited since the only related work is that in [118, 119]. Therefore, from various examples and theoretical results we have examined the theoretical properties of out-of-causal reversibility and demonstrated that out-of-causal-order reversibility is able to create new states unreachable by forward-only execution.

Most works in the literature discuss out-of-causal reversibility when creating of bonds rather than destructing bonds. In our model, where states are more elaborate since they preserve token evolution, we were able to observe that it is not possible to reverse a transition in out of causal order whose effect no longer exists in the system. This shows that out-of-causal order is not as flexible as one might initially believe. This applies and should be considered in other formal models, such as process calculi and event structures, independently of how abstract their states are.

Additionally, we establish the relationship between the three forms of reversing and define a transition relation that can capture each of the three strategies modulo the enabledness condition for each strategy. This allows us to provide a uniform treatment of the basic theoretical results.

We continue exploring these reversible strategies in extensions of reversing Petri nets, Multi Reversing Petri Nets (MRPNs), by allowing multiple tokens of the same type to exist in a model and developing reversible semantics in the presence of bond destruction. Our aim was to generalize reversing Petri nets in a setting where multiple tokens that have identical behavioural capabilities can occur in a system. However, allowing multiple instances of identical tokens results in ambiguities when it comes to causal dependencies. In fact, we have distinguished the different ways of introducing reversible behaviour into causal systems with multiple tokens and we explore two directions, namely, the individual token interpretation defined based on partial order [24, 137] and the collective token interpretation defined based on disjunctive causality.

We have proposed reversible semantics that follow the individual token philosophy and therefore achieve precise correspondence between the token instances and their past. The individuality of identical tokens can be imposed by their causal path which allows identical tokens to fire the same transition when going forward, however when going backwards tokens will be able to reverse only the transitions that they have fired. We have also provided the reversible semantics for out-of-causal-order reversibility in the presence of bond destruction. Finally, we show that the expressive power of multi reversing Petri nets is equivalent to the expressive power of single reversing Petri nets (SRPNs). However, there is blow-up on the size of SRPNs as multiple transitions in SRPNs can be represented by the same transition in MRPNs as long as the type equivalence between the required tokens and variables is the same.

We have also presented a more relaxed form of reversibility following the collective token philosophy and we have given the associated semantics for the respective firing rule. This approach considers all tokens of a certain type to be identical, disregarding their history during execution, and is particularly applicable in the context of resource-aware systems. In the collective token interpretation when multiple tokens of the same type reside in the same place then these tokens are not distinguished. This means that all that is known by the model is the amount of token occurrences of a specific type and their location in the marking. We have shown how this firing rule relates to the firing rules of the individual token interpretation and how the robustness of this mechanism can be applied in an application from biochemistry known as the autoprotolysis of water.

A subsequent extension of our formalism, called Controlled Reversing Petri Nets (CRPNs), considers approaches for controlling reversibility as for instance in [78, 81, 118]. While various frameworks make no restriction as to when a transition can be reversed (uncontrolled reversibility), it can be argued that some means of controlling the conditions of transition reversal is often useful in practice. For instance, in biological phenomena where environmental conditions change or when dealing with fault recovery where reversal is triggered when a fault is encountered. We therefore have extended our research by proposing conditional executions that indicate the pattern and direction of computation as well as irreversible actions or less likely executable actions. In fact, we have extended MRPNs with conditions that control reversibility by determining the direction of transition execution. We then provide the main behavioural properties of our controlled model as the specification of models according to these properties can be useful towards their analysis and verification.

Finally, we show the robustness of our control mechanism and the associated behavioural properties by modelling an example from telecommunications of a recently-proposed distributed algorithm for antenna selection. Our application illustrates the ability of CRPNs to not only formalise complex distributed systems, but also to naturally capture controlled execution and conservation of information in a system. The ability of the CRPN solution to act asynchronously and converge fast with minimal computational burden enables real time application of the algorithm even in high mobility scenarios. We have shown how the reversible structure of CRPNs is amenable to implementations from wireless communications in terms of distributed antenna selection and is expressive enough to encode reversible processes.

6.2 Current and Future Work

The simplicity of the basic user interface of Petri nets has easily enabled extensive tool support over the years, particularly in the areas of model checking, graphically oriented simulation, and software verification. Recently, Petri nets have been associated with a novel paradigm, known as Answer Set Programming (ASP) [53, 89], which is a declarative programming language with competitive solvers that solve a problem by devising a logic program such that models of the program provide the answers to the problem. ASP applies declarative logic programming techniques that run multiple simulations and parallel evolutions in order to analyse the properties of various modelling domains. Various subclasses of Petri Nets have been translated to ASP, such as regular Petri Nets [107], Simple Logic Petri Nets [17], 1-safe Petri nets [59], general Petri Nets [4], timed Petri nets [57], as well as high-level Petri nets [3].

