University of Cyprus

DEPARTMENT OF MATHEMATICS AND STATISTICS

THE L^p-SPECTRUM OF THE LAPLACIAN ON FORMS OVER WARPED PRODUCTS AND KLEINIAN GROUPS

DOCTOR OF PHILOSOPHY DISSERTATION

PETROS SIASOS

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PETROS SIASOS

A Dissertation Submitted to the University of Cyprus in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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*The present Doctoral Dissertation was submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy at the **Department of Mathematics and Statistics** and was approved on April 10th, 2024 by the members of the **Examination Committee**.*

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DECLARATION OF DOCTORAL CANDIDATE

The present doctoral dissertation was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy of the University of Cyprus. It is a product of original work of my own, unless otherwise mentioned through references, notes, or any other statements.

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Περίληψη

Στην παρούσα διατριβή, γενικεύουμε το σύνολο των πολλαπλοτήτων για τις οποίες το L^p-φάσμα της Λαπλασιανής σε k-μορφές εξαρτάται απο το p. Στο πρώτο μέρος, θεωρούμε γινόμενα πολλαπλότητες των οποίων η μετρική είναι στρεβλωμένο γινόμενο στο άπειρο και αποδεικνύουμε ότι το L^p-φάσμα της Λαπλασιανής σε k-μορφές περιέχει ένα παραβολικό χωρίο το οποίο εξαρτάται από τα k, p και την οριακή καμπυλότητα a_0 στο άπειρο.

Στο δεύτερο μέρος, θεωρούμε πηλίκα του υπερβολικού χώρου M με την ομάδα πηλίκο να είναι γεωμετρικά πεπερασμένη έτσι ώστε η M να έχει άπειρο όγκο και να μην έχει ακίδες. Αποδεικνύουμε οτι το L^p-φάσμα της Λαπλαισιανής σε k-μορφές είναι ακριβώς ένα παραβολικό χωρίο μαζί με ένα σύνολο από μεμονωμένες ιδιοτιμές της πραγματικής ευθείας.

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Abstract

In this thesis, we generalize the set of manifolds over which the L^p-spectrum of the Laplacian on k-forms depends on p. In the first part, we consider warped products at infinity, and we prove that the L^p-spectrum of the Laplacian on k-forms contains a parabolic region which depends on k, p and the limiting curvature a_0 at infinity.

In the second part, we consider manifolds M which are quotients of the hyperbolic space with a geometrically finite group, and such that M has infinite volume and no cusps. We prove that the L^p-spectrum of the Laplacian on k-forms over M is exactly a parabolic region together with a set of isolated eigenvalues on the real line.

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Chapter 1

Introduction

In this thesis, we study the L^p spectral theory for the Laplacian on forms on Riemannian manifolds. In particular, in the first part, we study the L^p-spectrum of the Laplacian on forms on manifolds which are warped products at infinity. In the second part, we compute the L^p-spectrum of the Laplacian on forms on a class of Kleinian groups.

The study of the L^p-spectrum of the Laplace-Beltrami operator on Riemannian manifolds is an active research area in the last decades. As the following historical references show, the L^p-spectrum of the Laplace-Beltrami operator on Riemannian manifolds may depend on p and this dependence can reflect the geometric structure and properties of the manifold. It is for example connected to the volume growth of the manifold.

In the case of operators on functions we have two guiding examples. We know that the spectrum of the Laplace-Beltrami operator on Euclidean space is p-independent, whereas the spectrum on the Hyperbolic space depends on p. Hempel and Voigt [HV86] studied the L^p-spectrum on Schrodinger operators on Euclidean spaces and found sufficient conditions on the potential so that the spectrum is independent of p. Sturm [Stu93] studied the L^p-spectrum for a class of uniformly elliptic operators on functions over open manifolds and showed the L^p-independence of the spectrum on such spaces, whenever the volume of the manifold has uniformly sub-exponentially volume growth and Ricci curvature bounded below. In the case of negatively curved manifolds, Davies, Simon and Taylor [DST88] examine the L^p-spectrum on quotients of the Hyperbolic space $\mathbb{H}^{N+1}/\Gamma$ and showed the L^p-dependence of the spectrum. Specifically, under the assumptions that $\mathbb{H}^{N+1}/\Gamma$ is of finite volume or has no cusps, they completely deter-

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mined the $\mathbb{H}^{N+1}/\Gamma$-spectrum being a parabolic region of the Complex plane for $p \neq 2$ together with a finite set of isolated eigenvalues and this reduced to a closed subset of $\mathbb{R}$, for $p = 2$ which is an interval with a finite set of eigenvalues.

Other examples of negatively curved manifolds where the L^p-spectrum of the Laplacian on functions depends on p can be found in the work of Taylor [Tay89] and Weber [Web07]. They studied the L^p-spectrum of the Laplace-Beltrami operator for quotients of symmetric spaces. These spaces are called locally symmetric spaces. More recently, Charalambous and Rowlett [CR24] studied the L^p-spectrum on conformally compact manifolds.

The L^p-spectrum has also been studied for the Laplacian on forms on Riemannian manifolds. Donnelly in [Don81] computed the L^2-spectrum of the Laplacian on forms of Hyperbolic space. Mazzeo in [Maz88] computed the L^2 essential spectrum of the Laplacian on forms of a conformally compact metric. Antoci also computed in [Ant04] the L^2 essential spectrum of the Laplacian on forms, for a class of warped product metrics.

At the same time, Charalambous [Cha05] proved the L^p-independence of the spectrum of the Laplacian on forms on non-compact manifolds, for $1 \leq p \leq \infty$. More specifically, she showed that under the assumptions that the Ricci curvature is bounded below, the volume growth is uniformly subexponential and the Weitzenbock tensor is bounded below the L^p-spectrum of the Laplacian on forms is independent of p for $1 \leq p \leq \infty$. More recently, Charalambous and Lu [CL24] computed the L^p-spectrum of the Laplacian on forms of Hyperbolic space, for $1 \leq p \leq \infty$, and proved that the form spectrum is p-dependent on this negatively curved space.

In this thesis our main goal is to generalize the set of manifolds over which the spectrum of the Laplacian on forms depends on p. We will consider the case of manifolds that are warped products at infinity and certain quotients of Hyperbolic space. The Laplacian on k-forms has a strong connection to the geometry and topology of a manifold, and hence is is usually more difficult to obtain results for the spectrum on forms in comparison to the spectrum on functions. This is the main reason why we initially concentrate on manifolds with a more rigid structure.

This thesis is organized as follows: In Chapter 2, we present some of the background results and definitions that will be used in the thesis, for notational and reference reasons. In Chapter 3, we decompose the Laplacian on forms over warped product metrics following [Ant04]. These are product manifolds $M = \mathbb{R} \times N$ with the warped

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product metric $g = dr^2 + f^2(r)g_N$. We will use the decomposition of the Laplacian on certain classes of k-forms over these manifolds which will be useful in the computation of their spectrum.

In Chapter 4, we focus our study on warped products at infinity. These are manifolds such that outside a compact set K, $M \setminus K$ is of the form $(c_0, \infty) \times N$ with metric

$$g = dr^2 + f^2(r)g_N,$$

where $f \in C^\infty(c_0, \infty)$ is the warping function and N is an $(n-1)$-dimensional compact manifold. We consider the class of warped product metrics with warping function f in the following class:

$$\begin{aligned} B = \{f \in C^2(a, \infty) : \frac{f''}{f} &= a_0 + o(1), \\ \left(\frac{f'}{f}\right)^2 &= a_0 + o(1), \text{as } r \to \infty, \text{with } a_0 > 0 \\ &\text{and } f \to \infty, \text{as } r \to \infty\}. \end{aligned}$$

In Theorem 4.1.2 and Proposition 4.2.3 we prove that over such M the L^p-spectrum of the Laplacian on k-forms contains a parabolic region which depends on k, p and a_0, and is at the same time contained in a parabolic region that also depends on the bottom of the L^2-spectrum. In the particular case that $f(r) \sim ce^{\sqrt{a_0}r}$ and the Laplacian on k-forms has no isolated eigenvalues of finite multiplicity, we prove that the L^p-spectrum is a parabolic region.

In the last Chapter, we study the L^p-spectrum of the Laplacian on forms on quotients of Hyperbolic space $M = \mathbb{H}^{N+1}/\Gamma$. In [DST88] Davies, Simon and Taylor studied the L^p-spectrum of the Laplace-Beltrami operator Δ_Γ over functions on non-compact quotients $M = \mathbb{H}^{N+1}/\Gamma$, where Γ is a geometrically finite group. Specifically, if M is either of finite volume or cusp-free, they determine explicitly the L^p-spectrum of Δ_Γ for $1 \leq p \leq \infty$, proving that it is a parabolic region together with a finite set of isolated eigenvalues.

In this chapter, we generalize their theorem to the Laplacian on forms $\vec{\Delta}_\Gamma$ in the case when M has no cusps. In order to do this we generalize many of the results from [DST88] concerning properties of the heat semigroup, the heat kernel and the resolvent operator for the Laplacian on functions to those corresponding to the Laplacian on forms.

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In Theorem 5.2.2 we prove that over $M = \mathbb{H}^{N+1}/\Gamma$ with Γ a geometric finite group such that M has infinite volume and no cusps, the L^p-spectrum of the Laplacian on k-forms is a exactly a parabolic region together with a set of isolated eigenvalues on the real line. For technical reasons we must assume that the corresponding L^2-spectrum has no eigenvalues that accumulate to the infimum of the essential spectrum, but we expect that this should always be the case.

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Chapter 2

Preliminaries and Background

In this chapter we present some of the background results and definitions that will be used in the thesis, for notational and reference reasons. Since we are interested in the spectral properties of the Laplacian on Riemannian manifolds we first begin with some spectral theory and then move on to present analytic geometric aspects of the Laplacian operator.

2.1 Spectral Theory

We start with some background on spectral theory. Let X be a normed space. By a linear operator in X, we shall mean a linear map T whose domain $D(T)$ is a linear subspace of X and whose range $R(T)$ is in X. When $D(T)$ is dense in X we will say that T is a densely defined linear operator in X.

Definition 2.1.1. *Let T be a linear operator in a normed space X. The resolvent set of T, denoted by $\rho(T)$, is defined to be the set of all points $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is a bijection from $D(T)$ onto X, which has bounded inverse. The analytic operator valued function*

$$R(\lambda)(T) = (T - \lambda I)^{-1},$$

where $\lambda \in \rho(T)$ is called the resolvent of T. The set $\mathbb{C} \setminus \rho(T)$ is called the spectrum of T, denoted by $\sigma(T)$.

Remark 2.1.1. *If T is a closed linear operator in a Banach space, then $\lambda \in \rho(T)$ is equivalent to $T - \lambda I$ being a bijection.*

In the case that $R(\lambda)(T)$ fails to be injective, then the value λ is called an eigenvalue for the operator T.

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Definition 2.1.2 ([Rud91], 12.17 Definition)**.** *Let Σ be a σ-algebra in a set Ω, and let H be a Hilbert space. In this setting, a resolution of the identity is a map*

$$E : \Sigma \to B(H),$$

where $B(H)$ denotes the Banach algebra of all bounded linear operators on H, with the following properties:

(i) $E(\emptyset) = 0$, $E(\Omega) = I$.

(ii) Each $E(\omega)$ is an orthogonal projection.

(iii) $E(\omega' \cap \omega'') = E(\omega')E(\omega'')$.

(iv) If $\omega' \cap \omega'' = \emptyset$, then $E(\omega' \cup \omega'') = E(\omega') + E(\omega'')$.

(v) For every $x \in H$ and $y \in H$, the set function $E_{x,y}$ defined by

$$E_{x,y}(\omega) = (E(\omega)x, y)$$

is a complex measure on Σ.

Remark 2.1.2. *Resolutions of the identity are also called spectral measures or projection-valued measures [[RS81] p.263] in the bibliography.*

Theorem 2.1.3 ([Rud91], 13.23 Lemma, 13.24 Theorem)**.** *Let E be a resolution of the identity, on a set Ω. To every measurable $f : \Omega \to \mathbb{C}$ corresponds a densely defined closed operator $\Psi(f)$ in H, with domain*

$$D(\Psi(f)) = \{x \in H : \int_\Omega |f|^2 dE_{x,x} < \infty\},$$

which is characterized by

$$(\Psi(f)x, y) = \int_\Omega f dE_{x,y}, \ \ x \in D(\Psi(f)), \ y \in H$$

and which satisfies

$$||\Psi(f)x||^2 = \int_\Omega |f|^2 dE_{x,x}, \ \ x \in D(\Psi(f)).$$

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Also, for every measurable $f : \Omega \to \mathbb{C}$

$$\Psi(f)^* = \Psi(\bar{f}).$$

Definition 2.1.3. *Let A be a densely defined linear operator on a Hilbert space H. A is called self-adjoint if-f $A = A^*$. Here A^* is the adjoint operator of A with respect to the Hilbert space pairing.*

The following Theorem is knowing as Spectral Theorem.

Theorem 2.1.4 ([Rud91], 13.30 Theorem)**.** *To every self-adjoint operator A in H corresponds a unique resolution of the identity E, on the Borel subsets of the real line, such that*

$$(Ax, y) = \int_{\mathbb{R}} \lambda \, dE_{x,y}(\lambda), \;\; x \in D(A), \; y \in H.$$

This E will be called the spectral decomposition of A.

The spectral decomposition Theorem allows us to prove various boundedness properties for operators related to a self-adjoint operator A using Theorem 2.1.3. We will prove the following Lemma which will apply in Chapter 5 to the Laplacian.

Lemma 2.1.5. *Let A be a self-adjoint operator with its spectrum $\sigma(A)$ in $[0, \infty)$. Then $(A + z^2)^{-1}$ is bounded for all z with $Re(z) > 0$. Replacing z with iz and z with $-z$ we get that $(A - z^2)^{-1}$ is bounded for all z with $|Im(z)| > 0$.*

Proof. Let E be the spectral decomposition of A (see Theorem 2.1.4). Now, Theorem 2.1.3 with $f(\lambda) = \frac{1}{\lambda + z^2}$ gives

$$||(A + z^2)^{-1} x||^2 = \int_0^\infty |f(\lambda)|^2 dE_{x,x}(\lambda).$$

Since $dE_{x,x}([0, \infty)) = ||x||^2$, we compute

$$\begin{aligned}\int_0^\infty |f(\lambda)|^2 dE_{x,x}(\lambda) &\leq \sup_{\lambda \in [0,\infty)} \left| \frac{1}{z^2 + \lambda} \right|^2 dE_{x,x}([0, \infty)) \\ &\leq \sup_{\lambda \in [0,\infty)} \left| \frac{1}{z^2 + \lambda} \right|^2 ||x||^2.\end{aligned}$$

Let $\epsilon > 0$, and $\lambda \in [0, \infty)$. We write $z = a + ib$, and thus $Re(z) > \epsilon$ is equivalent to $a > \epsilon$. Now, to estimate $\frac{1}{|z^2 + \lambda|^2}$ we take two cases. The one is $b = 0$ and the other is

$b \neq 0$. In both cases we easily find an upper bound for $\frac{1}{|z^2+\lambda|^2}$. Thus, we conclude that for every $\epsilon > 0$

$$||(\Delta + z^2)^{-1}x|| \leq c(z, \epsilon)||x||,$$

where $c(z, \epsilon)$ is a constant that dependes on z, ϵ, and the Lemma follows. □

Definition 2.1.4 ([RS81] p.236,253)**.** *Let A be a self-adjoint operator in a Hilbert space H and E_A be its spectral decomposition defined above. We say $\lambda \in \sigma_{ess}$, the essential spectrum of A, if-f $E_A(\lambda - \epsilon, \lambda + \epsilon))$ is infinite dimensional for all $\epsilon > 0$. If $\lambda \in \sigma(A)$, and $E_A(\lambda - \epsilon, \lambda + \epsilon))$ is finite dimensional for some $\epsilon > 0$, we say $\lambda \in \sigma_{disc}(A)$, the discrete isolated spectrum of A. Here the dimension of E_A means the dimension of the range $R(E_A)$.*

Remark 2.1.6. *Let us note that from the definition of the essential and discrete isolated spectrum that they form a disjoint union of the spectrum.*

We also have the following equivalent definitions about the discrete isolated, and essential spectrum.

Theorem 2.1.7. *[RS81] p.236]Let A be a self-adjoint operator on a Hilbert space, $\lambda \in \sigma_{disc}(A)$ if and only if both the following hold:*

(a) λ is an isolated point of $\sigma(A)$, that is, for some $\epsilon > 0$,$(\lambda - \epsilon, \lambda + \epsilon) \cap \sigma(A) = \{\lambda\}$.

(b) λ is an eigenvalue of finite multiplicity, i.e., the corresponding eigenspace is finite dimensional.

Moreover, $\sigma_{ess}(A)$ is a closed set, and, $\lambda \in \sigma_{ess}(A)$ if and only if one or more of the following holds:

(a) λ is a limit point of the set of all eigenvalues.

(b) λ is an eigenvalue of infinite multiplicity, i.e., the corresponding eigenspace is infinite dimensional.

(c) λ belongs to the continuous spectrum (see p. 231 [RS81] for definition).

For an operator on a Banach space, one usually does not make a distinction between the essential spectrum and the discrete isolated spectrum.

2.2 Semigroup Theory

Definition 2.2.1 ([EBN+99] p.36)**.** *A family* $(T(t))_{t\geq 0}$ *of bounded linear operators on a Banach spaces* X *is called a strongly continuous one-parameter semigroup if it satisfies the following:*

(i) $T(t+s) = T(t)T(s)$ *for all* $t, s \geq 0$.

