University of Cyprus

DEPARTMENT OF MATHEMATICS AND STATISTICS

Numerical Solution of *Fredholm* Integral and Integro-Differential Equations of the second type

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Contents

1 Introduction

In this thesis we study the numerical approximation of the solution to *Fredholm* integral and integro-differential equations of the second type, as these will be defined in the next chapter. Integral equations can be used to model population growth [5], biological systems [9], elastohydrodynamic lubrication [4], quantum scattering [3], and heat transfer [6]. Furthermore, ordinary and partial differential equations (ODEs and PDEs), which also arise in a wide range of physical problems, can be reformulated as integral equations. The advantage of the integral equation reformulation is that associated boundary and initial conditions (BCs and ICs) are incorporated within the integral equation, in contrast to ODEs and PDEs on which BCs and ICs are imposed.

It all began in 1823 when *Abel* proposed a generalization of the tautochrone problem whose solution involved the solution of an integral equation which was later designated as an integral equation of the first type, and in 1837 *Liouville* showed that the determination of a particular solution of a second order linear differential equation depends on solving an integral equation of a different type, called the integral equation of the second kind. After that, the work of *Volterra*, *Fredholm*, and *Hilbert*, starting in 1896, has made integral equations a discipline of its own, so to speak. In this thesis, we focus on *Fredholm* integral equations of the second type, and the approximation of their solution.

In Chapter 2, we give an overview of integral equations, including integro-differential equations; we present their definition, the different categories they are divided in, existence and uniqueness of a solution, as well as some other general results.

In Chapter 3, we consider numerical approximations by means of classical composite quadrature rules and collocation, an approach called *Nyström's* method in the literature [2]. In particular, we utilize the composite Trapezoidal and *Simpson's* Rules for the approximation of the integral in the equation, in conjunction with collocation at the same nodal points.

In Chapter 4, we consider *Fredholm* integro-differential equations of the second type, and present the numerical approximation for their solution by the composite Trapezoidal rule, along with centered finite differences for the approximation of the derivative. In Section 4.1 we consider **singularly perturbed** *Fredholm* integro-differential equations of the second type, and present the numerical approximation for their solution, using the popular *Shishkin* mesh [8].

We close with some concluding remarks in Chapter 5.

2 Integral Equations

An equation that contains the unknown function $u(x)$ within an integral is called an *integral equation*. For example

$$f(x) = \int_a^b K(x,t)u(t)dt, x \in [a,b], \tag{1}$$

where we assume that the so-called *kernel* $K(x,t)$ is a given function on the interval $[a,b] \times [a,b]$, satisfying

$$\int_a^b \int_a^b |K(x,t)|^2 dxdt < \infty, \tag{2}$$

and that $f(x)$ is a given, continuous function on $[a,b]$. Since the limits of integration are constant, (1) is called an integral equation of *Fredholm* type. The integral equation

$$f(x) = \int_a^x K(x,t)u(t)dt, x \in [a,b],$$

is of *Voltera* type, since it has a variable limit of integration. We will only consider *Fredholm* integral equations in this thesis.

If $u(x)$ appears inside and outside the integral, the *Fredholm* integral equation is called of the second type:

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt, x \in [a,b]. \tag{3}$$

The celebrated *Fredholm Alternative* [2] tells us when an integral equation has a unique solution: If the homogeneous integral equation

$$y(x) = \int_a^b K(x,t)y(t)dt, x \in [a,b],$$

has only the zero solution $y(x) = 0$, then the corresponding non-homogeneous integral equation

$$y(x) = f(x) + \int_a^b K(x,t)y(t)dt, x \in [a,b],$$

always has a unique solution. Conversely, if the homogeneous equation has non-zero solutions, then the non-homogeneous integral equation either has no solutions or has infinite solutions, and this depends on the given function $f(x)$ (see [2] for more details).

We mention that in the so-called case of a *separable* kernel, i.e. $K(x,t) = k_1(x)k_2(t)$, for some smooth functions k_1, k_2, integral equations may be solved analytically, in most cases (see Example 2.1 below). However, in the more general case of a smooth kernel considered in this thesis, a numerical method is necessary. In Chapter 3 we will consider the numerical approximation of the solution to (3).

Example 2.1: Separable Kernel

We consider the problem

$$y(x) = f(x) + \int_a^b k_1(x)k_2(t)y(t)dt, x \in [a,b],$$

which may be written as

$$y(x) = f(x) + k_1(x)\int_a^b k_2(t)y(t)dt.$$

Multiplying by $k_2(x)$ and integrating from a to b, gives

$$\int_a^b k_2(x)y(x)dx = \int_a^b k_2(x)f(x)dx + \int_a^b k_2(x)k_1(x)dx \int_a^b k_2(t)y(t)dt$$
$$\left(1 - \int_a^b k_2(x)k_1(x)dx\right)\int_a^b k_2(t)y(t)dt = \int_a^b k_2(x)f(x)dx,$$

from which we may calculate

$$C^* := \int_a^b k_2(t)y(t)dt = \frac{\int_a^b k_2(x)f(x)dx}{1 - \int_a^b k_2(x)k_1(x)dx} \in \mathbb{R},$$

provided

$$1 - \int_a^b k_2(x)k_1(x)dx \neq 0.$$

Hence, the solution is given by

$$y(x) = f(x) + C^*k_1(x).$$

One other type of equation we will consider involves the derivative of the unknown function as well, e.g.,

$$u'(x) = f(x) + \int_a^b K(x,t)u(t)dt, x \in (a,b], \quad (4)$$
$$u(a) = u_a \in \mathbb{R}. \quad (5)$$

This is an *integro-differential equation* (of first order in this case, since only first derivatives appear) of *Fredholm* type, and it includes the initial condition (5) in order to have a unique solution. We could have higher order integro-differential equations, but we will focus only on the type given by (4). In Section 4.1 we will consider the numerical approximation of the solution to (4)–(5).

A more general version of (4)–(5), not considered in this thesis, would be

$$\begin{aligned} u'(x) + c(x)u(x) &= f(x) + \int_a^b K(x,t)u(t)dt, x \in (a,b], \\ u(a) &= u_a \in \mathbb{R}, \end{aligned}$$

with, additionally, $c > 0$ a given sufficiently smooth function. The methods presented in this thesis could easily be extended to this case with minor modifications.

