

SYMBOLIC CALCULUS IN CALCULATING CHARACTERISTIC AND ACYCLIC POLYNOMIALS OF GRAPHS

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ABSTRACT

There are many polynomials closely connected with graphs and the theory of graphs. These polynomials are useful both in theory and its application as they enable to determine some properties of graphs and relations among graphs by the methods of algebra. In our contribution we will illustrate the use of symbolic calculus of Matlab software package as an efficient assistant in calculating the characteristic and the acyclic polynomial of various types of graphs.

KEYWORDS

Symbolic calculus, graphs, acyclic polynomial, characteristic polynomial

INTRODUCTION

There are several kinds of polynomials that enable to characterize the structure of graphs in a certain way. These polynomials belong to the graph theory. Some of the polynomials were introduced on the base of investigating special objects mainly in some branches of physics and chemistry. A graph G can be most precisely described by its adjacency matrix which will be denoted by A or $A(G)$. The characteristic polynomial of a graph G , which will be defined as the characteristic polynomial of its adjacency matrix, i.e. it is defined by the formula $P(G,x) = \det(xI - A)$, where I is the unit matrix of the same size as the matrix A . The roots of the characteristic polynomial are called the eigenvalues of the matrix A . They form the spectrum of the graph G . As the calculation of the characteristic polynomial using the above mentioned definition is rather complicated, the following very useful Sachs theorem is often used for these calculations.

Let us have a graph G , without loops and multiple edges, with n vertices. Its characteristic polynomial has the form

$$P(G, x) = \sum_{j=0}^n a_j x^{n-j},$$

where the relation

$$a_j = \sum_{s \in S_j} (-1)^{c(s)} 2^{r(s)}$$

holds for all j , $1 \leq j \leq n$, and $a_0 = 1$ by definition. Further, S_j is the set of all Sachs subgraphs s of the graph G having j vertices, $c(s)$ denotes the number of components of s and $r(s)$ is the number of cycles in s .

The acyclic polynomial was introduced to solve concrete problems in natural science. A systematic mathematic theory of the acyclic polynomial was established by Gutman [1] in 1977.

Let us have a graph with n vertices. Its acyclic polynomial is defined by the relation

$$\beta(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k p(G, k) x^{n-2k},$$

where $p(G, k)$ is the number of the subgraphs of the graph G which are created by exactly k disjoint edges (complete graphs K_2). The above mentioned subgraphs of the graph G having k independent edges are called k -matchings. Then the number $p(G, k)$ determines the number of different k -matchings of the graph G and sometimes it is called the k -th matching number of the graph G . If the number of edges of the graph G is $\frac{n}{2}$, then the related matching is called the complete matching of the graph G . It presents a linear factor of the graph G .

Sachs theory appears to be very useful in determining the form of the acyclic polynomial of a graph. Acyclic Sachs graphs are assumed to be the subgraphs of a graph G having only complete graphs K_2 as their components.

SPECIAL METHOD USING MATLAB

In our contribution we will only deal with the acyclic polynomial of a graph as calculations for the characteristic polynomial are very similar.

Matlab software package and in our case especially symbolic calculations of Matlab give us a very rapid and efficient tool for calculations of these polynomials. It is neither possible nor purposeful to give here a formally precise definition of various classes of graphs which we have chosen for performing calculation of adequate acyclic polynomials. We will illustrate the topology of each class of graphs using the following figures.

A path L_n is a graph defined by Fig 1.

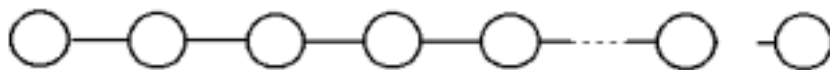


Fig.1

A cycle C_n is a graph defined by Fig 2.

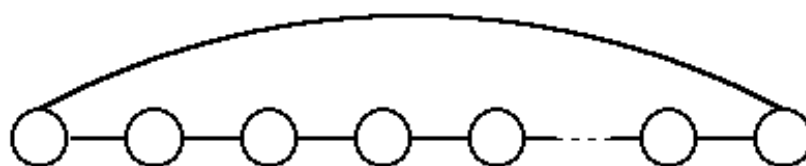


Fig.2

An open ladder Z_n defined by Fig 3.

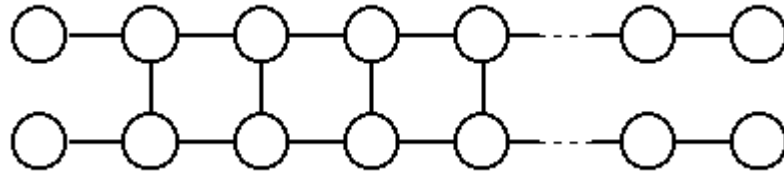


Fig. 3

A half closed ladder G_n defined by Fig 4.