Given that RPNs are able to model discrete event systems with well-formed semantics, they can also be used for specifying and manipulating the states of a system. Based on that we are currently exploring how ASP can be used to encode reversing Petri nets in an intuitive manner while preserving the modelling power and analyzability of decision problems in Petri net theory [39]. Our implementation allows the enumeration of all possible evolutions of a reversing Petri net simulation as well as the ability to carry out additional reasoning about these simulations. Our long term goal is the development of an ASP-based framework for reasoning about RPN models. The visual nature of Petri nets in combination with reversible computation can help in understanding reversibility through various case studies and explore how reversibility can help in specification, verification, and testing.

In order to further understand how reversibility affects computation, we need to investigate the expressiveness relationship between models that are equipped with reversibility and the forms of traditional models. The trade-offs between traditional models and reversible models, the relations between the several reversible approaches as well as a clarification and classification of the different notions are of particular interest, and need to be studied in depth. This will show whether the expressive power of the traditional model improves from the added feature of reversible computation and whether it affects the decidability problems discussed in [15], regarding reachability and coverability.

As such, another translation of RPNs investigates the expressiveness relationship between RPNs and coloured Petri nets where a subclass of RPNs with trans-acyclic structures has been translated into coloured Petri Nets (CPNs) by encoding the structure of the net along with the execution [13, 14]. The more typical challenges are related with the complexity and the cost of increasing (exponentially) the size of the net. Specifically, we have proposed a structural translation from RPNs to CPNs, where for each transition we consider both forward and backward instances. Furthermore, the translation relies on storing histories and causal dependencies of executed transition sequences in additional places (Figure 6.1). We have tested the translation on a number of examples, where the CPN-tools [64] was employed to illustrate that the translations conform to the semantics of reversible computation. As a result, we conclude that the principles of reversible computation in the presence of cyclic behaviour can be encoded in the traditional model.

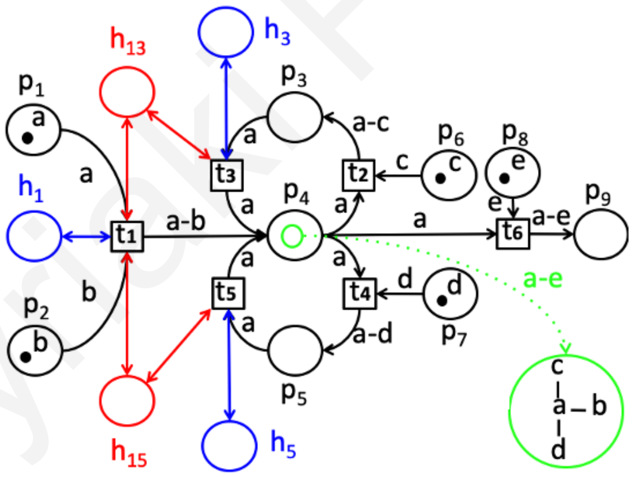


Figure 6.1: Translation from reversing Petri nets to coloured Petri nets

As an extension of our work with coloured Petri nets we aim to provide and prove the correctness of our translation and analyse the associated trade-offs in terms of Petri net size. We intend to investigate the expressiveness relationship between reversing Petri nets and coloured Petri nets and explore how reversible computation affects the expressive power of various subclasses of Petri nets. As a general aim, we plan on implementing an algorithmic translation that transforms RPNs to CPNs in an automated manner using the proposed transformation techniques.

From a more practical point of view, we believe that our framework can be applied in fields outside Computer Science, since the expressive power and visual nature offered by Petri nets coupled with reversible computation has the potential of providing an attractive setting for analysing systems (for instance in biology, chemistry or electrical engineering). Our application of RPNs in the antenna selection problem is a pioneering one, and we believe that the RPN approach can be expanded to other resource management problems in electrical engineering, drawing benefits from both the conservation properties of RPNs as well as the ability to run the networks, or their parts, in reverse direction to recover from faults and handle inherently reversible communication phenomena (e.g. receiver/transmitter duality).

Specifically, intelligent monitoring of the electric power system requires careful specification and modelling in order to analyse its constraints and deliverables, as well as to avoid hazards, wastage of resources and security threats. In order to observe the global behaviour of the power system, a global model representing the different smart grid components and the different modes of communication between these components is needed. The selected modelling formalism should be intuitive and should support the specification of the appropriate abstraction level where in a higher level we should be able to represent the various actors in a smart grid and in a lower level we should be able to include consumers/prosumers as well as their electrical appliances. The simulation of the model should be able to support all the features of the smart grid and incorporate all the technologies used in the smart grid; the model can therefore be used by the smart grid engineers and designers for what-if analysis and experimentation. We believe that reversing Petri nets satisfy the above requirements and that they can be used in order to represent the dynamic and complex behaviour associated with a smart grid covering all its functionalities ranging from the generation of power to intelligent billing mechanisms.