3 Numerical Solution of *Fredholm* Integral Equations of the 2^{nd} type

Our goal in this chapter is to approximate the solution u of (3). We begin by constructing a grid $a = x_1 < ... < x_{m-1} < x_m = b$ and substitute each x_i in equation (3), i.e. we use collocation:

$$u(x_i) = f(x_i) + \int_a^b K(x_i, t)u(t)dt, \ i = 1, ..., m. \tag{6}$$

The integral in (6) is approximated by numerical quadrature, based on the nodal points $x_1, x_2, ..., x_m$. Recall that in numerical integration, we use

$$\int_a^b g(t)dt \approx \sum_{j=1}^{N} w_j g(t_j),$$

where the w_j are called *weights* and the t_j *nodes*, (N is the number of terms). The different integration rules result from different choices for w_j and t_j. We take $t_j = x_i$, $N = m$, when approximating the integral in (6), in order to have only the unknown values $u(x_i)$ without introducing new ones. This corresponds to the linear system

$$U_i = f(x_i) + \sum_{j=1}^{m} w_j K(x_i, x_j)U_j, i = 1, ...m, \tag{7}$$

where $U_i \approx u(x_i)$. Solving the system results in the approximate values $u(x_1), ..., u(x_m)$. In matrix form, the linear system (7) is written as

$$A\vec{U} = \vec{b},$$

where

$$[A]_{i,j} = \begin{bmatrix} 1 - w_1K(x_1, x_1) & -w_2K(x_1, x_2) & ... & -w_NK(x_1, x_m) \\ -w_1K(x_2, x_1) & 1 - w_2K(x_2, x_2) & ... & -w_NK(x_2, x_m) \\ \vdots & \vdots & ... & \vdots \\ -w_NK(x_m, x_1) & -w_2K(x_m, x_2) & ... & 1 - w_NK(x_m, x_m) \end{bmatrix} \tag{8}$$

or $A = (I - M)$ with I the identity matrix, $M_{ij} = w_jK(x_i, x_j)$, and

$$\vec{b} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_m) \end{bmatrix}.$$

The above method is known as *Nyström's* method in the literature [2].

For example, if we were to use the **Composite Trapezoidal Rule** as our numerical integration scheme, the weights w_j and nodes x_j in (7) are

$$\vec{w} = [w_1, ..., w_m]^T = \frac{1}{2}\left(\frac{b-a}{m}\right)[1, 2..., 2, 1],$$

$$x_j = \frac{b-a}{m} j,\ j = 1, ..., m.$$

The integration scheme is

$$\int_a^b g(x)dx \approx \frac{1}{2}\left(\frac{b-a}{m}\right)[g(x_1) + 2(g(x_2) + ... + g(x_{m-1})) + g(x_m)]$$

and the system matrix in (8) is

$$[A]_{i,j} = \begin{bmatrix} 1-K_{1,1} & -2K_{1,2} & ... & -2K_{1,m-1} & -K_{1,m} \\ -K_{2,1} & 1-2K_{2,2} & ... & -2K_{2,m-1} & -K_{2,m} \\ \vdots & \vdots & ... & \vdots & \vdots \\ -K_{m,1} & -2K_{m,2} & ... & -2K_{m,m-1} & 1-K_{m,m} \end{bmatrix}, \tag{9}$$

where we used the notation $K_{i,j} = K(x_i, x_j)$.

Similarly, for the **Composite *Simpson's* Rule**, where the weights w_j and nodes x_j in (7) are

$$\vec{w} = [w_1, ...w_m]^T = \frac{1}{3}\left(\frac{b-a}{m}\right)[1, 4, 2, 4, ..., 2, 4, 1],$$

$$x_j = \frac{b-a}{m} j,\ j = 1, ..., m,$$

the numerical integration scheme is

$$\int_a^b g(x)dx \approx \frac{1}{3}\left(\frac{b-a}{m}\right)[g(x_0)+4g(x_1)+2g(x_2)+4g(x_3)+...+2g(x_{m-2})+4g(x_{m-1})+g(x_m)]$$

and the system matrix in (8) is

$$[A]_{i,j} = \begin{bmatrix} 1-K_{1,1} & -4K_{1,2} & -2K_{1,3} & ... & -2K_{1,m-2} & -4K_{1,m-1} & -K_{1,m} \\ -K_{2,1} & 1-4K_{2,2} & -2K_{2,3} & ... & -2K_{2,m-2} & -4K_{2,m-1} & -K_{2,m} \\ \vdots & \vdots & \vdots & ... & \vdots & \vdots & \vdots \\ -K_{m,1} & -4K_{m,2} & -2K_{m,3} & ... & -2K_{m,m-2} & -4K_{m,m-1} & 1-K_{m,m} \end{bmatrix}. \tag{10}$$

We illustrate the above in the following elementary example, with most calculations being displayed in detail.

Example 3.1:

Let $a = 0, b = 1, f(x) = x^2, K(x,t) = e^{|x-t|}$, and divide $[0,1]$ into 4 subintervals, using the nodal points $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$. Using the Composite Trapezoidal Rule, we write the equations satisfied by $u(x_i)$, as

$$u(x_i) = x_i^2 + \frac{1}{2}\left(\frac{1-0}{4}\right)[e^{|x_i-x_0|}u(x_0) + 2e^{|x_i-x_1|}u(x_1) + 2e^{|x_i-x_2|}u(x_2) + 2e^{|x_i-x_3|}u(x_3) + e^{|x_i-x_4|}u(x_4)],$$

where $i = 0, 1, 2, 3, 4$, or equivalently as

$$\begin{bmatrix} \frac{7}{8} & -\frac{1}{4}e^{\frac{1}{4}} & -\frac{1}{4}e^{\frac{1}{2}} & -\frac{1}{4}e^{\frac{3}{4}} & -\frac{1}{8}e^{1} \\ -\frac{1}{8}e^{\frac{1}{4}} & \frac{3}{4} & -\frac{1}{4}e^{\frac{1}{4}} & -\frac{1}{4}e^{\frac{1}{2}} & -\frac{1}{8}e^{\frac{3}{4}} \\ -\frac{1}{8}e^{\frac{1}{2}} & -\frac{1}{4}e^{\frac{1}{4}} & \frac{3}{4} & -\frac{1}{8}e^{\frac{1}{4}} & -\frac{1}{8}e^{\frac{1}{2}} \\ -\frac{1}{8}e^{\frac{3}{4}} & -\frac{1}{4}e^{\frac{1}{2}} & -\frac{1}{4}e^{\frac{1}{4}} & \frac{3}{4} & -\frac{1}{8}e^{\frac{1}{4}} \\ -\frac{1}{8}e^{1} & -\frac{1}{4}e^{\frac{3}{4}} & -\frac{1}{4}e^{\frac{1}{2}} & -\frac{1}{4}e^{\frac{1}{4}} & \frac{7}{8} \end{bmatrix} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{16} \\ \frac{1}{4} \\ \frac{9}{16} \\ 1 \end{bmatrix}.$$

Solving the system we find the approximate values of $u(x_0), u(x_1), u(x_2), u(x_3), u(x_4)$ as $u(0) = -1.2309, u(0.25) = -0.9738, u(0.5) = -0.6817, u(0.75) = -0.6118, u(1) = -0.4698$.