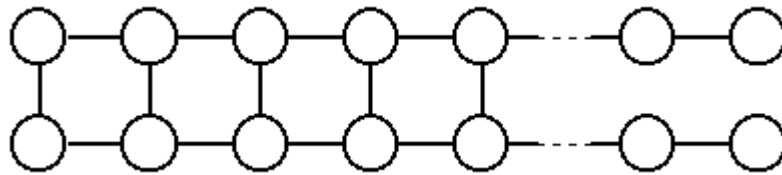


Fig.4

A closed ladder H_n defined by Fig 5.

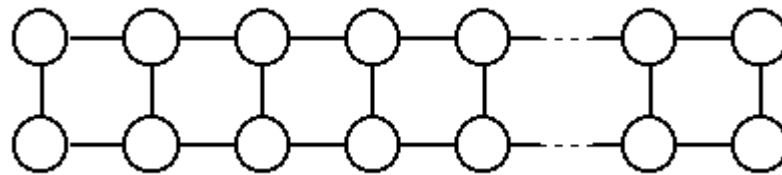


Fig. 5

A wheel M_n is a graph defined by Fig 6.

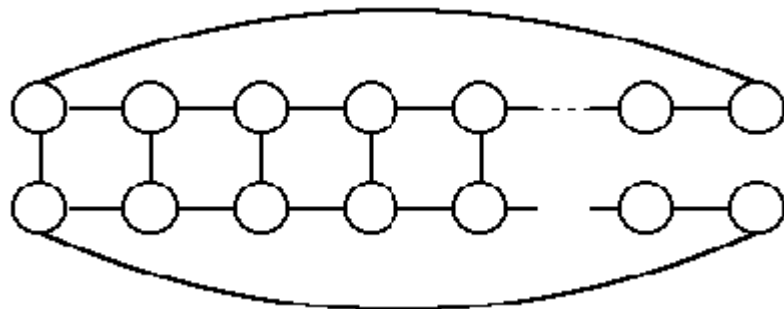


Fig. 6

Let us bring in without proof (see i.e. [2]) the recurrent relations for the calculation of the acyclic polynomials of special classes of graphs selected in our contribution. These formulas were used by software Matlab to calculate the acyclic polynomials of these classes of graphs.

These relations are the substantial part of the following script (in Matlab called M file) for the calculation of the first n acyclic polynomials for paths, cycles, ladders, wheels. Consequently the connection between the above relations and rows of the adequate M file is clearly seen

$$\begin{aligned}
\beta(L_1) &= x \\
\beta(L_2) &= x^2 - 1 \\
\beta(L_n) &= \beta(L_1)\beta(L_{n-1}) - \beta(L_{n-2}), n \geq 3 \\
\beta(C_n) &= \beta(L_n) - \beta(L_{n-2}), n \geq 3 \\
\beta(Z_1) &= \beta(L_4) - \beta(L_2) \\
\beta(Z_n) &= \beta(H_{n-1}) - \beta(Z_{n-1}), n \geq 2 \\
\beta(G_0) &= \beta^2(L_3) - \beta^4(L_1) \\
\beta(G_n) &= \beta(H_n) + \beta(H_{n+1}), n \geq 1 \\
H_i &\equiv H_{i,1} \\
\beta(H_{1,j}) &= \beta(L_{2(j+2)}) - \beta^2(L_j)\beta(L_2), j \geq 1 \\
\beta(H_{i+1,j-1}) &= \beta(H_{i,j}) - \beta(Z_i)\beta^2(L_{j-1}), i \geq 1, j \geq 2 \\
\beta(M_{1,j}) &= \beta^2(C_{j+1}) - \beta^2(L_j), j \geq 2 \\
\beta(M_{2,j}) &= \beta(M_{1,j+1}) - \beta(L_{2(j+1)}), j \geq 1 \\
\beta(M_{i+1,j-1}) &= \beta(M_{i,j}) - \beta(H_{i-1,j-1}), i \geq 2, j \geq 2 \\
\beta(M_n) &= \beta(M_{n-1,1}) - \beta(Z_{n-2}), n \geq 3
\end{aligned}$$

%Here given M-file enables recurrent calculation of the acyclic polynomial of certain classes of graphs using symbolic calculus of Matlab.