(ii) $T(0) = I$

(iii) the map $t \to T(t)$ *is strongly continuous (with respect to the strong operator topology).*

Two important constructions of semigroups that we will need in Chapter 5 are

Definition 2.2.2 ([EBN+99] p.43)**.** *If* Y *is a closed subspace of* X *such that* $T(t)Y \subset Y$ *for all* $t \geq 0$ *i.e., if* Y *is* $(T(t))_{t\geq 0}$*-invariant, then the restrictions*

$$T(t)_{\restriction} = T(t)_{\restriction Y}$$

form a strongly continuous semigroup $(T(t)_{\restriction})_{t\geq 0}$*, called the subspace semigroup, on the Banach space* Y*.*

Definition 2.2.3 ([EBN+99] p.43)**.** *For a closed* $(T(t))_{t\geq 0}$*-invariant subspace* Y *of* X*, we consider the quotient space* $X_/ = X/Y$ *with the canonical quotient map* $q : X \to X_/$*. The quotient operators* $T(t)_/$ *given by*

$$T(t)_/ q(x) = q(T(t)x), \quad x \in X, t \geq 0,$$

are well-defined and form a strongly continuous semigroup, called the quotient semigroup $(T(t)_/)_{t\geq 0}$ *on the Banach Space* $X_/$*.*

Definition 2.2.4 ([EBN+99] p.49)**.** *The generator* $A : D(A) \subset X \to X$ *of a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X *is the operator*

$$Ax = -\lim_{h\to 0^+} \frac{1}{h}(T(h)x - x)$$

defined for every x *in its domain*

$$D(A) = \{x \in X : \lim_{h\to 0^+} \frac{1}{h}(T(h)x - x) exists\}.$$

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Proposition 2.2.1 ([EBN+99] p.61). *Let Y be a $(T(t))_{t\geq 0}$-invariant closed subspace of X, and suppose that A is the generator of $T(t)$. Then,*

(i) the generator of $(T(t)_{\restriction})_{t\geq 0}$ is

$$A_{\restriction} y = Ay$$

with domain

$$D(A_{\restriction}) = D(A) \cap Y.$$

(ii) the generator $(A_{/}, D(A_{/}))$ of the quotient semigroup $(T(t)_{/Y})_{t\geq 0}$ on the quotient space $X_{/} = X/Y$ is given by

$$A_{/} q(x) = q(Ax)$$

with domain

$$D(A_{/}) = q(D(A)).$$

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2.3 The Laplacian and heat kernel on a Riemannian manifold

Let (M, g) be a smooth non compact Riemannian manifold of dimension n, where g is the Riemannian metric. We assume that M is complete with empty boundary. We will denote by $L^2(M)$ the set of square integrable functions on M. Note that $L^2(M)$, is a Hilbert Space with respect to the inner product

$$(f, g) = \int_M fg \, dV_g,$$

where dV_g denotes the volume element with respect to the Riemannian metric g. Now, we define the Laplace operator Δ on (M, g). On smooth functions the Laplacian is given by the second order operator $\Delta = -div \circ \nabla$, where div and ∇ are the divergence and the gradient respectively. Note that here we consider the analyst's Laplacian so that we end up with a non-negative operator. By Green's formulas the Laplacian is symmetric on the space of smooth functions with compact support $C_0^\infty(M)$, since $(\Delta u, v) = (u, \Delta v)$ for every $u, v \in C_0^\infty(M)$. Since $C_0^\infty(M)$ is dense in $L^2(M)$, Δ is a densely defined operator in $L^2(M)$. It is well known that this operator is symmetric but it is not a self adjoint operator on $C_0^\infty(M)$. By considering the quadratic form $Q(f) = (\Delta f, f) \geq 0$ corresponding to the Laplacian for all $f \in C_0^\infty(M)$, we take the Friedrichs extension of Δ on $L^2(M)$ which is given by the self adjoint operator which corresponds the closure of Q on $L^2(M)$ (see Theorem 1.2.8 [Dav89]). We will refer to this Laplacian as the Laplacian on $L^2(M)$. On complete manifolds this extension is unique.

Let E be the spectral decomposition of a self-adjoint operator A in a Hilbert space H. Then by Theorem 2.1.3 with $f(\lambda) = e^{-t\lambda}$ and $t \geq 0$ some fixed number we get the self-adjoint operator

$$\Psi(f) = \int_{\mathbb{R}} e^{-t\lambda} dE(\lambda).$$

It is common to denote $\Psi(f)$ by e^{-tA} and thus we write

$$e^{-tA} = \int_{\mathbb{R}} e^{-t\lambda} dE(\lambda).$$

The above definition give us a one-parameter family of self-adjoint operators. This family is actually a strongly continuous semigroup of linear operators on H with A its

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generator (see [Gri09] Theorem 4.9). Since the Laplacian on $L^2(M)$, is a self adjoint operator its semigroup is well defined and we denote it by $e^{-t\Delta}$. By definition, for any $f \in L^2(M)$, $e^{-t\Delta}f$ is also an element of $L^2(M)$. It can in fact be shown that $e^{-t\Delta}f$ will be smooth, by the smoothing properties of the heat operator (see Theorem 7.6 [Gri09]). The action of the semigroup on $L^2(M)$ can be described by an integral operator, which in turns corresponds to what is known as its integral kernel.

Definition 2.3.1. *Let T be an operator on $L^2(M)$. If there exists a function $T(x, y)$ such that*

$$Tf(x) = \int_M T(x, y)f(y)dy,$$

for all $f \in L^2(M)$, then $T(x, y)$ is called the integral kernel of T. Here for simplicity we denote dV_g by dy.

The existence of the kernel for the heat semigroup on complete smooth manifolds is well known.

Theorem 2.3.1 (Theorem 7.7[Gri09])**.** *Let (M, g) be a complete smooth Riemannian manifold. For any $x \in M$ and for any $t > 0$, there exists a unique function $p_{t,x} \in L^2(M)$ such that, for all $f \in L^2(M)$,*

$$e^{-t\Delta}f(x) = \int_M p_{t,x}(y)f(y)dy.$$

As noted in Remark 7.8 [Gri09] the function $p_{t,x}(y)$ is in general defined for all $t > 0, x \in M$ but for almost all $y \in M$. This is not a problem because after a some work it can be regularized to obtain a smooth function of all three variables t, x, y (see Theorem 7.20 [Gri09]). Now, we can define the heat kernel.

Definition 2.3.2. *For any $t > 0$ and all $x, y \in M$ set*

$$p(t, x, y) = (p_{t/2,x}, p_{t/2,y})_{L^2}.$$

The function $p(t, x, y)$ is called the heat kernel of M.

The main properties of $p(t, x, y)$ are stated in the following Theorem.

Theorem 2.3.2. *(Theorem 7.13[Gri09]) On a smooth complete Riemannian manifold (M, g) the heat kernel satisfies the following properties*

(i) *Symmetry:* $p(t,x,y) = p(t,y,x)$ *for all* $x, y \in M$ *and* $t > 0$.

(ii) *For any* $f \in L^2$, *and for all* $x \in M$ *and* $t > 0$,

$$e^{-t\Delta} f(x) = \int_M p(t,x,y) f(y) dy.$$

(iii) $p(t,x,y) \geq 0$ *for all* $x, y \in M$ *and* $t > 0$, *and*

$$\int_M p(t,x,y) dy \leq 1,$$

for all $x \in M$ *and* $t > 0$.

(iv) *The semigroup identity: for all* $x, y \in M$ *and* $t, s > 0$,

$$p(t+s,x,y) = \int_M p(t,x,z) p(s,z,y) dz.$$

(v) *For any* $y \in M$, *the function* $u(t,x) = p(t,x,y)$ *is* C^∞ *smooth in* $(0,\infty) \times M$ *and satisfies the heat equation*

$$\frac{\partial u}{\partial t} + \Delta u = 0.$$

(vi) *For any function* $f \in C_0^\infty(M)$,

$$\int_M p(t,x,y) f(y) dy \to f(x) \text{ as } t \to 0,$$

where the convergence is in $C^\infty(M)$.

The heat kernel has several symmetric properties reflecting the symmetries of the underlying space. For example, $p(t,x,y)$ is invariant under the isometry group $I(M)$ of the manifold. This property becomes particularly useful when we consider the heat operator on symmetric spaces such as the Hyperbolic space. Finally let us note that one can define the heat kernel on (M,g) as a fundamental solution to the heat equation.

Definition 2.3.3. *([Cha84] p. 135) A fundamental solution of the heat equation on* M *is a continuous function* $p = p(t,x,y)$ *defined on* $(0,\infty) \times M \times M$, *which is* C^2 *with respect to* x, C^1 *with respect to* t, *and which satisfies*

$$(\partial_t + \Delta_x) p = 0 \text{ and } \lim_{t \to 0^+} = p(t,\cdot,y) = \delta_y,$$

where δ_y is the Dirac delta function, that is, for all bounded continuous functions f on M we have, for every $y \in M$,

$$\lim_{t \to 0^+} \int_M p(t,x,y) f(x) dx = f(y).$$

These two definitions of the heat kernel are equivalent (see Theorem 9.5 [Gri09]).

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2.4 The Heat Semigroup on L^p spaces

Our aim is to define the Laplacian Δ as an operator on $L^p(M)$, and consider its spectrum in this more general setting. Until now, we have defined Δ on $L^2(M)$ and through its spectral resolution-decomposition we define the heat semigroup on $L^2(M)$ with Δ its generator.

Now, we proceed in the opposite direction. We will first define the Heat semigroup on $L^p(M)$ and then the Laplacian will be defined as the generator of this semigroup. This is the classical construction through which the Laplacian on $L^p(M)$ is defined on a complete manifold.

Definition 2.4.1 ([Dav89] p.22)**.** *Let Ω be a set with a countably generated σ-field and a σ-finite measure dx. If $A \geq 0$ is a real self-adjoint operator on $L^2(\Omega)$ satisfying*

(a) e^{-At} is positivity-preserving for all $t \geq 0$.

(b) e^{-At} is a contraction on L^∞ for all $t \geq 0$.

Then A is called a Dirichlet form and e^{-At} is called a symmetric Markov semigroup.

One can check that $e^{-t\Delta}$ on $L^2(M)$ is a symmetric Markov semigroup and as a result we have the following.

Theorem 2.4.1 ([Dav89] Theorem 1.4.1)**.** *The set $L^1(M) \cap L^\infty(M)$ is invariant under the semigroup $e^{-t\Delta}$ on $L^2(M)$, and $e^{-t\Delta}$ may be extended from $L^1(M) \cap L^\infty(M)$ to a positive one-parameter contraction semigroup $T_p(t)$ on $L^p(M)$ for all $1 \leq p < \infty$. These semigroups are strongly continuous if $1 \leq p < \infty$, and are consistent in the sense that*

$$T_p(t)f = T_q(t)f, \quad f \in L^p(M) \cap L^q(M).$$

Definition 2.4.2. *The Laplacian Δ_p on $L^p(M)$, $1 \leq p < \infty$ is defined as the generator of the semigroup $T_p(t)$ on $L^p(M)$ and for $p = \infty$ it is the adjoint of Δ_1, with L^∞ defined as $(L^1)^*$.*

We denote the spectrum of the Laplacian on L^p-integrable functions by $\sigma(p, 0, \Delta)$. Observe that $\sigma(p, 0, \Delta) = \sigma(p', 0, \Delta)$ for $1/p + 1/p' = 1$, since the adjoint operator of Δ_p is $\Delta_{p'}$.

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2.5 The Laplacian and heat kernel on forms

In this section we further assume that (M, g) is orientable. We consider the space of k-forms on M, denoted by $\Lambda^k(M)$, which are smooth sections of the bundle $\Lambda^k(T^*M)$. The Riemannian metric induces a pointwise inner product on $\Lambda^k(M)$ as follows. If we fix a point $p \in M$, this inner product is defined firstly on elements of the form $a_1 \wedge \cdots \wedge a_k$ by

$$< a_1 \wedge \cdots \wedge a_k, b_1 \wedge \cdots \wedge b_k >= \det(< a_i, b_j >),$$

where $a_1, \ldots, a_k, b_1, \ldots, b_k \in T_p^*(M)$ and $<,>$ is the induced inner product on $T_p^*(M)$ from the Riemannian metric g. Then by linearity we can extend it to the whole space $\Lambda^k(T_p^*M)$. Now we can define a global inner product on $\Lambda^k(M)$ using this pointwise inner product by

$$(a, b) = \int_M < a, b > dV_g.$$

In consequence, one can also define $L^2(\Lambda^k(M))$ as the k-forms with finite L^2-norm.

Definition 2.5.1. *Let n be the dimension of M. The Hodge star operator $*$ is defined as*

$$* : \Lambda^k(M) \to \Lambda^{n-k}(M)$$

$$a \mapsto *a$$

where $*a$ is the unique element of $\Lambda^{n-k}(M)$ s.t. $a \wedge b =< *a, b > dV_g$ for every $b \in \Lambda^{n-k}(M)$.

Let us note that the pointwise inner product on $\Lambda^k(M)$ is related to the Hodge star operator by $< a, b > dV_g = a \wedge *b$.

We have the following operators on k-forms. $d^k : \Lambda^k(M) \to \Lambda^{k+1}(M)$ is the exterior differentiation and $\delta^{k+1} : \Lambda^{k+1}(M) \to \Lambda^k(M)$ is the co-differential operator defined by $\delta^{k+1} = (-1)^{nk+1} * d *$, with $*$ the Hodge star operator and $n = \dim M$. Note that d is independent of the metric, but δ does depend on the metric since $*$ does. The Laplacian on k-forms on (M, g) is defined as follows:

Definition 2.5.2. *([Ros97] p. 33) The Laplacian on smooth k-forms on a (M, g) is given by $\Delta^k = \delta^{k+1} \circ d^k + d^{k-1} \circ \delta^k$. For simplicity, we will often write Δ instead of Δ^k.*

Remark 2.5.1. *The Laplacian on 0-forms is the Laplacian on functions and* $\Delta^0 = \delta d$.

The Laplacian is a symmetric operator on $C_0^\infty(\Lambda^k(M))$ since using integration by parts we have

$$\int_M < \Delta a, b >= \int_M < da, db > + < \delta a, \delta b >= \int_M < a, \Delta b > .$$

Note that δ is the adjoint operator to d with respect to the inner product $\int_M < a, b >$. Again we define Δ on $L^2(\Lambda^k(M))$ via the Friedrichs extension theorem using this quadratic form. This is closed, self-adjoint operator which is nonnegative on $L^2(\Lambda^k(M))$. The heat semigroup is defined for the Laplacian on $L^2(\Lambda^k(M))$ in a similar manner through its spectral resolution.

To define the heat kernel on forms we use the equivalent way that we described before as a fundamental solution of the heat equation. Now, the heat equation will be on forms.

Definition 2.5.3. *([Ros97] p.34) A double form* $\vec{p}(t,x,y)$ *is a smooth section of* $(0,\infty)\times \Lambda^k(M)\otimes\Lambda^k(M)$ *and it is called the heat kernel for the Laplacian on k-forms if*

(i) $(\partial_t + \Delta_x^k)\vec{p}(t,x,y) = 0$ *and*

(ii) $\lim_{t\to 0^+}\int_M < \vec{p}(t,x,y), \omega(y) >_y dy = \omega(x)$ *for every smooth with compact support k-form* ω.

Locally, if we have a chart $\{x^1,\dots,x^n\}$ on (M,g), then $\{dx^{i_1}\wedge\cdots\wedge dx^{i_k} : 1\le i_1 < \cdots < i_k \le n\}$ is a basis for $\Lambda^k(M)$. For simplicity, we will use the multi-index notation $I = (i_1,\dots,i_k)$ to express $dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ as dx^I. Thus, the heat kernel takes the form

$$\vec{p}(t,x,y) = \sum_{I,J} f_{I,J}(t,x,y)dx^I\otimes dy^J, \tag{2.5.1}$$

locally, where I, J run over a local basis of k-forms for some $f_{I,J}$. Thus, to determine the heat kernel on forms locally it is enough to determine the $f_{I,J}$ for a pointwise basis of k-forms . Let us note that $f_{I,J}$ form a rectangular matrix of dimension equal to $\binom{n}{k}$ for the case of k-forms. So, for the special case of1-forms we have that the $f_{I,J}$ form an $n\times n$ - matrix. Finally, using (2.5.1) we have the following expression for the integral

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in (ii)

$$\int_M < \vec{p}(t,x,y), \omega(y) >_y dy =$$

$$\sum_I \left[\int_M < \sum_J f_{I,J}(t,x,y) dy^J, \sum_K \omega_K(y) dy^K > dy \right] dx^I. \quad (2.5.2)$$

As an example of a heat kernel on forms we will compute it for the case of 1-forms over the Euclidean space $(\mathbb{R}^n, g_0)$. Let $\{x^1, \ldots, x^n\}$ be the Cartesian coordinates of $\mathbb{R}^n$ and $\{dx^1, \ldots, dx^n\}$ be the corresponding co-frame. The heat kernel with respect to this co-frame is written as

$$\vec{p}(t,x,y) = \sum_{i,k} e_{i,k}(t,x,y) dx^i \otimes dy^k.$$

We will show that $e_{i,k} = 0$ for $i \neq k$ and $e_{i,i}$ are all equal with the heat kernel on functions. Any 1-form ω can be written as $\omega = \sum_i f_i dx^i$ Since $\Delta^1_{\mathbf{R}^n}(f dx^i) = (\Delta_{\mathbf{R}^n} f) dx^i$ we have that the heat equation

$$\begin{cases} (\partial_t + \Delta^1_{\mathbb{R}^n})\omega(t,x) = 0 \\ \omega(x,0) = \omega_0 \end{cases}$$

where

$$\omega(t,x) = \sum_i f_i(t,x) dx^i, \;\; \omega_0(x) = \sum_i f_i^0(x) dx^i$$

is equivalent to

$$\begin{cases} (\partial_t + \Delta_{\mathbb{R}^n}) f_i(t,x) = 0 \\ f_i(x,0) = f_i^0(x), \end{cases} \quad , i = 1, \ldots, n,$$

From the above it is clear that the initial condition

$$\omega_0(x) = f_1^0(x) dx^1 + 0 \cdot dx^2 + \cdots + 0 \cdot dx^n$$

has solution

$$\omega(t,x) = f_1(t,x) dx^1 + 0 \cdot dx^2 + \cdots + 0 \cdot dx^n.$$

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At the same time (2.5.2) gives

$$\omega(t,x) = \sum_{i=1}^{n} [\sum_{k=1}^{n} \int_{\mathbb{R}^n} e_{i,k}(t,x,y) f_k^0(y) dy] dx^i = \sum_{i=1}^{n} [\int_{\mathbb{R}^n} e_{i,1}(t,x,y) f_1^0(y) dy] dx^i.$$

Thus, if we set

$$a_i = \int_{\mathbb{R}^n} e_{i,1}(t,x,y) f_1^0(y) dy,$$

we have that $a_2 = 0, \ldots, a_n = 0$. Since $f_1^0(x) \neq 0$ was arbitrary we get that $e_{i,1}(t,x,y) = 0$ for $i = 2, \ldots, n$. Similarly, one can prove $e_{i,2}(t,x,y) = 0$ for $i = 1, 3, \ldots, n$ and for the remaining cases. Finally, from

$$f_1(t,x) = \int_{\mathbb{R}^n} e_{1,1}(t,x,y) f_1^0(y) dy,$$

we have that $e_{1,1}(t,x,y)$ is equal to the heat kernel on functions. The same holds for every $e_{i,i}(t,x,y)$.

Thus, for the case of 1-forms the heat kernel with respect to the standard basis on $\mathbb{R}^n$ it is a diagonal matrix where each diagonal element is the heat kernel on functions. Similarly, one can show that the heat kernel on k-forms on $\mathbb{R}^n$, can be described as a rectangular matrix of dimension $\binom{n}{k}$, with its entries being the heat kernel on functions. So in Euclidean spaces, the heat kernel on forms is the "same" as the heat kernel on functions.

Now we will define the Weitzenbock tensor, a curvature tensor which will give us a way to find upper bounds for the heat kernel on k-forms.

Definition 2.5.4 (Definition 1 [Cha05])**.** *Let (M,g) be an oriented Riemannian manifold and V_i be a locally frame field and ω^i be its dual coframe field. We denote the tensor $W^k = -\sum_{i,j} \omega^i \wedge i(V_j) R_{V_i V_j}$ acting on k-forms, as the Weitzenbock tensor on k-forms, where $R_{XY} = D_X D_Y - D_Y D_X - D_{[X,Y]}$ is the curvature tensor and $i(Z)\theta$ is the interior multiplication by the vector field Z of the k-form θ.*

For 1-forms W^1 coincides with Ricci curvature, this is not true for higher order k.

Proposition 2.5.1. *[[Cha05] Theorem 4, [HSU77]] Let M be a complete manifold with Ricci curvature bounded below and Weitzenbock tensor on k forms bounded bellow $W^k \geq -K_2$. Then, the heat kernel on k forms $\vec{p}(t,x,y)$ has the following pointwise bound*

$$|\vec{p}(t,x,y)| \leq e^{K_2 t} p(t,x,y),$$

where p(t,x,y) is the heat kernel of the Laplacian on functions.