Other than electrical engineering, reversibility attracts much interest for its potential in many other application areas ranging from cellular automata, programming languages, circuit design to quantum computing. Of a particular interest is quantum computing which is a form of computing that performs based on quantum mechanical phenomena such as superposition and entanglement. Many of the components in quantum computers, such as databases or modular exponentiation, obey the fundamental laws of physics which are inherently reversible making quantum computations also reversible [1]. Our aim is to transition from quantum theory to quantum engineering by formally presenting the fundamental rules governing quantum systems, along with methodologies for verification of correctness, safety and reliability of these systems. Due to some essential differences between classical and quantum systems, classical model-checking techniques cannot be directly applied to quantum systems. Therefore, an interesting direction would be to extend RPNs to be able to model the behaviour of systems that combine classical and quantum communication and computation. The aim of this study could be to extend the application area of reversing Petri nets, which have been very successfully used to model classical engineering systems, by modelling the use and operation of Feynman's quantum computer [44]. Future research could develop model-checking techniques that can be used not only for quantum communication protocols but also for general quantum systems, including physical systems and quantum programs. We envisage that quantum model-checking techniques can be applied for checking physical systems, verification of quantum circuits, analysis and verification of quantum programs, and verification of security of quantum communication protocols. Similar work was done in process calculi, the most prominent being qCCS, a natural quantum extension of CCS [43] and CQP [52], a combination of the communication primitives of pi-calculus with primitives for measurement and transformation of quantum states.

Bibliography

[1] S. Abramsky. A structural approach to reversible computation. *Theoretical Computer Science*, 347(3):441–464, 2005.

[2] T. Altenkirch and J. Grattage. A functional quantum programming language. In *Proceedings of LICS 2005*, pages 249–258. IEEE Computer Society, 2005.

[3] S. Anwar, C. Baral, and K. Inoue. Encoding higher level extensions of Petri nets in answer set programming. In *Proceedings of LPNMR 2013*, LNCS 8148, pages 116–121. Springer, 2013.

[4] S. Anwar, C. Baral, and K. Inoue. Encoding Petri nets in answer set programming for simulation based reasoning. *TPLP*, 13(4-5-Online-Supplement), 2013.

[5] T. Araki and T. Kasami. Some decision problems related to the reachability problem for Petri nets. *Theoretical Computer Science*, 3(1):85–104, 1976.

[6] H. B. Axelsen. Clean translation of an imperative reversible programming language. In *Proceedings of CC, ETAPS 2011*, LNCS 6601, pages 144–163. Spinger, 2011.

[7] H. B. Axelsen. Time complexity of tape reduction for reversible turing machines. In *Proceedings of RC 2011*, LNCS 7165, pages 1–13. Springer, 2011.

[8] H. B. Axelsen. Reversible multi-head finite automata characterize reversible logarithmic space. In *Proceedings of LATA 2012*, LNCS 7183, pages 95–105. Springer, 2012.

[9] G. Bacci, V. Danos, and O. Kammar. On the statistical thermodynamics of reversible communicating processes. In *Proceedings of CALCO 2011*, LNCS 6859, pages 1–18. Springer, 2011.

[10] J.-L. Baer and C. S. Ellis. Model, design, and evaluation of a compiler for a parallel processing environment. *IEEE Transactions on software engineering*, (6):394–405, 1977.

[11] F. Barbanera, M. Dezani-Ciancaglini, and U. de'Liguoro. Compliance for reversible client/server interactions. In *Proceedings of BEAT 2014*, EPTCS 162, pages 35–42, 2014.

[12] F. Barbanera, M. Dezani-Ciancaglini, I. Lanese, and U. de'Liguoro. Retractable contracts. In *Proceedings of PLACES 2015*, EPTCS 203, pages 61–72, 2015.

[13] K. Barylska, A. Gogolinska, L. Mikulski, A. Philippou, M. Piatkowski, and K. Psara. Reversing computations modelled by coloured Petri nets. In *Proceedings of ATAED 2018*, CEUR Workshop Proceedings 2115, pages 91–111, 2018.

[14] K. Barylska, A. Gogolinska, A. Philippou, and K. Psara. Cycles in reversing computations modelled by coloured Petri nets. (In preparation).

[15] K. Barylska, M. Koutny, L. Mikulski, and M. Piatkowski. Reversible computation vs. reversibility in Petri nets. *Science of Computer Programming*, 151:48–60, 2018.

[16] K. Barylska, L. Mikulski, M. Piatkowski, M. Koutny, and E. Erofeev. Reversing transitions in bounded Petri nets. In *Proceedings of CS&P 2016*, CEUR Workshop Proceedings 1698, pages 74–85. CEUR-WS.org, 2016.