For the Composite *Simpson's* Rule, we have

$$u(x_i) = x_i^2 + \frac{1}{3}\left(\frac{1-0}{4}\right)[e^{|x_i-x_0|}u(x_0) + 4e^{|x_i-x_1|}u(x_1) + 2e^{|x_i-x_2|}u(x_2) + 4e^{|x_i-x_3|}u(x_3) + e^{|x_i-x_4|}u(x_4)],$$

where $i = 0, 1, 2, 3, 4$, or equivalently as

$$\begin{bmatrix} \frac{11}{12} & -\frac{1}{3}e^{\frac{1}{4}} & -\frac{1}{6}e^{\frac{1}{2}} & -\frac{1}{3}e^{\frac{3}{4}} & -\frac{1}{12}e^{1} \\ -\frac{1}{12}e^{\frac{1}{4}} & \frac{2}{3} & -\frac{1}{6}e^{\frac{1}{4}} & -\frac{1}{3}e^{\frac{1}{2}} & -\frac{1}{12}e^{\frac{3}{4}} \\ -\frac{1}{12}e^{\frac{1}{2}} & -\frac{1}{3}e^{\frac{1}{4}} & \frac{5}{6} & -\frac{1}{3}e^{\frac{1}{4}} & -\frac{1}{12}e^{\frac{1}{2}} \\ -\frac{1}{12}e^{\frac{3}{4}} & -\frac{1}{3}e^{\frac{1}{2}} & -\frac{1}{6}e^{\frac{1}{4}} & \frac{2}{3} & -\frac{1}{12}e^{\frac{1}{4}} \\ -\frac{1}{12}e^{1} & -\frac{1}{3}e^{\frac{3}{4}} & -\frac{1}{6}e^{\frac{1}{2}} & -\frac{1}{3}e^{\frac{1}{4}} & \frac{11}{12} \end{bmatrix} \begin{bmatrix} u(x_0) \\ u(x_1) \\ u(x_2) \\ u(x_3) \\ u(x_4) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{16} \\ \frac{1}{4} \\ \frac{9}{16} \\ 1 \end{bmatrix}$$

Solving the system we find the approximate values of $u(x_0), u(x_1), u(x_2), u(x_3), u(x_4)$ as $u(0) = -1.2343, u(0.25) = -0.9507, u(0.5) = -0.7659, u(0.75) = -0.5845, u(1) = -0.4485$.

As a second example, we illustrate the method for a larger number of nodal points.

Example 3.2:

Let $a = -1, b = 1, f(x) = 1, K(x,t) = xt + x^2t^2$. We implemented the method in MATLAB and present the results in Figure 1 below. The result of both the Composite Trapezoidal and *Simpson* rule is shown (for 10 and 20 nodal points), along with the exact solution $u(x) = 1 + \frac{10}{9}x^2$; the agreement is as expected.

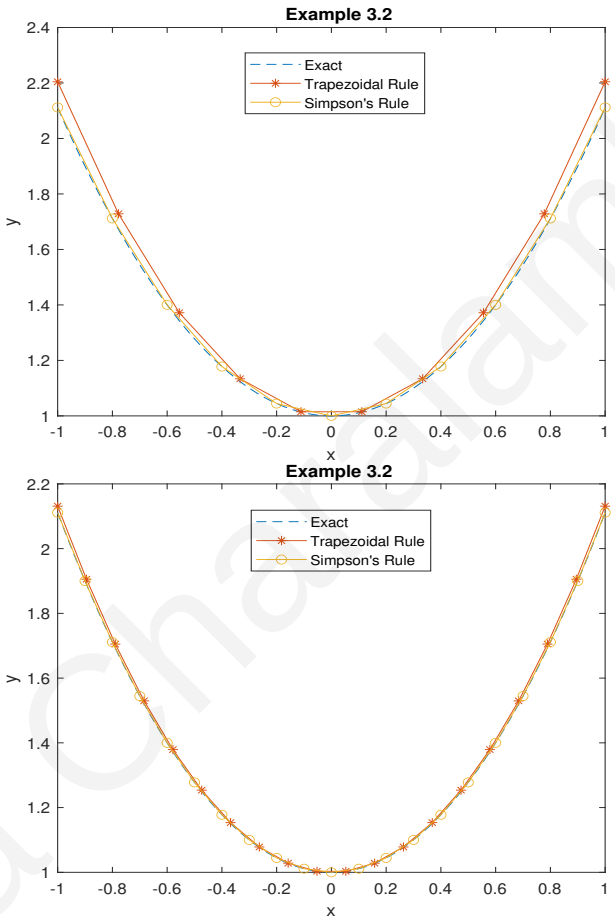

Figure 1: Approximation: Number of nodal ponts: 10 (top) and 20 (bottom).

In terms of the convergence of the method, we will study the algorithm with the two composite methods used above. We know from [1] that the Composite Trapezoidal Rule (for numerical integration) converges with order $O(h^2)$, and the Composite *Simpson's* Rule with order $O(h^4)$, as the meshwidth h tends to 0. This carries over to the numerical solution of (3), as we illustrate in the following example.

Example 3.3:

Let $a = 0, b = \pi/2, f(x) = \sin(x), K(x,t) = \sin(x)\cos(t)$. The exact solution is $u(x) = 2\sin(x)$, so we may calculate the errors reliably. We show in Figure 2 the convergence of the method, with the Composite Trapezoidal and *Simpson* Rules, as the number of nodal points m, increases. It is a logarithmic plot, and since we get straight lines with slope approximately –2, –4, respectively, we conclude that the methods converge with orders $O(m^{-2}), O(m^{-4})$, respectively.