Corollary 2.5.1. *Let M non-compact, complete, with Ricci curvature bounded from below and Weitzenbock tensor on k forms bounded from below. Then,*

$$\omega(x,t) = \int_M \vec{p}(t,x,y) \wedge *\omega_0(y)dy$$

is the unique solution to the heat equation on forms,

$$\begin{cases} (\partial_t + \Delta^k)\omega = 0 \\ \omega(\cdot, 0) = \omega_0 \end{cases}$$

Let us note that the same holds for compact manifolds without any further assumption (see [Cha84] p.338).

Finally, we write the following Propositions that we will need in Chapter 5. The first one is a generalization of the invariance of the integral under the isometry group, that is, if J is an isometry, then

$$\int_M f(J)dV_g = \int_M f dV_g$$

(see [Gri09] Lemma 3.27).

Proposition 2.5.2. *Let ω_1, ω_2 be k-forms and J be an isometry. Then*

$$\int_M < \omega_1(Jx), \omega_2(Jx) > dx = \int_M < \omega_1(x), \omega_2(x) > dx.$$

The second one is the invariance of the heat kernel on k-forms under the isometry group, whose proof follows from the previous proposition, and Corollary 2.5.1. For the case of heat kernel on functions this is a standard fact, see for example [Gri09] Theorem 9.12) .

Proposition 2.5.3. *Let J be an isometry. Then the heat kernel on forms satisfies*

$$\vec{p}(t,x,y) = \vec{p}(t,Jx,Jy).$$

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2.6 Spectrum on forms

Analogously to Section 2.4, in this section we will define the Laplacian on $L^p(\Lambda^k(M))$ as the infinitesimal generator of a certain semigroup in order to consider its spectrum.

As we saw, for any $t \geq 0$, the heat operator $e^{-t\Delta}$ is a bounded operator on $L^2(\Lambda^k(M))$. Moreover, when the Weitzenbock tensor is bounded bellow, i.e., $W^k \geq -K_2$ it can be proved that the L^2 to L^2 bound of $e^{-t\Delta}$ is at most e^{tK_2}. Even though it might no longer be a Markov semigroup, it can be extended similarly to Theorem 2.4.1 for Δ^0 to a semigroup of operators on $L^p(\Lambda^k(M))$ for any $1 \leq p \leq \infty$ (see for example [CL24]).

Definition 2.6.1. *The Laplacian Δ_p on $L^p(\Lambda^k(M))$, $1 \leq p < \infty$ is defined as the generator of the semigroup $e^{-t\Delta}$ on $L^p(\Lambda^k(M))$ and for $p = \infty$ as the adjoint operator to Δ_1, with $L^\infty = (L^1)^*$.*

We denote the spectrum of the Laplacian on L^p integrable k-forms by $\sigma(p, k, \Delta)$. Again we have that $\sigma(p, k, \Delta) = \sigma(p^*, k, \Delta)$ for $1/p + 1/p^* = 1$, since the adjoint operator of Δ_p is Δ_{p^*}. Moreover, because $*\Delta = \Delta*$ and $| * \omega| = |\omega|$ pointwise we get that $\sigma(p, k, \Delta) = \sigma(p, n - k, \Delta)$. This is called Poincare duality.

The following result will be useful for the computation of the L^p-spectrum.

Proposition 2.6.1. *[CL24] A complex number λ is in the spectrum of $\Delta = \Delta_p$, if-f one of the following holds*

(i) For any $\epsilon > 0$, there is a k-form $\omega \in Dom(\Delta_p)$ such that

$$\|\Delta\omega - \lambda\omega\|_{L^p} \leq \epsilon \|\omega\|_{L^p}$$

or,

(ii) For any $\epsilon > 0$, there is a k-form $\omega \in Dom(\Delta_{p^*})$ such that

$$\|\Delta\omega - \lambda\omega\|_{L^{p^*}} \leq \epsilon \|\omega\|_{L^{p^*}}$$

where p^* satisfies $1/p + 1/p^* = 1$.

As a result, to show that $\lambda \in \sigma(p, k, \Delta)$, it suffices to find a sequence of approximate

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eigenforms i.e. for every $\epsilon > 0$ there exists ω_ϵ such that

$$\|\Delta\omega_\epsilon - \lambda\omega_\epsilon\|_{L^p} \leq \epsilon \|\omega_\epsilon\|_{L^p}.$$

Moreover, as we will see, there is a relationship between the L^p-spectrum of the Laplacian depending on p. The Stein Interpolation result as stated in [Dav89]] Section 1.1.6, will be important in illustrating this. We state it below.

Lemma 2.6.1. *[The Stein Interpolation Theorem] [[Dav89] p.3] Let* $1 \leq p_0, p_1 \leq \infty$ *and* $S = \{0 \leq Rez \leq 1\}$. *Suppose that for all* $z \in S$, $T(z)$ *is a linear operator from* $L^{p_0}(\Lambda^k(M)) \cap L^{p_1}(\Lambda^k(M))$ *to* $L^{p_0}(\Lambda^k(M)) + L^{p_1}(\Lambda^k(M))$. *Furthermore, assume that*

(a) $< T(z)\omega, \eta >$ *is uniformly bounded and continuous on* S *and analytic in the interior of* S *whenever* $\omega \in L^{p_0}(\Lambda^k(M)) \cap L^{p_1}(\Lambda^k(M))$ *and* $\eta \in L^{p_0^*}(\Lambda^k(M)) \cap L^{p_1^*}(\Lambda^k(M))$.

(b) For all $y \in \mathbb{R}$
$$||T(iy)\omega||_{p_0} \leq M_0||\omega||_{p_0},$$
for all $\omega \in L^{p_0}(\Lambda^k(M)) \cap L^{p_1}(\Lambda^k(M))$.

(c) For all $y \in \mathbb{R}$
$$||T(1+iy)\omega||_{p_0} \leq M_1||\omega||_{p_0},$$
for all $\omega \in L^{p_0}(\Lambda^k(M)) \cap L^{p_1}(\Lambda^k(M))$.

Then for each $t \in (0,1)$ *and* $\omega \in L^{p_0}(\Lambda^k(M)) \cap L^{p_1}(\Lambda^k(M))$

$$||T(t)\omega||_{p_t} \leq M_0^{1-t}M_1^t||\omega||_{p_t}$$

where $\frac{1}{p_t} = \frac{t}{p_1} + \frac{(1-t)}{p_0}$. *Hence,* $T(t)$ *can be extended to a bounded operator on* $L^{p_t}(\Lambda^k(M))$ *with norm at most* $M_0^{1-t}M_1^t$.

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2.7 Riemannian Geometry

2.7.1 Curvatures in warped products

In this section we will describe the curvature of the manifold M when it is of the form $M = \mathbb{R} \times N$, where (N, g_N) is an $(n-1)$ dimensional compact, complete smooth manifold and M is endowed with the warped product metric

$$g_M = dr^2 + f^2(r) g_N.$$

Here $f(r)$ is a smooth function in r, called the warping function.

One expects that the curvatures of M will depend on the warping function $f(r)$ as well the curvature of N. This can be established and we provide the exact formulas.

Note that we will also be considering warped products in a slightly more general form of the type $(a, b) \times N$ with $(a, b) \subset \mathbb{R}$, but the curvature formulas will be the same.

Before we consider the general case, let as describe an example of such a manifold. We will look at the special case of the Hyperbolic space, since it will play an important part in this thesis.

There are four classical ways that one can define the Hyperbolic Space $\mathbb{H}^n$, the half-space model, the Poincare ball model, The Beltrami-Klein model and the Hyperboloid model. Actually these four models are equivalent since they are isometric as Riemannian manifolds (see [Lee18] Theorem 3.7).

We start with the Hyperboloid model for $\mathbb{H}^n$, $n \geq 2$ and define polar coordinates on it, following [Gri09]. The Hyperboloid H is given by the equation $(x^{n+1})^2 - \tilde{x}^2 = 1$, where $\tilde{x} = (x^1, \dots, x^n) \in \mathbb{R}^n$ and $x^{n+1} > 0$, is a submanifold of $\mathbb{R}^{n+1}$. We define the Riemannian metric g_H on H obtained as the restriction of the Minkowski pseudometric on $\mathbb{R}^{n+1}$

$$g_{\text{Mink}} = (dx^1)^2 + \cdots + (dx^n)^2 - (dx^{n+1})^2.$$

The Hyperboloid H with g_H is the Hyperboloid model of $\mathbb{H}^n$. We take $p = (0, \dots, 0, 1) \in \mathbb{H}^n$ and take polar coordinates on $\mathbb{H}^n$ such that

$$\cosh r = x^{n+1}, \;\; \theta = \frac{\tilde{x}}{|\tilde{x}|}, \;\; x \in \mathbb{H}^n \setminus \{p\}, r > 0, \theta \in S^{n-1}.$$

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One can prove that with respect to these coordinates g_H takes the form

$$g_{\mathbb{H}^n} = dr^2 + \sinh^2 r g_{S^{n-1}},$$

where $g_{S^{n-1}}$ denotes the Riemannian metric of the unit sphere S^{n-1}. Thus, $\mathbb{H}^n$ can be equivalently defined as the manifold $M = [0, \infty) \times S^{n-1}$ with the warped product metric $g_{\mathbb{H}^n}$ above. Note that the smoothness properties of f at $r = 0$ allow us to show that M is a complete smooth manifold.

We now go back to the general case of $(\mathbb{R} \times N, dr^2 + f^2(r)g_N)$. In ([Li12], Appendix A) Li computes the sectional curvatures of M using Cartan's structural equations (for these equations see for example [Boo86] p.380). Let $e_1 = \frac{\partial}{\partial r}$ and $\widetilde{e_a}$, for $a \in (2, \dots, n)$ be an orthonormal frame on N. Then setting $e_a = \frac{1}{f}\widetilde{e_a}$ for $a \in (2, \dots, n)$ we have that e_a for $a \in (1, \dots, n)$ is an orthonormal frame on M. Then, the sectional curvature of the planes $\pi_{1,\alpha}$ spanned by e_1 and e_α is given by

$$sec(\pi_{1,\alpha}) = -\left((logf)'' + ((logf)')^2\right) = -\frac{f''}{f}, \tag{2.7.1}$$

and the sectional curvature of the planes $\pi_{\alpha,\beta}$ spanned by e_α and e_β is given by

$$sec(\pi_{\alpha,\beta}) = \frac{\widetilde{sec}(\pi_{\alpha,\beta})}{f^2} - ((logf)')^2 = \frac{\widetilde{sec}(\pi_{\alpha,\beta}) - (f')^2}{f^2}, \tag{2.7.2}$$

where $\widetilde{sec}(\pi_{\alpha,\beta})$ denote the sectional curvature of the corresponding plane in N.

Now, we will show that all sectional curvatures of M take values between

$$-\frac{f''}{f}, \ \frac{\widetilde{sec}(\pi_{\alpha,\beta}) - (f')^2}{f^2}.$$

Let us recall that the curvature operator

$$\mathcal{R} : \Lambda^2(TM) \to \Lambda^2(TM),$$

is the self-adjoint operator uniquely defined by the relation

$$g_M(\mathcal{R}(W \wedge X, Y \wedge Z) = -R(W, X, Y, Z).$$

We have

Proposition 2.7.1 ([Pet06], Prop. 4.1.1)**.** *Let e_i be an orthonormal basis for T_pM. If*

$e_i \wedge e_j$ *diagonalize the curvature operator*

$$\mathcal{R}(e_i \wedge e_j) = \lambda_{ij} e_i \wedge e_j,$$

then for any plane π *in* T_pM *we have* $sec(\pi) \in [min\lambda_{ij}, max\lambda_{ij}]$.

So, since (2.7.1), (2.7.2) tell us that

$$\mathcal{R}(e_1 \wedge e_\alpha) = -\frac{f''}{f} e_1 \wedge e_\alpha,$$

$$\mathcal{R}(e_\alpha \wedge e_\beta) = \frac{\widetilde{sec}(\pi_{\alpha,\beta}) - (f')^2}{f^2} e_\alpha \wedge e_\beta,$$

we have the following

Proposition 2.7.2. *Let* $M = \mathbb{R} \times N$ *be the product manifold endowed with the warped product metric* $g_M = dr^2 + f^2(r) g_N$, *where* g_N *is a metric on* N. *Then all sectional curvatures of* M *are between*

$$-\frac{f''}{f}, \quad \frac{\widetilde{sec}(\pi_{\alpha,\beta}) - (f')^2}{f^2}.$$

Corollary 2.7.1. *If we take* $N = S^{n-1}$, *then since* $\widetilde{sec}(\pi_{\alpha,\beta}) = 1$ *we get that all sectional curvatures of* M *are between*

$$-\frac{f''}{f}, \quad \frac{1 - (f')^2}{f^2}.$$

Remark 2.7.1. *Let us note that in [Section 4.2 [Pet06]] the same result is obtained by another approach, but only for the special case of* $N = S^{n-1}$. *The technique there uses the Hessian operator of the radial distance function.*

2.7.2 Comparison Theorems

Let (M, g) be a complete Riemannian manifold of dimension n. Let $p \in M$, then the volume element in geodesic polar coordinates at p, can be expressed as

$$J(r, \theta) dr \wedge d\theta,$$

where $d\theta$ is the area element of the unit $(n-1)$ - sphere. An explicit expression of $J(r, \theta)$ can be found by the Jacobian of the exponential mapping, see [[Sak96] p.154],

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[[SY94] p.8-9].

Let $sn_K(r)$ be the solution to

$$u''(r) + Ku(r) = 0,$$

with the initial conditions $u(0) = 0$, $u'(0) = 1$. An explicit calculation gives

$$sn_K(r) = \begin{cases} \frac{1}{\sqrt{K}} sin\sqrt{K}r & \text{if } K > 0 \\ r & \text{if } K = 0 \\ \frac{1}{\sqrt{|K|}} sinh\sqrt{|K|}r & \text{if } K < 0 \end{cases} . \tag{2.7.3}$$

The above function is related to the space forms as follows: Let us denote by M_K^n the complete simply connected space of constant curvature K. Then the corresponding volume element of M_K^n is

$$J_K(r,\theta)dr \wedge d\theta = sn_K^{n-1} dr \wedge d\theta. \tag{2.7.4}$$

Now, we state the Bishop Volume Comparison Theorem.

Theorem 2.7.2. *[[Li12] p.14] Let M be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by a constant $(n-1)K$. Let $A_p(r)$ be the area of the geodesic sphere $\partial B_p(r)$ centered at $p \in M$ of radius r and $\bar{A}(r)$ be the area of the geodesic sphere $\partial \bar{B}_K(r)$ of radius r in M_K^n. Then, for $0 \leq r_1 \leq r_2 < \infty$ we have*

$$A_p(r_2)\bar{A}(r_1) \leq A_p(r_1)\bar{A}(r_2)$$

and

$$V_p(r_2)\bar{V}(r_1) \leq V_p(r_1)\bar{V}(r_2),$$

where $V_p(r)$ and $\bar{V}(r)$ are the volumes of $B_p(r)$ and $\bar{B}(r)$, respectively.

The theorem essentially follows by the comparison properties of the Riccati equation which J satisfies. Let us recall that

$$A_p(r) = \int_{S^{n-1}} J(r,\theta)d\theta$$

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and

$$V_p(r) = \int_0^r A_p(r')dr'.$$

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Chapter 3

The Hodge Laplacian on warped products metrics

Let $M = \mathbb{R} \times N$ be the product manifold, with the warped product metric

$$g = dr^2 + f^2(r) g_N,$$

as in the previous section. For the manifold N, the following two theorems are well known.

Theorem 3.0.1 (Hodge decomposition Theorem, Theorem 1.37 [Ros97]). *Suppose that N is compact complete and oriented smooth manifold. Then we have*

$$\Lambda^k(N) = dC^\infty(\Lambda^{k-1}(N)) \oplus \delta C^\infty(\Lambda^{k+1}(N)) \oplus H^k(N),$$

where

$$dC^\infty(\Lambda^{k-1}(N)) = \{\omega \in \Lambda^k(N) : \omega = d\eta_1, \text{ for some } \eta_1 \in \Lambda^{k-1}(N)\},$$

$$\delta C^\infty(\Lambda^{k+1}(N)) = \{\omega \in \Lambda^k(N) : \omega = \delta\eta_2, \text{ for some } \eta_2 \in \Lambda^{k+1}(N)\},$$

$$H^k(N) = \{\omega \in \Lambda^k(N) : \Delta\omega = 0\}.$$

Theorem 3.0.2 (Hodge Theorem for forms, Theorem 1.30 [Ros97]). *Let N as in Theorem 3.0.1. Then there exists an orthonormal basis of $L^2(\Lambda^k(N))$ consisting of eigenforms of the Laplacian on k-forms.*

As Antoci illustrates in [Ant04], the above theorems can be used to also provide

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the full decomposition of the Laplacian on L^2- integrable forms.

In our study of the L^p-spectrum in the next chapter we will need a large class of approximate eigenforms for the Laplacian. It turns out that certain types of k-forms that stem from this decomposition will be useful. So, we will write in detail the action of the Laplacian in these cases only, without providing the full decomposition of $L^2(\Lambda^k(M))$, since it is not required for our results.

Firstly, we write the action of the Hodge star $*$ operator on k-forms on M of the type $\omega = \omega_1 + \omega_2 \wedge dr$, where ω_1 and ω_2 are respectively a k-form and a $(k-1)$-form on N but which might also depend on r.

Lemma 3.0.3 ((3.3) in Section 3 from [Ant04]). *Let $\omega_1 = h_1(r)\eta_1$ and $\omega_2 = h_2(r)\eta_2$, where η_1 is a k-form on N, and η_2 is a $(k-1)$-form on N, and h_1, h_2 are smooth functions of r. Then, the k-form $\omega = \omega_1 + \omega_2 \wedge dr$ on M, satisfies*

$$*\omega = (-1)^{n-k} h_1 f^{n-2k+1}(*_N \eta_2) + h_2 f^{n-2k-1}(*_N \eta_1) \wedge dr,$$

where $*_N$ is the star operator on (N, g_N).

In the sequel, we write the action of the operators d, δ on k-forms on M of the form $\omega = \omega_1 + \omega_2 \wedge dr$.

Lemma 3.0.4 ((3.4) and (3.5) in Section 3 from [Ant04]). *Let $\omega_1 = h_1(r)\eta_1$, $\omega_2 = h_2(r)\eta_2$, where η_1 is a k-form on N and η_2 is a $(k-1)$-form on N, and h_1, h_2 are smooth functions of r. Then, the k-form $\omega = \omega_1 + \omega_2 \wedge dr$ on M, satisfies*

$$d\omega = h_1 d_N \eta_1 + [(-1)^k h_1' \eta_1 + h_2 d_N \eta_2)] \wedge dr,$$

$$\delta\omega = h_1 f^{-2} \delta_N \eta_1 + (-1)^k (h_2 f^{n-2k+1})' f^{-n+2k-1} \eta_2 + h_2 f^{-2} (\delta_N \eta_2) \wedge dr.$$

So, now we are in position to completely determine the action of Δ on k-forms on M of the form $\omega = \omega_1 + \omega_2 \wedge dr$, where again ω_1 and ω_2 are respectively a k-form and a $(k-1)$-form on N depending on r.