[17] T. M. Behrens and J. Dix. Model checking multi-agent systems with logic based Petri nets. *Annals of Mathematics and Artificial Intelligence*, 51(2-4):81–121, 2007.

[18] C. H. Bennett. Logical reversibility of computation. *IBM journal of Research and Development*, 17(6):525–532, 1973.

[19] G. Berry and G. Boudol. The chemical abstract machine. *Theoretical Computer Science*, 96(1):217–248, 1992.

[20] A. Bérut, A. Arakelyan, A. Petrosyan, S. Ciliberto, R. Dillenschneider, and E. Lutz. Experimental verification of landauer's principle linking information and thermodynamics. *Nature*, 483(7388):187–189, 2012.

[21] M. L. Blinov, J. Yang, J. R. Faeder, and W. S. Hlavacek. Graph theory for rule-based modeling of biochemical networks. pages 89–106, 2006.

[22] G. Brown and A. Sabry. Reversible communicating processes. In *Proceedings of PLACES 2015*, EPTCS 203, pages 45–59, 2015.

[23] R. Bruni, J. Meseguer, U. Montanari, and V. Sassone. A comparison of Petri net semantics under the collective token philosophy. In *Proceedings of ASIAN 1998*, LNCS 1538, pages 225–244, 1998.

[24] R. Bruni and U. Montanari. Zero-safe nets: Comparing the collective and individual token approaches. *Information and Computation*, 156(1-2):46–89, 2000.

[25] H. Buhrman, J. Tromp, and P. M. B. Vitányi. Time and space bounds for reversible simulation. In *Proceedings of ICALP 2001*, LNCS 2076, pages 1017–1027. Springer, 2001.

[26] M. Calder, S. Gilmore, and J. Hillston. Modelling the influence of RKIP on the ERK signalling pathway using the stochastic process algebra PEPA. In *Transactions on Computational Systems Biology*, LNCS 4230, pages 1–23. Springer, 2006.

[27] L. Cardelli and C. Laneve. Reversible structures. In *Proceedings of CMSB 2011*, pages 131–140. ACM, 2011.

[28] S. Chen, W. K. Fuchs, and J. Chung. Reversible debugging using program instrumentation. *IEEE Transactions on Software Engineering*, 27(8):715–727, 2001.

[29] I. Cristescu, J. Krivine, and D. Varacca. A compositional semantics for the reversible pi-calculus. In *Proceedings of ACM/IEEE 2013*, pages 388–397, 2013.

[30] V. Danos, J. Feret, W. Fontana, R. Harmer, and J. Krivine. Rule-based modelling of cellular signalling. In *Proceedings of CONCUR 2007*, LNCS 4703, pages 17–41. Springer, 2007.

[31] V. Danos and J. Krivine. Reversible communicating systems. In *Proceedings of CONCUR 2004*, LNCS 3170, pages 292–307. Springer, 2004.

[32] V. Danos and J. Krivine. Transactions in RCCS. In *Proceedings of CONCUR 2005*, LNCS 3653, pages 398–412. Springer, 2005.

[33] V. Danos and J. Krivine. Formal molecular biology done in CCS-R. *Electronic Notes in Theoretical Computer Science*, 180(3):31–49, 2007.

[34] V. Danos, J. Krivine, and P. Sobocinski. General reversibility. *Electonic Notes in Theoretical Computer Science*, 175(3):75–86, 2007.

[35] D. de Frutos-Escrig, M. Koutny, and L. Mikulski. Reversing steps in Petri nets. In *proceedings of PETRI NETS 2019*, LNCS 1152, pages 171–191. Springer, 2019.

[36] E. de Vries, V. Koutavas, and M. Hennessy. Communicating transactions - (extended abstract). In *Proceedings of CONCUR 2010*, LNCS 6269, pages 569–583. Springer, 2010.

[37] E. de Vries, V. Koutavas, and M. Hennessy. Liveness of communicating transactions (extended abstract). In *Proceedings of APLAS 2010*, LNCS 6461, pages 392–407. Springer, 2010.

[38] J. B. Dennis and S. S. Patil. *Speed independent asynchronous circuits*. Massachusetts Institute of Technology, Project MAC, 1971.

[39] Y. Dimopoulos, E. Kouppari, A. Philippou, and K. Psara. Encoding reversing Petri nets in answer set programming. In *Proceedings of RC 2020*, LNCS 12227, pages 264–271. Spinger, 2020.

[40] R. Drechsler and R. Wille. Reversible computation. In *Proceedings of IGSC 2015*, pages 1–5. IEEE Computer Society, 2015.

[41] C. Dufourd, A. Finkel, and P. Schnoebelen. Reset nets between decidability and undecidability. In *Proceedings of ICALP 1998*, LNCS 1443, pages 103–115. Springer, 1998.