%Variable POCET gives the number of derived polynomials

POCET=10

%Calculation of the polynomials for the class of graphs called paths (variable betaL, BetaL)

Calculation of the polynomials for the class of graphs called cycles(variable betaC, BetaC)

c1 = [1 0];

c2 = [1 0 -1];

betaL(1)=poly2sym(c1);

BetaL(1)=betaL(1);

betaL(2)=poly2sym(c2);

BetaL(2)=betaL(2);

for n =3:2*POCET+4

betaL(n)=betaL(1)*betaL(n-1)-betaL(n-2);

betaC(n)=betaL(n)-betaL(n-2);

BetaL(n)=collect(betaL(n));

BetaC(n)=collect(betaC(n));

end

%Calculation of the polynomials for the class of graphs called (variable betaZ, BetaZ)

%Calculation of the polynomials for the class of graphs called (variable betaG, BetaG)

%Calculation of the polynomials for the class of graphs called (variable betaH, BetaH)

%Here are calculated auxiliary polynomials with two parameters betah2(i,j)

betaZ(1)=betaL(4)-betaL(2);

```

for j=1:POCET
    betaH2(1,j)=betaL(2*(j+2))-betaL(j)^2*betaL(2);
end
for i=1:POCET
    for j=2:POCET
        betaH(i)=betaH2(i,1);
        if (i>1)
            betaZ(i)=betaH(i-1)-betaZ(i-1);
        end
        betaH2(i+1,j-1)=betaH2(i,j)-betaZ(i)*(betaL(j-1))^2;
    end
end
for n=1:POCET-1
    betaG(n)=betaH(n)+betaH(n+1);
end
for n=1:POCET-1
    BetaZ(n)=collect(betaZ(n));
    BetaG(n)=collect(betaG(n));
    BetaH(n)=collect(betaH(n));
end
%%Calculation of the polynomials for the class of graphs called (variable betaM,BetaM)
% Here are calculated auxiliary polynomials with two parameters betaM2(i,j)
for j=2:POCET+1
    betaM2(1,j)=betaC(j+1)*betaC(j+1)-betaL(j)*betaL(j);
end
for j=1:POCET
    betaM2(2,j)=betaM2(1,j+1)-betaL(2*(j+1));
end
for i=2:POCET
    for j=2:POCET
        betaM2(i+1,j-1)=betaM2(i,j)-betaH2(i-1,j-1);
    end
end
for n=3:POCET
    betaM(n)=betaM2(n-1,1)+betaZ(n-2);
end

```

Let us show the output of the performed symbolic calculations for the above mentioned classes of graphs for the first 9 polynomials of all the investigated classes. These calculations require calculation of 24 polynomials for graphs belonging to the class path, cycle and other auxiliary polynomials as it is seen from the recurrent formulas. The software gives us the solution for each class of polynomials in two variables. We will mention only the second of them which contains simplified symbolic expression of the polynomials and the identifier of which starts with capital letter. For simplification the function collect of Matlab was used.

The acyclic polynomials for 24 graphs of the class path:

```

BetaL =[
x,
x^2-1,
x^3-2*x,
x^4-3*x^2+1,
x^5-4*x^3+3*x,

```

$x^6-5x^4+6x^2-1,$
 $x^7-6x^5+10x^3-4x,$
 $x^8-7x^6+15x^4-10x^2+1,$
 $x^9-8x^7+21x^5-20x^3+5x,$
 $x^{10}-9x^8+28x^6-35x^4+15x^2-1,$
 $x^{11}-10x^9+36x^7-56x^5+35x^3-6x,$
 $x^{12}-11x^{10}+45x^8-84x^6+70x^4-21x^2+1,$
 $x^{13}-12x^{11}+55x^9-120x^7+126x^5-56x^3+7x,$
 $x^{14}-13x^{12}+66x^{10}-165x^8+210x^6-126x^4+28x^2-1,$
 $x^{15}-14x^{13}+78x^{11}-220x^9+330x^7-252x^5+84x^3-8x,$
 $x^{16}-15x^{14}+91x^{12}-286x^{10}+495x^8-462x^6+210x^4-36x^2+1,$
 $x^{17}-16x^{15}+105x^{13}-364x^{11}+715x^9-792x^7+462x^5-120x^3+9x,$
 $x^{18}-17x^{16}+120x^{14}-455x^{12}+1001x^{10}-1287x^8+924x^6-330x^4+45x^2-1,$
 $x^{19}-18x^{17}+136x^{15}-560x^{13}+1365x^{11}-2002x^9+1716x^7-792x^5+165x^3-10x,$
 $x^{20}-19x^{18}+153x^{16}-680x^{14}+1820x^{12}-3003x^{10}+3003x^8-1716x^6+495x^4-55x^2+1,$
 $x^{21}-20x^{19}+171x^{17}-816x^{15}+2380x^{13}-4368x^{11}+5005x^9-3432x^7+1287x^5-220x^3+11x,$
 $x^{22}-21x^{20}+190x^{18}-969x^{16}+3060x^{14}-6188x^{12}+8008x^{10}-6435x^8+3003x^6-715x^4+66x^2-1,$
 $x^{23}-22x^{21}+210x^{19}-1140x^{17}+3876x^{15}-8568x^{13}+12376x^{11}-11440x^9+6435x^7-2002x^5+286x^3-12x,$
 $x^{24}-23x^{22}+231x^{20}-1330x^{18}+4845x^{16}-11628x^{14}+18564x^{12}-19448x^{10}+12870x^8-5005x^6+1001x^4-78x^2+1]$