Proposition 3.0.1. *Let $\omega_1 = h_1(r)\eta_1$, $\omega_2 = h_2(r)\eta_2$, where η_1 is a k-form on N and η_2 is a $(k-1)$-form on N, and h_1, h_2 are smooth functions of r. Then, the k-form*

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$\omega = \omega_1 + \omega_2 \wedge dr$ *on* M*, satisfies*

$$\begin{aligned}\Delta\omega &= h_1 f^{-2}\Delta_N\eta_1 + h_2 f^{-2}(\Delta_N\eta_2)\wedge dr + (-1)^k 2h_1 f' f^{-3}(\delta_N\eta_1)\wedge dr \\ &+ (-1)^k 2h_2 f' f^{-1} d_N\eta_2 - [h_1'' + (n-2k-1)h_1' f' f^{-1}]\eta_1 \\ &- [h_2'' + (n-2k+1)(h_2 f' f^{-1})']\eta_2 \wedge dr.\end{aligned}$$

As we already mention, in our study in the next chapter we will need a large class of k-forms on M for which will need to know the action of Δ on them. The following Corollary summarizes the action of the Laplacian on the specific forms we will use.

Corollary 3.0.1. *(a) Let η be a co-closed k-eigenform over N with $\Delta_N\eta = \lambda\eta$. Then, the k-form $\omega = h_1(r)\eta$ over M satisfies*

$$\Delta(h_1(r)\eta) = \Delta_1(h_1(r))\eta,$$

where

$$\Delta_1(h_1(r)) = -\left[h_1''(r) + (n-2k-1)h_1'(r)\frac{f'(r)}{f(r)}\right] + \lambda\frac{h_1(r)}{f^2(r)}.$$

(b) Let η be a closed $(k-1)$-eigenform over N with $\Delta_N\eta = \lambda_0\eta$. Then, the k-form $\omega = h_2(r)\eta \wedge dr$ over M satisfies

$$\Delta(h_2(r)\,\eta\wedge dr) = \Delta_2(h_2(r))\,\eta\wedge dr,$$

where

$$\Delta_2(h_2(r)) = -\left[h_2''(r) + (n-2k+1)\left(h_2(r)\frac{f'(r)}{f(r)}\right)'\right] + \lambda_0\frac{h_2(r)}{f^2(r)}. \qquad (3.0.1)$$

Proof. We consider the k-form $\omega = h_1(r)\eta$ in M where η is a k-eigenform of Δ_N with $\Delta_N\eta = \lambda\eta$ and it is coclosed. In the decomposition

$$\omega = \omega_1 + \omega_2 \wedge dr,$$

this corresponds to the case $\omega = \omega_1$ and $\omega_2 = 0$. Therefore, Proposition 3.0.1 with

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$\omega_1 = h_1(r)\eta$ and $\eta_2 = 0$ and $\delta_N \eta_1 = 0$ gives

$$\begin{aligned}\Delta(\omega) = \Delta(h_1(r)\eta) =& h_1 f^{-2}\lambda\eta - [h_1'' + (n-2k-1)h_1' f' f^{-1}]\eta \\ =& [h_1 f^{-2}\lambda - [h_1'' + (n-2k-1)h_1' f' f^{-1}]]\eta \\ =& \Delta_1(h_1)\eta.\end{aligned}$$

Similarly, we consider the k-form $\omega = h_2(r)\eta \wedge dr$ in M where η is a $(k-1)$-eigenform of Δ_N with $\Delta_N \eta = \lambda_0 \eta$ which is closed. Taking $\omega_1 = 0$ and $\omega_2 = h_2(r)\eta$ in Proposition 3.0.1 we get

$$\begin{aligned}\Delta(\omega) = \Delta(h_2(r)\eta \wedge dr) =& + h_2 f^{-2}\lambda_0 \eta \wedge dr - [h_2'' + (n-2k+1)(h_2 f' f^{-1})']\eta \wedge dr \\ =& [h_2 f^{-2}\lambda_0 - [h_2'' + (n-2k+1)(h_2 f' f^{-1})']]\eta \wedge dr \\ =& \Delta_2(h_2)\eta \wedge dr.\end{aligned}$$

□

As we will see in the next chapter, the approximate eigenforms for the Laplacian, will be of the form $(\phi f^\mu)\eta \wedge dr$, with ϕ a function of r and f the warping function. So, for future reference we compute the following.

By (3.0.1) we have

$$\Delta_2 f^\mu = f^\mu \left[-(\mu + n - 2k + 1)\frac{f''}{f} - (\mu - 1)(\mu + n - 2k + 1)\left(\frac{f'}{f}\right)^2 + \frac{\lambda_0}{f^2} \right] \tag{3.0.2}$$

and

$$\begin{aligned}\Delta_2(\phi f^\mu) =& \phi \Delta_2(f^\mu) - \left[\phi'' f^\mu + 2\phi'(f^\mu)' + (n-2k+1)\phi' f^{\mu-1} f'\right] \\ =& - f^\mu \left[(\mu + n - 2k + 1)\phi \frac{f''}{f} + (\mu - 1)(\mu + n - 2k + 1)\phi \left(\frac{f'}{f}\right)^2 \right. \\ & \left. + \phi'' + (2\mu + n - 2k + 1)\phi' \frac{f'}{f} - \lambda_0 \frac{\phi}{f^2} \right].\end{aligned} \tag{3.0.3}$$

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Chapter 4

The L^p-spectrum for a class of warped product metrics

In this chapter, the space M will be a generalization of warped product manifolds. More specifically we will study manifolds which are warped products at infinity. Let us be more precise.

Definition 4.0.1. *We say that M is a warped product at infinity, if outside a compact set K, $M \setminus K$ is of the form $(c_0, \infty) \times N$ with metric*

$$g = dr^2 + f^2(r) g_N,$$

where $f \in C^\infty(c_0, \infty)$ is the warping function and N is an $(n-1)$-dimensional compact manifold.

The type of warping function f determines the geometry of the manifold at infinity, and hence its spectrum. For example if $c_0 > 0$, $f = r$ with $N = S^{n-1}$, then M is Euclidean at infinity and if $f = sinhr$ with $N = S^{n-1}$ as well it is hyperbolic. We will provide a large class of functions f for which the L^p-spectrum contains a parabolic region. We will consider the set of functions:

$$\begin{aligned} B = \{ f \in C^2(a, \infty) : \frac{f''}{f} &= a_0 + o(1), \\ \left(\frac{f'}{f}\right)^2 &= a_0 + o(1), \text{as } r \to \infty, \text{with } a_0 > 0 \\ &\text{and } f \to \infty, \text{as } r \to \infty \}. \end{aligned}$$

A prototype example of such f are $f = e^{\sqrt{a_0} r}$ and $f = sinh(\sqrt{a_0} r)$, which give us a manifold which is hyperbolic with Ricci curvature in the radial direction $-(n-1)a_0$. In other words, we consider manifolds which are asymptotically hyperbolic and with infinity volume since $f \to \infty$.

4.1 The L^p-spectrum for a class of warped product metrics part I

Let $P_{p,k}$ be the curve in the complex plane,

$$P_{p,k} = \left\{ -a_0 \left[\frac{n-1}{p} - k + is \right] \left[(n-1)\left(\frac{1}{p} - 1\right) + k + is \right], s \in \mathbb{R} \right\}$$

with $a_0 > 0$. Denote the parabolic region to the right of the curve $P_{p,k}$ by $Q_{p,k}$. In this section we construct approximate eigenforms in order to prove that the L^p-spectrum of the Hodge Laplacian on k-forms contains the parabolic region $Q_{p,k}$ whenever M is a warped product at infinity with warping function f.

In the following Lemma we find an equivalent expression for $Q_{p,k}$.

Lemma 4.1.1. *The parabolic region $Q_{p,k}$ can be expressed as,*

$$Q_{p,k} = \{ a_0 \left(\frac{n-1}{2} - k \right)^2 + z^2 : |Im(z)| \le \sqrt{a_0}(n-1) \left| \frac{1}{p} - \frac{1}{2} \right| \}.$$

Proof. We rewrite the parametric equation for the parabola as

$$-a_0 \left[\frac{n-1}{p} - k \right] \left[(n-1)\left(\frac{1}{p} - 1\right) + k \right] + a_0 s^2 - 2a_0(n-1)\left(\frac{1}{p} - \frac{1}{2}\right) is,$$

for any $s \in \mathbb{R}$, which is equal to

$$-a_0 \left[\left(\frac{n-1}{p}\right)^2 - \frac{(n-1)^2}{p} + k(n-1) - k^2 \right] + a_0 s^2 - 2a_0(n-1)\left(\frac{1}{p} - \frac{1}{2}\right) is,$$

for any $s \in \mathbb{R}$. Since

$$\left(\frac{n-1}{p}\right)^2 - \frac{(n-1)^2}{p} + k(n-1) - k^2 = -\left(\frac{n-1}{2} - k\right)^2 + (n-1)^2 \left(\frac{1}{p} - \frac{1}{2}\right)^2,$$

setting

$$y = -2a_0(n-1)\left(\frac{1}{p} - \frac{1}{2}\right) s,$$

we get that

$$P_{p,k} = \left\{ x+iy : x = a_0 \left(\frac{n-1}{2} - k \right)^2 - a_0(n-1)^2 \left(\frac{1}{p} - \frac{1}{2} \right)^2 \right.$$
$$\left. + \frac{y^2}{a_0 \left[2(n-1)(\frac{1}{p} - \frac{1}{2}) \right]^2} \right\}.$$

Note that for $z = a + ib$ and μ, λ any constants, we have

$$\{z^2 + \mu : b = \lambda\} = \{x + iy : x = \mu + \frac{y^2}{(2\lambda)^2} - \lambda^2\}.$$

Taking $\lambda = \sqrt{a_0}(n-1)(\frac{1}{p} - \frac{1}{2})$ and $\mu = a_0 \left(\frac{n-1}{2} - k \right)^2$ we see that $\partial Q_{p,k} = P_{p,k}$. The conclusion follows by noting that the region to the right of $P_{p,k}$ corresponds to $\{z^2 + \mu : b < \lambda\} = \{z^2 + \mu : |b| < \lambda\}$ □

Theorem 4.1.2. *Let M be a warped product at infinity where the warping function $f \in B$. For any $0 \leq k \leq \frac{n}{2}$ and $1 \leq p \leq 2$, the L^p-spectrum of the Laplacian, $\sigma(p, k, \Delta)$ contains $Q_{p,k}$. The remaining cases for p and k are given by duality as $\sigma(p, k, \Delta) = \sigma(p, n-k, \Delta)$ for $n/2 \leq k \leq n$ and $\sigma(p, k, \Delta) = \sigma(p^*, k, \Delta)$ whenever $\frac{1}{p} + \frac{1}{p^*} = 1$.*

Proof. Let us note that by our assumption $f \to \infty$. Let μ be a complex number. Note that in the particular case

$$\frac{f''}{f} = a_0, \quad \left(\frac{f'}{f} \right)^2 = a_0,$$

using (3.0.2) we compute,

$$\Delta_2 f^\mu = -f^\mu[a_0\mu(\mu + n - 2k + 1)] + \frac{\lambda_0 f^\mu}{f^2}.$$

If $f \to \infty$, the last term is of lower order. Hence, the candidate points for the spectrum are

$$\lambda = -a_0\mu(\mu + n - 2k + 1) = -a_0(\mu + n - 2k + 1) - a_0(\mu - 1)(\mu + n - 2k + 1). \quad (4.1.1)$$

We will show that any λ such as the one above belongs to $\sigma(p, k, \Delta)$. To achieve this we need to show that for every $\epsilon > 0$ there exists a k-form ω such that $||(\Delta - \lambda)\omega||_p \leq \epsilon ||\omega||_p$

by Proposition 2.6.1. We consider approximate eigenforms of the type $\omega = \phi f^{\mu} \eta \wedge dr$ where $\phi = \phi(r)$ has compact support in (c_0, ∞), $\mu \in \mathbb{C}$ and η is a smooth closed $k-1$ eigenform on N with eigenvalue λ_0. Set λ as in (4.1.1). By Corollary 3.0.1 $\Delta(\omega) = \Delta_2(\phi f^{\mu})\, \eta \wedge dr$. Using (3.0.3) and the triangle inequality we get

$$
\begin{aligned}
\|\Delta_p \omega - \lambda\omega\|_p^p &= \|\Delta_2(\phi f^{\mu})\eta \wedge dr - \lambda \phi f^{\mu} \eta \wedge dr\|_p^p \\
&= \Bigg\| f^{\mu}\eta \wedge dr \left[-(\mu-1)(\mu+n-2k+1)\phi \left(\frac{f'}{f}\right)^2 - (\mu+n-2k+1)\phi \frac{f''}{f} \right] \\
&- f^{\mu}\eta \wedge dr \left[\phi'' + (2\mu+n-2k+1)\phi' \frac{f'}{f} - \lambda_0 \frac{\phi}{f^2} \right] \\
&+ a_0(\mu+n-2k+1)\phi f^{\mu}\eta \wedge dr + a_0(\mu-1)(\mu+n-2k+1)\phi f^{\mu}\eta \wedge dr \Bigg\|_p^p \\
&= \Bigg\| -(\mu-1)(\mu+n-2k+1)\phi \left[\left(\frac{f'}{f}\right)^2 - a_0 \right] f^{\mu}\eta \wedge dr \\
&- (\mu+n-2k+1)\phi \left(\frac{f''}{f} - a_0 \right) f^{\mu}\eta \wedge dr \\
&- f^{\mu}\phi''\eta \wedge dr - f^{\mu}\eta \wedge dr(2\mu+n-2k+1)\phi' \frac{f'}{f} + f^{\mu}\eta \wedge dr \lambda_0 \frac{\phi}{f^2} \Bigg\|_p^p \\
&\leq |(\mu-1)(\mu+n-2k+1)| \left\| \phi \left[\left(\frac{f'}{f}\right)^2 - a_0 \right] f^{\mu}\eta \wedge dr \right\|_p^p \\
&+ |\mu+n-2k+1| \left\| \phi \left(\frac{f''}{f} - a_0 \right) f^{\mu}\eta \wedge dr \right\|_p^p \\
&+ \|\phi'' f^{\mu}\eta \wedge dr\|_p^p + |2\mu+n-2k+1| \|\phi' \frac{f'}{f} f^{\mu}\eta \wedge dr\|_p^p + \|\lambda_0 \frac{\phi}{f^2} f^{\mu}\eta \wedge dr\|_p^p \\
&= I + II + III + IV + V.
\end{aligned}
\tag{4.1.2}
$$

We set $\mu = -\frac{n-1}{p} + (k-1) + is$, for $s \in \mathbb{R}$. Note that

$$|\eta \wedge dr|_M = |\eta|_N f^{-(k-1)},$$

hence

$$|\omega|_M = f^{\mathrm{Re}(\mu)-k-1}|\phi||\eta|_N = f^{-\frac{n-1}{p}}|\phi||\eta|_N, \tag{4.1.3}$$

for our μ. Fix $\epsilon > 0$. Let A_ϵ ,B_ϵ such that $B_\epsilon > A_\epsilon > C+1$.We will take cut-off functions $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$ such that $\phi_\epsilon \in C^{\infty}(\mathbb{R})$, $\mathrm{spt}\phi_\epsilon \subset [A_\epsilon - 1, B_\epsilon] \subset (c_0, \infty)$, $\phi_\epsilon = 1$ on $[A_\epsilon, B_\epsilon]$

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and $|\phi'_\epsilon|, |\phi''_\epsilon| \le C$ for every $\epsilon > 0$. Observe that,

$$\int_0^\infty |\phi'_\epsilon|^p dr \le \int_{A_\epsilon - 1}^{A_\epsilon} |\phi'_\epsilon|^p dr + \int_{B_\epsilon}^{B_\epsilon + 1} |\phi'_\epsilon|^p dr \le C$$

and

$$\int_0^\infty |\phi''_\epsilon|^p dr \le \int_{A_\epsilon - 1}^{A_\epsilon} |\phi''_\epsilon|^p dr + \int_{B_\epsilon}^{B_\epsilon + 1} |\phi''_\epsilon|^p dr \le C,$$

where C are uniform constants independent of ϵ. From now on C will denote a uniform constant which does not depend on ϵ, and might differ from one line to the next. Note that the volume element on $M \setminus K$ is $dV_g = f^{n-1} dr d\sigma$ where $d\sigma$ is the volume element on N. Using (4.1.3) we compute

$$\begin{aligned} \|\omega_\epsilon\|_p^p &= \int_M |\phi_\epsilon|^p f^{p(\mathrm{Re}\mu)} |\eta \wedge dr|^p dV_g = \int_{A_\epsilon - 1}^{B_\epsilon + 1} \int_N f^{p(\mathrm{Re}\mu)} |\phi_\epsilon|^p |\eta \wedge dr|^p f^{n-1} dr d\sigma \\ &= \int_{A_\epsilon - 1}^{B_\epsilon + 1} C |\phi_\epsilon|^p dr > \int_{A_\epsilon}^{B_\epsilon} C dr = (B_\epsilon - A_\epsilon) C. \end{aligned} \tag{4.1.4}$$

We will show that each one of the five terms in (4.1.2) is uniformly bounded for our ω. First we get

$$\begin{aligned} IV &\le C \int_M |\phi'_\epsilon|^p \left(\frac{f'}{f}\right)^p f^{p(\mathrm{Re}\mu)} |\eta \wedge dr|^p dV_g = \int_{\mathrm{spt}\phi'_\epsilon} C |\phi'_\epsilon|^p \left(\frac{f'}{f}\right)^p dr \\ &\le C \int_{[A_\epsilon - 1, A_\epsilon] \bigcup [B_\epsilon, B_\epsilon + 1]} |\phi'_\epsilon|^p dr \le C, \end{aligned} \tag{4.1.5}$$

where we have used that $\left(\frac{f'}{f}\right)^p$ is bounded at infinity and hence it is uniformly bounded on (c_0, ∞). Similarly, $|\phi''_\epsilon|$ is bounded and supported on a set of length 2, hence $III \le C$.