[42] S. I. Feldman and C. B. Brown. Igor: A system for program debugging via reversible execution. In *Proceedings of the ACM SIGPLAN SIGOPS 1988*, pages 112–123. ACM, 1988.

[43] Y. Feng, R. Duan, Z. Ji, and M. Ying. Probabilistic bisimulations for quantum processes. *Information and Computation*, 205(11):1608–1639, 2007.

[44] R. P. Feynman. Quantum mechanical computers. *Foundations of physics*, 16(6):507–531, 1986.

[45] J. Field and C. A. Varela. Transactors: a programming model for maintaining globally consistent distributed state in unreliable environments. In *Proceedings of ACM SIGPLAN-SIGACT, POPL 2005*, pages 195–208. ACM, 2005.

[46] M. P. Frank. *Reversibility for efficient computing*. PhD thesis, Massachusetts Institute of Technology, Department of Electrical Engineering and Computer Science, 1999.

[47] M. P. Frank. Back to the future: The case for reversible computing. *arXiv preprint arXiv:1803.02789*, 2018.

[48] E. Fredkin and T. Toffoli. Conservative logic. In *Collision-based computing*, pages 47–81. Springer, 2002.

[49] E. F. Fredkin and T. Toffoli. Design principles for achieving high-performance submicron digital technologies. In *Collision-based computing*, pages 27–46. Springer, 2002.

[50] X. Gao, O. Edfors, F. Tufvesson, and E. G. Larsson. Massive MIMO in real propagation environments: Do all antennas contribute equally? *IEEE Transactions on Communications*, 63(11):3917–3928, 2015.

[51] Y. Gao, H. Vinck, and T. Kaiser. Massive mimo antenna selection: Switching architectures, capacity bounds, and optimal antenna selection algorithms. *IEEE Transactions on Signal Processing*, 66(5):1346–1360, 2018.

[52] S. J. Gay and R. Nagarajan. Communicating quantum processes. In *Proceedings of ACM, SIGPLAN-SIGACT, POPL 2005*, pages 145–157. ACM, 2005.

[53] M. Gebser, R. Kaminski, B. Kaufmann, and T. Schaub. *Answer set solving in practice*. Morgan Claypool Publishers, 2012.

[54] M. Ghazel and M. Jmaiel, editors. *Proceedings of VECoS 2016*, volume 1689 of *CEUR Workshop Proceedings*. CEUR-WS.org, 2016.

[55] U. Goltz and W. Reisig. The non-sequential behavior of Petri nets. *Information and Control*, 57(2/3):125–147, 1983.

[56] R. Grishman. The debugging system AIDS. In *Proceedings of AFIPS 1970*, AFIPS Conference Proceedings 36, pages 59–64. AFIPS Press, 1970.

[57] G. Havur, C. Cabanillas, J. Mendling, and A. Polleres. Automated resource allocation in business processes with answer set programming. In *Proceedings of BPM 2015, Revised Papers*, LNBIP 256, pages 191–203. Springer, 2015.

[58] A. Heindl and R. German. Performance modeling of IEEE 802.11 wireless lans with stochastic Petri nets. *Performance Evaluation*, 44(1-4):139–164, 2001.

[59] K. Heljanko and I. Niemelä. Bounded LTL model checking with stable models. *TPLP*, 3(4-5):519–550, 2003.

[60] M. Holzer, S. Jakobi, and M. Kutrib. Minimal reversible deterministic finite automata. In *Proceedings of DLT 2015*, LNCS 9168, pages 276–287. Springer, 2015.

[61] J. Hoydis, S. t. Brink, and M. Debbah. Massive MIMO in the UL/DL of Cellular Networks: How Many Antennas Do We Need? *IEEE Journal on Selected Areas in Communications*, 31(2):160–171, Feb. 2013.

[62] D. A. Huffman. Canonical forms for information-lossless finite-state logical machines. *IRE Transactions on Information Theory*, 5(5):41–59, 1959.

[63] R. P. James, A. Sabry, and J. Street. Theseus: A high level language for reversible computing. 2014.

[64] K. Jensen. *Coloured Petri nets - Basic concepts, analysis methods and practical use - Volume 1, second edition*. Monographs in Theoretical Computer Science. An EATCS Series. Springer, 1996.

[65] J. R. Jump and P. Thiagarajan. On the equivalence of asynchronous control structures. In *Proceedings of SWAT 1972*, pages 212–223. IEEE, 1972.

[66] P. Kaye, R. Laflamme, and M. Mosca. *An introduction to quantum computing*. Oxford University Press, 2006.

[67] T. Koju, S. Takada, and N. Doi. An efficient and generic reversible debugger using the virtual machine based approach. In *Proceedings of VEE 2005*, pages 79–88. ACM, 2005.