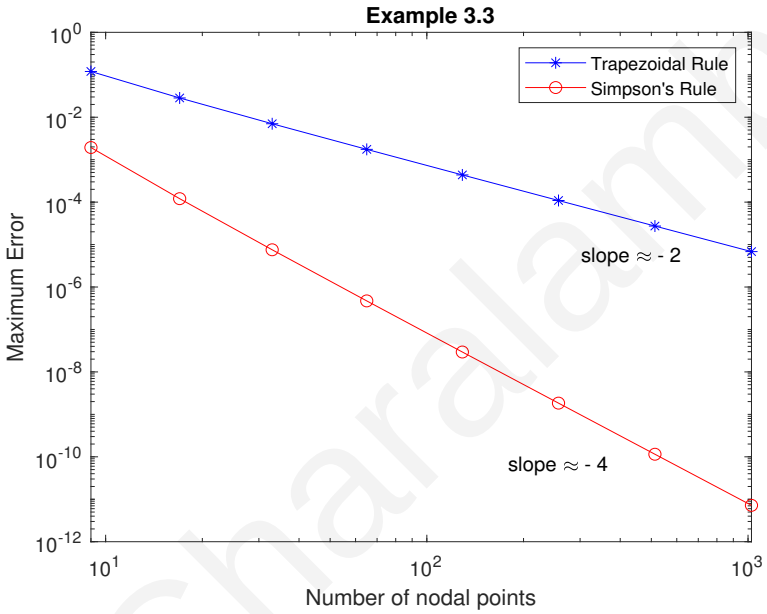

Figure 2: The convergence of the methods.

The fact that the error in Nyström's method depends on the order of the integration scheme as well as the smoothness of the kernel $K(x,t)$ may be seen as follows (see also [2]): subtracting (7) from (6) , we have

$$|u(x_i) - U(x_i)| = |\int_a^b K(x_i,t)u(t)dt - \sum_{j=1}^{m} w_j K(x_i,x_j)U(x_j)| = O(m^{-p}),$$

where p is the order of the quadrature scheme (assuming $K(x,t)$ is sufficiently smooth). In the Composite Trapezoidal Rule, $p = 2$, and in the Composite *Simpson's* Rule, $p = 4$.

4 Numerical Solution of *Fredholm* Integro-Differential Equations of the 2^{nd} type

Equation (4) defines a *Fredholm* integro-differential equation of the 2^{nd} type; our goal, again, is to approximate u. We will consider the approximation of the solution to the problem

$$\begin{aligned} u'(x) &= f(x) + \int_a^b K(x,t)u(t)dt, \\ u(a) &= u_a \in \mathbb{R}. \end{aligned} \tag{11}$$

We begin by constructing a grid $a = x_1 < ... < x_{m+1} = b$ and substitute x_i in equation (11):

$$u'(x_i) = f(x_i) + \int_a^b K(x_i,t)u(t)dt, i = 1, ..., m+1 \tag{12}$$

The integral in (12) is approximated by numerical quadrature, based on the nodal points $x_1, x_2, ..., x_{m+1}$:

$$\int_a^b K(x_i,t)u(t)dt \approx \sum_{j=1}^{m+1} w_j K(x_i,x_j)u(x_j).$$

The derivative $u'(x_i)$ is approximated using *finite differences*. Assuming the subdivision is uniform with meshwidth

$$h = x_{i+1} - x_i, \ \forall \ i = 1, \ldots, m,$$

we use the centered difference

$$u'(x_i) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h} \ , \ i = 2, \ldots, m,$$

for the internal nodal points, and for the endpoints we use

$$u'(x_1) \approx \frac{u(x_2) - u(x_1)}{h} \ , \ u'(x_{m+1}) \approx \frac{u(x_{m+1}) - u(x_m)}{h},$$

which are the forward and backward differences, respectively.

Denoting $U_i \approx u(x_i)$, we substitute the above in (12) and obtain

$$\begin{aligned} \frac{U_2 - U_1}{h} &= f(x_1) + \sum_{j=1}^{m+1} w_j K(x_1,x_j)U_j, \\ \frac{U_{i+1} - U_{i-1}}{2h} &= f(x_i) + \sum_{j=1}^{m+1} w_j K(x_i,x_j)U_j \ , \ i = 2, \ldots, m, \\ \frac{U_{m+1} - U_m}{h} &= f(x_{m+1}) + \sum_{j=1}^{m+1} w_j K(x_{m+1},x_j)U_j. \end{aligned}$$

In matrix form, the above system may be written as

$$A\vec{U} = \vec{F},$$

where $\vec{F} = [hf(x_1), 2hf(x_2), \ldots, 2hf(x_m), hf(x_{m+1})]^T$, $\vec{U} = [U_1, \ldots, U_{m+1}]^T$, and the coefficient matrix A is given by

$$A = J - M,$$

with

$$J = \begin{bmatrix} -1 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 1 \end{bmatrix}, \; M_{i,j} = \begin{cases} -hw_j K(x_1, x_j), i = 1, j = 1, \ldots, m+1 \\ -2hw_j K(x_i, x_j), i = 2, ..., m, j = 1, \ldots, m+1 \\ -hw_j K(x_{m+1}, x_j), i = m+1, j = 1, \ldots, m+1 \end{cases}$$

We note that the value of U_1 is already known from the initial condition ($U_1 = u(a) = u_a$), so the first row and column of A, as well as the first element in $\vec{F}$, are deleted before solving the linear system, yielding a vector $\vec{U} \in \mathbb{R}^m$ (as opposed to $\mathbb{R}^{m+1}$); the approximate values for $u(x_i), i = 1, \ldots, m+1$ are given by $[u_a, U_2, \ldots, U_{m+1}]^T \in \mathbb{R}^{m+1}$.

Example 4.1:

We illustrate the method for the following problem:

$$\begin{aligned} u'(x) &= xe^x + e^x - x + \int_0^1 xu(t)dt, \\ u(0) &= 0, \end{aligned}$$

with exact solution $u(x) = xe^x$. We use the composite Trapezoidal Rule, and in Figure 4 we show the exact and approximate solutions, showing excellent agreement. In Figure 4, we show the convergence as the number of nodal points is increased, in a log-log scale. The slope of the resulting line gives the (expected) convergence rate of -2.