Now we estimate (I).

$$\begin{aligned} I &= C \left\| \phi_\epsilon \left[\left(\frac{f'}{f}\right)^2 - a_0 \right] f^\mu \eta \wedge dr \right\|_p^p = C \int_M \phi_\epsilon^p \left| \left(\frac{f'}{f}\right)^2 - a_0 \right|^p f^{p(\mathrm{Re}\mu)} |\eta \wedge dr|^p dV_g \\ &\le C \sup_{[A_\epsilon - 1, B_\epsilon + 1]} \left| \left(\frac{f'}{f}\right)^2 - a_0 \right|^p \int_M \phi_\epsilon^p f^{p(\mathrm{Re}\mu)} |\eta \wedge dr|^p dV_g \\ &= C \sup_{[A_\epsilon - 1, B_\epsilon + 1]} \left| \left(\frac{f'}{f}\right)^2 - a_0 \right|^p \|\omega_\epsilon\|_p^p. \end{aligned} \tag{4.1.6}$$

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Similarly,

$$II \leq C \sup_{[A_\epsilon-1,B_\epsilon+1]} \left| \frac{f''}{f} - a_0 \right|^p \|\omega_\epsilon\|_p^p,$$

and

$$V \leq C \sup_{[A_\epsilon-1,B_\epsilon+1]} \left(\frac{1}{f^{2p}} \right) \|\omega_\epsilon\|_p^p.$$

By our assumptions $\frac{f''}{f} \to a_0$, $\left(\frac{f'}{f}\right)^{1/2} \to a_0$ and $f \to \infty$, hence we can find A_ϵ large enough such that

$$\sup_{[A_\epsilon-1,B_\epsilon+1]} \left| \left(\frac{f'}{f}\right)^2 - a_0 \right|^p, \sup_{[A_\epsilon-1,B_\epsilon+1]} \left| \frac{f''}{f} - a_0 \right|^p, \sup_{[A_\epsilon-1,B_\epsilon+1]} \left(\frac{1}{f^{2p}} \right) \leq \epsilon.$$

As a result

$$I + II + V \leq C\ \epsilon \|\omega_\epsilon\|_p^p,$$

with C independent of ϵ. In addition, for any A_ϵ we can chose B_ϵ large enough such that

$$B_\epsilon - A_\epsilon > \frac{1}{\epsilon}.$$

Hence, by (4.1.4) and (4.1.5)

$$III + IV \leq C = C\ \epsilon\ \frac{1}{\epsilon} \leq C\ \epsilon \|\omega_\epsilon\|_p^p.$$

Since, ϵ was arbitrary we conclude that

$$I + II + III + IV + V \leq \epsilon \|\omega_\epsilon\|_p^p.$$

So, by Proposition 2.6.1 we have shown that the points

$$\lambda = -a_0 \left(-\frac{n-1}{p} + (k-1) + is \right) \left(-\frac{n-1}{p} + (n-k) + is \right),\ s \in \mathbb{R}$$

belong to $\sigma(p, k, \Delta)$. Setting $k = n - m$, for $0 \leq m \leq n/2$, in the above equation and changing sign in both brackets we get

$$\lambda = -a_0 \left[(n-1) \left(\frac{1}{p} - 1 \right) + m - is \right] \left[\frac{n-1}{p} - m - is \right],\ s \in \mathbb{R},$$

which are exactly the points of $P_{p,m}$. Thus, for $0 \leq m \leq n/2$ we have shown that $P_{p,m} \subset \sigma(p, n-m, \Delta) = \sigma(p, m, \Delta)$, where the equality follows by Poincare duality.

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Observe that $\bigcup_{p\le q\le 2} P_{q,m} = Q_{p,m}$. Moreover, by $\sigma(q,m,\Delta) \subset \sigma(p,m,\Delta)$ for all $1 \le p \le q \le 2$ we get $Q_{p,m} \subset \sigma(p,m,\Delta)$ for any $1 \le p \le 2$ and $0 \le m \le \frac{n}{2}$. For a proof of $\sigma(q,m,\Delta) \subset \sigma(p,m,\Delta)$, see the first part in Lemma 5.2.6. $\square$

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4.2 The L^p-spectrum for a class of warped product metrics part II

In this section, we show that the L^p-spectrum is contained within a parabolic region. This is a result that depends on the rate of the volume growth of the manifold at infinity, which is defined as follows.

Definition 4.2.1. *The exponential rate of volume growth of M, denoted by γ, is the infimum of all real numbers β satisfying the property: for any $\epsilon > 0$, there is a constant C, depending only on ϵ and the dimension of M, such that for any $p \in M$ and any $R \geq 1$, we have*

$$(V_p(R)) \leq C(V_p(1))e^{(\beta+\epsilon)R},$$

where $V_p(R)$ denotes the volume of the ball of radius R at p.

Our main analytic tool is the following recent result, obtained in [CL24], which says that for a complete manifold M over which the Ricci curvature and the Weitzenbock tensor on k-forms are bounded below, the resolvent set of $(\Delta - (\lambda_1 + z^2))^{-1}$ is the region to the left of the parabola $|\mathrm{Im}(z)| = \gamma|\frac{1}{p} - \frac{1}{2}|$ in the complex plane, where γ is the exponential rate of volume growth of M and λ_1 is the bottom of the L^2 spectrum of the Laplacian on k-forms. More precisely we have:

Theorem 4.2.1. *[CL24] Let M be a complete manifold over which the Ricci curvature and the Weitzenbock tensor on k-forms are bounded below. Denote by γ the exponential rate of volume growth of M and λ_1 the infimum of the spectrum of the Laplacian on L^2 integrable k-forms. Let $1 \leq p \leq \infty$, and z be a complex number such that*

$$|Im(z)| > \gamma|\frac{1}{p} - \frac{1}{2}|.$$

Then

$$(\Delta - (\lambda_1 + z^2))^{-1}$$

is a bounded operator on $L^p(\Lambda^k(M))$.

Now let us return to the set of the manifolds M that we study in this Chapter, that is, warped products at infinity with warping function $f \in B$. In order to apply the previous Theorem in our case, we have to show first that for M the Ricci curvature and the Weitzenbock tensor on k-forms are bounded below and in the sequel to compute γ.

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In the following Proposition we prove that the Ricci curvature of M is bounded below. We compute also an upper bound for γ.

Proposition 4.2.1. *Let M be a warped product at infinity where the warping function $f \in B$. Then the Ricci curvature of M is bounded below, and the exponential of volume growth of M is at most $(n-1)\sqrt{a_0}$.*

Proof. The technique that we use is based on the proof of Proposition 1 in [Stu93]. Here, we follow the proof in [CR24] to show that the technique can be generalized to the case when the Ricci curvature is asymptotically bounded below by a negative constant, whereas Sturm only considers asymptotically nonnegative Ricci curvature. By Proposition 2.7.2 we have that all sectional curvatures of M lie between

$$-\frac{f''}{f}, \quad \frac{\bar{C}-(f')^2}{f^2},$$

where $\bar{C}$ reflects the sectional curvatures of N. As a result, on $M \setminus K$ the sectional curvatures tend to $-a_0$ as $r \to \infty$. This implies that, for any $\epsilon > 0$, there is a compact set $K_\epsilon \subset M$ with smooth boundary, such that the sectional curvatures on $M \setminus K_\epsilon$ are bounded below by $-(a_0+\epsilon)$. Therefore, the Ricci curvature is bounded below by $-(n-1)(a_0+\epsilon)$ on $M \setminus K_\epsilon$. Let $R = R_\epsilon$ be the diameter of K_ϵ.

For a point $x \in M$, let $s = s_\epsilon(x) = d(x, K_\epsilon)$ and $t = t_\epsilon(x) = s + R_\epsilon$. By definition, $t - s$ does not depend on $x \in M$. Now, by the compactness of K_ϵ, since the Ricci curvature is uniformly bounded from below on $M \setminus K_\epsilon$, we get that it is uniformly bounded from below on all of M. Thus, there exists a $K > 0$ such that the Ricci Curvature is bounded from below by $-(n-1)K^2$ on all of M. Clearly, $K \geq \sqrt{a_0+\epsilon}$.

We introduce the Sturm-Liouville equation,

$$u''(r) + q(r)u(r) = 0, \tag{4.2.1}$$

with initial conditions $u(0) = 0$, $u'(0) = 1$, where q is defined by

$$q(r) = \begin{cases} -(a_0+\epsilon) & \text{for } r \in [0, s) \\ -K^2 & \text{for } r \in [s, t) \\ -(a_0+\epsilon) & \text{for } r \in [t, \infty) \end{cases}. \tag{4.2.2}$$

Denote by M_q^n the complete simply connected space with piecewise constant cur-

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vature as defined by q. Following the notation in Section 2.7.2, the volume element $J_q(r,\theta)dr \wedge d\theta$ of M_q^n will be $u(r)^{n-1}dr \wedge d\theta$, where $u(r)$ is the solution of (4.2.1).

From Theorem 2.7.2 we have

$$\frac{V_x(r)}{V_x(1)} \leq \frac{\bar{V}(r)}{\bar{V}(1)}$$

where $V, \bar{V}$ are the volumes of the geodesic ball of M and M_q^n respectively. We compute

$$\bar{A}(r') = \int_{S^{n-1}} J_q(r',\theta)d\theta = \int_{S^{n-1}} u(r')^{n-1}d\theta = u(r')^{n-1}\int_{S^{n-1}} d\theta$$

and

$$\bar{V}(r) = \int_{S^{n-1}} d\theta \int_0^r u(r')^{n-1}dr'.$$

Therefore,

$$V_x(r) \leq V_x(1)\frac{\int_0^r u(r')^{n-1}dr'}{\int_0^1 u(r')^{n-1}dr'}. \tag{4.2.3}$$

The estimate of the above fraction will give us the upper volume estimate of $V_x(r)$.

By solving (4.2.1) with q as in (4.2.2) and the properties of the Sturm-Liouville's equation we get that

$$u(r) \geq \frac{1}{\sqrt{a_0+\epsilon}} sinh(r\sqrt{a_0+\epsilon}),$$

on $(0,\infty)$, uniformly on M. Thus,

$$\int_0^1 u(r')^{n-1}dr' \geq \int_0^1 \left(\frac{1}{\sqrt{a_0+\epsilon}} sinh(r'\sqrt{a_0+\epsilon})\right)^{n-1} dr' = C_\epsilon.$$

Combining with (4.2.3) we find

$$V_x(r) \leq V_x(1)C'_\epsilon \int_0^r u(r')^{n-1}dr'.$$

Thus it remains to estimate

$$\int_0^r u(r')^{n-1}dr'.$$

By solving the Sturm-Liouville's equation iteratively on each interval, we have

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$$u(r) \leq \begin{cases} \frac{1}{\sqrt{a_0+\epsilon}} e^{r\sqrt{a_0+\epsilon}} & \text{if } r \in [0,s) \\ \frac{1}{\sqrt{a_0+\epsilon}} e^{s\sqrt{a_0+\epsilon}} e^{K(r-s)} & \text{if } r \in [s,t) \\ \frac{K}{a_0+\epsilon} e^{s\sqrt{a_0+\epsilon}} e^{K(t-s)} e^{(r-t)\sqrt{a_0+\epsilon}} & \text{if } r \in [t,\infty) \end{cases} . \tag{4.2.4}$$

Note that

$$\begin{aligned} e^{s\sqrt{a_0+\epsilon}} e^{K(t-s)+(r-t)\sqrt{a_0+\epsilon}} &= e^{(K-\sqrt{a_0+\epsilon})(t-s)} e^{r\sqrt{a_0+\epsilon}} \\ &= e^{(K-\sqrt{a_0+\epsilon})R_\epsilon} e^{r\sqrt{a_0+\epsilon}} \end{aligned}$$

for $r \in [t,\infty)$. Moreover,

$$e^{s\sqrt{a_0+\epsilon}} e^{K(r-s)} \leq e^{r\sqrt{a_0+\epsilon}} e^{KR_\epsilon},$$

for $r \in [s,t)$. As a result

$$u(r) \leq C_\epsilon e^{r\sqrt{a_0+\epsilon}},$$

for all $r \in [0,\infty)$, since R_ϵ is independent of x. We can now estimate

$$\int_0^r u(r')^{n-1} dr' \leq C_\epsilon \int_0^r \left(e^{r'\sqrt{a_0+\epsilon}}\right)^{n-1} dr' = C_\epsilon e^{(n-1)r\sqrt{a_0+\epsilon}}.$$

Combining the above, we get that the fraction in (4.2.3) is uniformly bounded above in M by $C_\epsilon e^{(n-1)r\sqrt{a_0+\epsilon}}$. Thus, since $\epsilon > 0$ was arbitrary, we get that the exponential rate of volume growth, γ satisfies $\gamma \leq (n-1)\sqrt{a_0}$. □

The above Proposition combined with our computation of the L^p-spectrum will give us the exact value of the exponential rate of volume growth γ of M. More precisely we have.

Proposition 4.2.2. *Let M be a warped product at infinity where the warping function $f \in B$. Then $\gamma = (n-1)\sqrt{a_0}$.*

Proof. We proceed as in [CR24]. By Proposition 4.2.1, the Ricci curvature of M is bounded below, and by Proposition 2.7.1 the curvature operator on M is bounded below. Hence by Corollary 2.6 in [GM75] the Weitzenbock tensor on k-forms is also bounded below. By Theorem 4.2.1 we find that the resolvent set of the Hodge Laplacian

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acting on L^p k-forms contains the set

$$A = \{\lambda_1 + z^2 : |\mathrm{Im}(z)| > |\frac{1}{p} - \frac{1}{2}|\gamma\}, \tag{4.2.5}$$

where λ_1 is the bottom of the L^2 spectrum of the Laplacian on k-forms. Thus, we have

$$\sigma(p, k, \Delta) \subset \mathbb{C} \setminus A = A^{\complement} = \{\lambda_1 + z^2 : |\mathrm{Im}(z)| \leq |\frac{1}{p} - \frac{1}{2}|\gamma\}. \tag{4.2.6}$$

From Theorem 4.1.2 we have,

$$Q_{p,k} \subset \sigma(p, k, \Delta). \tag{4.2.7}$$

We proceed by contradiction. Assuming $\gamma < (n-1)\sqrt{a_0}$ we will show that this forces certain points of $Q_{p,k}$ and hence $\sigma(p, k, \Delta)$ to lie outside $\mathbb{C} \setminus A$ giving the desired contradiction. For simplicity we take $p = 1$. In this case, we have,

$$A^{\complement} = \{\lambda_1 + z^2 : |\mathrm{Im}(z)| \leq \frac{\gamma}{2}\}.$$

Setting $z = t + is$ this set can be expressed as

$$A^{\complement} = \{\lambda_1 + t^2 - s^2 + 2its : s^2 \leq \frac{\gamma^2}{4}, t \in \mathbb{R}\}$$

and setting $y = 2ts$ we get

$$\begin{aligned} A^{\complement} &= \{\lambda_1 + \frac{y^2}{4s^2} - s^2 + iy : s^2 \leq \frac{\gamma^2}{4}, y \in \mathbb{R}\} \\ &= \{x + iy : x \geq \lambda_1 - \frac{\gamma^2}{4} + \frac{y^2}{\gamma^2}, y \in \mathbb{R}\}. \end{aligned} \tag{4.2.8}$$

On the other hand, by Lemma 4.1.1,

$$Q_{1,k} = \{a_0 \left(\frac{n-1}{2} - k\right)^2 + z^2 : |\mathrm{Im} z| \leq \frac{\sqrt{a_0}(n-1)}{2}\}$$

which can similarly be expressed as

$$Q_{1,k} = \{x + iy : x \geq a_0 \left(\frac{n-1}{2} - k\right)^2 - \frac{a_0(n-1)^2}{4} + \frac{y^2}{a_0(n-1)^2}, y \in \mathbb{R}\}.$$

In other words both $A^{\complement}$ and $Q_{1,k}$ are parabolic regions to the right of parabolas of the

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type $x = x_0 + by^2$. For $A^{\complement}$, the constant b is $b_1 = \frac{1}{\gamma^2}$, and for $Q_{1,k}$ it is $b_2 = \frac{1}{a_0(n-1)^2}$. If $\gamma < (n-1)\sqrt{a_0}$ then $b_1 > b_2$ and hence the parabola of $A^{\complement}$ is strictly contained in $Q_{1,k}$ when $x + iy$ has x large enough. This gives us the contradiction. As a result we must have $\gamma = (n-1)\sqrt{a_0}$. $\square$

Now that we have found the exponential rate of volume growth γ of M, we are in position to determine the L^p-spectrum of the Laplacian on k-forms, if we make the assumption that the bottom of the L^2 spectrum of the Laplacian on k-forms is $\lambda_1 = a_0(\frac{n-1}{2} - k)^2$.

Proposition 4.2.3. *Let M be a warped product at infinity where the warping function $f \in B$. Assume that the bottom of the L^2 spectrum of the Laplacian on k-forms is*

$$\lambda_1 = a_0(\frac{n-1}{2} - k)^2,$$

then $\sigma(p, k, \Delta) = Q_{p,k}$.

Proof. Let A as in (4.2.5). For $\gamma = (n-1)\sqrt{a_0}$ and $\lambda_1 = a_0(\frac{n-1}{2} - k)^2$ we observe that

$$A^{\complement} = Q_{p,k}.$$

By Theorem 4.2.1 and the proof of Proposition 4.2.2 we have

$$\sigma(p, k, \Delta) \subset A^{\complement}.$$

At the same time, by Theorem 4.1.2 we have

$$Q_{p,k} \subset \sigma(p, k, \Delta) \subset A^{\complement} = Q_{p,k},$$

which give us that $\sigma(p, k, \Delta) = Q_{p,k}$. $\square$

One may ask if there exists any space satisfying the assumptions of Proposition 4.2.3. We will see that there exist various manifolds where this is the case, and the key factor so that the assumption on λ_1 holds, is the isolated discrete spectrum of Δ.

We begin by looking at a particular class of warped product at infinity which are conformally compact. Lets recall the definition of a conformally compact manifold.

Definition 4.2.2. *[Bor01] Let X be a smooth manifold with boundary ∂X, equipped with an arbitrary smooth metric $\bar{g}$. A boundary-defining function on X is a function*

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$x \geq 0$ such that $\partial X = \{x = 0\}$ and $dx \neq 0$ on ∂X. A conformally compact metric on the interior of X is a metric of the form

$$g = \frac{\bar{g}}{x^2}.$$

Borthwick proves the following structure theorem for conformally compact manifolds.

Proposition 4.2.4 (Proposition 3.1 [Bor01]). *Let X be a compact manifold with g a conformally compact metric. Then, there exists a product decomposition (x, y) near ∂X such that*

$$g = \frac{dx^2}{a(y)^2 x^2} + \frac{h(x, y, dy)}{x^2} + O(x^\infty) \tag{4.2.9}$$

Here $-a(y)^2$ is the limiting curvature at infinity.

In the above Proposition we denote by $O(x^\infty)$ any tensor $A(x)$ such that for any $n \in \mathbb{N}$, there exists a constant C_n for which $|A(x)| \leq C_n x^n$, in a neighborhood of 0.

Lemma 4.2.2. *Let M be a warped product at infinity where the warping function $f \in B$ is restricted to satisfy $f(r) \sim ce^{\sqrt{a_0}r}$, for some $a_0 > 0$, as $r \to \infty$. Then the metric $g = dr^2 + f^2(r)g_N$, on $M \setminus K = (c_0, \infty) \times N$ is a conformally compact metric, with limiting curvature $-a_0$ at infinity.*

Proof. By Proposition 4.2.4 it suffices to show that the metric g can be expressed as

$$g = \frac{dx^2}{a_0 x^2} + \frac{h(x, y, dy)}{x^2} + O(x^\infty).$$

Setting

$$r = -\frac{lnx}{\sqrt{a_0}} \Leftrightarrow x = e^{-\sqrt{a_0}r}$$

we get $dr^2 = \frac{1}{a_0 x^2} dx^2$. Therefore, the metric g can be rewritten as:

$$\begin{aligned} g &= dr^2 + f(r)^2 g_N \\ &= \frac{1}{x^2 a_0} dx^2 + \frac{[f(\frac{-lnx}{\sqrt{a_0}})x]^2}{x^2} g_N \end{aligned}$$

Since $f(r) \sim ce^{\sqrt{a_0}r}$, as $r \to \infty$ we have

$$
\begin{aligned}
c &= \lim_{r\to\infty} f(r)e^{-\sqrt{a_0}r} \\
&= \lim_{x\to 0} f\left(\frac{-lnx}{\sqrt{a_0}}\right)x
\end{aligned}
$$

Thus, if we set

$$
h = f^2\left(\frac{-lnx}{\sqrt{a_0}}\right)x^2 g_N
$$

we get

$$
h \to c\, g_N \text{ as } x \to 0
$$

which tells us that M is a conformally compact manifold with boundary $\partial X = N$ at infinity. $\square$

Theorem 4.2.3 ((1.3) Theorem [Maz88])**.** *For the conformally compact metric g in (4.2.9), if $-a_0$ is the maximum limiting curvature at infinity for some $a_0 > 0$, then the essential spectrum of Δ the Laplacian on L^2-integrable k-forms is $[a_0\frac{(n-2k-1)^2}{4}, \infty)$, $\{0\} \cup [\frac{a_0}{4}, \infty)$, $[a_0\frac{(n-2k+1)^2}{4}, \infty)$ for $k < \frac{n}{2}$, $k = \frac{n}{2}$, $k > \frac{n}{2}$ respectively.*

By this Theorem and Lemma 4.2.2 we have an immediate result for the bottom of the essential spectrum of M (in our case $-a_0$ is the maximum limiting curvature at infinity).