[68] V. Koutavas, C. Spaccasassi, and M. Hennessy. Bisimulations for communicating transactions - (extended abstract). In *Proceedings of FOSSACS 2014*, LNCS 8412, pages 320–334. Springer, 2014.

[69] S. Kuhn, B. Aman, G. Ciobanu, A. Philippou, K. Psara, and I. Ulidowski. Reversibility in chemical reactions. In *Reversible Computation: Extending Horizons of Computing - Selected Results of the COST Action IC1405*, LNCS 1270, pages 151–176. Spinger, 2020.

[70] S. Kuhn and I. Ulidowski. A calculus for local reversibility. In *Proceedings of RC 2016*, LNCS 9720, pages 20–35. Springer, 2016.

[71] M. Kutrib and A. Malcher. Reversible pushdown automata. In *Proceedings of LATA 2010*, LNCS 6031, pages 368–379. Springer, 2010.

[72] M. Kutrib and A. Malcher. One-way reversible multi-head finite automata. *Theoretical Computer Science*, 682:149–164, 2017.

[73] M. Kutrib, A. Malcher, and M. Wendlandt. Reversible queue automata. *Fundamenta Informaticae*, 148(3-4):341–368, 2016.

[74] M. Kutrib, A. Malcher, and M. Wendlandt. When input-driven pushdown automata meet reversiblity. *RAIRO - Theoretical Informatics and Applications*, 50(4):313–330, 2016.

[75] R. Landauer. Irreversibility and heat generation in the computing process. *IBM Journal of Research and Development*, 5(3):183–191, 1961.

[76] I. Lanese, M. Lienhardt, C. A. Mezzina, A. Schmitt, and J. Stefani. Concurrent flexible reversibility. In *Proceedings of ESOP 2013*, LNCS 7792, pages 370–390. Springer, 2013.

[77] I. Lanese and D. Medic. A general approach to derive uncontrolled reversible semantics. In *Proceedings CONCUR 2020*, LIPIcs 171, pages 33:1–33:24. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[78] I. Lanese, C. A. Mezzina, A. Schmitt, and J. Stefani. Controlling reversibility in higher-order pi. In *Proceedings of CONCUR 2011*, LNCS 6901, pages 297–311. Springer, 2011.

[79] I. Lanese, C. A. Mezzina, and J. Stefani. Reversing higher-order pi. In *Proceedings of CONCUR 2010*, LNCS 6269, pages 478–493, 2010.

[80] I. Lanese, C. A. Mezzina, and J. Stefani. Controlled reversibility and compensations. In *Proceedings of RC 2012*, LNCS 7581, pages 233–240. Springer, 2012.

[81] I. Lanese, C. A. Mezzina, and J. Stefani. Controlled reversibility and compensations. In *Proceedings of RC 2012*, LNCS 7581, pages 233–240. Springer, 2012.

[82] I. Lanese, C. A. Mezzina, and J. Stefani. Reversibility in the higher-order π-calculus. *Theoretical Computer Science*, 625:25–84, 2016.

[83] I. Lanese, C. A. Mezzina, and F. Tiezzi. Causal-consistent reversibility. *European Association for Theoretical Computer Science*, 114, 2014.

[84] I. Lanese, I. C. C. Phillips, and I. Ulidowski. An axiomatic approach to reversible computation. In *Proceedings of FOSSACS 2020*, LNCS, pages 442–461. Spinger, 2020.

[85] J. S. Laursen, U. P. Schultz, and L. Ellekilde. Automatic error recovery in robot assembly operations using reverse execution. In *Proceedings of IEEE/RSJ, IROS 2015*, pages 1785–1792. IEEE, 2015.

[86] Y. Lecerf. Récursive insolubilité de l'équation générale de diagonalisation de deux monomorphisms de monoïdes libres ϕ x=ψ x. *Comptes rendus de l'Académie des Sciences Paris*, 257:2940–2943.

[87] B. Lewis and M. Ducassé. Using events to debug java programs backwards in time. In *Proceedings of ACM, SIGPLAN, OOPSLA 2003*, pages 96–97. ACM, 2003.

[88] M. Lienhardt, I. Lanese, C. A. Mezzina, and J. Stefani. A reversible abstract machine and its space overhead. In *Proceedings of IFIP, FMOODS, IFIP, FORTE 2012*, LNCS 7273, pages 1–17. Springer, 2012.

[89] V. Lifschitz. *Answer Set Programming*. Springer, 2019.

[90] M. A. Marsan. Stochastic Petri nets: an elementary introduction. In *European Workshop on Applications and Theory in Petri Nets*, pages 1–29. Springer, 1988.

[91] M. A. Marsan, G. Balbo, and G. Conte. Performance models of multiprocessor systems. 1986.