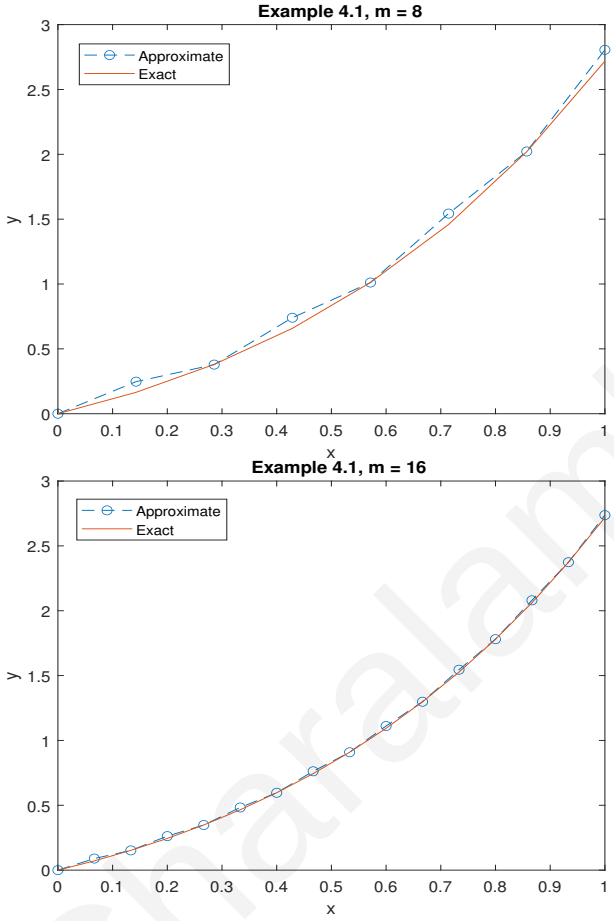

Figure 3: Exact and approximate solutions for Example 4.1.

4.1 Singularly perturbed *Fredholm* Integro-Differential Equations of the 2^{nd} type

Suppose now that there is a parameter $\varepsilon \in (0, 1]$ multiplying the derivative in (11), and we allow $\varepsilon \to 0$. The problem becomes

$$\begin{aligned} \varepsilon u'(x) &= f(x) + \int_a^b K(x,t)u(t)dt, \\ u(a) &= u_a \in \mathbb{R}. \end{aligned} \tag{13}$$

We begin, as usual, by constructing a grid $a = x_1 < ... < x_{m+1} = b$ and substitute x_i in equation (13):

$$\varepsilon u'(x_i) = f(x_i) + \int_a^b K(x_i,t)u(t)dt, i = 1, ..., m+1. \tag{14}$$

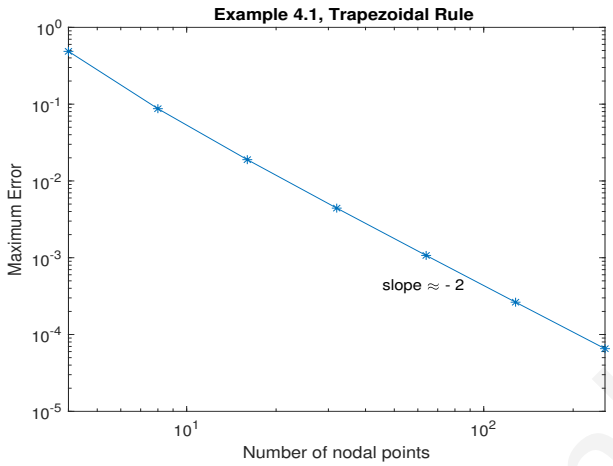

Figure 4: The convergence of the method for Example 4.1.

The approximation of the integral in (14) is

$$\int_a^b K(x_i,t)u(t)dt \approx \sum_{j=1}^{m+1} w_j K(x_i,x_j)u(x_j),$$

and the derivative $u'(x_i)$ is approximated using *finite differences*. Assuming the subdivision is uniform with meshwidth h, we use centered differences for the internal nodal points, and for the endpoints we use forward and backward differences, as was done before.

Denoting $U_i \approx u(x_i)$, we substitute the above in (14) and obtain

$$\begin{aligned}
\varepsilon\frac{U_2-U_1}{h} &= f(x_1) + \sum_{j=1}^{m+1} w_j K(x_1,x_j)U_j, \\
\varepsilon\frac{U_{i+1}-U_{i-1}}{2h} &= f(x_i) + \sum_{j=1}^{m+1} w_j K(x_i,x_j)U_j \ , \ i=2,\ldots,m, \\
\varepsilon\frac{U_{m+1}-U_m}{h} &= f(x_{m+1}) + \sum_{j=1}^{m+1} w_j K(x_{m+1},x_j)U_j.
\end{aligned}$$

In matrix form, the above system may be written as

$$A\vec{U} = \vec{F},$$

where $\vec{F} = [hf(x_1), 2hf(x_2), \ldots, 2hf(x_m), hf(x_{m+1})]^T$, $\vec{U} = [U_1,\ldots,U_{m+1}]^T$, and the coefficient matrix A is given by

$$A = \tilde{J} - M,$$

with

$$\tilde{J} = \begin{bmatrix} -\varepsilon & \varepsilon & & & \\ -\varepsilon & 0 & \varepsilon & & \\ & \ddots & \ddots & \ddots & \\ & & -\varepsilon & 0 & \varepsilon \\ & & & -\varepsilon & \varepsilon \end{bmatrix}, \; M_{i,j} = \begin{cases} -hw_j K(x_1, x_j), i = 1, j = 1, \ldots, m+1 \\ -2hw_j K(x_i, x_j), i = 2, ..., m, j = 1, \ldots, m+1 \\ -hw_j K(x_{m+1}, x_j), i = m+1, j = 1, \ldots, m+1 \end{cases}$$

As before, the value of U_1 is already known from the initial condition ($U_1 = u(a) = u_a$), so the first row and column of A, as well as the first element in $\vec{F}$, are deleted before solving the linear system; the approximate values for $u(x_i), i = 1, \ldots, m+1$ are given by $[u_a, U_2, \ldots, U_{m+1}]^T \in \mathbb{R}^{m+1}$.

We illustrate the numerical difficulties that arise when $\varepsilon \to 0$, in the following example.

Example 4.2:

Consider the following problem:

$$\begin{aligned} \varepsilon u'(x) &= f(x) + \int_0^1 xtu(t)dt, \\ u(0) &= 0, \end{aligned}$$

with f chosen so that the exact solution is $u(x) = 1 - e^{-x/\varepsilon}$. We use the Composite Trapezoidal Rule for the approximation, and in Figure 5 we show the results. For large values of ε, the method performs well; however, as $\varepsilon \to 0$, the approximations become inaccurate.