The acyclic polynomials for 24 cycles:

BetaC = [
0,0,
 $x^3-3x,$
 $x^4-4x^2+2,$
 $x^5-5x^3+5x,$
 $x^6-6x^4+9x^2-2,$
 $x^7-7x^5+14x^3-7x,$
 $x^8-8x^6+20x^4-16x^2+2,$
 $x^9-9x^7+27x^5-30x^3+9x,$
 $x^{10}-10x^8+35x^6-50x^4+25x^2-2,$
 $x^{11}-11x^9+44x^7-77x^5+55x^3-11x,$
 $x^{12}-12x^{10}+54x^8-112x^6+105x^4-36x^2+2,$
 $x^{13}-13x^{11}+65x^9-156x^7+182x^5-91x^3+13x,$
 $x^{14}-14x^{12}+77x^{10}-210x^8+294x^6-196x^4+49x^2-2,$
 $x^{15}-15x^{13}+90x^{11}-275x^9+450x^7-378x^5+140x^3-15x,$
 $x^{16}-16x^{14}+104x^{12}-352x^{10}+660x^8-672x^6+336x^4-64x^2+2,$
 $x^{17}-17x^{15}+119x^{13}-442x^{11}+935x^9-1122x^7+714x^5-204x^3+17x,$
 $x^{18}-18x^{16}+135x^{14}-546x^{12}+1287x^{10}-1782x^8+1386x^6-540x^4+81x^2-2,$
 $x^{19}-19x^{17}+152x^{15}-665x^{13}+1729x^{11}-2717x^9+2508x^7-1254x^5+285x^3-19x,$
 $x^{20}-20x^{18}+170x^{16}-800x^{14}+2275x^{12}-4004x^{10}+4290x^8-2640x^6+825x^4-100x^2+2,$
 $x^{21}-21x^{19}+189x^{17}-952x^{15}+2940x^{13}-5733x^{11}+7007x^9-5148x^7+2079x^5-385x^3+21x,$

$$\begin{aligned}
& x^{22}-22x^{20}+209x^{18}-1122x^{16}+3740x^{14}-8008x^{12}+11011x^{10}-9438x^8+ \\
& 4719x^6-1210x^4+121x^2-2, \\
& x^{23}-23x^{21}+230x^{19}-1311x^{17}+4692x^{15}-10948x^{13}+16744x^{11}-16445x^9+ \\
& 9867x^7-3289x^5+506x^3-23x, \\
& x^{24}-24x^{22}+252x^{20}-1520x^{18}+5814x^{16}-14688x^{14}+24752x^{12}-27456x^{10}+ \\
& 19305x^8-8008x^6+1716x^4-144x^2+2]
\end{aligned}$$

The acyclic polynomials for 9 closed ladders:

$$\begin{aligned}
\text{BetaZ} = [& \\
& x^4-4x^2+2, \\
& x^6-7x^4+11x^2-3, \\
& x^8-10x^6+29x^4-26x^2+5, \\
& x^{10}-13x^8+56x^6-94x^4+56x^2-8, \\
& x^{12}-16x^{10}+92x^8-234x^6+263x^4-114x^2+13, \\
& x^{14}-19x^{12}+137x^{10}-473x^8+815x^6-667x^4+223x^2-21, \\
& x^{16}-22x^{14}+191x^{12}-838x^{10}+1982x^8-2504x^6+1577x^4-424x^2+34, \\
& x^{18}-25x^{16}+254x^{14}-1356x^{12}+4115x^{10}-7191x^8+7018x^6- \\
& 3538x^4+789x^2-55, \\
& x^{20}-28x^{18}+326x^{16}-2054x^{14}+7646x^{12}-17266x^{10}+23431x^8-18336x^6+ \\
& 7622x^4-1444x^2+89]
\end{aligned}$$