[92] W. Mauerer. Semantics and simulation of communication in quantum programming. *arXiv preprint quant-ph/0511145*, 2005.

[93] H. C. Melgratti, C. A. Mezzina, I. Phillips, G. M. Pinna, and I. Ulidowski. Reversible occurrence nets and causal reversible prime event structures. In *Proceedingds of RC 2020*, LNCS 12227. Spinger, 2020.

[94] H. C. Melgratti, C. A. Mezzina, and I. Ulidowski. Reversing P/T nets. In *Proceedings of COORDINATION 2019*, LNCS 11533, pages 19–36. Springer, 2019.

[95] H. C. Melgratti, C. A. Mezzina, and I. Ulidowski. Reversing place transition nets. *Logical Methods in Computer Science*, 16(4), 2020.

[96] L. Mikulski and I. Lanese. Reversing unbounded Petri nets. In *Proceedings of PETRI NETS 2019*, volume 11522 of *LNCS*, pages 213–233. Springer, 2019.

[97] R. Milner. *Communication and concurrency*. Prentice hall New York etc., 1989.

[98] H. Mlnařík. *Quantum Programming Language LanQ*. PhD thesis, Masarykova Univerzita, Fakulta Informatiky, 2007.

[99] K. Morita. Reversible computing and cellular automata - A survey. *Theoretical Computer Science*, 395(1):101–131, 2008.

[100] K. Morita. Two-way reversible multi-head finite automata. *Fundamenta Informaticae*, 110(1-4):241–254, 2011.

[101] T. Murata. Relevance of network theory to models of distributed/parallel processing. *Journal of the Franklin Institute*, 310(1):41–50, 1980.

[102] T. Murata. Petri nets: Properties, analysis and applications. *Proceedings of the IEEE*, 77(4):541–580, 1989.

[103] C. Okasaki. *Purely functional data structures*. Cambridge University Press, 1999.

[104] B. Ömer. A procedural formalism for quantum computing. 1998.

[105] A. Ozgur, O. Lévêque, and D. Tse. Spatial degrees of freedom of large distributed mimo systems and wireless ad hoc networks. *IEEE Journal on Selected Areas in Communications*, 31(2):202–214, 2013.

[106] J. Palsberg and M. Abadi, editors. *Proceedings of ACM SIGPLAN-SIGACT, POPL 2005*. ACM, 2005.

[107] J. A. N. Pérez and A. Voronkov. Encodings of bounded LTL model checking in effectively propositional logic. In *Proceedings of CADE-21*, LNCS 4603, pages 346–361. Springer, 2007.

[108] M. A. Perkowski, M. Chrzanowska-Jeske, A. Mishchenko, X. Song, A. Al-Rabadi, B. Massey, P. Kerntopf, A. Buller, L. Józwiak, and A. J. Coppola. Regular realization of symmetric functions using reversible logic. In *Proceedings of Euro-DSD 2001*, pages 245–253. IEEE Computer Society, 2001.

[109] K. S. Perumalla. *Introduction to reversible computing*. CRC Press, 2013.

[110] C. A. Petri. Kommunikation mit automaten. *PhD Thesis 1962*.

[111] A. Philippou and K. Psara. Reversible computation in Petri nets. In *Proceedings of RC 2018*, LNCS 11106, pages 84–101. Springer, 2018.

[112] A. Philippou and K. Psara. Reversible computation in cyclic Petri nets (under submission). 2020.

[113] A. Philippou, K. Psara, and H. Siljak. Controlling reversibility in reversing Petri nets with application to wireless communications. In *Proceedings of RC 2019*, LNCS 11497, pages 238–245. Springer, 2019.

[114] A. Philippou, K. Psara, and H. Siljak. A collective-interpretation semantics for reversing Petri nets. (In preparation).

[115] I. Phillips and I. Ulidowski. Reversibility and models for concurrency. *Electronic Notes in Theoretical Computer Science*, 192(1):93–108, 2007.

[116] I. Phillips and I. Ulidowski. Reversibility and asymmetric conflict in event structures. *Journal of Logical and Algebraic Methods in Programming*, 84(6):781–805, 2015.

[117] I. Phillips and I. Ulidowski. Reversing algebraic process calculi. In *Proceedings of FOSSACS 2006*, LNCS 3921, pages 246–260. Springer, 2016.

[118] I. Phillips, I. Ulidowski, and S. Yuen. A reversible process calculus and the modelling of the ERK signalling pathway. In *Proceedings of RC 2012*, LNCS 7581, pages 218–232. Springer, 2012.

[119] I. Phillips, I. Ulidowski, and S. Yuen. Modelling of bonding with processes and events. In *Proceedings of RC 2013*, LNCS 7947, pages 141–154. Springer, 2013.

[120] J. Pin. On reversible automata. In *Proceedings of LATIN 1992*, LNCS 583, pages 401–416, 1992.