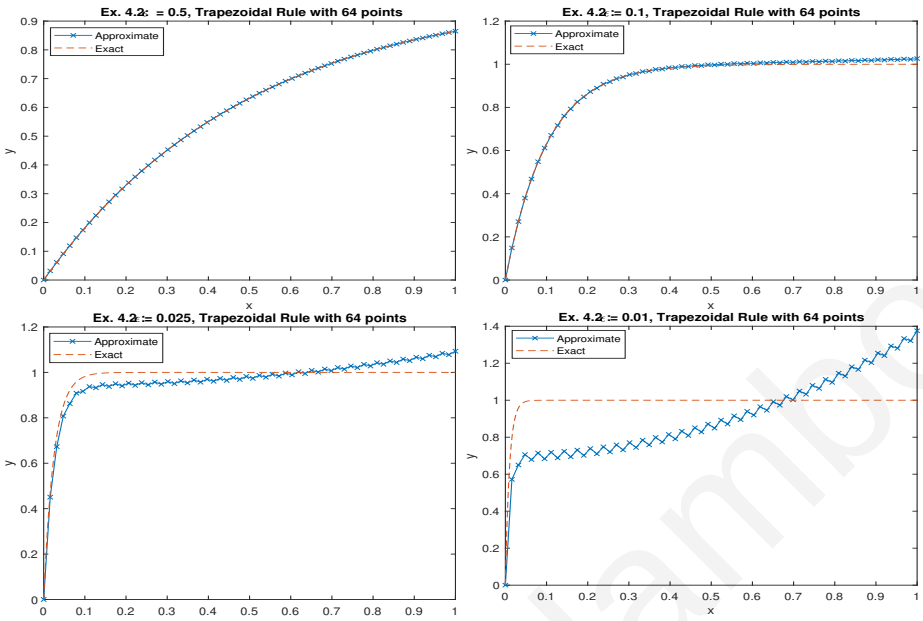

Figure 5: The exact and approximate solutions for Example 4.2.

Clearly, the method fails to provide an accurate approximation as $\varepsilon \to 0$, unless the stepsize h is $O(\varepsilon)$. A *uniform* mesh is *not* appropriate for singularly perturbed problems.

This deterioration is due to the presence of *boundary layers* in u, which are rapidly varying solution components, in the vicinity of the boundary (or initial point in our case). We **assume** that the solution u of (11) may be decomposed as

$$u = u_S + u_{BL} + u_R, \tag{15}$$

where the *smooth* part u_S is as smooth as the input data, the *boundary layer* part u_{BL} behaves like the function $e^{-t/\varepsilon}$, and the remainder u_R is negligible. This assumption is based on the analogous decomposition for singularly perturbed problems [8].

One non-uniform, layer adapted mesh is the so-called *Shishkin* mesh, which is a piecewise unform mesh commonly used in the literature for singularly perturbed problems [8]. To define the *Shishkin* mesh, we let

$$\tau = \min\{(b-a)/2, 2\varepsilon \ln 2m\}, \tag{16}$$

where $2m$ is the number of subintervals we wish to use in the interval $[a, b]$. Then, $[a, b]$ is subdivided into $[a, a+\tau], [a+\tau, b]$, and in each subinterval, a uniform partition (of m subintervals) is used. We set $h = \tau/m, H = (b-a-\tau)/m$ as the two different meshwidths. (See Figure 6 below.)

Collocating at $x_i, i = 1, \ldots, 2m+1$, gives

$$\varepsilon u'(x_i) = f(x_i) + \int_a^b K(x_i, t)u(t)dt, i = 1, \ldots, 2m+1. \tag{17}$$

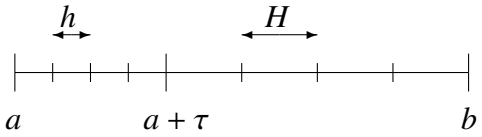

Figure 6: The *Shishkin mesh*.

For the approximation of the integral in (17) we have

$$\int_a^b K(x,t)u(t)dt = \int_a^{a+\tau} K(x,t)u(t)dt + \int_{a+\tau}^b K(x,t)u(t)dt,$$

with

$$\int_a^{a+\tau} K(x_i,t)u(t)dt \approx \sum_{j=1}^{m+1} w_j K(x_i, x_j)u(x_j),$$

$$\int_a^{a+\tau} K(x_i,t)u(t)dt \approx \sum_{j=1}^{m+1} W_j K(x_i, x_{j+m})u(x_{m+j}),$$

where w_j, W_j are the weights in the chosen composite quadrature rule on $[a, a+\tau]$, $[a+\tau, b]$, respectively. The discrete equations become

$$\varepsilon u_i' = f(x_i) + \sum_{j=1}^{m+1} w_j K(x_i, x_j)u_j + \sum_{j=1}^{m+1} W_j K(x_i, x_{j+m})u_{m+j}. \tag{18}$$

The derivative u_i' above is approximated using finite differences as before – forward difference for the first point, backward difference for the last point, and centered difference for the internal points. However, for $u'(x_{m+1})$ the meshwidth is different in the two intervals surrounding $x_{m+1} = a + \tau$ (see Figure 6), hence the centered difference for this point is modified as follows [10]:

$$u'_{m+1} \approx \frac{u_{m+2} - (H/h)^2 u_m - (1 - (H/h)^2)u_{m+1}}{H(1 + H/h)}.$$

The resulting system, in matrix form, may be written as

$$A\vec{U} = \vec{F},$$

where $\vec{F} = [hf(x_1), 2hf(x_2), \ldots, 2Hf(x_{2m}), Hf(x_{2m+1})]^T$, $\vec{U} = [U_1, \ldots, U_{2m+1}]^T$, j=1,. . . ,2m+1, and the coefficient matrix A is given by

$$A = \begin{bmatrix} J_1 - M_1 & \vec{a}_1 & -M_2 \\ \vec{a}_2 & \alpha & \vec{a}_3 \\ -M_3 & \vec{a}_4 & J_2 - M_4 \end{bmatrix},$$

where

$$J_1 = \begin{bmatrix} -\varepsilon & \varepsilon & & & \\ -\varepsilon & 0 & \varepsilon & & \\ & \ddots & \ddots & \ddots & \\ & & -\varepsilon & 0 & \varepsilon \\ & & & -\varepsilon & 0 \end{bmatrix} \in \mathbb{R}^{m\times m}, \; J_2 = \begin{bmatrix} 0 & \varepsilon & & & \\ -\varepsilon & 0 & \varepsilon & & \\ & \ddots & \ddots & \ddots & \\ & & -\varepsilon & 0 & \varepsilon \\ & & & -\varepsilon & 0 \end{bmatrix} \in \mathbb{R}^{m\times m},$$