Proposition 4.2.5. *Let M be a warped product at infinity where the warping function $f \in B$ is restricted to satisfy $f(r) \sim ce^{\sqrt{a_0}r}$, as $r \to \infty$. Then, the L^2 essential spectrum of the Laplacian on forms on M is $[a_0\frac{(n-2k-1)^2}{4}, \infty)$ for $k < \frac{n}{2}$, $\{0\} \cup [\frac{a_0}{4}, \infty)$, $[a_0\frac{(n-2k+1)^2}{4}, \infty)$ for $k < \frac{n}{2}$, $k = \frac{n}{2}$, $k > \frac{n}{2}$ respectively.*

In other words, the L^2 spectrum of the Laplacian on k-forms over such a manifold, consists of isolated eigenvalues in the interval $[0, a_0\frac{(n-2k-1)^2}{4})$ together with the interval $[a_0\frac{(n-2k-1)^2}{4}, \infty)$. The bottom of the essential spectrum $\lambda_1 = [a_0\frac{(n-2k-1)^2}{4}]$ coincides with the vertex of the parabola $Q_{p,k}$ in Lemma 4.1.1.

Interpreting the above warped products as conformally compact manifolds, allowed us to obtain a simple proof for what their essential spectrum should be. Compare for example with the more intricate arguments in [Ant04].

By the above Proposition, if we have a warped product at infinity M with warping function $f \in B$ such that $f(r) \sim ce^{\sqrt{a_0}r}$, as $r \to \infty$, and we assume that the spectrum

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of the Laplacian on L^2-integrable k-forms does not have isolated eigenvalues, we have that $\sigma(2,k,\Delta) = [a_0\frac{(n-2k+1)^2}{4}, \infty)$. Thus, M satisfies the assumption of Proposition 4.2.3 and we immediately get the following result.

Theorem 4.2.4. *Let M be a warped product at infinity where the warping function $f \in B$ is restricted to satisfy $f(r) \sim ce^{\sqrt{a_0}r}$, for some $a_0 > 0$, as $r \to \infty$. Let $k \neq \frac{n}{2}$ and assume that the L^2 spectrum of the Laplacian on k-forms over M has no isolated eigenvalues of finite multiplicity. Then the L^p-spectrum of the Laplacian on forms on M is $Q_{p,k}$.*

In the particular case when M is in addition an Einstein manifold we have a more precise result about its L^p-spectrum.

Corollary 4.2.1. *Suppose that M is a warped product of negative curvature, which is in addition an Einstein manifold, and such that the Yamabe invariant of N is non-negative. Then the L^p-spectrum of the Laplacian on functions $\sigma(p,0,\Delta)$ is precisely $Q_{p,0}$, with $-a_0$ the curvature of the warped product at infinity.*

Proof. Let us note the following two results, that will give us the proof of the corollary. In [Lee94] Lee, show that the L^2 spectrum of the Laplacian on M as in our assumptions has no isolated eigenvalues if M is Einstein. From [Bes07] 9.109, 9.110 we have that M is Einstein with negative curvature if and only if the warping function $f(r)$ is equal to the one of the following three functions: $cosh(\sqrt{a_0}r)$, $e^{\sqrt{a_0}r}$, $sinh(\sqrt{a_0}r)$, for some $a_0 > 0$. Since all three of these functions belong to B, the corollary follows from Theorem 4.2.4. $\square$

Finally, for the case when there are isolated eigenvalues we have proved the following.

Theorem 4.2.5. *Let M be a warped product at infinity where the warping function $f \in B$ is restricted to satisfy $f(r) \sim ce^{\sqrt{a_0}r}$, for some $a_0 > 0$, as $r \to \infty$. Assume that there are isolated eigenvalues $\lambda_m < a_0\frac{(n-2k-1)^2}{4}$, for $k < \frac{n}{2}$ and $\lambda_m < a_0\frac{(n-2k+1)^2}{4}$, for $k > \frac{n}{2}$. Then, the L^p-spectrum of the Laplacian on forms on M contains the set $Q_{p,k}$, and the L^p-spectrum of the Laplacian on forms on M is contained in the set*

$$\{x+iy : x < \lambda_1 + \frac{y^2}{\gamma^2} - \frac{\gamma^2}{4}, y \in \mathbb{R}\}.$$

Proof. The first inclusion follows from Theorem 4.1.2 . The second inclusion follows from (4.2.6) and (4.2.8). □

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4.3 Examples

A subset of the set B can be characterized by the asymptotic solutions of a certain Riccati differential equation. We have

Lemma 4.3.1. *Let* $q : (a, \infty) \to \mathbb{R}$ *be monotonic and*

$$\int_{T_0}^{\infty} |q(s)| ds < \infty,$$

for some $T_0 > a$. *Let* $f \in C^2(a, \infty)$ *be a solution of the differential equation* $f'' - (a_0 + q)f = 0$ *with* $a_0 > 0$. *Then,*

$$\frac{f''}{f} = a_0 + o(1), \quad \left(\frac{f'}{f}\right)^2 = a_0 + o(1), \text{ as } t \to \infty.$$

Let us note that the assumption $\int_{T_0}^{\infty} |q(s)| ds < \infty$ together with that q is monotonic implies that as $q \to 0$ as $t \to \infty$. An example of a function satisfying the assumptions of Lemma 4.3.1 is $f(r) = c sinh(\sqrt{a_0} r)$, $a_0 > 0$, $c \in \mathbb{R}$ with $q = 0$.

The proof of Lemma 4.3.1 is based on the following.

Proposition 4.3.1 (Exercise 9.9 (a) [Har02])**.** *Let* $\lambda > 0$ *and* $q(t)$ *be a continuous complex-valued function for large* t *such that*

$$Q_\lambda(t) = \int_t^{\infty} q(s) e^{-2\lambda s} ds \text{ exists,} \tag{4.3.1}$$

$$\int^{\infty} Q_\lambda(t) e^{2\lambda t} dt \text{ exists,} \tag{4.3.2}$$

and

$$\int^{\infty} |Q_\lambda(t)|^2 e^{4\lambda t} dt \text{ exists.} \tag{4.3.3}$$

Then $u'' - (\lambda^2 + q(t))u = 0$ *has a pair of solutions satisfying,*

$$u \sim e^{\pm \lambda t}, \quad \frac{u'}{u} = \pm\lambda + e^{2\lambda t} Q_\lambda(t) + o(1), \text{ as } t \to \infty.$$

Proof of Lemma 4.3.1

By the above Proposition it suffices to show that the function $q : (a, \infty) \to \mathbb{R}$ satisfies the assumptions (4.3.1), (4.3.2), (4.3.3) and $e^{2\lambda t} Q_\lambda(t) \to 0$ as $t \to \infty$. By the

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monotonicity of q we have

$$\begin{aligned} |Q_\lambda(t)| \leq \int_t^\infty |q(s)| e^{-2\lambda s} ds \leq & |q(t)| \int_t^\infty e^{-2\lambda s} ds \\ = & |q(t)| \frac{1}{2\lambda} e^{-2\lambda t} < \infty, \end{aligned} \tag{4.3.4}$$

for every t. Now (4.3.4) gives

$$\int_{t_0}^\infty |Q_\lambda(t)| e^{2\lambda t} dt \leq c \int_{t_0}^\infty |q(t)| dt < \infty \tag{4.3.5}$$

and

$$\begin{aligned} \int_{t_0}^\infty |Q_\lambda(t)|^2 e^{4\lambda t} dt \leq & \int_{t_0}^\infty |q(t)|^2 e^{-4\lambda t} e^{4\lambda t} dt \\ \leq & \int_{t_0}^\infty c|q(t)|^2 dt \leq c + \int_{T_0}^\infty |q(t)| dt. \end{aligned}$$

The last inequality follows, since in a neighborhood of ∞ we have $|q| < 1$, which gives $|q|^2 \leq |q|$. Furthermore, (4.3.4) also gives $Q_\lambda(t) e^{2\lambda t} \to 0$ as $t \to \infty$, since it implies

$$|e^{2\lambda t} Q_\lambda(t)| \leq c|q(t)|,$$

for every t and by our assumption $q \to 0$ as $t \to \infty$.

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Chapter 5

The L^p-spectrum over Kleinian Groups

In [DST88] Davies, Simon and Taylor studied the L^p-spectrum of the Laplace-Beltrami operator Δ_Γ on non compact quotients $M = \mathbb{H}^{N+1}/\Gamma$, where Γ is a geometrically finite group. Under the additional assumptions that M is either of finite volume or cusp-free, they determine explicitly the L^p-spectrum of Δ_Γ for $1 \leq p \leq \infty$. More precisely they proved.

Theorem 5.0.1 ([DST88] Theorem 8,9)**.** *Let $\{E_0, \ldots, E_m\}$ be a finite set of eigenvalues of $\Delta_{\Gamma,2}$ such that $E_j < \frac{N^2}{4}$. Then, if M has no cusps or has finite volume then*

$$\sigma(p, 0, \Delta_\Gamma) = \{E_0, \ldots, E_m\} \cup Q_p,$$

where Q_p is a parabolic region in the complex plane.

In this chapter our main goal is to generalize the above Theorem for the Laplacian on forms $\vec{\Delta}_\Gamma$ in the case where M has no cusps. In order to do this we define and study the Laplacian on forms $\vec{\Delta}_\Gamma$ on quotient spaces $M = \mathbb{H}^{N+1}/\Gamma$. Let us note that for simplicity, throughout this chapter where it is clear from the context, we denote the Laplacian on forms $\vec{\Delta}_\Gamma$ by Δ_Γ also. The main idea of the proof and which is based on the argument in [DST88], is to split the Laplacian on forms into two operators, one corresponding to the span of eigenforms with eigenvalues in the discrete isolated spectrum, and the second one acting on the quotient. The result we prove is the following:

Theorem. *Let $M = \mathbb{H}^{N+1}/\Gamma$, where Γ be a geometrically finite group and M be of*

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infinite volume with no cusps. In addition, assume that the set of isolated eigenvalues in the spectrum of Laplacian on L^2-integrable k-forms is finite, and consists of the points $\{E_0, \ldots, E_m\}$ for any $k \neq \frac{N+1}{2}$. Then, for $1 \leq p < \infty$ and $k < \frac{N+1}{2}$

$$\sigma(p, k, \Delta_\Gamma) = \{E_0, \ldots, E_m\} \cup Q'_{p,k}$$

and for $k > \frac{N+1}{2}$

$$\sigma(p, k, \Delta_\Gamma) = \sigma(p, n-k, \Delta_\Gamma).$$

Let us note that in this chapter where it is clear from the context we use L^p to denote $L^p(\Lambda^k(M))$. Also, in this chapter, we will use $n = N+1$ for the dimension of hyperbolic space to coincide with other literature.

We start by introducing the Laplacian and heat semigroup on M in Section 5.1 together with some basic properties, and then proceed to the technical parts for the proof of theorem in Section 5.2.

5.1 Γ-invariant forms

In the case of submanifolds and products of Riemannian manifolds the induced Riemannian metric is defined with respect to the original metrics. This is not the case for quotients of Riemannian manifolds. In order to establish a Riemannian metric on the quotient space we have to impose extra conditions. We have the following.

Theorem 5.1.1 ([Lee18] Proposition 2.32)**.** *Suppose (M, g) is a Riemannian manifold, and Γ is a discrete Lie group acting smoothly, freely, properly and isometrically on M. Then M/Γ has a unique Riemannian metric such that the quotient map $q : M \to M/\Gamma$ is a normal Riemannian covering.*

Now we will define the spaces that we will study in this chapter. Let $\mathbb{H}^{N+1}$ and Γ be a discrete group of $\mathrm{Isom}(\mathbb{H}^{N+1})$ such that Γ acts freely and properly on $\mathbb{H}^{N+1}$. Then by Theorem 5.1.1 the orbit space $M = \mathbb{H}^{N+1}/\Gamma$ is a Riemannian manifold such that the quotient map $q : \mathbb{H}^{N+1} \to \mathbb{H}^{N+1}/\Gamma$ is a Riemannian covering map. Recall that a Riemannian covering map is a smooth covering which is also a local isometry.

Our aim in this section is to define Γ-invariant k-forms on $\mathbb{H}^{N+1}/\Gamma$. For the convenience to the reader we proceed firstly with 0-forms.

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Let $f : \mathbb{H}^{N+1} \to \mathbb{C}$, f is called Γ-invariant if $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$ and $x \in \mathbb{H}^{N+1}$. For any Γ-invariant function f on $\mathbb{H}^{N+1}$ we define the induced function

$$\tilde{f} : \mathbb{H}^{N+1}/\Gamma \to \mathbb{C}$$

by

$$\tilde{f}(\Gamma x) = f(x),$$

where Γx is the orbit of x. If we set

$$\mathcal{F}_\Gamma(\mathbb{H}^{N+1}) = \{\text{all } \Gamma\text{-invariant functions } \; f : \mathbb{H}^{N+1} \to \mathbb{C}\}$$

this induces the map

$$\begin{aligned} T : \mathcal{F}_\Gamma(\mathbb{H}^{N+1}) &\longrightarrow \mathcal{F}(\mathbb{H}^{N+1}/\Gamma) \\ f &\longmapsto \;\; \tilde{f}. \end{aligned}$$

Let us note the following relation of T and $q : \mathbb{H}^{N+1} \to \mathbb{H}^{N+1}/\Gamma$ holds

$$T(f)(q(x)) = \tilde{f}(\Gamma x) = f(x).$$

Denote by Δ the Laplacian on $\mathbb{H}^{N+1}$ and by Δ_Γ the Laplacian on $\mathbb{H}^{N+1}/\Gamma$. The Laplacian Δ_Γ is well defined due to the Γ-invariance of the Laplacian under the isometries, $\Delta\gamma^* = \gamma^*\Delta$, $\gamma \in \mathrm{Isom}(\mathbb{H}^{N+1})$, here $\gamma^* f = f \circ \gamma$ is the pull back.

Now we see how the above extends to k-forms. A k-form ω is Γ-invariant if $\gamma^*\omega = \omega$. Here the pull back on forms is defined by

$$\begin{aligned} \gamma^* :&\Lambda^k(M) \to \Lambda^k(M) \\ (\gamma^*(\omega))_x &= \omega_{\gamma(x)} \circ (d\gamma_x \times \cdots \times d\gamma_x), \end{aligned}$$

(see [Tu11]). Also we denote by Δ the Laplacian on forms on $\mathbb{H}^{N+1}$ and by Δ_Γ the Laplacian on forms on $\mathbb{H}^{N+1}/\Gamma$. Similarly, Δ_Γ is well-defined due to the Γ-invariance of Δ under isometries, $\Delta\gamma^* = \gamma^*\Delta$, $\gamma \in \mathrm{Isom}(\mathbb{H}^{N+1})$ (see [CPR01] Proposition 2.11).

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For any Γ-invariant form ω on $\mathbb{H}^{N+1}$ we define the induced form

$$\tilde{\omega} : \mathbb{H}^{N+1}/\Gamma \to \mathbb{C}$$

by

$$\tilde{\omega}(\Gamma x) = \omega(x),$$

where Γx is the orbit of x. If we set

$$\mathcal{A}_\Gamma(\mathbb{H}^{N+1}) = \{\text{all } \Gamma\text{-invariant forms } \omega \text{ on } \mathbb{H}^{N+1}\}$$

this induces the map

$$\begin{aligned} T : \mathcal{A}_\Gamma(\mathbb{H}^{N+1}) &\longrightarrow \mathcal{A}(\mathbb{H}^{N+1}/\Gamma) \\ \omega &\longmapsto \ \tilde{\omega}. \end{aligned}$$

Note that the following relation between T and $q : \mathbb{H}^{N+1} \to \mathbb{H}^{N+1}/\Gamma$ holds

$$T(\omega)(q(x)) = \tilde{\omega}(\Gamma x) = \omega(x).$$

It is well-known that the relationship

$$e^{-t\Delta_\Gamma}(Tf) = T(e^{-t\Delta} f),$$

holds for the heat semigroups corresponding to Δ and Δ_Γ on Γ-invariant functions f over $\mathbb{H}^{N+1}$ (see Corollary 3 in [DM88], Lemma 2.14 in [Web07]). A similar relationship also holds for Γ-invariant forms.

Proposition 5.1.1. *Let ω be a continuous Γ-invariant k-form on $\mathbb{H}^{N+1}$ such that the restriction of ω to a fundamental domain $F \subset \mathbb{H}^{N+1}$ for Γ has compact support, then*

$$e^{-t\Delta_\Gamma}(T\omega) = T(e^{-t\Delta}\omega).$$

Proof. The proposition is based on the uniqueness, of solutions as stated in Corollary 2.5.1 of the heat equation on forms. Let ω be a continuous Γ-invariant k-form on $\mathbb{H}^{N+1}$ such that the restriction of ω to a fundamental domain $F \subset \mathbb{H}^{N+1}$ for Γ has compact

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support. We define $\omega_t(x)$, $\eta_t(x)$ to be solutions of the Cauchy problems

$$\begin{cases} (\partial_t + \Delta)\omega_t(x) = 0 \\ \omega_0(x) = \omega(x) \end{cases} \tag{5.1.1}$$

on $\mathbb{H}^{N+1}$, and

$$\begin{cases} (\partial_t + \Delta_\Gamma)\eta_t(\tilde{x}) = 0 \\ \eta_0(\tilde{x}) = T\omega(\tilde{x}) \end{cases} \tag{5.1.2}$$

respectively on $\mathbb{H}^{N+1}/\Gamma$. The solutions are given by

$$\omega_t(x) = (e^{-t\Delta}\omega)(x) \tag{5.1.3}$$

and

$$\eta_t(\tilde{x}) = (e^{-t\Delta_\Gamma}T\omega)(\tilde{x}). \tag{5.1.4}$$

So, if $\omega_t(x)$ were Γ-invariant, then, by applying T on both sides of (5.1.3) would have

$$\omega_t(x) = (T\omega_t) \circ q(x) = T(e^{-t\Delta}\omega) \circ q(x). \tag{5.1.5}$$

Moreover, if

$$\omega_t(x) = \eta_t \circ q(x) \tag{5.1.6}$$

holds, in other words if $\eta_t \circ q(x)$ solves (5.1.1), then by the uniqueness of solutions

$$\eta_t \circ q(x) = (e^{-t\Delta_\Gamma}T\omega)(q(x)) = \omega_t(x) = T(e^{-t\Delta}\omega) \circ q(x). \tag{5.1.7}$$

In other words, we get the desired equality

$$e^{-t\Delta_\Gamma}(T\omega) = T(e^{-t\Delta}\omega).$$

So it is enough to prove that the solution $\omega_t(x)$ is Γ-invariant and $\omega_t(x) = \eta_t \circ q(x)$. Let us begin by showing that $\omega_t(x)$ is Γ-invariant. By the definition of the heat kernel

$$\omega_t(x) = \int_{\mathbb{H}^{N+1}} < \vec{p}(t, x, y), \omega(y) > dy,$$

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where $\vec{p}(t,x,y)$ is the heat kernel on forms on $\mathbb{H}^{N+1}$. We compute

$$\begin{aligned}\gamma^*\omega_t(x) = \omega_t(\gamma x) &= \int_{\mathbb{H}^{N+1}} < \vec{p}(t,\gamma x,y), \omega(y) > dy \\ &= \int_{\mathbb{H}^{N+1}} < \vec{p}(t,x,\gamma^{-1}y), \omega(y) > dy \\ &= \int_{\mathbb{H}^{N+1}} < \vec{p}(t,x,y), \omega(\gamma y) > dy \\ &= \int_{\mathbb{H}^{N+1}} < \vec{p}(t,x,y), \omega(y) > dy\end{aligned}$$

where in the second equality we have used Proposition 2.5.3, in the third equality we have used Proposition 2.5.2 and the last equality follows by the Γ-invariance of ω.