The acyclic polynomials for 9 open ladders:

$$\begin{aligned}
\text{BetaG} = [& \\
& x^8-8x^6+16x^4-8x^2+1, \\
& x^{10}-11x^8+37x^6-43x^4+15x^2-1, \\
& x^{12}-14x^{10}+67x^8-132x^6+104x^4-28x^2+2, \\
& x^{14}-17x^{12}+106x^{10}-302x^8+403x^6-235x^4+51x^2-3, \\
& x^{16}-20x^{14}+154x^{12}-580x^{10}+1128x^8-1108x^6+506x^4-92x^2+5, \\
& x^{18}-23x^{16}+211x^{14}-993x^{12}+2576x^{10}-3700x^8+2825x^6-1051x^4+164x^2- \\
& 8, \\
& x^{20}-26x^{18}+277x^{16}-1568x^{14}+5125x^{12}-9874x^{10}+11031x^8- \\
& 6804x^6+2123x^4-290x^2+13, \\
& x^{22}-29x^{20}+352x^{18}-2332x^{16}+9234x^{14}-22546x^{12}+33899x^{10}- \\
& 30563x^8+15674x^6-4194x^4+509x^2-21, \\
& x^{24}-32x^{22}+436x^{20}-3312x^{18}+15443x^{16}-45928x^{14}+87872x^{12}- \\
& 106816x^{10}+79904x^8-34840x^6+8136x^4-888x^2+34]
\end{aligned}$$

The acyclic polynomials for 9 open ladders:

$$\begin{aligned}
\text{BetaH} = [& \\
& x^6-6x^4+7x^2-1, \\
& x^8-9x^6+22x^4-15x^2+2, \\
& x^{10}-12x^8+46x^6-65x^4+30x^2-3, \\
& x^{12}-15x^{10}+79x^8-178x^6+169x^4-58x^2+5, \\
& x^{14}-18x^{12}+121x^{10}-381x^8+581x^6-404x^4+109x^2-8, \\
& x^{16}-21x^{14}+172x^{12}-701x^{10}+1509x^8-1689x^6+910x^4-201x^2+13, \\
& x^{18}-24x^{16}+232x^{14}-1165x^{12}+3277x^{10}-5209x^8+4514x^6- \\
& 1961x^4+365x^2-21, \\
& x^{20}-27x^{18}+301x^{16}-1800x^{14}+6290x^{12}-13151x^{10}+16240x^8- \\
& 11318x^6+4084x^4-655x^2+34, \\
& x^{22}-30x^{20}+379x^{18}-2633x^{16}+11034x^{14}-28836x^{12}+47050x^{10}- \\
& 46803x^8+26992x^6-8278x^4+1164x^2-55]
\end{aligned}$$

The acyclic polynomials for 8 wheels:

BetaM = [
0,0,
 $x^6 - 7x^4 + 10x^2$,
 $x^8 - 10x^6 + 28x^4 - 22x^2 + 3$,
 $x^{10} - 13x^8 + 55x^6 - 87x^4 + 43x^2 - 1$,
 $x^{12} - 16x^{10} + 91x^8 - 224x^6 + 232x^4 - 80x^2 + 4$,
 $x^{14} - 19x^{12} + 136x^{10} - 460x^8 + 757x^6 - 559x^4 + 143x^2 - 3$,
 $x^{16} - 22x^{14} + 190x^{12} - 822x^{10} + 1888x^8 - 2250x^6 + 1254x^4 - 250x^2 + 7$,
 $x^{18} - 25x^{16} + 253x^{14} - 1337x^{12} + 3976x^{10} - 6692x^8 + 6089x^6 -$
 $2669x^4 + 430x^2 - 8$,
 $x^{20} - 28x^{18} + 325x^{16} - 2032x^{14} + 7453x^{12} - 16396x^{10} + 21263x^8 -$
 $15344x^6 + 5459x^4 - 732x^2 + 15]$

CONCLUSION REMARK

In our contribution we aimed to show one of the effective ways how to use some features of the symbolic calculus of Matlab software package. It is evident that this method can be used to calculate each polynomial of a graph for which a decomposition formula holds. Such formula is called Heilbronner formula for the characteristic and the acyclic polynomial of a graph.

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