[121] W. Reisig. Petri nets with individual tokens. *Theoretical Computer Science*, 41:185–213, 1985.

[122] W. Reisig. *Understanding Petri nets: modeling techniques, analysis methods, case studies.* Springer, 2013.

[123] G. Rozenberg and R. Verraedt. Subset languages of Petri nets part i: The relationship to string languages and normal forms. *Theoretical Computer Science*, 26(3):301–326, 1983.

[124] U. P. Schultz. Towards a general-purpose, reversible language for controlling self-reconfigurable robots. In *Proceedings of RC 2012*, LNCS 7581, pages 97–111. Springer, 2012.

[125] U. P. Schultz, M. Bordignon, and K. Støy. Robust and reversible execution of self-reconfiguration sequences. *Robotica*, 29(1):35–57, 2011.

[126] U. P. Schultz, J. S. Laursen, L. Ellekilde, and H. B. Axelsen. Towards a domain-specific language for reversible assembly sequences. In *Proceedings of RC 2015*, LNCS 9138, pages 111–126. Springer, 2015.

[127] H. Siljak, K. Psara, and A. Philippou. Distributed antenna selection for massive MIMO using reversing Petri nets. *IEEE Wireless Commununication Letters*, 8(5):1427–1430, 2019.

[128] D. G. Stork and R. J. van Glabbeek. Token-controlled place refinement in hierarchical Petri nets with application to active document workflow. In *Proceedings of ICATPN 2002*, pages 394–413, 2002.

[129] D. Tabak and A. H. Levis. Petri net representation of decision models. *IEEE Transactions on Systems, Man, and Cybernetics*, (6):812–818, 1985.

[130] G. Thieler-Mevissen. *The Petri net calculus of predicate logic*. Ges. für Mathematik und Datenverarbeitung, Inst. für Informationssystemforschung, 1977.

[131] F. Tiezzi and N. Yoshida. Reversible session-based pi-calculus. *Journal of Logical and Algebraic Methods in Programming*, 84(5):684–707, 2015.

[132] F. Tiezzi and N. Yoshida. Reversing single sessions. *CoRR*, abs/1510.07253, 2015.

[133] J. Timler and C. S. Lent. Maxwell's demon and quantum-dot cellular automata. *Journal of Applied Physics*, 94(2):1050–1060, 2003.

[134] T. Toffoli. Reversible computing. In *Proceedings of ICALP 1980*, LNCS 85, pages 632–644. Springer, 1980.

[135] I. Ulidowski, I. Lanese, U. P. Schultz, and C. Ferreira, editors. *Reversible Computation: Extending Horizons of Computing - Selected Results of the COST Action IC1405*, volume 12070 of *Lecture Notes in Computer Science*. Springer, 2020.

[136] I. Ulidowski, I. Phillips, and S. Yuen. Concurrency and reversibility. In *Proceedings of RC 2014*, LNCS 8507, pages 1–14. Springer, 2014.

[137] R. J. van Glabbeek. The individual and collective token interpretations of Petri nets. In *Proceedings of CONCUR 2005*, LNCS 3653, pages 323–337. Springer, 2005.

[138] R. J. van Glabbeek, U. Goltz, and J. Schicke. On causal semantics of Petri nets. In *Proceedings of CONCUR 2011*, LNCS 6901, pages 43–59. Springer.

[139] R. J. van Glabbeek and G. D. Plotkin. Configuration structures. In *Proceedings of IEEE Symposium on Logic in Computer Science 1995*, pages 199–209. IEEE Computer Society, 1995.

[140] K. Voss. Using predicate/transition-nets to model and analyze distributed database systems. *IEEE Transactions on Software Engineering*, (6):539–544, 1980.

[141] J. Wang. *Timed Petri nets: Theory and application*, volume 9. Springer Science & Business Media, 2012.

[142] T. Yokoyama. Reversible computation and reversible programming languages. *Electronic Notes in Theoretical Computer Science*, 253(6):71–81, 2010.

[143] T. Yokoyama, H. B. Axelsen, and R. Glück. Reversible flowchart languages and the structured reversible program theorem. In *Proceedings of ICALP 2008*, LNCS 5126, pages 258–270. Springer, 2008.

[144] T. Yokoyama, H. B. Axelsen, and R. Glück. Towards a reversible functional language. In *Proceedings of RC 2011*, LNCS 7165, pages 14–29. Springer, 2011.

[145] D. A. Zaitsev. Toward the minimal universal Petri net. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 44(1):47–58, 2014.

[146] M. V. Zelkowitz. Reversible execution. *Communications of the ACM*, 16(9):566, 1973.

[147] L. Ziarek and S. Jagannathan. Lightweight checkpointing for concurrent ML. *Journal of Functional Programming*, 20(2):137–173, 2010.