Finally, by the Γ-invariance of Δ and that q is a Riemannian covering we get

$$\Delta(\eta_t \circ q(x)) = (\Delta_\Gamma\, \eta_t) \circ q(x).$$

Combining this with (5.1.2) we get

$$-\Delta(\eta_t \circ q(x)) = \partial_t(\eta_t \circ q(x))$$

Thus, $\eta_t \circ q(x)$ is a solution to the heat equation on $\mathbb{H}^{N+1}$. Here we use the fact that since ω is Γ-invariant it satisfies the initial condition also. The proposition follows by the uniqueness of (5.1.2). □

For the heat kernel of the Laplacian on functions over quotients spaces, it is well-known that

$$p_\Gamma(t,q(x),q(y)) = \sum_{\gamma\in\Gamma} p(t,x,\gamma y),$$

(see Corollary 3 in [DM88], Theorem 2.13 in [Web07]). We show that a similar result also holds for the heat kernel on forms.

Proposition 5.1.2. *Let $\vec{p}(t,x,y)$ be the heat kernel on forms for the Laplacian on $\mathbb{H}^{N+1}$ and $\vec{p}_\Gamma(t,x,y)$ be the heat kernel on forms on $M = \mathbb{H}^{N+1}/\Gamma$. Then,*

$$\vec{p}_\Gamma(t,q(x),q(y)) = \sum_{\gamma\in\Gamma} \vec{p}(t,x,\gamma y),$$

where $q : \mathbb{H}^{N+1} \to \mathbb{H}^{N+1}/\Gamma$ is the covering map.

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Proof. Let ω be a continuous Γ-invariant k-form on $\mathbb{H}^{N+1}$ with compact support in a Fundamental domain F on $\mathbb{H}^{N+1}$. We compute

$$\begin{aligned}
T(e^{-t\Delta}\omega)(q(x)) =& e^{-t\Delta}\omega(x) = \int_{\mathbb{H}^{N+1}} < \vec{p}(t,x,y), \omega(y) > dy \\
=& \sum_{\gamma\in\Gamma} \int_{\gamma F} < \vec{p}(t,x,y), \omega(y) > dy \\
=& \sum_{\gamma\in\Gamma} \int_{F} < \vec{p}(t,x,\gamma y), \omega(\gamma y) > dy \\
=& \int_{F} < \sum_{\gamma\in\Gamma} \vec{p}(t,x,\gamma y), \omega(y) > dy.
\end{aligned}$$

since ω is Γ-invariant. By definition

$$e^{-t\Delta_\Gamma}(T\omega)(q(x)) = \int_F < \vec{p}_\Gamma(t, q(x), q(y)), \omega(y) > dy.$$

The above two equalities together with Proposition 5.1.1 give

$$\int_F < \sum_{\gamma\in\Gamma} \vec{p}(t,x,\gamma y), \omega(y) > dy = \int_F < \vec{p}_\Gamma(t, q(x), q(y)), \omega(y) > dy$$

Now, the Proposition follows if we show that $\sum_{\gamma\in\Gamma} \vec{p}(t,x,\gamma y)$ is continuous on $(0,\infty) \otimes \Lambda^k(\mathbb{H}^{N+1}) \otimes \Lambda^k(\mathbb{H}^{N+1})$. For this, it suffices to show that $\sum_{\gamma\in\Gamma} |\vec{p}(t,x,\gamma y)|$ is continuous on $(0,\infty) \times \mathbb{H}^{N+1} \times \mathbb{H}^{N+1}$.

The procedure is the same as in the proof of Theorem 2.13 [Web07]. In order to show that $\sum_{\gamma\in\Gamma} |\vec{p}(t,x,\gamma y)|$ is continuous we have to show that the series converges uniformly on any $[a,b] \times B \times B$, with $[a,b] \subset \mathbb{R}$ and B be a compact set in $\mathbb{H}^{N+1}$. As observed in [Web07] the set

$$\Gamma(B,R) = \{\gamma \in \Gamma \ : d(B, \gamma B) \leq R\}$$

is finite for any $R > 0$. So, equivalently we have to show that $\sum_{\gamma\in\Gamma} |\vec{p}(t,x,\gamma y)|$ converges uniformly on $[a,b] \times B \times B$ to zero as $R \to \infty$. Using Proposition 2.5.1 we have

$$|\vec{p}(t,x,y)| \leq e^{K_2 t} p(t,x,y).$$

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Thus on $[a, b]$ we get

$$|\vec{p}(t, x, y)| \leq e^{K_2 b} p(t, x, y). \tag{5.1.8}$$

Weber proves that

$$\sum_{\gamma \in \Gamma \backslash \Gamma(B,R)} p(t, x, \gamma y) \leq c a^{-\frac{n}{2}} \sum_{m=1}^{\infty} e^{(n-1)\sqrt{-k}R} e^{\left(\frac{-m^2 R^2}{2Db}\right)}$$

on $[a, b]$ for some $D > 2$ where c depends only on D and on k (see Lemma 2.15 in [Web07]). As a result the last series is uniformly convergent with respect to $R > 1$, independently of (t, x, y), and converges to zero as $R \to \infty$. Thus using (5.1.8) and the last argument we have finally that

$$\sum_{\gamma \in \Gamma \backslash \Gamma(B,R)} |\vec{p}(t, x, \gamma y)|$$

converges uniformly on $[a, b] \times B \times B$ to zero as $R \to \infty$. $\square$

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5.2 The L^p-spectrum on a class of Kleinian Groups

In this section the spaces under consideration will be the orbit space $M = \mathbb{H}^{N+1}/\Gamma$, where Γ is a geometrically finite group and M is of infinite volume with no cusps. For the sake of completeness we give the following definitions.

In order to define the concept of the geometrically finite group we will use the open unit model B^{N+1} of the hyperbolic space $\mathbb{H}^{N+1}$. Recall that the isometry group of the hyperbolic space $\mathbb{H}^{N+1}$ and the group $M(B^{N+1})$ of Mobius transformations of B^{N+1} are isomorphic (see Corollary 1 p.130 in [RAR94]).

Definition 5.2.1 ([RAR94] p.603)**.** *A convex polyhedron P in B^{N+1} is geometrically finite if and only if for each point x of $\bar{P} \cap S^N$ there is an open neighborhood N of x in the Euclidean space E^{N+1} that meets only the sides of P incident with x.*

Definition 5.2.2 ([RAR94] p.612)**.** *A discrete subgroup Γ of $M(B^{N+1})$ is geometrically finite if and only if Γ has a geometrically finite, exact, convex, fundamental polyhedron.*

As noted in [RAR94] on p.612, in dimensions $1, 2, 3$ one can define Γ to be geometrically finite if the polyhedra are finite-sided instead of geometrically finite. But in dimensions 4 and above, there are examples of polyhedra with an infinite number of sides (see Example 5 in [RAR94] p. 618). As shown in [RAR94] in dimensions $1, 2, 3$ the two definitions coincide. Finally, we recall that a Kleinian group Γ is defined to be a (countable) discrete subgroup of the isometry group of the hyperbolic space.

On L^p integrable k-forms $\Delta_{\Gamma,p}$ is defined via its semigroup as in the previous section. For $p = 2$ we will use the notation Δ_Γ for simplicity. Keeping similar notation with the previous Chapter we will denote by $P'_{p,k}$ the parabolic curve

$$P'_{p,k} = \{-(\frac{N}{p} - k + is)[N(\frac{1}{p} - 1) + k + is], s \in \mathbb{R}\},$$

and the parabolic region to the right of the curve $P'_{p,k}$ by

$$Q'_{p,k} = \{\left(\frac{N}{2} - k\right)^2 + z^2 : |\mathrm{Im} z| \leq N|\frac{1}{p} - \frac{1}{2}|\}. \tag{5.2.1}$$

These are the parabolas from the previous Chapter which corresponds to a manifold of dimension $n = N + 1$ and with limiting curvature at infinity $-a_0 = -1$, as is the case of $\mathbb{H}^{N+1}$.

Mazzeo and Phillips [MP90] computed the L^2 spectrum of Laplacian Δ_Γ for quotients $M = \mathbb{H}^{N+1}/\Gamma$, where Γ is a geometrically finite group, proving the following result.

Theorem 5.2.1 ([MP90] Theorem 1.11)**.** *When M is not compact the essential spectrum consists of the entire interval $[(N/2-k)^2, \infty)$, when $k \leq \frac{N+1}{2}$ and $[(N/2-k+1)^2, \infty)$, when $k \geq \frac{N+1}{2}$. The only exception is when $k = \frac{N+1}{2}$ and M has infinite volume. In that case in addition to the above, 0 is an eigenvalue of infinite multiplicity.*

By definition every point outside the essential spectrum in an isolated eigenvalue of finite multiplicity. Lax and Phillips [LP82] showed that for the Laplacian on functions for quotients $M = \mathbb{H}^{N+1}/\Gamma$, where Γ is a geometrically finite group, $\sigma(2,0,\Delta_\Gamma) \setminus \sigma_{\text{ess}}(2,0,\Delta_\Gamma)$ must be a finite set of eigenvalues of finite multiplicity. We expect that a similar result must hold for the Laplacian on forms, but this is still an open problem. We will see that if we make this additional assumption on the set of isolated eigenvalues of finite multiplicity for the manifold we can precisely compute the L^p-spectrum of the Laplacian on k-forms.

In this section we compute the L^p-spectrum of the Laplacian on k-forms. We will prove the following result.

Theorem 5.2.2. *Let $M = \mathbb{H}^{N+1}/\Gamma$, where Γ is a geometrically finite group and M has infinite volume and no cusps. In addition, assume that the set of isolated eigenvalues in the spectrum of Laplacian on L^2-integrable k-forms is finite, and consists of the points $\{E_0, \ldots, E_m\}$ for any $k \neq \frac{N+1}{2}$. Then, for $1 \leq p < \infty$ and $k < \frac{N+1}{2}$*

$$\sigma(p,k,\Delta_\Gamma) = \{E_0, \ldots, E_m\} \cup Q'_{p,k}$$

and for $k > \frac{N+1}{2}$

$$\sigma(p,k,\Delta_\Gamma) = \sigma(p,n-k,\Delta_\Gamma).$$

Our proof extends the method defined in [DST88], and relies on various properties of the resolvent operator and the heat kernel which we develop below. We will prove the result for $0 \leq k \leq \frac{N}{2}$, and for $k > \frac{N+1}{2}$ it will follow by Poincare duality.

Here we will be a little bit more precise and keep the sub index Δ_p for the Laplacian on $L^p(\Lambda^k(\mathbb{H}^{N+1}))$ and write $\Delta_{\Gamma,p}$ for the Laplacian on $L^p(\Lambda^k(M))$.

Lemma 5.2.3. *Let $0 \leq k \leq N/2$. Whenever $|Imz| > N/2$, the resolvent operator on forms $(\Delta_1 - (N/2-k)^2 - z^2)^{-1}$ is bounded on $L^1(\Lambda^k(\mathbb{H}^{N+1}))$.*

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Proof. This is obtained immediately from Theorem 4.2.1 as shown in [CL24], since the Hyperbolic space $\mathbb{H}^{N+1}$ has constant negative curvature -1 and rate of volume growth $\gamma = N$. Thus $|Imz| > \gamma|1/p - 1/2|$ reduces to $|Imz| > N/2$. $\square$

Proposition 5.2.1. *Let $0 \leq k \leq N/2$. Whenever $|Imz| > N/2$, the resolvent operator $(\Delta_{\Gamma,1} - (N/2-k)^2 - z^2)^{-1}$ is bounded on $L^1(\Lambda^k(M))$.*

Proof. The map $T : L^1(\mathbb{H}^{N+1}) \to L^1(M)$ has an adjoint which is an one to one map from $L^\infty(M) \to L^\infty(\mathbb{H}^{N+1})$. As a result, T is an onto map from $L^1(\mathbb{H}^{N+1})$ to $L^1(M)$. For any $\lambda \notin \sigma(k, 1, \Delta_{\Gamma,1})$ we have the functional analytic formula

$$(\Delta_{\Gamma,1} - \lambda)^{-1} = \int_0^\infty e^{-t\Delta_{\Gamma,1}} e^{\lambda t} dt,$$

and a similar formula for $(\Delta_1 - \lambda)^{-1}$ for any $\lambda \notin \sigma(k, 1, \Delta)$. By Proposition 5.1.1 after restricting the heat kernel on L^1, we get

$$(\Delta_{\Gamma,1} - \lambda)^{-1} T\omega = T(\Delta_1 - \lambda)^{-1}\omega$$

for any $\lambda \notin \sigma(k, 1, \Delta)$. Using the fact that

$$||(\Delta_{\Gamma,1} - \lambda)^{-1} T\omega||_{L^1(M)} \leq ||(\Delta_{\Gamma,1} - \lambda)^{-1}||_{\{L^1 \to L^1\}}\, ||T\omega||_{L^1(M)}$$

we have

$$||(\Delta_{\Gamma,1} - \lambda)^{-1}||_{\{L^1 \to L^1\}} \leq ||(\Delta_1 - \lambda)^{-1}||_{\{L^1 \to L^1\}}$$

for all $\lambda \notin \sigma(k, 1, \Delta)$. The proposition follows by combining this with Lemma 5.2.3. $\square$

We will now prove that the heat operator corresponding to Δ_Γ is bounded from L^2 to L^∞, it is in other words ultracontractive.

Proposition 5.2.2. *There exists $c(t) = c > 0$ such that*

$$||e^{-\Delta_\Gamma t}\omega||_\infty \leq c||\omega||_2, \text{ for every } \omega \in L^2(\Lambda^k(M)).$$

Proof. The proof is similar to Proposition 4 in [DST88]. Firstly, we will show that $e^{-t\Delta_\Gamma} : L^1(\Lambda^k(M)) \to L^\infty(\Lambda^k(M))$ is bounded. Let us note that

$$||e^{-\Delta_\Gamma t}\omega||_\infty \leq \left\| \int_M |\vec{p}_\Gamma(t, z, w)|\, |\omega(w)| dw \right\|_\infty, \tag{5.2.2}$$

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hence it suffices to show that $|\vec{p}_\Gamma(t,z,w)|$ is uniformly bounded with respect to z, w. Since on $\mathbb{H}^{N+1}/\Gamma$ the Weitzenbock tensor on k-forms is bounded below by a negative constant $-K_2$, by Proposition 2.5.1 we have

$$|\vec{p}_\Gamma(t,z,w)| \leq e^{tK_2}|p_\Gamma(t,z,w)|, \tag{5.2.3}$$

where p_Γ is the heat kernel on functions over $\mathbb{H}^{N+1}/\Gamma$. So it suffices to estimate the heat kernel on functions $p_\Gamma(t,z,w)$. This computed in Proposition 4 in [DST88]. For the sake of completeness we work out the details. If $0 < t \leq 1$, then the heat kernel on functions for $\mathbb{H}^{N+1}$ satisfies the estimate

$$0 < p(t,z,w) \leq c_0 t^{-\frac{N+1}{2}} e^{-\frac{N\rho}{2}} e^{-\frac{\rho^2}{4t}} (1+\rho)^{\frac{N}{2}}$$

as shown in [DM88], where ρ is the hyperbolic distance from z to w. Thus,

$$0 < p_\Gamma(t,z,z) \leq c_1 t^{-\frac{N+1}{2}} \sum_{\gamma\in\Gamma} e^{-\frac{\rho(z,\gamma z)^2}{4t}}.$$

Since M has no cusps and has constant curvature, it has bounded geometry (injectivity radius uniformly bounded below), therefore, the sum is bounded independently of z by [DM88], [Pat76]. So, the above with the well-known property of the heat kernel

$$p(t,z,w) \leq \sqrt{p(t,z,z)p(t,w,w)},$$

(see Exercise 7.21 in [Gri09]) gives

$$0 < p_\Gamma(t,z,w) \leq ct^{-\frac{N+1}{2}}.$$

Now, we show

$$e^{-t\Delta_\Gamma} : L^\infty(\Lambda^k(M)) \to L^\infty(\Lambda^k(M))$$

is bounded. From (5.2.2) it suffices to show

$$\sup_z \int_M |\vec{p}_\Gamma(t,z,w)|dw \leq c(t),$$

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where $c(t)$ is a uniform constant depending on t. By (5.2.3) and

$$\sup_z \int_M |p_\Gamma(t,z,w)| dw \leq 1,$$

we get

$$\sup_z \int_M |\vec{p}_\Gamma(t,z,w)| dw \leq \sup_z \int_M e^{K_2 t} |p_\Gamma(t,z,w)| dw \leq c(t),$$

where $c(t)$ is any constant depending on t. Since $e^{-t\Delta_\Gamma}$ is bounded from $L^1(\Lambda^k(M))$ to $L^\infty(\Lambda^k(M))$ and from $L^\infty(\Lambda^k(M))$ to $L^\infty(\Lambda^k(M))$, using interpolation (see Lemma 2.6.1) we get that

$$e^{-t\Delta_\Gamma} : L^2(\Lambda^k(M)) \to L^\infty(\Lambda^k(M))$$

is also bounded. □

Remark 5.2.4. *Proposition 5.2.2 can be obtained alternatively by using Proposition 5.1.2 in combination with the same estimates for $p_\Gamma(t,z,w)$ as proved in Proposition 4 [DST88].*

Corollary 5.2.1. *If ω is an L^2 k-eigenform for the Laplacian on $\mathbb{H}^{N+1}/\Gamma$, then $\omega \in L^\infty(\Lambda^k(M))$.*

Proof. Suppose that $\omega \in L^2$ be an eigenform of Δ_Γ with eigenvalue E. Then ω satisfies $e^{-t\Delta_\Gamma}\omega = e^{-tE}\omega$ for every $t > 0$. For $t = 1$, and using Proposition 5.2.2 we have

$$||e^{-E}\omega||_\infty = ||e^{-\Delta_\Gamma}\omega||_\infty \leq c||\omega||_2.$$

Hence

$$||\omega||_\infty \leq e^E c||\omega||_2,$$

which gives the corollary. □

Let us note that if $\mathbb{H}^{N+1}/\Gamma$ has cusps but it is of infinite volume, then as noted in [DST88], Fourier analysis shows that ϕ_0 the L^2 eigenfunction corresponding to the first eigenvalue, diverges to ∞ in each cusp. Although this is not known, it is expected that this would also happen for eigenforms. Our assumption that $\mathbb{H}^{N+1}/\Gamma$ is cusp-free, seems to therefore be necessary to conclude that every L^2- eigenform must also belong to L^p for all $p \in [2, \infty]$. We will now move to address the case $p \in [1, 2]$.