$$[M_1]_{i,j} = \begin{cases} hw_j K(x_1, x_j), i = 1, j = 1, \ldots, m \\ 2hw_j K(x_i, x_j), i = 2, ..., m, j = 1, \ldots, m \end{cases}$$

$$[M_2]_{i,j} = \begin{cases} hW_{j+1} K(x_1, x_{j+m+1}), i = 1, j = 1, \ldots, m \\ 2hW_{j+1} K(x_i, x_{j+m+1}), i = 2, ..., m, j = 1, \ldots, m \end{cases}$$

$$[M_3]_{i,j} = \begin{cases} 2Hw_j K(x_{m+1+i}, x_j), i = 1, j = 1, \ldots, m \\ Hw_j K(x_{2m+1}, x_j), i = m, j = 1, \ldots, m \end{cases}$$

$$[M_4]_{i,j} = \begin{cases} 2HW_{j+1} K(x_{m+1+i}, x_{m+1+j}), i = 1, \ldots, m-1 \\ , j = 1, \ldots, m \\ HW_{j+1} K(x_{2m+1}, x_{m+1+j}), j = 1, \ldots, m \end{cases}$$

and

$$\vec{a}_1 = [c_1, \vec{V}_1, c_2]^T, \vec{a}_2 = [\vec{V}_2, c_3], \vec{a}_3 = [c_4, \vec{V}_3], \vec{a}_4 = [c_5, \vec{V}_4, c_6]^T$$

$$[\vec{V}_1]_k = -2h(w_{m+1} + W_1) K(x_{k+1}, x_{m+1}), k = 1, \ldots, m-2$$

$$[\vec{V}_4]_k = -2H(w_{m+1} + W_1) K(x_{k+2+m}, x_{m+1}), k = 1, \ldots, m-2$$

$$[\vec{V}_2]_j = -H(1 + H/h) w_j K(x_{m+1}, x_j), j = 1, \ldots, m-1$$

$$[\vec{V}_3]_j = -H(1 + H/h) W_{j+2} K(x_{m+1}, x_{j+m+2}), j = 1, \ldots, m-1$$

$$c_1 = -h(w_{m+1} + W_1) K(x_1, x_{m+1})$$

$$c_2 = \varepsilon - 2h(w_{m+1} + W_1) K(x_m, x_{m+1})$$

$$c_3 = -\varepsilon(H/h)^2 - H(1 + (H/h) w_m + W_1) K(x_{m+1}, x_m)$$

$$c_4 = \varepsilon - H(1 + (H/h) W_2 K(x_{m+1}, x_{m+2})$$

$$c_5 = -\varepsilon - 2H(w_{m+1} + W_1)K(x_{m+2}, x_{m+1})$$
$$c_6 = -H(w_{m+1} + W_1)K(x_{2m+1}, x_{m+1})$$
$$\alpha = -\varepsilon(1 - (H/h)^2) - H(1 + (H/h)W_{m+1}K(x_{m+1}, x_{m+1})$$

We re-visit the problem in Example 4.2 in the following.

Example 4.3

Consider the problem:

$$\begin{aligned} \varepsilon u'(x) &= f(x) + \int_0^1 xtu(t)dt, \\ u(0) &= 0, \end{aligned}$$

with f chosen so that the exact solution is $u(x) = 1 - e^{-x/\varepsilon}$. We use the Composite Trapezoidal Rule for the approximation, on the *Shishkin* mesh, as described above. In Figure 7 we show the results for $\varepsilon = 0.025, 0.01$, and no deterioration is observed (as was the case in Example 4.2).

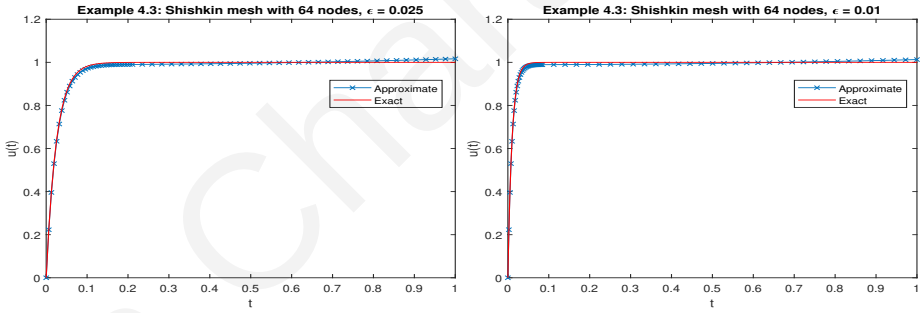

Figure 7: The exact and approximate solutions for Example 4.3.

We also show the convergence of the method for various values of ε, in Figure 8. We first observe that the method is *robust* (with respect to ε). Next, the slope calculated is slightly lower than -2, as was observed for non-singularly perturbed problems. The reason for this is that the *Shishkin* mesh yields a convergence rate of order $O(m^{-2}\ln^2(m))$ as opposed to $O(m^{-2})$, with the logarithmic term *not* removable (see, e.g. [8]).

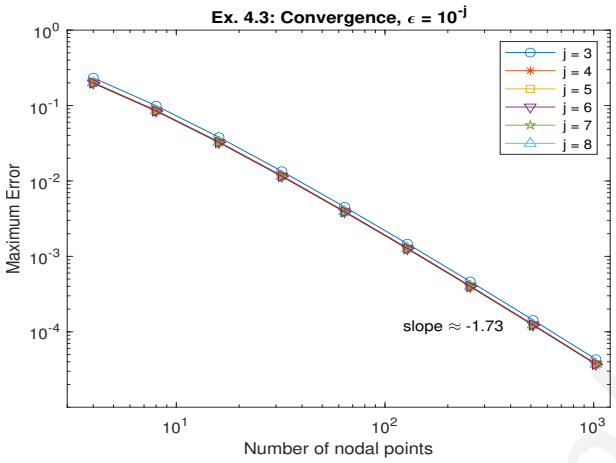

Figure 8: Convergence of the method for Example 4.3.

As a final example, we consider the following.