In the following Lemma we will compute one inclusion which the spectrum of Δ_Γ on $L^p(\Lambda^k(M))$ satisfies.

Lemma 5.2.5. *If $1 \leq p \leq 2$, then*

$$\sigma(k,p,\Delta_\Gamma) \subset \{E_0,\ldots,E_m\} \cup Q'_{p,k}.$$

Proof. Denote by ϕ_r the L^2-eigenform corresponding to the eigenvalue E_r. By Corollary 5.2.1 $\phi_r \in L^\infty$. Thus, using interpolation we also have that $\phi_r \in L^q$ for all $q \geq 2$.

Let $1 \leq p \leq 2$. Since $\phi_r \in L^q$ for every $q \geq 2$, we also have that $\phi_r \in L^{p^*}$ for $\frac{1}{p}+\frac{1}{p^*}=1$. So for any $\omega \in L^p$ the $L^p - L^{p^*}$ pairing

$$(\omega,\phi_r) = \int <\omega,\phi_r> dV_g$$

is well defined and gives an operator $\tilde{\phi}_r \in L^{p^*}$ such that $\tilde{\phi}_r(\omega) = (\omega,\phi_r)$. Define the subspace L^p_1 of L^p by

$$L^p_1(\Lambda^k(M)) = \{\omega \in L^p(\Lambda^k(M)) : \int <\omega,\phi_r> dV_g = 0, \text{for all } r \in \{0,\ldots,m\}\}.$$

Since L^p_1 is a closed subspace of L^p, we have that L^2_1 is a Hilbert space. We will show that L^p_1 is invariant under $e^{-t\Delta_\Gamma}$. Let $\omega \in L^p_1$. Then $e^{-t\Delta_\Gamma}\omega \in L^p$ since the heat operator is bounded on L^p. Since $\tilde{\phi}_r \in L^{p^*}$,

$$\begin{aligned}
(\tilde{\phi}_r, e^{-t\Delta_\Gamma}\omega) &= \int <\phi_r, e^{-t\Delta_\Gamma}\omega> dV_g \\
&= \int <e^{-t\Delta_\Gamma}\phi_r, \omega> dV_g \\
&= \int e^{-tE_r} <\phi_r,\omega> dV_g = 0.
\end{aligned}$$

for all r, where we have used that the heat operator on L^{p^*} is the adjoint of the heat operator on L^p. Therefore $e^{-t\Delta_\Gamma}\omega \in L^p_1$. By the previous claim we have that, $e^{-\Delta_\Gamma t} \restriction$ is the subspace semigroup on L^p_1 (see Definition 2.2.2), and we define $\Delta_{p,1}$ to be its generator. By Proposition 2.2.1 $\Delta_{p,1}$ is the restriction of $\Delta_{\Gamma,p} \restriction_{L^p_1}$ with domain the intersection of the domain of $\Delta_{\Gamma,p}$ with L^p_1. For $p=1$ we have $\Delta_{1,1} = \Delta_{\Gamma,1} \restriction_{L^1_1}$. As a result, the bound of $\Delta_{\Gamma,1}$ from Proposition 5.2.1 holds on $\mathcal{D}(\Delta_{\Gamma,1}) \cap L^1_1$. This gives that

$$(\Delta_{1,1} - (N/2-k)^2 - z^2)^{-1}$$

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is bounded on L^1_1 for $|\mathrm{Im}(z)| > N/2$. By replacing z with iz we get that

$$(\Delta_{1,1} - (N/2-k)^2 + z^2)^{-1}$$

is bounded for $Rez > N/2$.

Now for the quotient $L^p_2 = L^p(\Lambda^k(M))/L^p_1$, we define the quotient semigroup $e^{-t\Delta_\Gamma}/$ (see Definition 2.2.3) and denote its generator by $\Delta_{p,2}$. Let us note that $\dim(L^p_2) = m+1$. This follows easily, if we set

$$\begin{aligned} T : L^p &\longrightarrow \mathbb{R}^{m+1} \\ \omega &\longmapsto \quad (\tilde{\phi}_0(\omega), \cdots, \tilde{\phi}_m(\omega)) \end{aligned}$$

and notice that $\mathrm{Ker}T = L^p_1$ and $\dim(\mathrm{Im}T) = m+1$. As a result

$$\sigma(p, k, \Delta_{p,2}) = \{E_0, \dots, E_m\}.$$

We will now demonstrate that the spectra of $\Delta_{\Gamma,p}$, $\Delta_{p,1}$ $\Delta_{p,2}$ are related in the following way

$$\sigma(p, k, \Delta_{\Gamma,p}) = \sigma(p, k, \Delta_{p,1}) \cup \sigma(p, k, \Delta_{p,2}). \tag{5.2.4}$$

To show this, first observe that since L^p_2 is finite dimensional, there exists an isomorphism $L^p(\Lambda^k(M)) = L^p_1 \oplus L^p_2$. Define

$$R_0 : L^p(\Lambda^k(M)) \to L^p(\Lambda^k(M))$$

by

$$R_0 = (\Delta_{\Gamma,1} + 1)^{-1},$$

and observe that R_0 leaves L^p_1 invariant, just as the heat operator does. Now define R_1 to be the restriction of R_0 to L^p_1 and R_2 to be the induced operator on the quotient space L^p_2. The isomorphism $L^p(\Lambda^k(M)) = L^p_1 \oplus L^p_2$ implies that

$$\sigma(R_0) = \sigma(R_1) \cup \sigma(R_2). \tag{5.2.5}$$

Then using the spectral mapping theorem for generators of one-parameter contraction

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semigroups [DST88] we get that (5.2.5) implies (5.2.4)

To complete the proof of the lemma it remains to compute $\sigma(p,k,\Delta_{p,1})$. Theorem 5.2.1 and the definition of $\Delta_{2,1}$ give

$$\sigma(2,k,\Delta_{2,1}) = [(N/2-k)^2, \infty).$$

By Lemma 2.1.5, since $H = \Delta_{2,1} - (\frac{N}{2}-k)^2$ is non-negative self-adjoint and its spectrum is contained in $[0,\infty)$ we get that

$$(\Delta_{2,1} - (N/2-k)^2 + z^2)^{-1}$$

is bounded on L^2_1 whenever Re$z > 0$. As we have shown above,

$$(\Delta_{1,1} - (N/2-k)^2 + z^2)^{-1}$$

is bounded on L^1_1 for $Rez > \frac{N}{2}$. We are now ready to show

$$\sigma(p,k,\Delta_{p,1}) = Q'_{p,k}, \text{ for } p \in [1,2]$$

This is a standard interpolation argument which we include for the sake of completion. By the above we have

$$\left(\Delta_{p,1} - (\frac{N}{2}-k)^2 + (x+iy)^2\right)^{-1} \tag{5.2.6}$$

is bounded on L^2_1 for $x > 0$ and $y \in \mathbb{R}$ and

$$\left(\Delta_{p,1} - (\frac{N}{2}-k)^2 + (x+iy)^2\right)^{-1} \tag{5.2.7}$$

is bounded on L^1_1 for $x > \frac{N}{2}$ and $y \in \mathbb{R}$. We fix $\epsilon > 0$ and $\alpha \in \mathbb{R}$ and define the operator

$$T(x+iy) = \left(\Delta_{p,1} - (\frac{N}{2}-k)^2 + \frac{N^2}{4}(x+\epsilon+iy+i\alpha)^2\right)^{-1}. \tag{5.2.8}$$

For $y \in \mathbb{R}$ since $\frac{N}{2}\epsilon > 0$ and $\frac{N}{2}(y+\alpha) \in \mathbb{R}$, (5.2.6) gives that $T(iy)$ is bounded on L^2_1 for every $y \in \mathbb{R}$. Also, for $y \in \mathbb{R}$ since $\frac{N}{2}+\epsilon > \frac{N}{2}$ and $\frac{N}{2}(y+\alpha) \in \mathbb{R}$, (5.2.7) gives that $T(1+iy)$ is bounded on L^1_1 for every $y \in \mathbb{R}$. Now, we fix any $p \in (1,2)$ and define the unique $t \in (0,1)$ such that

$$\frac{1}{p} = t + \frac{1}{2}(1-t). \tag{5.2.9}$$

The Stein Interpolation Theorem (see Lemma 2.6.1) with $p_0 = 2$, $p_1 = 1$, and setting $x = t$ and $y = 0$ in (5.2.8) gives that

$$T(t) = \left(\Delta_{p,1} - (\frac{N}{2} - k)^2 + \frac{N^2}{4}(t + \epsilon + i\alpha)^2\right)^{-1}$$

is bounded on L^p_1, $1 < p < 2$. Since $\epsilon > 0$ and $\alpha \in \mathbb{R}$ were arbitrary, we have that

$$\left(\Delta_{p,1} - (\frac{N}{2} - k)^2 + z^2\right)^{-1}$$

is bounded in L^p_1, for $1 < p < 2$ whenever $Rez > \frac{N}{2}t$. Since $t = \frac{2}{p} - 1$ we have that

$$\left(\Delta_{p,1} - (\frac{N}{2} - k)^2 + z^2\right)^{-1}$$

is bounded in L^p_1, for $1 < p < 2$, whenever $\mathrm{Re} z > (\frac{2}{p} - 1)\frac{N}{2} = (\frac{1}{p} - \frac{1}{2})N$. Replacing z with iz and z with $-z$ we get that

$$\sigma(p, k, \Delta_{p,1}) \subset \left\{ \left(\frac{N}{2} - k\right)^2 + z^2 : |\mathrm{Im} z| \leq \left(\frac{1}{p} - \frac{1}{2}\right) N \right\} = Q'_{p,k}.$$

$\square$

Now in order to get the reverse inclusion from the one given in the above Lemma, we need the following result.

Lemma 5.2.6. *If $1 \leq p \leq q \leq 2$, then the curve $P'_{p,k}$ which is the boundary of $Q'_{p,k}$ is contained in $\sigma(p, k, \Delta_{\Gamma,p})$.*

Proof. As we saw in the proof of Proposition 5.2.2 $e^{-\Delta_\Gamma t} : L^p \to L^q$ is bounded for every $1 \leq p \leq q \leq \infty$. Following the same analytical argument as in Proposition 3.1 in [HV86] we can show that for any $1 \leq p \leq q \leq 2$

$$\rho(\Delta_{\Gamma,q}) \supset \rho(\Delta_{\Gamma,p}).$$

As a result we have

$$\sigma(q, k, \Delta_{\Gamma,q}) \subset \sigma(p, k, \Delta_{\Gamma,p}),$$

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for any $1 \leq p \leq q \leq 2$. As in the proof of Theorem 4.1.2, it again suffices to prove

$$P'_{q,k} \subset \sigma(q, k, \Delta_{\Gamma,q}).$$

In other words, for any $\lambda \in P'_{q,k}$ and $\epsilon > 0$ we have to construct approximate eigenforms $\omega \in \Lambda^k(\mathbb{H}^{N+1}/\Gamma)$ such that

$$||\Delta_\Gamma - \lambda)\omega||_q \leq \epsilon||\omega||_q.$$

We let $\mathbb{H}^{N+1} = [0, \infty) \times S^N$ with metric $g = dr^2 + (sinhr)^2 d\sigma^2$. Since Γ is geometrically finite, and has infinite volume, there is a region $\Omega \subset S^N$ and $b > 0$ such that (b, ∞) is contained in a fundamental domain of $\mathbb{H}^{N+1}/\Gamma$.

On this region we consider approximate eigenforms of the type

$$\omega = \phi f^\mu(\chi(\theta)\eta_0) \wedge dr,$$

with $f(r) = sinhr$, $\chi(\theta) \in C_0^\infty(\Omega)$, $\phi = \phi(r) \in C_0^\infty((b, \infty))$, $\mu \in \mathbb{C}$ and $\eta_0 =$ closed $(k-1)-$ eigenform on S^N with $\Delta_S \eta_0 = \lambda_0 \eta_0$, where Δ_S is the Laplacian on S^N.

The procedure is the same as in Theorem 4.1.2. Firstly, we have to compute the action of Δ on $\omega = \phi f^\mu(\chi(\theta)\eta_0) \wedge dr$. Let $\eta_2 = \chi(\theta)\eta_0$, and $h(r) = \phi f^\mu$. Since η_2 is no longer a closed $(k-1)$-eigenform of Δ_S, we not can use Corollary 3.0.1. However, by Proposition 3.0.1 with $\omega_1 = 0$ and $\omega_2 = h(r)\eta_2$ we have

$$\begin{aligned}\Delta\omega = &+ hf^{-2}(\Delta_S\, \eta_2) \wedge dr + (-1)^k 2hf' f^{-1} d_S \eta_2 \\ &- [h'' + (N - 2k + 2)(hf' f^{-1})']\eta_2 \wedge dr\end{aligned}$$

Now using the formula

$$\Delta_S(\chi(\theta)\eta_0) = (\Delta_S \chi(\theta)) - 2\nabla_{\nabla\chi(\theta)}\eta_0 + \chi(\theta)(\Delta_S \eta_0)$$

and the fact $d_S(\chi(\theta)\eta_0) = (d_S \chi(\theta)) \wedge \eta_0$ we get

$$\begin{aligned}\Delta\omega =& hf^{-2}(\Delta_S \chi(\theta))\eta_0 \wedge dr + hf^{-2}(-2\nabla_{\nabla\chi(\theta)}\eta_0) \wedge dr \\ &+ hf^{-2}\chi(\theta)\lambda_0\eta_0 \wedge dr + (-1)^k 2hf^{-1} f' d_S(\chi(\theta)) \wedge \eta_0 \\ &- [h'' + (N - 2k + 2)(hf' f^{-1})']\chi(\theta)\eta_0 \wedge dr.\end{aligned}$$

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In other words,

$$\begin{aligned}\Delta\omega =&\Delta_2(\phi f^{\mu})\chi(\theta)\eta_0 \wedge dr\\ &+\phi f^{\mu-2}(\Delta_S\chi(\theta))\eta_0 \wedge dr - 2\phi f^{\mu-2}(\nabla_{\nabla\chi(\theta)}\eta_0) \wedge dr\\ &+(-1)^k\phi f^{\mu}(f^{-1}f')d_S(\chi(\theta)) \wedge \eta_0 \\ =&\Delta_2(\phi f^{\mu})\chi(\theta)\eta_0 \wedge dr\\ &+A_1 + A_2 + A_3.\end{aligned} \tag{5.2.10}$$

We will choose ϕ_ϵ as in the proof of Theorem 4.1.2, but $\chi(\theta)$ will be the same for every $\epsilon > 0$. Here μ, λ we take the values $\mu = -\frac{N}{p} + (k-1) + is$ for $s \in \mathbb{R}$ and $\lambda = -\mu(\mu + N - 2k + 2)$. As a result

$$\begin{aligned}||\Delta\omega - \lambda\omega||_q^q \leq &||(\Delta_2(\phi f^{\mu})\eta_0 \wedge dr - \lambda\phi f^{\mu}\eta_0 \wedge dr)\chi(\theta)||_q^q\\ &+||A_1||_q^q + ||A_2||_q^q + ||A_3||_q^q.\end{aligned}$$

Since $\chi(\theta)$ is a bounded function on S^N with

$$C_2 \leq \int_{S^N} |\eta_0|_{S^N}\chi(\theta) \leq C_1$$

the first term is estimated exactly as in the proof of Theorem 4.1.2 by finding the appropriate A_ϵ, B_ϵ for the domain of ϕ_ϵ. So, it remains to bound the last three terms. Now by

$$\int_{S^N} |(\Delta_S\chi(\theta))\eta_0 \wedge dr|_M = C\int_{S^N} |\eta_0|_S f^{-(k-1)},$$

$$\int_{S^N} |(\nabla_{\nabla\chi(\theta)}\eta_0) \wedge dr|_M = C\int_{S^N} |\eta_0|_S f^{-(k-1)},$$

and due to the additional factor of $f^{\mu-2}$ in front of them in (5.2.10), the estimates for A_1, A_2 are of the same type as V in Theorem 4.1.2. To estimate A_3 we observe

$$\int_{S^N} |d_S(\chi(\theta)) \wedge \eta_0|_M = C|\eta_0|_S f^{-k}.$$

As a result A_3 is also similar with V but with an additional factor of $f^{\mu-1}$ instead of $f^{\mu-2}$ in (5.2.10), which still allows us to make it as small as we want by taking the support of ϕ_ϵ as large as we want.

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So, by Proposition 2.6.1 we have shown that the points

$$\lambda = -\left(-\frac{N}{p} + (k-1) + is\right)\left(-\frac{N}{p} + (N+1-k) + is\right), \ s \in \mathbb{R}$$

belong to $\sigma(p, k, \Delta)$. Setting $k = N + 1 - m$, for $0 \leq m \leq (N+1)/2$, in the above equation and changing sign in both brackets we get

$$\lambda = -\left[N\left(\frac{1}{p} - 1\right) + m - is\right]\left[\frac{N}{p} - m - is\right], \ s \in \mathbb{R},$$

which are exactly the points of $P_{p,m}$. Thus, for $0 \leq m \leq (N+1)/2$ we have shown that $P_{p,m} \subset \sigma(p, N+1-m, \Delta) = \sigma(p, m, \Delta)$, where the equality follows by Poincare duality.

$\square$

Finally, we are ready to give the proof of Theorem 5.2.2.

Proof of Theorem 5.2.2. If $1 \leq p \leq 2$, then Lemma 5.2.5 gives

$$\sigma(k, p, \Delta_{\Gamma,p}) \subset \{E_0, \ldots, E_m\} \cup Q'_{p,k}.$$

and

$$\{E_0, \ldots, E_m\} \subset \sigma(k, p, \Delta_{\Gamma,p}).$$

Now Lemma 5.2.6 gives

$$Q'_{p,k} \subset \sigma(k, p, \Delta_{\Gamma,p})$$

and the proof of Theorem 5.2.2 follows.

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Conclusions

In the present thesis we studied the L^p-spectrum of the Laplacian on k-forms over Riemannian manifolds. More precisely, we computed the L^p-spectrum of the Laplacian on k-forms over certain Riemannian manifolds and as a consequence we have shown that the L^p-spectrum of the Laplacian on k-forms over these spaces is dependent on p.

In the first part, we dealt with the case where the Riemannian manifolds were warped products at infinity, and we proved that the L^p-spectrum of the Laplacian on k-forms contains a parabolic region which depends on k, p and the limiting curvature a_0 at infinity.

In the second part, we considered Riemannian manifolds M which were quotients of the hyperbolic space with a geometrically finite group, and such that M had infinite volume and no cusps. We have shown that the L^p-spectrum of the Laplacian on k-forms over M is exactly a parabolic region together with a set of isolated eigenvalues on the real line.

By the above results, we have extended the class of Riemannian manifolds, over which the L^p-spectrum of the Laplacian on k-forms depends on p. Since, the above Riemannian manifolds were negatively curved, we expect that the same phenomenon may occur in other similar negatively curved spaces.

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