Example 4.4

$$\begin{aligned} \varepsilon u'(x) &= f(x) + \int_0^1 (\cos(x) + \sin(t))u(t)dt, \\ u(0) &= 0, \end{aligned}$$

with f chosen so that the exact solution is $u(x) = \cos(\pi x) - e^{-x/\varepsilon}$. We use the Composite Trapezoidal Rule for the approximation, on the *Shishkin* mesh, and in Figure 9 we show the result for $\varepsilon = 0.01$.

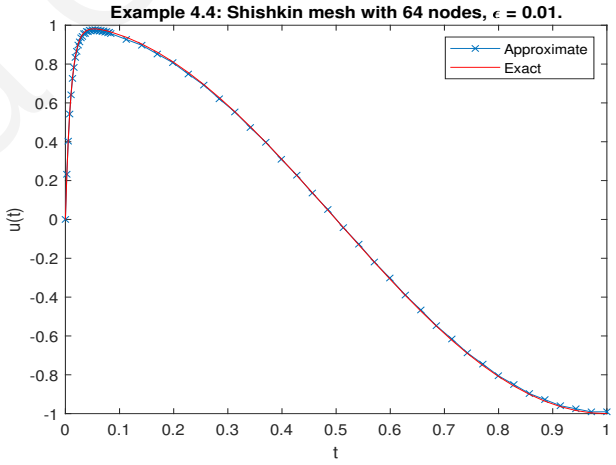

Figure 9: Exact and approximate solutions for Example 4.4.

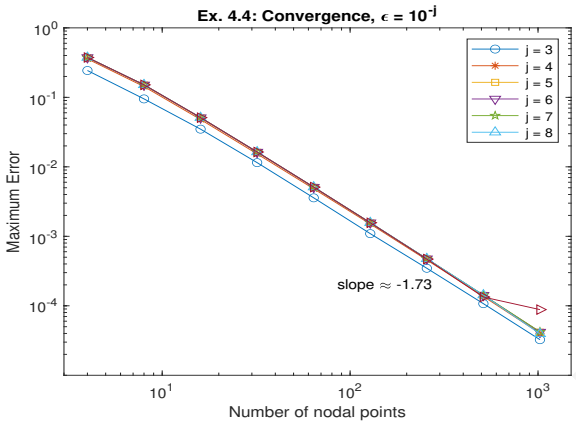

Figure 10: Convergence for Example 4.4.

The convergence of the method for various values of ε, is shown in Figure 10. The observations are exactly the same as the previous example.

Example 4.5

We finally consider a problem without a known exact solution:

$$\begin{aligned} \varepsilon u'(x) &= 1 - e^{-x/e} + \int_0^1 (t + e^{-x/e}) u(t) dt, \\ u(0) &= 0. \end{aligned}$$

In Figure 11 we show the approximate solution for various ε, and we see that the method produces the expected results.

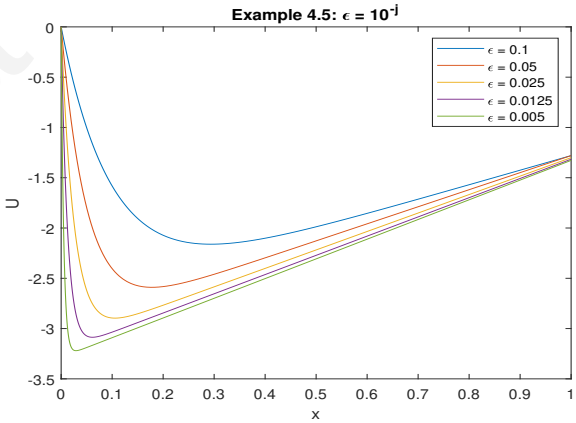

Figure 11: Approximate solution for Example 4.5.

5 Closing Remarks

In this thesis we considered *Fredholm* integral and integro-differential equations of the second type, and the numerical approximation of their solution using *Nyström's* method of collocation, in conjunction with the composite Trapezoidal and *Simpson's* rules – we mainly utilized the Trapezoidal Rule.

For *Fredholm* integral equations, we implemented the method and performed numerical experiments for problems with smooth data (hence smooth solution). The convergence rates observed, for both composite quadrature rule choices, namely $O(m^{-2})$ and $O(m^{-4})$, respectively, where m is the number of nodal points, were the expected ones.

We also considered integro-differential equations of first order, and for the derivative approximation we used finite differences. In particular, we used centered differences for the internal nodal points, and forward and backward differences for the first and last points, respectively. For smooth data, the method was observed to yield the expected $O(m^{-2})$ convergence rate, when the Trapezoidal Rule was used for the integral.

The above convergence rate did not hold when we considered *singularly perturbed* integro-differential equations. The method "broke down" due to the fact that we were using a uniform mesh to approximate a solution which includes *boundary layers*. This was rectified with the use of a layer-adapted, piecewise uniform *Shishkin* mesh [8]. We verified through numerical experiments that with this choice of the mesh points, *Nyström's* method yields uniform convergence, independently of ε, as the number of mesh points is increased, at the rate $O(m^{-2}\ln^2 m)$ (for the Trapezoidal Rule).

The next step in the study of this method for such problems is to prove the observed convergence rates. Once this is done, the case of *systems* could be considered, with either one or different singular perturbation parameters.

One could also study *Galerkin* variational methods [7] for the numerical approximation of the solution to such problems. In the *Galerkin* method for, e.g (11), we multiply the equation with a so-called *test function* $v(x)$ and integrate over the interval $[a, b]$, to get

$$\int_a^b v(x)\varepsilon u'(x)dx = \int_a^b v(x)f(x)dx + \int_a^b \int_a^b v(x)K(x,t)u(t)dtdx. \tag{19}$$

The variational problem to be solved is: find $u \in U$ such that (19) holds for $v \in V$, where U, V are appropriate function spaces (e.g. derivatives are squared integrable functions, etc.). At the discrete level, we seek $u_N \in U_N \subset U$ such that (19) holds for all $v \in V_N$, where U_N, V_N are appropriate finite dimensional subspaces of U, V, respectively. Assuming

$$U_N = span\{\phi_1, \ldots, \phi_N\}\ ,\ V_N = span\{\psi_1, \ldots, \psi_N\}$$

with ϕ, ψ polynomials, we have

$$u_N = \sum_{i=1}^{N} \xi_i \phi_i \ , \ v_N = \sum_{i=1}^{N} \zeta_i \psi_i.$$

Substituting the above expressions in (19), we obtain a linear system for the coefficients $\xi_i, i = 1, \ldots, N$, hence we obtain the approximate solution u_N. This study will appear elsewhere.

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